Transcendental Logic and Mathematics

On Kant's logical theory of mathematical concept-formation, proof, and construction

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τὸ δὲ ἓν ποιοῦν ἕκαστον, τοῦτο ὁ νοῦς.

Aristotle [*On the soul*, 430b5f.].

But now philosophers have claimed — and Kant is the most outstanding, the classic proponent of this standpoint — that besides logic and experience we also require certain cognitions *a priori* about reality. That mathematical cognition ultimately rests on a kind of intuitive insight, that even for the constitution of number theory we require a certain intuitive modus and if you will *a priori* insight, that the applicability of the mathematical perspective to objects of perception is an essential condition for the possibility of exact cognition of nature, this seems to me certain.

David Hilbert [44, pp. 87-88, 1922].

The presuppositions of [Hilbert's] finitisitc attitude present themselves at the same time as conditions for the possibility of the theoretical knowledge of nature, quite in the sense of the Kantian formulation of the problem. If this connection comes to be generally recognized, then the possibility arises in this way that the leading thoughts of Kant's *Critique of Pure Reason* will come to life again in a new form, freed from the special forms of its historical relativity, from whose fetters theoretical science has freed us.

Paul Bernays [4, pp. 144-45, 1928].

Indeed, there is hardly any later direction that does not somehow relate to Kant's ideas. On the other hand, however, because of a lack of clarity and in a strictly literal sense incorrectness of many of Kant's formulations, quite divergent directions have developed out of Kant's thought, none of which, however, really did justice to the core of Kant's thought. [...] But now, if the misunderstood Kant has already lead to so much that is interesting in philosophy, and also indirectly in science, how much more can we expect it from Kant rightly understood?

Kurt Gödel [36, p. 387, 1961].

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Chapter 1

An interpretation of Kant's philosophy of mathematics

"For better or worse, almost every philosophical development of significance since 1800 has been a response to Kant."[1] In Alberto Coffa's studies of the development of modern philosophy of mathematics, particularly of the emergence of logicism in the works of Bolzano, Dedekind, Frege and others, Kant's part is "THE ENEMY" in their quest for the "elimination of pure intuition from scientific knowledge."[2] The theory of pure intuition is indeed central to Kant's philosophy. It was Kant's response to his more fundamental insight — which despite all efforts of the logicists turns out to be correct — that the basic propositions of mathematics cannot all be analytic, that is, truths of logic.[3] But the theory of pure intuition is *not* Kant's most fundamental contribution to the philosophy of mathematics, though like the theory of intuition, this contribution is intimately connected to the non-logical status of certain mathematical propositions.

As Kant considered mathematics a paradigm of *a priori* knowledge (necessary and independent of experience), he required a justification of its distinguished epistemological status, given the non-tautologous character of some of its basic propositions. This raised the problem of a foundation of mathematics in a distinctly modern form. It is the particular way of *framing the problem*, deriving from a subtle analysis of the form of theoretical concept-formation, that I consider Kant's most lasting

Note on citation style: References to the *Critique of Pure Reason* are in the conventional A 123 / B 134 format, "A" referring to the pagination of the first edition, "B" to that of the second. All other references to works of Kant will be to the "Akademie Ausgabe" [52] using the format (11:110), the first number referring to the volume, the second to the page.

[1] Coffa [18, p. 8].

[2] Coffa [17, p. 680-81]. Note that I would hesitate to subsume Dedekind under the label "logicist", especially if this is understood in the sense associated with Frege [29], Russell [76], Carnap [14], or Hempel [39].

[3] In this, Kant directly opposed Leibniz, who claimed:

> The great foundation of mathematics is the *principle of contradiction or identity*, that is, that a proposition cannot be true and false at the same time and that therefore A is A and cannot be non-A. This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles. (Leibniz, III. letter to Clarke, [53, pp. 677f.])

contribution. His attempt at *solving* the problem of the non-analytic *a priori*, his theory of pure intuition, is interesting less for its particular account of the forms of sensible intuition space and time than for the systematic place that it assigns to a non-logical source of knowledge (and, secondly, for the difference it draws between constructive and non-constructive forms of reasoning). In particular, this extra-logical source was to substantiate the *existential assumptions* of mathematics. Kant thereby pinpointed one of the essential difficulties still dominating modern foundational debates. Many important theoretical moves of the debates to come were prefigured, naturally in nascent or pre-mature form yet often with surprising clarity, in Kant's thought. Regarding his solutions, a simple "Back to Kant!" is no longer viable today – our difficulties in the foundations of mathematics are far more subtle than those in his days. But recognition is due both for his challenges, and for the conceptual innovations that allowed to pose those challenges in the first place. In the conclusion of this thesis, I will sketch an argument that both Kant's challenge to justify non-analytic yet non-empirical judgements, and the general form of Kant's solution retain relevance today — but to actually make that case would require another book.

Overview of content

This thesis aims to develop an original reading of Kant's philosophy of mathematics through the critical discussion of one of its most influential contemporary interpretation, that of Michael Friedman.[4] With the exception of my take on Kant's treatment of arithmetical proofs involving the least number principle, and certain aspects of my discussion of arithmetical identities like $1+1=2$, most of my claims have predecessors in the works of earlier readers of Kant, in particular Gordon Brittain Jr., Charles Parsons, Jaakko Hintikka and Brigitte Falkenburg (although they have deep disagreements between each other, together they provide many essential aspects of what seems to me the correct view). But in view of Friedman's far-reaching criticism, a much more sustained defence is required. In the conclusion, I outline a modern approach to the foundations of mathematics that is inspired by Kant, but goes beyond him in essential points. Although it in part derives from very rich and suggestive comments by Paul Bernays, to my knowledge this view has not been presented in the literature so far.

Section 1.1 outlines the core features of Kant's philosophy of mathematics, reaching maturity in the second edition of the *Critique of Pure Reason*. The remainder of this introductory chapter, beginning with section 1.2, will present the arguments against Friedman's, and for my own interpretation. In the following chapters, the necessary exegetical tasks substantiating these arguments are carried out in detail.

In subsection 1.1.1, Kant's notion of "transcendental logic" as a conceptual framework for articulating propositions about the basic structural and relational properties of "objects in general" is introduced. It is not, in the first place, a *theory* that makes specific claims, but a rather something like a *language* in which various theories can be formulated, and types of structured objects and domains of objects described.

Kant famously distinguished between two radically different components of knowledge, concepts (in particular, the 'pure concepts of the understanding' of transcendental logic) and intuition (in particular, the 'pure forms of intuition' of the transcendental aesthetics). The crucial task is to determine their relationship.

[4]Developed in chs. 1 and 2 of *Kant and the Exact Sciences* [30] and further elaborated in "Kant on geometry and spatial intuition' [32].

Subsection 1.1.2 introduces Friedman's basic thesis, that Kant's logical theory was insufficient to represent concepts of structure, and his main conclusion, that it therefore required an *extension* of its expressive resources that was provided by pure intuition. Against this I maintain that intuition did not serve to *extend* the range of possible representations, but to *delimit it*: it singled out, from a wider range of conceptually possible structures, a certain subclass about which it grounds synthetic knowledge. In particular, it provides knowledge of the "real possibility" of the existential assumptions of arithmetic and Euclidean geometry.

Section 1.2 analyzes Friedman's basic thesis in more detail. From the fact that monadic predicate logic cannot differentiate between finite and infinite structures (subsection 1.2.1) and Kant's alleged claim that infinity cannot be conceptually represented (a misinterpretation, as chapter 2 shows), Friedman draws the consequence that Kant's logical theory should to be reconstructed as "essentially monadic first-order predicate logic" (subsection 1.2.2). Sub-subsection 1.2.2.1 considers an immediate objection, which is waived to get to the core issue Friedman raises, Kant's theory of the representation of mathematical content.

Subsection 1.2.3 determines more precisely Friedman's central claims by differentiating between the *conditions of expressing* mathematical propositions, and the *conditions of recognizing their truth*. Friedman holds that for mathematics, Kant had no principled way to distinguish them; he was bound to systematically conflate an abstract description with a concrete instantiation, both regarding propositional content and regarding logical inference; in the latter case, Kant could not distinguish between the capacity of carrying out of a (constructive) operation and the logical rule that corresponds to such an operation. Friedman appeals to Kant's notion of a "schema" in his interpretation; this is shown to be a misinterpretation in chapter 3.

After distinguishing Kant's notions of general and transcendental logic (subsection 1.2.4), Friedman's interpretation of transcendental logic is analyzed (subsection 1.2.5), in which pure intuition is treated as formally analogous to additional logical constants, which remain invariant across all interpretations of the non-logical symbols of a formal theory. This reading proves inconsistent with Kant's actual theory (not on grounds of a possible anachronism in comparing Kant's system to modern formal systems, but because the *content* expressed by the system Friedman presents does not match Kant's account).

Section 1.3 seeks to refute Friedman's reading by presenting an alternative interpretation that captures Kant's actual views. In subsection 1.3.1, three basic theses of the Transcendental Analytic (the first part of the Transcendental Logic) are isolated:

(I) Representation of, and reasoning with, synthetic operations *presupposes* basic concepts of structure.

(II) The basic concepts of structure are based in the understanding, and are *independent* from any specific form of intuition.

(III) These pure concepts of the understanding are representationally *exhaustive*: the structural and relational properties of any (system of) object(s) that are thinkable at all are representable by these concepts.

As a consequence, pure spatio-temporal intuition provides a delimitation of all representable structures, not an extension.

Kant tried to systematically introduce the pure concepts by reflecting on the semantic presuppositions of the basic forms of logical judgment, as the basic concept involved in the meaning of the logical particles. E.g., the notions of individual, plurality, and totality are involved in the meaning of the quantifiers (subsection 1.3.2). Subsection 1.3.3 discusses Kant's concept of "intuition in general", by which he means the general notion of *any* concrete individual object or structured manifold, and which is entirely independent from the specifically human forms of intuition. Kant repeatedly emphasizes that forms of intuition different from spatio-temporality are logically possible (though unimaginable to us), i.e., different manners of substantiating existential assumptions are possible, for which the pure concepts of the understanding would remain adequate basic notions to describe their structural and relational properties. The arguments for Kant's theses (I), (II), (III) are presented. (The question how contingent human forms of sensibility could ground necessary, *a priori* knowledge is taken up in the concluding chapter 4.)

Subsection 1.3.4 develops an important theme in Kant's logical theory that has received very little attention so far: his formal/contentual distinction as being orthogonal to his theoretical/practical distinction. Formally, theoretical propositions are existential and/or express states of affair as independent from an acting subject: "between any two points, *there exists* exactly one straight line"; "every natural *has* a unique prime factorization"; "every bounded set of real numbers *has* a least upper bound". Formally practical propositions are operational and refer to an action: "given two points, *to draw* a line"; "for a given number, *to generate* its unique prime factorization". The former are associated only with proofs (which, depending on the status of the existential premises and the rules of inference, need not correspond to an *effective* procedure for building up the objects whose existence was proven); the latter involve both a method or algorithm and a proof of its effectiveness. Surprisingly Kant argues that in the mathematical sciences, regarding its content, the theoretical manner of representation is fundamental: all practical sentences of mathematics, including those referring to constructive algorithms, presuppose and are, in fact, *reducible* to theoretical propositions. This means that in mathematics, *existence is prior to action*. It will be argued, more fully in the conclusion, that for Kant mathematical existence corresponds to "real possibility", which is associated with positive evidence of the *possible existence* of instances of a certain type of structure. In particular, possibility of object existence is prior to possibility of object-generating action. This corresponds to the representational priority of concepts of structure over concepts of synthetic operation. Note that this does not imply that Kant admitted as epistemically justified non-constructive (in the modern sense) existence assumptions. It means, however, that by 'construction', he did not mean 'effective action in time', but, more abstractly, the possibility of the exhibition of the relevant object, which must be described[5] in theoretical-structural-existential terms, prior to the practical-generative manner of representation. In the conclusion, we return to the philosophical significance of this modal claim ("must"), particularly its relevance to the platonism-constructivism debate.

A crucial thesis of Friedman's is that Kant was unable to represent reasoning about infinite structures without reliance on spatio-temporal intuition. In particular, he was unable to differentiate between the abstract conceptualization of a type of structure and the concrete operations generating that structure

[5] Or more precisely, "must be describ*able*".

(perhaps only by infinite approximation). This is in direct contradiction with Kant's account of the Dialectics of Reason, as subsection 1.3.5 shows. In fact, Kant emphasizes the coherence of "Platonic" concept-formation and reasoning, which takes as given (through an abstract structural description) an infinite domains of individual elements between which certain relations obtain, and which reasons logically about this domain. Kant's point concerning constructive proof and the restriction to potential infinity is not that non-constructive and Platonic concepts are unthinkable, but that the admissibility of various types of reasoning depends on how the domain about which one reasons is given. This is shown in subsection 1.3.6.

Subsection 1.4 develops Kant's philosophy of arithmetic on the basis of the foregoing analysis. Subsection 1.4.1 connects the basic concepts of arithmetic to Kant's general theory of concept-formation. Subsection 1.4.2 briefly sketches Friedman's interpretation of Kant's theory of arithmetic. In subsection 1.4.3 we show that Kant's fundamental concepts of arithmetic, as important sub-species of his general categories, are likewise independent from the intuitions of space and time.

In subsection 1.4.4 we raise the question why Kant considered arithmetical propositions as synthetic. We first discuss an idea of Paul Bernays, which we claim does not fully capture Kant's fundamental reason, i.e., the existential presuppositions underlying mathematics. Subsection 1.4.5 shows that Kant's reason for the synthetic character of arithmetic is correct: it presupposes the possible existence of domains of objects satisfying basic cardinal and ordinal properties. This discussion also brings out the intricate relationship between logical and arithmetical reasoning, and the precise role played by existential assumptions. On the one hand, the derivations of arithmetical identities and proofs of theorems are entirely analytic, in that each inference step is purely logical. On the other hand, (the non-trivial part of this) reasoning appeals to existential presuppositions that are themselves non-logical.

In addition to the argumentative objections presented in this chapter, chapter 2 provides a textual refutation of the basic premises of Friedman's interpretation.

Chapter 3 contains detailed textual justifications of various aspects of my interpretation.

Both chapter 2 and 3 may be skipped without loss of continuity. The concluding chapter 4 provides a self-contained development of the central arguments of this thesis, trying to integrate what I take to be Kant's lines of thought into a unified picture that is argumentatively convincing. The chapter also contains the outlines of an argument for the continued relevance of certain Kantian ideas in the foundations of mathematics today.

Appendix 1 analyses Kant's distinction of abstract conceptualisation and intuitive exhibition, and correspondingly his differentiation of conceptual inconsistency from impossibility of intuitive exhibition, as anticipated in the *Inaugural dissertation*.

Appendix 2 is a sketch of Kant's highly innovative but sadly underdeveloped theory of formal logic.

1.1 Programmatic outline of the interpretation

An interpretation is always also an explication of terminology. For the sake of clarity, I here anticipate my understanding of two crucial Kantian terms, which already involves a controversial positioning, and which can be justified only by actually developing the interpretation.

analytic, synthetic A proposition is *analytic* if it is a truth of logic, i.e., if its negation leads, via logical inferences and appealing only to definitions and other analytic truths, to a logical contradiction. A proposition is *synthetic* if it is not analytic. Because the primitive concepts used in definitions include relational-structural concept, the analytic sentences go beyond monadic first-order logic in the modern sense.[6]

intuition An *intuition* is the immediate and singular representation of an individual object or manifold.[11] Sensible intuitions are always given as manifolds. Although cognizing them *as having the property of being* a manifold involves conceptual mediation, e.g., via the general concept of quantity (the synthetic unity of a homogeneous manifold), in terms of which, say, space is recognized as a unified manifold. The representation of synthetic unity (which determines it *as* a manifold structured in a certain way) is conceptual, but the manifoldness itself is something intuitive.[12]

NB: Intuition [Anschauung] does not mean gut-feeling, vague premonition, or subconscious or inarticulalable knowledge. Intuition is, in the first place, intuition *of*, i.e., the immediate singular representation of an object or manifold, and only in the second place intuition *that*, i.e., the source of knowledge that this object or manifold has certain properties. Although it is helpful to think of intuition in relation to perceptual acquaintance with objects, the general notion of "intuition as such" [Anschauung überhaupt][13] is not tied to the human forms of perceptual intuition, the "pure intuitions" space and time. The latter are for Kant anthropological constants, and as such of a less general status than the pure concepts of the understanding, which as concepts of the structural unity and relational determination of individual objects relate merely to 'intuition as such' (this expression acts as kind of placeholder, analogous to an uninterpreted individual term '*x*' in modern logic); the pure concepts could be shared by hypothetical intelligent beings with radically different forms of intuition. At the most

[6] *Comment:* Kant provides two explanations of analyticity. The explanation in terms of conceptual "containment"[7] can mislead for two reasons: (*a*) it appears to commit Kant to a subjectivist or even psychologistic notion of analyticity (a propositions is analytic if *I must think* the predicate whenever I think the subject – but when is that?); and (*b*) it appears to commit him to a theory of logical form restricted to monadic subject-predicate sentences. (*a*) is inconsistent with Kant's explicit anti-psychologism.[8] About (*b*) more below. Kant's official explanation is that analytic propositions are true in virtue of the principle of contradiction alone.[9] As this principle also governs all logical inferences (an inference is valid if negating its conclusion and affirming its premises is contradictory), analytic propositions are derivable by involved arguments if all premises are themselves analytic. Dually, a proposition derived by purely analytic inferences remains synthetic if in any derivation of it requires at least one synthetic premise.[10]

Controversy (examples): Friedman: Kantian analyticity is restricted to propositions representable as valid sentences in modern first-order monadic predicate calculus (see section 1.2.2). Inferences involving polyadic predicates are generally non-analytic. Tait: Kantian analyticity is wider than the tautologies in any standard modern calculus, e.g., first-order polyadic predicate calculus [82]. Bernays: Kantian analyticity is connected to intensional (non-)identity of concepts and meaning explication; an implication $(A_1 \wedge \ldots \wedge A_n \rightarrow B)$ established by a complicated logically valid deduction D can nevertheless be synthetic, if somebody who understands the meanings of the A_i and the rules of inference employed in D, but who does not know D itself, is not forced by this understanding to accept B (e.g., understanding the axioms of arithmetic and the rules of logic only forces me to accept a theorem once I have understood a proof of it). Bernays position is difficult to make precise without either abolishing analyticity completely or else making it context- and subject-dependent. Note that the notion of analyticity I ascribe to Kant is different from Bernays': understanding the meaning of a proposition and is different from knowing all the logical consequences of its meaning (see section 1.4.4).

[11] Comp. A 09: 91:

> The intuition is the singular representation (*repraesentatio singularis*), the concept the general (*repraesentatio per notas communes*) or reflected representation (*repraesentatio discursiva*.)

[12] B 136f.:

> Space and time and all their parts are **intuitions**, thus individual representations along with the manifold that they contain in themselves. (B 137 fn.)

[13] For references sections 1.3.3, 3.4.1.

> general level, Kant operates with a completely unspecific notion of intuition, which indicates the place in logical concept-formation for concrete individuals that need to be given extra-conceptually, and which is not tied to any particular form of intuition. Note that "sensible intuition is either pure intuition (space and time) or empirical intuition of that which, through sensation, is immediately represented as real in space and time".[14]

I now summarize my interpretation. Detailed arguments will be developed in the following sections.

Kant strictly distinguishes the means of *concept- and proposition-formation and logical reasoning*, on the one hand, from the grounds of *evidence of propositions*, especially the conditions of exhibiting objects (or domains of objects) corresponding to existential propositions, on the other hand. This corresponds to his distinction between the faculty of pure understanding and the faculty of pure intuition. Both are necessary for cognition, but neither can take over the function of the other.

In particular, the means to define the content of *all* mathematical concepts and propositions, as well as the forms of inference constituting mathematical proofs, are supposed to be located *entirely* on the side of the pure understanding: Kant considered it possible (and indeed necessary) to define mathematical concepts, and deductively systematize mathematical theories, in a completely non-intuitive *logical* manner.[15] These logical means are necessarily presupposed to conceptualize the "synthetic unity" (structure) of whatever can be exhibited in intuition; but regarding the range of possible structures that they allow to characterize, they far transcend what can be thus exhibited. Kant's framework not only strictly separates abstract structural concepts and concrete individual instances, but also requires an explicit distinction between constructive and non-constructive methods of reasoning, and thus a problematization of their relation, respective justifications and systematic roles.

Above, "logical" concept-formation refers to the system of pure, non-intuitive[16] concepts of the understanding [reine Verstandesbegriffe] of Kant's *transcendental logic*,[17] which aims to provide the foundations of theoretical knowledge in general; as such, some of its parts are prior – in a sense to be explained – even to *formal logic*. Determining the relation of transcendental logic to formal logic, on the one hand, and to the transcendental aesthetic (the theory of pure intuition), on the other hand, is the crucial problem for an interpretation of the *Critique of Pure Reason*, and my basic point of disagreement with Friedman.

The elementary pure concepts, the categories, are independent of all intuition. They are abstract concepts of the synthetic unity of a manifold, i.e., concepts for describing the *structural composition* and *relational determinations* of "objects in general".[18] Kant tries to introduce them systematically by showing that they contain the same contentual notions that are *presupposed* by the logical functions of judgment. His idea is *not* that the categorically representable structures are those definable in terms of the

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[14] B 147.

[15] "Non-intuitive" here and elsewhere means without reference to the pure intuitions of space and time, let alone empirical intuition. There remains a reference to "intuition in general", which indicates nothing but the reference to (so far merely hypothetical) *individual* objects (and structures constituted by individuals), without any regard to the "form of intuition" in terms of which these individuals are – or even could be – given. See section 1.3.3. The choice of the word "logical" will be explained in the next paragraph, and more fully in subsection 1.1.1. That Kant did not fully *carry out* his project is a different matter: he indeed did not succeed in giving a thoroughgoing and rigorous analysis of the basic concepts of mathematics; this task that was not fully accomplished for another century.

[16] Comp. footnote 15.

[17] A 55ff. / B 79ff. The expression "transcendental", in contradistinction from "immanent" and "transcendent", refers to the *conditions of possibility*, or *foundations*, of a given field of knowledge.

[18] A 12 / B 26, A 93 / B 126, A 115, A 130, B 150, A 248 / B 305 *et passim*.

logical forms of formal logic; rather, the basic notions underlying the meaning of the logical particles (notions such as *individual*, *plurality*, *totality*, *existence*, *ground-consequence*, etc.) are introduced as contentual determinations of objects. For example: the logical form of a universal categorical judgment, in Kant's analysis, is

> 'Any x of type A is B.'

This logical form, invariant across all interpretations of 'A' and 'B', presupposes as understood the meaning of the notions of an individual (as subject of predication), of a plurality of in a specific regard homogeneous individuals, and of the totality or the class of all individuals of some kind.[19] These notions are so to speak 'in the background' of formal logic; in transcendental logic, they 'come to the fore' as contentually interpreted notions. Thus, transcendental logic allows to form propositions such as

> 'a is an individual object',
>
> 'b is a plurality of objects of type B',
>
> 'a is an element of the plurality b', or
>
> 'For any plurality x of individuals of type A, there is a y of type B in x'.

I claim that Kant maintained — from the fully mature period of the second edition of the *Critque* onward consistently and with great emphasis — that the pure non-intuitive concepts are sufficient to formally describe the *structural* aspects of *all* objects and domains of objects that are at all thinkable (which includes but is not exhausted by the objects exhibitable in pure intuition); in particular, that all mathematical concepts could ultimately be defined in terms of the pure "Stammbegriffe".[20] Note that I will throughout use the word "structural" (which Kant uses only on rare occasions [21]), in the sense of Paul Bernays' notion of "formal abstraction":

> [W]hat may be called *formal* or *mathematical abstraction* consists in emphasizing and exclusively taking into account the structural moments of an object — "object" here is taken in its widest sense — that is, the manner of its composition from its constituent parts [Art seiner Zusammensetzung aus Bestandteilen]. (Bernays [7, p. 23, transl. p. 238])

This unpretentious sense, 'the manner of an object's composition from its constituent parts, object understood in the widest sense', corresponds to Kant's abstract notion of synthetic unity, and it is in this sense, in particular, that I argue that he considered the basic concepts of arithmetic (including the numbers,

[19] Kant actually drew the distinction between a totality, as a plurality of well-distinguished objects considered one object, and a genus or class (as the extension of a concept) which need not be considered as being given as one completed object. See subsections 1.3.2 and 1.3.6.1 below.

[20] A 81 / B 107.

[21] For example, in the third *Critique*:

> [...] the dissectors of plants and animals, to investigate their structure and understand the reasons why and to what end such parts, why such a relative positioning and connection of the parts, why just this inner form was given to them, [...] (05:376)

Note that Kant's term "construction" is an obvious cognate of "structure".

the concept of progressive iteration and the specification of algorithmic procedures) to be non-intuitive concepts of structure based in the pure understanding[22] — independent, in particular, from the intuition of time! (I don't claim that Kant anticipated, in any immediate way, the details of the modern structural conception of mathematics (think: Galois, Dedekind, Hilbert, Noether, Bourbaki); but I do claim that he contributed significantly to the conceptual stage-setting for the shift from concrete computational to abstract conceptual mathematics during the long 19th century, and for all the philosophical birth pangs that this transition involved: it is no coincidence that the philosophical vocabulary employed by thinkers as diverse as Gauss, Riemann, Kronecker, Dedekind, Frege, Cantor, Brouwer, or Hilbert derives to a great extent from Kant, albeit often in distorted form; a stylistic and etymological analysis of the writings of this period is a desideratum.)

One could object that Kant, working with traditional Aristotelian syllogistic, could not coherently claim the possibility of logically defining mathematical concepts, let alone formalizing mathematical inferences, as he was lacking the necessary logical forms (non-monadic predicates and nested generalities). Friedman, who holds this view, maintains that the following inference was not logically valid for Kant (and crucially for his interpretation, that Kant was fully aware of this):

> For any point x and any length y there exists a curve z such that z is a circle with center x and radius y.
>
> a is a point.
>
> Therefore, for any length y there exists a curve z such that z is a circle with center a and radius y.

I argue that this objection fails. Kant actually sketched the (admittedly rough) outlines of a theory of logical forms that allowed him to handle concepts and inferences about relational structures, and which integrates seamlessly with his transcendental logic. His analysis of logical form explicitly emphasizes individual terms, which allows to handle pronominal cross-reference, substitution, and to distinguish the active variable from the individual parameters in an inference involving a quantifier.[23] The open sentence "for any length y there exists a curve z such that z is a circle with center __ and radius y" corresponds to the predicate "__ is the midpoint of circles with arbitrary radius". In this way one can make sense of the

[22] Here too I follow Bernays who, after having rejecting the Frege/Russell account of the numbers as properties of predicates, e.g., 2 being the property of having an extension containing two objects, i.e., that there there exists exactly one thing x and a distinct thing y to which the predicate applies, he characterizes the numbers as elementary structural concepts:

> The conceptual content of "two things" is not logically dependent on one of the other two conceptual components. "Two things" already means something, even without the claim of existence of two things, and also without reference to a predicate, which applies to the two things; it means, simply, "a thing and one more thing".
>
> With respect to this simple definition, the Number concept turns out to be an elementary *structural concept*. [...]
>
> We have defined mathematics as the knowledge that rests on the formal (structural) consideration of objects. The numbers, however, as counting numbers [Anzahlen] constitute the *simplest formal determinations* and as ordinal numbers the *simplest formal objects*. [... . T]he determinations of number [*Anzahlbestimmungen*] concern the composition from components of a total complex of that which is given or represented, that is, exactly what constitutes the structural side of an object. (Bernays [7, p. 29-31, transl. p. 242f.])

[23] Comp. Hodges [49], who argues that pre-Fregean "traditional" logicians were quite capable of handling inferences as the one above.

appearance that Kant's logic only deals with monadic predicates and inferences involving these.[24] Note that in modern logic, the transition from monadic to polyadic predicate calculi involves no introduction of new elementary rules of inference.[25] Rather, it involves admitting additional forms of concept-formation, in particular, symbols corresponding to relational concepts. But the pure categories of synthetic unity are precisely such concepts. (NB: I am not claiming that Kant developed a formal calculus with a neatly specified recursive syntax. If one wanted to use the modern distinction, one could say that his conception of logic was primarily semantic: it concerned the basic contentual notions like 'object', 'class', 'extension', 'relation', 'consequence', in terms of which formal calculi are interpreted, and without which they remain meaningless rules for scribbling on paper. More needs to be said about this, and about the sense of calling transcendental logic "logic"; for a start, subsection 1.1.1).

From the fact that there are existential propositions amongst the premises of mathematical theories, Kant inferred the necessity of an extra-logical source of knowledge; for "every existential proposition is synthetic",[26] i.e., not a truth of logic. Concerning existential propositions generally, he states:

> I may thus confine myself to a single fair demand: that one provide a general justification [...] how one should make a start to secure the objective reality of any concept we have thought out. In whatever manner the understanding may have arrived at a concept, the existence of its object is never, by any process of analysis, discoverable within it; for the knowledge of the e x i s t e n c e of the object consists precisely in the fact that the object is posited in itself, o u t s i d e t h e m e r e t h o u g h t o f i t. (A 639 / B 667).

An *intuition* [Anschauung] is an immediate and singular representation of a concrete object, which is given prior to abstract thought. Humans have specific "pure forms" of sensible intuition (space and time), but Kant emphasizes that the categories are not restricted to these; it is conceivable that there are other intelligent beings, with radically different forms of intuition, which nevertheless operate with the same pure categories, including the concepts of arithmetic. Kant considered the synthetic (structural) unity of such intuitive forms, and of all types of objects and relations exhibitable in them, to be abstractly definable by pure concepts; but the "objective reality" of these concepts, i.e., the existence of objects corresponding to them, required exhibition, i.e., *construction*:

> [M]athematical cognition [is rational cognition] from the c o n s t r u c t i o n of concepts. But to c o n s t r u c t a concept means: to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, a n o n - e m p i r i c a l intuition is required, which consequently, qua intuition, is an i n d i v i d u a l object. (713 / 741 ff.)

Note that "to construct a concept" always means to exhibit an object (or prove the explicit exhibitability or "real possibility" of an object) that instantiates the concept; it does not mean to 'make' the concept

[24] Kant thus stands in the tradition of "local formalizing" described by W. Hodges [49, Sec. 5, pp. 594ff.], which contrasts with the "global formalizing" of modern predicate calculi.

[25] In a Gentzen-style natural deduction calculus, all the introduction and elimination rules governing the logical constants remain the same. In a Hilbert calculus, no new logical axioms for relational predicates are introduced.

[26] A 598 / B 626

itself (knowing this distinction was considered by Kantians a litmus test for whether someone understood the *Critique*[27]).

> The objective reality of this concept [here Kant speaks of the concept 'parabola'], i.e., the possibility of the existence of a thing with these properties, can be proven in no other way than by providing the corresponding intuition. (8: 191)

The most basic characteristic that Kant requires of intuitive construction is that it provide concrete, directly apprehendable and manipulable instances; in the case of complex types, this is a step-by-step rule of construction together with a proof of its effectiveness. He is not concerned with contingent psychological limitations of human visualization, but with an idealized 'in principle' constructibility; thus chiliagons or even megagons are considered intuitively exhibitable, since effective rules of construction can be given, regardless of the fact that (presumably) nobody has or will ever draw them or could distinguish a 999999-gon from a 1000000-gon by visual inspection.[28] The in-principle exhibitability of discrete collections of arbitrarily large finite cardinality is secured by *a priori* intuition, even though for very large collections, while their "progressive apprehension" one by one and their "*comprehensio logica* in a numerical concept" is considered possible, their "*comprehensio aesthetica*" in one view of the imagination (let alone perception) is not. "The logical estimation of quantity progresses without hindrance into infinity", i.e., beyond any given finite bound; not before

> reason demands totality, thus comprehension into o n e intuition and e x h i b i t i o n of all the members of a progressively increasing number series [...], to think of the infinite as g i v e n i n i t s e n t i r e t y,

does intuitive constructibility reach its limit.[29] Thinking and reasoning about completed infinite totalities is possible without contradiction according to Kant (section 1.3.6), but exhibiting a totality of well-distinguished individuals, thereby securing the objective reality of such a concept, is not. Thus, although intuitive constructibility (by humans) is restricted to the potentially infinite, the actual infinite is not on that account ruled out as inconsistent. Kant did emphasis very strongly the distinction between conceptual or logical possibility, according to which the actually infinite is possible, and intuitive exhibitability (in terms of the human forms of intuition); the impossibility of exhibiting an actually infinite according to the latter does not imply its conceptual inconsistency. It only implies that we have no positive reasons that justify reference to an existing actual infinity in the premise of a mathematical argument.[30]

The above quoted passage continues, that although the construction of a mathematical concept,

> qua intuition, is an i n d i v i d u a l object, it must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure

[27] Comp. Christian Gottfried Schütz's letter to Kant from 13. Nov. 1785, (10:422, lines 30-34).

[28] AA 11:46, 8:211f. Note that neither regular chiliagons nor regular megagons are constructible by straight edge and compass. Defining a method for constructing non-regular chiliagons, on the other hand, is trivial.

[29] (5: 254).

[30] Comp. Appendix A below.

> intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having had to borrow the pattern for it from any experience. The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for regarding this empirical intuition one takes account only of the operation of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. (713 / 741 ff.)

As has been discussed by E. W. Beth[31] and J. Hintikka,[32] one form of proof Kant has in mind in this passage is formally analogous to free variable arguments of the form,

$$\dfrac{\dfrac{\begin{array}{c}[P(a)]\\ \vdots \\ Q(a)\end{array}}{P(a) \to Q(a)}}{\forall x.P(x) \to Q(x),}$$

i.e., to prove that $\forall x.P(x) \to Q(x)$ we assume some a with *P(a)*; we then derive *Q(a)* from this and discharge the assumption establishing $P(a) \to Q(a)$ unconditionally. But since we chose a arbitrarily, the desired general conclusion holds. Hintikka further emphasizes the role of intuition in grounding propositions containing, and thus proofs involving the instantiation of, existential quantifiers, in particular in the context of auxiliary constructions in geometry. As Parsons points out, all these forms of argument

> turn on the use of a free variable which indicates *any* one of a given class of objects, so that an argument concerning it is valid for *all* objects of the class. They thus have a formal analogy with the appeal to pure intuition, in that a *singular* term is used in such a way that what is proved of it can be presumed generally valid. Moreover, the manner in which this generality is assured, namely by not allowing anything to be assumed about a except what is explicitly stated in premises, is reminiscent of a statement of Kant about the role of a constructed figure in a proof: "If he is to know anything with *a priori* certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept." (Parsons, [62, p. 55f.])

The big problem is that, mathematically speaking, these methods of establishing generality alone cover little ground. Arguably (following Beth and Hintikka), one can account for the general theorems of classic Euclidean geometry. But already many ancient results of elementary number theory (beginning with books VII-X of the *Elements*) require different ways to establish their generality. As Kant himself points out, in arithmetic, intuitive constructions of the kind discussed so far can only establish *singular* propositions like "$7+5=12$" (in fact he maintains that all true singular arithmetical sentences, no matter how large the involved numbers, are synthetic and require appeal to intuition in their proof; the impatient reader is invited to skip to sections 1.4.4 and 1.4.5 for a discussion of this crucial point). But by such finitary

[31] Beth [8], [9].
[32] Hintikka [45], [46], [47], [48] .

constructions neither the classic theorem that the square root of every natural number is either natural or irrational, nor the decomposition of numbers into prime factors, nor even the proof of the Euclidean algorithm can be established. For all of them, each individual instance is intuitively verifiable, but the general theorem is not.

Friedman actually argues that Kant considered these propositions unprovable, strictly speaking not even *stateable*. I maintain that Kant did consider them provable, *but not by appeal to intuition*. General arguments in number theory that involve mathematical induction or infinite descent, and that do not involve any computation with concrete numerical values, were for Kant purely logical consequences of the general concept of finite number. In particular, I would argue, induction in the form of the least number principle was considered by Kant not a synthetic axiom but an analytic consequence of the pure general concept of finite number.[33] Intuition comes into play, according to Kant, only at the level of instantiating concrete numerical values, and in particular, in *executing* general computation procedures applied to concrete parameters. The general definition of arithmetical relations, the specification of arithmetical algorithms as well as the proofs of their correctness do not depend on (temporal or any other form of) intuition regarding their conceptual content and validity, though they *do* depend on some kind of intuition (which need *not* be spatio-temporal) for their non-vacuity.

[33] Note that mathematical induction is not an existential proposition. One form congenial to Kant is

$$\exists n\varphi(n) \to \exists n(\varphi(n) \wedge \forall m < n\, \neg\varphi(m)),$$

if there is a number satisfying φ, then there is a least such number. This merely implies the existence of a number *on the hypothesis* that there is a number; it does not claim the existence of any number categorically. It can be shown that this principle ought to be considered analytic on Kant's concept of finite number, and that Kant actually made use of this fact (see the quotes in this footnote below).

The equivalence of this principle to that of 'complete induction' follows by substituting "$\neg\varphi$" for "φ", contraposing, and using the equivalence of "$\neg\exists\neg$" and "$\forall$":

$$\forall n(\forall m < n\, \varphi(m) \to \varphi(n)) \to \forall n\varphi(n).$$

Note that this form of induction is more general than the customary statement with '$\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))$' in the antecedent, since it is valid for *any* well-founded $<$, and therefore applies also to Cantor's transfinite ordinals.

The following passage is a key source for this interpretation. Kant first sketches a proof using the least number principle of the theorem cited in the quote, and then argues:

> <u>If</u> it were not possible to *a priori* prove or explain: *that if the root of a given quantity cannot be expressed in whole numbers, then it cannot be expressed in fractions (though it can be approximated with arbitrary precision)*; <u>then</u> this would be a phenomenon of the relation of our power of imagination to the understanding, which we can perceive through experiments with numbers, but cannot explain from concepts of the understanding. But now the former is in fact possible; consequently, this conclusion does not follow. (11:210)

The interpretation of this is quite unambiguous: general arithmetical theorems can be proven and explained *a priori* from concepts of the understanding; their "ground" does not reside in temporal imagination, but in the pure concept of number. The following is from a draft to this letter, which makes the same point perhaps even more forcefully. The "reason why" the square root of any number that is not a perfect square cannot be represented by a rational fraction

> is indeed a reason that does not at all rest on the faculty of imagination (as a faculty so-to-speak organized by the understanding for the representation of the irrational), but rests rather on a condition which the understanding itself puts into its concept of the number, namely that the assumed square is not the square of a whole number, and consequently also not the square of any fully specifiable fraction [...]. <u>If</u> one could <u>not</u> prove *a priori* that in such cases the mean proportional is an irrational number, but would rather find this merely empirically — <u>only then</u> one would have reason to speculate about <u>some particular ground not contained in the concept of number of the understanding</u>, thus some subjective ground in an uncharted nature of the faculty of imagination, whose nature would bring about <u>a fact that the understanding cannot, by thinking, measure up to</u>.

But the understanding *can* measure up to it, and it does so by making use of a general defining criteria of numbers, i.e., the least number principle.

This final aspect of my interpretation remains its most tentative part. Even if I am right and Kant did hold this position, he at most made gestures toward carrying it out as a philosophical research program. All the same, I hope to make plausible that given Kant's general philosophical theory (from the period of the second edition of the *Critique* onward) and his applications thereof in his discussions of mathematics, it is the most plausible reading. A thorough treatment of this issue is lies beyond the scope of the present thesis, and is material for future work.

1.1.1 In what sense is the transcendental logic a logic?

To a modern reader, the very name "transcendental logic" might appear ill-chosen: "Kant's transcendental logic does not seem to be a logic in the modern sense at all: no syntax, no semantics, no inferences."[34] We will see that the transcendental logic is neither formal logic nor a formal system that requires an external interpretation to give meaning to its primitive terms. Rather, it is supposed to be the general conceptual framework within which even formal logic first acquires its significance, and in terms of which various abstract concepts and theories of types of objects can be formulated, some of which can then be interpreted and supplied with objective significance by reference to pure or empirical intuition.

The transcendental logic thus contains meaningful, *contentual* notions, the elementary concepts of synthetic unity, which are none-the-less *abstract*, in that they are not based in any intuitive or experiential source, and in that their reference to concrete objects is still problematic; but they are not *empty* or *meaningless* in the way that a completely uninterpreted formal calculus is.

The part of transcendental logic that is prior to the "schematization" of the categories (their application to the pure forms of sensible intuition),[35] which treats the non-intuitive[36] "intellectual synthesis" of pure thought,[37] is logic in the sense in which mathematicians such as Bernhard Riemann and Richard Dedekind in the 19th century, *before* Frege, Russell, Hilbert, Bernays, *et al.*, understood the word. It contains those "general concepts and such activities of the understanding *without* which no thinking whatsoever is possible."[38] Amongst these are the concepts of a totality or "system of individuals or elements" (which underlies an extensional understanding of predicates[39]) and of a "certain relation or order amongst such elements", both understood – in a state of pre-Cantorian and -Russellian innocence – as purely logical notions.[40] José Ferreirós shows convincingly that Riemann and Dedekind's conceptions of "logic" stand firmly in the Kantian tradition.[41] (If a counter-historic thought-experiment is permitted: were Kant to read Dedekind's 1887 *Was sind und was sollen die Zahlen*, he would not find it to contradict his notion of logic. His first real point of divergence would be Dedekind's attempt to prove proposition 66, the existence of a completed infinite totality. Kant, I believe, would maintain that Dedekind's brilliant structural description of the totality of finite numbers is conceptually adquate, but that his attempt at

[34] Achourioti, van Lambalgen [1, p. 254]
[35] A 137ff. / B 176ff., section 3.4.2.
[36] Comp. footnote 15.
[37] B 150ff., section 1.3.3.
[38] Dedekind, letter to Keferstein [22, p. 185].
[39] Schulthess [78, Ch. 1, pp. 11ff., pp. 83ff., 112ff.], Férreiros [26, p. 51]. Kant: "The *sphaera notionis* actually refers to the collection of objects which fall under a concept, as a *nota communis*." (24: 257)
[40] Dedekind [22, pp. 53, 63ff.], [22, p. 186].
[41] [26, II.2, pp. 47ff., VII, pp. 215ff.]

proving the existence of a completed totality corresponding to this structure fails. Kant would opt for a constructive grounding of arithmetic, *without* discarding the abstract conceptualization as a regulative ideal — Kant would disagree with Gauss', Kronecker's or Brouwer's rejections of infinitary concepts as meaningless, or even contradictory. See sections 1.3.5 and 1.3.6.)

I add here two interesting quotes that show that Kant considered the need to provide a symbolic, indeed even algorithmic (see second quote) formalization of transcendental logic.

> [...] and I have not lost the hope that a mathematical consideration of its methods and heuristic principles, and all its remaining desiderata, can shed further light on the critique and survey of pure reason, and provide it with new methods for the exhibition of its abstract concepts, indeed even something similar to Leibniz's *ars universalis characteristica combinatoria*. For the table of categories as well as the table of ideas (where among the latter the cosmological ideas show a certain similarity to the impossible roots [42]) are fully numbered and in regard to all possible use of reason by concepts so determinate as mathematics could only demand, in order to at least try how much, if not extension, at least clarification it could bring into them. (11: 29, 1791)

In the same vein, in a reflection from the period of conception of the *Critique*, Kant formulates the desideratum of an algorithmic formalization of transcendental logic. As we will see, by the "technical method" Kant means the specification, firstly, of elementary operations in terms of types of basic individuals ("the given") and of the basic kinds of relations between them ("synthetic unity"), and secondly, of complex operations defined by iterative combinations of elementary operations (sections 1.3, 1.3.4, and chapters 3 and 6 below):

> It is of greatest importance to make a science of reason technical. [...] Only regarding quantities, the inventors to the a l g o r i t h m were successful. Should this not also be possible in the critique of pure reason, not for the extension of cognitions but for their purification? By the technical method one can in that relation give to every concept its function, or rather, express the f u n c t i o n s themselves, by themselves and in relation to each other. (Algebra defines its functions only relative to each other, perhaps the same is the case in the t r a n s c e n d e n t a l a l g o r i t h m. Errors and omissions can be prevented only in this way.) (18:34, ca.1776-78)

[42] In the footnote Kant explains:

> If according to the principle '*in the series of appearances everything is conditioned*' I would yet strive for the unconditioned and the highest ground of the totality of the series [i.e., the object of a "cosmological idea"], then that is somewhat like searching for the $\sqrt{-2}$ [among the reals].

Note that Kant denied that "cosmological ideas" were "constitutive" concepts of real objects (because although not *internally* inconsistent they cannot consistently be combined with his concept of an object of real experience), but that he assigned them an instrumentalist role as "regulative principles", as an "o b j e c t i n t h e i d e a", which is "really only a schema for which no object is given, not even hypothetically, but which serves only to represent other objects to us, in accordance with their systematic unity, by means of the relation to this idea." As such, it "serves only to preserve the greatest systematic unity in the empirical use of our reason, in that one derives the object of experience, as it were, from the imagined object of this idea." (A 670 / B 698) In this regard too, thus, the use made of cosmological ideas (in relation to "real objects") mirrors the purely instrumentalist use made of imaginary numbers (in relation to real numbers) prior to the works of C. F. Gauß and W. R. Hamilton. Kant was quite familiar with Euler's and especially Lambert's mathematical works, and thus knew about importance of complex numbers. Also note that the distinction between real and ideal objects introduced by Kant prefigures that of Hilbert.

1.1.2 The disagreement with Friedman

Friedman's main thesis is that Kant could not *conceptually* represent mathematical structures. To represent them, he required intuition not merely as a source of evidence for the basic propositions (particularly to ensure existence and/or step-by-step constructibility), but as a 'quasi-logical' means to define their structural content. Intuitive exhibitability was thus no *restrictive* condition placed on concepts in order to delimit, within the range of all logically possible concepts, those whose objective realizability is beyond any doubt (as an absolute epistemological minimum). Rather, intuition served to *extend* the range of representable structures. Consequently Kant could not, e.g., separate the abstract notion of a dense order from a concrete intuitive realization of it (sections 1.2.5, 1.3.6). With the distinction between abstract concepts of structure and concrete realizations, Friedman at once abolishes those aspect of Kant's philosophy that makes it relevant to the debates of the 19th and 20th century: firstly to the debate of various kinds of 'Platonists' and 'constructivists', whose disagreements run precisely along the axis of *which* abstract concepts are admissible as describing objective mathematical ideas, and closely connected, the distinction between constructive and non-constructive forms of reasoning. Secondly, it makes unintelligible Kant's distinction between "real" and "ideal" notions, which in the hands of Hilbert turned out so fruitful.[43]

Closely connected is another thesis crucial to Friedman's interpretation: that Kant (if he were faithful to his own philosophy) would have to consider as literally non-sensical propositions containing genuine quantification over numbers and sets of numbers, *even if the quantifiers are interpreted constructively*.[44] Kant's discussions were already hardly 'up-to-date' with the mathematics of his own day. For example, he rarely even mentions the infinitesimal calculus. But taken seriously, Friedman's interpretation would literally throw Kant back to *before* classic Greek antiquity. Both the quantification over collections of discrete multiples (e.g., in Eudoxos theory of proportions, *Elements* book V. Def. 5) and the appeal to the least number principle (book VIIff.) were prized possessions of ancient mathematics. According to Friedman, Kant's "science of number"[45] would be no stronger than Robinson's arithmetic Q without "true quantifiers",[46] which is very weak indeed and hardly deserving of the title "science" as Kant understood it.[47] (Note that Friedman tries to soften this blow by admitting that Kant unconsciously smuggled in ideas that contradicted his official philosophy.)

None of this refutes Friedman, though it provides serious occasion to doubt the adequacy of his interpretation. My fundamental disagreement with Friedman, however, is quite simply that his is not a defensible interpretation of the *Critique of Pure Reason*, and in particular, of Kant's idea of transcendental logic. This, I claim, can be established on purely text-exegetical grounds.

[43] Comp. Hilbert's "Über das Unedliche" [**?** , p. 190]. In discussions of this idea, Hilbert consistently refers to Kant.

[44] Friedman [30, 113 fn. 31]. That Friedman really needs to take the hard-headed stance, that general number theoretical propositions aren't properly speaking even *thinkable* to Kant, to make his interpretation consistent will be argued below.

[45] (10:557).

[46] Friedman [30, 113 fn. 31].

[47] (4:467).

1.2 Transcendental logic and the representation of structure: Friedman's account

Friedman's interpretation is best approached through his claim that Kant could not conceptually represent infinity (subsection 1.2.1). This follows from his basic thesis that Kant's logic is (at most) equivalent to first-order monadic predicate logic (subsection 1.2.2), from which he concludes that all types of structures not definable therein were not for Kant capable of being conceptually "thought or represented"[48] (though some could be re-captured by pure intuition). After determining more precisely Friedman's position in section 1.2.3, I will describe in detail his interpretation of Kant's transcendental logic (subsection 1.2.5). In the following section, I present my criticism and the defense of my own interpretation, which go hand in hand.

1.2.1 The problem of representing infinity

Infinity enters mathematics, practiced as a science, in its earliest pages. Whether one accepts or shuns the actual infinite, the discovery of incommensurable magnitudes required mathematicians to conceptualize and reason with and about notions such as infinite sequences or never-terminating procedures.

In a logical theory with relational or functional expressions, finitely statable conditions unsatisfiable by finite domains are formulated easily and naturally. For example, let variables x, y, z range over some non-empty domain of elements, and specify the binary relation R on this domain by the following three conditions:

(R1) For any x and y, if $R(x,y)$ then not $R(y,x)$. (Asymmetry)

(R2) For any x, y, and z, if $R(x,y)$ and $R(y,z)$ then $R(x,z)$. (Transitivity)

(R3) For any x, there exists a y such that $R(x,y)$.

These conditions cannot be satisfied in any finite aggregate of elements, regardless of what binary relation on them "R" is interpreted as.[49] One can thus say that (R1) – (R3) jointly represent or define the concept of a wide class of (not necessarily isomorphic) infinite structures in purely logical terms. For no assumption is made about the 'content' of R, or the 'nature' of the objects it operates on; merely the meaning of the logical constants ("any", "exists", "if...then", etc.) in combination with the concepts of "domains of elements" and "binary relations" are required for this representation.

[48] Friedman [30, p. 122, p. 64].

[49] For assume that there is only a finite number n of such elements. Take any element a. By (R3) and (R1), there is a different element b with aRb. By the same token, for each thus obtained element there exists a further element different from all previously obtained (otherwise closing a 'loop' $aRbR...RkRa$ contradicting asymmetry and transitivity). After at most n steps, we must admit that there exist more than n elements after all, contradicting the assumption. Note that this informal argument uses notions not explicitly defined (or even mentioned) in (R1) – (R3), particularly the general concept of a finite number and the finite iteration of a proof-step. We will return to the status of such meta-theoretical arguments and the notions employed therein below.

1.2.2 Friedman's basic thesis: Kant's logic is equivalent to first-order monadic predicate logic

Friedman argues that Kant's logical theory did not allow this method of concept formation. Like Leibniz,[50] Kant supposedly limited logical or conceptual thought — the faculty of the "pure understanding" — to the forms of *monadic* subject-predicate logic, excluding relational expressions like "$R(x,y)$". But unlike Leibniz, he further confined logical forms to "finitary representations", by which Friedman means representations that are *themselves* constituted by a strictly finite number of terms or expressions, explicitly ruling out infinitary logical forms such as infinite conjunctions, or proofs constituted by infinitely many inferences.[51] This makes it impossible to logically formulate conditions satisfiable *exclusively* by infinite structures: any consistent finite set of sentences in a "finitary" monadic first-order language has a finite model, a significant insight that Friedman believes Kant (in his own way) understood.[52] Thus Kant could not, by any purely conceptual means available to him, represent structures that are necessarily infinite; to do so required methods beyond what he considered to be "logic". I shall refer to this Friedman's *basic thesis*: we should

> equate Kant's conception of logic with, at most, monadic (or perhaps 'essentially monadic'[53]) quantification theory plus identity [= first-order monadic predicate logic, henceforth "FOMPL"]. [30, p. 63, fn.9]

I understand this simply as the claim that all modes of concept-formation and inference that would be recognized as "logical" by Kant can be formally reproduced in FOMPL used as a 'neutral' heuristic devise to measure the representational strength of Kant's actual logic (thus providing an 'upper bound' to what Kant considered as logically representable). Obviously, no commitment to any properties particular to the modern theory should be ascribed to Kant.[54]

Friedman presents two arguments for his basic thesis, one direct, one abductive. The latter, presented as the main argument, interprets the passage *CPR* B40 (and two related passages) as claiming that no purely conceptual representation can be adequate for the representation of infinity.[55] This is taken as

[50] This reading of Leibniz, which seems to go back to Russell's [74], has been strongly contested since. See Ishiguro [51, ch.?]. It is not necessary for our purpose to take a side, but I admit my sympathy for the anti-Russellian reading.

[51] Friedman [30, pp. 63f., 66-71], [32, pp. 238f.]. I emphasize that I do not accept Friedman's interpretation of Leibniz! Also note that Friedman's use of "finitary" is not Hilbert's. It means that concepts, propositions, and proofs *themselves* must be finite entities, without any prima facie restriction on the kinds of *objects* to which they refer. In particular, Friedman doesn't ascribe to Kant the claim that "finitary" monadic concepts can refer *exclusively* to finite objects. Rather, Kant couldn't formulate conditions that can be fulfilled *exclusively* by structures that are (in the relevant aspect) non-finite. This does not preclude that the structures describable by pure concepts, some are infinite in the relevant aspect; *however, there will always be possible instances that are finite in that regard, and thus no conceptual demarcation of infinite structures was possible.*

[52] See [30, p. 68]. In what sense one could 'formulate' such conditions in an infinitary language without already presupposing a concept of infinity I leave for the reader to ponder.

[53] By "essentially monadic" predicate logic, Friedman means a logic that contains relational expressions but (for some reason) disallows quantifiers binding variables in non-monadic matrices. By this Friedman tries to account for the fact that Kant plainly admits as logically well-formed propositions such as '$a > b$' and '$a + b > c$' while avoiding the fatal consequence that Kant should therefore also admit the logical form of propositions like 'there is an x such that $x > b$, or 'for any x there is a y such that $x > y$', which would undermine his basic thesis. I refrain from further comments on this clearly ad hoc move.

[54] We see below that this 'neutrality' cannot really be maintained, as the basic form of predication for Kant involves the explicit typing of all quantified variables, which leads to a fatal asymmetry between Kant's and FOMPL undermining Friedman's interpretation. See 2.1 and ch. 6.

[55] Friedman [30, p. 63f., 68].

evidence that Kant considered his logic as restricted in the explained way and drew the correct conclusions (abductive because one argues from the explicit conclusion 'no conceptual representation of infinity' to the implicit premise 'monadic logic'). I show in chapter 2 that no such evidence is forthcoming at B40 *et al.*: a careful reading of these passages shows that Friedman's interpretation of them is not merely unfounded, but false. The second argument, hidden in a footnote but in my view more powerful, is the claim that unless logic was essentially monadic "Kant's Table of Judgments makes no sense",[56] combined with the observation that this table occupies a crucial place in the system of the "transcendental logic". To this we return to in chapter 6 and throughout this introduction.

1.2.2.1 Immediate objection: unwarranted restriction to first-order quantification

Friedman's basic thesis has an immediate objection: it unjustifiably restricts Kant's notion of logic to first-order quantification. Because this objection at first sight reveals little about Kant's concrete views on mathematics; because Friedman's thesis may possibly be modified to defuse it; and because his interpretation remains intrinsically interesting even if its basic premise *is* compromised, I will now present the objection but afterwards assume that it may be evaded. However, the theme of higher-order quantification and operations of class-, set-, or type-forming return in my interpretation of Kant's theory of logical and mathematical concept formation; in light of this, the present objection regains its pertinence.

In the introduction to the "Transcendental Logic", Kant states that "general logic [...] considers only the logical form in the relation of cognitions [in particular: concepts and judgments[57]] to one another".[58] With this in mind, consider the proposition

$$\forall P\, \exists Q\, (\forall x\, (Qx \to Px) \land \exists x\, (Px \land \neg\, Qx)), \tag{S}$$

i.e., for any concept P some concept Q is such that all Q are P, but some P are not Q.[59] Kant formulated (S) as a "principle of specification" in a hypothetical "ideal" theory of concepts, in which every consistent concept corresponds to a sub-class of infinite extension of a continuum of logically possible objects (strict numerical identity "=", which would allow defining concepts with finite extension would be excluded as non-logical, comp. A 263f. / B 319f.).[60] Clearly the truth of (S) would entail the existence of an

[56] Friedman [30, p.63, fn.9].

[57] Comp. B 377. "Cognition" is here definitively used in logical, not a psychological sense.

[58] A 55 / B 79.

[59] I use the formalism of modern predicate logic here simply to state this proposition unambiguously. From a modern perspective, its status obviously depends on what one takes the first- and second-order variables to ranges over. Clearly it is false on any interpretation that assigns to the second-order terms well-founded sets, which excludes infinite descending sequences $a_1 \ni a_2 \ni a_3 \ni \dots$.

[60] Kant clearly considered (S) logically consistent, as he explicitly states it as part of a hypothetical theory and discusses a model for it (described in spatial terms), in which the individuals are *possible* rather than *actual* objects. (Note that Kant is *not* saying in the following quote that in the theory under consideration there are no individuals at all in the "Umfang" or extension of a concept; rather he uses the more specific notion of the "*logical* horizon" of a concept as referring, not to the collection of individual instances but to the collection of its subspecies):

> One can regard every concept as a point, which, as the standpoint of an observer, has its horizon, i.e., a set of things that can be represented and, so to speak, surveyed from it. Within this horizon a set of points must be able to be given to infinity, each of which in turn has its narrower field of view; i.e., every species contains subspecies in accordance with the principle of specification, and the *logical* horizon consists only of smaller horizons (subspecies), but not of points that have no domain (individuals). (A 659 / B 687)

infinite number of individuals (assuming there are non-empty concepts): any non-empty concept has non-empty proper sub-concepts, which in turn have non-empty proper sub-concepts. (Note that if one interprets the second-order variables as ranging over open spatial regions and the individual variables over points, the propositions (S) states that each region contains a proper sub-region; Kant uses this topological interpretation for illustrative purposes,[61] but (S)'s logical form clearly does not depend on it). Yet (S) employs only monadic predicates, and by quantifying over them expresses a "relation of [concepts] to one another".[62] Why *ought* Kant to have rejected (S) as logically statable,[63] or deny that it logically entails an infinity of objects?; that is, why should he have limited logic to *first-order* quantification? The strict separation of first- from higher-order logic and certainly the customary restriction to the former are 20th century phenomena.[64] If Leibniz did not wield second-order quantification, his law of the identity of indiscernibles

$$x = y \longleftrightarrow_{def} \forall P\,(Px \leftrightarrow Py), \tag{LL}$$

i.e., two objects are numerically identical exactly if they have *all properties* in common, could not even be stated. But admitting second-order quantification is independent from rejecting "infinitary" concepts in Friedman's sense, e.g., infinite conjunctions. If the argument is really that Kant's logic is the same as Leibniz's except for these,[65] then Kant should plainly have accepted (S) as an admissible logical form—as indeed he did.[66] And as with (LL), his questions was not whether this logical form exists or is "thinkable" in terms of the pure understanding, but whether it is consistent and, the individual terms being taken to range over objects of a specified type, e.g., objects of pure intuition or of empirical reality, whether it is true. Kant certainly denied that existential propositions like (S) (applied to concrete objects) could be *justified* by logical principles alone (if at all). [67] It does not follow that he could not state them logically or derive their consequences analytically.[68] I therefore submit that Kant did not reject the possibility to

Note that Kant denies that this theory (which besides (S) contains further principles) could be realized in empirical reality with the individuals being interpreted as *real objects*, rather than as possibilia, as emphasized here:

> But it is easy to see that this continuity of forms is a mere idea, for which a corresponding object can by no means be displayed in experience, [...] because the species in nature are really partitioned and therefore in themselves have to constitute a *quantum discretum*, and if the steplike progression in their affinity were continuous, they would also have to contain an actual infinity of intermediate members between any two given species, which is [physically] impossible. (A 661 / B 689)

[61] See fn. 60.

[62] Comp. footnotes 57 and 58. The objection that (S) is really a statement of a many-sorted first-order theory with the binary predicate "Xx" would even strengthen our position: the logical form of (S) simply does not depend on spatio-temporal intuition, so our objector would unwittingly admit that Kant did have access to relational predicates after all.

[63] He *de facto* did not reject it. But the force of Friedman's argument is that, if Kant were consistent with his own principles, he should have.

[64] Shapiro [79, Ch. 7].

[65] See Friedman [30, p.??] and especially [32, p.238f.] for an unequivocal statement of this position.

[66] See fn. 60.

[67] A 598 / B 626.

[68] There is no reasonable justification for ascribing to Kant the today common restriction to first-order logic, which is in large parts a reaction to the metatheoretical theorems of Skolem and Gödel. On the contrary, as Brigitte Falkenburg has shown, Kant's formal ontology, framed by his logical theory, involves what she calls an "ambiguity" from the modern perspective at this point, by *explicitly employing both* but not apparently differentiating rigorously between quantification over logical individuals and over relations between and collections of such individuals. The "transcendental logic" incorporates finite *type forming* operations, by which new types of individuals are introduced in terms of (finite) relations between or (finite) collections of previously given individuals:

logically formulate conditions requiring the existence of infinitely many objects. This refutes Friedman's basic thesis.[69]

Nevertheless, I shall accept his unwarranted restriction to first-order logic until further notice, as it focuses the discussion on the core issue of Kant's theory: the elementary forms of representing and reasoning about combinations, relations, and orderings of individual objects, and the basic modi of unifying, manipulating, and generating concretely given manifolds. It is also conceivable that although Kant did not exclude higher-order quantification or even the possibility of logically representing some infinite structures, he nevertheless considered his logical framework inadequate for representing the specific operational and relational concepts of mathematics. In that case Friedman's reconstruction of Kant's theory of mathematics might remain adequate even if (the unrestricted formulation of) his basic premise proves untenable.

We thus return to the method of defining concepts exemplified by (R1) – (R3). Such representations depend on logical forms containing ineliminable quantifier-dependencies like "for any x, there exists a y such that $-x-y-$" in (R3). The second quantifier requires, for every admissible instantiation a of x, the existence of an object b relating to it in the manner specified in "$-a-b-$", and this dependence between individuals instantiating the quantified variables is generally not eliminable by logical transformation.[70] As Friedman points out, albeit with an incorrect example,[71] in monadic first-order logic such dependencies cannot occur: $\forall x \exists y(Px \to Qy)$ is equivalent to $(\forall x \neg Px) \vee (\exists x Qx)$, where $\exists x$ does not depend on $\forall x$, and generally, any quantifier nesting can similarly be eliminated.

$\forall\exists$-propositions are central to mathematical concept formation because they allow to abstractly represent one defining formal property of functions and operational rules, the existence of a value/output for any admissible argument/input (the other defining property is of course uniqueness).[72] The major thesis

> Jede relationale Struktur, die über einem endlichen Bereich empirischer Gegenstände definiert ist, die Grundsätze des reinen Verstandes erfüllt und somit ein *endliches Modell seines* [Kants] *Naturbegriffs* darstellt, darf wiederum als *logisches Individuum* fungieren, das zum empirischen Anwendungsbereich des Naturbegriffs zählt. [...] Was als einzelnes Erfahrungsobjekt, was als endliche Gesamtheit *mehrerer* Erfahrungsobjekte zählt, ist vom Standpunkt der kantischen Theorie der Erfahrung aus willkürlich. [Note that although Falkenburg speaks of a "theory of experience" and an "empirical domain of application" here, she is explicitly discussing the logical framework of Kant's theory in these passages!] (Falkenburg [25, Ch. 5.3, 5.4, esp. pp. 197-9])

The correct interpretation of this alleged "ambiguity", which we discuss below, is simply that Kant had a conceptual totality operator ("Allheit", B 111) in his transcendental logic, which allows to address a multiplicity (structured or unstructured) as *one* object ("Vielheit als Einheit", *ibid.*). But he placed strict limitations upon which multiplicities may actually be instantiated as unified totalities (as opposed to open multiplicities).

For another view on Kant's relation to higher-order logic, comp. Voullemin [87]. Here too, Kant's (partial) rejection of principles stated in terms of higher-order logic is not explained by his inability to logically formulate (or "think") such principles, but by his particular ontological and epistemological restrictions on the conditions of objective knowledge.

[69] That the restriction of Kant to first-order logic is also problematic for other reasons was emphasized in William Tait's discussion [82] of Friedman's interpretation.

[70] "Ineliminable" is thus to be taken in the wider sense that the expressed dependency-relation, rather than the precise formulation, is not eliminable by logical transformation. For example, in a classical logic with LEM, $\forall x \exists y\, Qxy$ is of course equivalent to, say, $\neg\exists x \neg\exists y\, Qxy$ ("for any x there is a y such that Qxy" is equivalent to "there is no x for which there is no y such that Qxy") — but this does not change the essential character of the dependency between the bound variables.

[71] His example at [30, p.63], reproduced by Falkenburg [25, p.326 fn.43], that $\forall x \exists y(Px \to Qy)$ is logically equivalent to $(\forall x Px) \to (\exists y Qy)$, is blatantly false. The first sentence requires the existence of a Q already if only a single object in the domain of discourse satisfies P, while the second requires the existence of a Q only if all objects in the domain are P.

[72] And more generally, because the combination of $\forall$ and $\exists$ over polyadic matrices allow to abstractly represent extremely highly complex dependency relations and structures in purely logical terms. For set theory, we only need one binary relation and increasingly deep quantifier nesting to generate representations of structures of arbitrary complexity.

of Friedman's reconstructions both of Kant's theory of geometry and of arithmetic is that the logical form of propositions expressing the formal properties of such mathematical objects (functions), respectively the form of well-defined operational rules, and crucially, *the methods of reasoning with these notions*, could not be "thought" in Kant's logic or in terms of the "pure concepts of the understanding" based on it. The impossibility of logically representing infinity is just an important symptomatic consequence of this fundamental restriction.[73]

1.2.3 More precise determination of Friedman's position: concept-formation and reasoning vs. objective realization

The construction postulates of classical geometry are Friedman's paradigm cases of this thesis. Thus,

> the existential proposition corresponding to the construction — that for any point and any line there is a circle with the given point as center and the given line as radius — cannot be conceptually expressed for Kant. In mere syllogistic logic, this existential proposition cannot, strictly speaking, even be stated (as we would now put it, it involves the form of quantificational dependence $\forall\forall\exists$). The only way even to think or represent this proposition — so as, in particular, to engage in rigorous geometrical reasoning thereby — is by means of the construction itself. (Friedman [30, p.126])

Before turning to the "construction itself", let us consider what "thinking", "representing", and "engaging in rigorous reasoning" with such a proposition amounts to. Friedman's claim is *emphatically not* that an intuition of the constructive operation is needed merely to ensure the *truth* of the proposition (*that it is in fact the case that* for any point and line there exists a circle) — it is required, rather, "even to think" the proposition. Thus, Kant could not hypothetically formulate the propositions and derive its logical implications, *and in an independent, separate act* determine (perhaps by appeal to intuition) whether what it states is in fact the case under some interpretation of the words "circle", "center", etc. Concerning "reasoning", Friedman is explicitly[74] saying that for Kant, inferences of the following form (corresponding to applications of operations on "given" objects of two types) were not logically valid:

(C1) For any point $\boldsymbol{x}$ and any straight line $\boldsymbol{y}$, there exists a unique circle with center $\boldsymbol{x}$ and radius $\boldsymbol{y}$.

(C2) $\boldsymbol{a}$ is a point, $\boldsymbol{b}$ is a straight line.

(C3) Therefore, there exists a unique circle with center $\boldsymbol{a}$ and radius $\boldsymbol{b}$.

[73] To be precise: the impossibilities of logically representing functional relations and of representing infinite structures are both direct consequences of the limitation to monadic first-order logic (so it is not quite correct to say that the second is a consequence of the first). However, any representation of any type of infinite structure in first-order logic *must* make use of at least one proposition logically equivalent to a $\forall\exists$-proposition (or an even more complex quantificational dependency), unless the non-logical vocabulary contains term-forming function symbols, in which case it is possible to avoid such propositions *inside the formalism*. E.g., if the non-logical vocabulary includes a constant term $\mathbf{a}$ and a term-forming operator f, the conditions $\forall x(\mathbf{a} \neq fx)$ and $\forall x \forall y(fx = fy \rightarrow x = y)$ suffice to force any model to be infinite. However, the $\forall\exists$-dependence is then simply 'outsourced' to the syntax of the formalism containing an inductive definition that includes a rule equivalent to the informal (i.e., not formalized) $\forall\exists$-sentence "for any term t, there exists a term ft".

[74] Friedman [30, p.82f.].

(C1) involves an expression, "circle with center x and radius y", that does not have the logical form of a monadic predicate, thus could not be thought in terms of "pure concepts" or represented in formal logic. Recognizing the inference as *valid* in virtue of its logical form was therefore impossible (which does not preclude that Kant recognized this particular inference as *correct* in another way, though not on formal-logical grounds). Clearly, the geometric content of the inference (C1), (C2) ⇒ (C3) is irrelevant here. According to Friedman's view, Kant would deny the logical validity of the following inference for exactly the same reasons:

(O1) For any object x of type A, there is an object y of type B such that y is related to x in such-and-such a way.

(O2) a is an object of type A.

(O3) Therefore, there is an object y of type B such that y is related to a in such-and-such a way.

The crucial distinction is between, *on the one hand*, conceptually representing a certain content, e.g., a feature of some relation as expressed in (O1) (not some interpretation of "in such-and-such a way", but the feature of a relation that for each A there is a B standing to it in that relation), or looked at differently, the property ("external determination"[75]) of objects of type A that each has a B related to it; and, *on the other hand*, the evidence that the thus represented propositional content is actually realized. In Kant's parlance, the former is to "think" the concept of a certain kind of relation, property, "external determination", or structure; the latter is to "cognize" the "objective reality" of that concept.[76] The former concerns the means of *concept formation* and *general representation* of objectual relations; the latter concerns the *existence* of objects exemplifying them and our *knowledge* thereof.

Friedman speaks strictly in the former sense when claiming Kant's inability "even to think or represent" relational concepts without appeal to intuition: not just the objective reality but the form of the concept itself presupposes intuition,[77] and this concerns not just the meaning of (C1) with "point", "line" etc. taken in the standard interpretation, but also the content of (O1) about a formal property of an unspecified relation. It is also not the merely psychological observation that we *de facto* could never make ourselves the conception, say, of a circle without spatial intuition — rather, it is the more basic logical (or 'meta-logical') claim that even given such an initial intuitive acquaintance, we could never move beyond the intuitive level to an *abstract description* of its formal-relational properties. Kant could thus never recognize (O1), (O2) ⇒ (O3) as logically valid, for that amounts to moving beyond any intuitive interpretation of "y is related x in such-and-such a way" to regarding the formal-relational content of (O1) in abstraction from this original interpretation, *which is precisely what he allegedly could not do*. Any abstraction allowing to recognize the correctness of (O1), (O2) ⇒ (O3) independently from a concrete intuitive interpretation of "y is related to x ...", that is, to recognize its logical validity, is fundamentally excluded. This distinction between abstract logical form and intuitive realization is precisely what Kant simply could not draw.

Consequently, Kant could not conceptualize most logically possible structures and reason about them hypothetically; for example, non-Euclidean geometries were for Kant not merely wrong descriptions of

[75] B 361ff.

[76] B 116, B 146 - 150, B 150, A 155f. / B 194f..

[77] Friedman [30, p.123].

intuitive space, not merely un*imagine*able as the ambient space of our life-world, but literally un*think*able: "Kant has no notion of possibility on which both Euclidean and non-Euclidean geometries are possible."[78] Whereas *our* notion of 'logically possible' could be characterized (Friedman claims) by "all models of consistent first-order (or second-order) theories", for Kant no such means were available. (There is the risk of conflating an alleged logical impossibility of the concept of an abstract structure described by the axioms of a non-Euclidean geometry, with uninterpreted primitives, and the incapacity of imagining a *space* with this structure; for Friedman's Kant, this distinction did not exist, an consequently, the failure of imagination is identified with the alleged conceptual impossibility. In reality, Kant rigorously distinguished between conceptual (in)consistency and intuitive (un)imaginablity,[79] and repeatedly emphasized the contingency of our having just the forms of intuition we do.[80])

Friedman identifies the "schemata" of mathematical concepts as the key notion of Kant's theory of mathematical reasoning: they "not only serve to contribute towards the objective reality of such concepts, but are also essential to our rigorous representation of the concepts themselves",[81] providing in particular "the possibility of a kind of rigorous reasoning with" them.[82] A schema is an intuitive representation, originating in our spatio-temporal imagination, of a "general procedure for producing" any and all instance of a certain concept; thus, "the schema of the concept of a triangle is just the Euclidean construction of the triangle of Proposition I.22."[83] While we can today distinguish between a concrete capacity of operating in a certain way (note that this capacity is still *general* relative to the individual acts it makes possible) and the abstract rule-form of which this capacity is an instance, Friedman's Kant had to — and did — defend the *complete identity* of the capacities of the productive imagination and the general rule-concepts describing constructive operations: the schemata, literally and simply, *are* these rules.[84] For it was just this distinction between an intuitive capacity to operate in a certain way, and the rule taken as an abstract conceptual or logical description of this way of acting, that was unavailable. We show below and in chapter 3 that such a reading completely misses Kant's reason for introducing schemata in the first place: after conceptualizing "general representations of synthesis" (including rules of operation) in terms of pure concepts of the understanding, he then required a clarification of the *conditions of their applicability*. The careful distinction between, on the one hand, logical representations of structures and rules (explicitly including specifications of algorithms) and, on the other hand, the conditions of possibility of their application for the cognition of real objects, is at the very heart of the argument in the "Transcendental Analytic", the first part of the "Transcendental Logic".

1.2.4 What is "transcendental logic"?

Transcendental logic, as the foundation of Kant's critical epistemology, is supposed to systematically present and justify the *a priori* conceptual frame for the representation of and reasoning with the "concept

[78] Friedman [30, p. 93].
[79] Inaug. Diss.
[80] below
[81] Friedman [30, p.123].
[82] Friedman [30, p.126]
[83] Friedman [30, p.124].
[84] Comp. Friedman [32, pp.236-9].

of an object in general",[85] including the objects of pure mathematics. In this and the following section, I sketch Friedman's and my own interpretation of it.

It is crucial to understand how Kant determines the relationship between transcendental and "general logic".[86] General logic deals with the logical forms of propositions and inferences and "abstracts from all content".[87] Its "highest principle" is the principle of non-contradiction,[88] which acts as the universal *negative* truth-criterion for all judgments,[89] as well as the principle "according to which all inferences of the mathematicians", and generally all apodictic inferences, "proceed" (Kant's point is that the *validity* of an inference is based on the principle of contradiction; this has nothing to do with whether the premises or conclusion by themselves are truths of logic).[90] The key tasks of general logic are thus to exhibit the basic forms of concepts, propositions, and inferences, and to determine whether a given proposition is contradictory, consistent, or tautologous (its negation contradictory) and whether a given inference is logically valid (its premises and the negation of its conclusion contradictory).[91] The relation of logical entailment is defined in terms of the notion of contradiction.

Unlike general logic, transcendental logic concerns a specific content, "the rules of the pure thinking of an object",[92] the elementary forms of conceptualizing and reasoning about the structural and relational determinations of objects in the widest sense. As a fundamental principle of Kant's epistemology, the "relation to the object"[93] in any cognition requires intuition (recall that "intuition" should not be read in a psychologistic sense of 'gut instinct', but in the formal sense of 'immediate relation to the object'; we specify this below). This means, not that cognition by abstract reasoning about objects is impossible, but that reference to these objects and the basic evidences for non-tautologous judgments about them must ultimately be grounded in an intuitive base (either empirical perception or *a priori* intuition). Scientists may develop highly abstract theories, but for these to become genuine cognition, their existential assumptions and the basic non-tautologous determinations of their objects must (in *some* to-be-specified) way be grounded in intuition. Hence, the basic notion that transcendental logic must develop is the concept of an "object of intuition in general".[94]

To explicate Kant's idea of a transcendental logic and determine its position in the system of the *Critique*, it is thus necessary to determine more precisely its relationship to general logic on the one hand, and to the theory of the "pure forms of intuition" space and time ("Transcendental Aesthetic") on the other hand. In the next subsection, Friedman's proposal will be discussed and rejected. In the following section, my own interpretation will be sketched (detailed arguments for its central claims follow in chapter 3).

[85] Comp. quotes below, A 55 / B 80.
[86] A 50-64 / B 74-88.
[87] A 54 / B 78.
[88] A 151ff. / B 190ff. For an in-depth discussion of the status of the principle of (non-)contradiction in Kant's theory, see Wolff [90, ch. 2].
[89] A 59 / B 84, A 151ff. / B 190ff.
[90] B 14.
[91] A 303 / B 359.
[92] A 55 / B 79f.
[93] A 55 / B 79f.
[94] B 150.

1.2.5 Friedman's reconstruction of transcendental logic

A seemingly plausible account of the relationship between general logic, transcendental logic, and the transcendental aesthetic is suggested by a misreading of *CPR* §10.[95] Kant appears to claim that the "content" of the "pure concepts of the understanding" (the pure categories of transcendental logic) depends on the *a priori* intuition of space and time.[96] Taken together with Kant's explicit reference to the Aesthetic, this easily leads to the conclusion that *transcendental logic is general logic supplemented with contentual representations based on a priori intuitions of spatio-temporal relations or procedures* — on Friedman's reading, identifying general logic with FOMPL, that it is the formal theory of monadic syllogistic logic enriched by predicates or operators whose objectual interpretation *and* formal behavior derive from the pure intuitions of space and time and are irreducible to the forms of general logic. This seems to be further supported by Kant stating that the cognition of objects always requires a "synthesis", an "operation" or "action of putting different representations together with each other and comprehending their manifoldness in one cognition":[97] the emphasis on its active or "spontaneous"[98] character easily suggests that synthesis is an essentially temporal procedure.

Friedman's reconstruction is based on this (as we shall see false) interpretation. I will try to spell it out in some detail. (If the following seems far too formalistic and reliant on the modern notion of a rigid formal calculus to the reader, I can only answer that this is the only way I see to make sense of what Friedman actually says, if one tries to spell it out in detail; that Friedman indeed tends to confuse the modern notion of a formal system with abstract conceptualization will be argued in section 1.3.6.)

Friedman equates transcendental logic to first-order monadic predicate logic supplemented with special term-forming operators (corresponding to intuitive actions of spatio-temporal manipulation) which (*i*) may be applied to given individual terms and (*ii*) whose variables may *not* be bound by quantifiers. Though they play similar grammatical roles, these operators are *not* direct analogues of the schematic predicate and function symbols of modern logic. The latter are (*i*) open for re-interpretation and (*ii*) their formal properties (syntactic rules and admissible interpretations) are independent from their properties definable inside the system (thus a modern two-place predicate symbol like '*R(x,y)*' must always stand for a binary relation, but whether to each object another one stands in that relation depends on whether this is explicitly stated *within* the formalism). In both regards, the additional operators are analogous, rather, to the logical constants ('all', 'not', etc.): their interpretation as well as their specific properties and behavior are not definable *inside* the formalism but are constitutive of it, remaining invariant under all its admissible applications.[99] On this account, spatio-temporal intuition is thus not a particular field of *application* for an abstract theory-form, providing it with a 'model' by making true its axioms under the appropriate interpretation of the non-logical symbols; rather, it is an additional "form of rational representation", of

[95] A 76ff. / B 102ff.

[96] This is due to an ambiguous use of "content": in the present passage, Kant actually means the "reference [Beziehung] to the object", rather than the propositional or conceptual content, the representation of the basic structural properties of an object. His claim, as we will see, is that the *application* of the pure concepts to actual objects requires intuition, but that their conceptual content is not dependent on any particular kind of intuition.

[97] A 77f. / B 103.

[98] A 68 - B 93, A 77 / B 102, B 130.

[99] We here ignore the crucial philosophical debate about the semantics of the logical constants, e.g., whether they ought to be interpreted model-theoretically or proof-theoretically.

"rational argument and inference" supplementing the inadequate means of monadic logic.[100] By acting like Skolem-functions,[101] the 'intuitive constants' operators allow for some representations of relational structure unavailable in purely monadic logic.

Crucial to Friedman's reconstruction is that there are no "true quantifiers";[102] that is, in his reconstruction of Kant's theory, no logical quantifier may bind the variables in the intuitive operators. Whereas in polyadic logic one could thus represent the idea of infinite divisibility by the sentence

' For all x there is a y such that y is a proper part of x ',

in Kant's reconstructed theory, the same idea is conveyed by something like

'$f_B(x)$ is a proper part of x ',

where $f_B(x)$ stands for a bisection operation yielding, say, the left-most half segment of any straight line substituted for x.[103] While in the former, the combination of the quantifiers 'does the work' of representing the iterative structure of infinite division, in the latter it is emphatically not the logical form of the sentence (the concept in question is "not capturable by a formula or sentence"[104]), but the specific *logically irreducible* meaning of the operator $f_B(x)$ in combination with the equally irreducible intuitive capacity to repeat its application. No quantifier may bind the free variable x. Generality and existence are expressed by the substitutability of any singular term of appropriate type, i.e., an operation *on* the formalism and not a proposition formulatable *inside* it. This mirrors the thesis that the well-definedness of functional operations was not logically or theoretically representable by Kant, but constituted an irreducibly *intuitive* capacity (restricted to a few elementary operations), a transcendental condition of theoretical cognition. Infinity is not represented by any logical form ("capturable by a formula or sentence"), but by the intuitive capability of arbitrarily iterating a concrete action: for a given a singular term t, we may apply the operator to get the singular term '$f_B(t)$'; applying it again, we get '$f_B(f_B(t))$', '$f_B(f_B(f_B(t)))$', and so on.

On Friedman's reading, intuition thus plays two roles. Firstly, to determine the specific content of these operators: $f_B(x)$ refers to the intuition of Euclidean bisection and cannot be reinterpreted as, say, a non-Euclidean operation or some function from an all-together different domain. And this content cannot be further analyzed, so that, e.g., the abstract concept of dense order could be regarded independently from the spatial intuition of infinite divisibility, or on a more general level, that the formal character shared by the geometrical bisection function, the arithmetical successor function, and the biological 'father of x' function could be regarded *in abstracto*.[105] Secondly, to ground the general capacity of *operating with* these operators, especially the possibility of iterating them (this being the specific contribution of temporal

[100] Friedman [30, p.94-5]
[101] Friedman [30, p.65], comp. Enderton [24, pp.287ff.].
[102] Friedman [30, 113 fn. 31].
[103] Friedman [30, p.71 and fn. 20]
[104] Friedman [30, p.66]
[105] Note that this formal similarity is, in the first place, their *functionality*, i.e., that for each element in their domain there exists exactly one element etc.; the iterability of functions, which Friedman surprisingly calls "the general form" that is "common to all functional operations", and which supposedly is based on temporal intuition, is really *not* identical to functionality. There are functions that are not recursively definable, and there are functions that are not arbitrarily iterable as they may assign values outside their domain of definition: $f(x) = \frac{1}{1-x}$ for example cannot, beginning with $x = 0$, be iterated more than once. This was certainly not beyond Kant's ken, and at any rate illustrates that functionality $\neq$ iterability. (Iteration is in fact more closely connected to the notion of well-foundedness.)

intuition), which, to repeat the crucial point, cannot be theoretically represented and reasoned about *inside* the formalism ($\cong$ Kant's frame of theoretical cognition), but is a constitutive "transcendental" condition of it.

All of this amounts to an awfully complicated way of saying that Kant (consciously yet obscurely) denied the possibility of *logically* describing complex relations (or "external determinations"), functionality, infinity and (potentially) infinite processes via nested generalities, and that he therefore made pure intuition play a role analogous to (though a lot less expressive than) the combinations of quantifiers in modern logic. Apart from the soon to be demonstrated fact that it is simply untenable on exegetical grounds, the basic problem with this interpretation is that, once one strips away all the formal machinery, it amounts to saying that Kant would not recognize the *purely logical* form shared by the propositions "every human has a mother", "every effect has a cause", "every natural number has a predeccessor" (but *not* shared by "every apple is poisonous" or "every human is an animal"), nor the *logically uniform* facts that these sentences, together with the additional assumptions that their respective subject-terms are non-vacuous and that "every mother of a human is a human", "every cause is an effect", and "every predecessor of a natural number is a natural number", respectively imply that either there are infinitely many humans, effects, and natural numbers preceding any given natural number, or else that some humans are their own ancestors, some effects their own causes, and some natural numbers precede themselves.[106] Kant *did* recognize these forms and consequences as *purely logical* and *not dependent on intuition*. Friedman's position, at base, reduces to denying this simple and ultimately rather plain fact.

1.3 Refutation of Friedman and alternative account of transcendental logic

With his reconstruction, Friedman solves the problem of determining the relationship between general logic, transcendental logic, and pure intuition, and provides an explanation of Kant's "synthetic *a priori*". But unfortunately his solution is untenable: it is simply not the case that Kant's conception of transcendental logic is that of monadic logic plus the pure intuitions of space and time. In section 1.3.1 we introduce Kant's notion of synthesis and synthetic unity, and formulate three basic thesis about the transcendental logic. Section 1.3.2 discusses the relationship between the concepts of synthetic unity and the logical functions of judgment. Section 1.3.3 explains the role of intuition in the transcendental logic. The following section sketches some ideas of Kant's logical theory, including his account of the logical representation of algorithms. The remaining sections discuss the role status of infinitary concepts.

1.3.1 Synthesis and synthetic unity: general representations of operations and the independence and exhaustiveness of the primitive concepts of structure

The categories are the "primitive" concepts of *synthesis* or *combination*, i.e., the *composition* of a complex whole from simple parts (where what is taken as simple is at this point open) and the *connection* of

[106] Note that the distribution of truth-values across these premises and conclusions is not uniform, but the form of the respective consequence-relation is.

elements into a relational nexus.[107] The word "synthesis" is used in this wide and in a more narrow sense: in §10, *synthesis* narrowly construed is differentiated from *synthetic unity*, the act of generating structure from the structure itself; it is then claimed that general representations of synthetic operations *presuppose* concepts of synthetic unity. §15 further discusses this act/object ambiguity: the combination – say of a manifold of discrete individuals into one aggregate or into one sequential order – might refer to the *act* of combining or to its *product*, the generated unity, relation or order regarded independently from the act. Kant claims that "all combination", the synthesis of *any* manifold (intuitive or not) into a unified structure, is always an operation of the understanding, but then (playing with the ambiguity of the word "Verbindung") emphasizes:

> But in addition to the concept of the manifold and of its synthesis, the concept of combination also carries with it the concept of the unity of the manifold. Combination is the representation of the s y n t h e t i c unity of the manifold. The representation of this unity cannot, therefore, arise from the combination; rather, by being added to the representation of the manifold, it first makes the concept of combination possible. (§15, B 130f.)

§10 and §15 announce the basic theses of Kant's theory of pure concepts: (*I*) general representations of synthetic operations (and reasoning with them) *presuppose* elementary concepts of synthetic unity; (*II*) these are "pure" or "abstract" concept based entirely in the understanding, *independent* from specific forms of intuition; (*III*) the pure concepts are *exhaustive*, i.e., every type of objectual structure and every type of operation of combination that is thinkable at all is definable in terms of these primitive concepts. In the following subsection we sketch Kant's elaboration of these theses (relegating the more involved exegesis of his arguments to ch. 3).

Remarks: ad (*I*). (*I.i*) The minimal requirements for "generally representing" a synthetic operation are the specification of the type of the given manifold and of the to-be-generated relation from or amongst its elements or parts (their synthetic unity). In general it is *neither* necessary that such definitions specify a step-by-step procedure (algorithm) for bringing this relational unity about, *nor* that the synthesis operate on a finite manifold or that it could even in principle be completed by a finite procedure (both of these additional requirement are however crucial components of the *cognition* of real objects). See 1.3.3, 1.3.4 and 1.3.5 below. (*I.ii*) the definability of synthetic operations in terms of structural concepts implies that in mathematics, the content of "practical" of "technical" propositions referring to (possible) actions generating objects (including algorithm specifications) are contentually reducible to "theoretical" propositions about properties and relations of objects independently from such actions, see again 1.3.4. This is also reflected in Kant's notion of "mathematical construction".

1.3.2 The primitive concepts of synthetic unity are found by reflecting on the conceptual presuppositions of the logical functions of judgment

In this section we discuss the idea behind Kant's notion of the pure concepts of the understanding or the "categories, which are the primitive concepts of objectual structure or synthetic unity.

[107] B 201f. fn., also comp.:

> The relation of the many amongst each other, insofar as they are comprehended into one, is the **combination**. (18:343)

The basic concepts are found by reflecting on what is conceptually presupposed by the "logical functions of judgments" (the quantifiers, modes of affirmation and negation, propositional connectives, and modal operators).[108] Frede and Krüger illustrated Kant's struggle to provide an adequate interpretation of the particular quantifier "some" (with some lenience one could say that he was oscillating between the function nowadays assigned to '∃', "there is at least one", and that of a plural quantifier roughly similar to those discussed by George Boolos, e.g., "*some* critics admire only each other").[109] Later we also encounter Kant's subtle discussion of the semantics of the universal quantifier. Given this apparent uncertainty about the meaning of the logical constants, one may doubt the stability of the "metaphysical deduction" of the categories from the functions of judgment, but for our current purposes this worry is secondary. I will now describe Kant's idea with reference to the concepts most relevant to our discussion, the categories of quantity.

In a judgment of the form 'all x of type A are B' — the explicit use of both the individual variable x and its type A is Kant's[110] — the functions of universal quantification and categorical predication have a determinate content invariant under all admissible interpretations of 'A' and 'B'. Understanding the quantifier presupposes the notions of an *individual* as subject of predication, of a *plurality* or *multiplicity* of distinct yet (in respect to type 'A') *homogeneous* individuals, and of a *totality* or *class* of such individuals as the domain of quantification. These are elementary (irreducible) concepts for the description of objectual structure. In general logic they remain in the background, so to speak, *implicitly presupposed* by the logical functions but not entering explicitly as contentual determinations of the objects of discourse. In transcendental logic they become thematic as contentually interpreted predicates of synthetic unity, allowing to *explicitly* refer to objects *as* individuals, or *as* multiplicities or totalities of individuals, or *as* elements of a class of objects of some type.[111] Thus, 'a is an individual of type B', 'b is a multiplicity of individuals of type B', 'c is an element of the totality d', and 'for each totality of individuals of type A, there is an individual of type B in that totality' are meaningful propositions of transcendental logic, involving the primitive categories of quantity.

Kant sometimes differentiates the categories, as concepts of an object in general, from the logical functions by the fact that the former are referred to an "object of intuition in general" whose synthetic unity, i.e., its structural properties, they represent. After a brief digression, I will discuss what Kant means by this, and whether or to what extend this makes transcendental logic dependent on his theory of pure intuition.

Digression: The categories as the "elementary", "original", "primitive", or "ancestral" concepts of the understanding are themselves not definable in terms of more fundamental notions.[112] Incidentally, they are in this regard analogous to Gödel's concept of set:

> The operation "set of x's" (where the variable "x" ranges over some given kind of objects) cannot be defined satisfactorily (at least not in the present state of knowledge), but can only be paraphrased

[108] §10 A 70 / B 95ff., §15 B 131, §16 B133f. fn., §19 B 104f..
[109] Boolos [10].
[110] See below 2.1, chapter 6.
[111] Here I follow the discussion of the category of quantity in Michael Frede and Lorenz Krüger [28].
[112] A 81f. / B 107f., B 109, A 241ff.

> by other expressions involving again the concept of set, such as: "multitude of x's", "combination of any number of x's", "part of the totality of x's".[113]

Gödel (himself an avid reader of Kant) pointed out this analogy:

> Note that there is a close relationship between the concept of set explained in footnote 11 [previous quote] and the categories of pure understanding in Kant's sense. Namely, the function of both is "synthesis", i.e., the generating of unities out of manifolds.[114]

Gödel's biographer and philosophical interlocutor Hao Wang further develops this Kantian theme in the influential chapter "The concept of set" of [88], reprinted in [89], where he states that the "range of variability" of a multitude must be "in some sense intuitive"; in order to "form a set" from such a manifold we "look through or run through or collect together" its elements [89, p.531]. This is an almost verbatim paraphrase of Kant's explanation of synthesis, that a manifold of intuition must "be gone through, taken up, and combined" and its various elements "collected together" into one (A 77 / B 102f). The analogy reaches even a bit further: like Gödel's iterative conception of set, the categories must be applied to manifolds that are assumed as prior given (see below; Gödel assumes a base level of discrete individuals, Kant mere unstructured manifolds from which representations of individuals must be synthetically generated); and the thus generated unities are again possible objects for further combination. At the same time, the categorically definable syntheses and types of structure are not restricted by some particular way in which manifolds are given, say spatio-temporally, allowing for concepts that far transcend what is in principle exhibitable in space and time. But this is also where the analogy ends: while Gödel affirmed an intuition of infinite sets and operations on them, Kant required for the *objective reality* of concepts (i.e., that they can justifiably be taken to refer to actual objects) that what they represent be exhibitable as finite objects in at most potentially infinite domains. Concepts of absolutely infinite structures were not for him *unthinkable* (as they were explicitly definable in terms of the categories), but the assumption that their object *exists* lacked any justification: Kant consequently rejected them as "constitutive" concepts of real objects, although he still assigned them a crucial role as "regulative" instrumentalist principles for the systematic unity of scientific theories (prefiguring Hilbert's treatment of "ideal objects", who often referenced Kant in this context).

1.3.3 "*Synthesis intellectualis*" and the "concept of an object of intuition in general" — the categories are contentually independent from spatio-temporal intuition and representationally exhaustive

We now turn to an aspect of Kant's explanation of the categories that may appear to contradict the above explanation: the category is defined as a primitive "concept of an object of intuition in general". This might seem to contradict the above thesis (*II*) that they are based entirely in the understanding, independently from all forms of intuition. We now show that no such a contradiction exists.

By "intuition" ("Anschauung", "intuitio") in the widest sense, Kant means *immediate* and *singular representations* of individual objects, in contradistinction from "concept" as mediate and general.[115] Scholars disagree about the relationship between the immediacy and singularity criteria.[116] In my opinion, Kant considered them materially equivalent: immediate acquaintance with something is acquaintance

[113] Gödel [37, p. 475, fn. 11]. This footnote is garbled in the revised and expanded version in [70, p. 470ff.] (without effect on content), comp. the original in the *American Mathematical Monthly*, 54 (9), 1947.

[114] Gödel [37, p. 484, fn. 26].

[115] A 320 / B 377, (9:91, §1).

[116] Hintikka [45], [46], [47], [48], in opposition to Parsons [64]. See the introduction in Posy [68] for the context of the debate.

with just *this one* thing (imm. ⇒ sing.); and representations containing no immediate relation to objects (by exhibition, production, or deixis) cannot numerically individuate but at most determine structural or qualitative identity of objects (sing. ⇒ imm.).[117] In any case, to use Parsons' slogan, Kantian intuition is in the first place "intuition *of*" ("intuition de re"), not "intuition *that*" ("intuition de dicto"):[118] its primary function is to provide an immediate contact (or at least the real possibility of such contact) with concrete objects; only secondarily does it ground propositional knowledge about their properties or other facts (we return to this at the end of this section).

We saw that the categories are based on the same primitive notions of combination also presupposed by the logical functions of judgment. The specific difference between them is that the categories refer these notions to an "object of intuition in general", their "transcendental content" consisting in "the synthetic unity of the manifold in intuition in general, on account of which they are called pure concepts of the understanding that pertain to objects *a priori*".[119] It might easily seem that "intuition in general" refers to the *a priori* forms of human sensibility, space and time (on these more below), but this is certainly not the case: the essential characteristics of "intuition in general" are the immediacy and singularity criteria discussed in the previous paragraph *and nothing more*, as I will now show.

In *CPR* §21 Kant recapitulates the first step of his argument to establish the pure concepts as the necessary pre-conditions of all possible cognition. In this argument,

> since the categories arise **independently from sensibility** merely in the understanding, I had to abstract from the way in which the manifold for an empirical intuition is given, in order to attend only to the unity that is added to the intuition through the understanding by means of the category.[120]

The concept of this unity of composition, "(including its inverse, the simple) is a concept not abstracted from intuition" but "a basic concept [Grundbegriff], notably *a priori*, ultimately the only basic concept *a priori*, which originally lies in the understanding as the ground of all concepts of objects of sensibility". There lie "as many concepts *a priori* in the understanding [...] as there are types of composition (synthesis)".[121] Only from one, single point, §21 continues,

> I could not abstract in the above proof, namely, from the fact that the manifold for intuition must be **given** prior to the synthesis of understanding and independently from it; how, however, is here left undetermined.

Transcendental logic furnishes the elementary concepts and rules of reasoning about the composition and relations of objects in general, the logical means to define the *content* of all theoretical knowledge (what could be called its 'formal' or 'structural content') in abstraction from *how* objects are given to us: it

[117] Kant thus denied the possibility of definite descriptions containing only general terms. A 263-4 / B 319-20, A 271-2 / B 327-8. Comp. Falkenburg [25, ch. 3], and Parsons [64, §1].

[118] Parsons [62], [65, p. 139].

[119] A 79 / B 104f.

[120] B 144f. For an account of the "transcendental deduction" consistent with the present discussion,see Dieter Henrich [40], [41] and [43]. For a thorough discussion of the literature about the "transcendental deduction of the pure concepts of the understanding", see Konrad Cramer [20] and Tilmann Pinder's critical review [66] thereof.

[121] Preisschrift

"concerns the cognitions of the understanding regarding their content, but independently from how objects are given" (17:651).[122] It merely presupposes *that* immediate and singular representations of individual objects can be given for application; but the pure categories

> – or predicaments, as they are also called – do not presuppose a specific kind of intuition (as for example the only kind possible to humans) such as space and time, which is sensible; the categories are forms of thought for the concept of an object of intuition in general, of whatever kind that intuition may be, even if it were a supersensible intuition of which we [humans] cannot have a specified concept. For we must always make ourselves a concept of an object through the pure understanding, if we want to judge something *a priori* of it, even if we later find that it is transcendent and cannot be given objective reality — thus the category by itself is not dependent on the forms of sensibility, space and time, but may also have other, for us not at all thinkable, forms for its application. (*Preisschrift*)

I suggest to interpret this as follows: "intuition in general" plays a logical role analogous to that of uninterpreted singular terms.[123] It plays the role of (as of yet merely hypothetical) concrete individuals in concept-formation, and is not tied to a particular form of intuition. The instantiation or existence of such concepts must in the final analysis be ensured extra-conceptually, by interpreting the individual terms as referring, e.g., to objects of spatial intuition. Insofar as we merely *define* (or describe) the formal synthetic unity of structures (what we suggested to call the 'formal content' of a theoretical cognition), "mak[ing] ourselves a concept of an object through the pure understanding [in order to] judge something *a priori* of it", the forms of human intuitions are not yet required. Only to ensure the "objective reality" of the concept, by exhibiting an instance, appeal to intuition is essential (and for humans, spatio-temporal intuition as the *a priori* forms of our specific kind of perception, is the only kind of intuition available).

The independence of abstract concept-formation from the intuitions of space and time is discussed in more detail in the crucial §§24-26, *CPR*. The pure concepts "refer through the mere understanding to objects of intuition in general", which is there again explicitly distinguished from spatio-temporal intuition; they are thus purely abstract structural concepts of possible individual objects, the "synthesis or combination of the manifold in them purely intellectual".[124] At that level, the categories are "mere forms of thought, through which no particular object is yet cognized" — this corresponds to our interpretation of "objects of intuition in general" as uninterpreted individual terms. But, Kant continues, since in human cognition "a specific form of sensible intuition *a priori* is fundamental", the understanding can determine manifolds of sensible representations in accordance with its pure concepts of combination,

> and thus think *a priori* synthetic unity of the manifold of sensible intuition, as the condition under which all objects of our (human) intuition must necessarily stand, through which then

[122] Note the ambiguity of the word "content"; on the one hand, Kant uses it to refer to the actual relation to objects of cognition, or even these objects themselves; on the other hand, he uses it as the purely conceptual or propositional content that is said *about* such objects, which concerns their categorically definable, structural properties; it is this latter sense that is claimed to be completely independent from intuition.

[123] An interpretation along these lines was defended by Jaakko Hintikka. Comp. footnote 181.

[124] A xxx/ B xxx. Concerning the vocabulary "intellectual", it is instructive to read the first two sections of Kant's 1770 Inaugural Dissertation, in light of which the correctness of the here give interpretation becomes even more obvious.

> the categories, as mere forms of thought, acquire objective reality, i.e., application to objects that can be given to us in intuition.

On this basis he distinguishes between *intellectual* and *figurative synthesis*, both *a priori*, but the latter merely the application of the former to a specific kind of intuition:

> This synthesis of the manifold of sensible intuition[125] that is possible and necessary *a priori*, may be called **figurative** (*synthesis speciosa*), as distinct from that synthesis which would be thought in the mere category in regard to the manifold of an intuition in general, and which is called combination of the understanding (*synthesis intellectualis*)

Kant dedicates the remainder of §24 to arguing that this application to sensible intuition provides no additional conceptual, e.g., relational, content, no additional forms for defining synthetic unity: *that the definable types of synthetic unity are exhausted by the pure concepts*. These alone provide the necessary representational resources to determine structural properties of intuitive objects. For example, the concept of iteration is not something that is made possible by temporal intuition, but quite on the contrary, the pure abstract concept of a progressive series of elements ordered according to a principle of progression makes it first possible to represent temporal succession.[126] Similarly, the concepts of geometry, regarding their purely structural moment of synthetic unity, are definable in terms of pure categories. Kant actually developed this idea in some detail, as we will see in 3.3.3.1.[127] In chapter 3, I will try to show that this

[125]Recall that "sensible intuition is either pure intuition (space and time) or empirical intuition of that which, through sensation, is immediately represented as real in space and time", B 147.

[126]The reader who is skeptical about this particular example, which is of fundamental importance to the interpretation of Kant's philosophy of arithmetic, is invited to skip to section 1.4.3, especially the quotation on p. 55 f..

[127]The following sequence of quotes corroborates this interpretation very strongly. It will be discussed in more detail in the chapter 3, but I add it here for convenience.

> Space, represented as an **object** (as is really required in geometry), contains more than the mere form of intuition, namely the **comprehension** of the manifold, given in accordance with the form of sensibility, into one **intuitive** representation, so that the **form of intuition** merely gives the manifold, but the **formal intuition** gives unity of the representation. (B 161 fn.)

In German it is unambiguous that what is given "in accordance with the form of sensibility" is the mere manifold, *not* its comprehension.

> Thus even the **unity of the synthesis** of the manifold, outside or within us, hence also a **combination** with which everything that is to be represented as determined in space or time must agree, is already given *a priori*, along with (**not in**) these intuitions, as condition of the synthesis of all **apprehension**. But this synthetic unity can be none other than that of the combination of the manifold of a given **intuition in general** in an original consciousness, in agreement with the categories, only applied to our sensible intuition [where "sensible intuition" includes the *a priori* forms of sensibility]. (B 161f.)

> [T]he **necessary unity** of space and of outer sensible intuition in general, and I, so to speak [gleichsam], draw its shape in accordance with this synthetic unity of the manifold in space. This very same synthetic unity, however, if I abstract from the form of space, has its seat in the understanding, and is the category of the synthesis of the homogeneous in an intuition in general, i.e., the category of **quantity**. (B 162.)

The synthesis of apprehension

> must necessarily be according to the synthesis of apperception, which is intellectual and contained in the category entirely *a priori*. (B 162 fn.)

> If one considers the properties of the circle by which this figure unifies in a universal rule at once so many arbitrary determinations of the space within it, one cannot refrain from ascribing a nature to this geometrical thing. Thus, in particular, two lines that intersect each other and also the circle, however they happen to be drawn, nonetheless always partition each other in a regular manner such that the rectangle from the parts of one line is equal to that from the other. Now I ask: "Does this law lie in the circle, or does it lie in the understanding?" i.e., does this

interpretation is the only one consistent with Kant's arguments and theoretical principles. At any rate, it is what he explicitly and repeatedly states.

To emphasize the intimate connection, I now anticipate some of Kant's statements about the nature of pure arithmetic, which we will discuss in more detail below. They were written during precisely the same period as all of the above passages, at the time of preparation of the second edition of the *CPR*, and therefore ought to be taken to represent Kant's mature position.[128] Please pay attention to the complete agreement in terminology.

> Arithmetic [...] has actually no *quantum*, i.e., no [particular] object of intuition regarded as a quantity, but mere quantity itself, i.e., the concept of a thing in general through quantitative determination, for its object.[129]
>
> [...]
>
> Time, as you rightly remark, has *no* influence on the properties of numbers (as pure quantitative determinations)—in the way that it has, say, on the properties of any alteration (as a *quantum*), which is itself only possible relative to a specific constitution of our inner sense and its form (time)—and the science of number, despite the succession that any construction of quantity requires, is a *pure intellectual synthesis* that we represent to ourselves in thought. (Letter to J. F. Schulz, 25. Nov. 1788, 10:554ff.)[130]

The interpretation of Kant's theory of arithmetic that emerges, briefly stated, is as follows: all the concepts of arithmetic, including the specification of its computational procedures (algorithms, the general representations of arithmetical constructions in Kant's sense), are definable in terms of pure concepts of

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> figure, independent of the understanding, contain the basis for this law in itself, or does the understanding, **since it has itself constructed the figure according to its [the understanding's] own concepts (namely, the equality of the radii)**, at the same time insert into it the law that chords cut one another in geometrical proportion? If one traces the proofs of this law, one soon sees that **it can be derived only from the condition on which the understanding based the construction of this figure, namely, the equality of the radii**. (*Proleg.* §38, 4:320f.)
>
> [T]hat which determines space into the figure of a circle, a cone, or a sphere is the understanding, insofar as it contains **the basis for the unity of the construction of these figures**. The bare universal form of intuition called space is therefore certainly the substratum of all intuitions determinable upon particular objects, and, admittedly, the condition for the possibility and variety of those intuitions lies in this space; but **the unity of the objects is determined solely through the understanding, and indeed according to conditions that reside in its own nature**. (*Proleg.* §38, 4:321f.)

[128] All the quotes in the present section are from the same t the Schulz letter (1788), the second edition of the *Critique* (published 1789) and the *Preisschrift* (prepared for the 1791 essay competition) were all written in close temporal proximity and at the height of Kant's philosophical power. The complete agreement in terminology and argumentation make inescapable the conclusion that here Kant formulated his most mature thoughts. We will see that his theory of arithmetic fits seamlessly into it and provides strong confirmation of our interpretation.

[129] Kant's use of three different words for object/subject matter, "Gegenstand", "Ding", and "Object", make the traslation difficult. Here is the original: "Die Arithmetik hat ... eigentlich kein Quantum, d.i. keinen Gegenstand der Anschauung als Größe, sondern bloß die Quantität, d.i. einen Begriff von einem Dinge überhaupt durch Größenbestimmung zum Objecte."

[130] I would like to point out the striking parallel between this passage to one from W. Tait's analysis of finitism:

> We are considering the generic form of a finite sequence, Number. We discern finite sequences as such in our everyday experience, and this is what gives meaning to Number in the broad sense: it is the source of our ability to apply the number concept. But Number also has a purely formal content, independent of our experience. This is why the number concept (in contrast with the concept of motion, for example, which also derives from a kind of structure discerned in experience) is a part of mathematics. [83, p. 26].

the understanding. The form of arithmetical synthesis is purely intellectual, and derives in no way from particular forms of intuition, e.g., the intuition of temporal progression (rather, conversely, the abstract notion of progression or iteration necessary to conceptually *determine* temporal intuition according to its form is a pure concept, logically prior to any particular form of intuition). Nevertheless, intuition plays a crucial role in *instantiating* numerical concepts (and *executing* algorithms), and thereby substantiates their existential presuppositions. Equations like "$5+7=12$" or "$\sqrt{\sqrt[4]{6561}}=3$" are *statable in purely categorical, non-intuitive*[131] *terms*, but they are *not truths of logic*, since, no matter how one analyzes the involved concepts, some exhibition or existential assumption (if only about a 'possible existence') must come in in their evaluation. This interpretation will be explicated and defended below.

We now return to the task, left at the beginning of this section, of characterising Kant's notion of pure intuition developed in the Transcendental Aesthetic (which is more specific than the "intuition in general" referred to by the pure categories).[132] Space and time are not general concepts but pure intuitions, singular representations *a priori* (there is only one space, only one time).[133] They are not properties, relations, or 'containers' of things in themselves, but *a priori* forms (conditions of possibility) of human perception. In theoretical cognition they fulfill, not a formal-logical but a semantic role by providing "schemata" for generating objects instantiating pure concepts, for generating finite or potentially infinite domains of individuals.[134] As we just saw, the abstract description of such objects and domains as well as the specification of procedures for generating them (where these exist) were possible purely conceptually for Kant (also see the discussion in chapter 4). Intuition only makes possible the *application* of these concepts, ensuring the objective reality of their existential assumptions by grounding the possibility of exhibition of instances.

Note that pure intuition by itself does not provide manifolds already articulated into (finite or infinite) collections of discrete elements; the domains of individuals to which concepts refer are not given by mere intuition but are generated by applying the synthetic operations of the understanding to it. To summarize: pure intuition is a non-discursive faculty for (*i*) imagining space and time as mere unstructured manifolds, and (*ii*) *in combination with the synthetic operations of the understanding*, to comprehend these manifolds as unified wholes and to generate from them the objects or individuals constituting the domains of application of pure concepts.[135]

Remarks: (*i*) and (*ii*) correspond to Kant's distinction between space and time as *forms of intuition* and as *formal intuitions* (cf. *CPR* §26). As the former they are "mere manifolds", as the latter they are comprehended as *one* object and require a synthesis of the understanding. It would be a misunderstanding to read this as Kant contradicting his own injunction against allowing infinite totalities as unified objects of intuitive cognition. Kant only denies our capacity to intuit a structure constituted by an infinite totality of well-distinguished individuals as its elements or parts; but such an atomistic view, though perfectly well *thinkable* in terms of Kant's categories

[131] Comp. footnote 15.
[132] The characterization of pure intuition in this paragraph closely follows Falkenburg [25, p. 325-6].
[133] A 25 / B 39, A 31f. / B 47.
[134] Falkenburg [25, p. 326]
[135] Comp. Falkenburg [25, p. 325-6]. This interpretation of Kantian intuition, which is in conflict with the Cohen/Cassirer/Friedman interpretation(s), will be fully justified by our interpretation of the transcendental logic.

(see section ??), does not match Kant's intuitive conception of space and time. (Also note that already all finite object of intuition have infinitely many aspects.)

1.3.4 Theoretical versus practical propositions and the logical representation of algorithms: the priority of existence over generation

(*Nota bene:* Following Kant's Latin, I translate "Satz" as "proposition".[136] Compare Per Martin-Löf's discussions in [58, pp.12ff.] (also [57]) of the pitfalls and etymological intricacies of this terminology, particularly the danger of ambiguity, allegedly due to Kant's usage of the term "Urteil", between the affirmation of a proposition and the affirmed proposition.)

In various places,[137] Kant distinguishes between the logical form of "theoretical" and "practical" propositions:

> Theoretical propositions refer to and determine an object, what properties it has or does not have; practical propositions on the other hand express an action, through which, as a necessary condition of it, an object becomes possible.
>
> Remark: logic is concerned only with the *form* of practical propositions, as distinguished from the form of theoretical propositions. Propositions that are practical in regard to their *content* are distinguished from speculative propositions and belong to moral philosophy. (9:110, §32)

The remark stresses that there are really two theoretical-practical distinctions, one on the formal-logical and one on the contentual level.[138] A proposition may be theoretical regarding its content yet practical in form. To avoid ambiguity, Kant suggests to call formally practical but contentually theoretical propositions "technical".[139] For instance, a geometrical proposition asserting a construction like "from a given line and a given right angle, to construct a square"[140] is practical regarding its form, but contentually "it contains nothing but a theoretical proposition".[141] To understand what exactly Kant means here we need to further fix his terminology. Formally practical propositions are again subdivided:

> A postulate is a practical, immediately evident proposition or a basic proposition that determines a possible action, of which is presupposed that the manner of executing it is immediately evident.
>
> Problems (*problemata*) are demonstrable practical propositions requiring an instruction, or such propositions expressing an action whose manner of execution is not immediately evident.

[136] E.g., 9:120.
[137] Published, unpublished, and correspondence. We cite from all these sources in this section. They are completely consistent with each other, and span over a long period of time.
[138] Kant elaborates this in the *First Introduction to the Critique of Judgment*, for details see chapter 6. In the following three paragraphs, I only sketch the key aspects of this claim relevant to Kant's philosophy of mathematics.
[139] (20: 200). I shall use the word "technical" strictly in Kant's sense of a formally practical, contentually theoretical expression. This use of terminology is widespread in Kant's writing, e.g., comp. footnote ??.
[140] Comp. 20: 198
[141] 20: 196

> [...]
>
> Remark 2: To a problem belongs: (*1*) the *quaestio*, which states what is to be done, (*2*) the resolution, which contains the manner and procedure for executing what is demanded, and (*3*) the demonstration, that if I proceed this way then what is required will be achieved. (9:112, §38)

A complete practical proposition (in other words, a justifiably asserted practical proposition) contains not only (*1*) the 'demand' to do something (suggested by the traditionally used infinitive form), but also (*2*) a specification of the required procedure, and (*3*) a demonstration of its effectiveness. Thus "problems" require an explicitly specified procedure for generating or finding a (type of) object or relation[142]—for "technical" problems, this is the specification of an "algorithm" (see p. 15 above)—as well as a proof of its correctness. (Analogously, "theorems" are "theoretical propositions capable of and in need of a proof", and the "essential and general moments of every theorem are the thesis and the demonstration".[143]) We can now see that Kant's claim, that although technical propositions

> certainly differ in the manner of representation from theoretical propositions (which represent the possibility of objects and their determinations), they need not on that account differ from them with respect to their content (20:196),

has two distinct, albeit intimately connected, aspects:

(*A*) To every technical assertion of a generating action always corresponds a theoretical assertion of the existence or possibility and/or determination of a certain (type of) object (without reference to an action). To the practical 'from a given line and a given right angle, to construct a square' corresponds the theoretical 'for any line and any right angle there is a square'.[144] And crucially, the latter is more fundamental than the former: mathematical technical propositions, "the content of which concerns merely the possibility of a represented object (through voluntary generation), are just applications of a complete theoretical cognition".[145] They are

> nothing but the theory of the nature of objects, applied to the way in which these can be generated according to a principle, i.e., the possibility of the object is represented through a voluntary action, (20:196)[146]

and may be thus be reduced to formally theoretical expressions (comp. (*2*) below). This is also true, in particular, for technical postulates, representing operations "of which we presuppose that the manner of execution is immediately evident". These too ultimately "contain nothing but a theoretical proposition". Thus the theoretical propositional content of Euclid's third postulate is independent of its practical formulation — regarding *both* its logical function as an element in formal derivations, e.g., as a premise of

[142] In fact, the "objects" of such procedures are typically specified by a relational structure. Kant is explicit about this at 20:196, where the to-be-generated object is a certain relation between two quantities.
[143] 9:112, §39.
[144] Or 'any line and any right angle uniquely determine a square'.
[145] 20:198
[146] Meant is not applied mathematics, but the practical formulation of constructive or computational procedures in pure mathematics! Applied mathematics receives its own separate treatment, see 20:198.

formal inferences independently of its truth, *and* regarding its evidence. The circle "is defined as a curve all of whose points are equidistant from a point"; the assumption of the existence of a unique circle for any given center and radius is sufficient for rigorous logical demonstrations, and even this proposition's evidence is given "even though the practical proposition that follows from it, viz., '*to describe a circle*' (as a straight line is rotated uniformly about a point), has not even been considered yet"[147] : under presupposition of the purely theoretical specification we can rigorously

> demonstrate all the properties of the circle. [...] I assume: they, the points of the circumference, are equidistant from the centre. The proposition '*to describe a circle*' is a practical corollary (or so-called postulate), which could never be demanded at all if the possibility—yes even the manner of possibility of the figure—were not already given

through the purely theoretical specification.[148] In the same vein:

> But if, according to the property of this conic section [parabola] derived from its definition, viz., the property that the semiordinate is the mean proportional line of the parameter and the abscissa, the following problem is posed: *let the parameter be given, to draw the parabola* (i.e., how the ordinates apply to the given diameter), then this belongs, as Borelli rightly states, to the technical art that is the practical corrollarium following from and after the science; for the science is concerned with the properties of the object, not with the way to generate it under given conditions. (11:43)

The priority of cognizing the *possible existence of an object* and its *properties* following from its definition over the operation of generating the object is also pointed out in Kant's explanations concerning his use of the expression "postulate" in the second *Critique*. In mathematics, postulates concern existence:

> Pure geometry has postulates as practical propositions which, however, contain nothing further than the presupposition that one could do something if it were required that one should do it, and these are the only propositions of pure geometry that concern an existence. (*CPrR* 5:31)

And the recognition of the possible existence of the object is prior to the possible action of generating it, i.e., regarding mathematical content, theoretical is prior to practical:

> The expression postulate [in practical philosophy] can lead to misunderstanding if one confounds it with the meaning that postulates of pure mathematics have, which have apodictic certainty. The latter postulate the possibility of an action, the object of which has beforehand been theoretically cognized *a priori* with complete certitude as possible. (*CPrR* 5:11 fn. *)

Fundamentally, mathematics is concerned with *possible existence*, which is conceptualized in purely theoretical, non-operational terms. The possibility of defining "the way in which [an object] can be

[147] 11:43, 1789
[148] 11:53, 1789.

generated according to a principle" depends on the prior demonstration of the "manner of possibility" of this object; the former is both formally and contentually reducible to the latter.

In sum, all technical propositions (problems and postulates) are fully reducible to theoretical propositions, both in terms of their content and regarding their evidence.[149] And this immediately entails the second aspect of Kant's claim:

(*B*) The logical development of a field of pure mathematics, including the specification of all construction and computation algorithms and their respective proofs of correctness, can be carried out in purely theoretical terms. This follows immediately from the above, as an algorithm can be specified by the sequential combination of elementary operations expressed as postulates, and thus by a combination of purely theoretical propositions.[150] Thus, the "specific instruction for solution" of construction-problems and its correctness are both "a pure consequence [reine Folgerung] of the theory".[151] Both the *specification* of an algorithm and the *demonstration of its correctness* are reducible to purely theoretical propositions and inferences. Practical propositions and the reference to actions are thus not essential to the logical representation of mathematical contents in a systematic scientific development of a theory of pure mathematics.

The deeper reason for this are the dependence and reducibility of representations of synthetic operations on, resp. to, concepts of synthetic unity, discussed in sections 1.3.1 - 1.3.3 above. The latter correspond to a structural and existential, *theoretical* form of representation, the former to an operational, *practical* form.

[149]The converse does not hold, as the class of theoretical sentences is strictly wider: the negation of a practical sentence need not be a practical sentence in the sense of specifying an effective operation. Thus to every practical sentence there corresponds a theoretical translation, but not vice versa. For in-depth discussion of the history of logical methods for representing algorithms, from Euclid over Descartes, Leibniz, Newton to modern type theories and computer science, see Menäpää [60], Menäpää [61], and Menäpää & van Plato [59].

[150]Note however that there is a crucial difference between traditional geometric constructions and arithmetic computation procedures: the former are specified by the combination of a *constant* number of applications of elementary operations that does not depend on the given parameters, e.g., the construction of *any* triangle according to *Elements* I.21 takes 31 steps, and analogously for all construction problems of Euclidean geometry. In arithmetic algorithms, the number of steps usually *varies* with the parameters: calculating a greatest common divisor may take fifty or five trillion steps, depending on the input. Thus, while a geometric construction procedure can be fixed by explicitly specifying each step individually, arithmetic algorithms contain instructions like "repeat this operation *n* times" or even "repeat until ...", and their proofs of correctness typically appeal to induction or well-ordering principles such as "every decreasing sequences of natural numbers terminates". The latter are emphatically *not* associated with an elementary and immediately obvious operation in the same way the proposition "for every point and line there is a circle" arguably is. We do *not* have an intuition corresponding to the general concept of number that is formally analogous to our intuition corresponding to the general concept of circle: every natural number is *different* from every other precisely regarding its step-by-step build-up (that is just the criterion of identity and difference for numbers), while every circle (or every regular tetrahedron) is identical to every circle (tetrahedron) in regard to this build-up. This difference (which corresponds to the difference between proofs of general propositions by free-variable argument versus by mathematical induction) turns out to be crucial in Kant's philosophy of mathematics, as it points to a fundamental difference in the *role of intuition* in the specification of algorithms and their correctness-proofs in arithmetic versus geometry: while in geometry, intuitive evidence is given for the premises of proofs of general propositions, in arithmetic the same type of evidence is given only for individual propositions. Friedman takes this as corroboration of this claim that for Kant, there are no genuine general arithmetic theorems (as they couldn't even be formulated according to his reading); I will show that it rather points to the fact that Kant considered 'pure' general arithmetic theorems (which do not involve any numerical calculation) to be provable from the *pure concept* of number by purely logical inference; a 'purely universal' arithmetic propositions does not make any categorical existential assertions, but at most hypothetically assert, e.g., "for every number *m*, there exists a number *n* such that..." or "if there exists a number *m* such that, then there exists ..."; these, insofar as they are provable, are analytic consequences of the concept of number; only existential propositions and computations are synthetic and require intuition.

[151]20:196.

1.3.5 Infinite series in the "Transcendental Dialectic": abstract conceptual representation vs. intuitive realization

As we saw, Kant repeatedly emphasizes that the categories, as primitive concepts of the formal structural unity of manifolds, are independent of the *a priori* forms of human intuition, space and time. He affirms the possibility (though literally un*imagin*ability for us) of finite beings with forms of perception essentially different from ours who nevertheless use the same elementary concepts of transcendental logic. Space and time, determining the form of all possible objects of our intuition, *restrict* rather than *extend* the class of objectual forms thinkable by the categories. Intuition provides, not additional means to conceptualize synthetic unity, but *basic evidence of the (possible) existence of objects* of certain types, grounding the existential principles of mathematics. Intuition is strictly required only for *cognition*, for only through it are objects actually given to us:

> To t h i n k an object and to c o g n i z e an object are thus not the same. For two components belong to cognition: first, the concept, through which an object is thought at all (the category), and second, the intuition, through which it is given; for if an intuition corresponding to the concept could not be given at all, then it would be a thought as far as its form is concerned, but without any object, and by its means no cognition of anything would be possible, since, as far as I would know, nothing would be given nor could be given to which my thought could be applied. (B 146)

A *pure* conceptual representation of an object makes no reference to spatio-temporal properties in describing its relational aspects. The categories provided a formal 'scaffolding' that can be 'dressed up' with intuitive notions (and indeed require this for objective cognition) but did not depend on such interpretations for the abstract representation of synthetic unity.

Kant's treatment of infinite series of discrete elements ordered by an asymmetric relation of conditional dependency illustrates this point. These concepts are defined by assuming an initial element and by specifying a "principle of progression" (or regression) with the logical form '*for any...there exists an...*', or '*if there exists an ..., then there exists an...*', (partially[152]) determining an iterative structure of progressively (or regressively) ordered sequences of elements. We will see that Kant, repeatedly and with much care, distinguishes the abstract concepts of such structures from their concrete (at least potential) *instantiations* or *executions* in intuition.

The "principles of progression" are definable in terms of those categories "in which the synthesis constitutes a series, and indeed a series of conditions subordinated (not coordinated) one to another for any conditioned".[153] By "subordination" Kant consistently means, not *subsumption* of an individual (Fido) or a particular concept ('dog') under a more general concept ('animal'), but the abstract form of an asymmetric ordering relation.

> The concept of the understanding underlying these ideas [of infinite series, i.e., the concept of the relation between adjacent elements] contains either solely a s y n t h e s i s o f h o m o -

[152] As is well-known today, for a complete determination of some structure, even 'up to isomorphism', much more is required. Nevertheless, all structures satisfying Kant's principles are of the described type.

[153] A 409 - B 436

> g e n e o u s t h i n g s (which is presupposed in the case of every quantity, in its composition as well as its division), or or else a synthesis of t h i n g s n o t h o m o g e n e o u s, which must be at least admitted in the case of the dynamical synthesis, in causal connection as well as in the connection of the necessary with the contingent. (A 530 / B 558)

Subordination is thus the abstract form of such relations as part-whole, element-aggregate, cause-effect, necessary condition-contingent conditioned, etc., i.e., asymmetric order inducing relations between individuals (of the same or of varying types).

> The whole is either a totality of *subordination*: series (sequential order) [Reihe (Reihenfolge)] or of *coordination*: aggregate (collection). (17:397)

Kant carefully distinguishes the concepts of infinite series defined in terms of subordination relations from the conditions of their (partial or potential) intuitive realizations. Only the latter require the execution of a synthesis in time, successively apprehending or generating the structure piece by piece. The former are completely independent from such intuitions:

> If one represents everything through mere pure concepts of the understanding, without the conditions of sensible intuition, then one can say directly: that for a given conditioned the whole series of conditions subordinated one to another is given; for the former is given only through the latter. But with appearances a special limitation is encountered in the way conditions are given, namely through the successive synthesis of the manifold of intuition. (A 416 / B 444)

In the former case,

> the synthesis of the conditioned with its conditions is a synthesis of the mere understanding [a "pure intellectual synthesis"], which represents things a s t h e y a r e without paying attention to whether and how we might achieve acquaintance with them [ob und wie wir zur Kenntnis derselben gelangen können]. (A 498 / B526f.)

The categories allow purely intellectual representations of types of objects, solely regarding their relational structure ("as they are"), regardless of the conditions of exhibiting them in concrete intuition.[154] While the categories are necessary conditions for representing all temporal operations of synthesis of manifolds of sensible intuition, of concrete acts of combining intuitively given elements, *qua* abstract concepts of synthetic unity they in no way depend on such acts of generating or exhibiting intuitive configurations. One "takes the conditioned in the transcendental meaning of a pure category", independently of the particular forms of intuition; in abstract conceptual representation defined in terms of these notions,

> if something is given as conditioned, we presuppose [...] the conditions and their series as it were u n s e e n, for this is nothing but the logical requirement of assuming complete premises for a given conclusion, and no time-order is to be found in the connection of the conditioned with its condition. (A 500 / B 528)

[154]Comp. Refl. 4089:

Subordinatorum multitudo est series. Series est vel sensitive vel intellectualiter subordinatorum. (17:412)

Thus the synthetic unity of infinite series, the concept of the ordered totality of their elements, is represented purely logically, and as such completely independent from temporal intuition:

> The synthesis of the conditioned with its condition and the whole series of the latter [as represented in terms of pure categories] carries with it no limitation through time and no concept of succession. The empirical synthesis, on the contrary, and the series of conditions in appearance [...] is necessarily given successively and is given only in time, one member after another; consequently here [empirical synthesis] I could not presuppose the absolute t o t a l i t y of synthesis and the series represented by it, as I could in the previous case [intellectual synthesis], because there all members of the series are given in themselves (without time-condition), but here they are possible only through the successive regress, which is given only through one's actually executing it. (A 500f. / B 528f.)

Note that this confirms our claim that "synthesis" is neither identical with 'act in time', nor, as a synthesis of the pure understanding, is it restricted to the conjunction of monadic predicates or the combination of strictly finite manifolds. The general meaning of synthesis is the objectual combination of manifolds (finite or not, continuous[155] or discrete) into unified relational structures, and this is a basic and purely conceptual, abstract representation. The operational sense of synthesis is not derived from the intuition of time. Time is the universal medium in which all acts of human subjectivity take place, and in which all concrete objects of cognition must be exhibitable. But the basic categories which allow to formally describe these acts and structures are completely independent of the intuition of time.

1.3.6 The "correct platonic concept" of infinite structures and the modus of existence of intuitive constructions

The above discussion contradicts Friedman's main argument, that Kant knowingly lacked the logical forms necessary for abstract conceptual representations of mathematical structures, and therefore took recourse to intuition *not merely as a source of evidence for propositions (esp. exhibition of instances of existential propositions) but as a compensation for the "impossibility" even "to think", to conceptually formulate, certain propositions*. The immediate consequence Friedman draws is that Kant was unable to make the distinction, crucial to modern foundational debates, between an abstract concept of structure and its realizations (or models[156]):

> The notion of infinite divisibility or denseness, for example, cannot be represented by any such formula as ['for any a and b, if $a < b$ then there is a c such that $a < c$ and $c < b$']: this logical form simply does not exist. Rather, denseness is represented by a definite fact about

[155] In the traditional sense of continuity that is captured in modern mathematics by *smooth infinitesimal analysis* rather than by the standard set theoretical account that reduces the real line to a set of dimensionless points. Comp. Bell [3].

[156] Talk of models in this context is somewhat problematic. We construct models of theories *inside* more general theories, for example, models of various geometries inside the theory of real and complex numbers. But for foundational theories such as arithmetic, analysis, or set-theory, in terms of which models for other theories are constructed, this is no longer possible. Specifying models for such theories requires that we posses an even stronger metatheory, in which the same kind of abstract concepts, e.g., that of set, occur again, or else reference to some extra-mathematical instance, whether it be intuition, a platonic heaven, or physical reality. Against the "Model-in-the-sky" fallacy of 'absolute' (or 'vulgar') mathematical platonism, see Bernays [5], [6], Dummett [23], Tait [83].

> my intuitive capacities: namely, whenever I can represent (construct) two distinct points *a* and *b* on a line, I can represent (construct) a third point *c* between them.
>
> [...]
>
> Our modern distinction [...] between an uninterpreted formal system and an interpretation that makes such a system true cannot be drawn here. In particular, the only way to represent the theory of linear order 1-6[157] is to provide, in effect, an interpretation that makes it true. The idea of infinite divisibility or denseness is not capturable by a formula or sentence, but only by an intuitive procedure that is itself dense in the appropriate respect. [...T]he proposition that space is infinitely divisible is *a priori* because its truth—the existence of an appropriate "model"—is a condition for its very possibility. One simply cannot separate the idea or representation of infinite divisibility from what we would now call a model or realization of that idea. (Friedman [30, pp. 64, 66])

Thus Kant could distinguish relational concepts neither from their instances nor from the constructive procedures generating its elements. (Note that Friedman's offhand identification of abstract concepts of relational structure with "uninterpreted formal systems" is admissible only if one recognizes that such systems *presuppose* basic contentual notions *including* that of relational structure; they are in truth only *partially uninterpreted*, as the meaning of the logical constants and the restrictions on the interpretation of the non-logical symbols presuppose the basic notions of individual, relation, etc. (the general concept of "relation" is emphatically not definable without circularity in first- or higher-order logic): not Kant's lack of some uninterpreted *formalisms* in the sense of modern proof theory, but the conceptual resources *formalized* by them is the issue. The relevant distinction is that between the abstract representation defined in purely conceptual (non-intuitive[158]) terms and its realization; that Kant had *this* distinction is clear, as we saw in the previous sections. [159])

[157] For example in the following form. It is crucial that "<" is taken as a variable predicate that may be interpreted (yielding either true or false interpretations) as "to the left of", "later than", "brother of", "greater than", or what have you.

1. Never $a < a$.
2. If $a < c$ and $c < b$ then $a < b$.
3. $a < b$ or $b < a$ or $a = b$.
4. For any a there is a b such that $a < b$.
5. For any b there is an a such that $a < b$.
6. For any a and any b, if $a < b$ then there is a c such that $a < c$ and $c < b$.

[158] Comp. footnote 15.

[159] Identifying an uninterpreted formal system and an abstract concept of a type of structure represented by it is admissible only of on remembers that an abstract concept is not therefore *meaningless* or *purely formal* in the sense of lacking all content. An "uninterpreted formalism" is always only *partially* uninterpreted, if it is to represent anything (as opposed to being merely an aggregate of arbitrary signs on paper). If it is to do the work to which Friedman puts it (the conceptual representation of structure), the meaning of its logical constants ("all", "there is", "if ... then ...", etc.) as well as the restrictions on admissible interpretations of its non-logical symbols (the predicate and relation signs) must be determined, *for only in terms of these contentually determinate meanings* the concept is definable. Take the formalism in footnote 157: "<" may refer to any binary relation (in that sense it is uninterpreted), but it may refer *only* to binary relations, and this concept of relation, abstract and formal as it may be, is a meaningful notion, indeed so fundamental that it arguably cannot be further defined without circularity (there is certainly no point in trying to define it, say, in set-theoretic terms; the basic notion of set-inclusion is itself relational, and thus any such attempt would be blatantly circular). The direction of explanation/definition is from these basic notions to the formal system, and not the other way around: one needs to understand "all", "if then", and what a relation is if one is to understand what the uninterpreted formalism represents. Our discussion of Kant's "transcendental logic" showed that his basic

A passage from the *Metaphysical Foundations of Natural Science*[160] illustrates Kant's careful distinction between the abstract "platonist" conceptualization of infinite structures and their constructive realizability. It shows clearly that he considered "platonist" concept-formation (his term) and logical reasoning with it to be coherent and admissible, so long as it is not uncritically applied to objects whose existence is tied to their intuitive constructibility. The problem Kant discusses is thus, again, the *applicability* of certain forms of reasoning to certain types of objects, and not the *wholesale unavailability* of elementary forms of definition and inference.

Theorem 4 of chapter II of the *Metaphysical Foundations* is that "matter is divisible to infinity, and, in fact, into parts such that each is matter in turn."[161] In the proof, Kant shows that the geometrical property of the infinite divisibility of space is transferable to (his particular conception of) physical matter. Remarks 2 then deals with the question whether the infinite divisibility of space and matter imply that they must therefore "*consist of an infinite collection of parts* [eine unendliche Menge von Teilen]". The "dogmatic metaphysician" will argue as follows (Kant's wording, rearranged):

(1) A whole always already contains within itself the totality of all the parts [as well-distinguished independently existing individuals] into which it can be divided.

(2) Let a whole be divisible to infinity, into parts each of which is in turn divisible into parts.

(3) Then this whole consists of an infinite collection of [independently existing] parts.

Kant recognises the logical validity of this inference,[162] but for reasons discussed in a moment he denies that (3) is true if the given "whole" is spatial or material: "one cannot admit that matter, or even space, *consists of infinitely many parts*". Since (2) applies to space and by theorem 4 also to matter, and since mathematics "cannot be sophistically argued away", he concludes that premise (1) must be false for both matter and space (as Kant's discussion of matter and space is completely parallel, I will focus only on space henceforth).

Kant's reason for denying that space "consists of an infinite collection of [well-distinguished independently existing] parts" is that the concept of such an infinite collection is incompatible with the possibility of intuitively presenting its object as a whole, whereas space *is* thus presented to us (for Kant space is present to our intuition *as one whole* but *not* as an infinite totality of individual parts). A reading perhaps suggested by Kant's wording, that he considered the concept of an actual or completed infinity logically incoherent, is immediately refuted: that something is constituted by an infinity of independently existing elements "can perfectly well be thought by our reason, but it cannot be made intuitive and be constructed". Thus the point is rather that presenting a completed infinity of individual spatial objects is incompatible with the "subjective form of our sensibility": the modus of existence of spatial representations is essentially

concept of "synthetic unity" is meant to capture just such a basic relational or structural concept. Capturing it in a recursive formalism is a *further step*, one that he indeed contemplated, comp. section 1.1.1.

[160] (4:505-508). All direct quotes in this section are from this passage.

[161] Footnote 160.

[162] Note that Kant operates coherently, though not always fully consciously, with two different notions of combination, that of a merelogical whole-part relation, and that of a collection or set ["Menge"] of well-distinguished individuals. The argument is obvious, but to formulate so as to bring out its logical structure takes some effort. But that it was considered as purely logical by Kant is beyond dispute.

tied to their being intuitively representable by us; they may thus be of an *at most potentially* infinite character. That

> which exists only by being given to our [intuitive] representation, of that nothing *more* is given than what is found in this representation — nothing further, that is, than the progression of representations reaches. Therefore, one can only say of appearances, whose division proceed to infinity, that there are just so many parts in the appearance as we wish to provide, that is, so far as we may chose to divide. For the parts, as belonging to the existence of an appearance, exist only in thought, namely, in the division itself. Now, the division certainly proceeds into infinity, but yet it is never given as infinite. Thus it does not follow from the fact that its division proceeds into infinity that the divisible contains an infinite collection of parts in itself.[163]

Kant's point is that the consequence rejected in the last sentence would require an additional premise to be logically valid: that the parts exist prior to and independently of the division. This assumption (1) is "indubitably certain for every whole, insofar as it is *a thing in itself*": "the *composited in things in themselves* must certainly consist of the simple; for here the parts must be given prior to all composition".[164] Thus by thinking of space as a property of the things in themselves, the "dogmatic metaphysician was free to compose space out of points" and treat of it as a completed infinite totality of individual elements. This atomistic conception is *not* internally inconsistent; however, it cannot be consistently combined with Kant's theory of the formal conditions of possible cognition of real objects (pure intuition).

> The cause of confusion is a poor understanding of the *monadology* developed by Leibniz, which does not at all belong to the explanation of natural phenomena but is rather an intrinsically correct *Platonic* concept of the world, insofar as it is considered not as an object of the senses but as thing in itself, merely as an object of the pure understanding.

[163]This is precisely the same argument that Paul Bernays made in his 1930 "Die Philosophie der Mathematik und die Hilbertsche Beweistheorie":

> Es gilt in diesem Sinne der Kantische Satz, daß die reine Anschauung die Form der empirischen Anschauung ist. [...] Man könnte [...] der Meinung sein, daß wir tatsächlich einer anschaulichen Erkenntnis des Aktual-Unendlichen fähig sind. [...] Insbesondere wird man versucht sein, auf die geometrische Anschauung zu verweisen [...]. Man täuscht sich hier leicht dadurch, daß man das Anschaulich-Räumliche im Sinne einer existentialen Auffassung interpretiert. Eine Strecke z. B. ist ja anschaulich nicht als eine geordnete Mannigfaltigkeit von Punkten, sondern als ein einheitliches Ganzes gegeben, allerdings ein ausgedehntes Ganzes, innerhalb dessen *Stellen* unterscheidbar sind. Die Vorstellung einer Stelle auf der Strecke ist eine anschauliche, aber die Gesamtheit *aller Stellen* auf der Strecke ist nur ein gedanklicher Inbegriff. Anschaulich kommen wir hier nur zum Potentiell-Unendlichen, indem jeder Stelle auf der Strecke eine Zerlegung in zwei Teilstrecken entspricht, bei der jede Teilstrecke wieder in Teilstrecken verlegbar ist. pp.34, 36.

Also compare Kant's discussion of this point in the following section 1.3.6.1.

[164]Cf. Refl. 5896:

> The completed totality of that which can exclusively be thought as a progressus into the infinite is impossible. But what is thought as a quantum merely by a concept of the understanding may very well be thought as a given infinite; for it is thought as given prior to the progressus. (18:378)

The idea here is that, while we may form the concept of an infinite totality, we may *not* think of the parts of space as being given as such a totality, so long as we regard space merely as a form of intuition. That we must regard space thus is an essential proposition of Kant's transcendental idealism, but from this it does *not* follow for Kant that the very concept of an actually infinite structure could not be thought.

The conception of actual infinities of individuals is thus not inherently unthinkable, the logical and conceptual methods of definition and inference adequate for them not intrinsically inconsistent (let alone non-existent for Kant, as Friedman would have us believe)—rather, these must simply not be applied to types of objects with whose mode of existence, e.g., intuitive constructibility, they are incompatible: a structure constituted by infinitely many independently existing elements "can perfectly well be thought by our reason, but it cannot be made intuitive and be constructed".[165] That this was also the point of Kant's "antinomies" of infinity was already observed by Ernst Zermelo in response to George Cantor's "not very profound polemic" against it.[166] Brigitte Falkenburg showed with much textual evidence from writings leading up to the antinomies that Kant had an adequate understanding of both the distinction between potential and actual as well as between a (rudimentary) ordinal and a cardinal conception of infinity,[167] concluding:

> Zermelos Entgegnung gegen Cantors Kritik [...] ist damit voll und ganz berechtigt. Die kosmologische Antinomie von 1781 entzündet sich nicht am Begriff des mathematischen Unendlichen *per se*, sondern an dessen metaphyischer Anwendung auf das Weltganze, genauer: auf die Welt als Ganzes in Raum und Zeit. Eine mathematische Theorie des Aktual-Unendlichen, wie sie später Cantor entwickelte, hätte für den kritischen Kant einen ähnlichen Status besessen wie die nicht-euklidische Geometrie [... Er hätte] eine Theorie des Aktual-Unendlichen wie die nicht-euklidische Geometrie als logisch möglich und symbolisch darstellbar, aber als unbrauchbar für die Kosmologie bewertet, weil sie den formalen sinnlichen Bedingungen der Möglichkeit kosmologischer Erkenntnis nicht gerecht wird. (Falkenburg [25, p. 171f.])

Kant:

> One could say that the object of a merely transcendental idea is something of which we have no concept, even though this idea is generated in an entirely necessary way by reason according to its original laws. [...] But we would express ourselves much better, and with

[165] Also comp. AA 02: 388 fn.2 (note that "number" here clearly has the meaning "finite natural number"; in light of this, Kant's discussion is completely correct even from a modern perspective, comp. fn. 167 below):

> Those who reject the actual mathematical infinite do not take much trouble. They frame a definition of the infinite from which they can shape out some contradiction. The infinite is said by them to be a quantity than which none greater is possible, and the mathematical infinite the multiplicity—of an assignable unit—than which none greater is possible. Having substituted greatest for infinite they easily conclude against an infinite of their own making, as a greatest multiplicity is impossible; or, they call an infinite multiplicity an infinite number, and show this to be absurd; which is plain enough, but a battle with their own fancy only. But if they would conceive of a mathematical infinite as a quantity which being referred to measure as unity is a multiplicity greater than all number; if, furthermore, they would take note that mensurability here denotes only the relation to the smallness of the human intellect, to which it is given to attain to a definite concept of multiplicity only by the successive addition of unit to unit, and to the sum total called number only by going through with this progress within a finite time, they would gain the clear insight that what does not fall in with a certain law of some subject does not on that account exceed all intellection; since an intellect may exist, though not a human one, perceiving a multiplicity distinctly by a single insight, without the successive application of measurement.

[166] Cantor, Zermelo [13, p. 377 remark 1], as quoted by Falkenburg [25, p. 165].

[167] That is, a conception of an infinite quantity in terms of an ordering relation of elements, resp., in terms of the amount of elements of a collection. E.g., in footnote 165, we find that Kant defines a quantity q to be infinite if, for all natural numbers n, $q > n$. This is exactly the definition of infinite ordinals we use in modern set theory. There is, of course, not even an inkling of different transfinite ordinal numbers, or of different infinite cardinalities in Kant's writing.

> less danger of misunderstanding, if we said: that we can have no acquaintance with an object that corresponds to an idea, even though we can have a problematic concept of it. (A338f. / B 396f.)

It should be added that Kant did ascribe to infinitary concepts an important role as instrumentalist "regulative principles" for the "systematic unity" of theories.[168] For the justification of their regulative use it is 'only' necessary to show that they can never lead to a contradiction with cognitions of real objects. (The exact status of these consistency demonstrations in Kant's theory naturally remained completely vague. Nevertheless, the historical connection to Hilbert's program of consistency proofs is clear. We know since 1931 that the 'only' in the sentence before this parenthesis is a lot less innocent than it must have appeared to Kant.)

1.3.6.1 Excursus: quantifiers and existential presuppositions in mathematical definitions

The above discussion raises a problem we should briefly touch upon. The definitions of many mathematical concepts contain quantifiers ("all", "some") ranging over non-finite domains. Thus we can define the circle as a closed planar curve such that *all* straight lines drawn from it to a distinguished point are equally long. But there are infinitely many possible such lines. Are we not then presupposing an independently existing infinite totality of objects in this definition? This consequence was drawn by Salomon Maimon in his discussion of Kant's philosophy, leading him to call notions like the circle "ideas" — as opposed to concepts — "of the understanding", precisely because they allegedly referred to structures constituted by completed infinite totalities. Kant's reply is extremely interesting.

> In the concept of a circular line is thought that *all* straight lines from it to a singular point (the center) are equal; this is the mere logical function of a general judgment in which the concept of a straight line is the subject, and refers only to *any one* line, not the *totality* of all lines that can be drawn in a plane from a given point. Otherwise, by the same reasoning, every line would be an *idea* of the understanding, because it would contain as parts all lines between any two conceivable points on it, whose collection similarly goes to infinity. That the line can be divided to infinity, that is no *idea*, for it signifies only a progress of division not bounded by the magnitude of the line; but to regard this infinite division with regard to its totality and thus as completed, *that* is an idea of reason of an absolute totality of conditions (of composition) that would be demanded of an object of sensibility, which is impossible because in appearances the unconditioned cannot be found.

Thus, Kant interprets the universal quantifier here, not as presupposing its domain as previously given as a completed totality of independently existing objects, but as referring to a conceptually specified type or genus of objects, such that *each* object of this type satisfies a given condition, independently from whether the totality of such objects can be given as a completed totality. It is thus compatible with a constructivist interpretation of mathematical concepts, as is indeed suggested by Kant's discussion of (what we would today call the *type-theoretical*) forms of judgment, and especially his explanation of

[168] A 642ff. / B 670ff., A 669ff. / B 697ff., A 832ff. / B 860ff.

universally quantified judgments as specifying a *rule* for determining a certain type of object (see above section 2.1 and chapter 6). Note, however, that this does not exclude non-constructive concepts and forms of inference as incoherent. To repeat the point, Kant did not reject actual-infintary concepts and modes of reasoning *tout court*; rather, he insisted that the respective forms of definition and reasoning must be adequate to the given subject matter.

1.4 Kant's foundation of arithmetic

In this section, we connect our general discussion of Kant's transcendental logic to his philosophy of arithmetic. As a bridge from the previous section, we first show that his basic concepts of arithmetic are based on the very same categorical foundation as his discussion of infinite series. Next, we briefly review Friedman's interpretation of Kant's theory. Then we refute this interpretation by a more detailed discussion of the status of arithmetic concept formation in the context of the transcendental logic. In the section that follows after the present one, we then look more closely at the actual role of temporal intuition in Kant's philosophy of mathematics.

1.4.1 The concept of number is a pure concept of the understanding

The purpose of this subsection is to show that Kant *de facto* employed the same conceptual resources when describing and reasoning about arithmetic that he used for the infinite series in the Transcendental Dialectic; thus they *ought to* be afforded the same independent status. In 1.4.2, we contrast this with Friedman's interpretation, which requires that Kant did *not actually* make this connection, and even that he should have rejected it. In 1.4.3, we show that Kant *consciously did* make this connection, refuting Friedman and further corroborating our interpretation.

The concepts of infinite series discussed above are closely related to our modern structural understanding of arithmetic and the number series — and so they were for Kant. The "mathematical"[169] "pure concept of the understanding underlying" the ideas of these series is that of "a synthesis of homogeneous things (which is presupposed in the case of every quantity, in its composition as well as its division)".[170] Thus Kant's arithmetic concepts are based on the same categorical foundation as his ideas of infinite series. Kant's basic concept of arithmetic, which we discuss in more detail below, is a general representation of the structural determinations of a "whole" constituted either by one, or by a plurality of, discrete homogeneous individuals.

> The whole is either one of subordination: series (sequential order), or of coordination: aggregate (accumulation).
>
> The whole, insofar as its parts are homogeneous, is a quantity. (17:397)

Kant thus considers both the cardinal aspect (the size or magnitude of an aggregate) and the ordinal aspect (serial order) as essential to the concept of discrete quantities, and connects it directly to his general conception of series in terms of pure concepts of the understanding. The careful distinction between

[169] B 201, incl. fn. 1.
[170] A 530 / B 558.

abstract conceptual representations of series of discrete homogeneous individuals defined in terms of purely "intellectual" concepts of subordination (asymmetric order), and potential realizations of such structures in sensible intuition, ought to be fully operative here, too:

> *Subordinatorum multitudo est series. Series est vel sensitive vel intellectualiter subordinatorum.* (17:412)

Kant indeed held that his distinction between abstract conceptualization and intuitive executability of infinite series in the Transcendental Dialectic applies equally to his arithmetical concepts. We will see below how he argued for this, and that he developed his philosophy of mathematics accordingly.

In connection with Kant's attention to both the cardinal and ordinal aspects of concepts of quantity[171] one should also note his emphasizing the importance, in "rigorous proof", of specifying whether one thinks of a set [Menge] of numbers as unordered, "*sparsim*", or as internally structured, "*conjuctim*", according to some condition:[172] "One may think the set of numbers either as scattered [zerstreut] or else as combined according to a particular rule, e.g., that they progress in the natural order of the numbers (from 0 by continued addition of 1)."[173] Thus one may form different existential propositions involving totalities of numbers, depending on which concept of them one presupposes:

> In the infinite set [Menge] of all possible numbers thought as unordered (*sparsim*), there exist [triples of numbers satisfying a] rational relations [i.e., Pythagorean triples] other than 3, 4, and 5.
>
> In the infinite series [Reihe] of all numbers progressing in the natural order beginning from 0 by continued addition of 1, there exist among the numbers immediately following each other and thus thought as so combined (*conjunctim*) none other than [...] such that [...]. (23:204)

Let this suffice as evidence that Kant based his theory of arithmetic on exactly the same categorical foundation as his discussion of infinite series in the Dialectic, and that he therefore should have allowed the same forms of logical reasoning about finite and infinite collections (sets) and structures (e.g., series) in both cases. Consequently, he ought to have ascribed to the concepts of arithmetic the same status as to his pure concepts of the understanding generally: logical independence from intuition regarding their conceptual content, but reliance on intuition for (and with that restrictions on) the possibilities of cognizing corresponding objects. They are abstract elementary concepts of the structural determinations of "objects in general";[174] intuition comes into play only at the level of the *exhibition* of objects instantiating these concepts, and of providing evidence for the *existential* claims made by mathematical theories.

Before we look more closely at Kant's characterization of the concepts of arithmetic as pure concepts of the understanding and their relation to intuition, let us briefly review Friedman's interpretation of Kant's philosophy of arithmetic.

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[171] Falkenburg [25, section 4.5, pp. 164-172] illustrates on further examples Kant's differentiated concepts of ordinal and cardinal numbers, and his sophisticated treatment of the concept of infinity.

[172] AA 23:197ff.

[173] AA 23:199, AA 8:409.

[174] See section 1.4.3 for the relevant quotes.

1.4.2 The basic thesis of Friedman's interpretation of Kant's theory of arithmetic

For Friedman's Kant, the primary function of intuition in arithmetic was not the "objective reality" of its concepts, whether their existential supposition may be satisfied, i.e., objects instantiating the defined structures exhibited. Rather, without the *a priori* intuition of time,

> the basic idea underlying the science of arithmetic, the idea of progressive iteration, could not even be thought or represented — whether or not this idea has a corresponding object or model. [30, p.122]

That is, "the successive iteration made possible by the pure intuition of time" is

> a necessary condition for our possession of the *concept* of magnitude (*quantity*) itself: without such iteration we would be quite unable even *to think*

quantitative determinations of objects.[175] For "to describe [finite iteration] in abstract terms" without recourse to the imagination of temporal succession was "quite beyond" Kant's logical means: *the very concept* of iteration, and thereby the concept of quantity, depended crucially on the intuition of time, not merely its actualization (or at least 'in principle' executability[176]) in concretely exhibitable objects, but the logical form of the concepts themselves, their definitional content, their mere logical "thinkability". Temporal intuition is thus "a necessary condition for our possession" of this concept *not only* in the undisputed sense that time is the first and only medium in which we are acquainted with concrete processes and objects instantiating iterative structures, *but also* in that we cannot move beyond such intuitions to a general representation of their abstract relational features in terms of independent and logically prior notions. And all this is the consequence, fully grasped by Kant, of his incapacity to formulate and integrate into a formal system of logical proof as elementary a proposition as "for any natural number x, there exists a unique successor of x".

1.4.3 The independence of the concept of iteration from temporal intuition

The idea of progressive iteration is indeed essential to Kant's concept of discrete quantity and his theory of arithmetic. The act/object ambiguity of expressions like 'something is posited repeatedly' easily creates the impression of a reliance on temporal intuitions. However, Kant asserted both the logical priority of the *structural* conception of a relation (synthetic unity) over the *operational* conception of a generating act (synthesis), and the independence of representations of operation from the intuition of temporal progresses. *Both* the structural and the operational are modi of representing forms of objects, the latter from the perspective of their genetic build-up, the former from their structural being determined thus-and-so. The genetic aspect shows a strong affinity to our temporal intuition because it allows to describe types of possible actions (and because all human action take place in time); but as Kant's notion of *synthesis intellectualis* and his discussion of the abstract concepts of infinite series shows, the logical representation

[175] *Ibid.*, Friedman's italics.

[176] Kant's presentability in intuition has a decidedly modal character, the 'in principle' executability, probles yes but Chiliagon he's in good company

of iterable operations and in particular the inductive or recursive definitions of infinite series are as little dependent on temporal intuition as other kinds of existential axiomatic definitions.[177]

In this section, I show that Kant *explicitly* took the position that the basic concept of arithmetic, the iterative conception of discrete quantity, is independent of temporal intuition. We begin with his informal description of this concept.

> Something can either be simply posited, or posited over and over (*iterative*) [sic!], in order to constitute the representation of an object; in the latter case it is Plurality, in the former One. All plurality is homogeneous, and the repeated position is addition. The object whose representation originates from the plurality of given [elements] is a *quantum*; and the representation of it *as* an object that contains a plurality in itself is the representation of quantity. In all quantity is composition. That out of which is composed is called unit, and is relatively (i.e., relative to that same unit) not composed, thus simple. Thus all units are relatively [beziehungsweise] simple, but in themselves they may very well again be composed, i.e. quantities.
>
> We can construct concepts of quantity, i.e., we can exhibit [darstellen] it in *a priori* intuition. (Handwritten reflections no. 5726 and 5727, 18:337f., ca. 1785-1789)

The category of quantity thus concerns an elementary structural feature of an object, the way it is composed from elements that are (for the given purpose regarded as) homogeneous. At its basis lies the concept of the synthetic unity of iterated composition from discrete homogeneous units. Kant is very clear that numbers are not themselves (finite) collections, but general structural concepts of collections or ordered sequences, representing their "quantitative determination".[178]

The basic concept of mathematical *composition* or *combination* thus concerns the structural build-up of an object from constituent elements and the induced ordering amongst these elements. Kant is utterly clear that iteration is of the essence of his basic concept of quantity; but as we saw above, he also held that *abstract* concepts of iterative structure are representable, e.g. by progressive or recursive definitions, in terms of pure categories "in which the synthesis constitutes a series" of elements subordinated on to another.[179] And such concepts of iterative composition we "can construct", i.e., we can "exhibit in *a priori* intuition" concrete structures realizing them, at least in the case of finite iteration. [180] Note that a

[177]The act-object ambiguity between relational structure and generating operation is analyzed in chapter 3 section 3.3.3; we will find (*a*) that notwithstanding Kant's emphasis on the *active* nature of the understanding, a basic concept of relational structure is logically prior to the representation of any operation; (*b*) that the active nature of the understanding *does not* imply for Kant a logical dependence on the intuition of time; contrariwise, any representation of temporal succession presupposes abstract relational categories; these allow the logically rigorous conceptual specification of such objects that are intuitively exhibitable — *as well as of such that are not*.

[178]AA 10:554ff.

[179]A 409 / B 436.

[180]Recall that by "to construct a concept" Kant always means, not to bring about the concept itself, say by logically combining other concepts, but to *exhibit an object corresponding to it*; to construct the concept Triangle means to exhibit *an individual triangle*, not to define or otherwise bring about the concept itself. There is a subtlety here because Kant argues that giving a "real definition" of a mathematical concept amounts to its construction: in contrast to merely "nominal definitions" they specify the means for exhibiting individuals falling under the defined concept. We will see that a real definition amounts to defining an operation for generating such individuals from given parameters, which is then recognized to be well-defined, i.e., realizable, on intuitive grounds. The rigorously defined concept, but not its objective reality, is logically prior to its exhibition in concrete intuitively presentable structures. See section **??** and chapters 3 and 6.

lot remains vague here; in particular, whether, and if so how, Kant intended to *define* the general notion of finitude. It is clear that his notion of discrete plurality was meant to cover both finite and infinite collections; but it is not clear how he intended to rigorously distinguish between them.

The abstract status of the basic category of composition — the general type of the asymmetric relation between the (relatively) simple and the composited whole — and its relationship to intuition is lucidly discussed in the 1791 *Prize Essay on the Progress in Metaphysics since the Time of Leibniz*. (For completeness I partly repeat some points of section 1.3.3 here.) To begin with, the basic concepts of synthesis are not *abstracted* from spatio-temporal forms of sensible intuition, but are given independently from and merely *applied* to them:

> The representation of a composited, as such, is not mere intuition, but requires the concept of a composition, which is applied to the intuition in space and time. This concept thus (including its inverse, the simple) is a concept that is not abstracted from intuitions, as a partial representation contained in these, but is a basic concept [ein Grundbegriff], notably *a priori*, ultimately the only basic concept *a priori*, which originally lies at the ground of all concepts of objects of sensibility in the understanding.
>
> [...]
>
> There will lie as many concepts *a priori* in the understanding, under which the objects that can be given to sensibility must stand, as there are types of composition (synthesis). (*Preisschrift über die Fortschritte der Metaphysik*, 20:271f., 1791)

These are general concepts of synthetic unity (relational structure and order) of or amongst individual elements assumed as given. The categories thus always refer to, or presuppose as given, a manifold of individuals. Individuals however can only be given in intuition, indeed, being a singular and immediately given representation of an object is the defining characteristics of *intuition in general*, "Anschauung überhaupt". But crucially, the intellectual concepts of synthetic unity do not presuppose anything additional about these elements whose ordering, combination and unity they abstractly represent, that is, anything above the formal properties of concreteness (or "immediacy of givenness") and individuality.[181] In particular, Kant continues:

> these categories – or predicaments, as they are also called – do not presuppose a specific kind of intuition (as for example the only kind possible to humans) such as space and time, which is sensible; they are only forms of thought for the concept of an object of intuition in general, of whatever kind that intuition may be, even if it were a supersensible intuition, of which we [*qua* humans] cannot have a specific concept. For we must always make ourselves a concept of an object through the pure understanding, if we want to judge something *a priori* of it, even if we later find that it is transgressing [our limits of intuitive instantiability and thus our possible cognition] and cannot be given objective reality — thus the category by itself is not

[181] I am thus in full agreement with J. Hintikka [45], [46], [47], [48] on the status of intuition in Kant's philosophy of mathematics, in opposition to C. Parsons [64]. See the introduction to Posy [68] for the context of this disagreement between these two great logicians and Kant scholars.

> dependent on the forms of sensibility, space and time, but may also have other, for us not at all thinkable, forms as a substratum [zur Unterlage] [...]. (*Ibid.*)

And all this is in particular true, Kant adds, for the basic categories and the predicaments of the science of number. Hence we can, in terms of the arithmetical notions of the pure understanding, "make ourselves a concept" not only of intuitively realizable finite structures, but also of transfinite structures. E.g., one can define in terms of the categories a kind of object that is constituted by infinitely many discrete elements ordered by a recurrently defined relation, such all 'local' aspects of its structure are intuitively realizable (Kant speaks of the "distributive unity",[182] the qualitative identity of the repeated relation between adjacent elements in such series), but that its 'global structure' (the "collective unity" of all the elements[183]), although it can be rigorously *thought* and reasoned about, cannot be *exhibited* in intuition, and therefore the existence and "objective reality" of this structure as One unified object (completed infinite totality) may not be assumed, but can function at most as a regulative idea.[184]

Further unambiguous statements of this status of the intellectual concepts of arithmetic are provided in many places. In the *Critique of Pure Reason*, where the general account of the pure concepts, their relation to intuition, and the notion of *intellectual synthesis* are developed in detail, the arithmetic concepts as subalternate "predicables" are treated only in passing. But here too, their status as non-intuitive[185] concepts of the pure understanding derived from the *unschematized*[186] categories is clear:

> The categories, as the true **ancestral concepts** of pure understanding, also have their equally pure **derivative** concepts, which could by no means be passed over in a complete system of transcendental philosophy, but with the mere mention of which I can be satisfied in this merely critical essay.
>
> Allow me to call these pure, but derivative concepts of the understanding the **predicables** of pure understanding (in contrast to the predicaments). If one has the original and primitive concepts, the derivative and subalternate ones can easily be added, and the family tree of pure understanding fully worked out (A 81f. / B 107f.)

In particular,

> the concept of a **number** (which belongs to the category of totality) (B111)

is such a predicament. Note that

> **allness** (totality) is nothing other than a **plurality** considered as a **unity**. (*Ibid.*)

In particular, totality, as a concept of the synthetic unity of a manifold, is a concept of *objects*, of concrete pluralities considered as *one*. Arithmetic treats, not of particular totalities, but of the various quantitative determinations of totalities.

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[182] A 643 / B 671.

[183] *Ibid.*

[184] Note that the contradictions investigated in the antinomies-chapter are said to emerge, not from the perfectly admissible logical conceptualization of infinite structures, but from the conflation of this abstract concept formation and the assumption of empirical verifiability and exhibitability of the infinite structures as one object.

[185] Comp. footnote 15.

[186] See above section 1.3.3 and below section 3.4.2.

> Arithmetic [...] has actually no *quantum*, i.e., no object of intuition regarded as a quantity, but mere quantity itself, i.e., the concept of a thing in general through quantitative determination, for its object.[187]

This allows us to infer how Kant defined natural numbers. Totality is a category of quantity, viz., the concept of a plurality of individual objects regarded as one object. Kant distinguishes between discrete and continuous quantities.[188] A discrete totality is thus a plurality of well-distinguishable individuals (which for Kant, ignorant of the distinction between a densely and a totally ordered field, here means that they can be apprehended as a totality that is not densely ordered), which is represented as *one* object. A natural number is a sub-species of the concept 'totality', and in particular, a concept of discrete totalities. For example, the number three, in its cardinal sense, would be the concept of such totalities a that there exists an individual b, a distinct individual c, another distinct individual d, and no other individual in a. Numbers are thus structural concepts, "quantitative determinations", of a very wide genus of individual objects (discrete totalities).[189]

Elsewhere, Kant sketched in more detail the system of the mathematical predicaments as "pure derivative concepts" of the category of quantity, and afterwards forcefully underlines, in no uncertain terms, their fully independent status.

> Categories of quantity (*quantitas*). 1. *Unit[y]* [Einheit] (mathematical, not qualitative, measure [...]; 2. *Multiplicity* [Vielheit]: collections [...] indeterminate multiplicities/collections [...] indefinite progressions; 3. *Totality*. Number [...] comprehension of a collection [...] Infinite quantity [...] Regression into infinity, continuity, the infinitely small $1/\infty$.
>
> [...]
>
> The concept of quantity is not a concept derived from experience. It lies entirely *a priori* in the understanding. [...] The concept of quantity contains that which the understanding does for itself, *viz.* to bring about a whole representation through the synthesis of repeated addition; therefore, it does not contain anything that requires perception [...] Thus it can be applied to *a priori* intuitions, space and time. But also from these it is not derive, but is only applied to them and gains objective reality through them of the things in space and time. It contains nothing but the synthetic unity of consciousness, which is required for a concept of an object in general, and in that it is an element of cognition, though not yet a cognition except through application to pure or empirical intuition. (18:661, ca. 1794-95)

[187] Kant's use of three different words for object/subject matter, "Gegenstand", "Ding", and "Object", make the traslation difficult. Here is the original: "Die Arithmetik hat ... eigentlich kein Quantum, d.i. keinen Gegenstand der Anschauung als Größe, sondern bloß die Quantität, d.i. einen Begriff von einem Dinge überhaupt durch Größenbestimmung zum Objecte."

[188] Comp. Frede Krüger [28].

[189] Frege criticized this notion of number as a first-order property of *objects* (totalities as individuals), insisting instead that a number should be thought of as a second-order property of *concepts* (e.g., the number of x that fall under concept P). Besides that fact that by using his inconsistent comprehension axiom, he associated with every concept an *individual object* (its extension), Bernays correctly critized that Frege's analysis of, say, the number 2 as the property "being a concept P such there exist an object x and another object y and no other objects such that $P(x)$ and $P(x)$" for involving the notions of existence and properties, which are not part of the essential property of cardinal Two-ness: "one object, and another object". Comp. fn. 22 above.

Kant again emphasizes the iterative nature of quantity *and in the same breath* denies that it derives from temporal intuition. The now obvious explanation for this is that he claims a "basic concept" of synthetic unity of *composition* or *combination*, a relational structure-concept that allows him to do precisely what Friedman believes impossible: to describe iteratively defined structures and procedures in abstract, non-intuitive terms. Now, certainly, Kant did not write *Paradoxien der Unendlichkeit* (Bolzano, 1851), nor *Was sind und was sollen die Zahlen?* (Dedekind, 1888), let alone *Grundgesetze der Arithmetik* (Frege, 1893): he never attempted a thoroughgoing rigorous logical analysis of the concept of finite number. All the same, he undeniably considered numerical concepts to be definable in terms of the purely "intellectual" (= non-intuitive) concepts, in particular the categories of an individual, a discrete multiplicity of homogeneous individuals, and a totality of such individuals which, besides existence, non-existence, and "limitation" (non-discrete graduation), are the "mathematical" categories of transcendental logic. It is quite indubitable that he considered it possible to "think" arithmetical concepts — in Friedman's strong sense of "rigorous reasoning with" — independently from the *a priori* intuition of time and logically (though not psychologically) prior to any cognition of intuitively exhibitable objects. To repeat the crucial quote from Kant's letter to his, after J. H. Lambert's death, certainly most capable mathematical collaborator, J. F. Schultz (who was at the same time preparing his groundbreaking and sadly ignored *Versuch einer genauen Theorie des Unendlichen*):

> Time, as you rightly remark, has no influence on the properties of numbers (as pure quantitative determinations)—as it does on the properties of any alteration (as a *quantum*), which is itself only possible relative to a specific constitution of our inner sense and its form (time)—and the science of number, regardless of the succession that any construction of quantity requires, is a pure intellectual synthesis that we represent to ourselves in thought. (Letter to J. F. Schulz, 25. Nov. 1788, 10:554ff.)

Thus what Kant generally argued for — concerning the abstract non-intuitive[190] representation of and reasoning about structures in terms of unschematized categories, the notion of *synthesis intellectualis* and the general representation of operations, and the status of "intuition in general" — he explicitly applied to the concepts of arithmetic in particular.

But now the question arises: what role *does* intuition play in Kant's theory? What exactly does the requirement of the intuitive evidence of every construction of quantity amount to? To this question we now turn.

> The imagination progresses in the composition necessary for the representation of magnitudes, into infinity; the understanding however leads it with its number concepts, for which the imagination must provide the schema; [...] this procedure belong[s] to the logical estimation of magnitude [...]. (5: 253)

[190]Comp. footnote 15.

1.4.4 The role of intuition in arithmetic, or: Why is $1+1=2$ not an analytic judgment?

In the previous sections, we have come to the following conclusions about Kant's transcendental logic and the theory of arithmetic based on it.

- The general concept of number, on which pure arithmetic is based, is a pure concept of the understanding independent from specific forms of intuition.
- The definition of both structural or relational concepts ("synthetic unity") and of operations and algorithms ("*synthesis intellectualis*") is possible in terms of such purely intellectual concepts.
- Logical reasoning (including such not formalizable in monadic first-order logic) with such concepts is possible independently from the existence or possible executability ("objective reality") of the structures or operations so defined.

These lead to the question: does it not follow that arithmetic is a purely analytic discipline, deriving all its propositions logically from definitions? For, if all its notions are definable conceptually, and all reasoning about them can be carried out logically, including even the description and the correctness proofs of its computational algorithms, what makes arithmetic truths like $5+7=12$ synthetic? And how does pure intuition come into play?

Bernays' explanation of syntheticity in terms of intensional difference

Paul Bernays,[191] following E. W. Beth,[192] provided one explanation of Kant's statement "that 7 **should** be added to 5 I have, to be sure, thought in the concept of the sum $= 7+5$, but not that this sum is equal to the number 12".[193] He interprets this using the distinction between (in Frege's words) the sense and the reference of an expression, i.e., intensionality and extensionality. "7+5" is an individual concept that can be expressed, using Russell's iota operator for "the unique x such that...", as $\iota x(x=7+5)$, which is different in sense from the individual concept $\iota x(x=12)$. According to Bernays, analyticity is tied to the "clarification of meaning", i.e., sense.[194] Thus, he claims, even pure logic contains a non-analytic "combintorial" or "synthetic element", which he illustrates using an analogy of Hermann Weyl: any given chess configuration C ('black pawn on B3, white queen on...') together with the rules of the game determine purely logically whether "possible mate in 10 moves" is true. Nevertheless, this proposition does not follow from the description of C and the rules "analytically", because it is not intensionally contained in them. Only by a specific combination of applications of the rules, their "actual execution", and not an "analysis of the meaning" of the description of the configuration and the rules, can this proposition be demonstrated: the content of the description of C and of the rules of chess can be grasped without knowing whether or not "mate in 10 possible"; my failing to see a possible checkmate does not prevent me from fully (in some sense of "fully") understanding the description of the chessboard configuration and the rules of the game. Analogously, logical derivations constituted by combinations

[191] Bernays [7, p. 138].
[192] Beth [8].
[193] B 14.
[194] Bernays [7, p. 25].

of inferential steps contain a non-analytic moment: from the content of the inference rules and of the premises cannot be immediately "read of" whether they imply a certain conclusion, but only "from that structure which is produced by the application of the rules, i.e., the execution of the inferences".[195] Embedding arithmetic calculations into a logical formalism does not therefore change their non-analytic nature.

Bernays' interpretation is supported by a passage in which Kant discusses the intensional-extensional distinction in connection with analyticity. He distinguishes the "subjective" and "objective" identity of concepts: the former concerns the "manner of synthesis" in terms of which the objects of a concept are given, the latter merely concerns its objective reference. Thus "3+5", "12-4", "2×4", "2^3", and "8" are objectively identical, but subjectively distinct (as they represent different synthetic operations by which the object 8 is given). He then argues that the identity of "3+4" and "7" cannot be an analytic truth, for by the transitivity of identity, that of "3+4" and "12-5" would have to be analytic, too. But as the "manner of synthesis" that they represent is clearly distinct, this cannot be the case. The non-analyticity of arithmetical identities is thus explained in terms of the intensional difference of the terms.

I believe that Bernays' explanation is not entirely false, but fails to go to the heart of the matter. It is true that Kant connected analyticity to matters of meaning in these passages; but the explanation does not pay sufficient attention to the crucial role of construction, the exhibition of concrete instances, in Kant's account.

1.4.5 The status of $1 + 1 = 2$

Let's measure Kant's thesis that "$1 + 1 = 2$" is synthetic against later objections. Below is a version of the Frege-Russell-Carnap logicist account of addition,[196] spelled out using modern symbolism. "$[x : \varphi(x)]$" symbolizes the concept of objects with the property expressed by the open sentence φ. "$x\ \varepsilon\ R$" means that x falls under the concept R. All lower-case variables range over individual objects. The relation "$y \in x$" means that x is a discrete aggregate or totality (which according to Kant's explanation is a discrete plurality considered as *one* object[197]), and that y is one of the individuals in that totality. "$\cap$", "$\cup$", and "$\varnothing$" have their known meaning. I use the square brackets [ ] over the { } customary in modern set theory and the two relations "ε" and "$\in$" to indicate that I am not presupposing some specific framework, but most importantly to emphasize the difference between concepts and individual objects (including discrete totalities treated as individuals), which Kant drew very explicitly.[198]

Let us define

$1 := [x : \exists y\, (y \in x \ \wedge\ \forall u(u \in x \to u = y)]$,

$2 := [x : \exists y\, \exists z\, (y \in x \ \wedge\ z \in x \ \wedge\ y \neq z \ \wedge\ \forall u\, (u \in x \ \to\ u = y \vee u = z)]$,

[195] Bernays [7, p. 27].

[196] Frege [29, pp. 76ff.], Russell [76, Ch. 2, pp. 13-23], Carnap [14, pp. 92-3]. This is not Frege's most sophisticated version (see Zalta [91]), but the one that Carnap presented at the famous 1930 Königsberg Foundation of Mathematics symposium at the peak of the "Grundlagenkrisis". But my discussion applies to (neo-)Fregean attempts. In particular, the fundamental problem of the syntheticity of existential propositions cannot be overcome.

[197] B 110.

[198] The reader is free to compare concepts with classes and objects with sets and other individuals; note that, in a set theoretic context, the natural numbers as defined here would indeed be *proper* classes.

and

$$P+Q=R \;:\longleftrightarrow\; \forall x\,\forall y\,(x\,\varepsilon\,P \;\wedge\; y\,\varepsilon\,Q \;\wedge\; x\cap y=\varnothing \;\rightarrow\; x\cup y\,\varepsilon\,R).$$

From these definitions, $1+1=2$ can easily be derived:

Assume $a\,\varepsilon\,1$ and $b\,\varepsilon\,1$, that is, assume that $\exists y\,(y\in a \;\wedge\; \forall u(u\in a\rightarrow u=y)$, and similarly for b. Assume further that $a\cap b=\varnothing$. Then for $c=a\cup b$, we obtain by easy† manipulation $\exists z\,\exists w\,(z\in c \;\wedge\; w\in c \;\wedge\; z\neq w \;\wedge\; \forall u\,(u\in c \;\rightarrow\; u=z\vee u=w)$, that is, $c\,\varepsilon\,2$. Discharging the assumptions, we have the implication $a\,\varepsilon\,1 \;\wedge\; b\,\varepsilon\,1 \;\wedge\; a\cap b=\varnothing \;\rightarrow\; a\cup b\,\varepsilon\,2$. But a and b were chosen arbitrarily. Therefore, by universal generalization, $\forall x\,\forall y\,(x\,\varepsilon\,1 \;\wedge\; y\,\varepsilon\,1 \;\wedge\; x\cap y=\varnothing \;\rightarrow\; x\cup y\,\varepsilon\,2)$, and thus by definition, $1+1=2$. $\dashv$

It is worth taking a closer look at this argument to understand its relevance to Kant's thesis, by reformulating it in words that Kant would immediately understand:

1 is defined as the concept of such discrete aggregates x for which there exists an object y in x, and all objects w in x are identical to y. 2 is the concept of such aggregates x for which there exist an object y and a distinct object z both in x such that all objects in x are either y or z. We define $P+Q=R$ to mean that for all disjoint aggregates x and y, if x falls under P and y falls under Q, then the combination of x and y falls under R. To prove that 1+1=2, assume that a is an aggregate falling under 1, and b is a different aggregate also falling under 1. That is, assume that there exists a unique object in a, and that there exists a distinct unique object in b. Let† d and e be these objects, respectively. Then there exists an aggregate c, the combination of a and b, that contains exactly d and e; thus, c is such that there exists an object z and a distinct object w and no further object in c. Hence the combination of a and b satisfies the definition of 2. Since a and b were chosen arbitrarily, we conclude that, in general, if *any* disjoint aggregates x and y both fall under 1, then their combination always falls under 2. Thus by definition, $1+1=2$. $\dashv$

The first thing to notice, upon spelling it out in natural language, is that this line of reasoning would be perfectly understandable to Kant. For a start, the definitions of the numbers correspond to his own explanation. One and Two, taken in their cardinal sense as answering the question "How many?", are quantitative determinations of discrete totalities, just as Kant claimed: Two is the concept of discrete totalities that contain one object and another object and no other object. Similarly, I claim that Kant would accept the definition of addition.

Next, note that the argument involves existential reasoning in three places. Firstly, (*i*) we assume at the outset that there exist the disjoint aggregates a and b both falling under 1. The argument proceeds with a and b as concrete instances that are manipulated to show that their combination must fall under 2. Finally we recognize that the choice of a and b was arbitrary (save for their disjointedness and their both falling under 1), and conclude that the combination of *any* pair of disjoint 1-aggregates falls under 2. This is *precisely* one of the forms of free-variable argument that Kant identifies with proof by construction (see pages 10-12 above). Secondly, (*ii*) at the points marked with a dagger †, we have to instantiate the existential quantifier in the definition of $a\,\varepsilon\,1$,

†This step will be discussed in more detail.

$$\frac{\exists y\,(y \in a \ \wedge\ \forall w(w \in a \rightarrow w = y))}{d \in a \ \wedge\ \forall w(w \in a \rightarrow w = d),}$$

and similarly for b. This is the kind of inference-step isolated by Hintikka as the other essential point where intuition (not necessarily spatio-temporal) enters for Kant.[199] The thus instantiated individual objects d and e are then collected into the aggregate c, whose existence is the third existential moment in this argument (*iii*). *But do these existential aspects make the conclusion synthetic?*

Starting with (*iii*), the proposition that for any d and e, there exists the totality $c = \{d, e\}$ (as one unified object) is indeed a non-logical principle whose negation is not contradictory. The unconditional existential form makes this an obvious case of Kant's general dictum that "every existential proposition is synthetic".[200] This dictum occurs in Kant's discussion of the ontological proof of the existence of God, and refers to existential claims in general. Its applicability to mathematics in particular is clear from the fact that Kant specifically emphasizes that the unification of a multiplicity into one totality is an independent act of synthesis that is not a logical consequence of the concepts of unity and plurality, and which need not always be possible.[201] There are cases in which the existence of each element of a plurality may be asserted, while the existence of their totality as one unified object may not be asserted. From a modern perspective, one might object that the axioms of pairing and of union are part of an implicit definition of the very concept of set, but one then needs to account for the existential presuppositions of set theory, even if one is restricted to a 'finitary' set theory like ZFC with the axiom of infinity replaced by its negation, which at any rate is bi-interpretable with first-order arithmetic.[202] (Carefully formulated, arithmetic and finite set theory in fact turn out to be *synonymous*.[203]) At any rate, modern set theory provides a good example for the fact that the existence of each element of a class does not logically imply the existence of that class as an individual object: the axioms of ZFC logically require of each set whose existence they imply that it exists as an individual in the range of the first-order quantifiers; but they do not imply the existence of the totality of those sets as such an individual (in fact they even imply its non-existence in this sense).

For the other, perhaps more important cases (*i*) and (*ii*), the apparently obvious answer to our question seems to be: no. The initial assumption of the existence of a and b (on which also the dagger†-marked existential instantiations in the middle of the argument depend) is not a categorical assertion, but a hypothetical stipulation later discharged. We infer the proposition that *if* $a\ \varepsilon\ 1$ and $b\ \varepsilon\ 1$, *then* $a \cup b\ \varepsilon\ 2$. By the meaning of material implication (which we use for the next three paragraphs for reasons explained below), this is always true: if a and b do exist, then the argument directly establishes that their combination has the desired property; and if they do not exist, then the antecedent is false and thus the implication vacuously true.

But this last point provides a hint for understanding Kant's thesis properly. Observe that on the above alleged reduction of arithmetical truths to logical truths, $2+2=5$ *is not logically false.* One can recognize

[199] Hintikka [45], [46], [47], [48].
[200] A 598 / B 626.
[201] B 110-1, (10:366-7), comp. Schubring [77, pp. 451-2].
[202] Lindström [54]. Note that the old-school logicists operated with propositional disjunction and an application of the general comprehension principle, instead of the axiom of pairing or the axiom of union.
[203] Kaye, Wong [71].

this from two (ultimately equivalent) perspectives. Firstly, assume that there exist only three objects. Then the antecedent of $2+2=5$, 'for any disjoint collections x and y, *if* x and y both fall under 2, then their combination falls under 5' would be false, and hence the whole proposition, by the meaning of the material implication "if...then...", would be true.[204] Secondly, as $2+2=5$ has the form "$\forall x\ldots$", its negation $2+2\neq 5$ is an existential proposition: its demonstration requires the exhibition of two disjoint 2-collections whose combination contains exactly 4 (and not 5) elements. This, I want to argue, is not an accident of our particular reconstruction but a necessary feature that will reappear, in one shape or another, in every presentation of arithmetic: any such presentation, implicitly or explicitly, *presupposes* the (possible) existence of arbitrarily large finite collections, an *a priori* presupposition that no amount of definition and logical analysis can account for.

We thus have the following situation. In a model or possible world where there are only 3 individuals, the proposition "$1+2=3$", "$2+2=4$", "$2+2=16$", "$3+3=99$", "$6+4=10$", "$6+6=10$" ..., interpreted in the above sense, would all come out true, and the propositions "$2+2\neq 4$", "$2+2\neq 5$", "$5+5\neq 10$", "$5+5\neq 11$", ..., are all false. For example, since there are no 6-element sets, it is vacuously true that all combinations of disjoint 6-element sets contain 10 elements, or 13, or any other number of elements greater than 3. One might argue that this is an artificial consequence of the interpretation of "if...then..." as the material implication, but this is not the case. The material implication rather serves as a particularly useful tool for indicating the grounds for distinguishing true from false arithmetical propositions. In the 3-world, $2+2=5$ evaluates as true because of the vacuity of the antecedent, not because of the existential reasoning in the above proof; similarly, $2+2\neq 5$ evaluates as false for vacuity reasons, not because the combination of two 2-sets is exhibitable that does not contain 5 elements. On the other hand, in large enough domains, true and false sentences evaluate correctly because of the relevant underlying existential assumptions. Hence, the type of reasoning that allows to differentiate true from false arithmetical propositions requires the exhibition of domains with sufficiently many elements. And although the existence of such domains is merely a hypothetical assumption later discharged, the concrete operating with these hypothetical or ideal elements is an essential condition for distinguishing arithmetic truth and falsity; and even their merely 'possible existence' is not a purely logical necessity.

In general, in a world with only n many individuals, every proposition of the form "$p+q=r$" would come out true if, *in actuality*, $p+q$ is greater than n (and dually for negations). What allows us to distinguish true from false arithmetical propositions? As I have argued, the problem does not lie with our analysis of arithmetical concepts, nor with the associated derivations presented above; these correspond, in an entirely natural way, to our actual arithmetical reasoning, which is representable in purely logico-conceptual forms, and is thus a "pure intellectual synthesis" in Kant's sense. The problem lies with the existential assumptions that substantiate this reasoning, in particular, the non-trivial part of the proof of true propositions (in a 3-world, both "$2+2=5$" *and* "$2+2=4$" come out true because of the vacuity of the antecedent, not because of the reasoning that operates with the individuals; in a world with enough individuals, "$2+2=4$" and "$2+2\neq 5$" both come out true precisely because of this non-trivial

[204] Different interpretation of implication are discussed two paragraphs below. However, using the modern interpretation of classical material implication is useful to highlight Kant's essential idea.

reasoning[205]). These basic existential assumptions, the synthetic *a priori* of arithmetic, must be strictly distinguished from the means of concept-formation and rules of formal reasoning that they substantiate.

The analysis of $1+1=2$ above may also be interpreted as: 'in every *possible* situation in which there are two disjoint 1-collections, their combination will be a 2-collection' (where the consequence relation need not be the material implication of modern classical logic that we presupposed above). But in a two-fold sense, recognizing the truth of this proposition requires intuition in Kant's very general sense of the possibility of exhibiting concrete objects realizing this situation (which, to repeat, is *not* necessarily tied to the particular forms space and time, comp. sections 1.3.3, 1.4.3): firstly, in establishing the *possibility* of the existence of discrete objects. As elementary as this appears, it is simply not a truth of logic that the existence of an object is even so much as a possibility: how do you know that one, or a million objects could exist?[206] And secondly, as we just saw, to distinguish the true arithmetical sentences from the infinitely many false ones. In order to carry out the above proof, we need to operate, if only hypothetically, with collections of concrete individuals; this is required both for establishing the consequent on assumption of the antecedent, and to rule out the false identities that are true in the hypothetical worlds in which the antecedent is false.

On conclusion, I submit that Kant's thesis can be made sense of. In the first place, the aspects of arithmetical reasoning (as discussed above) are captured precisely by the forms of argument that Kant connected with construction, especially the instantiation of existential assumptions (see pages 10-12 and section 4.1.6); this should be admitted and appreciated even if one rejects his notion of "pure intuition". More generally, I would argue that Kant was quite right in pointing out that the existential assumptions of mathematics, even if one only operates with a 'soft' notion of *possible existence*, is a non-analytic presupposition that requires an extra-logical source of evidence. Whether it be Kant's principle that any finite discrete quantity can be constructed in pure spatio-temporal intuition, the postulate that for each natural number there exists its successor, or some axiom of infinity (say, that the class of finite von Neumann ordinals is a set), a restricted form of set-comprehension, or Frege's so-called "Hume's principle"[207] — any principle implying mathematical existence requires a justification that goes beyond mere conceptual analysis. Even if one subscribes to the dictum, sometimes uttered by Hilbert, that mere consistency implies mathematical existence, one needs to account for consistency; and any such account, whether via a syntactic proof or via the description, construction or intuition of a model, will involve a non-analytic component (one should add that Hilbert and Bernays were under no illusion on this point).[208]

[205] This is one reason why using the material implication (which Kant did not interpret in our way) is useful here: it puts the finger on precisely where the essential arithmetical reasoning comes in, and illustrates the necessity of existential assumptions precisely for this part of the reasoning. That $2+2=4$ is true in a 3-world has nothing to do with actual arithemtical facts, but is a consequence of the material implication!

[206] You may reply: because I can coherently form the concept of a million objects. To which I reply: how do you know that your concept is not inconsistent? Any answer to this will either already presuppose arithmetic (leaving the question begged), or amount to what is essentially equivalent either to an outright *stipulation*, or else to an intuitive construction.

[207] Zalta [91].

[208] Bernays [7, pp. 17ff., pp. 92ff.]. Consistency proofs have somewhat come into disrepute in the wake of Gödel's second incompleteness theorem, but one should recognize, *(1)* that there *are* genuinely informative relative consistency proofs, e.g., the Gödel-Gentzen double-negation translation of PA into HA, or Gentzen's consistency proof of PA using quantifier-free transfinite induction up to ε_0, both of which secure *classical* number theory on the basis of *constructivist* principles, and which deliver, in the latter case, extremely profound insights into the fine-structure and transfinite content of proofs in arithmetic; the nature of Michael Rathjen's consistency proof for Π^1_2-Comprehension is also indicative of the deep connections between highly infinitary abstract and computationally explicit concrete mathematics; and *(2)* that the *recognition* of the consistency of a conceptual

In the conclusion of this thesis, I try to make the case that Kant was right not only about the synthetic character of mathematics (as I have a argued above) but also that his arguments for its aprioricity bear great foundational importance. The abstract character of the existential assumptions underlying arithmetic (and ultimately set theory, as the natural continuation of arithmetic) that we discussed in this section are either non-empirical, or entirely groundless. However, as will be discussed in the final section of the conclusion, modern reflections make it necessary to introduce a distinction *within* the non-empirical, between the properly speaking *a priori* (the necessary conditions of possibility of empirical experience) that must be knownable independently from any particular experience, and such principles which, although they cannot depend on empirical experience in the mundane sense of spatio-temporally localized physical and psychological events and the laws or law-like regularities governing them, nevertheless depend on another kind of experience that Bernays called "geistige Erfahrung". By holding fast to two essential components of Kant's overall philosophical tendency, the respect for the "factum of science" (which today must include the factum of non-constructive transfinite mathematics) and the insight into the non-analytic yet non-empirical nature of mathematics, we face the challenge to develop a more radical account of the possiblity of mathematics. Or in other words: we must go, with Kant, beyond Kant.

[Return ticket to page 11 for impatient readers.]

framework is still requirement even if a strict relative consistency proof is lacking (as it is and – until some radically new idea for a line of attack is discovered – will remain the case for full analysis and set theory).

Chapter 2

Exegesis: textual refutation of the arguments for Friedman's interpretation of Kant's philosophy of mathematics

It is a common view that before the predicate calculus of Frege's *Begriffsschrift* (1879), there was no logical framework that allowed systematic and rigorous reasoning about relational structures. These being essential to mathematics, Bertrand Russell went so far as to claim that "there probably did not exist, in the eighteenth century, any single logically correct piece of mathematical reasoning".[1] After explicating the differences in reasoning about symmetrical and asymmetrical relations, he locates the essential limitation of traditional Aristotelian logic (restricted to subject-predicate sentences) in its failure to deal with the latter.[2] It was Kant's achievement, according to Russell, to have first realized the vital importance of asymmetrical relations to mathematics, as well as the impossibility of representing them in a formalism restricted to monadic predicates.[3] Understanding the limitations of his own formal logic, Kant consequentially "thought that the actual *reasoning* of mathematics was different from that of logic".[4] To account for this reasoning (and for its *a priori* character) he invoked another faculty of the mind, leading to his theory of pure intuitions:

> What is essential, from the logical point of view, is, that the *a priori* intuitions supply methods of reasoning and inference which formal logic does not admit; and these methods, we are told, make the figure (which may of course be merely imagined) essential to all geometrical proofs.[5]

Russell paints a sympathetic picture of Kant's philosophy, solving the vexed problem of determining the precise role of pure intuition in it, while simultaneously establishing an apparently unimpeachable reason for its rejection: scientific progress in logic.

[1] Russell [75, p. 463]
[2] Russell [75, p.220ff.]
[3] Russell [75, p.229f.]
[4] Russell [75, p. 464], original italics.
[5] Russell [75, p. 463]

Like Russell, Friedman sees the primary role of Kantian intuition, *not* in providing concrete realizations of abstractly defined structures, or in making true the basic principles of some, and ruling out other, logically possible axiomatic theories, but in supplementing and first making possible *methods of rigorous mathematical reasoning* unavailable in traditional logic: pure intuition does not provide an object or model to concepts or theories formulated in logical terms, but makes the very representation of and reasoning about such objects possible in first place. In particular, it provides the primitive forms of representing operations and iterative procedures that monadic logic cannot capture. For more details, see sections 1.2.5 and 1.4.2 and below.

The argument, again, is that Kant understood that traditional logic cannot express the essential structural features of mathematical objects. Friedman analyzes the type of reasoning allegedly provided by intuition—constructive operations based on substitutability and iterable applicability of functional terms—and the representation of mathematical content based on it in Kant's system. Crucially, he identifies Kant's notion of a concept's "schematism" with the intuitive constructive function generating individuals instantiating it, which goes beyond the representative means of general logic. This involves a reduction of the capacity of representing individual terms, functional operations, iteration, and substitution in mathematics generally to the foundations of arithmetic, unifying seemingly disparate aspects of Kant's theory.

In this section I evaluate Friedman's arguments for this interpretation. Some constructive criticism is first offered with the goal of rendering his position in its most convincing form. I show that its two most important aspects are (*i*) the argument for the thesis that Kant explicitly regarded his logic to be (equivalent in representational strength to) 'essentially monadic' (predicate logic), and (*ii*) the reduction of the capacity to operate with functional terms to a foundation of arithmetic in the pure intuition of time. These are the major points of attack for my critique of Friedman, beginning in section **??**.

2.1 Basic argument: Kant's theory of concepts

Russell and Friedman explain the alleged limitations of traditional logic by the absence of certain logical forms, which can be constructed in modern predicate logic using quantifiers and polyadic predicate (relation) symbols or functional terms. These logical forms allow the fully abstract representation of relational structures, most importantly *ordered* and *infinite* structures. A classical example for a representation of both kinds of structures is the set of the sentences

$$\begin{gathered}\forall x \neg R(x,x),\\ \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow R(x,z)), \text{ and}\\ \forall x \exists y R(x,y),\end{gathered}$$

which express, respectively, the irreflexivity and transitivity of the relation R and, for every individual in the domain of predication, the existence of at least one individual standing to it in that relation. It is easy to see that this set of sentences enforces an asymmetrical ordering between R-related elements,[6] and that it is unsatisfiable in any finite domain, i.e. that all its models, under any interpretation of R, must be infinite structures.[7] One can thus say that the sentences jointly *represent* the ordering and the infinity of its models in a purely formal way.[8]

2.1.1 The argument at B40

Friedman's interpretation is based on the premise that Kant knew about the limitations of traditional logic, and that this knowledge was a crucial factor in his philosophy of mathematics, and his foundation of theoretical cognition in the *Critique of Pure Reason* in general. The reasoning Friedman ascribes to Kant is the following.

Friedman's reconstruction of Kant's argument (FrK)

(1) Basic concepts have the logical form of monadic predicates.
(2) Finite combinations of monadic predicates are insufficient for representing infinite structures.
(3) Concepts are always finite combinations of representations.
(4) Therefore, purely conceptual representations are insufficient for representing infinite structures.

This reasoning, Friedman claims, motivated Kant's theory of pure intuition. His primary evidence is a passage at B40 in §2 of the "Transcendental Aesthetic", the "metaphysical exposition" of the concept of space, where Kant argues that the "original" representation of space, an infinite structure, cannot be a concept. I will sketch Friedman's interpretation of this passage and show that it misrepresents Kant's argument. The passage runs as follows:

> 4) Space is represented as an infinite **given** magnitude. Now one must, to be sure, think of every concept as a representation that is contained in an infinite set of different possible representations (as their common mark), which thus contains these **under itself**; but no concept, as such, can be thought as if it contained an infinite set of representations **within itself**. Nevertheless space is so thought (for all the parts of space, even to infinity, are simultaneous). Therefore the original representation of space is an ***a priori*** **intuition**, not a **concept**.[9]

[6] I.e., $\forall x \forall y \neg(xRy \wedge yRx)$. Observe that transitivity and $xRy \wedge yRx$ imply $R(x,x)$, contradicting irreflexivity. This excludes the possibility of 'loops' in the order.

[7] Assume for *reductio* that the sentences are satisfied in a finite domain with n elements. Since for every element x there is at least one y with $R(x,y)$, we can pick an arbitrary element a and move to some b with $R(a,b)$, then to some c with $R(b,c)$, etc.. After at most n steps, we will encounter an element already encountered before. Contradiction (comp. previous footnote).

[8] This could be called *extensional representation*, since it only requires that all models must exhibit the to-be-represented properties, without any requirement that the representation must, in a more precise intensional sense, *express* the meaning of the involved notions. While extensionality is insufficient for a coherent account of mathematical representation and semantics (comp. [**?**], [**?**], [**?**] [**?**]), it is certainly a necessary condition.

It should be noted that first-order theories with exclusively infinite models can never pick out a unique structure, or even characterize their structures up to isomorphism ('categoricity'). For this, some quantification over properties is necessary (but not necessarily full-blown second-order logic, comp. Isaacson's Thesis) Note that no first-order system can fully represent up to isomorphism, requires second order, essentially Leibnizian, $\forall x \forall y \forall P(x = y \leftrightarrow (P(x) \leftrightarrow P(y)))$

[9] *CPR* B 40

Friedman interprets this as an enthymeme:

(1) Space is represented as infinite.

(2) Concepts are always finite combinations of representations.

(3) Therefore no representation of space as infinite can be purely conceptual.

From this we would interpolate that Kant implicitly presupposed premises (1) and (2) of (FrK) as unstated premises. Thus, Kant appears to argue that (monadic) concepts are inadequate for representing infinity.

But this evidently is not what B40 says. Kant is not arguing that space is *represented as infinite* and therefore, no representation of it *as infinite* can be a concept, but rather that *the original representation itself* of space *is infinite* and thus not a concept: Kant states (contra Leibniz) that no concept, *qua* concept, can contain an infinite number of representations "within itself"; but our original representation of space does contain such an infinity; therefore, our original representation of space is not a concept.[10] This argument neither presupposes nor implies that conceptual representations cannot adequately characterize infinite structures, but simply that concepts *themselves* are not infinite. Confusion may arises because Kant seems to slur over the difference between the representation space, and the representation *of* space. But this is just the point of the "Aesthetic": space, as a (form of) intuition, is not a thing in itself, but merely a (form of) representation.

Kant defends four propositions in §2: 1) Space (itself!) is not an empirical concept. 2) Space is a necessary representation *a priori*. 3) Space is not a discursive concept, but a pure intuition. 4) Space is originally given as an infinite magnitude. Thus, space itself is a representation—not a concept, but an intuition. The specific differences are, firstly that concepts are general and intuitions individual representations, secondly that unlike intuitions, concepts cannot contain an infinity of representations "within themselves": considered by themselves, concepts are finite structures.

All of this is consistent with the existence of purely conceptual representations of infinity, e.g. containing abstract relational or functional expressions, or higher-order quantification. Indeed, their key advantage over strictly first-order monadic predicates is precisely that they allow to represent infinite structures using only finitely many expression. To argue, as Kant does, that a representation is not a concept because it is *itself* an infinite structure, does not exclude the possibility that there are concepts, themselves finite, which *represent* infinite structures. Kant could be making the very same arguments in §2 if he did admit, say, polyadic predication or functional expressions as belonging to pure logic. All he is saying in B40 is that concepts (unspecified whether functional, relational, monadic, first-order, higher-order, ...) are finite, and our intuitive representation of space is not.

One might argue that the statement that "every concept [is] a representation that is contained in an infinite set of different possible representations (as their common mark)" limits concepts to the general representation of *properties*, and therefore monadic predicates. This is not a tenable reading.

[10] Notice Kant's formulation: "no concept, as such, can be thought [...]. Nevertheless, space is so thought." Kant is comparing the internal structure of concepts to the internal structure of our original representation of space, and concludes that they are essentially different. He does not conclude that finite conceptual representations of infinity are impossible.

1. Quantity	2. Quality	3. Relation	4. Modality
Universal	Affirmative	Categorical	Problematic
Particular	Negative	Hypothetical	Assertoric
Singular	Infinite	Disjunctive	Apodictic

Table 2.1 The logical functions of judgment, A 70 / B 95.

If Kant's conclusion were identical with an instance of (4) of (FrK), one could read B40 as an enthymematic argument and infer the unstated premise that concepts are stricly first-order monadic. I have tried to show that Kant did not in fact argue for this, and that it is highly unlikely that he intended to.[11]

Friedman is quite aware that reference to B40 alone is insufficient to conclude that Kant denied the possibility of logically representing infinity. What is needed is a systematic reason for Kant to restrict his theory of conceptual representation to monadic predication. In light of that, B40 would provide powerful circumstantial evidence that Kant understood the limitations of such a logic.

2.1.2 The argument from the table of logical functions of judgment

Perhaps the strongest argument for Friedman's thesis occurs in a footnote:

> If we do not limit ourselves to the logical forms of traditional syllogistic logic, Kant's Table of Judgments makes no sense. It is ... plausible, I think, to equate Kant's concept of logic with, at most, *monadic* (or perhaps "essentially monadic") quantification theory plus identity.

This, I submit, gives us the most convincing version of Friedman's argument: Kant's basic logical form of judgment is the categorical subject-predicate sentence, in which a property is ascribed to an individual, or generally ascribed to a class of individuals, as a formally monadic predicate. All other logical forms are generated from this basis by the functions of hypothetical and disjunctive judgment, as

[11] If my interpretation in chapter 3 is correct, then he certainly did not. But in any case, it seems clear that Kant never meant to argue for (FrK 4) in the 'transcendental aesthetic'. He certainly held that knowledge of the infinitude of space and time cannot be derived from pure concepts—not, however, due to a lack of conceptual means to *represent* infinity (in Friedman's sense that the very idea of infinity "could not even be thought"), but because intuition is required to ensure the objective validity and universality of these synthetic *a priori* judgments: "These principles [Grundsätze] could not be drawn from experience, for this would yield neither strict universality nor apodictic certainty. We would only be able to say: This is what common perception teaches, but not: This is how matters must stand" (*CPR* B 47). The other passages that Friedman provides as evidence (A 25, deleted in the second edition, and *Prolegomena* §12) rather seems to confirm this. Kant is not worried there about the logical means of representation, but about the (synthetic *a priori*) *validity* of judgments about the infinitude of space and time, e.g., that we can legitimately "require" *that* a line can be drawn indefinitely long.

well as quantification, negation, and modal operators. This limits the available logical forms to those of 'essentially monadic' Aristotelian syllogistics.

Furthermore, Kant explicitly excludes other basic logical forms in the *Critique*: "the understanding is completely exhausted and its capacity entirely measured by these functions".[12] And since the categories are based on this allegedly complete table of logical forms, the limitation to monadic logic is inscribed into the very foundation of Kant's theoretical philosophy. Hence, Kant's logical theory of concepts is that of (essentially) monadic predication. Kant further restricts concepts to finite combinations of representations; any pure concept is therefore equivalent to a finite conjunction of simple concepts or their negations. And as B40 now strongly suggests, Kant understood that in this logical system, the purely conceptual representation of infinity is indeed impossible.

The problem with this line of reasoning is the following: even if one grants that Kant's formal logic is adequately reconstructed in terms of monadic first-order predicate logic, it does not follow that he was incapable of defining mathematical structures conceptually, and derive theorems logically. The reason is that transcendental logic, which provides the basic resources of concept-formation, is not correctly analysed as a framework whose conceptual resources extend at most to the structures definable in terms of monadic logic. Rather, it corresponds (roughly) to the contentual language in terms of which the semantics of formal logic are formulated. For more details, see section 4.1.5.1. For present purposes, it suffices to point out that what is conceptually representable in Kant's system is not restricted what is definable *inside* monadic predicate logic, but rather contains what is definable in terms of those concepts necessary to *understand* the primitive logical functions of monadic predicate logic. Friedman's argument is a non-sequitur.

2.2 Friedman's reconstruction of Kant's foundation of mathematics

The explicit representations of infinities of objects possible in quantification theory (predicate logic) are conceptually unavailable As a result of the inadequacy of monadic logic to represent an infinity of objects, the eighteenth-century mathematician relied on intuition for the representations necessary for mathematical reasoning. Friedman argued that Kant understood the inadequacy of his logical system to represent infinite structures, and consequently developed his theory of pure intuition as supplement. Thus without the intuition of time, it was impossible for Kant to represent *or even think* the idea of progressive iteration

> [T]he pure intuition of time is still presupposed by the science of arithmetic, not to provide that science with an object or a model, as it were, but rather to make the science itself possible in the first place. Without the pure intuition of time, according to Kant, the basic idea underlying the science of arithmetic, the idea of progressive iteration, could not even be thought or represented — whether or not this idea has a corresponding object or model. (p. 122)

[12] *CPR* B 105 - A 79.

The successor function expresses the general form of iteration fundamental to all functional operations; time alone, as the 'mere form of iteration in general', guarantees the existence and well-definedness of the successor function, and thus of all functional operations

> But the point, I think, is that the successor function is not a specific function at all for Kant; rather, it expresses the general form of succession or iteration common to all functional operations whatsoever. So it is not necessary to postulate any specific initial functions in arithmetic: whatever initial functions there may be, the existence and well-definedness of the successor function is guaranteed by the mere form of iteration in general (that is, time). (p.89)

2.2.1 Textual evidence for Friedman's interpretation

The crucial piece of evidence for Friedman's thesis that Kant derived the basic concepts of arithmetic – the iterated application of the successor operation in the synthesis of intuition, and with it the possibility of representing iterated function application in general – from the formal properties of temporal intuition is found in the chapter "On the schematism of the pure concepts of the understanding". As we saw, the notion of a schema is fundamental to Friedman's interpretation.

> The schema is in itself always only a product of the imagination; but since the synthesis of the latter has as its aim no individual intuition but rather only the unity in the determination of sensibility, the schema is to be distinguished from an image. Thus, if I place five points in a row, ... , this is an image of the number five. On the contrary, if I only think a number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand) in accordance with a certain concept than the image itself, which in this case I could survey and compare with the concept only with difficulty. Now this representation of a general procedure of the imagination for providing a concept with its image is what I call the schema for this concept. [...]
>
> The pure image of all magnitudes (quantorum) for outer sense is space; for all objects of the senses in general, it is time. The **pure schema of magnitude** (quantitatis), however, as a concept of the understanding, is **number**, which is a representation that summarizes the successive addition of one (homogeneous) unit to another. Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition. [...]
>
> Now one sees from all this that the schema of each category contains and makes representable: in the case of magnitude, the generation (synthesis) of time itself, in the successive apprehension of an object [...].[13]

[13] *CPR* B 179-184 / A 140-145

2.2.2 Schematic reasoning and the foundation of mathematics

Friedman is not claiming that arithmetic is necessarily about temporal objects according to Kant. Arithmetic strictly speaking does not have objects on this conception; it is not a theory, a logically organized body of truths about a specific domain of objects, but a collection of techniques of calculation for solving particular problems occuring in many different domains (containing temporal or non-temporal objects). Not the objects of arithmetic depend on temporal intuition (strictly speaking, there are no such objects), but the methods of representation, calculation, and reasoning that comprise it. The very "*concept* of magnitude (*quantity*) itself" cannot even be thought independently of the pure intuition of time, which alone allows us to represent finite iteration and infinite progressions.

The idea of elements of some type forming a simple infinite progression is represented purely conceptually, firstly stating that there is a distinguished first element of that type, that any element of that type has a unique successor also of that type, and that nothing else is of that type, and secondly by embedding such descriptions into a sufficiently powerful framework of rules of inference.

Friedman's claims are that Kant could neither formulate, in abstract terms, ideas like 'every number has a unique immediate successor', and especially, that he could not embed them into such a frame of reasoning.

According to Friedman, Kant was unable to conceptually express the contents of basic mathematical propositions. In particular, propositions that involve multiple generalities, and that express determination-relations between different individuals. For Kant, "this type of thinking essentially exceeds the bounds of purely conceptual, purely intellectual thought".[14]

As a consequence, Kant was unable to render inferences of the following kind as logically valid inferences:

> For any point $\boldsymbol{x}$ and any straight line $\boldsymbol{y}$, there is a circle with center $\boldsymbol{x}$ and radius $\boldsymbol{y}$.
>
> $\boldsymbol{a}$ is a point, $\boldsymbol{b}$ is a straight line.
>
> Therefore, there is a circle with center $\boldsymbol{a}$ and radius $\boldsymbol{b}$.

It is of vital importance to realise that it is just this kind of inference of which Friedman claims (and must claim) that Kant would have to hold that one can maintain the premises while denying the conclusion without entering into a logical contradiction.[15] Friedman is even committed to saying, not only that Kant was *de facto* unable to render this inference in his logical theory, but that he also would have explicitly denied its logical character.[16] But would Kant really have denied this? Would he have accepted that the conjunction of the premises and the negation of the conclusion can be affirmed without logical contradiction?

[14] Friedman [31, p. 293].

[15] Friedman [30, p. 82f.]. Although untenable, this thesis is not as immediately absurd as it might appear when presented in this light. Consider that this—formally quite complex—inference is not simply, say, *modus ponens*, even if one were to read the first premise as a conditional "*IF* point $\boldsymbol{x}$ and...$\boldsymbol{y}$...are given *THEN* circle with...$\boldsymbol{x}$...$\boldsymbol{y}$...is given", since in the conclusion, the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are particularized to $\boldsymbol{a}$ and $\boldsymbol{b}$, making the consequent of the major premise different from the conclusion, and thus the inference not a valid instance of *modus ponens*. It is precisely this management of quantified variables that Kant, according to Friedman, could not represent logically. I should add that *I do not mean to suggest that Kant would have considered this inference as a simple instance of* modus ponens*, or its major premise as a* hypothetical *judgment.* See chapter **??** for a discussion of Kant's formal treatment of such inferences.

[16] Show this

> For since one found that the inferences [Schlüsse] of the mathematicians all proceed in accordance with the principle of contradiction (which is required by the nature of any apodictic certainty), one was persuaded that the basic propositions [Grundsätze] could also be cognized from the principle of contradiction, in which, however, they erred; for a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself. (B 14)

Friedman reads this as Kant *not* claiming, at the outset of the passage, that every inferential step in a mathematical proof proceeds according to the principle of contradiction, but merely, that mathematical proofs contain *amongst others* also logical steps — the above inference to the existence of the circle with center $\boldsymbol{a}$ and radius $\boldsymbol{b}$ being an example of a strictly speaking *non-logical* step.[17] The English translation of B 14 might seem to provide some room for this interpretation; for example, are inferences that "proceed in accordance with the principle of contradition" actually valid in virtue of that principle, or simply not in violation of it? The German original does not afford this comfortable ambiguity. "Schlüsse" are not mathematical proofs, they are the individual inferences or proof steps. And "Schlüsse [die] nach dem Satze des Widerspruchs fortgehen" are inferences that enforce their conclusion, if the premises are accepted, on pain of logical contradiction. (Apart from the fact that one would thus have to assume the grossest negligence in Kant's formulations, Friedman's reading also makes nonsense of Kant's line of thought. For if not even all mathematical inferences (proof steps) are valid in virtue of the principle of contradiction, why should anybody deceive themselves in thinking that the basic propositions are?[18])

In the 'Transcendental Doctrine of Method', Kant writes: "Only an apodictic proof, insofar as it is intuitive, can be called a [mathematical] demonstration."[19] The distinction is that between intuitive and non-intuitive apodictic proofs, which lies in the difference of the "Beweisgründe", i.e., the sources of *truth and evidence of the premises* of the respective proofs.[20] But concerning the form of the inference-steps in a proof, according to B 14, it "is required by the nature of any apodictic certainty" that all of them are *logically* valid; thus "intuitive apodictic demonstrations" are logically valid proofs appealing to intuitively evident premises, and *not* arguments including formal-logically invalid but nevertheless intuitive steps. Assuming that Kant is consistent in his terminology, it follows that he regarded all steps in mathematical demonstrations as logically representable, and furthermore regarded some of these steps (those corresponding to the applications of constructive operations) as *sanctioned* by intuition, in that intuition ensures the truth and evidence of their premises (the paradigmatic example is still the inference about the circle we discussed above).

The assumption of consistency in Kant's formulations is validated by the section "On the logical use of reason" in the "Transcendental Dialectic", where he illustrates the notion of a theorem derived by explicitly *logical* inferences with his favorite example, *Elements* book 1, proposition 32 (angle sum of triangles).[21] The specifically "intuitive" character of mathematical demonstrations, which distinguishes them from

[17] Friedman [30, p. 82f.].
[18] The point was made by Tait [82].
[19] A 734 / B 762.
[20] *Ibid.*
[21] A 303 / B 359ff.

other proofs, thus lies in the fact that its synthetic premises are sanctioned by intuition. Concerning their form they must be apodictic, i.e., rigorous logical derivations. This contradicts the basic thesis of Friedman's interpretation.

I close this chapter by indicating that what I just argued for is quite consistent with those of Kant's statements about mathematical demonstrations that might seem to strongly support Friedman's interpretation. For example, Kant writes:

> In this fashion, through a chain of inferences led [geleitet] throughout by intuition, [the mathematician] arrives at a solution of the problem that is simultaneously fully evident [einleuchtend] and general. (A 717 / B 745)

Intuition provides according to Kant a "lead" for the inferences of the mathematician in (at least) two ways: logically, it sanctions the premises (in particular the existential assumptions) of her proofs and in fact provides intuitive operations *corresponding to* key-inferences in these proofs; and heuristically, it suggests possible solutions to problems by providing concrete structures that exhibit the relations represented logically by the involved concepts, which often makes it easier to 'see' the right way forward. (Although one might suggest that, in the latter regard, it leads us astray just as often...)

Chapter 3

Exegesis: textual justification of our interpretation of Kant's transcendental logic and his theory of mathematics

3.1 Preliminary remarks

This chapter contains my interpretation of Kant's "transcendental logic", more specifically, its theory of the *basic content* of objective representations based in the "pure understanding", as well as its role in the foundation of mathematics. As we saw in chapter 2, Friedman correctly points out that Kant realized that the general problem of the representation of rule-based iterative operations and of relational structure in mathematics can be reduced to the question concerning the foundations of arithmetic, because regarded formally, Euclid-style geometrical constructions do not go beyond arithmetically representable procedures.[1] According to Friedman, the representation of such procedures exceeds the resources that Kant ascribes to the 'pure understanding'. The aim of this chapter is to show that this is false, and to provide an alternative interpretation.

In section 3.2, I discuss the oft-cited 1788 letter to Johann Schulz, in which Kant defends his thesis that the judgments of arithmetic are synthetic while maintaining that temporal intuition "has no influence on the properties of numbers", that the "science of numbers" is "a pure intellectual synthesis, which we represent to ourselves in thought".[2] This allows to formulate, in the context of the foundations of arithmetic, the two central problems of this chapter: firstly, Kant's theory of the representation of operations (with individuals as 'input' and 'output') and relational structure; secondly, its relation to the pure intuitions of time and space.

In sections 3.3 and 3.4, I present two interpretation hypotheses answering to these problems. According to the first, Kant aims in the *Critique* at a theory that integrates two different logical aspects of the

[1] Friedman [30, p.88ff.]. Note that this does not mean that Kant believed that all of mathematics could be reduced to arithmetic. On the contrary, he regarded geometric *evidence* as indispensable, for example concerning the *existence* of irrational magnitudes. The above point refers solely to the formal properties of constructive procedures, not to whether the existential assumptions made at the outset of such procedures can in fact be 'cashed in', i.e., provided with objective reality. Comp. chapter 1.

[2] 10:554.

representation of objects: the *genetic representation* of structure through generating operations governed by *algorithmic rules*; and the *abstract representation* of structure on the basis of *primitive general concepts*, e.g., the general concept of a finite collection or sequence. According to the second hypothesis, *both* aspects belong to the theory of the "pure concepts of the understanding", which makes reference to "intuition in general" in the sense of concrete individual elements or manifolds assumed as given, but is *entirely independent of spatio-temporal forms of intuition*. Thus, in particular, the conceptual means for the definition of algorithmic rules do *not* depend on the intuition of time. Rather, intuitions are necessary to sanction the *existential presuppositions* ("objective reality") of mathematical representations.

Sections 3.3 and 3.4 are the center pieces of this chapter: they present the case for the first and the second hypothesis respectively. Section 3.5 contains additional textual evidence.

3.2 The letter to Johann Friedrich Schulz

In a draft of his *Prüfung der Kantischen Critik* (first edition 1789), Johann Friedrich Schulz (1739-1805) contradicted Kant by arguing that all judgments of pure arithmetic are analytic. Kant's rejoinder, in a letter from the 25th of November 1788,[3] is a crucial piece of evidence for the interpretation of his mature theory of mathematics. Written shortly after the publication of the second edition of the *Critique*, it interlocks seamlessly—argumentatively and terminologically—with the completely re-written core of the 'Transcendental Logic', the 'Analytic of Concepts',[4] thus providing the opportunity to formulate precisely the problems facing an interpretation of that theory in relation to the foundations of mathematics.

For Kant, quantity [Größe] is a general structural determination ("synthetic unity") of objects concerning their composition from homogeneous constituent parts.[5] In the Schulz letter, Kant states that the subject matter of arithmetic are not *quanta*, objects of intuition regarded as magnitudes, but *quantity*, "the concept of an object in general through quantitative determination". This distinction has a very precise sense. Quantitative determinations are general structural representations, i.e., concepts of formal structural properties of objects. For example, the number 5 is a general representation of a structural property of certain concrete objects, *viz.* either of the order-structure of all series—each regarded as *one* series—with five members, or the size-structure of all aggregates—each again regarded as *one* aggregate—with five elements (Kant does not give ultimate priority to either the ordinal or the cardinal aspect of natural numbers). But the "object" (in the sense of subject matter) of arithmetic, now, are not the concrete objects of intuition (the concrete series or collections) that fall under these concepts, but these concepts of quantitative determination themselves. Thus, as representations of intuitive objects, 8 or 90×2 are general concepts (concerning their structural composition); but for arithmetic, they are individual terms that refer to singular quantities, and *these* are the basic objects of arithmetic representation.

Kant also distinguishes the "subjective" and "objective" identities of concepts of quantities, i.e., their contentual or intensional, and their referential or extensional identity. Extensionally, concepts of quantities refer to quantitative determinations; in this respect, $3+5$, $12-4$, 2×4, 2^3 are identical. But

[3] 10:554-558. All unmarked citations of this section are from this letter.
[4] B 90-169.
[5] 18:337-8, B 201 fn., B 203f. Comp. Parsons [63].

contentually, such concepts are "made" by specifying various "manner[s] of synthesis" that determine different procedures of calculation. These intensional differences between concepts of quantities are the reason for the non-analytic character of (non-trivial) judgments of arithmetic, because the result of a to-be-executed operation is not contained in its specification—not the analysis of the specifications of these procedures (the algorithms), but their *execution* allows to assert or negate the extensional identity of intensionally different terms. The content of concepts of quantities are given by specifying "manners of synthesis"; their extensional identity (or difference) is established by "constructing" these concepts, i.e., by executing these manners of synthesis.

It might seem plausible to identify the distinction between "subjective" (intensional) and "objective" (extensional) with the distinction "computation/operational" and "structural", and this in turn with the distinction "intuitive" and "logical/intellectual". But this would be a mistake.

A key point of this chapter is to show that "synthesis", understood both active/operationally and static/structurally, belongs to the concept formation of the understanding. While this is commonly regarded as a standard move in Kant's analysis of cognition, Friedman's interpretation relies on its partial denial, particularly regarding the representation of mathematical operations. Showing that both the structural and the operational sense of the representation of "synthesis" in mathematics firmly belongs to the pure understanding allows to make sense of Kant unambiguous claim, following the above discussion, that

> Time, as you rightly remark, has no influence on the properties of numbers (as pure quantitative determinations), as it does on the properties of any alteration (as a *quantum*), which is itself only possible relative to a specific constitution of our inner sense and its form (time), and the science of number, regardless of the succession that any construction of quantity requires, is a pure intellectual synthesis that we represent to ourselves in thought. (10:555)[6]

3.3 Hypothesis I: Kant's theory of object representation

Kant's transcendental logic is to provide a general theory of the *content* of cognition by systematically exhibiting the elementary conditions of the representation of objects, a "logic that contain[s] merely the rules of the pure thinking of an object".[7] A common assumption is that transcendental logic is essentially formal logic supplemented with, or applied to, representations based on the *a priori* intuitions of space and time discussed in the Aesthetic.[8] Indeed, although in transcendental logic "we isolate the understanding [...] and elevate from our cognition merely the part of our thought that has its origin solely in the understanding",

[6] Though interesting, it won't be necessary to discuss Friedman's attempt to explain this passage. Stated briefly, he argues that Kant regarded arithmetic and algebra, not as theories about specified domains of objects, e.g. $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$, but as calculation techniques that are independent of any particular domain. Thus, its objects are independent of time, but its methods of representation and calculation are not. Kant is thus not saying that pure arithmetic is independent of temporal intuition, but only that, since it is not tied to any particular domain of objects, no assumption concerning the temporality or atemporality of the objects to which it can be applied is necessary. Since our basic disagreement concerns precisely the *representation* of arithmetic contents, we need not go deeper into this reasoning. Though he cites it at the beginning of this argument, Friedman understandably never returns to the passage above. The "properties of numbers", e.g. 'being a prime', are unthinkable without reference to the kinds of operations that Friedman argues were necessarily tied to temporal intuition. He cannot explain Kant's statement concerning their independence of time; at most he can explain it away.

[7] A 55 / B 80.

[8] Comp. Friedman [30, pp.96f., 104]. The best 'evidence' for this can be found at A76f. / B 102.

> [t]he use of this pure cognition [...] depends on this as its condition: that objects are given to us in intuition, to which it can be applied. (A 62 / B 87)

The *form* of these intuitions however—*how* they are given—is entirely irrelevant at the outset of the transcendental logic. As will be argued at length in section 3.4, the pre-supposed "intuitions in general"[9] are concrete individual representations. Apart from this, no further initial assumptions concerning their formal properties are made at the outset of Kant's theory of the representation of objective structure.

> The transcendental logic concerns the cognitions of the understanding with respect to their content, but independently from how objects are given. (17:651)

The spatio-temporal form only becomes relevant in a second step. I should emphasize, however, that I will not simply *presuppose* this. In particular, I will not presuppose that Kant regarded the representation of synthetic operations on the basis of algorithmic rules as independent from temporal intuition. Rather, this will emerge as a natural consequence of the interpretation developed in the present section. Section 3.4 then aims secure this conclusion against all doubts by a close exegesis of a systematically decisive passage.

The present section thus focuses on only the first step of Kant's theory, specifically its attempt at integrating two different aspects of the logical representation of objective structure: the *genetic representation* through generating operations on the basis of algorithmic rules; and the *abstract representation* on the basis of (primitive) general concepts. In section 3.3.1, I will introduce Kant's "concept of an object in general" by reference to the first version of the "Transcendental Deduction" (henceforth: A-deduction). We will see that the logical relationship between *concepts* of synthetic unity (structure) and *rules* for generating such structure remains unclear, which will be discussed in section 3.3.2. Only in the B-deduction, Kant reached an understanding that adequately reflects the logical complexity of this relationship, as will be argued in section 3.3.3.

3.3.1 The concept of an object in general in the A-deduction

The concept of an object that Kant develops in the Transcendental Deductions lies at the basis of his 'formal' nature concept (*natura formaliter spectata*), which is to provide a foundation for the nomological conception of empirical reality as relational structure governed by mathematically representable laws:[10]

> By nature (in the empirical sense) we understand the interconnection [Zusammenhang] of appearances as regards their existence, in accordance with necessary rules, i.e., in accordance with laws. (A 216 / B 236)

Accordingly, the basic concept of an object is that of a unity constituted wholly by and embedded into such law-governed relations.

> The inner determinations of a *substantia phaenomenon* [...] are nothing but relations, and it is itself entirely a sum total of mere relations. (A 265 f. / B 321)

[9] A 79 / B 104f., B 144ff. (§21), B 150ff. (§24-26).

[10] Comp. Cassirer [16, Ch. 2], Buchdahl [11], Schulthess [78, pp. 1-10], Falkenburg [25, pp.192ff.].

The constitution of such an object, its structural unity, is represented (at least in part) mathematically. But transcendental logic is not presupposing mathematics as an unproblematic frame of representation, rather, the conditions of possibility of mathematical representation itself are to be clarified.[11] In other words, transcendental logic is to account even for the representation of the objects of pure mathematics. In the A-deduction, Kant argues that the cognition of any object rests on a rule-based synthesis that determines the generation of its structural unity, and that makes possible a concept of this unity:

> Hence we say that we cognize the object if we have effected synthetic unity in the manifold of intuition. But this is impossible if the intuition could not have been produced through a function of synthesis in accordance with a rule that makes the reproduction of the manifold necessary *a priori* and a concept in which this manifold is united possible. [...] This **unity of rule** determines any manifold, [...] and the concept of this unity is the representation of the object = *X*. (A 105)

For example, we can "think a triangle as an object" through the general rule of construction operating on any three given straight lines pairwise greater than the remaining (*Elements*, book 1 proposition 22).[12] The rules that Kant has in mind are general representations of uniform procedures for generating "synthetic unity"—order or structure—amongst, or on the basis of, given individual elements of specified types. Such rules cannot be adequately represented *formally* by monadic propositions of the form 'All *A* are *B*' or 'if *p* then *q*', chiefly, because these cannot logically express (quantified) dependency relations between individuals. For example, even for a simple rule like Euclid's first postulate, whether it be formulated theoretically as "a straight line is uniquely determined by two points", or "given two points *a* and *b*, there is exactly one straight line *c* between them", or practically as "to draw a straight line from any point to any point", the determination relation between the individuals (the points and the line) is beyond monadic formulations.[13] The question thus arises: what exactly is the status of such rules in Kant's transcendental logic?

3.3.2 Rules and concepts in logical representation

Terminological convention: For convenience, and only in this section 3.3.2, I mean by 'concept' the general representation of a kind of object in terms of characteristic properties. An explicit reference to a generating procedure is prima facie *not* presupposed, and neither is any restriction on the means of concept-formation. By 'rule', I mean the representation of a manner of generating objects of a certain kind through an operation—acting on given elements of specified types—that produces the characteristic structure of such objects (the generality of such a rule lies in the generality of the specification of the types of the 'given'). We will come back to what the representation of such actions actually consist in. (Note that this does not capture the full meaning of rule concept, even as Kant used it.)

[11] A 149f. / B 188f., A 160 / B 199, A 733 / B 761.

[12] A 105.

[13] For example, the hypothetical proposition "if (*Point(a)* and *Point(b)*) then *Line(c)* simply fails to express that *c* is the line between *a* and *b*.

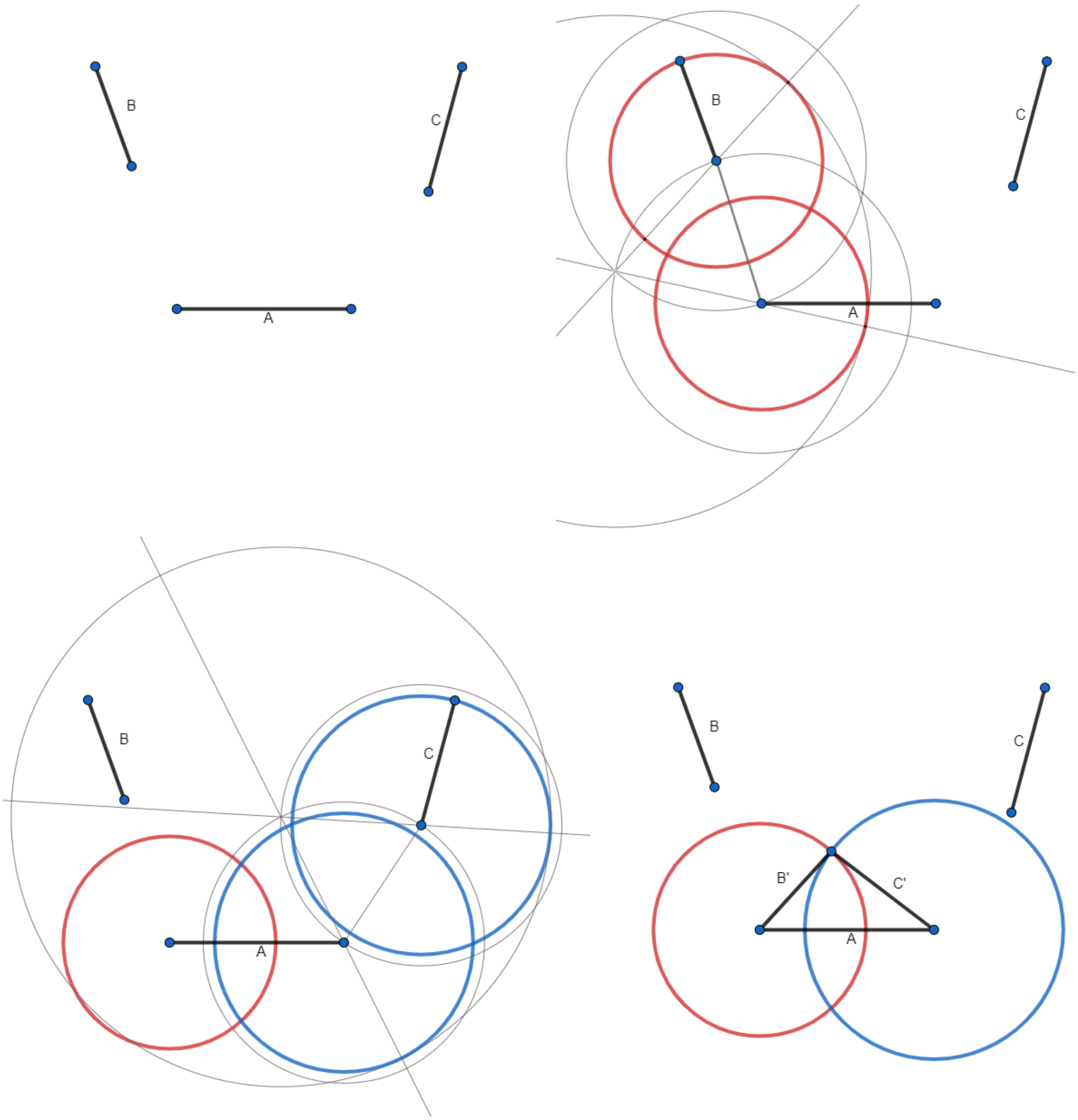

Fig. 3.1 Construction of a triangle from three given straight lines pairwise greater than the third according to Euclid's *Elements*, book 1, proposition 22.

As we saw, Kant argues in the A-deduction that a determinate conceptual cognition of an object rests on the representation of a synthetic operation generating the structural unity of the object. A difficulty in interpreting this passage is that Kant does not seem to distinguish clearly between concepts *of* "synthetic unity" and rules for *generating* such unity. On the one hand, he argues that rules "make" concepts "possible",[14] that concepts "rest" on functions.[15] On the other hand, "as far as its form is concerned", a concept "*is* always something general, and something that serves as a rule", and it "can *be* a rule [...] only if it represents the necessary reproduction of the manifold of given intuitions".[16] Thus, the "identity of the function by means of which this manifold is synthetically combined" consists in the "necessary unity of the synthesis [...] *in accordance with concepts, i.e., in accordance with rules* that not only make [the appearances] necessarily reproducible, but also thereby determine an object for their intuition".[17] Within a few sentences, Kant both distinguishes and identifies concepts and rules (I emphasize again that Kant's prime example of "rule" in this context is the general triangle-construction, fig. 3.1). What then is the relationship between concepts and rules?

There are of course very good reasons, well-known to Kant, to distinguish rules and concepts, and to not regard the possibility of the latter as absolutely dependent on the former. For one, a purely structural definition of an object is quite independent of a generating rule: understanding the concept of an isosceles triangle, a regular pentagonal dodecahedron, or the smallest odd perfect number does not require knowing the respective construction; indeed, evaluating the correctness of such rules already presupposes acquaintance with the concept.[18] Furthermore, the existence of a rule depends on what basic rules are allowed; and if this determined, then for any constructible structure there are in principle arbitrarily many different constructions, i.e., concept and rule do not stand in a one-to-one relation.[19]

The fundamental question, however, is what the general representations of structure-generating operations in terms of rules actually consist in. *What is a rule?* A complex rule like Kant's example of the triangle-construction (figure 3.1) results from linking elementary rules in an algorithm that determines the identity of a function of synthesis or "unity of action".[20] But what about the representation of the basic operations in terms of elementary rules?

There are, it seems, two alternatives. Either, an elementary rule is identified with a quasi-visual, kinematic, schematic representation of a type of (spatio-)temporal action that intuitively exhibits the general determination-relation between the given and their synthetic unity (e.g., rotating a line around a fixed point to generated a circle), and this "general procedure of the imagination for providing a concept with its image"[21] is taken as fundamental in the sense that its 'logical form' is not again representable in

[14] A 105
[15] A 68 / B 93.
[16] A 106, my italics.
[17] A 108, my italics.
[18] Here Kant's distinction between a *nominal* and a *real* definition of a concept is relevant. A nominal definition of a concept is a sufficient condition for determining whether a *given* object falls under the concept. A real definition is a sufficient condition to establish the "objective reality" of a concept, i.e., to exhibit a corresponding object. In the case of mathematical concepts, this is usually associated with the possession of a construction. Comp. A 241f.
[19] Just as any mathematical theorem has an in principle unlimited number of proofs. Comp. A 787 / B 815.
[20] A 68 / B 93, comp. B 153 discussed in section 3.4.1 below.
[21] A 140 / B 179f.

non-intuitive terms.[22] Or else, the determination-relation is expressed logically by specifying the types of the given and of their combination, the synthetic unity, in terms of abstract relational concepts, e.g. 'an object y of type B is determined by objects $x_1,\ldots,x_n$ of type $A_1,\ldots,A_n$ such that $R(x_1,\ldots,x_n,y)$', where (to avoid an infinite regress) the assumption that any $x_1,\ldots,x_n$ of type $A_1,\ldots,A_n$ can be combined in the way specified by $R(\cdot,\ldots,\cdot,\cdot)$ to determine a B is not again represented logically (and may be taken to depend on non-logical evidence); even the most simple of such logical representation are obviously beyond the means of monadic predicate logic.

Friedman of course ascribes the first alternative to Kant, and on reading the A-deduction, this might appear somewhat plausible. But there are serious arguments against this conception, both as an exegesis of Kant's position and as a general philosophical thesis. At the end of the A-deduction, Kant explicitly characterizes the understanding, in contradistinction from sensibility, as "the faculty of rules".[23] The "necessary unity of" and "all formal unity in" the synthesis of the imagination, i.e., the rules governing it (in particular including the formal representation of algorithms like the triangle-construction), derives from and is represented by the "pure understanding".[24] But there are also more philosophically substantial arguments against the idea that the logical rule-form is based in the intuitive imagination of kinematic acts. I will first present two of them, and then show that Kant indeed does justice to them in the second edition of the *CPR*.

3.3.2.0.1 (1) If some (spatio-)temporal manner of acting is to have any theoretical significance, i.e., enter into scientific concept formation and demonstration, then the *unity of this type of action* itself has to be conceptually comprehended, i.e., the given and the synthetic unity ('input' and 'output'), and possibly the inner structure of the act, have to be generally represented. But if this is possible at all, then conceptual resources that Friedman denies Kant must already be available; a capacity for imagining an operation can only provide an *evidential basis* for the objective reality or applicability of a rule-form, not the logical (though possibly the psychological) condition for thinking or formulating it or indeed for abstracting it from individual applications of the operation—on the contrary, the determinate 'thinkability' of any action is itself dependent on such conceptual means. (Versions of this argument made by Kant can be found *in abstracto* in the B-deduction, see section 3.3.3, *in concreto* in two letters, see section **??**.)

3.3.2.0.2 (2) Not every conceptual representation of objective structure can be reduced to rules of synthesis in the above sense (and thus *a fortiori* not to procedures of the imagination). Instead, the representation of objects in terms of such rules presupposes primitive concepts that cannot themselves rest on such rules, but rather provide the "general condition for rules", i.e., the means to conceptualize or define rules.[25] As a consequence, I claim, only the second of the above alternatives remains as an viable analysis of elementary rules. I will first develop the arguments for these claims independently from an interpretation of Kant; in the following section 3.3.3, I will show that Kant's does justice to them.

[22] 'Logical form' is placed in scare-quotes since on this account, the form of rules is not of course properly speaking *logical* at all.
[23] A 126.
[24] A 119, 125. In the B-deduction, Kant makes this claim even more forcefully, see section 3.4.1.
[25] A 135 / B 174.

Perhaps Kant was, for a time, convinced that *every* generality in conceptual object representations is based on, or at least always associated with, a rule defining an "identity of synthesis" that determines an isomorphism between the generation procedures for the various concrete objects falling under the concept (in the sense in which the Euclidean construction-rule for a triangle induces an isomorphism (modulo the lengths of the given lines) between all acts of constructing triangles according to that rule).[26] This analysis of 'generality' reaches its limit, however, when elementary rules, and in particular, when the foundations of arithmetical representations are in question (it also flounders on a general concept like 'geometrical figure'). A key example for this is the general concept of a finite number or a finite sequence. To see the difference, consider the generality of the concept 'regular pentagon inscribed in given circle', if we associate its "real definition" with the algorithm of *Elements* book 4, prop. 11, and the generality of the concept 'finite number'. In the former case, two distinct figures fall under the same concept due to the "identity of the synthesis" in their respective generation, only applied to different parameters. But the reason that 4 and 5 both fall under the same concept 'number' is precisely not because they can be generated by the same procedure. Rather, they are different *numbers* precisely because the respective synthesis generating them are not identical. It would be circular to reduce the generality of 'finite number' to the operation $x+1$, only applied to different parameters, because the variable x must already be presupposed to range precisely over the finite numbers. But what status then does the general concept of finitude, of finite sequence, or of natural number have? (I take it for granted that if either of these concepts is assumed, the others are easily defined.)

The following argument is based on Gaisi Takeuti's discussion at [84, p. ??] Let's assume that the general concept 'natural number' is to be explained by the following rules (obviously, only (N2) is an instance of the narrower sense of 'rule' used above):[27]

(N1) 1 is a natural number.

(N2) If a is a natural number, then $a1$ is a natural number.

(N3) Only the objects obtained by (N1) and (N2) are natural numbers.

It can be shown that such a 'definition' cannot explain the concept of a natural number without already implicitly presupposing it, or an equivalent general concept like 'finite iteration of a concrete operation' or 'finite sequence'. The variable a in (N2) ranges over *all* natural numbers. A criterion for what objects this rule is applicable to, and thus an account of the meaning of the rule, can only be given by an explanation in the fashion of "exactly those that, starting from 1, can be reached by repeated application of the operation (N2)", where "repeated application" already implies "*finite number* of repetitions". For imagine someone who claims not to understand (N2), in particular its condition "a is a natural number". A didactic

[26] According to some interpretations, Kant believes that the generality of determinate conceptual representations like 'Triangle', 'Conic Section', and even 'Dog', resides in functions with various parameters, such that variation of the parameters allows to run through all representations of possible individuals falling under the respective concept. The generality consists what remains invariant when varying the parameters, and this invariance is based precisely on the identity of the generation procedure that defines the operation. For example, take the function for constructing a triangle from three given straight lines. If one varies the lengths of the given lines, one can run through an infinite number of different triangles. This idea plays a big role in the Kant-interpretation of Ernst Cassirer and, in its tradition, Karl-Norbert Ihmig, who connects it to Poncelet's work in projective geometry, comp. Ihmig [50] and the works referenced therein.

[27] The following discussion is strongly influenced by Takeuti [84, pp.88f.] and Tait [83, pp.25f.].

explanation of the form "By (N1), 1 is a n.n.; thus, by (N2), 11 is a n.n.; thus, by (N2), 111; thus,..." will invariably end with "and so on", which leads us back to the above explanation in terms of the *general* notion of 'finite iteration', i.e., the concept covering *all* finite iterations or sequences.[28] The moral is that "we must accept or presuppose to some extent the notion of finite sequence (or finite iteration of an operation) as our basic notion."[29] This general notion cannot, in turn, be reduced to a system of rules of action ("rules" in the strict sense of the terminological convention on page 81). At most, such a system can bring out clearly various aspects and implications of this basic notion. Along similar lines, William Tait argued that the general concept or "generic form" of a finite sequence is logically prior to representations in terms of rules like (N1-3) above. Accordingly, our understanding e.g., of $n+2$ does not consist in our understanding each of the instances $1+2$, $2+2$, *and so on*, individually, but in our understanding of what it means in general for a finite sequence to be the two-element extension of another.[30]

We have, or at least we presuppose, the primitive and general concepts of an individual unit, of a collection of discrete units (indeterminate), and of a determinate, finite collection of discrete units, as well as the concept of adding a unit to a collection of units. In terms of these basic *contentual* concepts, we can define the concepts of arithmetic. These concepts are certainly *formal* in the sense that they do not specify what kind of objects or units we are dealing with—merely the 'concept of an individual object in general'—but they nevertheless have a *content*, concerning formal (structural) determinations of objects in general, such as the composition of a complex whole from given homogeneous units; this content cannot be defined in terms of more basic, primitive notions without circularity. It is of course possible to define the natural numbers in set-theory, but this clearly just pushes the buck down the line, and if we are after a conceptual *foundation*, might quite plausibly be circular.[31]

One conclusion is that I do not believe that an implicit definition in predicate logic of the concept of natural number, or of finite iteration, does what Friedman seems to thinks it does. Friedman argues that, because Kant could not formulate propositions of the form $\forall x \exists y(y = Sx)$, he "was quite unable even to *think*" the concept of a natural number without invoking intuition. Leaving aside Kant until the next section, one can say generally that the capacity to formulate such propositions in a formal calculus does nothing towards making it possible for us to think the concept of natural number; rather on the contrary, if we could not think the concept of natural number, we could not formulate such propositions—*not simply because we wouldn't know which axioms to formulate, but because the metatheory and semantics of the formal language, in terms of which the axioms represent structures, presuppose the general concepts of finitude or natural number.*

---

[28] Notice that an axiomatization of the theory of natural numbers in predicate logic is of no help here. A formalism can only represent a structure via an *interpretation*. Thus, in first-order predicate logic—apart from the fact that an extensionally definite characterization of the natural numbers is impossible anyways as no first-order theory individuates the natural numbers up to isomorphism—to recognize essential properties of the represented structure, e.g. the infinity of an infinite progression, requires *metatheoretical* arguments, which are only possible in terms of the semantics or interpretation of the formalism, to which the discussed argument remains applicable. In second-order predicate logic, the much stronger assumptions of the general concept of *all* sets of natural numbers is presupposed.

[29] Takeuti [84, p.89].

[30] Tait [83, pp.25f.].

[31] Comp. Putnam [69], Field [27]. Given the paradoxes surrounding the referential indeterminacy of the concept 'set' as well as the profoundly mystifying behaviour of sets with large cardinals, it seems save to say that, if we are worried about a foundation of our mathematical knowledge, the concept of finitude or natural number is in far less need of clarification.

If one grants that the axiomatization of arithmetic does not properly speaking *define* its basic concepts — not only because unless one employs higher-order quantification it fails to uniquely determine their extensions, but because it turns out the general notion of finitude is a necessary precondition for both logical and arithmetic thought, rather than something that can be 'thought in terms of something else' itself in less need of a foundation — one might then be inclined to suggest that this basic notion must therefore be an affordance of intuition (Gödel thought this about the much more problematic concept 'set'). But we will see that at least on Kant's technical notion of intuition, this does not work; the concepts of arithmetic are "pure concepts of the understanding", and its synthesis is "a pure intellectual synthesis that we represent to ourselves in thought".[32]

3.3.3 "The possibility of a combination in general", the basic concepts of arithmetic, and the representation of elementary geometrical operations

I will now show that in the re-worked foundation of the transcendental logic, the B-deduction, Kant does justice to both arguments **(1)** and **(2)** of the previous section.

Recall that according to **(1)**, the representation of an elementary operation of synthesis *presupposes* a concept of the generated synthetic unity (as well as a specification of the 'given'). The mere capacity to act in a certain way is not a sufficient basis for making cognitive use of this capacity, e.g., in mathematical proofs. In Kantian words, required is the "consciousness" of the synthesis, particularly of the "unity of the action", which consists precisely in a conceptual specification of the given and the structure generated from it (the "synthetic unity of the manifold")[33]—only this allows us to "think" the operation. These purely intellectual conditions to think abstractly various types of structures, rules, and procedures are not made possible by the action, but are rather the conditions for thinking the action, and provide thus, to co-opt Friedman's wording, the possibility of a kind of rigorous representation of—or more precisely, the possibility of a kind of rigorous reasoning with—such operations.[34]

In the first paragraph of the B-deduction (§15, B 129-131), Kant develops his theory "On the possibility of a combination in general". He points out that *any* combination of a manifold in general ("**Verbindung** (*conjunctio*) eines Mannigfaltigen überhaupt") is an "action of the understanding, which we would designate with the general title **synthesis**".[35] One example of such synthesis is the composition of some geometrical figure from two distinct given figures,[36] but also that of an infinite totality from abstract individuals *assumed* as given.[37] (The crucial point is that the operation of combining given individuals is a uniform action of the understanding, whether these individuals are spatial, temporal, or something

[32] 10:558.

[33] B 138. Comp. also B 151, 154, discussed in section 3.4.1, 20: 271f., see section **??**, and 12:222, see section 3.5. Interestingly, this mirrors Kant's views on the notion of "praxis" in practical philosophy, not merely as a mechanical process or the unconscious acting of an organic being, but as the effecting of a purpose through and on the basis of a conscious representation of the determinate principles governing the procedure, comp. Henrich [42, p.11]. Also comp. Tiles: "Kant distinguishes the rational (and thereby also moral) being from the non-rational on the basis of its capacity to act not merely according to a rule (or law) but according to its conception of the rule [reference: 4:412]. This is the capacity on which the possibility of logic, mathematics, scientitic knowledge and morality depend. [...] It connects thought with action and action to thought via the thought of action", [86, p.232].

[34] Comp. Friedman [30, p. 126]

[35] B 129f.

[36] B 201 fn. On this particular example, see section **??**.

[37] See section 1.3.5 above.

else.) Kant consciously plays with the act-object ambiguity of the word "Verbindung" between the act of combining and the resulting combination. Initially, he seems to identify the two aspects; then, to move fully to the act-side ("all combination[...] *is* an action[...] which we would [name] synthesis"); and almost immediately following this, we finally read:

> But in addition to the concept of the manifold and of its synthesis, the concept of combination also carries with it the concept of the unity of the manifold. Combination is the representation of the **synthetic** unity of the manifold. The representation of this unity cannot, therefore, arise from the combination; rather, by being added to the representation of the manifold, it first makes the concept of combination possible. (B 130f.)

The basic representation of the synthetic unity of the manifold, of the "combination" in the objectual or structural sense is thus not a product of the action of synthesis, but a condition of it. The basic or primitive concept of synthetic unity, which is presupposed in the representation of any operation of synthesis, is that of a *unified relational structure* constituted from various elements or parts of a manifold.

> The relation of the many amongst each other, insofar as they are comprehended into one, is the **combination** [Verbindung]. The combination according to a rule: **order**. (18:343)

Kant illustrates this with the example of the relations between individual musical notes in time, constituting the synthetic unities of harmony and melody.[38] But note again, that the synthesis is repeatedly and explicitly claimed to be independent of *specific forms* of intuition, including the temporal form!

The basic representation of a relational structure constituting the unity of a manifold is a *general concept of structure*, logically prior to its various specifications. Although it is a *formal* representation—in the sense that it does not yet contain specifications, neither of the manifold, e.g., concerning its or its elements' qualities or quantities, nor of the kinds of relationships amongst them that constitute their synthetic unity—this must not be taken to mean that it does not have a *content*. Quite on the contrary, its content is just that of a unified relational structure, a unity constituted by given elements; this is the basic 'formal content' in terms of which the contents of more specific representations of synthetic unity can be defined. More on these possibilities of concept-formation in a moment.

In his seminal paper on set-theory, Gödel noted the "close relationship" between his iterative conception of set and Kant's notion of combination, "the function of both" being "the generating of unities out of manifolds" previously given.[39] Among the main differences between the two notions are, firstly, that unlike Gödel, Kant does not presuppose a basic domain of well-defined *discrete* individuals, rather, the constitution of such individuals from the undifferentiated manifold of pure intuition (the 'carving out', so to speak) is itself a product of synthesis (section **??**); and secondly, that while Gödel's sets are uniquely determined by their elements ("extensionality"), for Kant it is precisely the relation *amongst* the parts of

[38] He also points out that, in the human mind, all order is apprehended temporally. That this must not be taken to suggest that he regarded the concepts of combination and order as derived from temporal intuition will be argued at length in section 3.4.

[39] Gödel [37, p. 484 fn. 26, see also pp. 474f., p. 475 fn.10-12]. Gödel distinguishes his notion of sets obtained by iterated application of the basic operation "set of x's" starting from a well-defined domain of discrete individuals, from that of sets obtained by "dividing the totality of all existing things into two categories", i.e., using unrestricted comprehension principles, which famously leads to Russell's and related anatomies. Interestingly, Gödel remarks that the operation of set-formation is indefinable ("at least [...] in the present state of knowledge") without appeal to expressions again involving the concept of set, such as "multitude", "combination", "part", comp. [37, p. 475 fn.11].

the manifold and their unification into *one* structure (of a specific kind) that is essential to the "synthetic unity". The same elements might thus constitute different synthetic unities, depending on how they are combined.

The key argument of the Transcendental Deductions is that the primitive representation of the synthetic unity of a manifold (of numerically distinct individual objects connected into one unified structure) is a necessary condition of the "identity of apperception", i.e., the numerical identity of the self-conscious subject. Dieter Henrich's influential reconstruction of it is guiding our discussion below.[40] Note that

the concept of the **composited** in general is no particular category. Rather, it is contained in every category (as **synthetic** unity of apperception). For the composited cannot, as such, be **intuited**; rather the concept or the consciousness of the **compositing** [des Zusammensetzens] (a function that, as synthetic unity of apperception, underlies all categories as their foundation) *must be presupposed* [vorhergehen] in order to think the manifold given in intuition as combined in one consciousness, i.e., in order to think the object as something composited. (12:222)

Synthetic unity is the general condition of the structural determination of a manifold, the condition according to which the to-be-unified manifold is specified, e.g., the manifold of points with distance x from point y; the comprehension (unification) of this homogeneous manifold constitutes the representation of the circle with radius x and center y.

> Rule is: the generality of the condition in the determination of the manifold.
>
> Or it is the unity of the determination, under which something is determined in a generally valid way.

3.3.3.1 Example: the rule for constructing a circle

Kant indeed extends to geometry the claim that all synthesis is an operation of the understanding, and that the concept of the to-be-generated synthetic unity — as a condition for the representation of and reasoning with, rather than a product of, the operation — is entirely based in the pure understanding. — run through three version of this argument, finally: show on the example of circle how exactly

In §17 (B 136-39), the basic principle of the synthetic unity is explained as entirely independent of all sensible conditions. Space, as a "mere form", only gives an indeterminate manifold without any representation of its structure ("unity"). To cognize something in space requires a determinate combination of the given manifold to be produced. The "unity of this action" is based in the unity of apperception, specifically, in the concept of the to-be-generated structure. Thus, the basic representation of synthetic unity, in terms of which a spatial structure is determined, is explicitly identified with the unity of apperception, which is introduced as independent of all particularly spatial aspects.[41]

Virtually the same argument is repeated in §26 (B 159-163). Kant distinguishes space and time regarded as "mere forms of sensible intuition" (containing only indeterminate manifolds without any structural determination) from space and time regarded as unified structures:

[40] Henrich [40], [43].

[41] B 137f.

> Space, represented as an **object** (as is really required in geometry), contains more than the mere form of intuition, namely the **comprehension** of the manifold, given in accordance with the form of sensibility, into one **intuitive** representation, so that the **form of intuition** merely gives the manifold, but the **formal intuition** gives unity of the representation. (B 161 fn.)

The German wording makes it unambiguous that what is given "in accordance with the form of sensibility" is the manifold, not its comprehension. This unification, which allows to think space *as an object*, is the specific contribution of the understanding:

> Thus even the **unity of the synthesis** of the manifold, outside or within us, hence also a **combination** with which everything that is to be represented as determined in space or time must agree, is already given *a priori*, along with (not in) these intuitions, as condition of the synthesis of all **apprehension**. But this synthetic unity can be none other than that of the combination of the manifold of a given **intuition in general** in an original consciousness, in agreement with the categories, only applied to our sensible intuition [where "sensible intuition" includes the *a priori* forms of sensibility].

At the basis of the comprehension of a determinate figure in space lies the conception of

> the **necessary unity** of space and of outer sensible intuition in general, and I, so to speak [gleichsam], draw its shape in accordance with this synthetic unity of the manifold in space. This very same synthetic unity, however, if I abstract from the form of space, has its seat in the understanding, and is the category of the synthesis of the homogeneous in an intuition in general, i.e., the category of **quantity**. (B 162)

The "drawing" that provides a unified spatial representation is *based on*, "in accordance with", the synthetic unity of the manifold; and the "very same synthetic unity", if one abstracts from the particularly spatial form, is entirely based in the understanding. In this "way it is proven" that the synthesis of generating determinate figures in space

> must necessarily be according to the synthesis of apperception, which is intellectual and contained in the category entirely *a priori*. It is one and the same spontaneity that, there under the name of imagination and here under the name of understanding, brings combination into the manifold of intuition. (B 162 fn.)

The point is thus emphatically not that the understanding's operations are essentially based in the imagination. Rather, it is the other way around: in the synthesis of the imagination, the determining factor is the essentially intellectual spontaneity, which is logically independent of that imagination, being merely *applied* to the particular forms of sensible intuition. Keeping in mind Kant's principles about the "combination in general" from §15, firstly, that the representation of *any* synthetic operation *presupposes* the representation of the synthetic unity of the generated structure, secondly, that the act of generating such structure is "exclusively an operation of the understanding",[42] and thirdly, that the representation of

[42] B 130, B 135.

the synthetic unity is defined in terms of the pure categories, we must conclude from the above that Kant holds that the definition of geometrical concepts, regarding their formal relational or structural contents, must be defined entirely in terms of the categories.

And indeed, this is precisely what Kant argues for. In the *Prolegomena* (§38), Kant repeats the above argument again in perhaps even clearer words (my **emphases**):

> If one considers the properties of the circle by which this figure unifies in a universal rule at once so many arbitrary determinations of the space within it, one cannot refrain from ascribing a nature to this geometrical thing. Thus, in particular, two lines that intersect each other and also the circle, however they happen to be drawn, nonetheless always partition each other in a regular manner such that the rectangle from the parts of one line is equal to that from the other. Now I ask: "Does this law lie in the circle, or does it lie in the understanding?" i.e., does this figure, independent of the understanding, contain the basis for this law in itself, or does the understanding, **since it has itself constructed the figure according to its [the understanding's] own concepts (namely, the equality of the radii)**, at the same time insert into it the law that chords cut one another in geometrical proportion? If one traces the proofs of this law, one soon sees that **it can be derived only from the condition on which the understanding based the construction of this figure, namely, the equality of the radii**. (*Proleg.* §38, 4:320f.)

Kant again distinguishes between the sensible aspect of the form of space as providing a mere indeterminate manifold, the operation of synthesis by the understanding (construction), and the conceptual representation of the synthetic unity, the understanding.

The general concept of a circle, the equality of the radii, is "the condition on which the understanding based the construction of this figure", which is the basis for the unity of the construction of the various figures (comp. "rule" general condition in the determination of the manifold). The understanding thus constructs the figure according to "its own concept", and thereby brings about not only the unity of the figure, but also the possibility for deriving general laws about it, and of embedding it into a theoretical context (series of connections to conics, etc.) The condition of the construction is the *concept* of the synthetic unity, the presupposed representation of the to-be-generated structure, which is a **real** (as opposed to nominal) definition of the type of structure.

The theorems of geometry thus do not "lie in space", rather, "they lie in the understanding and in the way in which it determines space according to the conditions of the synthetic unity toward which its concepts are one and all directed":

> [T]hat which determines space into the figure of a circle, a cone, or a sphere is the understanding, insofar as it contains **the basis for the unity of the construction of these figures**. The bare universal form of intuition called space is therefore certainly the substratum of all intuitions determinable upon particular objects, and, admittedly, the condition for the possibility and variety of those intuitions lies in this space; but **the unity of the objects is determined solely through the understanding, and indeed according to conditions that reside in its own nature**. (*Proleg.* §38, 4:321f.)

How then does Kant define the general concept of the synthetic unity "circle" *in terms of the categories*, such that it can be employed in the representation of the elementary operation of constructing circles, and such that this representation can play its part in the logical proofs of geometry? Given our thesis, that the basic concept underlying all categories is that of a relational structure constituting the unity of a given manifold, this is not difficult to see. Roughly, circles are defined as a type C of object determined by two objects of different types P and L, such that all parts of C stand in the same quantitative relation (determined by L) to their P. Dressing this definition up with geometrically interpreted terms, one can arrive at the intuitive idea of a circle, which provides objective reality.

3.4 Hypothesis II: the independence from spatio-temporal intuition

Against the idea that the structure-concept of discrete sequence or aggregate of units (or the finite iteration of an operation) is based on *a priori* intuition in Kant's sense, at least two general philosophical objections can be made, which I briefly discuss before leading over to the justification of the second interpretation hypothesis.

Firstly, the problem of 'generality' discussed in section 3.3.2 rears its head again. A rule like 'every finite sequence of units can be extended by one unit', which is necessarily connected to the concept of finite sequence (expressing the well-definedness of the successor operation), obviously refers to *all* finite sequences. But this kind of generality is not afforded by Kantian forms of intuition; for these only provides general representations of various types of *individual* sequences. For example, the form of the intuition of the sequence 111111111111 is general in the sense that is the basis of an equivalence relation (similarity type) between all instances of the type 'sequence of twelve ones' (or, if we abstract from the kind of unit, twelve concrete elements or moments). What we do not have is an intuition, a concrete and immediate representation, of *all* types of finite sequences.

What is true is that *any* type of finite sequence can be exhibited in intuition. But what is not true is that the basic concept of finite sequence, which covers all finite sequences, is a concept derived from intuition.

To achieve *intuitive* generality for the rule, 'any finite sequence can be extended by one', one would have to imagine an *arbitrary* finite sequence, which raises a far more difficult variation on Locke's and Berkeley's puzzlement about the 'general idea of a triangle' ("neither oblique nor rectangle nor...").[43] Neither imagining a sequence vaguely, "without imagining its internal structure clearly enough" to determine a particular n of units and "see[ing] the irrelevance of this internal structure", nor yet taking some particular string for "paradigm" really works here.[44] In the second case, it is unclear how the intuition of some concrete sequence could provide knowledge about all sequences whatsoever. In the first case, what can the "irrelevance" consist in? The generality of geometrical constructions lies in the irrelevance of particular determinations, e.g., lengths, angles, of the given elements (so long as they fulfill the stated conditions) to the procedure of step-by-step construction and reasoning. [45]

[43] Locke [56, Bk IV ch7. sec.9].
[44] Parsons [62, p. 106].
[45] Hale [38, p.].

Secondly, the fatal flaw of attempts to base the original representation of sequences on the intuition of time is that to recognize a temporal succession of discrete events requires to recognize it *as* a sequence, *as* a unified structure constituted by a multiplicity of discrete units of homogeneous type, i.e., the general concept of finite sequence is logically prior to the representation of temporal succession. This is Russell's objection to Kant's alleged thesis that the basic concepts of arithmetic are drawn from temporal intuition: "it seems to be refuted by the simple observation that time must have parts, and therefore plurality, whole and part, are prior to any theory of time";[46] it is also Tait's argument against Brouwer and Kant:

> To experience succession in time is to experience a particular sequence of events *as* a sequence. And it is this, our understanding of the idea of a finite sequence and our ability to apply it in experience, that is basic to our reasoning about and application of numbers. (Tait [81, p.176])

And ironically, it is Kant's own argument when, in §24 of the B-deduction, he says of "the concept of succession" that "such a combination of the manifold is not found" in temporal intuition, but that it is "first brought about" when that intuition is determined by the application of the pure concepts of the understanding. That very same argument is later repeated in a letter by Tieftrunk to Kant, and in his response, Kant takes it as the starting point for re-explaining the relationship between pure concepts and schemata in his theory (see section 3.5).

3.4.1 Intellectual synthesis in the B-Deduction: a close reading

In this section, I want show that Kant regarded the formal representation of operations of generating structure – essential both to arithmetical calculation and geometrical construction – as being based entirely in the pure understanding: the formal aspect of the rules governing such operations can be reduced to the *synthesis intellectualis* that is thought in the non-schematized categories. Diametrically opposed, Friedman's main thesis is that for Kant, only by the *schematization* of *a priori* concepts "the possibility of a kind of rigorous representation of – or more precisely, the possibility of a kind of rigorous reasoning with – these concepts" is provided, and in particular, that it is the non-logical *intuitive* aspect of the schematism which alone can provide the "form of rational representation", of "rational argument and inference" that is characteristic of mathematics.[47] Friedman and I agree that when, in the context of the "schemata" of mathematical object-concepts, Kant speaks of a "rule of synthesis", he means a constructive rule for generating the elements in the extension of the concept.[48] For example, a schema of the concept 'Circle' is a general operation that yields a circle from any given point and straight line. This way of speaking, however, slurs over a crucial distinction: between the rule-form as a logical representation, and the properly so-called *geometrical* operation on a manifold of spatial intuition. Of course, that Kant could not possibly draw this distinction is exactly Friedman's thesis; and consequentially, he plainly *identifies* the schema with the rule.[49]

I deny that Kant regarded the formal aspect of these rule-forms as a contribution of the intuitive (spatio-)temporal aspect of the schematism. My strategy to show that there is a direct and continuous

[46] Russell [75, p. 464]
[47] Friedman [30, pp. 94f., 126]
[48] Friedman [32, pp. 236f.]
[49] Friedman [32, pp. 236f.]

line of argument that can be traced from §15 through §21 and especially §24 to the Schematism-chapter, which definitively refutes Friedman's reading. Seeing this requires a somewhat pedantic exegesis of rather difficult pieces of text; the pay-off is that Friedman's interpretation can be seen—in my opinion conclusively—to be inconsistent with the official and argued-for doctrine of the *Critique*.

3.4.1.1 *Synthesis intellectualis* and the transcendental synthesis of the imagination in §24

Kant evidently intended §24 to serve as a link between the theory of the *synthesis of the pure understanding*, i.e., the first part of the B-deduction, and the theory of the *schematism*: in the latter, the "transcendental synthesis of the imagination" plays the key role of facilitating the application of the categories to empirical objects—and in §24, this notion is provided its theoretical foundation. Kant here also introduces his term "*synthesis intellectualis*", which we encountered in our discussion of the Schulz-letter in section 3.2. Close attention to §24 is thus doubly crucial for our controversy with Friedman. Firstly, because of the central role Friedman ascribes to the schematism in Kant's philosophy of mathematics. And secondly, because of Kant's explicit characterization of "the science of number" as a "pure intellectual synthesis that we represent to ourselves in thought", whose subject matter is "the concept of an object in general through quantitative determination" on which "our inner sense and its form (time)" has "no influence"[50] —these claims must be interpreted in their proper context, and that is the §24. (I invite the reader to pay particular attention to Kant's choice of words in this quotation, and to compare it to that of §24; also note the close proximity of the times of writing of these passages.) We will see that the two issues, schematism and intellectual synthesis, are most intimately connected.

The text analyzed in this section is very dense, yet I found it unavoidable to quote it extensively. For the convenience of the reader, and because of the importance to my argument, I therefore added a concise bullet-point abstract of the core argument of §24 using Kant's own words at the end of this section (p. 95). I hope this will alleviate some of the befuddlement that is the natural reaction to reading the B-deduction.

> Now all intuition that is possible for us is sensible (Aesthetic), thus for us thinking of an object in general through a pure concept of the understanding can become cognition only insofar as this concept is related to objects of the senses. Sensible intuition is either pure intuition (space and time) or empirical intuition B146f.

Kant begins §24 by re-iterating the point of §21 concerning the independence of the categories from our forms of sensible intuition.

> The pure concepts of the understanding are related/refer [beziehen sich] through the mere understanding to objects of intuition in general, without it being determined whether this intuition is our own or some other but still sensible one, but they are on this account mere **forms of thought**, through which no determinate object is yet cognized. (B 150)

It is undeniable that "intuition in general", to which the pure concepts relate, is entirely independent from temporal (let alone spatial) forms. As we saw above, what is given in an "intuition in general" is

[50] 10:554ff.

characterized merely by the formal properties of immediacy and singularity. The pure understanding only provides the forms of thinking, ordering, and combining objects of such intuitions, but cannot produce out of itself the existence of such objects, i.e., “no determinate object is yet cognized”.

Kant then introduces the terminology ‘pure intellectual synthesis’, and links it to the first argument-step of the B-deduction.

> The synthesis or combination of the manifold in [the pure concepts] was related/ referred merely to the unity of apperception, and was thereby the ground of the possibility of cognition *a priori* insofar as it rests on the understanding, and was therefore not only transcendental but also merely purely intellectual. (B 150)

Thus, there are forms of synthesis, means of representing operations of combination, i.e., rule-forms, that are entirely independent from anything ‘outside’ the pure understanding. Obvious as this may be, I emphasize it because, as we will see, Kant will presently argue that this intellectual synthesis already exhausts all possible forms (of rules) of synthetic *a priori* determinations of intuitive manifolds. The *a priori* forms of sensible intuition provide a field of *application* for these forms of thought, but the means of the pure understanding for representing the generation of synthetic unity are *not* thereby extended—no additional “form of rational representation”, of “rational argument and inference”[51] is provided (recall that this is the fundamental thesis of Friedman’s interpretation).

> But since in us a certain form of sensible intuition *a priori* is fundamental, which rests on the receptivity of the capacity for representation (sensibility), the understanding, as spontaneity, can determine the manifold of given representations according to [gemäß] the synthetic unity of apperception, and thus think *a priori* synthetic unity of the apperception of the manifold of **sensible intuition**, as the condition under which all objects of our (human) intuition must necessarily stand, through which then the categories, as mere forms of thought, acquire objective reality, i.e., application to objects that can be given to us in intuition. (B 150f.)

The forms of sensible intuition *a priori* mediate the application of the pure concepts to actual objects, providing them *objective reality*. Before expanding on the precise relationship between understanding and sensibility, Kant fixes some terminology.

> This **synthesis** of the manifold of sensible intuition, which is possible and necessary *a priori*, can be called **figurative** (*synthesis speciosa*), as distinct from that which would be thought in the mere category in regard to the manifold of an intuition in general, and which is called combination of the understanding (*synthesis intellectualis*); both are **transcendental**, not merely because they themselves proceed *a priori* but also because they ground the possibility of other cognition *a priori*. (B 151)

Paradigmatic examples of “figurative synthesis” are the non-empirical, schematic representations of drawing a line and describing a circle. The notion of figurative synthesis is further specified:

[51] Friedman [30, pp.94f., 126]

> Yet the figurative synthesis, if it pertains merely to [goes merely toward, geht auf] the original synthetic unity of apperception, i.e., this transcendental unity, which is thought in the categories, must be called, as distinct from the merely intellectual combination, the **transcendental synthesis of the imagination.** ***Imagination*** is the faculty for representing an object even **without its presence** in intuition. (B 151)

Crucially, it is exactly this "transcendental synthesis of the imagination" that underlies the schematism (see section 3.4.2). The above passage must not be misconstrued as saying that besides the "transcendental synthesis", there might be further forms of *a priori* figurative synthesis, *a priori* manners of determining spatio-temporal intuition, that do not "go merely toward" the synthetic unity of the pure understanding, and that *these* putative additional forms of synthesis are what underlies mathematical operations. Rather, the figurative synthesis that underlies, e.g., geometrical construction, is *nothing but the application of intellectual synthesis to* a priori *spatial intuition*, and the rule-form of this determination, the "unity of the action" is entirely based in the understanding.

Most of the remainder of §24 is devoted to definitively ruling out the possibility of misunderstanding on just this point.

> Now since all of our intuition is sensible, the imagination, on account of the subjective condition under which alone it can give a corresponding intuition to the concepts of understanding, belongs to **sensibility**; but insofar as its synthesis is still an exercise of spontaneity, which is determining and not, like sense, merely determinable, and can thus determine the form of sense *a priori* according to [gemäß] the unity of apperception, the imagination is to this extent a faculty for determining the sensibility *a priori*, and its synthesis of intuitions, **according to the categories**, must be the transcendental synthesis of the **imagination**, which is an effect of the understanding on sensibility and its first application (and at the same time the ground of all others) to objects of the intuition that is possible for us. (B 151f.)

Insofar as the imagination is a faculty of operating according to rules, generating structure, determining a given manifold, the form of this synthesis, the "unity of the action", is *entirely* based in the understanding. The "transcendental synthesis of the imagination", as Kant just defined it, "pertains merely to [...] this transcendental unity, which is thought in the categories". In particular, this is true for the determination of temporal intuition:

> That which determines the inner sense is the understanding and its original faculty of combining the manifold of intuition, i.e., of bringing it under an apperception (as that on which its very possibility rests). Now since in us humans the understanding is not itself a faculty of intuitions [...] <u>its synthesis, considered in itself alone, is nothing other than the unity of the action, of which it is conscious, as such, even without sensibility, but through which it is capable of itself determining sensibility internally with regard to the manifold</u>, which may be given to [the understanding] in accordance with the form of [sensible] intuition.[52] Under

[52] I amended the Cambridge translation slightly to capture the references of the terms rendered ambiguous by the translation, expressed in German by grammatical gender and declension.

> the designation of a **transcendental synthesis of the imagination**, it [the understanding] therefore exercises that action on the **passive** subject, whose **faculty** it is, about which we rightly say that the inner sense is thereby affected. (B 153)

Despite everything, it remains tempting to say that the very word "action" implies a necessary dependence on temporal intuition, that any act must necessarily take place in time. This, however, is a fact of human psychology, not of transcendental philosophy. Despite §15 having unambiguously introduced the meaning of the term "act" as entirely independent of temporal intuition, and despite the even sharper focusing of this point in the terminology of *synthesis intellectualis*, Kant recognizes the danger of confusion. He chastises psychologists that tend to identify the understanding with, or even subordinate it to, inner temporal sense, "which we carefully distinguish". To preclude any misunderstanding, he underlines again:

> Apperception and its synthetic unity is so far from being the same as the inner sense that the former, rather, as the source of all combination, relates to the manifold of **intuitions in general** under the name of the categories, to objects in general, prior to all sensible intuition;[53] inner sense, on the contrary, contains the mere **form** of intuition, but without combination of the manifold in it, and thus it does not yet contain any **determinate** intuition at all, which is possible only through the consciousness of the determination of the manifold through the transcendental action of the imagination (synthetic influence of the understanding on the inner sense), which I have named the figurative synthesis.

Once we link back this conclusion to the beginning of §24, and in particular, to the theory of combination developed in §15, we must admit that the only possible way to read the B-deduction is as stating, unambiguously, that the representation of synthetic operations in terms of rules is entirely based on the *intellectual synthesis* of the pure understanding, independently from all spatio-temporal intuition. **For recall that...** Because of the density of §24, I have summarized its contents once more with Kant's own words, organized in bullet-points (**highlights** are mine).

Summary of the core argument in §24 in Kant's own words

3.4.1.1.1 • The "pure concepts of the understanding" relate/refer [beziehen sich] "through the mere understanding to objects of intuition in general", independently from whether spatio-temporal or not. The "synthesis or combination of the manifold in [the pure concepts] was related/referred merely to the unity of apperception" and is called **pure intellectual synthesis**.

3.4.1.1.2 • The understanding can also **determine the manifold of** those representations given under the **spatio-temporal** forms of **intuition** *a priori*— the conditions "under which all objects of our (human) intuition" necessarily stand— "**according to the synthetic unity of apperception**", by which "the

[53] This is mistranslated in the Cambridge edition (Guyer), which has "...applies to all sensible intuition of objects in general, to the manifold of **intuitions in general**, under the name of the categories...". The word "applies" is also problematic, as Kant is not using the verb that is correctly translated by "applies", "wird angewendent", but the more vague "gehen auf". He obviously is not changing his mind about the condition of *application* of the categories, but that they refer to a wider class of possible objects that those given in human sensibility.

categories, as mere forms of thought, acquire objective reality, i.e. **application** of objects that can be given to us". "This **synthesis of the manifold of sensible intuition**, which is possible and necessary *a priori*, can be called **figurative** (*synthesis speciosa*)".

3.4.1.1.3 • **Insofar** as figurative synthesis "**pertains merely to the** original synthetic unity of apperception, i.e., this transcendental **unity**, which is **thought in the categories**", it is called the "**transcendental synthesis of the imagination**".

3.4.1.1.4 • Thus, **insofar** imagination "is **determining** and not, like sense, merely determinable, and can thus determine the form of sense *a priori* according to the unity of apperception", it "**does not belong to sensibility**", but is a "faculty for determining the sensibility *a priori*, and its synthesis of intuitions, according to the categories, must be the transcendental synthesis of the imagination, which is an effect of the understanding on sensibility".

3.4.1.1.5 • In particular, the "understanding and its original faculty of combining the manifold of intuition" can determine inner sense. The **synthesis of the understanding**, "**considered in itself alone**", i.e., intellectual synthesis, "is nothing other than the **unity of the action**, of which it is conscious, as such, even without sensibility, but through which it is capable of itself determining sensibility internally with regard to the manifold".

3.4.1.1.6 • "Under the designation of a transcendental synthesis of the imagination", the understanding "therefore exercises that action on the passive subject".

3.4.1.1.7 • The understanding "as the **source of all combination**, relates to the manifold of intuitions in general under the name of the categories, to objects in general, prior to all sensible intuition".

3.4.1.1.8 • **Inner sense** "contains the mere form of intuition, but **without combination** of the manifold in it, and thus it does not yet contain any determinate intuition at all". That is **possible only through** "the **consciousness of the determination of the manifold through the transcendental action of the imagination** (synthetic influence of the understanding on the inner sense), which I have named the figurative synthesis".

Immediately following this argument, Kant discusses some examples that include statements like "[w]e cannot think a line without **drawing** it", and more generally, that the unity of geometrical structures must be thought in terms of "motion", not "of an object", but "as a **description** of space", a "pure *actus* of successive synthesis of the manifold in outer intuition in general through productive imagination", which Friedman takes as evidence for his interpretation that only in terms of kinematic acts, the representation of structure in terms of functional rules is possible. But by contextualizing Kant's remarks, it can be seen that this interpretation cannot be correct. Before I show this in the next section, I first want to demonstrate that the §24 connects *immediately* to the Schematism-chapter.

3.4.2 The schematism of the pure concepts of the understanding and the "science of numbers"

> [T]he schematism of the understanding through the transcendental synthesis of imagination comes down to nothing other than the unity of all the manifold of intuition in inner sense, and thus indirectly to the unity of apperception, as the function that corresponds to inner sense (to a receptivity). Thus the schemata of the concepts of pure understanding are the true and sole conditions for providing them with a relation to objects.

> there must be a third thing, which must stand in homogeneity with the category on the one hand and the appearance on the other, and makes possible the application of the former to the latter. This mediating representation must be pure (without anything empirical) and yet **intellectual** on the one hand and **sensible** on the other. Such a representation is the **transcendental schema**.

> The concept of the understanding contains pure synthetic unity of the manifold in general. Time, as the formal condition of the manifold of inner sense, thus of the connection of all representations, contains an *a priori* manifold in pure intuition. Now a transcendental time-determination is homogeneous with the **category** (which constitutes its unity) insofar as it is universal and rests on a rule *a priori*. But it is on the other hand homogeneous with the appearance insofar as time is contained in every empirical representation of the manifold.

A concept "cannot have any significance, where an object is not given either for them themselves or at least for the elements of which they consist". But

> the modification of our sensibility is the only way in which objects are given to us; and, finally, that pure concepts *a priori*, in addition to the function of the understanding in the category, must also contain *a priori* formal conditions of sensibility (namely of the inner sense) that contain the general condition under which alone the category can be applied to any object. We will call this formal and pure condition of the sensibility, to which the use of the concept of the understanding is restricted, the schema of this concept of the understanding.

The 'division of labour' is unambiguous. The representation of synthesis as an operation of combination is a function of the understanding, while

> The schema of a pure concept of the understanding [...] is rather only the pure synthesis, in accord with a rule of unity according to concepts in general, which the category expresses, and is a transcendental product of the imagination, which concerns the determination of the inner sense in general in accordance with conditions of its form (time) in regard to all representations, insofar as these are to be connected together *a priori* in one concept in accord with the unity of apperception.

Friedman's whole interpretation rests on the *identification* of the schema and the rule.

> The schemata are therefore nothing but *a priori* **time-determinations** in accordance with rules, and these concern, according to the order of the categories, the **time-series**, the **content of time**, the **order of time**, and finally the **sum total of time** in regard to all possible objects.

3.5 Additional textual justification: the correspondence with J. H. Tieftrunk

In a letter from the 5th of November 1797, Johann H. Tieftrunk writes to Kant a long letter in which he tries to reconstruct the theory of the representation of objects of the transcendental logic; Kant reacted to it only a few days later. Complete English translations of these letters have not been published so far; the Cambridge edition only contains some passages, and the translations are somewhat misleading. I provide here an extract of my translation of this correspondence, which provides strong additional justification to the above interpretation. The extant fragment of Tieftrunk's letters begins thus:

> ... and **succession** lies not in the category of quantity as such, but emerges first through the influence of apperception on sensibility, in that the former determines the latter according to its form (its manner of being determinable). The consequence of this determination mediated by imagination is the generation of space and time, as formal intuitions; through which the representation of [spatial] separation and [temporal] succession etc. become possible.
>
> The πρωτον ψευδος is the following. Sense, imagination and apperception **coincide** in the generation of the formal intuition (space and time), and because of this, one believes that the category (quantity) as such consists in nothing but the representation (*actu*) space and time.
>
> But it is **possible** to become aware of the fact that the original, pure apperception obtains by itself and independently of all sensibility, a unique function of the intellect, notably its highest function, from which all knowledge begins, although it does **not** produce out of itself **everything** that belongs to our knowledge. The specific feature of the category of quantity (through which it is differentiated from the forms of sensibility, space and time) is the *actus* of unifying (*synthesis intellectualis*) the homogeneous manifold. The fundamental condition of this *actus* of unity is the synthesis into one, through this becomes possible the synthesis of one to one, i.e. of the plurality, and the plurality, again combined as one, is totality. Here nothing of space and time or a real quantum is contained; merely the rule or the condition is given, under which a quantum must be apperceived: a homogeneous manifold must be capable of being synthesized into one, many, or all. (12:212, 5. Nov. 1797)

In his (uncharacteristically timely) reply, Kant does not only not disagree with Tieftrunk's portrayal of his theory, but places even more emphasis on the independent status of the general structure concept of a combination or composition, which is the fundamental condition of all object representations, underlying all categories.

> The concept of the **composited** in general [des Zusammengesetzten überhaupt] is no particular category. Rather, it is contained in every category (as **synthetic** unity of apperception). For the composited cannot, as such, be **intuited**; rather <u>the concept or the consciousness of</u>

> the **compositing** [des Zusammensetzens] (a function that, as synthetic unity of apperception, underlies all categories as their foundation) *must be presupposed* [vorhergehen] in order to think the manifold given in an intuition as combined in one consciousness, i.e., in order to think the object as something composited, which takes place through the schematism of the power of judgment, in that the **compositing** is consciously *applied* to inner sense, according to the representation of time. [...]
>
> The categories all refer to something *a priori* composited and contain, if it is homogeneous, mathematical functions, if it is heterogeneous, dynamical functions; e.g., regarding the first: the category of extensive quantity refers to: One in Many; quality or intensive quantity refers to Many in One. The former the **collection** of the homogeneous (e.g., square inches in an area); the latter the **degree** (e.g., illumination in a room) [...]
>
> The synthesis of the **composition** of the manifold requires an intuition *a priori* if the pure concepts of the understanding are to have an object, and these [intuitions] are space and time. [If one abstracts from space and time] the concept of the composition, which underlies all categories, by itself alone is without sense, i.e. one cannot see that any object corresponds to it [...]: for [the categories] are mere forms of composition (the synthetic unity of the manifold in general), and belong to thinking, not to intuition —Now there are indeed synthetic judgments *a priori*, which are grounded on intuition *a priori* (space and time); to which thus corresponds an object in a non-empirical representation (under the forms of thinking can be placed forms of intuition, which give sense and reference to the former) [...]
>
> The difficult passage *CPR* [B] 177ff. [the schematism chapter] is to be understood thus: — The logical subsumption of a concept under a higher concept takes place according to the rule of **identity**: the lower concept must be thought as **homogeneous** with the higher one. The **transcendental** [subsumption], on the other hand, i.e. the subsumption of an empirical concept under a pure concept of the understanding, [...] under which something contentually **heterogeneous** [to it] is thus subsumed, which would be contrary to logic if it were to take place immediately, is yet possible, if an empirical concept is subsumed under a pure concept of the understanding *via* a mediating concept, *viz.* that of the **composition** of representations of the subject's inner sense, insofar as they are, in accordance with temporal conditions, composited *a priori* according to a general rule, which is homogeneous with the concept of a composition in general (such is the category), and thus, under the name **schema**, [this mediating concept] makes possible the subsumption of appearances under the pure concepts of the understanding according to their synthetic unity (the act of composition). — The examples given in the text [i.e., *CPR* B 176 - 187] make it impossible to miss this conception. (12:222, 11. Dec. 1797)

I leave this without further commentary. I take Kant's clear formulations to speak for themselves, and to strongly corroborate the interpretation offered in this chapter.

Chapter 4

Conclusion: the Factum of Science – Kant's philosophy of mathematics then and now

> The empirical derivation, however, to which [Locke and Hume] resorted, cannot be reconciled with the reality of the scientific cognition *a priori* that we possess, that namely of pure mathematics and general natural science, and is therefore refuted by the factum. (B 128)

This concluding chapter contains two parts. Firstly, a self-contained summary of the main argument of this thesis, which may otherwise be difficult discern from the amount of material presented. I will try to present it in such a way as to make Kant's line of thought argumentatively convincing. And secondly, the sketch of an argument for the continued relevance of an approach inspired by Kant to the foundations of mathematics, an approach whose motto might well be: Going with Kant beyond Kant.

4.1 Summary of the core argument

The following questions are answered from the standpoint of Kant's mature philosophy: Why are mathematical judgements synthetic and *a priori*? Why is construction essential to mathematical reasoning, and what exactly *is* construction? What is the relationship of concepts and intuition?

4.1.1 Analyticity, first pass

A proposition is *analytic* for Kant if it is a truth of logic, which following Leibniz he explains as *truth in virtue of the principle of contradiction*.[1] A proposition being in this sense logically true could be explained either as being true in virtue of the meaning of the involved concepts, or as resting *exclusively* on definitions and logical inference. On the former account, a proposition is analytic if negating it contradicts the meaning of the involved notions. This leaves us with explaining the meaning of 'meaning' and the logical consequence relation that entails the contradiction. On the second account, we must specify the

[1] A 151/B 191.

rules of logical inference, and we must give an account of the primitive concepts used in the definitions (thus the extension of 'analytic sentence' depends on which forms of concept-formation are admissible, particularly on the status of relational predicates or functional operators). Both, again, involve appeal to meaning.[2] We return to Kant's basic units of meaning and logical consequence (and inference) when discussing the categories.

We will find in section 4.1.7 that Kant's analytic sentences correspond, in modern terms, to a *proper superset* of the valid sentences of first-order predicate logic, but also that the analytic mathematics sentences are a *proper subset* of the class of all true mathematical sentences. As a first step towards establishing this, we must turn to the complement of analyticity.

4.1.2 Why Are Mathematical Judgements Synthetic?

A proposition is *synthetic* for Kant if it is not analytic. I argue that

(1) Kant's fundamental reason for the synthetic character of mathematical judgements is that they involve or presuppose existential claims.

This is the case for geometric construction postulates like 'for any two distinct points, to draw a line', and – as argued in section 1.4.5 – for singular arithmetical identities like '$5+7=12$', '$\sqrt[4]{6561}=3^2$', '$\sqrt{3}=1.7320508...$', or '$135664+37863=173527$'. It is, however, not the case for all mathematical propositions: the mathematical induction principle (whose synthetic *a priori* character Henri Poincaré maintained[3]) would not be synthetic on Kant's account, but an analytic or definitorial constituent of the concept of finite number (note that the induction principle is not a theorem of first-order predicate logic). In general, we may distinguish definitions of objects or structures ('synthetic unity of manifolds') and existential postulates that ensure that these definitions are non-vacuous.

The existential postulates are the *only* irreducible source of non-analyticity, and the entry point for what Kant calls "intuition" or intuitive "construction" (these technical terms have a precise meaning, see sections 1.3.3, and 4.1.5 below), which is the evidential basis of the existential assumptions. In particular, the claim that non-analyticity can be focused exclusively in the existential assumptions means that

(2) mathematical content, specifically the definition of the structural properties of mathematical objects, relations and domains and the propositional content of mathematical judgements, are representable in purely conceptual, non-intuitive terms,

and

(3) mathematical reasoning, including the description, execution, and proof of correctness of constructive procedures, is representable in purely logical terms.

[2] In the case of the undefined primitive concepts, this may appear clear, yet it involves an important subtlety: one might take the logically formulated axioms of a theory as an 'implicit definition' of the primitive terms, comp. Giovannini, Schiemer [34]. This leads to two crucial issues, both directly relevant to the interpretation of Kant: 1. Such definitions cannot account for all elementary concepts, which, as we will see, involve an irreducible kernel of meaning, what we have called 'formal content'. Two such irreducible kernels of meaning are the closely connected *general concepts of object* and *of relation*. And 2., how do we know that the axioms of the theory are mutually consistent? This issue is in the background of section 4.1.7 below, but requires further discussion.

[3] Poincaré [67].

This implies that conditional judgements of the form '*if A then T*', where T is a theorem that has been proven from the axioms whose conjunction is A, are invariably analytic for Kant, even when the proof involved intuitive construction; while at the very same time, if at least one premise in any proof of T must be synthetic, then T is itself synthetic.[4] In short: the syntheticity of mathematics resides *entirely* in the non-analyticity of its basic sentences, not in the reliance on non-logical forms of reasoning; and the non-analyticity of these sentences is *essentially* tied to existential assumptions. Establishing (1) - (3) will require a clarification of the Kantian notion of mathematical existence and of mathematical representation. First we consider some immediate objections.

Against (1) it will be objected that Kant's arguments in the *Critique*[5] for the non-analytic character of mathematics do no explicitly refer to existence. One can even find quotes from important passages that seem to definitively rule out this interpretation:

> In mathematical problems the question is not [...] about existence as such at all. (A 719/B 747)

Rather, it will be continued, non-analyticity lies in the "extension" [Erweiterung] of concepts by determinations that are not logically "contained" in them,[6] e.g., the predicate 'x is a triangle' being further determined by 'x's inner angle sum = two right angles', where the latter is explicitly said *not* to be an analytic consequence of the definition of the former:

> I am not to look at what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it. (A 718/B 746)

This extension proceeds by the "construction of the concept": by carrying out a certain procedure, one exhibits an object instantiating the concept.[7] Applying further (auxiliary) constructive operations, one manipulates the object in such a way that one can deduce that the desired property holds of it. Finally, if the specific parameters determining the particular instance, e.g., the triangle's side-lengths and position, are irrelevant to the general "operation of construction" and proof, one concludes that the property holds universally for all objects falling under the initial concept.[8]

[4] B 14. This formulation (quantifying over demonstrations) was chosen because there being a proof of T from synthetic premises does not imply that there could not be another proof from purely analytic premises. However, it is not necessary to "check" every proof of a proposition to show that it is synthetic: making an unconditional existential claim or having a non-contradictory negation are sufficient critera for syntheticity.

[5] Especially B 14-17.

[6] B 13.

[7] A 713-4/B 741-2.

[8] Ibid. This method of establishing general propositions can be captured in a modern natural deduction calculus by inferences operating with a free variable, e.g.,

$$\dfrac{\dfrac{\begin{array}{c}[P(a)]^1\\ \vdots \\ Q(a)\end{array}}{P(a) \to Q(a)}\,1}{\forall x.P(x) \to Q(x).}$$

The free variable a corresponds to the construction of the concept P. The square brackets $[\,]^1$ indicate that the initial assumption is discharged before applying universal generalisation.

Contra (1), our objector thus concludes, for Kant mathematics does not concern the existence of objects but the non-analytic extension of concepts; and *contra* (3), it proceeds not exclusively by logically deriving consequences from definitions and basic propositions, but by the synthetic activity of intuitive construction, an activity described in explicit contradistinction to logical reasoning. *Contra* (2), Friedman in particular will maintain that Kant did not have the logical resources to define the contents of mathematics purely conceptually.

My aim is not to declare the above quotations irrelevant or misleading, but to articulate their sense. We will find, *pace* our objector, (*i*) that Kant's case for syntheticity is crucially connected to a specific kind of *mathematical existence*, and (*ii*) that characterising his notions of mathematical concept-formation and constructive proof as *non-logical* – in the sense of not being *representable* in terms of logical concepts and rules – involves a conflation of two senses of 'non-logical': employing non-analytic principles vs. not being logically representable. This conflation obscures the fundamental insight and lasting relevance of Kant's philosophy of mathematics. In particular, we will see that there is no contradiction in claiming that mathematical reasoning involves synthetic construction and that it is representable in purely logic terms.

Our objector may grant, *ad* (*i*), that there is some existential aspect to construction (as it involves exhibition of objects), but resist the claims that this is existence of a particular mathematical kind, or that it is the existential import or content of mathematical propositions that makes them synthetic. *Ad* (*ii*) Friedman in particular would insist, explicitly and deliberately espousing the above-mentioned 'conflation' of two senses of 'non-logical', that Kant drew the line between logic and mathematics *within* what we today call logic; that he considered forms of reasoning as non-logical and synthetic that we today consider logical; and that this non-logical component lies not in the affirmation of the existence of mathematical objects, but in an idealised pattern of an intuitive spatio-temporal activity of object-construction – supplementing a lack of properly speaking logical and conceptual methods – that became theoretically (if not heuristically) obsolete with the advent of modern logic in the late nineteenth century.

To resolve these issues, we must look to the foundations of Kant's theoretical philosophy, the Transcendental Logic.[9]

4.1.3 Synthesis And Synthetic Unity

Synthesis is the "action of putting different representations together with each other and comprehending their manifoldness in one cognition"; manifolds of intuition, in particular, must be "be gone through, taken up, and combined", its various elements "collected together", "combined and ordered", into a unified structure.[10] Such synthesis, Kant maintains, is a necessary condition for the rational representation and cognition of any objects.[11]

[9]More specifically to its first division, The Transcendental Analytic (A 50ff./ B 74ff.), and more specifically still, to Book One, The Analytic of Concepts (A 65ff./ B90ff.).

[10]A 77/ B102-3, B145. On Kant's notion of intuition more below.

[11]Note that Kant is concerned with "Erkenntnis", the 'mature' cognition required for discursive deliberation and science, not with minimal conditions of objective representation of the environment that animals and young children are capable of. The terminus technicus "cognition" ["Erkenntnis"], refers to an "objective concious representation whose (actual) objective validity can in principle be established through argument, by the individual with the cognition", Burge [**?** , pp.154-6]. It is also crucial to recognise that Kant tries to develop, not a psychological but logical theory: he is concerned with the general concepts and rules

Transcendental logic deals with the basic forms of synthesis. The function of its categories, the "pure concepts of the understanding", is the general representation of "pure synthesis"; pure synthesis is "synthesis resting on a ground of synthetic unity *a priori*" or equivalently[12] "synthesis according to/following [nach] concepts".[13] Note the curious circularity in this explanation: the categories, as "pure synthesis, generally represented", are introduced as *concepts of* synthetic operations which *rest on*, or proceed *in accordance with*, *concepts* of "synthetic unity".[14] But are not these latter concepts the categories themselves?

4.1.4 The Categories

The categories are the primitive concepts ("Stammbegriffe") of the "synthetic unity of a manifold",[15] which is Kantian for *the structural and relational determinations of manifolds*, by means of which it is possible to form the concept of an object, or of a system of interrelated objects, whose parts are given by the manifold.[16] The categories of quantity, for example, are the *general concepts* of a unit (an individual), of a multiplicity of individuals, and of a totality, i.e., a multiplicity considered again as one (completed, unified) individual object.[17] These are concepts of ways objects *are* and *relate to each other* in and amongst themselves, not of how they are generated by an action, nor of our epistemic access to them.[18] It is perfectly Kantian to speak of the pure concepts of the understanding as elementary concepts of a "Platonic" manner of concept-formation, which represents objects and their relational structures "as they are" in themselves, independently from a knowing subject.[19]

necessarily presupposed in any objective cognition, valid even for reasoners with forms of perception radically different from those of humans, and thus independent of human psychology. See A 54-7/ B 78-82. Also cf. B 148 and B 150, discussed below.

[12] The equivalence is addressed in the paragraphs 4.1ff..

[13] A 78/B 104.

[14] A 78/B 104.

[15] A 79/ B 105, A 81, 107. The complete expression is "synthetic unity of a manifold *of intuition in general*". We come to the roles of "intuition" and "manifold" shortly.

[16] A 80/ B 106, A 93/ B 125, B 137, A 162/ B 201(fn.1.).

[17] B 111:

> So ist die A l l h e i t (Totalität) nichts anders als die Vielheit als Einheit betrachtet.

In wise anticipation, Kant points out that not all multiplicities may, without further ado, be regarded as totalities (comp. footnote 18 below). Note the striking parallel to Cantor's clarification:

> Unter einer [...] "Menge" verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken läßt, d. h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann. [**?** , p. 204]

[18] Comp. A 498/ B 526-7:

> [A] synthesis of the mere understanding, who represents things a s t h e y a r e, without regard to whether and how we might achieve acquaintance with them.

Also comp. AA 18: 343:

> The relation of the many amongst each other, insofar as they are comprehended into one, is the combination.

Note that "combination" [Verbindung], objectually (not operationally) understood, "*conjuctio*", and "synthetic unity" are synonyms. Though Kant distinguishes ("inner") structural from ("outer") relational determinations of objects, both are essentially relational notions: one concerning the internal relational structure of an object, the other the external relations between objects. The move from external to internal is pragmatically embodied in the (not always admissible and thus necessarily synthetic) move from multiplicity to totality. See B 111, B 201(fn. 1), B 131f., A 265/ B 321. Also comp. Falkenburg [25, p. page number].

[19] For Kant's use of the term "Platonic", see *Metaphysical Foundations of Natural Science* AA 4: 507. Comp. footnote 18 above.

Here one might protest: Weren't the categories precisely the way *we get to know* objects, concepts of appearances and not of The Thing In Itself? And isn't this way of accessing objects, according to Kant, inseparably connected with *how we generate* representations of them? Didn't we just say that Kant introduced the categories as "synthesis, generally represented", and is this synthesis not something that *we* must be capable, at least in principle, of carrying out?

In the following three subsections, we review the arguments for the theses that the categories are independent from the intuitions of space and time, and exhaustive regarding the representation of relational structure (subsection 4.1.5); that the concepts of operation always presuppose conceptual representation of structure (subsection 4.1.6); finally, we show how this informs Kant's view on analyticity, syntheticity and existence in mathematics (subsection **??**). The questions raised in the previous paragraph and at the end of section 4.1.2 can then be answered in subsection 4.1.6; the main questions asked at the beginning of this conclusion, and the issues raised at the end of section 2 can be answered after subsection 4.1.7.

4.1.5 Independence from space and time and exhaustiveness of the categories

By the 'content' of a concept, we mean its intension, its contribution to the propositional content of a judgement. The independence of the categories means that their content is not derived from the intuitions of space and time, and that their applicability is restricted neither to objects in space and time, nor to the structure and substructures of space and time themselves.

> Space and time are valid, as conditions of the possibility of how objects can be given to us, no further than for objects of the senses, hence only for experience. Beyond these boundaries they do not represent anything at all, for they are only in the senses and outside of them have no reality. The pure concepts of the understanding are free from this limitation and extend to objects of intuition in general, whether the latter be similar to our own or not. (B 149)

'Intuition in general' means the singular representation of a concrete individual, as opposed to a general concept.[20] Objects in space and time as well as space and time themselves are concrete individuals that are give according to the forms of human intuition. Kant emphasises that the extension of the "concept of an object in general",[21] whose structural properties are describable in terms of the categories, encompasses but is not restricted to those forms of intuition.

> The pure concepts of the understanding are related through the mere understanding to objects of intuition in general, without it being determined whether this intuition is our own or some other but still sensible one, but they are on this account mere forms of thought, through which no determinate object is yet cognized. (B 150)

[20] See section 1.3.3, A 320/ B 377. Cf. 9:91, §1:

> All cognition, i.e., all conscious representations that refer to an object, are either intuitions or concepts. Intuition is a singular representation (*repraesentatio singularis*), concept a general (*repraesentatio per notas communes*) or reflected representation (*representatio discursiva*).

[21] A 95/ B 128.

These 'forms of thought' of an as of yet merely hypothetical object are to the originary content of the categories, the "rules of the pure thinking of an object",[22] i.e., the synthetic or structural unity of a manifold in general. In the context of the last quote, Kant then distinguishes between *synthesis intellectualis* and *synthesis speciosa*. The former is the purely categorical representation of synthetic unity; the latter is the *application* of this to particular forms of intuition, such as space and time. As we saw in chapter 3, Kant unambiguously maintains that the application of the categories to a particular form of intuition does not add additional representational capacities regarding structural properties. We will see that the categories, as concepts of relational structure, must indeed be prior to the representation of the structure of space and time when we consider Kant's arguments for the independence thesis in sections 4.1.5.2 and 4.1.5.3. This implies that the intuitions of space and time *cannot* play the role Friedman envisions for them, and it substantiates Kant's thesis of their representational exhaustiveness, i.e., the claim that the concept of *any* relational structure that is definable at all must be definable in purely categorical terms.[23]

> The pure categories– or predicaments, as they are also called – do not presuppose a specific kind of intuition (as for example the only kind possible to humans) such as space and time, which is sensible; the categories are forms of thought for the concept of an object of intuition in general, of whatever kind that intuition may be, even if it were a super-sensible intuition of which we [humans] cannot have a specified concept. For we must always make ourselves a concept of an object through the pure understanding, if we want to judge something *a priori* of it, though we may later find that it is transcendent and cannot be given objective reality. (AA 20: 272)

The fact that the pure categories specifically concern the *relational* properties of objects is of crucial importance. Yet, up to this point, Friedman's thesis may remain attractive still: the categories may be independent of space and time and *necessary but insufficient* conditions to conceptualise mathematical content – precisely the *relational* content of representations requires spatio-temporal intuition.

This thesis is already refuted by the fact that Kant considers the "ideas of reason", e.g., the concept of an infinite series considered as a completed totality, as definable in purely categorical terms, explicitly emphasising that these ideas do *not* depend on spatio-temporal notions.[24] It is also refuted by the fact that Kant characterises the categories as pure concepts of "synthetic unity", "combination", "conjuction", "nexus", "types of composition [...], i.e., types of synthetic unity", etc. It is further refuted by Kant explicitly denying that temporal intuition makes any contribution to the content of arithmetical concepts, which exclusively contain a "pure intellectual synthesis", for which spatio-temporal intuition merely provides a field of application (even though it is the only field accessible to humans).[25]

Notwithstanding this unambiguous evidence, the continued attraction of Friedman's thesis is a natural consequence of Kant's close association of the categories with the functions of formal logic, which Kant

[22] A 55/ B 80.

[23] It actually only substantiates this exhaustiveness relative to spatio-temporal intuition, i.e., the latter cannot add any representational capacities to the categories; their complete exhaustiveness would follows from Kant's claim that besides spatio-temporal intuition and the pure categories, there can be no third source of possible representation. The problematic nature of this claim will briefly concern us in the final section of this conclusion.

[24] See section 1.3.5.

[25] See sections 1.3.3 and 1.4.

tied stronger to Aristotelian term logic than his own original logical investigations should have allowed him.[26] We consider this issue before turning to Kant's positive arguments for the independence and exhaustiveness of categorical representation in sections 4.1.5.2 and below.

4.1.5.1 Formal logic and the categories

As a sufficiently close approximation, one may reconstruct Kant's *formal* logic as monadic first-order predicate logic.[27] As opposed to traditional term logic, Kant analysed the form of general propositions not as "all A are B", but as "all x of type A are B", highlighting the distinction of individual and general terms and making explicit the role of individuals as what the quantifiers range over. Since Kant claims an essential link between the categories and the logical functions of judgement, the interpretation that the categorically representable structures are those representable in first-order monadic predicate logic naturally suggests itself. But the categories are not those concepts that are definable in terms of formal logic, but those *contentual* notions that are presupposed by the primitive notions of formal logic. The universal quantifier, for example, presupposes the concept of an individual, and of a class of individuals. Transformed into an object concept, the latter notion gives rise to the concept of a totality, a plurality or class of individuals considered as one object. Thus, if like Friedman one wishes to draw an analogy to modern logic, the following can by said: from a model-theoretic point of view, the categorically definable concepts do not correspond to the different structure-types of models distinguishable by theories formulated *within* first-order monadic predicate logic (Friedman's thesis); rather, the categories correspond to the concepts that are necessary to articulate the very *notion of a model* of such theories, e.g., the concept of a totality of individuals satisfying certain properties (which is a concept that is not definable *in* first-order monadic logic, but is required to understand the semantics of this logic). What the categories provide is thus analogous to what the *language* of set theory provides: a general conceptual framework for defining types of structure.[28] If one considers the trajectory from Kant's conception of transcendental logic to the modern structural conception of mathematics, with the crucial waypoint of Bernhard Riemann's 1854 *Habilitationsvortrag* on the theory of manifolds, the historical adequacy of this analogy becomes evident.[29]

A note on formalisation: from the point of view of Kant's formal logic, "x is an element of the totality y" has the logical form "x is $\big($an element of the totality $y\big)$", i.e., "x is P". The relational character of the predicate is part of its *content*, which is the topic of transcendental logic (particularly the Transcendental Analytic of Concepts) and does not depend on spatio-temporal intuition (as emphasised, it rather corresponds to the informal set-theoretic notions underlying the semantics of formal logic). It can be shown that in this way, especially making use of the formal machinery of individual terms, all the reasoning necessary for mathematics can be carried out using the standard rules of inference available to Kant (note that monadic and polyadic predicate logic share the same rules of inference). In particular, this

[26] Schulthess

[27] Appendix 2. PL vs TT

[28] It is important to distinguish the language of set theory from the axioms of set theory, in particular, the existential axioms. One can use the language to define any structure one might be interested in, say, an infinite set, the totality of all Dedekind cuts, a Hilbert space, or a supercompact cardinal, without being committed to the *existence* of such a structure.

[29] See Ferreirós [26, p.47ff.].

works by the method of "local formalization" that Wilfrid Hodges discusses in [49], which differs from the "global formalization" of post-Fregean logic.

4.1.5.2 First argument for independence: priority and generality

The above discussion establishes that Kant espoused the categories' independence from and representational exhaustiveness relative to the intuitions of space and time. In this and the next subsection, we consider two concrete arguments for these claims implicit in Kant's writing.

The first argument for the independence of the categories uses the fact that the same basic notions of order, measure, and structure are used for describing different forms of intuition, and even different relations within the same form of intuition.[30] For example, from the fact that the description of the order-type of a given spatial configuration is possible independently from the description of the order-type of a given temporal configuration (and *vice versa*), it follows that if the same basic notions are used in both, then these cannot depend on the particular forms of intuition in either. That which is a condition of possibility of two mutually independent descriptions is logically prior to and independent from either. This argument appeals to the following consequence of the transitivity of conceptual presupposition, which we accept without argument: if concept X is presupposed by description A and description B, but if A neither presupposes B nor B A, then X is independent of both.[31]

To be more concrete, lets assume that there is an order-isomorphism between the points on a line (regarding "to the left of") and the moments in time (regarding "earlier than"). As the description of the general properties of this shared order-type depend neither on the specifically spatial nor the specifically temporal character of the two structures (since each can be described without reference to the other[32]), the primitive notions of order needed to articulate these general properties cannot depend on the spatial or temporal aspects. Incidentally, where we say "isomorphism" or "homomorphism" Kant would say "analogy", explaining that

> analogy means, not as commonly understood an imperfect similarity between two objects, but rather a perfect similarity between two relations of completely non-similar objects. (*Proleg.* §58, AA 4: 357)

[30] E.g., positional and size relations.

[31] Here the following objection could be raised. Suppose that "description A presupposes concept X" is explicated as "to fully understand the meaning of A, one must understand the meaning of X". Suppose further that both A and B presuppose X, where A and B are distinct and neither presupposes the other. It might still be the case that one could not understand the meaning of X unless one understood at least one of A or B (either one being sufficient). For example, might Kant not have maintained that to understand the notion of a dense order, on must have either the intuition of space or time (either one being sufficient)? Independently of whether we accept the explication of "presupposition", this argument fails. As soon as we are capable of describing identical structural properties of two objects whose descriptions are mutually independent, we *must* be capable of articulating a general structural concept that is applicable to each of them, such that this application does not depend on any reference to the respective other. As soon as I am capable of articulating a general structural property that obtains in two very different domains, which allows me to say that these domains share an identical feature, then the abstraction

[32] Here it is important to distinguish description from knowledge. A well-versed reader of Kant may object that Kant tied spatial and temporal intuition much closer together as I am allowing, e.g., in the "Refutation of Idealism" (B 274 ff.), where it is claimed that one cannot determine one's position in time without a persistent reference point in spatial intuition. But this concerns our determination and knowledge of an actual temporal fact, not the mathematical description of the structure of time itself.

Analogy, just as isomorphism, is a second-order notion that concerns the structural equality between relational structures that may obtain within completely different domains of objects, e.g., spatial points and temporal moments (the analogy is not a first-order relation between points, or between points and moments, but a second-order relation between two relations, one obtaining between points, the other between moments). The categories allow reasoning on the level of abstraction of perfect similarity relations between relations obtaining within completely different domains of objects, independently from the specific features of those domains.[33]

Our forms of intuition determine the particular ways that objects are given to us. The content of our theoretical cognitions concerns the properties and relations of these objects.

> Transcendental logic treats of cognitions of the understanding regarding their content, but independently from the way how objects are given. (17:651)

Everything presenting itself in our intuition is purely relational,[34] indeed, the object of sensible intuition is a "sum-total of mere relations",[35] of internal and external determinations. Since transcendental logic treats the content of our cognitions, it must concern the general relational properties of objects of intuition, but independently from the particular forms of intuition.

> Only of one thing we could not abstract, namely, that the manifold of intuition must be g i v e n, prior to the synthesis of the understanding [...]. But how, that here remains undetermined. [The categories] are only rules for an understanding whose entire capacity consists in thinking, i.e., in the action of bringing the synthesis of the manifold that is given to it in intuition from elsewhere to the unity of apperception, which therefore cognizes nothing at all by itself, but only combines and orders the material for cognition, the intuition, which must be given to it through the object.

In other words, it concerns the general principles for determining the order, structure, and measure of manifolds of "intuition in general". Abstracting from the particular forms of intuition, one is left with the abstract concepts of relational structure. One way of making this abstractness evident is by the reflection from the beginning of this section: if space and time share a certain structural property, and if their respective partaking in this property in no way depends on that of the respective other, than this property cannot concern something that is either inextricably spatial or inextricably temporal.

Kant's exhaustiveness thesis[36] that the *synthesis speciosa*, the application of the *synthesis intellectualis* of the categories to a particular form of intuition, does not provide additional representational means beyond those already afforded by the *synthesis intellectualis*, is a direct consequences of his claim that "everything that belongs to our our intuition [...] contains nothing but mere relations"[37] combined with his

[33] As Brigitte Falkenburg [25] has pointed out, they also allow to treat collections of and relations between objects again as individual objects. By itself, this is not a type-confusion, but corresponds to the move of the "set of" operation emphasised by Gödel, i.e., the move from of a plurality of individuals to this plurality considered as an individual, Kant's category of totality. Comp. section 4.1.4, esp. foonote 17.

[34] A 49f. / B 66ff.

[35] A 265 / B 321.

[36] See sections 1.3.3 and 3.4.1.1.

[37] A 49 / B 66.

claim that all those relational properties and structures are independently definable in terms of the pure concepts of synthetic unity.

What such application *can* and *does* provide is evidence that certain categorically representable relational structures can be instantiated; in particular, it substantiates the existential assumptions of mathematics.

4.1.5.3 Second argument: objectivity and other forms of intuition

The fact that the same primitive notions are used to describe the structure of space and time plainly does not imply that there is a complete isomorphism between space and time. But the fact that these primitive notions are capable of describing different types of structures immediately implies the possibility, constantly pointed out by Kant, of types of intuition radically different from our own, in which nevertheless the same categories would find application. Conversely, the fact that Kant repeatedly stresses the conceptual possibility of other types of intuition to which the categories could be applied shows that the latter cannot depend on spatio-temporal intuition for their intensional content (though they *do* depend on it for their instantiation).

Connected to this complex of issues there is a separate, second argument for the categories' independence from and exhaustiveness relative to spatio-temporal forms, which we mention for illustrative purposes, but without entering into the subtle philosophical difficulties that it raises. This argument relies on the categories' function in making *objective* cognition possible. The point has been eloquently made by Frege in the *Grundlagen*:

> Space, according to Kant, belongs to appearance. For other rational beings it might present itself in a wholly different manner. Indeed, we cannot even know whether it appears the same to one man as to another; for we cannot lay one man's intuition of space next to another's, in order to compare them. And yet there is something objective in it; all recognise the same geometrical axioms, if only by behaviour, and must do so to if they are to find their way about in the world. What is objective in it is the lawful, conceptual, what is subject to judgement, what is expressible in words. [29, *GL* §26][38]

Says Kant:

> Cognitions and judgements must, together with the conviction that accompanies them, be generally communicable; for otherwise they would have no correspondence with the object: they would all be a merely subjective play of the powers of representation, just as Skepticism insists. (AA 5: 298)

Intersubjective communicability is a *conditio sine qua non* for objectivity according to Kant. Yet as Frege points out, Kant considers the immediately given intuitions of space and time, by themselves, as subjective.[39] For intuition to contribute *anything* to cognition at all, those aspects of it which do

[38] Echoing Kant's famous "intuitions without concepts are blind" (A 51 /B 75), Frege concludes: "What is exclusively intuitive is not communicable."

[39] A 42/ B 59.

so contribute *must* be capable of being articulated in terms of inter-subjective, communicable general concepts, which can only be defined, in the final analysis, in terms of the pure categories, and which cannot, in turn, be derived from intuition.

> Without sensibility no object would be given to us, and without understanding none would be thought. Thoughts without content are empty, intuitions without concepts are blind. It is thus just as necessary to make the mind's concepts sensible (i.e., to add an object to them in intuition) as it is to make its intuitions understandable (i.e., to bring them under concepts). Further, these two faculties or capacities cannot exchange their functions. The understanding is not capable of intuiting anything, and the senses are not capable of thinking anything. (A 51/ B 75.)

Note that 'the thought' is not the subjective psychological process but precisely that which is intersubjectively communicable, for both Frege and Kant.

Incidentally, Kant is even more radical than Frege is in *Grundlagen* §26: Frege imagines two subjects whose spatial intuitions satisfy only the projective laws. By the principle of duality, they may agree on all geometric propositions even if one of them privately experiences what they both call 'points' in the way that the other experiences lines, and vice versa.[40] What is objective in their spatial intuition are the structural facts that they can inter-subjectively agree upon, even if their private "immediate quality of given-ness" (to use Carnap's phrase[41]) differ. Kant not only admits the possibility of such differently constituted subjects that still agree on the same geometric laws, but even such beings whose *forms* of intuition are different, i.e., who might disagree about the geometrical or chronometrical axioms, or live on completely different 'planes' of perception. Certainly, what is still missing in Kant's discussions of spatial intuition is an unambiguous and principled distinction between abstract geometry (as a purely structural discipline), applied physical geometry, and perceptual geometry, distinctions that could be drawn with complete rigour only after Hilbert's *Grundlagen der Geometrie*. That being said, Kant did have the distinction between an abstract structural characterisation and concrete instantiations. And the same foundational issues raised by Hilbert's work point right back to the fundamental issues discussed by Kant.

An example illustrating that Kant's talk of radically different forms of intuition was no idle speculation may be instructive. The young Kant argued that the geometrical structure of space, particularly its dimensionality, is the consequence of general relations between spatial elements (which he then thought were determined by a contingent physical fact). This was later transmuted into the insight that the range of possible conceptually representable types of structures is far greater than what accords with the human forms of intuition of space and time, and more generally, into the thesis that the axioms of mathematics are not logical truth, i.e., that they are synthetic.

[40] In the real projective plane, every definition remains significant, ever proof valid, and thus every theorem true, when the words "point" and "line", resp. "join" and "intersection" are interchanged, as in "for any two points, there exists a line that is their join" and "for any two lines, there exists a point that is their intersection". One can then develop the theory in a two-column format, and always get two theorems for the price of one proof (cf. Coxeter [19, p. 15]). This is one of the most illustrative examples of the difference between abstract structure and concrete instantiation.

[41] Carnap [15, §66].

> I maintain that substances in the existing world, of which we are a part, have essential forces of such a kind that they propagate their effects in union with each other according to the inverse-square relation of the distances; secondly, that the whole to which this gives rise has, by virtue of this law, the property of being three-dimensional: thirdly, that this law is arbitrary, and that God could have chosen another, e.g., the inverse-cube, relation; fourthly, and finally, that an extension with different properties and dimensions would also have resulted from a different law. A science of all these possible kinds of spaces would undoubtedly be the highest geometry that a finite understanding could undertake. The impossibility we notice in ourselves of representing to ourselves a space of more than three dimensions seems to me to stem from the circumstance that our soul likewise receives impressions from without according to the inverse square relation of distances, and because its nature is itself constituted so as not only to be thus affected, but also to act external to itself in this way (Kant 1747, AA 1:24-25).

A space is here conceived as an "entire collection of substances",[42] i.e., in an atomistic sense as an infinite whole constituted by simple elements, which, in older Kant's words is an entirely consistent, "intrinsically correct Platonic" conception.[43] Depending on the relations obtaining between the elements of this totality, the overall structure of the totality is determined, e.g., regarding its dimensionality. Different relations give rise to different types of space. This manner of concept-formation, which allows for the description of actual infinite structures, i.e., infinite collections of elements coordinated with each other by certain general relational conditions obtaining between them, is precisely what the unschematized, pure concepts of the understanding make possible.[44] Note that Kant is here light-years ahead of Leibniz, who considered three-dimensionality and the axioms of Euclidean geometry logical necessities that hold true in all possible worlds.[45] Kant not only saw that the axioms of Euclidean geometry are not logically necessary (that after all is the meaning of "synthetic"), but recognised explicitly the conceptual possibility of other types of spaces.[46]

(This raises the question, which will concern us in the final section, of how Kant could maintain the aprioricity of Euclid's axioms. Here we only note that if one substitutes for Euclid's axioms those for the linear continuum, the problem remains exactly the same: as Dedekind observed, there is no logical reason why space could not be discontinuous; but then what is the status of his completeness axiom?)

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[42] AA 1:24

[43] AA 4:507-8 In the present context, the elements are physical substances, a space their totality including the sum-total of all their relations; in later discussions (see next footnote), Kant allows for a less relationist notion of space, where the elements are points rather than substances; for the logical structure of the concept, this is irrelevant.

[44] As pointed out in the previous footnote, one should distinguish here between two levels: the mathematico-conceptual level of describing a type of structure as a collection of elements with relations operating on them, and the question in the foundations of physics concerning the relationship between dynamical laws and the geometry of space. Unlike his older Self, the young Kant was a relationist holding that space and spatial relations are derivative upon substances and the forces between them, which are metaphysically prior, see Callender [12, p. 134]. Clearly the availability of the logico-conceptual means that we are concerned with here is prior to and independent from these metaphysical questions. At any rate, it is certainly not the case that the older Kant changed his mind or even forgot about the purely conceptual representability of infinite relational structures, as is quite indubitably show by the evidence presented in sections 1.3.5 and especially 1.3.6 above.

[45] De Risi [73, pp. 40-49, esp. pp.44-5 and p.49].

[46] Concerning the link between the inverse square law and three-dimensionality, see Callender [12], who argues that Kant was essentially correct. For a critical discussion, see Gatzia and Ramsier [33].

4.1.6 The logical representation of synthetic operation

About the representation of synthetic operations, there are two trivial-seeming points to be made, which have a significant impact on the debate with Friedman. Consider again the following inference, which corresponds to carrying out the operation of constructing a circle.

> For all points and all lengths, there is a circle with that point as centre and that length as radius.
>
> *a* is a point, *r* is a length.
>
> Therefore, there is a circle with radius *a* and length *r*.

The two points are the following. Firstly: the synthetic aspect of this inference lies entirely in the synthetic nature of its premises; the inference itself is purely logical. Secondly: as stated, the first premise represents the mathematical content of the construction principle for circles adequately; the practical formulation "to describe a circle", even with the added specification "by rotating a straight line around a fixed point", is of derivative, secondary status: it is neither the case that the propositional content of the first premise can only be represented via this practical formulation, nor is it the case that the evidence of this premise depends on a primary representation of the act of carrying a line around a point. Conversely, the theoretical formulation employs basic non-operational, structural concepts that are necessarily presupposed by the operational representation; and it articulates the fundamental condition upon which the possibility of the action depends.

The first point appears so blatantly obvious that it takes a brilliant mind to deny. Friedman's ingenious interpretation – I say this without a shred of sarcasm – ultimately depends on Kant's conscious denial thereof.[47] This could be explained only if Kant held, against the second point, that the general representation of synthetic operations is irreducibly based on the primacy of the practical representation of an action, and furthermore, that this representation of an action is irreducibly non-conceptual, so that it cannot enter into the properly speaking logical deductive frame of reasoning. None of this is the case: Kant espoused both points as stated above.

Given our discussion of the exhaustiveness of the categories above, it will suffice to focus on the second point. We begin with the representational aspect, and turn to the evidential aspect afterwards.

The representation of an object generating action presupposes a prior concept that determines the 'given' elements the action operates on, as well as a concept of the structure of the to-be-generated object, the 'synthetic unity' or relation brought about amongst or determined by the elements.[48] This is the sense of Kant's claim mentioned in 4.1.3, that the concept *of* an operation of synthesis "rests on" a prior concept of "synthetic unity".[49] One will caution that besides 'input' and 'output', the determinate concept of an action should also specify the *manner in which* one gets from the former to the latter (an algorithm); this

[47] See the quotes in chapters 1 and 2.

[48] B 133.

[49] A 78/ B 104. On this point, comp. Tiles [86, p. 232]:

> Kant distinguishes the rational (and thereby also moral) being from the non-rational on the basis of its capacity to act not merely according to a rule (or law) but according to its conception of the rule [ref: AA 4: 412]. This is the capacity on which the possibility of logic, mathematics, scientific knowledge and morality depend. [...] It connects thought with action and action to thought via the thought of action

becomes particularly pressing in view of the fact, emphasised by Kant,[50] that the same structure may be generated from the same elements (and the same theorem proven from the same axioms) in many different ways.[51]

In the specification of an algorithmic construction, one can replace every practical sentence appealing to an operation for generating some object y with certain properties from given objects $x_0, \dots x_n$ with the corresponding theoretical proposition that asserts, for all $x_0, \dots x_n$, the existence of such a y.[52] Carrying out an operation on a given set of objects $a_0, \dots a_n$ corresponds to the purely logical inferences of instantiating the universal quantifier(s) to a_i and then the existential quantifier to yield a b.[53]

[...] Given the points a and b, draw the line C between them. Now describe the circle D with centre a and radius equals the distance between a and b. [...]	[...] a and b are distinct points. For all distinct points x and y, there exists a straight line between them. Therefore, there exists a straight line between a and b. Let it be called C. Let D be the distance between the points a and b. For every point x and every distance y, there exists a circle with centre x and radius y. Therefore, there exists a circle D with [...]

For complex algorithms defined by specifying each step individually, this clearly works. But some questions remain. 1. What about the elementary operations? Is there not some specifically operational content that is lost when we replace the expression of an elementary operation by its corresponding 'for all ... exists ...' sentence? 2. If nothing essential is lost by translating algorithmic into theoretical language, why is there still a difference between computable and non-computable?[54]

But the conception of a rule governing an object-generating action presupposes a *prior* conception of the (conditions of possibility of the) outcome of the action. Cf. AA 5: 11(fn.). This is not the case for the universal "formal laws" of practical philosophy, most notably, the categorical imperative, which are *second-order laws*, formal principles governing rules which in turn govern actions, where the outcome of the actions themselves are not directly considered. Note that these are not algorithms, but laws of freedom, i.e., of unconstrained – by external rules if not by circumstances – action in conscious accordance with the universal law that reason gives to itself. AA 5: 30-1.

[50] A 787/ B 815.

[51] Exercise 13.26 of Ireland and Rosen [**?**] famously demands:

> Count the number of proofs of [Gauss'] law of quadratic reciprocity given thus far in this book and devise another one. (p. 202)

According to the latest count, the number of all published proofs of Gauss' theorem is < 245 (http://www.rzuser.uni-heidelberg.de/ hb3/fchrono.html). Comp. Pieper's *Variationen über ein zahlentheoretisches Thema von C. F. Gauss* [**?**]. Note that the problem of providing determinate criteria of identity for proofs, required for solving the 24th item on Hilbert's famous list (but not mentioned during his 1900 Paris ICM address) remains open, comp. Thiele [**?**]. Although we can't always decide whether two proofs are 'essentially' the same or not, we can often tell when two proofs (making use of different methods, ideas, or even axioms) are definitely *not* the same.

[52] Of course one first needs to do this recursively for all complex subroutines, so that all called operations are primitive.

[53] I here ignored the crucial role of the types of various objects. If one were to fully formalise such reasoning, a type-theoretical language with constructors and eliminators captures the intensional content better than a one- or even a many-sorted predicate logic. In fact this corresponds closer to Kant's own logical theory.

[54] A third question is: What about algorithms that contain operations that are to be iterated n times, where n varies for different inputs? Since the logical structure of such appeals are entirely reducible to arithmetical principles, the issue does not go beyond those we discussed in our treatment of Kant's theory of arithmetic in chapter 1. We briefly touch upon the issue of the least-number principle in section 4.1.7 below.

The second question concerns the status of the existential axioms. If the "for all ... exists..." sentences corresponding to the basic operations are *constructive* (we return to this notion shortly), and if the iterations and recursions are organised in such a way that no step depends on previously completing an infinite task, then the procedure is effectively computable (these effective procedures also include certain infinite approximations); otherwise, it might not be.[55] At any rate, the claim is only that every practically specified procedure can be expressed in terms of a theoretical proof, not that every theoretical proof corresponds to a practically specifiable procedure.

Concerning elementary operations, two points are to be made. Firstly, the purely theoretical expression is *sufficient* for the deductive development of the theory, including the representation of the complex constructive operations. It is true that this manner of expression makes not even an implicit reference to action, say, the motion of a line around a point. But neither is this necessary to establish the geometric facts contained in the theorems. Kant is unambiguous that the content of a theory of pure mathematics is entirely theoretical; the practical is a mere "corolloary" that is in no way required for the development of the theory (see quotes in section 1.3.4 and below).

This leaves us with the question of evidence. Is it the case that we only know the truth of the theoretical sentence because of our capacity to *draw* circles? Here too, Kant is perfectly unequivocal: the theoretical proposition is evident "even though the practical proposition that follows from it, viz. 'to describe a circle' (as a straight line is rotated uniformly about a point), has not even been considered yet."[56] "The proposition 'to describe a circle' is a practical corollary (or so-called postulate), which could never be demanded at all if the possibility – yes, even the manner of possibility – of the figure were not already given."[57]

> The expression postulate [in practical philosophy] can lead to misunderstanding if one confounds it with the meaning that postulates of pure mathematics have, which have apodictic certainty. The latter postulate the possibility of an action, the object of which has beforehand been theoretically cognized *a priori* with complete certitude as possible. (*CPrR* 5:11 footnote)

Imagine an idealised mathematician, which, for any given centre and radius, can trace out the respective circle. The circle does not come into existence by tracing it. If the constructor is capable of doing the tracing at all, it is because the possibility of her action was already given. This possibility does not depend on her tracing out; and recognising the general possibility of her tracing depends on first recognising the possible existence of a circle for her to trace.

But what then is the meaning of "construction" as used by Kant, and how does one differentiate basic constructive from non-constructive existence sentences? In particular, does this metaphor of the idealised constructor that moves along 'tracks' whose possibility 'was already there' not betray that Kant, on his own principles, should have been a Platonist regarding the completed infinite? If the curve 'already needs to be there to be traced out', does this not imply that an actual infinity of objects (the various circles)

[55] A precise demarcation of course only became possible with the works of Turing, Church, and Gödel. But the distinction is clearly available prior to this, since we can provide unambiguous examples for cases on both sides.

[56] 11: 43, 1789.

[57] 11:53, 1789.

already need to be there? And more extremely, consider the continuum of points on a single circle — don't they *all* need to be there already, prior to the drawing? And if so, must we not acknowledge their actual uncountable infinity?

These questions lead to deep puzzles, but the short answer is: no. Firstly, the difference between potential and actual infinity can be made sense of without temporal intuition or the metaphor of action. The basic conceptual distinction relevant here is that between plurality and totality, i.e., the second and third of Kant's categories. A case in point, from contemporary discussions, is that there is a coherent philosophical approach to set theory, called potentialism, which considers the universe of sets as an inherently potential infinity, while each individual infinite set is considered as an actual infinity.[58] Yet it is plain that this cannot be cashed in in terms of the temporal metaphor: as Charles Parsons (who is at times sympathetic to potentialism) repeatedly pointed out, the actual infinities of set theory soon[59] grow so large that there is simply *no* meaningful sense to be made by thinking of them as being embedded in a temporal procedure.[60] Potentiality is simply the impossibility of treating a plurality as one unified whole; one reason for this may be a principle of indefinite extensibility governing the plurality, which allows, for any definite totality of elements of the plurality, to define a new element not in that totality (the prime example are of course the introduction principles for Cantor's ordinal numbers, with the rule that every set of ordinals has a least upper bound). Time has nothing to do with this.

Secondly, Kant states that we have intuitive knowledge only of individuals. Space and time are give as singular manifolds, but not as totalities of well-distinguished points. Similarly, for each given centre and radius, we have intuitive evidence that their circle exists. Theses circles form an open plurality. But we do not, on Kant's account, have the intuitive means to exhibit this plurality as *one* object, the completed totality of infinitely many well-distinguished objects. It is this being given in intuition that grounds construction, i.e., the possibility of exhibition of instances. And Kant considered it a fundamental limitation of the human mind that it can only exhibit finitely many instances in one intuition, though no upper bound on how large that finite number is exists.[61]

For Kant, the fundamental content of existential postulates is a distinctively mathematical kind of existence, the "real possibility of a concept", which consists in evidence for the possible existence of an instance:

> The objective reality of this concept, i.e., the possibility of the existence of a thing with these properties, can be proven in no other way than by exhibiting the corresponding intuition. (AA 8: 191)

The evidential basis for this kind of knowledge consists in the pure forms of intuition, which, as such, do not refer to action or motion:

> In space considered in itself there is nothing movable [...]. In the same way the transcendental aesthetic cannot count the concept of alteration among its *a priori* data; for time itself does not alter, but only something that is within time. (A 41/ B 58)

58 Linnebo [55].
59 It's hard to get out of the temporal manner of speaking.
60 Parsons [62].
61 Recall the quotes on page 11.

And yet these intuitions are to provide the evidential ground for the synthetic *a priori* judgements of mathematics.

4.1.7 Analyticity and syntheticity, final pass

Kant had the resources to define mathematical structures in purely conceptual terms. He also had the resources to logically derive all mathematical theorems from the basic propositions, where the logically representable derivations include the construction procedures. The synthetic content of mathematics is thus entirely contained in its basic propositions.

> As they found that all the inferences of mathematicians go according to the principle of contradiction (which the nature of every apodictic certainty requires), so they persuaded themselves that the basic propositions too could be known from the principle of contradiction, in which they erred; for a synthetic proposition can certainly be known by the principle of contradiction, but only in this way, that another synthetic proposition is presupposed from which it can be deduced, but never in itself (B 14)

Some of the basic propositions of mathematics are existential, where the notion of existence is distinctively mathematical. It concerns not the actual existence of empirical objects, but the objective reality of mathematical concepts, i.e., the *real possibility* of the existence of instances. But however this modal notion of mathematical existence is understood in detail, it is clear that Kant's principle applies: "every existential proposition is synthetic".[62] This raises

> the single fair demand: that one provide a justification [...] how one secures the objective reality of a concept we have thought out. In whatever manner the understanding may have arrived at the concept, the existence of its object can never be found analytically in the concept. (A 639/ B 677)

The categorical existence assumptions of mathematics are thus irreducibly synthetic.

In section 4.1.2 the stronger claim was made, that the existential assumptions are *the only irreducible* source of non-analyticity. This, I will argue, is indeed a consequence of the fact that mathematical contents and reasoning are logically representable. To be sure, the basic propositions of mathematical theories usually contain sentences that are non-existential yet non-tautologous in the sense of first-order predicate logic. A prime example is the principle of mathematical induction, which Henry Poincaré famously maintained as the prototypical synthetic *a priori*. Let's discuss this example before returning to the general case.

There is strong evidence that Kant considered induction, especially in the form of the least number principle, as expressing an analytic, even definitorial aspect of the general concept of finite number. Most concretely, he unambiguously argues that a proof employing infinite descent (an immediate application of the least number principle) thereby only appeals to grounds completely "contained in the understanding's

[62] A 598 / B 626. See the discussion on page 60 on the relevance of this statement to mathematics. Obviously, Kant's principle only refers to *categorical* existential sentences, not simply to any sentence that includes an existential quantifier. For example, a conditional sentences that says that some objects exists under certain conditions, or a negated existential sentence, may very well be analytic.

concept of number".[63] It is easy to see Kant's point: it is clearly essential to understanding the meaning of the concept "natural number" that whatever may fall under it cannot possess infinitely many predecessors in the "natural ordering"[64]. Consequently a property cannot hold of any number, unless there is a smallest number of which it holds.[65]

Returning to the general case, suppose that a theory about some as of yet hypothetical domain or structure has been set down in general relational terms. Challenged about the evidence for the basic principles of the theory, a proponent might reply: "This is simply what I choose to speak about: some domain of objects, ideal to be sure, which satisfy these conditions. The axioms are merely descriptions of this abstract type of relational structure; they are certainly not satisfied in every domain, but that I never demanded. They simply define this particular type of structure that I am interested in, and are thus analytic constituents of its definition. In mathematics we say: 'I wish to think these relations; which rules follow from them according to logical form?'"[66] The Kantian would answer: "I will gladly accept your definition. But amongst your basic principles, some make categorical existential claims. Though they also contribute to the description of the structural properties you are hoping to define, I cannot accept that these sentences are analytic. And indeed, they point to the fundamental issue: how do you know that the structure you are defining *really is* possible at all?" Kant would grant that all the non-existential, descriptive propositions may be considered part of the subject matter one chooses to define; this indeed raises the issue, which Kant already pondered in the 1760s,[68] that

[63] See footnote 33.

[64] See section 1.4.1.

[65] Note that this is not a categorical existential sentence: "if there is a number with property P, then there is a smallest such number". In view of the second incompleteness theorem, the question of the analyticity of the induction axioms actually becomes highly subtle. I do not have the space here to defend this claim, but I would argue that the non-analytic component still does not lie with the induction axiom, but rather, *firstly*, with the assumption of structures instantiating the axioms of arithmetic, and *secondly*, with the assumptions about the existence of various subsets of natural numbers over which (in the second-order case) the second-order quantifier in the induction axioms ranges, resp. which (in the first-order case) correspond to the extensions of the substitution instances of the predicate-place in the first-order axiom-schema.

[66] The final sentence of this fictional speaker is a direct quote of Kant, Refl. 3979, (17: 374). For illustrative purposes, I added the following reflections of Kant on the issue of the freedom of concept-formation in mathematics. 17/369:

> Our reason contains nothing but *relationes* [sic]. If these are not given through the relations of space and time, and also not through the iteration and composition of a One from a Many in pure mathematics, then they are not *relationes* that refer to an object, but only the relations of concepts according to laws of reason.

Refl. 3973, 17/371:

> No purely voluntary [willkürlich] concepts of pure reason can emerge in us except those of iteration, i.e., concepts of numbers and of magnitude.[67] All concepts of reason that contain a different kind of relation require that this relation is given by experience or by the nature of the understanding or reason. Empirical relations, however, presuppose a fundamental relation concept [fundamentalen Verhältnisbegriff].

Refl. 3979, 17/374:

> A dogmatic science of pure reason has voluntary [willkürliche] concepts at its basis, and their relations reside in logical form. But no arbitrary relations can be thought other than those of iteration, i.e., of numbers, and consequently, this science is mathematics. One can also investigate the universal law governing freedom [Willkür] itself, and this is moral philosophy. (cont. next page)
>
> Where concepts are given neither through experience nor through the nature of reason, there freedom is their ground, either regarding their form, when no matter (object) is given (the free formation of concepts is purely mathematical), or where pure freedom as such is itself both matter (object) and form of the concept; in either case the judgements are derived according to logical form.
>
> In mathematics we say: I wish to think these relations; which rules follow from them according to logical form?

[68] As discussed in the the classical article by L. W. Beck [**?**].

> synthetic judgements would change into analytic. It is a question how much arbitrariness there is.[69]

If one chooses to relegate all descriptive properties to the definition of the subject matter, then what remains problematic is only whether the existence of an object or structure instantiating this description is possible: all the synthetic content can be focused in the existential assumptions. As Schultz wrote in direct collaboration with Kant:

> One may put as many properties into the concept of the subject as one wants, to prove that the predicate can be derived from the subject by the mere principle of contradiction. This trick will not help. The *Critique of Pure Reason* gladly permits such an analytic judgement, but now it considers the concept of the subject itself, and asks: how did you come to put these different properties into it, that it already contains synthetic propositions?; first prove the objective reality of your concept: first of all, prove that any one of its properties really belong to a possible object; and then further prove that this same object, which has this first property, also has the others, which are not contained in the first.

This points to two basic tasks: separating and presenting in isolation the logically independent basic determinations of the subject matter; and showing that, taken together, they define a real possibility. By focusing on the problem of the objective reality of mathematical concepts, Kant put the finger on the central problem in the foundations of mathematics today: which existential assumptions of mathematics can be justified, and how?

If mathematics today is thought of as the science of possible structures, then a fundamental philosophical question is: which structures are possible? It is thus entirely natural that the issue of consistency of our basic theories became recognised as the crucial issue: for if *we know* that a theory is consistent, what prevents us from claiming that it describes a possible domain of objects? Of course we know today that the problem of consistency is maximally difficult: although we have a huge host of relative consistency results, we can have no absolute insurance that there isn't a contradiction in our basic theories, perhaps lurking just behind the horizon of humanly accessible proofs (there is no way to exclude, with Cartesian certainty, the unpleasant scenario that the axioms of Peano arithmetic imply $0 = 1$ but its shortest proof requires more steps than the volume of the observable universe measured in cubic Planck-lengths[70]). Insofar as we believe, as I think we ought to, in the consistency of our foundational theories — primitive recursive arithmetic (the paradigmatic theory of finitism[71]), arithmetic (the paradigmatic theory of the countable), analysis (the paradigmatic theory of the continuum), set theory (the paradigmatic theory of the higher infinite), and the philosophically relevant possible extensions of set theory (either Ultimate L or forcing axioms like Martin's Maximum) — for all of them this belief must rest on grounds that are not reducible to mere logical truths. What do they rest on?

[69] Refl. 3928, quoted by Beck.

[70] Note that for any formal theory, there must exist a minimal length of a proof of contradiction if there is one at all. It is one of the standard lines of attack on the consistency issue, e.g., in Gentzen's relative consistency proof of first-order arithmetic, to show that any proof of a contradiction could be shortened to a strictly shorter one. If this can be established, then it at once follows that there cannot be any proof of a contradiction.

[71] I accept Tait's thesis [83].

4.2 The continued relevance of Kantian ideas to the philosophy of mathematics

We saw that the Kantian challenge, suitably generalised, is still of crucial relevance today. That Kant was capable of posing this question at all is due to his sophisticated analysis of theoretical concept-formation. But one question from the beginning of section 4.1 remains as of yet unanswered: we saw why Kant considers mathematical sentences to be synthetic, but we did not yet touch on his claim that they are *a priori*. It is a tempting thought that, fundamentally, the tension between syntheticity (or non-logicality) and aprioricity (or non-empiricity, universality, necessity[72]) is the motor that generates the philosophical difficulties in the foundations of mathematics. An all too brief reflection on this vast subject will illustrate my claim that Kantian ideas, suitably adapted, retain their relevance to the foundations of mathematics today.

The discovery of the problem of the non-empirical yet non-analytic status of mathematics, and the location of the fundamental problem of the existential assumptions is the primary source of Kant's continued relevance. As intimated above, the issues are still with us. A central point of controversy between Friedman's and my own interpretation is the question of the conceptual definability of mathematics in Kant's system. Today it is of course uncontroversial that the definition of mathematical structure and the derivation of mathematical theorems can be done in purely logico-conceptual terms (at any rate the intensional content of the concepts of set theory are not tied to spatio-temporal intuition). What remains as problematic as ever is the significance of the existence assumptions concerning these structures, even if existence is taken in the 'weak' sense of consistency. The relation of consistency and existence was a central point of disagreement in the unfortunate misunderstanding between Frege and Hilbert. I fully subscribe to Walter Dean's analysis, (*i*) that "there is a precise sense in which Frege was correct to maintain that the general problem of demonstrating the consistency of a set of axioms is as difficult as it can be", and (*ii*) that "there is a precise sense in which Hilbert was correct to maintain that the general problem of demonstrating the existence of a structure satisfying a set of axioms is as easy as it can be conditional on their proof-theoretic consistency".[73] This shows is that calling consistency a 'weak' sense of mathematical existence is tantamount to missing the point.

Kant suggested that the existence issue could be solved in terms of modality, particularly, spatio-temporal possibility. Due to our intuitions of them, we have *a priori* evidence for the real possibility of structures instantiating the axioms of geometry and arithmetic. The basic form of his line of thought was as follows:

(*1*) *De facto*, we have the science of pure mathematics. Our job as philosophers is to take this as our starting point, and explain how it is possible. (*2*) The propositions of mathematics are both universal and necessary. Consequently, they cannot be derived empirically. The only way to explain both (*1*) and (*2*), Kant argued, is that the principles of mathematics must articulate the necessary conditions of possibility of our empirical experience. They are prior to experience, and thus *a priori*.

[72]Clearly, none of these are synonymous! Kant argued that strict universality and necessity must imply non-empiricity. For our present purposes it is not necessary to delve deeper into this issue.

[73]Dean [21].

Without entering any deeper into Kant's reasoning, I want to make the following suggestion: together with Kant, we should maintain both (*1*) and (*2*): the respect for the "factum of science", and the argument for the non-empirical nature of mathematics. But in light of our knowledge of mathematics today, we cannot accept his conclusion.

Kant's attempted solution of *a priori* intuitions as conditions of possible empirical experience *may* work for finitary arithmetic. Arguably, it *might* even work for Euclidean geometry, in the sense that it gives us knowledge about what is necessary and possible within idealized Euclidean space, even though this does not allow extrapolation to the global features of physical space-time. It certainly *does not* work for classical real analysis: although it is hard to deny that we have powerful 'intuitions' of the continuum, they neither satisfy Kant's stringent criteria on "reine Anschauung", nor can the claim be justified that the classical continuum constitutes a necessary condition of possibility of empirical experience, as even the two greatest champions of the continuum, Dedekind and Hilbert, pointed out. And that it doesn't work for the higher infinities of set theory hardly needs mentioning.

The respect for the "factum of science" prohibits that we reject, on these or similar philosophical grounds, the justification of infinitary mathematics. Some of the most exciting, difficult, and rigorous mathematical work of the last 140 years is unthinkable without the ventures into the lofty reaches of the higher infinite. And the phenomenon of non-conservativity discovered by Gödel implies that these higher abstractions may have a direct bearing to the concrete domain of even elementary number theory. There are Diophantine equations in the integers for which the question whether they have a root cannot be decided without appeal to set-existence axioms that go far beyond ZFC. Admittedly, we haven't found any 'natural examples' of tanglible incompleteness (as Harvey Friedman calls this phenomenon) yet, only such that have been constructed specifically for the purpose of showing that they are independent. For all that we know, all sentences about the natural numbers that mathematicians (not logicians) have so far considered worthy of their attention may have elementary proofs or refutations. For example, there is no reason to think that Wiles' proof of Fermat's Last shouldn't be reducible, if not to Peano arithmetic then at least to a relatively weak system of real analysis (to account for the functional analysis and the use of dependent choice in the Langlands/Tunnell part of the proof), which in turn should have a proof-theoretic reduction to some constructivistically justifiable theory. But even if this should be so, it does not change the basic point. Firstly, the algebraic number theory wielded in Wiles' proof is *de facto* unthinkable without going beyond elementary arithmetic, indeed making use of the full conceptual power of ZFC.[74] Without going to these levels of abstraction, the proof simply makes no cognitive sense; even if it can later be 'engineered down', this stripped down version will be utterly incomprehensible without understanding the original, infinitary story. Another example of the same phenomenon is described by Michael Rathjen in his ground-breaking work on the ordinal analysis of Π^1_2-CA :

> Extensive ordinal representation systems are difficult to understand from a purely syntactical point of view, often to such an extent that it makes no sense to present an ordinal representation system without giving some kind of semantic interpretation. For ordinal representation systems in impredicative proof theory it is essential to understand the projection functions

[74] For this sentence I rely on personal communication with Kevin Buzzard.

> (also called collapsing functions) which they encapsulate. In this section we will indicate a model for the projection functions, employing rather sweeping large cardinal axioms, in that we shall presume the existence of certain cardinals, featuring a strong form of indescribability.
>
> Large cardinals have been used quite frequently in the definition procedure of strong ordinal representation systems, and large cardinal notions have been an important source of inspiration. In the end, they can be dispensed with, but they add an intriguing twist to the relation between set theory and proof theory. The advantage of working in a strong set–theoretic context is that we can build models without getting buried under complexity considerations.[75]

The point here is that the syntactic representation system is a purely constructive object that in no way relies, for its technical definition, on actual infinities. Yet to make any sense of it, it is indispensable to provide an interpretation of it in terms of extremely large cardinals, which go far beyond what can be proven to exist on the basis of ZFC alone.

And even if we were to ignore this relevance of infinitary mathematics to the finite numbers, there still remains the 'factum' of set theory itself. This is a vibrant field of highly sophisticated research – one may even say that there is renewed progress on its original trauma, the continuum problem: see Asperó & Schindler [2]. The job of philosophers is to explain its possibility, not to explain it away.

By holding fast to Kant's argument for the non-empirical nature of mathematics and his respect for the factum of science, we thus face the following situation. Putting the two together, we find that we can neither admit that mathematics is empirical in the sense of depending on the observation of physical or psychological regularity, nor that it is *a priori*: not *a priori* in the sense of articulating the implicit structure of possible empirical cognition, and not *a priori* in the sense of being independent of *any kind* of experience. This is the final point that I want to make here.

The signature move of infinitary mathematics is that of transcending any uniquely defined theoretical frame, by considering it from a higher standpoint. This is already what the incompleteness theorems force us to do, and it is also what motivates the dialectical ascent of set theory (indeed, as Gödel saw, the two are intimately connected: "The true reason for the incompleteness, which clings to all formal systems of mathematics, lies [...] therein, that the formation of ever higher types can be continued into the transfinite"[76]). As Zermelo put it beautifully:

> The "ultrafinite antinomies of set theory", to which scientific reactionaries and anti-mathematicians appeal in their fight against set theory with such eager passion, are only apparent "contradictions", due only to a confusion between *set theory itself*, which is non-categorically determined by its axioms, and the individual *models* representing it: what in one model appears as an "ultrafinite non- or overset", is already a fully valid "set" with cardinal number and ordinal type in the next higher model and, in turn, serves itself as the bed-stone in the construction of the new domain. To the unlimited series of Cantorian ordinal numbers there corresponds a likewise unlimited double series of essentially different set-theoretic models in each of which the entire classical theory finds its expression. The two diametrically

[75] Rathjen [72], comp. http://www1.maths.leeds.ac.uk/ rathjen/pime.pdf
[76] Gödel [35, fn. 48a].

> opposed tendencies of the thinking mind, the ideas of creative *progress* and of synoptic *completion*, which form also the basis of Kant's "antinomies", find their symbolic exhibition and their symbolic reconciliation in the transfinite number series, which rests upon the notion of well-ordering and which, though lacking in true completion on account of its boundless progressing, possesses relative way stations, namely those "boundary numbers", which separate the higher from the lower model types. Thus, instead of leading to constriction and mutilation, the set-theoretic "antinomies" lead, when understood correctly, to an as yet unforeseeable development and enrichment of the mathematical science.[77]

What this expresses is not only the essential open-endedness of the universe of sets, but an essential open-endedness of mathematical concept-formation itself. Though we do not necessarily need to extend our primitive notions, we can never be in a position to have exhausted all *priniciples* governing fundamental mathematics. Or as Tait says:

> In the foundations of set theory, Plato's dialectician, searching for the first principles, will never go out of business.[78]

Consequently, it cannot be reduced to *a priori* principle that we can set down, once and for all. Paul Bernays' writings make a beginning for bringing together Kantian ideas and reflections on modern foundational issues. The following texts provide the suitable starting point.

- Zur Frage der Anknüpfung an die Kantische Erkenntnistheorie (1947)
- Die Erneuerung der Rationalen Aufgabe (1949)
- Die Mathematik als ein zugleich Vertrautes und Unbekanntes (1955)
- Bemerkungen zur Philosophie der Mathematik (1969)

What Bernays claims is that besides empirical experience and primitive intuition (structured by the categories), there is the third, essentially open field of what he calls "geistige Erfahrung". Its essential open-endedness (which is forced by the set-theoretic antinomines and the non-conservativity phenomena discovered by Gödel) implies that the principles governing it can never be completely characterised *ab ovo* and once and for all: neither *restricting* it to some minimal or 'safe' basis (as various constructivist would), nor trying to characterize it in its absolute totality is possible. Evidence acquired in this third domain, by experimental 'spiritual experience', is neither empirical nor *a priori* in the traditional senses of these words – and it is in this open field that higher mathematics takes place. Bernays' remarks in the above and some further essays are extremely exciting. Their consequences still lie in wait to be worked out.

Motto: *Ignoramus. Placet experiri.*

[77] Zermelo [92].
[78] Tait [83].

Chapter 5

Appendix I: Kant on infinitary concepts in the Inaugural Dissertation

Kant's 1770 inaugural dissertation, *De mundi sensibilis atque intelligibilis forma et principiis* is a crucial source for understanding Kant's technical terminology and conceptual framework. In this appendix, we show that already here, Kant had in place the distinctions between, on the one hand, logical consistency and definability, and, on the other hand, intuitive exhibitability. Especially interesting is his discussion of the abstract concept of infinitary structures. In particular, Kant proposes the correct definition of the ordinally infinite. Friedman's interpretation can be refuted by reference to this passage in the dissertation alone. Note that one cannot argue that Kant must have changed his mind about the impossibility of defining mathematical notions purely conceptually after 1770: except for a minor terminological point,[1] the part of the dissertation that we here discuss is entirely in agreement with the argument of the *Critique*; and more importantly, Kant is not merely *claiming* the abstract definability of infinitary notions in purely intellectual terms, but actually does it.

Kant begins by distinguishing abstract intellectual concepts of structure from the intuitive exhibition of instances. His essential point, as we shall see shortly, is that an impossibility to do the latter does not imply conceptual inconsistency.

Aliud enim est, datis partibus compositionem totius sibi concipere, per notionem abstractam intellectus, aliud, hanc notionem generalem, tanquam rationis quoddam problema, exsequi per facultatem cognoscendi sensitivam, h.e. in concreto eandem sibi repraesentare intuitu distincto. Prius fit per conceptum compositionis in genere, quatenus plura sub eo (respective erga se invicem) continentur, adeoque per ideas intellectus et universales; pos-	It is one thing, the parts being given, to conceive the composition of the whole by an abstract intellectual notion, and another thing to follow out this general notion, as a problem [posed by] reason, by the cognitive sensuous faculty, i.e., to represent it in the concrete by a distinct intuition. The former is done through the general concept of composition, as several things are contained either under it, or [are composed] with each other, and hence by

[1] In the dissertation, "synthesis" refers exclusively to effective action in finite time. In the *Critique*, Kant distinguished *synthesis intellectualis* from *synthesis speciosa*; the former refers to the abstract conceptual definability of a relational structure, and is presupposed as a necessary condition for the representation of the synthesis of an intuitive manifold (as by an action taking place in time). See section 1.3.3 above.

> terius nititur condicionibus temporis, quatenus, partem parti successive adiungendo, conceptus compositi est genetice i.e. per SYNTHESIN possibilis, et pertinet ad leges intuitus.
>
> intellectual and universal ideas. The latter rests on the conditions of time, inasmuch as the concept of a composite is possible genetically, i.e., by *synthesis*, by the successive union of part to part, and falls under the laws of intuition.

Kant applies this distinction to the intellectual concepts of the continuum and of an infinite quantity, neither of which can be completely exhibited, regarding the totality of their parts as well-distinguished individuals, in human intuition.

> [...] nec totum in priori casu secundum leges intuitus quoad compositionem, nec in posteriori compositum quoad totalitatem complete cogitari possunt. Hinc patet, qui fiat, ut, cum *irrepraesentabile* et *impossibile* vulgo eiusdem significatus habeantur, conceptus tam continui quam infiniti a plurimis reiiciantur, quippe quorum, secundum leges cognitionis intuitivae, repraesentatio plane est impossibilis. Quanquam autem harum e non paucis scholis explosarum notionum, praesertim prioris causam hic non gero**), maximi tamen momenti erit monuisse: gravissimo illos errore labi, qui tam perversa argumentandi ratione utuntur.
>
> [...] hence neither the whole in the first case as to composition, nor the composite in the latter case as to totality can be thought completely in accordance with the laws of intuition. *Unrepresentable* and *impossible* being vulgarly deemed to have the same meaning, it is plain why most reject the concepts of the continuous and of the infinite, since the representation of both according to the laws of intuitive cognition is impossible. Though I do not venture here to defend these notions, esp. not the first, which by many schools of thought are considered exploded,**) still this reminder is of the greatest import: for those who use so perverse an argumentation have fallen into a grave error.

Kant chastises the conflation of abstract conceptual definability of an object with its exhibitability in concrete intuition; correspondingly he distinguishes conceptual inconsistency from the impossibility of carrying out the exhibition – which may be an infinite task – in finite time. His criticism in the footnote**) of a common yet fallacious 'definition' of an infinite quantity and his own, correct definition, which we discuss in more detail in a moment, confirms our thesis that 'conceptual definability' is *not* merely the conjunction of interpreted monadic predicates, but the abstract formal description of a relational structure, here that of the ordinally infinite (in fact the very same we still use in set theory). The "grave error", Kant continues, rests precisely in the conflation the two notions he just distinguished, logical inconsistency and the human limits of intuitive exhibitability:

> Quicquid enim repugnat legibus intellectus et rationis, utique est impossibile; quod autem, cum rationis purae sit obiectum, legibus cognitionis intuitivae tantummodo non subest, non item. Nam hic dissensus inter facultatem sensitivam et intellectualem (quarum indolem mox exponam) nihil indigitat, nisi, quas mens ab intellectu acceptas fert ideas abstractas, illas in concreto exsequi et in in-
>
> For whatever is repugnant to the laws of the intellect and reason is of course impossible; but for something that is the object of pure reason, which merely does not fall under the laws of intuitive cognition, this is not so. For here the disagreement between the sensuous and the intellectual faculties (more on this shortly), indicates nothing except that the abstract ideas that the mind has received

tuitus commutare saepenumero non posse.

from the intellect can often not be followed out in the concrete and translated into intuitions.

A defender of Friedman's interpretation may yet remain sceptical. What is the status of these abstract definitions? Do they actually *characterise* the structural aspects of the definienda, or are they merely nominal definitions? Friedman obviously does not deny that Kant could *talk* about infinity, using concept-words. Rather his point must be — and this is indeed the fundamental thesis of his interpretation — that the *logical form* and the *abstract non-intuitive content* of concepts, as understood by Kant, cannot carry any information about the relevant structural properties; only and exclusively the generative intuitive constructions can provide this. In the above mentioned footnote**), Kant rejects the faulty definition of the infinite as the "greatest quantity", and proposes instead the correct definition of a 'multiplicity greater than all finite number':

Si vero infinitum mathematicum conceperint ceu quantum, quod relatum ad mensuram tanquam unitatem est multitudo omni numero maior, si porro notassent, mensurabilitatem hic tantum denotare relationem ad modulum intellectus humani, per quem, nonnisi successive addendo unum uni, ad conceptum multitudinis definitum et, absolvendo hunc progressum tempore finito, ad completum, qui vocatur numerus, pertingere licet: luculenter perspexissent, quae non congruunt cum certa lege cuiusdam subiecti, non ideo omnem intellectionem excedere, cum, qui absque successiva applicatione mensurae multitudinem uno obtutu distincte cernat, dari possit intellectus, quanquam utique non humanus.

If they had instead conceptualised a mathematical infinite as a quantity which, in relation to a measure as unit, is a multiplicity greater than any number; if, furthermore, they would recognise that mensurability here denotes only a relation to the modalities of the human intellect, which can attain a definite concept of multiplicity only by the successive addition of unit to unit, and to the sum total called number only by going through with this progress within a finite time, they would clearly perceive that that which does not agree with a certain law of some subject need not therefore exceed all intellection; since an intellect may exist, though not a human one, perceiving a multiplicity distinctly by a single insight, without the successive application of measurement.

Essentially, our disagreement with Friedman is about whether Kant's concept 'x is a quantity greater than all finite numbers' (which is a correct definition of the ordinally infinite) has the logical form

$$Q(x) \wedge G(x), \tag{I}$$

or whether it has the form

$$Q(x) \wedge \forall n(\mathit{Finite}(n) \Rightarrow x > n). \tag{II}$$

According to Friedman, the quantificational and relational form of (II) could not be conceptually expressed by Kant. He could press the point further by arguing that this definition contains the general concept of finitude (which, ranging over *all* finite numbers, somewhat paradoxically carries the relevant non-finite content), and claim that Kant explained finitude with reference to temporal iteration.[2] This reply fails for two reasons: (*1*) if Kant *did* have the logical resources for expressing (II), relations and quantifiers,

[2]Friedman [30, p.???].

then he could already introduce infinitary concepts without recourse to the prior definition of finitude; but Kant really did recognise the formal difference between 'x is red' and 'x is greater than y', as well as the difference between 'for all x there is a y' and 'there is a y for all x';[3] (*2*) in any case, Kant does *not* base the notion of finitude on the notion of time (and rightly so!). Conversely, the notion of "finite time", which he actually uses, already presupposes the pure concept of finitude; the concepts of *unit*, *plurality*, *greater than*, are all purely intellectual concepts. A more complete explanation of Kant's notion of infinite quantity could be the following:

> 0 is a finite number. For any finite number, there is a finite number greater than it. If the number n is greater than m, and o is greater than n, then o is greater than m. No number is greater than itself. α is an infinite quantity if it is greater than any finite number.
>
> (III)

In contrast to (II), the expression "finite" does not have to be interpreted contentually for this characterisation of infinity to work: it implies, by logical form alone, that α is proceeded by infinitely many elements in the 'greater than'-order, even if "finite number" is replaced by "P". There cannot be any serious doubt that Kant would have considered (III) as capturing the "abstract intellectual notion" discussed in the quotations above. Note that if one abstracts from the meaning of "finite" by replacing it by "P", (III) remains a sufficient but not necessary condition for the structural property of being infinite.[4]

[3] In essence this is Kant's distinction between "*omintudo distributiva*" and "*omnitudo collectiva*". Three examples of this logical difference: in theology, Kant distinguishes two criteria of universality of a "church": for every individual, there existing a 'private church' provided by rational religion ("*universitas vel omnitudo distributiva*"); and there existing one universal church for all individuals ("*omnitudo collectiva*") (AA 06: 157). Regarding "general property", Kant distinguishes between for every person there existing some land that is their rightful property ("*omnitudo distributiva*"), and there being land that is the rightful property of all ("*omnitudo collectiva*", AA 23: 320). Finally, in the *Critique*, this distinction is used in two places: to differentiate the 'first-order' relations between individual elements within a structure (an infinite sequence), and the 'second-order' relation of the elements to the structure as a whole (A xxx: B xxx.); and in the discussion of the rational concept of God (the "ideal prototype"), in the distinction between 'for every perfection there is a (possible) being that has it' and 'there is a being that has every perfection' (A xxx: B xxx.).

As a side-remark. The logical form of this last distinction is mirrored in the Bernays-Lévy Reflection Axiom in set theory, which (very roughly) asserts that for every property of the universe of sets, there is a set that has that property, while avoiding to assert the antinomic principle that there is a set that has every property of the universe of sets.

[4] *Nota Bene*: (III), considered exclusively regarding its logical form (abstracting from the meaning of 'finite'), expresses a condition that is sufficient but not necessary for α to be infinite, i.e., to have infinitely many predecessors in the "greater than" relation: if one replaces "finite number" by "P", all but the final sentence are satisfiable by *all* (finite and transfinite) ordinals. Therefore, one cannot use (III) as a definition of "infinite", let alone get a definition of "finite" through the backdoor via negation. These sentences (excluding the last) state necessary but insufficient conditions for characterising "finite number"; consequently, neither could we have defined ω, i.e., the *least* ordinally infinite quantity; both requires much more involved definitions, which Kant did not possess, but – this is the key point – he would have no problem in understanding them, in terms of the pure categories. One may wonder: if we haven't *defined* infinity or finitude, how can we claim that we introduced infinitary concepts? The first part of (III), i.e., all but the final sentence, states conditions that are satisfied by all ordinals (finite and transfinite), but leaves open (qua logical form, i.e., if we ignore the contentual meaning of "finite") whether we mean to refer to the finite ordinals exclusively, or whether we are also referring to (some or all) transfinite ordinals (more conditions are needed to determine this). The final sentence states that, whatever were are talking about (which definitively includes all of the finite ordinals but may include more if we abstract from the meaning of 'finite' in (III)), α is greater than all of them. Hence α definitively falls beyond the boundary between finite and infinite, even though we haven't clearly defined this boundary itself. For this we need Dedekind; incidentally, in a round-about way, the introduction to *Was sind und was sollen die Zahlen* supports our reading of Kant: Dedekind emphasises the Kantian distinction between "Anschauung" and "reine Verstandesbegriffe" (Dedekind: "allgemeinere Begriffe und solche Tätigkeiten des Verstandes, ohne welche überhaupt kein Denken möglich ist"), and argues that the concepts of number are such purely intellectual concepts that do not require, for their definition, any reference to intuition. Like Kant, he also recognizes the necessity to show that these concepts are *instantiated*. The difference to Kant is that he tries to prove the existential assumptions by reference to a variation on the Kantian "transcendental unity of apperception", i.e., the capacity of accompanying every object of thought with the "I think"; this argument Kant would not allow (as for him the unity

of apperception is merely a formal structuring principle of experience, *not* a principle grounding the *existence* of objects), and rightly so: Dedekind's 'proof' of proposition 66 makes essential use of what would later be called 'unrestricted comprehension', which allows to introduce the extension of any predicate as a set, and which leads straight into the antinomies of 'naïve' set theory, as Dedekind, made aware of it by Cantor, grudgingly had to acknowledge in the preface to the third edition of *Zahlen*.

Chapter 6

Appendix II: Kant's logical theory – a sketch

> *"Ziel dieses Teils ist es, Kants vielzitiertes Diktum, daß die Logik seit Aristoteles keinen Schritt vorwärts und rückwärts getan habe, zu widerlegen: einen großen Schritt vorwärts hat Kant selbst in seiner logischen Reflexion getan."*
>
> Peter Schulthess [78, p.11]

6.1 The silent revolution: object and concept in logic

Friedman accepts the thesis, which he takes from Manley Thompson, that Kant made a significant advance in logical theory by replacing the traditional *logic of terms* with a *logic of objects and concepts*, "a logic in which the form of predication is 'Fx' and not '*S is P*''". But Friedman disagrees with Thompson's further claim that "the general logic required by Kant's transcendental logic is thus at least first-order quantificational logic plus identity", insisting instead on the restriction to monadic predicates.[1]

According to Thompson himself, the advance was only *implicit*, as Kant's official logical theory demanded the traditional form of predication as a relation between two general terms. But from his theoretical deliberations, we can infer that "the concept that Kant [actually] wants" for the subject of predication is that of an arbitrary individual to which the predicate applies, "represented by the 'x' in 'Fx' rather than 'S' in '*S is P*''".[2]

In fact, as Peter Schulthess showed, Kant carried out such a transition quite explicitly, in conscious opposition to tradition. Concepts, in virtue of their logical form "as predicates of possible judgments", necessarily relate to indeterminate (or better: partly determinate) individual objects.[3] This reference to the

[1]Friedman [30, p. 63, fn. 9]. The quotes are his of Thompson [85, p. 342].

[2]Thompson [85, p. 342].

[3]Schulthess argues that the role of intuition as concrete individuating representations lead to this change in Kant's logical theory [78, p. 79 *et passim*]. His book is a masterpiece of meticulous exegesis that traces Kant's logical writings chronologically (up to the *Critique*), situating them in their historical context. Some of the many piece of evidence Schulthess discusses, firstly, *CPR* B 94 / A69:

indeterminate individual extension-element is essential to the logical form of predication. In his logical writings, Kant rendered propositions accordingly by using individual variables, e.g.,

> Every ***x***, to which the concept of body (***a+b***) applies, attractiveness (***c***) applies.[4]

The vital fact that Friedman, Thompson and Schulthess ignore is that Kant did not actually anticipate the form of predication 'Fx' of modern first-order logic. In their 1917-18 lectures *Prinzipien der Mathematik*, which Wilfried Sieg called "the real beginning of modern mathematical logic", Hilbert and Bernays gave the today standard development of the syntax and semantics for propositional, monadic, first-order, and ramified second-order logic, and treated metamathematical independence and completeness problems. A crucial aspect of their approach is that formal theories are always accompanied by an informally defined semantics that specifies the domains, usually many-sorted structures, in which the first-order variable-, predicate-, and function-symbols are interpreted[5] (this connects to their method of "existential axiomatics" that assumes the existence of systems of objects satisfying the structural conditions defined by the axioms; therein "lies something so-to-speak transcendental for mathematics" according to Bernays[6]). That the semantics, or as Kant would have it, "relation to the object"[7] is assumed as given independently is reflected on the formal-syntactic level, in particular in the form 'Fx': it contains no parameter for the type or domain of the first-order variable 'x', since it is presupposed that this domain is specified informally (outside the formalism). Accordingly, a simple universally quantified proposition has the form '$\forall x(Fx)$'; the range of the quantifier '$\forall$', i.e., the range of possible values that x may assume, is a 'formally hidden' *fixed* parameter.

In the course of a formal derivation in first-order logic, changing the domain of quantification is strictly prohibited, and the formation of new types (not merely a *restriction* of the domain of quantification, but the introduction of a new kind of individual) is syntactically impossible (Hodges calls this "global formalizing", and distinguishes it from the "local formalizing" of traditional logicians, who customarily changed the domains of discourse in the course of an argument, say, from individuals objects as the subject of predication to pairs of objects[8]). Consequently, the basic form of predication need not include an explicit specification of the type of the subject, and the traditional forms of categorical sentences, which explicitly specify this type, are rendered not as primitive logical forms but as composite, non-uniform constructions (this comes out strongest in the case of the singular sentence, where the specific function of the subject term, to fix the domain of predication, is obliterated):

> Concepts, however, as predicates of possible judgments, are related to some representation of a still undetermined object. [...] It is therefore a concept only because other representations are contained under it by means of which it can be related to objects. It is therefore the predicate for a possible judgment.

Secondly, AA 17, p. 346:

> A predicate does not represent a part of an entity but the entity itself as partly determined. Through a predicate I do not represent a part of the entity, or have a concept of that part, but I represent the object itself, and have of it a partial conception.

Finally, AA 24, p. 257:

> The *sphaera notionis* actually refers to the collection of objects which fall under a concept, as a *nota communis*.

[4] AA 09, p. 111 and AA 16, p. 671 (R 3127) .

[5] Sieg [80, p. 356].

[6] Bernays [**?** , p. ?].

[7] *CPR* B ?? / A ??.

[8] Hodges [49]

All humans are mortal	$\forall x\,(Hx \to Mx)$
Some humans are mortal	$\exists x\,(Hx \wedge Mx)$
This human is mortal	$Ha \wedge Ma$

If these propositions are to be rendered instead as $\forall x(Mx)$, $\exists x(Mx)$, and Ma, then the domain of the variables and singular terms must be specified informally—'invisible' on the formal level—to 'Human'. None of this should be taken as a criticism of first-order predicate logic; it merely serves to bring out the differences to Kant.

For Kant, on the other hand, the categorical form specifying the concepts of both subject and object is a primitive form of judgement. It would be false to take his above-cited rendering of a universal categorical judgement, 'all ***x*** to which ***a*** applies, ***b*** applies', as an instance of a composite form like $\forall x\,(Fx \to Gx)$ build up from the more primitive forms Fx, $\varphi \to \psi$, and $\forall x\varphi$. Rather, the occurrence of two general terms in universally and existentially quantified categorical judgments is a consequence of the fact that *in Kant's logical theory, all objects are required to be explicitly typed.* In part, this is certainly due to tradition as well as the grammatical features of natural language. But Kant's experiments with the logical form of predication during his so-called silent years (the period of conception of the *Critique*) show that he developed well-wrought systematic reasons for this convention.

The logical forms of judgment always involve the specification of the type of object. '*x:A*' is the form of a singular judgment; *x* is a concrete individual, and *A* is it's type, e.g., this is a triangle, Socrates is a human.

> ***x*** is thus the determinable (object), which I think through the concept ***a***, and ***b*** is its determination (or manner of determining it). In mathematics ***x*** is the construction of ***a***, in experience the *concretum*. [9]

A quantified judgment, on the other hand, does not have the form 'all *x:A*', but rather, 'all (*x:A*)*B*'. It is false to identify this with the '*Fx*' of predicate logic, which implictly presumes a domain of the quantifier, which corresponds to the subject term *A*; it makes no sense to say 'all *x:A*'.

In the case of mathematical theorems, Kant holds that *A* corresponds to the *real definition* of the type of *x*, i.e., it determines the method of constructing *x*, and *x* is an individual *a priori* intuition constructed according to this method, "the construction" in the objetual sense. He also writes that the "a general concept of a sensible *dati*" expresses "the action of determining an object according to sensible conditions".

> Now ***x*** is this determinable that contains the condition of determination; ***a*** signifies merely the action of determination in general. It is no surprise, if beyond that action of determination more is contained in ***x***, which will be expressed by ***b*** [...], *e.g.* in space, besides the general action of constructing a triangle, also the degree of the angles [...]. These determinations are found in ***x*** in intuition, through the construction of ***a***, e.g. 'Triangle' [...]. Thus the relation,

[9] 17:645, ca. 1773 - 75)

> which is thought through $\boldsymbol{a}$, is further determined by the real condition of the subject [$\boldsymbol{x}$] [...].[10]
>
> $\boldsymbol{x}$ always signifies the object of the concept $\boldsymbol{a}$.[11]
>
> For the formation of a rule three pieces are required: 1. $\boldsymbol{x}$ as the *datum* for a rule (object of sensibility or rather sensible real representation). 2. $\boldsymbol{a}$, the *aptitudo* for a rule or the condition, under which it can be referred to a rule. 3. $\boldsymbol{b}$, the exponent of the rule. [12]
>
> The basic principle: everything, that can be thought, stands under a rule, for only through the rule is it an object of thought.
>
> The **principium of synthesis** contains rules of thinking *a priori*, insofar as it is **determined on objects**. Thus, in it is 1. the pure thinking ($\boldsymbol{a}$) and its rule, 2. the condition of the object, i.e. that under which something is given (or brought) to thought as an object ($\boldsymbol{x}$), 3. the determination of the thought from this relation ($\boldsymbol{b}$).[13]

6.2 Concept formation, real definitions, and the logical form of complex judgments

On this basis, we can give an account of Kant's views on the logical form of mathematical propositions. But first, we need two more components: Kant's views on the *definition* of mathematical concepts; and on primitive *relational* expressions in mathematics. Beginning with the latter, it is quite clear that Kant's distinction between propositions formulated in terms of pure concepts of the understanding and mathematical propositions appealing to intuitive concepts does not match our distinction between monadic and polyadic predication. Kant's notion of "external determinations" of an object covers both intuitive, e.g., spatial relations, and relations entirely based in the understanding, e.g., causality.

> Mathematics, on the contrary, is capable of axioms, e.g., that three points always lie in a plane, because by means of the construction of concepts in the intuition of the object it can connect the predicates of the latter *a priori* and immediately. A synthetic principle from mere concepts, on the contrary, e.g., the proposition: everything that happens, has its cause, can never be immediately evident.

The difference between these two propositions is not that they contain fundamentally different logical forms of predication, but their respective source of evidence. The basic relational expressions ('point A lies in plane B', 'event A is the cause of event B') are not differentiated by being made possible

For the former, consider Kant's discussion of the definition of the concept of a circular line.[14]

[10] 17:643-4, ca. 1773 - 75)

[11] 17:644, ca. 1773 - 75)

[12] 17:656, ca. 1773 - 75)

[13] 17:661, ca. 1773 - 75)

[14] In his comments on Salmon Maimon's *Versuch über die Transzendentalphilosophie*, communicated via a letter to Hertz in 1789, Kant reacted to the argument that even on the level of elementary cognitions of the understanding, we are employing *ideas*, i.e. conceptual representations of infinite totalities (Maimon pointed out that e.g. the definition of a circle refers to an infinite totality of radii).The disagreement with Maimon will be in the next section.

> In the concept of a circular line is thought nothing but that **all** straight lines from it to a single point (the center) are equal to each other. This is a merely logical function of generality of the judgment, in which the concept of a line constitutes the subject, and signifies only **any one** line, not the **totality** of lines, that can be described from a given point in a plane.[15]

The definition of the concept is given by an open sentence: '*a line such that all straights line falling upon it from a point are equal to each other*'. Thus, in the above rendering, the definition "a circle is *a line such that...*" can be represented by *(x:Circle)B(x)*, where *B(x)* corresponds to the defining open sentence printed in cursive.

In the natural language rendering of this definition, the logical role of individual terms, here represented in particular by the variable *x*, are fixed via pronominal cross-reference ("falling upon *it*"). But the use of singular names or variables, which Kant appealed to in his formal analysis of propositions (recall his analysis of a categorical judgment using the quantifier "every ***x***, such that..." above), can bring out these dependencies in a sufficiently rigorous manner. There is absolutely nothing that prevented Kant from managing individual variables by phrasing the definition as '*a line* ***x*** *such that all straight lines falling upon* ***x*** ...'; for as we saw above, this is exactly what Kant had in mind in his experiments in the *Duisburg Nachlass*. One can easily formalize mathematical propositions in this way (NB: Contrary to Kant's use, I employ the word "line" for straight line, and "curve" for line; this is simply for reasons of readability):

(*x: Line*) is equal to *y: Line*. Cat., open (*x, y*)

A straight line *x* is equal to a straight line *y*.

(*x: Line*) falls from *z: Point* onto *w:Curve*. Cat., open (*x, z, w*)

A straight line *x* falls from a point *z* onto a curve *w*.

(*x: Line* falling from *z: Point* to *w:Curve*) *x* is equal to *y: Line*. Cat., open (*x, y, z, w*)

A straight line *x* falling from a point *z* onto a curve *w* is equal to a straight line *y*.

(**all** *x: Line* falling from *z: Point* to *w:Curve*) *x* is equal to *y: Line*. Univ. cat., open (*y, z, w*)

All straight lines *x* falling from a point *z* onto a curve *w* are equal to a straight line *y*.

(*w: Curve*) such that [(**all** *x: Line* falling from *z: Point* to *w:Curve*) *x* is equal to *y: Line*.] Cat., open (*y, z, w*), **concept for definition of circle**

[15] 11:52, 1789

A curve *w* such that all straight lines *x* falling from a point *z* onto *w* are equal to a straight line *y*.

(**some** *w: Curve*) is such that [(**all** *x: Line* falling from *z: Point* to *w:Curve*) *x* is equal to *y: Line*.] Part. cat., open (*y, z*)

Some curve *w* is such that all straight lines *x* falling from a point *z* onto *w* are equal to a straight line *y*.

(**all** *z:Point*) {(**some** *w: Curve*) is such that [(**all** *x: Line* falling from *z: Point* to *w:Curve*) *x* is equal to *y: Line*.]} Univ. cat., open (*y*)

For all points *z* there is some curve *w* such that all straight lines *x* falling from the point *z* onto *w* are equal to a straight line *y*.

(**all** *y: Line*) ⟨(**all** *z:Point*){(**some** *w: Curve*) is such that [(**all** *x: Line* falling from *z: Point* to *w:Curve*) *x* is equal to *y: Line*.]}⟩ Univ. cat., closed, **theoretical formulation of Euclid's third postulate**

For all lines *y*, all points *z*, there is some curve *w* such that all straight lines *x* falling from the point *z* onto *w* are equal to a straight line *y*.

The expected objection to this is that, *of course*, by allowing elementary propositions of the form '(*x: Line*) falls from *z: Point* onto *w:Curve*', the whole exercise becomes a triviality: if you admit what are essentially polyadic predicates, it is obviously possible to define mathematical concepts logically. But that is just what Friedman argues Kant did not admit. So are we not just begging the question here?

The immediate reply is that, as we showed in section 3.3.3, Kant thought that the real definition of mathematical concepts (the representation of the synthetic unity of the manifold), and in particular also the concept of a circle, is a contribution of the understanding, in the sense that it can be defined by formal means independent from intuition; and using only logical instruments that Kant explicitly discusses, the above is a demonstration that he was in the position to do what he claimed.

The second point to note is that it is certainly true that merely the categorical form of the mentioned proposition does not represent its relational content. The property which the predicate ascribes to the indeterminate line *x* is, in Kant's terminology, an "external determination", i.e., a determination of an object relative to other objects. But as we argued in chapter 3, the *formal* relational content of such a predicate is not derived from intuition, but is based on the fundamental representation of synthetic unity of a manifold in general by the pure understanding ("the highest point to which one must affix all use of the understanding, even the whole of logic"[16]). This *contentual* representation plays a role very much like that of the semantics presupposed by polyadic predicate logic: it is not the formal language, in terms of which these *basic contentual* representations are defined, rather, these basic representations provide the meanings of the elementary logical functions, in terms of which more specific formal representation can be defined (the behaviour of a polyadic predicate cannot be defined in the formal language to whose vocabulary it belongs, rather, the underlying semantics determines this behavior and thereby allows the representations of structures by regimenting and combining this basic content into increasingly complex descriptions—to put it in a nutshell: the concept "relation" cannot be defined in relational predicate logic.)

Finally, the point of the above exercise was not merely to show that, if you allow propositions like '(*x: A*) relates to *y: B* and *z: C* in such-and-such way', it is possible to define mathematical concepts. The point rather was to show how the basic forms of judgment remain intact, how Kant could continue to claim that the table of judgment contains all elementary such forms. While (in virtue of the basic representation of synthetic unity, *not* in virtue of intuition), predicates can contain any number of formal parameters, there is only one *active variable*, which is the subject of predication; in this way, Kant's theory of logical inferences also remains intact. The use of individual variables allows to manage the substitution of individual terms in logical inferences. For example:

> (*all x:Event*) [(some *y: Event*) *y* caused *x*]
>
> *a:Event*
>
> (some *y: Event*) *y* caused *a*

In the same way Kant would render the inference that Friedman argued he could not regard as logical, i.e., by two applications of this type of inference, first for *a:Point*, then for *b:Line*.

> For any point ***x*** and any straight line ***y***, there is a circle with center ***x*** and radius ***y***.
>
> ***a*** is a point, ***b*** is a straight line.
>
> Therefore, there is a circle with center ***a*** and radius ***b***.

To sum up: the reason why Kant can maintain the completeness of his table of logical functions is that a basic (non-hypothetical, non-disjunctive) judgment, in particular when occurring in the context of a logical inference, always has exactly one *active* variable, whose type is specified by the subject term. But this (type of) object is further determined by a predicate that may contain any number of parameters (which can themselves become the active variable by a logical transformation or in the course of a deduction). The characteristic feature of a categorical judgment is thus not that its predicate term is a strictly monadic predicate, but that one concrete object or type of object is singled out, of which something is asserted categorically, i.e., in an unconditional manner; what is asserted of it may be a relational or a non-relational determination. The formal relational content of a predicate with parameters is definable in terms of the basic concepts of synthetic unity (which are not derived from intuition but constitute the fundamental *contentual* representations of the pure understanding) in combination with the logical functions of judgment. The categorical form thus covers both relational and monadic forms of judgments.

[16] B 133f. fn

Using only logical methods discussed by Kant himself, I have shown that this is sufficient for a precise representation of the formal structural content of mathematical propositions. In the next section, I will discuss the relationship between theoretical and practical propositions, and thus in particular the status of the logical representation of algorithmic procedures in Kant's logical theory.

6.3 Theoretical and practical propositions

The propositions discussed above are all *theoretical propositions* with regard to their *logical form*. The distinction theoretical-practical is made on two levels: formal and contentual.

> But while practical propositions certainly differ in regard to the manner of representation [Vorstellungsart] from theoretical propositions (which represent the possibility of objects and their determinations), they need not on that account differ from the latter with respect to their content. [Practical propositions in a theoretical domain] are nothing but the theory of the nature of objects, applied to the way in which these can be generated by us according to a principle, i.e., the possibility of the object is represented through a voluntary action. [...] Thus the solution to the problem in mechanics: "*for a given force, that is to be in equilibrium with a given weight, to find the relation of the respective lengths of the lever arms*", is certainly expressed as a practical formula, but it contains nothing but the theoretical proposition, that "*the lengths of the arms are in inverse proportion to the force and the weight if these are in equilibrium*"; only that this relation, regarding its origin, is represented as possible through a cause (our choice), whose determining ground is the representation of that relation. It is exactly the same with all practical propositions that concern merely the generation of objects. (*First Introduction to the Critique of Judgment*, 20:196) return to introduction

Kant's notion of an object comes out nicely: the to-be-generated object in his example is a determinate ratio between the lengths of levers More crucially still: in a practical proposition, we represent an operation *whose determining ground is the representation of the to-be-generated relational structure* — this is precisely the point made in 3, viz.: the representation of a generating operation requires the definition of the to-be-generated relational structure as its "determining ground" or "condition of construction"

> Practical propositions, therefore, the content of which concerns merely the possibility of a represented object (through voluntary generation), are merely applications of a complete theoretical cognition and cannot constitute a special part of a science. A practical geometry, as a separate science, is an absurdity, although ever so many practical propositions are contained in this pure science, most of which, as problems, require a special instruction for their solution. The problem: "from a given line and a given right angle, to construct a square", is a practical proposition, but a pure consequence [reine Folgerung] of the theory. (*First Introduction to the Critique of Judgment*, 20:198)

The construction postulates are thus merely consequences of the theoretical definition of the basic concepts, viz. problems that *do not* requires a special instruction for their solution.

> If a circle is defined as a curve all of whose points are equidistant from a point (the center): is then not this concept also given in intuition? And that even though the practical proposition that follows from this, viz., "*to describe a circle*" (as a straight line is rotated uniformly about a point), is not even considered yet. (11:43, 1789)

> For any center and radius, it is possible to draw a line such that all straight lines falling on it from the center are equal to the radius.

> Furthermore, the possibility of a circle is not merely **problematical** prior to, and, as it were, dependent on the practical proposition: "to describe a circle by the motion of a straight line around a fixed point"; rather, the possibility is **given** in the definition of the circle, for the circle is constructed through the definition itself, i.e., it is exhibited in intuition, while not on paper (in empirical intuition) but in imagination (*a priori*). For I may always, with a free hand and chalk, draw a circle on the blackboard and put a point in it, and on it I can demonstrate—under the assumption of that (so–called) nominal definition, which is in fact a real definition—all the properties of the circle, even if it never be congruent with a curve described by the rotation of a straight line around a point. I assume: they, the points of the circumference, are equidistant from the centre. The proposition: "to describe a circle" is a practical corollary (or so-called postulate), which could never be demanded at all if the possibility—yes even the manner of possibility of the figure—were not already given in the definition. (11:53, 1789)

This points straight back to our discussion of the "conditions of construction", whose representation depends entirely on the concepts of pure understanding, as we discussed in chapter 3.

6.4 The role of intuition in Kant's foundation of geometry

Friedman argues that spatial intuition does not play the role of "providing a model" for some, and ruling out other, abstractly formulated axiom systems by allowing to "inspect" given geometrical figures to determine which logically possible conditions, Euclidean or non-Euclidean, they fulfill. He follows Kitcher in arguing that our perceptual apparatus could not possibly be used to somehow "see" whether a given triangle is Euclidean or not: "Our capacity for visualizing figures has neither the generality nor the precision to make the required distinctions." And neither, he continues, was it intended for this by Kant; rather, the primary role of intuition "is to underwrite the constructive procedures" used in the synthetic part of proofs:

> It is extremely unlikely, however, that in appealing to intuition at A25/B39 Kant is imagining any such process of "visual inspection". It is much more plausible that, in precise parallel to his discussion of the angle-sum property at A715-717/B743-745, he is referring to the Euclidean *proof* of this proposition (Prop. I.20). We consider a triangle *ABC* and prolong *BA* to point *D* such that *DA* is equal to *CA* (see Figure 6). We then draw *DC*, and it follows that $\sphericalangle ADC = \sphericalangle ACD$ and therefore $\sphericalangle BCD > \sphericalangle ADC$. Since the greater angle is subtended

> by the greater side (Prop. 1.19), $DB > BC$. But $DB = BA + AC$; therefore $BA + AC > BC$. Q.E.D. Intuition is required, then, not to enable us to "read off" the side-sum property from the particular figure *ABC*, but to guarantee that we can in fact prolong *BA* to *D* by Postulate 2.

I am in full agreement with Friedman about the role of intuition with respect to the postulates as described in this passage (but concerning the discrimination between different axiom-systems, see below). When I read Kant as saying that intuition provides objective reality to principles whose logical form *can* be expressed abstractly, this is just what I mean: it "guarantee[s] that we can **in fact** prolong" a line as demanded by postulate 2. But note that Friedman's point here is completely independent from his main argument, from whether Kant did or did not have the conceptual resources to represent the relational content of the postulate abstractly; the question is *whether or not what the postulate demands can be done*, not whether its logical form "exists". In this sense, pure intuition indeed "underwrites constructive procedure", precisely by sanctioning their existential presuppositions:

> Pure geometry has postulates as practical propositions which, however, contain nothing further than the presupposition that one **could** do something if it were required that one **should** do it, and these are the only propositions of pure geometry that concern an existence. They are thus practical rules under a problematic condition. (*CPrR* AA 5:31)

Each proof-step that invokes such a postulate in its major premise corresponds to, represents the execution of, and is sanctioned by, an intuitively evident construction, i.e., the basic evidence for the existence of the required object; thereby, the intuitive evidence of mathematical demonstrations is secured. We already saw above how Kant thought about the relationship between theoretical and practical sentences; the practical form is not essential to the contents of the postulates. They express objective (theoretically statable) general determination-relations involving existential presuppositions, whose necessity and universality is guaranteed by *a priori* intuition, which makes first possible to do what the postulate demands, i.e., to *exibit* an object of the required type, given arbitrary parameters.

Now, Friedman's argument, that intuition could not play the role of deciding between different logically possible theories because it could not possibly decide whether triangles are Euclidean or non-Euclidean, is somewhat strange. For Kant most certainly thought that the fact that the angle-sum of any triangle is equal to two rights is *true* in virtue of intuition (note that Friedman emphasizes that for Kant, pure geometry is not "a body of truths" but merely a "form of reasoning"). But surely, this does not imply that Kant believed that this (presumed) truth could be "seen" without proof, or that its (presumed) knowledge is the result of simply looking at or mentally picturing various triangles. Rather, it is demonstrated by rigorous and completely general proof, whose premises (Kant believes) are intuitively evident, which makes it possible to recognized each individual proof-step not only as logically valid but as intuitively sound, and therefore sanction its conclusion with the same evidence. None of this implies that Kant believed that he had the super-human capacity to determine whether the angle-sum of an empirically given triangle $= 180.0^\circ$ or $= \pm 180.000000000001^\circ$.[17] But if intuitions makes true the premises of a theory by sanctioning its

[17] Here one will rightly remark that the alleged intuitive evidence of the basic premises of classical geometry is no less problematic than an immediately evident angle-sum property, as these reduce *de facto* to the same problem. For example, how could we possibly discriminate intuitively between a Euclidean plane, in which there is exactly one parallel to a given straight

existential presuppositions, does it then not "provide a model"? (Obviously, from our modern perspective it is highly doubtful that we have intuitive certainty about the basic premises of classical geometry (which is why we provide it with numerical models, pushing the problem further down the line); but that is not the issue here.[18]) Neither does this imply that Kant couldn't formulate logically possible conditions under which it can be proven that the angle-sum theorem fails. Could Kant not state the axioms of spherical geometry?[19] Yes, he regarded spherical geometry as a part of classical geometry, rather than a separate system; but nothing, certainly not his logic, prevented him from stipulating the requisite conditions on antipodal points, calling great circles "straight lines" and stating its principles as a separate theory that allows the logical derivation of propositions inconsistent with the concept of a straight line in Euclidean plane geometry.

In conclusion, Friedman's argument that Kant did not regard intuition as a source for the truth and evidence of the principles of geometry, particularly its existential presuppositions (in Friedman's words: providing a model) is not tenable. To argue that, because intuition was meant to "underwrite the constructive procedure[s]" of geometry (which it certainly was), we must therefore reject the idea that Kant used intuition to provide evidence for the (presumed) truth of the principles of one theory (thereby 'selecting' it from a range of logically possible alternatives), or even that it reflects his inability to express in logical terms — or regard as a basic constituent of the pure understanding — the concept of relations e.g. of the form "given objects of the types A and B, an object of type C is uniquely determined by them", is not convincing when confronted with the textual evidence.

line through a given point not on it, and a hyperbolic geometry, where there is an infinity of such parallels, but where, given a line d and a point A one meter away from d, the angle of parallelism is one-tenth of a part per trillion less than a right angle? But I think this is really an argument *against* Kant, not an argument that could be ascribed to him for the sake of interpreting his philosophy of mathematics. *In any case, I believe that Kant's repeated attempts to prove the parallel-postulate show quite conclusively that he was himself not at all convinced of its* immediate *intuitive evidence!*

[18] Friedman would probably object that the reason that we do no longer have this false certainty is precisely our capacity to formulate alternative axiomatic systems in modern logic, and our realization that intuition does little to nothing to help us choose between them. But it seems to me quite an obvious historiographical ὕστερον πρότερον to say that our capacity to think different geometrical systems is a result of the progress in logical theory, rather than that the formulation of these new systems by Lambert, Gauss, Bolyai, Lobachevsky, *et al.* were driving forces in the development of modern logic. It was obviously possible to *think* alternative geometries before Frege and Hilbert, whose calculi are simply *formalizations* or *models* of logical reasoning. The fundamental disagreement then comes down, again, to the question of whether Kant admitted non-monadic reasoning as part of the discursive structure of the understanding. If he did not, then his theory is made obsolete not by the invention of predicate logic, but reduced *ad absurdum* already by his contemporaries and near-contemporaries pioneering non-classical geometry.

[19] Comp. AA 4:285, 2:381.

References

[1] Achourioti, T. and Van Lambalgen, M. (2011). A formalization of Kant's transcendental logic. *The Review of Symbolic Logic*, 4(2):254–289.

[2] Asperó, D. and Schindler, R. (2021). Martin's maximum++ implies woodin's axiom. *Annals of Mathematics*, 193(3):793–835.

[3] Bell, J. L. (2017). Continuity and infinitesimals. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2017 edition.

[4] Bernays, P. (1928). Über nelsons stellungnahme in der philosophie der mathematik. *Die Naturwissenschaften*, 16(9):142–45.

[5] Bernays, P. (1935). On platonism in mathematics. *Philosophy of mathematics: selected readings, 1964*, pages 274–286.

[6] Bernays, P. (1957). Mathematische existenz und widerspruchsfreiheit. *Journal of Symbolic Logic*, 22(2).

[7] Bernays, P. (1976). *Abhandlungen zur Philosophie der Mathematik*. Wissenschaftliche Buchgesellschaft, Darmstadt, 2nd edition.

[8] Beth, E. W. (1953). Kants einteilung der urteile in analytische und synthetische. *Algemeen Nederlands Tijdschrift voor Wijsbegeerte en Psychologie*, 46:253–64.

[9] Beth, E. W. (1956). Ueber lockes allgemeines dreieck. *Kant-Studien*, 48:361.

[10] Boolos, G. (1984). To be is to be a value of a variable (or to be some values of some variables). *The Journal of Philosophy*, 81(8):430–449.

[11] Buchdahl, G. (1969). *Metaphysics and the Philosophy of Science : The Classical Origins, Descartes to Kant*. Basil Blackwell, Oxford.

[12] Callender, C. (2005). Answers in search of a question:'proofs' of the tri-dimensionality of space. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 36(1):113–136.

[13] Cantor, G. (2013). *Gesammelte Abhandlungen: mathematischen und philosophischen Inhalts*. Springer-Verlag.

[14] Carnap, R. (1931). Die logizistische grundlegung der mathematik. *Erkenntnis*, 2(1):91–105.

[15] Carnap, R. (1998). *Der logische aufbau der welt*, volume 514. Felix Meiner Verlag.

[16] Cassirer, E. (1922). *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit*, volume 2. Wissenschaftliche Buchgesellschaft, Darmstadt, 3rd edition.

[17] Coffa, A. (1982). Kant, balzano, and the emergence of logicism. *The journal of Philosophy*, 79(11):679–689.

[18] Coffa, J. A. (1993). *The semantic tradition from Kant to Carnap: To the Vienna station*. Cambridge University Press.

[19] Coxeter, H. S. M. (1992). *The real projective plane*. Springer, 3rd edition.

[20] Cramer, K. (1985). *Nicht-reine synthetische Urteile a priori - Ein Problem der Transzendentalphilosophie Immanuel Kants*. Carl Winter Universitätsverlag.

[21] Dean, W. (2020). On consistency and existence in mathematics. *Proceedings of the Aristotelian Society*, 120(3).

[22] Dedekind, R. (2017). *Was sind und was sollen die Zahlen? Stetigkeit und Irrationale Zahlen*. Springer-Verlag.

[23] Dummett, M. (1963). The philosophical significance of gödel's theorem. *Ratio*, 5(2).

[24] Enderton, H. B. (2001). *A mathematical introduction to logic*. Elsevier, 2 edition.

[25] Falkenburg, B. (2000). *Kants Kosmologie: Die wissenschaftliche Revolution der Naturphilosophie im 18. Jahrhundert*. Vittorio Klostermann, Frankfurt am Main.

[26] Ferreirós, J. (2008). *Labyrinth of thought: A history of set theory and its role in modern mathematics*. Springer Science & Business Media.

[27] Field, H. (1998). Do we have a determinate conception of finiteness and natural number? In Schirn, M., editor, *The Philosophy of Mathematics Today*, pages 99–130. Claredon Press, Oxford, 2nd edition.

[28] Frede, M.; Krüger, L. (1970). Über die Zuordnung der Quantitäten des Urteils und der Kategorien der Grösse bei Kant. *Kant-Studien*, 61:28–49.

[29] Frege, G. (1884, 1988). *Die Grundlagen der Arithmetik: eine logisch mathematische Untersuchung über den Begriff der Zahl*. Meiner: Philosophische Bibliothek, Hamburg.

[30] Friedman, M. (1992a). *Kant and the Exact Sciences*. Harvard University Press.

[31] Friedman, M. (1992b). *Kant and the Exact Sciences*. Harvard University Press.

[32] Friedman, M. (2012). Kant on geometry and spatial intuition. *Synthese*, 186:231–255.

[33] Gatzia, D. E. and Ramsier, R. D. (2020). Dimensionality, symmetry and the inverse square law. *Notes and Records*.

[34] Giovannini, E. N. and Schiemer, G. (2019). What are implicit definitions? *Erkenntnis*, pages 1–31.

[35] Gödel, K. (1931). Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für mathematik und physik*, 38(1):173–198.

[36] Gödel, K. (1995). *Kurt Gödel: Collected Works: Volume III: Unpublished Essays and Lectures*. Oxford.

[37] Gödel, K. (1983). What is Cantor's continuum problem? In Putnam, H.; Benacerraf, P., editor, *Philosophy of Mathematics: Selected Readings*, pages 470–485. Cambridge University Press, 2nd edition.

[38] Hale, B. (2011). The problem of mathematical objects. In Sommaruga, G., editor, *Foundational Theories of Classical and Constructive Mathematics*, volume 76 of *The Western Ontario Series in Philosophy of Science*, pages 71–84.

[39] Hempel, C. (1945). On the nature of mathematical truth. *The American Mathematical Monthly*, 52(10):543–556.

[40] Henrich, D. (1976). *Identität und Objektivität: Eine Untersuchung über Kants transzendentale Deduktion*. Carl Winter - Universitätsverlag, Heidelberg.

[41] Henrich, D. (1979). 'identität' - begriffe, probleme, grenzen. *Mechanics of Cohesive-frictional Materials*, 1(4):385–403.

[42] Henrich, D. (1982). *Selbstverhältnisse*. Philipp Reclam jun., Stuttgart.

[43] Henrich, D. (1988). Die Identität des Subjekts in der transzendentalen Deduktion. In Oberer, H.; Seel, G., editor, *Kant: Analysen, Probleme, Kritik*, pages 39–70. Königshausen Neumann, Würzburg.

[44] Hilbert, D. (1988). *Wissen und Mathematisches Denken*. Vorlesungen, Winter-Semester 1922/23. Published by Mathematisches Institut, Göttingen.

[45] Hintikka, J. (1972). Kantian intuitions. *Inquiry: An Interdisciplinary Journal of Philosophy*, 15.

[46] Hintikka, J. (1981). Kant's theory of mathematics revisited. *Philosophical Topics*, 12(2):201–215.

[47] Hintikka, J. (1984). Kant's transcendental method and his theory of mathematics. *Topoi*, 3(2):99–108.

[48] Hintikka, J. (1992). Kant on the mathematical method. In *Kant's Philosophy of Mathematics*, pages 21–42. Springer.

[49] Hodges, W. (2009). Traditional logic, modern logic and natural language. *Journal of Philosophical Logic*, 38(6):589.

[50] Ihmig, K.-N. (1993). Reine Anschauung und Reihenbegriff. Zu Cassirers Rezeption von Kants Theorie der Geometrie. *Dialektik*, 1:112–128.

[51] Ishiguro, H. (1990). *Leibniz's philosophy of logic and language*. Cambridge University Press.

[52] Kant, I. (1900ff.). *Gesammelte Schriften ("Akademieausgabe")*. Königlich Preußische Akademie der Wissenschaften / de Gruyter (since 1922), Berlin.

[53] Leibniz, G. W. (1969). *Philosophical Papers and Letters*. D. Reidel Publishing Company, Dordrecht.

[54] Lindström, P. (1997). *Aspects of incompleteness*. Cambridge University Press.

[55] Linnebo, O. (2013). The potential hierarchy of sets. *The Review of Symbolic Logic*, 6(2).

[56] Locke, J. (1690 / 1979). *An essay concerning human understanding*. The Clarendon edition of the works of John Locke. Oxford University Press.

[57] Martin-Löf, P. (1987). Truth of a proposition, evidence of a judgement, validity of a proof. *Synthese*, 73(3):407–420.

[58] Martin-Löf, P. (1996). On the meanings of the logical constants and the justifications of the logical laws. *Nordic journal of philosophical logic*, 1(1):11–60.

[59] Mäenpää, P.; von Plato, J. (1990). The Logic of Euclidean Construction Procedures. In Haaparanta, L., Kusch, M., and Niiniluoto, I., editors, *Language, Knowledge, and Intentionality: Perspectives on the Philosophy of Jaakko Hintikka*, volume 49 of *Acta Philosophica Fennica*, pages 275–293. Helsinki.

[60] Mäenpää, P. (1993). *The Art of Analysis. Logic and History of Problem Solving*. PhD thesis, University of Helsinki.

[61] Mäenpää, P. (1997). From Backward Reduction to Configurational Analysis. In Otte, M.; Panza, M., editor, *Analysis and Synthesis in Mathematics: History and Philosophy*, volume 196 of *Boston Studies in the Philosophy of Science*, pages 201–226. Kluwer Academic Publishers, Dordrecht.

[62] Parsons, C. (1980). Mathematical intuition. *Proceedings of the Aristotelian Society*, 80:145–168.

[63] Parsons, C. (1992a). Arithmetic and the categories. In Posy, C. J., editor, *Kant' s Philosophy of Mathematics*, pages 135–158.

[64] Parsons, C. (1992b). Kant's philosophy of arithmetic. In *Kant's philosophy of mathematics*, pages 43–79. Springer.

[65] Parsons, C. (2007). *Mathematical thought and its objects*. Cambridge University Press.

[66] Pinder, T. (1988). Nicht-reine synthetische urteile a priori: Ein problem der transzendentalphilosophie immanuel kants (review). *Zeitschrift für philosophische Forschung*, 42:154–159.

[67] Poincaré, H. (1983). On the nature of mathematical reasoning. In Putnam, H.; Benacerraf, P., editor, *Philosophy of Mathematics: Selected Readings*. Cambridge University Press, 2nd edition.

[68] Posy, C. J. (2013). *Kant's philosophy of mathematics: modern essays*. Springer.

[69] Putnam, H. (1983a). Models and reality. In Putnam, H.; Benacerraf, P., editor, *Philosophy of Mathematics: Selected Readings*, pages 421–444. Cambridge University Press, 2nd edition.

[70] Putnam, H.; Benacerraf, P. (1983b). *Philosophy of Mathematics: Selected Readings*. Cambridge University Press, 2nd edition.

[71] R. Kaye, T. L. W. (2007). On interpretations of arithmetic and set theory. *Notre Dame Journal of Formal Logic*, 48(4):497–501.

[72] Rathjen, M. (2005). An ordinal analysis of parameter free π 1 2-comprehension. *Archive for Mathematical Logic*, 44(3):263–362.

[73] Risi, V. D. (2016). *Leibniz on the Parallel Postulate and the Foundations of Geometry: The Unpublished Manuscripts*. Springer.

[74] Russell, B. (1900). *A Critical Exposition of the Philosophy of Leibniz, with an Appendix of Leading Passages*. Cambridge University Press.

[75] Russell, B. (1903/2010). *Principles of Mathematics*. Routledge.

[76] Russell, B. (1920). *Introduction to mathematical philosophy*. Allen & Unwin.

[77] Schubring, G. (1982). Ansätze zur begründung theoretischer terme in der mathematik—die theorie des unendlichen bei johann schultz. *Historia Mathematica*, 9(4):441–484.

[78] Schulthess, P. (1981). *Relation und Funktion: eine systematische und entwicklungsgeschichtliche Untersuchung zur theoretischen Philosophie Kants*. Walter de Gruyter, Berlin.

[79] Shapiro, S. (1991). *Foundations without foundationalism: A case for second-order logic*. Clarendon Press.

[80] Sieg, W. (2013). *Hilbert's Programs and Beyond*. Oxford University Press.

[81] Tait, W. (1983). Against intuitionism. *Journal of Philosophical Logic*.

[82] Tait, W. (1991). Reflections on the concept of *a priori* truth and its corruption by Kant. In Detlefsen, M., editor, *Proof and Knowledge in Mathematics*.

[83] Tait, W. (2005). *The Provenance of Pure Reason: Essays in the Philosophy of Mathematics and its History*. Oxford University Press.

[84] Takeuti, G. (1987). *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam.

[85] Thompson, M. (1972). Singular terms and intuitions in Kant's epistemology. *Review of Metaphysics*, 26:314–343.

[86] Tiles, M. (2009). A Kantian perspective on the philosophy of mathematics. In Irvine, A. D., editor, *Philosophy of Mathematics*, Handbook of the Philosophy of Science, pages 231–270. Elsevier, Amsterdam.

[87] Vuillemin, J. (1972). Das Problem der Identität in der Beweistheorie und die kantische Fragestellung. *Kant-Studien*, 63(1-4):289–304.

[88] Wang, H. (1974). *From Mathematics to Philosophy*. Routledge.

[89] Wang, H. (1983). The concept of set. In Putnam, H.; Benacerraf, P., editor, *Philosophy of Mathematics: Selected Readings*, pages 530–570. Cambridge University Press, 2nd edition.

[90] Wolff, M. (2017). *Der Begriff des Widerspruchs: Eine Studie zur Dialektik Kants und Hegels*. Minerva Verlag, Berlin, 3 edition.

[91] Zalta, E. N. (2020). Frege's Theorem and Foundations for Arithmetic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2020 edition.

[92] Zermelo, E. (1930). Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre. *Fundamenta mathematicae*, 16.