Collapsing theorem for Delaunay complexes in non-general position and symmetry

Master thesis

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Abstract

Assuming *general position*, Bauer and Edelsbrunner show that the Čech and Delaunay complexes in $\mathbb{R}^n$ collapse into each other. By allowing non-unique solutions to certain minimal spheres, we bypass this assumption. Furthermore, using recursion, we prove an equivariant version of the collapsing theorem restricted to the plane assuming the point cloud has symmetry. We also discuss a simple symmetrical point cloud configuration in the plane that induces an indecomposable reducible representation on the barcode decomposition of a symmetric Delaunay complexes.

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Contents

Summary

Section 1 motivates the reader to read this thesis by introducing topological data analysis (TDA). This thesis studies a particular problem in TDA, and motivation for the research and exact content is given. Related works are discussed together with remarks about the methods used.

Section 2 introduces the reader to simplicial complexes' geometric and combinatorial nature. Images and discussion is provided for readers not familiar with the topic.

Section 3 defines the main example complexes of interest: the Čech, $\check{\text{Čech}}_r(X)$, Delaunay, $\text{Del}_r(X)$, and E-selective Delaunay, $\text{Del}_r(X, E)$, complexes. Figures show a few examples and an equivalent method to compute the complexes. This method is defined at the hand of minimal spheres called MEES.

Section 4 dives into the computation of MEES, which are equivalent to solving a quadratic optimisation problem. A theorem turns this quadratic optimisation into an affine combination problem with constraints. The implications for the MEES are discussed.

Section 5 describes a method to reduce the size of a simplicial complex while retaining the (simple) homotopy type. Such a reduction is called a collapse. The collapses are characterised in the existence of specific acylic pairings of elements in the complex called a discrete vector field.

Section 6 shows an example of reducing a selective Delaunay complex into a smaller one using a discrete vector field. Afterwards, the main theorem, stating that this procedure can be done more generally, is proven. A corollary is that the Čech complex collapses into the Delaunay complex.

Section 7 repeats the theory of the previous sections 2 to 5, specialised in the symmetric/equivariant case.

Section 8 is the symmetric/equivariant analogue of section 6. An example is showcased of how the general proving structure works, and the equivariant version of the main theorem is proven.

Section 9 shows corollaries of the main theorem and describes an example calculation of the persistent homology types of Delaunay complexes in a simple configuration. A question is raised about non-symmetric data which have a proper symmetric subset.

1 Introduction

1.1 Topological Data Analysis: intuitively

Extraction of information from large data sets that are incomplete and noisy is generally challenging. Topological data analysis (TDA) provides a general framework to analyze data in a manner that is robust to noise. As the observed data is not random, it is reasonable to assume that the data inherits some notion of shape.

An example of TDA

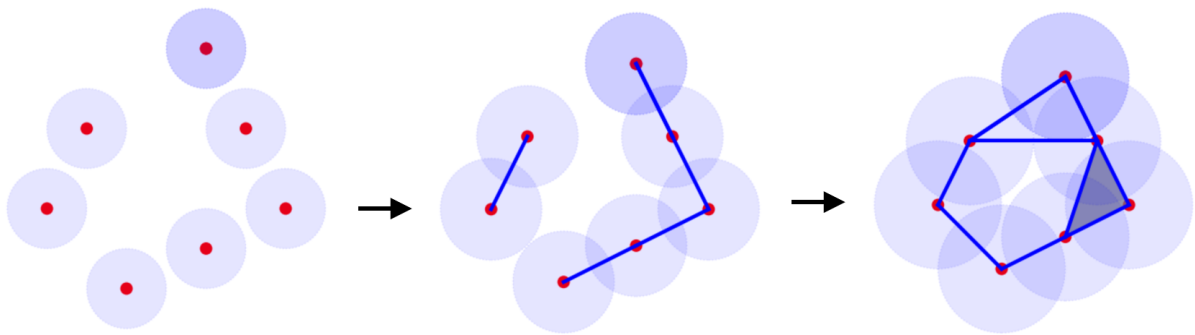


Figure 1: *The* Čech *method*

In Fig. 1 there are seven data points. It looks like the points have the shape of a circle, how can we make that concrete? By enlarging the size of the points, visualised as 'thickening', the resulting collection of disks looks like a thick circle. When two disks start to overlap, indicate this by a line. If three disks overlap, draw a triangle. The resulting shape is a collection of lines and triangles which approximately look like a version of the circle. Intuitively, it might be apparent that moving the points slightly, leads to a slightly different construction. This slight change makes the method robust to noise in the data.

Topology

One determinant of a shape is the number of its holes. In topology, this determinant is abstractly captured by the *homotopy type* [1] Chapter 4. The homotopy type is preserved when we allow the shape to be continuously deformed into another shape. In this sense, the disk and a point are considered equivalent, as the disk can be continuously shrunk into its centre. These deformations allow the shapes associated with data, to be shrunk into, or at least approximated by, simpler more tractable forms. These shapes consist of combinations of vertices, edges, triangles, tetrahedra and higher dimensional analogues, see Fig. 1. There are multiple approaches to connecting points with edges and triangles. In figure Fig. 1, the Čech approach is illustrated. The arising shape is called the Čech complex. The Čech complex is straightforward to compute but the downside is its size. However, there exists another similar complex, called the Delaunay complex, which negates the size issue of the Čech complex if the data satisfies some *general position* requirements.

The Delaunay approach

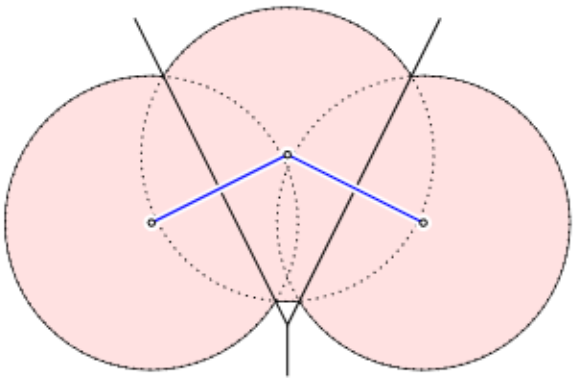


Figure 2: Figure is taken from [2] Fig. 2

Consider the three points in Fig. 2 which have been 'thickened' to create three disks similar to Fig. 1. This time, a restriction is put on the area of each disk; the disk cannot pass the line in between points. A blue edge is drawn if two restricted disks intersect. This is the Delaunay approach, where the disks are restricted/truncated, and the resulting figure is the Delaunay complex. This is different from the Čech complex because all three points have intersecting balls, but no triangle appears as the restricted disks do not intersect. Therefore, the Delaunay complex is smaller.

Topological equivalence

Surprisingly, the Čech and Delaunay approach, see Fig. 1 and Fig. 2 respectively, always result in the same number of holes. The first part of the thesis proves this by showing that the Čech complex can be deformed into the Delaunay complex while preserving the number of holes. The second part of this thesis is that this is also true if the points have symmetry.

1.2 Motivation for research

Topological data analysis: applications

The topic of this thesis is closely related to the field of Topological Data Analysis (TDA). It connects data analysis, algebraic topology and discrete geometry. An introduction can be found in [3]. The field has widespread applications in biology, chemistry, physics, psychology and many more. For example, consider that flocking of birds can be described as collective motion. These bird flocks have complicated 3D shapes which can be investigated using TDA [4].

Topological data analysis: theoretical tools

The theoretical most precise tool to find holes is to compute the homotopy groups, as in [1] chapter 4. These homotopy groups can be complicated to compute for high dimensional objects. Especially for the first homotopy group it is in general undecidable whether any given representation is isomorphic to another representation as it is equivalent to the group isomorphism problem, which is undecidable. Another way of detecting holes is through *homology theory* [1] Chapter 2. The homology theory can be quite abstract but is significantly simplified for CW and simplicial complexes. This simplification also leads to much easier computational implementations.
Unfortunately, the homology type of a particular complex is often unstable under perturbation and noise. A sequence of complexes $T_1 \subset T_2 \subset \dots$ ordered by inclusion at the hand of parameter $r = 1, 2, \dots$ can capture the changes in homology instead.[1] This is much more stable and called the *persistent homology type* [3] Chapter 5. The persistent homology (if finite) can be displayed in a canonical way, called the barcode [5] Theorem 1.2. In practice, having more parameters i, j allows more flexibility. The resulting persistent homology type is, unsurprisingly, called *multi-parameter* persistent homology. The downside

[1] One such example is seen in Fig. 1, where the black arrows indicate an inclusion.

is that these are not necessarily displayable in a canonical way, see [6] Chapter 4 for an introduction and overview.

Reduction in computation time

The computation time that algorithms take to compute persistent homology scales with the size of the complex. Therefore, since the Čech is much larger than the Delaunay complex it could help to first reduce the Čech complex to the Delaunay complex for computing persistent homology. This reduction has to be done in a way that does not change the persistent homology type. One way to reduce complexes is to sequentially remove pairs of simplices, called collapsing, in a specific manner that retains the homotopy type. That the Čech complex can be collapsed to the Delaunay complex is a result found in [7] Theorem 5.10. However, the result requires that the data is in 'General position', explained below.

'General position' assumptions

Often preconditions of the data are assumed when proving that theoretical methods work. In practice, these requirements are mild in the sense that the data is at most an arbitrarily small perturbation away from satisfying the requirements of general position. For example, points in a concyclic configuration are a small perturbation away of being not concyclic see Theorem 4.1 [8]. The generic name of these assumptions is 'general position' where the exact meaning of the term depends on the context. In our situation of data in Euclidean space $\mathbb{R}^p$, it means the following.

Definition 1.1. A set $X \subset \mathbb{R}^p$ is in **general position** if for at most $p+1$ points $\{p_0, \ldots, p_n\} = P \subset X$.

- **No Concyclicity**: The minimal circumsphere of P contains no points of $X - P$.
- **No affine dependence**: $P' = \{p_1 - p_0, \ldots, p_n - p_0\}$ is linearly independent.

In the plane, the first condition translates into circumcircles and the second into no colinearity. A non-example for points in the plane is seen in Figure Fig. 3. However, deciding whether there is a subset D of X in $\mathbb{R}^2$ with at least k points that satisfies the second condition is NP-complete, see Theorem 1 [9]. This signifies that the problem is at least as hard to solve computationally as other NP-hard problems.

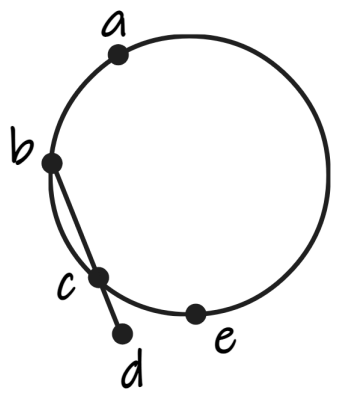


Figure 3: The points abc have the circumcircle which goes through e. Therefore, $abce$ is not in general position. Furthermore, bcd all lie on the same line. Therefore, they are affinely dependent. This is therefore a point set very much not in general position.

General position and symmetry

General position, particularly the second condition of concyclicity, excludes the case of possible rotational symmetry in the data. For example, it excludes a point set in a perfect square configuration, the orbit of a rotation group of 4 elements. Symmetrical configurations appear in applied areas, usually because of energy minimising properties. For example, fullerene is a carbon configuration which is symmetrical, whose topology and geometry have been studied at the hand of measurement data [10].

1.3 Extensions and contribution to the existing theory

The result for collapsing the Čech complex into the Delaunay complex is extended in this thesis by handling the case of non general position. The extension provides an affirmative answer to an open question posed in [7] on page 19.

Extension. Let $X \subset \mathbb{R}^n$ be finite and r non-negative. Then there exists a collapse

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X)$$

The approach used is similar as in [7] and consists of finding a discrete Morse function that induces the collapse, see Corollary 6.9.1 for the proof.
Unfortunately, if X has rotational or reflection symmetries, basic examples of data in non-general position, then the collapse is in general not symmetrical itself. To accommodate, the specific case of orthogonally symmetric planar data $X \subset \mathbb{R}^2$ is considered to show that the collapse can be symmetrical as well. The proof relies on a symmetric, more precisely *equivariant*, version of discrete Morse theory as discussed by [11].

Contribution. Let G be a finite group acting orthogonally, i.e. reflections and rotation, on $\mathbb{R}^2$. Let X be a finite G-set in $\mathbb{R}^2$ and r non-negative. Then there exists a G equivariant collapse

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow_G \mathrm{Del}_r(X)$$

This statement is proven in Corollary 8.19.1.[2]
We do not explore higher dimensions than the planar case. A complication, but not necessarily an obstruction in higher dimensions, is the much larger number of Euclidean symmetries that can appear.

Similar to [7] Theorem 5.9, both theorems arise as corollaries proving a similar result holds for complexes in between the Delaunay and Čech complexes. These complexes are called *E selective Delaunay complexes*, denoted as $\mathrm{Del}_r(X, E)$ where $E \subset X$. A corollary of these statements, as noted in [7] page 18, is an inclusion-like map between two Delaunay complexes with different point clouds $X \subset Y$. Specifically, we have the following commutative diagram.

$$\begin{tikzcd}
Del_r(X) \arrow[r, hook] \arrow[d, dashed] & Del_r(Y, X) \\
Del_r(Y) \arrow[ur, hook, "\sim"'] &
\end{tikzcd}$$

This map is natural in r, which also indicates that there is an induced map of filtrations and, hence, on the persistent homology and thus on the barcode.

A barcode decomposition for equivariant Delaunay complexes are computed. The decomposition is compared to the decomposition induced by the representation of the group on homology. The resulting group representation is reducible but indecomposable.

We end with a question on whether persistent homology data of a symmetrical set X is captured in a set $X \subset Y$ which is not symmetric.

1.4 Related Work

This thesis is mainly focused on extending results of [7]. The methods are similar, and, therefore, this thesis is similar. Another thesis describing Theorem 5.10 in [7] is found at [12]. The method of background exposition in chapters 1,2 and 4 of [12] and chapters 2 and 4 of [13] served as inspiration for the exposition in this thesis.

[2] The notation in Corollary 8.19.1 is different but logically equivalent.

Related complexes

To study the topology of data, a collection of hypothetical spaces is constructed out of the data. In short, the general TDA pipeline is: obtain data, construct a simplicial complex (or general space) out of data and then compute (multi-parameter) persistent homology. Focusing on the second step, there are many simplicial complexes that can be constructed out of data. A non-exhaustive list of well-known examples are the Čech, Delaunay (or alpha shape), Vietoris Rips, Witness complex [14] and Wrap complex [15]. Applications and introductions of the Čech, Vietoris Rips, and Witness complex can be found in Chapter 2 of [16]. There is also a notion of selective Delaunay complex, which lies in between the large Čech and smaller Delaunay complex, as introduced by [7] on page 7 and further generalised in Definition 2.21 in [12]. In practise, there is no 'best' complex, it all depends on the situation at hand. We expect that most discussion of this thesis translates well to the general version of the complexes defined in Definition 2.21 [12], although no claim is made that this is the case.

In this thesis, the most important complexes are the Čech and Delaunay complex. Both complexes accurately capture the homotopy type of the union of balls of a fixed radius around points. This result follows from a version of the nerve lemma, Theorem 3.1 in [17]; or, for the Delaunay complex, directly if X is in general position, see Theorem 3.2 in [18]. Between the Čech and Delaunay complex, the Delaunay complex has better properties if the points are in general position; it is bounded by the dimension of $\mathbb{R}^n$ [19] and it automatically provides a triangulation of the space. Both facts are often far from true for the Čech complex.

The Delaunay and Čech complexes are homotopy equivalent at a single radius r, Corollary 5.1.1. However, to conclude that their persistent homology types are isomorphic requires a particular identification to stay consistent with different choices of r. Such an explicit homotopy equivalence, in the case of general position, is found in [7] Corollary 6.1.

Discrete Morse theory

Discrete Morse theory can be used to shrink a simplicial complex into a smaller one while retaining the homotopy type. Specifically, it posits that certain functions, called discrete Morse functions, characterise when a CW/simplicial complex can be shrunk into a sub-complex via special deformation retractions. The special retractions are called *collapses*. These are retractions which remove a pair of cells/simplices such that the lower dimensional cell/simplex is not contained in any other cell/simplex. The homotopy type is retained as a collapse encodes a deformation retract.

Discrete Morse theory was first extensively introduced in [20] for CW complexes, and an accessible introduction is found in [21]. The "Morse" part comes from its analogy from the smooth case [22]. Therefore, many definitions find their name counterparts in smooth/continuous theory. Whereas a smooth Morse function turns a smooth manifold into a CW/simplicial complex, a discrete Morse function collapses a CW/simplicial complex into a sub-complex. Because algorithms that compute homology depend on the size of the complex, this theory substantially reduces overhead by first reducing the complex size [23]. This gives another way to view the main result of the thesis. Indeed, the Čech is much larger than the Delaunay complex, and, when computing homology type, overhead can be avoided by first collapsing to the Delaunay complex.

Actually, a collapse preserves another invariant, called the *simple homotopy type*. The study of the simple homotopy type is related to the question whether every complex that is contractible has a contraction induced by a series of elementary/simple retracts. The answer is negative, the Dunce hat [24] is a counter-example of a contractible simplicial complex which has no collapse. The study of the simple homotopy type for complexes was first (extensively) done in [25]. For more reading on simple homotopy theory, see [26] and [27].

2

Simplicial Complexes

In the previous section, motivation is given to study spaces that arise out of finite data X in $\mathbb{R}^n$. A straightforward approach is to consider collections of edges, triangles, tetrahedra, etc that connect points in X. These are called simplicial complexes.

Overview

This section discusses simplicial complexes with two important interpretations: geometrical and abstract. The advantage is that the abstract simplicial complex inherits a natural notion of topology via its geometric realisation. This section discusses literature, such as chapter 2 in [13], or section 3 in [28], and can be skipped if the reader is familiar with abstract simplicial complexes and their geometric realisation.

Figures supplement the discussion to make the reasoning more accessible.

Notation and assumptions

Every simplicial complex in this thesis is finite. Furthermore, a set finite set of points such as $\{a, b, c\}$, is denoted by abc. Lastly, we use the notation $Q + x$ to denote $Q \cup \{x\}$ and the notation $Q - x$ to denote the set of elements in Q with x removed. Note that sometimes $Q + x = Q$ and $Q - x = Q$.

2.1 Geometric simplicial complex

Geometric simplicial complexes arise as generalisations of polyhedra glued together. We first describe figures such as points, edges, triangles, tetraedra etc, see Fig. 4.

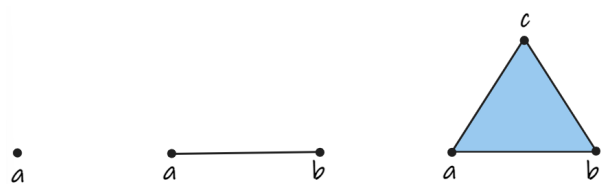


Figure 4: The geometric 0,1,2-simplices are drawn respectively from left to right. The simplices are created out of taking all convex combinations, i.e. the convex hull, of $\{a\}, \{a, b\}$ and $\{a, b, c\}$ respectively. Higher n-simplices can be constructed by adding a point in a linearly independent manner. A 3-simplex is a tetraeder and can not be embedded in $\mathbb{R}^2$ as 4 points can not be affinely independent in $\mathbb{R}^2$.

A set $B = \{v_0, \ldots, v_n\} \subset \mathbb{R}^p$ is said to be an **affine basis** if $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent. The **convex hull** of a set $X \subset \mathbb{R}^n$, denoted by $hull(X)$, is the following set of combinations of elements in X.

$$hull(X) = \{\sum_{x \in X} \lambda_x x | \lambda_x \geq 0, \quad \sum_{x \in X} \lambda_x = 1\}$$

A **geometrical p-simplex** Q is the convex hull of an affine basis of $p+1$ points P in $\mathbb{R}^p$. This simplex has the subset topology of $\mathbb{R}^p$. A **face** of Q is the convex hull of any subset F of the points P. Notice that a 2-simplex implicitly has three 1-simplices as faces and three 0-simplices as faces.

A collection of geometrical simplices defines a geometric simplicial complex as follows.

Definition 2.1. Let K be a collection of geometrical simplices in $\mathbb{R}^d$. Suppose further that

- If Q is a simplex and D a face, then $D \in K$, i.e. K is closed under taking faces.
- The intersection of two geometrical simplices in K is either a face of both simplices or empty.

Then K is called a **geometrical simplicial complex**

Fig. 5 illustrates an example of a geometrical simplicial complex in $\mathbb{R}^2$, while Fig. 6 provides a non example of a geometrical simplicial complex.

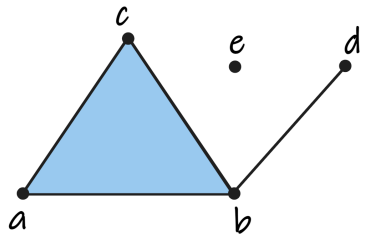


Figure 5: The geometric simplicial complex here contains one 2-simplex $\{abc\}$, four 1 simplices {ab,ac,bc,bd} and five 0-simplices $\{a, b, c, d, e\}$. This geometric simplicial complex is drawn in the plane, and each set of 3 vertices is affinely independent.

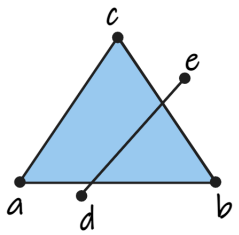


Figure 6: The simplex $\{bc\}$ and $\{de\}$ intersect at a point halfway through $\{bc\}$. If the collection of simplices was a geometric simplicial complex, then this point has to be included in the complex. Therefore, this is not a depiction of a geometric simplicial complex in $\mathbb{R}^2$. However, if we imagine $\{de\}$ to point outwards of the paper, then this defines a geometric simplicial complex in $\mathbb{R}^3$.

If K is a geometric simplicial complex, then $|K|$ is the union of its simplices. Note that $|K|$ inherits the subset topology from $\mathbb{R}^m$. The set of n-simplices in a simplicial complex K is called the **n-skeleton** of K and is denoted by K_n. The 0 skeleton K_0 is also called the vertex set.

Notice that a geometric p-simplex defines a geometric simplicial complex canonically. This is the **standard** p-simplex, denoted as Δ^p.[3]

Remark 2.2. Geometric simplicial complexes are generalised by CW complexes, see [1] Chapter 0 for more information on CW complexes. Part of the theory in Section 5 and Section 7 generalises to CW complexes. However, this thesis, does not consider the more general theory of CW complexes. If necessary to state a result, we occasionally refer to CW complexes.

[3] A geometrical simplicial complex K can be realised be seen as the result of an series of gluing operations of simplices. More precisely, let K be a geometric simplicial complex with vertex set $K_0 \subset \mathbb{R}^m$. We can construct the n-skeleton K_n out of $n-1$ skeleton K_{n-1}, by gluing all the n-simplices. Consider a single n-simplex $P \in K_n$. Suppose P is determined by $\{p_0, \ldots, p_n\}$ and identify these with the vertices of Δ^n. Notice that $|\Delta^n_{n-1}| \subset |\Delta^n|$. By our identification of P with the vertices of Δ^n we see that $\Delta^n \subset \mathbb{R}^m$. Thus, $\Delta^n_{n-1} \subset \Delta^n \subset \mathbb{R}^m$ and by our identification we see that $\Delta^n_{n-1} \subset K_{n-1}$. Defining $K_{n-1} \cup \Delta^n$ shows how to glue the n simplex P into K along its boundary. After repeated glueings, all n-simplices are added such that the new complex is the n-skeleton K_n

2.2 Abstract simplicial complex

If just the relations between the vertices is important, i.e. which vertices define a simplex, then the following definition is appropriate.

Definition 2.3. Given a finite set X, then an **abstract simplicial complex** $\mathbb{K}$ is a collection of subsets of X which is closed under taking subsets.

Example 1. An example of an abstract simplicial complex on $X = \{a, b, c, d, e\}$ is the following collection.

$$\mathbb{K} = \{a, b, c, d, e, \{a, b\}, \{b, c\}, \{a, c\}, \{b, d\}, \{a, b, c\}\}$$

Notice that this abstract simplicial complex describes the relations between vertices in Fig. 5.

Similar to geometric simplicial complexes, an element $P \in \mathbb{K}$ is a p**-simplex**, where $p = |P| - 1$ is called its **dimension**. If $D \subset P$, then D is a **face** of P and P is a **coface** of D. The **dimension** of an abstract simplicial complex is the same as the largest dimension of all of its p-simplices. A 0-simplex is called a **vertex**. The set of all 0-simplices is called the **vertex set** and denoted by $\mathbb{K}_0$.

Structure of abstract simplicial complexes

Abstract simplicial complexes are related to each other via certain maps which preserve their simplex structure.

Definition 2.4. A **simplicial morphism** $f : \mathbb{K} \to \mathbb{K}'$ between two abstract simplicial complexes $\mathbb{K}, \mathbb{K}'$ is a map between the vertex set $f_0 : \mathbb{K}_0 \to \mathbb{K}'_0$ such that $f_0(P) \in \mathbb{K}'$ if $P \in \mathbb{K}$.[4]

An abstract simplicial complex $\mathbb{K}$ is a partially ordered set or poset for short. The partial order is induced by inclusion, i.e. $D \leq P$ iff $D \subset P$. This poset structure can be drawn as a diagram, called the **Hasse diagram** $H(\mathbb{K})$. The diagram is a directed graph whose vertices are simplices in $\mathbb{K}$, and there is a directed edge from a p-simplex to each of its $p - 1$ faces, see Fig. 7. This thesis identifies an abstract simplicial complex and its Hasse diagram.

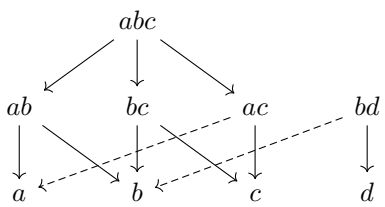


Figure 7: The Hasse diagram of the abstract simplicial complex in Example 1 is shown. Note that the size of the graph explodes quickly as we add larger n-simplices. The pointed arrow between a and ac, and bd and b are dotted to indicate that this graph is not planar. There are no intersections between any of the arrows.

2.3 Translating between the abstract and geometric

Geometry to combinatorial

Loosely speaking, a geometrical simplicial complex is an abstract complex by forgetting the geometry. More precisely, let K be a geometrical simplicial complex. Define the abstract simplicial complex $\mathbb{K}$ as all $P \subset K_0$ such that the convex hull of P is in K.

[4] Abstract simplicial complexes and simplicial morphisms define a category, called ASC. Formally, the fact that every simplicial complex has an induced poset structure is represented as a functor between the category of simplicial complexes and the category of posets. There is also another more general structure of simplicial sets at play.

Combinatorial to geometrical

The machinery to translate an abstract simplicial complex into a geometrical one is non-trivial, but not complicated. Suppose $f : \mathbb{K}_0 \to \mathbb{R}^m$ is a map of vertices into the Euclidean space $\mathbb{R}^m$. The convex hull of the image of $P \in \mathbb{K}$ under f might have intersections of simplices that are not faces of either, see Fig. 6. This is an example of what can go wrong for the simplicial $\mathbb{K}$ in Fig. 7.

Definition 2.5. A **geometric realisation** $|\mathbb{K}|$ of an abstract simplicial complex $\mathbb{K}$ is any geometric simplicial complex whose associated abstract simplicial complex is simplicially isomorphic to $\mathbb{K}$.

The following theorem shows how to construct a geometrical simplicial complex out of an abstract one, see [13] page 33 for more information.

Theorem 2.6 (Geometric realisation)**.** let $\mathbb{K}$ be an abstract simplicial complex. Then there exists a geometric realisation $|\mathbb{K}|$. [5]

Proof. Consider the Euclidean space $\mathbb{R}^{\mathbb{K}_0}$, generated on the vertices of $\mathbb{K}$. Identify $\mathbb{K}_0$ with its image in $\mathbb{R}^{|\mathbb{K}_0|} = \mathbb{R}^m$ under the map $p \mapsto e_p$. Define K as the collection of geometric simplices which are the convex hull of $P \in \mathbb{K}$ under this identification. More precisely, let $f : \mathbb{K} \to \mathbb{P}\mathbb{R}^m$, where $\mathbb{P}(\mathbb{R}^m)$ is the powerset, denote the map that sends a simplex $P = \{p_0, \ldots, p_n\}$ in $\mathbb{K}$ to the convex hull of $\{p_0, \ldots, p_n\} \subset \mathbb{R}^m$. Define K as the image of the map f.
Note that each simplex in K has an associated simplex in $\mathbb{K}$ which maps to it. Let P, D be two simplices in K and their associated simplices P', D' in $\mathbb{K}$. Then the intersection $P \cap D$ is precisely the convex hull of the points $P' \cap D'$. This is the image of $P' \cap D' \in \mathbb{K}$ under f by definition. Therefore K is a geometric simplicial complex. By construction, its associated abstract simplicial complex is isomorphic to $\mathbb{K}$. This completes the proof. □

Nomenclature

From this point, any abstract simplicial complex is called a simplicial complex, i.e. the word "abstract" is left out. We might refer to a simplicial complex, as just a complex. Other 'types' of complexes, such as CW complexes, will be explicitly called CW complexes.

[5] Note that the appropriate space $\mathbb{R}^m$ should at least be as large as the dimension of $\mathbb{K}$. Otherwise, any simplex associated with this dimension has vertices which cannot form an affine basis of the space. However, the dimension in which the realization lies can be significantly smaller. It is only required that there is a map $f_0 : \mathbb{K} \to \mathbb{R}^m$ such that every set of $d + 1$ vertices is affinely independent. This can always be achieved in $\mathbb{R}^{2d+1}$, where d is the dimension of $\mathbb{K}$.

3

Euclidean Data Complexes

The previous section introduced simplicial complexes, both abstract and geometrical. In particular, it was shown that all abstract simplicial complexes have an associated topology induced by their geometric realisation.

Overview

In this section, the complexes, called Čech, Delaunay, and E selective Delaunay, are defined using data $X \subset \mathbb{R}^n$ via the *nerve* method. The section shows that the Delaunay complex is often smaller than the Čech complex if X satisfies a non-degeneracy condition. Complexes, called E selective Delaunay complexes, that lie between the Čech and Delaunay complexes are introduced. The rest of the sections almost exclusively studies the E selective Delaunay complexes. In the next section, an alternative way to characterise these complexes is discussed, see Lemma 4.6.

Pictures and images supplement the discussion to ensure the reader can visualise the complexes. The first part of this section can be skipped if one is familiar with section 3 of [28], and the latter part can be skipped if one is familiar with the complexes in [7]. The notion of general position is as in [7] Definition 4.2, and notation is derived from that paper.

Notation and assumption

We assume that the data $X \subset \mathbb{R}^m$ is finite and fixed throughout sections 3 to 6 unless otherwise stated. Furthermore, r is assumed to be some non-negative real number. Lastly, we denote the standard Euclidean distance as $d(-,-) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, where $d(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

3.1 The nerve construction

First, we discuss nerves of covers as all complexes of interest arise in this way.

Covers

The following notion formalises the notion of 'covering' a topological space with sets.

Definition 3.1. Let T be a topological space and $\mathcal{U}$ a collection of closed sets in T. If $T \subset \bigcup_{U \in \mathcal{U}} U$, then $\mathcal{U}$ is called a **cover** of T.

Covers provide a method to understand topological spaces by first understanding small patches U, then the overlaps $U \cap V$ and then patching them together.[6] [7]

[6] The van Kampen theorem is an example of this [1] Theorem 1.20.

[7] In turn, this approach can become difficult if the patches U or the overlaps are complicated. For that reason, some assumptions on the cover are usually made to ensure that the overlaps are better understood.

Remark 3.2. As mentioned in the introduction, we can construct many spaces/simplicial complexes from the data X. One example, the main one of interest, arises from the union of (closed) balls around the points of a fixed radius. The balls themselves are subspaces which make up the union.

Nerves

The intersection data of sets in the cover can be described using a simplicial complex. Indeed, closeness can be quantified by calling a set of points is close if their associated neighbourhoods in the cover intersect. If two points are close, draw a 1-simplex, i.e. edge, between the points. For three points, if every pair is close, then this produces three 1-simples between them, which is an empty triangle. If all three points are close, then the 2-simplex is added, i.e. the triangle is filled in. Continuing this process gives a simplicial complex and motivates the following definition.

Definition 3.3. Let T be a topological space and $\mathcal{U} = \{U_i | i \in I\}$ a cover. Then the **nerve**, $\mathcal{N}(\mathcal{U})$ of this cover is the following simplicial complex.

$$\mathcal{N}(\mathcal{U}) = \{J \subset I | \bigcap_{j \in J} U_j \neq \emptyset\}$$

The complexes considered in this section arise as the nerves of covers of the same topological space as we shall see.

Remark 3.4. In effect, discrete data is transformed into continuous data using covers and then into combinatorial data using the nerve. Note that there is no guarantee (yet) that the topology of the simplicial complex in any way captures the topology induced by the cover. In a later section, a theorem is stated which shows that the homotopy type is captured under certain conditions Theorem 5.1. These conditions are satisfied for all the complexes relevant to the discussion. Furthermore, note that computers handle combinatorics well. Therefore, nerves allow us to turn a topological problem into a task suitable for computers.

3.2 Čech and Delaunay complexes

Čech complex

Denote by $B_r(x)$ the (closed) ball of radius r around $x \in \mathbb{R}^n$, i.e. $B_r(x) = \{y \in \mathbb{R}^n | d(x, y) \leq r\}$. For the set $X \subset \mathbb{R}^n$, the collection of balls of radius r is denoted as $\mathcal{B}_r = \{B_r(x) | x \in X\}$. The union of the collection of balls defines a topological space $T = \bigcup_{x \in X} B_r(x)$. The collection $\mathcal{B}_r$ is a cover for T. The nerve of this cover is defined as follows.

Definition 3.5. The Čech **complex (at radius r)** is the following simplicial complex.

$$\check{\text{C}}\text{ech}_r(X) = \{Q \subset X | \bigcap_{x \in Q} B_r(x) \neq \emptyset\}$$

Fig. 8 depicts an example of the $\check{\text{C}}\text{ech}_r(X)$ complex. Note that $\check{\text{C}}\text{ech}_r(X)$ is often a 'proper' abstract simplicial complex. Indeed, Fig. 8 shows that the complex is not a geometrical simplicial complex even if X is small. Fortunately, the Čech complex is relatively straightforward to define and to conceptualise.[8]

[8] Historically, Čech, after which the complex is named, studied how covers of topological spaces can be used to determine their cohomology. The arising construction is called the Čech cohomology. Formally, this cohomology can be defined as the categorical colimit of refinements of covers. Informally, this method determines which spaces are allowed to be approximated by a cover of open sets. The Čech complex we use is defined by a cover of a set. A reference of Čech's work can be found in [29].

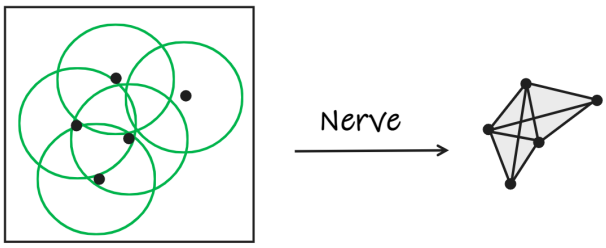


Figure 8: The Čech complex is drawn on the right out of the nerve of the cover of circles. Note that the disc defined by the four top points intersect at a common point. Hence, the 3-simplex is added to the complex. Note that this Čech complex is not a geometrical simplicial complex as depicted in the plane, see Definition 2.1

Voronoi diagram

The following definition captures the notion of the area closest to a point with respect to other points.

Definition 3.6. Then **the Voronoi cell**, $Vor(x, X)$ of $x \in X$ is defined as

$$Vor(x, X) = \{y \in \mathbb{R}^m | d(x, y) \leq d(z, y),\ \forall z \in X\}$$

Fig. 9 depicts some examples of Voronoi cells, such as the one in light blue. Note that every Voronoi cell is *closed and convex*. The Voronoi cell of x is precisely the region around x that consist of all points closest to x with respect to other points in X. In this sense, the Voronoi cell of x is the closest area associated to x.

To compute $Vor(x, X)$, first compute an affine hyperplane $H_z = \{y | d(y, x) = d(y, z)\}$ for each point $z \in X$. An hyperplane H_z defines two half-spaces. The **positive hyperplane** consists of all the points y such that $d(y, x) \leq d(y, z)$. The Voronoi cell of x is then the intersection of all positive hyperplanes around x. In Fig. 9, segments of the hyperplanes which lie on the intersection of two or more Voronoi cells are depicted as black lines between the points.

Definition 3.7. The collection of all Voronoi cells $\{Vor(x, X) | x \in X\}$ is called the **Voronoi diagram**

Fig. 9 depicts a Voronoi diagram.

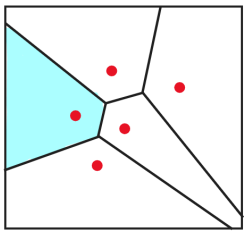


Figure 9: Of the 5 points in the plane, the Voronoi cell of the centre-left point is coloured light blue. The other Voronoi cells are depicted in white. Since all the Voronoi cells have been drawn, this depicts the Voronoi diagram of the points. All Voronoi cells are convex and closed. Although the Voronoi cells do not have to be bounded, for explanatory purposes, the Voronoi diagram is restricted to the box.

Delaunay complex

The dimension of the Čech complex is equal to the cardinality minus one, $|X| - 1$, of X as $r \to \infty$. Indeed, as $r \to \infty$, the intersection $\bigcap_{x \in X} B_r(x)$ is never empty if $X \neq \emptyset$. Therefore, $X \subset \text{Čech}_r(X)$ which implies that the dimension is $|X| - 1$. This is also the maximal dimension it can obtain. Furthermore, $|X| - 1$ bounds all $\text{Čech}_{r'}(X)$ for each r' since $\text{Čech}_{r'}(X) \subset \text{Čech}_r(X)$ for $r' \leq r$.

By truncating the balls at their Voronoi cells, intersections can be avoided. This motivates the following definition.

Definition 3.8. The **Voronoi cell (at radius r)** is the following set.

$$Vor_r(x, X) = B(r, x) \cap Vor(x, X)$$

Fig. 10 depicts an example of a Voronoi cell at radius r in light blue. The radius is the same as the circles in Fig. 8.

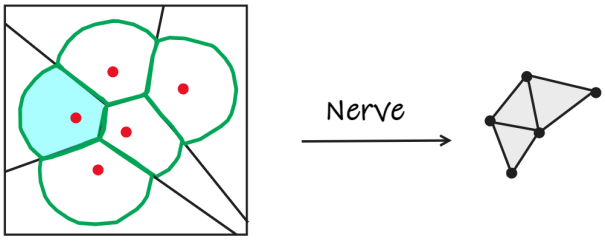


Figure 10: The intersection of each Voronoi cell with a ball of radius r is depicted by a green circle. The Voronoi cell at radius r for the centre-left point is highlighted in light blue. The intersections define the Delaunay complex on the right via the nerve construction. Note that the Delaunay complex is smaller and included in the Čech complex at the same radius, see Fig. 8 .

Denote the collection of Voronoi cells at radius r with $\mathcal{V}_r = \{Vor_r(x, X) | x \in X\}$. Their union defines a subspace T of $\mathbb{R}^n$. Therefore, $\mathcal{V}_r$ defines a cover of that space. Similar to the Čech complex, see Definition 3.5, this construction defines a simplicial complex by taking the nerve.

Definition 3.9. The **Delaunay complex (at radius r)** [9], $\mathrm{Del}_r(X)$, is the following simplicial complex.

$$\mathrm{Del}_r(X) = \{Q \subset X | \bigcap_{x \in Q} Vor_r(x, X) \neq \emptyset\}$$

In other words, the Delaunay complex at radius r is the nerve of the cover $\mathcal{V}_r$. Fig. 10 illustrates the image of a Delaunay complex at a radius r where r is the same radius as in Fig. 1.

Remark 3.10. In the literature, the Delaunay complex $\mathrm{Del}_r(X)$ is sometimes called the *Alpha-shape* or alpha complex, see Definition 4.12 in [12] or Chapter 3.4 in [28]. This thesis follows the convention as in [7]. The reason is that the $\check{\mathrm{C}}\mathrm{ech}_r(X)$ and $\mathrm{Del}_r(X)$ complexes are regarded as special cases of the more general 'selective Delaunay complexes', see Definition 3.15.

Notice that the Delaunay complex, $\mathrm{Del}_r(X)$, depicted in Fig. 10 is significantly smaller then the Čech complex at the same radius. This is not by accident. Indeed, the dimension of the Delaunay complex is bounded by the dimension of the ambient space $\mathbb{R}^n$ [19] (or prop 2.10 [12]) if X has a non-degeneracy condition called *general position*.

3.3 General position

General position for Delaunay complexes is motivated at the hand of an example. Recall that the largest dimension any simplicial complex with vertices X can have is $|X| - 1$. This dimension is always obtained by $\check{\mathrm{C}}\mathrm{ech}_r(X)$ if $r \to \infty$, and it can also be obtained by $\mathrm{Del}_r(X)$ in the following situation.

[9] We often leave out the "at radius r" part. However, in the literature such as in Chapter 3.3 in [28], the **Delaunay complex**, $\mathrm{Del}(X)$, refers to the complex we obtain if we take the nerve of $\mathcal{V}$ instead of the nerve of $\mathcal{V}_r$. A reader with the necessary background should be aware that this thesis considers $\mathrm{Del}(X)$ as a special case of $\mathrm{Del}_r(X)$ for $r \to \infty$ (as X is finite).

Example 2. Let $S^1 \subset \mathbb{R}^2$ be a circle in the plane and $X \subset S^1$ a collection of 4 points on the circle as in Fig. 11. The centre z of the circle has the same distance from every point. Thus, all Voronoi cells intersect at the centre z. This implies that $X \in \mathrm{Del}_r(X)$ if r is large enough. Such a Delaunay complex is drawn in Fig. 11. Notice that, in this case, $\mathrm{Del}_r(X) = \check{\mathrm{C}}\mathrm{ech}_r(X)$, and, that, in particular, $\mathrm{Del}_r(X)$ is not a geometrical simplicial complex in the plane as depicted.

This example can be generalised to higher dimensions $\mathbb{R}^n$ by considering the S^{n-1} sphere with radius δ with points X on it. If $\delta \leq r$ and $|X| > n+1$, then the Delaunay complex is not geometrical in $\mathbb{R}^n$.

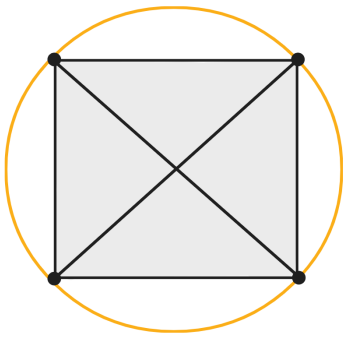


Figure 11: The Delaunay complex, $\mathrm{Del}_r(X)$, is depicted for X the 4 corner points of the square, which lie on the same circumsphere. The Delaunay complex contains the 3 simplex. The Delaunay complex is not a geometrical simplicial complex in the plane as depicted. This is in contrast to Fig. 10. If one point is removed, then it forms a geometric simplicial complex in $\mathbb{R}^2$ again. This is also the case if we had added another point inside the circle. Note that the Delaunay complex in this figure is equal to the Čech complex.

If X is assumed to have no set of at most $n+1$ points on the same circumsphere, then, at least, the situation depicted in Example 2 is excluded. This motivates the following condition for X when considering Delaunay complexes.

Definition 3.11. A set $X \subset \mathbb{R}^n$ is in **general position** if for at most $n+1$ points P

- The minimal circumsphere of P contains no points of $X - P$.
- P is affinely independent. [10]

Remark 3.12. Note that this condition is very mild. Already an arbitrarily small perturbation of the point set X can put the points in a configuration that satisfies the first condition of Definition 3.11, see Theorem 4.1 [8].

In principle, only the first condition is necessary to bound [19] (or proposition 2.10 [12]) the dimension of the Delaunay complex. Specifically, the bound on the dimension is n, where $X \subset \mathbb{R}^n$. In this thesis, no general position assumption is assumed on X.

3.4 Selective Delaunay complexes

The difference between the sets in the covers associated with the Čech and Del complex is that the latter consists of truncated balls. Specifically, the balls were truncated with the Voronoi cells. The Voronoi cells were defined by the constraints $d(x,y) \leq d(z,y)$ for each $z \in X$. However, in principle, we could require that only $d(x,y) \leq d(z,y)$ for some $E \subset X$. This motivates the following definition.

[10] For a given set X of points in the plane, deciding whether there is a subset D of cardinality at least $k \in \mathbb{N}$ that satisfies the second condition of general position, i.e. no three points colinear, is NP-hard, see theorem 1 in [9].

Definition 3.13. Let E be a subset of X, then the E **selective Voronoi cell**, $Vor(x, E)$, of $x \in X$, is defined as the following set.

$$Vor(x, E) = \{y \in \mathbb{R}^n | d(y, x) \leq d(y, z) \ \forall z \in E\}$$

Fig. 12 illustrates a few selective Voronoi cells.

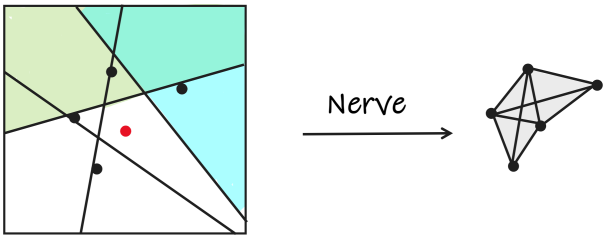


Figure 12: The E-selective Voronoi diagram is drawn for E as the red point. For the upper and rightmost point, the E-selective Voronoi cells are coloured green and light blue respectively. On the right is the selective Delaunay complex obtained by the nerve construction.

Similar to the Voronoi diagram, we define the **selective Voronoi balls (at radius r)**, $Vor_r(x, E)$, by intersecting the selective Voronoi cell of x by a ball of radius r around x, i.e.

$$Vor_r(x, E) = Vor(x, E) \cap B_r(x)$$

The union of these selective Voronoi balls form a space with the selective Voronoi balls as a cover, $\mathcal{V}_r(E) = \{Vor_r(x, E) | x \in X\}$. This space is the same as the union of balls of radius r as shown in the following lemma.

Lemma 3.14. Given $E \subset X$, then $\bigcup_{x \in X} Vor_r(x, E) = \bigcup_{x \in X} B(x, r)$.

Proof. Notice that a point y in $\bigcup_{x \in X} B(x, r)$ lies in a (or multiple) selective Voronoi cell $Vor(x, E)$. Indeed, X is finite, so $\{d(y, z) | z \in E\}$ is a finite set and thus has a minimum, say, at x. Therefore, $y \in Vor_r(x, E)$ and thus $y \in \bigcup_{x \in X} Vor_r(x, E)$. The converse implication follows trivially as each $Vor_r(x, E)$ is defined as a subset of $B_r(x)$.

□

Similar to the $\check{\text{C}}\text{ech}_r(X)$ and $\text{Del}_r(X)$ complex, the following definition is the nerve associated to the cover $\mathcal{V}_r(E)$.

Definition 3.15. For $E \subset X$ the **E selective Delaunay complex (at radius r)**, $\text{Del}_r(X, E)$, is the following simplicial complex.

$$\text{Del}_r(X, E) = \{Q \subset X | \bigcap_{x \in Q} Vor_r(x, E) \neq \emptyset\}$$

Fig. 13 provides an illustration of an E selective Delaunay complex where E is the red point.

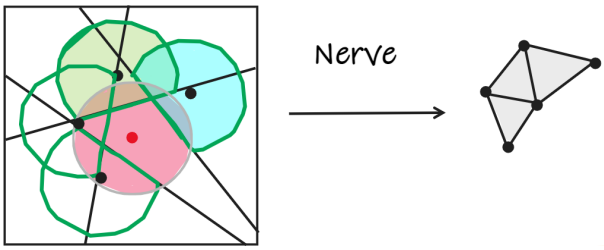


Figure 13: The E-selective Voronoi cells at radius r for E the red point are drawn as green circles. The E-selective Voronoi cells at radius r for the centre, upper and rightmost points are drawn in red, green and light-blue, respectively. Notice that the Voronoi cell at radius r of the red dot is the ball, as there are no constraints for its radius r selective Voronoi cell other than the radius.

Note that the selective Delaunay complex is smaller than the Čech complex Fig. 8 and (larger or) equal to the Delaunay complex Fig. 10.

Remark 3.16. In this thesis, the subset E is also called the **constraint set**. Section 4 further motivates why this nomenclature is chosen.

The E selective Delaunay complexes generalise both the Čech and Delaunay complex. Indeed, for $E = \emptyset$ we have that $\mathrm{Del}_r(X, \emptyset) = \check{\mathrm{C}}\mathrm{ech}_r(X)$, while for $E = X$, we have that $\mathrm{Del}_r(X, X) = \mathrm{Del}_r(X)$.

Notation and Nomenclature

This thesis mainly considers E-selective Delaunay complexes. If the E is clear from context, then it is left out of the "E selective Delaunay complex". Furthermore, if it is clear from the context that the complex referred to is a selective Delaunay complex then the word "selective" is left out. Therefore, the reader should be aware that a Delaunay complex can refer to both an E-selective Delaunay complex and a 'regular' Delaunay complex. If a reference to the 'regular' Delaunay complex is made, then this is made explicit in the text and in notation expressed as $\mathrm{Del}_r(X)$.

4

Computing selective Delaunay simplices as constrained minimal enclosing spheres

In the previous section, the Čech, Delaunay and selective Delaunay complexes are introduced using the nerve construction. Equivalently, each of these complexes are associated to a type of minimal sphere. The last observation is crucial for the main theorem.

Overview

This section first introduces notation and terminology to handle constrained minimal spheres. After that, it is shown that minimal spheres that include sets Q and exclude sets E are an equivalent way to describe Delaunay complexes. The section then shows that computing certain constrained affine combinations is, in some sense, equivalent to computing these minimal spheres. Afterwards, the degeneracy of these affine combinations and notation is discussed. The theory is due to [7], and the content of this section is similar to their discussion.

4.1 Minimal Enclosing Exluding Spheres (MEES)

A sphere S with centre z and radius r is identified with the pair (z, r), i.e. $S = (z, r)$.

Definition 4.1. Let S be some sphere in $\mathbb{R}^n$ with centre z and radius r, and $x \in \mathbb{R}^n$.

- x is **included** if $d(x, z) \leq r$.
- x is **excluded** if $d(x, z) \geq r$.
- x is **on** if $d(x, z) = r$, e.g. it is both included and excluded.

The sets of these points are denoted by $\mathrm{incl}(S), \mathrm{excl}(S)$ and $\mathrm{on}(S)$ respectively.

Definition 4.2. If S is a sphere in $\mathbb{R}^n$, then a point x is **strictly included** if $x \in \mathrm{incl}(S) - \mathrm{excl}(S)$. It is **strictly excluded** if $x \in \mathrm{excl}(S) - \mathrm{incl}(S)$.

A minimal sphere enclosing a set of points is unique [30] if the set is bounded. With a compactness argument in $\mathbb{R}^n$ and a computation of the intersection of spheres, a similar result holds when additionally a set E needs to be excluded.[11]

Lemma 4.3. Let $Q, E \subset \mathbb{R}^n$ be finite sets. Suppose that the set of spheres which encloses Q and excludes E, $\mathcal{S}$, is non-empty. Under this condition, a sphere with minimal radius exists in $\mathcal{S}$. Additionally, this sphere is unique.

Proof. See Lemma A.1 for existence and Lemma A.2 for uniqueness respectively in the appendix. □

Of course, there might be no sphere which encloses Q and excludes E. The following function formalises this notion.

[11] No reference has been found with a proof that the solution is unique if it exists. With additional conditions, the quadratic minimization problem is known to have a unique solution, but these conditions are not necessary.

Definition 4.4. Let Q, E be finite subsets of $\mathbb{R}^n$, then the **Minimal Enclosing Excluding Sphere** (MEES)[12] of the pair (Q, E), is the minimal S^{n-1} sphere $S = (z, r) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$Q \subset \text{incl}(S) \quad E \subset \text{excl}(S)$$

This sphere is denoted as $S(Q, E)$. If no sphere is exists, then $S(Q, E) = (\emptyset, \infty)$ by convention.

Notice that $S(Q, E)$ exists if and only if the following quadratic optimisation program of $(z, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ has a solution.

$$\begin{array}{lll} \min_{(z,r)} & r^2 & \\ \text{s.t.} & d(q,z)^2 \leq r^2 & q \in Q \\ & d(e,z)^2 \geq r^2 & e \in E \end{array} \tag{1}$$

Suppose that $Q' \subset Q$, $E' \subset E$ and $S(Q, E)$ exists. Then $S(Q', E')$ also exists. Indeed, $S(Q, E)$ includes Q' and excludes E', and $s(Q', E') \leq s(Q, E)$. This shows the structure of a simplicial complex, where a simplex Q is included if all its faces Q' are.

Alternative definition of the Delaunay complex as MEESs

The radius of the MEES, i.e. the second component of $S(Q, E)$, is of particular importance to us.

Definition 4.5. The second component of the MEES function $S(-, -)$ is called the **minimal enclosing excluding sphere radius**. This function is denoted by $s(-, -)$.

Notice that the convention that $S(Q, E) = (\emptyset, \infty)$, if the MEES of the pair (Q, E) does not exist, implies that $s(Q, E) = \infty$. The importance of this radius function is due to the following lemma.

Lemma 4.6. A simplex Q is in $\text{Del}_r(X, E)$ if and only if $s(Q, E) \leq r^2$.

Proof. See [7] Lemma 3.1. □

All in all, Lemma 4.6 gives us a translation tool to transform the language of simplices $Q \in \text{Del}_r(X, E)$ to radii $s(Q, E)$ of MEES in $\mathbb{R}^n$. This alternative characterisation of the Delaunay complex also has the following immediate corollary.

Corollary 4.6.1. For $E \subset X$, and $x \in X$, the following relation.

$$\text{Del}_r(X, E + x) \subset \text{Del}_r(X, E)$$

Proof. By Lemma 4.6, we may write

$$\text{Del}_r(X, E) = \{Q \subset X | s(Q, E) \leq r\}$$

and

$$\text{Del}_r(X, E + x) = \{Q \subset X | s(Q, E + x) \leq r\}$$

Since $s(Q, E) \leq s(Q, E + x)$, it follows that $\text{Del}_r(X, E + x) \subset \text{Del}_r(X, E)$. □

The following examples 3, 4 and 5 for the Čech, Delaunay ($\text{Del}_r(X)$) and then E selective Delaunay complex respectively illustrate in what manner the simplices in the Delaunay complexes $\text{Del}_r(X, E)$ are computed as MEES, see Lemma 4.6.

Example 3. Notice that the intersection of balls $\bigcap_{x \in Q} B_r(x)$ is not empty if and only if there is some point z such that $d(z, x) \leq r$ for all $x \in Q$. In other words, $Q \in \text{Čech}_r(X)$ is equivalent to the existence of a sphere of radius less than r which encloses Q. In turn, a sphere that encloses Q exists if and only if such a sphere with minimum radius exists [13]. Thus, computing whether a simplex Q is in $\text{Čech}_r(X)$ is equivalent to computing the radius of the Minimal Enclosing Sphere (MES) of Q. Fig. 14 illustrates this alternative way of computing the Čech complex in Fig. 1.

[12] The Minimal Enclosing Excluding Sphere (MEES), generalises the notion of Minimal Eclosing Circle (MEC), respectively. Solving MEC's can be arithmetically formulated as a quadratic minimization problem [31]. Since 1900, much research has been done to quickly find an approximate solution. Applications of MEC's are plentiful. For example, in the military, bombs have a circular detonation zone. The challenge is to swiftly find a point such that the bomb destroys as many targets as possible. To exclude unwanted collateral damage, however, such as civilian centers, we would need to compute the MEES.

[13] Actually, it is only necessary that Q is bounded, and not necessarily that it is finite [30]

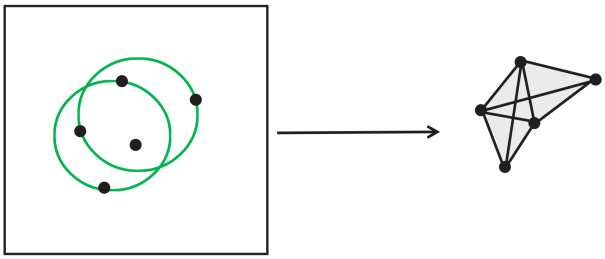


Figure 14: The Čech complex, $\check{\text{C}}\text{ech}_r(X)$, on the right is also constructed by computing enclosing circles whose radius cannot exceed the radius r. The upper four points are enclosed by a circle whose radius is no larger then r. Therefore, the corresponding 3 simplex is added in the complex. The same holds for the four left-most points.

Example 4. Similar to the Čech complex in Example 3, the intersection of Voronoi cells $\bigcap_{x \in Q} Vor_r(x, X)$ is not empty if and only if there is a $z \in \bigcap_{x \in Q} Vor_r(x, X)$ such that the following three criteria are satisfied.

- z is equidistant from all points in Q, i.e. $d(z, -)$ is constant on Q.
- z is not strictly contained in the interior of any Voronoi cell, i.e. $d(z, q) \leq d(z, x)$ for all $x \in X$ and $q \in Q$
- z is no further then r from Q, i.e. $d(z, q) \leq r$ for all $q \in Q$.

These criteria imply that $d(z, q) = \delta \leq r$ for all $q \in Q$. Therefore, the existence of z is equivalent to the existence of a sphere (z, δ) with centre z and radius $\delta \leq r$ such that Q lies on the sphere and $X - Q$ is excluded. Fig. 15 illustrates this alternative way of computing $\text{Del}_r(X)$ in Fig. 10.

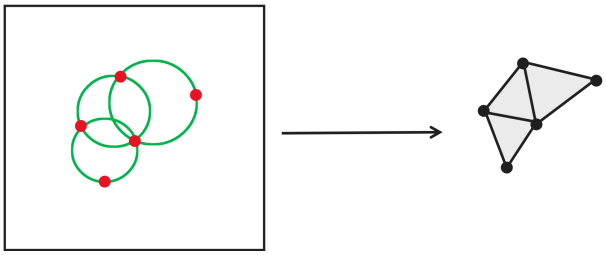


Figure 15: The Delaunay complex, $\text{Del}_r(X)$, on the right is constructed by computing circum-circles whose radius cannot exceed r. The upper four points do not have such a circle, therefore this 3 simplex is not added in the complex. This is in contrast to Fig. 14, where this 3 simplex is added in the Čech complex.

Example 5. Similar to Čech and Delaunay complexes in Example 3 and Example 4 respectively, the intersection of selective Voronoi cells $\bigcap_{x \in Q} Vor_r(x, E)$ is nonempty if and only if there exists a point z such that $d(z, q) \leq r$ and $d(z, q) \leq d(z, e)$ for all $q \in Q$ and $e \in E$. The point z defines a sphere S with radius $\delta \leq r$ such that Q is contained in the sphere and E is excluded from the sphere. Furthermore, such a sphere exists if and only if a minimal sphere exists, see Lemma 4.3. Fig. 16 illustrates an alternative computation of $\text{Del}_r(X, E)$ in Fig. 13.

4.2 Solving minimal spheres by affine combinations

Intuitively, to compute $S(Q, E)$, the centre of the sphere should be be as close as possible to the included points. At the same time, the centre should be as far as possible from the excluded points. Lastly, only the points on the boundary should have influence. Indeed, otherwise, there is still room to move the MEES around without violating included/excluded conditions. This is formalised by Theorem 4.7, which is Theorem 4.1 in [7].

Theorem 4.7 (Special KKT Conditions)**.** Let $Q, E \subset X$ be finite sets and S a sphere with centre z. Then S is the MEES including Q and excluding E, i.e. $S = S(Q, E)$, if and only if z is an affine

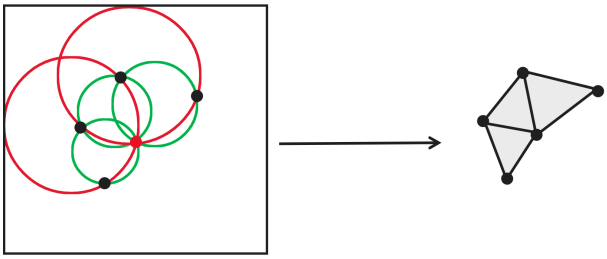


Figure 16: The enclosing circles that include points and exclude the red point are drawn on the left. This corresponds to the E-selective Delaunay complex at radius r for E the red point. The red circles are too large and therefore their corresponding simplices are not added to the complex.

combination of points in $Q \cup E$,

$$z = \sum_{x \in Q \cup E} \lambda_x x \quad \& \quad \sum_{x \in Q \cup E} \lambda_x = 1$$

with the condition that

- $\lambda_x \geq 0$ if $x \in Q - E$
- $\lambda_x \leq 0$ if $x \in E - Q$
- $\lambda_x = 0$ if $x \notin on(S)$

Proof. See Theorem 4.1 in [7]. [14] □

Together, Theorem 4.7 and Lemma 4.6 imply that $Q \in \mathrm{Del}_r(X, E)$ if and only if there exists an affine combination of the points $\sum_{x \in Q \cup E} \lambda_x x = z$, such that $\lambda_x \geq 0$ if x is not excluded, $\lambda_x \leq 0$ if x is not included and $\lambda_x = 0$ if x is not included and excluded. Lastly, the coefficients should sum to 1.

If X is in general position, then the affine combination of Theorem 4.7 is unique. The uniqueness is not true if X is not in general position; see Example 6 below.

[14] Informally, one takes the derivative of the quadratic optimisation problem Eq. (1). This results in an affine problem. Such a derivative is formalised by the Karush-Kahn-Tucker (KKT) conditions, which are a generalisation of Lagrange multipliers. See Chapter 5.5.3 in [32] for more details on KKT conditions.

Example 6. Let $Q = E = X = \{a, b, c, d\}$, where a, b, c, d are the corner points of a square labelled in an counter clockwise manner shown below Example 6. The circumsphere of the square is $S = (z, r)$ where $z = \frac{1}{4}(a + b + c + d)$. This affine combination satisfies the special KKT conditions, Theorem 4.7, and thus $S = S(Q, E)$.

However, there are 4 variables $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ and only 3 equations. Specifically, 2 equations come from $z = \sum_{x \in Q \cup E} \lambda_x x$ and 1 equation is from $\sum_{x \in Q \cup E} \lambda_x = 1$. Since $Q = E$, there are no other conditions on the coefficients. Therefore, there is one free variable and, hence, an infinite number of solutions. We conclude that the affine combination of points as in Theorem 4.7 need **not** to be unique, in contrast to when general position is assumed as in [7].

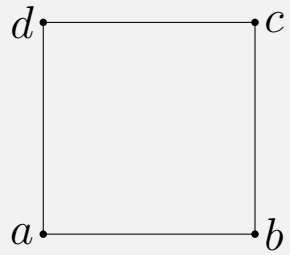


Figure 17: Four points depict a situation of non-general position. The centre of the circumcircle is an affine combination of the corner points in an infinite number of ways.

The previous discussion motivates us to introduce notation to keep track of different possible affine combinations satisfying Theorem 4.7. The following convention is introduced.

Definition 4.8. Suppose that $S(Q, E)$ exists with centre z, then denote by $S_{Q,E} = \{\lambda_x | x \in Q \cup E\}$ any set of coefficients which satisfies the conditions of Theorem 4.7 for $S(Q, E)$. Such a set is called a **solution**.

Notice that $S_{Q,E}$ is a convention on how to denote solutions, as the object $S_{Q,E}$ is not unique. Fortunately, this convention helps us to develop a kind of 'calculus' in Section 6. For this 'calculus' only statements about a particular $S_{Q,E}$ or all $S_{Q,E}$ are relevant for that discussion.

Recall that the special KKT conditions formalise the idea that we move closer to points to be included and further from points to be excluded. This gives motivation for the following naming convention.

Definition 4.9. Suppose that $S_{Q,E}$ is a solution, then

$$\text{front}(S_{Q,E}) = \{\lambda_x > 0 | \lambda_x \in S_{Q,E}\}$$

$$\text{back}(S_{Q,E}) = \{\lambda_x < 0 | \lambda_x \in S_{Q,E}\}$$

This convention is the same as used in [7] page 11 and the front is always contained in the included set, while the back is contained in the exclude set.

Remark 4.10. If $S(Q, E)$ exists and $S_{Q,E}$ is a solution, then there is a bijection between coefficients in $S_{Q,E}$ and elements in $Q \cup E$. More exactly, the correspondence $\lambda_x \mapsto x$ for $\lambda_x \in S_{Q,E}$ and $x \in Q \cup E$ is a bijection. Therefore, the notation $S_{Q,E} \subset D$ denotes the statement that, for all $\lambda_x \in S_{Q,E}$ we know that $x \in D$. Similar convention is used for the $\text{front}(S_{Q,E})$ and $\text{back}(S_{Q,E})$.

Lemma 4.11. Suppose $S(Q, E)$ exists with solution $S_{Q,E}$. If $\lambda_x \in \text{front}(S_{Q,E})$, then $x \in Q$.

Proof. Since $\lambda_x \in S_{Q,E}$, it follows that $x \in Q \cup E$. If $x \in E - Q$, then $\lambda_x \leq 0$ by condition 2 of Theorem 4.7. This is a contradiction as $\lambda_x > 0$. Therefore, $x \in Q$, which proves the statement. □

A similar statement with analogous proof holds for $\lambda_x \in \text{back}(S_{Q,E})$ implies $x \in E$.

Lemma 4.12. Suppose $S(Q, E)$ exists with solution $S_{Q,E}$. If $\lambda_x \in \text{back}(S_{Q,E})$, then $x \in E$.

The solutions are useful to measure which simplices have the same MEES as the following lemma states.

Lemma 4.13. If $S = S(Q, E)$ exists with solution $S_{Q,E}$ and $\text{front}(S_{Q,E}) \subset Q'$ and $\text{back}(S_{Q,E}) \subset E'$, then $S(Q', E') = S(Q, E)$

Proof. Suppose that $S_{Q,E} = \{\lambda_x | x \in Q \cup E\}$ is a solution for $S(Q, E)$ as in the premise. Define the set $S_{Q',E'}$ as $\{\lambda_x \in S_{Q,E} | x \in Q' \cup E'\}$, i.e. $S_{Q',E'}$ consists of the coefficients in $S_{Q,E}$ which belong to elements in $Q' \cup E'$. The centre z of S is an affine combination of points in $Q' \cup E'$, where the coefficient of $x \in Q' \cup E'$ is λ_x in $S_{Q',E'}$. Furthermore, the coefficients in $S_{Q',E'}$ satisfy the conditions of Theorem 4.7. Therefore, $S_{Q',E'}$ is a solution. By Theorem 4.7, we conclude that S is the MEES of the pair (Q', E'), i.e. $S = S(Q', E')$. □

Lemma 4.13 can be better explained at the hand of Example 6 in the following example.

Example 7. For the points $Q = E = X = \{a, b, c, d\}$ as in Example 6, take the solution $S_{Q,E}$ defined by the coefficients $\lambda_a = \lambda_c = \frac{1}{2}$ and $\lambda_d = \lambda_b = 0$. Let $Q' = \{a, c\}$ and $E' = \emptyset$. To check whether $S(Q, E) = S(Q', E')$, it is sufficient to know that the centre $z = \frac{1}{4}(a+b+c+d)$ is an affine combination of points in $S_{Q',E'} := S_{Q,E} - \{\lambda_b, \lambda_d\}$ such that the conditions in Theorem 4.7 are satisfied. In this case, the coefficients $S_{Q',E'}$ define a solution for the MEES $S(Q', E')$. Therefore, we can conclude that $S(Q, E) = S(Q', E')$.

Remark 4.14. If $z = \sum_{x \in Q \cup E} \lambda_x x$ and $\lambda_y = 0$, then $z = \sum_{x \in Q \cup E,\ x \neq y} \lambda_x x$. This implies that the coefficients in the solution $S_{Q,E}$ still affinely sum to z even if the zero coefficients have been removed. In Lemma 6.2 and Lemma 6.3, this observation is crucial as it indicates that a MEES changes only if points get positive or negative coefficients for every new solution (or the MEES does not exist).

5

Discrete Morse theory

In the previous sections, the E selective Delaunay complex for $E \subset X$ was introduced as a simplicial complex from the nerve construction of the cover of E selective Voronoi cells. Furthermore, there was an inclusion $\mathrm{Del}_r(X, E + x) \hookrightarrow \mathrm{Del}_r(X, E)$ which was directly seen from the alternative characterisation of these complexes as the radius of Minimal Enclosing Excluding Spheres (MEES). This inclusion implies that $\mathrm{Del}_r(X, E + x)$ can be seen as a subspace of $\mathrm{Del}_r(X, E)$.

Overview

This section introduces the notion of a collapse of simplicial complexes by showing how they encode a deformation retract on the geometric realisation. Thereafter, discrete Morse theory is introduced which characterises collapses at the hand of a special partition. This characterisation is used to find special partitions on the Delaunay complex $\mathrm{Del}_r(X, E)$ such that the main theorem can be proven in the next section Theorem 6.9. Lastly, a few lemmas help us to find these special partitions.

This section is supplemented with figures to motivate definitions. This section can be skipped if one is familiar with discrete Morse theory for simplicial complexes such as in Chapter 4 of [13].

5.1 Homotopy theory

More information on homotopy theory and deformation retracts can be found in [1] Chapter 4 and Chapter 0, respectively. We briefly discuss theory necessary to be able to state what an homotopy equivalence is.

Homotopies

An **homotopy** H between two continous maps $f, g : M \to N$ is a continuous map $H : [0, 1] \times M \to N$ such that $H(0, -) = f(-) : M \to N$ while $H(1, -) = g(-) : M \to N$. If such a homotopy exists, then f and g are called homotopic.

Deformation retracts

A basic type of continuous deformation of a space is one that shrinks the space. Such a continuous deformation can be formalised as follows. Let T be a space and Y a subspace, then an homotopy H such that $H(0, z) = z$ for all $z \in T$, $H(1, z) \in Y$, and such that Y is fixed by H, i.e. $H(t, y) = y$ for all $t \in [0, 1]$ is called a **deformation retract** of T into Y. The map $H(1, -) : T \to Y$, which arises from a deformation retract, is called a **retract**. Fig. 18 depicts a deformation retract for a triangle.

Homotopy equivalence

Two spaces M, N are called **homotopy equivalent** if there are two maps $f : M \to N$ and $g : N \to M$ such that $f \circ g$ is homotopic to the identity on N and $g \circ f$ is homotopic to the identity on M. The maps f, g are called **homotopy equivalences**. A retract is a homotopy equivalence between a space and a

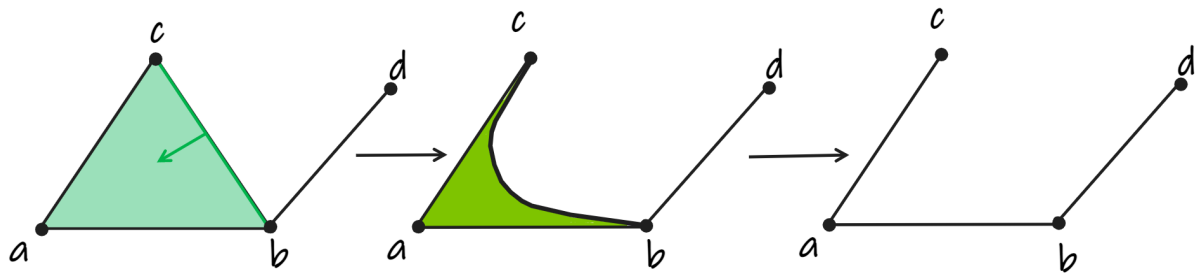


Figure 18: A deformation retraction is visualised as pushing a larger space into a smaller subspace. This specific deformation retract is called a collapse resulting from pushing the 1-simplex $\{bc\}$ into $\{abc\}$ along the arrow. The 1-simplex $\{bc\}$ is called a free face of the 2-simplex as it is not the face of any other simplex in the complex.

subspace. Homotopy equivalent spaces have isomorphic homotopy and homology groups induced by the homotopy equivalence.

Nerve lemma

As seen in Lemma 3.14 each Delaunay complex $\mathrm{Del}_r(X, E)$ is the nerve of a cover of the same space, i.e. the union of balls of radius r. The following theorem provides a sufficient condition for the geometric realisations of the Delaunay complexes to be homotopy equivalent.

Theorem 5.1. [Nerve lemma] Let $\mathcal{C} = (C_i)_{i \in I}$ be a finite collection of compact and convex subsets of R^d, and let T be their union. Then the geometric realisation of $\mathcal{C}$ is homotopy equivalent to T.

Proof. See [17] Theorem 3.1 □

The following corollary is proven by showing that the cover defining the $\mathrm{Del}_r(X, E)$ for different $E \subset X$ satisfy the conditions in Theorem 5.1.

Corollary 5.1.1. The geometric realisations of the complexes $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, F)$ are homotopy equivalent for $E, F \subset X$.

Proof. Recall the definition of $\mathrm{Del}_r(X, E)$, Definition 3.15, as the nerve of the selective Voronoi cells of radius r, $Vor_r(x, E)$. As proven in Lemma 3.14, the collection of E selective Voronoi cells at radius r, denoted as $\mathcal{V}_r(E)$, is equal to the union of balls at radius r. Therefore, it is sufficient to prove that $|\mathrm{Del}_r(X, E)|$ and $|\mathrm{Del}_r(X, F)|$ are homotopy equivalent to the union of balls $T = \bigcup_{x \in X} B_r(x)$.

Furthermore, both a selective Voronoi cell and a ball of radius r are closed convex sets, therefore their finite intersection is also a closed convex set. Since a ball of radius r is bounded, so is a selective Voronoi cell at radius r. Therefore, a selective Voronoi cell at radius r is a convex closed bounded set of $\mathbb{R}^n$, and, hence, convex and compact. The nerve lemma 5.1 then implies that $|\mathrm{Del}_r(X, E)|$ and $|\mathrm{Del}_r(X, F)|$ are homotopy equivalent to the union of balls at radius r. This proves that geometric realisations of $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, E + x)$ are homotopy equivalent. □

Encoding retracts as collapses

The geometrical realisation associates a complex with a topology. A retract reduces a topological space into a smaller sub-space. However, not every deformation retract on the geometric realisation $|\mathbb{K}|$ guarantees that the resulting subspace is a geometrical simplicial complex corresponding to a sub-complex $\mathbb{K}'$ of $\mathbb{K}$. First, we consider a single 2-simplex where we investigate how to encode a deformation retract as the removal of simplices in the complex.

First, consider an example where this is the case.

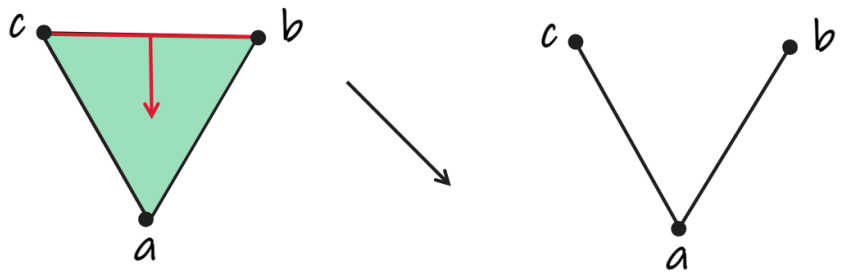


Figure 19: A retract of the geometric 2-simplex abc on the left into the complex on the right is shown.

Example 8. Denote the standard 2-simplex on vertices $X = \{a, b, c\}$ by $\mathbb{K}$. A geometric realisation $|\mathbb{K}|$ is depicted in Fig. 19.
Consider the face $Q = \{b, c\}$ and the deformation retract which pushes the interior of $|Q|$ into the (topological) interior of $|P|$. To visualise this, imagine that the green part is a sheet supported by the stick bc. If the support is removed, then the sheet falls. The resulting retract maps $|\mathbb{K}|$ into $|\mathbb{K}'| = |\mathbb{K}| - (int(|Q|) \cup int(|P|))$, where $int(-)$ denotes the topological interior.

The simplicial complex associated with $|\mathbb{K}'|$ is the following complex.

$$\mathbb{K}' = \{a, b, c, \{a, b\}, \{a, c\}\}$$

Thus the impact of the retract on the complex $\mathbb{K}$ is the removal of the pair $\{a, b, c\}$ and $\{b, c\}$ from $\mathbb{K}$. This pair is highlighted red in the following Hasse diagram of $\mathbb{K}$ and $\mathbb{K}'$. The arrow $\mathbb{K} \searrow \mathbb{K}'$ denotes the removal of this pair of simplices.

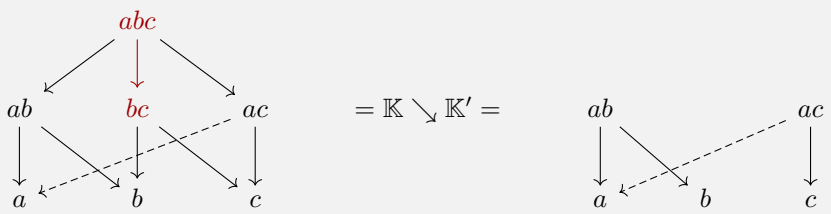


Figure 20: The diagram exemplifies the removal of a pair of $\mathbb{K}$ induced by the retract of Fig. 19.

In general, consider the n simplicial complex, denoted by Δ^n, and a $n-1$ face Q. There is a deformation retract of $|\Delta^n|$ into $|\Delta^n| - (int(|\Delta^n|) \cup int(|D|))$ similar to the case above. That this is true is not formally proven in this section as it follows from a more general discussion in Section 7, specifically, Lemma 7.13.

Since a complex is simply a collection of simplices, we expect to be able to iteratively apply the collapse in Example 8, see for example Fig. 21.

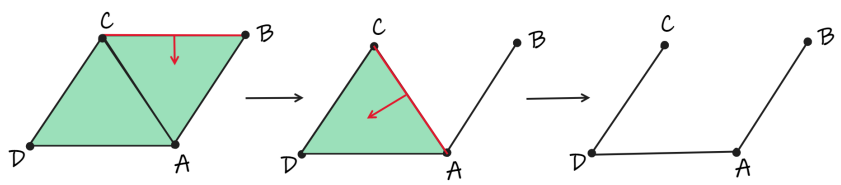


Figure 21: Repeated application of the collapse depicted in Fig. 19 is illustrated.

However, we need to be careful as we cannot expect that every removal of simplices corresponds to a correct deformation retract, see for example Fig. 22. To avoid this, we need to ensure that, of the pair $\{Q, P\}$ which is to be removed, Q is not attached to another simplex besides P.

Definition 5.2. If $\mathbb{K}$ is a complex, then $Q < P$ is called a **free face** of P in $\mathbb{K}$ if Q is not the face of

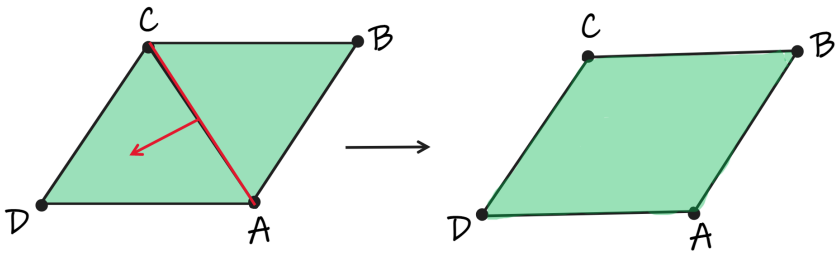


Figure 22: The homotopy which pushes the line segment into the triangle as depicted is a valid continuous map. However, the resulting space is not a simplicial complex, as it is not a collection of simplices.

another simplex, i.e. $Q < D$ implies $D = P$.[15]

The following definition states in what way the existence of a retract of a geometrical simplex in a geometrical simplicial complex is encoded in the associated abstract simplicial complex.

Definition 5.3. Suppose $\mathbb{K}$ has a n simplex P with a free $n-1$ face Q, then $\mathbb{K}$ has an **elementary collapse** into $\mathbb{K}' = \mathbb{K} - \{P, D\}$; we denote this as $\mathbb{K} \searrow \mathbb{K}'$.

An elementary collapse was depicted in Example 8. As illustrated in Fig. 21, we can iteratively apply elementary collapses.

Definition 5.4. A **collapse** between $\mathbb{K}$ and a subcomplex $\mathbb{K}'$, denoted $\mathbb{K} \searrow \mathbb{K}'$ is a sequence of elementary collapses $\mathbb{K} \searrow \ldots \searrow \mathbb{K}'$.

Simple homotopy theory

Although a collapse preserves the homotopy type, not all homotopy equivalences are induced via collapse. One example is the Dunce hat [24]. It has no free face, but it is contractible to a point. The collapses do preserve another notion called the **simple homotopy group**. This is first (extensively) introduced in [25]. If two spaces are related via a sequence of collapses and expansions, then they are simple homotopy equivalent. Therefore, Corollary 6.9.1 implies that the complexes are also simple homotopy equivalent. See [27] and [26] for more information in simple homotopy theory.

5.2 Characterising collapses

Discrete vector fields

An elementary collapse is the removal of a free pair of simplices. In a collapse, no pair can be removed twice. This motivates encoding a collapse with a partition such that every class is either a singleton or a pair of two adjacent nodes.

Definition 5.5. If P is a poset, then a partition $\sim$ is called a **(partial) matching** if it consists of singletons and pairs of adjacent nodes.

Since a collapse is an *ordered sequence* of elementary collapses, the matching should have no ambiguity in which order the pairs should be removed. This motivates the following notion.

Definition 5.6. A partition $\sim$ of a poset P is **acyclic** if the quotient $P/\sim$, with the inherited pre-order, is a poset.

To keep the geometric intuition of the collapse encoding a sequence of retracts depicted by arrows as in Fig. 21, we define the following.

Definition 5.7. An acyclic matching V is called a **discrete (gradient) vector field**. The paired simplices are called **non-critical**, and the others are **critical**

[15] Occasionally we say that the pair $\{Q, P\}$ is free since both Q and P are free if and only if Q is free

Remark 5.8. A matched pair $\{Q, P\}$ with $Q < P$ in a discrete vector field can be visualised in a geometric realisation by arrows pointing from the "midpoint", formally called the *barycentre*, of Q to P. The retract in Fig. 19 and Fig. 18 is indicated by such an arrow. Being a partition means that no simplex is the head or tail of two arrows. Acyclicity means that every sequence of arrows and faces can never form a cycle.

The following theorem shows that discrete vector fields can, indeed, encode collapses. The theorem is also found in [12] as Theorem 4.12 and it is a simplicial version of Theorem 3.5 in [20].

Theorem 5.9 (Discrete Morse theorem)**.** Suppose that $\mathbb{K}$ is a simplicial complex with a discrete vector field $\sim$ whose set of critical simplices is a sub-complex $\mathbb{K}'$, then $\mathbb{K} \searrow \mathbb{K}'$.

Proof. It suffices to prove that there exists a pair $\{Q, P\}$ in $\sim$ such that Q is a free face of P in $\mathbb{K}$. Indeed, then $\mathbb{K} \searrow \mathbb{K} - \{Q, P\}$ is an elementary collapse to a sub-complex. The discrete vector field $\sim$ induces a new discrete vector field $\sim'$ whose critical simplices form the sub-complex $\mathbb{K}'$ in $\mathbb{K} - \{Q, P\}$. Therefore, we can iteratively apply such elementary collapses. This implies that $\mathbb{K} \searrow \mathbb{K}'$.

To see that such a pair exists, consider that there is some pair $\{Q, P\}$ of non-critical simplices which is maximal in the poset $\mathbb{K}/\sim$. If no such maximal pair exists, then $\mathbb{K} = \mathbb{K}'$ and we are done. We claim that Q is free. Suppose in contrary that Q is not free, i.e. that $Q < D$ for some $D \in \mathbb{K}$. Then $\{Q, P\} < [D]$ in $\mathbb{K}/\sim$ and either $[D]$ is critical or non-critical. Because $\{Q, P\}$ is maximal, it follows that D must be critical. However, the set of critical simplices $\mathbb{K}'$ is a subcomplex. Therefore, since $Q < D$, it follows that $Q \in \mathbb{K}'$. This is a contradiction since Q is non-critical. It follows that Q is a free face of P in $\mathbb{K}$ which implies the theorem as argued before. □

Remark 5.10. Finding a discrete vector field is not necessarily hard, but finding one whose set of critical simplices is minimal is. Therefore, sometimes random algorithms are employed to reduce the memory storage of a simplicial complex [33].

Discrete Morse functions

Discrete vector fields can be characterised by the existence of monotone decreasing functions which attest to their acyclicity. This holds for each acyclic partition and is the content of the following lemma.

Lemma 5.11. Let $\mathbb{K}$ be a simplicial complex. A partition $\sim$ is acyclic iff there exists a poset P and an order-preserving function $f : \mathbb{K} \to P$ such that f is constant on equivalence classes.

Proof. $\Longrightarrow$ Define $P := \mathbb{K}/\sim$ and $f : \mathbb{K} \to P$ as the projection map. This satisfies the lemma.

$\Longleftarrow$ Suppose that $[a] > [b] > [a]$ for equivalence classes $[a], [b]$ in $\mathbb{K}/\sim$. Then $f(a) = f([a]) > f(b) = f([b]) > f([a]) = f(a)$ because f is order preserving. However, P is a poset set and thus this a contradiction. Therefore, the induced pre-order on $\mathbb{K}/\sim$ is anti symmetric, and is thus a partial order. □

Any partially ordered set P can always be linearly extended. Because $\mathbb{K}$ is finite, $\mathbb{K}/\sim$ is finite. Since the P from the lemma is $\mathbb{K}/\sim$, its linear extension is finite. This implies that P can be embedded in $\mathbb{R}$ via an order preserving function. Therefore, without loss of generality, we may assume that $P = \mathbb{R}$. A discrete vector field is thus equivalent to the existence of a real-valued function $f : \mathbb{K}/\sim \to \mathbb{R}$ by Lemma 5.11. Any such associated order preserving function f is called a **discrete Morse function** (DMF) with discrete gradient V.

Refining acyclic partitions

To find discrete vector fields, it helps to first search for large acyclic partitions and then refine them into smaller partitions. The following lemma guarantees that such an approach works.[16]

[16] In the literature, this is a well known result called the *cluster lemma*. A proof for acyclic matchings can be found in Lemma 4.2 in [13].

Lemma 5.12 (Cluster lemma)**.** Let $\mathbb{K}$ be a simplicial complex and $\sim$ an acyclic partition on $\mathbb{K}$. Let $\sim_{[k]}$ be an acyclic partition on each equivalence class $[k]$. Then $\sim' = \bigcup_k \sim_{[k]}$ is an acyclic partition on $\mathbb{K}$.

Proof. The relation $\sim'$ is a partition. Denote the order-preserving function of $\sim$ with f. Each $\sim_{[k]}$ defines an order preserving function $f_k : [k] \to P_k$. We define the set $P' = \bigsqcup_{[k]\in\mathbb{K}/\sim} P_k$. Let $x \in [k], y \in [k]'$, then define the order $<'$ on P' as $x <' y$ if $[k] \neq [k]'$ and $f(x) < f(y)$. If $[k] = [k]'$, then $x <' y$ if $f_k(x) < f_k(y)$. Then the function $f' = \sqcup f_k$ is order-preserving by construction and constant on equivalence classes of $\sim'$. $\square$

Let the **intersection** $\sim$ of two partitions $\sim_1, \sim_2$ be the partition $x \sim y$ if $x \sim_1 y$ and $x \sim_2 y$. This intersection preserves acyclicity by the following lemma.

Lemma 5.13. If $\sim_1, \sim_2$ are two acyclic partitions on $\mathbb{K}$, then their intersection is acyclic also.

The proof is straightforward.

6

Collapsing theorems

The previous sections explained how Voronoi balls define Delaunay complexes via the nerve construction. The existence of simplices Q in the Delaunay complex $\mathrm{Del}_r(X, E)$ is equivalent to solving the Minimal Enclosing Excluding Sphere (MEES) of the pair (Q, E). These MEES give rise to an affine optimisation problem. The corresponding solutions are unique if the points are in general position, but not necessarily otherwise. Furthermore, it was shown that $\mathrm{Del}_r(X, E + x) \subset \mathrm{Del}_r(X, E)$. A characterisation of when a complex collapses into a sub-complex was proven in Theorem 5.9.

Overview

This section shows that the Delaunay complex $\mathrm{Del}_r(X, E)$ collapses into $\mathrm{Del}_r(X, E+x)$, i.e. $\mathrm{Del}_r(X, E+x) \searrow \mathrm{Del}_r(X, E)$ for finite sets $E \subset X \subset \mathbb{R}^n$. The proof consists of finding a discrete vector field such that $\mathrm{Del}_r(X, E + x)$ is the set of critical simplices. Specifically, a pairing lemma states that a simplex $Q \in \mathrm{Del}_r(X, E)$ is removed in $\mathrm{Del}_r(X, E + x)$ if and only if this also holds for $Q + x$ and $Q - x$. Pairing $Q - x$ with $Q + x$ defines a discrete vector field which induces the collapse. A corollary is that the Čech complex collapses into the Delaunay complex.

Remark 6.1. The proof of this statement for X in general position is due to [7] Theorem 5.10. The proof of the non-general position case in this section is almost identical and is supplemented with an example.

Notation

From this point onwards, we fix sets $E \subset X \subset \mathbb{R}^n$, an $x \in X$, and a non-negative $r \in \mathbb{R}_{\geq 0}$ unless otherwise specified. Often a simplex $Q = \{a, b, c\}$ is denoted as the letter combination abc. Furthermore, we use the notation $Q + x = Q \cup \{x\}$ and denote with $Q - x$ the set Q with x removed. Lastly, the notation $Q \pm x$ refers to the fact that the statement holds for both $Q + x$ as $Q - x$.

6.1 Example of collapse

Let $X = \{a, b, c, d, e\}$ in $\mathbb{R}^2$ as indicated in Fig. 23. Notice that X is not in general position as the points a, b, c, e lie on the same circle, e.g. are concyclic, while the points bcd lie on one line, e.g. are not affinely independent. We remind the reader of notation and conventions used for computing MEES, see Definition 4.4.

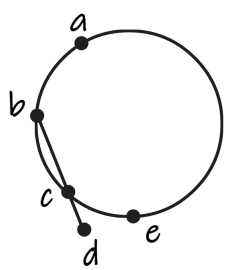


Figure 23: A configuration of points in the plane not in general position.

Computing $\mathrm{Del}_r(X, \emptyset)$

To calculate $\mathrm{Del}_r(X, \emptyset)$, the MEES $S(Q, \emptyset)$ of the subsets $Q \subset X$ are drawn in Fig. 24. The circles in green are those considered to have a radius δ smaller or equal to r. Therefore, *bcde*, *bcd* and *abc* are included in the complex $\mathrm{Del}_r(X, \emptyset)$.

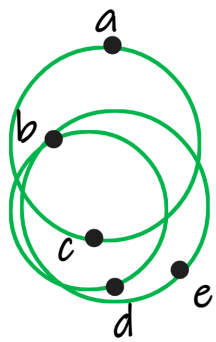


Figure 24: The MEES $S(Q, \emptyset)$ are drawn in green for subsets Q of $X = \{a, b, c, d, e\}$. The MEES $S(ae, \emptyset)$ and $S(ad, \emptyset)$ are not drawn since their radius is considered too large. Thus *ad* and *ae* are not in $\mathrm{Del}_r(X, \emptyset)$.

The complex $\mathrm{Del}_r(X, \emptyset)$ is drawn in the plane in Fig. 25. This visualisation is similar to the manner used before, see Fig. 6.

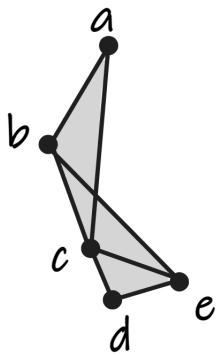


Figure 25: The complex $\mathrm{Del}_r(X, \emptyset)$.

We recall to the reader the definition of a geometrical simplicial complex, see Definition 2.1. The complex $\mathrm{Del}_r(X, \emptyset)$ is not drawn as a geometrical simplicial complex in Fig. 25. Indeed, both *bcd* and *bcde* are simplices whose vertices are not affinely independent. A geometric realisation of $\mathrm{Del}_r(X, \emptyset)$ is drawn in Fig. 26. Suddenly, the simplex *bcd* becomes visible. The 3-simplex *bcde* is in the complex and therefore the complex is 3 dimensional.

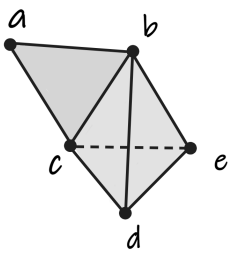


Figure 26: A geometric realisation of $\mathrm{Del}_r(X, \emptyset)$ is visualised in $\mathbb{R}^3$. Specifically, the 3-simplex, i.e. tetraeder, *abce* is drawn together with the 2 simplex, i.e. triangle, *abc*, in $\mathbb{R}^3$.

Computing $\mathrm{Del}_r(X, c)$

Since $\mathrm{Del}_r(X, c)$ is a subcomplex of $\mathrm{Del}_r(X, \emptyset)$, see Corollary 4.6.1, it suffices to calculate the MEES $S(Q, c)$ for each $Q \in \mathrm{Del}_r(X, \emptyset)$. This is shown in Fig. 27. The green circles indicate simplices in $\mathrm{Del}_r(X, c)$. The red circle denotes that $s(bce, \emptyset) \leq r < s(bce, c) < \infty$. Note that also $s(be, \emptyset) \leq r < s(be, c)$. In other words, the simplices bce and be are included in $\mathrm{Del}(X, \emptyset)$ but not present in $\mathrm{Del}_r(X, c)$. The purple circles similarly point out that $s(bd, \emptyset) \leq r < s(bd, c)$, but additionally that $s(bd, c) = \infty$. Since $s(bd, c)$ is a lower bound for $s(bde, c), s(bcd, c)$ and $s(bcde, c)$, those are also infinite. Therefore, the simplices bd, bde, bcd and $bcde$ are in $\mathrm{Del}_r(X, \emptyset)$, but they are not present in $\mathrm{Del}(X, c)$. Simplices that are included in $\mathrm{Del}_r(X, c)$ are abc and cde, as indicated by the green circles.

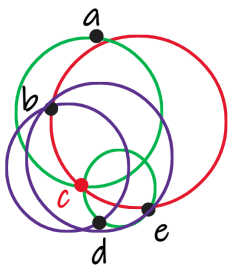


Figure 27: The MEES $S(Q, c)$ are drawn in green, red and purple for $Q \subset X$ such that $Q \in \mathrm{Del}_r(X, c)$, $Q \notin \mathrm{Del}_r(X, c)$ and $Q \notin \mathrm{Del}(X, c)$ respectively. Notice that the simplices corresponding to the red and purple MEES come in pairs $Q \pm c$.

The complex $\mathrm{Del}_r(X, c)$ is drawn in Fig. 28. The complex is defined by the simplices abc and cde, and is smaller than $\mathrm{Del}_r(X, \emptyset)$.

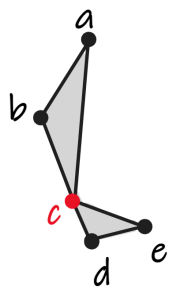


Figure 28: Depicting the complex $\mathrm{Del}_r(X, c)$ in the plane defines a geometric simplicial complex, in contrast to $\mathrm{Del}_r(X, \emptyset)$ which is not a geometric simplicial complex in the plane.

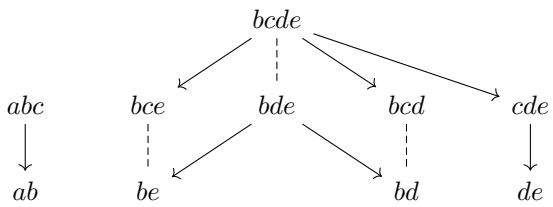
Constructing discrete vector field

Now that the complexes $\mathrm{Del}_r(X, \emptyset)$ and $\mathrm{Del}_r(X, c)$ are determined, the difference can be collapsed at the hand of a discrete vector field.

A discrete vector field can be constructed by observing that a simplex $Q \in \mathrm{Del}_r(X, \emptyset) - \mathrm{Del}_r(X, c)$ if and only if $Q \pm c \in \mathrm{Del}_r(X, \emptyset) - \mathrm{Del}_r(X, c)$. This can be measured by the radius of their MEES. For example, notice that

$$s(bde, \emptyset) = s(bcde, \emptyset) \leq r < s(bde, c) = s(bcde, c)$$

The observation that simplices in $\mathrm{Del}_r(X, \emptyset) - \mathrm{Del}_r(X, c)$ come in pairs $\{Q - c, Q + c\}$ is of a more general nature and formally proven in by the pairing lemma, Lemma 6.6. The pairing $\{Q - c, Q + c\}$ is represented in the following diagram, where a dotted line between simplices indicates paired simplices.



Note that this pairing is acyclic. That this pairing is acyclic in general follows from Lemma 6.7. Notice also that not all simplices present in $\mathrm{Del}_r(X, \emptyset)$ are drawn to ensure the Hasse diagram is not too large.

This acyclic pairing defines a discrete vector field. By the discrete Morse theorem, Theorem 5.9, the discrete vector field induces a collapse from $\mathrm{Del}_r(X, \emptyset)$ into the complex $\mathrm{Del}_r(X, c)$. This collapse is indicated in Fig. 29. The simplices which need to be removed are coloured red, while the subcomplex $\mathrm{Del}_r(X, c)$ is visualised in grey. Note that *bcde* and *bde* are paired, and the arrow indicates this at the midpoint of *bde* which points inwards *bcde*.

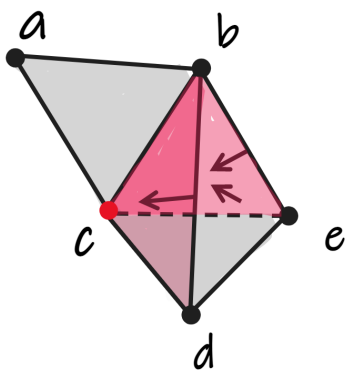


Figure 29: A collapse of $\mathrm{Del}_r(X, \emptyset)$ into $\mathrm{Del}_r(X, c)$ is shown.

The collapsing theorem, Theorem 6.9, is proven by showing that the procedure in this example can always be applied.

6.2 Preparatory lemmas

From our example, it seemed natural to consider pairs of the form $\{Q - x, Q + x\}$ to induce the collapse $\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E + x)$. An essential observation is that simplices had to be removed in pairs. Essentially, the general reason that this pairing occurs is that a sphere $S(Q, E)$ has $s(Q, E) < s(Q, E+x)$ if and only if $x \in \mathrm{on}\, S(Q, E+x)$ (or it does not exist). To prove that $s(Q, E) < s(Q, E+x)$ implies that $x \in \mathrm{on}\, S(Q, E + x)$ (if it exists) is precisely the reason that Theorem 4.7 has been introduced.

6.2.1 Same sphere lemmas

We refer the reader to the definition of a MEES $S(Q, E)$, see Definition 4.4, and of solutions $S_{Q,E}$, see Definition 4.8 for their definition and notation conventions. Specifically, a solution $S_{Q,E}$ is used to denote any set of coefficients $\{\lambda_x | x \in Q \cup E\}$ satisfying Theorem 4.7 for the MEES $S(Q, E)$. The following lemma is the more general version of Lemma 5.4 [7] with the same name.

Lemma 6.2 (Same sphere lemma)**.** Let $Q \in \mathrm{Del}(X, E)$ and $x \in X$, then

$$S(Q, E) = S(Q - x, E) = S(Q + x, E)$$

if and only if there exists some solution $S_{Q+x,E}$ for $S(Q + x, E)$ such that $\lambda_x \notin \mathrm{front}(S_{Q+x,E})$.

An example of this lemma is found in Example 7.

Proof. $\Longrightarrow$
If $S = S(Q - x, E) = S(Q + x, E)$, then $S(Q - x, E)$ exists since $Q \in \mathrm{Del}(X, E)$. Therefore, there is a solution $S_{Q-x,E}$. There are two cases, either $x \in Q - x \cup E$, or $x \notin Q - x \cup E$.
Case 1
Suppose that $x \in Q - x \cup E$, and, thus, $\lambda_x \in S_{Q-x,E}$. Since the solution $S_{Q-x,E}$ satisfies Theorem 4.7 for the pair $(Q - x, E)$ and $x \notin Q - x$, it holds that $\lambda_x \leq 0$ by condition 2 of Theorem 4.7. Because $\lambda_x \leq 0$, it follows that $\lambda_x \notin \mathrm{front}(S_{Q-x,E})$. Furthermore, because $S(Q - x, E) = S(Q + x, E)$, it holds that $S_{Q+x,E} := S_{Q-x,E}$ is a solution for $S(Q + x, E)$. Indeed, it defines an affine combination for the centre of $S(Q + x, E)$ whose coefficients satisfies Theorem 4.7. Therefore, $S_{Q+x,E}$ is a solution to $S(Q + x, E)$ such that $\lambda_x \notin \mathrm{front}(S_{Q+x,E})$.
Case 2
Suppose that $x \notin Q - x \cup E$, and, thus, $S_{Q-x,E}$ contains no coefficient λ_x. Define $\lambda_x := 0$ and let $S_{Q+x,E}$ denote the set of coefficients of $S_{Q-x,E}$ with λ_x added, i.e. $S_{Q+x,E} := S_{Q-x,E} \cup \lambda_x$. Note, $S_{Q+x,E}$ is a solution for $S(Q + x, E)$. Indeed, $Q + x$ is included by $S(Q - x, E)$ and $\lambda_x = 0$ satisfies the conditions of Theorem 4.7 in addition to the other coefficients of $S_{Q-x,E}$. Furthermore, $\lambda_x = 0$ implies that $\lambda_x \notin \mathrm{front}(S_{Q+x,E})$.

In both cases it follows that there exists a solution $S_{Q+x,E}$ such that $\lambda_x \notin \mathrm{front}\, S_{Q+x,E}$.

$\Longleftarrow$
Conversely, suppose $\lambda_x \notin \mathrm{front}(S_{Q+x,E})$. Notice that $S_{Q+x,E}$ is a solution for $S(Q + x, E)$ implicitly assumes that $S(Q + x, E)$ exists. Furthermore, because $\mathrm{front}(S_{Q+x,E}) \subset Q + x$ by Lemma 4.11, we conclude $\mathrm{front}(S_{Q+x,E}) \subset Q - x$. Now, either $\lambda_x = 0$ or $\lambda_x < 0$.
Case 1
If $\lambda_x = 0$, then the affine combination induced by $S_{Q+x,E}$ for z, the centre of $S(Q + x, E)$, still satisfies the conditions of Theorem 4.7 without x. Therefore, $S_{Q-x,E} := S_{Q+x,E} - \lambda_x$ defines a solution for $S(Q - x, E)$. This implies that $S(Q - x, E) = S(Q + x, E)$.
Case 2
If $\lambda_x < 0$, then $x \in E$ by Lemma 4.12. Define $S_{Q-x,E}$ to be equal to $S_{Q+x,E}$. Note that $S_{Q-x,E}$ consists of coefficients in $Q - x \cup E$ such that the affine combination is z, the centre of $S(Q + x, E)$, and they satisfy Theorem 4.7 for the pair $(Q - x, E)$. Therefore, $S_{Q-x,E}$ is a solution to $S(Q - x, E)$.

By Theorem 4.7, we conclude that in both cases $S(Q, E) = S(Q - x, E) = S(Q + x, E)$. □

A similar lemma regarding the constraint set is analogously proven.

Lemma 6.3. Let $Q \in \mathrm{Del}(X, E)$ and $x \in X$, then

$$S(Q, E) = S(Q, E - x) = S(Q, E + x)$$

if and only if there exists some solution $S_{Q,E+x}$ for $S(Q, E + x)$ such that $\lambda_x \notin \mathrm{back}(S_{Q,E+x})$.

Remark 6.4. Informally speaking, a MEES $S(Q, E)$ the MEES of the pair $(Q - x, E)$ if there is a solution $S_{Q,E}$ that does not require the coefficient λ_x to be positive. In this sense, there is some freedom in the choice of the solution $S_{Q,E}$. This freedom of choice means that multiple solutions can yield the same MEES. Therefore, some points can be removed from the constraints without changing the MEES, as another affine combination still exists that satisfies Theorem 4.7.

6.2.2 Pairing lemma

If x is excluded by $S(Q, E)$, then $S(Q, E)$ is also the MEES of the pair $S(Q, E + x)$. If x is strictly included, then, necessarily, $s(Q, E) < s(Q, E + x)$. The same sphere lemma, Lemma 6.3 then states that $\lambda_x \in \mathrm{back}(S_{Q,E+x})$ for all solutions of $S_{Q,E+x}$ (assuming it exists). In particular, this implies that $x \in \mathrm{on}\, S(Q, E + x)$. Therefore, $S(Q, E + x)$ is also the MEES of the pair $(Q + x, E + x)$. This indicates that $S(Q, E) = S(Q \pm x, E)$ and $S(Q, E + x) = S(Q \pm x, E + x)$, see Fig. 30. However, pre-caution is required as the MEES $S(Q, E + x)$ may not exist. The pairing lemma, Lemma 6.6, with the same name as Lemmas 5.5 and 5.7 in [7], formalises that $S(Q, E) = S(Q \pm x, E)$ and $S(Q, E + x) = S(Q \pm x, E + x)$

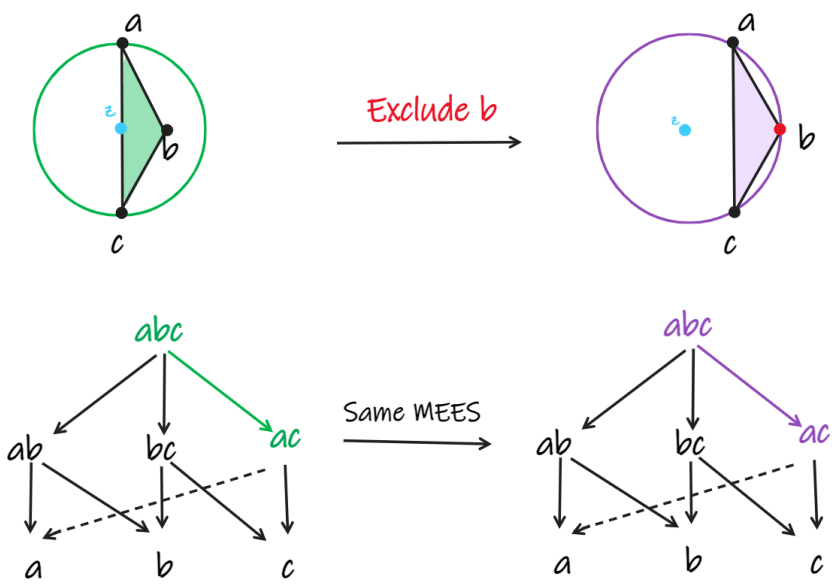


Figure 30: The pairing lemma is illustrated with an example of three points. Note that $\{abc, ac\}$ have constant MEES for both constraint sets $E = \emptyset$ and $E = \{b\}$. Such a pair hints at creating a discrete vector field which induces the collapse.

Remark 6.5. If $x \notin \mathrm{excl}(S(Q,E))$ (or $x \notin \mathrm{incl}(S(Q,E))$), then in particular $x \notin \mathrm{on}(S)$, hence $\lambda_x \notin \mathrm{front}(S_{Q,E}) \cup \mathrm{back}(S_{Q,E})$ for every solution $S_{Q,E}$ by Theorem 4.7.

Lemma 6.6 (Pairing lemma)**.** Suppose that $Q \in \mathrm{Del}(X,E)$ and x is strictly included by $S(Q,E)$, then

$$s(Q,E) = s(Q \pm x, E) \tag{2}$$
$$s(Q,E+x) = s(Q \pm x, E+x) \tag{3}$$

Proof. Because $S(Q,E)$ includes x, the MEES $S(Q+x,E)$ exists. Therefore, a solution $S_{Q+x,E}$ exists. Because x is strictly included, in particular, $x \in \mathrm{incl}(S(Q+x,E)) - \mathrm{front}(S_{Q+x,E})$ for every solution $S_{Q+x,E}$. By Lemma 6.2 for the solution $S_{Q+x,E}$, the equality $S(Q,E) = S(Q \pm x, E)$ holds. This proves the first statement.

If $s(Q,E+x) \geq s(Q-x,E+x) = \infty$ then we are done. Suppose that $s(Q-x,E+x) < s(Q,E+x)$. Then, in particular, $S(Q-x,E+x)$ exists and it strictly excludes x. Therefore, $x \notin \mathrm{on}(S(Q-x,E+x))$, and, in particular, $\lambda_x \notin \mathrm{back}(S_{Q-x,E+x})$ for all solutions $S_{Q-x,E+x}$. Since $S(Q-x,E+x)$ exists, there is a solution $S_{Q-x,E+x}$. By Lemma 6.3, we can conclude that $S(Q-x,E+x) = S(Q-x,E)$. Because the spheres are equal, the latter sphere also strictly excludes x. In the first part, we concluded that $s(Q-x,E) = s(Q+x,E)$. The latter sphere includes x and the former sphere strictly excludes x. However, the spheres are equal, and, therefore, this is a contradiction. We conclude that $s(Q,E+x) = s(Q-x,E+x)$.

Note that $s(Q,E+x) \leq s(Q+x,E+x)$. If $s(Q,E+x) = \infty$, then $s(Q+x,E+x) = \infty$ and equality follows.
If $s(Q+x,E+x) > s(Q,E+x)$, then the sphere $S(Q,E+x)$ exists and strictly excludes x. Since, by assumption, $s(Q,E) < s(Q,E+x)$, it follows that $S(Q,E+x)$ has $\lambda_x \in \mathrm{back}(S_{Q,E+x})$ for all solutions $S_{Q,E+x}$. Indeed, otherwise $S(Q,E) = S(Q,E+x)$ by Lemma 6.3. It follows that $x \in \mathrm{on}(S(Q,E+x)) \subset \mathrm{incl}(S(Q,E+x))$. However, $S(Q,E+x)$ both strictly excludes x and includes x. This is a contradiction, and, therefore, $S(Q,E+x) = S(Q+x,E+x)$.

We conclude that $s(Q,E+x) = s(Q \pm x, E+x)$. □

For Delaunay complexes, the pairing lemma, Lemma 6.6, has the following corollary.

Corollary 6.6.1. For $E \subset X \subset \mathbb{R}^n$ finite and $x \in X$, then

$$Q + x \in \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + x)$$

if and only if

$$Q - x \in \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + x)$$

Remember that the notation $Q \pm x$ refers to the statement $Q + x$ and $Q - x$. In this sense, both implications are given at the same time in the following proof.

Proof. By Lemma 4.6, $Q \pm x \in \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + x)$ translates into

$$s(Q \pm x, E) \leq r < s(Q \pm x, E + x)$$

This means that $s(Q \pm x, E) < s(Q \pm x, E + x)$. In turn, this implies that $S(Q \pm x, E)$ strictly includes x. By the pairing lemma, Lemma 6.6, we can conclude that $s(Q \mp x, E) = s(Q \pm x, E) \leq r$ and $r < s(Q \pm x, E + x) = (Q \mp x, E + x)$. This proves the corollary. □

From this corollary, we obtain a partition of $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + x)$ into pairs $\{Q + x, Q - x\}$. If we can prove that the collection of these pairs is acyclic, then these pairs define a discrete vector field that induces the collapse $\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E + x)$.

6.2.3 Vertex refinement

Notice that $\{x - x, x\} = \{\emptyset, x\}$ is not a pair in any Delaunay complex as $\emptyset$ is not in a Delaunay complex. However, since the vertex set is in every Delaunay complex, this is not relevant for any of the discrete vector fields that are examined in this thesis.

Lemma 6.7 (vertex refinement)**.** Let $\mathbb{P}(X)$ be the powerset of X. Then the partition into pairs $\{Q - x, Q + x | Q \in \mathbb{P}(\mathbb{X})\}$ is acyclic.

Proof. If $\{Z - x, Z + x\} > \{Q - x, Q + x\}$ in the quotient with the induced pre-order, then $Z + x > Q - x$. Either $Z + x > Q + x$ or $Q + x$ and $Z + x$ are incomparable. In the former case, there exists some $y \neq x$ in $Z + x$ which is not in $Q + x$. This y must also be in $Z - z$ and not in $Q - x$. We conclude that $|Z - x| > |Q - x|$.

In the latter case, there is at least one $y \neq x$ in $Z + x$ which is not in $Q - x$, which implies that y is in $Z - x$. We conclude again that $|Z - x| > |Q - x|$.

We conclude that the function $\{Z - x, Z + x\} \mapsto |Z - x|$ strictly decreases along the pre-order. This implies that the pre-order is a partial order; hence the partition is acyclic. □

Remark 6.8. In principle, x in Lemma 6.7 can be replaced by any nonempty subset $Y \subset X$. This is done in Lemma 8.1 later.

6.3 Proof of collapsing theorem

We now prove the collapse of $\mathrm{Del}_r(X, E)$ into $\mathrm{Del}_r(X, E + x)$. The collapsing theorem, Corollary 6.9.1, follows from successive applications of these collapses.

For this proof, we use the following notation.

$$\{Q + x, Q - x\}_r = \{Q + x, Q - x\} \cap \mathrm{Del}_r(X, E)$$

Theorem 6.9. For $E \subset X \subset \mathbb{R}^n$ finite point clouds and r non negative, then

$$\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E + x)$$

Proof. Construct the partition $\{\{Q+x, Q-x\}_r | Q \in \mathrm{Del}_r(X,E)\}$. The partition is acyclic due to Lemma 6.7. Therefore, due to the cluster lemma, Lemma 5.12, it suffices to prove that each of the $\{Q+x, Q-x\}_r$ has an acyclic partition such that the pairs are in $\mathrm{Del}_r(X,E) - \mathrm{Del}_r(X,E+x)$ and the singletons in $\mathrm{Del}_r(X,E+x)$.

Corollary 6.6.1 shows that $Q \in \mathrm{Del}_r(X,E) - \mathrm{Del}_r(X,E+x)$ is equivalent to $Q \pm x \in \mathrm{Del}_r(X,E) - \mathrm{Del}_r(X,E+x)$.

This gives two cases: either $\{Q+x, Q-x\}_r = \{Q+x, Q-x\} \subset \mathrm{Del}_r(X,E) - \mathrm{Del}_r(X,E+x)$, or $\{Q+x, Q-x\}_r \subset \mathrm{Del}_r(X,E+x)$

In the former case, define $\{Q+x, Q-x\}_r$ as an equivalence class in the partition.
In the latter case, partition $\{Q+x, Q-x\}_r$ into singletons. Such a (trivial) partition has the usual pre-order and is acyclic.

The classes of the defined acyclic partitions on the $\{Q-x, Q+x\}_r$ are singletons or pairs and, hence, form an acyclic matching. As argued before, the resulting partition on $\mathrm{Del}_r(X,E)$, by taking unions of the partitions on the $\{Q-x, Q+x\}_r$, defines a discrete vector field. Furthermore, the non-critical simplices precisely make up $\mathrm{Del}_r(X,E) - \mathrm{Del}_r(X,E+x)$. We can conclude that the partition is a discrete vector field whose set of critical simplices is $\mathrm{Del}_r(X,E+x)$. By Theorem 5.9 it follows that $\mathrm{Del}_r(X,E) \searrow \mathrm{Del}_r(X,E+x)$. This was what had to be proven. □

The following corollary is obtained by iteratively applying the collapsing theorem.

Corollary 6.9.1 (Delaunay collapsing theorem)**.** Let $\emptyset \leq x_1 \leq \ldots \leq x_n$ be some chosen order, then

$$Cech_r(X) = \mathrm{Del}_r(X,\emptyset) \searrow \mathrm{Del}_r(X,x_1) \searrow \ldots \searrow \mathrm{Del}_r(X,X) = \mathrm{Del}_r(X)$$

Remark 6.10. Note, adding more points at a time, i.e. collapsing $\mathrm{Del}_r(X,E) \searrow \mathrm{Del}_r(X,F)$ with $E \subset F$, can also be done by choosing an order of $E - F$ and applying Theorem 6.9 iteratively.

7

Specialisation to symmetric data

The previous sections introduced selective Delaunay complexes $\mathrm{Del}_r(X, E)$ for $E \subset X \subset \mathbb{R}^n$ whose simplices can be computed in terms of Minimal Enclosing Excluding Spheres, see Definition 4.4. Further, a series of collapses, induced by discrete vector fields, existed between two selective Delaunay complexes. This implied the collapsing theorem, see Corollary 6.9.1.

Overview

In this section, the nomenclature is introduced to formally capture what is meant by 'symmetry'. Thereafter, the theory of Section 3 and Section 5 is repeated for the special case of a symmetrical data set X. The symmetry considered is 'geometrical' in the sense that it is a symmetry of $\mathbb{R}^n$ of which X is a subset. At the end of this section, equivariant discrete Morse theory characterises collapses that are symmetrical.

7.1 Nomenclature

A symmetry on any set A is formalised by a group G **acting** on A. This means that there is a group homomorphism $\rho : G \to Aut(A)$, where $Aut(A)$ is the group of *automorphisms*, i.e. bijective functions, of A. The latter is a group via function composition $f \circ h$. The action ρ is implicit and understood from the context. In this sense, we identify G with its image $\rho(G)$ in $\subset Aut(A)$, i.e. we say $G \subset Aut(A)$. In this thesis, only finite groups are considered.

Not all actions preserve structures present in the set A. Actions which preserve structure are often given special names. In the following definitions, we define a few names.

Definitions

Definition 7.1. If A is set and there is an action $\rho : G \to Aut(A)$, the bijective functions of A, then A is called a G **set**.

Note that a *simplicial morphism* $f : \mathbb{K} \to \mathbb{K}'$, see Definition 2.4, is a map f_0 between their vertices such that f sends simplices in $\mathbb{K}$ to simplices in $\mathbb{K}'$. The simplicial automorphism group of the simplicial complex $\mathbb{K}$, denoted by $S - Aut(A)$ consists of all simplicial morphisms $f : \mathbb{K} \to \mathbb{K}$ whose vertex map f_0 is bijective and the inverse also defines a simplicial morphism.

Definition 7.2. A simplicial complex $\mathbb{K}$ is a **simplicial** G **complex** if there is an action $\rho : G \to S - Aut(A)$.

Furthermore, recall from linear algebra that the set of invertible linear maps on a vector space V is called the **general linear group**, denoted as $GL(V)$. In other words, the 'linear versions' of automorphisms are the general linear maps.

Definition 7.3. If V is a finite dimensional vector space over a field k, then an action $\rho : G \to GL(V)$ is called a **(linear) representation** of G over V.

Remark 7.4. For the current and following section, the field k used is the set of real numbers, $\mathbb{R}$. This makes all representations *real representations*. When studying barcodes in a later section, we use a different field, specifically, the field $\mathbb{F}_2$.

Notice that a simplicial G complex and a vector space V with a linear representation are both, in particular, G sets.

If X is a G-set, then the **orbit** of $x \in X$, denoted by Gx, is the set of all y such that $g(x) = y$ for some $g \in G$. If $g(Q) = Q$ for any $Q \subset X$ then g is said to **fix** Q. For $Q \subset X$, the notation G^Q denotes the set of all $g \in G$ such that $g(Q) = Q$, i.e. the set of group elements fixing Q. The set is called the **stabiliser subgroup** of Q (with respect to G).

Symmetrical data

Recall that the Delaunay complex $\mathrm{Del}_r(X, E)$ is defined by the ambient space $\mathbb{R}^n$, where $X \subset \mathbb{R}^n$. Therefore, we expect it to be a simplicial G complex if there is a linear representation on $\mathbb{R}^n$ of G such that X is symmetric under this representation. However, not all linear representations preserve MEES/Euclidean metric. Since $\mathrm{Del}_r(X, E)$ is defined by the standard Euclidean metric, we restrict ourselves to representations which preserve the Euclidean metric. More exactly, we require that the representation ρ of G restricts to the linear orthogonal group, i.e. $\rho : G \to O(n)$. Indeed, linear maps in $O(n)$ preserve the standard Euclidean metric.
The terminology has been introduced to consider in what way X is considered to be 'symmetric'.

Definition 7.5. A finite set $X \subset \mathbb{R}^n$ is said to be G **symmetric**, with respect to group G, if there is a representation $\rho : G \to O(n)$ such that the induced action on X makes X into a G-subset of $\mathbb{R}^n$.

Since an action g preserves the euclidean metric of $\mathbb{R}^d$, such actions are isometries. Notice that a G symmetric set X, in the sense of Definition 7.5, is (usually) not in general position, see Definition 3.11. For example, the four points in Fig. 11 form a G symmetric set where G acts as rotation by $1/2\pi$. In general, this is due to the action preserving MEES whose centre is fixed under the action, see Lemma 7.11.

A few of these types of symmetrical data sets X are considered in [34]. In the paper, a perturbation method, which puts symmetrical sets in general position, is shown to be biased towards providing certain Delaunay triangulations over others. Furthermore, the most common type of triangulations might not be 'nice looking', i.e. they might contain many small triangles. It is unknown to me whether other perturbation methods, such as Theorem 4.1 [8], have the same bias behaviour. Delaunay triangulations are, however, outside of the scope of this thesis and more information can be found in [35] Section 5.5 and 5.6.

The natural notion of a 'symmetry preserving' function $f : X \to Y$ between G sets X and Y, is a function that transforms the symmetry on X into the symmetry of Y. More exactly, the function should commute with the actions $g \in G$.

Definition 7.6. Let G be a group, X and Y both G sets, and $f : X \to Y$ a function. The function f is G-**equivariant** if $gf(x) = f(gx)$ for all $g \in G$ and $x \in X$.

Remark 7.7. In categorical language, any category in which all maps are invertible is called a **groupoid**. A group G then is defined as a groupoid category with a single object. A homomorphism is a functor $F : G \to C$ such that the only object is sent to X. In this language, the previous definitions turn into the following.

- If $C = Set$, then X is a G-**set**.
- If $C = Simp$, the category of simplicial complexes, then X is a **simplicial** G **complex**.
- If $C = Vect_k$, then F is called a **representation of** G.

Assumption

Throughout this and the following section, we assume that there is a representation $\rho : G \to O(n)$ of a finite group G. If X is a G-set, then it is G symmetric set with respect to this representation.

Furthermore, unless noted otherwise, it is assumed that E and X are G symmetrical subsets of $\mathbb{R}^n$ and $E \subset X$.

7.2 Equivariant Delaunay complex

The following lemma proves that the Delaunay complex $\mathrm{Del}_r(X, E)$ is a simplicial G complex, if both X and E are G symmetric.

Lemma 7.8. The Delaunay complex $\mathrm{Del}_r(X, E)$ is a simplicial G complex.

Proof. The vertices of $\mathrm{Del}_r(X, E)$ is X which is a $G - set$. Let Q be some simplex in $\mathrm{Del}_r(X, E)$ and $g \in G$, then

$$gVor(x, E) = g\{y \in \mathbb{R}^n | d(x, y) \leq d(y, z) \forall z \in E\} \tag{4}$$
$$= \{y \in \mathbb{R}^n | d(gx, y) \leq d(y, z) \forall z \in E\} = Vor(gx, E) \tag{5}$$

This equality holds since $gE = E, g\mathbb{R}^n = \mathbb{R}^n$ and implies that $g : Vor(x, E) \mapsto Vor(gx, E)$ permutes the Voronoi balls.

For any simplex $Q \in \mathrm{Del}_r(X, E)$, by definition, $\bigcap_{x \in Q} Vor_r(x, E) \neq \emptyset$. Since each of the g is a bijection and $g(Vor(x, E)) = Vor(gx, E)$, we can conclude that

$$g(\bigcap_{x \in Q} Vor_r(x, E)) = \bigcap_{x \in Q} Vor_r(g(x), E) \neq \emptyset$$

Thus, $g(Q) = \{g(q_1), \ldots, g(q_n)\} \in \mathrm{Del}_r(X, E)$, which means that $\mathrm{Del}_r(X, E)$ is a simplicial G complex. □

The structure of how an element $g \in G$ permutes simplices is reminiscent of *equivariance* of functions, i.e. $gf(x) = f(gx)$. This motivates the following name.

Definition 7.9. If $\mathrm{Del}_r(X, E)$ is a simplicial G-complex, then it is called the **equivariant Delaunay complex**.

7.3 MEES and G-actions

A Delaunay complex is equivalently described at the hand of the MEES, Lemma 4.6, and see Definition 4.4 for the definition of a MEES. Not surprisingly, the MEES function is equivariant, that is, $gS(Q, E) = S(gQ, gE)$. This is the content of the following lemma.

Lemma 7.10. The MEES function is equivariant, i.e. $gS(Q, E) = S(gQ, E)$ for all $g \in G$.

Proof. Since $\mathrm{Del}_r(X, E)$ is a simplicial G complex, a simplex Q is in $\mathrm{Del}_r(X, E)$ if and only if gQ is in $\mathrm{Del}_r(X, E)$ for all $g \in G$. In particular, this implies that $S(gQ, E)$ exists. Furthermore, since each $g \in G$ is an isometry, it preserves spheres. Let $S(Q, E) = (z, r)$, then each $q \in Q$ has $d(q, z) \leq r$ and so $gS(Q, E)$ has $d(gq, gz) = d(q, z) \leq r$. Therefore, $gS(Q, E)$ includes gQ and excludes E. If $s(Q, E) < s(gQ, E)$, then $g^{-1}S(gQ, E)$ includes Q, excludes E and has strictly smaller radius then $S(Q, E)$. This is a contradiction and we conclude that $gS(Q, E) = S(gQ, E)$. Thus the MEES function is equivariant. □

Another simple observation is that for any sphere S, $\mathrm{incl}(S), \mathrm{excl}(S)$ and $\mathrm{on}(S)$ are also equivariant.

Lemma 7.11. Let S be a sphere in $\mathbb{R}^n$, then

$$g\,\mathrm{incl}(S) = \mathrm{incl}(gS) \tag{6}$$
$$g\,\mathrm{excl}(S) = \mathrm{excl}(gS) \tag{7}$$
$$g\,\mathrm{on}(S) = \mathrm{on}(gS) \tag{8}$$

Proof. This follows directly from the definition and the fact that g acts as an isometry. □

7.4 Equivariant discrete Morse theory

The symmetric specialisation of discrete Morse theory, called equivariant discrete Morse theory, was first introduced in [11]. An equivariant collapse between two simplicial G complexes encodes an equivariant deformation retract similar to a 'regular' collapse. In turn, equivariant discrete Morse theory characterises equivariant collapses between simplicial G complexes.

7.4.1 Equivariant homotopy equivalences

Recall that a homotopy equivalence between X, Y are two maps $f : X \to Y, g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. If this map is G equivariant in the sense that $gH(x,t) = H(gx,t)$ for all $g \in G$, then the homotopy equivalence is called G equivariant. Furthermore, we remind the reader of a nerve, see Definition 3.3. There is an G equivariant homotopy equivalence between $Del_r(X,E)$ and $Del_r(X, E + Gx)$. Such a result can be concluded from an equivariant version of the nerve lemma [36] Lemma 2.5.

Lemma 7.12. Let $\mathbb{K}$ be a simplicial G complex and let $\mathcal{U} = \{U_i | i \in I\}$ be a G-invariant covering of $\mathbb{K}$. If every nonempty finite intersection $\cap_{i \in Q} U_i$ for $Q \subset I$, is G^Q-contractible, then $|\mathbb{K}|$ and the nerve $\mathcal{N}(\{U_i | i \in I\})$ are G-homotopy equivalent, i.e. $|\mathbb{K}| \sim_G \mathcal{N}(\mathcal{U})$

Proof. See Lemma 2.5 in [36]. □

Here G-equivariant covering means that the induced action of G on the cover permutes the sets of the cover. In other words, I is a G-set. The good covering is satisfied for closed convex subsets of $\mathbb{R}^2$. The notion of G^Q -contractible means that the homotopy H of the deformation retraction is equivariant with respect to G^Q.

Corollary 7.12.1. Let $E \subset X \subset \mathbb{R}^n$ be G symmetrical, $x \in X$ and $r \in \mathbb{R}_{\geq 0}$. Then the complex $\mathrm{Del}_r(X,E)$ is G-equivariantly homotopy equivalent to $\mathrm{Del}_r(X, E + Gx)$, i.e.

$$Del_r(X,E) \simeq_G Del_r(X, E + Gx)$$

Proof. Note that $\mathrm{Del}_r(X,E)$ and $\mathrm{Del}_r(X, E + Gx)$ are both the nerve of the union of Voronoi balls $Vor_r(y,E)$ and $Vor_r(y, E + x)$ of radius r around points $y \in X$ respectively. The union of Voronoi balls is independent of the excluded set E by Lemma 3.14. Therefore, if we can prove that $\mathrm{Del}_r(X,E) \simeq_G \bigcup_{x \in X} B_r(x) \simeq_G \mathrm{Del}_r(X, E + Gx)$, then we are done. The equivariant nerve lemma, see Lemma 7.12, implies the wanted equivalence.

Note that G permutes the Voronoi balls by Lemma 7.8, i.e. the cover is G invariant. It remains to prove that the intersection, $U = \cap_{y \in Q} Vor_r(y,E)$ corresponding to a simplex Q is equivariantly contractible with respect to G^Q.

Since the intersection of convex sets is convex, U is convex. Let $q \in U$ and construct the following fixed point $z = \frac{1}{|G^Q|} \sum_{g \in G^Q} g(q)$. This point is fixed under G^Q and lies inside U since it is a convex combination of points in U. Define the linear homotopy $H(x,t) = (1-t)x + tz$. Since $H(x,t)$ is, for all x, t a convex combination of points in U, it is a homotopy in U. It defines an homotopy between the identity and the constant map $x \mapsto y$. The homotopy is also equivariant since $gH(x,t) = (1-t)gx + tgz = (1-t)gx + tz = H(gx,t)$. We conclude that U is G^Q contractible. By Lemma 7.12 the proof can be concluded. □

This approach has similar downsides as the nerve lemma in the non-equivariant case: It is unclear whether this homotopy equivalence is stable under the radius and whether it is induced by collapses.

7.4.2 Geometrical equivariant collapse

The main theorem Corollary 8.19.1 relies on an equivariant version of discrete Morse theory. This is the work of [11]. Therefore, similar nomenclature to that work is chosen in this thesis.

For a geometric n-simplex W and a face D, let $P_W(D)$ denote the union of all $n-1$ faces of W that contain D. Furthermore, define the dual of D in W, denoted as D^*, to be the set of vertices of W not contained in D, i.e. $D^* = W_0 - D_0$.

Lemma 7.13. If W is a geometric n-simplex and D is a face, then W retracts into $P_W(D^*)$. Moreover, if g is a permutation of W_0 such that $g(D_0) = D_0$, then the retract $H : W \times I \to W$ can be chosen equivariant with respect to g, i.e. $gH(x,t) = H(gx,t)$ for all x, t.

Proof. See proposition 4.1 [11] □

This lemma is best supplemented by Fig. 31, similar to Figure 4 in [11].

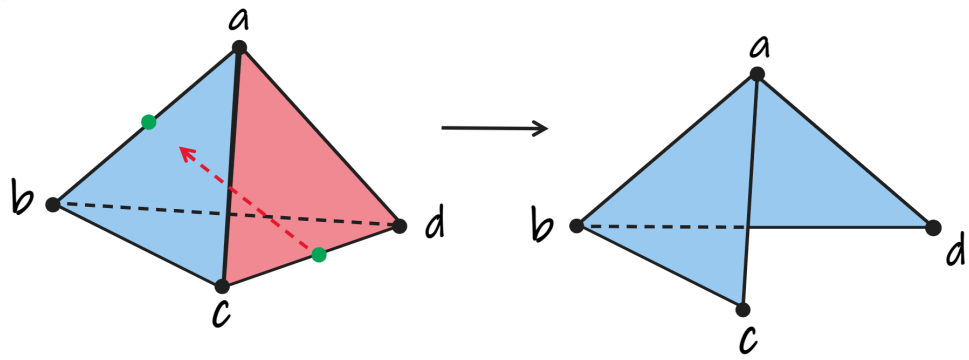


Figure 31: The 3-simplex, i.e. tetrahedron, on the left is retracted into the complex consisting of two triangles abd and abc. If g is the reflection of the hyperplane through the centre of cd and ab which sends a to b and c to d, then this retract can be done equivariantly. The retraction can be imagined as pushing the centre of cd, the green dot, into the centre of ab, and the rest into the triangles defined by abd and abc. This is indicated by the red arrow. The simplices which need to be collapsed are coloured red.

From Fig. 31 can be seen that the resulting abstract simplicial complex is precisely $\mathbb{K} - [cd, abcd]$. In other words, an interval has been removed. This suggests replacing acyclic matchings by acyclic partitions of intervals.

7.4.3 Simplicial equivariant collapse

Lemma 7.13 motivates to consider acyclic partitions of intervals, similar to acyclic matching, as a way to encode equivariant collapses.

A simplex W in a complex $\mathbb{K}$ is an **interval free face** of Q if it is only a face of simplices in $[W, Q]$. Furthermore, a complex $\mathbb{K}$ is said to have an **elementary interval collapse** to a subcomplex $\mathbb{K}'$ if $\mathbb{K}' = \mathbb{K} - [W, Q]$ where W is an interval free face of Q. To extend collapses to be equivariant, notice that the subcomplex $\mathbb{K}$ is only a G subcomplex if all intervals in the orbit $G[W, Q]$ are removed. This motivates the following definition.

Definition 7.14. A G complex $\mathbb{K}$ has a G **equivariant elementary collapse** to a G subcomplex $\mathbb{K}'$, if $\mathbb{K}' = \mathbb{K} - G[W, Q]$ where W is an interval free face of Q.

Remark 7.15. An G equivariant elementary collapse $\mathbb{K}$ to $\mathbb{K}'$ models a G equivariant retract between the geometric realisations $|\mathbb{K}|$ and $|\mathbb{K}'|$ by Lemma 7.13. Therefore, the spaces $|\mathbb{K}|$ and $|\mathbb{K}'|$ are G homotopy equivalent.

Similar to non-equivariant case, a G equivariant collapse is simply a sequence of elementary equivariant interval collapses.

Definition 7.16. A simplicial G complex $\mathbb{K}$ is said to G equivariantly collapse into a simplicial G subcomplex $\mathbb{K}'$ if there is a sequence of G equivariant elementary collapses of $\mathbb{K}$ into $\mathbb{K}'$, i.e.

$$\mathbb{K} \searrow_G \dots \searrow_G \mathbb{K}'$$

Such a collapse is denoted by $\mathbb{K} \searrow_G \mathbb{K}'$.

The notion of matching is generalised in the following definition.

Definition 7.17. A partition $\sim$ of a poset P is a **generalised matching** if the equivalence classes consist of intervals $[Q, D]$ where $Q, D \in P$.

Similar to the non-equivariant case, for a collapse $\mathbb{K} \searrow_G \mathbb{K}'$, the orbit $G[W, Q]$ is only removed once in a collapse. This motivates to encode such a collapse via a partition.

Definition 7.18. Let $\sim$ be a partition on a G-set X. The partition is **equivariant** if $B \in\sim$ if and only if $g(B) \in\sim$ for all $g \in G$ where $g(B) = \{g(b)|b \in B\}$.

Furthermore, the sequence of elementary G equivariant collapses forms a partition of intervals of the difference $\mathbb{K} - \mathbb{K}'$. Equivariant discrete Morse theory encodes the existence of a collapse $\mathbb{K} \searrow_G \mathbb{K}'$ by a G equivariant acyclic partition of intervals.

Definition 7.19. If $\sim$ is an acyclic partition of intervals on a simplicial complex $\mathbb{K}$ such that the singletons form a subcomplex, then $\sim$ is a **generalised Morse matching**.

Similar to a matching, the singleton classes are called **critical** and others are **non-critical**.

Remark 7.20. In contrast to a discrete vector field, the nomenclature generalised Morse matching is chosen to stay in line with [11]. Alternatively, we could have chosen the name of a G equivariant generalised discrete vector field.

Remark 7.21. An example of an equivariant interval in $\mathbb{P}(X)$, the powerset of X, is $[P - G^P x, P + G^P x]$ for G^P, the stabiliser subgroup of P. This interval reappears in the equivariant version of the pairing lemma, Lemma 8.5.

Theorem 7.22 below is a straightforward corollary of the proof of Theorem 4.2 in [11] for CW complexes.

Theorem 7.22 (Equivariant simplicial Morse lemma)**.** Let $\mathbb{K}$ be a simplicial G complex and $\mathbb{K}'$ a simplicial G sub-complex. Let $\sim$ be an G equivariant generalised Morse matching such that $\mathbb{K}'$ is the set of critical simplices, then

$$\mathbb{K} \searrow_G \mathbb{K}'$$

Proof. Similar to Theorem 5.9, let $I = [W, Q]$ be a maximal non-critical interval in $\sim$. If no such I exists $\mathbb{K}' = \mathbb{K}$ and we are done. Suppose instead that W is not an interval free face of Q, i.e. there exists a $P > W$ such that $P \notin [W, Q]$. If P is critical, then W is too because $\mathbb{K}'$ is a subcomplex. This is a contradiction. The other possibility is that P is non-critical and contained in an interval $I' \in\sim$. This implies that $I' > I$, and, therefore, I is not maximal in $\sim$. We conclude that W is an interval free face of Q. Because gI is also a non-critical interval for all $g \in G$ since $\sim$ is equivariant, it follows that $\mathbb{K} - GI$ is an equivariant elementary interval collapse.

Since $\mathbb{K} - GI$ is another G complex with subcomplex $\mathbb{K}'$ and $\sim$ is still an equivariant generalised Morse matching such that $\mathbb{K}'$ is critical, it follows that the previous steps can be reapplied. Therefore, we conclude that $\mathbb{K} \searrow_G \mathbb{K}'$. $\square$

8

Equivariant collapsing theorem

The previous sections introduced and proved a collapsing theorem between the Čech and Delaunay complex for $X \subset \mathbb{R}^n$. Afterwards, the theory to prove Corollary 6.9.1 was specialised to the symmetric/equivariant version where $X \subset \mathbb{R}^n$ is symmetric, see Definition 7.5. In particular, equivariant discrete Morse theory was discussed which characterised symmetric collapses by equivariant acyclic partitions of intervals.

Overview

This section specialises the approach used to prove the Delaunay collapsing theorem 6.9.1 to the case where $X \subset \mathbb{R}^2$ is symmetric, see Definition 7.5. Notice that this is only the planar case, as symmetries in higher dimensions severely complicate the approach used. A partial computation of an example case is drawn to indicate how the proof works. Unfortunately, in contrast to the non-equivariant case, non-trivial examples drawn on paper happen to explode in size very quickly. This necessitates a more abstract discussion of the simplicial G structure first rather than directly calculating the complexes.

Assumptions and notation

We call in the following list all the notation that is used in this section.

- G is a finite group with a real representation $\rho : G \to O(2)$ on $\mathbb{R}^2$.
- $E \subset X \subset \mathbb{R}^2$ are finite and G symmetric, while $x \in X$.
- r is a non-negative (radius) real number.
- For $Q \subset X$ and G a group, the notation G^Q denotes all $g \in G$ such that $g(Q) = Q$. It is called the **stabiliser subgroup** of G (with respect to Q).
- All groups in this section, often denoted as H, are subgroups of G. These have corresponding representations, $\rho|_H : H \to O(2)$, inherited from G. Note that this implies that Z is an H-set if Z is a G-set and $H \subset G$ is a subgroup.
- For an interval I, e.g. $I = [Q, Q + Gx]$, the notation I_r denotes $I \cap \mathrm{Del}_r(X, E)$.
- The notation $Q + x$ denotes $Q \cup \{x\}$ and $Q - x$ denotes the set Q with x removed.

The section explicitly states where it deviates from these assumptions.

8.1 Preparatory lemmas

8.1.1 Common Vertex Refinement

First, consider the non-equivariant collapsing theorem, Theorem 6.9, where $\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E+A)$ for $A \subset X$ arose as a sequence $\mathrm{Del}_r(X, E) \searrow \ldots \searrow \mathrm{Del}_r(X, E + A)$. Critical simplices are never paired, and every non-critical simplex Q is matched to $Q \pm x$ for some $x \in A$. Thus, a priori, the Q is paired inside the interval $[Q - A, Q + A]$. Fortunately, the collection of sets of the form $[Q - A, Q + A]$ is an acyclic partition. Therefore, in general, it suffices to search for an acyclic matching within $[Q - A, Q + A]$ by Lemma 5.12. The following lemma generalises Lemma 6.7.

Lemma 8.1 (Common vertex refinement)**.** Suppose X is a (not necessarily symmetrical) set and $Y \subset X$. Then

$$\{[Q - Y, Q + Y] | Q \in \mathbb{P}(X)\}$$

is an acyclic partition of intervals of $\mathbb{P}(X)$, the power set of X.

Proof. For $Q \in \mathbb{P}(X)$, $Q \in [Q - Y, Q + Y]$ by definition, hence the intervals cover $\mathbb{P}(X)$.

If $Q - Y \subset H \subset Q + Y$ for some simplex H, then $H - (Q - Y) \subset Y$, because, $(Q + Y) - (Q - Y) = Y$. Therefore, the difference between H and $Q - Y$ lies precisely in Y, i.e. $H - Y = Q - Y$. By analogous argument, also $H + Y = Q + Y$ holds. We conclude that $H \in [Q - Y, Q + Y]$ implies that $[H - Y, H + Y] = [Q - Y, Q + Y]$.

It follows that two intervals I_1, I_2 are equal if they overlap. Indeed, if $H \in I_1 \cap I_2$, then both $I_1 = [H - Y, H + Y] = I_2$ by the earlier argument. Therefore, the intervals are disjoint. We conclude that the intervals define a partition. We give this partition the symbol $\sim$.

Let $I_Q = [Q - Y, Q + Y] < [P - Y, P + Y] = I_P$ in the quotient $P(X)/\sim$ with induced pre-order. The inequality implies that $P + Y > Q - Y$. We know that some $x \in (P + Y) - (Q - Y)$ with $x \notin Y$ exists. Indeed, otherwise, $Q - Y + Y = P + Y$ and this is a contradiction as $I_Q \neq I_P$.

Note that $Q - Y \subset P - Y$ and $x \in P - Y$ but not in $Q - Y$. Therefore, $|P - Y| > |Q - Y|$. We conclude that the function $f(I_P) = |P - Y|$ strictly decreases on equivalence classes. Therefore, $\sim$ is acyclic. □

Example 9. The common vertex refinement can be drawn as a ladder-like diagram. This is done below for $X = abcdef$ and $Y = ef$. Even if we can freely move between simplices connected by dotted lines, there is no cycle.

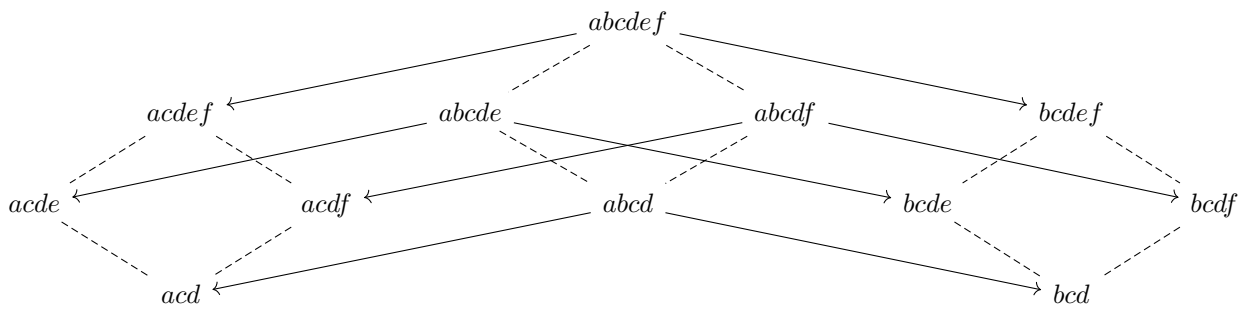


Figure 32: An example of a Common vertex refinement, Lemma 8.1, for $X = abcdef$ and $Y = ef$. Simplices that are connected by a dotted lines are identified in the quotient $\mathbb{P}(X)/\sim$.

Remark 8.2. Any generalised matching can be refined into a standard matching using the common vertex refinement, Lemma 8.1, and the cluster lemma, Lemma 5.12. More exactly, if $I = [Q, P]$ is an interval, then using Lemma 8.1 for $Y = \{x\} \subset P - Q$ gives an acyclic matching.

8.1.2 Equivariant pairing lemma

Informally, the (non equivariant) pairing lemma, Lemma 6.6, essentially depended on the statement that, for simplex $Q \in \mathrm{Del}_r(X, E)$, $s(Q, E) < s(Q, E + x)$ implies that $x \in \mathrm{on}\, S(Q - x, E + x)$ (if the MEES exists). However, when more than one constraint is added, i.e. $s(Q, E) < s(Q, E + Y)$ for $Y \subset X$, then, generally speaking, we can only conclude that some $y \in Y$ lies on $S(Q, E + Y)$.

To accommodate, observe that, if $Y = Gx$, then $gy \in \mathrm{on}(S(Q, E + Gx))$ for all g such that $gQ = Q$. Remember that G^Q denotes the set of g such that $g(Q) = Q$. The observation implies that $G^Q y \subset \mathrm{on}(S(Q, E + Gx))$ for at least some y. Note that $G^Q y$ is not necessarily equal to Gx.

Remember that it is assumed that X is assumed to be G symmetric.

Definition 8.3. Let $H \subset G$ be a subgroup. Then $X/H := \{Hx_1, \ldots, Hx_m\}$ denotes the collection of disjoint H orbits of elements in X. If $m = 1$, then the action of H is called **transitive** on X.

Remark 8.4. Even if H is a proper subgroup of G it can happen that $X/H = X/G$.

We can now state and prove the equivariant version of the pairing lemma.

Lemma 8.5 (Equivariant pairing lemma)**.** Let H be a subgroup of G. Suppose $E \subset X$ are H symmetric, and $Q \in \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$. Furthermore, assume that $Y_1 < \ldots < Y_m$ is some ordering of Hx/H^Q. Then there exists a $Y = Y_j$ such that $s(-, E)$ is constant on $[Q-Y, Q+Y]$ and $s(-, E+Hx) > r$ on $[Q - Y, Q + Y]$.

The proof is similar to Lemma 6.6 and Lemma 5.7 in [7].

Proof. Let $F_Q = \mathrm{excl}\, S(Q, E) \cap (E + Hx)$ be all elements of $E + Hx$ already excluded by $S(Q, E)$. Notice that, by construction, $S(Q, E) = S(Q, F_Q)$. Define A_j, for $1 \leq j \leq m$, as the following set.

$$A_j = F_Q + \bigcup_{i \leq j} Y_i$$

Both F_Q and Y_i are H^Q symmetric. Therefore, A_j is also H^Q symmetric. There must exist some smallest j such that $s(Q, A_j) > r \geq s(Q, A_{j-1})$. This implies that $Q \in \mathrm{Del}_r(X, A_{j-1}) - \mathrm{Del}_r(X, A_j)$. We show that $Y = Y_j$ satisfies the conclusion as in the lemma.

Notice that Y lies in the complement of F_P by construction. Because F_P are all the points in $E + Hx$ that are excluded by $S(Q, E)$, it follows that Y is strictly included by $S(Q, E)$. Therefore, $S(-, E)$ is constant on $[Q-Y, Q+Y]$. More exactly, this follows by repeated applying Lemma 6.2 as every coefficient λ_y for $y \in Y$ is zero in every solution for the sphere $S_{Q+Y,E}$. We conclude the first part.

By Lemma 6.3, there exists at least one $y \in Y$ that is strictly included in $S(Q, A_{j-1})$. Furthermore, H^Q preserves the sphere $S(Q, A_{j-1})$ as Q and A_{j-1} are H^Q symmetric. It follows that hy is strictly included by $S(Q, A_{j-1})$ for each $h \in H^Q$. By definition, Y is an orbit of H^Q. We conclude that Y is strictly included by $S(Q, A_{j-1})$. Therefore, $S_{A_{j-1}}$ is constant on $[Q - Y, Q + Y]$.

Suppose that $s(Q - Y, E + Hx) \leq r$. Then, by monotonicity of $s(-, -)$, it follows that $s(Q - Y, A_j) \leq r < s(Q, A_j)$. This implies that some $z \in Y$ is strictly excluded by $S(Q - Y, A_j)$. Notice that $Q - Y$ and A_j are H^Q symmetric. Thus, hz is strictly excluded by $S(Q - Y, A_j)$ for all $h \in H^Q$. Therefore, Y is strictly excluded by $S(Q - Y, A_j)$. By repeated application of Lemma 6.3, we conclude that $S(Q - Y, A_j) = S(Q - Y, A_{j-1})$. Indeed, the former sphere already strictly excludes Y.

The argued equalities put together to form the following equality.

$$S(Q - Y, A_j) = S(Q - Y, A_{j-1}) = S(Q + Y, A_{j-1})$$

The first sphere strictly excluded Y, whereas the latter sphere includes Y. This is a contradiction.

Hence, we conclude that $s(Q - Y, E + Hx) > r$. It follows that $s(-, E + Hx) > r$ on $[Q - Y, Q + Y]$ by monotonicity of $s(-, E + Hx)$. This proves the second statement.

This proves the lemma. □

Remark 8.6. Notice that the equivariant pairing lemma 8.5 is also for $n \neq 2$, i.e. $E \subset X \subset \mathbb{R}^n$. Indeed, nowhere in the proof is a reference to specific properties of the plane made. However, this generality is not necessary for this thesis and therefore not proven.

Remark 8.7. If G had no nontrivial subgroups, then the Y as in the equivariant pairing lemma is unique. An example of such a case is when G is a cyclic prime group, i.e. isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for p a prime.

Remark 8.8. Notice that simplices Q such that $G^Q = G$ can exist. Namely, pick $x \in X$ with the smallest norm. The sphere $(0, \|x\|)$ is a candidate for the MEES of $S(Gx, E)$ for all E, since it excludes X by construction. Therefore, $S(Gx, E)$ exists and $Gx \in \mathrm{Del}_r(X, E)$ if $r \geq \|x\|$. The simplex Gx has $G^{Gx} = G$.

8.1.3 Intermediate partition lemma

In general, while an Y for the equivariant pairing Lemma, 8.5, exists for any fixed simplex Q (locally). It is unclear how to consistently select Y for all Q (globally). The unclear choice suggests building the partition step-wise. The construction of intermediate steps is reflected in the following lemma.

Lemma 8.9 (Intermediate partition lemma)**.** Let $H \subset G$ be a subgroup. Suppose that $E' \subset E$ are H symmetric. Define A by the equality

$$A = (E' - E)/H = \{Hx_1 > \ldots > Hx_n\}$$

for some ordering of H orbits. Denote by $\bar{A}$ the set $A \cup \{\infty\}$ where ∞ is the maximal element of $\bar{A}$. Further, write E_i for $E + \bigcup_{j \leq i} Hx_j$.
There exists an order preserving H-equivariant function $\psi : \mathrm{Del}_r(X, E) \to \bar{A}$ such that

- $\psi^{-1}(Hx_i) = \mathrm{Del}_r(X, E_{i-1}) - \mathrm{Del}_r(X, E_i)$ for all i.
- $\psi^{-1}(\infty) = \mathrm{Del}_r(X, E')$.

Proof. The lemma already prescribes the conditions for the function. Let ψ be the following map.

- $\psi : Q \mapsto Hx_i$ if $Q \in \mathrm{Del}_r(X, E_{i-1}) - \mathrm{Del}_r(X, E_i)$
- $\psi : Q \mapsto \infty$ if $Q \in \mathrm{Del}_r(X, E')$

This map is H-equivariant since $\mathrm{Del}_r(X, E_i)$ are H complexes. We need to show it is order-preserving.

If $Q < D$, then $s(Q, E_i) \leq s(D, E_i)$ for each i as $s(-, E_i)$ is increasing. Therefore, if $Hx_i = \psi(Q) > \psi(D) = H_{x_j}$ for $j > i$, then $s(Q, E_i) > r \geq s(D, E_i)$. This is a contradiction and so ψ is order-preserving. We conclude the lemma. □

Remark 8.10. The function of Lemma 8.9 gives rise to a H equivariant partition on $\mathrm{Del}_r(X, E)$ whose classes consist of subsets of $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$ or are $\mathrm{Del}_r(X, E + Hx)$.

8.1.4 Orbit lemmas

Suppose that $Q \in \mathrm{Del}_r(X, E)$ such that G^Q is transitive on Gx and $Q \notin \mathrm{Del}_r(X, E + Gx)$. Then, the equivariant pairing lemma, Lemma 8.5, implies that

$$Q \in \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Gx)$$

only if

$$[Q - Gx, Q + Gx] \subset \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Gx)$$

In contrast to the non-equivariant case Corollary 6.9.1, there is no reason to assume that the converse holds, i.e. that some $P \in [Q - Gx, Q + Gx]$ such that $P \in \mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$ implies that Q has the same property. It can happen that only a proper subset of $[Q - Gx, Q + Gx]$ lies outside $\mathrm{Del}_r(X, E + Gx)$. In such a case, the orbit lemma, Lemma 8.16, provides a desired partition in this case on $[Q - Gx, Q + Gx]$. The proof of the orbit lemma can be done in two cases. The first case is Lemma 8.11 where we assume that $|Hx| \leq 2$, the second case is Lemma 8.15 where we assume that $|Hx| \geq 3$.

The case where $|G^Q x| \leq 2$, the situation above does reduce to the if and only if statement.

Remember that for any interval I, such as $[Q, Q + Hx]$, I_r denotes the following.

$$I_r = I \cap \mathrm{Del}_r(X, E)$$

Lemma 8.11 (Small orbit lemma)**.** Suppose H is a subgroup of G, and $E \subset X$ are H symmetric, and $|Hx| \leq 2$. Suppose that $Q \in \mathrm{Del}_r(X, E)$ where $Q = Q - Hx$, and H^Q acts transitively on Hx. Then there exists a H^Q-equivariant acyclic generalised matching on $[Q, Q + Hx]_r$ such that the critical intervals lie in $\mathrm{Del}_r(X, E + Hx)$ and the non critical intervals in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$.

Proof. First, suppose $|Hx| = 1$, i.e. $Hx = \{x\}$. This means $[Q, Q+Hx]_r = \{Q-x, Q+x\}_r = \{Q, Q+x\}_r$ since $Q = Q - Hx$. The proof of the (non-equivariant) collapsing theorem, Corollary 6.9.1, implies the existence of a acyclic generalised matching on $\{Q, Q + x\}_r$. Furthermore, every $h \in H^Q$ has $h[Q, Q + x]_r = [Q, Q + x]_r$ as $h(x) = x$ by assumption. Therefore, the partition on $\{Q, Q + x\}_r$ is H^Q-equivariant. This proves the case $|Hx| = 1$.

Now, suppose $|Hx| = 2$, i.e. $Hx = \{x, y\}$. Denote by D_1 and D_2 the complexes $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, E + Hx)$ respectively. If $[Q, Q + Hx]_r \subset D_2$, then take the partition of singletons. If Q or $Q+Hx$ lie in $D_1 - D_2$, then the equivariant pairing lemma, Lemma 8.5, can be applied. Indeed, H^Q acts transitively on Hx, and E is H symmetric which implies that $[Q, Q + Hx]_r = [Q, Q + Hx] \subset D_1 - D_2$. In this case, $[Q, Q + Hx]$ is an interval for the partition that is H^Q-equivariant.

The remaining case is that there exists a $P \in [Q, Q + Hx]_r$ such that $P \neq Q, Q + Hx$ and $P \in D_1 - D_2$. Impose the order $x < y$ in Hx. By Corollary 6.6.1, there exists some $z \in Hx$ such that $\{P - z, P + z\} \subset D_1 - D_2$. Either P contains z, in which case $P + z = P$ and $P - z = Q$, or P does not contain z, in which case $P - z = P$ and $P + z = Q + Hx$. In both cases, it follows that Q or $Q + Hx$ lies in $D_1 - D_2$. Both cases have already been solved by the prior argument.

By construction, every simplex P such that $P \in D_1 - D_2$ is in a non singleton interval of the partition. Furthermore, the set of critical simplices lies in D_2. □

Remark 8.12. Notice that this lemma does not use any properties of the plane $\mathbb{R}^2$. Therefore, this lemma also holds for $X \subset \mathbb{R}^n$. However, this is not proven nor used in this thesis.

In contrast to Lemma 8.11, the proof of Lemma 8.16 is rather long. Therefore, it is supplemented with Example 10. This should help navigate the special case of one orbit $Gx = X$.

Lemma 8.13 (Big orbit lemma)**.** Suppose H is a subgroup of G, and $E \subset X$ are H symmetric, and $|Hx| \geq 3$. Suppose that $Q \in \mathrm{Del}_r(X, E)$ where $Q = Q - Hx$, and H^Q acts transitively on Hx. Then there exists a H^Q-equivariant acyclic generalised matching on $[Q, Q+Hx]_r$ such that the critical intervals lie in $\mathrm{Del}_r(X, E + Hx)$ and the non critical intervals in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$.

Example proof of the big orbit lemma

The proof of the big orbit lemma, Lemma 8.15, is not very insightful and also long. Therefore, an example case of a single orbit is provided below in Example 10.

Example 10. The quadruplet of figures, Fig. 33, Fig. 34, Fig. 35 and Fig. 36, help to illustrate the big orbit lemma, Lemma 8.15, for the case $G = H = D_8$, $X = Gx$, and $E = \emptyset$ and $[Q, Q + Hx]_r = (\emptyset, Hx]_r$. That is, we consider only a single orbit (which corresponds in notation) to the case $[\emptyset, Hx]_r$, i.e. $Q = \emptyset$. Here, D_8 refers to the dihedral group of order eight, which we represent as the is the symmetry group of the 8-gon. More exactly, the group D_8 with generators r, i is represented algebraically as

$$D_8 =< r, i | r^8 = e, i^2 = e, r^n i = i r^{-n} >$$

The representation $\rho : G \to O(2)$ sends the **rotation** generator r to a rotation of $1/4\pi$ and the **reflection** generator i to a reflection through the vertical axis. This representation makes G into the symmetry group of the 8-gon. The orbit of the vector $(0, 1)$, denoted 1, is depicted in Fig. 33. The orbit $G1$ is the set X, i.e. $X = G1$.

The example is structured in the following way:

- Compute a few simplices Q in $\mathrm{Del}_r(X, \emptyset)$ by computing the MEES $S(Q, \emptyset)$.
- Compute $\mathrm{Del}_r(X, Gx) = \mathrm{Del}_r(X, X)$ by computing the MEES $S(Q, Gx)$.
- Find a collapse/generalised Morse matching on the few simplices Q computed for $\mathrm{Del}_r(X, \emptyset)$.
- Define the G-equivariant generalised Morse matching that induces the collapse.

Computing $\mathrm{Del}_r(X, \emptyset)$

The MEES of $S(\{1, i\}, \emptyset)$, for $i = 2, 3, 4, 5$, are depicted in Fig. 33 in black, blue, green and red, respectively.

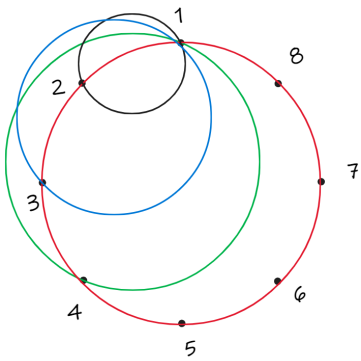


Figure 33: For G the Dihedral group of order 8, a single orbit of points numbered 1 to 8 is drawn $G1 = X$. The MEES of $S(\{W_i\}, \emptyset)$ for $W_i = \{1, i\}$ with $i = 2, 3, 4, 5$ is depicted in black, blue, green, red, respectively.

Secondly, we should indicate what radius r to choose as to make the example interesting. To do this, note that $S(\{1, 5\}, \emptyset) = S(Gx, \emptyset) = S(Gx, Gx)$, which is the red circle in Fig. 33. Therefore, if r is large enough such that $s(\{1, 5\}, \emptyset) \leq r$, then $\mathrm{Del}_r(X, \emptyset) = \mathbb{P}(X) - \emptyset = \mathrm{Del}_r(X, Gx)$, where $\mathbb{P}(X)$ is the power set. Therefore, the radius r should be smaller as the example is otherwise trivial. The radius r is selected such that $s(\{1, 4\}, \emptyset) \leq r$, the green circle in Fig. 33, while $r < s(\{1, 5\}, \emptyset)$.

Notice that, by Lemma 6.2, if $s(Q, E) < s(Q + x, E)$, then $x \in \mathrm{on}\, S(Q + x, E)$ if it exists. If $V = \{a, b, c\} \subset G1$ is such that $s(V, \emptyset) > s(Y, \emptyset)$ for every proper subset Y of V, then, by repeated application of Lemma 6.2, we know that $V \subset \mathrm{on}\, S(V, \emptyset)$. Because no three points in $G1$ are colinear, in particular, V is not colinear. Therefore, $V \subset \mathrm{on}\, S(V, \emptyset)$ implies that $S(V, \emptyset)$ is the circumcircle of V. The circumcircle of V is the same as $G1$ as $G1$ is concyclic by construction. Therefore, $S(V, \emptyset) = S(Gx, \emptyset)$ and as such, $V \notin \mathrm{Del}_r(X, \emptyset)$. A concrete example of such a simplex V is $V = \{1, 4, 6\}$.

The simplices which are included in $\mathrm{Del}_r(X, \emptyset)$ are those whose points are included in the MEES of $S(\{1, i\}, \emptyset)$ for $i = 2, 3, 4$. Let us denote $W_i = \{1, i\}$ for $i = 2, 3, 4, 5$. Note that $S(W_i, \emptyset)$ includes W_j for $1 \leq j \leq i$.

Computing $\mathrm{Del}_r(X, G1)$

Notice that all simplices W_i for $i \neq 2$ are not in $\mathrm{Del}_r(X, G1)$. Indeed, because j is strictly included in $S(W_i, \emptyset)$ for $1 < j < i$, there must be some j such that $j \in \mathrm{on}\, S(W_i, G1)$ as $s(W_i, \emptyset) < s(W_i, G1)$. Furthermore, $W_i \subset \mathrm{on}\, S(W_i, G1)$ by definition as $W_i \subset G1$ and therefore $V = W_i \cup \{j\} \subset \mathrm{on}\, S(W_i, G1)$. By our earlier argument, it follows that $S(W_i, G1)$ is the circumsphere of $G1$. This sphere had radius strictly bigger then r. We conclude that W_i is in $\mathrm{Del}_r(X, G1)$ only if $i = 2$. The complex $\mathrm{Del}_r(X, G1)$, thus, only consists of the edges $\{1, 2\}, \{2, 3\}, \ldots, \{1, 8\}$ as depicted black in Fig. 34.

Computing a few collapses

Because W_i satisfies $W_i \in \mathrm{Del}_r(X, \emptyset) - \mathrm{Del}_r(X, Gx)$ for $i = 3, 4$, this is also true for all $P \in [W_i, \mathrm{incl}(S(W_i, \emptyset))]$. To simplify notation, denote by A_{W_i}, for $i = 3, 4$, all the points included in the MEES $S(W_i, \emptyset)$. The simplices A_{W_i} are depicted in Fig. 34 in blue and green, respectively. Therefore, the generalised matching that we choose should contain these as non singleton intervals.

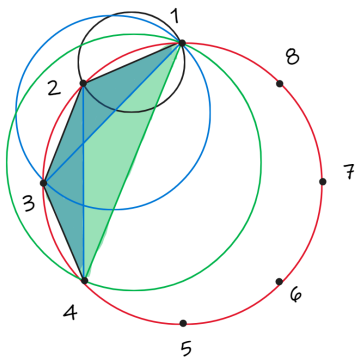


Figure 34: The simplices $A_{W_i} = \mathrm{incl}(S(W_i, \emptyset)$ have been drawn for $i = 1, 2$ in black, $i = 3$ in blue and $i = 4$ in green. Additionally the simplices $\{2, 3, 4\}$, $\{2, 3\}$ and $\{3, 4\}$ have been drawn.

Define the partition $\sim$ of intervals $\{[W_i, A_{W_i}] | i = 3, 4\}$ and the other simplices are singletons. Note that this partition is acyclic as $s(W_j, \emptyset) < s(W_i, \emptyset)$ for $j < i$ and $s(-, \emptyset)$ is constant on the interval $[W_i, A_{W_i}]$. The resulting collapse is indicated in Fig. 35. Notice that the arrows point outwards from the centre towards the black edges.

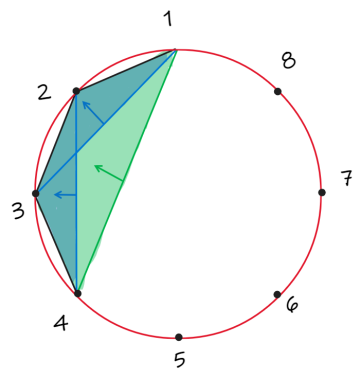


Figure 35: The generalised acylic matching is drawn for the intervals $[W_i, A_{W_i}]$.

The generalised matching

Define the partition consisting of the intervals $g[W_i, A_{W_i}]$ for each $g \in G$ and each $i = 3, 4$. The resulting partition is well-defined since

$$gA_{W_i} = g\,\mathrm{incl}(S(W_i, \emptyset)) = \mathrm{incl}(S(gW_i, \emptyset)) = A_{gW_i}$$

Furthermore, the partition is acyclic. Indeed, two elements P, Q in intervals I, J respectively have $Q < P$ if $s(Q, \emptyset) < s(P, \emptyset)$ by construction. This implies that $s(-, \emptyset)$ is a decreasing function on the intervals. Furthermore, the partition consists of intervals. Lastly, the partition is G equivariant by construction. The resulting collapse is depicted in Fig. 36.

This ends the example.

8.1.5 Proof of the big orbit lemma

To prove that the big orbit lemma works, we need to ensure that the simplex $Q + x$ is critical if and only if Q is assuming the action is transitive.

Lemma 8.14. Suppose H is a subgroup of G, and $E \subset X$ are H symmetric. Suppose that $Q \in \mathrm{Del}_r(X, E + Hx)$ with $Q = Q - Hx$ and H^Q acts transitively on Hx. Then $Q + x \in \mathrm{Del}_r(X, E + Hx)$ also.

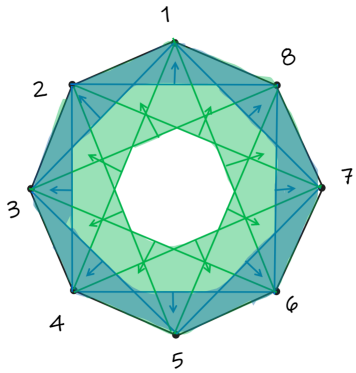


Figure 36: The equivariant generalised Morse matching is depicted that induces the collapse of $\mathrm{Del}_r(X, \emptyset) \searrow_G \mathrm{Del}_r(X, Gx)$.

Proof. Suppose that $Q + x \notin \mathrm{Del}_r(X, E + Hx)$. Because $Hx = H^Q x$, it follows that $Q + y \notin \mathrm{Del}_r(X, E + Hx)$ for all $y \in Hx$. Choose a total order on $Hx = \{x_1 < \ldots < x_n\}$. By the proof of the (non equivariant) collapsing theorem, Corollary 6.9.1, there exists an acyclic matching that pairs $Q + x$. Specifically, the matching pairs elements $Q + y$ to elements in $(Q, Q + Hx]_r$, since Q is in $\mathrm{Del}_r(X, E + Hx)$. By our earlier argument, all $Q + y$ need to be matched. Let $f : Hx \to Hx$ be the bijection such that $f(x_i)$ has that $Q + x_i$ is matched to $Q + x_i + f(x_i)$. Thus, there are matched pairs $\{Q + x_i, Q + x_i + f(x_i)\}$, for $i = 1, \ldots, n$. Then observe the following sequence of inequalities in the quotient by the partition

$$\{Q + x_1, Q + x_1 + f(x_1)\} > \{Q + f(x_1), Q + f(x_1) + ff(x_1)\} > \ldots$$

Because Hx is finite and f is a bijection, there is some $m \in \mathbb{N}$ such that $f^m(x_1) = x_1$. In other words, the sequence repeats. This implies that $\sim$ is not acyclic. We conclude a contradiction, and therefore $Q + x \in \mathrm{Del}_r(X, E + Hx)$ and the statement holds. □

Notice that circle is uniquely determined by 3 points on its boundary if the points are not colinear. Furthermore, any orthogonal action g preserves a circle S around zero, i.e. $gS = S$. This implies that an orbit of points from an orthogonal action always lie on the same circle around zero. No three points on the circle are colinear, and, therefore, if the orbit consists of three points, the points are not colinear. The previous two observations are the reason why only the plane is considered for the equivariant case. It is also the reason why the case $|Hx| \geq 3$ works for Lemma 8.15.

Recall that I_r, for an interval I, denotes the set $I \cap \mathrm{Del}_r(X, E)$.

Lemma 8.15 (Big orbit lemma)**.** Suppose H is a subgroup of G and $E \subset X$ are H symmetrical, and $|Hx| \geq 3$. Suppose further that $Q \in \mathrm{Del}_r(X, E)$ where $Q = Q - Hx$, and H^Q acts transitively on Hx. Then there exists a H^Q-equivariant acyclic generalised matching on $[Q, Q + Hx]_r$ such that the critical intervals lie in $\mathrm{Del}_r(X, E + Hx)$ and the non critical intervals in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$.

Proof structure
This proof rests on examining whether simplices of the form $Q, Q + Hx, Q + y, Q + \{y, z\}$ or $Q + \{h, y, z\}$ lie in D_1, D_2 or $D_1 - D_2$ where $y, z, h \in Hx$. In the end, we shall see that the only relevant case is $Q + \{y, z\}$. This is also indicated in Example 10 by the A_{W_i}.

Proof. Denote by D_1 and D_2 the complexes $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, E + Hx)$ respectively.

Case Q,Q+Hx
Notice that $Q \notin D_1$ implies that $[Q, Q + Hx]_r = \emptyset$, so this is a trivial case. If either $Q + Hx$ or Q lie in $D_1 - D_2$ then the entire interval has this property by Lemma 8.5. Specifically, the lemma can be applied and $Hx = H^Q x$ as H^Q acts transitively on Hx by assumption. Therefore, $[Q, Q + Hx]_r = [Q, Q + Hx] \subset D_1 - D_2$. Taking the entire interval $[Q, Q + Hx]$ defines an H^Q equivariant acyclic generalised matching satisfying the conclusion. So, assume that $Q, Q + Hx \notin D_1 - D_2$.

If $Q + Hx \in D_2$, then the entire interval is in D_2 since $s(P, E + Hx) \leq s(Q + Hx, E + Hx) \leq r$ for all $P \in [Q, Q + Hx]_r$. In this case, the partition of singleton intervals defines a trivial acyclic H-equivariant partition. So assume that $Q + Hx \notin D_2$.

Since $Q + Hx \notin D_1 - D_2$ it follows that $Q + Hx \notin D_1$ since $Q + Hx \notin D_2$ by assumption. Listing the assumptions so far shows that $Q \in D_2$ and $Q + Hx \notin D_1$, or, equivalently,

$$s(Q, E) \leq s(Q, E + Hx) \leq r < s(Q + Hx, E)$$

Notice that this implies that $s(Q, E) < s(Q + y, E)$ for some $y \in Hx$. Because $h \in H^Q$ satisfies $hs(Q, E) = s(Q, E) < hs(Q + y, E) = s(Q + hy, E)$ and H^Q acts transitively on Hx, it follows that $s(Q, E) < s(Q + y, E)$ for all $y \in Hx$. Lastly, note that, in the following proof, the interval $[\emptyset, Hx]_r$ is simply the interval $(\emptyset, Hx]_r$. Therefore, this case is also examined.

Case Q+y
If $Q + y \notin D_1$ for any $y \in Hx$. Then, since the action H^Q is transitive on Hx, it follows that $Q + y \notin D_1$ for all $y \in Hx$. Therefore, $[Q, Q + Hx]_r$ is the singleton $\{Q\}$. A singleton partitions the interval trivially. Thus, this case is solved.

Assume that $Q + y \in D_1$. By Lemma 8.14 it follows that $Q + y \in D_2$ because $Q \in D_2$.

Case Q+{y,z,h}
Let $V = \{y, z, h\}$ denote three distinct points in Hx. If $S(Q+V, E+Hx)$ does not exist, then $Q+V \notin D_2$. Suppose then that $S(Q + V, E + Hx)$ exists. Observe that $V \subset \text{on}\, S(Q + V, E + Hx)$. Since V consists of three non colinear points, we know that $S(Hx, Hx) = S(Q + V, E + Hx)$, as the MEES must be the circumsphere of Hx. Furthermore, the only fixed point of H is 0 as it is not just a reflection in a line since $|Hx| \geq 3$. It follows that $S(Hx, Hx) = (0, \|x\|)$ and this MEES exists. If $S(Hx, Hx)$ included Q, excluded E, and $s(Hx, Hx) \leq r$, then $S(Hx, Hx) = S(Q + Hx, E + Hx)$ and thus $Q + Hx \in D_2$. This is a contradiction as, by assumption, $Q + Hx \notin D_2$, and therefore $S(Hx, Hx)$ does not include Q, or exclude E or $s(Hx, Hx) > r$. We conclude that $Q + V \notin D_2$, as $S(Q + V, E + Hx)$ should include Q, exclude E or have radius less then r. Since $s(-, E)$ is monotonically increasing, the same conclusion holds for any set V' of more than three distinct points.

Furthermore, if $V \subset V' \subset Hx$, then $Q + V' \notin D_2$ since $s(-, -)$ is increasing.

Now, $S(Q + V, E)$ is determined by two points, i.e. $S(Q + V, E) = S(Q + W, E)$ for $|W| \leq 2$, or $Q + V \notin D_1$ (these cases are not exclusive). Indeed, if there is no $W \subset V$ with $|W| = 2$ such that $s(Q + W, E) = s(Q + V, E)$, then $s(Q + W, E) < s(Q + V, E)$ for all $W \subset V$. This implies that $V \subset \text{on}\, S(Q + V, E)$. Again, it follows that $S(Q + V, E) = S(Hx, Hx)$. As argued before, the sphere $S(Hx, Hx)$ does not include Q or exclude E or have $s(Hx, Hx) \leq r$ (it can be one or more of these options). Since $S(Q+V, E)$ has to satisfy all three conditions if $Q+V \in D_1$, we conclude that $Q+V \notin D_1$.

Case Q+{y,z}
The remaining cases to consider is for simplices of the form $Q + W$ where $|W| = 2$, and those simplices B such that $S(B, E) = S(Q + W, E)$. Once these cases have been investigated, we shall end up with non-singleton intervals contained in $D_1 - D - 2$ of the form $[Q + W, B]$, where the B is $\text{incl}(S(Q + W, E))$. First, we consider simplices $Q + W$ where $|W| = 2$. The following argument shows that $S(Q + W, E) = S(Q + W, E + W)$.

Let $W = \{y, z\} \subset Hx$, then we shall show that $W \,\text{on}\, S(Q + W, E)$. There are two cases, either $s(Q + y, E) = s(Q + W, E)$ or $s(Q + y, E) < s(Q + W, E)$.
Suppose, first, that $s(Q + y, E) = s(Q + W, E)$ holds. Let $h \in H^Q$ be such that $h(y) = z$. The h exists as H^Q acts transitively on Hx. It follows that $hs(Q + y, E) = s(Q + z, E)$ which implies that $s(Q + z, E) = s(Q + W, E)$. Because $s(Q, E) < s(Q + y, E)$, we know that $y \in \text{on}\, S(Q + y, E)$. Similarly, this holds for z, i.e. $z \in \text{on}\, S(Q + z, E)$. Since $S(Q + y, E) = S(Q + W, E) = S(Q + z, E)$ the sphere $S(Q + W, E)$ has y and z on it, and, therefore, $W \subset \text{on}\, S(Q + W, E)$.

Suppose now that $s(Q+y,E) < s(Q+W,E)$ holds. We can already conclude that $z \in \text{on}\, S(Q+W,E)$. Because $s(Q+z,E) = s(Q+y,E)$ by similar argument as in the case above for $s(Q+y,E) = s(Q+W,E)$, it follows that $y \in \text{on}\, S(Q+W,E)$. This means that $W \subset \text{on}\, S(Q+W,E)$.

In both cases, we can conclude that $S(Q+W,E) = S(Q+W,E+W)$ as W is excluded by $S(Q+W,E)$.

If $S(Q+W,E)$ excludes Hx, then $Q+W$ is in D_2. Otherwise, $s(Q+W,E+W) < s(Q+W,E+Hx)$ and there is some $h \in Hx - W$ that is not excluded by $S(Q+W,E+W)$. In this case, the set $L := \{h\} \cup W$ has $L \subset \text{on}\, S(Q+W,E+W+Hx)$. By the argument presented for the case $Q+\{y,z,h\}$, it follows that $S(Q+W,E+W+Hx)$ must be the circumsphere $S(Hx,Hx)$ which implies that $Q+W \notin D_2$.

Construction of generalised matching
From our previous discussion, the only simplices B that can satisfy $B \in D_1 - D_2$ are those such that $S(B,E) = S(Q+W,E)$ for $|W| = 2$. Let $Q+W \in D_1 - D_2$, then, denote by A_W all the simplices in $[Q, Q+Hx]_r$ that are included in $\text{incl}\, S(Q+W,E)$, i.e. $A_W = [Q, Q+Hx]_r \cap \text{incl}\, S(Q+W,E)$. Such intervals are not singletons as we saw that $Q+W \in D_1 - D_2$ implies that there exists an $h \in Hx - W$ which is strictly included in $S(Q+W,E)$. We shall prove that $\{[Q+W, A_W] | |W| = 2, Q+W \in D_1 - D_2\}$ is the wanted partition.

Notice that each $[Q+W, A_W]$ is contained in a distinct pre-image of $S(-,E)$. Therefore, together with the singletons sets, these intervals $[Q+W, A_W]$ define a partition $\sim$. Furthermore, notice that, in the quotient $D_1/\sim$, if two intervals are related as $[Q+W, A_W] < [Q+W', A_{W'}]$, then $s([Q+W, A_W], E) < s([Q+W', A_{W'}], E)$ by construction. This implies acyclicity of the partition. The H^Q equivariance follows from the following equation, where $h \in H^Q$ is arbitrary.

$$h[Q+W, A_W] = [hQ + hW, hA_W] = [Q + hW, A_{hW}]$$

This concludes the proof as the partition defined is an H^Q equivariant generalised matching as in the conclusion of the lemma. □

The big and small orbit lemma, Lemma 8.15 and Lemma 8.11 respectively provide the general case.

Lemma 8.16 (Orbit lemma)**.** Suppose H is a subgroup of G and $E \subset X$ are H symmetrical. Suppose further that $Q \in \text{Del}_r(X,E)$ where $Q = Q - Hx$, and H^Q acts transitively on Hx. Then there exists a H^Q-equivariant acyclic generalised matching on $[Q, Q+Hx]_r$ such that the critical intervals lie in $\text{Del}_r(X, E+Hx)$ and the non critical intervals in $\text{Del}_r(X,E) - \text{Del}_r(X, E+Hx)$.

This lemma provides the case for the Delaunay collapsing theorem as long as the action G^Q acts transitively on Gx for every $Q \in \text{Del}_r(X,E)$. A direct corollary is the case where the group G is cyclic prime, i.e. has no nontrivial subgroups.

Corollary 8.16.1. Let $E \subset X$ be G-sets and $x \in X$. If G is a cyclic prime group, i.e. $G = \mathbb{Z}/p\mathbb{Z}$, then there exists a G-equivariant collapse

$$\text{Del}_r(X,E) \searrow_G \text{Del}_r(X, E+Gx)$$

Proof. Consider the partition $\{[Q, Q+Gx]_r | Q = Q - Gx\}$ of $\text{Del}_r(X,E)$. The partition is acyclic due to Lemma 8.1. Pick from each orbit class of $[Q, Q+Gx]_r$ a representative (such as $[Q, Q+Gx]_r$ itself). By the orbit lemma, Lemma 8.16, there is a G^Q equivariant acyclic generalised matching on $[Q, Q+Gx]_r$ such that the critical intervals lie in $\text{Del}_r(X, E+Hx)$ and the non critical intervals in $\text{Del}_r(X,E) - \text{Del}_r(X, E+Hx)$. Denote this partition by $\sim_Q$ and let $[Q', Q'+Gx]_r$ be another element in the orbit of $[Q, Q+Gx]_r$. Since G has no non-trivial subgroups, there is exactly one $g \in G$ such that $g(Q) = Q'$. Therefore, if we define the $G^{Q'}$ equivariant partition $\sim_{g(Q)}$ on $[Q', Q'+Gx]_r$ as the partition of sets $g(L)$ where $L \in \sim_Q$. Then each interval in the orbit class of $[Q, Q+Gx]_r$ has an acyclic partition. The constructed partition is a G equivariant partition of intervals. Furthermore, the partition is acyclic by the Lemma 5.12. We conclude that there is an G equivariant generalised Morse matching such that critical simplices lie in $\text{Del}_r(X, E+Gx)$ and the non critical simplices in $\text{Del}_r(X,E) - \text{Del}_r(X, E+Gx)$. By Theorem 7.22, we conclude that the statement holds.

□

However, in general, there is no reason to assume that H^Q acts transitively on Gx for some simplex $Q \in \mathrm{Del}_r(X, E)$. The orbit reduction lemma Lemma 8.18 reduces the problem recursively such that the orbit lemma, Lemma 8.16, can be applied.

8.2 Example of the solution structure

The example focuses on exemplifying the orbit lemma, Lemma 8.16, that reduces the problem into smaller chunks.

Example setup

Example 11. Similar to Example 10, the group G is the dihedral group D_8 of order 8. The group is represented in the following way.

$$G = D_8 =< r, i | r^8 = e, i^2 = e, r^n i = i r^{-n} >$$

The group is the symmetry group of the 8-gon. In Fig. 37, two orbits of points under the group action G are drawn. We denote by 1 the point $(0, 1)$. The outer 8 red points, denoted as $G1$, and the inner black and blue eight points. The element i acts by reflection in the vertical axis, which is the line through $1, 5$. The element r acts as a counter-clockwise rotation by $1/4\pi$ sending 1 to 2 to 3, etc. Further assume that $E = \emptyset$.

The simplex Q denotes the four black points $\{a, b, c, d\}$. The stabiliser subgroup of Q in G is isomorphic to D_4, the symmetry of the square. More precisely, $G^Q = G = D_8 =< r^2, i | r^8 = e, i^2 = e, r^n i = i r^{-n} >$ is the representation of G^Q. The simplex Q' define the four blue points. The simplices Q and Q' are chosen such that a rotation $r \in G$ has $r(Q) = Q'$.

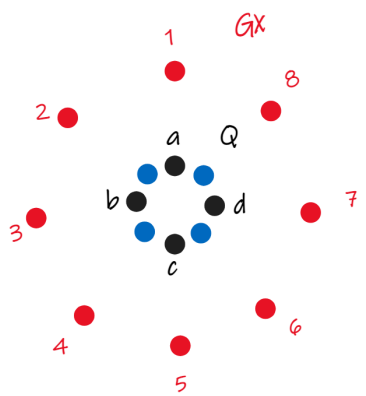


Figure 37: Two orbits of points with symmetry group $G = D_8$ are drawn. The orbit $G1$ is drawn and numbered from 1 to 8.

Remark 8.17. The number of points in this example might seem unnecessarily large. However, at minimum two orbits are required to distinguish this from the Example 10. Furthermore, the order of G needs to be high enough, e.g. greater than eight, to ensure that subtle arguments involving non-trivial stabilizer subgroups become clear.

The structure of this example is as follows:

1. Formulate what is sufficient to prove.
2. Reduce the problem.
3. Repeat steps 1 and 2.
4. General remark on how to iteratively apply steps.

Problem

Find G equivariant generalised Morse matching on $\mathrm{Del}_r(X, E)$ such that the non singleton intervals lie in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + G1)$.

Problem reduction

Remember that $[Q, Q + Gx]_r$ denotes $[Q, Q + Gx]_r = [Q, Q + Gx] \cap \mathrm{Del}_r(X, \emptyset)$. Notice that the set $(\emptyset, G1]_r$ can be done by the orbit lemma, Lemma 8.16. More details on this precise case are given in Example 10.

We expect that partitions can be found in sets of the form $[Q, Q + Gx]_r$ where $Q = Q - Gx$. Indeed, any acyclic matching from the non-equivariant version of the collapsing theorem, Theorem 6.9, suggests as much. Due to Lemma 8.1, it is also sufficient to find appropriate partitions in $[Q, Q + Gx]_r$. However, it makes no sense to find a G-equivariant partition on $[Q, Q + Gx]_r$, as $r[Q, Q + Gx]_r = [Q', Q' + Gx]$, which is a different set. Instead, we should attempt to find an appropriate G^Q-equivariant partition on $[Q, Q + Gx]_r$. Fortunately, as the following argument shows, this suffices.

Suppose that $\sim_Q$ is a G^Q equivariant partition on $[Q, Q + Gx]_r$. Then, this partition can be translated to one on $[Q', Q' + Gx]_r$ by $[r] \in G/G^Q$, the cosets of G under G^Q. Indeed, define $\sim_{Q'}$ to be the partition $r(I) \in \sim_{Q'}$ if and only if $I \in \sim_Q$. If instead of r, the action r^3 was taken, then $r^3 r^1(I) = r^4(I) \in \sim_Q$ since $\sim_Q$ is G^Q equivariant and $r^4 \in G^Q$. In other words, the action r^3 simply permutes the classes in $\sim_{Q'}$. Therefore, the problem restricts to finding a G^Q equivariant matching on $[Q, Q + Gx]_r$ with the required properties.

New problem

Find G^Q equivariant generalised matching on $[Q, Q + Gx]_r$ such that the non singleton intervals lie in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + G1)$.

Problem reduction

The orbit $G^Q 2$ and $G^Q 1$ can be seen in Fig. 38. Unfortunately, G^Q does not act transitively on $G1$, i.e. $G^Q 2 = \{2, 4, 6, 8\} \neq G1$. Therefore, the orbit lemma, Lemma 8.16, cannot be applied directly.

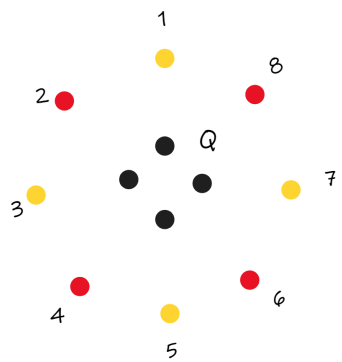


Figure 38: The orbits of 1 and 2 under G^Q is depicted in yellow and red respectively.

However, if, instead, there exists two G^Q equivariant generalised matchings such that the first has non-singleton intervals in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + G^Q 1)$ and the second has non-singleton intervals in

$$\mathrm{Del}_r(X, E + G^Q 1) - \mathrm{Del}_r(X, E + G^Q 1 + G^Q 2) = \mathrm{Del}_r(X, E + G^Q 1) - \mathrm{Del}_r(X, E + Gx)$$

Then they are acyclic due to Lemma 8.9. Furthermore, they can be combined to create a larger G^Q equivariant generalized matching due to Lemma 5.12. Therefore, we can focus on finding a G^Q equiv-

ariant generalized matching whose critical simplices are $\mathrm{Del}_r(X, E + G^Q 1)$, as the other case is similar.

New problem

Find G^Q equivariant acyclic generalised matching on $[Q, Q + Gx]_r$ such that the non singleton intervals lie in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + G^Q 1)$.

Problem reduction

Notice that the problem is similar to how this example started. Instead of partitioning $[Q, Q + Gx]_r$ directly, the set $[Q, Q+Gx]_r$ can be partitioned into sets of the form $[B, B+G^Q 1]_r$ where $B = B - G^Q 1$. This is acyclic due to Lemma 8.1. Notice that, finally, the orbit lemma, Lemma 8.16, can be applied to $[Q, Q + G^Q 1]_r$. We only need to consider the remaining sets $[B, B + G^Q 1]_r$ where $B \neq Q$. To avoid overusing the + symbol, we denote by Q^J the set $Q + J$.

Consider, for example, the set $[B, B + G^Q 1]_r = [Q^{24}, Q^{24} + G^Q 1]_r$, where $Q^{24} = Q + \{2, 4\}$, for which $G^{Q^{24}} =< r^4, i | r^8 = e, i^2 = e, ir^4 = r^4 i >$, see Fig. 39. This stabiliser subgroup is isomorphic to the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$ as it is abelian with two generators of order 2.

Note that $G^{Q^{24}} 1 = \{1, 5\}, G^{Q^{24}} 3 = \{3\}, G^{Q^{24}} 7 = \{7\}$. Since $G^Q 1 \neq G^{Q^{24}} 1$, the orbit lemma,

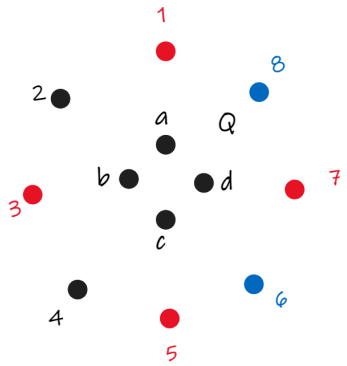


Figure 39: The black dots represent Q^{24}. The red points are the orbit $G^Q 1$. The orbit of $G^{Q^{24}}$ on $G^Q 1$ is not transitive. Indeed, $G^Q 1$ consists of the orbits $G^{Q^{24}} 1, G^{Q^{24}} 3$ and $G^{Q^{24}} 7$ respectively.

Lemma 8.16, cannot be applied. However, similar to the case for Q and Q', any $G^{Q^{24}}$-equivariant partition on $[Q^{24}, Q^{24} + G^Q 1]_r$ defines, via a reflection i, an $G^{Q^{68}}$-equivariant partition on $[Q^{68}, Q^{68} + G^Q 1]_r$. This is well defined similar to the previous argument for $[Q, Q + Gx]_r$ and $[Q', Q' + Gx]_r$.

New Problem

Find a $G^{Q^{24}}$ equivariant acyclic generalised matching on $[G^{Q^{24}}, G^{Q^{24}} + G^Q 1]_r$ such that the non singleton intervals are in $\mathrm{Del}_r(X, E + G^Q 1)$.

Solution structure

The previous steps can be re-applied to this new problem. Specifically, construct an acyclic generalised matching on $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + G^{Q^{24}} 3)$, then on $\mathrm{Del}_r(X, E + G^{Q^{24}} 3) - \mathrm{Del}_r(X, E + G^{Q^{24}} 3 + G^{Q^{24}} 7)$ and then on $\mathrm{Del}_r(X, E + G^{Q^{24}} 3 + G^{Q^{24}} 7) - \mathrm{Del}_r(X, E + G^Q 1)$.

This process must be finite, as the size of the orbit $G1, G^Q 1, G^{Q^{24}} 1$ is reduced every time. Therefore, eventually, the orbit lemma, Lemma 8.16, can be applied. That this procedure works in general is formally proven in the orbit reduction lemma, Lemma 8.18.

8.2.1 Orbit reduction lemma

Lemma 8.18 (Orbit reduction lemma)**.** Suppose H is a subgroup of G and $E \subset X$ are H symmetrical. If $Q \in \mathrm{Del}_r(X, E)$ with $Q = Q - Hx$, Then there exists a H^Q-equivariant acyclic generalised matching on $[Q, Q + Hx]_r$ such that the critical intervals lie in $\mathrm{Del}_r(X, E + Hx)$ and the non critical intervals in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Hx)$.

Because recursion is applied in the following proof, we need to distinguish between the 'current' and 'next level' in the recursion. This is done by adding a $'$ to the relevant notation.

Proof. Denote by D_1 and D_2 the complexes $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, E + Hx)$ respectively. Further, let $Hx/H^Q = \{H^Q x_1 < \ldots < H^Q x_m\}$ be an ordering. If $m = 1$, then H^Q acts transitively on Hx. Therefore, the conditions of the orbit lemma, Lemma 8.16, are satisfied. The orbit lemma supplies the wanted generalised matching.

If $m > 1$, then denote by E_i the union of orbits $H^Q x_j$ for $j \leq i$, i.e. $E_i = \bigcup_{j \leq i} H^Q x_i$. By Lemma 8.9, for H^Q sets $E, E + Hx$, there is an acyclic H^Q equivariant partition on $[Q, Q + Hx]_r$ such that the classes are contained in $\mathrm{Del}_r(X, E + Hx)$ or in $\mathrm{Del}_r(X, E_{i-1}) - \mathrm{Del}_r(X, E_i)$ for some i. Therefore, due to Lemma 5.12, it suffices to provide an H^Q equivariant acyclic generalised matching on $[Q, Q + Hx]_r$ such that every non-trivial class is in some $\mathrm{Del}_r(X, E_{i-1}) - \mathrm{Del}_r(X, E_i)$ and the singleton classes in $\mathrm{Del}_r(X, E_i)$. Indeed, the wanted partition on D_1 then consists of all the non-singleton intervals of the partitions on the $\mathrm{Del}_r(X, E_{i-1}) - \mathrm{Del}_r(X, E_i)$, for all i, where the singleton intervals consist of $\mathrm{Del}_r(X, E + Hx)$. For every $1 \leq i \leq m$, we construct an H^Q equivariant acyclic generalised matching on $[Q, Q + Hx]_r$ such that the non singleton intervals lie inside $\mathrm{Del}_r(X, E_{i-1}) - \mathrm{Del}_r(X, E_i)$ and the critical intervals inside $\mathrm{Del}_r(X, E_i)$.

First, define the following notation.

$$D_1' = \mathrm{Del}_r(X, E_{i-1}), \quad D_2' = \mathrm{Del}_r(X, E_i), \quad E' = E_{i-1}, \quad x' = x_i$$

Further, denote by $[B, B + H^Q x']_{r'}$, for $B \in [Q, Q + Hx]_r$, the simplices in $[B, B + H^Q x'] \cap D_1'$, i.e.

$$[B, B + H^Q x']_{r'} = [B, B + H^Q x'] \cap D_1'$$

Notice that $[B, B + H^Q x']_{r'} \subset [Q, Q + Hx]_r$ by construction.

Consider the following partition, which we denote by $\sim$, of $[Q, Q + Hx]_r \cap D_1$.

$$\{[B, B + H^Q x']_{r'} | B \in [Q, Q + Hx]_r, \quad B = B - H^Q x'\}$$

This partition is acyclic due to Lemma 8.1 where $Y = H^Q x'$. Notice that H^Q acts on the partition $\sim$. Indeed, for $h \in H^Q$, $h[B, B + H^Q x'] = [h(B), h(B) + H^Q x']$ and $h(B) = h(B) - H^Q x'$ by construction. Let $[B, B + H^Q x]_{r'}$ be a representative of its orbit class of $\sim$ under the action H^Q. Suppose there is an H^B equivariant partition, denoted $\sim_B$, on $[B, B + H^Q x]_{r'}$. Then we can define an $H^{B'}$-equivariant partition on any of the intervals $[B', B' + H^Q x']$ in its orbit. Indeed, for $h \in H^Q$, define $\sim_{hB}$ as the partition of sets $h(L)$ where $L \in \sim_B$. Notice that $\sim_{hB} = \sim_B$ if $h \in H^B$ since $\sim_B$ is H^B-equivariant. Therefore, the choice of h depends only on its class in H^Q/H^B. Furthermore, for two elements h, g with different classes in H^Q/H^B, we note that $h(B) \neq g(B)$, for otherwise $h^{-1}g \in H^B$. Therefore, defining $\sim_{hB}$ as $\sim_B$ is well defined irrespective of choice of element $h \in H^Q$ such that $h(B) = B'$.

Therefore, for providing a H^Q-equivariant generalised matching on $[Q, Q + Hx]_r$ such that non-trivial intervals lie in $D_1' - D_2'$, it is sufficient to define an H^B-equivariant generalised matching on $[B, B + H^Q x]_{r'}$ such that the non-trivial intervals lie in $D_1' - D_2'$ and the singleton intervals in D_2' for one representative of every orbit class in $\sim$.

Notice, however, the above steps can be applied again. Indeed, E' is H^Q symmetric, $x' \in X$, and B is in $\mathrm{Del}_r(X, E')$ with $B = B - H^Q x'$, which is similar to the situation where the proof started. More exactly, redefine $H := H^Q$, $E := E'$, $x := x'$, $Q := B$, $D_1 := D_1'$ and $D_2 := D_2'$, then apply the steps above again.

Therefore, this process can be iteratively applied. However, this process is finite as each step reduces the orbit size $|H^Q x| < |Hx|$. We conclude that the lemma holds. □

8.3 Proof of equivariant collapsing theorem

The equivariant Delaunay collapsing theorem, Corollary 8.19.1, can now be proven.

Theorem 8.19 (Equivariant Delaunay collapsing theorem)**.** Let $E \subset X$ be G symmetric and $x \in X$. Then there exists a G-equivariant collapse

$$\mathrm{Del}_r(X, E) \searrow_G \mathrm{Del}_r(X, E + Gx)$$

Proof. By Theorem 7.22 it suffices to prove there is a G equivariant generalised Morse matching on $\mathrm{Del}_r(X, E)$ whose non singleton intervals are $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Gx)$. Let $\{[Q, Q + Gx]_r | Q \in \mathrm{Del}_r(X, E), \quad Q = Q - Gx\}$ be the acyclic partition induced by Lemma 8.1 of $\mathrm{Del}_r(X, E)$. This partition is G equivariant. Choose a representative $[Q, Q + Gx]_r$ of each equivalence class under the G orbit. By Lemma 8.18 there exists an G^Q equivariant generalised matching on $[Q, Q + Gx]_r$ such that the non singleton intervals are in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Gx)$. Similar to the proof in Lemma 8.18, the matching on $[Q, Q + Gx]_r$ can be defined uniquely on each of the intervals in its orbit class under G. Due to Lemma 5.12, the union of the partitions is acyclic. Therefore, this defines a G-equivariant generalised Morse matching on $\mathrm{Del}_r(X, E)$ whose non critical simplices are in $\mathrm{Del}_r(X, E) - \mathrm{Del}_r(X, E + Gx)$ while the critical simplices are in $\mathrm{Del}_r(X, E + Gx)$. This implies the theorem as argued at the beginning of the proof. □

Similar to Corollary 6.9.1, by iteratively applying Theorem 6.9 the equivariant Delaunay collapsing theorem follows.

Corollary 8.19.1. Let $X \subset \mathbb{R}^2$ be G symmetric, then there exists a G equivariant collapse between the $\check{\mathrm{C}}\mathrm{ech}_r(X)$ and $\mathrm{Del}_r(X)$ complex, i.e.

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow_G \mathrm{Del}_r(X)$$

9

A corollary of the equivariant collapsing theorem

In the previous chapters, Delaunay complexes were introduced and using discrete Morse theory, a collapsing theorem between them was proven. All theory was specialised to the equivariant case where the data $X \subset \mathbb{R}^2$ was assumed to be symmetric with respect to an orthogonal action of a finite group G.

Overview

In this section, the definitions relevant to persistent homology are recalled and corollaries of the main theorem are discussed. Specifically, two corollaries, Corollary 6.1 and the diagram on page 18 found in [7], of isomorphic persistent homology and a map $\mathrm{Del}_r(X) \to Del_r(Y)$ for $X \subset Y$, are repeated for the equivariant case. An overview of discrete Morse theory for computations of persistent homology can be found in [23]. Furthermore, two simple examples illustrate the computation of the barcode decomposition of an equivariant Delaunay complex. This decomposition is not invariant under symmetry. The last part poses two questions about equivariant persistent homology and persistent homology of non-symmetric data sets Y which have a symmetrical subset X.

The section assumes that the reader is familiar with (simplicial) homology, found in, for example, Chapter 3 of [13] or general homology theory, see Chapter 2 of [1].

Notation

For this section, we assume that k is some field. Furthermore, $X \subset \mathbb{R}^n$ unless it is clear from the context that X is G symmetric, see Definition 7.5, for some finite group G. In that case, it is assumed that $X \subset \mathbb{R}^2$. Furthermore, this section assumes that $E \subset X$ and $r \geq 0$.

9.1 Persistent homology of Delaunay complexes

If the discussion is too abstract, the reader is advised to first skip to the example below, see Section 9.2.

Homology of Delaunay complexes

Similar to [13] section 3, denote the n-th homology group over a field k of a simplicial complex $\mathbb{K}$ as $H_n(\mathbb{K}; k)$. Recall that a collapse between $\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E+F)$ implies that there exists a series of deformation retracts between their geometric realisations, i.e. $|\mathrm{Del}_r(X, E+F)| \hookrightarrow |\mathrm{Del}_r(X, E)|$ is a homotopy equivalence. This fact implies, in particular, that the homology of $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, E+F)$ are isomorphic.

The equivariant collapsing theorem 8.19.1 states that the deformation retracts can be chosen to be G equivariant if $X \subset \mathbb{R}^2$ is G symmetric. This implies that the homologies are isomorphic as representations of G. Diagrammatically, this is captured in the following commutative diagram where the horizontal

maps are the homotopy equivalences.

$$\begin{tikzcd} |\operatorname{Del}_r(X, E+F)| \arrow[r, hook, "\simeq"] \arrow[d, "g"'] & |\operatorname{Del}_r(X, E)| \arrow[d, "g"] \\ |\operatorname{Del}_r(X, E+F)| \arrow[r, hook, "\simeq"] & |\operatorname{Del}_r(X, E)| \end{tikzcd}$$

More algebraically, the diagram denotes the fact that the inclusion $i : |\operatorname{Del}_r(X, E+F)| \hookrightarrow |\operatorname{Del}_r(X, E)|$ commutes with every g, i.e. $i \circ g = g \circ i$.

However, we can conclude an even stronger result, as there is an equivalence between their *persistent homologies*, discussed below.

Persistent homology

To measure how homology 'persists through time', a notion of a 'sequence of topological spaces' is required.

Definition 9.1. A **filtration** [17] is a collection of topological spaces $F = (F_r)_{r \in \mathbb{R}_{\geq 0}}$ such that

$$F_r \subset F_{r'}, \quad r \leq r'$$

Example 12. All simplicial complexes introduced this far are part of a filtration. Indeed, $\operatorname{Del}_r(X, E) \subset \operatorname{Del}_{r'}(X, E)$ for every $r \leq r'$. In particular, the $\check{\text{C}}\text{ech}_r(X) = \operatorname{Del}_r(X, \emptyset)$ and Delaunay $\operatorname{Del}_r(X) = \operatorname{Del}_r(X, X)$ complexes are part of filtrations.

Furthermore, these filtrations are *discrete*, in the sense that there is a finite set $\{r_1, \ldots, r_n\}$ such that $\operatorname{Del}_{r_i}(X, E)$ is equal to $\operatorname{Del}_{r'}(X, E)$ for all $r' < r_{i+1}$ and not otherwise. The reason for this is that X is finite.

A map between filtrations $\mu : F \to L$ is a map $\mu_r : F_t \to L_t$ at each t, such that it stays 'the same' map on the subspace $F_t \subset F_{t'}$ for $t \leq t'$. This is the following definition.

Definition 9.2. A **natural transformation** of two filtrations $(F_i)_{t \in T}$ and $(L_i)_{t \in T}$ is a collection of maps $(\mu_t)_{t \in T}$ such that

$$\begin{tikzcd} X_r \arrow[r, "\mu_r"] \arrow[d, hook] & Y_r \arrow[d, hook] \\ X_{r'} \arrow[r, "\mu_{t'}"] & Y_{r'} \end{tikzcd}$$

where the vertical maps are inclusions.

Example 13. The following diagram illustrates that the inclusion induces a natural transformation between $\operatorname{Del}_*(X, E+F)$ and $\operatorname{Del}_*(X, E)$.

$$\begin{tikzcd} \operatorname{Del}_r(X, E+F) \arrow[r, hook] \arrow[d, hook] & \operatorname{Del}_r(X, E) \arrow[d, hook] \\ \operatorname{Del}_{r'}(X, E+F) \arrow[r, hook] & \operatorname{Del}_{r'}(X, E) \end{tikzcd}$$

Remember that the homology $H_*(\mathbb{K}; k)$ is a vector space.

[17] A more general definition to enable multi-parameter persistence homology is to consider directed graphs for T, called quivers. See [37] for more information.

Definition 9.3. The homology of a filtration $H_*(F;k)$ is called the **persistent homology**, denoted $PH(F;k)$, of that filtration. The inclusion maps $F_r \hookrightarrow F'_r$ induce linear maps

$$H_*(F_r;k) \xrightarrow{f_r^{r'}} H_*(F_{r'}(X);k)$$

where $f_{r'}^{r''} \circ f_r^{r'} = f_r^{r''}$

The persistent homology is an example of a *persistence module.*

Definition 9.4. A **persistence module**[18] V is a sequence of vector spaces over k, $(V_r)_{r\geq 0}$ with a linear maps $f_{r\to r'} : V_r \to V_{r'}$ such that $f_{r'}^{r''} \circ f_r^{r'} = f_r^{r''}$.

Example 14. Let $I = [a,b)$ be an interval in $\mathbb{R} \cup \{\infty\}$, that is, we consider $[a,\infty)$ a valid interval. A basic example of an persistence module is when $V_r = 0$ if $r \notin I$, and $V_r = k$ otherwise. Furthermore, the linear maps are identities $f_r^{r'} = id$ for $[r,r') \subset I$ and zero otherwise. Such a persistence module is called an **interval module** and denoted by $I^{[a,b)}$.

The notion of a map between persistence modules is similar to the natural transformation of filtrations.

Definition 9.5. A map of persistence modules $\gamma : V \to W$ is a collection of linear maps $\gamma_r : V_r \to W_r$ such that the following diagram commutes for all $f_r^{r'}$ and $g_r^{r'}$.

$$\begin{tikzcd} V_r \arrow[r, "\gamma_r"] \arrow[d, "f_r^{r'}"'] & W_r \arrow[d, "g_r^{r'}"] \\ V_{r'} \arrow[r, "\gamma_{r'}"] & W_{r'} \end{tikzcd}$$

If the horizontal maps γ_r are isomorphisms, then the persistence modules are said to be **isomorphic**.

Isomorphic persistent homology of Delaunay complexes

A natural transformation between filtrations F and L induces a map, via homology, between their persistent homology. The Delaunay collapsing theorem 6.9 implies that there is a retract $|\operatorname{Del}_r(X,E)|$ into $|\operatorname{Del}_r(X,E+F)|$, where the latter complex is seen as a subspace of the former complex. In turn, this implies that the inclusion $|\operatorname{Del}_r(X,E+F)| \hookrightarrow |\operatorname{Del}_r(X,E)|$ is a homotopy equivalence. Now observe the following diagram, in which the horizontal maps are inclusions and thus homotopy equivalences by the previous observation.

$$\begin{tikzcd} |\operatorname{Del}_r(X,E+F)| \arrow[r, hook, "\simeq"] \arrow[d, hook] & |\operatorname{Del}_r(X,E)| \arrow[d, hook] \\ |\operatorname{Del}_{r'}(X,E+F)| \arrow[r, hook, "\simeq"] & |\operatorname{Del}_{r'}(X,E)| \end{tikzcd}$$

This diagram is commutative, and is thus a map of persistence modules, as it consists of the inclusion of subspaces into (a subspace of) $|\operatorname{Del}_{r'}(X,E)|$. Since the horizontal maps are (simple) homotopy equivalences, these maps are isomorphisms in the persistence homology. This implies that the persistent homologies are naturally isomorphic. This is stated as the following corollary.

Corollary 9.5.1. For all $E, F \subset X \subset \mathbb{R}^m$, the persistent homology of $\operatorname{Del}_r(X,E)$ and $\operatorname{Del}_r(X,E+F)$ are isomorphic, in particular, this holds for $\operatorname{Del}_r(X)$ and $\check{\text{C}}\text{ech}_r(X)$.

If there is a finite group G and $X \subset \mathbb{R}^2$ is symmetric in the sense of Definition 7.5, then the persistence homologies are additionally isomorphic as representations of G as a corollary of Corollary 8.19.1. Indeed, consider the following cubical-shaped diagram where $\operatorname{Del}_r(X,E+F)$ and $\operatorname{Del}_r(X,E)$ are denoted by V_r, W_r, respectively.

[18] The "module" refers to the fact that the theory can be done more generally with modules over a Principal Ideal Domain (PID) such as $\mathbb{Z}$ instead of a field k. However, for non-field coefficients, torsion can be encountered in the persistent homology, see [38]. This thesis only considers coefficients in a field as it is relevant for the barcode defined later.

$$
\begin{tikzcd}
 & V_r \arrow[rr, hook] \arrow[d, hook] & & W_r \arrow[d, hook] \\
V_r \arrow[ur, "g"] \arrow[d, hook] \arrow[rr, hook, bend right] & V_{r'} \arrow[rr, hook, bend right] & W_r \arrow[ur, "g"] \arrow[d, hook] & W_{r'} \\
V_{r'} \arrow[ur, "g"] \arrow[rr, hook] & & W_{r'} \arrow[ur, "g"] &
\end{tikzcd}
$$

The diagram is again commutative as the complex $\mathrm{Del}_r(X, E+F)$ is a G subcomplex of $\mathrm{Del}_{r'}(X, E)$ at every radius $r \leq r'$. Corollary 8.19.1 implies that there are G equivariant deformation retracts of $\mathrm{Del}_r(X, E)$ into $\mathrm{Del}_r(X, E+F)$ for each r, and therefore, the horizontal maps are G equivariant homotopy equivalences. Furthermore, the commutativity implies that the natural transformation $\mathrm{Del}_*(X, E+F)$ to $\mathrm{Del}_*(X, E)$ commutes with the action of G. Thus, the homology of $\mathrm{Del}_r(X, E)$ and $\mathrm{Del}_r(X, E+F)$ are isomorphic as representations over G at any fixed r.

Remark 9.6. Categorically, a filtration is functor $F : P \to Top$, where P is a poset, such that the morphisms become mono-morphisms $F(i \to j) = F(i) \hookrightarrow F(j)$. A functor $F : P \to Vect_k$ of a poset P to vector spaces over k is then called a persistence module. Recall that homology, $H(-; k)$, is a functor from $Top \to Vect_k$. Therefore, composing a filtration with homology gives a persistence module. All the drawn diagrams above state that the corresponding constructions are functorial.

The barcode

Two persistence modules V, W can be composed as a direct sum in the usual way $V \oplus_i W = V_i \oplus W_i$ where the maps are the induced maps from the individual modules. The following theorem states a converse results, i.e. is a decomposition theorem for persistent homology.

Corollary 9.6.1. If V is a persistence module such that $dim(V_r) < \infty$ for all $r \in T$, then the persistence module decomposes into a direct sum of interval modules.

Proof. See Theorem 1.2 in [5] □

Note that the interval modules might contain repeats of the same interval. To correctly index the intervals, the notion of a multiset is handy.

Definition 9.7. A **multiset** is a set with multiplicity; elements can be repeated.

Example 15. The multiset $\{1, 2, 3, 3, 3\}$ is not equal to the multiset $\{1, 2, 3\}$ even though they both define the same set.

If the persistence module is the persistent homology of a filtration F, then the decomposition, unique to reordering, is called the **barcode** of F. The decomposition is written in the following way.

$$PH(F) = \oplus_{[a,b)] \in \mathcal{B}} I^{[a,b)}$$

Here, $\mathcal{B} = \{[a_1, b_1), \ldots, [a_m, b_m)\}$ is a multiset of intervals.

Remark 9.8. To compute the barcode, we can choose generators of the homology at each filtration parameter r such that the 'oldest' homology generator is preserved if possible. This preference is called the **elder rule**, as found in Chapter 7.1 in [28]. For a more comprehensive overview on the computation of persistent homology see [38].

9.2 Example of persistent homology of equivariant Delaunay complex

The Delaunay complexes $\mathrm{Del}_r(X, E)$ are simplicial G complexes if X is G symmetric as in Definition 7.5 and G is finite. The homology of the complex also has an action of G by linear maps. Remember that a linear action of G into a vector space is called a group representation of G over V. Similar to the barcode, group representations also have a notion of decomposability.

Group representation theory

Two basic notions in group representation theory are *decomposability* and *reducibility* [39]. The former asks the question: "Do the G-invariant subspaces decompose as a direct sum?". The latter asks the question: "Are there subspaces which are G-invariant?".

More exactly, a representation $\rho : G \to Aut(V)$ over a vector space V has a **sub-representation** $\rho' : G \to Aut(W)$ over a subspace W, if $W \subset V$ is a proper non-zero subspace and ρ restricted on the codomain to $GL(W)$ is ρ', i.e. $\rho(g)|_W = \rho'(g)$ for all $g \in G$.

If ρ can be written as the direct sum of two sub-representations, i.e. $\rho(G) = \rho_0(G) \oplus \rho_1(G)$, then ρ is called **decomposable**.

Decomposability is equivalent to finding a basis such that every action g is in the same diagonal block form (with diagonal entries being square matrices). In other words, every g has the following form in this basis.

$$\begin{pmatrix} A_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_m \end{pmatrix}$$

Here, $A_1, \dots A_m$ are square matrices of possibly different sizes. This, in turn, corresponds to every action working on a direct sum of spaces.

If ρ has a sub-representation, then ρ is called **reducible**. Reducibility is equivalent to finding a basis such that each action g is represented in upper triangular form with (possibly) matrix entries.

Remark 9.9. Decomposable implies reducible, and the converse holds if $k = \mathbb{C}$ [39].

Example computation of barcode of equivariant Delaunay complexes

We have seen that the persistent homology of the filtration defined by the Delaunay complex $\mathrm{Del}_*(X, E)$ has a decomposition into barcodes. Such a decomposition gives a basis for the homology $H(\mathrm{Del}_r(X, E); k)$ at every r. If $\mathrm{Del}_r(X, E)$ is a simplicial G complex, then the homology $H(\mathrm{Del}_r(X, E); k)$ also has a notion of decomposability with respect to the G action. In the example below, it is shown that the representation induced by the decomposition into barcodes need not make the action of G on the homology decomposable.

In this example, the barcode decomposition of the first persistent homology for a Delaunay complex with a G action using the elder rule (see Remark 9.8), i.e. we compute $PH_1(\mathrm{Del}_r(X), \mathbb{F}_2)$.[19]

[19] The choice of the field $k = \mathbb{F}_2$ is not unusual for persistent homology computations. See [40] for more information on the consequences of the choice of a field.

Consider a slightly deformed square of 4 points a, b, c, d in $\mathbb{R}^2$, see Example 16.

$$a = (0, 1) \tag{9}$$
$$b = (\epsilon, \epsilon) \tag{10}$$
$$c = (1, 0) \tag{11}$$
$$d = (1 - \epsilon, 1 - \epsilon) \tag{12}$$

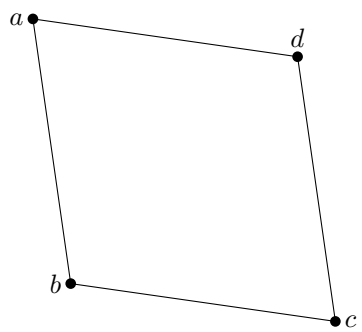
where $1 >> \epsilon > 0$. Computing the distances for $d(a, b), d(b, d)$ and $d(a, c)$ respectively shows that

- $d(a, b) = \sqrt{(1 - \epsilon)^2 + \epsilon^2} = \sqrt{1 - 2\epsilon + 2\epsilon^2}$
- $d(b, d) = \sqrt{(1 - 2\epsilon)^2 + (1 - 2\epsilon)^2} = \sqrt{2}|1 - 2\epsilon|$
- $d(a, c) = \sqrt{2}$

Notice that the $d(a, b) = d(a, d) = d(b, c) = d(c, d)$ equalities hold by construction. Lastly, the inequalities $d(a, b) < d(b, d) < d(a, c)$ also hold for ϵ small enough.

Remark 9.10. Notice that two points q, p in the Čech complex $\check{\text{Cech}}_r(X)$ define a 1-simplex if $d(q, p) \leq 2r$. Indeed, then the balls $B_r(p) \cap B_r(q)$ intersect in at least $\frac{1}{2}(p + q)$.

If we compute the Delaunay complex at radius $2r = d(a, b)$, then we obtain the following Figure 16.

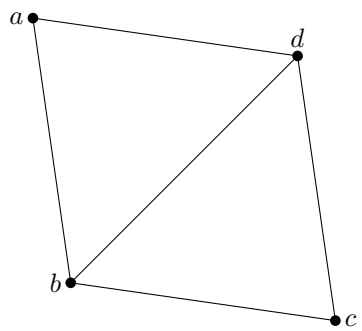


Example 16.

Furthermore, consider an action of $G = \mathbb{Z}/2\mathbb{Z}$ by reflection in the line spanned by bd. Hence, on the vertices, this action permutes a and c.

The simplicial complex is homotopic to S^1 and therefore has the same homology type. A generator for the homology H_1 is $ab + bc + cd + ad$. Therefore $H_1(\text{Del}_r(X); \mathbb{F}_2) \cong \mathbb{F}_2$. The induced action on homology sends the generator to itself.

If a ball around b intersects the ball around d at precisely one point, then this is the midpoint between b, d. The midpoint between b and d is $(0.5, 0.5)$, which is also the midpoint of a and c. At radius $r' = \frac{1}{2}d(b, d)$, the balls $B_{r'}(b)$ and $B_{r'}(d)$ intersect precisely at 0. However, the balls of a, c respectively do not intersect at zero at that radius since $2r' < \sqrt{2} = d(a, c)$. Therefore, there is no triple intersection of balls around a, b, d at radius $r' = d(b, d)$, but the balls of b and d do intersect. This means that the Delaunay complex at radius $r = d(b, d)$ is given by the edges of the triangles abd and bcd.



Example 17.

This space is homotopic to the wedge of two circles. Therefore, the first homology is $H_1(\text{Del}_{r'}(X); \mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2$. The elder rules suggests keeping $ab + bc + cd + ad$ as a generator. The other generator is chosen

as $ab + bc + ac$.

The next step at which the filtration changes, is when r is large enough such that there is a triple intersection of balls around abd and bcd. The resulting space is convex and thus contractible to a point. Therefore, the homology is trivial.

We conclude that the first persistent homology is isomorphic to the following persistence module.

$$PH_1(\mathrm{Del}_r(X); \mathbb{F}_2) \cong \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$$

The generators that were chosen induce the following barcode decomposition.

$$\mathbb{F}_2 \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \mathbb{F}_2 \oplus \mathbb{F}_2$$

On the homology, the g action fixes $ab + bc + cd + ad$, and permutes $ab + bc + ac$ with $bc + bd + bd$. In terms of the basis from the barcode decomposition, $bc + bd + bd$ is represented as $(1, 1)$ in $\mathbb{F}_2 \oplus \mathbb{F}_2$. Therefore, $g : \mathbb{F}_2 \oplus \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$ acts on the second term $\mathbb{F}_2 \oplus \mathbb{F}_2$ of the persistence module as follows.

$$g = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

The barcode together with the g action is summarised in the following diagram.

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "\mathrm{id}"] & \mathbb{F}_2 \oplus \mathbb{F}_2 \arrow[d, "\begin{pmatrix}11\\01\end{pmatrix}"] \\
\mathbb{F}_2 \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathbb{F}_2 \oplus \mathbb{F}_2
\end{tikzcd}$$

The diagram implies that the subspace generated by $ab + bc + cd + ad$ is invariant under g, that is $g(ab + bc + cd + ad) = ab + bc + cd + ad$. Therefore, the representation is reducible.

However, the representation is not decomposable. If the representation was decomposable, then there exist two coefficients $\lambda_1, \lambda_2 \in \mathbb{F}_2$ such that the following holds for some invertible matrix B.

$$B^{-1}gB = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}$$

Since g is invertible, the matrix on the right-hand side is also invertible. This implies that $\lambda_i = 1$, for $i = 1, 2$, which is the only invertible element in $\mathbb{F}_2$. However, this implies that $g = id$ as both B and B^{-1} can be taken to the right-hand side.

We conclude that the barcode decomposition of the persistent homology induces a reducible indecomposable representation of G. In this sense, the notions of barcode decomposition and group representation decomposition do not coincide.

To ensure that the representation as in Example 16 is decomposable, a different choice of the coefficient field would suffice. Indeed, if $k = \mathbb{C}$, then a reducible representation is decomposable [39]. Therefore, the representation discussed would be decomposable. However, the field $\mathbb{C}$ is not finite, which makes it less usable for computational purposes.

We wonder whether persistent homology can be adapted to the equivariant case to ensure that the decomposition of the barcode is also a decomposition of the representation even if the field is not $\mathbb{C}$. One such direction could be an equivariant version of persistent homology, where the homology itself is equivariant.

9.3 A question for properties induced map of persistent homology

Induced map of Delaunay complexes

As noted in [7] page 18, the collapsing theorem 6.9.1 induces an inclusion-like map between the geometric realisation of two Delaunay complexes with different point clouds $X \subset Y$. The map follows from the following commutative diagram, where the map from $|\operatorname{Del}_r(Y, X)|$ to $|\operatorname{Del}_r(Y, Y)|$ is the retract induced by Theorem 6.9.

$$\begin{tikzcd}
|Del_r(X,X)| \arrow[r, hook] \arrow[d, dashed] & |Del_r(Y,X)| \\
|Del_r(Y,Y)| \arrow[ur, hook, "\simeq"'] &
\end{tikzcd}$$

This map induces a map from $\operatorname{Del}_r(X, X)$ to $\operatorname{Del}_r(Y, Y)$. For equivariant Delaunay complexes where X, Y are G sets, we know that the retract can be chosen to be equivariant by Theorem 8.19. The induced map on persistent homology is also equivariant.

However, suppose that Y is *not* symmetric, but X is symmetric. Does the map $\operatorname{Del}_r(X, X) \to \operatorname{Del}_r(Y, Y)$ map capture some notion of symmetry of the persistent homology of Y?

This is relevant because it could happen that only subsets of data is symmetrical, even if the whole data is not. In particular, this might help with clustering algorithms, which deal with local clumps of data, although no claim is made on the topic.

A

Existence and uniqueness of a Minimal Enclosing Excluding Sphere (MEES)

Suppose that Q, E are two finite sets in $\mathbb{R}^n$. This appendix proves the claim that there is none or exactly one sphere S_ρ which encloses Q and excludes E whereby ρ is minimal. Note that, if Q is a point, then $\{Q\}$ is considered a minimal enclosing sphere.

First, we show that a minimal sphere exists if there exists at least one. Afterwards, we shall show that the minimal sphere is unique.

Existence of minimal sphere

Lemma A.1. Let $Q, E \subset \mathbb{R}^n$ be finite sets. Suppose that the set of spheres which encloses Q and excludes E, $\mathcal{S}$, is non-empty. Under this condition, a sphere with minimal radius exists in $\mathcal{S}$.

Proof. Let R be the set $\{r \in \mathbb{R}_{\geq 0} | \exists (z, r) = S \in \mathcal{S}\}$, and δ the infimum of R in $\mathbb{R}$. There is a sphere $(z_m, r_m) \in \mathcal{S}$ with radius arbitrarily close to δ, i.e. $r_m \in [\delta, \delta + 1/m]$, for each m. By construction it holds that $\|z_m - q\| \leq r_n \leq \delta + 2$ for some $q \in Q$. This implies that the sequence $(z_m)_{n \in \mathbb{N}}$ is bounded, since $\|z_m\| - \|q\| \leq \|z_m - q\| \leq \delta + 2$. Therefore, the sequence $(z_m)_{m \in \mathbb{N}}$ has a convergent subsequence with limit z. We shall show that the sphere (z, δ) encloses Q and excludes E.

Suppose that (z, δ) does not enclose a point $q \in Q$, i.e. $\delta < d(z, q)$. Then the following inequality holds for all (z_m, r_m).

$$\delta < d(z, q) \leq d(z, z_m) + d(z_m, q) \leq d(z, z_m) + r_m$$

However, $d(z, z_m) + r_m$ converges to δ by construction. This implies that there exists some (z_N, r_N), for $N \in \mathbb{N}$ large, such that q is not enclosed by (z_N, r_N). This is a contradiction.

Suppose that (z, δ) does not include a point $e \in E$, i.e. $\delta > d(z, e)$. Then an analogous proof follows from the following inequality.

$$\delta > d(z, e) \geq d(z_m, e) - d(z, z_m) \geq r_m - d(z, z_m)$$

Here, $r_m - d(z, z_m)$ converges to δ. We conclude that (z, δ) is in $\mathcal{S}$. Therefore, there exists a minimal sphere in $\mathcal{S}$. □

Uniqueness of minimal sphere

Consider two different spheres S_R and S_r with radius R, r respectively. Suppose that both spheres include Q and exclude E. We will construct a sphere S_ρ with a strictly smaller radius $r, R > \rho$ such that all points of Q are included and of E are excluded by S_ρ. Notice that this implies that a minimal sphere that includes Q and excludes E must be unique. The proof is based on algebraic manipulation, and the

line of reasoning is from [41].

First we construct a new sphere, then we show that this new sphere has the required properties. [20]

Construction of a new sphere

Lemma A.2. Suppose S_R and S_r are two spheres in $\mathbb{R}^n$ with radius R and r respectively, which both include Q and exclude E for finite sets Q and E. Then there exists a sphere S_ρ that includes Q, excludes E and ρ is strictly smaller than both r and R.

Proof. Notice that any sphere $S = (z, r)$ with centre z and radius r uniquely defines the ball $B_r(z)$ with centre z and radius r. In this proof, we shall provide a construction of a ball B_ρ from the two balls B_R and B_r defined by S_R, S_r respectively that defines a sphere S_ρ which fulfils the criteria of the lemma.

By rotation and transformation, we can choose a convenient coordinates. Let the two balls of radii R and r be located along the x_1-axis centered at $(0, \ldots, 0)$ and $(d, 0, \ldots, 0)$, respectively. Without loss of generality, we can assume that $r \leq R$ and $d > 0$.
By construction, it is clear that $B_R \cap B_r = \emptyset$ if $R < d - r$, and if $R = d - r$, then both balls only share one common point $(R, 0, \ldots, 0)$.
By Pythagoras, if $d^2 + r^2 \leq R^2$ then the ball B_r is contained in the ball B_R and hence smaller. If there is no other n-dimensional ball such that all points of Q are enclosed and of E are excluded, then B_r and thus S_r is unique. The only case left to consider is when $R^2 < d^2 + r^2$.

The equations of the two corresponding boundary spheres S_R and S_r are

$$\begin{aligned} x_1^2 + x_2^2 + \ldots + x_n^2 &= R^2 \\ (x_1 - d)^2 + x_2^2 + \ldots + x_n^2 &= r^2. \end{aligned} \tag{13}$$

We can write $x_2^2 + \ldots + x_n^2 = R^2 - x_1^2$ and combine this with the second equation to get $(x_1 - d)^2 + (R^2 - x_1^2) = r^2$. Solving for x_1 gives $x_1 = \frac{1}{2d}(d^2 - r^2 + R^2)$. Let's denote this point in the x_1-axis with c. Note that by construction $0 < c < d$. Indeed, since $R^2 < d^2 + r^2$ we infer that $-r^2 + R^2 < d^2$ from which it follows that $2dc = d^2 - r^2 + R^2 < 2d^2$.

Plugging c back into Eq. (13) gives

$$x_2^2 + \ldots + x_n^2 = R^2 - c^2 = R^2 - \big(\frac{1}{2d}(d^2 - r^2 + R^2)\big)^2 = \frac{1}{4d^2}\big(4d^2R^2 - (d^2 - r^2 + R^2)^2\big) \tag{14}$$

which is the equation of a n-1 sphere with radius

$$\rho = \frac{1}{2d}\sqrt{(2dR)^2 - (d^2 - r^2 + R^2)^2} \tag{15}$$

and centre $(c, 0, \ldots, 0)$.
By the symmetry of the construction it holds that $\rho = \frac{1}{2d}\sqrt{(2dr)^2 - (d^2 + r^2 - R^2)^2}$. From a geometrical perspective, ρ is smaller than r, and thus R. This follows also algebraically from the fact that $4d^2(r^2 - \rho^2) = (d^2 + r^2 - R^2)^2 > 0$.

This S^{n-1} sphere defines an n ball B_ρ via the condition $(x_1 - c)^2 + x_2^2 + \ldots + x_n^2 \leq \rho^2$. Note that the dimension n does not play a role in describing the characteristics $\big((c, 0, \ldots, 0), \rho\big)$ of the new ball.

We shall now prove that this ball satisfies the properties as the lemma claims.

[20] We remark that, for the purposes of proving that the minimal sphere is unique, we can assume that $r = R$. However, we prove the more general case where $R \geq r$ is allowed.

Properties of B_ρ

If B_ρ is contained in the union of the two balls with radius R and r respectively, then $E \subset \text{excl}(B_R \cup B_r) \subset \text{excl}(B_\rho)$. If the intersection of both spheres is in B_ρ, then $Q \subset \text{incl}(B_r \cap B_R) \subset \text{incl}(B_\rho)$.

We shall prove that B_ρ is contained in the union of the two balls and contains their intersection. First, observe that a point $x = (c, x_2, \ldots, x_n)$ which satisfies Eq. (13), i.e. x lies in the intersection of S_R and S_r, implies the following equalities for ρ.

$$c^2 + x_2^2 + \ldots + x_n^2 = c^2 + \rho^2 = R^2$$
$$(c-d)^2 + x_2 + \ldots + x_n^2 = (c-d)^2 + \rho^2 = r^2$$

Union

Let $p = (p_1, \ldots, p_n)$ be a point on the boundary of the new ball B_ρ, i.e. $p \in S_\rho$. The question is whether $p_1^2 + p_2^2 + \ldots + p_n^2 \leq R^2$ or $(p_1 - d)^2 + p_2^2 + \ldots + p_n^2 \leq r^2$? We show that these cases correspond to $p_1 \leq c$ and $p_1 \geq c$, respectively.

Suppose that $p_1 \leq c$. From $(p_1 - c)^2 + p_2^2 + \ldots + p_n^2 \leq \rho^2$, it follows that $p_1^2 - 2cp_1 + c^2 + p_2^2 + \ldots + p_n^2 \leq \rho^2$. The following equalities show that $p \in B_R$.

$$\begin{aligned} p_1^2 + \ldots + p_n^2 &\leq \rho^2 + 2cp_1 - c^2 \\ &= (R^2 - c^2) + 2cp_1 - c^2 \\ &= R^2 + 2c(p_1 - c) \leq R^2 \end{aligned}$$

The last inequality uses the assumption $p_1 \leq c$. This proves that p lies inside B_R.

Assume now that $p_1 \geq c$. Similar to the previous case, we know that $(p_1 - c)^2 + p_2^2 + \ldots + p_n^2 = \rho^2 = r^2 - (c-d)^2$. Furthermore, the following equality holds by expanding the square.

$$(p_1 - d)^2 + p_2^2 + \ldots + p_n^2 = (p_1 - c)^2 + p_2^2 + \ldots + p_n^2 + 2p_1c - c^2 - 2p_1d + d^2$$

The following inequalities show that $p \in B_r$.

$$\begin{aligned} (p_1 - d)^2 + p_2^2 + \ldots + p_n^2 &= (p_1 - c)^2 + p_2^2 + \ldots + p_n^2 + 2p_1c - c^2 - 2p_1d + d^2 \\ &= \rho^2 + 2p_1(c-d) - c^2 + d^2 \\ &\leq \rho^2 + 2c(c-d) - c^2 + d^2 \\ &= r^2 - (c-d)^2 + c^2 - 2cd + d^2 \\ &= r^2 \end{aligned}$$

The inequality on the third line holds since $c - d < 0$ and $p_1 \geq c \geq 0$.

This proves that B_ρ is contained in the union $B_R \cup B_r$.

Intersection

Let p be a point in $B_r \cap B_R$, then:

(i) $p_1^2 + p_2^2 + \ldots + p_n^2 \leq R^2$, and

(ii) $(p_1 - d)^2 + p_2^2 + \ldots + p_n^2 \leq r^2$.

From (i) we have $(p_1 - c)^2 + p_2^2 + \ldots + p_n^2 \leq (p_1 - c)^2 + R^2 - p_1^2 = -2cp_1 + c^2 + R^2 = -2cp_1 + 2c^2 + \rho^2$ as $\rho^2 = R^2 - c^2$ by (2). If $p_1 \geq c$ then $p \in B_\rho$.

From (ii) we have

$$(p_1 - c)^2 + p_2^2 + \ldots + p_n^2 \leq (p_1 - c)^2 + r^2 - (p_1 - d)^2 = -2cp_1 + c^2 + r^2 + 2dp_1 - d^2 = 2(d-c)p_1 + c^2 + r^2 - d^2$$

As $\rho^2 = R^2 - c^2$, we get

$$2(d-c)p_1 + c^2 + r^2 - d^2 = \rho^2 - R^2 + 2(d-c)p_1 + 2c^2 + r^2 - d^2 = \rho^2 - R^2 + 2(d-c)(p_1 - c) + 2dc + r^2 - d^2$$

Recall that c is defined such that $2dc = d^2 - r^2 + R^2$, hence

$$(p_1 - c)^2 + p_2^2 + \ldots + p_n^2 \leq \rho^2 - R^2 + 2(d-c)(p_1 - c) + R^2 = \rho^2 + 2(d-c)(p_1 - c)$$

As $d > c$, we conclude that $p \in B_\rho$ if $p_1 \leq c$.

Conclusion

If there exist two different balls B_R and B_r in $\mathbb{R}^n$ such that all points of Q are included and of E are excluded, then there exists a ball with a smaller radius with the same properties. Therefore, the minimal ball with these properties is unique. We conclude that the Minimal Enclosing Excluding Sphere (MEES) of the pair (Q, E) is unique. $\square$

B

A descriptive explanation of the Minimal Enclosing Excluding Sphere (MEES)

Most conclusions of the thesis, such as Lemma 6.2, depend on the algebraic properties of a solution $S_{Q,E}$ of the Minimal Enclosing Excluding Sphere (MEES) $S(Q, E)$ for the pair (Q, E), see Definition 4.8 and Definition 4.4 respectively. Therefore, we provide some insights on how a solution is derived if it exists. The Karush-Kahn-Tucker (KKT) conditions mentioned in section 4 have some subtle behaviour. In this appendix a computation is made to show for a simple example that Theorem 4.7 is correct.

Finding the MEES is about solving linear and quadratic equations with boundary conditions (Quadratic Programming, QP), and the theory described in this appendix is standard. That is why a limited reference is made to the accompanying literature [31], chapter 2. Relevant is the idea that Quadratic Programming sometimes reduces to Linear Programming (LP).

Convex hull

Consider a finite data set Q in $\mathbb{R}^n$. In geometry, the convex hull of a shape is the smallest convex set that contains it. Algebraically, the convex hull of Q is the set $Hull(Q) = \{\sum_{x \in Q} \lambda_x x | \lambda_x \geq 0, \sum_{x \in Q} \lambda_x = 1\}$. For a convex hull, the **vertices** is the set of extreme points of the polytope. This is best indicated in Fig. 41.

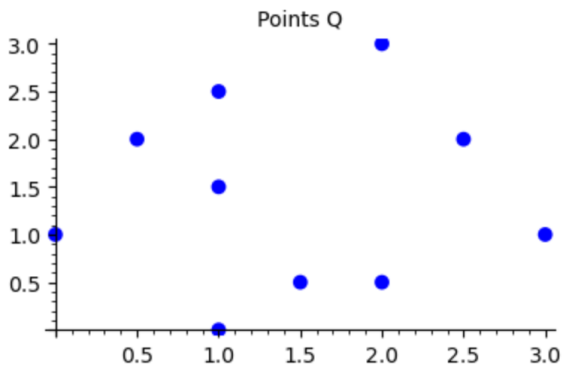


Figure 40: A finite set of points in $\mathbb{R}^2$.

Note that in this example the set of vertices is $V = \{(0.0, 1.0), (0.5, 2.0), (1.0, 0.0), (1.0, 2.5), (2.0, 3.0), (3.0, 1.0)\}$.

Minimal Enclosing Circle (MEC)

Any circle which encloses all points Q encloses also the convex hull of Q. A well-known result from algebraic geometry is that the minimal enclosing sphere only depends on the vertexes of the convex hull

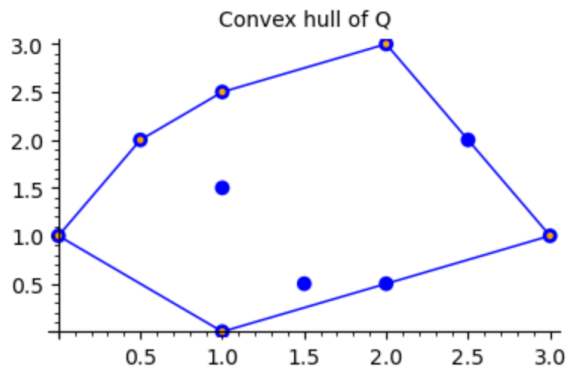


Figure 41: Boundary of the convex hull of Q where the vertices that determine the convex hull, i.e. the corner points of the polytope, are highlighted.

of Q. In other words, all other points of Q can be disregarded to determine the minimal enclosing sphere.

The minimal enclosing sphere can algebraically be formulated as a QP problem by minimizing the square of the radius as already seen in Eq. (1).

$$\begin{array}{ll} \min_{(z,r)} & r^2 \\ \text{s.t.} & d(q,z)^2 \leq r^2 \quad q \in Q \end{array} \tag{16}$$

Here a sphere S is identified with its centre z and radius r, i.e. $S = (z, r)$. Given the set Q we get the following visualisation of the Minimal Enclosed Circle (MEC).

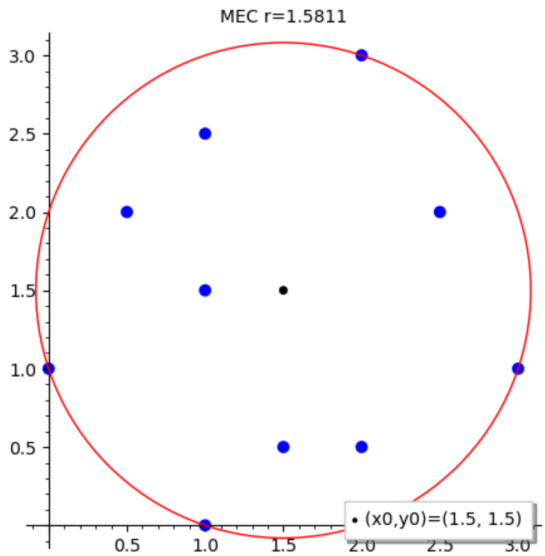


Figure 42: Minimal Enclosed Circle of Q

Note that, in general, three points determine uniquely a circle. However, in this example we have a fourth point of Q on the boundary, the points consist of $\{(0.0, 1.0), (1.0, 0.0), (2.0, 3.0), (3.0, 1.0)\}$.

There is another approach to determining the MEC. The center of minimal enclosing sphere z lies in the convex hull of Q, and hence z can be written as a linear combination of the vertices $z = \sum_I \lambda_i v_i$ whereby $\forall i\, \lambda_i \geq 0$ and $\sum \lambda_i = 1$. Instead of minimizing r^2, we can solve the dual problem expressed whose objective is expressed in the coefficients λ_i.

$$\begin{array}{ll} \max_{\lambda_i} & \sum_i \lambda_i d(v_i, z) \\ \text{s.t.} & \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1 \end{array} \tag{17}$$

According to Theorem 4.7, a property of the optimal solution is that the number of non-zero lambdas is in general at most $n+1$. For a circle, we therefore would expect three positive lambda-values. However, as there are four points on the minimal circle, there is extra freedom of choice for the coefficients λ_i.
In the table below, two sets of four lambdas, the sets denoted as $\{\lambda\}_1$ and $\{\lambda\}_2$ respectively, are shown which describe the minimal including circle in Fig. 42. Note that only the second pair of lambdas consists of two non-zero values. This demonstrates that $n+1$ is an upper bound.

$\{\lambda\}_1$	$\{\lambda\}_2$	**vertex points on the boundary**
0.2802	0.0000	(0.0,1.0)
0.1638	0.5001	(1.0,0.0)
0.3318	0.4999	(2.0,3.0)
0.2242	0.0000	(3.0,1.0)

Note that the lambdas are positive, hence the center $c = \sum_i \lambda_i v_i = (1.5, 1.5)$ lies in the convex hull of Q, and their sum is equal to 1.0.

The example demonstrates that for calculating the optimal solution general position is not a requirement. The general position condition only ensures that there is no freedom of choice for the lambdas.

Minimal Enclosing Excluding Sphere (MEES)

We now consider the case where we exclude one point denoted by x. There are four excluding possibilities: the point x lies

i) outside the MES

ii) on the MES

iii) in the MES but not in the convex hull

iv) in the convex hull

There are three simple cases.
Ad i) If the point x is outside the MES, then the MEES is equal to the MES.
Ad ii) If the point x is on the MES, then also the MEES is equal to the MES.
Ad iv) If the point x is in the convex hull, no MEES exists as it always has to contain the convex hull.

The case that the point x lies in the MES but not in the convex hull is solvable. The geometrical argument is that there is a hyperplane which separates the point x and the convex hull (hyperplane separation theorem). Hence, we can start with a very large sphere from point x which encloses the MES, and let the radius shrink until the sphere excludes point from Q.

In the following figure the red punt $x = (0.5, 2.5)$ is excluded which is an example of case (iii).

The center has moved from $(1.5, 1.5)$ to $(1.75, 1.25)$. Note that in general the new center does not have to lie in the convex hull of Q. For instance, as the point x approaches the edge of the enclosing polygon, the MEES becomes larger and larger.
A fundamental observation is that in case (iii), the point x lies not in the convex hull, and hence by definition, can only be written as $x = \sum_i \lambda_i v_i$ if at least one of the coefficients lambda is negative. Note

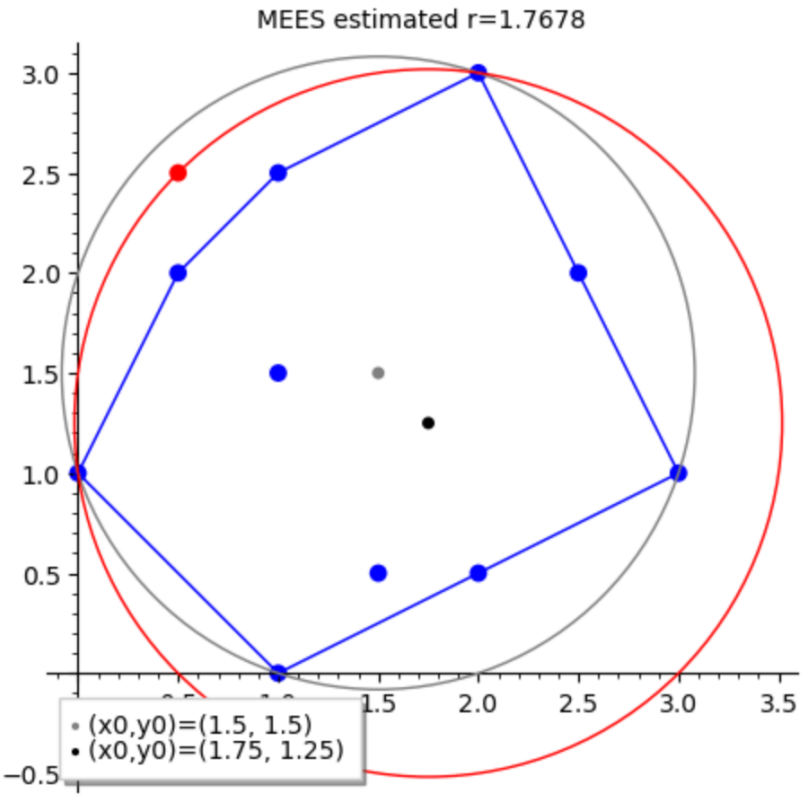


Figure 43: Minimal Enclosed Excluding Circle of Q, x

that there is no boundary on how negative a lambda parameter can be.

But there is also another crucial observation: the point x lies on the boundary of the MEES. It is a well-know fact that from the points on the surface one can determine the center and radius of a circle. For instance in $\mathbb{R}^2$ for points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, the circle equations are:

$$
\begin{aligned}
&r_1^2 = x_1^2 + y_1^2 \\
&r_2^2 = x_2^2 + y_2^2 \\
&r_3^2 = x_3^2 + y_3^2 \\
&A = x_1(y_2 - y_3) - x_1(x_2 - x_3) + x_2y_3 - x_3y_2 \\
&B = r_1^2(y_3 - y_2) + r_2^2(x_1 - y_3) + r_3^2(y_2 - x_1) \\
&C = r_1^2(x_2 - x_3) + r_2^2(x_3 - x_1) + r_3^2(x_1 - x_2) \\
&D = r_1^2(x_3y_2 - x_2y_3) + r_2^2(x_1y_3 - x_3x_1) + r_3^2(x_2x_1 - x_1y_2)
\end{aligned}
$$

If $A \neq 0$ then using A, B, C we have

$$c = (\frac{-B}{2A}, \frac{-C}{2A}) \text{ and } r^2 = \frac{B^2 + C^2 - 4AD}{(2A)^2}.$$

These points uniquely define (c, r) but they also uniquely define the non-zero lambdas. The reason is that $A \neq 0$ implies that the condition of general position holds, in which case the dual solution is unique.

If $A = 0$, then the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on a line. As only the vertices are relevant, only two points uniquely define (c, r) via $c = (\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)$ and $r^2 = \frac{1}{4}\big((x_1 - x_2)^2 + (y_1 - y_2)^2\big)$. In this 'non-general position' case, there are various possibilities for lambdas.

Because there is an explicit formula for how to calculate the circle parameters when the boundary points are known, we have the following diagram with a one-to-one relation between the solution of the above QP problem. As shown with an example, there can be many dual solutions (e.g feasible sets of lambdas)

and at least one of the sets satisfies the conditions in Theorem 4.7.

$$\begin{array}{ccc} Explicit\ formula\ (boundary\ points) & & \\ \Updownarrow & & \\ QP(c,r) & \longrightarrow & Dual\,QP(\lambda_i) \end{array}$$

Conclusion

General position of the data points implies that the set of coefficients $\{\lambda_i | i \in Q\}$ is unique to describe the MEES. In the general case however, there are several sets possible to describe the MEES. Note that the MEES is always unique, e.g. its center and radius, but the dual solution is not, as we demonstrated by an example. However, in those cases there exists a solution as indicated in Theorem 4.7. If such a solution would not exist, the original QP-solution in terms of (c, r) would also not exist.

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