# Generalized Global Symmetries and the Swampland

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# Abstract

The recent discovery of a large class of generalized global symmetries has significantly impacted our understanding of many phenomena in quantum field theory, ranging from anomalies to confinement with applications to both low- and high-energy physics. It is widely believed however, that exact global symmetries are forbidden in consistent theories of quantum gravity. By studying how generalized global symmetries are avoided in string theory, we may discover mechanisms which turn out to be general features of quantum gravity.

In this thesis we will follow this approach and apply it to a large class of effective field theories obtained from Calabi-Yau compactification of type IIB string theory. After a review of these effective theories, we give an accessible introduction to generalized global symmetries with a focus on their role in the swampland program. Using asymptotic Hodge theory we then extract the most general limiting form of the action near boundaries in the complex structure moduli space. With these in hand, we enumerate the global symmetries that may emerge in these limits and explore how they are broken by stringy effects.

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# Introduction

Few ideas have been as consequential to the way we think about nature as the idea of symmetry in physics. From snowflakes to honeycombs, from crystals to molecules, symmetries are ubiquitous in nature. In each case, the physical system is invariant under some transformation such as a rotation or a translation, and recognizing the symmetries of a system has long proven to be an indispensable tool for physicists. It allows us to reduce our description of the problem, rendering solvable that which would otherwise be intractable. However, it would not be until the twentieth century that symmetries would begin to take their rightful place, at the heart of the formulation of our physical laws.

Indeed, it is by demanding invariance not just of an object or system, but of physical laws themselves that the full extent of their significance becomes apparent. It is precisely this thinking that led Einstein to his celebrated general theory of relativity, to this day our most successful theory of gravity. Later, the advent of quantum theory brought along new ways we use symmetries. With quantum states now vectors in a Hilbert space, the mathematical ideas of representation theory would make symmetries the essential building blocks of our physical theories, leading eventually to our modern framework of quantum field theory.

One of the central problems of modern high energy physics has been the mending of these two pillars. General relativity is a classical field theory, and its prediction of, yet failure to describe black holes has been an important clue that this theory is incomplete. Nevertheless, naive attempts to embed it into the QFT framework along with the other forces of nature fail dramatically. The resulting theory is plagued by infinities of the kind not readily amenable to standard renormalization methods. These problems all point towards the fact that our modern theories of physics should eventually be subsumed in a new framework of quantum gravity.

## Enter String Theory

String theory is one of the most promising candidates for such a fully-fledged theory of quantum gravity. It proposes to replace the point-particles of conventional quantum mechanics by extended strings as the basic excitations of the theory. The extended nature of these strings regulates the short-distance singularities that are so common in ordinary quantum field theory, rendering the theory “UV-complete”. Despite springing from such a humble idea, string theory has since proven to be one of the richest physical theories we know of. Its spectrum naturally includes fermions, gauge fields and chirality, each of which constitute major hurdles for alternative theories of quantum gravity.

These remarkable features of string theory come with their own baggage, however. Most people familiar with string theory will have heard of its predictions of extra dimensions. These arise as a consistency condition for the stability of the string theory vacuum. Indeed, despite, or rather because of its richness, string theory is a deeply constrained theory. Its promise of being a UV-complete theory means that it has nowhere to hide its flaws, so that the constraints posed by mathematical consistency weigh much heavier than for its low-energy counterparts. While we know of several consistent string theory constructions, there are ample clues to suggest that these correspond to different weakly coupled descriptions of a single unified theory.

Nevertheless, despite the apparent elegance of a uniquely determined fundamental theory, we must eventually come to grips with the fact that our world is four-dimensional. Though methods exist to extract from a higher-dimensional theory a lower-dimensional one in the infrared, they bring with them a tremendous amount of ambiguity. The resulting low-energy degrees of freedom propagate on a background painted by the frozen high-energy modes. The many parameters that define a given quantum field theory must be determined by such a choice of background, but the number of choices is huge and without a systematic way of rooting out the good from the bad, it seems hopeless to pick out our universe from the vast landscape of string theory vacua.

## The Landscape and the Swampland

What all of this means is that a truly realistic string-theoretic description of nature has remained out of reach. The failure of such a top-down approach to quantum gravity has led physicists to consider more bottom-up approaches. These efforts have culminated in what is known as the *swampland program* [1]. In its most basic form it is an attempt at bridging the massive gap between the infrared physics we probe and the extreme ultraviolet where quantum gravity steps in. The idea is that although quantum gravity will not be probed directly in the foreseeable future, the constraints imposed by self-consistency in the UV should still leave their mark on the physics in the IR. Uncovering the structures that underlie these imprints may give us clues into the true nature of string theory, and guide us forward in our search for physics beyond the standard model.

More concretely, these proposed consistency conditions are formulated in terms so-called *swampland conjectures*, which range from constraints on the infrared spectrum, to bounds on the cosmological constant. A recurring theme and important organizing principle has been the role of symmetries. Perhaps the purest manifestation of this is the No Global Symmetry conjecture, which states that global symmetries are forbidden in any consistent theory of quantum gravity, thus recognizing the centrality of gauge symmetries in fundamental physics. Viewed through this lens, several swampland conjectures can be thought of as manifestations of this idea and providing concrete ways quantum gravity avoids global symmetries.

While it has long been realized that global symmetries play a central role in guiding swampland conjectures, recent years have seen a revolution in our understanding of such symmetries. A key step was the discovery that our usual notions of global symmetries are really a special case of an enlarged class of *generalized global symmetries* [2]. From the point of view of traditional quantum field theory, they provide us with a unifying perspective on many phenomena in gauge theories ranging from anomalies to confinement. Since this seminal work, an extensive literature has been built up trying to understand theories which exhibit such generalized global symmetries and their broad range of application has made them a thriving topic of research.

Their relevance has been observed crucially within the swampland program, where demanding their absence has led to new and interesting connections between various swampland conjectures. A fruitful line of attack has been to study low-energy theories and asking how their global symmetries are broken in the UV. However, the focus has often been either on field theory constructions, or higher-dimensional string theory setups where the theories are often far more constrained. The goal of this thesis will be to push these efforts into the four-dimensional domain. To this aim, we will study the generalized global symmetries that could appear in four-dimensional theories which are explicitly realized in string theory. In particular, we will investigate a specific corner of the string landscape, namely Calabi-Yau compactifications of the type IIB superstring. In the end, we find that these theories manage to avoid global symmetries through a delicate interplay of non-trivial EFT couplings and charged objects.

## Outline of this Thesis

To set the stage for our later discussions, we begin in chapter 1 by discussing some elementary string theory. In particular, we will make the journey from ten dimensions down to the four-dimensional world we see around us, through a procedure known as compactification. In particular, by assuming that the extra dimensions of string theory are very small, we may derive an effective description that is four-dimensional. The resulting effective theory depends strongly on the geometry of these extra dimensions for which we consider a well-studied class of six-dimensional spaces known as Calabi-Yau manifolds. As a manifestation of the ambiguity associated with this reduction, we arrive at a low-energy theory with a proliferation of so-called *moduli* fields which live in a *moduli space*. These parameterize the shape and size of the Calabi-Yau manifold and control the coupling constants of the low-energy theory. Though we focus only on a particular sub-sector of this theory, we will see that even this sub-sector admits a remarkable degree of structure and the final part of this chapter is dedicated to understanding this structure.

Having firmly established our setting, we then turn to the meat of this thesis in chapter 2, namely generalized global symmetries in the swampland program. We begin our discussion with a tour of the swampland, with a particular focus on the role of global symmetries. While we cover only a small corner of the web of swampland conjectures, the conjectures we consider here form the backbone of the program. Understanding how they intertwine forms a key step to disentangling this web and our exposition emphasizes the role global of symmetries. After our first encounter with the swampland, we will switch gears and give an introduction to the new notions of global symmetry alluded to earlier. After a brief review of ordinary global symmetries in QFT, we recast these familiar ideas in a more modern language which will form the starting point for further generalizations. The bulk of this chapter is dedicated to developing generalized symmetries by means of various examples, largely from a field theory perspective. These provide us with a controlled setting in which we can begin to phrase the sorts of questions which we wish to eventually answer in a string theory context. Armed with these new concepts, we then revisit the swampland to see how they have led to new connections and new conjectures.

Chapter 3 is dedicated to introducing the mathematical framework on which the main results of this thesis rely. Despite only focusing on a very particular class of four-dimensional theories, these are often still too complicated to be studied in full generality. It turns out however, that by restricting our attention to special limits in the moduli space certain universal structures emerge. These are described in terms of a variation of Hodge structure, which allows us to track the essential geometric properties of the Calabi-Yau as we move around the moduli space. The special limits occur whenever the Calabi-Yau develops a singularity. Crucial mathematical results that allow us to study these limits include the Nilpotent Orbit

Theorem and the $SL(2)$-Orbit Theorem [3]. It turns out that for Calabi-Yau that are described by a single modulus, we have sufficient control over the limiting structures to write down the most general form of the action near these singularities. Compared to previous investigations we will be careful to keep track of the quantization of charges and fluxes so as to retain full control over the resulting low-energy action.

Finally, in chapter 4, we will study the global symmetries that arise in the effective theories we derived in the previous chapter. Our discussion is largely exploratory, as the question of how global symmetries are broken in a given theory is a broad one. After enumerating the global symmetries that could be arise in each of the examples we consider, we explore how stringy ingredients can be used to break them, preventing inconsistency with the no global symmetry conjecture. Moreover, we investigate what happens to these symmetries as we approach the singularity in the moduli space. Along the way we highlight connections to different swampland conjectures, and present bottom-up interpretations of their effects.

# Chapter 1

# Type IIB Compactification

At the most basic level, string theory starts from the action of a propagating relativistic string which is then quantized using relatively standard quantization procedures. Nevertheless, the truth is that for many applications of string theory, phenomenological ones in particular, the real starting point is the result of a series of approximations to this basic picture.

To set the stage for our discussion of global symmetries in string theory, we will begin this chapter in section 1.1 with a bird's-eye view of the chain of approximations that leads us to the starting point of our investigation, the type IIB low energy effective action, as laid out in section 1.2. In the process we will encounter one of the most famous facts about string theory, namely that it requires our universe to be ten-dimensional. Resolving the apparent tension of this prediction with our observed four-dimensional universe will require a procedure called compactification, which will allow us to extract an effective four-dimensional description from the higher dimensional string theory. The basic idea is to assume that the extra dimensions are real, but small. As we discuss in section 1.3 by means of some basic examples, the specific geometry of these small extra dimensions determines the physics of the effective four-dimensional theory. Phenomenological and technical constraints on this effective theory will lead us to a special class of spaces known as Calabi-Yau manifolds, which are discussed in section 1.4. Finally, in section 1.5, we perform the compactification of type IIB string theory to obtain the basic four-dimensional theory studied in this work, whose properties we go on to discuss in some detail in section 1.6.

## 1.1 From Strings to Fields

For string theory to make contact with phenomenologically relevant applications we require a reduction of the basic picture of scattering and propagating strings to a low energy effective field theory in line with more typical particle physics models. Conceptually, this reduction is possible when the extended nature of the strings, as quantified by the string-scale $l_s$, becomes negligible compared to the energy scales being probed. In this case, their description reduces to one of point-particles and the framework of local quantum field theory becomes relevant again. A key step is to identify the nature and interactions of the resulting particles, which is then summarized in a low energy effective action of the string theory.

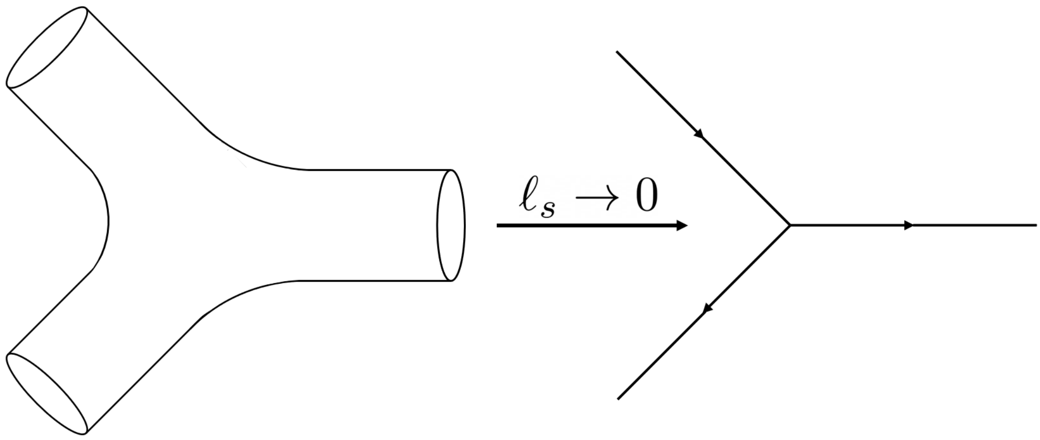



Figure 1.1: As we take the string-scale to zero the strings are well-approximated by point-particles, described by ordinary quantum fields. Their properties depend on the vibrational mode of the string.

### 1.1.1 Strings and the Worldsheet

The starting point for this program is the action for a propagating relativistic string. Whereas point-particles trace out one-dimensional world-lines in the ambient spacetime through which they propagate, strings trace out two-dimensional world-sheets. The action describing this propagation is a generalization of the analogous action for the particle, and is precisely given by the invariant area which this world-sheet sweeps out.

In practice however, we adopt a slightly different perspective of this action. Measuring the area of a world-sheet involves an integral over that world-sheet. This naturally gives the action an interpretation as a field theory *on* the world-sheet. The fields of this field theory are nothing but the embedding coordinates of the world-sheet into spacetime, and it enjoys a number of gauge symmetries. Upon quantizing this theory we have to demand that this gauge symmetry is respected, which in practice fixes the dimension of the spacetime through which our string propagates. Indeed, the requirement of vanishing gauge anomalies is the origin of the famous result that $d = 10$ in string theory!

The actual quantization of the world-sheet theory can be done through any one of the standard methods, i.e. path integral quantization or canonical quantization. A proper treatment of the quantized degrees of freedom of the string reveals an expansion in terms of oscillator modes of the string. The masses of these excited states are proportional to the string scale, and once we take the low energy limit where this becomes negligible we can ignore the massive excitations of the string. Among the modes predicted by the basic string theory sketched above is a tachyonic particle, i.e. a particle with negative mass-squared. This signals an instability of the vacuum, which although not technically fatal, often poses enough technical difficulties for people to discard the theory. Superstring theory provides a cure by including fermionic fields on the worldsheet theory. In one fell swoop this removes the tachyonic instability of the bosonic string, while also introducing fermions into the picture, something obviously necessary from a phenomenological perspective.

Once supersymmetry is included, and the dust has settled, one is left with five types of superstring theories. Of particular interest to us will be the type IIB superstring, whose low energy effective action describing the interactions of the massless modes of this string theory will be introduced explicitly in section 1.2. While we will not consider the details of how to derive this action, we note that its equations of motion are precisely the consistency conditions that ensure gauge invariance of the world-sheet theory. Indeed, this is often how these equations are derived in the first place. A more formal way of deriving the effective action is to compute explicit string scattering amplitudes for various in- and out-going excited states of the string and reverse engineering the low-energy action that reproduces them. Computing such scattering amplitudes is a difficult task and in general can only be performed for strings propagating through very special ambient spacetimes. In practice we typically assume spacetime curvature to be sufficiently small that our theory is well-approximated by the effective action derived in a flat background.

### 1.1.2 Branes

In our lightning review of string theory we have only focused on strings. However, despite what the name would suggest string theory is more than just a theory of propagating strings. Our first indication of this fact comes from the existence of open strings. The world-sheet theory for these strings will necessarily involve boundaries and should therefore be supplemented by a set of boundary conditions for the fields that live on the world-sheet. These boundary conditions can schematically be summarized by specifying whether the string end-points are fixed or free to move in the ambient spacetime. In general, one can fix their end-points onto some sub-manifold of spacetime, and these sub-manifolds define what are known as D-branes.

Though one might not guess it from their definition, D-branes are in fact dynamical objects in string theory, on equal footing with the fundamental strings. Nevertheless, their tension is non-perturbative in the string coupling, so that at weak string coupling, where a perturbative, diagrammatic expansion in scattering strings is appropriate, branes are heavy solitonic objects. This means that we will primarily view them as part of the background through which our strings, and eventually our fields propagate.

## 1.2 Type IIB Effective Action

Of the five types of string theory, the work in this thesis will focus only on one of these five types, namely type IIB string theory. We therefore begin by reviewing the effective action for type IIB string theory. It turns out that this is one of the two *unique* $\mathcal{N} = 2$ supergravity theories in ten dimensions (the other being the low energy effective action of type IIA string theory).

### 1.2.1 Bulk Theory

To begin with, let us first familiarize ourselves with the players, that is, the massless bosonic fields in our theory. In the NS-NS sector we have a set of universal fields common to all string theories, given by

- A symmetric, traceless tensor $g_{\mu\nu}$. It is interpreted as the metric tensor, or more specifically, the fluctuations of the metric tensor around a background value. These fluctuations decompose into spin-2 particles, which by definition, are gravitons.
- An anti-symmetric tensor $B_{\mu\nu}$, i.e. a 2-form.
- A scalar $\phi$. It is called the dilaton and it has a rather special interpretation in 10D. Its expectation value is precisely the string coupling constant that controls the strength of

string-string scattering processes.

In addition to these fields, there are the bosonic fields in the superstring R-R sector.

- A set of anti-symmetric tensors $(C_p)_{\mu_1 \ldots \mu_p}$, i.e. $p$-forms, for $p = 0, 2, 4$.

All of the $p$-form fields, including the 2-form $B$-field, have the interpretation of generalized gauge fields whose gauge parameters are now $(p-1)$-forms rather than the typical 0-forms of ordinary Maxwell theory. As in the case of Maxwell theory, the action is therefore expressed in terms of their associated field strengths, which are given by[1]

$$F_{p+1} = dC_p, \quad H_3 = dB_2. \tag{1.1}$$

In a marked deviation from ordinary gauge theory however, the gauge transformations of the lower-degree gauge fields also affect those of higher degree. The meaning of these mixed gauge transformations will be the subject of subsequent chapters, albeit in a somewhat different context. For the time being we only list the full set of gauge transformations here

$$\begin{aligned} B_2 &\to B_2 + d\zeta_1, \quad C_4 \to C_4 + \frac{1}{2} dC_2 \wedge \zeta_1, \\ C_2 &\to C_2 + d\Lambda_1, \quad C_4 \to C_4 - \frac{1}{2} dB_2 \wedge \Lambda_1, \\ C_4 &\to C_4 + d\Lambda_3, \quad C_0 \to C_0 + 1. \end{aligned} \tag{1.2}$$

It is then convenient to re-define the following field strengths

$$F_3 \to F_3 = dC_2 - C_0 dB_2, \qquad F_5 \to F_5 = dC_4 + \frac{1}{2}\left(C_2 \wedge H_3 - B_2 \wedge F_3\right). \tag{1.3}$$

The low-energy action describing the dynamics of these fields (or in other words, the action that reproduces the relevant string scattering amplitudes) is given by [4]

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int \bigg( R * 1 - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{-\phi} H_3 \wedge * H_3 - \frac{1}{2} e^{2\phi} F_1 \wedge * F_1 - \frac{1}{2} e^{\phi} F_3 \wedge * F_3 \\ - \frac{1}{4} F_5 \wedge * F_5 - \frac{1}{2} C_4 \wedge H_3 \wedge F_3 \bigg), \end{aligned} \tag{1.4}$$

with $\kappa_{10}^2$ the gravitational coupling, given in terms of the string scale by $\frac{1}{4\pi}(4\pi^2 l_s^2)^4$.

The 4-form gauge field is somewhat special as a proper string theory computation shows that it should have the degrees of freedom of a *self-dual* gauge field. That is to say that on-shell its field strength should satisfy $*F_5 = F_5$. This carries with it the problem that when this is imposed off-shell, the canonical kinetic term for such a field vanishes by virtue of the fact that the wedge product is anti-symmetric for 5-forms. The action (1.4) is therefore a *pseudo-action* whose equations of motion are to be supplemented by an auxiliary self-duality constraint for the 5-form field strength. The extra factor of $\frac{1}{2}$ in the relevant kinetic term compensates for the fact that off-shell it has twice the number of degrees of freedom.

### 1.2.2 Brane Action

As mentioned, string theory also contains dynamical objects known as branes. Their masses are inversely proportional to the string coupling so that they are non-perturbative objects.

[1] Throughout this thesis we will employ a differential form notation. We refer to appendix A for a summary of our conventions and some useful identities regarding such forms.

Despite their large mass, they do have low energy dynamics described by the brane effective action. This action captures the dynamics of massless deformations of the brane[2], as well as its coupling to the (closed string) bulk fields. In particular, branes carry Ramond-Ramond charge and therefore couple to the $p$-form fields via their world-volume $\mathcal{W}^{p+1}$. This is captured by the Chern-Simons action, whose leading contribution is given by

$$S_{CS} \supset \mu_p \int_{\mathcal{W}^{p+1}} C_{p+1}. \tag{1.5}$$

The factor that sits out front is the charge of the brane, which is equal to its tension $\mu_p = T_p = \frac{2\pi}{l_s^p}$. Our treatment of these branes will largely be semi-classical, viewing them as sources for the RR-fields, while otherwise ignoring any spacetime curvature they induce. Indeed, this will be the main role the branes will play in our story so that we do not go into the details of the corresponding brane action. A good analog to keep in mind is a black hole, whose formation nor core can be described within GR, but whose effect on the fields outside is well-understood (in fact, depending on their dimension, branes sometimes *are* black holes).

## 1.3 Compactification

The framework used to reduce a ten-dimensional string theory to a four-dimensional one is based on an old idea called Kaluza-Klein compactification. Although originally proposed as a way of unifying gravity and gauge theory in the 1920's [5, 6], it has since resurfaced as a convenient way of reducing the unwanted number of dimensions predicted by string theory. The basic idea is that although the theory is defined in ten dimensions, only four of these are extended while the six extra dimensions are assumed to be compact so that they can be attributed a well-defined "size". As the radius of these extra directions decreases, the minimum wavelength for fluctuations in these directions shrinks as well, increasing their energy. After integrating out these suppressed fluctuations, one is left with an effective four-dimensional description of a theory that lives in ten dimensions. The goal of this subsection will be to make this idea more precise by means of some simple examples, before subsequently applying it to the effective ten-dimensional supergravity description of type IIB string theory.

Here we should make a crucial distinction distinction however, between theories which couple to gravity and those that do not. Indeed, because compactification involves imposing a non-trivial geometry for the space on which our theory is defined, the story plays out rather differently for theories where this geometry is dynamical compared to those where it is not. We will start by considering an example where gravity decouples which will introduce us to the first key concept of this section, which is the notion of zero-modes. Afterwards, we will consider a gravitational theory, where we will have our first encounter with moduli fields. We round up our discussion of Kaluza-Klein compactification with a more general example that will set the stage for our discussion of Calabi-Yau compactifications in section 1.5 and close with some remarks about how stringy effects affect the compactification procedure.

### 1.3.1 Massless Scalar Field

Let us start by being more specific about how we impose the compactification on our spacetime manifold $M_5$. The key assumption will be that $M_5$ decomposes as a direct product

[2]These include the translational degrees of freedom of the brane in the directions transverse to its world-volume. They appear as massless scalar fields living *on* the brane world-volume and can be interpreted as the Goldstone bosons associated with broken translational invariance.

into a non-compact "external" piece, typically taken to be some maximally symmetric spacetime, such as Minkowski space $\mathbb{R}^{1,3}$, and a compact "internal" piece $X_1$. We therefore write $M_5 = \mathbb{R}^{1,3} \times X_1$ and implement this ansatz[3] by writing the metric tensor as

$$G_{MN}(x^M) = \begin{pmatrix} g_{\mu\nu}(x^\mu) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}. \tag{1.6}$$

For our first example we will consider a five-dimensional spacetime where $X_1 = S^1$ such that the metric can be written as

$$ds^2 = G_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu + R^2 d\theta^2, \tag{1.7}$$

where we have denoted the compact circle coordinate $y = \theta$ and $R$ denotes the radius of the circle. On this background geometry we place a massless scalar field whose action is given by

$$S = -\int d^5x \sqrt{-G}\, G^{MN}\partial_M\varphi\partial_N\varphi. \tag{1.8}$$

The equations of motion of this action, evaluated for the metric (1.7), are then given by

$$\partial_\mu\partial^\mu\varphi + R^{-2}\partial_\theta^2\varphi = 0. \tag{1.9}$$

We can solve these equations of motion by performing a mode expansion in the compact dimension, which in our case simply corresponds to a Fourier series in $\theta$

$$\varphi(x,\theta) = \frac{1}{\sqrt{2\pi R}}\sum_n \varphi_n(x)e^{in\theta}. \tag{1.10}$$

Inserting this into equation (1.9) and using the orthogonality properties of the Fourier modes, we find that the equations of motion for the different modes decouple

$$\left(\partial_\mu\partial^\mu - \frac{n^2}{R^2}\right)\varphi_n(x) = 0. \tag{1.11}$$

What this tells us is that the single five-dimensional scalar decomposes into an infinite tower of four-dimensional scalars called Kaluza-Klein (KK) modes, which in turn obey a Klein-Gordon equation with mass $n/R$. The mode with $n = 0$ is of particular interest, as it is the only mode whose mass remains small as we reduce the circle radius to zero. In this limit, the low energy dynamics of our theory will be dominated by this massless mode. We can therefore obtain a low energy description of the theory by integrating out the massive KK modes keeping only the massless zero-mode. Indeed, inserting the expansion (1.10) into the action and using the orthogonality of the Fourier modes to perform the integral over the compact space, the action (1.8) reduces to

$$\begin{aligned} S &= -\sum_{m,n}\int d^4x \int R d\theta \left(\eta^{\mu\nu}\partial_\mu\varphi_m\partial_\nu\varphi_n - \frac{mn}{R^2}\varphi_m\varphi_n\right)\frac{e^{i(n+m)\theta}}{2\pi R} \\ &= -\sum_{n\geq 0}\int d^4x \left(\eta^{\mu\nu}|\partial_\mu\varphi_n|^2 + \frac{n^2}{R^2}|\varphi_n|^2\right), \end{aligned} \tag{1.12}$$

[3] For some string theory applications one should allow a more general ansatz whereby the external metric is allowed to depend on the internal coordinates through a so-called warp-factor. These will however not be important to us and all ansatze will essentially be of the form (1.6).

where we have imposed the reality condition $\varphi_{-n} = \bar{\varphi}_n$. At energy scales much smaller than $R^{-1}$ we can obtain an effective theory by integrating out all the massive modes as their lowest lying excitations are beyond the cutoff. Because the modes are completely decoupled, we can simply discard these excited modes such that the action (1.12) reduces to a four-dimensional action of a single massless scalar field $\varphi_0$.

What this toy example shows is that the geometry of the compact space, here captured by the radius $R$, affects the physics in the lower dimensional theory. Before taking the limit $R \to 0$, it controls the masses of the tower of states that appears when the theory is placed on a circle. Moreover, we found that the states in this tower corresponded to eigenfunctions of the Laplace operator on the internal space. Once we took the limit $R \to 0$, we therefore found that the field-content of the low energy theory was dictated by the zero-modes of this internal Laplacian. This concept of zero-modes carries over directly to the more complicated cases we will consider in subsequent sections, where the low energy four-dimensional field content will be given by the zero-modes of generalized differential operators on the compact space. We therefore close this section with the following general message

> The topology of the compactification manifold determines the low-energy field content of the compactified theory. Compactification introduces an infinite tower of massive modes whose masses depend inversely on the compactification scale, and some number of massless zero-modes.

### 1.3.2 Gravitational Theories and Moduli

While the previous example highlights the role of the geometry of the compactification space, it ignores the fact that the geometry, and by extension the radius $R$, are dynamical in a theory of gravity. A proper treatment of the compactification of a gravitational theory requires us to consider the Einstein-Hilbert term, whose action is given by

$$S = M_P^3 \int d^5x \sqrt{-G}\, R^{(5)}, \tag{1.13}$$

where we have explicitly included the Planck mass to ensure the action is dimensionless, while the superscript on $R$ emphasizes that this is the 5-dimensional Ricci scalar. When we compactify the gravitational theory, the metric ansatz (1.6) has the interpretation of a *background* around which we expand the now-dynamical metric field. Indeed, the compactification procedure can in principle be performed around any vacuum expectation value for the fields, including the scalar $\varphi$, as long as this background is consistent with the equations of motion. In case we set the vev for the non-metric fields to zero this simply reduces to the condition that the background metric should be Ricci-flat.

Let us illustrate this using the simple case of a circle compactification of the action (1.13). In the following, we will consider variations of the metric $G_{MN}$ around the background (1.7), which we point out is Ricci-flat

$$G_{MN} = \bar{G}_{MN} + h_{MN}. \tag{1.14}$$

Just as we have decomposed the background metric into internal and external parts, we can do the same for the variations around it. This leads to the following decomposition of the metric variations

$$h_{MN} = \begin{pmatrix} h_{\mu\nu} & h_{\mu\theta} \\ h_{\mu\theta} & h_{\theta\theta} \end{pmatrix}. \tag{1.15}$$

Based on their index structure, we can see that from a four-dimensional perspective these components have the interpretation of a tensor, a vector and a scalar. The linearized Einstein equations can now be shown to take the form of a simple Laplace equation for each of these components, so that we may again perform a Fourier expansion

$$h_{MN}(x,\theta) = \frac{1}{\sqrt{2\pi R}} \sum_n h^{(n)}_{MN}(x) e^{in\theta}, \tag{1.16}$$

The massless fields then correspond to variations which are constant with respect to the compact coordinate $\theta$. We can thus parameterize these massless deformations by allowing the background metric to depend on the non-compact coordinates. We then choose to decompose the resulting metric as[4]

$$ds^2 = G_{MN}(x)dx^M dx^N = \phi^{-1/3}(x)\left(g_{\mu\nu}(x)dx^\mu dx^\nu + \phi(x)(A_\mu(x)dx^\mu + R_0 d\theta)^2\right). \tag{1.17}$$

Written this way the four-dimensional fields $g_{\mu\nu}(x)$, $\phi(x)$ and $A_\mu(x)$ admit natural interpretations. The tensor part is simply the dynamical metric of the dimensionally-reduced four-dimensional gravity, while the off-diagonal components descend to gauge fields, which as we shall see below indeed reproduce electromagnetism. In fact, the appearance of this gauge field was the original motivation for Kaluza and Klein to consider a five-dimensional theory of gravity as a way of unifying gravity with electromagnetism. Most relevant for us however, is the scalar part which now has the interpretation of a varying compactification radius! Indeed, computing the compact radius using this metric we obtain

$$\int_0^{2\pi} \sqrt{G_{\theta\theta}}\, d\theta = R_0 \int_0^{2\pi} \phi^{1/3}(x)\, d\theta = 2\pi\phi^{1/3}(x) R_0 \equiv 2\pi R(x). \tag{1.18}$$

This is our first example of a modulus and they play a very important role in the compactified theory. In general, such parameters that control the size and shape of the compact manifold show up as massless scalar fields in the compactified theory. In our case, we can see this explicitly by inserting the ansatz (1.17) into the action and expanding the Ricci scalar, yielding the well-known result

$$M_P^3 \int d^5x \sqrt{-G}\, R^{(5)} = 2\pi R M_P^3 \int d^4x \sqrt{-g}\left(R^{(4)} - \frac{1}{6}\phi^{-2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu}\right). \tag{1.19}$$

Here, $R^{(4)}$ denotes the four-dimensional Ricci scalar which is computed from the four-dimensional metric $g_{\mu\nu}$. We see that the modulus $\phi$ indeed appears as a massless scalar field in the theory, but equation (1.19) shows us that we can in fact say more. Indeed, the action for the massless scalar takes the form of a non-linear sigma model with a field-space metric given by $\frac{1}{3}\phi^{-2}$. This field space is precisely the *moduli space* of the circle. We will encounter more examples of such spaces in later sections and we will see that one can assign a canonical metric to such moduli spaces, which, in the case of the circle modulus is precisely given by the field space metric $\frac{1}{3}\phi^{-2}$. We therefore close this section with the following general message

[4] While this ansatz may appear somewhat odd, we point out that the choice (1.17) gives the vector field $A_\mu$ parameterizing the off-diagonal metric variations $h_{\mu\theta}$ a natural interpretation as a gauge field. Indeed, we see that for the five-dimensional metric to be invariant under general coordinate transformations of the compact space $\theta \to \theta + \xi(x)$, the gauge field should undergo the compensating transformation $A_\mu \to A_\mu + \partial_\mu\xi$. The overall factor of $\phi^{-1/3}$ is present to obtain the correct normalization in 4D.

Geometric moduli that control the size and shape of the compactification space show up as massless fields in the compactified theory. Their field-space corresponds to the geometric moduli space. More generally, we refer to any massless scalar fields in an EFT as (physical) moduli.

### 1.3.3 Compactification in the General Case

The simple example of a circle compactification highlights the key lessons that we will carry with us to the next sections. Nevertheless, its simplicity hides some of the details that will become important when we consider actual string theory compactifications. Indeed, our toy model contained only two types of field, a scalar and a metric, but in situations relevant to string theory we also encounter $p$-form gauge fields. Moreover, the simple geometry of the circle resulted in a compactification that bordered on triviality. We will ameliorate both issues by considering the compactification of a $p$-form gauge field on a generic background manifold. This will allow us to simultaneously illustrate the compactification procedure for this important class of fields, while also highlighting what more general zero-modes may look like. Meanwhile, we postpone a more general discussion of moduli and moduli spaces to the next section.

To illustrate how the compactification procedure goes through for a $p$-form on a general background manifold, consider a (non-gravitational) theory of a massless $p$-form gauge field on a $D = d_e + d_i$ dimensional spacetime (assuming $d \geq p$)

$$\int_{\mathcal{M}} \left( -\frac{1}{2} F_{p+1} \wedge *F_{p+1} \right), \tag{1.20}$$

where we have switched back to a differential form notation. In general, the equations of motion and Bianchi identity for this action are given by

$$d * F_{p+1} = 0, \quad dF_{p+1} = 0, \tag{1.21}$$

Next, we wish to identify the zero-modes of the gauge field $A_p$. We should be careful to exclude modes which are pure gauge, which we do by performing a gauge fixing. A suitable gauge fixing condition is given by $*d * A_p = 0$ (generally called Harmonic gauge), in which case we can rewrite the equation of motion for $A_p$ as

$$0 = *d * F_{p+1} = *d * dA_p = (*d * d + d * d*)A_p \equiv \Delta A_p. \tag{1.22}$$

The operator that acts on $A_p$ on the right is called the Laplacian, which generalizes the usual Laplacian we encountered in the circle compactification. If the metric, which enters the problem through the Hodge star, decomposes into a product structure, the Laplacian similarly splits into an internal and external part, and the equations of motion for the gauge field become a direct generalization of the free-field equation for the scalar

$$(\Delta_{ext} + \Delta_{int})A_p = 0. \tag{1.23}$$

Identifying the zero-modes of $A_p$ now corresponds to finding the zero-modes of the internal Laplacian $\Delta_{int}$. Such $p$-forms are called Harmonic forms and the Hodge decomposition theorem tells us that such Harmonic forms are in 1-to-1 correspondence to the generators elements of $H^p(X)$. What this means is that if we denote by $\{\alpha_i^r\}_{i=1,\ldots,b_r}$ a basis of the $r$'th cohomology of $X$ where $b_r$ is the rank of $H^r(X)$ (also called the Betti number), then the first terms in a mode expansion for $A_p$ are given by

$$A_p = \hat{A}_p + \sum_{i=1}^{b_r} \sum_{r=1}^{p} A^i_{p-r} \wedge \alpha^r_i. \tag{1.24}$$

A more explicit way of seeing how the lower-form modes $A^i_{p-r}$ come about is by performing a similar tensor-decomposition as we did for the metric. That is to say,

$$A_p = (A_p)_{\mu_1 \ldots \mu_p} \oplus (A_p)_{\mu_1 \ldots \mu_{p-1} m_p} \oplus \ldots \oplus (A_p)_{m_1 \ldots m_p}. \tag{1.25}$$

The terms in this decomposition correspond from left-to-right to zero through $p$-forms on the internal manifold. To summarize, we close this sub-section with the following general lesson

> Zero-modes of $p$-form fields are in one-to-one correspondence with the elements of cohomology groups $H^{r \leq p}(X)$ of the compactification manifold.

### 1.3.4 Compactification in String Theory

While the previous sub-sections serve to illustrate the most relevant aspects of Kaluza-Klein compactification, the astute reader may wonder what happens when we go beyond the field theory approximation of string theory. Indeed, the field theory approximation from section 1.2 relied crucially on the assumption that the string scale $l_s$ was negligible, so that we are justified in taking the point-particle limit. However, when the radius of curvature of the compactification manifold becomes of the order of the string scale we have to contend with the fact that strings probe this internal geometry rather differently than point particles do. Once the field theory approximation breaks down we are forced to consider strings propagating through different backgrounds, which can only be studied explicitly for very special examples.

One such example is that of a circle-compactified string theory. One of the most striking effects is the appearance of a *second* tower of states, called winding modes, whose masses are proportional to the compactification radius $R$. These modes have a natural physical interpretation: in addition to the momentum modes around the compact dimensions, strings can also non-trivially wrap around the compact directions of the background manifold. As we shall see, these modes play a crucial role in the swampland program, but from a practical perspective they mean that we only have control over the compactification whenever $R \gg l_s$. Simultaneously however, the compactification radius cannot become too large, as otherwise the Kaluza-Klein modes with $M_{KK} \sim \frac{1}{R}$ become important at our energy scale of interest. Hence, for a given momentum scale $p$, we obtain a hierarchy

$$l_s \ll R \ll p^{-1}, \tag{1.26}$$

which is required for the Kaluza-Klein compactification to be valid. More generally, such "winding" states may arise whenever the compactification manifold has non-trivial cycles for the string to wrap around. In this more general context such states are referred to as worldsheet instantons. They lead to corrections to physical quantities which are suppressed so long as the (appropriately generalized) hierarchy (1.26) is satisfied.

More generally still, we may also consider branes that wrap non-trivial cycles on the background geometry. Indeed, while we again may not be able to produce these states at low energy, branes wrapping non-contractible cycles are stable. A $(p+1)$-dimensional D$p$-brane wrapped in a $q$-cycle then leads to a $(p-q+1)$-dimensional state, charged under the dimensionally reduced Ramond-Ramond fields. Of particular interest to us will be the case $p = q = 3$, which are the so-called D3-particles, to be introduced in section 1.5.2.

# 1.4 Calabi-Yau Compactification

Though simple, the examples in the previous section illustrate how the geometry of the compactification manifold is crucial for understanding the lower-dimensional physics. This begs the question of what a suitable compactification manifold for string theory should look like. The key to answering this question lies in an aspect of string theory that we have mostly sidestepped until now, namely, supersymmetry. Indeed, just as compactification affects the bosonic field content of a theory, so too does it affect the fermionic field content and, by extension, the degree of supersymmetry enjoyed by the compactified theory. Highly symmetric spaces such as the circle, torus or their higher dimensional generalizations tend to leave unbroken all supersymmetry generators present in the higher dimensional theory. This leads to a four-dimensional theory with so much supersymmetry that it constrains the effective action beyond the point of being interesting, let alone realistic.

Nevertheless, for both practical and phenomenological reasons, preserving some degree of supersymmetry is desirable. From a practical perspective, the presence of supersymmetry gives us a great deal of control over the resulting field theories. This is especially crucial in the context of string theory, where it oftentimes allows us to even extract some non-perturbative information. As a general rule of thumb, introducing more supercharges leads to a a more constrained (and thus more controlled) theory.[5]

On the phenomenological side of things, minimal supersymmetry has traditionally been viewed as a natural candidate for physics beyond the standard model. Indeed, it offers a natural cure to the hierarchy problem, generically includes dark matter candidates and recovers exact gauge coupling unification in the UV. Nevertheless, while these phenomenological reasons may be powerful drivers for the study of supersymmetric theories, the non-detection of supersymmetry at the LHC has gradually increased the scale at which it is spontaneously broken, which poses challenges for realistic model building in string theory.

In light of these challenges, our motivations for considering supersymmetric theories are more humble. As will be discussed at length in chapter 2, swampland ideas have made investigations into consistent QGs, supersymmetric or otherwise relevant in their own right. From this point of view, much work has already been done for theories with more than eight super-charges, where exact results are often provable. The next step down in this hierarchy concerns theories with exactly eight super-charges, which correspond to four-dimensional theories with $\mathcal{N} = 2$ supersymmetry. The aim of this section will be to show how we can obtain such theories from type IIB supergravity.

[5] In fact, we already encountered this fact in section 1.2, where we mentioned that there are only two $\mathcal{N} = 2$ supergravity theories in ten dimensions. Since supersymmetry is a fermionic symmetry, its generators are spinors which have more components in higher dimensions. Hence, we can introduce more supercharges either by adding generators (i.e. increasing $\mathcal{N}$), or by increasing the dimension. Consequently, there is only one unique supergravity theory in 11D!

### 1.4.1 Calabi-Yau Manifolds And Supersymmetry

The question we should answer therefore is exactly *how* the compactification manifold determines the supersymmetry of the lower-dimensional theory. The relevant mechanism is equivalent to spontaneous symmetry breaking in gauge theories, i.e. the Higgs mechanism. Indeed, in supergravity, supersymmetry is a gauge symmetry and it may occur that the chosen background is not invariant under some, or all, of the symmetry generators. The resulting theory may then be a supergravity theory with fewer supercharges than the original, uncompactified theory. To identify the degree of supersymmetry preserved by the vacuum, we should therefore consider the supersymmetry variations of the background ansatz for our Kaluza-Klein reduction.

For the compactifications we consider in this thesis, we will set the background for all fields other than the metric to zero. In this case, the only non-trivial supersymmetry transformations of type IIB supergravity are those of the two gravitinos

$$\delta_\epsilon \Psi_M^A = \nabla_M \epsilon^A \stackrel{!}{=} 0, \quad M = 0, \dots, 9, \quad A = 1, 2. \tag{1.27}$$

Here, $M$ is the 10-dimensional vector index carried by the spin-3/2 gravitino, while $\nabla$ is the covariant derivative with respect to the (Levi-Civita) spin-connection of the background metric. In much the same way that $p$-forms were decomposed in section 1.3.3, the diagonal split of the metric induces a similar split of the spin-bundles over spacetime. Avoiding the details of this decomposition, the internal part of equation 1.27 boils down to the existence of a covariantly constant six-dimensional spinor on the compactification manifold $X_6$

$$\nabla_m \epsilon_i = 0, \quad m = 1, \dots, 6. \tag{1.28}$$

Every such spinor will correspond to two supersymmetry generators $\epsilon_i^A(x, y) \sim \epsilon_4^A(x) \otimes \epsilon_i(y)$ in the four-dimensional theory. In this way we have reduced the determination of the lower-dimensional supersymmetry to a similar zero-mode counting problem as in the previous section. In particular, if we are interested in $\mathcal{N} = 2$ supersymmetry in 4D, we require the existence of exactly one covariantly constant spinor $\epsilon_i$ on $X_6$.

The condition that such spinors exist imposes very strong constraints on the space $X_6$. On the one hand, it can be shown that equation 1.28 implies that the manifold $X_6$ is Ricci-flat. This is consistent with the fact that the background manifold should obey the string equations of motion, which for vanishing background fields simply reduce to Ricci-flatness. In fact, the observation that supersymmetric backgrounds satisfy the string equations of motion holds quite generally, which greatly simplifies the analysis for more complicated backgrounds.

Meanwhile, the existence of a spinor that is merely non-vanishing is already a non-trivial condition, related to the structure group of the spin-bundle on our manifold. Indeed, spinors are sections of the spin-bundle so that if this bundle is non-trivial the best we can do is define these spinors locally, with transition functions relating spinors on overlaps of local patches. Nevertheless, if these transition functions are such that they leave some non-zero spinors invariant, then these locally-defined spinors can be consistently patched together into a global section. A minimal condition that allows for this is to demand that the transition functions on $X_6$ are restricted to $SU(3)$, in which case we say that our manifold has $SU(3)$-structure. In particular, this ensures the existence of exactly one globally-defined non-vanishing spinor, while we would need to further restrict our structure group if we demand multiple such spinors.

While $SU(3)$-structure is sufficient to ensure that we at least have one candidate-spinor, it is not enough to ensure that it is covariantly constant. Given this spinor, equation 1.28

can be viewed as a condition on the spin-connection on our manifold. A convenient way of phrasing this condition is in terms of the parallel transport induced by this connection. In particular, for non-flat spin-bundles over $X_6$, parallel transport along loops will induce a non-trivial holonomy transformation on the transported spinor[6]. For a connected manifold, these holonomy transformations form a group $\mathcal{H}$ called the holonomy group of $X_6$[7], and the condition 1.28 is equivalent to the condition that $\mathcal{H}$ acts trivially on $\epsilon_i$. Much as invariance under transition functions demanded $SU(3)$-structure, invariance under the holonomy group requires the holonomy group of the connection to be $SU(3)$. Likewise, the existence of more than one covariantly constant spinor requires us to further restrict our holonomy group, while we will refer to manifolds with $\mathcal{H}$ precisely equal to $SU(3)$ as manifolds with $SU(3)$-holonomy. With that, we are now finally in a position to define the central object of study for the remainder of this section

> A compact, $2n$-dimensional Riemannian manifold with $SU(n)$-holonomy is called a Calabi-Yau manifold.

### 1.4.2 Calabi-Yau Geometry

While the discussion in the previous section has emphasized the connection between Calabi-Yau manifolds and supersymmetry, the definition in terms of SU(3)-holonomy is rather abstract for the purposes of Kaluza-Klein compactification. This sub-section aims to bridge this gap by introducing some of the additional structures present on Calabi-Yau manifolds. We will necessarily lean rather heavily on some basic results in complex and Kahler geometry, for which we refer to the textbooks [7, 8].

The first property that will allow us to get a handle on these manifolds is the fact that manifolds with SU($n$)-holonomy are Kahler manifolds. Hence, Calabi-Yau manifolds are complex manifolds of complex dimension $n$, whose Hermitian metric defines a real, closed Kahler form

$$\mathcal{K} = \frac{i}{2} g_{i\bar{j}} dx^i \wedge d\bar{x}^{\bar{j}}, \quad i, \bar{j} = 1, \ldots, n. \tag{1.29}$$

In particular, this Kahler form endows $X$ with a symplectic structure which is compatible with the metric. Closedness of $\mathcal{K}$ implies that the metric can locally be obtained from a Kahler potential $K$ via

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K. \tag{1.30}$$

Just as the Hermitian metric defines a closed (1,1)-form, so too does the associated Ricci-tensor

$$\mathcal{R} = i R_{i\bar{j}} dx^i \wedge d\bar{x}^{\bar{j}}, \tag{1.31}$$

which for the special class of Kahler manifolds with SU($n$)-holonomy vanishes by virtue of Ricci-flatness. One can show that the first Chern class of any Kahler manifold is represented by $\frac{1}{2\pi}\mathcal{R}$, which implies that those Kahler manifolds with $SU(n)$-holonomy also have vanishing first Chern class. The converse statement is generically false however, and true only if $X$ is simply-connected. Nevertheless, a useful result known as Yau's theorem, whose precise statement we do not give, ensures that for compact Kahler manifolds with vanishing Chern class,

[6]Note that although $X_6$ is necessarily Ricci-flat, it need not be flat. In the latter case, the Levi-Civita spin-connection is automatically flat as well and we can only have non-trivial holonomy around non-contractible cycles.

[7]More precisely, $\mathcal{H}$ is the holonomy group of the connection on $X_6$, but it is understood that this refers to the Levi-Civita connection when we discuss Riemannian manifolds (as we do in this thesis).

there always exists a Ricci-flat Kahler metric whose Kahler form is unique in its cohomology class $[\mathcal{K}]$.

In addition to the Kahler form, which is present for Kahler manifolds in general, Calabi-Yau manifolds also admit another important globally defined differential form. In particular, it can be shown that there exists a no-where vanishing holomorphic $(n, 0)$-form, usually denoted $\Omega$.[8] Any $\Omega$ with these properties is unique up to constant re-scaling so that we will usually refer to this form as the unique holomorphic $(n, 0)$-form. The significance of this form will become clear in later sections where we will see that it encodes the complex structure on $X$. Moreover, it follows from the properties of Kahler manifolds that any holomorphic $(n, 0)$-form is also harmonic, which by the Hodge theorem implies that it is a representative of $H^n(X)$.

This brings us to the final topic of interest, namely the cohomology classes of Calabi-Yau manifolds. Just as the complex structure on $X$ can be used to decompose differential forms by their $(p, q)$-type, so too can we use it to decompose the cohomology classes of $X$ by their type. This leads to the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X). \tag{1.32}$$

The dimensions of the $(p, q)$-components of the cohomology groups define the so-called Hodge numbers

$$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}(X). \tag{1.33}$$

These Hodge numbers assemble into a Hodge diamond which classifies the topology of the underlying manifold. The properties of Calabi-Yau manifolds mean that not all $h^{p,q}$ are independent and in fact, for Calabi-Yau 3-folds $h^{1,1}$ and $h^{2,1}$ completely determine the corresponding Hodge diamond

$$\begin{matrix} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{matrix} \quad = \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \ . \tag{1.34}$$

Of the remaining non-zero entries above, $h^{3,3}$ and $h^{0,0}$ correspond to the volume form and constant function on $X$, respectively. Meanwhile, we saw that the holomorphic $(3, 0)$-form represents a class in $H^3(X)$, which we can now refine to a class in $H^{3,0}(X)$. Its uniqueness implies that $h^{3,0} = 1$ and its complex conjugate $\bar{\Omega}$ is a anti-holomorphic $(0, 3)$-form that spans $H^{0,3}$.

### 1.4.3 Calabi-Yau Moduli Spaces

Having discussed some basic properties of Calabi-Yau manifolds, the next matter to discuss is the moduli space of Calabi-Yau manifolds. We have argued in section 1.3.2 that parameters controlling the size and shape of the compactification manifold show up as massless

[8] As a an aside we also remark that $\Omega$ is a trivializing section for the canonical bundle $\wedge^n T^{*1,0}(X)$. In fact, one can and commonly does define Calabi-Yau manifolds as compact Kahler manifolds with trivial canonical bundle.

fields in the compactified theory. In general they correspond to deformations of the metric on the compact space, but for the simple circle compactification we found that these could be adequately described by allowing for a internal metric that depends on the external coordinates. In the more complicated case of Calabi-Yau compactifications the relevant metric deformations will generically depend on the internal coordinates so that we will need to more careful about identifying the zero-modes.

To do so, we again consider variations $h$ of the metric on $X_6$ and consider how they affect the equations of motion. If we again set any non-metric fields to zero, the gravitational part of the equations of motion of type IIB supergravity are simply given by the Einstein field equations in a vacuum. The massless deformations $h$ are then precisely those that solve the internal part of these equations of motion, i.e. those that leave $X_6$ Ricci-flat [9]

$$R_{mn}(g+h) = 0. \tag{1.35}$$

This fact serves to highlight the relationship between physical and geometric moduli. In particular, we find that variations $h$ that satisfy the physical condition (1.35) are precisely those that preserve the Calabi-Yau property of the background metric on $X_6$ and so the deformations $h$ can be interpreted as parameterizing all possible Calabi-Yau manifolds. As we saw in section 1.3.3 however, we have to be careful not to over-count modes that arise from gauge transformations as these do not lead to physically distinct configurations. For the metric tensor, these arise from general coordinate transformations on $X_6$ and we can account for these by performing a gauge fixing by imposing

$$\nabla^m h_{mn} = 0, \quad m, n = 1, \ldots, 6, \tag{1.36}$$

where $\nabla$ is the Levi-Civita connection of the background metric $g$. By expanding equation 1.35 to linear order in $h$ and combining with equation 1.36, it can be shown that both conditions reduce to the so-called Lichnerowicz equation

$$\nabla^k \nabla_k h_{mn} + 2R_{m\ n}^{\ k\ l} h_{kl} = 0. \tag{1.37}$$

So far we have treated $X_6$ as a real manifold, but it is of course natural to decompose $h_{mn}$ according to the index-decomposition induced by the complex structure on $X_6$. These lead to two types of deformations, $h_{i\bar{j}}$ and $h_{ij}$, which by virtue of a similar decomposition for the Riemann tensor separately satisfy equation 1.37. Their solutions lead to two classes of moduli which we separately discuss in the following.

### Kahler Moduli

The first set of deformations are those with an index structure matching that of the metric, i.e. $h_{i\bar{j}}$. These moduli carry the name of Kahler moduli because they parameterize deformations of the Kahler form on $X_6$. Indeed, like the metric, the deformations $h_{i\bar{j}}$ can be viewed as the components of a $(1,1)$-form $h$ on $X_6$

$$h := \frac{i}{2} h_{i\bar{j}} dx^i \wedge d\bar{x}^{\bar{j}}. \tag{1.38}$$

In this way they can exactly be interpreted as deformations of the Kahler form $\mathcal{K}$. The Lichnerowicz equation can then be shown to be equivalent to

$$\Delta h = 0. \tag{1.39}$$

We find that deformations of the Kahler form which satisfy equation 1.37 are in one-to-one correspondence to harmonic $(1,1)$-forms on $X_6$, which are in turn enumerated by $h^{1,1}$. These are precisely the deformations of the Kahler form which change the cohomology class of $J$, and Yau's theorem then tells us that there exists a unique new Ricci-flat metric with the deformed $J$ as its Kahler form. Turning this on its head we can also use Yau's theorem to deduce that non-trivial deformations of the Kahler metric which leave it Ricci-flat are precisely those that change the cohomology class of the associated Kahler form, as the original Kahler metric is unique in its Kahler class. This means that our conclusions are valid beyond the infinitesimal analysis which led to equation 1.37. We can parameterize this space by expanding $h$ in terms of a real basis $\{\omega_\alpha\}_{\alpha=1,\dots,h^{1,1}}$ for $H^{1,1}$ as

$$h = \sum_{\alpha=1}^{h^{1,1}} v^\alpha \omega_\alpha, \quad v^\alpha \in \mathbb{R}, \tag{1.40}$$

and identify the Kahler moduli with the expansion coefficients $v^\alpha$. It should be noted here that not all values of $v^a$ correspond to valid transformations of the metric. Indeed, the resulting metric should still be positive-definite which restricts $v^\alpha$ to a subset of $\mathbb{R}^{h^{1,1}}$ called the Kahler cone. While we have included them here for the sake of completeness, the Kahler moduli will mostly not be of interest to us in this thesis.

### Complex Structure Moduli

In addition to the Kahler moduli, deformations with an index structure of the form $h_{ij}$ will generate additional moduli called complex structure moduli. As the name suggests, these will indeed parametrize the complex structure of the Calabi-Yau manifold. To see this, note that deforming the metric by $h_{ij}$ will add off-diagonal terms that break the Hermitian structure of the metric $g$. For the resulting manifold to again be Kahler we need to change the complex structure that controls the index structure of the metric. Identifying those deformations that also leave the resulting manifold Ricci-flat is not as straight-forward as for the Kahler moduli, but it turns out that there is still a way of reducing the Lichnerowicz equation to a cohomology problem[9]. In this case, we can use $h_{ij}$ to define a complex $(2,1)$-form by contracting with the holomorphic $(3,0)$-form $\Omega$ [9]

$$\frac{1}{2}\Omega_{ij}{}^{\bar{l}} h_{\bar{k}\bar{l}} dx^i \wedge dx^j \wedge d\bar{x}^{\bar{k}}. \tag{1.41}$$

One can show that the Lichnerowicz equation implies that this form is harmonic, so that these deformations correspond to elements of the cohomology group $H^{2,1}(X_6)$. We can expand these in a basis of harmonic $(2,1)$-forms $\{\chi^a_{ij\bar{k}}\}_{a=1,\dots,h^{2,1}}$

$$-\frac{1}{2}\Omega_{ij}{}^{\bar{l}} h_{\bar{k}\bar{l}} = \sum_{a=1}^{h^{2,1}} z_a \chi^a_{ij\bar{k}}, \quad z_a \in \mathbb{C}, \tag{1.42}$$

which, because $\Omega_{ijk}$ is non-vanishing, we can invert to obtain an expansion for the variations $h_{ij}$ themselves

$$h_{ij} = -\frac{1}{\|\Omega\|^2} \sum_{a=1}^{h^{2,1}} \bar{z}^a (\bar{\chi}_a)_{i\bar{k}\bar{l}} \Omega^{\bar{k}\bar{l}}{}_j \equiv \sum_{a=1}^{h^{2,1}} \bar{z}^{\bar{a}} (\bar{b}_{\bar{a}})_{ij}, \tag{1.43}$$

[9]Once one realizes that the $h_{ij}$ correspond to changes in complex structure, one could also reason directly in terms of deformations of the associated map and demanding that the resulting almost complex structure is again integrable.

| Cohomology Group | Dimension | Basis |
|---|---|---|
| $H^{1,1} = H^2$ | $h^{1,1}$ | $\omega_\alpha$ |
| $H^{2,2} = H^4$ | $h^{1,1}$ | $\tilde{\omega}^\beta$ |
| $H^{2,1} \subset H^3$ | $h^{2,1}$ | $\chi_a$ |
| $H^3$ | $2h^{2,1} + 2$ | $(\alpha_I, \beta^I)$ |

Table 1.1: Bases of Harmonic forms on $X_6$ used to decompose the ten-dimensional $p$-forms into zero modes.

where we have defined

$$(\bar{b}_{\bar{a}})_{ij} := \frac{1}{\|\Omega\|^2}(\chi_a)_{i\bar{k}\bar{l}}\Omega^{\bar{k}\bar{l}}{}_j, \qquad \|\Omega\|^2 := \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}. \tag{1.44}$$

Here, $h_{ij}$ can be interpreted as an element of $H^1(X_6, T^{(1,0)}X_6)$ via $h_{ij}dz^j$.[10] Note that unlike the Kahler moduli however, it is not obvious that the infinitesimal deformations that satisfy the Lichnerowicz equation actually integrate to finite changes of complex structure. Nevertheless, it is possible to show that this is indeed the case [10]. As we shall see, the geometry of this moduli space will be of particular importance, and we will encounter it from several perspectives throughout this thesis. In the present chapter, we will mainly consider these spaces from a more physical perspective, in terms of field spaces of massless scalar fields. However, the correspondence of these moduli spaces to the geometrical notion of a moduli space allows for the application of many powerful mathematical techniques and this is the view that we will take in chapter 3.

## 1.5 Calabi-Yau compactification of Type IIB

At last it is time to put the pieces together and perform the Calabi-Yau compactification of the type IIB supergravity action from section 1.2 [11]. We will start this section by discussing the field-content of the four-dimensional theory using our results from sections 1.3 and 1.4. Because the resulting theory (by construction) enjoys $\mathcal{N} = 2$ supersymmetry, it is natural to assemble the low-energy field content into irreducible multiplets. This will allow us to write down the most general four-dimensional $\mathcal{N} = 2$ supergravity action and finally give the Kaluza-Klein reduced action in this form.

### 1.5.1 Field Content

The field content of the lower-dimensional theory is determined by the zero-modes of the relevant differential operators. As discussed at length in the previous section, the massless modes that descend from the metric are precisely the moduli introduced in section 1.4.3. The ten-dimensional metric can be decomposed in terms of these moduli according to

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{i\bar{j}}(y)dy^i d\bar{y}^{\bar{j}} + v^\alpha(x)(\omega_\alpha)_{i\bar{j}}(y)dy^i d\bar{y}^{\bar{j}} + \bar{z}^{\bar{a}}(x)(\bar{b}_{\bar{a}})_{ij}(y)dy^i dy^j + \text{c.c.}, \tag{1.45}$$

where the background Calabi-Yau metric $g_{i\bar{j}}(y)$ is fixed. Meanwhile, the scalar dilaton $\hat{\phi}$ and the 0-form $\hat{C}_0$ simply reduce to four-dimensional scalars which do not depend on the

[10] Indeed, this is the appropriate cohomology to consider. A more formal treatment of the complex structure deformations would involve the Kodaira-Spencer map which relates the *tangent space* of the Moduli space (more precisely deformation space) to the cohomology group $H^1(X_6, T^{(1,0)}X_6)$. For Calabi-Yau 3-folds, the holomorphic $(3,0)$-form yields an isomorphism $H^1(X_6, T^{(1,0)}X_6) \cong H^{2,1}(X_6)$ via equation 1.41.

compact coordinates. Note that throughout this section we use hats to distinguish ten- and four-dimensional objects. The remaining $p$-form fields are more involved however, as they admit non-trivial zero-modes corresponding to various cohomology classes of the Calabi-Yau. We encountered the non-trivial cohomology classes of Calabi-Yau manifolds in section 1.4.2. To perform an expansion as in equation 1.24 we assign bases for these cohomology groups which are listed in table 1.1. Note that we choose two sets of bases for $H^3$ and $H^{2,1} \subset H^3$ as we require the latter sub-set for the expansion of the metric, while we need the full third cohomology to expand the $\hat{C}_4$-field. For reasons that will become clear shortly, we choose to split the basis for $H^3$ into two sets of $h^{2,1}+1$ elements. Putting everything together we find the following decomposition for the non-metric fields fields

$$\begin{aligned} \hat{\phi}(x,y) &= \phi(x), \qquad \hat{B}_2(x,y) = B_2(x) + b^\alpha(x)\omega_\alpha(y), \\ \hat{C}_0(x,y) &= C_0(x), \quad \hat{C}_2(x,y) = C_2(x) + c^\alpha(x)\omega_\alpha(y), \\ \hat{C}_4(x,y) &= C_4(x) \ + D_2^\alpha(x)\wedge\omega_\alpha(y) + \rho_\beta(x)\tilde{\omega}^\beta(y) \\ &\qquad\qquad + A^I(x)\wedge\alpha_I(y) - V_I(x)\wedge\beta^I(y). \end{aligned} \tag{1.46}$$

Together with the four-dimensional metric/graviton field $g_{\mu\nu}(x)$ and moduli fields $v^i$, $z^a$, these constitute the basic field content of the compactified four-dimensional supergravity action. Before we proceed however, three remarks are in order.

- Firstly, recall that the dynamics of the four-form gauge field are not fixed by the action, since we should supplement the resulting equations of motion by the self-duality constraint $*\hat{F}^5 = \hat{F}^5$. This condition descends to a set of relations between the lower-dimensional components of $\hat{C}_4$, which similarly should be imposed at the level of the (four-dimensional) equations of motion. However, unlike for the ten-dimensional action we will find that the four-dimensional theory allows us to impose this condition at the level of the action, because the condition now relates two different sets of fields. Consequently it is possible to eliminate half of the degrees of freedom coming from the pair $(\rho_a, D_2^a)$ and likewise for $(A^I, V_I)$, which is why we chose a split basis for $H^3$. In our discussion of SUSY multiplets we will assume that this has been done and keep only the first set of fields in each pair (although this particular choice is arbitrary).
- Unrelated to the self-duality constraint we also emphasize here that while the expansion of $\hat{C}_4(x,y)$ also allows for a four-dimensional 4-form $C_4(x)$, this 4-form has no degrees of freedom in four dimensions so that we drop it in the following.
- Lastly, we would like to highlight that the off-diagonal metric components do not generate additional vector fields, because Calabi-Yau three-folds have trivial first cohomology $H^1$.

### Supersymmetry Multiplets

To organize the subsequent discussion of the action describing the dynamics of these fields, we next discuss how they organize into so-called supersymmetry multiplets. Restricting to fields of spin $\leq 2$, four-dimensional $\mathcal{N}=2$ supergravity has four such multiplets. These are called the gravity, vector, hyper- and tensor multiplets. Their bosonic field content is summarized in table 1.2 and we spend a few words describing them to establish some nomenclature.

Firstly, it should be clear from the table that the names of these multiplets refer to the highest-spin bosonic field content. The gravity multiplet is a universal multiplet present for any supergravity theory and contains a graviton and a gauge field called the graviphoton.

| Multiplet | # | Fields |
|---|---|---|
| Gravity | 1 | $(g_{\mu\nu}, A^0)$ |
| Vector | $h^{2,1}$ | $(A^a, z^a)$ |
| Hyper | $h^{1,1}$ | $(v^\alpha, b^\alpha, c^\alpha, \rho_\alpha)$ |
| Tensor | 1 | $(B_2, C_2, \phi, C_0)$ |

Table 1.2: Assignment of the four-dimensional fields to their respective supersymmetry multiplets

The others are optional and may or may not be present in a consistent supergravity theory. Moreover, note that despite their ostensibly different field contents, the tensor and hypermultiplets have the same spin content. In fact, it is possible to reduce a theory containing both hypermultiplets and tensor multiplets to one containing only hypermultiplets through a procedure called dualization which we will discuss at some length in subsequent sections. For the time being we simply assume that this procedure has been performed and include only hypermultiplets in the theory.

The basic $\mathcal{N} = 2$ supersymmetric action we will consider using these multiplets is given by

$$S = \int \left( \frac{1}{2} R * 1 - h_{uv} dq^u \wedge * dq^v - K_{a\bar{b}} dz^a \wedge * d\bar{z}^{\bar{b}} + \frac{1}{2} \mathrm{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \mathrm{Re} \mathcal{N}_{IJ} F^I \wedge F^J \right), \\ a, b = 1, \ldots n_V, \quad I = 0, \ldots n_V, \quad u, v = 1, \ldots, 4n_H. \tag{1.47}$$

Here, $n_V$ and $n_H$ denote the number of hyper- and vector multiplets, respectively. The scalars in these multiplets are described by non-linear sigma models. Their target manifolds are constrained to be of a special type to be discussed at length in the next section. Moreover, $F^I$ are the field strengths of the vector fields in the vector and gravity multiplets. Note that because there is always one gravity multiplet, there is also always one vector which is why the index $I$ starts at zero. Lastly, the coupling matrices for the gauge fields generically depend on the scalars in the vector multiplets in a non-trivial way and their properties are linked to the scalar manifold geometry via supersymmetry.

### 1.5.2 Performing the Reduction

We are now finally in a position to perform the dimensional reduction of the type IIB action to 4D. Upon inserting the expansion (1.46) into the IIB action and performing the integral over the internal manifold, we obtain the compactified action. Because the resulting theory has $\mathcal{N} = 2$ supersymmetry we expect the final result to take the general form (1.47). Indeed, it can be shown that it does and table 1.2 shows how the four-dimensional field content assembles into the four-dimensional supersymmetry multiplets. As alluded to in the introduction, this work will mainly be concerned with the dynamics of the resulting vector multiplets which as we can see include the complex structure moduli. Because these fields will be so central to our discussion in later chapters we explicitly perform the dimensional reduction for the fields in these multiplets, while we do not give any details, nor state the result, for the hypermultiplets.

The fields in the vector multiplets descend from the ten-dimensional metric, which supplies the scalars, and the four-form RR field, which provides the vector fields, and the relevant terms in the ten-dimensional action are given by

$$S_{\text{vector}} = \int \left( \frac{1}{2} \hat{R} * 1 - \frac{1}{4} \hat{F}_5 \wedge * \hat{F}_5 \right). \tag{1.48}$$

Note that the gauge fields for $\hat{F}_5$ also enter the ten-dimensional action through the Chern-Simons term in equation 1.4, but the non-zero terms that descend from this upon compactification only include the scalar part $\rho_a$ in the expansion (1.46) and hence, does not contribute to the vector sector of the theory. In the following, we will compactify both of these terms, starting with the second as it is simpler. We will not try to give all the details but choose to highlight some intermediate results which will provide us with the necessary insights.

### Compactification of the Gauge Fields

In order to discuss the compactification of the gauge kinetic term in equation 1.48, we should start by being more concrete about the basis $(\alpha_I, \beta^I)$. So far its main purpose has been to define a split that allows us to identify the physical modes of the self-dual 4-form $\hat{C}_4$. However, for the purposes of the dimensional reduction that follows, there exists a useful canonical choice for this basis. To define it, let us note that we can define a natural anti-symmetric pairing on the vector space $H^3$, defined by

$$\langle \alpha, \beta \rangle := \int_{X_6} \alpha \wedge \beta. \tag{1.49}$$

We call this the intersection form[11] and it defines an alternating non-degenerate bi-linear (i.e. symplectic) form on the vector space $H^3(X, \mathbb{C})$. This fact will be very important to us later when we discuss the geometry of four-dimensional $\mathcal{N} = 2$ supergravity, but for now the most important thing it allows us to do is to define a real symplectic basis $(\alpha_I, \beta^I)$[12] for $H^3(X_6, \mathbb{C})$ which satisfies

$$\eta_{IJ} := \langle \alpha_I, \alpha_J \rangle = 0 = \langle \beta^I, \beta^J \rangle =: \eta^{IJ}, \quad \eta_I^J := \langle \alpha_I, \beta^J \rangle = \delta_I^J. \tag{1.50}$$

Explicitly, this allows us to write the symplectic pairing (1.49) as a matrix with respect to the basis $(\alpha_I, \beta^I)$, which has the canonical form

$$\eta = \begin{pmatrix} 0 & 1_{n_V+1} \\ -1_{n_V+1} & 0 \end{pmatrix}, \tag{1.51}$$

where we point out that this basis is unique up to basis transformations that preserve this form of $\eta$, i.e. transformations in $Sp(2n_V + 2, \mathbb{R})$. We can also use these properties to extract the components of a general element $Q \in H^3$ with respect to the symplectic basis using the intersection form

$$Q := p^I \alpha_I - q_I \beta^I \quad \Rightarrow \quad p^I = \langle Q, \beta^I \rangle \equiv \int_{B_I} Q, \quad q_I = \langle Q, \alpha_I \rangle \equiv \int_{A^I} Q, \tag{1.52}$$

where we have introduced a poincare dual Homology basis $(A_I, B^I)$. In the following we assume that the basis used to expand $\hat{C}_4$ is such a symplectic basis.

With these preliminaries out of the way, we turn to expanding the gauge kinetic term in equation 1.48 using the expansion (1.46) for $\hat{C}_4$. We note that we only need to keep the part

[11] Under Poincare duality it maps to the intersection product in homology.

[12] Note that any symplectic vector space admits a symplectic basis, defined by the properties (1.50).

valued in $H^3$, as it can be checked that the mixed vector-hypermultiplet terms all vanish. The result is given by

$$\begin{aligned}\int_{X_6} \hat{F}_5 \wedge *_{10}\hat{F}_5 &= \int_{X_6} (F^I \wedge \alpha_I - G_I \wedge \beta^I) \wedge *_{10}(F^I \wedge \alpha_I - G_I \wedge \beta^I) + \text{Hypermultiplet Terms} \\ &= \left(\int_{X_6} \alpha_I \wedge *_6\alpha_J\right) F^I \wedge *_4 F^J - 2\left(\int_{X_6} \alpha_I \wedge *_6\beta^J\right) F^I \wedge *_4 G_J + \left(\int_{X_6} \beta^I \wedge *_6\beta^J\right) G_I \wedge *_4 G_J.\end{aligned} \tag{1.53}$$

To arrive at the second line we have used the following general identity, valid for product manifolds of the form we consider

$$*_{10}(A_p^{(4)} \wedge B_q^{(6)}) = (-1)^{pq}(*_4 A_p^{(4)}) \wedge (*_6 B_q^{(6)}), \tag{1.54}$$

with $A_p^{(4)}$ and $B_q^{(6)}$ four- and six-dimensional differential forms, respectively, while $*_4$ and $*_6$ denote the Hodge star with respect to the external and internal metrics, respectively. We see that the gauge couplings between the four-dimensional fields $F^I$ and $G_I$ are determined by the components of the Hodge star of the Calabi-Yau acting on the basis elements for $H^3$. To simplify the notation in the following, we write this in matrix-form as

$$(\mathcal{M}_{\alpha\alpha})_{IJ} = \int_{X_6} \alpha_I \wedge *_6\alpha_J, \quad (\mathcal{M}_{\alpha\beta})_I^J = \int_{X_6} \alpha_I \wedge *_6\beta^J, \quad (\mathcal{M}_{\beta\beta})^{IJ} = \int_{X_6} \beta^I \wedge *_6\beta^J, \tag{1.55}$$

which we assemble into a matrix $\mathcal{M}$

$$\mathcal{M} = \begin{pmatrix} \int_{X_6} \alpha_I \wedge *_6\alpha_J & \int_{X_6} \alpha_I \wedge *_6\beta^J \\ \int_{X_6} \beta^I \wedge *_6\alpha_J & \int_{X_6} \beta^I \wedge *_6\beta^J \end{pmatrix} \equiv \begin{pmatrix} (\mathcal{M}_{\alpha\alpha})_{IJ} & (\mathcal{M}_{\alpha\beta})_I^J \\ (\mathcal{M}_{\beta\alpha})_J^I & (\mathcal{M}_{\beta\beta})^{IJ} \end{pmatrix}. \tag{1.56}$$

As discussed at the end of section 1.5.1, we still have to impose the self-duality constraint on $\hat{F}_5$. Inserting the expansion of $\hat{F}_5$ in terms of $F^I$ and $G_I$ into the self-duality constraint $*\hat{F}_5 = \hat{F}_5$ and using the identity (1.54), we find after integrating over $X_6$ that the former reads

$$*G_I = ((\mathcal{M}_{\alpha\alpha})_{IJ} - (\mathcal{M}_{\alpha\beta})_I^K(\mathcal{M}_{\beta\beta}^{-1})_{KL}(\mathcal{M}_{\alpha\beta})_J^L)F^J + (\mathcal{M}_{\beta\beta}^{-1})_{IJ}(\mathcal{M}_{\alpha\beta})_K^J * F^K. \tag{1.57}$$

Next, we note that the symmetry properties of $\mathcal{M}$ along with the fact that $*^2 = -1$ for three-forms (in six dimensions) allow us to derive the following relationship between its blocks[13]

$$(\mathcal{M}_{\beta\beta}^{-1})_{IJ} = ((\mathcal{M}_{\alpha\alpha})_{IJ} - (\mathcal{M}_{\alpha\beta})_I^K(\mathcal{M}_{\beta\beta}^{-1})_{KL}(\mathcal{M}_{\alpha\beta})_J^L), \tag{1.58}$$

simplifying (1.57) to

$$*G_I = (\mathcal{M}_{\beta\beta}^{-1})_{IJ}F^J + (\mathcal{M}_{\beta\beta}^{-1})_{IJ}(\mathcal{M}_{\alpha\beta})_K^J * F^K. \tag{1.59}$$

As promised, this can be imposed at the level of the action by including a total derivative term

$$\int_{M_4} \left( -\frac{1}{4}(\mathcal{M}_{\alpha\alpha})_{IJ}F^I \wedge *_4F^J + \frac{1}{2}(\mathcal{M}_{\alpha\beta})_I^J F^I \wedge *_4 G_J - \frac{1}{4}(\mathcal{M}_{\beta\beta})^{IJ} G_I \wedge *_4 G_J + \frac{1}{2} F^I \wedge G_I \right). \tag{1.60}$$

[13] More precisely, it is easy to show that the Hodge star, viewed as a matrix acting on $H^3$ is symplectic with respect to the pairing $\eta$. The former is given in terms of $\mathcal{M}$ as $-\eta\mathcal{M}$ and enforcing that the latter is symplectic yields the given relation.

Indeed, it can be checked that the equation of motion for $G_I$ one derives from this action is precisely the self-duality constraint (1.59). At this point one may integrate out the field $G_I$ imposing its equations of motion on the action, which yields the final four-dimensional action

$$S_4 = \int_{M_4} \left( -\frac{1}{2}(\mathcal{M}^{-1}_{\beta\beta})_{IJ} F^I \wedge *F^J + \frac{1}{2}(\mathcal{M}_{\alpha\beta})_I^K (\mathcal{M}^{-1}_{\beta\beta})_{KJ} F^I \wedge F^J \right). \tag{1.61}$$

Comparing to the action (1.47) we identify the gauge kinetic functions in terms of the matrix $\mathcal{M}$ as

$$\mathrm{Im}\mathcal{N}_{IJ} = -(\mathcal{M}^{-1}_{\beta\beta})_{IJ}, \quad \mathrm{Re}\mathcal{N}_{IJ} = (\mathcal{M}_{\alpha\beta})_I^K (\mathcal{M}^{-1}_{\beta\beta})_{KJ}. \tag{1.62}$$

We see that the kinetic functions are defined in terms of the matrix elements of the Hodge star acting on the middle cohomology, which, as we as we shall see in chapter 4, depends explicitly on the complex structure on $X_6$. For both emphasis and future reference, we also invert equation 1.62 (using (1.58)) to give an expression for $\mathcal{M}$ in terms of $\mathcal{N}$

$$\mathcal{M} = \begin{pmatrix} -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -(\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \tag{1.63}$$

### Compactification of the Ricci Scalar

The second term that contributes to the vector multiplet sector is the Einstein-Hilbert term in equation 1.48. Its compactification is a rather involved exercise so we do not include it here, instead referring to e.g. [12, 13] for details. The upshot of the computation is that the Ricci scalar decomposes according to

$$\int_{M_4\times X_6} \frac{1}{2}\hat{R} * 1 = \int_{M_4} \left( \frac{1}{2} R * 1 - K_{a\bar{b}} dz^a \wedge * d\bar{z}^{\bar{b}} + \text{Kahler Moduli} \right), \tag{1.64}$$

where $K_{a\bar{a}}$ is given by

$$K_{a\bar{b}} = \frac{1}{4\mathcal{V}} \int_{X_6} (b_a)^{ij} (\bar{b}_{\bar{b}})_{ji} *_6 1, \tag{1.65}$$

with $\mathcal{V}$ the volume of the background CY manifold[14]. Equation 1.65 now gives an explicit expression for the metric on the complex structure moduli space. If we recall the definition of the $b_a$ in equation 1.44, we see that after some rearranging this metric can be written as

$$K_{a\bar{b}} = -\frac{\int_{X_6} \chi_a \wedge \bar{\chi}_{\bar{b}}}{\int_{X_6} \Omega \wedge \bar{\Omega}}. \tag{1.66}$$

It turns out that this metric on the moduli space in fact is itself Kahler! To see this, we show how it can be obtained from a Kahler potential. In the process we can begin to see the relationship between the complex structure and the holomorphic three-form $\Omega$. Under changes of complex structure we have that $\Omega$, which starts out life as a $(3,0)$-form, will generally not remain of type $(3,0)$. Instead, if we decompose $\Omega$ with respect to the new complex structure we generically expect it to have components along other parts of the new Hodge decomposition of $H^3$. Turning this on its head, we find that the holomorphic $(3,0)$-form will have to change inside $H^3$ if it wishes to remain of type $(3,0)$ under changes of complex structure. These ideas are quantified by the so-called Kodaira formula[15]

$$\partial_{t^a}\Omega = k_a \Omega + \chi_a, \quad k_a = k_a(t^a), \quad \chi_a \in H^{2,1}, \tag{1.67}$$

[14] Intuitively, this appears because of the integral over the compact manifold in the term $R^{(4)} \wedge *_6 1_6$.

[15] As we shall see shortly, the holomorphic $(3,0)$-form can be interpreted as a section of a line bundle over the moduli space. Viewed in this light, the somewhat heuristic equation 1.67 should be interpreted as evaluating the canonical connection on that bundle in a given frame.

where $k_a$ is a function of the moduli and $\chi_a$ are moduli-independent elements of $H^{2,1}$. As a corollary of the proof of equation 1.67, one can also show that the $\chi_a$ correspond precisely to the basis elements of $H^{2,1}$ used also in (1.42). Using these facts, it is easy to show that equation 1.66 can be obtained from the Kahler potential

$$K := -\log\left(i\int \Omega\wedge\bar{\Omega}\right) = -\log\left(i\langle\Omega,\bar{\Omega}\rangle\right), \tag{1.68}$$

where we have used the symplectic pairing (1.49) to rewrite our result. Moreover, evaluating the derivative $\partial_{t^a}K$ we also readily make the identification that $k_a = -\partial_{t^a}K$. This way of viewing $\Omega$ as a function of the moduli will be essential in the next section and will be worked out further in chapter 3.

**Branes Wrapping Cycles**

As we mentioned briefly in section 1.3.4, string theory compactifications may also involve branes. Although these states may be too heavy to be produced at low energies, the UV-complete theory contains them and once present they will appear as sources for the $p$-form gauge fields of the theory. Similarly, their dimensionally reduced counterparts will appear as sources for the dimensionally reduced gauge fields. Of particular interest to us will be the D3-branes. These couple directly to the $\hat{C}_4$ R-R field in 10D, from which the fields in the four-dimensional vector sector descend.

In particular, a D3-brane wrapping a 3-cycle will span a one-dimensional curve in the non-compact space. This curve can then be interpreted as the world-line of a particle or a black hole which is charged under the gauge fields in (1.61). Let us assume that the world-volume splits according to the $\mathcal{W}_4 = \mathcal{C}^1\times L^3$ for $L^3\subset X_6$ a homologically non-trivial cycle in the Calabi-Yau and $\mathcal{C}^1\subset M_4$ the particle world-line. The resulting charges depend only on the homology class of $L^3$ which we may represent by its Poincare dual 3-form. Expanding the latter with respect to the symplectic cohomology basis (1.50) we therefore schematically write $L^3\sim p^I\alpha_I - q_I\beta^I$, where $\mathbf{q} := (p^I, -q_I)^T$ is the so-called charge vector. We can then dimensionally reduce the Chern-Simons term

$$\begin{aligned}\int_{\mathcal{C}^1\times L^3}\hat{C}_4 = \int_{\mathcal{C}^1}\int_{L^3}\left(A^I\wedge\alpha_I - V_I\wedge\beta^I\right) &= \int_{\mathcal{C}^1}\int_{X_6}\left(A^I\wedge\alpha_I - V_I\wedge\beta^I\right)\wedge\left(p^I\alpha_I - q_I\beta^I\right)\\ &= \int_{\mathcal{C}^1} p^I V_I - A^I q_I = \int_{\mathcal{C}^1}(A^I, -V_I)\eta(p^I, -q_I)^T.\end{aligned} \tag{1.69}$$

This derivation should be taken with a grain of salt. The reason is the self-duality of $\hat{C}^4$, which means that half of the degrees of freedom above are redundant. However, the self-duality relation becomes a non-local integral relation on the gauge fields $A^I$ and $V_I$ so that we cannot impose it at the level of the action here. In the next section we will see that the fields $F^I$ and $G_I$ correspond to electric and magnetic fields, so that a D3-particle with charge vector $(p^I, -q_I)^T$ has electric charge $q_I$ and magnetic charge $p^I$ (see also appendix A).

A final very important property of these wrapped branes is that their mass depends on the volume of the cycle they wrap. Intuitively, this can be understood by noting that the resulting mass is obtained by integrating the brane tension over the world-volume of the brane. This can be made precise by explicitly performing the dimensional reduction for the brane action, which we have omitted here. Unlike their charge however, this is not a topological property, in that it depends on the particular representative the brane wraps in its homology class. Nevertheless, for so-called special Lagrangian cycles [14, 15], which minimize the volume in their respective homology class, the resulting volume, and hence particle mass, can be computed explicitly and is given by [16]

$$M^2 = \frac{\left|\int_{L^3} \Omega\right|^2}{\int_{X_6} \Omega \wedge \bar{\Omega}} = e^K \left|\int_{L^3} \Omega\right|^2 = e^K |\langle \Omega, \mathbf{q} \rangle|^2, \tag{1.70}$$

from which we explicitly see that the mass depends on the moduli $z^a$.[16] More generally, this expression is a lower bound on the mass of a D3-particle. These states are special for another reason. Like the branes they descend from, they are BPS states, meaning that they preserve half of the supersymmetry of the EFT (equivalently are annihilated by half of the supersymmetry generators). While a thorough discussion of such BPS states is beyond the scope of this thesis, this does allow us to import various supergravity results and apply them to our D3-particles, something which we will do in chapter 4.

## 1.6 Vectors, Duality, and Special Geometry

So far we have mostly considered the scalar and vector parts of the action separately. Nevertheless, it should not come as a surprise that these sectors are deeply intertwined through supersymmetry. In particular, the structure of the gauge sector imposes constraints on the scalar manifold. While we have already seen that the latter is Kahler, it turns out that we can in fact say more and the resulting geometry is known as *special Kahler geometry*. To define it we will first introduce the notion of electric-magnetic duality, specifically in the context of the $\mathcal{N} = 2$ vector sector, and highlight the underlying symplectic structure of the theory. We then define special geometry in a way that highlights the role of this symplectic structure. To emphasize the bottom-up approach we take in later chapters, the material in this section will take a more general supergravity perspective, in that we do not specialize the discussion to theories obtained from Calabi-Yau compactifications. We will however comment on how these structures emerge in this particular setting.

### 1.6.1 Electric-Magnetic Duality

Electric magnetic duality is a powerful and rather general feature of gauge theories. In its simplest form it is an observation about the symmetries of Maxwell's equations under interchange of electric and magnetic fields. Indeed, given the basic Maxwell action

$$S = \int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 \right), \tag{1.71}$$

the Maxwell equations (without sources) read[17]

$$d\left(\frac{1}{e^2} * F_2\right) = 0, \quad \frac{1}{2\pi} dF_2 = 0. \tag{1.72}$$

[16] Let us take this opportunity to make an important general remark regarding the Planck mass. Eventually we will be interested in D3-particles which become light relative to the Planck mass. Just as in the circle compactification examples, the four-dimensional Planck mass depends on the over-all volume of the Calabi-Yau. In type IIB Calabi-Yau compactifications, this is determined by the Kahler moduli, which decouple from the vector sector we consider. As such, we are free to assume the Planck mass to be fixed as we vary the complex structure moduli and it suffices to study the mass (1.70).

[17] Although this factor will not play a role in this section, we include a normalizing factor of $2\pi$ to match the conventions of chapter 2. See also A for our conventions on flux quantization.

These Bianchi identities tell us that $F_2$ is closed and hence, by Poincaré's Lemma, locally exact[18]. This is nothing more than the statement that we can introduce a gauge field $F_2 = dA_1$ which, up to gauge redundancy, describes the true dynamical degrees of freedom of the theory. However, in absence of electric sources, the same is true about $\frac{1}{e^2} * F_2$. One may therefore be tempted to introduce a "dual" field strength

$$G_2 := \frac{2\pi}{e^2} * F_2, \tag{1.73}$$

which therefore satisfies its own Bianchi identity so that it can be written in terms of a "dual" gauge field $G_2 = dV_1$. The resulting symmetry of the equations of motion under the exchange of $G_2 \leftrightarrow F_2$ suggests the possibility of a dual description where $G_2$ assumes the role of the fundamental electromagnetic field strength. Indeed, we can explicitly obtain such a dual description by what is essentially a Legendre transform. To this aim, we forget the Bianchi identity for $F_2$ so that it becomes an arbitrary 2-form field and reinstating it via a Lagrange multiplier $V_1$ (this should be reminiscent of the manipulations we performed between equations 1.59-1.61)

$$S = \int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 - \frac{1}{2\pi} dF_2 \wedge V_1 \right). \tag{1.74}$$

Integrating out $V_1$ can be done exactly, both classically as well as quantum mechanically: $V_1$ enters algebraically and is hence non-dynamical. Performing the path integral over $V_1$ yields a delta-function that enforces $dF_2 = 0$, which simply takes us back to our original theory where $F_2$ can be expressed in terms of a dynamical gauge field $A_1$. Alternatively, one may choose to integrate out $F_2$ instead. Notice that in this case, the Lagrange multiplier may be integrated by parts to read $\int F_2 \wedge dV_1$. We find that like $V_1$ did before, $F_2$ now enters algebraically and moreover quadratically in the action. Just as we could integrate out $V_1$ exactly, we can also perform the Gaussian integral over $F_2$ exactly. For future reference we note that this is equivalent to solving the equations of motion for $F_2$ (note that we consider variations of $F_2$ rather than the non-existent gauge field $A_1$)

$$F_2 = -\frac{e^2}{2\pi} * dV_1, \tag{1.75}$$

and inserting this into the action to obtain

$$S = \int \left( -\frac{1}{2\tilde{e}^2} \tilde{G}_2 \wedge \tilde{G}_2 \right), \tag{1.76}$$

where we have introduced the field strength $\tilde{G}_2 = dV_1$ for $V_1$ and the "dual" gauge coupling $\tilde{e} = 2\pi/e$. As promised, we have obtained a dual description of the original Maxwell theory in terms of the dual field strength (1.73). Indeed, solving equation 1.75 for $\tilde{G}_2$, we find that it coincides exactly with the dual field strength $G_2$ introduced above.

Interestingly, we find that the dual description has a coupling constant related to the original by $e \to 2\pi/e$. In particular, this means that it interchanges strongly and weakly coupled descriptions of the theory. This makes it an example of a so-called S-duality (with

[18] We emphasize the importance of the word locally here. As we shall explore at length in chapter 2, many of the most interesting phenomena in gauge theory exist only on backgrounds with non-trivial topology, be it due to the presence of charges or due to the intrinsic spacetime geometry, where closedness of $F_2$ does not imply exactness. This is particularly true of abelian gauge theories, whose "field space" has a particularly simple topology.

the S referring to strong in strong coupling). While this duality was rather easily shown to be exact for free Maxwell theory, it is a highly non-trivial fact about supersymmetric gauge theories that they often admit such dualities even for interacting non-abelian theories. In line with our general rule that supersymmetric theories are generally under greater computational control, exact dualities are often only provable for theories with enough supercharges. In fact, the conjectural Seiberg duality in four-dimensional $\mathcal{N} = 1$ Yang-Mills theory does not even claim to be an exact duality, but rather states that both theories flow to the same theory in their respective infrared limits. Nevertheless, having a weakly coupled description of a strongly coupled interacting gauge theory is an incredibly powerful tool to have and it is one of the primary motivations for studying such supersymmetric gauge theories.

In the following sub-section, we will see how electric-magnetic dualities play out for the vector multiplet sector of four-dimensional $\mathcal{N} = 2$ supergravity. Indeed, just as in the case of Maxwell theory, the gauge fields admit a similar duality structure. However, unlike in non-abelian gauge theory, its main purpose is not to probe the strongly coupled physics of the theory. Instead, the power of electric-magnetic duality comes from the fact that the gauge couplings are now functions of the scalars in the theory. A full duality transformation should account for this fact and consequently intertwines the geometry of the scalar manifold with the duality structure of the gauge vectors. Understanding the resulting duality structure will serve us well when we come to discuss the symmetries of these theories in chapter 4.

### 1.6.2 Duality in Supergravity

Recall from section 1.5.2 that the action for the vector fields in $\mathcal{N} = 2$ supergravity is given by

$$S = \int \left( \frac{1}{2} \mathrm{Im} \mathcal{N}_{IJ} F^I \wedge *F^J + \frac{1}{2} \mathrm{Re} \mathcal{N}_{IJ} F^I \wedge F^J \right). \tag{1.77}$$

The detailed properties of the coupling matrix $\mathcal{N}$ will not be important to us here, but we emphasize that it should be symmetric with negative-definite imaginary part and may generically depend on the complex structure moduli $z^i$. The equations of motion for this action are given by

$$d(\mathrm{Im}\mathcal{N}_{IJ} * F^J + \mathrm{Re}\mathcal{N}_{IJ} F^J) = 0, \qquad dF^I = 0. \tag{1.78}$$

In our discussion of duality in Maxwell theory we restricted our attention to the basic $\mathbb{Z}_2$ duality that exchanged the electric and magnetic fields. It turns out however, that the story is slightly richer [17]. Making the full extent of this structure explicit, requires the introduction of some auxiliary fields. Following [18, 19], let us start by splitting the degrees of freedom of the gauge fields into imaginary (anti-)self-dual parts as follows

$$F^I_\pm = \frac{1}{2}(F^I \pm i * F^I), \quad *F^I_\pm = \mp i F^I_\pm, \quad F^I = F^I_+ + F^I_-. \tag{1.79}$$

The action can then be rewritten as follows

$$\begin{aligned} S &= \int \left( \frac{1}{2} \mathcal{N}_{IJ} F^I_- \wedge F^J_- + \frac{1}{2} \bar{\mathcal{N}}_{IJ} F^I_+ \wedge F^J_+ \right) \\ &= \int \mathrm{Re} \left( \mathcal{N}_{IJ} F^I_- \wedge F^J_- \right). \end{aligned} \tag{1.80}$$

Next, we will introduce the dual field strength $G_I$, whose closedness will correspond to the equation of motion for $A^I$. Let us therefore define

$$G^-_I := \frac{\delta S[F_-, F_+]}{\delta F^I_-} = \mathcal{N}_{IJ} F^J_-, \quad G^+_I := \frac{\delta S[F_-, F_+]}{\delta F^I_+} = \bar{\mathcal{N}}_{IJ} F^J_+. \tag{1.81}$$

By (anti-)self-duality of $F^I_\pm$, these fields can again be thought of as the (anti-)self-dual parts of a dual field strength $G_I$, given by

$$G_I = G_I^- + G_I^+ = 2\mathrm{Re}G_I^- = 2\mathrm{Re}\mathcal{N}_{IJ}F_-^J = \mathrm{Im}\mathcal{N}_{IJ} * F^J + \mathrm{Re}\mathcal{N}_{IJ}F^J. \tag{1.82}$$

As promised, we see that we can identify the equations of motion of $F^I$ with a Bianchi identity for the dual field $G_I$. The action can now be rewritten entirely in terms of the self-dual components $F^I_-$, $G_I^-$ as

$$S_{vec} = \int \mathrm{Re}\left(F^I_- \wedge G_I^-\right), \tag{1.83}$$

however we stress that $F_-$ and $G^-$ are not independent but strictly related via (1.81). The equations of motion can similarly be rewritten as

$$d(\mathrm{Re}G_I^-) = 0, \quad d(\mathrm{Re}F^I_-) = 0. \tag{1.84}$$

As an aside we note that the same equations hold for the anti-self-dual fields $F_+$ and $G^+$. Written in this way it is clear that it may be possible to rotate the two fields $F_-$ and $G^-$ into each other while preserving the equations of motion of the theory. We formalize this by introducing the a vector $(F_-, -G^-)^T$ and subjecting it to general rotations as

$$\begin{pmatrix} F'_- \\ -G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F_- \\ -G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F_- \\ -G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F_- \\ -\mathcal{N}F_- \end{pmatrix} = \begin{pmatrix} (A - B\mathcal{N})F_- \\ (C - D\mathcal{N})F_- \end{pmatrix}, \tag{1.85}$$

where we have used the relationship 1.81 to rewrite $G^-$ in terms of $F_-$. It is clear that for any real, constant, invertible matrix $\mathcal{S} \in GL_n(2n, \mathbb{R})$, the transformed vector satisfies equivalent equations of motion as the old one (i.e. the transformation (1.85) maps solutions to solutions). This is only part of the story, however. Firstly, we should recall that the actual equations of motion of the system were given by equation 1.78. The equations

$$d(\mathrm{Im}\mathcal{N}_{IJ} * F^J + \mathrm{Re}\mathcal{N}_{IJ}F^J) = 0 \quad \Leftrightarrow \quad dG_I = 2d(\mathrm{Re}G_I^-) = 0, \tag{1.86}$$

were only equivalent because of the relationship (1.81) between $F^-$ and $G_-$. However, applying a generic rotation of the form (1.85) to the action (1.80) need not lead to a similar relationship between the transformed vectors $F'_-$ and $G'^-$. It then follows that for the transformed equations of motion to actually be the same as the old, the transformed fields should exhibit this same relationship. To see when this is the case, we consider how the action transforms under arbitrary transformations (1.85). Note that because we are looking for a *duality* rather than a symmetry, we should allow for a non-trivial transformation of the gauge kinetic matrix $\mathcal{N} \to \mathcal{N}'$, just as we transformed $e^2 \to 1/e^2$ in the case of Maxwell theory. This leads to

$$\begin{aligned} S \to S' &= \int \left(\frac{1}{2}\mathcal{N}'_{IJ}F'^I_- \wedge F'^J_- + \frac{1}{2}\bar{\mathcal{N}}'_{IJ}F'^I_+ \wedge F'^J_+\right) \\ &= \int \left(\frac{1}{2}\mathcal{N}'_{IJ}((A - B\mathcal{N})F_-)^I \wedge ((A - B\mathcal{N})F_-)^J + \frac{1}{2}\bar{\mathcal{N}}'_{IJ}((A - B\bar{\mathcal{N}})F_+)^I \wedge ((A - B\bar{\mathcal{N}})F_+)^J\right) \end{aligned} \tag{1.87}$$

We then demand that we obtain $G'^-$ by varying with respect to $F'^I_-$. Using the transformation (1.85) for $G'^-$, we then obtain the condition that

$$G'^- = -(C - D\mathcal{N})F_- \stackrel{!}{=} \frac{\delta S'[F'_-, F'_+]}{\delta F'_-} = \mathcal{N}'(A - B\mathcal{N})F_-. \tag{1.88}$$

From this we conclude that the kinetic matrix $\mathcal{N}$ must transform as

$$\mathcal{N} \to \mathcal{N}' = -(C - D\mathcal{N})(A - B\mathcal{N})^{-1}. \tag{1.89}$$

Note that in order to derive the variation 1.88 we assumed that the matrix $\mathcal{N}'$ was again symmetric. This is non-trivial and indeed restricts the form of $\mathcal{S}$ we can allow. A brief computation reveals that we require

$$A^T C = C^T A, \quad D^T B = B^T D, \quad A^T D - C^T B = 1. \tag{1.90}$$

Matrices of the form above define an element of the symplectic group $Sp(2n_V + 2, \mathbb{R})$. In principle, this is the duality group for the vector fields in $n_V$ vector multiplets. There is a further restriction, however, related to the scalar fields in the vector multiplets. Indeed, unlike in the simple Maxwell theory considered above, we are not free to arbitrarily transform the kinetic matrix $\mathcal{N}$ as it depends on the scalar fields. For the new theory to be equivalent to the old we then demand that the transformation of $\mathcal{N}$ above is induced by an appropriate transformation of the scalars. In particular, we require the new theory to again have $\mathcal{N} = 2$ supersymmetry, which means that the transformed scalar action should again be of the form

$$\int K_{a\bar{b}} dz^a \wedge * d\bar{z}^{\bar{b}}. \tag{1.91}$$

These transformations are nothing but the diffeomorphisms of the scalar manifold. Hence, for the a given $\mathcal{S} \in Sp(2n_V + 2, \mathbb{R})$ to be a duality of the full theory, we require that the corresponding transformation of the kinetic functions is induced by a diffeomorphism of the scalar manifold. More precisely, we should have a homomorphism

$$\iota : \mathrm{Diff}(\mathcal{M}) \to Sp(2n_V + 2, \mathbb{R}), \tag{1.92}$$

such that for a given diffeomorphism $\xi : \mathcal{M} \to \mathcal{M}$, the vector sector transforms covariantly according to [18]

$$\begin{aligned} &z^a \to \xi(z^a), \\ &V \to \iota(\xi) V, \\ &\mathcal{N}(z^a) \to \mathcal{N}'(z^a) = \mathcal{N}(\xi(z^a)), \end{aligned} \tag{1.93}$$

with $\mathcal{N}'$ given by (1.89).

### Duality from Calabi-Yau Compactification

In the following sub-section we will show how these covariance properties are encoded in the geometry of the scalar manifold $\mathcal{M}$. In particular, we will see that the existence of certain compatible structures on $\mathcal{M}$, required by supersymmetry, will allow us to lift the action of $\xi$ to the full vector multiplet sector of $\mathcal{N} = 2$ supergravity. Before we do so however, let us first briefly discuss how the duality structure is realized in the Calabi-Yau compactification we performed in section 1.5.2. In fact, while the origin of the symplectic duality transformations is somewhat subtle in the more general supergravity setting, they become rather obvious in the Calabi-Yau setting. Indeed, recall that in section 1.5.1, we defined the fields $F^I$ and $G_I$ as the Kaluza-Klein zero-modes of the ten-dimensional field strength $\hat{F}_5$

$$\hat{F}_5 = F^I \wedge \alpha_I - G_I \wedge \beta^I + \dots. \tag{1.94}$$

The self-duality of the latter gave us a relationship between these two field strengths, which we used to obtain a description only in terms of $F^I$. Notice however, that the very definition of the fields $F^I$ and $G_I$ depends on the symplectic basis $(\alpha_I, \beta^I)$ we choose for $H^3(X, \mathbb{C})$. This choice is essentially arbitrary however, and any other symplectic basis would have given us a perfectly equivalent lower-dimensional theory. The duality structure which we deduced by studying the equations of motion then follow readily by considering how the low-energy action transforms under symplectic transformations of this basis. In particular, the non-trivial duality transformation (1.89) of the kinetic matrix $\mathcal{N}_{IJ}$ follows directly from the transformation properties of the matrix (1.56) under symplectic changes of basis.

### 1.6.3 Special Kahler Geometry

As we have seen in section 1.5.2, the scalar manifold of the scalars in the vector multiplets we obtained from Calabi-Yau compactification was a Kahler manifold. It turns out that this is a general fact about four-dimensional theories with eight supercharges, irrespective of whether we derive it from a Calabi-Yau compactification. In fact, supersymmetry places much stronger conditions on $\mathcal{M}$ than this. Indeed, it is well-known that supersymmetry requires the scalar field spaces of four-dimensional $\mathcal{N} = 2$ supersymmetric theories to be *special Kahler*. The aim of this section is to introduce this notion and show how it leads to the construction from equation 1.93. There are many equivalent definitions, each of which emphasize different aspects. The traditional definition is in terms of a so-called pre-potential. It states that the Kahler manifold $\mathcal{M}$ comes equipped with a set of homogeneous coordinates[19] $X^I(z^i)$ for $I = 0, \ldots, n_V$ and a holomorphic pre-potential $F(X^I)$, that is homogeneous of degree 2 in the $X^I$, such that the Kahler potential on $\mathcal{M}$ is locally given by

$$K(z, \bar{z}) = -\log i \left( \bar{X}^{\bar{I}} \partial_I F(X) - X^I \partial_{\bar{I}} \bar{F}(\bar{X}) \right). \tag{1.95}$$

It is a well-known fact that the existence of a pre-potential is in fact not a frame independent notion, that is, it depends on the duality frame we choose for a given theory. Instead, we choose to give a more geometric definition [20], which we will in the end relate back to the pre-potential formulation given above. Let us emphasize that the goal here is not a full top-down derivation of special Kahler geometry, but rather cast the definition in its natural geometric setting, and show how it leads to the duality structure of the vector multiplets.

Before we can define what a special Kahler manifold is we need one preliminary definition. A Kahler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a holomorphic line bundle $\mathcal{L} \to \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kahler 2-form $\mathcal{K} := \frac{i}{2\pi} \partial \bar{\partial} K$

$$c_1(\mathcal{L}) = [\mathcal{K}]. \tag{1.96}$$

The condition that the scalar manifold $\mathcal{M}$ be Hodge is easiest to understand from the perspective of $\mathcal{N} = 1$ supergravity. In this context, the significance of this bundle comes from the fact that the $\mathcal{N} = 1$ supergravity action is uniquely fixed by specifying a globally defined holomorphic section of this line bundle, which is referred to as the superpotential of the theory. While a discussion of $\mathcal{N} = 1$ supergravity is beyond the scope of this thesis, we point out that any $\mathcal{N} = 2$ supergravity theory should be consistent with all constraints arising from the $\mathcal{N} = 1$ theory as we can obtain the latter by simply ignoring half of the supercharges. Hence we should demand that our scalar manifolds are also Hodge. For our present purposes however, this property encodes the geometry of $\mathcal{M}$ (via the Kahler form $\mathcal{K}$) in the bundle $\mathcal{L}$. In particular, $\mathcal{L}$ carries a canonical connection $\partial K = \partial_a K dz^a$ whose curvature is just the Kahler form on $\mathcal{M}$ representing its first Chern class.

[19] Recall that homogeneous coordinates on a projective space of dimension $n$ are a set of $n+1$ coordinates, which are defined up to an over-all rescaling by a non-vanishing constant.

The second ingredient of special Kahler geometry is the symplectic structure required by the duality structure of the vector sector. The natural language to lift scalar diffeomorphisms to symplectic transformations is to view them as arising from transition functions of a symplectic bundle over the moduli space $\mathcal{M}$. The fact that the transition functions are constant (with respect to the moduli) means that we have a flat vector bundle. Hence we demand that the moduli space admits such a rank $2n_V + 2$, flat vector bundle $\mathcal{V} \to \mathcal{M}$. The map $\iota$ from equation 1.93 is then obtained by pulling back the frame of this bundle by $\xi$, which induces a symplectic transformation on the fiber. Moreover, we have that it is endowed with a canonical symplectic pairing which is preserved by the structure group. By analogy with equation 1.49, we denote this pairing $\langle \cdot, \cdot \rangle$.

Special geometry arises when we simultaneously consider both of these structures. To this aim, we consider the tensor product bundle $\mathcal{H} = \mathcal{V} \otimes \mathcal{L}$ whose sections can generically be written as

$$\Pi = \begin{pmatrix} X^I \\ -\mathcal{F}_I \end{pmatrix}, \quad I = 0, \dots, n_V. \tag{1.97}$$

The transition functions between two local trivializations $U_i, U_j \subset \mathcal{M}$ of $\mathcal{L}$ and $\mathcal{V}$ respectively take values in

$$e^{f_{ij}(z)} \in \mathbb{C}, \qquad \mathcal{S}_{ij}(z) = \mathcal{S}_{ij} \in Sp(2n_V + 2, \mathbb{R}), \tag{1.98}$$

where we emphasize that $\mathcal{L}$ is holomorphic, while the transition functions on the flat bundle $\mathcal{V}$ are constant. Sections of $\mathcal{H}$ then transform as

$$\Pi_i(z) = e^{f_{ij}(z,\bar{z})} \mathcal{S}_{ij} \Pi_b(z). \tag{1.99}$$

The definition of special Kahler geometry is now phrased in terms of the compatibility of these two structures.

A Hodge-Kahler manifold $\mathcal{M}$ is called special Kahler if the bundle $\mathcal{H}$ admits a globally defined holomorphic section $\Pi(z^a)$ that satisfies

$$\langle \Pi, \nabla_a \Pi \rangle = \langle \Pi, \partial_a \Pi \rangle = 0, \qquad K = -\log\left(i\langle \Pi, \bar{\Pi}\rangle\right), \tag{1.100}$$

where $\nabla_i \Pi$ denotes the covariant derivative with respect to the canonical connection on $\mathcal{L}$

$$\nabla_a \Pi := (\partial_a + \partial_a K)\Pi = \begin{pmatrix} \nabla_a X^J \\ -\nabla_a \mathcal{F}_J \end{pmatrix}, \tag{1.101}$$

and the first equality in equation 1.100 follows from the anti-symmetry of the symplectic pairing.

Let us start by presenting a few remarks

- Firstly, note that $K$ is manifestly invariant under transition functions of the symplectic bundle.
- Conversely, we have that transition functions in $\mathcal{L}$ correspond precisely to Kahler transformations of $\mathcal{M}$. Indeed, under $\Pi \to e^{-f}\Pi$, it follows that

$$K(z,\bar{z}) \to K'(z,\bar{z}) = -\log(i\langle \Pi, \bar{\Pi}\rangle e^{-f-\bar{f}}) = K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}). \tag{1.102}$$

- A number of other useful identities follow readily from equation 1.100, of which we only give the relationship of $\Pi$ to the Kahler metric on $\mathcal{M}$

$$K_{a\bar{b}} = -i\langle \nabla_a \Pi, \nabla_{\bar{b}} \bar{\Pi} \rangle e^K. \tag{1.103}$$

- While for our purposes, the section $\Pi$ will allow us to derive a manifestly covariant kinetic matrix $\mathcal{N}$, it should be noted that the special geometry, as defined above, can equivalently and more directly be obtained as a condition imposed directly on the Kahler metric and its derived curvature/Weyl tensors [16]

$$R_{a\bar{b}c\bar{d}} = K_{a\bar{b}}K_{c\bar{d}} + K_{a\bar{d}}K_{c\bar{b}} - C_{ac\bar{e}}\bar{C}_{\bar{b}\bar{d}\bar{e}}K^{e\bar{e}}. \qquad (1.104)$$

It can be shown that this equation follows from the holomorphic section $\Pi(z^a)$ and equation 1.103.

### Special Kahler Geometry from Calabi-Yau Compactification

The structures encountered above arise naturally in Calabi-Yau compactifications. We will revisit the geometry of Calabi-Yau moduli spaces in chapter 3, but we can already comment on the relevant identifications here. In particular, we shall see that the middle cohomology $H^3(X_6, \mathbb{C})$ becomes the fiber of a flat symplectic vector bundle over the moduli space which inherits its symplectic structure from the pairing (1.49). From our discussion around equation 1.67 it follows that the harmonic $(3,0)$-form $\Omega$ depends on the moduli and can be viewed as a section of this bundle. The transition functions on this bundle naturally capture the fact that $\Omega$ is defined only up to rescaling. Its span inside $H^3$ traces out a line bundle $\mathcal{L}$. When expanded with respect to a symplectic 3-form basis the components of $\Omega$ define the so-called *period vector*

$$\Pi(z) := \begin{pmatrix} \int_{A_I}\Omega \\ -\int_{B^I}\Omega \end{pmatrix} \equiv \begin{pmatrix} X^I \\ -\mathcal{F}_I \end{pmatrix}, \qquad (1.105)$$

which corresponds to the holomorphic section required by special geometry. The condition that the Chern class of $\mathcal{L}$ is represented by the Kahler form on $\mathcal{M}$ can be viewed as the definition of the latter, and one checks that this yields the metric from section 1.5.2.

### Duality from Special Kahler Geometry

As promised, the holomorphic section $\Pi$ will allow us to construct a kinetic matrix $\mathcal{N}$ that transforms according to (1.85) for $\mathcal{S} \in Sp(2n_V+2,\mathbb{R})$. To this aim, we define a $(2n_V+2)\times(n_V+1)$ matrix

$$\begin{pmatrix} \bar{X}^I & D_aX^I \\ -\bar{\mathcal{F}}_I & -D_a\mathcal{F}_I \end{pmatrix}, \qquad (1.106)$$

whose columns manifestly transform as symplectic vectors. Indeed, by virtue of the fact that $\Pi$ was a section of the symplectic bundle $\mathcal{V}$, the covariant derivatives $D_i\Pi$ are likewise sections of this bundle. If we further assume that $\begin{pmatrix}\bar{X}^I & D_aX^I\end{pmatrix}$ is invertible, we can define the gauge kinetic matrix as

$$\bar{\mathcal{N}}_{IJ} = \begin{pmatrix}\bar{\mathcal{F}}_I & D_a\mathcal{F}_I\end{pmatrix}\begin{pmatrix}\bar{X}^J & D_bX^J\end{pmatrix}^{-1}, \qquad (1.107)$$

which manifestly transforms as (1.89) under diffeomorphisms of the scalar manifold.[20] Though it is clear that (1.107) transforms appropriately, it is not at all obvious that it is a well-defined

[20] While the equation above clearly has the desired properties, its definition in terms of a frame may be unsatisfactory. For a more global perspective [21], we note that any choice of $\mathcal{N}$ is defined by a map $\mu : \mathcal{M}_{SK} \to Sp(2n_V+2,\mathbb{R})/U(n_V+1)$. In the Calabi-Yau setting, this essentially corresponds to the fact that $\mathcal{N}$ is determined by the *matrix* $-\eta\mathcal{M}$ (1.63), which we noted is $Sp(2n_V+2,\mathbb{R})$-valued. This correspondence is not unique and $\mathcal{N}$ is invariant under an appropriate $U(n_V+1)$-action on $\mathcal{M}$. Moreover, the map $\mu$ is uniquely obtained from the so-called period map, which for our purposes encodes the period vector discussed here (but see also chapter 3, or [3]).

kinetic matrix. In particular, it is not obvious that it is symmetric with negative-definite imaginary part, nor is it obvious that $\left(\bar{X}^I \quad D_a X^I\right)$ is invertible. Nevertheless, all three of these facts follow directly from equations 1.100 and 1.103 and ultimately boil down to the statement that the Kahler metric is positive-definite and symmetric [19]. Finally, although we do not show it here, one can use the Kodaira formula (1.67) and the properties of the Hodge star on Kahler manifolds to show that evaluating equation 1.107 with the period vector (1.105) leads to the same kinetic function we derived in section 1.5.2.

**Special Coordinates**

Finally, we would like to close this sub-section by discussing a special set of coordinates on the scalar manifold $\mathcal{M}$ which allow us to make contact with the more conventional pre-potential formulation mentioned earlier. Recall that the transition functions on $\mathcal{L}$ act homogeneously on $\Pi$, such that in particular $X^I \to e^{-f(z)} X^I$, where $e^{-f(z)}$ is nowhere vanishing. By analogy with the homogeneous coordinates found on projective spaces, we can then view the $X^I$ as a set of $n_V + 1$ homogeneous coordinates on the $n_V$-dimensional space $\mathcal{M}$, provided that the vielbein

$$e_i^j(z) = \partial_i (X^j / X^0), \quad i, j = 1, \ldots, n_V, \tag{1.108}$$

is invertible. In this case we can write $\mathcal{F}_J(z) = \mathcal{F}_J(X(z))$. Expanding the first part of equation 1.100 into components, it follows that

$$0 = \langle \Pi, \partial_i \Pi \rangle = X^I \partial_i \mathcal{F}_J - F_J \partial_i X^I = 0. \tag{1.109}$$

Combining these two facts, we obtain that

$$X^I \partial_I \mathcal{F}_J(z) = \mathcal{F}_J(X). \tag{1.110}$$

The most general solution to this equation is that $\mathcal{F}_J(X)$ is the derivative of some homogeneous function of degree 2, i.e.

$$\mathcal{F}_J(X) \equiv \partial_J \mathcal{F}(X). \tag{1.111}$$

It follows that given this function, the Kahler potential is given by

$$K = -\log i \left( \bar{X}^I \partial_I \mathcal{F}(X) - X^I \partial_{\bar{I}} \bar{\mathcal{F}}(\bar{X}) \right), \tag{1.112}$$

as promised in the introduction to this section. It can then be shown that equation (1.107) can be expressed in terms of the pre-potential as

$$\mathcal{N}_{IJ} = \bar{\mathcal{F}}_{IJ} + 2i \frac{\mathrm{Im}\mathcal{F}_{IK} X^K \mathrm{Im}\mathcal{F}_{JL} X^L}{\mathrm{Im}\mathcal{F}_{MN} X^M X^N}. \tag{1.113}$$

### 1.6.4 Symmetries and Charges

So far we have discussed the *duality* structure of $\mathcal{N} = 2$ supergravity. This is to be distinguished from the symmetries of these theories. Indeed, while the combined transformation (1.93) preserves the equations of motion, it generically need not leave the action itself invariant. The sub-set of the transformations (1.93) that do leave the action invariant are genuine global symmetries of the theory.

To identify such symmetries, let us first consider the vector part of the action. Under symplectic transformations, we find that the action (1.83) transforms as

$$\begin{aligned} S \to S' = \int \mathrm{Re}\big[ & (A^T D - C^T B)_I{}^J F_-^I \wedge G_J^- - (A^T C)_{IJ} F_-^I \wedge F_-^J \\ & + 2(C^T B)_I^J F_-^I \wedge G_J^- - (B^T D)^{IJ} G_I^- \wedge G_J^- \big]. \end{aligned} \tag{1.114}$$

The first term above reduces to the original action by virtue of the fact that $A^T D - C^T B = 1$. The fate of the second term depends on whether one considers a classical theory or a quantum theory. In case of the former, we simply have to note that $\mathrm{Re} F_-^I \wedge F_-^J = \frac{1}{2} F^I \wedge F^J$, which is a total derivative and can therefore be discarded (ignoring possible boundary effects). Quantum mechanically, the situation is more subtle and we will revisit it shortly when we consider the role of charges.

As for the other two terms however, they in general vanish only if $B = 0$. Moreover, while every symplectic matrix with $B = 0$ leaves the vector action invariant, the scalar diffeomorphism that induces it need not leave the scalar action invariant. In particular, we find that in order to leave the scalar sigma-model invariant, we require that this diffeomorphism is in fact a isometry of the Kahler metric. Based on this, we can identify the classical global symmetry group as a sub-group of the maximal duality-group $Sp(2n, \mathbb{R})$ defined by the conditions that $B = 0$ and that it is generated by an isometry of the scalar manifold $\mathcal{M}$, i.e.

$$Sp(2n_V + 2, \mathbb{R})|_{B=0} \cap \iota(\mathrm{Iso}(\mathcal{M})) \subset Sp(2n_V + 2, \mathbb{R}), \tag{1.115}$$

with $\iota$ the homomorphism (1.93).

### Charges

As mentioned, the question of which dualities survive to the full quantum theory is more subtle. The problem lies in the combination of Dirac quantization and the path integral. While we will revisit this point in later chapters, we can already introduce the main idea. In its simplest form, Dirac quantization states that in a theory with both electric and magnetic charges, these charges should satisfy the so-called Dirac quantization condition. In particular, for a single $U(1)$ gauge field, with electric and magnetic charges $n^e$ and $n_m$, we have that

$$n^e n_m = 2\pi\mathbb{Z}. \tag{1.116}$$

More generally, these charges are measured by the flux integrals

$$\int_{S^2} \begin{pmatrix} F^I \\ G_I \end{pmatrix} = \begin{pmatrix} n_m^I \\ n_I^e \end{pmatrix}, \tag{1.117}$$

whose quantization is preserved under symplectic transformations with integral entries. If we wish for the dualities discussed above to be dualities, or even symmetries of the full theory, we therefore find that the discussion of the preceding sections should really be restricted to the discrete sub-group $Sp(2n_V + 2, \mathbb{Z}) \subset Sp(2n_V + 2, \mathbb{R})$.

# Chapter 2

# Generalized Global Symmetries and the Swampland

In our endeavor to connect string theory to phenomenologically relevant four-dimensional physics we were able to catch a glimpse of the vast landscape of string theory vacua. Indeed, despite string theory being essentially unique in the UV, the journey down to the IR introduces an enormous amount of freedom for low-energy model building. The latter typically involves an expansion around a consistent background, but as we have seen such a choice is far from unique.

While the choice for a Calabi-Yau background constrains the space of possibilities, there remain a huge number of different Calabi-Yau manifolds. Moreover, we saw that any particular choice of Calabi-Yau manifold typically introduces a large number of moduli fields, i.e. massless scalar fields. Physically, such fields would lead to new long-range forces which have not been observed so that any realistic string theory model should include a mechanism which stabilizes these moduli. This is typically done by turning on non-trivial background fluxes for the other fields in the theory, which generically induces a potential for the moduli fields. For a particular choice of Calabi-Yau one is then left with a discrete set of so-called "flux vacua" and estimates for the number of such vacua for a typical Calabi-Yau compactification run into the order of $10^{500}$ [22]. These along with other observations were some of the original motivations for string theorists to embark on the swampland program. In this chapter we will begin to explore concretely some of these ideas, by giving an overview of some of the most important swampland conjectures.

The goal of this chapter will be two-fold. The first will be to develop some familiarity with the most important swampland ideas, which will be the topic of section 2.1. The second is to give a reasonably self-contained introduction to generalized global symmetries. Our focus will be rather narrow in this respect, focusing in particular on so-called higher-form global symmetries in abelian gauge theories. These will be most relevant to us, but also provide a relatively simple setting in which to illustrate the basic ideas. Starting in section 2.2, we will first review ordinary global symmetries in quantum field theory from a perspective more amenable to generalization. In section 2.3, we will then introduce higher-form global symmetries by means of various (suggestive) examples which will serve to illustrate the basic ideas. In section 2.4, we discuss further generalizations to this theme, though not at the same level of detail as before. Finally, in section 2.5, we will revisit the swampland and discuss how generalized global symmetries have made an impact in this area, eventually setting up for our own application of these ideas in the type IIB EFTs introduced in the previous chapter.

## 2.1 A Tour of the Swampland

Up until now we have spoken of low-energy theories rather loosely, but to fully understand what the swampland program is about, we should make this idea more precise. As discussed, the modern view of quantum field theories is as effective field theories (EFTs) for a more fundamental UV theory. In particular, a quantum field theory is not complete without specifying a so-called cut-off scale $\Lambda_{\text{EFT}}$, up to which the theory remains valid. It follows that two theories with different cut-offs, but which are otherwise indistinguishable (same fields, coupling constants etc.), are to be viewed as distinct. Of course, this is familiar even in the more traditional view, as the renormalization group tells us that coupling constants depend on the energy scale (e.g. the cut-off).

This "bottom-up" view of EFTs is to be distinguished from the "top-down" perspective. Here one starts with a more fundamental UV-complete theory and successively integrates out modes to lower the cut-off. In this case it is clear that if we increase the cut-off, we somehow need to integrate these modes back in to remain consistent with the UV theory. This is precisely the perspective we adopted in chapter 1, where we derived a low-energy theory starting from string theory. Over the years, people have noticed that all effective theories derived in this way satisfy certain commonalities which suggests that despite the vastness of the string landscape, not every consistent quantum field theory can be obtained in this way.

These considerations lead us to make the following distinction

The **Swampland** is the set of apparently consistent EFTs that cannot be embedded in a consistent theory of quantum gravity. The **Landscape** are those that can.

The goal of the swampland program is to formulate a set of principles that allow us to distinguish whether an effective theory belongs to the swampland or the landscape. These are formulated in so-called swampland conjectures. Such conjectures typically involve, either implicitly or explicitly, the introduction of an additional energy scale $\Lambda_{swamp}$, at which point consistency with quantum gravity requires some modification of the theory. As we move towards the UV and the EFT cut-off approaches and surpasses the scale $\Lambda_{swamp}$, one finds that swampland conjectures become increasingly restrictive. This is illustrated in the figure 2.1. It is important to emphasize the role of gravity here. In particular, swampland constraints should disappear as the Planck mass $M_{pl}$ (i.e. the energy scale of gravity) is sent to infinity. For a typical swampland conjecture we then have that $\Lambda_{swamp} \to \infty$ as $M_{pl} \to \infty$. This means that without coupling to gravity, swampland constraints need not apply.

In the rest of this section we will encounter some of the most important swampland conjectures. The goal is not just to familiarize the reader with the conjectures that are out there, but also to highlight how the conjectures fit together into an interconnected web.

### 2.1.1 No Global Symmetries

Perhaps the most well-established swampland principle is the No Global Symmetries (NGS) conjecture, which states that

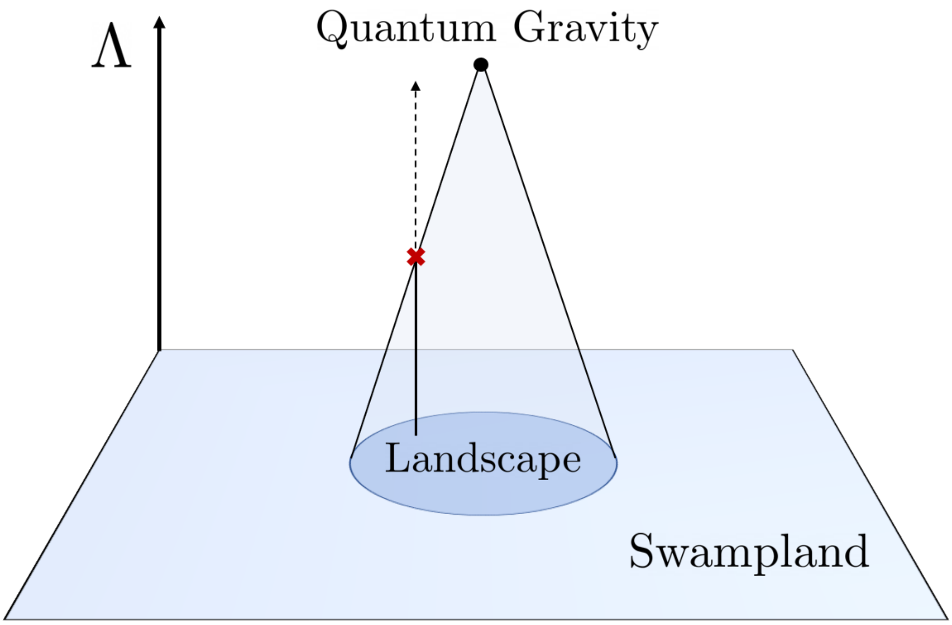



Figure 2.1: Cartoon of the landscape, sitting inside the swampland. While a theory with a given cut-off $\Lambda_0$ may lie in the landscape, we cannot arbitrarily increase this cut-off without modifying the theory. In this way, swampland constraints become stronger as we increase the cut-off.

**No Global Symmetries Conjecture:** A consistent theory of quantum gravity cannot have exact global symmetries.

It arguably predates much of the swampland program and simply cements the primacy of gauge symmetries in nature. However, also from a swampland perspective do there exist compelling arguments for why global symmetries cannot occur in a quantum theory of gravity. These arguments are bolstered by the fact that they can be made without committing to any particular UV completion, that is, independent of string theory. Indeed, one can see the problem with exact global symmetries making only a few assumptions about semi-classical phenomenology of any given theory of QG. The standard argument proceeds as follows [23].

Let us assume that there exists some exact continuous global symmetry. The first part of the argument aims to show that any such global symmetry can lead to so-called remnants, stable left-overs of evaporated black holes. Indeed, given a global symmetry, we can construct a black hole carrying some non-zero charge under the global symmetry by colliding charged particles. Once formed, the black hole will emit Hawking radiation, thereby losing mass. Hawking's calculation shows that it will decay to some remnant black hole of $\mathcal{O}(M_{pl})$ mass without losing its global charge (more precisely, the radiation will on average be neutral). We can construct remnants in arbitrarily many representations of the symmetry group by colliding different numbers of charged particles, thus leading to an infinite number of distinct stable charged states, all of mass below the Planck mass.

The second part of the argument seeks to show why this is a problem. Of course, having an infinite number of states below the Planck mass sounds problematic, and indeed may reasonably be assumed to be inconsistent, but showing that something goes wrong is rather non-trivial. One of the strongest arguments is based on entropy considerations, and in fact it is possible to phrase the entire argument in terms of black hole entropy. Indeed, the no hair theorem states that a black hole can be completely characterized in terms of its mass, angular momentum and *gauge* charge. Notably, this does not include its global charge, so that an outside observer has an infinite degree of uncertainty with respect to the microstate of the black hole. Hence we may associate an infinite entropy to the black hole which contradicts various entropy bounds, most notably the Bekenstein-Hawking formula [24]

$$S_{BH} = \frac{\text{Area}}{4L_P^2}, \tag{2.1}$$

which states that the entropy should be proportional to the horizon area (i.e. finite). The infinite entropy of the charged black holes can also be phrased in terms of the infinite remnants, but the remnant argument loses some credibility as the final stages of evaporation are likely to be strongly quantum gravitational, and hence not amenable to semi-classical reasoning. This is a generic problem with black hole arguments, but entropy bounds provide the most credible probe of this sort of trans-Planckian physics.

Importantly, the argument above holds only for continuous global symmetries where the machinery of Noether's theorem applies directly. Motivating an analogous statement about discrete symmetries has proven more difficult. Strikingly however, it has been observed in all manner of string theory constructions that any apparent discrete global symmetries are in fact gauge redundancies. Moreover, the absence of discrete global symmetries is attractive also because it allows one to connect various swampland conjectures, which is a theme we will return to throughout this chapter.

While the NGS conjecture may appear to be a strong condition on consistent QGs, a moments thought reveals that its IR implications are rather weak. Indeed, global symmetries are perfectly fine at low energies, as long as they are either approximate and somehow broken in the UV. One therefore requires full knowledge about the UV theory for one to be able to make statements about its consistency. Many swampland conjectures can be interpreted as a way of extending the NGS conjecture to a statement that applies directly to the IR, hence emphasizing a bottom-up perspective of swampland constraints.

### 2.1.2 Completeness Hypothesis

The second pillar of the swampland program is the completeness hypothesis. In contrast with the NGS conjecture, it is a statement about gauge symmetries. Its formulation goes back at least to Polchinski [25] and is in part motivated by the existence of D-branes in string theory. Its exact statement is

> **Completeness Hypothesis:** A theory with a gauge symmetry, coupled to gravity, must have states of all possible charges (consistent with Dirac quantization) under the gauge symmetry.

We wish to emphasize that the fact that one can probe gauge charge from outside a black hole means that there is no tension with the remnant argument presented above.

From a top-down perspective, with the realization that D-branes carry R-R charge came the observation that they in fact complete the spectrum of charges for these gauge fields in precisely the way the completeness hypothesis predicts. Moreover, this is a feature preserved by compactification. Indeed, as we have seen, compactification of $p$-form gauge fields leads to lower-degree gauge fields, which couple to branes wrapped on cycles[1]. Crucially, this conjecture includes gauge symmetries associated to higher-form gauge fields, which will be an important observation once we come to consider generalized global symmetries in section 2.3.

With the completeness and NGS conjectures in hand we can illustrate how the swampland conjectures interweave and combine to give *new* statements. Indeed, when we combine the two conjectures, we can obtain the compactness conjecture, which states

**Compactness Conjecture:** All continuous gauge groups must be compact.

For instance, applied to the abelian case, it states that the gauge group can be $U(1)$ but *not* $\mathbb{R}$ (viewed as an additive group). The distinction between these may be somewhat unfamiliar. The key point is that writing down the action for the gauge field only requires us to specify the *Lie algebra* of the gauge group. Since $U(1)$ and $\mathbb{R}$ have isomorphic Lie algebras, one would then expect them to be described by the same action. It turns out however, that specifying the action is not enough and a prescription for a gauge theory is only complete once we specify the allowed charges[2]. For $U(1)$ gauge theory (and compact theories more generally) it follows that the charges are all quantized. For non-compact groups such as $\mathbb{R}$ however, we can consider arbitrary irrational electric charges. Since there is no way for an irrationally charged particle to decay into rationally charged ones, we find that a theory with gauge group $\mathbb{R}$ exhibits a super-selection rule on the spectrum, which is characteristic of a global symmetry. By the NGS conjecture, this is forbidden.

The argument above suggests another, somewhat more heuristic argument in favour of this conjecture, which is to wonder how one could arrive at an *incomplete* spectrum. In the QFT context, we can always take the masses of particles of a given charge to infinity, decoupling them from the theory and leaving us with an incomplete, but consistent, spectrum. In quantum gravity, this procedure will necessarily produce a black hole, which, because the theory contains gravity, remains part of the spectrum. Of course, we may imagine a theory where such charges were never present to begin with. However, by the argument above, this amounts to a modification of the definition of the gauge group. In particular, a $U(1)$ gauge theory *is defined by* the existence of observables called *Wilson lines*

$$W_n(\gamma^1) = \exp\left(in \int_{\gamma^1} A_1\right), \tag{2.2}$$

which, as we discuss in section 2.3.1, can be interpreted as infinitely massive particles that probe the response of the gauge theory. In quantum gravity, these probe particles also automatically probe gravity through the integral $\sim \int \sqrt{-g}$, so that their inclusion (which is likewise automatic!) will lead to the existence of charged (finitely massive) black hole states which carry the "missing" charge, even if we neglected to include such a charge as a field in the theory.

The relationship between the completeness hypothesis and the absence of global symmetries goes much further than this however. Indeed, as we will explore in section 2.5.1, it

[1] While absent in the case of Calabi-Yau compactification, one might wonder what happens to e.g. gauge fields deriving from off-diagonal metric components. Here, higher-dimensional gravitational configurations exist [26] that appear as singular monopole solutions upon compactification. These so-called Kaluza-Klein monopoles admit an interpretation as charged solitons which fill out the spectrum of charges.

[2] The interested reader may wish to consult the discussion in [23] for a more detailed review of this rather technical fact and its swampland consequences, discussed in the next paragraph. We will briefly revisit it in section 2.4.2.

can sometimes be argued that the former follows from the latter. However, to understand this, we should better understand the notion of a generalized global symmetry, which will be developed in section 2.3.

While almost as well-established as the NGS conjecture, the completeness hypothesis need also not lead to low-energy bounds on the theory as it makes no statement about the masses of the charged states. These could in principle be arbitrarily heavy, in which case one would need to have a description of the full QG theory to check its validity. To actually constrain the spectrum of an EFT, one requires a non-trivial bound on the masses of these charges. Indeed, this is exactly what the next conjecture does for us.

### 2.1.3 Weak Gravity Conjecture

The weak gravity conjecture [27] is our first example of a conjecture that serves a bottom-up purpose. While the previous two conjectures are in a sense more so principles of quantum gravity, this conjecture is our first true swampland conjecture in the sense that it seeks to distinguish the (IR) swampland from the landscape. In particular, it imposes a bound on the mass of some of the states predicted by the completeness hypothesis. However, as is to be expected, this is intimately related to the NGS conjecture. Indeed, consider a four-dimensional $U(1)$ gauge theory, coupled to gravity, given by

$$S = \int_X \left( \frac{M_p^2}{2} R * 1 - \frac{1}{2e^2} F_2 \wedge * F_2 + \ldots \right), \tag{2.3}$$

where we have made the Planck mass explicit. We implicitly fix the normalization of the gauge fields such that their coupling to charges is controlled by integral charges (e.g. the covariant derivative of a charged scalar is given by $d + iqA_1$ for $q \in \mathbb{Z}$). Let us now consider what happens as we take the gauge coupling $e$ to zero. Recall that we argued that the completeness conjecture is consistent with the black hole remnant argument by virtue of the fact that we can measure the gauge charge of a black hole from outside its horizon. This ceases to be the case when we take $e$ to zero as the flux lines become weaker and weaker, eventually vanishing completely. In this case, the argument against global symmetries kicks in which tells us that this limit should be forbidden.

This behaviour can in fact explicitly be understood as the emergence of a global symmetry in the limit $e \to 0$, either as an un-gauging of the global symmetry the gauge fields once gauged, or in terms of higher-form global symmetries, to be discussed in later sections. These limits should therefore be forbidden, which is precisely what the second part of the weak gravity conjecture states. With this background in mind, let us now give the statement of the WGC, quoted from [28]

**Weak Gravity Conjecture:** Consider a theory, coupled to gravity, with a U(1) gauge symmetry with gauge coupling $e$.

- (Electric WGC) There exists a particle in the theory with mass $m$ and charge $q$ satisfying the inequality
$$m \lesssim eqM_p. \tag{2.4}$$
- (Magnetic WGC) The cutoff scale $\Lambda$ of the effective theory is bounded from above approximately by the gauge coupling
$$\Lambda \lesssim eM_p. \tag{2.5}$$

Let us start by discussing the second part. Indeed, in light of our previous discussion, it is the magnetic version that protects us in the limit of $e \to 0$, as here the cut-off goes to zero and the EFT description breaks down. Alternatively, it is a precise bound on just how approximate we can make a global symmetry at a given cut-off $\Lambda$. As a microscopic motivation for why the cut-off should appear in this expression, we present the following non-rigorous argument, which emphasizes some points about EFTs and dualities which we feel complement the theme of this thesis. To this aim, let us assume the electric WGC and dualize to a magnetic frame. Then the electric WGC becomes a statement about the masses $m_{mag}$ of the magnetic charges in the original theory, which should therefore satisfy

$$m_{mag} \lesssim M_p \, q/e, \tag{2.6}$$

where we have used the dualization relation $e \to 1/e$. By completeness of the spectrum we may assume that $q = 1$. The electromagnetic field of a point-like monopole is divergent at the location of the monopole. The EFT cut-off $\Lambda$ then shields us from this region, which is likely to be described by some UV-complete theory, so that the effective radius of the monopole is set by $R \sim \Lambda^{-1}$. Computing the energy stored in the field outside this radius, one obtains

$$m_{mag} \sim \frac{\Lambda}{e^2}, \tag{2.7}$$

which upon insertion in (2.6) leads to the magnetic WGC.

As for the electric part of this statement, it can most readily be understood microscopically in terms of extremal black holes. In particular, charged black holes generically admit two event horizons. The extremality bound tells us when these horizons coincide

$$M_{BH} \geq eqM_p, \tag{2.8}$$

where $M_{BH}$ is the mass of the black hole and $q$ its charge. Black holes which violate this bound have naked singularities, so that the electric WGC follows immediately from the cosmic censorship hypothesis which forbids these.

The weak gravity conjecture, as stated above applies only to the very limited context of a single $U(1)$ gauge field. It admits many generalizations and refinements which attempt to extend these observations to more complicated setups. One which is of particular interest to us is the generalization applicable to the gauge fields encountered in the vector sector of $\mathcal{N} = 2$ supergravity. To state it, let us first generalize the notion of the physical charge of a particle. A particle that couples to multiple gauge fields as e.g. the D3 particles introduced in section 1.5.2 should satisfy the weak gravity conjecture with respect to its combined charge under the gauge fields [29]. To make this precise, let us consider a particle with charge vector $\mathbf{q} = (p^I, -q_I)^T$ as in section 1.5.2. To construct a frame-independent expression for the strength of its coupling to the gauge fields, let us introduce the following matrix

$$\mathcal{M} = \begin{pmatrix} -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -(\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \tag{2.9}$$

Which we recall, corresponds to the Hodge star matrix when it is obtained from a Calabi-Yau compactification. The physical charge of the particle is then defined as

$$\mathcal{Q}_{\mathbf{q}}^2 := \frac{1}{2}\mathbf{q}^T \mathcal{M}\mathbf{q}. \tag{2.10}$$

The weak gravity conjecture in this case then states that [29]

**WGC for Multiple Gauge Fields:** There should exist a particle with mass $m$ and physical charge $\mathcal{Q}$, such that

$$\mathcal{Q}^2 M_p^2 \geq m^2. \tag{2.11}$$

### 2.1.4 Swampland Distance Conjecture

The swampland distance conjecture [30] is of a somewhat different flavour from the ones discussed so far. In particular, it is a statement about the moduli spaces one can encounter in any EFT obtained from quantum gravity. Nevertheless, as we shall see it can be understood as a statement about the *way* in which global symmetries are avoided in quantum gravitational EFTs. The precise statement, quoted from [28], is the following.

**Swampland Distance Conjecture:** Consider a theory, coupled to gravity, with a (physical) moduli space $\mathcal{M}$ which is parameterized by the expectation values of some field $z^i$ which have no potential.

- Starting from any point $p \in \mathcal{M}$ there exists another point $q_\infty \in \mathcal{M}$ such that the geodesic distance between $p$ and $q_\infty$, denoted $d(p, q_\infty)$, is infinite.
- There exists an infinite tower of states, with an associated mass scale $M$, that becomes exponentially light at any infinite distance limit in field space

$$M(q) \sim M(p) e^{-\alpha d(p,q)}, \qquad \text{as } d(p,q) \to \infty \tag{2.12}$$

where $\alpha$ is some positive constant.

The first part of this conjecture can be thought of as the statement that the moduli space is always non-compact, i.e. there always exist infinite distance limits. The second part describes what happens as we approach such limits. In particular, it predicts an eventual breakdown of the EFT as we can never describe an infinite number of massless particles within an EFT[3]. This prediction should again be understood as arising strictly from quantum gravitational consistency. Indeed, from a quantum field theory perspective, there is nothing wrong with infinite distance limits, which although unreachable certainly need not necessarily lead to a breakdown of the EFT as they are approached.

Of course, we have already encountered a manifestation of this conjecture in the context of Kaluza-Klein compactification in section 1.3.2. There we discovered a modulus $\phi$ parameterizing the compactification radius. We found that its field space metric was given by $\frac{1}{3}\phi^{-2}$ so that the geodesic distance from the reference point $\phi = 1$ to the point $\phi = \infty$, computed by

$$d(1, \infty) = \int_1^\infty \sqrt{\frac{1}{3}\phi^{-2}}\, d\phi = \frac{1}{\sqrt{3}} \log(\infty) \tag{2.13}$$

is infinite[4], hence satisfying the first part of the conjecture. Moreover, recall that the mass of the KK modes was inversely proportional to the radius-squared, which is given by $R_0^2 \phi^{2/3}$.

[3] This intuition is made precise by the species bound [31], which we do not discuss here. Rather we take for granted the fact that the SDC predicts a breakdown of the EFT.

[4] As a word of caution, we remark that only in this simple one-dimensional moduli space are we able to readily identify the relevant geodesic to show that the distance to $\phi = \infty$ is infinite along any path. In general, identifying points of infinite distance is much more difficult as it is essentially a statement about *all* paths to a given point.

Thus we have that for the mass-scale of the Kaluza-Klein tower

$$M_{KK}(\phi) \sim \frac{1}{R^2(\phi)} \sim \frac{1}{R^2(1)\phi^{2/3}} \sim M_{KK}(1)\phi^{-2/3} \sim M_{KK}(1)e^{-\frac{2}{\sqrt{3}}d(1,\phi)}, \tag{2.14}$$

where we see that the exponential scaling emerges due to the non-trivial field space metric $\frac{1}{3}\phi^{-2}$.

The computation above also reveals a second point of infinite distance, namely that at $\phi = 0$, corresponding to a vanishing compactification radius. In this limit, the Kaluza-Klein states become infinitely massive. This illustrates how the distance conjecture is really a statement about quantum gravity. It is clear that from a quantum field theory perspective, there is no reason to suspect any sort of EFT breakdown as we approach $\phi \to 0$, rather we simply approach a strictly four-dimensional theory. However, had we obtained this theory from string theory we would have encountered a second tower of massless modes, which are precisely the winding modes discussed in section 1.3.4. In fact, for a simple circle compactification it is not hard to show that the resulting modes in fact become exponentially light in the field distance, as predicted by equation 2.12. Indeed, this is the deeper evidence in favour of the SDC as framed above.

As promised, we now illustrate how the SDC can be viewed as a way of protecting a QG-consistent EFT from the emergence of global symmetries. Indeed, consider the behaviour of the scalar kinetic term

$$\int \left(-\frac{1}{6}\phi^{-2}d\phi \wedge *d\phi\right), \tag{2.15}$$

as we approach the limit $\phi \to \infty$. Were it not for the non-trivial field-space metric, this theory would have a shift-symmetry $\phi \to \phi + c$ for constant $c$. However, the factor of $\phi^{-2}$ explicitly breaks this symmetry. Nevertheless, as we take the background value $\phi_0$ to be larger and larger, the metric approaches a constant and becomes insensitive to small variations around $\phi_0$

$$\frac{1}{(\phi_0 + \delta\phi)^2} \sim \phi_0^{-2} - \mathcal{O}(\phi_0^{-3}). \tag{2.16}$$

We see that the theory develops an approximate global shift symmetry which becomes exact in the limit $\phi \to \infty$. The SDC prevents this by shielding us from such regions of field space. In fact, there is a second manifestation of this effect in the Kaluza-Klein model. Although we did not discuss it in much detail, we saw that the Kaluza-Klein gauge field had a kinetic term of the form

$$\int \left(-\frac{1}{2}\phi F_2 \wedge *F_2\right). \tag{2.17}$$

By comparison to the standard form we therefore identify the gauge coupling $e^2 = \phi^{-1}$ which we observe is now moduli-dependent. In particular, the limit $\phi \to \infty$ corresponds to the weak coupling limit, which by our discussion of the WGC is similarly forbidden. In this case, the SDC gives us the particular mechanism by which the WGC is satisfied.

### 2.1.5 The Swampland: Outlook

While this concludes our first tour of the swampland, the conjectures discussed here only scratch the surface of the swampland program. In particular we have emphasized conjectures which we will most directly encounter in later sections. Necessarily omitted were whole classes of conjectures that e.g. generalize and refine the conjectures presented here, but also those that make statements about the properties of individual string vacua. In section 2.5, we will revisit the swampland, armed with the notion of generalized global symmetries which we develop in the following sections. While we will expand somewhat our discussion from the previous section, we refer the interested reader to the excellent review articles in [28] and [32].

## 2.2 Global Symmetries

In the following sections we will begin to explore various generalized notions of global symmetries, most important of which are the so-called higher-form global symmetries. In fact, we have already encountered a clue that such symmetries should exist: the $p$-form gauge fields in supergravity enjoy higher-form *gauge* symmetries, whose gauge parameters were $(p-1)$-forms. It would seem only natural then that a global analog of these should exist as well. Nevertheless, despite higher-form gauge symmetries being part and parcel of string theory and gauge theory more generally, our understanding of higher-form *global* symmetries took far longer to mature [2]. At least part of the reason for this was likewise hinted at by the construction of higher-form gauge symmetries. Indeed, while the objects charged under ordinary global symmetries are the familiar point particles, whose one-dimensional world-lines act as sources for ordinary gauge fields

$$q \int_{\mathcal{C}^{(1)}} A_1, \tag{2.18}$$

the natural sources for higher-form gauge symmetries are the strings and branes of string theory (cf. equation 1.5). In the latter case the $p$-form gauge fields provide us with a proxy to study the gauged version of these symmetries. However, when discussing the global case we do not have this crutch and we should look for a more direct representation of such symmetries on the objects in question[5]. This is precisely what quantum field theory does for us: particles are the quanta of local quantum fields, which in turn carry the mathematical representation of the symmetry group. The analog for higher-form global symmetries would be a field theory on the space of e.g. loops, whose fundamental fields create strings in the theory. While there have been attempts at constructing such theories, most notably in the context of string field theory (but see also [33]), our understanding of them remains unsatisfactory for developing these ideas.

Fortunately for us however, there is another way of talking about extended objects that does not involve promoting them to the fundamental quanta of our theory. Instead, sticking with the framework of local quantum field theory, we can construct such objects as composite operators out of the usual local ones. If these objects are to be the true degrees of freedom of our theory, we should expect some redundancy in our description of them in terms of local quantum fields. This should smell an awful lot like gauge theory and indeed, gauge theory will be our most important tool to study generalized global symmetries. In particular, the canonical example of extended operators will be the Wilson lines in $U(1)$ (or more general) gauge theories, which are the gauge invariant (i.e. physical) degrees of freedom built out of the local gauge field $A_1$, but we will see more examples below.

In order to discuss generalizations of global symmetries, we should begin by formulating exactly what we mean by ordinary global symmetries in quantum field theory. The modern perspective on global symmetries is phrased in terms of the existence of certain topological

[5]Even when a global symmetry is not gauged it can still be instructive to study the theory coupled to *background* gauge fields for that symmetry. This is an alternative approach to studying higher-form gauge symmetries which we also briefly discuss later.

operators. The goal of this sub-section is to introduce this perspective starting from the more familiar notions of symmetries in both classical and quantum mechanics. The material in this sub-section in particular will be largely standard, but we present it here in hopes that it will provide useful context to make the ideas presented later more intuitive.

### 2.2.1 Classical Symmetries

Let us start by recalling some basic features of global symmetries in classical field theory. Of course, one of the hallmarks of continuous global symmetries in ordinary field theory is the existence of an associated conserved current. Given a (finite dimensional) connected Lie group $G$ with lie algebra $\mathfrak{g}$, the classical Noether's theorem gives us a recipe to identify such currents by considering the infinitesimal action on the fields in the action, schematically denoted $\Phi_i$ with $i$ enumerating the fields

$$\Phi_i \to \Phi_i' = \Phi_i + \alpha^a \delta_a \Phi_i, \quad a = 1, \ldots, \dim \mathfrak{g}. \tag{2.19}$$

The fact that this defines a classical symmetry means that it leaves the action invariant up to a possible total derivative term (which we ignore). If we now allow for spacetime varying Lie algebra parameters $\alpha^a(x)$, the transformation (2.19) will generically no longer be a symmetry of the action. Nevertheless, the induced variation should vanish once we restrict to constant $\alpha^a$ so that to leading order, this variation will be proportional to $\partial_\mu \alpha^a(x)$. Hence it is always possible to cast the corresponding variation of the action in the following form

$$S[\Phi_i] \to S[\Phi_i] - \int d^d x\, J^\mu_a(\Phi_i(x), \partial_\mu \Phi_i(x), \ldots)\, \partial_\mu \alpha^a(x), \tag{2.20}$$

for some local function $J^\mu_a$ of the fields. Finally, we note that any on-shell field configuration is a stationary point of the action, so that the action is in fact invariant under *any* infinitesimal variation of the fields and we conclude that when evaluated on-shell

$$\partial_\mu J^\mu_a = 0. \tag{2.21}$$

Associated to this conserved current is a conserved charge which is usually given in terms of an integral over a constant-time slice of the time-like component of this current

$$Q_a(t) = \int d^{d-1} x J^0_a(x, t), \tag{2.22}$$

which may generically be time-dependent. Its conservation follows immediately from the conservation law of the current via

$$\partial_0 Q_a = \int d^{d-1}x\, \partial_0 J^0_a = \int d^{d-1}x\, \partial_i J^i_a = \int d^{d-2}x\, n_i J^i_a = 0, \tag{2.23}$$

where in the last step we used Stokes' theorem and assumed that the fields vanish at infinity.

As usual when working in curved spacetime, these results are most naturally expressed in terms of the covariant differential form notation employed in the previous chapter. In this case, the conserved current $J^\mu_a$ now defines the components of a Lie algebra-valued 1-form $*J_{d-1}$ (we suppress the Lie-algebra indices in the following). The conservation law (2.21) is equivalent to the statement that this form is closed, i.e. it obeys

$$dJ_{d-1} = 0, \tag{2.24}$$

which is now true also for curved spacetime. To define the conserved charge we performed an integral over a constant-time slice of spacetime. It is clear that the natural generalization is to integrate the $(d-1)$-form $J_{d-1}$ over a $(d-1)$-dimensional sub-manifold. Hence we can define the (Lie algebra-valued) conserved charge as

$$Q(\Sigma^{d-1}) = \int_{\Sigma^{d-1}} J_{d-1}. \tag{2.25}$$

We see that rather than being time-dependent, the conserved charge is associated to the sub-manifold $\Sigma^{d-1}$. If we choose $\Sigma^{d-1}$ to be a Cauchy surface[6], we can always choose a time direction such that we recover our old expression in some appropriate set of coordinates on spacetime. More naturally however, we can also express charge conservation directly in terms of the dependence of $Q$ on $\Sigma^{d-1}$. Indeed, $Q$ is invariant under continuous deformations of $\Sigma^{d-1}$. Given any two such surfaces $\Sigma_1^{d-1}$ and $\Sigma_2^{d-1}$ that together bound a volume $\Sigma^d$, we compute

$$Q(\Sigma_2^{d-1}) - Q(\Sigma_1^{d-1}) = \int_{\Sigma_2^{d-1}} J_{d-1} - \int_{\Sigma_1^{d-1}} J_{d-1} = \int_{\partial\Sigma^d} *J_1 = \int_{\Sigma^d} dJ_{d-1} = 0. \tag{2.26}$$

### 2.2.2 Quantum Symmetries

The path integral versions of these statements are captured by Ward-Takahashi identities, which we quickly review here. In the Euclidean path integral formalism, a continuous global symmetry can be defined as a transformation of the fields as in (2.19), such that

$$\mathcal{D}\Phi_i e^{-S[\Phi_i]} = \mathcal{D}\Phi_i' e^{-S[\Phi_i']}, \tag{2.27}$$

where we now also have to consider the transformation of the functional measure. For symmetries which act linearly the latter transformation is trivial, but this generally need not be the case. To derive the quantum analog of equation (2.24), we now consider how the path integral transforms under (2.19) if we again allow for spacetime varying Lie algebra parameters $\alpha$. In general, neither the measure nor the action will be invariant, but the resulting variation is again expected to be proportional to $d\alpha$

$$\begin{aligned} \mathcal{Z} \to \mathcal{Z}' &= \int \mathcal{D}\Phi_i'\, e^{-S[\Phi_i']} \\ &= \int \mathcal{D}\Phi_i\, e^{-S[\Phi_i]} \left(1 + \int_{\mathcal{M}} J_{d-1}(\Phi_i, d\Phi_i, \ldots) \wedge d\alpha\right) \\ &= \mathcal{Z}\left(1 + \int_{\mathcal{M}} \langle J_{d-1}(\Phi_i, d\Phi_i, \ldots)\rangle \wedge d\alpha\right). \end{aligned} \tag{2.28}$$

The Ward identity follows by realizing that when both measure and action are transformed appropriately, the transformation (2.19) is just a change of dummy variable in the path integral, such that $\mathcal{Z} = \mathcal{Z}'$ for any choice of parameter $\alpha(x)$. We conclude that the extra term vanishes, again for arbitrary functions $\alpha$, which upon integrating by parts leads to the identity

$$\langle dJ_{d-1}\rangle = 0. \tag{2.29}$$

Note that if the measure transforms trivially for all choices of $\alpha$, the current $J_{d-1}$ coincides exactly with that obtained in the classical case. The argument above goes through largely

[6] For our purposes, we can think of this as an appropriate generalization of a constant-time slice. Importantly, we have that $\mathcal{M}$ is diffeomorphic to $\Sigma^{d-1} \times \mathbb{R}$ for any (smooth) Cauchy surface $\Sigma^{d-1}$, which effectively restricts the spacetime geometries we consider.

unchanged if instead of the partition function we consider correlation functions of arbitrary local operator insertions

$$\mathcal{O}_i := \mathcal{O}_i(\Phi_j(x_i), \partial_\mu \Phi_j(x_i), \ldots), \tag{2.30}$$

whose dependence on the fields $\Phi_i$ may lead them to transform non-trivially under the symmetry

$$\mathcal{O}_{i_1} \to \mathcal{O}'_{i_1} = \mathcal{O}_{i_1} + \alpha(x_{i_1})\delta\mathcal{O}_{i_1}. \tag{2.31}$$

We then consider a correlation function

$$\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\ldots\rangle := \frac{1}{\mathcal{Z}}\int \mathcal{D}\Phi_i e^{-S[\Phi_i(x)]}\,\mathcal{O}_{i_1}\mathcal{O}_{i_2}\ldots, \tag{2.32}$$

which by construction are automatically time-ordered (so that for bosonic operators, everything inside the correlation function commutes). We then evaluate the variation of this correlation function as

$$\begin{aligned}\langle \mathcal{O}'_{i_1}\mathcal{O}'_{i_2}\ldots\rangle' :&= \frac{1}{\mathcal{Z}'}\int \mathcal{D}\Phi'_i e^{-S[\Phi'_i(x)]}\,\mathcal{O}'_{i_1}\mathcal{O}'_{i_2}\ldots \\ &= \left\langle \left(1+\int_{\mathcal{M}} J_{d-1}\wedge d\alpha\right)\left(\mathcal{O}_{i_1}+\alpha(x_{i_1})\delta\mathcal{O}_{i_1}\right)\left(\mathcal{O}_{i_2}+\alpha(x_{i_2})\delta\mathcal{O}_{i_2}\right)\ldots\right\rangle.\end{aligned} \tag{2.33}$$

Using again that the effected transformation is a change of dummy variable and keeping only leading order terms, we arrive at the Ward identity

$$\int_{\mathcal{M}} \alpha\,\langle dJ_{d-1}\,\mathcal{O}_{i_1}\mathcal{O}_{i_2}\ldots\rangle = \sum_n \alpha(x_{i_n})\langle\delta_a\mathcal{O}_{i_n}\mathcal{O}_{i_1}\mathcal{O}_{i_2}\ldots\rangle, \tag{2.34}$$

where we have integrated by parts[7]. If we assume that none of the other operator insertions $\mathcal{O}_i$ coincide, we can extract more detailed information by judiciously choosing our transformation parameters $\alpha(x)$ to have support on some sub-manifold (with boundary) $\Sigma^d_n \subset \mathcal{M}$ containing only the point $x_i$ corresponding to the insertion $\mathcal{O}_i$ (e.g. by taking it to be constant on $\Sigma^d_n$ and zero outside). The sum on the right-hand side then picks out only the term involving the insertions inside $\Sigma^d_n$ so that we obtain

$$\int_{\Sigma^d_n} \langle dJ_{d-1}\,\mathcal{O}_i\ldots\rangle = \langle\delta\mathcal{O}_i\ldots\rangle. \tag{2.35}$$

This equation is the path integral representation of the familiar fact that conserved charges act as the generators of the symmetry on the local operators in the theory. Indeed, using Stokes' theorem, it can be rewritten in terms of the charges (2.25)

$$\begin{aligned}\langle\delta\mathcal{O}_i\ldots\rangle &= \int_{\Sigma^d_n}\langle dJ_{d-1}\,\mathcal{O}_i\ldots\rangle \\ &= \int_{\partial\Sigma^d_n}\langle J_{d-1}\,\mathcal{O}_i\ldots\rangle \\ &=: \left\langle Q(\Sigma^{d-1}_n)\,\mathcal{O}_i\ldots\right\rangle,\end{aligned} \tag{2.36}$$

where $\Sigma^{d-1}_i \equiv \partial\Sigma^d_i$ and it is again assumed that only $x_i \in \Sigma^d_i$. The last step should be viewed as a definition of the operator $Q(\Sigma^{d-1}_i)$ in terms of how its insertion affects correlation functions. The previous derivation also makes evident the topological nature of the operators $Q(\Sigma^{d-1}_i)$. The "master equation" (2.34) is valid for any choice of $\alpha$. Hence, equation 2.35 and those that descend from it are valid for any choice of volume $\Sigma^d$, provided that we do not deform it to include any other local operator insertions present in the correlation function.

[7]Strictly speaking, the derivative acts on the whole correlation function. However, it only does so via the dependence on the insertion location of $J_1$, so that we write $dJ_{d-1}$ as a short-hand.

### Basic Example

To break up the rather dry discussion, and to highlight the physical meaning of the operator $Q(\Sigma^{d-1})$, let us consider a very simple example: a complex scalar field

$$S = \int_{\mathcal{M}} d\phi^\dagger \wedge * d\phi. \tag{2.37}$$

It enjoys a global $U(1)$ symmetry given by phase rotations $\phi \to e^{i\alpha}\phi$, whose infinitesimal action on the fields is given by

$$\phi \to \phi + i\alpha\phi, \qquad \phi^\dagger \to \phi^\dagger - i\alpha\phi^\dagger, \tag{2.38}$$

where we identify $\delta\phi = i\phi$. Evaluating the variation of the action for spacetime varying $\alpha$, we obtain after a brief computation that

$$\delta S = \int_{\mathcal{M}} J_3 \wedge d\alpha, \qquad J_3 := i * \left(\phi^\dagger d\phi - \phi\, d\phi^\dagger\right), \tag{2.39}$$

from which it follows that $\langle dJ_3 \rangle = 0$. Let us now consider the following correlation function

$$\langle \phi^q(x) \rangle := \int \mathcal{D}\phi e^{-S} \phi^q(x), \tag{2.40}$$

and apply the procedure laid out above. In particular, we perform a variation with spacetime varying $\alpha$

$$\langle (\phi^q)'(x) \rangle' = \left\langle \left(1 + \int_{\mathcal{M}} J_3 \wedge d\alpha\right) (\phi^q(t,x) + iq\alpha(x)\phi(x)) \right\rangle = \langle \phi^q(x) \rangle, \tag{2.41}$$

from which we conclude that

$$\left\langle \left( \int_{\mathcal{M}} \alpha dJ_3 \right) \phi(x) \right\rangle = iq \left\langle \alpha(x)\phi(x) \right\rangle. \tag{2.42}$$

If we now choose our $\alpha(x)$ to have support in a neighbourhood $\Sigma^4_x$ of the point $x$ and apply Stokes' theorem, we obtain

$$\left\langle \left( \int_{\partial\Sigma^4_x} J_3 \right) \phi(x) \right\rangle \equiv \langle Q(\partial\Sigma^4_x)\phi^q(x) \rangle = iq\langle \phi^q(x) \rangle, \tag{2.43}$$

where we now explicitly see the action of the charge operator on the field $\phi(x)$. If we specialize further by choosing our spacetime to be of the form $\mathcal{M} = \mathbb{R}_t \times \Sigma^3$ and choose $x = (t, y)$ and $\Sigma^4 = (-\infty, t_0] \times \Sigma^3$, we obtain

$$\langle Q(\Sigma^{d-1}_{t_0})\phi^q(t,x) \rangle = iq\langle \phi^q(t,x) \rangle, \qquad \text{for } t < t_0. \tag{2.44}$$

This equation has a nice interpretation: the operator $\phi^q(t,x)$ creates $q$ particles with unit charge at time $t$, which then propagate and cross the slice $\Sigma^{d-1}_{t_0}$ at a time $t_0$. The charge operator counts the number of world-lines that cross its volume.

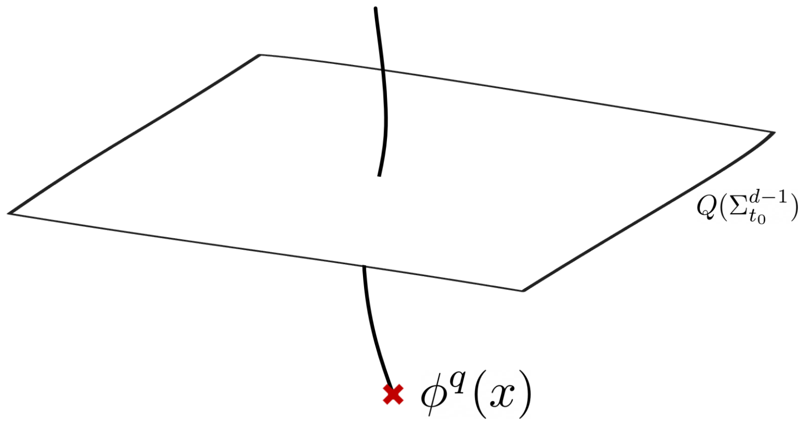



Figure 2.2: The operator $\phi^q(x)$ creates a particle of charge $q$ which then propagates and crosses the charge operator. Its insertion counts the charge passing through the surface.

### Finite Transformations

Our considerations so far have all been infinitesimal, which in the continuous case encodes the most detailed information about the symmetry. However, discrete symmetries need not have currents and charges associated to them, in which case it is useful to talk about finite transformations instead. In the continuous case, the generators $Q(\Sigma^{d-1})$ can be readily exponentiated to yield operators which effect finite transformations of the charged fields

$$U_g(\Sigma^{d-1}) := \exp(i\alpha Q(\Sigma^{d-1})), \tag{2.45}$$

where $g$ denotes the group element that corresponds to the Lie algebra parameters $\alpha$. In particular, suppose we have such an operator defined on a small sphere $S^{d-1}$ that encloses a single charged local operator inserted at a point $x$. We then have a Ward identity that tells us that it can be removed (by shrinking $S^{d-1}$)[8] at the cost of an action of $g$ in some representation $R_{\mathcal{O}_i}$

$$U_g(S^{d-1})\mathcal{O}_i(x) = R_{\mathcal{O}_i}(g)\mathcal{O}_i(x), \tag{2.46}$$

where this equation is to be understood as an operator equation valid inside correlation functions and away from any other operator insertions. A special case is when spacetime is non-compact, or has boundaries[9]. For instance, for a spacetime of the form $\Sigma^{d-1}\times\mathbb{R}$ with $\Sigma^{d-1}$ either compact or non-compact, our operators can be evaluated for any constant-time slice $\Sigma^{d-1}_t$. In this case, the Ward identities encode equal-time commutation relations

$$U_g(\Sigma^{d-1}_{t+\varepsilon})\mathcal{O}_i(x) = R_{\mathcal{O}_i}(g)\mathcal{O}_i(x)U_g(\Sigma^{d-1}_{t-\varepsilon}), \quad x\in\Sigma_t, \tag{2.47}$$

where we have made the relevant time-ordering explicit. Alternatively, keeping $U_g(\Sigma^{d-1})$ fixed, we can think of its insertion as introducing a discontinuity, induced by the group action, at $\Sigma^{d-1}$ as we move the other operator insertions across it. That is to say, for two points $x\in\Sigma^{d-1}$, $x' = x+\epsilon \notin \Sigma^{d-1}$ and a charged operator $\mathcal{O}$, we have for $\epsilon\to 0$

$$\mathcal{O}(x') = U_g(\Sigma^{d-1})\mathcal{O}(x') \neq U_g(\Sigma^{d-1})\mathcal{O}(x) = R_{\mathcal{O}}(g)\mathcal{O}(x), \tag{2.48}$$

which is illustrated in figure 2.3. Indeed, viewed as an abstract operator insertion, we can interpret this as the definition of the operators $U_g(\Sigma^{d-1})$. Importantly, this perspective remains valid even for discrete symmetries, which do not have an equivalent of equation 2.34.

[8]Here, and throughout this chapter we assume that the sphere $S^{d-1}$ is small enough that we may assume that it is contractible (i.e. there are no topological obstructions in the spacetime).

[9]spacetimes with spatial lead to rather rich structure which we do not go into here.

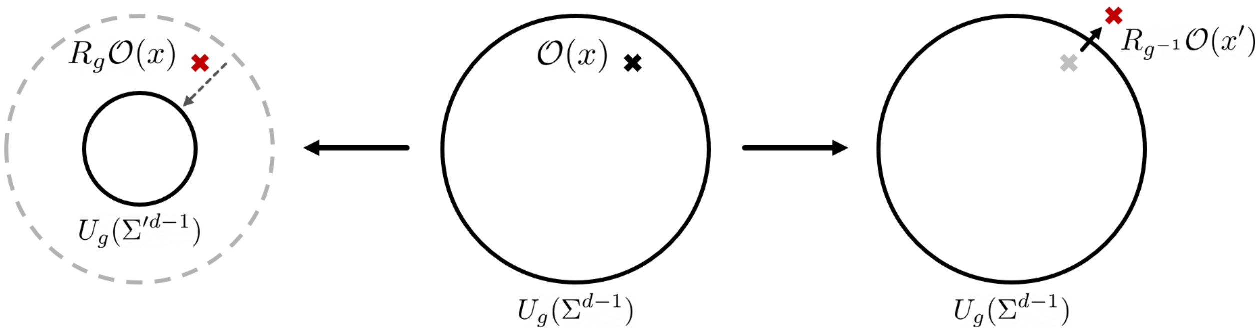



Figure 2.3: Two (equivalent) ways of thinking about the action of operator insertions $U_g(\Sigma^{d-1})$. One can either consider deforming $\Sigma^{d-1}$ across another charged operator insertion, yielding a group action on the latter, or move the defect across the local operator.

Using this definition, we can derive a number of other properties that these operators satisfy. For example, one finds that two such operators supported on the same manifold $\Sigma^{d-1}$ will satisfy a group law

$$U_g(\Sigma^{d-1})U_{g'}(\Sigma^{d-1}) = U_{gg'}(\Sigma^{d-1}). \tag{2.49}$$

In terms of local operators crossing these insertions, we can imagine first cross the insertion on the right, picking up a group action $R(g')$, followed by a second group action $R(g)$. Because $\Sigma^{d-1}$ is codimension one, we can always define a normal direction which specifies the ordering, so that equation 2.49 makes sense even for non-abelian symmetry groups. The inverse operators $U_g^{-1}(\Sigma^{d-1}) = U_{g^{-1}}(\Sigma^{d-1})$ can be obtained by inverting the orientation of $\Sigma^{d-1}$. In the continuous case this can be traced back to the appearance of a relative minus-sign in equation 2.36. For discrete symmetries we take this as part of the definition of what constitutes a global symmetry.

These fusion rules can also be used to construct non-trivial junctions of topological operators. For example, we can consider the insertion of three symmetry operators supported on co-dimension 1 manifolds $\Sigma_i^{d-1}$ which now share a single co-dimension 2 boundary. Such a configuration is shown in figure 2.4. The result is a consistent path integral insertion, provided the group elements on the components are related as in the figure. This follows e.g. by following the group action picked up as we move a local operator insertion around the configuration, and observing that it is path-independent (equivalently, the resulting configuration is topological up to local insertions as we can collapse the junction to a single operator $U_g$).

Before closing this section we would like to make some remarks about several important topics in the study of ordinary global symmetries which will admit natural generalizations once we consider generalized global symmetries in the next section.

- **Gauge Symmetries** Our discussion so far has not mentioned gauge symmetries at all and for good reason. Indeed, given a gauge symmetry one could in principle define topological operators $U_g$. A consistent gauge theory can, by definition, not have operators charged under that symmetry, so that the associated symmetry operators always turn out to be trivial.
- **Gauging Global Symmetries** Given a global symmetry it is natural to wonder whether or not one is allowed to gauge it. While the very meaning of "gauging" a discrete symmetry is somewhat subtle, in the continuous case this is always associated to a gauge field for that symmetry. As an intermediate step to gauging a global sym-

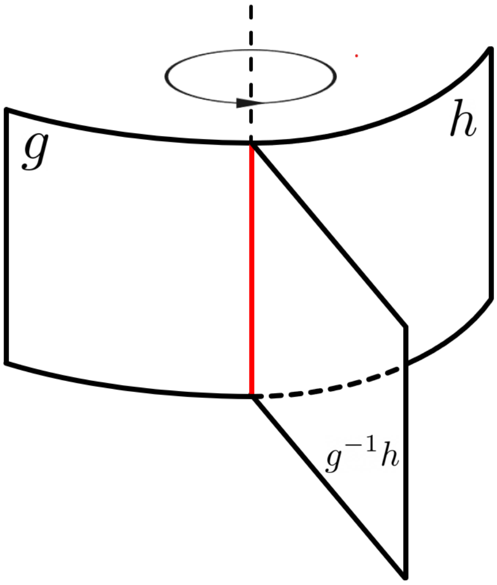



Figure 2.4: Junction of symmetry operators. The orientations of the operator follow the circular arrow. It can also be viewed as a manifestation of the fusion law (2.49), where $g$ and $h$ fuse to give $g^{-1}h$. Figure based on [34].

metry one can imagine turning on a background (non-dynamical) gauge field for the symmetry which couples (to linear order) to the conserved current as

$$\delta S = \int_{\mathcal{M}} A_1 \wedge J_{d-1}, \tag{2.50}$$

such that the partition function becomes a function of the gauge field $A_1$. Gauging the symmetry then corresponds to performing a path integral over this background gauge field. From this perspective, the intermediate partition function $\mathcal{Z}[A_1]$ can be used to detect obstructions to this gauging procedure. Indeed, the path integral over $A_1$ should be taken over equivalence classes of $A_1$ related by gauge transformations so that the integrand (in this case $\mathcal{Z}[A_1]$) must be invariant under gauge transformations of the background connection $A_1$. When this fails we say that the symmetry suffers an 't Hooft anomaly and these may sometimes be used to deduce properties of the UV physics [35]. Even if such anomalies are present, it may still be possible to consistently couple the theory to a background gauge field by including a so-called anomaly theory. We do not pursue this rich topic here, but briefly give a taste in section 2.3.2.

More generally, a theory may admit mixed ('t Hooft) anomalies. In this case, a theory has two global symmetries each of which is non-anomalous and hence may be gauged. However, once we gauge one of them the other develops an anomaly. The classic example of this is the chiral anomaly [36,37], whereby the path integral measure ceases to be invariant under the axial symmetry of a chiral fermion once one couples it to electromagnetism (i.e. gauge the $U(1)$-vector symmetry) [38].

- **Gauging Discrete Symmetries** For completeness, let us say a few words about gauging discrete symmetries. As in the continuous case, we should couple the system to a background connection. The first step is to specify a gauge bundle with structure group $G$. Since $G$ is discrete, this bundle is necessarily flat and since flat bundles carry canonical connections, we need only specify the bundle. The latter is done by choosing a contractible open cover (making sure higher overlaps are likewise contractible) and choosing transition functions on overlaps satisfying the usual cocycle condition. Since the bundle is flat, these transition functions are simply group elements of $G$.[10]

This construction can equivalently be expressed by a network of $U_g(\Sigma^{d-1})$ symmetry operators [2]. This is done by specifying a triangulation of the spacetime dual to the open cover and inserting symmetry operators implementing the transition functions on the $(d-1)$-dimensional faces of the triangulation, which are joined at $(d-2)$-triple intersections via junctions of the type described above [34] (cf. figure 2.4). Gauging the symmetry then corresponds to summing over all such operator networks. Dependence of the partition function on the background connection signals an 't Hooft anomaly.[11]

Though completely general, the construction above is often more involved than necessary. For simple situations such as when the discrete gauge group is abelian (e.g. $\mathbb{Z}$), the relevant gauge symmetry often admits a more concrete representation in terms of a 0-form gauge field. In this case, a particular bundle of the type described above corresponds to a state with a topological defect that is charged under the 0-form gauge field. We will see this explicitly below.

- **Selection Rules** The language of topological operators makes manifest another fundamental aspect associated to global symmetries, namely that they lead to selection rules on correlation functions [2]. Let us illustrate this with a very basic example, consider inserting a charged local operator $\mathcal{O}(x)$ on a spacetime with the topology of a 2-sphere. If the operator $\mathcal{O}(x)$ is charged under a global symmetry, we can attempt to compute the correlator $U_g(S^1_x)\mathcal{O}(x)$ for $S^1$ a circle surrounding the point $x$. One way is to contract the circle to a point, crossing the local operator $x$, in which case we obtain a group action $R_{\mathcal{O}_i}(g)$ on the local operator. Another is to contract the circle $S^1_x$ along the other hemisphere, in which case the action of $U_g(S^1_x)$ is trivial. To prevent ambiguity, it follows that the operator $\mathcal{O}(x)$ must have zero charge. Physically, this is just the familiar statement that the field lines emanating from $x$ have nowhere to end on the compact space $S^2$.

- **Spontaneously Broken Symmetries** Although not particularly relevant to us, spontaneous symmetry breaking is typically regarded as one of the more important applications of global symmetries. Given a global symmetry, one may encounter a phase where the vacuum of the theory is not invariant under the action of a symmetry of that same theory. In this case there is typically a so-called order parameter whose expectation value becomes non-vanishing in the spontaneously broken phase.

  The classic example of this occurs when we include a $U(1)$-invariant potential to the complex scalar field considered in equation 2.37, whose minimum is located at a non-vanishing value of $|\phi|$. In this case the vacuum expectation value $\langle\phi(x)\rangle$ is the associated order parameter which becomes non-vanishing in the spontaneously broken phase. The symmetry operators connect vacua with different expectation values for the order parameter. In the continuous case, we can consider infinitesimal variations generated by the charge operator $Q$. This leads to the emergence of a massless Goldstone boson which enjoys a characteristic (global) continuous shift symmetry (e.g. $\varphi(x) \to \varphi(x)+c$). This example will be considered explicitly in section 2.3.2.

[10] Mathematically, this is essentially the construction of Cech cohomology, which tells us that such bundles correspond to elements of $\hat{H}^1(M, G)$. In practice, we will take a more pedestrian approach and identify them with elements in $H^1_{dR}(M)$ with restricted holonomy.

[11] This construction can also be applied for continuous $G$, in which case we are only able to capture the sub-set of flat backgrounds.

## 2.3 Higher-Form Global Symmetries

While we have been rather explicit about introducing the notion that topological operators are associated to symmetries, we now get to enjoy the fruits of our labour. Indeed, given this perspective of global symmetries, it is in principle a straight-forward matter to generalize ordinary symmetries in various directions. The most natural (and indeed the first) generalization of this paradigm is to consider topological operators that are supported on more general $(d-p-1)$-dimensional manifolds $\Sigma^{d-p-1}$. Recall that we defined topological operators supported on $(d-1)$-dimensional manifolds via their effect on local operators inside correlation functions. We could do so precisely because we had a well-defined notion of $\Sigma^{d-1}$ enclosing local operator insertions. Preventing this sort of ambiguity means that local operators can only be charged under the usual $(d-1)$-dimensional symmetry operators.

This leads to the natural observation that the operators charged under $(d-p-1)$-dimensional symmetry operators should themselves be $p$-dimensional! In slightly more mathematical terms, this is the statement that the linking number can be defined for manifolds whose dimensions add up to $d-1$. While we post-pone a more concrete description of such charged extended operators to later sections, we can already give an abstract description of the corresponding topological symmetry operators. These can explicitly be defined in terms of their Ward identities. Given a $p$-form global symmetry with charged operators $\mathcal{O}(\mathcal{C}^p)$ supported on $p$-dimensional manifolds $\mathcal{C}^p$, the analog of equation 2.46 is obtained for a small sphere $S^{d-p-1}$ linking $\mathcal{C}^p$ exactly once

$$U_g(S^{d-p-1})\mathcal{O}(\mathcal{C}^p) = R_{\mathcal{O}}(g)\mathcal{O}(\mathcal{C}^p), \tag{2.51}$$

which should be interpreted as shrinking $\Sigma^{d-p-1}$ to a point, crossing $\mathcal{C}^p$ in the process. Such generalized operators satisfy many of the same properties as their $(d-1)$-dimensional counterparts, including a fusion law of the form

$$U_g(\Sigma^{d-p-1})U_{g'}(\Sigma^{d-p-1}) = U_{gg'}(\Sigma^{d-p-1}). \tag{2.52}$$

A crucial distinction however, is that there is no natural ordering on the left-hand side. Indeed, two codimension $p+1$ operators (for $p>0$) can always be deformed past each other without crossing so that the notion of time-ordering breaks down (one can compare this to the 0-form case, where the operators will always cross). Hence, the higher-form global symmetry group is necessarily abelian. In the continuous case, this means that the group is either $U(1)$ or $\mathbb{R}$, whose representations are all one-dimensional and are labeled by the charge $q$ of the operator (which is integer or real, respectively). The factor $R_{\mathcal{O}}(g)$ then represents the phase picked up by the charged operator, so that the most general Ward identity is given by

$$U_{e^{i\alpha}}(\Sigma^{d-p-1})\mathcal{O}_q(\mathcal{C}^p) = e^{iq\alpha\mathcal{I}\left(\mathcal{C}^p,\Sigma^{d-p-1}\right)}\mathcal{O}_q(\mathcal{C}^p), \tag{2.53}$$

where we account for the possibility that $\Sigma^{d-p-1}$ and $\mathcal{C}^p$ link more than once by including the intersection number $\mathcal{I}\left(\mathcal{C}^p,\Sigma^{d-p-1}\right)$. Just as in the 0-form case, *continuous* $p$-form symmetries are associated to conserved currents which define local operators in the theory. In the generalized case however, these now define $(d-p-1)$-form currents whose conservation law is the natural generalization of equation 2.29

$$dJ_{d-p-1} = 0. \tag{2.54}$$

These currents are integrated over $(d-p-1)$-dimensional manifolds and the associated charge operators measure the the flux passing through this manifold

$$Q(\Sigma^{d-p-1}) = \int_{\Sigma^{d-p-1}} J_{d-p-1}. \tag{2.55}$$

A charge $q$ operator will produce $q$ units of flux which is measured

$$U_{e^{i\alpha}}(\Sigma^{d-p-1}) = e^{i\alpha Q(\Sigma^{d-p-1})} \tag{2.56}$$

Although much more could be said about these symmetries by reasoning abstractly at the level of topological operators, we choose to keep this introductory discussion short. Instead, we will introduce these higher-form global symmetries by means of several examples, each of which will serve to illustrate different aspects of $p$-form global symmetries. Since the focus of this thesis will eventually be the vector sector of $\mathcal{N} = 2$ supergravity, the examples discussed here focus on generalized global symmetries appearing in abelian gauge theories. Despite the symmetries themselves being constrained to be abelian by equation (2.52), they may nonetheless appear in theories with non-abelian gauge groups. We briefly comment on this case occasionally, but leave this rich subject otherwise untouched. Finally, we emphasize that these examples run in increasing order of relevance to our eventual applications, with the first two being included to better illustrate the concepts involved.

### 2.3.1 Example: Maxwell Theory

The prototypical example of a theory exhibiting a higher-form global symmetry is Maxwell theory in four dimensions

$$S = \int_X \left( -\frac{1}{2e^2} F_2 \wedge *F_2 \right). \tag{2.57}$$

#### Electric Symmetries and Wilson Lines

This theory has a so-called electric 1-form symmetry with conserved 2-form current

$$J_e = \frac{1}{e^2} * F^2, \tag{2.58}$$

whose conservation law simply corresponds to the equations of motion. The associated conserved charge is given by

$$Q_e^{(1)} = \frac{1}{e^2} \int_{S^2} *F_2, \tag{2.59}$$

which, when expanded into electric and magnetic fields, measures precisely the electric flux passing through the surface $S^2$ (equivalently, the electric charge enclosed by $S^2$). In absence of any matter fields (we will discuss matter fields shortly), gauge field configurations with non-zero charge (2.59) are created by operator insertions known as Wilson loops (or lines). They modify the path integral by introducing a source which enforces the presence of a charge along a given curve, or equivalently a flux through a surface linking that curve. We can define them by the curve $\gamma^1$ on which they are supported as well as their charge $n$

$$W_n(\gamma^1) = \exp\left( in \int_{\gamma^1} A_1 \right). \tag{2.60}$$

In a theory without charged matter fields, it is these operators rather than the gauge fields $A_1$, which are the gauge invariant observables in the theory, provided the curve $\gamma^1$ is either closed or ends at infinity. If this curve extends along the time direction we can interpret it as the world-line of an infinitely massive probe charge.[12] Alternatively, we can localize this

[12] This is is completely analogous to how we use scalar sources to probe a scalar field theory by defining a generating function $\mathcal{Z}[J]$.

curve on a constant-time slice, in which case we can view these operator as creating a loop of flux. The distinction between these cases fades somewhat in Euclidean field theory and we will not keep track of it in the following, for now focusing on Wilson loops rather than lines.[13]

For another perspective on Wilson loops, recall that in $U(1)$ gauge theory, the gauge field $A_1$ is the connection form on a $U(1)$-bundle. In general, such a connection is determined completely by its holonomy (i.e. parallel transport around loops), so that in principle it should be possible to describe the dynamics of $A_1$ in terms of a map that assigns to a given loop $\gamma^1$ the $U(1)$-valued holonomy of $A_1$ around that loop. Of course, this is precisely what the Wilson loops (2.60) do for us. From this perspective it is clear that if we could write down a theory on loop-space, the fundamental quantum fields would be the Wilson loops and we would not have to make reference to any local gauge field $A_1$.[14] In lieu of such a description however, the gauge field $A_1$ gives us a useful, albeit redundant description of the degrees of freedom of the theory. For our purposes, it gives an explicit description of the electric 1-form global symmetry in terms of its action on the local field $A_1$. In particular, it acts according to

$$A_1 \to A_1 + \Lambda_1, \tag{2.61}$$

where $\Lambda_1$ is a closed 1-form so that it leaves $F_2$ invariant. Let us say a few words about the interpretation of equation 2.61 as a global symmetry. Indeed, one usually thinks of global symmetries as transformations whose transformation parameters are constant over spacetime, the prototypical example being a global shift symmetry $\phi(x) \to \phi(x) + c$. It may therefore be surprising that the form $\Lambda_1$ is generically *not* constant. One way to think about this is again in terms of the holonomy perspective put forward above. Indeed, if the gauge bundle is trivial, we can view $\Lambda_1$ as the connection form of a flat connection (i.e. with vanishing curvature). Its holonomy is non-trivial only around non-contractible loops, and moreover its holonomy depends only on the homotopy type of the loop. These statements are equivalent to saying that the symmetry (2.61) is locally constant on *loop space*. Compare this to the 0-form case, where the transformation parameters are only required to be *locally* constant on spacetime, that is, constant on each connected component of $X$, which are counted by the zeroth cohomology of $X$. Similarly, the symmetry parameters $\Lambda_1$ take values in the first cohomology group.

The complementary perspective is to view the Wilson loops as the fundamental objects charged under the symmetry. We can use the explicit transformation (2.61) to exhibit the transformation properties of these operators

$$W_n(\gamma^1) \to W_n'(\gamma^1) = \exp\left(in \oint_{\gamma^1} \Lambda_1\right) \exp\left(in \oint_{\gamma^1} A_1\right) = e^{in\alpha} \exp\left(in \oint_{\gamma^1} A_1\right), \tag{2.62}$$

where we wrote $\alpha = \int_{\gamma^1} \Lambda_1$ for the period of $\Lambda_1$ around the cycle. This can can be non-zero whenever $\gamma^1$ defines a non-trivial homology class and moreover, depends only on the homology class of $\gamma^1$. By analogy with the global U(1) charge of a complex scalar field, the integer $n$ multiplies the phase acquired by the charged object under transformations by $g = e^{i\alpha}$ so that we can indeed interpret it as the charge of $W_n(\gamma^1)$ under the 1-form symmetry. We see

[13] Non-compact spacetimes (or more generally spacetimes with boundaries) admit a number of interesting phenomena related to the non-trivial topology induced by the boundary. We will not cover these effects in detail but refer to e.g. [39] for more details.

[14] If one views gravity as a gauge theory, the analogous objects could also be treated as the fundamental degrees of freedom. Indeed, this is the approach taken by loop quantum gravity, one of the primary alternatives to string theory as a theory of quantum gravity.

that $\Lambda_1 \in H^1(X, \mathbb{R}/2\pi\mathbb{Z})$ as $\Lambda_1$ that satisfy $\int_{\gamma^1} \Lambda_1 \in 2\pi\mathbb{Z}$ act trivially on all operators in the theory. These are therefore understood to define additional gauge redundancies of $A_1$.

Based on these observations, we can now readily derive the Ward identity associated to the symmetry (2.61). Indeed, following the Noether procedure from section 2.2.2, we now introduce a spacetime dependent transformation parameter $\epsilon(x)$ (to distinguish from the phase $\alpha$)

$$A_1 \to A_1 + \epsilon(x)\Lambda_1 \tag{2.63}$$

and compute the infinitesimal variation of the Wilson line expectation value

$$\langle W_n'(\gamma^1)\rangle' = \left\langle \left(1 - i\int_{\mathcal{M}} J_2^e \wedge d\epsilon \wedge \Lambda_1\right)\left(W_n(\gamma^1) + in\left(\int_{\gamma^1} \epsilon\Lambda_1\right) W_n(\gamma^1)\right)\right\rangle \stackrel{!}{=} \langle W_n(\gamma^1)\rangle. \tag{2.64}$$

If we now take $\epsilon$ to have support only in a neighbourhood[15] $\gamma^1 \times \Sigma^3$ with boundary $\gamma^1 \times \Sigma^2$, we obtain

$$i\left\langle \left(\int_{\gamma^1\times\Sigma^2} J_2^e \wedge \Lambda_1\right) W_n(\gamma^1)\right\rangle = i\alpha\left\langle \left(\int_{\Sigma^2} J_2^e\right) W_n(\gamma^1)\right\rangle \stackrel{(2.64)}{=} in\alpha\left\langle W_n(\gamma^1)\right\rangle. \tag{2.65}$$

Identifying the charge operator $Q_e^{(1)}(\Sigma^2)$ as in (2.59), we find the infinitesimal version of the integrated Ward identity associated to the electric 1-form symmetry

$$\begin{aligned} U_{e^{i\alpha}}(S^2)W_n(\gamma^1) =& \exp\left(\frac{i\alpha}{e^2}\int_{S^2} *F_2\right)\exp\left(in\oint_{\gamma^1} A_1\right) \\ =& e^{in\alpha}\exp\left(in\oint_{\gamma^1} A_1\right). \end{aligned} \tag{2.66}$$

Importantly, this perspective remains valid even when the global symmetry does not admit a simple action on the local fields in the theory.

### Magnetic Symmetries and 't Hooft Lines

In fact, the simple Maxwell theory already contains a second 1-form symmetry that allows us to illustrate this phenomenon. Indeed, the Bianchi identities $dF_2 = 0$ suggest the existence of a second conserved current $J_m = \frac{1}{2\pi}F_2$ to which we may associate a second set of topological operators. The associated conserved charge

$$Q_m^{(1)}(S^2) = \frac{1}{2\pi}\int_{S^2} F_2, \tag{2.67}$$

physically measures the magnetic flux passing through the surface $S^2$. While possibly less familiar, the operators that create such configurations are well-known and are given by 't Hooft lines $T_m(\gamma^1)$. They too are line operators supported on 1-dimensional manifolds, but in contrast to the Wilson lines they do not admit a simple expression in terms of the elementary gauge field $A_1$. The standard definition of these operators is in terms of a restriction of the path integral to field configurations that carry non-zero magnetic charge along the curve $\gamma^1$. That is, we define it precisely by restricting to field configurations with a non-zero

[15] We make a number of simplifying assumptions to streamline the discussion. One can formalize and generalize the steps below by taking a tubular neighbourhood of $\gamma^1$ and choosing $\Lambda$ to be strictly proportional to the dual of the (integral) homology class of $\gamma^1$ (this latter assumption is not necessary but makes equation 2.65 immediate).

expectation value for the operator (2.67) for any $S^2$ that links the given curve $\gamma^1$. This defines an observable in the same way that more standard path integral insertions do, albeit without a simple expression for the relevant path integral modification. In more geometric terms we note that the current $\frac{1}{2\pi}F_2$ represents the first Chern class of the associated $U(1)$-bundle so that inserting an 't Hooft line can be interpreted as restricting the topology of the underlying gauge bundle (which we otherwise sum over in the path integral).

There is however, a more direct way of studying 't Hooft lines and the magnetic 1-form symmetry. Indeed, recall from section 1.6.1 that the two conserved currents we considered here are interchanged under the electric-magnetic duality transformation of Maxwell theory. Although it was mentioned only briefly, electric-magnetic duality also maps electric and magnetic charges to each other. For instance, in the dual theory, the old Wilson lines now create states of non-zero *magnetic* flux $\frac{1}{e^2} * F_2 \to -\frac{1}{2\pi}G_2$ and hence correspond to 't Hooft lines in the dual theory. More interestingly for us, the dual theory now grants us an explicit description of the old 't Hooft lines in terms of the Wilson lines for the dual gauge field $V_1$. In particular, in terms of the dual fields, we now have the analogous equation

$$\begin{aligned} U_g^m(S^2)T_m(\gamma^1) &= \exp\left(\frac{i\alpha}{\tilde{e}^2}\int_{S^2} *G_2\right)\exp\left(im\int_{\gamma^1} V_1\right) \\ &= e^{im\alpha}\exp\left(im\int_{\gamma^1} V_1\right). \end{aligned} \tag{2.68}$$

### Photons as Goldstone Bosons

A final aspect we would like to touch on briefly, but which is not of direct importance to our main discussion, is the possibility of spontaneously broken $p$-form global symmetries. Consider for instance the 1-form global symmetry (2.19). In this case, the relevant order parameter is measured by the expectation value of the Wilson loop $\langle W_1(\gamma^1)\rangle$, naturally generalizing the local operator $\langle\phi(x)\rangle$. Its role as an order parameter is somewhat more subtle however, as we use it to define phases not by the (non-)vanishing of $\langle W_1(\gamma^1)\rangle$ but rather by its long-distance behaviour as the size of the loop $\gamma^1$ grows. This is not too different from the local case as we could equivalently define a spontaneously broken phase by the long-distance behaviour of the expectation value

$$\langle\phi(x)\phi(y)\rangle \overset{|x-y|\to\infty}{\to} \langle\phi(x)\rangle\langle\phi(y)\rangle \neq 0. \tag{2.69}$$

In general, a Coulomb-type potential $V(r) \sim \frac{1}{r}$ (mediated by $A_1$) will lead to a long distance behaviour for the Wilson loops given by [40]

$$\langle W_n(\gamma^1(r))\rangle \sim e^{-V(r)} = e^{-1/r} \overset{r\to\infty}{\to} 1, \tag{2.70}$$

indicating spontaneous symmetry breaking. Intuitively, this behaviour can be understood as viewing the insertion of a Wilson loop as the creation of a particle-anti-particle pair (of opposite charge) which are then separated by the radius $r$ of the loop and experience a Coulomb potential $V(r)$. This is to be contrasted with the long-distance behaviour in the confining phase (which is absent here but present for non-abelian gauge theory), where $V(r) \sim r$ leads to a vanishing vev of the Wilson loop above.

These considerations lead to a very satisfying conclusion, namely that the massless photon is in fact the Goldstone boson associated to the spontaneously broken 1-form symmetry [2]. Moreover, the symmetry (2.61) now admits the interpretation of the remnant continuous shift-symmetry (the same conclusion holds for the magnetic 1-form symmetry and its associated dual gauge field $V_1$). While spontaneous symmetry breaking of generalized global symmetries is beyond the scope of this thesis, we refer the interested reader to [39] for a more detailed discussion.

#### Conclusions

Before closing this sub-section, let us recapitulate the main lessons we have learned

- Gauge theories provide ubiquitous realizations of higher-form global symmetries. In particular, any $p$-form gauge theory has both an electric $p$-form symmetry and a magnetic $(d-p-1)$-form symmetry.
- The operators charged under these symmetries are the (non-local) Wilson and 't Hooft operators, respectively.
- The generalized global symmetry need not admit a clear action on the local fields of the theory, but there is always a dual frame where this is the case. This highlights the fact that global symmetries are a property of the theory, *not* our description of it.
- When they do admit an action on the local fields, they can be treated much like ordinary global symmetries, a feature which we will exploit in the following sub-section.

In the next sub-section, we will continue our investigation of Maxwell theory, but generalize it in two ostensibly different ways. On the one hand, we will study what happens when we attempt to gauge the 1-form global symmetries discussed in this section. On the other hand, we will show how they may be broken by the addition of charged matter. As we shall see however, these two questions are in fact dual to one another and they provide complimentary perspectives of the same physics.

### 2.3.2 Example: BF Theory

At the end of section 2.2.2 we discussed how one may gauge a continuous 0-form global symmetry. To wit, this involved introducing a 1-form gauge field which coupled to the conserved current. Gauging a $p$-form global symmetry proceeds completely analogously and requires the introduction of a $(p+1)$-form gauge field. Of course, this is why we spoke of higher-form gauge fields as the gauged version of $p$-form global symmetries. To gauge the magnetic 1-form symmetry we therefore introduce a 2-form background gauge field

$$\delta S = -\int B_2 \wedge J_2^m = -\frac{1}{2\pi}\int B_2 \wedge F_2, \tag{2.71}$$

which we subsequently render dynamical by including a canonical kinetic term and performing the path integral over $B_2$. The resulting action is then given by

$$S_{BF} = \int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 - \frac{1}{2g^2} H_3 \wedge *H_3 - \frac{1}{2\pi} B_2 \wedge F_2 \right). \tag{2.72}$$

The new BF-coupling is our first example of a Chern-Simons coupling and these will play a very important role throughout the rest of this thesis.

#### 2-Form Symmetries

Before considering how gauging the magnetic symmetry affects our old results, let us first consider the new physics introduced by the 2-form gauge field. Ignoring the BF-coupling, we can repeat the arguments from the previous sub-section to see that we have an "electric" 2-form global symmetry that acts on the 2-form gauge field according to

$$B_2 \to B_2 + \Lambda_2, \tag{2.73}$$

where $\Lambda_2$ is now an element of the second cohomology. One way of thinking about this symmetry is to generalize our discussion about theories on loop spaces and view $B_2$ as a connection on a $U(1)$-bundle over loop space. This perspective is formalized by the notion of a circle 2-bundle with a connection, which although interesting, we do not pursue here.[16] Instead, we study the resulting 2-form global symmetry in more physical terms in the following sub-sections. Of particular interest to us will be the conserved current and charge associated to the symmetry (2.73), which follow readily from the equations of motion for $B_2$

$$J_1 = \frac{1}{g^2} * H_3, \qquad U_{e^{i\alpha}}(\Sigma^1) = \exp\left(\frac{i\alpha}{g^2}\int_{\Sigma^1} *H_3\right). \tag{2.74}$$

The objects carrying this charge are now supported on 2-dimensional manifolds, which naturally generalize the Wilson lines from section (2.3.1)

$$W_n(\Sigma^2) = \exp\left(in\int_{\Sigma^2} B_2\right). \tag{2.75}$$

These can similarly be thought of as either creating a string that propagates along $\Sigma^2$ and couples to the $B_2$ field via its world-line (in fact, this is exactly how string-theory strings couple to the NS-NS B-field from section 1.2.1), or as enforcing a non-zero $J_1$ flux along any loop that links the surface $\Sigma^2$. The symmetry operators $U_{e^{i\alpha}}(\Sigma^1)$ effect transformations of the form (2.73) provided the curve $\Sigma^1$ links with $\Sigma^2$ exactly once

$$\begin{aligned}\exp\left(\frac{i\alpha}{g^2}\int_{\Sigma^1} *H_3\right)\exp\left(in\int_{\Sigma^2} B_2\right) &= \exp\left(i\alpha n\right)\exp\left(in\int_{\Sigma^2} B_2\right) \\ &= \exp\left(in\int_{\Sigma^2} \Lambda_2\right)\exp\left(in\int_{\Sigma^2} B_2\right).\end{aligned} \tag{2.76}$$

Similarly, the Bianchi identity for $B_2$ motivates us to consider an additional conserved current

$$J_3 = \frac{1}{2\pi}H_3, \qquad U_{e^{i\alpha}}(\Sigma^3) = \exp\left(\frac{i\alpha}{2\pi}\int_{\Sigma^3} H_3\right), \tag{2.77}$$

whose charged operators are defined analogously to 't Hooft lines, i.e. they enforce a non-zero flux for $J_3$. One notices that this corresponds to a 0-form symmetry, which suggests they admit a more familiar interpretation as an ordinary global symmetry. We will see this more explicitly below.

### The BF-coupling

The story changes when we include the effect of the BF coupling. In this case the symmetry (2.73) ceases to be a good symmetry of the theory as the BF term is manifestly not invariant

[16] In this context, the gauge field $B_2$ is a local representative of the differential character $H_3$, the latter being defined precisely through its assignment of a holonomy to 2-cycles. The BF-coupling is then understood in terms of the product in differential cohomology. See e.g. [41].

under this shift[17] This is reflected in the equations of motion which now read

$$B_2: \ \frac{1}{g^2} d * H_3 = \frac{1}{2\pi} F_2, \qquad A_1: \ \frac{1}{e^2} d * F_2 = -\frac{1}{2\pi} H_3. \tag{2.78}$$

These equations reveal a general pattern. By gauging a $p$-form symmetry, here the 1-form symmetry with current $\frac{1}{2\pi}F_2$, we render its current exact, as is reflected by the equation of motion for $B_2$. At the same time however, its associated gauge field introduces a $(p+1)$-form symmetry, here $\frac{1}{g^2} * H_3$, which is explicitly broken by the coupling to the conserved current.

### Dualizing $A_1$

These observations are most clearly realized when we move to a magnetic dual frame where the (now-gauged) magnetic 1-form symmetry becomes electric. Indeed, performing the dualization along the lines of section 1.6.1 (accounting for the additional BF-coupling), we obtain a dual "BV"-theory

$$S_{BV} = \int \left( -\frac{1}{2\tilde{e}^2} (dV_1 - B_2) \wedge *(dV_1 - B_2) - \frac{1}{2g^2} H_3 \wedge * H_3 \right), \tag{2.79}$$

where now $F_2$ and $e^2$ are related to $V_1$ and $\tilde{e}^2$ via

$$F_2 = -\frac{e^2}{2\pi} * (dV_1 - B_2), \quad \tilde{e} = 2\pi/e. \tag{2.80}$$

The gauging of the 1-form global symmetry $V_1 \to V_1 + \Lambda_1$ is now manifest, as under gauge transformations of $B_2$ we have

$$B_2 \to B_2 + d\Lambda_1, \qquad V_1 \to V_1 + \Lambda_1, \tag{2.81}$$

which follows from gauge invariance of $F_2$ in equation 2.80. In fact, we see that locally $V_1$ is now pure gauge and we can always gauge fix locally to set $V_1$ to zero. Looking at the action, we see that this has the effect of generating a mass-term for $B_2$, which is consistent by virtue of the fact that its gauge symmetry is now fixed.

### Dualizing $B_2$

The equations of motion for $A_1$ reveal the dual side of this story. It too can be read two ways, stating either that the electric 1-form symmetry is explicitly broken by the coupling to $B_2$, or that the dual "magnetic" 0-form symmetry with current $J_3 = \frac{1}{2\pi} H_3$ is gauged by $A_1$. Indeed, while so far we have framed the BF coupling as gauging the magnetic 1-form symmetry of $A_1$, it is easy to see that we can integrate by parts to write it as an "AH-coupling"

$$S_{AH} = \int \left( -\frac{1}{2e^2} F_2 \wedge * F_2 - \frac{1}{2g^2} H_3 \wedge * H_3 - \frac{1}{2\pi} A_1 \wedge H_3 \right). \tag{2.82}$$

But this is just how we gauge ordinary 0-form symmetries! It follows that the action (2.82) can be viewed as a (rather convoluted) way of coupling Maxwell theory to charged matter. We

[17] Note that due to the quantization of $\frac{1}{2\pi}F_2$, the BF-coupling is in fact invariant if $\Lambda_2 \in H^2(X, 2\pi\mathbb{Z})$. However, as for the electric 1-form symmetry, equation 2.76 shows that these transformations act trivially on all operators in the theory and so these $\Lambda_2$ do not define a symmetry transformation, but are in fact additional gauge redundancies of $B_2$. We will encounter an exception to this shortly.

can make this explicit by dualizing the gauge field $B_2$ along the lines of section 1.6.1. Going through the motions once more, we promote the field strength $H_3$ to an arbitrary 3-form and include a Lagrange multiplier $-\frac{1}{2\pi}a\, dH_3$ to enforce closedness of $H_3$. Upon integrating out $H_3$ by enforcing its equation of motion

$$-\frac{1}{g^2} * H_3 - \frac{1}{2\pi}(da - A_1) = 0 \quad \Rightarrow \quad H_3 = -\frac{g^2}{2\pi} * (da - A_1), \tag{2.83}$$

we obtain the equivalent action

$$S_{aF} = \int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 - \frac{1}{2\tilde{g}^2} d_A a \wedge *d_A a \right), \tag{2.84}$$

where we have introduced the short-hand $d_A a = da - A_1$ and $\tilde{g} = 2\pi/g$.

### Abelian Higgs Model

The coupling $d_A a$ is known as a Stueckelberg coupling and it arises as the low-energy limit of the abelian Higgs model mentioned briefly at the end of section 2.2.2. To see this, consider the action of a complex scalar field with $U(1)$ gauge charge $q$ experiencing a typical "Mexican hat" potential

$$S = \int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 + \mathcal{D}_{qA}\phi^\dagger \wedge *\mathcal{D}_{qA}\phi - \mu \left( |\phi|^2 - v^2 \right)^2 \right), \tag{2.85}$$

where the scalar couples to $A_1$ via the covariant derivative $\mathcal{D}_{qA} \equiv d - iqA_1$. Expanding the potential we see that $\phi$ acquires a mass-squared given by $m^2 = -2\mu v^2$ which is negative for $\mu > 0$. This signals an instability in the field $\phi$ which subsequently rolls down to the stable minimum $|\phi| = |v|$. Expanding the field around this minimum as $\phi = (v + \rho)e^{i\theta}$ it is easy to see that the radial fluctuations $\rho$ acquire a now-positive mass $m^2 = 4\mu v^2$ while the angular fluctuations $\theta$ remain massless. At low energy (equivalently for large $v$) we can integrate out the massive radial fluctuations which decouple so that we are left with an effective action describing the gauge field and $\theta$

$$S = \int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 + v^2 (d\theta - qA_1) \wedge *(d\theta - qA_1) \right). \tag{2.86}$$

Upon identifying $v^2 = \frac{1}{2\tilde{g}^2}$, $\theta = a$ and $q = 1$, we recover the dualized BF theory (2.84).

### The Axion

The detour through the abelian Higgs model sheds some interesting light on the theory (2.84). Firstly, we recognize that the field $a$ is in fact a $2\pi$-periodic variable, emerging as the low-energy description of the phase of the complex field $\phi$. This shows up in the low-energy theory as the *discrete* gauge symmetry

$$a \to a + 2\pi. \tag{2.87}$$

Indeed, $a$ is best thought of as a 0-form gauge field whose "field strength" $da$ may admit non-trivial monodromy $\int_{\gamma^1} da \in 2\pi\mathbb{Z}$ around non-trivial cycles $\gamma^1$, induced by the identification 2.87. The closed form $\frac{1}{2\pi}da$ is just the dual description of the 1-form current $\frac{1}{g^2} * H_3$ and the

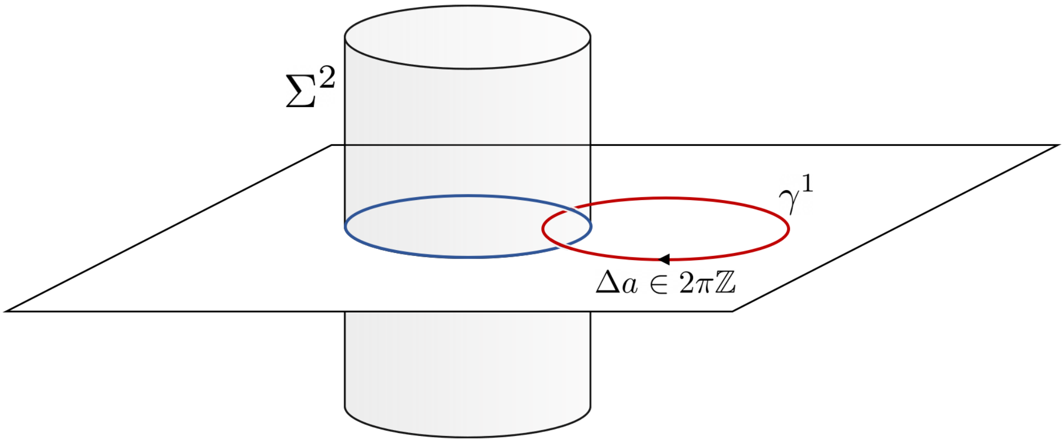



Figure 2.5: The "Wilson Surface" creates a string which propagates along the world-sheet $\Sigma^2$. An axion that encircles the string picks up a non-trivial monodromy controlled by the charge of the string.

monodromy[18] measures the charge carried by the strings created by the world-sheet operators $e^{in\int_{\Sigma^2} B_2}$ (see figure 2.5).

By analogy to the Wilson lines from section 2.3.1, the gauge invariant operators constructed out of $a$ are exponentials of the form $e^{ina(x)}$ which are well-defined local operators, supported on 0-dimensional manifolds (i.e. points) $x \in X$. The gauged 0-form symmetry predicted by equation 2.78 is nothing but the constant shift-symmetry $a \to a + c$ for constant $c$ under which these exponentials transform as

$$e^{ina(x)} \to e^{inc} e^{ina(x)}. \tag{2.88}$$

Here we explicitly see that $c$ is only required to be a locally-constant function, which is $2\pi$-periodic (i.e. $c+2\pi$ is related to $c$ by a gauge transformation of $a$) and is hence counted by the cohomology group $H^0(X, \mathbb{R}/2\pi\mathbb{Z})$. Importantly, this symmetry is to be distinguished from the discrete shift symmetry (2.87), which is *always* gauged, even in absence of the gauge field $A_1$. When we include the latter however, we can also see explicitly how the shift symmetry (2.88) is gauged. In particular, gauge transformations of $A_1$ now also affect the scalar $a$

$$A_1 \to A_1 + d\lambda_0, \qquad a \to a + \lambda_0, \tag{2.89}$$

which follows immediately from the usual gauge transformation of the complex scalar $\phi$ (or, more generally, from the dualization procedure). It follows that we may fix a gauge where $a$ vanishes and the second term in (2.84) reduces to a mass term for the gauge field $A_1$. As an aside, we remark that neither the action (2.79) nor the action (2.84) can be dualized to yield a theory in terms of the pair $(V_1, a)$. This is reflective of the fact that it is impossible to write down a local QFT including both electric and magnetic sources.

### The Operator Perspective

Up until now we have made ample use of of the availability of conserved currents and local representations of the symmetry action to discuss the symmetry content of the action (2.72).

[18] While we motivated the periodicity of $a$ via the Higgs model, *any* 0-form obtained by dualizing a 2-form (more generally a $(d-2)$-form) will admit non-trivial monodromy. This is required for the dualization to properly reproduce the quantization $\left[\frac{H_3}{2\pi}\right] \in H^3(X, \mathbb{Z})$, which can be achieved by adding a dualizing term $\frac{1}{2\pi}\int da \wedge H_3$ with $\left[\frac{da}{2\pi}\right] \in H^1(X, \mathbb{Z})$. This fact is general, and is the reason why e.g. the dual field strength $G_2$ was quantized in section 1.6.1. See also [42] or [21], section 1.6.1.

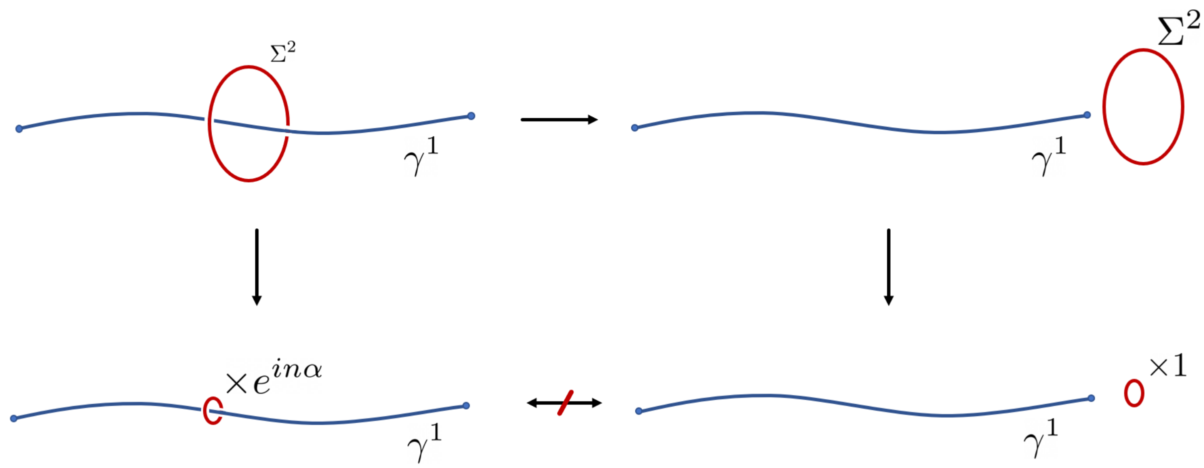



Figure 2.6: The action of the symmetry operator $U_{e^{i\alpha}}(\Sigma^2)$ becomes ambiguous when acting on an endable line operator. Consistency requires $e^{in\alpha} = 1$, so that only the trivial operator $U_1(\Sigma^2)$ remains.

Of course, this is rather natural for continuous global symmetries where we have these tools available to us. It is nonetheless interesting to consider how things play out in terms of the more abstract operator language. Consider for instance what happens to the Wilson lines of our theory once we gauge the magnetic 1-form symmetry. While previously such Wilson lines could only be defined for *closed* loops $\gamma^1$ (ignoring the case of bounded or non-compact spacetimes), we are now granted a new set of gauge invariant line operators supported on curves with boundaries. In particular, we can allow such curves to end on local insertions of $e^{ia(x)}$, which therefore lead to a new class of gauge invariant operators

$$\exp\left(i\int_{\gamma^1} A_1 - i\int_{\partial\gamma^1} a\right). \tag{2.90}$$

We say that the Wilson lines have become endable. We then consider how these new endable operators transform under the electric 1-form symmetry, in particular by considering the value of the correlator

$$U_{e^{i\alpha}}(S^2)\exp\left(in\int_{\gamma^1} A_1 - in\int_{\partial\gamma^1} a\right) = \exp\left(i\alpha\int_{S^2}\frac{1}{e^2}*F_2\right)\exp\left(in\int_{\gamma^1} A_1 - in\int_{\partial\gamma^1} a\right), \tag{2.91}$$

for any $S^2$ that surrounds $\gamma^1$. On the one hand, we may consider shrinking the $S^2$, eventually crossing the charge $n$ Wilson line so that we pick up a phase $e^{in\alpha}$. However, now that $\gamma^1$ has a boundary, we may alternatively unlink $S^2$ and $\gamma^1$ by sliding the former off of the end points (see figure 2.6). When we subsequently shrink $S^2$ down to a point we do not cross any operators and we do not pick up a phase. We conclude that the operator $U_{e^{i\alpha}}(S^2)$ is no longer topological which signals that the associated symmetry is now broken. Conversely, the operators charged under the 0-form global symmetry are now no longer gauge invariant, by the very fact that they are charged under the (now-gauged) symmetry. This means that the local operators $e^{ina(x)}$ can now *only* appear at the end-points of line operators as in (2.90). We say that the operators $e^{ina(x)}$ are no longer *genuine* local operators.

A perfectly analogous phenomenon occurs for the pair $B_2$ and $V_1$ in the dual frame. Indeed, we can now consider "Wilson surfaces" for $B_2$ corresponding to axionic strings, supported on surfaces with boundary, and at this boundary attach an 't Hooft line

$$\exp\left(i\int_{\Sigma^2} B_2 - i\int_{\partial\Sigma^2} V_1\right) \tag{2.92}$$

The combination is gauge invariant and hence a good operator in the theory. Any circle that links $\Sigma^2$ can be smoothly unlinked and hence the symmetry operators of the 2-form symmetry are no longer topological.

### 't Hooft Anomalies

This pattern of gauging and breaking of symmetries can be understood in terms of a mixed 't Hooft anomaly between the electric $p$-form and magnetic $(d-p-1)$-form symmetry associated to any $p$-form gauge field. Indeed, equation 2.78 tells us that in gauging the magnetic 1-form symmetry, we broke the electric 1-form symmetry, which is the hallmark of a mixed 't Hooft. In fact, we can study this anomaly directly in Maxwell theory by simultaneously including background (classical) gauge fields for both symmetries. For the magnetic 1-form symmetry, we have already seen that including the BF-coupling corresponds to a substitution $dV_1 \to dV_1 - B_2^{(m)}$ in the dual frame, thus rendering the magnetic symmetry $V_1 \to V_1 + \Lambda_1$ gauged for all $\Lambda_1$. Gauging the electric 1-form symmetry similarly involves substituting all instances of $F_2$ by $F_2 - B_2^{(e)}$ to render the action invariant under arbitrary shifts of $A_1$. At the end of this procedure, we are left with the action

$$\int\left(-\frac{1}{2e^2}\left(F_2-\hat{B}_2^{(e)}\right)\wedge *\left(F_2-\hat{B}_2^{(e)}\right)-\frac{1}{2\pi}\hat{B}_2^{(m)}\wedge\left(F_2-\hat{B}_2^{(e)}\right)\right). \tag{2.93}$$

The 't Hooft anomaly is now manifest in the final term where we see a coupling occur between the two background gauge fields, which is *not* gauge invariant under gauge transformations of $B_2^{(m)}$. Consequently, we cannot render both dynamical without suffering a gauge anomaly. As mentioned briefly at the end of section 2.2.2, we can still do so if we imagine our theory as living on the boundary $X$ of a five-dimensional spacetime $Y$. If we supplement the action above by a bulk theory with action

$$-\frac{1}{2\pi}\int_Y\left(\hat{B}_2^{(m)}\wedge d\hat{B}_2^{(e)}\right), \tag{2.94}$$

we find that the anomalous gauge transformation $\hat{B}_2^{(m)} \to \hat{B}_2^{(m)} + d\lambda_1^{(m)}$ now cancels among the two terms

$$\begin{aligned}\int_X\left(d\lambda_1^{(m)}\wedge\hat{B}_2^{(e)}\right)-\int_Y\left(d\lambda_1^{(m)}\wedge d\hat{B}_2^{(e)}\right)&=\int_X\left(d\lambda_1^{(m)}\wedge\hat{B}_2^{(e)}\right)-\int_Y d\left(d\lambda_1^{(m)}\wedge\hat{B}_2^{(e)}\right)\\&=\int_X\left(d\lambda_1^{(m)}\wedge\hat{B}_2^{(e)}\right)-\int_X\left(d\lambda_1^{(m)}\wedge\hat{B}_2^{(e)}\right).\end{aligned} \tag{2.95}$$

The action (2.94) is the anomaly theory which encodes the 't Hooft anomaly[19] and the mechanism by which a bulk anomaly cancels one on the boundary is known as anomaly inflow.

[19] We could have equivalently omitted the $\hat{B}_2^e$ from the final term in equation 2.93, in which case the theory would not have been invariant under electric gauge transformations (since $F_2 \to F_2 + d\lambda_1^{(e)}$). The relevant anomaly theory would then be given $-\int_Y\left(d\hat{B}_2^{(m)}\wedge\hat{B}_2^{(e)}\right)$ which reduces to (2.94) after integration by parts. See also the discussions in e.g. [43, 44] for a more formal treatment.

#### Partial Symmetry Breaking

Despite the mixed 't Hooft anomaly, there is no reason why we should choose to completely gauge one symmetry, while breaking the other. Indeed, it is clear from the dual picture that we could choose to couple $A_1$ to a field of generic (integral) charge $q$. In the original frame this corresponds to a non-trivial constant multiplying the BF-term. As we now show, this corresponds to breaking the global symmetry down to a discrete sub-group of the original continuous global symmetry groups. It is here that the operator formalism becomes essential.

Consider the gauge transformations of the modified theory. In the frame with the axion, these read

$$A_1 \to A_1 + d\lambda_0, \qquad a \to a + q\lambda_0. \tag{2.96}$$

We find that the operators (2.90) are now only gauge invariant if $q$ Wilson lines emanate from a single insertion of $e^{ia(x)}$. As a result, the unit charge Wilson line is no longer endable. If we repeat the argument from above, we now find that there remain some non-trivial operators $U^e_g(S^2)$ which remain topological. Consider for example the correlator

$$U^e_{e^{ip/q}}(S^2)\exp\left(iq\int_{\gamma^1} A_1 - i\int_{\partial\gamma^1} a\right) = \exp\left(\frac{ip}{q}\int_{S^2}\frac{1}{e^2}*F_2\right)\exp\left(iq\int_{\gamma^1} A_1 - i\int_{\partial\gamma^1} a\right). \tag{2.97}$$

As before, we may slide off the sphere $S^2$ and conclude that the phase-factor must vanish (see figure 2.6). What is new however, is that this result is not inconsistent with simply shrinking the sphere and crossing the Wilson lines. Indeed, in this latter case, we find that the phase factor is precisely given by $e^{iq(p/q)} = e^{ip}$ so that if $p \in 2\pi\mathbb{Z}$ (corresponding to $\alpha \in 2\pi\mathbb{Z}/q$) the composite operator is neutral under $U^e_{e^{ip/q}}$. We therefore find that the symmetry operators remain topological for a discrete sub-group $\mathbb{Z}_q$. Note moreover that closed Wilson lines are still allowed, so that there do still exist operators charged under the remnant $\mathbb{Z}_q$ symmetry. A perfectly analogous argument holds for the operators (2.92).

#### Conclusions

Before we close our discussion of BF theory, let us highlight some key lessons.

- Given a theory with a continuous $p$-form global symmetry, we can gauge it by coupling it to a $(p+1)$-form gauge field. When we gauge a global symmetry in this way, its associated conserved current will become exact, while the $(p+1)$-form global symmetry associated to the newly introduced $(p+1)$-form gauge field will in turn be broken.
- Alternatively, we may consider explicitly breaking a $p$-form global symmetry by coupling the associated gauge field to charged matter. In particular, charge $q$ matter breaks a $U(1)$ global symmetry down to a $\mathbb{Z}_q$ sub-group and renders the charged operators endable. This breaks the topological dependence of the associated symmetry operators.
- Of particular interest to us are the 0- and 1-form cases discussed here. The former corresponds to a periodic scalar field which has no local gauge transformations, but which admits a periodic identification corresponding to a large gauge transformation.

### 2.3.3 Example: Axion-Electrodynamics

As a final example of $p$-form symmetries in gauge theories, we will consider another modification of Maxwell theory, namely axion electrodynamics. This case will be of particular interest to us once we discuss type IIB EFTs. As a first step however, we first consider a more modest modification of Maxwell theory.

**Theta Angle and the Witten Effect**

Recall that the Maxwell action admits an additional topological $\theta$-term

$$\int \left( -\frac{1}{2e^2} F_2 \wedge *F_2 + \frac{\theta}{8\pi^2} F_2 \wedge F_2 \right) . \tag{2.98}$$

While a complete discussion of the role of the $\theta$-term is beyond the scope of this thesis, there are a number of interesting phenomena that we would like to highlight. Firstly, note that this term does not affect the classical theory, as it can locally be written as a total derivative. Nevertheless, this need not mean that it vanishes, even for spacetimes without boundary. Much like how the magnetic charge corresponded to the first Chern class of the gauge bundle, the $\theta$-term corresponds to the first Pontryagin class of the bundle. For $U(1)$-bundles, the latter is given by the square of the first Chern class so that the $\theta$-term is non-vanishing for topologically non-trivial configurations of the gauge field. In particular, this means that the $\theta$-term is quantized in the sense that

$$\int \left( \frac{1}{8\pi^2} F_2 \wedge F_2 \right) \in \mathbb{Z}, \tag{2.99}$$

which renders $\theta$, $2\pi$-periodic in the path-integral. The physical effect of the $\theta$-term is to modify the spectrum of the theory. In particular, the Noether procedure now leads to the modified electric current

$$\tilde{J}_2^e := \left( \frac{1}{e^2} * F_2 - \frac{K\theta}{4\pi^2} F_2 \right) . \tag{2.100}$$

To see the implication of this modification of the notion of electric charge, consider the correlator

$$\langle \tilde{Q}^e(S^2) T_m(\gamma^1) \rangle \equiv \left\langle \int_{S^2} \left( \frac{1}{e^2} * F_2 - \frac{\theta}{4\pi^2} F_2 \right) T_m(\gamma^1) \right\rangle , \tag{2.101}$$

where $S^2$ links $\gamma^1$ once. As before, the term $\frac{1}{e^2} * F_2$ acts trivially on the 't Hooft line, but the $\theta$-term now detects the fact that the 't Hooft line fixes the magnetic flux through $S^2$ to $m$. Thus, we find that the correlator above evaluates to

$$-\frac{m\theta}{2\pi} T_m(\gamma^1), \tag{2.102}$$

which indicates that the charge $m$ 't Hooft line now acquires a fractional electric charge proportional to $\theta$. This is known as the Witten effect [45], and we will encounter it in several guises in the remainder of this thesis. The Witten effect has another manifestation once we consider how the periodicity of $\theta$ affects our previous reasoning. In particular, while the partition function of the action (2.98) is invariant under shifts $\theta \to \theta + 2\pi$, equation 2.102 shows that it affects the 't Hooft lines. Indeed, the charge operator in (2.101) picks up an extra $\frac{1}{2\pi} \int_{S^2} F_2$, and hence, the charge $m$ 't Hooft line now acquires an additional $m$ units of electric charge, creating a state known as a *dyon*. Nevertheless, the spectrum as a whole is left invariant once we recognize that the most general line operator in Maxwell theory is in fact such a dyonic line operator $L_{n,m} = W_n T_m$, given by the simultaneous insertion of a charge $n$ Wilson line and a charge $m$ 't Hooft line. These line operators fill out a lattice $\mathbb{Z} \times \mathbb{Z}$, on which the shift $\theta \to \theta + 2\pi$ acts as $(n, m) \to (n + m, m)$, which leaves the lattice invariant.

### The $SL(2,\mathbb{Z})$ Duality Group

While the periodic $\theta$ comes with an obvious duality of Maxwell theory, we have also seen that Maxwell theory contains another, more subtle electric-magnetic duality. This duality extends to the theory including the theta angle. To make this manifest, we introduce the complex coupling

$$\tau = \frac{\theta}{2\pi} - \frac{2\pi i}{e^2}, \tag{2.103}$$

in terms of which we can rewrite the original action as

$$\frac{1}{2\pi}\int\left(\frac{1}{2}\mathrm{Im}\,\tau F_2\wedge *F_2 + \frac{1}{2}\mathrm{Re}\,\tau F_2\wedge F_2\right). \tag{2.104}$$

If we now perform the dualization procedure from section 1.6.1, we obtain the dual action

$$\frac{1}{2\pi}\int\left(\frac{1}{2}\mathrm{Im}(-\tau^{-1})G_2\wedge *G_2 + \frac{1}{2}\mathrm{Re}(-\tau^{-1})G_2\wedge G_2\right). \tag{2.105}$$

with

$$G_2 = -\left(\mathrm{Im}\,\tau * F_2 + \mathrm{Re}\,\tau F_2\right). \tag{2.106}$$

We see that in addition to the discrete shift-symmetry $\tau\to\tau+1$ it inherits from the $\theta$-angle, the theory has another duality which maps $\tau\to-\frac{1}{\tau}$. Together, these two transformations generate the *modular group* $SL(2,\mathbb{Z})\cong Sp(2,\mathbb{Z})$

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad \begin{cases} a,b,c,d\in\mathbb{Z},\\ ad-bd=1,\end{cases} \tag{2.107}$$

and it is the analog of the symplectic duality group from supergravity we encountered in section 1.6. In fact, it *is* the duality group of supergravity, since the theory above corresponds to the $\mathcal{N}=2$ vector sector for a theory with zero vector multiplets. In this case, the action for the remaining gravi-photon simply reduces to Maxwell theory. Moreover, we see explicitly how the classical duality group $Sp(2,\mathbb{R})$ is broken down to a discrete sub-group via the quantization of the Wilson/'t Hooft lines.[20]

### Axion Electrodynamics

The effects discussed above appear rather more dramatically once we promote the periodic $\theta$-angle to a dynamical field. Its periodicity then becomes a gauge transformation of this field, which we therefore identify as an axion. The resulting theory is given by

$$\int\left(-\frac{1}{2e^2}F_2\wedge *F_2 + \frac{K}{8\pi^2}aF_2\wedge F_2 - \frac{1}{2g^2}da\wedge *da\right), \tag{2.108}$$

whose equations of motion now read

$$A_1:\ \frac{1}{e^2}d*F_2 = \frac{K}{4\pi^2}da\wedge F_2,\quad a:\ \frac{1}{g^2}d*da = -\frac{K}{8\pi^2}F_2\wedge F_2. \tag{2.109}$$

We see that the coupling to the axion now explicitly breaks the electric 1-form current. Nevertheless, as in the previous section, a discrete symmetry, controlled by the value of $K$,

[20] Recall that their quantization *defined* the gauge group to be $U(1)$.

remains. Indeed, by analogy to the modified current (2.100), we can imagine defining a modified current

$$d\tilde{J}_2^e := d\left(\frac{1}{e^2} * F_2 - \frac{K}{4\pi^2} a F_2\right) = 0. \tag{2.110}$$

While conserved by virtue of the equations of motion, it is again not invariant under $a \to a + 2\pi$. This time however, this is a genuine gauge transformation of the theory, so that $\tilde{J}_2^e$ is not a well-defined current. Nevertheless, we can still use it to construct the associated symmetry operators, which remain well-defined for $g = e^{2\pi i p/K}$ [46]

$$U_{e^{2\pi i p/K}}(\Sigma^2) = \exp\left(\frac{2\pi i p}{K}\int_{\Sigma^2}\left(\frac{1}{e^2} * F_2 - \frac{K}{4\pi^2} a F_2\right)\right), \quad p \in \mathbb{Z}_K, \tag{2.111}$$

which is gauge invariant by virtue of the quantization of $\frac{1}{2\pi}F_2$. This phenomenon is perfectly analogous to what we encountered in our discussion of BF-theory, where we arrived at a similar conclusion based on the requirement that $U_g$ be topological. Here it can be understood as a direct consequence of the Witten effect, applied to a dynamical theta-angle. These conclusions can similarly be reached by moving to a dual frame. The dualization of the gauge field proceeds completely analogously, only now the $\theta$-angle is dynamical. It is clear that the would-be conserved magnetic current in this frame

$$\frac{1}{2\pi} G_2 = \frac{1}{e^2} * F_2 - \frac{aK}{4\pi^2} F_2, \tag{2.112}$$

is no longer gauge invariant, while the gauge invariant field strength

$$\frac{1}{2\pi}\tilde{G}_2 := \frac{1}{e^2} * F_2 = \frac{1}{2\pi} G_2 + \frac{aK}{4\pi^2} F_2, \tag{2.113}$$

is no longer conserved [21].

Similarly, the equation of motion for $a$ tells us that the axionic shift symmetry $a \to a + c$ is explicitly broken by the coupling to the gauge field. Nevertheless, for $|K| \neq 1$ a discrete shift-symmetry remains, which is most readily seen by noting that shifts $a \to a + 2\pi/K$ leave the action invariant, without corresponding to gauge transformations of $a$. From an operator perspective, this again also follows from the existence of gauge invariant topological operators

$$U_{e^{2\pi i p/K}}(\Sigma^3) = \exp\left(\frac{2\pi i p}{K}\int_{\Sigma^3}\left(\frac{1}{g^2} * da + \frac{K}{8\pi^2} A_1 \wedge F_2\right)\right), \quad p \in \mathbb{Z}_K. \tag{2.114}$$

### Chern-Weil Currents

While so far we have interpreted the equations of motion as broken global symmetries, following the logic from section 2.3.2 we may also interpret them as gauging the currents

$$J_3 = \frac{1}{4\pi^2} da \wedge F_2, \qquad J_4 = \frac{1}{8\pi^2} F_2 \wedge F_2. \tag{2.115}$$

These are examples of so-called *Chern-Weil* currents. Indeed, more generally we may consider arbitrary wedge products of gauge field strengths $F_{p+1}$

$$J := F_{p+1} \wedge \ldots \wedge F_{q+1}, \tag{2.116}$$

[21] The distinction between these currents has appeared in the literature under the terms Page and Maxwell currents. More specifically, the external current that couples to $\tilde{G}$ is the Maxwell current, while the current that couples to $G_2$ is the Page current. In light of our eventual applications, we remark that brane states couple (in four dimensions) to the latter.

which are automatically closed by virtue of their associated Bianchi identities. These were first discussed systematically in the literature in [47], to which we refer for a more thorough discussion of such currents in the non-abelian case.

While it is clear that such symmetries are not readily derived from any transformation of the fields, they nonetheless define quantized charge operators which exponentiate to topological operators of the theory. One might object that the current $J_4$ is a top-form and as such trivially conserved. Nevertheless, our discussion of the $\theta$-angle suggests that it should be interpreted as a topological charge associated to the full spacetime. In this sense we do not require the current to be conserved locally. In particular, the current $J_4$ is an example of a "(−1)-form" symmetry. Although breaking such a symmetry is slightly subtle, it is clear that absent other effects, gauging it corresponds to rendering the associated current cohomologically trivial on-shell. The situation is less subtle for the $J_3$ current, which equation 2.109 tells us is similarly gauged.

### Monopoles and Strings

Let us now consider how we can couple the theory (2.108) to magnetic charge. Up until now, we have essentially viewed magnetic charge as equivalent to electric charge, as we may always move to a dual frame where this is the case. However, the situation becomes more subtle in the presence of Chern-Simons terms. To see why, let us present an alternative definition of the latter. Absent monopoles, we can introduce an auxiliary five-manifold $Y$ with boundary $X$ to rewrite this term as

$$\frac{K}{8\pi^2}\int_X aF_2 \wedge F_2 = \frac{K}{8\pi^2}\int_Y da \wedge F_2 \wedge F_2, \tag{2.117}$$

where we also have to specify an extension of the fields on $X$ into the bulk of $Y$. The resulting (exponentiated) action is in fact independent of the choice of $Y$, since the difference between two such choices $Y_1$ and $Y_2$ is $2\pi$-quantized (thus dropping out of the path integral)

$$\frac{K}{8\pi^2}\left(\int_{Y_1} da \wedge F_2 \wedge F_2 - \int_{Y_2} da \wedge F_2 \wedge F_2\right) = \frac{K}{8\pi^2}\int_{Y_1 \cup Y_2^-} da \wedge F_2 \wedge F_2 \in 2\pi K\mathbb{Z}, \tag{2.118}$$

where $Y_2^-$ denotes $Y_2$ with the orientation reversed. Indeed, this is a generic manipulation for topological terms that could also have been applied to the BF term from section 2.3.2.

This construction fails in the presence of monopoles, due to the violation of the Bianchi identity

$$\frac{1}{2\pi}dF_2 = \delta_3(\gamma^1), \tag{2.119}$$

where $\delta_3(\gamma^1)$ is a delta function, localized on the world-line. Indeed, it is easy to see that in this case

$$\int_X aF_2 \wedge F_2 = \int_Y d\left(a \wedge F_2 \wedge F_2\right) = \int_Y da \wedge F_2 \wedge F_2 + 4\pi\int_Y aF_2 \wedge \delta_3(\gamma^1). \tag{2.120}$$

We can remedy this by assuming that the monopole carries its own dynamics, described by a world-volume action

$$S_{mono} = \int_{\gamma^1}\left(\frac{1}{2v^2}d_{KA}\sigma \wedge *_1 d_{KA}\sigma + \frac{a}{2\pi}d_{KA}\sigma\right), \qquad d_{KA}\sigma := d\sigma - KA_1, \tag{2.121}$$

where $\sigma$ is a charged scalar that transforms as

$$A_1 \to A_1 + d\lambda_0, \qquad \sigma \to \sigma + K\lambda_0. \tag{2.122}$$

When we add this last term to the Chern-Simons term, we find that it cancels the offending term, so that the combined action can be described by the (gauge-invariant) action

$$\int_X \left( \frac{K}{8\pi^2} a F_2 \wedge F_2 + \frac{a}{2\pi} d_{KA}\sigma \wedge \delta_3(\gamma^1) \right) = \int_Y \left( da \wedge F_2 \wedge F_2 + \frac{1}{2\pi} da \wedge d_{KA}\sigma \wedge \delta_3(\gamma^1) \right). \tag{2.123}$$

The basic idea here is that the Chern-Simons term becomes anomalous in the presence of monopoles, but this anomaly is canceled by the anomaly of the monopole action. The latter is nothing but the witten effect. Indeed, under shifts $a \to a + 2\pi$, the monopole action transforms as

$$S_{mono} \to S_{mono} - K \int_{\gamma^1} A_1, \tag{2.124}$$

which tells us that the monpole picks up $-K$ units of electric charge, thus becoming a dyon. Note morover that a coupling $\int_{\gamma^1} \frac{aK}{2\pi} A_1$, which would be sufficient to cancel this anomaly would not be gauge invariant under transformations of $A_1$, thus signaling the need for the charged world-volume field $\sigma$.

What is the interpretation of this field? The classic example where this mechanism plays out is for the Polyakov-'t Hooft monopole (see e.g. [48] ch. 15 for a review). In this case, we view the $U(1)$ gauge theory as the spontaneously broken phase of an $SU(2)$ gauge theory, coupled to an adjoint scalar with a potential

$$V(\phi) \sim \left(\mathrm{tr}(\phi^2) - v^2\right). \tag{2.125}$$

This leads to a vev

$$\phi \sim \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}. \tag{2.126}$$

We are left with a massless $U(1)$ gauge field associated to the generator that preserves this vev, while the remaining generators acquire a mass. At low energies, the effective $U(1)$ gauge theory now contains monopoles which arise as solitonic solutions of the $SU(2)$ gauge theory. Geometrically, the non-abelian gauge group contributes the non-trivial topology required by a monopole configuration of the $U(1)$ field. We can compare this to monopoles created by 't Hooft lines, where the singularity on the world-line effectively modifies the topology of spacetime. Indeed, the $SU(2)$ description of the monopole field configuration allows us to resolve its core, which is no longer singular.

The corresponding monopole solution now possesses two kinds of collective coordinates, which correspond to massless deformations of the monopole solution. These include translational degrees of freedom, but also a dyonic collective coordinate of the type described above. The latter corresponds to rotations of the monopole field configuration, inside its field space. More generally, we expect, based on the general argument presented above, that any $U(1)$ monopole should carry some anomalous world-volume theory to cancel the bulk anomaly.

Another way of seeing the need for this world-volume degree of freedom, is to note that monopoles break the magnetic 1-form symmetry, and by extension, break the Chern-Weil current $F_2 \wedge F_2$ which the axion coupling gauged [47]. When we include the monople action however, the axion equation of motion is modified to

$$\frac{1}{g^2} d * da = -\left( \frac{K}{8\pi^2} F_2 \wedge F_2 + \frac{1}{2\pi} d_A \sigma \wedge \delta_3(\gamma^1) \right), \tag{2.127}$$

which now tells us a linear combination of bulk and localized currents are gauged, while the bulk current itself can safely be broken.

Finally, let us mention the case of axionic strings. Recall from section 2.3.2 that these are states around which the axion picks up a non-trivial monodromy $\int_{S^1} da$, which is the zero-form analog of $\int_{S^2} F_2$. These too leads to an anomaly which should be canceled by an appropriate world-sheet theory. Alternatively, such strings violate the associated "Bianchi identity" $\frac{1}{2\pi} d(da) = \delta_2(\Sigma^2)$, so that we break the gauged Chern-Weil symmetry $da \wedge F_2$

$$0 \overset{d^2=0}{=} d\left( \frac{1}{e^2} d * F_2 \right) = \frac{K}{4\pi} d(da) \wedge F_2 = \frac{K}{8\pi^2} \delta_2(\Sigma^2) \wedge F_2 \neq 0. \tag{2.128}$$

In this case, we can include a chiral boson (i.e. $d\sigma = *d\sigma$) on the world-sheet which is electrically charged under the gauge field (see e.g. [49, 50] for details). This modifies the equation of motion for the latter, resolving the inconsistency.

### Conclusions

Finally, let us close this sub-section by highlighting some key lessons we can draw from this example

- The presence of three-field Chern-Simons terms of the kind above break both the electric 1-form and axionic shift-symmetries, potentially down to discrete sub-groups controlled by the integer $K$.
- This term also gauges two Chern-Weil symmetries, whose currents are given by various wedge products of field strengths.
- The Chern-Simons term induces the Witten effect on monopoles. The broken electric symmetry can also be understood as a consequence of this effect, which tells us that electric charge is only defined mod $K$.
- Monopoles lead to an anomaly in the Chern-Simons term, which requires it to carry an anomalous world-volume theory which cancels the bulk anomaly. The Witten effect can then be understood as the anomaly of the world-volume theory. A similar statement holds for axionic strings.

## 2.4 More Generalized Global Symmetries

While this in principle concludes our review of higher-form global symmetries, these are not the only generalizations of global symmetries that have been realized in recent years. In this section we would like to comment on some of the more recent developments regarding generalized global symmetries, in particular with a focus on their application in a pure QFT context while reserving a discussion of their role in the swampland program for the next section. Indeed, a remarkable body of literature has been built up that seeks to understand quantum field theories exhibiting such generalized global symmetries. While we cannot do this rich subject justice, we do wish to give a sense of what directions these further generalizations take here.

### 2.4.1 Higher-Group Symmetries

The abstract operator definition of global symmetries introduced in section 2.2 led to an immediate generalization in the form of $p$-form global symmetries. These satisfy all the usual properties of ordinary global symmetries, except that their associated symmetry operators are supported on $(d-p-1)$-dimensional manifolds (and consequently, act on $p$-dimensional operators). It is not hard however, to come up with further generalizations to this programme and indeed, several such variations exist.

The simplest such generalization is given by the so-called *higher-group symmetries*. In this case, we have that the symmetry operators of two distinct higher-form global symmetries mix to give a mathematical structure known as a higher-group. In particular given a $p$-form symmetry with group $G$ and a $q$-form symmetry with group $H$, their symmetry operators may not furnish a representation of the product group $G \times H$, but the operators of one may act non-trivially on the other. The simplest such mixing occurs when we have a 0-form symmetry and a 1-form symmetry leading to a 2-group symmetry.

A somewhat more down-to-earth perspective of this phenomenon is given in terms of the background gauge fields for these symmetries. Indeed, when we couple the symmetries that mix into a 2-group to background fields, the 2-group structure appears as a mixing of the gauge transformations for these fields. In particular, given the 1- and 2-form background fields $\hat{A}_1$, $\hat{A}_2$ for the 0- and 1-form symmetries, then a typical 2-group will appear as a mixed gauge transformation of the form

$$\hat{A}_1 \to \hat{A}_1 + d\lambda_0, \quad \hat{A}_2 \to \hat{A}_2 + d\lambda_2 + \alpha d\hat{A}_1 \wedge \lambda_0, \tag{2.129}$$

for some constant $\alpha$. Of course, viewed this way, the phenomenon above is nothing but the mixed gauge transformations (1.2) we encountered in type IIB string theory. Indeed, gauged higher-group symmetries are ubiquitous in string theory constructions, most famously in the Green-Schwarz mechanism. The primary distinction here is that the preceding discussion applies to non-dynamical background fields for global symmetries.

Let us make this idea more concrete by means of an example. In fact, we have already encountered such an example in the previous section: namely, axion electrodynamics. Following [46], we couple the theory to background gauge fields for its $p$-form global symmetries. Compared to the examples we have considered previously however, there are two subtleties to deal with. The first is the fact that the electric 1-form symmetry and the axionic shift symmetry are broken down to a discrete $\mathbb{Z}_K$ sub-group. Their background field should thus be understood to denote a discrete (hence flat) background connection specifying cohomology classes (cf. our discussion at the end of section 2.2.2)

$$\left[\frac{K}{2\pi}\hat{C}_1\right] \in H^1(X, \mathbb{Z}_K), \quad \left[\frac{K}{2\pi}\hat{B}_2^{(e)}\right] \in H^2(X, \mathbb{Z}_K), \tag{2.130}$$

which we choose to represent by continuous gauge fields $\hat{C}_1$ and $\hat{B}_2^{(e)}$ with restricted holonomy. The second subtlety is related to the axion coupling, which involves a bare axion, rather than its "field strength" $da$. As such we cannot simply substitute $da \to da - \hat{C}_1$ to ensure invariance under arbitrary shifts. Nevertheless, we can invoke the construction from section 2.3.3 to rewrite this term using an auxiliary five-manifold $Y$

$$\frac{K}{8\pi^2}\int_X aF_2 \wedge F_2 = \frac{K}{8\pi^2}\int_Y da \wedge F_2 \wedge F_2. \tag{2.131}$$

Written in this way, we can now readily couple the theory to background gauge fields as[22]

$$
\begin{aligned}
S = &\int_X \left( -\frac{1}{2e^2}(F_2 - \hat{B}_2^{(e)}) \wedge *(F_2 - \hat{B}_2^{(e)}) - \frac{1}{2g^2}(da - \hat{C}_1) \wedge *(da - \hat{C}_1) \right) \\
&+ \frac{1}{2\pi} \int_X \left( d\hat{B}_2^{(m)} \wedge A_1 - a\, d\hat{C}_3 \right) + \frac{K}{8\pi^2} \int_Y (da - \hat{C}_1) \wedge (F_2 - \hat{B}_2^{(e)}) \wedge (F_2 - \hat{B}_2^{(e)}),
\end{aligned}
\tag{2.132}
$$

where, for reasons that will become clear shortly, we have integrated the gaugings of $F_2$ and $da$ (first term, second line) by parts. The problem we are faced with is that in the presence of the background gauge fields, the action is no longer independent of the choice of $Y$. This can be fixed, up to terms involving only background gauge fields, by including yet more counter terms to cancel the offending parts of the last term[23]

$$
-\frac{K}{8\pi^2} \int_Y \left( da \wedge \hat{B}_2^{(e)} \wedge \hat{B}_2^{(e)} + 2\hat{C}_1 \wedge \hat{B}_2^{(e)} \wedge F_2 \right). \tag{2.133}
$$

By virtue of flatness of the background fields $\hat{C}_1$ and $\hat{B}_2^{(e)}$, we can apply Stokes' theorem to write it as an integral over $X$, which we can then combine with the gaugings of $F_2$ and $da$ to obtain

$$
\frac{1}{2\pi} \int_X \left( \left( d\hat{B}_2^{(m)} + \frac{K}{2\pi} \hat{C}_1 \wedge \hat{B}_2^{(e)} \right) \wedge A_1 - a \left( d\hat{C}_3 + \frac{K}{4\pi} \hat{B}_2^{(e)} \wedge \hat{B}_2^{(e)} \right) \right). \tag{2.134}
$$

The upshot of this computation is that we now see appear the combinations of gauge fields

$$
\hat{H}_3^{(m)} := d\hat{B}_2^{(m)} + \frac{K}{2\pi} \hat{C}_1 \wedge \hat{B}_2^{(e)}, \quad \hat{G}_4 := d\hat{C}_3 + \frac{K}{4\pi} \hat{B}_2^{(e)} \wedge \hat{B}_2^{(e)}. \tag{2.135}
$$

If we wish for the construction to remain gauge invariant, up to anomaly terms, we conclude that the gauge fields $\hat{B}_2^{(m)}$ and $\hat{C}_3$ must transform non-trivially under gauge transformations of $\hat{C}_1$ and $\hat{B}_2^{(e)}$

$$
\begin{aligned}
&\hat{B}_2^{(m)} \to \hat{B}_2^{(m)} + d\Lambda_1^{(m)} - \frac{K}{2\pi} \Lambda_0 \hat{B}_e^{(2)} - \frac{K}{2\pi} \hat{C}_1 \wedge \Lambda_1^{(e)} + \frac{K}{2\pi} d\Lambda^{(0)} \wedge \Lambda_e^{(1)}. \\
&\hat{C}_3 \to \hat{C}_3 + d\Lambda_2 - \frac{K}{2\pi} \hat{B}_2^{(e)} \wedge \Lambda_1^{(e)} - \frac{K}{2\pi} \Lambda_1^{(e)} \wedge d\Lambda_1^{(e)},
\end{aligned}
\tag{2.136}
$$

where the mixing of 0-, 1- and 2-form symmetry transformations in particular signals a 3-group. It is of course also interesting to consider this phenomenon directly in terms of the topological operators, which was analyzed in [51].

### 2.4.2 Non-Invertible Global Symmetries

The second important class of generalizations are the *non-invertible global symmetries* which involve operators of a fixed co-dimension, but which admit fusion rules that generalize ordinary group multiplication. In particular, this means that these operators no longer furnish a group representation, but rather satisfy a fusion algebra of the form

$$
U_A(\Sigma^{d-p-1}) \times U_B(\Sigma^{d-p-1}) = \sum_C N_{AB}^C\, U_C(\Sigma^{d-p-1}), \tag{2.137}
$$

[22]For the present discussion we are not interested in 't Hooft anomalies. We therefore work modulo terms involving only background fields, which simply encode said anomalies.

[23]Note that terms involving only one background field such as $\frac{2K}{8\pi^2} da \wedge F_2 \wedge \hat{B}_2^{(e)}$ and $\frac{K}{8\pi^2} \hat{C}_1 \wedge F_2 \wedge F_2$ are $2\pi\mathbb{Z}$ valued by virtue of equation 2.130. They therefore do not contribute to the path integral.

which reduced to the usual $p$-form algebra whenever $N_{AB}^{C}$ is non-zero only one operator $C$. For a simple example of how these may occur, consider the fusion law satisfied by the Wilson lines in gauge theory. For the abelian case we had already concluded that inserting two Wilson lines simply results in a new Wilson line whose charge equals the sum of the constituent lines. The picture becomes more interesting for non-abelian gauge groups, where Wilson lines are labeled by a representation $\rho$ of the gauge group and are written

$$W_\rho(\gamma^1) = \mathrm{Tr}_\rho \exp\left(i \int_{\gamma^1} A_1\right), \tag{2.138}$$

with the trace taken in the representation $\rho$.[24] There is no good way to add two Wilson lines in different representations so that they satisfy a more general fusion algebra [52]

$$W_\rho(\gamma^1) \times W_\nu(\gamma^1) = \sum_i W_{\mu_i}(\gamma^1) \tag{2.139}$$

which can be understood as the creation of a multi-particle state in the representation $\rho \otimes \nu$, which is subsequently decomposed into irreducible representations as $\rho \otimes \nu = \oplus_i \mu_i$. Now, Wilson lines are generically not topological and hence do not constitute a non-invertible symmetry by the above definition, but it turns out that for special choices of the gauge group $G$, a subset of the Wilson line operators can become topological, hence furnishing a non-invertible global symmetry.

Finally, though we do not discuss it in any detail here, it has been proposed that non-invertible global symmetries exist even in a setting quite familiar by now, namely axion electrodynamics [53]. A more careful analysis than ours reveals that the (partially) broken axionic shift-symmetry survives as a non-invertible global symmetry, where the relevant topological operators are constructed out of the usual shift-symmetry $U_g(\Sigma^3)$, together with the partition function of a topological field theory for which the gauge field acts as a background connection. The latter cancels the dependence of $U_g$ on $\Sigma^3$ for $g \neq e^{2\pi i n/K}$ but leads to non-trivial fusion rules of the kind discussed above.

## 2.5 Generalized Global Symmetries in the Swampland

Let us now return to the main topic of this chapter and indeed this thesis, namely the role of generalized global symmetries in the swampland program. As we have already alluded to, generalized global symmetries have proven to be a useful organizing principle within the swampland. While at first only leading to connections between existing conjectures, more recently they have been used to formulate new conjectures as well. The goal of this section is to revisit the ideas from section 2.1 and view them in the language of generalized global symmetries.

The main idea is to extend the No Global Symmetries conjecture to include generalized global symmetries, including discrete ones. That is to say, any generalized global symmetry should be either broken or gauged. The idea for this somewhat predates the formal notion of a generalized global symmetry [23], as it was well understood that branes in string theory always carry gauge charge, which allows an outside observer to detect them. Conversely, brane states

[24] In the abelian case, the charge $n$ can also be understood to label the representations of the gauge group, which are always 1-dimensional (so that the trace operation is trivial). In fact, this is the origin of the distinction between $U(1)$ and $\mathbb{R}$ gauge theory, as the representations of the latter are labeled by real numbers rather than discrete integers.

that do not carry such gauge charge would be undetectable away from their core, leading to problems akin to those associated with global charge. [25] In fact, the NGS conjecture for higher-form global symmetries can be understood as a result of the ordinary NGS conjecture under compactification. Indeed, consider for example a circle compactification of Maxwell theory from five to four dimensions. In this case, the gauge field, which has a five-dimensional 1-form global symmetry, splits according to

$$A_M \to A_\mu \oplus A_5. \tag{2.140}$$

The four-dimensional theory now has a new 1-form global symmetry associated to the four-dimensional gauge field, but also a 0-form global symmetry associated to the compactified component of the gauge field. If we wish to prevent this we should demand that higher-form global symmetries are forbidden to begin with. Any UV-completion should then either gauge, or break these symmetries.

### 2.5.1 New Connections

This new, stronger requirement leads to various new connections between the conjectures discussed in section 2.1. The reasoning often proceeds along similar lines. Namely, given an effective theory, valid up to some cut-off, one asks how its global symmetries may be gauged or broken by UV effects, allowing for a consistent UV-completion. This can then lead to the prediction of new states or objects which break global symmetries, or we may expect certain symmetry-breaking terms to be required in the effective theory. We present a selection of such examples from the literature below.

**Completeness Hypothesis**

The completeness hypothesis for continuous, connected gauge groups can now be understood as a condition to ensure that all ungauged higher-form global symmetries are broken. Indeed, recall from our discussion in section 2.3.2 that by including charge $q$ matter, a $U(1)$ $p$-form global symmetry is broken down to a subgroup $\mathbb{Z}_q$. If one demands that the symmetry is broken completely, one requires that the spectrum of charges is complete, including higher-dimensional objects.

Nevertheless, we have already seen that this cannot be the whole story. For example, even in our simple examples of theories with generalized global symmetries, we saw that the magnetic 1-form symmetry was gauged by the inclusion of the charged particle, hence removing the symmetry without the need to introduce monopoles. Similarly, in axion electrodynamics, the electric 1-form symmetry is broken purely by virtue of gauging the Chern-Weil current, i.e. without the need for charged particles. Both of these are examples of Chern-Simons terms, which spoil the naive one-to-one correspondence between completeness and NGS.

More generally however, it has been argued [52] that completeness of the spectrum can be understood as arising from the absence of *any* topological operators. That is to say, we extend the NGS conjecture to the non-invertible symmetries from section 2.4.2, which due to their fusion algebra lose their interpretation as a symmetry. As discussed, these may survive in the presence of Chern-Simons terms [53], such that a complete spectrum may be needed to break them. This remains an open question however, as their realization in string theory constructions remains poorly understood. Nevertheless, this enlarged notion of NGS could

[25] Somewhat amusingly, the authors refer to (ordinary) global symmetries as $(-1)$-form gauge symmetries.

prove to be the generalization necessary to get to the heart of why global symmetries are forbidden in QG.

#### Higher-Groups and the WGC

In addition to revealing new connections to the completeness hypothesis, generalized global symmetries have also shed new light on the WGC. In [54] it was argued that aspects of the WGC for theories with multiple $p$-form gauge fields can be understood in terms of the higher-group structure of the associated $p$-form global symmetries. Suppose for instance that, starting from a UV-complete theory, we obtain axion electrodynamics at low energies. In the UV we expect the presence of both electric charges (of mass $m_e$), that break the remnant $\mathbb{Z}_K$ electric 1-form symmetry and axionic strings (of tension $T_e$), which break the axionic 2-form symmetry. These states come with energy-scales $\Lambda_e \sim m_e$ and $\Lambda_{string} \sim \sqrt{T_s}$, respectively, below which the symmetry is restored.

If we assume that we are well-below $\Lambda_e$, then the electric 1-form symmetry is a good symmetry of the theory, and we are free to introduce a background connection for this symmetry as in section 2.4.1. However, as is clear from equation (2.135), any background $\hat{B}_2^{(e)}$ automatically turns on a background connection for $\hat{C}_3$ as well! Hence, if we are to consistently recover axion electrodynamics at low energies, this means that the axionic 2-form symmetry must be a good symmetry whenever the electric 1-form symmetry is a good symmetry. This directly leads to a bound [46]

$$\Lambda_e \lesssim \Lambda_{string}, \qquad \Rightarrow \qquad m_e \lesssim \sqrt{T_e}. \tag{2.141}$$

In [54], it is then further argued that by assuming that these strings satisfy their version of the WGC, and that the associated axion couples to instantons with action $S_{inst} \sim g^{-2}$ [50], this leads to the statement of the weak gravity conjecture for the excitations of the string.

#### Chern-Weil Currents

Finally, in [47] it has been shown that by including Chern-Weil currents in the analysis, one can deduce many distinctly stringy phenomena by demanding that all such symmetries are broken. For instance, low-energy theories obtained from string theory typically contain three-field Chern-Simons terms (see for instance the last term in equation (1.4)). These terms can be understood as gauging particular Chern-Weil currents, and consistently breaking the remaining symmetries constrains the interactions of the objects that do so. In this way one finds that branes should be able to end on branes, understood as branes becoming endable in the sense of section 2.3.2. Moreover, it was shown that the world-volume degrees of freedom of the brane can be understood as the degrees of freedom that cancel the anomaly inflow from the bulk theory. This observation can in fact be used to re-derive the full Chern-Simons action of the brane.

### 2.5.2 Cobordism Conjecture

In addition to providing new connections between existing swampland conjectures, accounting for generalized global symmetries has also led to new conjectures. One of the most significant developments to come out of this is the cobordism conjecture [55]. In topology, cobordisms are a very coarse way of classifying manifolds. They answer the question of when two manifolds can be realized as the boundary of a manifold of one higher dimension. Crucially, this construction should take into account any structure (e.g. orientation, spin, symplectic etc.)

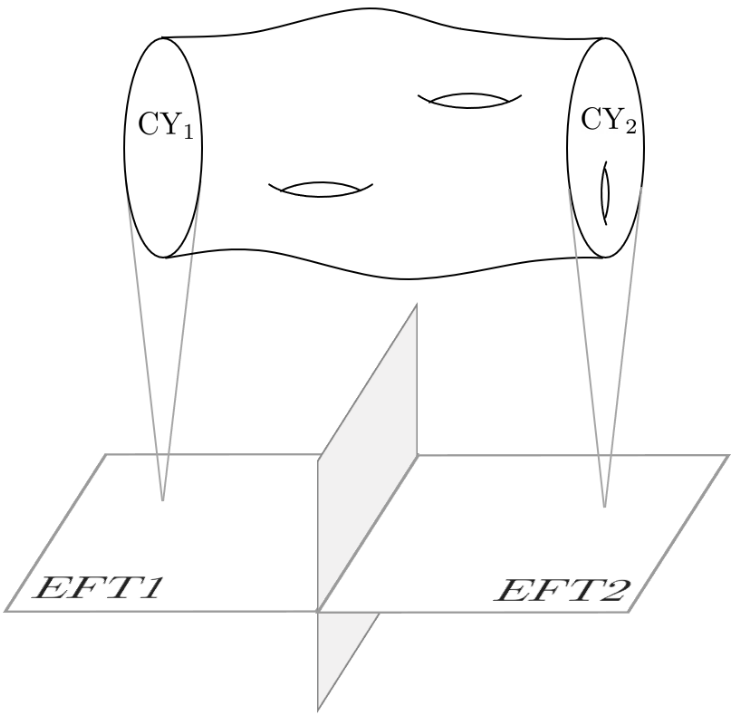



Figure 2.7: Illustration of a cobordism. Topology-changing transitions such as cobordisms should be allowed in the full theory to prevent a global symmetry. This implies the existence of defects which enable such transitions. Figure based on [32].

on these boundary manifolds and extend them into the "bulk" in such a way that we recover the original structure by restricting to the boundary (cf. our discussion around equation 2.131). This construction provides an equivalence relation between manifolds with a certain $G$-structure, which leads to a classification of such manifolds in terms of their equivalence class under this relation. The resulting set of equivalence classes is generically denoted $\Omega_d^G$ for a given dimension $d$ and $G$-structure, and it admits an abelian group structure.

The cobordism conjecture then states

**Cobordism Conjecture:** Consider some $D$-dimensional QG theory compactified on a $d$-dimensional internal manifold. All cobordism classes must vanish

$$\Omega_d^{QG} = 0. \tag{2.142}$$

Otherwise they give rise to a $(D-d-1)$-form global symmetry with charges $[M] \in \Omega_d^{QG}$.

The basic idea here is that any compactification manifold of e.g. string theory should carry a number of structures that make it a suitable string background. The collection of these structures is denoted $QG$ and manifolds with this structure can be collected into the cobordism group $\Omega_d^{QG}$. In the compactified theory, this background is allowed to fluctuate, which we recall gave rise to moduli. However, in the *full* theory, we should be able to transition between any two such backgrounds, even topologically distinct ones. A non-trivial cobordism class implies an obstruction to such transitions. From this perspective, we can therefore imagine the cobordism class as a $d$-form charge carried by the compact space, which should be forbidden by (a generalized) NGS conjecture.

In general however, we do not know what the full structure group of quantum gravity is. Rather, we can only proceed by approximation and consider e.g. backgrounds with a spin or $G$-bundle structure. These will not typically lead to a vanishing cobordism group, but we know that in the UV-complete theory, cobordisms between these ostensibly non-cobordant backgrounds should be possible. While the UV-theory therefore carries the structure necessary to trivialize these cobordism classes, from the low-energy perspective these transitions appear as defects/objects that interpolate between these non-cobordant backgrounds.

Reasoning based on the vanishing of cobordism classes has proven an effective tool to re-derive known properties of string theory, such as the existence of various brane states as necessary [56]. What makes this conjecture truly remarkable however, is that it has also led to new predictions. For instance, the original paper on the topic suggests the existence of undiscovered defects (analogous to brane states) that interpolate between otherwise non-cobordant backgrounds. Non-trivial cobordism groups have also been used to determine uncancelled anomalies of type IIB string theory [57] and subsequently proposing the relevant anomaly theory that restores the full duality group.

More dramatic still has been the proposal to take seriously the idea of a trivial cobordism class. In particular, it is well-known [58] that Kaluza-Klein compactification of e.g. five-dimensional gravity (cf. section 1.3.2), though classically stable[26], admits a non-perturbative instability by which the KK background tunnels to one with a hole, a region not covered by the metric solution. Worse still, one finds that once formed, this hole will grow leaving "nothing" behind. Such solutions are called bubbles of nothing and they rely crucially on the fact that the internal space shrinks to zero size as we approach the bubble. Thus a necessary condition for this to occur is that the background manifold can continuously be shrunk to zero size. In the past it has been argued that topological obstructions, such as the presence of spinors along with the metric background, can form a topological obstruction to such a pathological instability. This is essentially equivalent to the statement that the background defines a non-trivial cobordism class in $\Omega^{QG}$, which by the cobordism conjecture cannot be the case [59]. It follows that stability of a given compactification becomes a dynamical question, which is a far subtler matter (see e.g. [60, 61]) than the guaranteed protection afforded by topological obstructions. In a more general context, these bubbles-of-nothing are examples of end-of-the-world branes, domain walls which enact the cobordism to the trivial element in $[0] \in \Omega_d^{QG}$, which have garnered significant attention in recent years.

### 2.5.3 Symmetries in Type IIB Compactification

Finally, we turn to our case of interest, namely type IIB Calabi-Yau compactifications. These provide us with a concrete setting in which to study the gauging and breaking of (generalized) global symmetries. As we have seen, generalized global symmetries provide us with a new interpretation of phenomena predicted by many existing swampland conjectures. While we expect the global symmetry group to be trivial (once we account for the correct stringy effects), the way this is arranged in a given theory can be non-trivial and highlight interesting connections. This motivates us to consider global symmetries as they appear in explicitly realized string theory constructions.

In particular, we will investigate what global symmetries may appear in the vector sector

[26] Recall that small variations around the background produced precisely the tower of Kaluza-Klein modes. Had any of these modes appeared with a negative mass-squared, this would signal a classical instability.

of type IIB Calabi-Yau compactifications, which we introduced in chapter 1

$$S = \int \left( \frac{1}{2}\mathcal{I}_{IJ} F^I \wedge *F^J + \frac{1}{2}\mathcal{R}_{IJ} F^I \wedge F^J - K_{i\bar{j}} dz^i \wedge *d\bar{z}^{\bar{j}} \right). \tag{2.143}$$

We emphasize that the kinetic matrix $\mathcal{N} = \mathcal{R} + i\mathcal{I}$ and the Kahler metric $K_{i\bar{j}}$ are functions of the moduli. At generic points in the moduli space, these are expected to be very complicated functions, for which we do not have generic expressions. Nevertheless, near special limits in the moduli space the mathematical tools afforded to us by asymptotic Hodge theory will allow us to extract the limiting form of these functions, where they simplify considerably. This will provide us with concrete theories to study for their symmetry content, along the lines of this chapter.

Such special limits correspond to points on the moduli space where the corresponding Calabi-Yau manifold develops a singularity. These limits have been well-studied from both a mathematical, as well as a physical perspective. Regarding the latter, previous investigations of compactifications in these asymptotic regimes have shown that such limits often lie at infinite distance in field space. Moreover, they are associated with gauge couplings going to zero such that a global symmetry is restored, therefore providing connections to the WGC, the SDC and the NGS conjecture. In the following chapter, we will develop the tools needed to understand these results, and subsequently apply them to derive the explicit limiting form of the action (2.143).

# Chapter 3

# Asymptotic Hodge Theory

In this chapter we will present the mathematics necessary to study the vector sector of $\mathcal{N}=2$ supergravity near special limits in the Calabi-Yau moduli space. As we have already seen in section 1.6, these theories display a rich geometric structure. Important physical quantities are encoded in sections of various bundles over the scalar manifold. Indeed, for Calabi-Yau compactifications in particular, we encountered the following

- The Kahler potential, expressed in terms of the holomorphic $(3,0)$-form $\Omega$

$$e^{-K} = i\langle \Omega, \bar{\Omega} \rangle. \tag{3.1}$$

- The gauge kinetic functions, expressed in terms of the Hodge star matrix

$$\mathcal{M} = \begin{pmatrix} \langle \alpha_I, *\alpha_J \rangle & \langle \alpha_I, *\beta^J \rangle \\ \langle \beta^I, *\alpha_J \rangle & \langle \beta^I, *\beta^J \rangle \end{pmatrix} = \begin{pmatrix} -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -(\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \tag{3.2}$$

- The mass and physical charge of a (BPS) state $\mathbf{q}$, expressed in terms of the Hodge star matrix and $\Omega$, respectively

$$\mathcal{Q}^2_{\mathbf{q}} := \frac{1}{2}\mathbf{q}^T\mathcal{M}\mathbf{q} = \frac{1}{2}\langle \mathbf{q}, *\mathbf{q} \rangle, \qquad M^2_{\mathbf{q}} = e^K |\langle \Omega, \mathbf{q} \rangle|^2. \tag{3.3}$$

More generally we have seen the importance of the structure group of the symplectic bundle, which was intimately related to the dualities and symmetries of the vector sector. All of these structure arise naturally when studying the variation of the Hodge decomposition on the middle cohomology of the Calabi-Yau.

Starting in section 3.1, we therefore begin by reviewing the basic mathematics used to describe these variations, codified in a variation of Hodge structure (VHS). We will encounter the Hodge bundle, which is the flat symplectic bundle required by supersymmetry. Since this bundle is flat, its transition functions are constant and can be characterized by a *monodromy action*, which will be central for the rest of this chapter. In section 3.2 we then zoom in on regions near special limits in the moduli space where the Calabi-Yau manifold becomes singular. It is here that the monodromy group takes center stage. It turns out that these singularities are distributed along divisors in the moduli space, which generate an action of the monodromy group on the Hodge bundle. The nilpotent orbit theorem of Schmidt makes precise the way in which this monodromy action controls the VHS near these special limits. The algebraic properties of these monodromy matrices will be seen to control much of the physics, and we give a classification of such singularities in terms of so-called mixed Hodge structures. Along the way we re-derive basic physical results, deferring to the original literature for a more complete overview. In section 3.3, we review the single-variable $\mathrm{SL}_2$-orbit theorem. It gives a more detailed description of the structure present at these singularities and allows us to derive an approximate form of the Hodge star matrix near such limits. In section 3.4, we then finally use the results of this chapter to obtain the most general limiting form of the physical quantities enumerated above.

## 3.1 Hodge Structures

The goal of this section will be to give a more precise mathematical description of the geometric structures we have encountered in a more physical setting in chapter 1.

### 3.1.1 Pure Hodge Structures

Before graduating to describing variations over the moduli space, it will serve us well to first make precise the various structures present at a typical point in moduli space, that is, away from any singularities. This defines what is known as a polarized pure Hodge structure, whose definition summarizes the various structures encountered in chapter 1. While we are eventually interested in the setting of 3-folds, we formulate these definitions for arbitrary (complex) dimension $n$.

Given the real vector space $H_{\mathbb{R}} = H^n(X, \mathbb{R})$ with complexification $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$ and a contained lattice $H_{\mathbb{Z}} = H^n(X, \mathbb{Z}) \subset H_{\mathbb{R}}$, its associated Hodge decomposition defines a pure Hodge structure of weight $n$. More precisely, this is defined as the pair $(H_{\mathbb{R}}, F)$, where $F$ is a finite decreasing filtration of $H_{\mathbb{R}}$. That is, we have a sequence of sub-spaces $F^p$

$$F^n \subset F^{n-1} \subset \ldots \subset F^1 \subset F^0 = H_{\mathbb{C}}, \tag{3.4}$$

satisfying

$$H_{\mathbb{C}} = F^p \oplus \overline{F}^{n-p+1}. \tag{3.5}$$

This filtration is equivalent to the Hodge decomposition of $H_{\mathbb{C}}$, and the two are related by

$$H^{p,q} = F^p \cap \overline{F}^q, \quad F^p = \bigoplus_{r \geq p} H^{r,k-r}. \tag{3.6}$$

While possibly less familiar than the Hodge decomposition $H^{p,q}$, the filtration $F^p$ is the structure that most naturally describes the Hodge structure on $H_{\mathbb{R}}$ once we consider variations of this structure over the moduli space. As we have also seen in chapter 1, the vector space $H_{\mathbb{R}}$ carries two additional structures that will be of interest to us for the following discussion. The first follows from the fact that the Hodge star operator maps $n$-forms to $n$-forms on a Calabi-Yau $n$-fold. It therefore restricts to a well-defined homomorphism on the space $H_{\mathbb{C}}$, whose action decomposes along with the Hodge decomposition on $H_{\mathbb{C}}$

$$*\omega = i^{p-q}\omega, \quad \omega \in H^{p,q}. \tag{3.7}$$

In this way it is clear that the Hodge star depends on the complex structure which controls the Hodge decomposition. If we wish to abstract this construction away from the context of describing cohomology classes, the operator $*$ is typically written $C$ and is referred to as the Weil operator of the Hodge structure. In later sections we will occasionally adopt a similar notation. The second structure defined on $(H_{\mathbb{R}}, F)$ is inherited from the pairing we encountered in section 1.5.2. In particular, we have a bi-linear pairing

$$\langle \alpha, \beta \rangle \equiv S(\alpha, \beta) := \int_X \alpha \wedge \beta, \quad \alpha, \beta \in H_{\mathbb{C}}. \tag{3.8}$$

This pairing satisfies a number of nice properties with respect to the Hodge decomposition and Weil operator[1]

1. $S(\alpha, \beta) = (-1)^n S(\beta, \alpha)$,
2. $S(F^p, F^{n-p+1}) = 0, \quad \Leftrightarrow \quad S(H^{p,q}, H^{r,s}) = 0, \qquad (p, q) \neq (s, r)$,
3. $S(Cv, \overline{v}) > 0, \qquad v \in H^{p,q},\ v \neq 0$.

Note that this last property renders $S(C\cdot, \bar{\cdot})$ into an inner product on $H_{\mathbb{C}}$ and the associated vector norm $||v||^2 := S(Cv, \overline{v})$ is called the Hodge norm on $H_{\mathbb{C}}$. In general, any bilinear pairing on a pure Hodge structure is said to be a polarization if it satisfies the properties above. The existence of these additional structures on $(H_{\mathbb{R}}, F)$ places essential constraints on the Hodge structure. Though this is sometimes left implicit, these constraints are crucial to many of the results that follow.

### 3.1.2 Variation of Hodge Structure

Having discussed the structures present at a typical point in the moduli space, we can now study how this structure varies if we allow the space $X$ in question to vary. We will first do so from a global point of view, before moving on to a local description of the moduli space more suitable for the applications we will be interested in.

Given a Calabi-Yau manifold $X$, let us denote its complex structure moduli space $\mathcal{M}$. Each point $z \in \mathcal{M}$ then corresponds to a Calabi-Yau $X_z$, differing from each other only in their complex structure. Consequently, each of these $X_z$ have isomorphic cohomology groups $H^n(X_z, \mathbb{C}) \cong H_{\mathbb{C}}$, which suggests the possibility of assembling them into a bundle. It turns out that there is a canonical way of doing so, so that the moduli space naturally comes equipped with a flat vector bundle $\mathcal{H}$ whose fibers are identified point-wise as the cohomology group $H^n(X_z, \mathbb{C})$. Note however, that while this bundle is flat, the possibly non-trivial topology of $\mathcal{M}$ may prevent it from being trivial[2], which will be important to us later. The various sub-spaces ($H^{p,q}$, $F^p$, $H_{\mathbb{R}}$, $H_{\mathbb{Z}}$, ...) defined in the previous section can be realized as fibers of smooth sub-bundles of this vector bundle. Moreover, the fiber-wise polarization extends to a flat bilinear form on the bundle $\mathcal{H}$, which reduces the structure group of $\mathcal{H}$ to the automorphism group of $S(\cdot, \cdot)$. Because $\mathcal{H}$ is flat, it comes equipped with a canonical flat connection $\nabla$, referred to in this context as the Gauss-Manin connection. It is here that the importance of the Hodge filtration first becomes apparent. In particular, these sub-bundles admit two very nice properties

[1] Strictly speaking, this is only true for $n = 3$. More generally one should restrict the discussion to the so-called primitive part of $H^n$, defined as the kernel of the cohomology map $\mathcal{K} \wedge \cdot$ for $\mathcal{K}$ the Kahler form. For $n = 3$ this condition is automatically satisfied for any $\alpha \in H^3$ as $\mathcal{K} \wedge \alpha \in H^5$ which is empty. Since this is our case of interest, we do not keep track of this distinction.

[2] A more careful treatment would involve taking a family of Calabi-Yau manifolds $\pi : \mathcal{X} \to \mathcal{M}$ whose fiber over a point $z \in \mathcal{M}$ is the Calabi-Yau $\pi^{-1}(z) = X_z$. The cohomology groups of $X_z$ assemble into a sheaf over $\mathcal{X}$ and by taking the direct image sheaf under $\pi$ one obtains a local system on $\mathcal{M}$, which in the present case corresponds in the bundle flat $\mathcal{H}$. The discussion in the main text should be viewed as a heuristic motivation, but is otherwise correct once one recovers the bundle $\mathcal{H}$.

- While all of the sub-bundles $H^{p,q}$ are smooth, it turns out that the sub-bundles $F^p$ are in fact holomorphic.
- With respect to the flat connection on $\mathcal{H}$, we have that $\nabla F^p \subset F^{p-1} \otimes T^{(1,0)}\mathcal{M}$. This property is called Griffiths' transversality.

The collection of objects and properties listed above define precisely the abstract notion of a variation of Hodge structure (VHS) whose abstract definition we do not repeat here. Instead, we specialize further to the case of Calabi-Yau 3-folds, where we have more detailed information available to us. In particular, we have that $F^3 = H^{3,0}$ is one-dimensional, and holomorphic sections correspond to a holomorphic choice of $(3,0)$-form as a function of the moduli. Moreover, in the particular case of Calabi-Yau 3-folds we obtain a stronger version of Griffiths' transversality that states that we can recover the full Hodge filtration from such a section by repeated application of the connection $\nabla$. In particular, choosing holomorphic coordinates $z^i$ on $\mathcal{M}$, we have that

$$\Omega(z) \to \nabla_i \Omega(z) \to \nabla_i \nabla_j \Omega(z) \to \nabla_i \nabla_j \nabla_k \Omega(z). \tag{3.9}$$

Hence the full information about the sub-bundles $F^p$ can be derived from a holomorphic section $\Omega$ of $F^3$. It follows that if we understand how the section $\Omega$ behaves as a section of $\mathcal{H}$ that we can (at least in principle) reconstruct the full VHS.

To better understand the properties of $\Omega(z)$, let us switch to a more local description in terms of the period vector. As a first step, let us fix a base point $z_0 \in \mathcal{M}$ and choose a (real, integral[3]) basis $(\alpha_I, \beta^I)$ of the fiber $H^3(X_{z_0}, \mathbb{C})$. We can take this basis to be symplectic with respect to the polarization on the fiber over $z_0$

$$\eta_{IJ} = S(\alpha_I, \alpha_J) = 0 = S(\beta^I, \beta^J) = \eta^{IJ}, \quad \eta_I^J = S(\alpha_I, \beta^J) = \delta_I^J. \tag{3.10}$$

which fixes it uniquely up to symplectic transformations. We can extend this basis to a local frame of $\mathcal{H}$ which is flat with respect to $\nabla$. That is to say, if we take our holomorphic section $\Omega(z)$ and expand it with respect to the basis above

$$\Pi(z) := \begin{pmatrix} \int_{A^I} \Omega(z) \\ -\int_{B_I} \Omega(z) \end{pmatrix} \equiv \begin{pmatrix} X^I(z) \\ -F_I(z) \end{pmatrix}, \qquad \Omega(z) = X^I(z)\alpha_I - F_I(z)\beta^I, \tag{3.11}$$

the components of $\nabla_i \Omega(z)$ are simply given by $\partial_i \Pi(z)$. As mentioned, the basis $(\alpha_I, \beta^I)$ can only be used to define a frame *locally*. The extent of this failure is measured precisely by the *monodromy*. In particular, let $\tilde{\mathcal{M}}$ denote the universal cover of $\mathcal{M}$. We can view this space as the total space of a principal bundle over $\mathcal{M}$ with structure group $\pi_1(\mathcal{M})$ the fundamental group of $\mathcal{M}$

$$\pi_1(\mathcal{M}) \to \tilde{\mathcal{M}} \to \mathcal{M}. \tag{3.12}$$

Pulling back the Hodge bundle $\mathcal{H}$ by the projection $\pi$ we obtain a trivial bundle $\pi^*\mathcal{H} \cong \mathcal{M} \times H_{\mathbb{C}}$ where the frame $(\alpha_I, \beta^I)$ is now globally well-defined. It is a general fact that any flat vector bundle over $\mathcal{M}$ is the associated bundle to (3.12) by some representation $T : \pi_1(\mathcal{M}) \to GL(H_{\mathbb{C}})$. Hence we obtain an equivalent characterization of the sections of $\mathcal{H}$ as maps $\tilde{\mathcal{M}} \to H_{\mathbb{C}}$ that are equivariant under the action of $\pi_1(\mathcal{M})$. In particular, this means that we can lift the period vector $\Pi(z)$ to a vector defined on the universal cover that is equivariant under the action of $\pi_1(\mathcal{M})$

$$T_g^{-1}\Pi(gt) = \Pi(t), \qquad g \in \pi_1(\mathcal{M}). \tag{3.13}$$

[3]Recall that the space $H_{\mathbb{R}} \subset H_{\mathbb{C}}$ comes with a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, which is the image of $H^3(X_{z_0}, \mathbb{Z})$ in $H^3(X_{z_0}, \mathbb{R})$ (i.e. 3-forms with integral periods over integral cycles). An integral basis is a basis for $H_{\mathbb{R}}$ which is also a basis for $H_{\mathbb{Z}}$. We revisit this point in section 3.4.

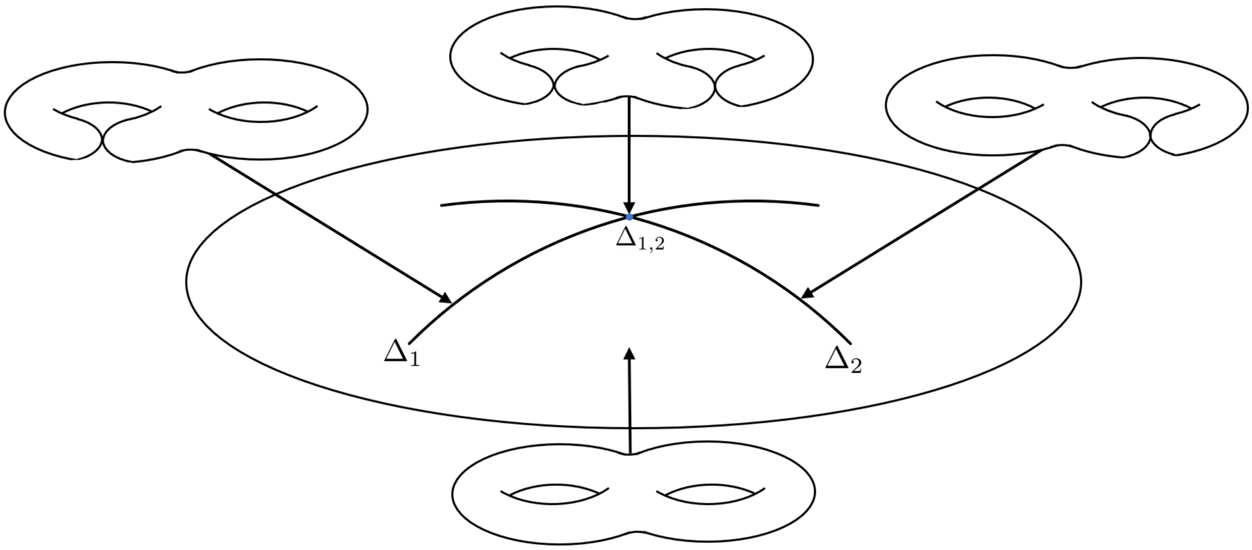



Figure 3.1: Cartoon of the moduli space of a double torus. The singular locus consists of two divisors $\Delta_1$ and $\Delta_2$ where one of the cycles of the torus pinches off.

Equation 3.13 precisely captures the extent to which the bundle $\mathcal{H}$ fails to be trivial, and it will play a crucial role throughout this chapter and the next. The image of $\pi_1(\mathcal{M})$ under $T$ inside $GL(H_{\mathbb{C}})$ is called the *monodromy group* and it is a discrete subgroup of the symplectic group, i.e. $T_g \in Sp(2n_V + 2, \mathbb{Z})$.

## 3.2 Singularities in Moduli Space

We now turn our attention to describing the singularities in moduli space. In our discussion of Calabi-Yau compactifications, we saw that we can describe this moduli space locally as a smooth manifold, with a tangent space spanned by the complex structure deformations of the Kahler form. However, the moduli space need not admit such a nice description globally. In particular, there will be singular points on this moduli space where the complex structure of the Calabi-Yau degenerates. Intuitively, this can be understood in terms of cycles in the Calabi-Yau shrinking to zero size, as is illustrated in figure 3.1.

While *a priori* these singular points may be distributed along the moduli space in a very complicated way, it can be shown that after a possible resolution [62], these singularities can be assumed to appear in very specific configurations. In particular, the complete set of singular points, called the singular locus $\Delta$ is built out of elementary building blocks called divisors $\Delta = \cup_i \Delta_i$. Locally, we can describe each of these divisors as the vanishing locus of one of the coordinates $z^i = 0$, meaning that they are of co-dimension one. Crucially, these divisors may intersect, but we may further assume that these crossings are always normal, i.e. points on the intersection of $k$ divisors can locally be described as the vanishing locus of $k$ of the coordinates $z^{i_1}, \ldots, z^{i_k} = 0$. Taken together, it follows that any given point on the singular locus is the intersection of $k$ divisors, such that we obtain a standard model for the local geometry near any singular point, given by

$$\mathcal{E} \cong (\mathbb{D}^*)^{n_k} \times (\mathbb{D})^{n-n_k}, \tag{3.14}$$

where

$$\mathbb{D} := \{z \in \mathbb{C} | |z| \leq 1\}, \qquad \mathbb{D}^* := \mathbb{D} \setminus \{0\}. \tag{3.15}$$

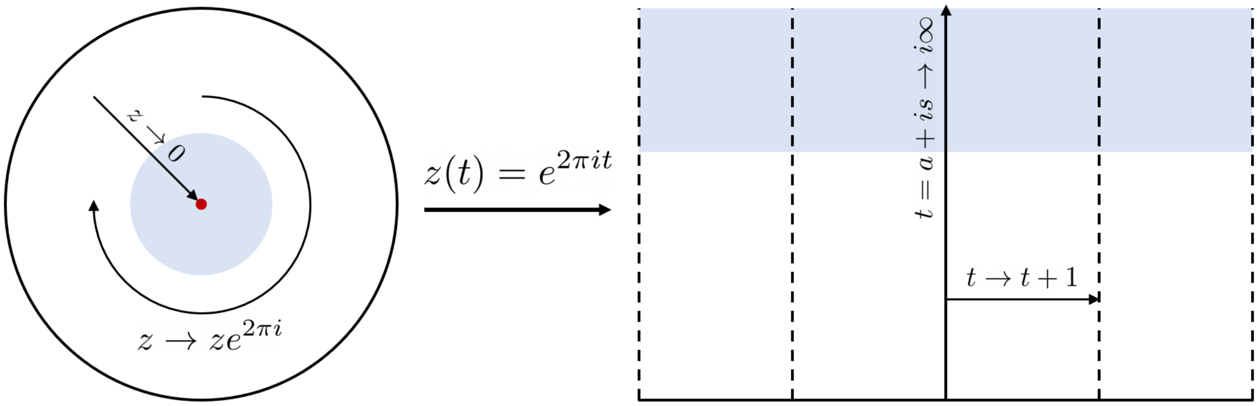



Figure 3.2: Illustration of the local model for a one-dimensional moduli space as a punctured disk, along with its universal cover, the upper half-plane. Encircling the singular locus corresponds to discrete shifts on the upper-half plane. Figure based on [63].

We can split the coordinates on this patch as $(z^i, \zeta^M)$ where $z^i$ are the coordinates on the punctured disk $\mathbb{D}^*$ (with the singularity located at $z^i = 0$). The remaining complex directions are parameterized by the $\zeta^M$, which we refer to as spectator moduli. These typically will not play an important role in our discussions so that we will often neglect to write them.

Throughout the rest of this chapter, it will be important to keep track of which regions of the moduli space we are studying. For this reason, let us introduce some notation that will help us in doing so. Firstly, we denote a $k$-fold intersection of divisors by

$$\Delta_{i_1 \dots i_k} := \Delta_{i_1} \cap \dots \cap \Delta_{i_k}, \tag{3.16}$$

where the $\Delta_i$ denote individual divisors labeled by an index $i$. while we denote the points on $\Delta_{i_1 \dots i_k}$ that do not lie on any *other* divisors by

$$\Delta^\circ_{i_1 \dots i_k} := \Delta_{i_1} \cap \dots \cap \Delta_{i_k} - \bigcup_{j \neq i_1, \dots, i_k} \Delta_{i_1 \dots i_k j}. \tag{3.17}$$

This generic state of affairs is likewise illustrated in figure 3.1. In view of the discussion in section 3.1.2, we also note that the universal cover of the patch $\mathcal{E}$ is given by

$$\tilde{\mathcal{E}} \cong (\mathbb{H})^{n_k} \times (\mathbb{D})^{n - n_k}, \tag{3.18}$$

where $\mathbb{H} := \{t \in \mathbb{C} \,|\, \mathrm{Im}\, t > 0\}$ denotes the upper-half plane. We will generically introduce coordinates $(t^i, \zeta^M)$ on the universal cover which are related to those on $\mathcal{E}$ via the projection $\pi : \tilde{\mathcal{E}} \to \mathcal{E}$ as

$$\pi : (t^i, \zeta^M) \mapsto (z^i(t^i), \zeta^M) := (e^{2\pi i t^i}, \zeta^M). \tag{3.19}$$

This implies that the singularities now lie at $t^i = i\infty$, while the $\zeta^M$ are again spectator moduli. It is natural to decompose the coordinates $t^i$ into real and imaginary parts

$$t^i = a^i + i s^i, \tag{3.20}$$

so that the singularity now lies at $s^i \to \infty$. For reasons that will become clear shortly, we refer to the real part $a^i$ as the axion and the imaginary part as the saxion.

### 3.2.1 The Nilpotent Orbit Theorem

As we have seen in section 3.1.2, the period vector efficiently encodes the information contained in the variation of Hodge structure. Moreover, we saw how the period vector can be viewed as a function on the universal cover, which descends to a multi-valued function on the moduli space. Crucially, the singularities in the moduli space lead to non-trivial cycles in $\mathcal{M}$ which act as generators for the fundamental group $\pi_1(\mathcal{M})$. It follows that encircling a singular divisor $\Delta^\circ_{i_1\ldots i_k}$ leads to a non-trivial monodromy action on the period vector $\Pi$. That is to say, as $z^i \to z^i e^{2\pi i}$ (eq. $t^i \to t^i + 1$), the period vector transforms as

$$\Pi(z) \to T_i \Pi(z). \tag{3.21}$$

As we show in this section, the algebraic properties of these monodromy matrices encode the essential information about a given singularity. We therefore begin by discussing some of these properties.

1. Quasi-unipotency

$$\exists\, m_i, n_i \in \mathbb{N}: \quad (T_i^{m_i} - \mathrm{Id})^{n_i+1} = 0. \tag{3.22}$$

2. Commutation

$$[T_i, T_j] = 0 \quad \forall\, i, j \in \{i_1, \ldots, i_k\}. \tag{3.23}$$

Both of these properties are non-trivial and in particular, commutativity only holds for those $T_i$ obtained by encircling a single fixed component $\Delta^\circ_{i_1\ldots i_k}$ of the singular locus. To further analyze the first property, we consider the Jordan–Chevalley decomposition of the matrix $T_i$

$$T_i = T_i^s T_i^u. \tag{3.24}$$

This is a unique decomposition of $T_i$ into semisimple $T_i^s$ and unipotent $T_i^u$ parts which commute. The failure of the $T_i$ to be unipotent is captured by the semisimple part, however one can show that it is always of finite order, i.e. $(T_i^s)^{m_i} = 1$. It turns out that the essential information about the singularity is contained in the unipotent part. While we will revisit the distinction between unipotency and quasi-unipotency in chapter 4, it only serves to clutter the notation for the time being. We therefore simplify the present discussion by redefining our coordinates as $z^i \to (z^i)^{m_i}$. The resulting monodromy matrices are all unipotent of order $n_i$. In particular, the unipotency of the $T_i$ implies that their logarithms are nilpotent so that we introduce the *log-monodromy matrices*

$$N_i := \log T_i \in \mathfrak{sp}(2n, \mathbb{Q}), \tag{3.25}$$

from which it follows that

$$N_i^{n_i+1} = 0. \tag{3.26}$$

With these in hand, we are now in a position to state our first result, due to Schmidt [3]

**Nilpotent Orbit Theorem:** Near a point $p \in \Delta^{\circ}_{i_1,\ldots,i_k}$, the Hodge filtration $F^p$ can be approximated by

$$F^p(t^i,\zeta^M) = \underbrace{\exp\left(\sum_{i=1}^{n_k} t^i N_i\right) F_0^p(\zeta^M)}_{F^p_{nil}} + \mathcal{O}(e^{2\pi i t^j}), \tag{3.27}$$

where $F_0^p(\zeta^M)$ is holomorphic in the spectator moduli $\zeta^M$. For sufficiently large $\operatorname{Im} t^i$ the leading approximation $F^p_{nil}(t^i,\zeta)$ is a well-defined Hodge filtration.

As we approach the singularities in the moduli space the Calabi-Yau manifold degenerates. As is to be expected, so too does the Hodge decomposition on the cohomology. The nilpotent orbit theorem essentially states that this degeneration is "mild" and that it is completely controlled by the log-monodromy matrices $N_i$. Here we wish to emphasize two points that follow from (3.27)

1. While the Hodge filtration becomes singular at $z^i = 0$, its singular behaviour is contained entirely in the oscillatory behaviour of the factor $\exp\left(\sum_{i=1}^{n_k} t^i N_i\right)$. In particular, it means that the limit

$$F_0^p(\zeta^M) = \lim_{t^i \to i\infty} \exp\left(-\sum_{i=1}^{n_k} t^i N_i\right) F^p(z^i,\zeta^M), \tag{3.28}$$

   exists and is holomorphic in the (non-singular) coordinates $\zeta^M$. Note that commutativity of the monodromy matrices implies that these expressions are unambiguous with respect to the ordering of the $N_j$.

2. As a representative of $F^3$, the period vector admits a similar expansion as

$$\Pi(t^i,\zeta^M) = \underbrace{\exp\left(\sum_{i=1}^{n_k} t^i N_i\right) \mathbf{a}_0(\zeta)}_{\Pi_{nil}} + \mathcal{O}(e^{2\pi i t^i}). \tag{3.29}$$

   It follows that the period vector similarly degenerates, but we can nonetheless extract a limiting piece that represents $F_0^3$

$$\mathbf{a}_0(\zeta^M) = \lim_{t^i \to i\infty} \exp\left(-\sum_{i=1}^{n_k} t^i N_i\right) \Pi(z^i,\zeta^M) \in F_0^3. \tag{3.30}$$

While it may be tempting to again work exclusively with the nilpotent orbit $\Pi_{nil}(t^i,\zeta^M)$, it need no longer the case that its derivatives span the full filtration $F^p_{nil}$. Indeed, it is not hard to see that derivatives with respect to the moduli $t^i$ pull down factors of $N_i$ from the exponent. However, these matrices are nilpotent so that the derivatives may vanish before we are able to span the full filtration. In this case, one is either forced to include exponential corrections to $\Pi_{nil}$ or work with the full filtration $F^p_{nil}$ to recover the complete information about the VHS.

Moreover, while the filtration $F^p_{nil}$ defines a good Hodge filtration near the singularity, the same need not be the case *at* the singularity. In general, the limiting filtration $F^p_0$ that is defined here will define a filtration, but it will generically fail to be Hodge in the sense that we lose compatibility with the polarization (this is equivalent to a breakdown of equation 3.6). Nevertheless, this means that the filtration at the singularity is *almost* Hodge in a precise sense. In the following section we will investigate what structure remains at these singularities. However, before we do so, let us illustrate the usefulness of the nilpotent orbit theorem by deriving conditions for a singularity to lie at infinite distance.

### Infinite Distance Limits

As a first application of the nilpotent orbit theorem, we recall the expression for the Kahler potential

$$K = -\log i\langle \Pi, \bar{\Pi}\rangle = -\log i\Pi^T \eta \bar{\Pi}. \tag{3.31}$$

Moreover, recall that the monodromy matrices $T_i$ preserve the polarization, so that the log-monodromies satisfy the Lie algebra relationship (cf. (3.10))

$$N_i^T \eta + \eta N_i = 0. \tag{3.32}$$

Inserting the nilpotent orbit approximation to the period vector, we obtain the following leading order expression for the Kahler potential as we take the limit $t^i \to i\infty$

$$e^{-K} = i\mathbf{a}_0^T \eta \exp\left(\sum_i (\bar{t}^i - t^i) N_i\right) \bar{\mathbf{a}}_0 + \mathcal{O}(e^{2\pi i t^i}). \tag{3.33}$$

By nilpotency of the $N_i$ the expansion of the exponential truncates at order $m_i$ in each of the moduli $t^i$ so that the leading order dependence on the $t^i$ is at most polynomial in $\operatorname{Im} t^i$. Moreover, the dependence on $\operatorname{Re} t^i$ is contained entirely in the exponential corrections to this result. Crucially however, the integers $m_i$ only impose an upper bound on the degree of this polynomial, as the pairing between $\mathbf{a}_0$ and $N_j^{d_j}\bar{\mathbf{a}}_0$ may vanish for $d_j < m_j + 1$. We will revisit this subtlety in the next subsection where we study conditions for when this may happen, but we may already derive some important physical consequences from the expression (3.33).

The first is that it allows us to identify a class of infinite distance limits in the moduli space, as is required by the swampland distance conjecture. Indeed, the singularities in the moduli space are natural candidates for such infinite distance points and the Kahler potential (3.33) allows us to write down a necessary condition for this to be the case. Intuitively, the metric derived from this potential should not decay too quickly as we approach the infinite distance point. In particular, a rather straight-forward computation shows that if $N_i \mathbf{a}_0 = 0$ for all $i$, then the leading term in (3.33) is exponential, and the distance to the singularity evaluated along the path $(t^i(\tau), \zeta^M(\tau)) = (i\tau, \ldots, i\tau, 0, \ldots, 0)$ is finite. Hence we obtain the completely general result that

$$p \text{ at infinite distance} \quad \Rightarrow \quad \exists\, N_i : \quad N_i \mathbf{a}_0 \neq 0. \tag{3.34}$$

The converse implication is more complicated however, because this involves showing infinite distance along *all* paths (or equivalently, identifying the correct geodesics). Nevertheless, if we specialize to the case that we send only one modulus to the boundary, i.e. we focus on individual divisors away from any intersections, it is possible to prove the converse implication. Here, path dependence is not an issue and we can determine whether we are at infinite distance based on the leading contribution to the Kahler metric. Evaluating the polynomial contribution to (3.33) explicitly for this case, we obtain

$$\begin{aligned} e^{-K} &= i\mathbf{a}_0^T \eta \sum_n \frac{1}{n!}\left(-2i\,(\operatorname{Im} t)N\right)^n \bar{\mathbf{a}}_0 \\ &= -\sum_n \frac{2^n}{n!}(\operatorname{Im} t)^n i^{3-n}\langle \mathbf{a}_0, N^n \bar{\mathbf{a}}_0\rangle. \end{aligned} \tag{3.35}$$

By computing the associated Kahler metric (which we do explicitly in section 3.2.3), one can then show that a sufficient condition for $p$ to lie at infinite distance is that this leading contribution is non-constant, i.e.

$$i^{3-d}\langle \mathbf{a}_0, N^d \mathbf{a}_0\rangle \equiv i^{3-d} S_d(\mathbf{a}_0, \bar{\mathbf{a}}_0) \neq 0, \tag{3.36}$$

for some $d > 0$. In section 3.2.3 we will see that the quantity defined above is in fact positive for some $d > 0$ if and only if $N\mathbf{a}_0 \neq 0$, so that for one-modulus degenerations we obtain the stronger result

$$p \text{ at infinite distance} \quad \Leftrightarrow \quad N\mathbf{a}_0 \neq 0. \tag{3.37}$$

This example highlights how the algebraic properties of the log-monodromy matrices encode useful physical information about the singularities in the moduli space *via* the nilpotent orbit theorem. It is also clear however, that if we wish to make more concrete statements we will need a handle on these algebraic properties. This requires the introduction of so-called mixed Hodge structures that generalize the pure Hodge structures that exist away from the singularity.

### 3.2.2 Mixed Hodge Structures

The nilpotent orbit theorem tells us that although the Hodge filtration becomes singular near the singularity, we can still extract a “principal part” of the degenerating Hodge structure. Intuitively, one expects that this limiting filtration contains information about the Calabi-Yau that has degenerated there. However, the degenerate nature of this space means that this information is no longer encoded in a pure Hodge structure. Instead, we obtain *several* pure Hodge structures of different weights, leading to the notion of a mixed Hodge structure. The first ingredient in the construction is the introduction of a weight filtration. Recall that we defined the weight of a pure Hodge structure as the integer $k$ such that

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q} \tag{3.38}$$

It turns out that the decomposition defined by the limiting filtration $F_0^p$

$$H_0^{p,q} = F_0^p \cap \bar{F}_0^q, \tag{3.39}$$

no longer satisfies the property (3.38), but one can recover it if one admits a sum over different weights $k$ as

$$H_{\mathbb{C}} = \bigoplus_k \bigoplus_{p+q=k} H_0^{p,q} \equiv \bigoplus_k \tilde{W}_k. \tag{3.40}$$

The *monodromy weight filtration* refines the above intuition of a weight decomposition $\tilde{W}_k$ to an increasing filtration $W_k$ of $H_{\mathbb{C}}$[4]

$$W_{-1} \equiv 0 \subset W_0 \subset W_1 \subset \ldots \subset W_{2n-1} \subset W_{2n} = H_{\mathbb{C}}. \tag{3.41}$$

[4]More precisely, the weight filtration is defined over $\mathbb{Q}$, meaning that the filtration of $H_{\mathbb{C}}$ is obtained from that on $H_{\mathbb{Q}}$ via complexification. Intuitively, this follows from the fact that $N_{(k)} \in \mathfrak{sp}(2n, \mathbb{Q})$, but we refer to the mathematical literature for details [3]. Though not relevant right now, this point will be important come section 3.4.

To define it, we require again the log-monodromy matrices which, through equation 3.28, control the deviation of the $F_0^p$ from the genuine (but approximate) Hodge filtration $F_{nil}^p$. In particular, we fix a point $p$ on the intersection of $n_k$ divisors $p \in \Delta^\circ_{i_1,\ldots,i_k}$, with associated log-monodromy matrices $N_{i_1}, \ldots, N_{i_k}$. The weight filtration is then defined as

$$W_l = \bigoplus_{j \geq 0, l-3} \ker N_{(k)}^{j+1} \cap \mathrm{im} N_{(k)}^{j-l+3}, \quad N_{(k)} := N_{i_1} + \ldots + N_{i_k}, \tag{3.42}$$

which can be shown to define a filtration of the type above. We remark that the resulting filtration $W_l$ is independent on the particular linear combination $\sum_i c_i N_i$ of log-monodromies we use, as long as all coefficients $c_i$ are positive [64] and use $N_{(k)}$ for the sum of the first $k$ log-monodromies. For simplicity, we will therefore always choose $c_i = 1$. We will occasionally write $W_l(N_{(k)})$ when we wish to emphasize the dependence of the spaces on these matrices. The weight filtration defined above is the *unique* filtration of $H_{\mathbb{C}}$ that satisfies

1. $N_{(k)} W_l \subset W_{l-2}$
2. $N_i : Gr_{3+j} \to Gr_{3-j}$ defines an isomorphism, where we have defined

$$Gr_l := W_l / W_{l-1}, \tag{3.43}$$

   the graded pieces of the limiting Hodge filtration.

Intuitively, these graded pieces $Gr_l$ capture the part of the weight filtration of weight precisely $l$ (i.e. without the parts lower in the filtration). This intuition is confirmed by the fact that the limiting filtration $F_0^p$ defines a pure Hodge structure of weight $l$ on $Gr_l$. In particular, we define

$$F_0^p Gr_l := (F^p \cap W_l)/(F^p \cap W_{l-1}), \tag{3.44}$$

in terms of which we obtain a pure Hodge structure on $Gr_l$ as

$$Gr_l = \bigoplus_{p+q=l} H_0^{p,q}, \quad H_0^{p,q} = F_0^p Gr_l \cap \overline{F_0^q Gr_l}. \tag{3.45}$$

The definition above subsumes the pure Hodge structure away from the singularity, in which case the weight filtration is trivial (i.e. non-zero only for one value of $l$) and the pure Hodge structure reduces to the ordingary Hodge filtration on $H_{\mathbb{C}}$. More generally, a *mixed Hodge structure* is defined as any two filtrations $(F_0^p, W_l)$ (with $W_l$ defined over $\mathbb{Q}$) such that for each $l$ the filtration (3.44) is a pure Hodge structure. As we shall see in section 3.4, the pure Hodge structures on the graded pieces admit a very elegant geometric interpretation. Nevertheless, the nature of the spaces $Gr_l$ as quotients makes explicitly working with them rather cumbersome. Let us therefore introduce an equivalent but more convenient way of encoding the mixed Hodge structure.

Rather than using the graded pieces to define a pure Hodge structure, we instead introduce an analogous splitting of, but whose elements are simply vectors, rather than

$$I^{p,q} := F_0^p \cap W_{p+q} \cap \left( \bar{F}_0^p \cap W_{p+q} + \bigoplus_{j \geq 1} \bar{F}_0^{q-j} \cap W_{p+q-j-1} \right). \tag{3.46}$$

The decomposition $I^{p,q}$ is called the Deligne splitting of $H_{\mathbb{C}}$. Though its definition is somewhat involved, it satisfies a number of properties which make it “the closest thing” to the decomposition above. In particular, it is the unique splitting that satisfies

$$F_0^p = \bigoplus_{r \geq p} \bigoplus_{s} I^{r,s}, \quad W_l = \bigoplus_{p+q \leq l} I^{p,q}, \tag{3.47}$$

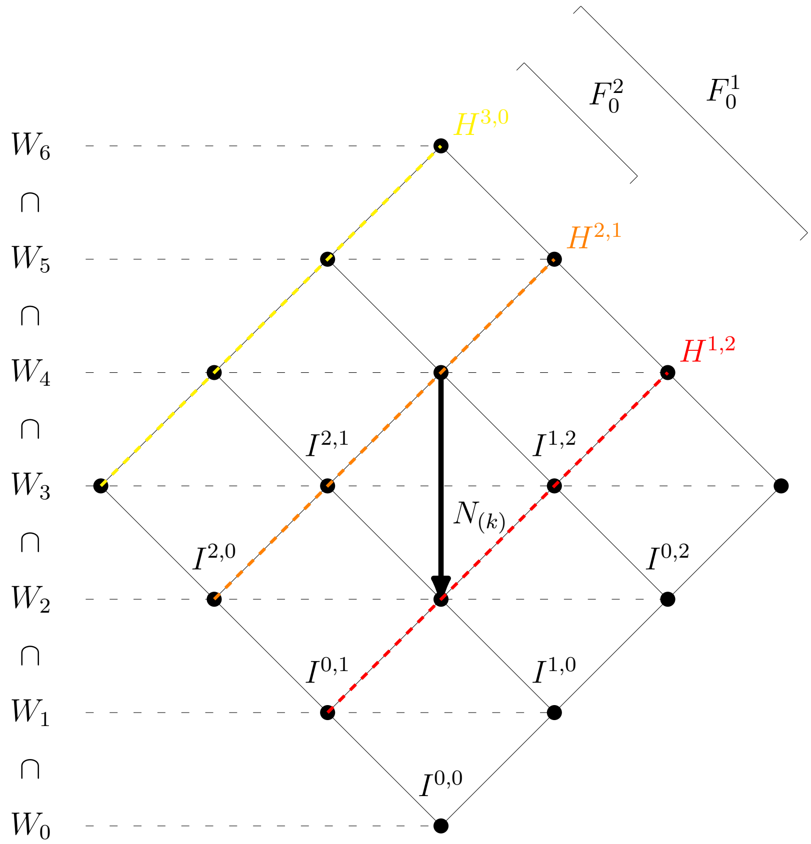



Figure 3.3: A Hodge-Deligne diamond illustrating how the spaces $F_0^p$, $W_l$, $H^{p,q}$ and $I^{p,q}$ are related. The arrow indicates a generic action of the log-monodromy matrix acting on these spaces.

and

$$\bar{I}^{p,q} = I^{q,p} \mod \bigoplus_{r<q,s<p} I^{r,s}. \tag{3.48}$$

Morally speaking, the $I^{p,q}$ correspond to the spaces $H_0^{p,q}$ in (3.45), but because we do not take the quotient they also contain all of the lower parts in the weight decomposition. Indeed, this is the meaning of the non-trivial conjugation rule (3.48), which measures the failure of the $I^{p,q}$ to define pure Hodge structures. Nevertheless, if it so happens that $F_0^p$ and $W_l$ conspire to give a Deligne splitting where $\overline{I}^{p,q} = I^{q,p}$, then the complicated definition (3.46) reduces to a straight-forward generalization of

$$I^{p,q} = F^p \cap \bar{F}^q \cap W_{p+q}. \tag{3.49}$$

We call Deligne splittings of the form above $\mathbb{R}$-split, and they will play an important role in section 3.3. The constructions above may be rather abstract, but by visualizing them in a so-called Hodge-Deligne diamond as in figure 3.3, their interrelationships become clear.

The log-monodromy matrix $N_{(k)}$ used to define the weight filtration also has a special role to play in the Deligne splitting. Indeed, it can be shown to act on the $I^{p,q}$ as

$$N_{(k)} I^{p,q} \subset I^{p-1,q-1}, \tag{3.50}$$

thus appearing as a "lowering operator" in the Hodge-Deligne diamond that moves a vector down by one row. It follows from the fact that $I^{p,q} = 0$ for $p, q > 3$ and $p, q < 0$ that

$$N_{(k)}^4 = 0, \tag{3.51}$$

providing an upper bound on the nilpotency of the log-monodromy matrices. Moreover, the property (3.50) allows us to further simplify the Deligne-splitting. In particular, we can define the primitive part of $I^{p,q}$ as

$$P^{p,q} := I^{p,q} \cap \ker N_{(k)}^{p+q-2}. \tag{3.52}$$

Intuitively, these are the parts of $I^{p,q}$ that cannot be obtained through the action of $N$ on any of the higher $I^{p,q}$. These primitive spaces contain the essential information of the Deligne splitting, as the latter can be recovered from the primitive parts via

$$I^{p,q} = \bigoplus_{l\geq 0} N^l P^{p+l,q+l}. \tag{3.53}$$

Their siginificance now comes from the fact that the polarization $S(\cdot,\cdot)$ on $H^3$ induces a polarization on the $P^{p,q}$ of fixed weight.

1. $$S_l(P^{p,q}, P^{r,s}) = 0, \quad r+s = 3+l = p+q, (p,q) \neq (s,r), \tag{3.54}$$

2. $$i^{p-q} S_l(v,\bar{v}) > 0, \qquad v \in P^{p,q},\ v \neq 0. \tag{3.55}$$

### 3.2.3 Classifying Singularities

The mixed Hodge structures introduced in the previous subsection provide a way of encoding more detailed information about the singularity involving both the log-monodromy matrices $N$ as well as the limiting filtration $F_0^p$ in terms of the new structure $I^{p,q}$. The coarsest possible classification of such a decomposition is given by the dimensionality of the spaces $I^{p,q}$, however it turns out that for many applications this coarse classification is already sufficient to fix the behaviour of physical quantities near a given singularity. These numbers can then be assembled into a diamond

$$\begin{array}{ccccccc} & & & i^{3,3} & & & \\ & & i^{3,2} & & i^{2,3} & & \\ & i^{3,1} & & i^{2,2} & & i^{1,3} & \\ i^{3,0} & & i^{2,1} & & i^{1,2} & & i^{0,3} \\ & i^{2,0} & & i^{1,1} & & i^{0,2} & \\ & & i^{1,0} & & i^{0,1} & & \\ & & & i^{0,0} & & & \end{array} \qquad i^{p,q} := \dim_{\mathbb{C}} I^{p,q}. \tag{3.56}$$

The fact that the $I^{p,q}$ arise as a finer splitting of the (moduli-dependent) $H^{p,q}$ decomposition of $H_{\mathbb{C}}$ means that the numbers $i^{p,q}$ are related to the Hodge numbers $h^{p,q}$. Indeed, it can be shown using the properties derived in the previous sub-section that these satisfy

$$\sum_q i^{p,q} = h^{p,3-p}. \tag{3.57}$$

By summing over $p$ we recover the expected result that the $I^{p,q}$ span the full middle cohomology. Moreover, one can show that they satisfy the following additional properties [65]

$$\begin{aligned} &i^{p,q} = i^{q,p} = i^{3-p,3-q}, && \text{for all } p,q, \\ &i^{p-1,q-1} \leq i^{p,q}, && \text{for } p+q \leq 3. \end{aligned} \tag{3.58}$$

For Calabi-Yau 3-folds these properties lead to a concise classification of the possible Hodge-Deligne diamonds that may occur. In particular, we have that for Calabi-Yau 3-folds $h^{3,0} = 1$. Evaluating equation 3.57 for $p = 3$, it follows that because the $i^{p,q}$ are all non-negative only one of $i^{3,q}$ can be non-zero. We label these possibilities by roman numerals $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ for $q = 0, 1, 2, 3$, respectively. Once this is fixed, the symmetries (3.58) imply that there remain only two independent Hodge-Deligne numbers, which we can choose to be $i^{2,1}$ and $i^{1,1}$. Equation 3.57 applied to $p = 2$ further constrains this to one independent Hodge-Deligne number once we fix $h^{2,1}$, although the particular form this constraint takes depends on which of the four $i^{3,q}$ is non-zero. We may always take the unfixed Hodge-Deligne number to be $i^{1,1}$ which we use to label the four singularity types

$$\mathrm{I}_{i^{1,1}}, \quad \mathrm{II}_{i^{1,1}}, \quad \mathrm{III}_{i^{1,1}}, \quad \mathrm{IV}_{i^{1,1}}. \tag{3.59}$$

The singularity types (3.59) thus completely specify the Hodge-Deligne numbers once we specify the Hodge number $h^{2,1}$, which we recall corresponds to the dimensionality of the complex structure moduli space via $\dim_{\mathbb{C}}\mathcal{M} = h^{2,1} + 1$. Table 3.1 lists all of the allowed singularity types for arbitrary values of $h^{2,1}$.

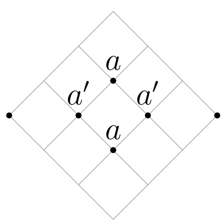
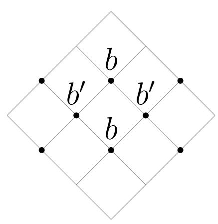
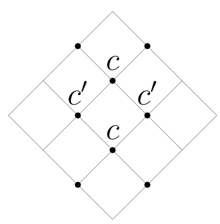
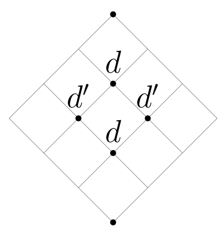
| singularity | $\mathrm{I}_a$ | $\mathrm{II}_b$ | $\mathrm{III}_c$ | $\mathrm{IV}_d$ |
|---|---|---|---|---|
| HD diamond |   |   |   |   |
| index | $a + a' = h^{2,1}$<br>$0 \leq a \leq h^{2,1}$ | $b + b' = h^{2,1} - 1$<br>$0 \leq b \leq h^{2,1} - 1$ | $c + c' = h^{2,1} - 1$<br>$0 \leq c \leq h^{2,1} - 2$ | $d + d' = h^{2,1}$<br>$1 \leq d \leq h^{2,1}$ |
| $\mathrm{rk}(N, N^2, N^3)$ | $(a,\, 0,\, 0)$ | $(2 + b,\, 0,\, 0)$ | $(4 + c,\, 0,\, 0)$ | $(2 + d,\, 2,\, 1)$ |
| eigvals $\eta N$ | $a$ negative | $b$ negative<br>2 positive | not needed | not needed |

Table 3.1: Classification of singularity types in complex structure moduli space based on the $4h^{2,1}$ possible different Hodge-Deligne diamonds. In each Hodge-Deligne diamond we indicated non-vanishing $i^{p,q}$ by a dot on the roster, where the dimension has been given explicitly when $i^{p,q} > 1$. In the last two rows we listed the characteristic properties of the log-monodromy matrix $N$ and the symplectic pairing $\eta$ that are sufficient to make a distinction between the types. Table taken from [66].

### Classifying Infinite Distance Limits

As an exercise in the application of mixed Hodge structures, we now use this classification to complete our classification of infinite distance limits initiated in section 3.2.1. Indeed, recall that whether a singularity laid at infinite distance or not was determined by the positivity of the pairing

$$i^{3-d} S_d(\mathbf{a}_0, \bar{\mathbf{a}}_0) > 0, \tag{3.60}$$

for $d > 0$. This now becomes a simple corollary of the results from the previous section. In particular, recall that $\mathbf{a}_0$ is a representative of $F_0^3$ and as such, belongs to one of the $I^{3,0}, \dots, I^{3,3}$ (of which only one was non-empty). If we let $d \leq 3$ denote the integer such that $\mathbf{a}_0 \in I^{3,d} = P^{3,d}$, equation 3.60 follows readily from the second property (3.55). In particular, we find that the integer $d$ is determined by the singularity type

$$\text{singularity is type I, II, III, IV} \quad \Leftrightarrow \quad d = 0, 1, 2, 3, \tag{3.61}$$

and that it vanishes if and only if the singularity is of type I. Equation (3.37) then becomes

$$p \text{ at infinite distance} \quad \Leftrightarrow \quad \text{Type II, III, IV singularity}, \tag{3.62}$$

which we stress holds for one-modulus degenerations. Of particular interest to us will be the limiting moduli-dependence of the Kahler metric, at least in the one-modulus case. For the infinite distance limits, the integer $3 \geq d > 0$ controls the leading power of $\mathrm{Im}\, t = (t - \bar{t})/2i$ that appears in the polynomial part of $e^{-K}$. This implies that the Kahler potential is well-approximated by

$$-K = \log\left(\alpha \mathrm{Im}\, t^d + \dots\right) \sim \log\left(\alpha\, \mathrm{Im}\, t^d\right) + \dots = d \log(t - \bar{t}) + \log(\alpha/(2i)^d) + \dots, \tag{3.63}$$

where $\alpha$ is an unimportant ($t$-independent) coefficient, and dots denote sub-leading terms. One then readily computes

$$K_{t\bar{t}} \sim \frac{d}{4s^2} + \mathcal{O}\left(e^{2\pi i t}\right), \tag{3.64}$$

from which the infinite distance behaviour follows. The type I case requires more care however, and we do not discuss it in full generality.

### BPS Masses

We can likewise derive a limiting expression for the mass of BPS states, i.e. D3 particles wrapping special Lagrangian 3-cycles. Since this mass constitutes a lower-bound, it is interesting to consider which states can *ever* become light in the asymptotic limit. Recall that for a state with charge vector $\mathbf{q}$, we gave the following general expression for the corresponding BPS mass

$$M^2 = e^K \left| \int_{L^3} \Omega \right|^2 = e^K |\langle \Pi, \mathbf{q} \rangle|^2 = \frac{|\langle \Pi, \mathbf{q} \rangle|^2}{i \langle \Pi, \bar{\Pi} \rangle}. \tag{3.65}$$

Using our limiting expression for the Kahler potential above, the denominator is given by

$$i \langle \Pi, \bar{\Pi} \rangle \sim s^{-d} + \mathcal{O}\left(e^{2\pi i t}\right), \tag{3.66}$$

Next, we insert the nilpotent orbit result for the period vector into the numerator to obtain

$$|\langle \mathbf{q}, \Pi \rangle|^2 \sim |\langle \mathbf{q}, e^{tN} \mathbf{a}_0 \rangle|^2 = \left| \sum_n \frac{t^n}{n!} \langle \mathbf{q}, N^n \mathbf{a}_0 \rangle \right|^2 \sim |t^{l_m}|^2 |S_{l_m}(\mathbf{q}, \mathbf{a}_0)|^2, \tag{3.67}$$

where the integer $l_m$ denotes the highest non-vanishing term in the expansion and we have identified the polarization form $S_{l_m}$ up to unimportant signs and imaginary coefficients. For infinite distance singularities, we may replace $t^{l_m}$ by the saxion as the axion-dependence becomes sub-leading once we divide by (3.66) (for the finite distance case, the leading contribution to (3.66) is constant, so axions may become important). Upon taking the quotient, we obtain our leading order form of the

$$M^2 \sim s^{2l_m - d}. \tag{3.68}$$

We emphasize that this result is again only valid for infinite distance limits, though we will revisit this point in the next chapter.

## 3.3 The Single-Variable SL$_2$-Orbit Theorem

As we have seen, the nilpotent orbit theorem teaches us that the key information about a given singularity is contained in the algebraic properties of the log-monodromy matrices. Moreover, while this algebraic data is *a priori* expected to be highly case dependent, the mixed Hodge structures from the previous section often give us sufficiently refined information about a given singularity to allow us to derive some physically relevant properties. Nevertheless, for our eventual goal of studying the global symmetries of Calabi-Yau compactifications, we require more detailed information still, specifically regarding the kinetic matrix. In particular, we should derive a similar approximation scheme for the Hodge star that determines it. The goal of this section will be to introduce the necessary results, in the form of the SL$_2$-orbit theorem[5]. In the following we specialize to one-modulus degenerations, so that we have only a single log-monodromy to keep track of. The multi-variable case is markedly more involved [67] and we do not discuss it here.

Roughly speaking, the SL$_2$ orbit theorem assigns to a given limiting filtration $F_0^p$, another, closely related filtration. The latter comes with a natural action of the $\mathfrak{sl}(2)$ Lie algebra on its associated mixed Hodge structure. The goal of this section is to introduce this machinery and use it to construct an approximation to the Hodge star near infinite distance points, which we will then use to compute the gauge kinetic functions.

### 3.3.1 Nilpotent Matrices and $\mathfrak{sl}(2)$

To set the stage for the SL$_2$-orbit theorem, we first provide a sketch of how the connection between the nilpotent orbit data and the representation theory of $\mathfrak{sl}(2)$ comes about. In the following, let $m := h^{2,1} + 1$. Recall that, for the case of Calabi-Yau 3-folds, the log-monodromy matrices $N_i$ are nilpotent elements of $\mathfrak{sp}(2m, \mathbb{R})$. Such elements are well-known to be classified by representations of $\mathfrak{sl}(2, \mathbb{R})$ as follows [68].

Given a nilpotent element $N \in \mathfrak{sp}(2m, \mathbb{R})$, a theorem of Jacobson and Morozov [69] tells us that one can always find operators $N^+, N^0 \in \mathfrak{sp}(2m, \mathbb{R})$, such that the triple $\{N^- := N, N^+, N^0\}$ generates the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra

$$[N^0, N^+] = +2N^+, \qquad [N^0, N^-] = -2N^-, \qquad [N^+, N^-] = N^0. \tag{3.69}$$

The operators $N^0$, $N^+$ and $N^-$ admit interpretations as weight, raising and lowering operators, respectively. Equation 3.69 tells us that the matrices $\{N^-, N^+, N^0\}$ furnish a Lie algebra representation of $\mathfrak{sl}(2, \mathbb{R})$ on the vector space $H_{\mathbb{C}}$. However, this representation will generically be reducible, which means that there exists a basis of $H_{\mathbb{C}}$ such that the triple admits the block-diagonal form

$$N^0, N^\pm = \begin{bmatrix} \boxed{\nu_1} & & & \\ & \boxed{\nu_2} & & \\ & & \boxed{\nu_3} & \\ & & & \ddots \end{bmatrix}, \tag{3.70}$$

with each block furnishing an irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$. Note that some of these blocks may correspond to the trivial representation, leading to zero entries on the diagonal.

[5]Our presentation is in some sense backwards here. The results below are in fact the *reason* that the filtration $W_l$ was such that it defined a mixed Hodge structure.

In the following, we then focus on a particular block in this decomposition, labeled by the index $\nu$ above, whose dimension we denote by $d_\nu + 1$. As follows from the representation theory of $\mathfrak{sl}(2,\mathbb{R})$, the weight operator $N^0_\nu$ defines a decomposition of the sub-space $H_\nu \subset H_\mathbb{C}$ on which it acts

$$H_\nu = H_\nu^{d_\nu} \oplus H_\nu^{d_\nu-2} \oplus \ldots \oplus H_\nu^{-d_\nu+2} \oplus H_\nu^{-d_\nu}, \qquad H^l_\nu = \{v \in H_\nu |\; N^0_\nu v = lv\}. \tag{3.71}$$

The raising and lowering operators $N^\pm_\nu$ then map between these spaces as $N^\pm_\nu : H^l_\nu \to H^{l\pm 2}_\nu$. Since the weight decomposition has $d_\nu$ levels, it follows that $N^\pm_\nu$ can at most raise/lower a vector $d_\nu$ times. This fixes the nilpotency order of the raising/lowering operators as

$$(N^-_\nu)^{d_\nu+1} = (N^+_\nu)^{d_\nu+1} = 0. \tag{3.72}$$

The full log-monodromy matrix is a direct sum of such blocks, so that the nilpotency of the original nilpotent operator $N$ places an upper bound on the dimension of each block. Moreover, the full vector space $H_\mathbb{C}$ can be decomposed into sub-spaces $H_\nu$ on which the irreducible representations act, so that we obtain

$$H_\mathbb{C} = H_{\nu_1} \oplus H_{\nu_2} \oplus \ldots \oplus H_{\nu_n}. \tag{3.73}$$

Each of these sub-spaces comes with its own weight space decomposition, so that we obtain a simultaneous decomposition by weight and irrep.

$$H_\mathbb{C} = \begin{bmatrix} H_{\nu_1}^{d_{\nu_1}} \\ \\ H_{\nu_1}^{d_{\nu_1}-2} \\ \\ \vdots \\ \\ H_{\nu_1}^{-d_{\nu_1}+2} \\ \\ H_{\nu_1}^{-d_{\nu_1}} \end{bmatrix} \bigoplus \cdots \bigoplus \begin{bmatrix} H_{\nu_n}^{d_{\nu_n}-2} \\ \\ H_{\nu_n}^{d_{\nu_n}-2} \\ \\ \vdots \\ \\ H_{\nu_n}^{-d_{\nu_n}+2} \\ \\ H_{\nu_n}^{-d_{\nu_n}} \end{bmatrix}. \tag{3.74}$$

### 3.3.2 The SL$_2$-Split

Clearly, the structures that emerge mirror those present in the Deligne splitting. There too did we obtain a weight decomposition and did the nilpotent log-monodromy matrix $N_{(k)}$ act as a lowering operator with respect to this decomposition. Morally speaking, each block in the decomposition above (equivalently each column in (3.74)) corresponds to a column in the Hodge-Deligne diamond. The precise identification of the triple $\{N^-, N^+, N^0\}$ and its action on the $I^{p,q}$ is slightly more subtle however. In the one-modulus case, the problem can essentially be traced back to the non-trivial conjugation relation (3.48). In particular, while we may attempt to define a naive weight operator which satisfies

$$\mathcal{N}^0 v^{p,q} = (p+q-3)v^{p,q}, \quad v^{p,q} \in I^{p,q}, \tag{3.75}$$

the conjugation relation (3.48) shows that this matrix can never be real. The single-variable SL$_2$-orbit theorem of Schmidt remedies this issue for one-modulus degenerations[6]. The procedure consists of two steps.

[6] The following construction is not in fact the one given in the original single-variable SL$_2$-orbit theorem in [3]. Instead we follow the discussion from [70]. This also extends more readily to the multi-variable case.

1. In the first, a result due to Deligne shows that given the Deligne splitting obtained from $(F_0^p, W_l(N))$, there is always another mixed Hodge structure $(\hat{F}_0^p, W_l(N))$ with the same weight filtration which is polarized by $N$. In particular, there exists an operator $\delta \in \mathfrak{sp}(2m, \mathbb{R})$ such that
$$\hat{F}_0^p := e^{-i\delta} F_0^p, \tag{3.76}$$
defines an $\mathbb{R}$-split Deligne splitting. To this new filtration, we can associate a new Deligne splitting $\hat{I}^{p,q}$ by evaluating equation (3.46) for the pair $(\hat{F}_0^p, W_l(N))$ (or, now that it is $\mathbb{R}$-split, 3.49). Its explicit construction is given in appendix B.

2. The second step involves a further rotation, by a second, uniquely defined operator $\zeta \in \mathfrak{sp}(2m, \mathbb{R})$
$$\tilde{F}_0^p := e^{\zeta} \hat{F}_0^p. \tag{3.77}$$
The resulting filtration defines a mixed Hodge structure $(\tilde{F}_0^p, W_l(N))$ with associated Deligne splitting $\tilde{I}^{p,q}$ that is again $\mathbb{R}$-split. We call the splitting the $\tilde{I}^{p,q}$ the $\mathrm{SL}_2$-splitting to distinguish it from the $\mathbb{R}$-split Deligne splitting above. Its explicit construction is likewise given in appendix B. Let us mention that for our applications, this second step is always trivial, we mention it here for the sake of completeness.

Once these steps are taken, we are left with a mixed Hodge structure $(\tilde{F}, W(N))$, distinct from but related to the original one associated to the singularity at $p \in \Delta^\circ_{i_1,\ldots,i_k}$. Following the construction from section 3.3.1, we know that for a given nilpotent $N$ we can always find an $\mathfrak{sl}(2, \mathbb{R})$-triple $\{N^- := N, N^+, N^0\}$, but this choice is not unique. The $\mathrm{SL}_2$-orbit theorem states that there exists a unique choice of such triples that is compatible with the $\mathrm{SL}_2$-split $\tilde{I}^{p,q}$ in the following sense.

1. The weight operator acts on the $\mathrm{SL}_2$-splitting according to
$$N^0 \tilde{I}^{p,q} = (p + q - 3)\tilde{I}^{p,q}. \tag{3.78}$$

2. The raising/lowering operators act as raising/lowering operators on the $\mathrm{SL}_2$-splitting
$$N^{\pm} \tilde{I}^{p,q} \subset \tilde{I}^{p\pm1,q\pm1}. \tag{3.79}$$

3. Associated to the $\mathrm{SL}_2$-splitting there is a pure Hodge structure in the sense of section 3.1. This is defined by
$$F_\infty^p := e^{iN^-} \tilde{F}_0^p, \quad C_\infty v^{p,q} = i^{p-q} v^{p,q}, \quad v^{p,q} \in H_\infty^{p,q} := F_\infty^p \cap \bar{F}_\infty^q. \tag{3.80}$$

4. Let $d$ denote the integer such that
$$N^d \neq 0, \quad N^{d+1} = 0. \tag{3.81}$$
Then the weight space decomposition of $H_\mathbb{C}$ induced by the weight operator $N^0$
$$H_\mathbb{C} = H^d \oplus H^{d-2} \oplus \ldots \oplus H^{-d}, \quad H^l := \{v \in H_\mathbb{C} |\ N^0 v = lv\}, \tag{3.82}$$
satisfies the following properties with respect to the polarization on $H_\mathbb{C}$
$$C_\infty : H^l \to H^{6-l}, \qquad \langle H^l, H^{l'} \rangle = 0 \quad \text{if } l + l' \neq 6. \tag{3.83}$$

We emphasize that each of these results is a non-trivial part of the full statement of the $\mathrm{SL}_2$-orbit theorem and we encourage the reader to review the original mathematical literature [3, 70]. The remainder of this section will be dedicated to a particular consequence of the $\mathrm{SL}_2$-orbit theorem which will allow us to approximate the Hodge star in the asymptotic regime.

### 3.3.3 The SL$_2$-Approximated Hodge Star

One of the non-trivial consequences of the SL$_2$-orbit theorem is that it provides us with an approximation to the Weil operator. This uses crucially the compatibility of the boundary pure Hodge structure with the $\mathfrak{sl}(2)$-triple of the mixed Hodge structure. We denote this approximation $C_{\mathfrak{sl}(2)}$. Starting from the boundary Weil operator $C_\infty$, we proceed in two steps [68]

- First, we introduce the leading saxion-dependence and set the axion $a = 0$. Analogous to the statement of the nilpotent orbit theorem, the limiting Weil operator is obtained from the following limit

$$C_\infty = \lim_{s\to\infty} e(s)Ce(s)^{-1}, \qquad e = \exp\left(\frac{1}{2}\log(s)N^0\right). \tag{3.84}$$

  The leading saxion-dependence is then re-instated by

$$C_{\mathfrak{sl}(2)}(a=0,s) = e^{-1}(s)C_\infty e(s). \tag{3.85}$$

- Next we re-introduce the axion dependence. By manipulating the results of the nilpotent orbit theorem, it follows that this axion dependence takes a very simple form

$$C_{\mathfrak{sl}(2)}(a,s) := e^{aN^-}C_{\mathfrak{sl}(2)}(a=0,s)e^{-aN^-} = e^{aN^-}e(s)^{-1}C_\infty e(s)e^{-aN^-}. \tag{3.86}$$

#### Weak Coupling Limits

As an immediate application of the result (3.86), let us derive a sufficient condition to assess whether states are weakly coupled in the asymptotic regime (see [71] for a far more extensive analysis which includes multi-moduli limits). What we mean by this is that given a D3 brane with charge vector $\mathbf{q} = (p^I, -q_I)^T$, its physical charge, defined in section 2.1.3 as

$$\mathcal{Q}_{\mathbf{q}} = \frac{1}{2}\mathbf{q}^T\mathcal{M}\mathbf{q}, \qquad \mathcal{M} = \begin{pmatrix} -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -(\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}, \tag{3.87}$$

goes to zero in the limit $s\to\infty$. In the geometric Calabi-Yau context, it follows from section 1.5.2 that the matrix $\mathcal{M}$ is precisely the Hodge star matrix which we approximated above. In particular, it is given by

$$\mathcal{M} = \eta C \sim \eta C_{\mathfrak{sl}(2)}, \tag{3.88}$$

such that the physical charge can be expressed in terms of the $\mathfrak{sl}(2)$-approximated Hodge norm as

$$\mathcal{Q}_{\mathbf{q}}^2 \sim \frac{1}{2}\langle \mathbf{q}, C_{\mathfrak{sl}(2)}\mathbf{q}\rangle = \frac{1}{2}\|\mathbf{q}\|^2_{\mathfrak{sl}(2)}, \tag{3.89}$$

where we have used the fact that $\mathbf{q}$ is real. Next, we decompose the charge vector $\mathbf{q}$ and the axion-dependent combination $\rho(a) := e^{-aN^-}\mathbf{q}$ according to the weight-decomposition of $H_{\mathbb{C}}$ induced by the weight operator $N^0$

$$\mathbf{q} = \sum_l \mathbf{q}^l, \qquad \rho(a) = \sum_l \rho^l(a), \qquad \mathbf{q}^l, \rho^l(a) \in H^l_{\mathbb{C}}. \tag{3.90}$$

The vector $\rho(a)$ can be used to simplify the expression for the Hodge norm as follows. Recall that $N^-, N^0 \in \mathfrak{sp}(4,\mathbb{Z})$ meaning that their exponents preserve the polarization. This allows us to rewrite (3.89) as

$$\begin{aligned}\langle C_{\mathfrak{sl}(2)}\mathbf{q},\mathbf{q}\rangle &= \langle e^{aN^-}e(s)^{-1}C_\infty e(s)e^{-aN^-}\mathbf{q},\mathbf{q}\rangle \\ &= \langle C_\infty e(s)e^{-aN^-}\mathbf{q}, e(s)e^{-aN^-}\mathbf{q}\rangle \\ &= \langle C_\infty e(s)\rho(a), e(s)\rho(a)\rangle.\end{aligned} \tag{3.91}$$

Next, we use that the operator $e(s)$ acts on weight-eigenstates as multiplication by $s^{\frac{l-3}{2}}$, along with the orthogonality property 3.83

$$\begin{aligned}\langle C_\infty e(s)\rho(a), e(s)\rho(a)\rangle &= \sum_{k,l} s^{\frac{k+l-6}{2}}\langle C_\infty\rho(a)^k,\rho(a)^l\rangle \\ &= \sum_l s^{l-3}\langle C_\infty\rho(a)^l,\rho(a)^l\rangle.\end{aligned} \tag{3.92}$$

We find that the leading order saxion dependence of the Hodge norm (3.89) is dictated by the highest-weight component of $e^{-aN^-}\mathbf{q}$ in (3.90). Moreover, since the lowering operator $N^-$ only produces states of lower weight, it follows that this highest-weight piece is the same as that of $\mathbf{q}$. If we denote the latter by the integer $l_m$, we obtain a succinct criterion to determine whether the state is weakly or strongly coupled in the asymptotic regime

$$\lim_{s\to\infty}\mathcal{Q}^2_{\mathbf{q}} = \begin{cases}\infty & \text{if } l_m > 3,\\ \text{const.} & \text{if } l_m = 3,\\ 0 & \text{if } l_m < 3.\end{cases} \tag{3.93}$$

In the next section, we will use this condition to identify a basis for $H_\mathbb{C}$ in which the gauge fields are weakly coupled, i.e. their gauge couplings go to zero in the limit $s\to 0$, which will ensure for us that we are justified in a semi-classical analysis.

#### A Note on Methods

Before we proceed with our analysis, let us comment on our motivation for using the method presented above. In particular, one could also have recovered approximate expressions for the gauge kinetic functions from the special geometry identity (1.107), applied to the nilpotent orbit approximation of the period vector. The reason for employing the $\mathrm{SL}_2$-approximated Hodge star is twofold. The first is that it extends more readily to the multi-modulus case. The second is that our method applies also to the type I case. A generic feature at these limits is that the nilpotent orbit $\Pi_{nil}$ does not contain enough information to recover the full Hodge filtration $F^p_{nil}$, as is required to reconstruct the Weil operator. This simply follows from the defining condition that $N\mathbf{a}_0 = 0$, which is precisely how derivatives act on $\Pi_{nil}$. In this case, one should include exponential corrections. If, however, one has available the full filtration $F^p_{nil} = e^{tN}F^p_0$ for all $p$, there is no need to re-package this information into a period vector. Indeed, this will be our goal in the following section, where we will use Hodge theoretic constraints to recover the full nilpotent orbit data. With this in hand, one could alternatively reconstruct the exponential corrections to $\Pi_{nil}$, which is the approach taken in [66].

## 3.4 Reconstructing the Hodge Star

Finally, we use the results from this chapter to construct an approximate form of the Hodge star operator near the singularities in the moduli space, from which we obtain the limiting

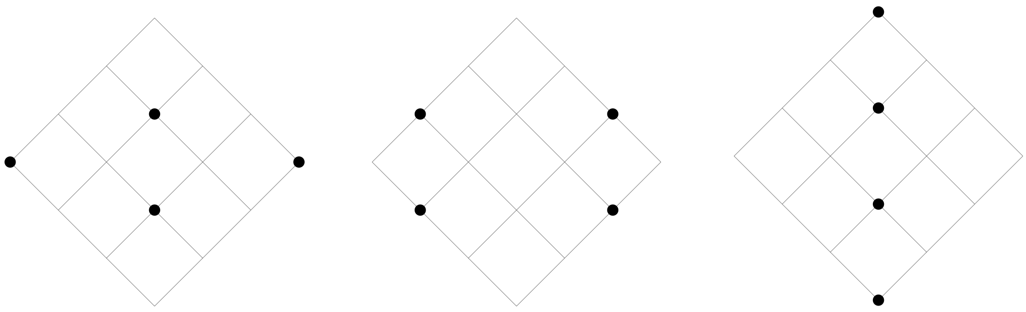



Figure 3.4: Hodge-Deligne diamonds for the three one-modulus degenerations for $h^{2,1}$. Black dots indicate non-empty $I^{p,q}$.

form of the gauge kinetic matrices. In particular we will do so for one-dimensional moduli spaces, i.e. $h^{2,1} = 1$. Looking back at the restrictions on the sub-type index in table 3.1, we see that for $h^{2,1} = 1$ we can never have a type III$_c$ singularity due to the restriction on $c$. Moreover, positivity of the sub-type index for the other cases means that there are only three possible cases for such one-dimensional moduli spaces

$$\begin{aligned} &\mathrm{I}_1: \quad \text{Conifold Point,} \\ &\mathrm{II}_0: \quad \text{Tyurin Degeneration,} \\ &\mathrm{IV}_1: \quad \text{Large Complex Structure Point,} \end{aligned} \tag{3.94}$$

whose corresponding Hodge-Deligne diamonds are shown in figure 3.4. The names correspond to the geometric setting in which each of these singularity types is realized. Indeed, it is easy to forget that these abstract mixed Hodge structures should arise from degenerating Calabi-Yau manifolds. While the three types above are *allowed* by the algebraic properties of the mixed Hodge structures, we are not guaranteed that all such singularity types are actually realized by an actual degenerating Calabi-Yau. Nevertheless, this is the case for the three one-modulus cases studied here and we will make some brief comments on their geometric properties as we go through the cases. In fact, these one-dimensional moduli spaces have been well-studied in the past, often *starting* from the geometric setting of an explicit family of Calabi-Yau. We instead start from the abstract classification of such limits and occasionally identify with the geometric case. Our method will capture all such geometric cases, whose details are encoded in terms of various unfixed parameters.

Our strategy in this section will consist, broadly speaking of two steps. The first is to construct the most general nilpotent orbit data associated to a given singularity type, consisting of the the triple $(N, F_0^p, W_l)$. With this in hand we can apply the procedure from the previous section to obtain the limiting form of the Hodge star. While this may appear to be a daunting task at first, it turns out that for the one-modulus cases considered here the algebraic constraints encountered throughout this chapter are sufficient to specify, to a reasonable level of detail, the most general possible data. This has been done in [72]. We will use their results and comment on the relevant procedure, but otherwise do not repeat it in full detail. Altogether this will give us the information we need to begin analyzing the symmetries that appear in the resulting EFT. With these general remarks out of the way, let us be more specific about how we will proceed.

1. The "zero'th" step will be to fix a basis $\{e_1, e_2, e_3, e_4\}$ for the space $H_{\mathbb{C}}$ with respect to which we will cast our results. This basis cannot be arbitrary however, and should satisfy three important properties.

- **Symplectic** The first is that we wish for it to be symplectic with respect to the pairing $\langle\cdot,\cdot\rangle$ on $H_{\mathbb{C}}$. While it is not strictly necessary that this pairing take the canonical form (3.10), it will be convenient when the time comes to discuss the physics, and we should at the very least know its explicit form. This means concretely that our basis elements can be identified with the symplectic basis elements as

$$\begin{aligned} e_1 &\cong \alpha_0, & e_3 &\cong \beta^0, \\ e_2 &\cong \alpha_1, & e_4 &\cong \beta^1. \end{aligned} \tag{3.95}$$

- **Integral** The second, arguably more important property is that it should be integral, which is to say that it should constitute a $\mathbb{Z}$-basis[7] for the lattice $H_{\mathbb{Z}} \subset H_{\mathbb{C}}$. What we mean by this is that the basis should be primitive with respect to the lattice $H_{\mathbb{Z}} \subset H_{\mathbb{C}}$. Recall that this lattice is given by $H^3(X,\mathbb{Z})$, which is poincare dual to $H_3(X,\mathbb{Z})$. In turn, this implies that D3-particles have integral charge vectors and that the associated field strengths have quantized fluxes, i.e.

$$\int_{S^2} \begin{pmatrix} F^I \\ G_I \end{pmatrix} \in \mathbb{Z}^{2h^{2,1}+2}. \tag{3.96}$$

- **Weak Coupling** The third property is more physically motivated. This is that the chosen basis should correspond to a weakly coupled frame for our gauge fields. This means that given a symplectic basis $(\alpha_I, \beta^J)$, we demand that charges which couple to the corresponding $F^I$ (i.e. the electric fields of the given frame) have vanishing physical charge in the limit $s \to \infty$. These correspond to states with charge vector $\mathbf{q} = (0,0,-q_0,-q_1)^T$ so that we demand that the basis elements $\beta^I$ satisfy $\|\beta^I\| \lesssim \mathcal{O}(1)$. According to our discussion in section 3.3.3, this can be arranged by choosing a basis so that the $\{\beta^I\} \subset W_{l<3}$.[8] As a direct consequence of the fact that $NW_l \subset W_{l-2}$ it also follows that monodromy transformations will preserve "electric" states.

In [72] it was shown that for each of the three cases (3.94), there always exists an integral symplectic basis which satisfies all three properties. More precisely, it was shown that for each case we can choose an integral symplectic basis that is compatible with the weight filtration in the sense that successive basis vectors span successive spaces in the filtration. This statement is non-trivial and uses crucially the fact that both the weight filtration and the polarization are defined over $\mathbb{Q}$. Together these ensure that we may choose an integral basis compatible with the weight filtration, and moreover that this basis may be chosen symplectic for a case-dependent form of the symplectic pairing $\eta$. We prefer to keep $\eta$ in the canonical form (3.10), in favour of sacrificing strict compatibility with the weight filtration.

2. Once the basis is fixed, one can then derive the most general form of the log-monodromy matrix, making sure to comply with the various consistency conditions encountered

[7] A $\mathbb{Z}$-basis is a basis $\{e_i\}$ for $H_{\mathbb{R}}$ such that any point in $H_{\mathbb{Z}}$ is a linear combination of the $e_i$ with integer coefficients. It follows that the elements $\{e_i\}$ are primitive, i.e. $\nexists n \in \mathbb{Z}, |n| \geq 2 : [e_i] = 0 \in H_{\mathbb{Z}}/nH_{\mathbb{Z}}$ (intuitively, $e_i$ is not an integer multiple of some other element in the lattice). That this is always possible follows from the fact that the lattice $H_{\mathbb{Z}}$ is torsion-free.

[8] More precisely, our discussion showed that we need $\{\beta^I\} \subset H^{l<3}$, which differs from the condition stated in the main text only by pieces of lower-weight. Note moreover that this condition is ambiguous for the type $I$ case since dim $W_{l<3} = 1$. This case will be treated separately.

throughout this chapter. These are given by

$$NW_l \subseteq W_{l-2}, \quad N^d = 0, \quad N^T\eta + \eta N = 0, \quad e^N \in Sp(4,\mathbb{Z}), \tag{3.97}$$

together with the case-dependent polarization conditions on the pure Hodge structures on the primitive spaces (equivalently the graded pieces). The latter enforce positivity conditions that will ensure that the resulting kinetic terms have the correct sign. We emphasize that the monodromy so-recovered only corresponds to the nilpotent part of the full monodromy $T$.

3. Finally, we write down the most general basis vectors for the limiting filtration $F_0^p$. These too should satisfy various compatibility conditions, both with the weight filtration as well as with the log-monodromy matrix. The details are somewhat case-dependent and we refer to [72] for further details. It will be convenient to express the filtration in terms of a *period matrix* whose columns successively span the elements of the limiting filtration

$$\Omega = \left(F_0^3 \quad \subset \quad F_0^2 \quad \subset \quad F_0^1 \quad \subset \quad F_0^0\right). \tag{3.98}$$

An important freedom we have is to redefine our coordinates on the moduli space. Indeed, in the definition of the limiting filtration entered crucially a choice of lifting defined by the coordinate $t = \frac{1}{2\pi i}\log(z)$. Under a divisor-preserving[9] re-definition $z \to \tilde{z}(z) = e^{f(z)}z$ for $f(z)$ arbitrary, one finds that (for large $\mathrm{Im}\, t$)

$$t = \frac{\log(z)}{2\pi i} \to \frac{\log(\tilde{z}(e^{2\pi i t}))}{2\pi i} \sim \frac{\log\left(0 + \tilde{z}'(0)e^{2\pi i t} + \mathcal{O}(e^{4\pi i t})\right)}{2\pi i} = \frac{f(0)}{2\pi i} + t, \tag{3.99}$$

so that [3, 72]

$$F_0^p := \lim_{t\to i\infty} e^{-tN}F^p(z) \to \lim_{t\to i\infty} e^{-\left(\frac{f(0)}{2\pi i}+t\right)N}F^p(z) = e^{-\frac{f(0)}{2\pi i}N}F_0^p. \tag{3.100}$$

For our purposes we can thus generate all relevant transformations of $F_0^p$ by constant re-scalings of the form $z \to e^{2\pi i\lambda}z$ which correspond to shifts of $t \to t+\lambda$. This freedom can be used to reduce the number of arbitrary parameters, or to bring them into a convenient form.

4. Having assembled the most general nilpotent orbit data which can arise in a single-variable degeneration, we can apply procedure from the previous sections to obtain approximate expressions for the period vector and gauge kinetic functions. The former follows directly from the nilpotent orbit theorem: one takes a representative $\mathbf{a}_0$ of $F_0^3$ constructed above (normalized to convenience) and recovers from it the nilpotent orbit

$$\Pi_{nil} = e^{tN}\mathbf{a}_0. \tag{3.101}$$

The kinetic functions are constructed from the $\mathrm{SL}_2$-approximated Hodge star. Following the procedure of section 3.3, we start by translating the mixed Hodge structure $(F_0^p, W_l)$ to the more convenient Deligne splitting $I^{p,q}$. We then construct the rotation operators that take us to the $\mathrm{SL}_2$-splitting where we have control over the action of the Hodge star. By identifying the pure Hodge structure associated to the boundary, we can extract the limiting form of the Weil operator, and extend it into the near-boundary region following section 3.3.3. Finally, we recover the gauge kinetic functions by inverting the matrix $\mathcal{M}$ in (3.87).

[9]That is to say $\tilde{z}(0) = 0$, while $\tilde{z}(z) \neq 0$ for $z \neq 0$.

### 3.4.1 Type $\mathrm{IV}_1$

We start by discussing the Type $\mathrm{IV}_1$ case. In order to illustrate the method, we will be quite detailed for this first case. We emphasize however, that for each of the three cases, the construction of the nilpotent orbit data was done in [72], so that we are relatively brief about this part.

#### Nilpotent Orbit Data

Following the steps outlined above, we start by fixing our integral symplectic basis. In [72] it was shown that this may always be done such that the weight filtration is spanned as follows

$$\text{Type IV}_1: \quad \begin{cases} W_6 = \text{span}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} \bigoplus W_4, \\ W_4 = \text{span}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix} \bigoplus W_2, \\ W_2 = \text{span}\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix} \bigoplus W_0, \\ W_0 = \text{span}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}. \end{cases} \tag{3.102}$$

In this basis one may then construct the most general log-monodromy matrix compatible with this filtration, as well as the constraints (3.97). The result is given by [72]

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 \\ c & b & 0 & -m \\ b & n & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \begin{cases} m, n \in \mathbb{Z}, \\ b + mn/2 \in \mathbb{Z}, \\ c - m^2 n/6 \in \mathbb{Z}, \\ m \neq 0,\ n > 0. \end{cases} \tag{3.103}$$

We note that the property (3.102) does not uniquely fix a basis. In particular, there remain integral symplectic transformations that preserves the weight filtration, whereby lower-weight pieces get rotated into the higher-weight pieces. These can be used to simplify some of the coefficients appearing in the log-monodromy matrix, but we do not do so here. It is then shown that, by an appropriate choice of coordinates $z$, the most general limiting filtration $F_0^p$ can be spanned by the following period matrix (3.98) [72]

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & c/(2m) & 0 & 1 \\ c/(2m) & b/m & 1 & 0 \end{pmatrix}, \quad \text{with } \xi \in \mathbb{C}. \tag{3.104}$$

#### $\mathfrak{sl}(2)$-orbit Data

With the nilpotent orbit in hand, we can now begin to compute the associated Deligne splitting by evaluating equation 3.46. It turns out that for this particular case the $I^{p,q}$ simply correspond to the columns of the period matrix above, so that we do not give their explicit expression here. Instead, we proceed right away to constructing the rotation operator that renders this splitting $\mathbb{R}$-split. Following the procedure in appendix B, we find the following operator

$$\delta = \delta_{-3,-3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \text{Im}\,\xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.105}$$

which we then use to rotate to the $\mathbb{R}$-split Deligne splitting

$$\begin{pmatrix} \hat{I}^{3,3} & \hat{I}^{2,2} & \hat{I}^{1,1} & \hat{I}^{0,0} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \operatorname{Re}\xi & c/(2m) & 0 & 1 \\ c/(2m) & b/m & 1 & 0 \end{pmatrix}. \tag{3.106}$$

The second rotation matrix is trivial so that we immediately arrive at the final splitting. In the following we will need the weight operator of the $\mathfrak{sl}(2)$-triple associated to this splitting. This is readily obtained by solving

$$N^0 \hat{I}^{p,q} = (p+q-3)\hat{I}^{p,q}. \tag{3.107}$$

Note that because the $\hat{I}^{p,q}$ span the full $H_{\mathbb{C}}$, this equation fixes $N^0$ unambiguously. The result is given by

$$N^0 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6\operatorname{Re}\xi & 2c/m & -3 & 0 \\ 2c/m & 2b/m & 0 & -1 \end{pmatrix}. \tag{3.108}$$

### Boundary Data

Next, we compute the limiting pure Hodge structure and associated Weil operator by applying the log-monodromy matrix. The former is given by a limiting Hodge filtration with associated period matrix given by

$$F^p_\infty = e^{iN} F^p_0 \quad \Leftrightarrow \quad \Omega^p_\infty = e^{iN}\Omega^p, \tag{3.109}$$

which evaluates to

$$\Omega_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ im & 1 & 0 & 0 \\ \frac{1}{6}i(m^2n+3c) + \operatorname{Re}\xi & c/(2m) + mn/2 & b/m + in & 1 \\ c/(2m) - mn/2 + ib & b/m + in & 1 & 0 \end{pmatrix}. \tag{3.110}$$

From it, one evaluates

$$H^{p,q}_\infty = F^p_\infty \cap \bar{F}^q_\infty, \tag{3.111}$$

which yields

$$\begin{aligned} H^{3,0}_\infty : &\quad \big(1,\ \ im,\ \ \tfrac{1}{6}i(m^2n+3c) + \operatorname{Re}\xi,\ \ c/(2m) - mn/2 + ib\big)\,, \\ H^{2,1}_\infty : &\quad \big(1,\ \ \tfrac{1}{3}im,\ \ -\tfrac{1}{6}i(m^2n - c) + \operatorname{Re}\xi,\ \ c/(2m) + mn/6 + ib/3\big)\,, \end{aligned} \tag{3.112}$$

with the other two spaces being related to these by complex conjugation. One sees that the pure Hodge structure at the boundary is related to the mixed Hodge structure in a rather non-trivial way. One can now solve for the limiting Weil operator using

$$C_\infty v = i^{p-q} v, \qquad v \in H^{p,q}_\infty, \tag{3.113}$$

which like (3.107) unambiguously defines $C_\infty$. The result is given by

$$C_\infty = \frac{1}{m^2 n}\begin{pmatrix} 6\operatorname{Re}\xi & 3c/m & -6 & 0 \\ mc & 2mb & 0 & -2m^2 \\ (m^4n^2 + 3c^2 + 36(\operatorname{Re}\xi)^2)/6 & c(b + 3\operatorname{Re}\xi/m) & -6\operatorname{Re}\xi & -mc \\ c(b+3\operatorname{Re}\xi/m) & m^2n^2/2 + 2b^2 + 3c^2/(2m^2) & -3c/m & -2mb \end{pmatrix}. \tag{3.114}$$

### Final Result

Finally, we can put everything together to compute the $\mathfrak{sl}(2)$-approximated Hodge star. Recall that the $\mathfrak{sl}(2)$-approximated Weil operator is obtained by evaluating

$$C_{\mathfrak{sl}(2)}(a,s) = e^{aN^-} e(s)^{-1} C_\infty e(s) e^{-aN^-}, \qquad e(s) = \exp\left(\frac{1}{2}\log(s) N^0\right) \tag{3.115}$$

where $N^-$ and $N^0$ are given in (3.103) and (3.108), respectively. By multiplying from the left with $\eta$, we recover the Hodge star matrix

$$\mathcal{M} = \eta C_{\mathfrak{sl}(2)} = \begin{pmatrix} \langle \alpha_I, *\alpha_J\rangle & \langle \alpha_I, *\beta^J\rangle \\ \langle \beta^I, *\alpha_J\rangle & \langle \beta^I, *\beta^J\rangle \end{pmatrix}. \tag{3.116}$$

For this case in particular, the expression obtained from (3.116) is too large to present explicitly here, but we will do so for the following two cases. Instead, we note that this matrix determines the kinetic matrix $\mathcal{N}$ via the identification

$$\mathcal{M} = \eta C_{\mathfrak{sl}(2)} = \begin{pmatrix} -\mathrm{Im}\mathcal{N} - \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -(\mathrm{Im}\mathcal{N})^{-1} \end{pmatrix}. \tag{3.117}$$

Thus, by inverting (minus) the bottom-right block of the resulting matrix, we recover the imaginary part of $\mathcal{N}$, while multiplying the latter by the bottom-left block yields the real part of $\mathcal{N}$. Rather than present the full matrix $\mathcal{M}$, we give here the bottom two blocks

$$\begin{aligned} &\langle \alpha_I, *\beta^J\rangle = -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} = \\ &\frac{-1}{m^3 n s^3}\begin{pmatrix} -mna^3 + 3(c/m)a + 6\,\mathrm{Re}\,\xi/m & 3a(2b/m + na) + 3c/m^2 \\ c(3a^2+s^2) - m^2na^2(a^2+s^2) + 6a\,\mathrm{Re}\,\xi & 3a(c/m + a(2b+mna)) + 2(b+mna)s^2 \end{pmatrix}, \\ &\langle \beta^I, *\beta^J\rangle = -(\mathrm{Im}\mathcal{N})^{-1} = \\ &\frac{1}{m^2 n s^3}\begin{pmatrix} 6 & 6ma \\ 6ma & 2m^2(3a^2+s^2) \end{pmatrix}. \end{aligned} \tag{3.118}$$

From these, we compute the real and imaginary parts of $\mathcal{N}$ to obtain our final result for the kinetic matrix

$$\mathcal{N} = \begin{pmatrix} -\frac{1}{3}m^2na^3 - \mathrm{Re}\,\xi & \frac{1}{2}mna^2 - \frac{c}{2m} \\ \frac{1}{2}mna^2 - \frac{c}{2m} & -na - \frac{b}{m} \end{pmatrix} + \begin{pmatrix} -\frac{1}{6}m^2ns(3a^2+s^2) & \frac{1}{2}mnas \\ \frac{1}{2}mnas & -\frac{n}{2}s \end{pmatrix}. \tag{3.119}$$

The corresponding period vector (in the nilpotent orbit approximation) given by

$$\Pi_{nil} = e^{tN}\mathbf{a}_0 = \begin{pmatrix} 1 \\ mt \\ -m^2nt^3/6 + ct/2 + \xi \\ mnt^2/2 + bt + c/(2m) \end{pmatrix}, \tag{3.120}$$

where $\mathbf{a}_0$ is a representative of $F_0^3$, for which we have used the first column of (3.104).

### Geometric Interpretation

As mentioned the type $\mathrm{IV}_1$ case is best understood from a mathematical perspective, but the reason for this actually has its origin in physics. In particular, one of the most famous interactions between math and physics has been the discovery of mirror symmetry. Although a proper discussion of mirror symmetry is beyond the scope of this thesis, let us say a few words here.

While we have not discussed it explicitly in this thesis, the type IIA effective action bears striking resemblance to that of type IIB, the most prominent difference being the appearance of *odd* $p$-form fields ($C_1$, $C_3$), rather than even. Likewise, when compactified on a Calabi-Yau, the resulting theory is also an $\mathcal{N} = 2$ supergravity theory. Here the most prominent difference is that the complex structure moduli now appear in the hyper-multiplets, whereas the Kahler moduli enter the vector multiplets. This suggests the possibility that given a Calabi-Yau compactification of type IIB on $X$, there exists a mirror Calabi-Yau $\hat{X}$ where the complex and Kahler moduli are exchanged whose type IIA compactification leads to the same four-dimensional theory. This idea has been formalized in the concept of mirror symmetry.

For some very special cases, this so-called mirror map has been constructed explicitly [73]. By matching the physics on both sides of the duality, we can then gain insight into the the geometry of the mirror pair. In particular, the un-fixed coefficients appearing in (3.120) can be matched to topological invariants on the mirror side. For instance, restricting to $m = 1$ for simplicity, one readily sees that the period (3.120) can be obtained from a pre-potential

$$\mathcal{F} = \frac{1}{6} n \frac{(X^1)^3}{X^0} + \frac{1}{2} bt (X^1)^2 + \frac{1}{2} c X^1 X^0 + \frac{1}{2} \xi (X^0)^2 \tag{3.121}$$

By matching with the mirror side, one finds that the coefficient $n$ can be identified with the triple intersection number of the mirror Calabi-Yau $\hat{X}$

$$n = \int_{\hat{X}} \omega \wedge \omega \wedge \omega, \tag{3.122}$$

where $\omega$ is an appropriately chosen harmonic $(1,1)$-form on $\hat{X}$. This is a topological invariant of the mirror $\hat{X}$. Similarly, the parameter $\xi$ can be identified with another topological invariant of the mirror $\hat{X}$, namely

$$\xi = \frac{\chi \zeta(3)}{(2\pi i)^3}, \tag{3.123}$$

where $\chi$ is the Euler characteristic of $\hat{X}$ and $\zeta(3)$ is the Riemann-Zeta function evaluated at 3, in which case one finds that the real part of $\xi$ vanishes.

### 3.4.2 Type $\mathrm{II}_0$

Next, we consider the type $\mathrm{II}_0$ case. We will not be quite as detailed as above, but do present intermediate results.

#### Nilpotent Orbit Data

For this case, it was shown that one can always find an integral symplectic basis such that the weight filtration (here consisting only out of $W_2 \subset W_4$) is spanned by [72]

$$\text{Type II}_0: \quad \begin{cases} W_4 = \text{span}\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \right\} \bigoplus W_2 \\ W_2 = \text{span}\left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \right\} \end{cases} . \tag{3.124}$$

Again, we have that equation 3.124 does not uniquely specify the symplectic basis. In particular, integral symplectic transformations of the form

$$\mathcal{S} = \begin{pmatrix} A^{-1} & 0 \\ C & A^T \end{pmatrix}, \tag{3.125}$$

with $AC$ symmetric preserve both the symplectic pairing $\eta$ as well as the form of the weight filtration (3.124). We will use this freedom to simplify some of the unfixed coefficients appearing below. The most general log-monodromy matrix compatible with the weight filtration above is given by

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m & k & 0 & 0 \\ k & n & 0 & 0 \end{pmatrix}, \quad \text{with} \begin{cases} m, n, k \in \mathbb{Z}, \\ m, n > 0. \end{cases} \tag{3.126}$$

Following [72], we will assume that $n/k$ is integer, in which case we may use a symplectic transformation of the kind (3.125) to set $k = 0$ and assume $m \geq n > 0$. We emphasize however, that this is not quite the most general choice, although we clearly cover a large subset of this type of limits class. In [72] it was then shown that the most general period matrix compatible with (3.124) and this choice of log-monodromy, is given by

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\alpha & 0 & 1 & 0 \\ \beta & 1 & 0 & 0 \\ i\alpha\delta & i/\alpha & \delta & 1 \end{pmatrix}, \quad \text{with} \begin{cases} \alpha := \sqrt{m/n} \in \mathbb{R}, \\ \beta, \delta \in \mathbb{C}. \end{cases} \tag{3.127}$$

Under rescalings $z \to e^{-2\pi i\lambda} z$ (cf. (3.100)) we have that

$$\beta \to \beta + m\lambda, \quad \delta \to \delta + n\lambda, \tag{3.128}$$

which we use to set $\beta = 0$ in the following. For ease of notation, we decompose the remaining parameter $\delta$ into real and imaginary parts as follows

$$\delta = p - iq/\alpha. \tag{3.129}$$

### $\mathfrak{sl}(2)$-Orbit Data

Using equation 3.46 we can derive the associated Deligne splitting, which is now non-trivially related to the period matrix above

$$\begin{pmatrix} I^{3,1} & I^{1,3} & I^{2,0} & I^{0,2} \\ 1 & 1 & 0 & 0 \\ i\alpha & -i\alpha & 0 & 0 \\ 0 & -i\alpha q & 1 & 1 \\ i\alpha\delta & -i\alpha p & i\alpha & -i\alpha \end{pmatrix}. \tag{3.130}$$

Following the procedure from appendix B we construct the relevant rotation operators to obtain the $\mathbb{R}$-split Deligne splitting. These are given by

$$\delta = \delta_{-1,-1} = -\frac{q}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \end{pmatrix}, \tag{3.131}$$

while the rotation matrix $\zeta$ is again trivial. The resulting $\mathbb{R}$-split Deligne splitting is given by

$$\begin{pmatrix} \hat{I}^{3,1} & \hat{I}^{1,3} & \hat{I}^{2,0} & \hat{I}^{0,2} \\ 1 & 1 & 0 & 0 \\ i\alpha & -i\alpha & 0 & 0 \\ \frac{1}{2}\alpha q & -\frac{1}{2}i\alpha q & 1 & 1 \\ \frac{q}{2}+i\alpha p & \frac{q}{2}-i\alpha p & i\alpha & -i\alpha \end{pmatrix}. \tag{3.132}$$

Moreover, the associated weight operator is given by

$$N^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & -1 & 0 \\ q & 2p & 0 & -1 \end{pmatrix}. \tag{3.133}$$

### Boundary Data

Rotating the associated filtration according to $F^p_\infty = e^{iN}\hat{F}^p_0$, we find the following period matrix

$$\Omega_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\alpha & 0 & 1 & 0 \\ i(m+q\alpha/2) & 1 & 0 & 0 \\ -n\alpha+q/2+ip\alpha & i\alpha & p-iq/(2\alpha)+in & 1 \end{pmatrix}. \tag{3.134}$$

The Hodge decomposition associated to this filtration is given by

$$\begin{aligned} H^{3,0}_\infty : &\quad \big(1,\ \ i\alpha,\ \ i(m+q\alpha/2),\ \ -n\alpha+q/2+ip\alpha\big)\,, \\ H^{2,1}_\infty : &\quad \big(1,\ \ i\alpha,\ \ i(-m+q\alpha/2),\ \ n\alpha+q/2+ip\alpha\big)\,, \end{aligned} \tag{3.135}$$

with the remaining two spaces following by complex conjugation. As before, we can solve for the limiting Weil operator which is now given by

$$C_\infty = \begin{pmatrix} 0 & q/(2m) & -1/m & 0 \\ q/(2n) & p/n & 0 & -1/n \\ m+q^2/(4n) & pq/(2n) & 0 & -q/(2n) \\ pq/(2n) & n+p^2/n+q^2/(4m) & -q/(2m) & -p/n \end{pmatrix}, \tag{3.136}$$

which one checks to have the correct action on the $H^{p,q}_\infty$ given above.

### Final Result

With these results, we can again construct the $\mathfrak{sl}(2)$-approximated Weil operator, given now by

$$\eta C_{\mathfrak{sl}(2)} = \frac{1}{s}\begin{pmatrix} q^2/(4n)+m(a^2+s^2) & aq+pq/(2n) & -a & -q/(2n) \\ aq+pq/(2n) & q^2/(4m)+(an+p)^2/n+ns^2 & -q/(2m) & -a-p/n \\ -a & -q/(2m) & 1/m & 0 \\ -q/(2n) & -a-p/n & 0 & 1/n \end{pmatrix}. \tag{3.137}$$

From the bottom two blocks, we can then obtain the final result for the kinetic matrix, given by

$$\mathcal{N} = -\begin{pmatrix} ma & \frac{q}{2} \\ \frac{q}{2} & na-p \end{pmatrix} - i\begin{pmatrix} ms & 0 \\ 0 & ns \end{pmatrix}. \tag{3.138}$$

Moreover, the nilpotent orbit approximation to the period vector is given by

$$\Pi_{nil} = e^{iN}\mathbf{a}_0 = \begin{pmatrix} 1 \\ i\alpha \\ mt \\ i\alpha(\delta + nt) \end{pmatrix}, \tag{3.139}$$

where, for $\mathbf{a}_0$, we again use the first column in equation 3.127 (recall that we set $\beta = 0$). We remark that this period vector is not in the special coordinates of section 1.6 (although this can always be achieved by a symplectic rotation).

#### Geometric Interpretation

Compared to the previous example, the geometric interpretation of this case is not as well-studied. Crudely speaking, under a Tyurin degeneration, the Calabi-Yau $X_6$ splits into two new 3-folds[10] $X_6^1$ and $X_6^2$, glued along a special type of divisor in each $X_6^i$ [74]. While the details of this are not important to us, this fact has a nice interpretation at the level of the associated mixed Hodge structures. In particular, the two pure Hodge structures on the graded pieces $Gr_4$ and $Gr_2 \cong NG_4$ admit the interpretation of the pure Hodge structures on the non-singular spaces $X_6^i$. The unfixed parameters above then capture the non-trivial gluing of these spaces so that the resulting mixed Hodge structure is more than the sum of its parts.

### 3.4.3 Type $\mathrm{I}_1$

This final case is slightly more involved, because it lies at finite distance. In particular, what this means is that the period vector in the nilpotent orbit approximation will not span the full Hodge filtration, which practically implies that to leading approximation the Kahler metric vanishes. We will address this issue shortly but for now we begin as before by writing down the most general nilpotent orbit data for this case.

#### Nilpotent Orbit Data

For this case it is always possible to pick an integral symplectic basis such that the weight filtration $W_2 \subset W_3 \subset W_4$ is spanned by [72]

$$\text{Type I}_1: \quad \begin{cases} W_4 = \text{span}\left\{\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\right\} \bigoplus W_3 \\ W_3 = \text{span}\left\{\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\right\} \bigoplus W_2 \\ W_2 = \text{span}\left\{\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}\right\}. \end{cases} \tag{3.140}$$

Similarly to the previous cases compatibility with the weight filtration does not uniquely fix our basis, as the former is invariant under integer symplectic transformations of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ 0 & b_2 & 1 & -b_1 \\ b_2 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c_{11} & 0 & c_{12} \\ 0 & 0 & b^{-1} & 0 \\ 0 & c_{21} & 0 & c_{22} \end{pmatrix}, \quad \begin{cases} 0 < b \in \mathbb{Z}, \\ \{c_{ij}\} \in SL(2,\mathbb{Z}). \end{cases} \tag{3.141}$$

[10] These new 3-folds are no longer Calabi-Yau, but admit a special structure in that they now become quasi-Fano varieties.

We will not make use of this freedom however. The most general log-monodromy matrix associated to this weight filtration is given by

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 0 < n \in \mathbb{Z}. \tag{3.142}$$

Meanwhile, we can choose our period matrix to be of the form

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \gamma & 0 & 0 \\ \delta - \tau\gamma & 0 & 1 & 0 \\ \tau & \delta & 0 & 1 \end{pmatrix}, \quad \begin{cases} \gamma, \delta \in \mathbb{R}, \\ \operatorname{Im} \tau \neq 0. \end{cases} \tag{3.143}$$

### $\mathfrak{sl}(2)$-Orbit Data

The Deligne-splitting obtained from this data is given by

$$\begin{pmatrix} I^{2,2} & I^{3,0} & I^{0,3} & I^{1,1} \\ 1 & 0 & 0 & 0 \\ \gamma & 1 & 1 & 0 \\ 0 & \delta - \gamma\tau & \delta - \gamma\bar{\tau} & 1 \\ \delta & \tau & \bar{\tau} & 0 \end{pmatrix}. \tag{3.144}$$

It follows that the resulting Deligne-splitting is already $\mathbb{R}$-split. The weight operator associated to this Deligne splitting is then given by

$$N^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & \delta & -1 & -\gamma \\ \delta & 0 & 0 & 0 \end{pmatrix}. \tag{3.145}$$

### Boundary Data

Evaluating $F^p_\infty = e^{iN} F^p_0$ we find the following period matrix

$$\Omega_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \gamma & 0 & 0 \\ \delta - \gamma\tau & in & 1 & 0 \\ \tau & \delta & 0 & 1 \end{pmatrix}, \tag{3.146}$$

with associated Hodge decomposition given by

$$\begin{aligned} H^{3,0}_\infty : &\quad \big(0, \;\; 1, \;\; \delta - \gamma\tau, \;\; \tau\big)\,, \\ H^{2,1}_\infty : &\quad \big(1, \;\; \gamma, \;\; in, \;\; \delta\big)\,. \end{aligned} \tag{3.147}$$

In the following we introduce the following short-hands

$$\tau = x + iy, \tag{3.148}$$

keeping in mind that $\operatorname{Im} \tau = y \neq 0$. We can then solve for the limiting Weil operator acting on the $H^{p,q}_\infty$, which now reads

$$C_\infty = \frac{1}{ny} \begin{pmatrix} 0 & -\delta y & y & \gamma y \\ n(\delta - \gamma x) & nx - \delta\gamma y & \gamma y & -n + \gamma^2 y \\ -n^2 y + n|\delta - \gamma\tau|^2 & n(\delta x - \gamma|\tau|^2) & 0 & -n(\delta - \gamma x) \\ n(\delta x - \gamma|\tau|^2) & -\delta^2 y + n|\tau|^2 & \delta y & -nx + \delta\gamma y \end{pmatrix}. \tag{3.149}$$

### Final Result

Finally, we compute the $\mathfrak{sl}(2)$-approximated Hodge star matrix, which is given by

$$\eta C_{\mathfrak{sl}(2)} = \frac{1}{s}\begin{pmatrix} \frac{s|\delta-\gamma\tau|^2}{y} - n(a^2+s^2) & \frac{\delta(sx-ay)-\gamma s(x^2+y^2)}{y} & a & \frac{a\gamma y+s\gamma x-\delta s}{y} \\ \frac{\delta(sx-ay)-\gamma s(x^2+y^2)}{y} & -\frac{\delta^2}{n}+\frac{s(y^2+x^2)}{y} & \frac{\delta}{n} & \frac{\delta\gamma}{n}-\frac{sx}{y} \\ a & \frac{\delta}{n} & -\frac{1}{n} & -\frac{\gamma}{n} \\ \frac{-\delta s+\gamma(sx+ay)}{y} & \frac{\delta\gamma}{n}-\frac{sx}{y} & -\frac{\gamma}{n} & -\frac{\gamma^2}{n}+\frac{s}{y} \end{pmatrix}. \tag{3.150}$$

Extracting the relevant blocks, we then find that the gauge kinetic matrix is given by

$$\mathcal{N} = -\begin{pmatrix} na-\gamma(\delta-\gamma x) & \delta-\gamma x \\ \delta-\gamma x & x \end{pmatrix} + i\begin{pmatrix} ns-\gamma^2 y & \gamma y \\ \gamma y & -y \end{pmatrix}, \tag{3.151}$$

where we can perform a shift $a \to a+\gamma(\delta-\gamma x)/n$ to simplify the real part (this could also be done at the level of the mixed Hodge structure, but its purpose is more clear here). The associated nilpotent orbit is given by

$$\Pi_{nil} = \begin{pmatrix} 0 \\ 1 \\ \delta-\gamma\tau \\ \tau \end{pmatrix}. \tag{3.152}$$

It is clear that the vanishing of $N\mathbf{a}_0$ implies that the associated nilpotent orbit is moduli-independent. As such it leads to a constant Kahler potential and a vanishing Kahler metric. Indeed, this was exactly what we used to argue that these limits are located at finite distance. While this means that we cannot naively use $\mathbf{a}_0 \in F_0^3$ to recover the full Hodge filtration, we have access to the full limiting filtration $F_0^p$. One expects that this should contain sufficient information to reconstruct an approximate period vector which does span the full filtration. Indeed, this was the strategy used in [66], where limiting expressions for the period vector were recovered that include the first sub-leading corrections. Although the authors do not work in an integral basis, we can nevertheless use their leading form for the Kahler metric for this case, as we are only interested in the leading axion/saxion dependence which is invariant under symplectic transformations. We note their leading order result here

$$K_{t\bar{t}} \sim ce^{-4\pi s}, \tag{3.153}$$

where $c$ is a model-dependent coefficient.

### Geometric Interpretation

This final class of singularities are called conifold singularities. Like the type $\mathrm{IV}_0$ case they have been extensively studied in geometric settings. A characteristic feature of such singularities is that some number (here only one) of three-cycles shrink to zero size. We refer to the literature (e.g. [75]) for further discussion on these types of singularities.

# Chapter 4

# Symmetries in One-Modulus Limits

With our limiting expressions for the monodromy, period vector and kinetic matrix in hand, we are now finally in a position to study the resulting EFTs and their associated global symmetries. We will consider separately the details of each of the one-modulus limits discussed in section 3.4. In particular, we will focus on the following questions.

1. Which symmetries are broken or gauged *without* the addition of stringy ingredients?
2. Which stringy effects are necessary to break the remaining symmetries?
3. Which symmetries emerge[1] as we approach infinite distance limits in field space?
4. Can the massless tower of states predicted by the SDC be understood as preventing an emergent symmetry?

To set the stage for this discussion, we start in section 4.1 with some general considerations about the symmetries of this class of EFTs. Afterwards, we will go through each of the three one-modulus limits from the previous chapter. Our analysis will start off rather broad with a discussion of type $\mathrm{II}_0$ limits in section 4.2. In section 4.3, we present the finite distance type $\mathrm{I}_1$ case as a counterpoint, though our analysis will be less in-depth. Finally, in section 4.4 we briefly discuss the type $\mathrm{IV}_0$ case, whose physics is markedly more involved than the previous two cases.

## 4.1 General Considerations

Let us recall one more time the action of the $\mathcal{N} = 2$ vector sector, specialized to the one-modulus case

$$S = \int \left( \frac{1}{2}\mathcal{I}_{IJ} F^I \wedge *F^J + \frac{1}{2}\mathcal{R}_{IJ} F^I \wedge F^J - K_{t\bar{t}} dt \wedge *d\bar{t} \right), \tag{4.1}$$

where we have chosen to write the scalar kinetic term in terms of the coordinate $t$. As we have seen, the moduli $t = a + is$ admit a shift symmetry $a \to a + 1$ which we interpreted as a gauge transformation. Consequently, the real part of the moduli will play the role of an axion. Motivated by these facts, we choose not to consider saxion dynamics. Unlike the axion, the saxion is a genuine scalar and it is not charged under any of the other gauge

[1] The use of the term emergent in this chapter is not to be confused with the term emergent as it appears in the emergence proposals from [76, 77], which we briefly mention in section 4.3, nor the emergent string proposal [78] which we mention in 4.2.

symmetries of the theory. Consequently, we do not expect it to affect the symmetry content of the theory in any interesting way, but rather clutter the analysis. We do comment on one possible exception to this below, but otherwise assume the saxion to be fixed at a value sufficiently large for the tools from chapter 3 to be applicable. The action then reduces to

$$S = \int \left( \frac{1}{2}\mathcal{I}_{IJ} F^I \wedge *F^J + \frac{1}{2}\mathcal{R}_{IJ} F^I \wedge F^J - K_{t\bar{t}} da \wedge *da \right). \tag{4.2}$$

With this caveat out of the way, let us first enumerate the possible global symmetries this model may possess. To this aim, we list here the equations of motion for the action (4.2)

$$\begin{gathered} d(\mathcal{I}_{IJ} * F^J + \mathcal{R}_{IJ} F^J) \equiv dG_I = 0, \\ K_{t\bar{t}} d * da + \partial_a K_{t\bar{t}} (da \wedge * da) + \frac{1}{2}(\partial_a \mathcal{I})_{IJ} F^I \wedge *F^J + \frac{1}{2}(\partial_a \mathcal{R})_{IJ} F^I \wedge F^J = 0, \end{gathered} \tag{4.3}$$

where we have identified the dual field strength $G^I \equiv \mathcal{I} * F^I + \mathcal{R} F^I$ as a short-hand for the the equations of motion. These equations are to be supplemented by the Bianchi identities

$$dF^I = 0, \quad d(da) = 0. \tag{4.4}$$

Based on these equations, we can distinguish three general classes of potential global symmetries of the theory

1. The electric and magnetic 1-form symmetries of the vector fields $F^I$, with associated currents
$$J^I = F^I, \quad J_I = G_I. \tag{4.5}$$

2. The continuous axionic shift symmetry and its dual 2-form symmetry, with associated currents
$$J^i = da, \quad J^i = K_{t\bar{t}} * da. \tag{4.6}$$

3. Chern-Weil global symmetries constructed out of the field strengths $F^I$ and $da$, with associated currents
$$J^{IJ} = F^I \wedge F^J, \quad J^I = da \wedge F^I. \tag{4.7}$$

In the following we discuss the fate of these global symmetries, and what factors may lead to their gauging or breaking.

### 4.1.1 Vector Symmetries

Recall from our discussion of electric magnetic duality in section 1.6 that the field strengths $F^I$ and their duals $G_I$ can be assembled into a symplectic vector, so that the conservation laws of the currents (4.5) can succinctly be written as

$$d \begin{pmatrix} F \\ G \end{pmatrix} = 0. \tag{4.8}$$

This vector similarly transforms under scalar isometries, in particular the axion monodromy that occurs as we encircle a divisor. Consequently, if $T$ denotes the associated monodromy transformation, then under $a \to a + 1$ the field strengths transform according to

$$\begin{pmatrix} F \\ -G \end{pmatrix} \to T \begin{pmatrix} F \\ -G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ -G \end{pmatrix}. \tag{4.9}$$

These monodromy transformations are gauged isometries of the scalar manifold (recall that the true scalar manifold is the moduli space parameterized by $z$), so that despite their closedness, these currents need not define good symmetries of the theory. Those currents that transform non-trivially under (4.9) are automatically broken, potentially down to a discrete sub-group. Indeed, this is a generalization of what we saw in axion electrodynamics, where the electric current $G$ transformed non-trivially under axion monodromy.

It follows that we can already give a crude classification of the number of un-broken symmetries, based solely on the properties of the monodromy matrix. In particular, since the latter is given by $T = e^N = 1 + N + \ldots$, currents that are invariant are annihilated by $N$.[2] Thus, the number of unbroken symmetries is bounded from below by the dimension of the kernel of $N$. In fact, this observation extends immediately to the multi-modulus case, where we obtain several monodromy matrices $T_i$, each of which effecting a gauge transformation of the theory. As such, we require the un-broken currents to be invariant under all of them, meaning that they should be annihilated by $N_{(k)} = N_1 + \ldots + N_k$. The dimension of the nullspace of $N_{(k)}$ is given by

$$\dim_{\mathbb{R}}(\ker N_{(k)}) = \dim_{\mathbb{R}}(H_{\mathbb{C}}) - \dim_{\mathbb{R}}(\mathrm{im} N_{(k)}). \tag{4.10}$$

In all cases, the dimension of $H_{\mathbb{C}}$ is given by $2h^{2,1} + 2$, while the rank of $N_{(k)}$ was given in table 3.1. Combining these two facts we obtain the following general classification

$$\dim_{\mathbb{R}}(\ker N_{(k)}) = \begin{cases} 2h^{2,1} + 2 - a & \text{Type I}_a, \\ 2h^{2,1} - b & \text{Type II}_b, \\ 2h^{2,1} - 2 - c & \text{Type III}_c, \\ 2h^{2,1} - d & \text{Type IV}_d. \end{cases} \tag{4.11}$$

Moreover, if we combine this with the upper bounds of the indices $a, b, c, d$, similarly given in table 3.1, it follows that axion monodromy is never sufficient to break all of the symmetries. Of course, this is nothing but the statement that a nilpotent matrix $N_{(k)}$ can never be full-rank.

### D3 Particles

It therefore follows that we will always require some charged states to break the remaining symmetries. This includes any remnant discrete symmetries which are not detected by the argument above. The natural charged objects that couple to these gauge fields are of course D3-branes wrapped on 3-cycles. There is a priori no obstruction for a brane to wrap a given cycle, so that away from any singularities, we obtain a complete spectrum of D3-particles, as one expects from the completeness hypothesis. One of the questions we wish to answer however, is to what extent these D3 branes are also *required* to break a given symmetry, and if we can relate their properties to the symmetries they break.

In particular, since we have chosen our basis to correspond to a weak coupling limit, some or all of the couplings may go to zero as we approach the singularity. This means that some of the charged states decouple as their physical charge goes to zero. If those charged states were necessary to break a current not already broken by monodromy, then a global symmetry appears. We expect to encounter some form of EFT obstruction to this phenomenon. This

[2] One might object that the currents are not simply arbitrary vectors in $H_{\mathbb{C}}$ but rather elements of an integral basis, so that it might be necessary to invoke the rationality of $N$ and the weight-filtration to make this argument general.

can also be understood directly from the magnetic WGC, which states that the EFT cut-off should go to zero with the gauge coupling. It is natural to expect this EFT obstruction to be related to the symmetry that emerges in this limit. In the following, we discuss a candidate for such EFT obstructions.

### Charge Orbits

One important property of the symmetry-breaking brane states is that their charge vectors $\mathbf{q} = (p^I, q_I)^T$ also transform under monodromy. From a top-down perspective, this simply follows because they denote the components of a 3-form (Poincare dual to the 3-cycle wrapped by the brane). From the perspective of the EFT, this can be understood as a direct consequence of the non-trivial monodromy of the conserved currents to which they couple. In particular, the equations of motion in the presence of sources now read

$$dF^I = p^I \delta_3(\gamma^1), \qquad dG_I = q_I \delta_3(\gamma^1), \tag{4.12}$$

where $\delta_3(\gamma^1)$ is a 3-form localized on the particle world-line. Consistency with the monodromy transformation (4.9) requires the charge vector to similarly transform under monodromy. We will later see explicitly how this can be interpreted as a manifestation of the Witten effect. For now however, let us briefly discuss the properties of the monodromy orbits generated by these states. These have been extensively studied in [76] and [79] and we review some of their most important results.

The first important observation is that near any singularity in the moduli space, it is always possible to find an infinite monodromy orbit. This follows from precisely the same reasoning we used to deduce that monodromy cannot break all of the symmetries. This presents a natural candidate for the EFT obstruction, required by the emergent global symmetry discussed above. We are then faced with two questions

- Does the infinite monodromy orbit become massless in the limit $g \to 0$?
- Is the monodromy orbit related to the emergent global symmetry?

The main tool we have to assess these is the BPS mass formula, for which we derived the limiting expression (3.68). The main difficulty one runs into is that it is hard to determine generically whether a given 3-cycle supports a BPS state, eq. a special Lagrangian cycle. This mass therefore gives only a lower-bound on the mass of a given state. Nevertheless, in special cases it is sufficient to assume that only a single state in the orbit is populated by a state that is indeed BPS, from which it then follows that the whole orbit is populated by BPS states which obey the mass formula.

In [76], the authors present a sufficient condition for this to be the case. The crux of the argument is to keep track of the assumed BPS state, say $\mathbf{q}$, as we adiabatically transition from $a$ to $a+1$. While we know that the spectrum is invariant under $a \to a+1$ up to a possible re-labelling[3], this only ensures for us that at $a+1$, the state labeled $T\mathbf{q}$ is BPS. If however, as we go from $a$ to $a+1$ the state $\mathbf{q}$ remains BPS, then we have two BPS states at $a+1$, namely $\mathbf{q}$ as well as $T\mathbf{q}$ which we had already concluded should be BPS (see also figure 4.1).

[3]It is important to distinguish the state from the charge vector to which it is associated. The latter is only a label and a state may change its label under monodromy (as long as the spectrum as a whole is left invariant). This is just the physical interpretation of the fact that the monodromy of $\mathbf{q}$ corresponds to a transition function on the Hodge bundle.

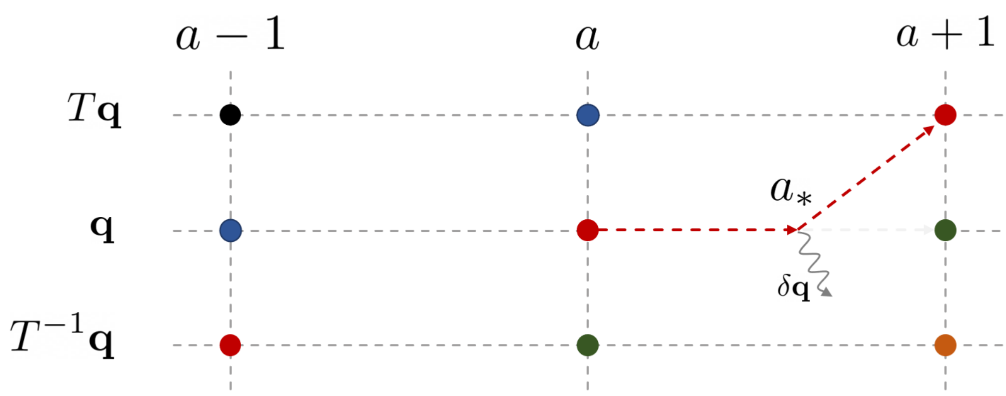



Figure 4.1: Diagram illustrating how states are mapped under monodromy. Physical states are colour coded, and their label transforms under shifts $a \to a+1$. The red dot denotes a BPS state at $a$ with label $\mathbf{q}$. Monodromy ensures that at $a+1$, we have at least one (red) BPS state, now at $T\mathbf{q}$. If the state with label $\mathbf{q}$ remains BPS, we may have two BPS states at $a+1$. However, it may occur that at some intermediate value $a_*$, the state at $\mathbf{q}$ ceases to be BPS (e.g. becomes unstable to decay into the new BPS state $T\mathbf{q}$), so that the new (green) state $\mathbf{q}$ at $a+1$ need not be BPS [76].

The obstruction for the state $\mathbf{q}$ to remain BPS is related to its stability. Indeed, it may occur that somewhere along the path $a \to a+1$ the BPS state $\mathbf{q}$ becomes unstable to decay, so that by the time we arrive at $a+1$ we no longer have a BPS state at $\mathbf{q}$. Below we present only a sufficient condition for $\mathbf{q}$ to remain stable, but otherwise do not motivate it, instead referring to [76] for details. In particular, if we define the two sets

$$\mathcal{M}_1 := \{\mathbf{q} \in H_{\mathbb{Z}} \,|\quad S_{l_m}(\mathbf{q}, a_0) \neq 0, \quad S_{l>l_m}(\mathbf{q}, a_0) = 0 \quad l_m < d/2\}, \tag{4.13}$$

$$\mathcal{M}_2 := \{\mathbf{q} \in H_{\mathbb{Z}} \,|\quad S_l(\mathbf{q}, a_0) = 0, \quad \forall\, l\}, \tag{4.14}$$

then the BPS mass formula

$$M_{\mathbf{q}}^2 \sim s^{-d} \sum_l s^{2l} S_l(\mathbf{q}, a_0) + \mathcal{O}(e^{2\pi i t}), \tag{4.15}$$

tells us that states in $\mathcal{M}_1$ become light polynomially in $s$, while those in $\mathcal{M}_2$ are exponentially light. If we then consider states in an infinite orbit and define $\delta\mathbf{q} := T\mathbf{q} - \mathbf{q}$, then the state $\mathbf{q}$ for which $\delta\mathbf{q} \in \mathcal{M}_2$ will become unstable to decay into $T\mathbf{q}$ and $\delta\mathbf{q}$, at least for some value of the moduli encountered as we encircle the divisor $a \to a+1$. To ensure that we obtain a genuine tower of massless states then, we should demand that $\delta\mathbf{q} \in \mathcal{M}_1$. We emphasize that this is a sufficient condition, since we are not guaranteed that the state $\delta\mathbf{q}$ is BPS. The upshot of the results in [76] is then that in one-modulus degenerations, stable massless towers of states were generated by monodromy only in the type IV case. The discussion presented here is a reduced version of the argument presented in [76] to which we refer for more detail as well as a microscopic interpretation of the condition $\delta\mathbf{q} \in \mathcal{M}_1$.

In the following, we will revisit these results and study how the monodromy orbits can be understood from the EFT perspective. Moreover, we will see why in some cases we can generate infinite towers of massless states, and in some cases we cannot.

### 4.1.2 Axionic Symmetries

The axion similarly has associated to it a pair of global symmetries. While the magnetic 1-form symmetry of the vector fields could be broken due to their non-trivial monodromy transformation, the analogous axionic current $da$ is manifestly invariant so that this symmetry should be broken some other way. The most obvious candidate for this are the axionic strings discussed in chapter 2. Unlike the D3-particles however, we do not have an obvious string-theoretic candidate for how these strings should be realized. While, for example, D3-branes wrapping 2-cycles will lead to strings, it is not clear that these are magnetically charged under the axion. More generally, branes (including NS5 branes) that lead to a string in 4D will not wrap a 3-cycle. Since our classification only focuses on the latter, we would need a more refined description of the Calabi-Yau to make concrete statements here. We will therefore naturally have less concrete things to say about such strings, but we will comment on what their properties might be.

We can however, readily study the continuous shift-symmetry of the axion, which can be broken through the coupling to the gauge fields. Indeed, any non-vanishing terms other than the first in equation 4.3 can lead to a broken axionic shift symmetry. The most well-studied example of this has been the exponential corrections to the Kahler metric, which are known to generate symmetry breaking terms as we move into the bulk of the moduli space. A sufficient condition to completely break the axionic symmetry is to include two exponential corrections at orders $p$ and $q$ for $p, q$ co-prime. We generically expect such corrections to exist, and as such the axionic symmetry is expected to always be approximate.[4]

However, as we have seen throughout the previous chapter, the leading order contribution to the Kahler metric is typically axion-independent. This means that as we take the limit $s \to \infty$, we are at risk of another global symmetry appearing. There is however, a second symmetry breaking effect, namely the coupling of the axion to the gauge fields. If $\mathcal{R} \sim a$, this is analogous to what happens in axion electrodynamics, and we will encounter such couplings here as well. These may prevent an emergent axionic symmetry in the limit $s \to \infty$. If the coupling to the gauge fields is not sufficient to break the shift symmetry however, we should expect EFT break-down whenever $s \to \infty$ corresponds to an infinite distance limit, barring other symmetry breaking effects.

Finally, let us comment on possible emergent symmetries appearing from the magnetic axionic symmetry. As we have seen, the Kahler metric always decays in the limit $s \to \infty$, with the rate of decay determining whether the limit is at finite or infinite distance. If we view the axion as a gauge field, this means that the limit $s \to \infty$ is a strong-coupling limit. The dual statement is that the dual 2-form field is weakly coupled, so that the strings which couple to the axion magnetically decouple in the limit $s \to \infty$. It is therefore plausible that these strings are responsible for the EFT to breakdown near such limits.

#### Saxionic Symmetries

While we do not consider any saxion dynamics in this work (saxion *dependence* will be important to determine coupling constants and particle masses), let us briefly comment on one place where this may become relevant. Suppose for the moment that we include the saxions as dynamical fields in our description. Following our discussion about dualities, we

[4] Towards the end of writing it was pointed out to us that there exist some Calabi-Yau with *no* exponential corrections. In section 4.2 we briefly mention a mechanism by which the axionic symmetry is broken through instanton effects induced by monopole loops [80]. It is possible that in this case (as well as all other cases) the axionic symmetry is still broken by this effect (specifically due to magnetic D3-branes).

know that the full duality group of our theory is generated by scalar diffeomorphisms. These diffeomorphisms include the saxions. In fact, for any infinite distance limit, we have already observed that the limiting form of the Kahler potential is given by

$$K_{t\bar{t}}(ds \wedge *ds + da \wedge *da) = \frac{d}{4s^2}(ds \wedge *ds + da \wedge *da). \tag{4.16}$$

Modulo constants, this metric is the two-dimensional euclidean anti-de Sitter metric and its isometry group is well-known to be given by the same modular group

$$t \to \frac{at+b}{ct+d}, \qquad ad - bc = 1, \tag{4.17}$$

which obviously covers axion monodromy with $a = d = b = 1$, $c = 0$. Importantly however, some of these isometries may lift to genuine global symmetries of the theory. The reason that we need not concern ourselves with this possibility is that apart from axion monodromy, the generator

$$t \to -\frac{1}{t}, \tag{4.18}$$

inverts the saxion. Not only does this typically correspond to an electric-magnetic duality (i.e. induces a symplectic transformation of the gauge fields with $B \neq 0$)[5], it also takes us away from the asymptotic limit where the approximate form of the metric (4.16) is valid. The associated duality is therefore only approximate, so that the saxions will not lead to additional symmetries of the theory.[6]

### 4.1.3 Chern-Weil Symmetries

Finally, there are the Chern-Weil symmetries. As we discuss below, these are typically gauged via the topological Chern-Simons term $\mathcal{R}_{IJ}F^I \wedge F^J$. While this does not require the introduction of additional elements to break them, their gauging forces us to be careful with how we break the other symmetries of the theory. In particular, breaking the magnetic 1- and axionic 2-form symmetries also breaks the Chern-Weil symmetries. This is perfectly analogous to what we observed in section 2.3.3, where three-term Chern-Simons couplings required the introduction of world-volume degrees of freedom. Genuinely identifying these for our case requires more detailed information than we have available to us. We will mention how the Chern-Weil symmetries are gauged/broken at the level of the action, but otherwise leave a more thorough investigation of these symmetries for future work.

## 4.2 Type $II_0$

The first case we consider is the type $II_0$ singularity, the reason being that it has a simpler structure than the type $IV_1$ limit, but unlike the type $I_1$ case it lies at infinite distance making it more interesting from a swampland perspective. The action for the vector fields is given

[5]While this means that the corresponding transformation is not a classical symmetry of the theory, we should still expect such transformations to be gauged in the quantum theory. In fact, this is the case for ten-dimensional supergravity when UV-completed into type IIB string theory.

[6]This can also be understood in terms of the statement that the scalar field space is a punctured disk in the $z$-plane, which retracts to the circle. The saxionic part can therefore never have interesting winding modes and its shift-symmetry is broken by the Kahler metric.

by

$$\int \Big( -\frac{ms}{2} F^0 \wedge *F^0 - \frac{ns}{2} F^1 \wedge *F^1 - \frac{ma}{2} F^0 \wedge F^0 - \frac{q}{2} F^0 \wedge F^1 \\ -\frac{na+p}{2} F^1 \wedge F^1 - \frac{1}{4s^2} da \wedge *da \Big) , \tag{4.19}$$

with associated equations of motion

$$d \underbrace{\left( ms * F^0 + maF^0 + qF^1 \right)}_{-G_0} = 0, \quad d \underbrace{\left( ns * F^1 + (na+p)F^1 + qF^0 \right)}_{-G_1} = 0, \tag{4.20}$$

where we have identified the $G_I$ as the conserved currents obtained from the Noether procedure. If we allow ourselves to use the Bianchi identity these equations simplify down to

$$sd\left( *F^0 \right) = -da \wedge F^0, \quad sd\left( *F^1 \right) = -da \wedge F^1. \tag{4.21}$$

We can also include the equation of motion for the axion, which reads

$$\frac{1}{2s^2} d * da = \frac{m}{2} F^0 \wedge F^0 + \frac{n}{2} F^1 \wedge F^1. \tag{4.22}$$

It is clear that this theory is a variation of the axion electrodynamics we considered in section 2.3.3. As such we will make generous use of our observations in earlier sections.

### 4.2.1 Global Symmetries

In the following we go over the three classes of symmetries discussed in the previous sections and determine whether they are broken or gauged at the level of the equations of motion. In the next sub-section we will comment on how the remaining symmetries can be broken.

#### Vector Currents

We can study the fate of the 1-form global symmetries following the line of argument from the previous section. The field strengths transform under monodromy according to

$$\begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} \to T \begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m & 0 & 1 & 0 \\ 0 & n & 0 & 1 \end{pmatrix} \begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} = \begin{pmatrix} F^0 \\ F^1 \\ -G_0 + mF^0 \\ -G_1 + nF^1 \end{pmatrix} . \tag{4.23}$$

In accordance with our general analysis, we find that $2h^{2,1} = 2$ symmetries are un-broken corresponding to the magnetic symmetries. Moreover, we find that the two electric symmetries are broken to discrete sub-groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$, controlled by the integers that appear in the log-monodromy matrix.

The interpretation of these results is clear. Due to the Chern-Simons coupling to the axion, the electric charges are only conserved mod $m, n$. This is nothing but the Witten effect, which implies that a magnetic charge becomes a dyon under monodromy and as such our flux-integrals are only well-defined up to said monodromy.

#### Axionic Symmetries

The axion equations of motion (4.22) above tell us that the axionic shift-symmetry (with associated current $*da$) is broken. In particular, we see that the terms responsible are the same Chern-Simons couplings that broke the electric symmetries above. The integers $n, m$ that controlled the remnant electric 1-form symmetries also control the axionic shift symmetry. Here we have two integers to keep track of however, and the remnant 0-form symmetry is given by $\mathbb{Z}_{\gcd(m,n)}$ where $\gcd(m, n)$ is the greatest common divisor of the integers $m, n$. Following the argument from section 4.1.2, we expect this discrete symmetry to be broken by exponential corrections to the Kahler metric.

#### Chern-Weil Symmetries

The broken electric and axionic symmetries serve to gauge the two types of Chern-Weil symmetries introduced above, which is read most directly from equations 4.21 and 4.22. More specifically, the axion gauges a linear combination $mF^0 \wedge F^0 + nF^1 \wedge F^1$, while leaving unbroken $mF^0 \wedge F^0 - nF^1 \wedge F^1$. Consequently, these need not be broken further.

### 4.2.2 Breaking the Symmetries

Next, we discuss charged states that may break the remaining symmetries.

#### D3 Particles

In order to break the remaining symmetries of the vector fields we should include charged D3 particles. Since the magnetic 1-form symmetry is never broken, let us start by considering magnetically charged states. In their presence, the Bianchi identity of the (electric) gauge fields is violated by the addition of a source term localized on the particle world-line

$$dF^I = p^I \delta_3(\gamma^1). \tag{4.24}$$

Following our earlier discussions, to break the magnetic symmetries completely we require at least a state with charge $p^0 = 1$ and a (possibly different) state with charge $p^1 = 1$.

If we assume these states to be pure monopoles, the Witten effects turns them into dyons with electric charges $m, n$. This is illustrated by the basic monodromy orbit

$$\ldots \xrightarrow{T} \begin{pmatrix} 1 \\ 1 \\ -m \\ -n \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 1 \\ m \\ n \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 1 \\ 2m \\ 2n \end{pmatrix} \xrightarrow{T} \ldots. \tag{4.25}$$

From this perspective it is clear that in order to consistently break the magnetic 1-form symmetry, the whole orbit should be populated by physical states. Nevertheless, the question of whether this orbit should be BPS is subtle as follows from our discussion in section 4.1.1.

It is clear however, that such magnetically charged states can never break any remaining electric 1-form symmetries. Of course, this is because the electric charge (measured by $\int G_I$) is only conserved mod $m, n$, which was necessary to allow for the Witten effect to occur in the first place. As a result, the monodromy cannot break the electric symmetry any further, so that whenever $m, n \neq 1$ we also require states with elementary electric charge $q_I = 1$.

**Chern-Weil Symmetries**

As was already pointed out, the coupling to the axion gauges some of the Chern-Weil currents discussed in the opening of this chapter. The subsequent breaking of the magnetic symmetries also breaks the Chern-Weil currents constructed out of them, so that we do not require any additional objects. It also means that consistency with their gauging requires some additional elements. Alternatively, we may think of this as an EFT implementation of the Witten effect experienced by the monopole.

Following section 2.3.3 (but see also [47, 50]), we could argue for the presence of world-volume degrees of freedom on the monopole worldline, however identifying such degrees of freedom would required detailed knowledge of the world-volume theory of the compactified D3-brane. Moreover, it is not obvious which brane degree of freedom would carry charge under the R-R 4-form (compare this to e.g. the KK monopole where any state that can be boosted along the compact direction automatically carries gauge charge). Here we simply state that the theory can be rendered consistent by including a world-volume degree of freedom on any magnetically charged state

$$S \sim -\int_{\gamma^1} a\left(d\sigma - mp^0 A^0 - np^1 A^1\right), \qquad \begin{cases} A^I \to A^I + d\lambda^I, \\ \sigma \to \sigma + mp^0\lambda^0 + np^1\lambda^1. \end{cases} \tag{4.26}$$

but otherwise do not claim to have a candidate for such a field. In this case, the axion gauges a linear combination of currents

$$\frac{1}{s^2} d * da = \left(\frac{m}{2}F^0 \wedge F^0 + \frac{n}{2}F^1 \wedge F^1 + \left(db^0 - mp^0 A^0 - np^1 A^1\right) \wedge \delta_3(\gamma^1)\right). \tag{4.27}$$

### 4.2.3 Emergent Symmetries and EFT Obstructions

While we always have the states necessary to break the symmetries, at least in the UV-completion where brane states are dynamical, we expect some of these symmetries to re-emerge in the limit where we take $s \to \infty$. As mentioned, the usual culprit is the axion, which develops a shift-symmetry as the exponentially suppressed corrections to the Kahler metric vanish and an axion-independent metric emerges. While this is indeed the case, we find that the Chern-Simons coupling may break the axionic symmetry *even* as we approach $s \to \infty$. In particular, we can canonically normalize the fields by

$$F^I \to F^I/\sqrt{s}, \qquad a \to sa. \tag{4.28}$$

The corresponding factors of $s$ cancel in the Chern-Simons term

$$\int aF^I \wedge F^I \to \frac{s}{(\sqrt{s})^2}\int aF^I \wedge F^I, \tag{4.29}$$

which we find persists even in the limit $s \to \infty$.[7] This point will be important below when we discuss axionic strings but for now, we observe that if $\gcd(m,n) = 1$, the axionic symmetry is broken for all values of $s$. Meanwhile, when $\gcd(m,n) \neq 1$ a discrete global shift symmetry may appear as the exponential corrections to the Kahler metric vanish.

Another potential emergent symmetry in this limit is related to the gauge coupling going to zero. Indeed, by construction, the electrically charged states decouple from the gauge field

[7] As we will see below, this is the case for all infinite distance limits considered here. There may be a general argument that this should be the case, either based on symmetries or Hodge theoretic reasoning, but we do not know of one.

in the limit $s \to \infty$ restoring the electric 1-form symmetry. However, here too we may be protected from such a symmetry if $m = n = 1$. This suggests that these emergent symmetries are somehow not as "fundamental". In the following however, we will assume that we are working in the general case where $m, n \neq 1$ and study possible EFT obstructions to the emergence of these symmetries.

### BPS Masses

A natural candidate for the massless tower of states that protects us in this limit is of course the monodromy orbit (4.25). Explicitly, we find that when we evaluate the BPS mass formula for a state with non-vanishing magnetic charge, the particle mass is linearly divergent in the limit $s \to \infty$

$$M^2 = \frac{|\langle \mathbf{q}, \Pi_{nil}\rangle|^2}{2s^2} = \frac{1}{4}\left(m(p^0)^2 + n(p^1)^2\right) s + \mathcal{O}(1). \tag{4.30}$$

Because the monodromy is induced by the Witten effect, it is only these magnetically charged states which can generate infinite monodromy orbits. These orbits can therefore never become light as we take the infinite distance limit. This is the same conclusion arrived at in [76], here viewed as a direct consequence of the fact that the tower is generated by the Witten effect (along with the fact that dyons are heavy).

As for the electrically charged states, our BPS mass formula tells us that there is no obstruction for these states to become light

$$M^2 = \frac{1}{4s}\left((q^0)^2/m + (q^1)^2/n\right) + \mathcal{O}(s^{-2}). \tag{4.31}$$

This could also be viewed as the statement that the electric states satisfy the WGC, in which case their asymptotic massless-ness follows from the vanishing physical charge. The problem here however, is that there is no mechanism by which we should expect an infinite tower of them to be present (although demanding that they satisfy the tower/lattice versions of the Weak Gravity Conjecture [81, 82] would lead to this conclusion).

While the results presented so far essentially follow from straight computation, in the remainder of this sub-section we will be more speculative. In particular, we will discuss what additional states may be present that break the remaining symmetries, what their properties may be, and how they may lead to an EFT break-down. These results are not intended to be conclusive and we provide this warning to indicate a drop in the level of rigor in the following paragraphs.

### Instanton Effects

Let us begin by briefly commenting on the case $\gcd(m, n) \neq 1$. Here, the limit $s \to \infty$ is associated with an axionic shift-symmetry, albeit only a discrete one. It has been argued that in the presence of a Chern-Simons coupling as in (4.19), monopoles loops induce instanton-effects which lead to a non-perturbative potential for the axion which breaks the emergent global symmetry [80]. Because we always have magnetic D3-particles in the theory (which remain strongly coupled in the limit) it is possible that this always breaks the axionic symmetry, even if $\gcd(m, n) \neq 1$.

### Axionic Strings

As we have seen, we can always use D3 branes to construct the states that break the symmetries associated with the gauge fields. However we have not said anything about the unbroken

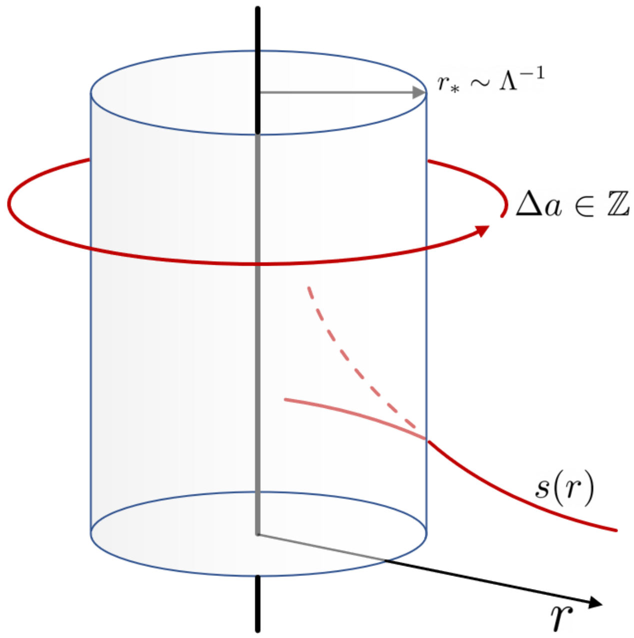



Figure 4.2: Illustration of an EFT string configuration (adapted from [83]). Going around the string induces a monodromy for the axion, while the saxion diverges as we approach it. The finite EFT cut-off means that the string has a finite size $r_* \sim \Lambda^{-1}$, which shrinks to zero as we try to take $\Lambda \to \infty$. We emphasize that we are not able to explicitly construct such solutions for our models.

axionic 2-form symmetry $da$. Although we do not have as clear of a picture on how these are potentially broken as for the gauge fields, we would still like to speculate on this issue here. Most likely we require the existence of some form of axionic string. While we do not have any clear candidate for such strings (i.e. it is not obvious which string theory state leads to the appropriate string in the EFT), we can comment on some of their properties

Up until now we have used a probe approximation to describe the D3-particles. It is well-known that this approximation breaks down for low co-dimension objects, such as strings of co-dimension two. In particular, a string charged under the axion is expected to similarly back-react on the saxion, that is, induce a non-trivial profile for $s$ outside its world-sheet. For fundamental[8] strings in particular, it has been argued [83, 84] that it is precisely the back-reaction of such strings that forces the saxion to probe the asymptotic regions of its field space, meaning that the saxion profile diverges near the string core. In fact, such solutions are constructed explicitly in [84], which they refer to as EFT strings. More generally, such configurations have been interpreted as end-of-the-world branes[9] [85].

The detailed arguments in [84] that lead to EFT string solutions assumes that these strings are BPS and that a continuous axionic shift symmetry is restored as we approach the string core (where $s \to \infty$). Our results show however, that this symmetry is always broken down to at most a discrete sub-group, even in the limit of large $s$. Extending their argument to our case requires additional care (and need not even be possible in general). This does not mean such states cannot exist, nor that more general axionic cannot strings exist.

[8] By this we mean strings which cannot be resolved into a smooth field configuration inside any EFT, meaning that their core remains singular (this is exactly how string theory D-branes appear from an EFT perspective).

[9] The strings can then be understood as defects that kill a non-trivial cobordism class, which corresponds to the non-contractible cycle in the moduli space [56, 84] (cf. footnote 6).

Barring these issues, we may still try to apply their results to our case, making additional assumptions if necessary. Since the large-saxion limit lies at infinite distance, the SDC implies that we expect a breakdown of the EFT as we try to resolve the string core. Alternatively, this breakdown can be understood as the emergence of a global symmetry. Indeed, since the axion is strongly coupled, its dual 2-form is expected to be weakly coupled. As we follow the saxion to infinity, the string decouples and the 2-form global symmetry is restored. It is argued in [84] that this limit is associated with the EFT string tension going to zero. Just like the fundamental strings from section 1.1, these strings exhibit an infinite tower of excited states, whose mass is proportional to the string tension. As the latter goes to zero, we encounter a massless tower of states that shields us from the infinite distance limit. This is analogous to a breakdown of the field theory approximation which led to the type IIB action from section 1.2, although we emphasize that the string becoming light in this case need not correspond to a type IIB superstring. It follows that the massless tower of states signals that we should move to a string-theoretic description of the physics, rather than a field-theory description.

All in all, these observations fit nicely with the emergent string proposal from [78], which states that any infinite distance limit is associated either with a decompactification limit (i.e. a tower of KK modes becoming light), or with the emergence of a weakly coupled string. More generally our models admit two natural candidates for the massless towers of states, namely one related to the monodromy orbit and one related to the axionic strings, which are needed to break the 1- and 2-form symmetries, respectively. The former may, through an appropriate duality, be related to a KK tower (see also our comments in section 4.4), while the latter may lead to an emergent critical string. We end our speculation on this topic here however, since genuinely identifying both the strings and more generally the tower which becomes light first requires more detailed information about the UV-completion than we have available to us. Nevertheless, these remarks should illustrate that the physics of axionic strings provides both interesting connections to various swampland conjectures and can quite naturally be viewed through the lens of global symmetry breaking.

Finally, let us note that the obstructions to directly importing the results from [84] are intimately related to the Witten effect. In particular, as we encircle an axionic string, the induced monodromy simultaneously leads to the Witten effect [49]. These strings should therefore be appropriately charged under the gauge field, which is used in [50] to argue that the light string states also satisfy WGC with respect to the vanishing gauge couplings (thus relating the SDC to the WGC along the lines of 2.1.4).

### Semi-Simple Monodromy

As a final subtlety, recall that when we introduced the monodromy matrix $T$, we used the Jordan–Chevalley decomposition to split it into semi-simple and nilpotent parts

$$T = T_s T_u. \tag{4.32}$$

We mentioned that the semi-simple part is always of finite order and moreover, that it does not enter into the nilpotent orbit theorem. However, in deriving the action (4.19) and the associated monodromy, we *started* from the nilpotent orbit data. As such we can never recover a possibly non-trivial semi-simple piece.

In principle, this semi-simple piece can have rather dramatic consequences on the analysis of the previous section. For instance, it may act non-trivially on some or all of the vector currents, breaking them completely without the need for additional brane states. Recovering the details of this effect requires one to start from the geometric setting e.g. by solving the Picard-Fuchs equations for the periods and mapping the result to an integral basis for comparison with our results. This is a highly non-trivial task in general and we can therefore not comment on it much further.

### 4.2.4 Conclusions

Before moving on we would like to take stock of the results from this sub-section. We will use the questions posed at the start of this section as a guide.

1. *Which symmetries are broken or gauged without the addition of stringy ingredients?*

   - The electric 1-form symmetry is broken by monodromy down to a $\mathbb{Z}_m$ and a $\mathbb{Z}_n$ sub-group. This is a consequence of the Witten effect, induced by the Chern-Simons couplings.
   - The axionic shift-symmetry is broken down to a $\mathbb{Z}_{\mathrm{gcd}(m,n)}$ sub-group by the Chern-Simons couplings. Exponential corrections to the Kahler metric are expected to break it completely.
   - The Chern-Weil currents $da \wedge F^I$ are both gauged.
   - The magnetic 1-form symmetries, the axionic 2-form symmetry and the off-diagonal combination $F^0 \wedge F^0 - F^1 \wedge F^1$ are all intact.

2. *Which stringy effects are necessary to break the remaining symmetries?*

   - D3-particles provide us with a full spectrum of charges that break all vector symmetries. These also break the associated Chern-Weil symmetries. Consistency with the latter's gauging requires world-volume degrees of freedom which we were not able to identify.

     When $m, n = 1$, we do not require a full set of charges, as the electric 1-form symmetries are already broken completely.
   - The axionic 2-form symmetry requires the inclusion of strings to be broken. Though we do not have a higher-dimensional candidate for such strings, we have discussed their properties.

3. *Which symmetries emerge as we approach infinite distance limits in field space?*

   - If $m, n \neq 1$, the electrically charged states which are required to break the electric 1-form symmetries decouple. These lead to emergent (discrete) symmetries. Notably, for $m = n = 1$, there is no emergent vector symmetry, as the Chern-Simons term persists in the limit $s \to \infty$.
   - If $\gcd(m, n) \neq 1$, the exponentially suppressed corrections to the Kahler metric vanish and lead to an emergent discrete shift symmetry for the axion. Notably, for $\gcd(m, n) = 1$, there is no emergent shift symmetry, as the Chern-Simons term persists in the limit $s \to \infty$. Otherwise, instanton effects as in [80] may break this symmetry in this limit.
   - Since the axion is strongly coupled, it is expected that the axionic strings are weakly coupled and decouple in the limit $s \to \infty$, leading to an emergent 2-form global symmetry.

4. *Can the massless tower of states predicted by the SDC be understood in terms of an emergent symmetry?*
   - The monodromy orbit, which is related to the broken 1-form symmetries, does not lead to a massless tower of states. We have framed this as a consequence of the fact that this orbit is generated by the Witten effect, which only acts on magnetically charged states. These are strongly coupled and heavy in the limit $s \to \infty$.
   - Electric BPS states do become light, but we cannot guarantee that we have an infinite tower of them. These states would be directly related to the emergent 1-form symmetry if we assume the states that broke this symmetry satisfy the tower/lattice WGC.
   - We have proposed that the massless tower of states in this case could arise due to the strings which break the 2-form symmetry becoming massless. Moreover, this is the only emergent symmetry to persist for arbitrary values of $m, n$ (specifically for $m = n = 1$, this is the only emergent symmetry), which could be viewed as a hint towards this direction.

## 4.3 Type $I_1$

Next, we turn to the type $I_1$ case, which we know occurs at finite distance. Moreover its action is rather similar to the previous case, allowing us to highlight the differences between such finite and infinite distance limits. Indeed, the relevant action is given by

$$\int \Big( \frac{ns}{2} F^0 \wedge *F^0 + \gamma y F^0 \wedge *F^1 - \frac{y}{2} F^1 \wedge *F^1 - \frac{na}{2} F^0 \wedge F^0 \\ -2\beta F^0 \wedge F^1 - \frac{x}{2} F^1 \wedge F^1 - \frac{1}{2g^2} da \wedge *da \Big) , \tag{4.33}$$

where we have shifted the saxion $s \to s + \gamma^2 y/n$ to simplify the first kinetic term and introduced the short-hand $\beta := \delta - \gamma x$ which is real. The equations of motion are given by

$$d \underbrace{\left(-ns * F^0 - \gamma y * F^1 + naF^0 + 2\beta F^1\right)}_{-G_0} = 0, \quad d \underbrace{\left(\gamma y * F^0 + y * F^1 + 2\beta F^1 + xF^1\right)}_{-G_1} = 0, \tag{4.34}$$

where we have again identified the $G_I$. If we allow ourselves to use the Bianchi identity these equations simplify down to

$$sd\left(*F^0\right) = -da \wedge F^0, \quad sd\left(*F^1\right) = 0 \tag{4.35}$$

We can also include the equation of motion for the axion, which reads

$$\frac{1}{g^2} d * da = \frac{n}{2} F^0 \wedge F^0, \tag{4.36}$$

where we emphasize that the coupling $g^{-1}$ is exponentially suppressed in $s$, but axion-independent.

In order not to repeat ourselves, the analysis here will be less detailed than for the previous case, as many observations carry over immediately. We will list the (un-)broken symmetries and how they can be broken, but otherwise focus on interesting differences compared to the previous example.

### 4.3.1 Global Symmetries

#### Vector Symmetries

The vector symmetries can again be read off from the monodromy action on the field strengths, this time given by

$$\begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} \to T \begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ n & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} = \begin{pmatrix} F^0 \\ F^1 \\ -G_0 + nF^0 \\ -G_1 \end{pmatrix} . \tag{4.37}$$

In accordance with our general discussion in section 4.1.1, we find that $2h^{2,1} + 1 = 3$ symmetries are left un-broken, while the electric current associated to $A^0$ is broken down to $\mathbb{Z}_n$. This can similarly be understood in terms of the Witten effect. However, unlike the previous case, the axion now only couples to one of the gauge fields.

#### Axionic Symmetries

The situation for the axion is the same as before, except that its shift symmetry is now broken only by its coupling to $A^0$. It is therefore simply broken down to $\mathbb{Z}_n$ at leading order, and broken completely by exponential corrections.

#### Chern-Weil Symmetries

Here too is the situation largely unchanged from before. There are some differences however. The first is that now only the $F^0 \wedge F^0$ current is gauged by the axion, rather than a linear combination of the two. Moreover, the current $da \wedge F^1$ is not gauged in this case, and requires the addition of charges to be broken.

### 4.3.2 Breaking the Symmetries

#### D3 Particles

As before, we find that we require magnetically charged states to break the magnetic 1-form symmetries. The magnetic charges that break $F^1$ also break the Chern-Weil current $da \wedge F^1$, and since the latter is not gauged, no complications arise from this. Meanwhile, the magnetic charge that breaks $F^1$ generates an orbit of charges by the Witten effect according to

$$\ldots \xrightarrow{T} \begin{pmatrix} 1 \\ 0 \\ -n \\ 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 0 \\ n \\ 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 \\ 0 \\ 2n \\ 0 \end{pmatrix} \xrightarrow{T} \ldots . \tag{4.38}$$

The monodromy orbit is necessarily magnetically charged, because it corresponds to the Witten effect experienced by the gauge field $A^0$. Moreover, for non-trivial $n$, we require both types of electric charges to break the remaining electric 1-form symmetries.

#### Axionic Strings

While we do not revisit these in as much detail as in the previous section, one can argue that similar considerations should hold regarding axionic strings, at least with respect to the gauge field $A^0$ which couples to the axion.

### 4.3.3 Emergent Symmetries and EFT Obstructions

In order to take the limit $s \to \infty$, let us again canonically normalize the fields as

$$F^0 \to F^0/\sqrt{s}, \qquad a \to ag \sim ae^{2\pi s}. \tag{4.39}$$

The resulting action now reads

$$\begin{aligned}\int \Big( \frac{n}{2} F^0 \wedge *F^0 + \frac{\gamma y}{\sqrt{s}} F^0 \wedge *F^1 - \frac{y}{2} F^1 \wedge *F^1 - \frac{ane^{2\pi s}}{2s} F^0 \wedge F^0 \\ - \frac{2(\delta - \gamma x)}{\sqrt{s}} F^0 \wedge F^1 - \frac{x}{2} F^1 \wedge F^1 - \frac{1}{2} da \wedge *da \Big), \end{aligned} \tag{4.40}$$

The mixed terms are now sub-leading so that we focus on the dynamics of the field $F^0$. We observe that like the previous case, the Chern-Simons term persists in the limit $s \to \infty$ and in fact becomes dominant.

Regarding the vector symmetries, only one of the gauge couplings vanishes in the limit $s \to \infty$ and this is the same gauge field that couples to the axion. Hence, a $\mathbb{Z}_n$ sub-group may emerge here. Meanwhile, the axionic shift symmetry is similarly restored up to the $\mathbb{Z}_n$ sub-group that emerges as the sub-leading exponential corrections to the Kahler potential are dropped.

#### A Comment on Finite Distance Singularities

Unlike the previous case, the type $\mathrm{I}_1$ limit is at finite distance. As such we do not expect an infinite tower of massless states. In fact, it has been argued that such finite distance singularities arise precisely because we have integrated out a state (more generally several states) which becomes massless at the singularity [86], and as such should have been included in our effective description. This inconsistency manifests itself as a singularity in the moduli space. We therefore do not expect an infinite tower of states to become massless, but rather a single state. From a geometric perspective, we briefly mentioned that conifolds are associated with a 3-cycle shrinking to zero size. Branes wrapping those cycles are then expected to become massless in the low-energy theory. More generally it has been proposed that infinite distance singularities are the result of integrating out an infinite tower of massless states [76, 77], namely those predicted by the SDC, but this is a topic otherwise beyond the scope of this thesis.

#### BPS Masses

Let us consider again the BPS masses of the D3-particles. As before, the charge that experiences the Witten effect is magnetic. Evaluating the BPS formula for states of charge $\mathbf{q} = (p^0, p^1, -q_0, -q_1)^T$, we find that

$$M^2 \sim \frac{(p^0\delta + q_1 + (p^1 - \gamma p^0)x)^2 + (p^1 - \gamma p^0)^2 y^2}{2y} + \mathcal{O}(e^{2\pi i t}). \tag{4.41}$$

As before, states with non-vanishing magnetic charge $p^0$ (which generate an infinte orbit) remain massive in the limit $s \to \infty$, although their mass now stays finite. In particular, states for which the leading order contribution vanishes can are given by

$$\mathbf{q} = p^0(1, \gamma, 0, \delta)^T + q_0(0, 0, -1, 0)^T \tag{4.42}$$

for $p^0, q_0$ arbitrary. One recognizes this vector as lying in the span of

$$\mathbf{q} \in \text{span}\left(I^{2,2} \oplus I^{1,1}\right), \tag{4.43}$$

from equation 3.144, where the second piece is left invariant under monodromy. Since the parameters $\gamma, \delta$ are real however, we generically expect (4.41) to vanish only for $\mathbf{q} = (0, 0, -q_0, 0)^T$, as the part along $I^{2,2}$ is not integrally quantized. We therefore identify the (single) electric BPS state which decouples in the limit to be the one responsible for the conifold singularity along the lines of our discussion above. Moreover, this is the same state which is responsible for breaking the emergent 1-form symmetry associated to $A^0$. We therefore expect that once we integrate this state back in, the emergent 1-form symmetry remains broken in the UV.

**Axionic Strings**

Finally, let us comment on the axionic string discussed in the previous section. These are again required to break the 2-form symmetry associated to the axion, but we should not expect these to lead to massless towers of states in the limit $s \to \infty$. Invoking again the arguments from [84], we argue that this is because the core of the string now lies at finite distances in field space. Indeed, a key part of the argument that lead to the conclusion that the string has an infinite tower of massless excitations was that the string was fundamental, namely that we could not resolve its core. However, when the core of the string is located at finite distance in field space, the divergent saxion need not lead to a singularity there. Instead we expect the axionic string which breaks the 2-form symmetry to be a solitonic object which can be resolved in an EFT.

### 4.3.4 Conclusions

As before we close this section by revisiting the questions from the start of this chapter. Since these limits lie at finite distance however, we do not address the last two questions but rather comment on what happens as we approach the singularity.

1. *Which symmetries are broken or gauged without the addition of stringy ingredients?*
   - One of the electric 1-form symmetries is broken by monodromy down to a $\mathbb{Z}_n$ sub-group. This is again a consequence of the Witten effect, induced by the Chern-Simons coupling.
   - The axionic shift-symmetry is broken down to a $\mathbb{Z}_n$ sub-group by the Chern-Simons couplings. Exponential corrections to the Kahler metric are expected to break it completely.
   - The Chern-Weil currents $da \wedge F^I$ and $F^0 \wedge F^0$ are gauged.
   - The electric 1-form symmetry, both magnetic 1-form symmetries, the axionic 2-form symmetry and the Chern-Weil currents $da \wedge F^1$ and $F^1 \wedge F^1$ are both intact.
2. *Which stringy effects are necessary to break the remaining symmetries?*
   - D3-particles provide us with a full spectrum of charges to break all vector symmetries. These also break the associated Chern-Weil symmetries. Consistency with the latter's gauging requires world-volume degrees of freedom which we were not able to identify.

When $n = 1$, we do not require a full set of charges, as the electric 1-form symmetry is already broken completely.

- The axionic 2-form symmetry again requires the inclusion of strings to be broken.

3. *What happens as we approach the finite distance singularity?*
   - If $n \neq 1$, the electrically charged state that was necessary to break the electric 1-form symmetry decouples. This leads to an emergent (discrete) symmetries. Notably, for $n = 1$, there is no emergent vector symmetry.
   - Likewise, if $n \neq 1$ the exponentially suppressed corrections to the Kahler metric vanish and lead to an emergent discrete shift symmetry for the axion. Notably, for $n = 1$, there is no emergent shift symmetry.
   - The monodromy orbit is now of finite mass. This differs somewhat from the analysis in [76] because we work in an integral basis which led us to exclude certain states for which the orbit became light, rather than have to invoke a stability argument. The conclusion is still the same however, namely that monodromy orbits do not lead to an infinite tower of states in this limit.
4. *Do any states become massless as we approach the singularity?*
   - There is a single electrically charged state which becomes massless in this limit, and it is the same state that breaks a potential emergent 1-form symmetry associated to the gauge field $A^0$. The finite distance singularity is related to this state becoming massless.
   - Because this singularity is at finite distance, the axionic strings which were necessary to break the 2-form symmetry are expected to be solitonic and of finite tension. They therefore do not lead to an infinite tower of massless states.

## 4.4 Type IV$_0$

The final model we consider is the type IV$_0$ case. Its structure is rather different from the previous two examples and as such provides a nice contrast to what we have seen so far. The action is given by the following

$$\begin{aligned}\int \Bigg( & -\frac{m^2ns(3a^2+s^2)}{12}F^0\wedge *F^0 + \frac{mnas}{2}F^0\wedge *F^1 - \frac{ns}{4}F^1\wedge *F^1 - \frac{m^2na^3}{6}F^0\wedge F^0 \\ & + \frac{mna^2-c/m}{2}F^0\wedge F^1 - \frac{na+b/m}{2}F^1\wedge F^1 - \frac{3}{4s^2}da\wedge *da\Bigg), \end{aligned} \tag{4.44}$$

where, for simplicity we have dropped the real part of the parameter $\xi$ (which is the case for models obtained from mirror symmetry). The corresponding equations of motion are given by

$$\begin{aligned} d\left(\frac{m^2ns}{6}(3a^2+s^2)*F^0 - \frac{mnas}{2}*F^1 + \frac{m^2na^3}{3}F^0 - \frac{mna^2-c/m}{2}F^1\right) &= 0, \\ d\left(\frac{ns}{2}*F^1 - \frac{mnas}{2}*F^0 + (na+b/m)F^1 - \frac{mna^2-c/m}{2}F^0\right) &= 0. \end{aligned} \tag{4.45}$$

As we shall see, the highly non-linear couplings of the axion lead to a non-standard form of the Witten effect. This will have rather dramatic consequences, and will eventually lead to the massless tower of states observed in [76].

### 4.4.1 Global Symmetries

For this last case, we focus mainly on the vector symmetries and briefly discuss axionic symmetries, leaving a discussion of the role of the Chern-Weil currents for future work.

#### Vector Symmetries

The monodromy transformation is now markedly more involved. Using the monodromy matrix from section 3.4.1, we find that the field strengths transform according to

$$\begin{aligned}\begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} \to T \begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ m & 1 & 0 & 0 \\ -\frac{m^2 n}{6} + c & -\frac{mn}{2} + b & 1 & -m \\ \frac{mn}{2} + b & n & 0 & 1 \end{pmatrix} \begin{pmatrix} F^0 \\ F^1 \\ -G_0 \\ -G_1 \end{pmatrix} \\ &= \begin{pmatrix} F^0 \\ F^1 + mF^0 \\ -G_0 - (\frac{m^2 n}{6} - c)F^0 - (\frac{mn}{2} - b)F^1 + mG_1 \\ -G_1 + (\frac{mn}{2} + b)F^0 + nF^1 \end{pmatrix}.\end{aligned} \tag{4.46}$$

We find that the monodromy leaves un-broken $h^{2,1} - 1 = 1$ symmetry. More interesting is the fact that now one of the *magnetic* 1-form symmetries is explicitly broken too. In particular, the integer $m$ controls the remnant discrete symmetry for the same reason as it did in the previous two examples. The precise fate of the two electric symmetries depends on the constellation of values the parameters $m, n, e, f$ take.

While the monodromy (4.46) tells us that the magnetic symmetry is broken, it is interesting to consider how this monodromy arises from the action itself. This can be traced back to the non-linear axion dependence in both the real and imaginary parts of the kinetic matrix $\mathcal{N}$. The action simplifies when we rewrite it in terms of the gauge-invariant combinations

$$F^0, \quad \tilde{F}^1 := F^1 - maF^0, \tag{4.47}$$

in terms of which the kinetic terms are now manifestly invariant

$$S_{kin} = \int \left( -\frac{m^2 n s^3}{12} F^0 \wedge *F^0 - \frac{ns}{4} \tilde{F}^1 \wedge *\tilde{F}^1 \right). \tag{4.48}$$

Moreover, computing the variation of the action under $a \to a + 1$, while also effecting the monodromy transformation $F^1 \to F^1 + mF^0$, we obtain

$$\delta S = \int \left( -\frac{1}{2} \big( (c - m^2 n/6) + m\,(b + mn/2) \big) F^0 \wedge F^0 - (b + mn/2)\, F^0 \wedge F^1 - \frac{n}{2} F^1 \wedge F^1 \right), \tag{4.49}$$

which is manifestly quantized by virtue of the constraints (3.103) on the parameters $m, n, e, f$. Note that this also means that the non-trivial constant parts of $\mathcal{R}$ are essential to ensure gauge invariance.[10]

#### Axionic Symmetries

The axion no longer has a simple shift symmetry, not even a discrete one, due to its appearance in the kinetic terms for the gauge fields. For special values of the parameters $m, n, e, f$

[10] A similar observation was made in [50] regarding a model with cubic $\theta$-terms.

there may be a discrete shift-symmetry where one also shifts $F^1$ by a term proportional to $F^0$ so that the gauge-invariant combination $\tilde{F}^1$ is left invariant. This is rather case-dependent however, and we do not present a general condition for this to be the case.

## 4.4.2 Breaking the Symmetries

### D3 Particles

While the magnetic 1-form symmetry associated to the current $F^1$ may be broken by the monodromy, we always require magnetically charged particles to break $F^0$. In addition, it is possible that there remain discrete symmetries, e.g. when $m \neq 1$ or when some of the integers in the monodromy matrix $T$ admit a common divisor, in which case additional particles are also necessary.

### Axionic Strings

As before we require axionic strings to break the 2-form symmetry. Though we again do not have an immediate UV origin for these strings, we will give them a higher-dimensional interpretation below.

## 4.4.3 Emergent Symmetries and EFT Obstructions

Upon re-scaling the gauge fields and axion according to

$$F^0 \to F^0/s^{3/2}, \qquad F^1 \to F^1/s^{1/2}, \qquad a \to as, \tag{4.50}$$

we find that the factors of $s$ drop out of the non-trivial couplings in the action, which now reads

$$\int \Big( -\frac{m^2 n s^3(3a^2+1)}{12} F^0 \wedge *F^0 + \frac{mna}{2} F^0 \wedge *F^1 - \frac{n}{4} F^1 \wedge *F^1 - \frac{m^2 n a^3}{6} F^0 \wedge F^0 \\ + \frac{mna^2 - c/(ms^2)}{2} F^0 \wedge F^1 - \frac{na + b/(ms)}{2} F^1 \wedge F^1 - \frac{3}{4} da \wedge *da \Big) . \tag{4.51}$$

This implies that emergent symmetries can appear only to the extent to which these symmetries were not already broken, which is exactly what we observed for the previous cases. In particular, we may encounter emergent 1-form symmetries associated to the electric charges decoupling if these are not already broken by the Chern-Simons coupling.

### BPS Masses

As discussed, it is known that for this class of limits it is possible to construct an infinite tower of massless BPS states (assuming one such state exists) from the monodromy orbit [76]. We can now see explicitly why this is possible for this type of limit, but not for the others. The key point here is that the monodromy now no longer admits an interpretation in terms of a classical Witten effect. In particular, due to the non-trivial gauge transformation of $F^1$, we find that electrically charged states may also experience a "Witten effect". In particular, we find the non-trivial monodromy orbit

$$\dots \xrightarrow{T} \begin{pmatrix} 0 \\ 0 \\ m \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 \\ 0 \\ -m \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 \\ 0 \\ -2m \\ 1 \end{pmatrix} \xrightarrow{T} \dots . \tag{4.52}$$

Evaluating the BPS mass formula for these states, we find

$$M^2 \sim \frac{3}{4ns}\left(1+\frac{k^2}{s^2}\right)+\mathcal{O}(s^{-4}), \tag{4.53}$$

where $k$ denotes the $k$'th state in the orbit. Firstly, we observe that the BPS masses of the states in this orbit all vanish in the limit. Moreover, the mass of the vector $\delta\mathbf{q}$ now manifestly lies in the polynomially massless set $\mathcal{M}_1$, and as such the states in the orbit remain stable. We thus find a massless tower of states, generated by the modified Witten effect, that acts on electrically charged states, provided there exists an appropriate seed-charge that is BPS.

### Mirror Interpretation

In contrast with the previous two cases, mirror symmetry can give us additional insight into the nature and origin of both the charge orbit (4.25) as well as the non-trivial gauge transformation (4.46). Indeed, as we have mentioned, type $\mathrm{IV}_0$ limits correspond to large complex structure limits which under mirror symmetry map to the large volume point. The different electric and magnetic states have a natural interpretation here. In particular, given a charge vector

$$\mathbf{q} = (p^0, p^1, -q_0, -q_1)^T, \tag{4.54}$$

the components of different charges map to branes in type IIA as [87]

$$\begin{aligned} p^0 &\to \text{D6-Brane wrapping the Calabi-Yau},\\ p^1 &\to \text{D4-Brane wrapping 4-cycle (divisor)},\\ q^1 &\to \text{D2-Brane wrapping 2-cycle (curve)},\\ q^0 &\to \text{D0-Brane on a point.} \end{aligned} \tag{4.55}$$

General charged states then correspond to bound states of the particles that descend from these brane states. From this perspective, the infinite charge orbit obtained above corresponds to bound states of D0- and D2-branes. The type IIA picture also suggests another perspective on this theory, namely as arising from a further circle compactification of M-theory compactified on the same Calabi-Yau.

### Axionic Strings

Finally, we revisit once more the axionic strings. In principle, we should expect such strings to be present here as well. Compared to the previous two cases however, the non-linear axion couplings complicates the explicit analysis from [84]. On the one hand, it is not obvious how, or even if the axion can be explicitly dualized to a 2-form. This only aggravates the issues we have sidestepped in our previous discussions, likely putting any explicit EFT string solution outside reach.

However, it is plausible that these strings are not the ones responsible for the infinite tower of states, in contrast to the type $\mathrm{II}_0$ case considered earlier. The motivation for this comes again from the mirror picture discussed above. Following the logic of the emergent string conjecture [78], this infinite distance limit is associated to either a decompactification limit, or the emergence of a tensionless string. In this case however, it is clear from the type IIA perspective that this limit is a decompactification limit, as it corresponds to an infinite volume Calabi-Yau [78]. This can be made explicit in terms of the M-theory perspective mentioned above.

Note that this does not mean a string cannot become tensionless. Indeed, this may well still be the case, but it is possible that, measured with respect to the Planck scale, the KK tower becomes light faster. This possibility was also accounted for in the EFT string proposals from [83, 84], where it is conjectured that the mass-scale $M_{SDC}$ of the SDC is related to the EFT string tension as

$$M_{SDC}^2 \sim M_{Pl}^2 \left(\frac{T_{\text{string}}}{M_{Pl}^2}\right)^w, \qquad w \in \{1, 2, 3\}. \tag{4.56}$$

When $w = 1$, we encounter the possibility of a stringy tower of states (though this still need not necessarily be the case), while cases with $w > 1$ correspond to decompactification limits, as another tower of states becomes light more quickly than those related to the string. Note moreover that there is an obvious similarity between the integer $w$ and the integer $d$ that entered our singularity type classification. Proving, or even checking such a one-to-one relationship is likely a very difficult task in general. However, it is interesting to note that if these integers agree then an emergent string in the type II$_0$ limit is indeed possible (since then $w = d = 1$).

Finally, we remark that the mirror picture of type IIA/M-theory compactifications can potentially be used to elucidate some of the properties of these axionic strings. The mirror dual of the complex structure axion is the scalar zero-mode of the NS-NS B-field, while the associated saxion is just the real Kahler modulus (the limit $s \to \infty$ therefore corresponds to the large volume point, as expected). It follows that the strings which magnetically couple to this axion descend from states magnetically coupled under the B-field which, by definition, are the NS5-branes (wrapped on 4-cycles to give four-dimensional strings). Studying the corresponding states could provide interesting insights into these axionic strings, which although only directly valid at the LCS point, might teach us more general lessons about axionic strings in the presence of Chern-Simons terms.

### 4.4.4 Conclusions

Finally, we address our four questions for the type IV$_0$ case.

1. *Which symmetries are broken or gauged without the addition of stringy ingredients?*

    - Both electric 1-form symmetries are broken due to the monodromy. Moreover, one of the magnetic 1-form symmetries is broken by monodromy. Comparison to the large volume point suggests we set $m = 1$, in which case the latter is broken completely.

    - The axionic shift-symmetry is likewise broken. It is possible a discrete subgroup remains, under which the gauge field $A^1$ transforms non-trivially as well. We have not checked this in full generality.

    - One magnetic 1-form symmetry and the axionic 2-form symmetry are both intact.

2. *Which stringy effects are necessary to break the remaining symmetries?*

    - D3-particles provide us with a full spectrum of charges that break all vector symmetries. These also break the associated Chern-Weil symmetries. Consistency with the latter's gauging requires world-volume degrees of freedom which we were not able to identify.

For specific values of the parameters $m, n, e, f$, we do not require a full set of charges, as both electric and one magnetic 1-form symmetry is already broken completely.

- The axionic 2-form symmetry requires the inclusion of strings to be broken. These strings are now mirror dual to NS5-branes wrapped on 4-cycles. Studying them from this perspective could give us insights into the role of axionic strings in the presence of such Chern-Simons terms.

3. *Which symmetries emerge as we approach infinite distance limits in field space?*

   - Any un-broken electric 1-form symmetries will re-emerge as we take the limit $s \to \infty$, while those broken by the Chern-Simons terms remain so.
   - The exponentially suppressed corrections to the Kahler metric vanish, which could potentially lead to an emergent shift-symmetry (under which the field $F^0$ would have to transform non-trivially).
   - Since the axion is strongly coupled, it is again expected that the axionic strings are weakly coupled and decouple in the limit $s \to \infty$, leading to an emergent 2-form global symmetry.

4. *Can the massless tower of states predicted by the SDC be understood in terms of an emergent symmetry?*

   - The monodromy orbit now leads to an infinite tower of massless states, provided one BPS seed-charge exists. This is attributed to a non-standard Witten effect, due to the non-linear axion couplings, under which electrically charged states transform non-trivially.
   - Though it is possible that there are also strings which become massless in this case, the mirror symmetric view of this limit suggests that it corresponds to a decompactification limit, with the relevant tower of states corresponding to the monodromy orbit.

# Chapter 5

# Summary and Outlook

## 5.1 Summary

In this thesis we have studied generalized global symmetries as they appear in four-dimensional effective field theories obtained from type IIB Calabi-Yau compactifications. As a first step in this direction we have focused on EFTs near particular limits in the complex structure moduli space. Here, the tools from asymptotic Hodge theory allow us to extract the limiting form of the action which gives us full control over the low-energy theory in these regions of the moduli space. By restricting to singularities in one-dimensional moduli spaces, we could reconstruct the most general asymptotic form of the action and monodromy matrices. Compared to previous investigations of these limits, we have been careful to work in an integral symplectic basis, as constructed in [72]. This gave us access to the various discrete symmetries that may arise and allowed us to keep track of the quantization of charges. For each class of limit, we enumerated the possible global symmetries and explored how they may be broken by stringy effects.

Of primary importance was the role of the monodromy around the singularity, which was responsible for breaking some subset of the global symmetries. From the EFT perspective, these were associated to a variety of Chern-Simons couplings, arising from the $\mathcal{R}F \wedge F$ term in the action. In simple cases, these reduce to ordinary axion couplings in terms of which the monodromy orbits of [76, 79] admit a bottom-up interpretation as a manifestation of the Witten effect. For the remaining symmetries, D3-particles provided us with the states necessary to break all global symmetries associated to the gauge fields. Meanwhile, exponential corrections to the Kahler metric are expected to break any axionic shift symmetry not already broken by the Chern-Simons terms. Finally, we argued that in each case, some type of axionic string should be present to break the associated 2-form symmetry, though we could not always identify the string theory states which lead to such strings.

Near each singularity, we encountered one or several gauge couplings which go to zero. For specific values of the parameters in the nilpotent orbit data, the Chern-Simons terms were sufficient to prevent the emergence of a global symmetry. More generally however, discrete 0- and 1-form symmetries could emerge. A completely general feature was the emergence of a 2-form global symmetry associated to the axion. We then investigated what EFT obstructions could shield us from these emergent global symmetries. It was already known that monodromy orbits could only lead to massless towers of states in the type $\mathrm{IV}_0$ case [76]. Here, we reproduced this result and interpreted it in terms of a modified Witten effect for electric charges. For the other class of infinite distance limit we proposed that axionic strings along the lines of [50, 84] could lead to the infinite tower of states that shields us from this limit, thus relating these models to the emergent string conjecture [78].

## 5.2 Outlook

As we have hopefully illustrated throughout this thesis, there remain many interesting avenues for further research. Most closely related to the work in this thesis is the extension of our investigation to two-moduli cases. In fact, if one abandons the condition that we work in an integral basis, our results can be readily extended following the methods of [66]. These computations have been performed as part of the work for this thesis, but were not included here as the subsequent analysis was incomplete. Moreover, keeping track of the integral structure proved useful as it allowed us to study discrete symmetries, but doing so is generally much more involved than for the one-modulus cases. On the other hand, the one-modulus cases already illustrate that there remain many interesting open questions.

In a similar vein, we have sporadically commented on how the un-fixed parameters appearing in the limiting mixed Hodge structures influence our results. It could be interesting to study in more detail what values these parameters take in geometric examples. In particular, we found that their values could determine whether or not a global symmetry emerged in a given limit. Since we have used the latter to argue in favour of such dramatic effects as infinite towers of massless states, it would be interesting to see if there is any relationship between the geometry, the symmetries and these massless towers. For instance, the tower that becomes light first (i.e. particles or strings) could depend on whether an electric 1-form symmetry appears. This would involve studying explicit geometric examples and therefore run somewhat counter to the more general spirit of asymptotic Hodge theory, where we want to describe the most general asymptotic form near a singularity, however, as we have seen, questions relating to symmetries often depend on model-dependent details. Finally let us also recall that our method could not recover a possible semi-simple part of the monodromy transformation. A good first step towards investigating its influence would likewise be to consider explicit geometric examples.

Departing somewhat from the specific context of this thesis, we have seen that some of the most interesting questions were related to the properties of the states that break the symmetries we have encountered. These included the nature of the axionic strings necessary to break its 2-form symmetry, the BPS spectrum of D3-particles required to break the 1-form symmetries and the relative masses of the associated towers of states. Moreover, the precise fate of the Chern-Weil currents depends strongly on the world-volume degrees of freedom on these states. Though one can argue on general grounds that the appropriate world-volume degrees of freedom should exist, it could nonetheless be insightful to study how these descend from the ten-dimensional theory. More generally, the properties mentioned here are all sensitive to the UV-completion of the theory, highlighting the more general pattern that breaking magnetic symmetries typically occurs in said UV-completion.

All of this naturally leads one to consider EFTs obtained from other string theory constructions, where either a geometric or microscopic picture of these objects is more clear. While one of the advantages of studying type IIB EFTs has been that the tools of Hodge theory give us very explicit, yet general expressions for the low-energy theory, our comments regarding mirror symmetry already highlight that dualities to other string theory settings could lead to a more microscopic understanding of these phenomena. Alternatively, taking a step back and considering more controlled higher-dimensional theories has likewise proven fruitful in the past [88].

Lastly, while we have primarily focused on the role of higher-form global symmetries, it should be clear from our general discussion in chapter 2 that these are far from the only class of generalized global symmetries. Of particular interest to the theories we have considered in this thesis are the non-invertible symmetries which we have already noted may be present in theories with Chern-Simons terms. Indeed, taking these into consideration may help elucidate the relationship between these terms and the symmetry breaking states discussed in this thesis, while also providing a concrete setting for studying their role in the swampland program [52].

# Appendix A

# Conventions

## A.1 Differential Forms

Whenever we work in Minkowski space, we take the metric signature $(-,+,+,\ldots,+)$. Differential forms are expanded with respect to a coordinate basis $\{dx^\mu\}$, $\mu = 0,1,\ldots,D-1$ according to

$$\omega_p = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}. \tag{A.1}$$

The exterior derivative of this form is defined as

$$d\omega_p = \frac{1}{p!}\partial_{[\sigma}\omega_{\mu_1\ldots\mu_p]}dx^\sigma\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}, \tag{A.2}$$

and satisfies

$$d(\alpha_p\wedge\beta_q) = d\alpha_p\wedge\beta_q = (-1)^p\alpha_p\wedge d\beta_q, \tag{A.3}$$

which leads for instance to the integration by parts formula

$$\int(d\alpha_p\wedge\beta_{D-q}) = (-1)^{p+1}\int(\alpha_p\wedge d\beta_{D-q})\,. \tag{A.4}$$

The Hodge dual of a $p$-form is the $(D-p)$-form defined by

$$*\omega_p = -\frac{\sqrt{-g}}{p!(D-p)!}\varepsilon_{\mu_1\ldots\mu_D}g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\omega_{\nu_1\ldots\nu_p}dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_D}. \tag{A.5}$$

Here, we have introduced $g = \mathrm{det}g_{\mu\nu}$ and the Levi-Civita symbol, defiend as

$$\varepsilon^{01\ldots(D-1)} = -\varepsilon_{01\ldots(D-1)} = 1. \tag{A.6}$$

It follows from our definition of the Hodge star that

$$*1 = \sqrt{-g}dx^0\wedge\ldots\wedge dx^{D-1}, \quad *(*\omega_p) = -(-1)^{p(D-p)}\omega_p, \tag{A.7}$$

with the extra minus-sign being absent for Euclidean signature. Moreover, given two $p$-forms $\alpha_p$, $\beta_p$, we have the following useful identities

$$\alpha_p\wedge*\beta_p = \frac{1}{p!}\sqrt{-g}\,\alpha^{\mu_1\ldots\mu_p}\beta_{\mu_1\ldots\mu_p}dx^0\wedge\ldots\wedge dx^{D-1} = \beta_p\wedge*\alpha_p, \tag{A.8}$$

from which follows that

$$*\alpha_p\wedge*\beta_{D-p} = -\alpha_p\wedge\beta_{D-p}, \tag{A.9}$$

where the minus-sign is again absent for Euclidean signature.

## A.2 Quantization

### "Stringy" Conventions

For most of this thesis, we employ conventions for charge quantization which follow directly from the action in section 1.2. Here, the coupling $\mu_{p-1}\int C_p$ leads to

$$\mu_{p-1}\int_{\Sigma^{p+1}} F_{p+1} \in 2\pi\mathbb{Z}. \tag{A.10}$$

If we work in units such that $l_s = 1$, then the charge-tension relation

$$T_p = \mu_p = 2\pi l_s^{-(p+1)}, \tag{A.11}$$

implies that the ten-dimensional gauge fields are quantized as

$$\int_{\Sigma^{p+1}} F_{p+1} \in \mathbb{Z}. \tag{A.12}$$

Upon compactification, we assume that we work in an integral symplectic homology basis $(A_I, B^I) \in H_3(X,\mathbb{Z})$ (see also section 3.4) which is dual to the cohomology basis $(\alpha_I, \beta^I)$. For a D3-particle wrapping the cycle $\mathcal{L}^J$, dual to $\alpha_J$, it then follows that its electric charge is quantized as

$$1 = \int_{S^2\times\mathcal{L}^J} F_5 = \int_{S^2\times X_6} \left(F^I\wedge\alpha_I - G_I\wedge\beta^I\right)\wedge\alpha^J = \int_{S^2} G_J. \tag{A.13}$$

Similarly, from a single D3-brane wrapping $\mathcal{L}_J$, dual to $\beta^J$, it follows that the associated magnetic charge is quantized as

$$1 = \int_{S^2\times\mathcal{L}_J} F_5 = \int_{\Sigma^2\times X_6} \left(F^I\wedge\alpha_I - G_I\wedge\beta^I\right)\wedge\beta^J = \int_{S^2} F^J. \tag{A.14}$$

Thus, in four-dimensional theories obtained from Calabi-Yau compactification of type IIB supergravity, we employ the quantization

$$\int \begin{pmatrix} F^I \\ G_I \end{pmatrix} \in \mathbb{Z}^{2n_V+2}. \tag{A.15}$$

### Field Theory Conventions

Throughout chapter 3 (and also section 1.6.1), we have employed a set of conventions which highlight the connection to geometry. In particular, we identify the gauge field with the connection form on the gauge bundle, so that its field strength is quantized as

$$\frac{1}{2\pi}\int_{\Sigma^{p+1}} F_{p+1} \in \mathbb{Z}. \tag{A.16}$$

Dualization relates this to the quantization of the electric flux

$$\frac{1}{e^2}\int_{\Sigma^{D-p-1}} *F_{p+1} \in \mathbb{Z}. \tag{A.17}$$

# Appendix B

# Construction of SL$_2$-Rotation Operators

Here we give the explicit expressions for the rotation operators that enter the $\mathfrak{sl}(2)$-orbit theorem from section 3.3. We follow the presentation in [68, 79].

### The Operator $\delta$

The first rotation operator rotates the original Deligne splitting, rendering it $\mathbb{R}$-split. One way to measure the failure of this splitting to be $\mathbb{R}$-split is to consider the "naive" weight operator from the main text

$$\mathcal{N}^0 v^{p,q} = (p+q-3)v^{p,q}, \qquad v^{p,q} \in I^{p,q}. \tag{B.1}$$

This operator is in general not real, but we can use its non-trivial conjugation to define the operator $\delta$

$$\bar{\mathcal{N}}^0 = e^{-2i\delta}\mathcal{N}^0 e^{2i\delta}. \tag{B.2}$$

One moreover shows [67] that the operator $\delta$ so-defined acts strictly as a lowering operator on the Deligne splitting $I^{p,q}$ (this follows from the fact that $\bar{I}^{p,q} = I^{q,p}$ fails only by a part that sit lower in the decomposition)

$$\delta I^{p,q} \subset \bigoplus_{r<p,s<q} I^{r,s}. \tag{B.3}$$

Furthermore, we have that

$$\delta \in \mathfrak{sp}(2h^{2,1}+2,\mathbb{R}) \quad \Leftrightarrow \quad \delta^T\eta + \eta\delta = 0, \tag{B.4}$$

and $\delta$ commutes with all log-monodromy matrices. We can obtain a more explicit expression for $\delta$ by decomposing it with respect to its action on the $I^{p,q}$. In particular, we define

$$\delta = \sum_{p,q\geq 1} \delta_{-p,-q}, \qquad \delta_{-p,-q} I^{r,s} \subset I^{r-p,s-q}. \tag{B.5}$$

The separate pieces are then given in terms of $\mathcal{N}^0$, which we decompose in a similar way, by [63]

$$\begin{aligned}
&\delta_{-1,-1} = \frac{i}{4}(\bar{\mathcal{N}}^0 - \mathcal{N}^0)_{-1,-1}\,, \quad \delta_{-1,-2} = \frac{i}{6}(\bar{\mathcal{N}}^0 - \mathcal{N}^0)_{-1,-2}\,, \quad \delta_{-1,-3} = \frac{i}{8}(\bar{\mathcal{N}}^0 - \mathcal{N}^0)_{-1,-3}\,,\\
&\delta_{-2,-2} = \frac{i}{8}(\bar{\mathcal{N}}^0 - \mathcal{N}^0)_{-2,-2}\,, \quad \delta_{-2,-3} = \frac{i}{10}(\bar{\mathcal{N}}^0 - \mathcal{N}^0)_{-2,-3} - \frac{i}{5}[\delta_{-1,-2}, \delta_{-1,-1}]\,,\\
&\delta_{-3,-3} = \frac{i}{12}(\bar{\mathcal{N}}^0 - \mathcal{N}^0)_{-3,-3} - \frac{i}{3}[\delta_{-2,-2}, \delta_{-1,-1}]\,,
\end{aligned} \tag{B.6}$$

with the rest following by complex conjugation.

**The operator $\zeta$**

Though we do not need it in this thesis, we give here also the expression for $\zeta \in \mathfrak{sp}(2h^{2,1}{+}2, \mathbb{R})$. Once we have rotated the original mixed Hodge structure according to

$$\hat{F}_0^p = e^{-i\delta} F_0^p, \tag{B.7}$$

we can evaluate equation 3.46 (or, equivalently (3.49), now that it is $\mathbb{R}$-split) to obtain the associated Deligne splitting $\hat{I}^{p,q}$. We can then again decompose $\delta$ according to its action on this new splitting $\hat{I}^{p,q}$

$$\delta = \sum_{p\geq 1, q\geq 1} \delta_{-p,-q}, \qquad \delta_{-p,-q} I^{r,s} \subset I^{r-p,s-q}. \tag{B.8}$$

The operator $\zeta$ is given in terms of its components

$$\zeta = \sum_{p\geq 1, q\geq 1} \zeta_{-p,-q}, \qquad \zeta_{-p,-q} I^{r,s} \subset I^{r-p,s-q}, \tag{B.9}$$

which are given in terms of those of $\delta$ by

$$\begin{aligned}
&\zeta_{-1,-2} = -\frac{i}{2}\delta_{-1,-2}\,, && \zeta_{-1,-3} = -\frac{3i}{4}\delta_{-1,-3}\,,\\
&\zeta_{-2,-3} = -\frac{3i}{8}\delta_{-2,-3} - \frac{1}{8}[\delta_{-1,-1}, \delta_{-1,-2}]\,, && \zeta_{-3,-3} = -\frac{1}{8}[\delta_{-1,-1}, \delta_{-2,-2}]\,,
\end{aligned} \tag{B.10}$$

with the rest either vanishing or being related to these by complex conjugation.

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