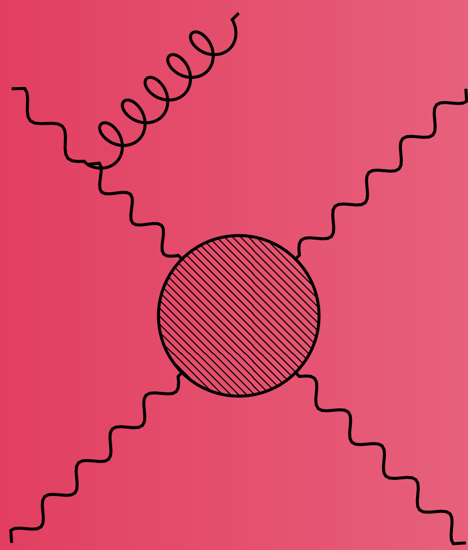
Infrared Divergences in Perturbative Quantum Gravity and String Field Theory



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Master Thesis

Infrared Divergences in Quantum Gravity and String Field Theory

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A thesis submitted in fulfilment of the requirements for the degree of MSc.Theoretical Physics

in the

School of Natural Sciences

July 10, 2022

Declaration of Authorship

I, Petros Agridos, declare that this thesis titled, 'Infrared Divergences in Quantum Gravity and String Field Theory' and the work presented in it is my own. I confirm that this work submitted for assessment is my own and is expressed in my own words. Any uses made within it of the works of other authors in any form (e.g., ideas, equations, figures, text) are properly acknowledged at any point of their use. A list of the references employed is included.

Signed: *Petros Agridos*

Date: 10/07/2022

"Ἓν οἶδα, ὅτι οὐδὲν οἶδα"
Σωκράτης

Abstract

The main purpose of this thesis is to better understand how the Faddeev-Kulish method works in a theory that contains two massless gauge fields. We begin with a review of infrared divergences associated with soft particles. Subsequently, we consider as a model the Einstein-Maxwell theory coupled to a scalar field, for which the Lagrangian and the asymptotic potential are constructed. It turns out that theories, for which the cancellation of soft divergences with the Faddeev-Kulish method is already known, can be partially investigated within this model. The dressing of photon states with soft gravitons and the cancellation of the corresponding divergences, is studied as another partial case. The dressing of scalar asymptotic states with both soft gravitons and photons is studied as well, with some interesting features appearing at second and higher order corrections. On the contrary, we show that if hard photon legs are also included, the method seems to provide a series of possibly divergent operators that did not appear in the previous cases. We close with a discussion on how infrared divergences are realized in string theory and point some steps that might lead to the extension of the Faddeev-Kulish method within the framework of string field theory.

Acknowledgements

First and foremost, I would like to thank my daily supervisor Dr. Nava Gaddam, for giving me the opportunity to collaborate and conduct research on this new and promising field. I am very grateful for all interesting discussions we had and for making this journey so beautiful. His professionalism and enthusiasm, together with his a passion to share his knowledge throughout this project have managed to affect me to the greatest extend.

I would also like to thank Prof. Stefan Vandoren and Prof. Umut Gursoy for taking the time to read and review my thesis.

I couldn't but express my deepest gratitude to my parents for their support and love throughout not only my academic years, but my entire life. The least I could do as a payback is to dedicate this thesis to them.

Unfortunately, during almost my entire master studies, humanity has been facing embarrassing and dark days. I would therefore like to thank my girlfriend Alexandra, for shedding some light to these days and for continuously supporting me.

Petros Agridos
Utrecht, the Netherlands
July 10, 2022

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Abbreviations

BCH	**B**aker-**C**ampbell-**H**ausdorff
BMS	**B**ondi-**M**etzner-**S**achs
BRST	**B**ecchi-**R**ouet-**S**tora-**T**yutin
CFT	**C**onformal **F**ield **T**heory
EMS	**E**instein-**M**axwell-**S**calar
FK	**F**addeev-**K**ulish
IR	**I**nfra-**R**ed
OPE	**O**perator-**E**xpansion-**P**roduct
PQG	**P**erturbative **Q**uantum **G**ravity
QED	**Q**unatum **E**lectro-**D**ynamics
QFT	**Q**unatum **F**ield **T**heory
SFT	**S**tring **F**ield **T**heory
UV	**U**ltra-**V**iolet

To my parents,
Konstantinos and Alexandra

Chapter 1

Introduction

This Chapter is the building block of this thesis and aims to introduce the reader to the topics discussed in the following chapters. We begin in Section 1.1 by motivating how the study of infrared divergences has given new insights in physics, followed by the main research questions that we aim to investigate in the present work. Details on the structure of the current document are given in Section 1.2, while in section 1.3, we provide the various conventions that were employed in this thesis.

1.1 Motivation

Infrared divergences appear in physical theories involving massless particles, typically originating from regions in integrals where the energy or momentum of a particle becomes zero; translating into a particle being soft. Usually, to avoid them, one defines an infrared cut-off as the minimal value of momentum/energy (equivalently the maximal value of the wavelength) to be taken into account in the calculations. However, there are processes in which soft particles play an important role and can therefore not be avoided. An example of such a process is the emission and absorption of low energy photons by an electron which undergoes acceleration. This process is known as Brehmsstrahlung or breaking radiation. In such cases, one has to find a systematic procedure that automatically removes them from the theory.

Such a procedure was proposed in 1937 by Bloch and Nordsieck [1]. It is known as the "inclusive formalism" and is the one exclusively discussed in standard quantum field theory (QFT) textbooks [2–5]. The main consideration in this procedure is that the infinite range interactions vanish at very large time (asymptotically). In other words, the asymptotic dynamics is governed by the free Hamiltonian and the physical states live in the standard Fock space. Then, infrared divergences can be dealt with by summing over all physically indistinguishable cross sections, order by order in perturbation theory. However, the S-matrix itself remains ill-defined. The standard way to avoid this subtlety is to consider that the S-matrix itself is not physical, since its elements are not observable. Instead, what is considered physical, is the total probability for a process to occur. This is measured by the cross section, which should always be finite.

In 1970, a different method giving finite results was proposed by Faddeev and Kulish [6] within the frame of quantum electrodynamics (QED). They built upon the argument that the range of massless field's is infinite. Therefore, in the presence of an elecromagnetic field, the electrons scattered by it would still feel an interaction potential, even at very large time. This means that the original consideration of Block and Nordsieck that the electron fields are asymptotically free is in fact faulty. The way to remove infrared divergences by modifying the asymptotic states had already been shown by Chung [7]. Faddeev and Kulish showed that Chung's construction of the asymptotic states is connected to the non-vanishing asymptotic potential they computed.

The picture that was formed is that in the presence of infinite range fields the asymptotic states (past and future ones) are not free anymore, but rather dressed by clouds of soft photons. Taking into account the contribution from these clouds leads to finite S-matrix elements at every loop order. This method was recently extended [8] to the case of perturbative quantum gravity (PQG), where scalar field states get dressed by a cloud of soft gravitons.

Despite the fact that this method works and provides finite S-matrix elements, it was left aside for many years. Recently, it captured the interest of many physicists due a series of papers by Andrew Strominger et al. [9–13], in which a close relation among Weinberg's soft theorem, asymptotic symmetries and the memory effect was demonstrated. The pairwise connection among them is depicted in Figure 1.1, adopted from [13]. For the case of PQG, the asymptotic symmetry group is the group of the so-called BMS supertranslations [14, 15], for which a connection with the Faddeev-Kulish amplitudes[1] was recently discovered [17].

A natural question to be asked, is whether and how this procedure for removing soft infrared divergences can be applied in string theory. It is well known that string theory suffers from infrared divergences [18], similar to those appearing in usual QFTs. Since in the standard worldsheet string theory there is no systematic way known to deal with infrared divergences, it might be considered reasonable to establish this question in string field theory (SFT). This is the second quantized formulation of string theory,

[1] In the case of QED the corresponding group is the one of Large Gauge Symmetries. We refer the interested reader to [16] for more details.

which allows one to treat it as a QFT. In that sense and similarly to how particles are promoted to fields in the standard QFT, the strings are promoted to string fields.

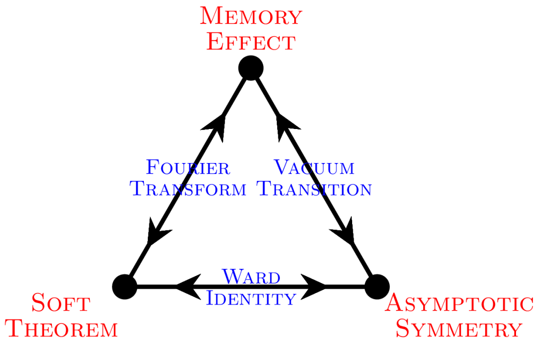


FIGURE 1.1: The "infrared triangle", indicating the connection among Weinberg's soft theorem, asymptotic symmetries and the memory effect.

Therefore, a string field consists of infinitely many massless modes. It is expected that these massless modes will contribute in the dressing of the asymptotic states. Investigating a model which contains two interacting massless gauge fields might be beneficial to determine how these interactions affect the dressing of the Fock states [19]. In this thesis, we aim to explore the Einstein-Maxwell-Scalar (EMS) field model.

1.2 Structure of the thesis

This thesis is organised as follows. In Chapter 2, we review some basic elements of worldsheet string theory. This aims to serve as a warm-up for Chapter 3, in which we introduce string field theory. It also serves to maintain this thesis as self-contained as possible. Next, in Chapter 4, we derive Weinberg's soft theorem for soft gravitons attached on (hard) photon legs. This derivation relies on the Lagrangian for the EMS model, constructed within the framework of PQG in Chapter 5. Apart from the construction of the aforementioned Lagrangian, we also use this chapter in order to calculate the various terms in the asymptotic potential. The outcomes of Chapters 4 & 5 will be implemented in the following chapters. In particular, in Chapter 6, we study the part of the asymptotic potential involving photon and graviton interactions and discuss how the Faddeev-Kulish method deals with IR divergences in such processes. Following that, on Chapter 7, more complicated processes, involving all kinds of interactions based on the total asymptotic potential, are discussed. In Chapter 8, an approach to a set up of how the Faddeev-Kulish method can be possibly extended to string field theory, is investigated. Finally, in Chapter 9, the outcomes of this thesis are displayed and discussed, followed by our suggestions for future research. Several Appendices have also been included in order to preserve the main chapters as clear as possible. The list of the references that were used can be found in the last pages of the document.

1.3 Conventions

In this thesis we work with a mostly plus metric of the form $(-1, 1, \ldots, 1)$. Hellenic letters are used for Lorentz indices and Latin letters otherwise. We use bold characters to denote 3-vectors. Contracted indices, for example $A_\mu B^\mu$, will be ususally denoted by a dot product, i.e. $A \cdot B$. The Einstein summation convention is employed throughout the thesis. We will mainly work with natural units, that is $\hbar = c = 2\kappa^2 = 8\pi G = 1$. Finally, the propagators and interaction vertices given in Appendix C carry factors $(2\pi)^{-4}$ and $(2\pi)^4$, which we have omitted to write. When one propagator is combined with exactly one vertex, these will automatically cancel, however, one has to be aware of their appearance in different cases.

Chapter 2

Elements of worldsheet string theory

In this chapter, we aim to review some basic elements and notions of worldsheet string theory, with a greater purpose to motivate the construction of a string field theory in Chapter 3. We focus on the closed bosonic string only. We begin with a short description of the Polyakov action and the string spectrum, in Section 2.1. Subsequently, in Section 2.2, we discuss string theory as a Conformal Field Theory. Finally, in Section 2.3, we introduce the Polyakov path integral and comment on how interactions are incorporated in string theory. We will not provide any detailed analysis, assuming that the reader is already familiar with the various concepts discussed. A more complete and detailed study of the worldsheet string theory, we refer the reader to [20–25].

2.1 The Polyakov action and dynamics

Consider a closed string propagating in a D-dimensional Minkowski space-time. Analogously to the worldline swept out by the motion of a point-particle, the moving string sweeps out a two-dimensional surface, called the worldsheet. Points on the worldsheet are parametrized by two coordinates $(\sigma^0 = \tau, \sigma^1 = \sigma)$, with $\sigma \in (0, 2\pi]$. The propagation of the string in the worldsheet is described by the Polyakov action:

$$S_P[X, g] = -T \int_M d^2\sigma \sqrt{-g} g^{ab} \eta^{\mu\nu} \partial_\alpha X_\mu \partial_b X_\nu \tag{2.1}$$

where $T := \frac{1}{2\pi\alpha'}$ denotes the tension of the string and we have defined $d^2\sigma = d\tau d\sigma$. The parameter α' in the definition of the string tension is known as the Regge slope.

The worldsheet geometry is encoded in the metric $g^{\alpha\beta}$, with $g = detg_{ab}$ and $\alpha, \beta = 0, 1$. The scalar fields X^μ, with $\mu = 0, \dots, D-1$, describe the embedding of the string in the target (Minkowski) spacetime, for which the metric is denoted as $\eta^{\mu\nu}$. The action of Eq.(2.1) possesses three main symmetries:

1. It is invariant under the global symmetry of the Poincare group, under which the worldsheet fields transform as

$$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu, \quad \delta g^{\alpha\beta} = 0 \tag{2.2}$$

2. It is invariant under the local (gauge) symmetry of worldsheet diffeomorphisms, under which

$$\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha, \quad g_{\alpha\beta}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha}\frac{\partial f^\delta}{\partial \sigma^\beta} g_{\gamma\delta}(\sigma') \tag{2.3}$$

3. There is another local symmetry called Weyl symmetry. Under this

$$g_{\alpha\beta} \to e^{\phi(\sigma,\tau)} g_{\alpha\beta}, \quad \delta X^\mu = 0 \tag{2.4}$$

Using the gauge invariance of Eq.(2.3) and Eq.(2.4), one can choose to fix the worldsheet metric to be $g^{\alpha\beta} = \eta^{\alpha\beta}$. The most general solution such that $\delta_X S_P[X, g] = 0$ can be written as a sum of left- and right-movers, which in turn can be expressed in terms of the string oscillating modes $a^\mu_n, \tilde{a}^\mu_n$ as

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau - \sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n} a^\mu_n e^{-2in(\tau-\sigma)} \tag{2.5a}$$

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau + \sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n} \tilde{a}^\mu_n e^{-2in(\tau+\sigma)} \tag{2.5b}$$

where x^μ is the center-of-mass position of the string and $p^\mu := \sqrt{\frac{2}{\alpha'}} a^\mu_0$ denotes its total momentum. We have also introduced the string length scale parameter, as $l_s = \sqrt{2\alpha'}$. Requiring that the functions X^μ_R and X^μ_L are real implies that x^μ and p^μ are real, while positive and negative modes are conjugate to each other

$$a^\mu_{-n} = (a^\mu_n)^\star, \quad \tilde{a}^\mu_{-n} = (\tilde{a}^\mu_n)^\star \tag{2.6}$$

The solutions given by Eq.(2.5a) and Eq.(2.5b) are subject to the constraints

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \tag{2.7}$$

where we have defined $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. These constraints come from the equation of motion obtained by requiring $\delta_g S_P[X, g] = 0$, upon relaxing the condition $g_{\alpha\beta} = \eta_{\alpha\beta}$. By defining stress-energy tensor

$$T_{\alpha\beta} = -\frac{2}{T\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}} \tag{2.8}$$

this equation of motion is simply $T_{\alpha\beta} = 0$. Recovering $g^{\alpha\beta} = \eta^{\alpha\beta}$, one obtains Eq.(2.7). Replacing Eq.(2.5a) and Eq.(2.5b) in it, leads to

$$(\partial_- X)^2 = \alpha' \sum_n L_n e^{-in(\tau-\sigma)} = 0 \tag{2.9a}$$

$$(\partial_+ X)^2 = \alpha' \sum_n \tilde{L}_n e^{-in(\tau+\sigma)} = 0 \tag{2.9b}$$

where we have defined

$$L_n = \frac{1}{2}\sum_m a_{n-m} a_m, \quad \tilde{L}_n = \frac{1}{2}\sum_m \tilde{a}_{n-m}\tilde{a}_m \tag{2.10}$$

Of primary importance are the constraints $L_0 = \tilde{L}_0 = 0$, since they include the square of the spacetime momentum p^μ, for which $p^2 = -M^2$. This means that they can be used to read off the mass of states in the closed-string spectrum

$$\alpha' M^2 = 4\sum_{n>0} a_n a_{-n} = 4\sum_{n>0} \tilde{a}_n \tilde{a}_{-n} \tag{2.11}$$

At the quantum level and for the first two mass levels, the physical states are:

1. The ground state $|0;k\rangle$, which is a tachyon with

$$M^2 = -\frac{4}{\alpha'} \tag{2.12}$$

2. A set of $24^2 = 576$ massless states of the form

$$|\Omega^{ij}\rangle = a^i_{-1}\tilde{a}^j_{-1}|0;k\rangle \tag{2.13}$$

corresponding to the tensor product of the massless vectors associated with the left-moving and the right-moving sector. This can be decomposed into three parts. The part of $|\Omega^{ij}\rangle$ that is symmetric and traceless in i and j transforms under $SO(24)$ as a massless spin-two particle, realized with the graviton $G_{\mu\nu}(X)$. The part carrying trace, $\delta_{ij}|\Omega^{ij}\rangle$, corresponds to a massless scalar, called the dilaton $\Phi(X)$. There is a last part, which transforms under $SO(24)$ as an anti-symmetric rank-2 tensor. This is known as the Kalb-Ramond field $B_{\mu\nu}(X)$.

2.2 Connection with conformal field theory

The discussion of Section 2.1 is based on the convention that the worldsheet metric carries Lorentzian signature. This convention is the appropriate one in order to describe a physically evolving string. It is convenient though, to consider a worldsheet metric with Euclidean signature. This can be achieved by performing a Wick rotation $\tau \to -i\tau$. Then, introducing complex coordinates

$$z = e^{2(\tau - i\sigma)}, \quad \bar{z} = e^{2(\tau + i\sigma)} \tag{2.14}$$

one can define a map from the worldsheet cylinder to the complex plane. This map is depicted in Figure 2.1, adopted from [23], and is related to the concept of conformal transformations on the worldsheet.

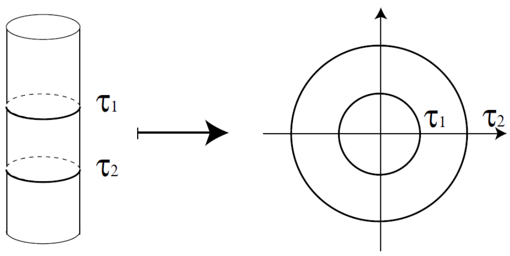


FIGURE 2.1: Conformal map from the cylinder to the plane.

By definition, a conformal transformation is a change of coordinates $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ such that the metric changes as

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma) \tag{2.15}$$

Then, a Conformal Field Theory (CFT) is defined as a field theory which is invariant under these transformations. Examples of 2-dimensional conformal transformations are:

- Translations:
$$z \to z + a \tag{2.16}$$

- Rotations (for $|\zeta| = 1$) and dilatations (for $\zeta \neq 1$):
$$z \to \zeta z \tag{2.17}$$

CFTs in 2 dimensions are important in string theory, since one can transmit the analysis from the cylinder to the complex plane, which is easier to handle. In the rest of this section, we present how the CFT analysis is employed in string theory.

At the quantum level we are interested in correlation functions. These involve products of local operators. A statement about what happens as local operators approach each other is given by the operator product expansion (OPE). The idea is that two local operators inserted at nearby points can be closely approximated by a string of operators at one of the two points. Denoting by $\mathcal{O}_i$ all the local operators of the CFT, this is written as [2]

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(z,\bar{z}) = \sum_k C^k_{ij}(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}) \tag{2.18}$$

The OPE's have singular behaviour as $z \to w$. It turns out to contain the same information as commutation relations, as well as telling as how operators transform under symmetries. If we know the OPE between an operator and the stress-tensor $T(z)$ and $\bar{T}(\bar{z})$[3], then we immediately know how the operator transforms under conformal symmetry. By definition, an operator $\mathcal{O}$ is said to have weight $(h,\tilde{h})$ if, under $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}$, $\mathcal{O}$ transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}) \tag{2.19}$$

where $\delta z = \epsilon z$ describes translations for $z = 1$ and rotations for $z \neq 1$. The terms $h\mathcal{O}$ and $\tilde{h}\mathcal{O}$ are special operators which are eigenstates of dilatations and rotations. Consider

[2]Such expressions are always to be understood as statements which hold as operator insertions inside time-ordered correlation function.

[3]Expressions of T for bosonic matter and ghosts can be found in Appendix E.1.

now the OPE of an operator with the energy-momentum tensor T or $\bar{T}$. If the OPE truncates at order $(z-w)^{-2}$ or $(\bar{z}-\bar{w})^{-2}$, respectively, that is

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h\mathcal{O}(w,\bar{w}}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-2} + \textit{non-singular} \tag{2.20a}$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \frac{\tilde{h}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})} + \textit{non-singular} \tag{2.20b}$$

then we call $\mathcal{O}$ a primary operator. A primary operator transforms, in general, as

$$\mathcal{O}(z,\bar{z}) \to \tilde{\mathcal{O}}(z,\bar{z}) = (\frac{\partial\tilde{z}}{\partial z})^{-h}(\frac{\partial\bar{\tilde{z}}}{\partial\bar{z}})^{-\tilde{h}}\mathcal{O}(z,\bar{z}) \tag{2.21}$$

The energy-tensor itself is a primary operator, since its OPE with itself expands as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \textit{non-singular} \tag{2.22a}$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \textit{non-singular} \tag{2.22b}$$

The constants c and $\tilde{c}$ are called the central charges. For the bosonic string theory, they are equal to $c = \tilde{c} = 26$.

2.3 The Polyakov path integral and interactions

Consider the string propagating in a Euclidean target space. The Polyakov action then, reads:

$$S_P = -T\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\delta^{\mu\nu}\partial_\alpha X_\mu\partial_\beta X_\nu \tag{2.23}$$

The path integral is obtained by integrating over all degrees of freedom, that is, over all embedding coordinates X^μ and all worldsheet metrics $g_{\alpha\beta}$:

$$Z = \frac{1}{\textit{Vol }G}\int \mathcal{D}g\mathcal{D}Xe^{-S_P[X,g]} \tag{2.24}$$

Notice the presence of the term *Vol G*. As we mentioned in Section 2.1, the Polyakov action possesses two local (gauge) symmetries, namely, diffeomorphisms and Weyl transformations. When integrating over all possible metrics and embeddings, without gauge fixing, one cannot distinguish among the ones that are physically equivalent. This will give us a divergent factor, equal to the volume of the group G of these equivalent degrees of freedom. By dividing with $Vol\, G$, we are able to cancel this factor. A procedure to completely fix the gauge and proceed to quantization was invented by Faddeev and Popov [26]. This procedure involves the introduction of two anti-commuting ghost fields b,c. These ghost fields do not correspond to real degrees of freedom. Their role is to cancel the unphysical degrees of freedom, leaving only the $D-2$ transverse modes of X^μ. The path integral is then written as

$$Z[g] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c e^{-S_P[X,g]-S_{gh}[b,c,g]} \tag{2.25}$$

with the ghost action S_{gh}, given by

$$S_{gh} = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}) \tag{2.26}$$

The (b,c) ghost system has its own stress-energy tensor and the discussion of CFT of the previous section can be further extended to it. The ghost action given by Eq.(2.26) possesses a $U(1)$ global symmetry, which infinitesimally reads:

$$\delta b = -ib, \quad \delta c = ic, \quad \delta\bar{b} = -i\bar{b}, \quad \delta\bar{c} = i\bar{c} \tag{2.27}$$

According to Noether's theorem, every global symmetry is generated by a conserved current. In our case, this current is called the "ghost current" and is given by:

$$j(z) = -:b(z)c(z):\,, \quad \bar{j}(\bar{z}) = -:\bar{b}(\bar{z})\bar{c}(\bar{z}) \tag{2.28}$$

This current, in turn, is associated to a conserved charge, which we will call "the ghost number". It is defined as

$$N_{gh} = N_{gh,L} + N_{gh,R} \tag{2.29}$$

with

$$N_{gh,L} = \oint \frac{dz}{2\pi i} j(z) \tag{2.30a}$$

$$N_{gh,R} = \oint \frac{d\bar{z}}{2\pi i} \bar{j}(\bar{z}) \tag{2.30b}$$

We now pass to the concept of interactions in string theory. The interactions of particles in perturbative string theory are summarized by a set of scattering amplitudes, known as the string S-matrix [27]. The external states, are the physical states of the theory are the primary states of the CFT with weight (+1,+1) and are represented by infinite semi-tubes attached to the surfaces. Under a conformal mapping, these semi-tubes can be mapped to points called punctures on the worldsheet. The prescription for computing amplitudes via the Polyakov path integral is to insert one vertex operator $V_i(\sigma)$ into the gauge-fixed path integral for each external particle:

$$V_\alpha(k_i) := \int d^2\sigma \sqrt{g(\sigma)} V_\alpha(k;\sigma) \tag{2.31}$$

The integration over the entire worldsheet is justified, since under the conformal mapping there is no preferred point on the worldsheet for the vertex to sit. The resulting space, at g loop, is a Riemann surface $\Sigma_{g,n}$ of genus g with n punctures. Then, for n vertex insertions, the amplitude can be organized as a sum over worldsheets of different topology:

$$\langle V_1 \dots V_n \rangle = \sum_{topologies} g_s^{-\chi} \int \mathcal{D}X\mathcal{D}g e^{-S_p(X,g)} V_1(p_1)\dots V_n(p_n) \tag{2.32}$$

This perturbative expansion of the amplitude in terms of Riemann surfaces is in Figure 2.2, adopted from [22]:

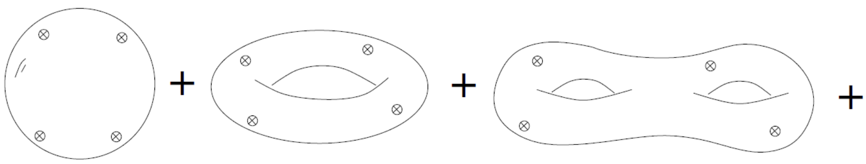


FIGURE 2.2: The string perturbative expansion is a sum over Riemann surfaces (n=4).

The space of all Riemann surfaces $\Sigma_{g,n}$ is called the moduli space and is as the quotient of the space of metrics, denoted as $Met(\Sigma_{g,n})$, by gauge transformations [28]:

$$\mathcal{M}_{g,n} := Met(\Sigma_{g,n})/G \tag{2.33}$$

The Riemann surfaces that belong to the moduli space are parametrized by the so-called moduli parameters t_i.

Chapter 3

Closed String Field Theory

In Chapter 2, we reviewed some general results of string theory based on the worldsheet formulation. However, this formulation encounters three main issues [29]. The first one is that the theory is defined only perturbatively with respect to the coupling constant. Moreover, it only allows the computation of on-shell scattering amplitudes, while off-shell quantities are not well-defined. Finally, since string theory is independently defined for each consistent background, it is not clear whether there is a universal set of degrees of freedom describing different backgrounds. One possible approach to the aforementioned issues[4] is the construction of a field theory of strings, which we will call String Field Theory (SFT). A very elegant SFT for the open bosonic string has been constructed by Witten [33]. This formulation has already given some results, with the most important being the explanation of the instability of the tachyonic vacuum of the bosonic string [34, 35]. Recent advances have managed to extend the formalism to closed bosonic strings. We wish to review part of these advances in this chapter. In Section 3.1, we introduce the concept of the string field. The free and interacting SFT are discussed in Section 3.2 and Section 3.3, respectively. This chapter is a review of [19, 36].

3.1 Introducing the string field

The idea in order to construct a String Field Theory, is to first introduce a string field, subsequently to find an action and the appropriate gauge symmetries and then to recover the perturbation theory from a gauge-fixed path integral [37]. The string field is the dynamical variable of the SFT. The starting point is to realize that since a string is a 1-dimensional extended object, then the string field, which we will denote as Ψ, must

[4] Complete proofs of background independence can be found in [30] for closed superstring field theory and [31] for closed bosonic string field theory. Off-shell amplitudes are discussed in [32].

depend on the spatial positions of each point of the string, collectively denoted as $X^\mu(\sigma)$. Hence, the string field is a functional $\Psi[X^\mu(\sigma)]$. Roughly speaking, a string field, after quantization, is an operator which creates or destroys a string at a given time. To make contact with the path integral quantization, we could take the string wavefunctional and elevate it to a field operator (second quantization). A different and more rigorous way to proceed to the quantization of the string field is to introduce a BRST (Becchi-Rouet-Stora-Tyutin) symmetry[5], by adding a gauge fixing term in the path integral given by Eq.(2.25). This reads

$$S_{g.f.}[g,\hat{g},B] = -\frac{i}{4\pi}\int d^2\sigma B^{\alpha\beta}(\sqrt{-g}g_{\alpha\beta} - \sqrt{-\hat{g}}\hat{g}_{\alpha\beta}) \tag{3.1}$$

where $B^{\alpha\beta}$ is an auxiliary field. Varying the action of Eq.(E.1) with respect to $B^{\alpha\beta}$, will procude the gauge fixed action $S_P[X,g] + S_{gh}[b,c,g]$. In this approach the string field will also depend on the ghosts: $\Psi[X(\sigma), c(\sigma)]$[6]. However, it turns out to be complicated enough to find an explicit expression for the string field as a functional of $X(\sigma)$ and $c(\sigma)$. It is often more convenient to work with the representation independent ket $|\Psi\rangle$, defined as

$$\Psi[X(\sigma), c(\sigma)] := \langle X(\sigma), c(\sigma)|\Psi\rangle \tag{3.2}$$

We can then define a string field as a vector in the Hilbert space of the worldsheet CFT, which we denote, together with the ghost insertions, by $\mathcal{H}$. Given a basis of states $\{|\phi_\alpha\rangle\}$, we may write a string field as an arbitrary linear superposition of the basis states [38] as

$$|\Psi\rangle = \sum_\alpha \psi_\alpha|\phi_\alpha\rangle \tag{3.3}$$

Here each target space field ψ_α is the component of the vector $|\Psi\rangle$ along the basis vector $|\phi_\alpha\rangle$. In momentum space, this basis can be chosen to be first-quantized off-shell states $\phi_\alpha(\vec{k}) = \mathcal{V}_\alpha(k;0,0,0)|0\rangle$, such that one can write

$$|\Psi\rangle = \sum_\alpha \int \frac{d^Dk}{(2\pi)^D}\Psi_\alpha(\vec{k})|\phi_\alpha(\vec{k})\rangle \tag{3.4}$$

[5] BRST summetry is briefly discussed in Appendix E.1

[6] Notice that there is no dependence on the b ghost, since it is the conjugate momentum of the c-ghost: $b(\sigma) \sim \frac{\delta}{\delta c(\sigma)}$.

where k is the D-dimensional momentum of the string and α denotes a collection of discrete quantum numbers.

3.2 Free BRST string field theory

Having introduced the string field, we would like to have an action for the theory. Since the fundamental structure of the string field is not known, the only reasonable way to derive it is from the equations of motion. In this section we will focus on the free theory, for which we need to find the kinetic term. To this end, we need an equation of motion and an appropriate inner product on $\mathcal{H}$. Employing the BRST quantization on the worldsheet theory, one finds that the string physical states $|\psi\rangle$ obey the BRST condition

$$Q_B|\psi\rangle = 0 \tag{3.5}$$

Considering the string field Ψ to be a linear combination of all possible one-string states $|\psi\rangle$, as Eq.(3.3) indicates, the BRST condition applies to the string field itself:

$$Q_B|\Psi\rangle = 0 \tag{3.6}$$

Next, we need to find an inner product $\langle\cdot,\cdot\rangle$ on the Hilbert space $\mathcal{H}$. A natural candidate is the BPZ inner product

$$\langle A, B\rangle := \langle A|c_0^-|B\rangle \tag{3.7}$$

where $\langle A| = |A\rangle^t$ is the BPZ conjugate of $|A\rangle$. Notice the presence of the c_0^- insertion in Eq.(3.7) and recall the ghost symmetry of Eq.(2.27). This symmetry is known to be anomalous in curved spaces [20]. For the case of closed bosonic string, the cancellation of the ghost number anomaly requires that the action has ghost number

$$N_{gh}(S) = 2\chi_{g,0} = 6 \tag{3.8}$$

Without c_0^-, the total ghost number would be[7]

$$N_{gh}(Q) + 2N_{gh}(\Psi) = 5 \tag{3.9}$$

[7] Ghost numbers are provided in Appendix E.

Therefore, the insertion of the c_0 ghost[8] is necessary for the ghost anomaly to cancel. Using the properties of the BPZ inner product[9] one obtains the action

$$S_{free} = \frac{1}{2}\langle\Psi, Q_B\Psi\rangle = \frac{1}{2}\langle\Psi|c_0^- Q_B|\Psi\rangle \tag{3.10}$$

which is equivalent to a 2-point correlation function on the sphere. The problem, however, is that the presence of c_0^- annihilates some part of the string field and renders the kinetic term non-invertible. Nevertheless, one can overcome this problem by imposing additional constraints on the string field:

$$b_0^-|\Psi\rangle = L_0^-|\Psi\rangle = 0 \tag{3.11}$$

In writing the action, only the condition that the states are BRST closed has been used. One needs to interpret the condition that the CFT states are not BRST-exact, that is,

$$|\psi\rangle \sim |\psi\rangle + Q_B|\chi\rangle \tag{3.12}$$

Uplifting this condition to the string field, the most direct interpretation is that it corresponds to a gauge invariance of the form:

$$|\Psi\rangle \to |\Psi'\rangle = |\Psi\rangle + \delta_\Lambda|\Psi\rangle, \quad \delta_\Lambda|\Psi\rangle = Q_B|\Lambda\rangle \tag{3.13}$$

This gauge invariance can be fixed in the Siegel gauge [39]:

$$b_0^+|\Psi\rangle = 0 \tag{3.14}$$

After gauge fixing and upon using the decomposition of the BRST operator, given by Eq.(E.8), the action reduces to

$$S_{free} = \frac{1}{2}\langle\Psi|c_0^- c_0^+ L_0^+|\Psi\rangle \tag{3.15}$$

The propagator is then obtained by inverting Eq.(3.15). It reads

$$\Delta = \frac{b_0^+}{L_0^+} \tag{3.16}$$

[8] For the c-ghost, we have $N_{gh} = 1$.
[9] See Appendix E.2

3.3 Feynman diagrams & closed interacting SFT

In Section 3.2, we wrote down the string field propagator. We would like to give it some meaning in terms of Riemann surfaces. To this end, it is convenient to express it in the Schwinger parameter representation:

$$\frac{1}{L_0^+} = \int_0^\infty dse^{-sL_0^+} \tag{3.17}$$

Since L_0^+ is the generator of dilatations for the closed string, the integrand may be seen as a piece of worldsheet; more precisely, a tube of length s and twisting angle θ. One can then use this tube in order to glue together parts of Riemann surfaces[10]. This operation is known as the plumbing fixture and is commonly denoted by #. An example of how this operation works is presented in Figure 3.1, adopted from [19]. In this example, a 4-punctured sphere is produced by the gluing of two 3-punctured spheres.

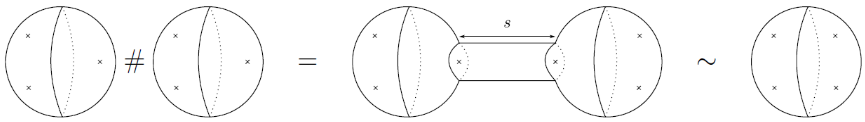


FIGURE 3.1: A 4-punctured sphere obtained by the gluing of two 3-punctured spheres.

The plumbing fixture is defined as an operation on moduli space by considering the plumbing fixture of all possible pairs of surfaces, permutations of punctures and $s \in [0, \infty), \theta \in [0, 2\pi]$. The space obtained by gluing 3-punctured spheres together is called the propagator region of the moduli space and is denoted by $\mathcal{F}_{0,4}$:

$$\mathcal{F}_{0,4} := \mathcal{M}_{0,3} \# \mathcal{M}_{0,3} \subset \mathcal{M}_{0,4} \tag{3.18}$$

Notice that not all 4-punctured spheres can be obtained from the plumbing fixture of 3-punctured spheres. There is a remaining region of the moduli space, called the fundamental region, which cannot be reached this way. It is denoted by $\mathcal{V}_{0,4}$ and defined as

$$\mathcal{V}_{0,4} := \mathcal{M}_{o,4} - \mathcal{F}_{0,4} \tag{3.19}$$

One can extend this construction to higher dimensional moduli spaces, yielding a series of fundamental regions $\mathcal{V}_{g,n}$ and propagator regions $\mathcal{F}_{g,n}$ for all $g, n \geq 0$. An interpretation

[10] These parts can belong to the same or different surfaces.

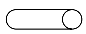
of string amplitudes in terms of Feynman diagrams can then be reached; Feynman diagrams containing propagators are connected to the region $\mathcal{F}_{g,n}$. Since elements of the region $\mathcal{V}_{g,n}$ do not contain propagators, they can be nicely interpreted as interaction vertices. It turns out that the gauge-fixed SFT action can be written in the form

$$S = \sum_{g,n\geq 0} \hbar^g \frac{g_s^{2g-2+n}}{n!} \mathcal{V}_{g,n}(\Psi^n) \tag{3.20}$$

where $\Psi^n := \Psi^{\otimes n}$. The kinetic term in Eq.(3.20) is realized as $\mathcal{V}_{0,2}$, while the rest of $\mathcal{V}_{g,n}$ represent contact interactions of all orders. The coupling constant g_s is determined by the tree-level cubic interaction. Terms with $g \geq 1$ are associated to quantum corrections of order indicated by the power of $\hbar$. The action of Eq.(3.20) can be written in the momentum space as

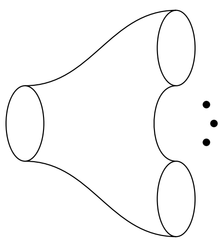
$$\begin{aligned} S = &- \int d^D k \Psi_\alpha(k) K_{\alpha\beta}(k) \Psi_\beta(k) \\ &- \sum_{n\geq 0} \int d^D k_1 \dots d^D k_n V^{(n)}_{\alpha_1 \dots \alpha_n}(k_1, \dots, k_n) \Psi_{\alpha_1}(k_1) \dots \Psi_{\alpha_n}(k_n) \end{aligned} \tag{3.21}$$

One can then use Eq.(3.21) to read off the Feynman rules. The propagator is represented as an infinite length tube and is given by



$$= K_{\alpha\beta}(k)^{-1} = -\frac{iM_{\alpha\beta}(k)}{k^2 + m_\alpha^2} Q_\alpha(k) \tag{3.22}$$

where $M_{\alpha\beta}(k)$ is a finite-dimensional matrix[11] giving the overlap of states of equal mass and Q_α a polynomial in k. The interactions are obtained by plugging the basis states $\{\phi_\alpha\}$ inside the vertices $\mathcal{V}_n$, given by Eq.(3.4):



$$\begin{aligned} &= iV^{(n)}_{\alpha_1 \dots \alpha_n}(k_1, \dots, k_n) := i\mathcal{V}_n\big(\phi_{\alpha_1}(k_1), \dots, \phi_{\alpha_n}(k_n)\big) \\ &\propto i \int dt e^{-g^{\{a_k\}}_{ij}(t) k_i k_j} P_{\alpha_1 \dots \alpha_n}(k_1, \dots, k_n; t) \end{aligned} \tag{3.23}$$

In Eq.(3.23) t denotes collectively the moduli parameters t_i and $P_{\{\alpha_i\}}$ is a polynomial in k.

[11] A definition of $M_{ab}(k)$ using arbitrary basis and dual basis states is given in Appendix E.

Chapter 4

Infrared divergences in perturbative Quantum Gravity

In this chapter we study the appearance of IR divergences in scattering amplitudes as a result of the addition of extra soft graviton lines to the photon external legs of an already finite scattering amplitude. In section 4.1, we derive Weinberg's soft theorem [3, 40], associated with real soft graviton attachments to external photon legs. The effect of attaching virtual soft gravitons to the external legs is explored in Section 4.2. Finally, in Section 4.3, we explore the phase divergences associated to the virtual attachments.

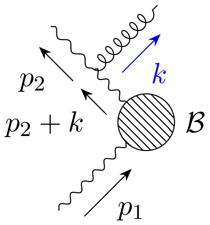
4.1 Weinberg's soft theorem

In this section we study the effect of attaching soft real gravitons to the external legs of a scattering process. In general, such attachments must be taken into account when computing any scattering process. This is because there is no possibility to experimentally distinguish the process in which an arbitrary soft zero charge particle is emitted from the one without such a particle [41]. Consider the diagrams of Figure 4.1, where a soft graviton line is attached to either external or internal[12] photon legs. We assume that the incoming and outgoing momenta are p_1 and p_2, respectively. Consider initially the case where the incoming line emits a soft graviton of momentum k (Figure 4.1b). The contribution to the S-matrix element then comes from the photon-photon-graviton vertex and the internal photon propagator of momentum $p_1 - k$. The situation is similar if the emission takes place on the outgoing leg (Figure 4.1a), with the difference that the contributing internal propagator carries momentum $p_2 + k$.

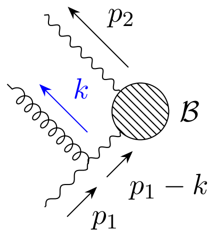
[12]Here by internal we refer to legs inside the blob $\mathcal{B}$.

To distinguish the two scenarios it is convenient to define:

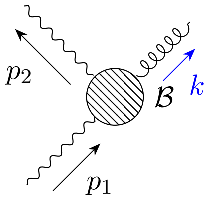
$$\begin{cases} \eta = +1 & \textit{outgoing} \\ \eta = -1 & \textit{incoming} \end{cases}$$



(a) Soft graviton attached to the outgoing leg.



(b) Soft graviton attached to the incoming leg.



(c) Soft graviton attached to an internal leg.

FIGURE 4.1: Real soft graviton attachments.

The photon propagator and photon-photon-graviton vertex are given in Appendix C[13]. It is more illuminating for our analysis to label the photons and graviton in the diagram of Figure 4.1a with their polarization indices. We mark as (ρ) the edge where the photon of momentum $p_2 + k$ attaches to the blob and by (β) the edge connected to the vertex. For the external photon, the point connected to the vertex is labelled by the index (γ) and its free endpoint by the index (σ). For the emitted soft graviton we mark its endpoint indices as μ and ν. Therefore, the contribution of this diagram reads

$$M_{\beta}^{\mu\nu\sigma} - i\frac{\eta_{\rho\beta}}{(p+k)^2 - \epsilon} \times \frac{i}{2}\left(I^{\mu\nu\rho\sigma} p_2 \cdot (p_2 + k) + \Lambda^{\mu\nu\rho\sigma}\right) \tag{4.1}$$

with $I_{\beta\gamma\mu\nu}$ and $\Lambda_{\beta\gamma\mu\nu}$ given by Eq.(D.10) and Eq.(D.11), respectively. In the limit $k \to 0$, the propagator is

$$\frac{\eta_{\rho\beta}}{p_2^2 + k^2 + 2p_2 \cdot k - i\epsilon} \approx \frac{\eta_{\rho\beta}}{2p_2 \cdot k - i\epsilon} \tag{4.2}$$

where p_2^2 vanishes on-shell. For similar reasons, the first term in the vertex parenthesis vanishes, while $\Lambda^{\mu\nu\rho\sigma}$ becomes

[13]The Feynman rules are obtained from the Lagrangian we construct in Chapter 5.

$$\Lambda^{\mu\nu\rho\sigma} = 2\eta^{\mu\nu}p_2^\rho p_2^\sigma - \eta^{\rho\nu}p_2^\mu p_2^\sigma + \eta^{\rho\sigma}p_2^\mu p_2^\nu - \eta^{\mu\sigma}p_2^\rho p_2^\nu - 3\eta^{\nu\sigma}p_2^\rho p_2^\mu + \eta^{\rho\sigma}p_2^\nu p_2^\mu - 2\eta^{\mu\rho}p_2^\nu p_2^\sigma$$

Then, Eq.(4.1) reduces to

$$\frac{1}{4p\cdot k}\left(2\eta^{\mu\nu}p_{2\beta}p_2^\sigma - \delta_\beta^\nu p_2^\mu p_2^\sigma + 2\delta_\beta^\sigma p_2^\mu p_2^\nu - \eta^{\mu\sigma}p_{2\beta}p_2^\nu - 3\eta^{\nu\sigma}p_{2\beta}p_2^\mu - 2\delta_\beta^\mu p_2^\nu p_2^\sigma\right) \tag{4.3}$$

Contracting Eq.(4.3) with the polarization vectors[14] $\epsilon_\sigma^\star$ and ϵ^β and using that $\epsilon^\mu p_\mu = 0$, we obtain

$$M^{\mu\nu} = \frac{p_2^\mu p_2^\nu}{p\cdot k - i\epsilon} \tag{4.4}$$

The evaluation of the diagram of Figure 4.1b is analogous and gives a factor of

$$-\frac{p_1^\mu p_1^\nu}{p_1\cdot k + i\epsilon} \tag{4.5}$$

The contributions of the two diagrams are to be summed over and we can elegantly write it as

$$\sum_i \eta_i \frac{p_i^\mu p_i^\nu}{p_i\cdot k - i\eta_i\epsilon} \tag{4.6}$$

Apart from these two contributing diagrams, there is also the process represented in Figure 4.1c, in which the soft graviton is attached to an internal line. However, internal lines are off-shell, meaning that the condition $p^2 = -m^2$ does not hold anymore. Hence the denominator is dominated by $p^2 + m^2$ and the factor $p\cdot k$ can be ignored in the limit $k \to 0$. The contribution generalizes to diagrams that involve more than one external legs. Consider, for example, an arbitrary diagram such that of Figure 4.2, which contains n external legs, with corresponding momenta $p_1, \ldots, p_n$. Then, the result of Eq.(4.6) becomes[15]

$$\sqrt{8\pi G}\sum_{i=1}^n \frac{\eta_i p_i^\mu p_i^\nu}{p_i\cdot k - i\epsilon} \tag{4.7}$$

[14]Notice that there are two options

[15]Notice that we have recovered Weinberg's factor: $2\kappa^2 = 1 \Rightarrow \kappa = \sqrt{8\pi G}$

where the sum is running over all external lines. It is indicated that the effect of attaching real soft gravitons to a scattering amplitude that did not initially contain any, turns out to multiply the initial amplitude with the factor given by Eq.(4.7). This is known as the gravitational soft Weinberg's theorem. The result we obtained is exactly the same as the one computed by Weinberg for external scalar legs, verifying the statement that it is does not depend on the spin of the external legs and endorsing the universality of the IR structure of gravity.

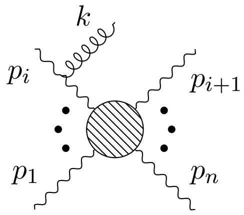


Figure 4.2: Real graviton emission in a diagram with n external legs.

Before we close this section, we should note that according to Weinberg, when the external leg is massless, there is another type of divergences, the so-called collinear divergences. These occur from the denominator $\frac{1}{p \cdot k}$, which diverges when $\mathbf{p}$ is parallel to $\mathbf{k}$. However, these divergences cancel for the case of soft gravitons [8, 42], so they will be ignored in the rest of our study.

4.2 Virtual infrared divergences

In the previous section, we studied when and how the insertion of a soft graviton in the case of photon external legs can contribute to divergent S-matrix elements. In this section, we wish to study the divergences that arise from the soft part of a virtual graviton attached to the external photon legs. The virtual graviton can connect either to the same leg, or to different legs as illustrated in Figure 6.2. The addition of one virtual soft graviton line to an external leg, results in multiplying the matrix element a pair of factors of the form of Eq.(4.6), connected by a graviton propagator[16]

$$-\frac{i}{2(2\pi)^4}\frac{I_{\mu\nu\rho\sigma}}{k^2 - i\epsilon} \tag{4.8}$$

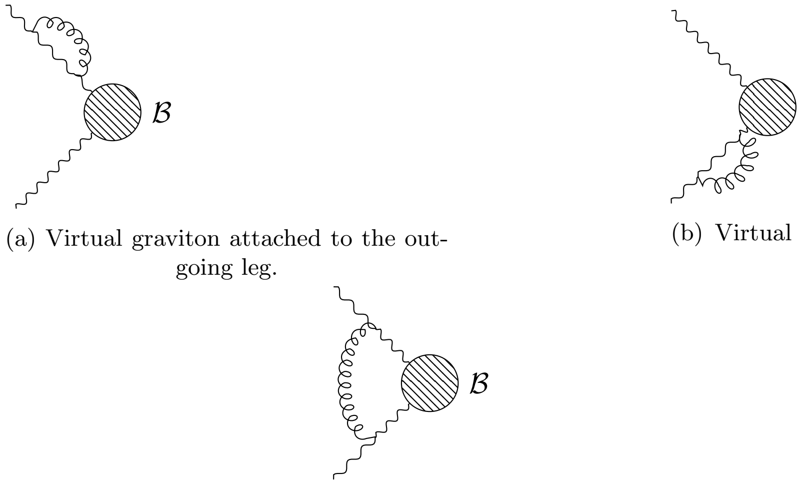
and then sum over polarization indices and integrate over k. The total contribution can be put in the form

$$\frac{1}{2}\int^{\Lambda} d^4k B(k) \tag{4.9}$$

[16] Notice that we have brought back the factor $(2\pi)^{-4}$.

with $B(k)$ given by

$$\begin{aligned} B(k) &= \frac{-i8\pi G}{2(2\pi)^4(k^2-i\epsilon)}\sum_{n,m}\frac{\eta_n\eta_m p_n^\mu p_n^\nu p_m^\rho p_m^\sigma(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma})}{(p_n\cdot k-i\eta_n\epsilon)(-p_m\cdot k-i\eta_m\epsilon)} \\ &= \frac{-i8\pi G}{(2\pi)^4(k^2-i\epsilon)}\sum_{n,m}\frac{\eta_n\eta_m\Big((p_n\cdot p_m)^2-\frac{1}{2}p_n^2p_m^2\Big)}{(p_n\cdot k-i\eta_n\epsilon)(-p_m\cdot k-i\eta_m\epsilon)} \\ &= \frac{i8\pi G}{(2\pi)^4(k^2-i\epsilon)}\sum_{n,m}\frac{\eta_n\eta_m(p_n\cdot p_m)^2}{(p_m\cdot k-i\eta_n\epsilon)(p_m\cdot k+i\eta_m\epsilon)} \end{aligned} \tag{4.10}$$



(a) Virtual graviton attached to the outgoing leg.

(b) Virtual graviton attached to the incoming leg.

(c) Virtual graviton connecting the incoming and outgoing legs.

FIGURE 4.3: Virtual soft graviton corrections.

The S-matrix can thus be written in terms of the initial, divergence-free S-matrix $S^0_{\beta\alpha}$ as

$$S_{\beta\alpha} = exp\Big\{\frac{1}{2}\int^\Lambda d^4k B(k)\Big\}S^0_{\beta\alpha} = exp\Big\{\frac{4\pi G}{(2\pi)^4}\sum_{nm}(p_n\cdot p_m)^2 n_n n_m J_{nm}\Big\}S^0_{\beta\alpha} \tag{4.11}$$

where all the divergences are now carried by the term

$$J_{nm} = i\int^\Lambda d^4k\frac{1}{(k^2-i\epsilon)(p_n\cdot k-i\eta_n\epsilon)(p_m\cdot k+i\eta_m\epsilon)} \tag{4.12}$$

The real part of J_{nm} is directly associated with IR divergences and will be used in Chapter 6 to compute the total virtual soft graviton contribution to a single photon scattering by an external gravitational field. For the time being, we will focus on the imaginary part of J_{nm}, which contains phase divergences.

4.3 Phase divergences in photon-graviton scattering

In this last section, we study the phase divergences associated to soft virtual gravitons, following the approach of [8]. The integrand in Eq.(4.12) has its poles on

- $k^2 - i\epsilon = 0 \Rightarrow -(k^0)^2 = |\mathbf{k}|^2 - i\epsilon = 0 \Rightarrow k^0 = \pm\sqrt{|\mathbf{k}|^2 - i\epsilon} \Rightarrow k^0 \approx |\mathbf{k}| - i\epsilon$ or $k^0 \approx -|\mathbf{k}| + i\epsilon$, and
- $p_m \cdot k - i\eta_m\epsilon = 0 \Rightarrow k^0 \approx \frac{\mathbf{p}_m\cdot\mathbf{k}}{p_m^0} + i\eta_m\epsilon$ or $k^0 \approx -\frac{\mathbf{p}_m\cdot\mathbf{k}}{p_m^0} - i\eta_m\epsilon$

Before performing the k^0-contour integration, we notice that if n_n, n_m are opposite, that is we have an incoming and an outgoing particle, the poles lie on the same side of the real axis and can both be avoided by integrating on the other half-plane. As a consequence, J_{nm} will be real. However, in the case of the particles being both incoming or both outgoing, the poles lie on opposite sides of the real axis and thus cannot both be avoided. Considering the case $n_n = n_m = +1$ and closing the contour in the upper half-plane, we pick up the contribution from the poles at $-|\mathbf{k}| + i\epsilon$ and $\mathbf{v}_m \cdot \mathbf{k} + i\epsilon$, where we have defined $\mathbf{v}_m = \frac{p_m}{p_m^0}$. Applying the residue theorem we get

$$\begin{aligned} J_{nm} = 2\pi i \sum_{j=1}^{2} Res\big[J_{nm}(j)\big] &= i\int^{\Lambda} d^3\mathbf{k}(2\pi i)\Big(\lim_{k^0\to|\mathbf{k}|+i\epsilon} \frac{k^0 + |\mathbf{k}| - i\epsilon}{(k^2 - i\epsilon)(p_n\cdot k)(p_m\cdot k)} \\ &+ \lim_{k^0\to\ \mathbf{v}_m\cdot\mathbf{k}-i\epsilon} \frac{k^0 - \mathbf{v}_m\cdot\mathbf{k} - i\epsilon}{(k^2 - i\epsilon)(p_n\cdot k)(p_m\cdot k)}\Big) \\ &= i\int^{\Lambda} d^3\mathbf{k}(2\pi i)\Big(\frac{1}{-2|\mathbf{k}|(\mathbf{p}_n\cdot\mathbf{k} - |\mathbf{k}|p_n^0)(\mathbf{p}_m\cdot\mathbf{k} - |\mathbf{k}|p_m^0)} \\ &+ \frac{1}{(|\mathbf{k}|^2 - (\mathbf{v}_m\cdot\mathbf{k})^2)(-p_m^0)(\mathbf{p}_n\cdot\mathbf{k} - p_n^0\mathbf{v}_m\cdot\mathbf{k} - i\epsilon)}\Big) \end{aligned} \tag{4.13}$$

The second term relates to the imaginary part, providing us the phase divergence. This can be written as

$$\mathbf{k}\cdot\mathbf{p}_n - p_n^0\mathbf{v}_m\cdot\mathbf{k} = p_n^0(\mathbf{k}\cdot\mathbf{v}_n - \mathbf{k}\cdot\mathbf{v}_m) = p_n^0\mathbf{k}\cdot\big(\mathbf{v}_n - \mathbf{v}_m\big) \tag{4.14}$$

Expressing the inner products as $\mathbf{k}\cdot\mathbf{v}_i = |\mathbf{k}||\mathbf{v}_i|\cos\theta_i$, where θ_i is the angle between the vectors $\mathbf{k}$ and $\mathbf{v}_i$, this becomes

$$p_n^0|\mathbf{k}|\big(|\mathbf{v}_n|\cos\theta_n - |\mathbf{v}_m|\cos\theta_m\big) = p_n^0|\mathbf{k}||\mathbf{v}_n - \mathbf{v}_m|\cos\gamma \tag{4.15}$$

where γ is the angle between the vectors $\mathbf{k}$ and $\mathbf{v}_n - \mathbf{v}_m$. Thus we can write

$$\Im(J)_{nm} = -\frac{2\pi}{p_n^0 p_m^0} \int^{\Lambda} d^3\mathbf{k} \frac{1}{|\mathbf{k}|^2(1-|\mathbf{v}_m|^2 \cos^2\theta_m)} \times \frac{1}{\big(|\mathbf{k}||\mathbf{v}_m - \mathbf{v}_n|(\cos\theta_m \cos\beta - \sin\theta_m \cos\theta_n \sin\beta) - i\epsilon\big)} \tag{4.16}$$

since

$$\cos\gamma = \cos\theta_m \cos\beta - \sin\theta_m \cos\theta_n \sin\beta \tag{4.17}$$

with β the angle between $\mathbf{v}_n$ and $\mathbf{v}_n$. The result must be Lorentz invariant and therefore can be transformed to a more convenient frame of reference. Since the photons have no mass, it does not make sense to consider a center of mass frame. However, the photons have non-vanishing momenta, so there will be a unique frame in which the total momentum of the system vanishes. In this frame, the velocities are back to back, meaning that $\beta = \pi$, thus one is left with

$$-\frac{2\pi}{p_n^0 p_m^0 |\mathbf{v}_n - \mathbf{v}_m|} \int^{\Lambda} d^3\mathbf{q} \frac{1}{|\mathbf{q}|^3(1-|\mathbf{v}_m|^2\cos^2\theta_m)(\cos\theta_m - i\epsilon)} \tag{4.18}$$

Choosing $\mathbf{v}_m$ to be along the first axis, the integrand is independent of the angles $\theta_1, \ldots, \theta_{n-1}$ [43]. Carrying out those integrals yields an overall factor of

$$\frac{(2\pi)^{5-n}}{p_n^0 p_m^0 |\mathbf{v}_n - \mathbf{v}_m|} \frac{2\pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2}-1)} \int^{\Lambda} dq q^{n-2} \frac{1}{q^3} \int_0^{\pi} d\theta \sin^{n-3}\theta \frac{1}{(1-|\mathbf{v}_m|^2\cos^2\theta)(\cos\theta - i\epsilon)} \tag{4.19}$$

Making the variable change $x = \cos\theta$ in the θ-integral, we have

$$dx = -\sin\theta d\theta \tag{4.20}$$

$$x_0 = 1, x_1 = -1 \tag{4.21}$$

$$\sin^{n-4}\theta = \sin^{2\left(\frac{n}{2}-2\right)}\theta = (1-x^2)^{\left(\frac{n}{2}-2\right)} \tag{4.22}$$

such that the integral takes the form

$$\int_{-1}^{1} dx \frac{(1-x^2)^{\frac{n}{2}-2}}{\big(1-|\mathbf{v}_m|^2 x^2\big)} \frac{1}{x - i\epsilon} \tag{4.23}$$

From Eq.(4.23), it is derived that on the one hand is odd and integrates to zero, while on the other hand the imaginary part $i\pi\delta(x)$ of the pole integrates to $i\pi$. To regulate the IR divergence in the k-integral, we should take $n > 4$, so

$$\int_0^\Lambda dk k^{n-5} = \frac{\Lambda^{n-4}}{n-4} \tag{4.24}$$

Altogether, we have

$$\frac{2^{6-n}\pi^{5-\frac{n}{2}}i}{p_n^0 p_m^0 |\mathbf{v}_n - \mathbf{v}_m| \Gamma(\frac{n}{2}-1)} \frac{\Lambda^{n-4}}{n-4} \tag{4.25}$$

Recovering the factor

$$p_n^0 p_m^0 |\mathbf{v}_n - \mathbf{v}_m| = \frac{1}{(p_n \cdot p_m)} \tag{4.26}$$

we read off the phase divergent factor from Eq.(4.11)

$$\begin{aligned}
&\exp\left\{\frac{4\pi G}{(2\pi)^4}\sum_{n,m}(p_n \cdot p_m)^2 \underbrace{\eta_n\eta_m}_{=1} \Im(J)_{nm}\right\} \\
&= \exp\left\{\frac{4\pi G}{(2\pi)^4}\sum_{n,m}(p_n \cdot p_m)^2 \frac{2\pi^3 i}{(p_n \cdot p_m)}\frac{2}{\epsilon}\right\} \\
&= \exp\left\{iG\sum_{n,m}(p_n \cdot p_m)\frac{1}{\epsilon}\right. \\
&= exp\left\{2iG(p_n \cdot p_m)\frac{1}{\epsilon}\right\}
\end{aligned} \tag{4.27}$$

In the last equality we multiplied with a factor of 2, since the sum counts the pairs (nm) and (mn) separately. Restoring the factor $2G = \frac{1}{8\pi}$, the divergent phase reads:

$$\phi_{mn}^{gp} = \frac{i}{8\pi}(p_n \cdot p_m)\frac{1}{\epsilon} \tag{4.28}$$

Chapter 5

The Einstein-Maxwell-scalar field model

In this chapter we study the EMS model. More precisely, in Section 5.1, we construct the Lagrangian within the framework of Perturbative Quantum Gravity (PQG). The asymptotic potential is calculated in Section 5.2. We close this chapter with Section 5.3, in which the asymptotic operators are briefly introduced. We will not enter into deeper technical details on PQG than the ones necessary for the purpose of this chapter. For further information on the topic the reader is referred to [44–48].

5.1 The Lagrangian for the EMS model

In this section, we wish to derive the gravitational Lagrangian for a graviton scattering with a massive scalar field, to which we associate a $U(1)$ symmetry, in the presence of the electromagnetic field. In the absence of gravity, the scalar QED Lagrangian reads

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi + \mathcal{L}_{gf} \tag{5.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative . To simplify the calculations, we choose to work in the Feynman-'t Hooft gauge:

$$\mathcal{L}_{gf} = -\frac{1}{2}(\partial \cdot A)^2 \tag{5.2}$$

Within the frame of linearized gravity, we want to couple this Lagrangian to the Einstein-Hilbert term

$$\mathcal{L}_{E\text{-}H} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R \tag{5.3}$$

with R being the Ricci scalar and $\kappa^2 = \frac{8\pi G}{c^4}$ the Einstein gravitational constant[17]. To this end, we follow the Gupta procedure [49], according to which, one introduces a symmetric tensor field $h_{\mu\nu}$ describing fluctuations around the Minkowski space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.4a}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{5.4b}$$

$$\sqrt{-g} = 1 + \frac{1}{2} h \tag{5.4c}$$

The Christoffel symbols[18] up to linear order in h are given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^{\rho}_{\nu} + \partial_\nu h^{\rho}_{\mu} - \partial^\rho h_{\mu\nu}\right) \tag{5.5}$$

Since the Christoffel symbols are linear in h, terms involving multiplication of Γ's in the Riemann tensor definition are of second order in h and can thus be ignored. The remaining ones give

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma}\right) \tag{5.6}$$

The Ricci tensor is obtained by contracting over the μ,ρ indices. We obtain

$$R_{\nu\sigma} = \frac{1}{2} \eta^{\mu\rho} R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho \partial_\nu h^{\rho}_{\sigma} + \partial_\sigma \partial_\rho h^{\rho}_{\nu} - \partial_\sigma \partial_\nu h - \partial^2 h_{\nu\sigma}\right) \tag{5.7}$$

where we defined $h = \eta^{\nu\sigma} h_{\nu\sigma} = h^{\nu}_{\nu}$. Contracting again over ν and σ yields the Ricci scalar

$$R = g^{\nu\sigma} R_{\nu\sigma} = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h \tag{5.8}$$

[17] In the following analysis we will take $2\kappa^2 = 16\pi G = c = 1$.

[18] We refer the reader to Chapters 3 and 7 of [50] for a review of notions of curved space-time.

Assuming that the fields ϕ and A_μ couple minimally to gravity, the Lagrangian takes the form

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^* g^{\mu\nu}(D_\nu\phi) - m^2\phi^*\phi - \frac{1}{2}(\nabla \cdot A)^2) + \mathcal{L}'_{gf} \tag{5.9}$$

where the gauge fixing term for the gravitational field is chosen to be

$$\mathcal{L}'_{gf} = \frac{1}{2}C^2, \quad C^\mu = \partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h \tag{5.10}$$

Notice that under the assumption of minimal coupling, the electromagnetic field tensor does not change, since the Christoffel symbols are symmetric under exchange of the two lower indices:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu}A_\rho - \partial_\nu A_\mu + \Gamma^\rho_{\nu\mu}A_\rho = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.11}$$

The Lagrangian of Eq.(5.9) can be further simplified and rewritten in terms of the fields $h_{\mu\nu}, A_\mu, \phi$ and $\phi^\star$. We will perform it for each term separately. The term $(\nabla \cdot A)^2$, is treated in Appendix A.1. Expanding then $\sqrt{-g} = 1 + \frac{1}{2}h$ and using Eq.(A.9), we obtain:

$$-\frac{1}{2}\sqrt{-g}(\nabla_\mu A^\mu)^2 = -\frac{1}{2}(\partial \cdot A)^2 - \big(-\frac{h}{4}(\partial \cdot A)^2 + h^{\mu\nu}A_\nu\partial_\mu(\partial \cdot A) - \frac{h}{2}A_\nu\partial^\nu(\partial \cdot A)\big) \tag{5.12}$$

For the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, we have

$$\begin{aligned}&-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)g^{\mu\rho}g^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\\&+\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\rho}h^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) + \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\nu\sigma}h^{\mu\rho}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\end{aligned} \tag{5.13}$$

Therefore, upon multiplying with $\sqrt{-g}$ and expanding it, this term becomes

$$\begin{aligned}-\frac{1}{4}\sqrt{-g}F_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} = &-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\\&+\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\rho}h^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\\&+\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\nu\sigma}h^{\mu\rho}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\\&-\frac{1}{8}\eta^{\mu\rho}\eta^{\nu\sigma}h(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\end{aligned} \tag{5.14}$$

In Eq.(5.14), the second and third line are equal upon some index renaming, so that we actually need only one of them. Next, we expand the covariant derivatives as

$$\begin{aligned} -(D_\mu\phi)^\star g^{\mu\nu}(D_\nu\phi) - m^2\phi^\star\phi &= -(\partial_\mu + ieA_\mu)\phi^\star g^{\mu\nu}(\partial_\nu - ieA_\nu)\phi - m^2\phi^\star\phi \\ &= -\partial_\mu\phi^\star g^{\mu\nu}\partial_\nu\phi - e^2 A_\mu\phi^\star g^{\mu\nu}A_\nu\phi - m^2\phi^\star\phi \\ &\quad + \partial_\mu\phi^\star g^{\mu\nu} ieA_\nu\phi - ieA_\mu\phi^\star g^{\mu\nu}\partial_\nu\phi \end{aligned} \tag{5.15}$$

The terms of Eq.(5.15) are expressed in terms of h in Appendix A.2. Replacing them in the Lagrangian of Eq.(5.9) together with Eq.(5.14) and Eq.(5.12) allows us to express it as

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \frac{1}{8}\partial_\alpha h\partial^\alpha h - \eta^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi - m^2\phi^\star\phi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &- \frac{1}{2}(\partial\cdot A)^2 - V_{tot} \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} V_{tot} = &-\frac{h}{4}(\partial\cdot A)^2 + h^{\mu\nu}A_\nu\partial_\mu(\partial\cdot A) - \frac{1}{2}hA_\nu\partial^\nu(\partial\cdot A) \\ &- \frac{1}{2}(\eta^{\mu\rho}h^{\nu\sigma} - \frac{1}{4}h\eta^{\mu\rho}\eta^{\nu\sigma})(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \\ &- h^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi + \frac{1}{2}\eta^{\mu\nu}h\partial_\mu\phi^\star\partial_\nu\phi + \frac{1}{2}m^2 h\phi^\star\phi - ie\eta^{\mu\nu}(\partial_\mu\phi^\star A_\nu\phi - A_\mu\phi^\star\partial_\nu\phi) \end{aligned} \tag{5.17}$$

In Eq.(5.17) we have only kept cubic vertex interactions. For scalar QED it has been shown in [51] that the four point interaction does not survive asymptotically. Despite that something similar has not been shown for other kinds of interactions appearing, we can simply assume that the argument still holds and simply ignore them. If one wants to take them into consideration, then it is possible that terms of order h^2 in the original expansion are also kept. However, the entire process becomes extremely complex and it is not clear if there is something to be achieved by doing so.

5.2 Calculation of the asymptotic potential

Having expressed the interacting part in terms of the fields $h_{\mu\nu}, A_\mu, \phi$ and $\phi^\star$, we would like to study the interactions at large time, working in the interaction picture. This can be accomplished by first expanding the fields in Fourier space:

$$h_{\mu\nu}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2k_0}} \big(a_{\mu\nu}(\vec{\mathbf{k}})e^{ik\cdot x} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-ik\cdot x}\big) \tag{5.18a}$$

$$A_\mu(\mathbf{x},t) = \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2p_0}} \big(b_\mu(\mathbf{p})e^{ip\cdot x} + b^\dagger_\mu(\mathbf{p})e^{-ip\cdot x}\big) \tag{5.18b}$$

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2r_0}} \big(c(\mathbf{r})e^{ir\cdot x} + d^\dagger(\mathbf{r})e^{-ir\cdot x}\big) \tag{5.18c}$$

$$\phi^\star(\mathbf{x},t) = \int \frac{d^3\mathbf{s}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2s_0}} \big(c^\dagger(\mathbf{s})e^{-is\cdot x} + d(\mathbf{s})e^{is\cdot x}\big) \tag{5.18d}$$

In order to simplify the analysis, it is convenient to split the interaction potential as a sum of three individual potentials

$$V_{tot} = V_{gs} + V_{ps} + V_{gp} \tag{5.19}$$

where

$$V_{gs} := -h^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + \frac{1}{2}\eta^{\mu\nu}h\partial_\mu\phi^*\partial_\nu\phi + \frac{1}{2}m^2h\phi^*\phi \tag{5.20}$$

describes the interactions among gravitons and scalars and

$$V_{ps} := ie\eta^{\mu\nu}(A_\mu\phi^*\partial_\nu\phi - \partial_\mu\phi^*A_\nu\phi) \tag{5.21}$$

characterizes the interactions among photons and scalars. The interactions among gravitons and photons are captured by

$$\begin{aligned} V_{gp} := &-\frac{1}{2}(\eta^{\mu\rho}h^{\nu\sigma} - \frac{1}{4}h\eta^{\mu\rho}\eta^{\nu\sigma})(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \\ &- \frac{h}{4}(\partial\cdot A)^2 + h^{\mu\nu}A_\nu\partial_\mu(\partial\cdot A) - \frac{1}{2}hA_\nu\partial^\nu(\partial\cdot A) \end{aligned} \tag{5.22}$$

We may study the asymptotic behavior of each individual potential and then simply gather the contributions.

5.2.1 Graviton-Scalar interaction potential

We start from the potential V_{gs} and in particular the mass term. Using the Fourier space expansions of Eq.(5.18a), Eq.(5.18c) and Eq.(5.18d), we have

$$h\phi^\star\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{s}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2k_0 2r_0 2s_0}}\left(a_{\mu\nu}(\mathbf{k})e^{ik\cdot x} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-ik\cdot x}\right)$$
$$\times\left(c(\mathbf{r})c^\dagger(\mathbf{s})e^{i(r-s)\cdot x} + c(\mathbf{r})d(\mathbf{s})e^{i(r+s)\dot{x}} + d^\dagger(\mathbf{r})c^\dagger(\mathbf{s})e^{-i(r+s)\cdot x} + d^\dagger(\mathbf{r})d(\mathbf{s})e^{-i(r-s)\cdot x}\right)$$

Two types of terms appear:

1. Terms that contain two scalar creation or two scalar annihilation operators.
2. terms that contain one scalar creation and one scalar annihilation operator.

Terms of the first kind carry time dependence

$$e^{-i(r^0+s^0\pm k^0)t} = e^{-i(\sqrt{\mathbf{r}^2-m^2}+\sqrt{(\mathbf{r}+\mathbf{k})^2-m^2}\pm k^0)t} \tag{5.23}$$

while terms of the second kind have time dependence

$$e^{-i(r^0-s^0\pm k^0)t} = e^{-i(\sqrt{\mathbf{r}^2-m^2}-\sqrt{(\mathbf{r}+\mathbf{k})^2-m^2}\pm k^0)t} \tag{5.24}$$

Using the Riemann-Lebesgue lemma[19], one can argue that terms of the first kind become highly oscillatory for large t, such that they average to zero when integrating over the momenta $\mathbf{p}$ and $\mathbf{k}$. For terms of the second kind, the oscillatory behavior is suppressed in the limit $\mathbf{k} \approx 0$, due to the presence of the minus sign. Therefore, only terms of this kind may be taken into consideration leaving us with

$$h\phi^\star\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{s}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2k_0 2r_0 2s_0}}\Big(a_{\mu\nu}(\mathbf{k})e^{i(\mathbf{r}-\mathbf{s}+\mathbf{k})\cdot\mathbf{x}}e^{-i(r^0-s^0+k^0)t}$$
$$+ a^\dagger_{\mu\nu}(\mathbf{k})e^{i(\mathbf{r}-\mathbf{s}-\mathbf{k})\cdot\mathbf{x}}e^{-i(r^0-s^0-k^0)t}\Big)\left(c(\mathbf{r})c^\dagger(\mathbf{s}) + d^\dagger(\mathbf{r})d(\mathbf{s})\right) \tag{5.25}$$

[19] See Appendix D. Notice that there is no integration over time so far, such that the Riemann-Lebesgue lemma applies. This comes later. It is convenient though to apply this argument here in order not to carry terms that will eventually vanish.

Integrating[20] over the spatial volume $\int d^3\mathbf{x}$ yields the delta functions[21] $\delta^{(3)}(\mathbf{r}-\mathbf{s}+\mathbf{k})$ and $\delta^{(3)}(\mathbf{r}\ -\mathbf{s}-\mathbf{k})$ for the exponential factors of $a_{\mu\nu}(\mathbf{k})$ and $a_{\mu\nu}(\mathbf{k})$, respectively.

These delta functions kill the s-integral such that we are left with[22]

$$h\phi^\star\phi = \int \frac{d^3\mathbf{k}d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}2r_0\sqrt{2k_0}}\Big(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{r_0}t}(c(\mathbf{r})c^\dagger(\mathbf{r}+\mathbf{k})+d^\dagger(\mathbf{r})d(\mathbf{r}-\mathbf{k})) + a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{r_0}t}(c(\mathbf{r})c^\dagger(\mathbf{r}-\mathbf{k})+d^\dagger(\mathbf{r})d(\mathbf{r}+\mathbf{k}))\Big) \tag{5.26}$$

which in the approximation $\mathbf{k}\ \approx 0$ becomes

$$h\phi^\star\phi = \int \frac{d^3\mathbf{k}d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}2r_0\sqrt{2k_0}}\big(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{r_0}t}+a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{r_0}t}\big)\big(c(\mathbf{r})c^\dagger(\mathbf{r})+d^\dagger(\mathbf{r})d(\mathbf{r})\big) \tag{5.27}$$

If we define $\rho_b(\mathbf{r})=c^\dagger(\mathbf{r})c(\mathbf{r})+d^\dagger(\mathbf{r})d(\mathbf{r})$ and multiply by the pre-factor containing the mass, we obtain

$$\frac{1}{2}m^2\eta^{\mu\nu}h_{\mu\nu}\phi^\star\phi = \frac{1}{2}m^2\eta^{\mu\nu}\int \frac{d^3\mathbf{k}d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}2r_0\sqrt{2k_0}}(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{r_0}t}+a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{r_0}t})\rho_b(\mathbf{r}) \tag{5.28}$$

The same approach is to be employed for the second term in Eq.(5.20). First we note that

$$\partial_\alpha\phi^\star = -is_\alpha\int\frac{d^3\mathbf{s}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2s_0}}\big(c^\dagger(\mathbf{s})e^{-is\cdot x}-d(\mathbf{s})e^{is\cdot x}\big) \tag{5.29a}$$

$$\partial^\alpha\phi = ir^\alpha\int\frac{d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2r_0}}\big(c(\mathbf{r})e^{ir\cdot x}-d^\dagger(\mathbf{r})e^{-ir\cdot x}\big) \tag{5.29b}$$

Following the computations performed for the mass term, we obtain:

$$\frac{1}{2}\eta^{\mu\nu}h_{\mu\nu}\partial_\alpha\phi^\star\partial^\alpha\phi = \frac{1}{2}\eta^{\mu\nu}\int\frac{d^3\mathbf{k}d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}2r_0\sqrt{2k_0}}r^2\big(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{r_0}t}+a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{r_0}t}\big)\rho_b(\mathbf{r}) \tag{5.30}$$

[20] Recall that we are working in the interaction picture.
[21] The delta function definition in Fourier space is given by Eq.(D.5)
[22] In order to pass from Eq.(5.25) to Eq.(5.26), we made use of Eq.(D.6).

Then, combining Eq.(5.30) with Eq.(5.28), we observe that the contribution of these two terms vanishes on-shell ($r^2 + m^2 = 0$). Therefore, the only contribution comes from the term $-h^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi$ in Eq.(5.20), for which we find

$$V^I_{as,gs} = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{r}}{2r_0\sqrt{2k_0}} r^\mu r^\nu \left(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{r^0}t} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{r^0}t}\right)\rho_b(\mathbf{r}) \tag{5.31}$$

5.2.2 Photon-Scalar interaction potential

The same strategy is to be applied for V_{ps}, given by Eq.(5.21). We find

$$ie\eta^{\mu\nu}A_\mu\phi^\star\partial_\nu\phi = -e\int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}}\frac{1}{2r_0\sqrt{2p_0}} r^\mu (b_\mu(\mathbf{p})^{i\frac{r\cdot p}{r^0}t} + b^\dagger_\mu(\mathbf{p})e^{-i\frac{r\cdot p}{r^0}t}\rho_c(\mathbf{r}) \tag{5.32}$$

which turns out to be equal to $-ie\eta^{\mu\nu}\partial_\mu\phi^\star A_\nu\phi$, such that

$$\mathcal{V}^I_{as,ps} = -2e\int \frac{d^3\mathbf{p}d^3\mathbf{r}}{(2\pi)^{\frac{3}{2}}2r_0\sqrt{2p_0}} r^\mu (b_\mu(\mathbf{p})e^{i\frac{r\cdot p}{r^0}t} + b^\dagger_\mu(\mathbf{p})e^{-i\frac{r\cdot p}{r^0}t})\rho_c(\mathbf{r}) \tag{5.33}$$

where we have defined $\rho_c(\mathbf{r}) = c^\dagger(\mathbf{r})c(\mathbf{r}) - d(\mathbf{r})d^\dagger(\mathbf{r})$.

5.2.3 Photon-Graviton interaction potential

Having seen how the existence of non-vanishing asymptotic interaction potentials occurs for scalar fields in the presence of gravitational and electromagnetic forces, we now wish to apply the same techniques for the interactions among photons and gravitons. In contrast to the previous cases, two massless fields are now involved. We thus have to decide what the external states are and which field is to be considered soft in our approximation. We can distinguish the following cases:

1. The two external legs correspond to photons and the soft particle corresponds to the graviton.
2. The one external leg corresponds to a graviton, the other to a photon and the soft particle is the photon.

We will focus only on the first case. The potential in Eq.(5.22) is already in the proper form[23], so we can proceed and follow the same steps as in Section 5.2.1. All but two of the terms are of the form $\sim h\partial A\partial A$, so instead of calculating each of them separately, we may compute the most general expression and then extract the information of a specific term of Eq.(5.22) by a simple substitution. The most general expression, after expanding the fields in Fourier space as in Eq.(5.18a) and Eq.(5.18b), reads

$$\begin{aligned}
h_{\mu\nu}\partial_\alpha A_\beta\partial_\gamma A_\delta &= \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2p_0 2k_0 2q_0}}\big(a_{\mu\nu}(\mathbf{k})e^{ik\cdot x} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-ik\cdot x}\big) \\
&\quad\times ip_\alpha\big(b_\beta(\mathbf{p})e^{ip\cdot x} - b^\dagger_\beta(\mathbf{p})e^{-ip\cdot x}\big)iq_\gamma\big(b_\delta(\mathbf{q})e^{iq\cdot x} - b^\dagger_\delta(\mathbf{q})e^{-iq\cdot x}\big) \\
&= -\int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2p_0 2k_0 2q_0}}p_\alpha q_\gamma\Bigg(\big(a_{\mu\nu}(\mathbf{k})\Big(b_\beta(\mathbf{p})b_\delta(\mathbf{q})e^{i(p+q+k)\cdot x} \\
&\quad - b_\beta(\mathbf{p})b^\dagger_\delta(\mathbf{q})e^{i(p-q+k)\cdot x} - b^\dagger_\beta(\mathbf{p})b_\delta(\mathbf{q})e^{i(q-p+k)\cdot x} + b^\dagger_\beta(\mathbf{p})b^\dagger_\delta(\mathbf{q})e^{-i(q+p-k)\cdot x}\Big) \\
&\quad + a^\dagger_{\mu\nu}(\mathbf{k})\Big(b_\beta(\mathbf{p})b_\delta(\mathbf{q})e^{i(p+q-k)\cdot x} - b_\beta(\mathbf{p})b^\dagger_\delta(\mathbf{q})e^{i(p-q-k)\cdot x} \\
&\quad - b^\dagger_\beta(\mathbf{p})b_\delta(\mathbf{q})e^{i(q-p-k)\cdot x} + b^\dagger_\beta(\mathbf{p})b^\dagger_\delta(\mathbf{q})e^{-i(q+p+k)\cdot x}\Big)\Bigg)
\end{aligned}$$

Again, following the same arguments as in Section 5.2.1 for large t and in the limit $\mathbf{k} \approx 0$, we can neglect terms containing two photon creation or annihilation operators. Repeating the same lines, we obtain

$$h_{\mu\nu}\partial_\alpha A_\beta\partial_\gamma A_\delta = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{p}d^3\mathbf{k}}{2p_0\sqrt{2k_0}}p_\alpha p_\gamma\big(a_{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\beta\delta}(\mathbf{p}) \tag{5.34}$$

where we have defined $j^{ph}_{\beta\delta}(\mathbf{p}) = b^\dagger_\beta(\mathbf{p})b_\delta(\mathbf{p}) + b_\beta(\mathbf{p})b^\dagger_\delta(\mathbf{p})$. Having computed the most general expression, it is now easy to come back and compute (almost) everything. It turns out that most of the terms of this form in Eq.(5.22) will produce terms involving either p^2 or $p^\alpha \cdot b_\alpha(\mathbf{p})$. These vanish by virtue of the on-shell and Lorentz conditions for photons and thus do not contribute to the asymptotic potential[24]. The only non-vanishing contribution comes from the term $-\frac{1}{2}\eta^{\mu\rho}h^{\nu\sigma}\partial_\nu A_\mu\partial_\sigma A_\rho$ and reads

$$\mathcal{V}^I_{as,gp}(t) = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}\big(a_{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)p^\mu p^\nu\rho^{ph}(\mathbf{p}) \tag{5.35}$$

[23]What we mean with the phrase ”proper form” is that no derivative acts on the gravitational field. This is important so that no vanishing occurs when the momentum of the graviton is taken to be soft.

[24]The corresponding vanishing expressions can be found in Appendix A.6

where we have defined $\rho^{ph}(\mathbf{p}) = b^{\dagger}_{\alpha}(\mathbf{p})b^{\alpha}(\mathbf{p})$ as the photon density. As we have already mentioned, there are two more terms in Eq.(5.22) that have a different form $\sim hA\partial\partial A$. Nevertheless, one can still repeat the process and find:

$$-\frac{1}{2}hA_{\alpha}\partial^{\alpha}\partial^{\rho}A_{\rho} = \frac{\eta^{\mu\nu}}{2}\int\frac{d^3\mathbf{k}d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}2p_0\sqrt{2k_0}}\left(a_{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\mu\nu}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\right)p^{\alpha}p^{\rho}j^{ph}_{\alpha\rho}(\mathbf{p}) \tag{5.36}$$

and

$$h^{\mu\nu}A^{\nu}\partial^{\mu}\partial^{\rho}A_{\rho} = -\int\frac{d^3\mathbf{k}d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}2p_0\sqrt{2k_0}}\left(a_{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\mu\nu}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\right)p^{\mu}p^{\rho}j^{\nu\ ph}_{\rho}(\mathbf{p}) \tag{5.37}$$

By virtue of the Lorentz condition, these contributions also vanish.

5.3 The asymptotic interaction picture

In the present section, we want to demonstrate how the non-vanishing asymptotic potential can be used in order define the asymptotic interaction picture [6]. First, recall that the ordinary interaction picture is an intermediate representation between the Schrodinger picture and the Heisenberg picture. The Hamiltonian is split up as

$$H = H_0 + V^I(t) \tag{5.38}$$

where H_0 describes the free Hamiltonian of the system and V_{as} captures the interacting part. Time dependence is carried by both operators and states. The time dependence of generic operators[25] and states is governed by H_0 and $V^I(t)$, respectively:

$$i\frac{d}{dt}\phi_I(t) = [\phi(t), H_0] \tag{5.39a}$$

$$i\frac{d}{dt}|\psi(t)\rangle_I = V^I(t)|\psi(t)\rangle_S \tag{5.39b}$$

The standard assumption is that the interacting part vanishes asymptotically, that is

$$V^I_{as} := \lim_{t\to\ \infty} V^I(t) \to 0 \tag{5.40}$$

The solutions of Eq.(5.39a) and Eq.(5.39b) are then the standard free fields and free Fock states. However, if Eq.(5.40) does not hold, one has to modify them. In other words,

[25] corresponding to fields

one has to define an asymptotic interaction picture. The dynamics in this asymptotic interaction picture is governed by the operator $U_{as}(t)$. This obeys the Schrödinger-like equation

$$i\frac{dU_{as}(t)}{dt} = H^I_{as}(t)U_{as}(t) \tag{5.41}$$

where $H^I_{as}(t) = H_0 + V^I_{as}(t)$. To proceed, one might choose to split the free evolution operator from the part due to the non-vanishing asymptotic interactions. We make the ansatz

$$U_{as}(t) = e^{-iH_0t}Z(t) \tag{5.42}$$

Upon substitution in Eq.(5.41), one obtains

$$i\frac{dZ(t)}{dt} = V^I_{as}(t)Z(t) \tag{5.43}$$

The formal solution to Eq.(5.43) is a time-ordered exponential [52] of the form

$$Z(t) = \mathcal{T}exp\Big\{-i\int^t dt_1 V^I_{as}(t)\Big\} \tag{5.44}$$

In the cases of photon-scalar and graviton-scalar interactions the asymptotic potentials are such that $V^I_{as}(t)$ commutes with $Q(t_1,t_2)$ for any t_1,t_2,t, where we have defined $Q(t_1,t_2) := [V^I_{as}(t_1), V^I_{as}(t_2)]$. This property allows us to write the solution of Eq.(5.43) as

$$Z(t) = exp\Big\{-i\int^t dt_1 V_{as}(t_1) - \frac{1}{2}\int^t dt_1\int^{t_1} dt_2 Q(t_1,t_2)\Big\} \tag{5.45}$$

The asymptotic operator $U_{as}(t)$ can be written as

$$U_{as}(t) = e^{-iH_0t}e^{i\Phi(t)}e^{R(t)} \tag{5.46}$$

by defining

$$R(t) = -i\int^t dt_1 V^I_{as}(t_1) \tag{5.47a}$$

$$\Phi(t) = \frac{i}{2}\int^t dt_1\int^{t_1} dt_2 Q(t_1,t_2) \tag{5.47b}$$

The operators defined by Eq.(5.47a) and Eq.(5.47b) turn out both to contain IR divergences. Both are particularly used in order to dress the Fock states. The so-called radiation operator $e^{R(t)}$ turns out to play a primary role in the cancellation of the IR divergences coming from the soft part of the virtual photon/graviton corrections to the S-matrix. The so-called phase operator $e^{i\Phi(t)}$ on the other hand, is used to cancel the phase divergences associated with these virtual soft particles.

Chapter 6

Finite S-matrix in Einstein-Maxwell theory

In the previous chapter we found explicit expressions of the asymptotic interaction potentials for the three types of interactions appearing in our model and wrote the total asymptotic potential for a scattering process involving external scalar and photon legs as a sum of these partial asymptotic potentials. A particular scattering process can be chosen from this general theory under the appropriate considerations. For example, the absence of an electromagnetic field amounts to setting $A_\mu = 0$. The theory reproduces the scattering of a scalar field by a gravitational field, for which the cancellation of infrared divergences at all loop orders, has been shown [8]. Similarly, switching off the gravitational field by setting $h_{\mu\nu} = 0$, reproduces the scattering of a scalar field by an electromagnetic field, for which the S-matrix has also be shown to be IR finite [6, 53]. In this chapter, we want to study the scattering of a hard photon in an external gravitational potential, which simply means to set $\phi = 0$. If this theory also turns out to be IR finite, the next question to be asked, is whether the FK method can remove IR divergences from the general theory (EMS model) involving all kinds of interactions.

6.1 Photon scattering in an external gravitational potential

Setting $\phi = 0$ reduces the EMS model to pure Einstein-Maxwell theory. In this case, the asymptotic potential is given by Eq.(5.35). Remarkably, the asymptotic potential has the same form as the one a scalar field would have if scattered by the same gravitational potential. Therefore, many properties of $V^I_{as,gs}(t)$ will also hold for $V^I_{as,gp}(t)$ and the entire analysis for the cancellation of infrared divergences will follow the same steps to that of [8]. For this purpose, we will consider the scattering of a photon by some gravitational potential, represented in Figure 6.1a. The first important property is that $V^I_{as,gp}(t)$ commutes with its own commutator at different times t_1, t_2, i.e.

$$[V^I_{as,gp}(t), Q_{gp}(t_1, t_2)] = 0 \tag{6.1}$$

Therefore, the solution of Eq.(5.43) acquires the form of Eq.(5.45) which allows us to write the asymptotic operator as in Eq.(5.46), i.e.

$$U_{as,gp}(t) = e^{-iH_0 t} Z_{gp}(t) = e^{-iH_0 t} e^{i\Phi_{gp}(t)} e^{R_{gp}(t)} \tag{6.2}$$

with

$$R_{gp}(t) = -\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{k} d^3\mathbf{p}}{\sqrt{k_0} p \cdot k} (a_{\mu\nu}(\mathbf{k}) e^{i\frac{p\cdot k}{p^0}t} - h.c.) p^\mu p^\nu \rho^{ph}(\mathbf{p}) \tag{6.3a}$$

$$\Phi_{gp}(t) = -\frac{1}{16\pi^3} \int d^3\mathbf{p} d^3\mathbf{q} (p \cdot q) : \rho^{ph}(\vec{p}) \rho^{ph}(\mathbf{q}) : I(t) \tag{6.3b}$$

The expression for $R_{gp}(t)$ is calculated straightforwardly by the substitution of $V^I_{as,gp}(t)$ in Eq.(5.47a), while the expression for $\Phi_{gp}(t)$ will be derived in Section 6.2. Up to that point we will focus on $R_{gp}(t)$ only. The interaction operator may be written as[26]

$$V_{as}(t) = -\int d^3\mathbf{x} h^{\mu\nu} T^{as}_{\mu\nu}(t, \mathbf{x}), \quad T^{as}_{\mu\nu} = \int d^3\mathbf{p} \frac{p_\mu p_\nu}{2p_0} \rho^{ph}(\mathbf{p}) \delta^{(3)}\left(\mathbf{x} - \frac{t\mathbf{p}}{p^0}\right) \tag{6.4}$$

We want to calculate the asymptotic field

$$A^\mu_{as}(t, \mathbf{x}) = Z^\dagger(t) A^\mu(t, \mathbf{x}) Z(t) \tag{6.5}$$

The operator $Z(t)$ includes contributions from the operators $e^{R_{gp}(t)}$ and $e^{i\Phi_{gp}(t)}$. We will first focus on the former. We have

$$V_{as}(t) = -\int d^3\mathbf{x} h^{\mu\nu}(t, \mathbf{x}) T^{as}_{\mu\nu}(t, \mathbf{x}) = -\int \frac{d^3\mathbf{p}}{2p_0} h^{\mu\nu}(t\mathbf{p}/p_0) p_\mu p_\nu \rho^{ph}(\mathbf{p}) \tag{6.6}$$

[26] In the case of QED, the interaction operator is written in the interaction picture as the spatial integral of the photon gauge field coupled to some current, i.e. $V = -e \int d^3\mathbf{x} j_\mu A^\mu$. In PQG we know that the gravitational field should couple to energy and the most natural choice is the energy-momentum tensor, which carries precisely two Lorentz indices.

The commutator of Eq.(6.6) with a photon creation operator is

$$\begin{aligned}[V_{as}(t), b_\sigma^\dagger(\mathbf{q})] &= -\int \frac{d^3\mathbf{p}}{2p_0} h^{\mu\nu}(t\mathbf{p}/p_0) p_\mu p_\nu [\rho^{ph}(\mathbf{p}), b_\sigma^\dagger(\mathbf{q})] \\ &= -\int \frac{d^3\mathbf{p}}{2p_0} h^{\mu\nu}(t\mathbf{p}/p_0) p_\mu p_\nu b_\sigma^\dagger(\mathbf{p}) \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ &= -h^{\mu\nu}(t\mathbf{p}/p_0) \frac{q_\mu q_\nu}{2q_0} b_\sigma^\dagger(\mathbf{q}) \end{aligned} \tag{6.7}$$

where we used Eq.(B.6). Moreover, since

$$[V_{as,gp}(t), \rho^{ph}(\mathbf{q})] \propto [\rho^{ph}(\mathbf{p}), \rho^{ph}(\mathbf{q})] \tag{6.8}$$

which vanishes upon integration over p, we can write using Eq.(D.8)

$$\begin{aligned} e^{-R_{gp}(t)} b_\sigma^\dagger(\mathbf{q}) e^{R_{gp}(t)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \overbrace{[R_{gp}(t), \dots, [R_{gp}(t), b_\sigma^\dagger(\mathbf{q})] \dots]}^{n} \\ &= b_\sigma^\dagger(\mathbf{q}) \sum_{n=0}^{\infty} \frac{1}{n} \Big(-i \int^t dt_1 h^{\mu\nu}(t\mathbf{p}/p_0) \frac{q_\mu q_\nu}{2q_0} \Big)^n b_\sigma^\dagger(\mathbf{q}) \\ &= exp\Big\{ -i \int^t dt_1 h^{\mu\nu}(t\mathbf{p}/p_0) \frac{q_\mu q_\nu}{2q_0} \Big\} b_\sigma^\dagger(\mathbf{q}) \end{aligned} \tag{6.9}$$

A similar computation applies for $b_\sigma(\mathbf{q})$, such that we can collectively write

$$e^{-R_{gp}(t)} b_\sigma^\dagger(\mathbf{q}) e^{R_{gp}(t)} = \exp\Big\{ -i \int d\tau h^{\mu\nu}(t\mathbf{q}/q_0) \frac{q_\mu q_\nu}{2q_0} \Big\} b_\sigma^\dagger(\mathbf{q}) \tag{6.10a}$$

$$e^{-R_{gp}(t)} b_\sigma(\mathbf{q}) e^{R_{gp}(t)} = \exp\Big\{ i \int d\tau h^{\mu\nu}(t\mathbf{q}/q_0) \frac{q_\mu q_\nu}{2q_0} \Big\} b_\sigma(\mathbf{q}) \tag{6.10b}$$

For the phase operator, we have

$$[: \rho^{ph}(\mathbf{p}) \rho^{ph}(\mathbf{q}) :, b_\sigma^\dagger(\mathbf{r})] = b_\sigma^\dagger(\mathbf{p}) \rho^{ph}(\mathbf{q}) \delta^{(3)}(\mathbf{r}-\mathbf{p}) + b_\sigma^\dagger(\mathbf{q}) \rho^{ph}(\mathbf{p}) \delta^{(3)}(\mathbf{r}-\mathbf{q}) \tag{6.11}$$

such that

$$[-i\Phi_{gp}(t), b_\sigma^\dagger(\mathbf{r})] = b_\sigma^\dagger(\mathbf{r}) \frac{i}{16\pi} I(t) \int d^3\mathbf{p} (p \cdot r) \rho^{ph}(\mathbf{p}) \tag{6.12}$$

Performing a similar calculation to that of Eq.(6.9) for the phase operator, gives us

$$e^{-i\Phi_{gp}(t)}b^\dagger_\sigma(\mathbf{r})e^{i\Phi_{gp}(t)} = b^\dagger_\sigma(\mathbf{r})\exp\Big\{\frac{i}{16\pi}\int d^3\mathbf{p}(p\cdot r)\rho^{ph}(\mathbf{p})I(t)\Big\} \tag{6.13a}$$

$$e^{-i\Phi_{gp}(t)}b_\sigma(\mathbf{r})e^{i\Phi_{gp}(t)} = \exp\Big\{-\frac{i}{16\pi}\int d^3\mathbf{p}(p\cdot r)\rho^{ph}(\mathbf{p})I(t)\Big\}b_\sigma(\mathbf{r}) \tag{6.13b}$$

Putting everything together and since $[R_{gp}(t),\Phi_{gp}(t)]=0$, we find

$$\begin{aligned}A^\mu_{as}(t,\mathbf{x}) = Z^\dagger(t)A^\mu(t,\vec{x})Z(t) = \int\frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2q_0}}\Bigg(\exp\Big\{i\int d\tau h^{\mu\nu}(\tau\mathbf{q}/q_0)\frac{q_\mu q_\nu}{2q^0}\Big\}\\ \times\exp\Big\{-\frac{i}{16\pi}I(t)\int d^3\mathbf{p}(p\cdot q)\rho(\mathbf{p})\Big\}b_\sigma(\mathbf{q})e^{iqx}+h.c.\Bigg)\end{aligned} \tag{6.14}$$

In Eq.(6.14), the first exponential contains contributions from the soft component of the gravitational field, while the second exponential the ones from the phase operator.

6.1.1 Construction of the Chung states

Using the operator of asymptotic dynamics $U_{as}(t)=e^{-iH_0t}Z(t)$ in place of the free time evolution operator, we may define the asymptotic S-matrix as

$$S_A = \lim_{t\to\infty} Z^\dagger(t)S_D Z(t) \tag{6.15}$$

The matrix elements of Eq.(6.15) can be viewed in two equivalent ways; either as the matrix elements of the asymptotic S-matrix S_A between the standard ("free") Fock states, or as the matrix elements of the standard Dyson S-matrix between the states of the asymptotic space $\mathcal{H}_{as}=Z(t)\mathcal{H}_{Fock}$. Notice that $Z(t)$ does not define a unitary transformation in the Fock space, since the operator $e^{R_{gp}(t)}$ creates unbounded number of low-energy gravitons. Instead of $R_{gp}(t)$, we can use the operator $\tilde{R}_{gp}$, given by

$$\tilde{R}_{gp} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{p}d^3\mathbf{r}}{2\sqrt{2k^0}}(f^{\mu\nu}(k,p)a^\dagger_{\mu\nu}(\mathbf{k})-h.c.)\rho_b(\mathbf{r}) \tag{6.16}$$

This operator is characterized by an IR function $f^{\mu\nu}(k,r)$, which can be used as long as it does not modify the space of asymptotic states $\mathcal{H}_{as}$. From the BCH formula, given by Eq.(D.7), we have

$$e^{\tilde{R}_{gp}} = e^{R_{gp}(t)+\tilde{R}_{gp}-R_{gp}(t)} = e^{R_{gp}(t)}e^{\tilde{R}_{gp}-R_{gp}(t)}e^{\frac{1}{2}[\tilde{R}_{gp},R_{gp}(t)]} \tag{6.17}$$

We observe that if the last two factors in Eq.(6.17) are well-defined and unitary within the Fock space then, indeed, an equivalent description of the space of asymptotic states is given by$\tilde{R}_{gp}$:

$$e^{\tilde{R}_{gp}}\mathcal{H}_F = e^{R_{gp}(t)}e^{\tilde{R}_{gp}-R_{gp}(t)}e^{\frac{1}{2}[\tilde{R}_{gp},R_{gp}(t)]}\mathcal{H}_F = e^{R_{gp}(t)}\mathcal{H}_F = \mathcal{H}_{as} \tag{6.18}$$

The requirement of unitarity imposes some constraints on $f_{\mu\nu}(k,p)$. We will first investigate the commutator factor. We have

$$\begin{aligned} \frac{1}{2}[\tilde{R}_{gp}, R_{gp}(t)] = \frac{1}{2(2\pi)^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2\sqrt{2k_0}}\int \frac{d^3\mathbf{l}d^3\mathbf{q}}{2\sqrt{2l_0}}\big[\big(f^{\mu\nu}(k,p)a_{\mu\nu}(\mathbf{k}) - h.c.\big)\rho^{ph}(\mathbf{p}), \\ \frac{q^\rho q^\sigma}{l\cdot q}\big(a_{\rho\sigma}(\mathbf{l})e^{i\frac{l\cdot q}{q^0}t} - h.c.\big)\rho^{ph}(\mathbf{q})\big] \end{aligned} \tag{6.19}$$

Upon computing the commutator[27], Eq.(6.19) turns out to contain the integral

$$\int \frac{d^3\mathbf{k}}{2k_0}\big(f^\star_{\mu\nu}\frac{q^{\mu\nu}}{k\cdot q}e^{-i\frac{k\cdot q}{q^0}t} - h.c\big) \tag{6.20}$$

Choosing $f_{\mu\nu}$ to be real suffices to render this integral convergent. This is the first constraint imposed on $f_{\mu\nu}$. The other one arises from the requirement of unitarity of the factor $e^{\tilde{R}_{gp}-R_{gp}(t)}$. Using again the BCH formula, we can extract the factor

$$\int \frac{d^3\mathbf{k}}{2k_0}\Big(|f_{\mu\nu}(k,p) - \frac{p_\mu p_\nu}{k\cdot p}e^{-i\frac{k\cdot p}{p^0}t}|^2 - \frac{1}{2}|f^\mu_\mu(k,p)|^2\Big) \tag{6.21}$$

the convergence of which will determine the low-energy behavior of $f_{\mu\nu}$. In the Fock space, the physical states satisfy the Gupta-Bleuler condition [49] :

$$\big(k^\mu a_{\mu\nu}(\vec{k}) - \frac{1}{2}k_\nu a^\rho_\rho(\vec{k})\big)|\Psi\rangle = 0 \tag{6.22}$$

We want this condition to also hold on $\mathcal{H}_{as}$. If we find an operator of the form $e^{\tilde{R}_{gp}}$ that commutes with Eq.(6.22), then the physical asymptotic states can be obtained by a transformation of the physical Fock states. This requires that

[27] Notice that the l-integral is killed by a delta function

$$
\begin{aligned}
&[k^\mu a_{\mu\nu} - \frac{1}{2}k_\nu \eta^{\rho\sigma} a_{\rho\sigma}, f^{\alpha\beta} a_{\alpha\beta}] = k^\mu f^{\alpha\beta}[a_{\mu\nu}, a^\dagger_{\alpha\beta}] - \frac{1}{2}k_\nu \eta^{\rho\sigma} f^{\alpha\beta}[a_{\rho\sigma}, a^\dagger_{\alpha\beta}] \\
&= k^\mu f^{\alpha\beta}\big(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\big) - \frac{1}{2}k_\nu \eta^{\rho\sigma} f^{\alpha\beta}\big(\eta_{\alpha\rho}\eta_{\sigma\beta} + \eta_{\alpha\sigma}\eta_{\rho\beta} - \eta_{\alpha\beta}\eta_{\rho\sigma}\big) \\
&= k^\mu f_{\mu\nu} + k^\mu f_{\nu\mu} - k_\nu f^\alpha_\alpha - \frac{1}{2}k_\nu f^\alpha_\alpha - \frac{1}{2}k_\nu f^\alpha_\alpha + \frac{1}{2}k_\nu 4 f^\alpha_\alpha = 2k^\mu f_{\mu\nu} = 0
\end{aligned}
$$

The convergence constraint of Eq.(6.21), requires $f_{\mu\nu}$ to have a singularity that will cancel against $\frac{p_\mu p_\nu}{k\cdot p}$, so we might choose

$$
f_{\mu\nu}(k,p) = \Big(\frac{p_\mu p_\nu}{k\cdot p} + c_{\mu\nu}\Big)\lambda(k,p) \tag{6.23}
$$

for some smoothing function $\lambda(k,p)$, with $\lambda = 1$ for small k. The functions $c_{\mu\nu}$ are chosen to obey three constraints

$$
k^\mu c_{\mu\nu} = -p_\nu \tag{6.24}
$$

$$
c_{\mu\nu}c^{\mu\nu} - \frac{1}{2}(c^\mu_\mu)^2 = 0 \tag{6.25}
$$

$$
c^{\mu\nu}\epsilon^\lambda_{\mu\nu} = 0 \tag{6.26}
$$

The first constraint comes by taking a contraction of Eq.(6.23) with k^μ, in combination with the requirement that $k^\mu f_{\mu\nu} = 0$. The second constraint is necessary to avoid additional singular terms, while, the last one, helps us express $f_{\mu\nu}$ as a linear combination of graviton polarization tensors, $\epsilon^\lambda_{\mu\nu}(\mathbf{k})$, which subsequently allows us to write a one-photon asymptotic state $|\Psi\rangle = e^{\tilde{R}_{gp}} b^\dagger_\sigma(\mathbf{p})|0\rangle$ in the Chung form [7]. This can be accomplished by making use of the residual freedom[28]

$$
f_{\mu\nu} \to f'_{\mu\nu} = f_{\mu\nu} + k_\nu v_\mu + k_\mu v_\nu - (k\cdot v)\eta_{\mu\nu} \tag{6.27}
$$

This residual freedom allows us to choose $f'_{\mu\nu}$ to be traceless and purely spatial and write it as a linear combination of graviton polarization tensors

$$
f'_{\mu\nu}(k,p) = F^{(1)}(k,p)\epsilon^{(1)}_{\mu\nu}(\vec{k}) + F^{(2)}(k,p)\epsilon^{(2)}_{\mu\nu}(\vec{k}) \tag{6.28}
$$

from which we extract

[28]Even if we used a gauge fixing term in Eq.(5.9), this does not completely fix the gauge. In fact, it only cancels longitudinal degrees of freedom. There is still a remnant symmetry to be exploited.

$$F^n = \frac{p^\mu p^\nu}{p \cdot k} \epsilon^n_{\mu\nu} \lambda(k,p) \tag{6.29}$$

by normalizing the polarization tensors as $\epsilon^n_{\mu\nu}\epsilon^{n\mu\nu} = 1$ and using that $k^\mu \epsilon_{\mu\nu} = c^{\mu\nu}\epsilon_{\mu\nu} = \eta^{\mu\nu}\epsilon_{\mu\nu}$. The asymptotic state for a photon is

$$|\Psi\rangle = e^{\tilde{R}'_{gp}} b^\dagger_\sigma(\mathbf{p})|0\rangle = \exp\Big\{\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{k}}{2\sqrt{k_0}} \big(f'_{\mu\nu}(k,p) a^{\dagger\mu\nu}(\mathbf{k}) - h.c\big)\Big\} b^\dagger_\sigma(\mathbf{p})|0\rangle \tag{6.30}$$

Using the BCH formula to normal order, we can write Eq.(6.30) as

$$\begin{aligned}
|\Psi\rangle &= \exp\Big\{-\frac{1}{2(2\pi)^3} \int \frac{d^3\mathbf{k} d^3\mathbf{l}}{2\sqrt{2k_0}2\sqrt{2l_0}} f'^{\mu\nu}(k,p) f^{\star\prime\rho\sigma}[a_{\rho\sigma}(\mathbf{l}), a^\dagger_{\mu\nu}(\mathbf{k})]\Big\} \\
&\quad \times \exp\Big\{\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{k}}{2\sqrt{2k_0}} f'_{\mu\nu}(k,p) a^{\dagger\mu\nu}(\mathbf{k})\Big\} \\
&\quad \times \exp\Big\{-\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{k}}{2\sqrt{2k_0}} f'^{\star}_{\mu\nu} a^{\mu\nu}(\mathbf{k})\Big\} b_\sigma(\mathbf{p})|0\rangle \\
&= \exp\Big\{-\frac{1}{2(2\pi)^3} \int \frac{d^3\mathbf{k}}{8k_0} \big(2|f_{\mu\nu}|^2 - |f^\mu_\mu|^2\big)\Big\} \\
&\quad \times \exp\Big\{\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{k}}{2\sqrt{2k_0}} f'_{\mu\nu}(k,p) a^{\dagger\mu\nu}(\mathbf{k})\Big\} b_\sigma(\mathbf{p})|0\rangle \\
&= \exp\Big\{-\frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{8k_0} \sum_n |F^n|^2\Big\} \\
&\quad \times \exp\Big\{\frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3\mathbf{k}}{2\sqrt{2k_0}} \sum_n F^n \epsilon^n_{\mu\nu} a^{\dagger\mu\nu}(\mathbf{k})\Big\} b^\dagger_\sigma(\mathbf{p})|0\rangle
\end{aligned} \tag{6.31}$$

where in the second equality we have omitted the third term, since after expanding the exponential, terms beyond zeroth order will give graviton annihilation operators that commute with $b_\sigma(\mathbf{p})$ and annihilate the vacuum. For the first term we simply expanded the commutator $[a_{\rho\sigma}(\mathbf{l}), a^\dagger_{\mu\nu}(\mathbf{k})]$. In the last equality, we used Eq.(6.28) and that

$$2|f'_{\mu\nu}|^2 - |f'^\mu_\mu|^2 = 2\sum_{n,n'} F^n F^{n'} 2\epsilon^n_{\mu\nu}\epsilon^{\mu\nu}_{n'} = 2\sum_{n,n'} F^n F^{n'} \delta_{n,n'} = 2\sum_n |F^n|^2 \tag{6.32}$$

It is convenient to define

$$S^n_i = \frac{1}{2\sqrt{(2\pi)^3 2k_0}} F^n(p_i) \tag{6.33}$$

such that the initial and final states can be written, to lowest order, as

$$|i\rangle = \Big(1 - \int d^3\mathbf{k} \sum_n |S_i^n|^2\Big)\Big(1 + \int d^3\mathbf{k} \sum_n S_i^n \epsilon^n_{\mu\nu} a^{\dagger\mu\nu}(\mathbf{k})\Big) b^\dagger_\sigma(\mathbf{p}_i)|0\rangle \tag{6.34a}$$

$$\langle f| = \langle 0|b_\sigma(\mathbf{p}_f)\Big(1 - \int d^3\mathbf{k} \sum_n |S_f^n|^2\Big)\Big(1 + \int d^3\mathbf{k} \sum_n S_f^n \epsilon^{\star n}_{\mu\nu} a^{\mu\nu}(\mathbf{k})\Big) \tag{6.34b}$$

6.1.2 Cancellation of the infrared divergences

After the construction of the dressed asymptotic states, we wish to demonstrate how they can be used to cancel the IR divergences associated to the soft part of the virtual graviton corrections to the tree level diagram of Figure 6.1a. We will work to one loop order corrections, that is to order k^2 on vertex. The corresponding contributions are displayed in Figure 6.2.

(a) Tree-level diagram

(b) Cloud-to-cloud contribution

FIGURE 6.1: Photon scattering in a gravitational potential.

As discussed in Chapter 4, one virtual soft graviton contributes a factor

$$\frac{1}{2}\int d^4k B(k)$$

with $B(k)$ given by

$$B(k) = \frac{1}{2(2\pi)^4} \sum_{nm} n_n n_m (p_n \cdot p_m)^2 J_{nm}(k)$$

The IR divergences are carried by J_{nm}, which as we have already seen is complex. The imaginary part contributes to phase divergences given by Eq.(4.28). For the real part we have

$$\Re \int d^4k J_{nm}(k) = -\pi \int \frac{d^3\mathbf{k}}{k_0} \frac{1}{(p_n \cdot k)} \frac{1}{(p_m \cdot k)} \tag{6.35}$$

Thus, the total contribution from the real part is

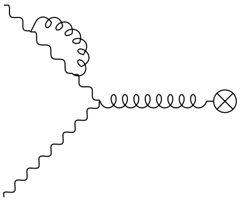
$$\frac{1}{2}\int d^4k[\Re B(k)] = -\frac{1}{2(2\pi)^4}\sum_{nm}\int \frac{d^3\mathbf{k}}{k_0}\frac{\eta_n\eta_m(p_n\cdot p_m)^2}{(p_n\cdot k)(p_m\cdot k)} \tag{6.36}$$

For one incoming particle with momentum p_1 and one outgoing particle with momentum p_2, this gives[29]

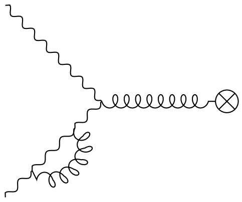
$$-\frac{1}{64\pi^3}\int \frac{d^3\mathbf{k}}{k_0}\Big(\frac{p_1^4}{(p_1\cdot k)^2}+\frac{p_2^4}{(p_2\cdot k)^2}-\frac{2(p_1\cdot p_2)^2}{(p_1\cdot k)(p_2\cdot k)}\Big) \tag{6.37}$$

Notice however, that the first two terms in Eq.(6.37) vanish on-shell, so that we are left with

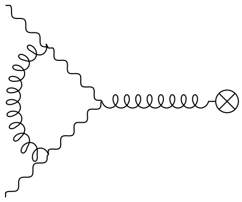
$$M_{virtual} = \frac{1}{32\pi^3}\int \frac{d^3\mathbf{k}}{k_0}\frac{(p_1\cdot p_2)^2}{(p_1\cdot k)(p_2\cdot k)} \tag{6.38}$$



(a) Virtual graviton attached to the outgoing leg.



(b) Virtual graviton attached to the incoming leg.



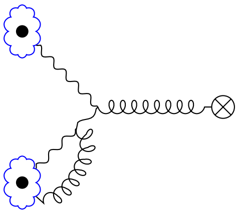
(c) Virtual graviton connecting the incoming and outgoing legs.

FIGURE 6.2: Virtual soft graviton contributions up to order κ^2

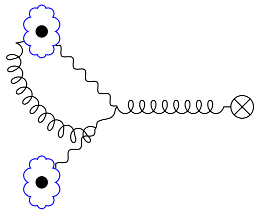
The contribution of Eq.(6.38) is precisely what we want to cancel out. Next, we explore how the dressing of states as in Eq.(6.34a) and Eq.(6.34b) leads to this. Consider first tree-level diagrams with one real soft graviton attached in either the initial or the final state. These diagrams are displayed in Figure 6.3. Using the results of the previous subsection, we can compute the factors corresponding to each of these diagrams. Consider for example, the diagram 6.3d, in which the internal photon propagator carries momentum $p_f - k$. This is to be thought of as the diagram evaluated in Chapter 4, with the difference that the open edge of the soft graviton emitted is now attached to the soft graviton cloud of the asymptotic state, thus providing another vertex κ. Let us denote with $M_{\rho\sigma}$ the amplitude computed in Chapter 4, where the free indices ρ and σ belong

[29]Recall that $\eta = -1(+1)$ for an incoming (outgoing) leg.

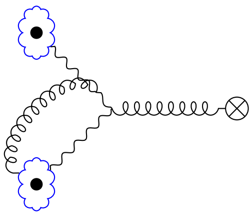
to the open edge of the soft graviton. This graviton is now to be attached to the soft cloud. We need to propagate its open edge to a point in the cloud, carrying indices μ and ν.



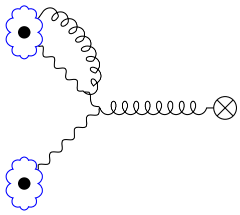
(a) Incoming leg's soft graviton, attached to the the initial state's cloud.



(b) Incoming leg's soft graviton, attached to the the final state's cloud.



(c) Outgoing leg's soft graviton, attached to the the initial state's cloud.



(d) Outgoing leg's soft graviton, attached to the the final state's cloud.

FIGURE 6.3: Real soft graviton contributions at order κ^2.

The diagram is evaluated as

$$\sum_n \int d^3\mathbf{k} S_f^n \frac{\epsilon_{\mu\nu}^n}{\sqrt{(2\pi)^3 2k_0}} I^{\mu\nu\rho\sigma} M_{\rho\sigma}(p,k) \tag{6.39}$$

In the limit $k \to 0$,we have $M^{\rho\sigma} = \frac{p_f^\rho p_f^\sigma}{p_f \cdot k}$. Contracting this with $I^{\mu\nu\rho\sigma}$ will provide a factor of 2, such that in total we obtain

$$2\sum_n \int d^3\mathbf{k} S_f^n \frac{\epsilon_{\mu\nu}^n}{\sqrt{(2\pi)^3 2k_0}} \frac{p_f^\mu p_f^\nu}{p_f \cdot k} = 2\sum_n \int d^3\mathbf{k} S_f^n S_f^n \tag{6.40}$$

where the equality is obtained by virtue of Eq.(6.33) and Eq.(6.29), with the smooth function $\lambda(k,p) \to 0$. The contributions coming from the rest of the diagrams of Figure 6.3 can be calculated in a similar way. Consider for example the diagram of Figure 6.3a, in which the soft graviton is attached to the incoming line. This is the analogous of Figure 4.1b. The minus sign is already included in the definition of S_i^n, so since it will appear twice, this contribution is overall positive and equal to

$$2\sum_n \int d^3\mathbf{k} S_i^n S_i^n \tag{6.41}$$

The contribution of the diagrams 6.3b and 6.3c connects the soft graviton of the incoming state to the soft cloud of the outgoing state and vise versa. These are equal to one another and will come with an overall minus sign, since S_i appears only once, in each case. We obtain

$$-4\sum_n \int d^3\mathbf{k} S_i^n S_f^n \tag{6.42}$$

Summing up the above contributions gives in total[30]

$$M_{soft} = -\frac{1}{16\pi^3}\int \frac{d^3\mathbf{k}}{k_0}\frac{(p_f\cdot p_i)^2}{(p_f\cdot k)(p_i\cdot k)} \tag{6.43}$$

Notice that the divergences do not cancel yet. We need to take into account two more contributions. The first one comes from the tree diagram, with no external gravitons, but with the κ^2-order term in the normalization of either the initial or the final state. This is identified with the second term in the first parenthesis in Eq.(6.34a) and Eq.(6.34b). It gives

$$M_{norm} = -\sum_n \int d^3\mathbf{k}(|S_i^n|^2 + |S_f^n|^2) \tag{6.44}$$

The other contribution at order κ^2 comes from the cloud-to-cloud propagating graviton of Figure 6.1b. This diagram contributes[31]

$$M_{cloud\text{-}cloud} = 2\sum_n \int d^3\mathbf{k} S_i^n S_f^n \tag{6.45}$$

These two extra contributions give in total

$$M' = -\sum_n \int d^3\mathbf{k}(S_f - S_i)^2 = \frac{1}{32\pi^3}\int \frac{d^3\mathbf{k}}{k_0}\frac{(p_f\cdot p_i)^2}{(p_f\cdot k)(p_i\cdot k)} \tag{6.46}$$

Putting all contributions together, we find that

$$M_{virtual} + M_{soft} + M' = 0 \tag{6.47}$$

that is, the IR divergences are canceled out.

[30] The calculation can be found in Section A.7

[31] The calculation for the cloud-to-cloud contribution and M' can be found in Section A.8

6.2 Phase divergent factor for the Einstein-Maxwell system

The phase divergent factor associated with the asymptotic potential $V_{gp}(t)$ can be found by making use of Eq.(5.47b). In Appendix A.5, we find that

$$\begin{aligned} Q_{gp} &= \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{q}}{2p_0 2q_0 2k_0} p^\mu p^\nu q^\alpha q^\beta 2i\sin\Big(k\cdot(\frac{p}{p^0}t_1 - \frac{q}{q^0}t_2)\Big) \\ &\qquad \times (\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\beta\mu}\eta_{\alpha\nu} - \eta_{\mu\nu}\eta_{\alpha\beta})\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p}) \\ &= \frac{i}{16\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{q}}{k_0 p_0 q_0}(p\cdot q)^2\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p})\sin\Big(k\cdot(\frac{p}{p^0}t_1 - \frac{q}{q^0}t_2)\Big) \end{aligned} \tag{6.48}$$

Using Eq.(D.3) to perform the k-integral, we obtain a term proportional to $\frac{1}{p\cdot q}$. Integrating over t_2, the asymptotic phase operator becomes

$$\Phi_{gp}(t) = -\frac{1}{16\pi^3}\int d^3\mathbf{p}d^3\mathbf{q}(p\cdot q) : \rho^{ph}(\mathbf{p})\rho^{ph}(\mathbf{q}) : \int^t \frac{dt_1}{|t_1|} \tag{6.49}$$

The asymptotic phase operator involves the integral

$$I(t) = \int^t \frac{dt_1}{|t_1|} \tag{6.50}$$

which diverges as $t \to \infty$. By making use of the relation [54]

$$\int d^m t_1 f(t_1) = \frac{2\pi^{\frac{1}{2}m}}{\Gamma\big(\frac{1}{2}m\big)}\int dr f(r) r^{m-1} \tag{6.51}$$

for $f(t_1) = \frac{1}{|t_1|}$, we can rewrite the integral[32] as

$$I(t\to\infty) = \frac{2\pi^{\frac{1}{2}(n-3)}}{\Gamma\big(\frac{1}{2}(n-3)\big)}\int dr r^{n-5} \tag{6.52}$$

In this case, to regulate the divergence at the upper limit, we should take $n < 4$, which gives:

$$\frac{2\pi^{\frac{n-3}{2}}}{\Gamma(\frac{n-3}{2})}\frac{1}{n-4} = \frac{2}{4-n} + \textit{IR finite} \tag{6.53}$$

[32] Note that $m = n - 3$ in our case

Then, as $t \to \infty$, Eq.(6.49) gives

$$\Phi_{gp}(\infty) = -\frac{1}{8\pi} \int d^3\mathbf{p} d^3\mathbf{q} (p \cdot q) : \rho^{ph}(\mathbf{p}) \rho^{ph}(\mathbf{q}) : \frac{1}{\epsilon}$$

where $\epsilon = 4 - n > 0$. This will produce, for each pair of particles (m, n) in the state it acts on, the phase factor

$$-\frac{i}{8\pi}(p_m \cdot p_n)\frac{1}{\epsilon} = -\phi^{gp}_{mn} \tag{6.54}$$

which exactly cancels out the phase divergent factor of Eq.(4.28) calculated in Chapter 4.

Chapter 7

Extension to the EMS model?

In Chapter 6, we have shown that the FK method also works for the Einstein-Maxwell theory. This means that the EMS model can reproduce three independent theories with finite S-matrix elements. It is then natural to ask, if the total asymptotic potential of Eq.(5.19) can also be used to obtain finite S-matrix elements for more general processes involving external scalar and photon legs interacting with both gravitational and electromagnetic potentials. We try to answer this question in section 7.1. Only the results of the calculations are present. We find some subtleties when trying to write the asymptotic operator in the form of Eq.(5.46). However, as we show in section 7.2, these subtleties are not present if one considers only scalar external legs.

7.1 The EMS model for photon and scalar external legs

The asymptotic potential for the EMS model is

$$
\begin{aligned}
V^I_{as,tot} = &-\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{r}}{2r_0\sqrt{2k_0}} r^\mu r^\nu \big(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{k^0}t} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{k^0}t}\big)\rho_b(\mathbf{r}) \\
&-\frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}} p^\mu p^\nu \big(a_{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^\dagger_{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t}\big)\rho^{ph}(\mathbf{p}) \\
&-\frac{2e}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{p}d^3\mathbf{r}}{2r_0\sqrt{2k_0}} r^\mu \big(b_\mu(\mathbf{p})e^{-i\frac{r\cdot p}{r^0}t} + b^\dagger_\mu(\mathbf{p})e^{-i\frac{r\cdot p}{r^0}t}\big)\rho_c(\mathbf{r})
\end{aligned} \tag{7.1}
$$

All of the individual theories possessed the important property that $V^I_{as,i}(t)$ commutes with its own commutator $Q_i(t_1,t_2)$ for all times t, t_1, t_2. Because of this property, we were able to write the solution Z(t) in the form of Eq.(5.45) and then distinguish the divergences associated with the radiation operator $e^{R(t)}$ and the phase operator $e^{i\Phi(t)}$.

It is then natural to start our investigation by asking if the general EMS model also satisfies this property. The operator $Q(t_1, t_2)$ now reads

$$\begin{aligned} Q(t_1,t_2) &= [V^I_{as,tot}(t_1), V^I_{as,tot}(t_2)] = [V^I_{as,gs}(t_1), V^I_{as,gs}(t_2)] + [V^I_{as,gs}(t_1), V^I_{as,ps}(t_2)] \\ &+ [V^I_{as,gs}(t_1), V^I_{as,gp}(t_2)] + [V^I_{as,ps}(t_1), V^I_{as,gs}(t_2)] + [V^I_{as,ps}(t_1), V^I_{as,ps}(t_2)] \\ &+ [V^I_{as,ps}(t_1), V^I_{as,gp}(t_2)] + [V^I_{as,gp}(t_1), V^I_{as,gs}(t_2)] + [V^I_{as,gp}(t_1), V^I_{as,ps}(t_2)] \\ &+ [V^I_{as,gp}(t_1), V^I_{as,gp}(t_2)] \end{aligned} \tag{7.2}$$

We aim to calculate explicitly the various commutators in Eq.(7.2). The results for the graviton-scalar and photon-scalar systems are already known [6, 8]:

$$Q_{gs,gs}(t_1,t_2) = \frac{i}{32\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{r}d^3\mathbf{s}}{k_0 r_0 s_0}\left(2(r\cdot s)^2 - m^4\right)\sin\left(k\cdot(\frac{r}{r_0}t_2 - \frac{s}{s_0}t_1)\right)\rho_b(\mathbf{s})\rho_b(\mathbf{r}) \tag{7.3}$$

$$Q_{ps,ps}(t_1,t_2) = \frac{ie^2}{8\pi^3}\int \frac{d^3\mathbf{p}d^3\mathbf{r}d^3\mathbf{s}}{p_0 r_0 s_0}(r\cdot s)\sin\left(p\cdot(\frac{r}{r_0}t_2 - \frac{s}{s_0}t_1)\right)\rho_c(\mathbf{r})\rho_c(\mathbf{s}) \tag{7.4}$$

For the graviton-photon system we have found that[33]

$$Q_{gp,gp}(t_1,t_2) = \frac{i}{16\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{q}}{k_0 p_0 q_0}(p\cdot q)^2\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p})\sin\left(k\cdot(\frac{p}{p^0}t_1 - \frac{q}{q^0}t_2)\right) \tag{7.5}$$

Moreover, as we show in Appendix A.4

$$Q_{gs,ps}(t_1,t_2) = [V^I_{as,gs}(t_1), V^I_{as,gp}(t_2)] = 0 \tag{7.6}$$

Similar calculations give us[34]

$$[V^I_{as,gs}(t_1), V^I_{as,gp}(t_2)] = \frac{i}{16\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{r}}{k_0 r_0 p_0}(r\cdot p)^2\sin\left(k\cdot(\frac{r}{r_0}t_1 - \frac{p}{p_0}t_2)\right)\rho^{ph}(\mathbf{p})\rho_b(\mathbf{r})$$

[33] Notice that we have replaced the operator Q_{gp} computed in Section 6.2 by $Q_{gp,gp}$. There is nothing mysterious with that, we simply do it to distinguish of which interaction potentials are involved.

[34] Only the final results are presented from now on since the calculations become lengthy enough.

Since

$$[V^I_{as,gp}(t_1), V^I_{as,gs}(t_2)] = -[V^I_{as,gs}(t_2), V^I_{as,gp}(t_1)] \tag{7.7}$$

we may write

$$\begin{aligned} Q_{gs,gp}(t_1,t_2) = -\frac{i}{16\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{r}}{k_0 r_0 p_0}\bigg(&\sin\Big(k\cdot(\frac{r}{r_0}t_2 - \frac{p}{p_0}t_1)\Big) - \sin\Big(k\cdot(\frac{r}{r_0}t_1 - \frac{p}{p_0}t_2)\Big)\bigg) \\ &\times (r\cdot p)^2\rho^{ph}(\mathbf{p})\rho_b(\mathbf{r}) \end{aligned} \tag{7.8}$$

where we have defined

$$Q_{gs,gp}(t_1,t_2) = [V^I_{as,gs}(t_1), V^I_{as,gp}(t_2)] - [V^I_{as,gs}(t_2), V^I_{as,gp}(t_1)] \tag{7.9}$$

Finally

$$\begin{aligned} [V^I_{as,gp}(t_1), V^I_{as,ps}(t_2)] = \frac{e}{32\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{r}}{p_0 r_0\sqrt{p^0k^0}}p^\mu p^\nu r^\sigma\Big(&-e^{-i\frac{p}{p^0}\cdot(kt_1+rt_2)}a_{\mu\nu}(\mathbf{k})b_\sigma(\mathbf{p}) \\ &+e^{-i\frac{p}{p^0}\cdot(kt_1-rt_2)}a_{\mu\nu}(\mathbf{k})b^\dagger_\sigma(\mathbf{p}) - e^{i\frac{p}{p^0}\cdot(kt_1-rt_2)}a^\dagger_{\mu\nu}(\mathbf{k})b_\sigma(\mathbf{p}) \\ &+e^{i\frac{p}{p^0}\cdot(kt_1+rt_2)}a^\dagger_{\mu\nu}(\mathbf{k})b^\dagger_\sigma(\mathbf{p})\Big)\rho_c(\mathbf{r}) \end{aligned}$$

Again, using that $[V^I_{as,ps}(t_1), V^I_{as,gp}(t_2)] = -[V^I_{as,gp}(t_2), V^I_{as,ps}(t_1)]$ and defining

$$Q_{ps,gp}(t_1,t_2) = [V^I_{as,gp}(t_1), V^I_{as,ps}(t_1)] - [V^I_{as,gp}(t_2), V^I_{as,ps}(t_1)] \tag{7.10}$$

we obtain

$$\begin{aligned} Q_{ps,gp}(t_1,t_2) = -\frac{e}{32\pi^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{r}}{r_0 p_0\sqrt{p_0k_0}}p^\mu p^\nu r^\sigma\bigg(&\Big(e^{i\frac{p}{p^0}\cdot(kt_1+rt_2)} - e^{i\frac{p}{p^0}\cdot(kt_2+rt_1)}\Big)a_{\mu\nu}(\mathbf{k})b_\sigma(\mathbf{p}) \\ &+\Big(e^{i\frac{p}{p^0}\cdot(kt_2-rt_1)} - e^{i\frac{p}{p^0}\cdot(kt_1-rt_2)}\Big)a_{\mu\nu}(\mathbf{k})b^\dagger_\sigma(\mathbf{p}) \\ &+\Big(e^{-i\frac{p}{p^0}\cdot(kt_1-rt_2)} - e^{-i\frac{p}{p^0}\cdot(kt_2-rt_1)}\Big)a^\dagger_{\mu\nu}(\mathbf{k})b_\sigma(\mathbf{p}) \\ &+\Big(e^{-i\frac{p}{p^0}\cdot(kt_2+rt_1)} - e^{-i\frac{p}{p^0}\cdot(kt_1+rt_2)}\Big)a^\dagger_{\mu\nu}(\mathbf{k})b^\dagger_\sigma(\mathbf{p})\bigg)\rho_c(\mathbf{r}) \end{aligned} \tag{7.11}$$

We have managed so far to write

$$Q(t_1,t_2) = Q_{gs,gs}(t_1,t_2) + Q_{ps,ps}(t_1,t_2) + Q_{gp,gp}(t_1,t_2) + Q_{gs,gp}(t_1,t_2) + Q_{ps,gp}(t_1,t_2) \tag{7.12}$$

We observe that since the asymptotic potential V_{gp} does not commute with V_{gs} and V_{ps}, two more terms will contribute to the phase operator. Since now the various contributions to $Q(t_1,t_2)$ are known, it is possible to check whether $[V^I_{as}(t), Q(t_1,t_2)] = 0$. We may write

$$[V^I_{as}(t), Q(t_1,t_2)] = [V^I_{as,gs}(t), Q(t_1,t_2)] + [V^I_{as,ps}(t), Q(t_1,t_2)] + [V^I_{as,gp}(t), Q(t_1,t_2)] \tag{7.13}$$

and check each term separately. Starting with $[V^I_{as,gs}(t), Q(t_1,t_2)]$, we find that

$$[V^I_{as,gs}(t), Q_{gs,gs}(t_1,t_2)] = 0 \tag{7.14}$$

$$[V^I_{as,gs}(t), Q_{ps,ps}(t_1,t_2)] = 0 \tag{7.15}$$

$$[V^I_{as,gs}(t), Q_{gp,gp}(t_1,t_2)] = 0 \tag{7.16}$$

$$[V^I_{as,gs}(t), Q_{gs,gp}(t_1,t_2)] = 0 \tag{7.17}$$

while there is a non-vanishing contribution

$$\begin{aligned}[V^I_{as,gs}(t), Q_{ps,gp}(t_1,t_2)] = -\frac{e}{4(2\pi)^{\frac{9}{2}}} \int \frac{d^3\mathbf{k} d^3\mathbf{p} d^3\mathbf{r} d^3\mathbf{s}}{r_0 k_0 s_0 p^0 \sqrt{2p_0}} (r\cdot p)^2 s^\sigma \Big((a_1(t_1,t_2) b_\sigma(\mathbf{p}) \\ + a_2(t_1,t_2) b^\dagger_\sigma(\mathbf{p})) e^{ik^0 t} - \big(a_3(t_1,t_2) b_\sigma(\mathbf{p}) + a_4(t_1,t_2) b^\dagger_\sigma(\mathbf{p})\big) e^{-ik^0 t}\Big) \rho_c(\mathbf{s}) \rho_b(\mathbf{r})\end{aligned} \tag{7.18}$$

In Eq.(7.18) we have for the sake of simplicity defined

$$\begin{aligned} a_1(t_1,t_2) &= e^{i\frac{p}{p^0}\cdot(kt_1+rt_2)} - e^{i\frac{p}{p^0}\cdot(kt_2+rt_1)} \\ a_2(t_1,t_2) &= e^{i\frac{p}{p^0}\cdot(kt_2-rt_1)} - e^{i\frac{p}{p^0}\cdot(kt_1-rt_2)} \\ a_3(t_1,t_2) &= e^{-i\frac{p}{p^0}\cdot(kt_1-rt_2)} - e^{-i\frac{p}{p^0}\cdot(kt_2-rt_1)} \\ a_4(t_1,t_2) &= e^{-i\frac{p}{p^0}\cdot(kt_2+rt_1)} - e^{-i\frac{p}{p^0}\cdot(kt_1+rt_2)} \end{aligned}$$

Repeating the computations for $[V^I_{as,ps}(t), Q(t_1,t_2)]$, we find

$$[V^I_{as,ps}(t), Q_{gs,gs}(t_1,t_2)] = 0 \tag{7.19}$$

$$[V^I_{as,ps}(t), Q_{ps,ps}(t_1,t_2)] = 0 \tag{7.20}$$

However

$$[V^I_{as,ps}(t), Q_{gp,gp}(t_1,t_2)] = -\frac{ie}{4(2\pi)^{\frac{9}{2}}}\int \frac{d^3\mathbf{p}d^3\mathbf{k}d^3\mathbf{r}d^3\mathbf{q}}{r_0k_0q_0p_0\sqrt{2p^0}}(p\cdot k)^2\Bigg(\sin\Big(k\cdot(\frac{q}{q^0}t_2-\frac{p}{p^0}t_1)\Big) + \sin\Big(k\cdot(\frac{p}{p^0}t_2-\frac{s}{s^0}t_1)\Big)\Bigg)r^\mu\Big(e^{i\frac{p\cdot r}{r^0}t}b_\mu(\mathbf{p})-e^{-i\frac{p\cdot r}{r^0}t}b^\dagger_\mu(\mathbf{p})\Big)\rho^{ph}(\mathbf{q})\rho_c(\mathbf{r}) \tag{7.21}$$

while

$$[V^I_{as,ps}(t), Q_{gs,gp}(t_1,t_2)] = \frac{ie}{2(2\pi)^{\frac{9}{2}}}\int \frac{d^3\mathbf{p}d^3\mathbf{k}d^3\mathbf{r}d^3\mathbf{s}}{k_0p_0r_0s_0\sqrt{2p_0}}(p\cdot s)\Bigg(\sin\Big(k\cdot(\frac{s}{s^0}t_2-\frac{p}{p^0}t_1)\Big) - \sin\Big(k\cdot(\frac{s}{s^0}t_1-\frac{p}{p^0}t_2)\Big)\Bigg)r^\mu\Big(b_\mu(\mathbf{p})e^{i\frac{p\cdot r}{r^0}t}-b^\dagger_\mu(\mathbf{p})e^{-i\frac{p\cdot r}{r^0}t}\Big)\rho_b(\mathbf{s})\rho_c(\mathbf{r}) \tag{7.22}$$

and

$$[V^I_{as,ps}(t), Q_{ps,gp}(t_1,t_2)] = -\frac{e^2}{4(2\pi)^{\frac{9}{2}}}\int \frac{d^3\mathbf{p}d^3\mathbf{k}d^3\mathbf{r}d^3\mathbf{s}}{r_0p_0s_0\sqrt{2k_0}}p^\mu p^\nu(s\cdot r)\Big(e^{ip^0t}(a_1(t_1,t_2)a_{\mu\nu}(\mathbf{k}) + a_3(t_1,t_2)a^\dagger_{\mu\nu}(\mathbf{k})) - \big(a_2(t_1,t_2)a_{\mu\nu}(\mathbf{k})+a_4(t_1,t_2)a^\dagger_{\mu\nu}(\mathbf{k})\big)e^{-ip^0t}\Big)\rho_c(\mathbf{s})\rho_c(\mathbf{r}) \tag{7.23}$$

Finally, for $[V^I_{as,gp}(t), Q(t_1,t_2)]$, we find

$$[V^I_{as,gp}(t), Q_{gs,gs}(t_1,t_2)] = 0 \tag{7.24}$$

$$[V^I_{as,gp}(t), Q_{ps,ps}(t_1,t_2)] = 0 \tag{7.25}$$

$$[V^I_{as,gp}(t), Q_{gp,gp}(t_1,t_2)] = 0 \tag{7.26}$$

$$[V^I_{as,gp}(t), Q_{gs,gp}(t_1,t_2)] = 0 \tag{7.27}$$

while

$$
\begin{aligned}
[V^I_{as,gp}(t), Q_{ps,gp}(t_1,t_2)] = &-\frac{e^2}{8(2\pi)^{\frac{9}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{l}d^3\mathbf{r}}{p_0^2 r_0\sqrt{2p_0 l_0 k_0}} p^\mu p^\nu p^\alpha p^\beta r^\sigma\Big(e^{i\frac{k\cdot p}{p^0}t}a_{\mu\nu}(\mathbf{k}) \\
&+ e^{-i\frac{k\cdot p}{p^0}t}a^\dagger_{\mu\nu}(\mathbf{k})\Big)\Big(\big(a_1 b_\sigma(\mathbf{p}) - a_2 b^\dagger_\sigma(\mathbf{p})\big)a_{\alpha\beta}(\mathbf{l}) + \big(a_3 b_\sigma(\mathbf{p}) + a_4 b^\dagger_\sigma(\mathbf{p})\big)a^\dagger_{\alpha\beta}(\mathbf{l})\Big)\rho_c(\mathbf{r}) \\
&- \frac{e^2}{4(2\pi)^{\frac{9}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{r}d^3\mathbf{q}}{p_0 q_0 k_0 r_0\sqrt{2q_0}}(p\cdot q)^2\Big(\big(a_2 b^\dagger_\sigma(\mathbf{q}) + a_1 b_\sigma(\mathbf{q})\big)e^{-i\frac{k\cdot p}{p^0}t} \\
&- \big(a_3 b_\sigma(\mathbf{q}) + a_4 b^\dagger_\sigma(\mathbf{q})\big)e^{i\frac{k\cdot p}{p^0}t}\Big)\rho_c(\mathbf{r})\rho^{ph}(\mathbf{p})
\end{aligned}
\tag{7.28}
$$

We therefore conclude that for the EMS model, $[V^I_{as}(t), Q(t_1,t_2)] \neq 0$, since there are 5 non-vanishing contributions, which also do not cancel each other. Therefore, we cannot write the solution in the form of Eq.(5.45), but instead we will have

$$
\begin{aligned}
Z(t) = exp\Big\{ &- i\int^t dt_1 V^I_{as}(t_1) - \frac{1}{2}\int^t dt_1\int^{t_1} dt_2 Q(t_1,t_2) \\
&- i\frac{1}{6}\int^t dt_1\int^{t_1} dt_2\int^{t_2} dt_3 W(t_1,t_2,t_3) + \dots\Big\}
\end{aligned}
\tag{7.29}
$$

where we have defined $W(t_1,t_2,t_3) = [V^I_{as}(t_1), Q(t_2,t_3)]$. It is not clear whether these terms include divergences or even, if they do, that these divergences can be canceled. Even worse, it is not even clear to what order in the Dyson series such terms disappear, since the computations become pretty cumbersome to be carried even for the next order.

7.2 The EMS model for scalar external legs

We found that in the presence of both scalar and photon external legs, we have more, possibly divergent, terms in the Dyson series expansion than those we had in the individual cases. The origin of these subtleties is unclear, so it would not make sense to continue this study without first having an answer of what they mean. Nevertheless, we might be able to address the same question by considering only scalar external legs. This amounts to setting $\rho^{ph}(\mathbf{p}) = 0$ in Eq.(5.19). The asymptotic potential then is

$$
V^I_{as,tot}(t) = V^I_{as,gs}(t) + V^I_{as,ps}(t) \tag{7.30}
$$

Because of Eq.(7.6), we have that

$$\begin{aligned} Q(t_1,t_2) &= [V_{as,gs}(t_1) + V_{as,ps}(t_1), V_{as,gs}(t_2) + V_{as,ps}(t_2)] \\ &= [V_{as,gs}(t_1), V_{as,gs}(t_2)] + [V_{as,ps}(t_1), V_{as,ps}(t_2)] \\ &= Q_{gs,gs}(t_1,t_2) + Q_{ps,ps}(t_1,t_2) \end{aligned} \tag{7.31}$$

where we have defined

$$Q_{gs,gs}(t_1,t_2) = [V_{as,gs}(t_1), V_{as,gs}(t_2)]$$
$$Q_{ps,ps}(t_1,t_2) = [V_{as,ps}(t_1), V_{as,ps}(t_2)]$$

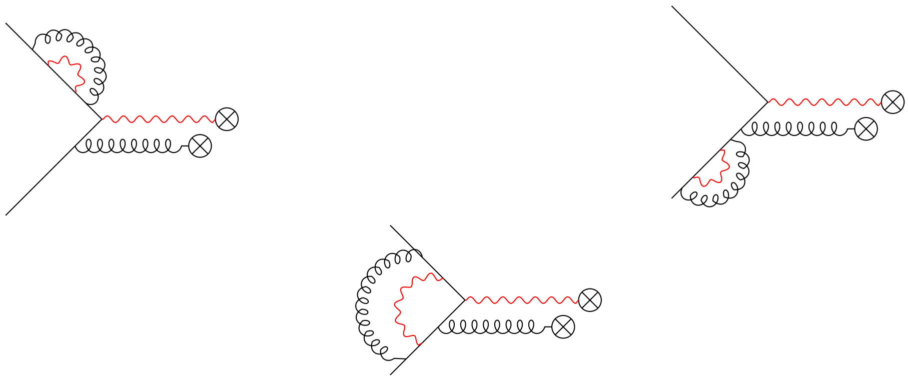


FIGURE 7.1: Soft contributions at order κ^2 (black loops) or e^2 (red loops).

The property

$$[V^I_{as}(t), Q(t_1,t_2)] = 0$$

is then straightforwardly recovered and the solution can be written as

$$Z(t) = e^{R(t)}e^{i\Phi(t)} = e^{R_{gs}(t)}e^{i\Phi_{gs}(t)}e^{R_{ps}(t)}e^{i\Phi_{ps}(t)} = Z_{gs}(t)Z_{ps}(t) \tag{7.32}$$

since all terms commute. Consider the scattering of a scalar particle first by a gravitational potential and subsequently by an electromagnetic potential. Then at order κ^2 or e^2, the virtual soft gravitons and photons that carry IR divergences, are represented in Figure 7.1 , where the virtual photon (red) and graviton (black) loops have been put together for the sake of space. We now focus on how the dressing takes place. Because of Eq.(7.32), the asymptotic scalar field is

$$\phi'(t,\vec{x}) = Z_{ps}^\dagger \underbrace{Z_{gs}^\dagger \phi(t,\mathbf{x}) Z_{gs}(t)}_{=\phi_{as}(t,\mathbf{x})} Z_{ps}(t) = Z_{ps}^\dagger(t)\phi_{as}(t,\mathbf{x})Z_{ps}(t) \tag{7.33}$$

In this sense, we dress with soft photons the already soft graviton-dressed states. Because of Eq.(7.6), we could have equivalently done the reverse. Similarly, the S-matrix element is written as

$$S'_A = Z_{ps}^\dagger \underbrace{Z_{gs}^\dagger S_D Z_{gs}(t)}_{=S_A} Z_{ps}(t) = Z_{ps}^\dagger(t) S_A Z_{ps}(t) \tag{7.34}$$

For the case of scalar-photon interactions, an analogous construction of the Chung states is known. We can conveniently define [53]

$$\tilde{F}^n = \frac{p^\mu \epsilon^n_\mu}{p \cdot q}\lambda(p,q) \tag{7.35a}$$

$$\tilde{S}^n_i \propto \tilde{F}^n(p_i) \tag{7.35b}$$

in analogy to Eq.(6.33) and Eq.(6.29). Again, $\lambda(p,q)$ represents a smoothing function, to be taken equal to 1, in the limit $p \to 0$. Since, as we have seen, the dressing occurs independently, we expect that the initial state $|i\rangle$ given by Eq.(6.34a), is now modified as

$$\begin{aligned} |i\rangle = &\Big(1 - \int d^3\mathbf{p} \sum_n |\tilde{S}^n_i|^2\Big)\Big(1 + \int d^3\mathbf{p} \sum_n \tilde{S}^n_i \epsilon^n_\mu b^{\dagger\mu}(\mathbf{p})\Big) \\ &\times \Big(1 - \int d^3\mathbf{k} \sum_n |S^n_i|^2\Big)\Big(1 + \int d^3\mathbf{k} \sum_n S^n_i \epsilon^n_{\mu\nu} a^{\dagger\mu\nu}(\mathbf{k})\Big) c^\dagger(\mathbf{q}_i)|0\rangle \end{aligned} \tag{7.36}$$

A similar modification applies to the final state $|f\rangle$.

Chapter 8

A string field theory perspective

In the previous chapters we have studied how the FK method is employed in PQG and how one arrives at finite S-matrix elements. The present chapter aims to examine these concepts in string theory. First, we review how IR divergences are realized in string theory. We then try to understand how analogous statements, based on the elements of Chapter 3 may be employed in order to carry out the procedure. We focus again only on the closed bosonic string field.

8.1 Infrared divergences in string theory

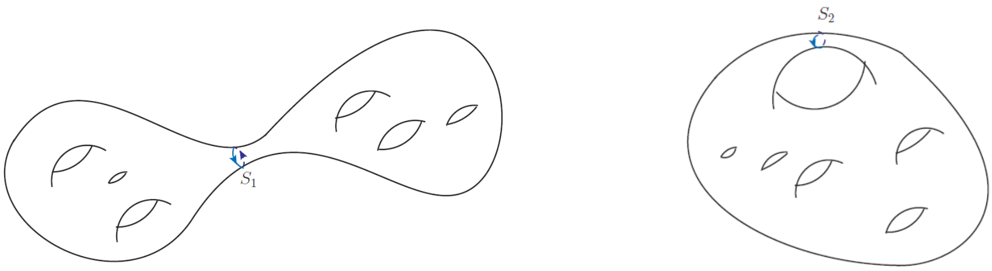
One might worry whether IR divergences are generally incorporated in string theory amplitudes. Intuitively, seeing a string as loop of non-zero size, divergences are expected to arise when long distances in space-time are involved [55]. As we have seen in Chapter 2, the worldsheet formulation of string theory provides an elegant expression for scattering amplitudes, with the perturbative expansion expressed as a sum over Riemann surfaces of all genera. The space of all Riemann surfaces is called the moduli space.

In general, divergences in QFTs arise from the integration over the Schwinger parameters. It is therefore illuminating to express the string theory amplitudes in the Schwinger parameter representation [18]. The propagators read

$$\frac{1}{p_i^2 + m_i^2} = \int_0^\infty ds_i e^{-s_i(p_i^2+m_i^2)} \tag{8.1}$$

One can distinguish two types of divergences; the ultraviolet (UV) and the Infrared (IR). The former arises from regions of integration, where loop momenta become large, while the latter comes from regions of integration where propagators have vanishing denominators. In the Schwinger parameter representation, UV divergences arise from the region where Schwinger parameters s_i vanish. In contrast, IR divergences come from regions where Schwinger parameters become infinite.

Since string theory amplitudes are written as integrals over the moduli parameters, one concludes that the divergences from the integral over the moduli spaces of Riemann surfaces typically arise from the regions where the Riemann surface becomes singular, that is, degenerate.Two types of degeneration can be distinguished. These are depicted in Figure 8.1, adopted from [56]. The first one is the separating type degeneration Figure 8.1a in which a Riemann surface breaks apart into two parts. The second one is the non-separating type degeneration presented in Figure 8.1b in which a Riemann surface reduces to a lower genus surface with two extra punctures.



(a) Separating type degeneration

(b) Non-separating case

FIGURE 8.1: Degeneration of Riemann surfaces.

In both types of degeneration, the Schwinger parameter s_i is taken to infinity. A direct comparison with QFT indicates that these are to be considered as IR divergences[35]. The connection to the soft divergences that have been extensively studied in this thesis, is realized in the region:

$$s_i \to \infty, \quad m_i = 0, \quad p_i \approx 0 \tag{8.2}$$

and for D=4 non-compactified spacetime dimensions. However, there is no known procedure in the worldsheet string theory that can remove these divergences.

8.2 Faddeev-Kulish in string field theory

In Chapter 3, we studied how string theory can be expressed in its second quantized formulation as a usual, more complicated though, QFT. The soft IR divergences discussed

[35]There are more types of IR divergences, for example, tadpole and mass renormalization divergences.

in the previous section can be addressed in this language. Before a possible set up for this problem is discussed, it would be constructive to first refer to what we have learned for the FK method. The main steps followed are actually two. The first one is the construction of the asymptotic potential. This is achieved by assigning a soft momentum to the massless particle[36] involved in the cubic interaction and subsequently study the behavior of the interaction potential at large time. This is done in the interaction picture.

The second step is to construct the Chung states. The cancellation of IR divergences associated to virtual soft photon/graviton particle corrections is achieved by picking up suitable diagrams, as indicated by the dressing of the states. It is also important to consider real photon/graviton contributions to make use of the appropriate Weinberg's soft theorem. But how does this all work in SFT?

A first question is, what is the analog of a virtual soft massless particle in SFT. Since the string field is defined[37] as a collection of infinitely many string modes, we can define a soft massless virtual string field as a collection of infinitely many off-shell soft massless modes. In the case of closed bosonic SFT, there are three massless modes in the spectrum, identified with the graviton $G_{\mu\nu}$, the dilaton Φ and the Kelb-Ramond field $B_{\mu\nu}$. In Chapter 7, we found that photon and graviton dress scalar external legs independently. This indicates that this might also be the case in SFT.

An extension of Weinberg's soft theorems in SFT was recently published by Ashoke Sen [57], for Type II superstring field theory, with external states belonging to the NS-NS sector. A task in our case would be to derive Weinberg's soft theorem, based on the interaction vertex of Eq.(7.11). Notice however that its form is far more complicated than anything we had to deal with in PQG.

The form of the interaction vertex however, might be useful in order to obtain the asymptotic potential. Again the expression might look complicated, however it is possible to obtain it by a suitable choice of the basis states $\{\phi_a\}$. The most challenging part seems to be the construction of the Chung states for the string field. Of course, this is because we have no idea about the asymptotic potential yet.

Despite the aforementioned challenges, string field theory has already been successful in dealing with IR divergences arising from mass renormalization [58, 59], so we expect that the FK method can also be successfully extended. It is useful, however, to first consider more simple string field scattering processes, for example, a cubic interaction involving one soft graviton and two massive scalars.

[36]or to one of the massless particles if more are involved.
[37]See Section 3.1

Chapter 9

Closing Remarks

This is the last chapter of this thesis. In Section 9.1, the findings of this thesis are presented and discussed. A more extended discussion based on our findings, as well as on other research advances takes place in Section 9.2.

9.1 Conclusions

In Chapter 4, we have derived Weinberg's soft theorem for gravitons attached to photon external legs. The result is exactly the same to the one we would obtain if the external legs were scalar particles or electrons etc. This is in complete accordance with Weinberg's statement [40] that soft theorems are universal and do not depend on the spin of the external particles.

In Chapter 5, we considered the EMS model, in order to apply the Faddeev-Kulish method in the presence of three cubic interactions, namely, scalar-graviton, scalar-photon and photon-graviton. The Lagrangian for this model was constructed and the asymptotic potential was found as a sum of the asymptotic potentials of each type of interaction.

Since the cancellation of IR divergences has been successfully carried out for scalar-graviton and scalar-photon interactions, we also investigated it for the scattering of a photon in the presence of a gravitational potential. This was discussed in Chapter 6. Since the form of the asymptotic potential in this case is the same as the one that external scalars would feel, the cancellation of IR divergences should follow the same lines. It turned out that this is the case, verifying the statement of [8], that their construction is also universal and does not depend on the spin of the external particles.

The Faddeev-Kulish method turns out to work for all individual types of cubic interactions. The next question we wanted to investigate, was whether it still works for more complicated scattering processes, involving scalar and photon external particles in the presence of both electromagnetic and gravitational potentials. This was done in Chapter 7. We found that a very crucial property that allowed us to write the asymptotic operator in terms of the radiation and phase operator in all individual cases does not hold in this general case. This property can be recovered if one only considers external scalar legs, in the presence of the aforementioned potentials. Due to the commutation relations, the dressing of the states occurs independently and one can work out the cancellation to all orders in κ and e.

Finally, in Chapter 8, we investigated how IR divergences arise in worldsheet string theory and tried to sketch how the Faddeev-Kulish method could be employed to remove them, working in its second-quantized formulation, string field theory.

9.2 Discussion

As mentioned earlier, we found some subtleties when trying to employ the FK method to a theory with both photon and scalar fields, in the presence of both gravitational and electromagnetic potentials. In the individual theories, for example the Einstein-Maxwell, the operator governing the asymptotic dynamics is found to precisely carry information about the radiation and phase divergences. These operators are used to cancel the two types of IR divergences associated with the virtual soft gravitons (see Chapter 4). The fact that this can be done relies on the property that the asymptotic potential commutes with its own commutator for all t, t_1, t_2. However, as we have shown in Section 7.1, this is no longer true for these more general scattering processes. Therefore, the asymptotic operator given by Eq.(5.46), has to be modified by the insertion of at least one extra operator, which is not clear if it is divergent or not. If it is not, then maybe these subtleties can be surpassed and one might be able to carry the Faddeev-Kulish method in the standard way. However, if it is divergent, then one might worry what the origin of these divergences is. The careful reader might have noticed that in the calculation of Eq.(7.11), the commutation relation on photon fields enforced the momentum of a hard external photon to vanish. Therefore, these divergences might be an indication that we are asking the wrong question. Or there may be a deeper reason behind their appearance. Since the origin of these extra operators and whether they diverge or not remains unclear, we propose it as a question for further investigation.

Another interesting question that has arisen, has again to do with the presence of the two potentials, but only with scalar external legs considered[38]. Consider the diagram of Figure 9.1. In this diagram, the outgoing leg emits a soft graviton, which in turn decays into two soft photons, which are absorbed by the same leg at some later time. A similar scenario involving three soft gravitons is not taken into consideration in the proof of cancellation of IR divergences. This is justified by the statement that the effective coupling for the emission of a soft graviton from another soft graviton of some energy E, is proportional to E. Then, an IR divergence is prevented by the vanishing of this coupling [41]. It is not clear however, whether this statement also holds in the present case. Moreover, from Eq.(7.36), as well as the very similar one for the final state, which we have not written down, it looks possible that the double-dressed states might be able to deal with divergences from such diagrams. We also leave this as an open question.

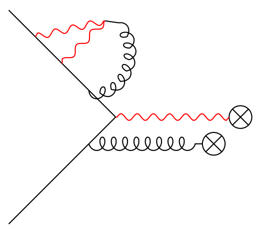


FIGURE 9.1: Soft photon-graviton interactions attached to a scalar external leg.

Closing with the EMS model, let us briefly refer to another case to be investigated. When studying the asymptotic potential related to the photon-graviton interactions, we only considered external photon legs and soft graviton attachments. However, since all three particles involved in the cubic interaction are massless, one could instead consider Eq.(5.22) and try to calculate the asymptotic potential for a soft photon field. A simple example, in close analogy to the one we studied in Chapter 6, is the scattering of a graviton in an electromagnetic potential under which it transmutes to a photon. However, in our theory, described by Eq.(5.9), cubic interactions of order h^2 and A^3 are prohibited. This means that the dressing of the states with soft photons is not possible in such a scenario. Therefore, the asymptotic potential is expected to vanish completely. The proof (or disproof) of this statement is also left as an open question.

The question on if and how the FK method might be extended to SFT has already been discussed in Chapter 8, so we will not state more here towards this direction. We will only point out the necessity of studying simplified models before we address the more general problem.

[38]See Section 7.2.

So, do we finally have an IR finite S-matrix for QED and PQG that can be possibly also obtained within the frame of SFT? The answer to this question turns out to be slightly obscure. At this point, we remind the reader the IR triangle of Figure 1.1. As we have already mentioned in Chapter 1, a connection of the FK states to the BMS group of supertranslations has already been found, while the connection to soft theorems has been extensively studied. The last corner of the triangle is the top one, related to the gravitational memory effect. This predicts the spatial displacement of pairs of masses due to the passing of a gravitational wave [60]. States with memory are unavoidable in gravitational scattering processes. The connection with IR divergences comes when one tries to represent them in the standard Fock space. But what about the asymptotic FK space of states? During our research, a recent paper of R.Wald et al. came to our attention [61]. In this paper it is clearly stated that in quantum gravity, these states are unphysical. The argument is that the FK dressing procedure would attempt to produce a Hilbert space of eigenstates of supertranslation charges at spatial infinity, while as they prove, only the vacuum can be such an eigenstate.

As we see, the construction of the FK states, despite its success on curing IR divergences and providing finite S-matrix elements, encounters issues when it comes to the memory effect corner. Whether this construction works successfully by coincidence or whether there are deeper reasons that have not yet been discovered, this remains to be seen.

Appendix A

Various calculations

A.1 Derivation of Eq.(5.12)

We wish to express $(\nabla \cdot A)^2$ in terms of partial derivatives. First, we notice that

$$g^{\mu\rho} g^{\nu\sigma} = (\eta^{\mu\rho} - h^{\mu\rho})(\eta^{\nu\sigma} - h^{\nu\sigma}) = \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\rho}h^{\nu\sigma} - \eta^{\nu\sigma}h^{\mu\rho} + O(h^2) \tag{A.1}$$

Using the above expression and the definition of the covariant derivative, we have

$$\begin{aligned}(\nabla \cdot A)^2 &= g^{\mu\rho} g^{\nu\sigma} \nabla_\rho A_\mu \nabla_\sigma A_\nu = g^{\mu\rho} g^{\nu\sigma} (\partial_\rho A_\mu - \Gamma^\lambda_{\rho\mu} A_\lambda)(\partial_\sigma A_\nu - \Gamma^\lambda_{\sigma\nu} A_\lambda) \\ &= [\eta^{\nu\sigma}\eta^{\mu\rho} - \eta^{\mu\rho}h^{\nu\sigma} - \eta^{\nu\sigma}h^{\mu\rho}](\partial_\rho A_\mu \partial_\sigma A_\nu - \Gamma^\lambda_{\rho\mu} A_\lambda \partial_\sigma A_\nu - \partial_\rho A_\mu \Gamma^\lambda_{\sigma\nu} A_\lambda) \\ &= (\partial \cdot A)^2 - \eta^{\nu\sigma}\eta^{\mu\rho}(\Gamma^\lambda_{\rho\mu} A_\lambda \partial_\sigma A_\nu + \partial_\rho A_\mu \Gamma^\lambda_{\sigma\nu} A_\lambda) - (\eta^{\nu\sigma}h^{\mu\rho} + \eta^{\mu\rho}h^{\nu\sigma})\partial_\rho A_\mu \partial_\sigma A_\nu\end{aligned} \tag{A.2}$$

up to linear order in h. Substituting the Christoffel connection in the terms of the second parenthesis,we obtain

$$\begin{aligned}-\eta^{\mu\rho}\eta^{\nu\sigma}\Gamma^\lambda_{\mu\rho} A_\lambda \partial_\sigma A_\nu &= -\frac{1}{2}\eta^{\nu\sigma}\eta^{\mu\rho}(\partial_\rho h^\lambda_\mu + \partial_\mu h^\lambda_\rho - \partial^\lambda h_{\mu\rho} A_\lambda \partial_\sigma A_\nu \\ &= -\frac{1}{2}(\partial_\rho h^{\rho\lambda} + \partial_\mu h^{\mu\lambda} - \partial^\lambda h) A_\lambda (\partial \cdot A) \\ &= -\partial_\mu h^{\mu\nu} A_\nu (\partial \cdot A) + \frac{1}{2}\partial^\nu h A_\nu (\partial \cdot A)\end{aligned} \tag{A.3}$$

where in the last step we renamed the indices $\rho \to \mu$ and $\lambda \to \nu$. The same computation applies also to the other term giving exactly the same result (again upon renaming indices), such that in total they sum up to

$$-2\partial_\mu h^{\mu\nu} A_\nu(\partial\cdot A) + \partial^\nu h A_\nu(\partial\cdot A) = -2(\partial_\mu h^{\mu\nu} - \partial^\nu h)A_\nu(\partial\cdot A) \tag{A.4}$$

Finally, the terms in the last parenthesis of Eq.(A.2) give,respectively

$$-\eta^{\mu\rho}h^{\nu\sigma}\partial_\rho A_\mu \partial_\sigma A_\nu = h^{\nu\sigma}\partial_\sigma A_\nu(\partial\cdot A) \tag{A.5}$$

$$-\eta^{\nu\sigma}h^{\mu\rho}\partial_\rho A_\mu \partial_\sigma A_\nu = h^{\mu\rho}\partial_\rho A_\mu(\partial\cdot A) \tag{A.6}$$

which upon some index renaming, sum up to

$$-2h^{\mu\nu}\partial_\nu A_\mu(\partial\cdot A) \tag{A.7}$$

Performing a partial integration on the terms of Eq.(A.4) and adding to Eq.(A.7), gives us

$$\begin{aligned} &2h^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial\cdot A) + 2h^{\mu\nu}A_\nu\partial_\mu(\partial\cdot A) - h(\partial\cdot A)^2 - hA_\nu\partial^\nu(\partial\cdot A) \\ &= 2h^{\mu\nu}A_\nu\partial_\mu(\partial\cdot A) - h(\partial\cdot A)^2 - hA_\nu\partial^\nu(\partial\cdot A) \end{aligned} \tag{A.8}$$

Therefore,Eq.(A.2) acquires the form

$$(\nabla\cdot A)^2 = (\partial\cdot A)^2 + 2h^{\mu\nu}A_\nu\partial_\mu(\partial\cdot A) - h(\partial\cdot A)^2 - hA_\nu\partial^\nu(\partial\cdot A) \tag{A.9}$$

A.2 Computation of Eq.(5.15)

$$\begin{aligned} -\sqrt{g}\partial_\mu\phi^\star g^{\mu\nu}\partial_\nu\phi &= -(1+\frac{1}{2}h)(\eta^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi - h^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi) \\ &= -\eta^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi + h^{\mu\nu}\partial_\mu\phi^\star\partial_\nu\phi - \frac{1}{2}\eta^{\mu\nu}h\partial_\mu\phi^\star\partial_\nu\phi + O(h^2) \end{aligned} \tag{A.10}$$

while the second one gives

$$\begin{aligned} -\sqrt{g}e^2A_\mu\phi^\star g^{\mu\nu}A_\nu\phi &= -(1+\frac{1}{2}h)(e^2\eta^{\mu\nu}A_\mu\phi^\star A_\nu\phi - e^2A_\mu\phi^\star h^{\mu\nu}A_\nu\phi) \\ &= -e^2\eta^{\mu\nu}A_\mu\phi^\star A_\nu\phi + e^2h^{\mu\nu}A_\mu\phi^\star A_\nu\phi - \frac{1}{2}e^2\eta^{\mu\nu}hA_\mu\phi^\star A_\nu\phi + O(h^2) \end{aligned} \tag{A.11}$$

The third term is

$$\sqrt{g}\partial_\mu\phi^\star g^{\mu\nu}ieA_\nu\phi == (1+\frac{1}{2}h)(\partial_\mu\phi^\star\eta^{\mu\nu}ieA_\nu\phi - \partial_\mu\phi^\star h^{\mu\nu}ieA_\nu\phi)$$
$$= ie\eta^{\mu\nu}\partial_\mu\phi^\star A_\nu\phi - ie\partial_\mu\phi^\star h^{\mu\nu}A_\nu\phi + \frac{1}{2}ie\eta^{\mu\nu}h\partial_\mu\phi^\star A_\nu\phi + O(h^2) \tag{A.12}$$

while for the fourth term we get

$$-\sqrt{g}(ieA_\mu\phi^\star g^{\mu\nu}\partial_\nu\phi) = -(1+\frac{1}{2}h)(ieA_\mu\phi^\star\eta^{\mu\nu}\partial_\nu\phi - ieA_\mu\phi^\star h^{\mu\nu}\partial_\nu\phi)$$
$$= -ieA_\mu\phi^\star\eta^{\mu\nu}\partial_\nu\phi + ieA_\mu\phi^\star h^{\mu\nu}\partial_\nu\phi - \frac{1}{2}iehA_\mu\phi^\star\eta^{\mu\nu}\partial_\nu\phi + O(h^2) \tag{A.13}$$

Finally,the last term is

$$-m^2\sqrt{g}\phi^\star\phi = -m^2\phi^\star\phi - \frac{1}{2}m^2h\phi^\star\phi \tag{A.14}$$

A.3 Computation of Eq.(6.20)

The integral of Eq.(6.19) involves the commutators:

1. $\left[f^{\mu\nu}(k,p)a^\dagger_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), \frac{q^\rho q^\sigma}{l\cdot q}a_{\rho\sigma}(\mathbf{l})e^{i\frac{l\cdot q}{q^0}t}\rho^{ph}(\mathbf{q})\right]$
2. $-\left[f^{\mu\nu}(k,p)a^\dagger_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), \frac{q^\rho q^\sigma}{l\cdot q}a^\dagger_{\rho\sigma}(\mathbf{l})e^{-i\frac{l\cdot q}{q^0}t}\rho^{ph}(\mathbf{q})\right]$
3. $-\left[f^{\star\mu\nu}(k,p)a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), \frac{q^\rho q^\sigma}{l\cdot q}a_{\rho\sigma}(\mathbf{l})e^{i\frac{l\cdot q}{q^0}t}\rho^{ph}(\mathbf{q})\right]$
4. $\left[f^{\star\mu\nu}(k,p)a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), \frac{q^\rho q^\sigma}{l\cdot q}a^\dagger_{\rho\sigma}(\mathbf{l})e^{-i\frac{l\cdot q}{q^0}t}\rho^{ph}(\mathbf{q})\right]$

From the 1^{st} commutator we have

$$[a^\dagger_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a_{\rho\sigma}(\mathbf{l})\rho^{ph}(\mathbf{q})] = a^\dagger_{\mu\nu}(\mathbf{k})[\rho^{ph}(\mathbf{p}), a_{\rho\sigma}(\mathbf{l})\rho^{ph}(\mathbf{q})] + [a^\dagger_{\mu\nu}(\mathbf{k}), a_{\rho\sigma}(\mathbf{l})\rho^{ph}(\mathbf{q})]\rho^{ph}(\mathbf{p})$$
$$= -a^\dagger_{\mu\nu}(\mathbf{k})a_{\rho\sigma}(\mathbf{l})[\rho^{ph}(\mathbf{q}), \rho^{ph}(\mathbf{p})] - [a_{\rho\sigma}(\mathbf{l}), a^\dagger_{\mu\nu}(\mathbf{k})]\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p})$$
$$= -a^\dagger_{\mu\nu}(\mathbf{k})a_{\rho\sigma}(\mathbf{l})\big(b^\dagger_\delta(\mathbf{q})b^\delta(\mathbf{p}) - b^\dagger_\delta(\mathbf{p})b^\delta(\mathbf{q})\big)\delta^{(3)}(\mathbf{p}-\mathbf{q})$$
$$-\big(\eta_{\rho\mu}\eta_{\sigma\nu} + \eta_{\rho\nu}\eta_{\sigma\mu} - \eta_{\mu\nu}\eta_{\rho\sigma}\big)\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p})\delta^{(3)}(\mathbf{k}-\mathbf{l})$$

Notice that the first term in the last line will vanish when integrate over q's. We are left with

$$-\frac{1}{4(2\pi)^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{q}}{2k_0}e^{i\frac{k\cdot q}{q^0}t}\frac{q^\mu q^\nu}{k\cdot q}f_{\mu\nu}(k,p) \tag{A.15}$$

The 2^{nd} and 3^{rd} commutator vanish, while a similar calculation for the 4^{th} one gives

$$\frac{1}{4(2\pi)^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}d^3\mathbf{q}}{2k_0}e^{-i\frac{k\cdot q}{q^0}t}\frac{q^\mu q^\nu}{k\cdot q}f^\star_{\mu\nu}(k,p) \tag{A.16}$$

Putting these together, one arrives at Eq.(6.20)

A.4 Computation of Eq.(7.6)

We have

$$Q_{gs,ps}(t_1,t_2) = \frac{2e}{(2\pi)^3}\int \frac{d^3\mathbf{k}d^3\mathbf{r}}{2r_0\sqrt{2k_0}}\int \frac{d^3\mathbf{p}d^3\mathbf{s}}{2s_0\sqrt{2p_0}}r^\mu r^\nu s^\sigma$$
$$\times\left[\left(a_{\mu\nu}(\mathbf{k})e^{i\frac{r\cdot k}{r^0}t_1}+a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{r\cdot k}{r^0}t_1}\right)\rho_b(\mathbf{r}),\left(b_\sigma(\mathbf{p})e^{i\frac{s\cdot p}{s^0}t_2}+b^\dagger_\sigma(\mathbf{p})e^{-i\frac{s\cdot p}{s^0}t_2}\right)\rho_c(\mathbf{s})\right]$$

This involves the commutators:

1. $[e^{i\frac{r\cdot k}{r^0}t_1}a_{\mu\nu}(\mathbf{k})\rho_b(\mathbf{r}),e^{i\frac{s\cdot p}{s^0}t_2}b_\sigma(\mathbf{p})\rho_c(\mathbf{s})]$
2. $[e^{i\frac{r\cdot k}{r^0}t_1}a_{\mu\nu}(\mathbf{k})\rho_b(\mathbf{r}),e^{-i\frac{s\cdot p}{s^0}t_2}b^\dagger_\sigma(\mathbf{p})\rho_c(\mathbf{s})]$
3. $[e^{-i\frac{r\cdot k}{r^0}t_1}a^\dagger_{\mu\nu}(\mathbf{k})\rho_b(\mathbf{r}),e^{i\frac{s\cdot p}{s^0}t_2}b_\sigma(\mathbf{p})\rho_c(\mathbf{s})]$
4. $[e^{-i\frac{r\cdot k}{r^0}t_1}a^\dagger_{\mu\nu}(\mathbf{k})\rho_b(\mathbf{r}),e^{-i\frac{s\cdot p}{s^0}t_2}b^\dagger_\sigma(\mathbf{p})\rho_c(\mathbf{s})]$

Let's compute the 1^{st} commutator (neglecting the time-dependence for the time-being). We have

$$\begin{aligned}[a_{\mu\nu}(\mathbf{k})\rho_b(\mathbf{r}),b_\sigma(\mathbf{p})\rho_c(\mathbf{s})] &= a_{\mu\nu}(\mathbf{k})[\rho_b(\mathbf{r}),b_\sigma(\mathbf{p})\rho_c(\mathbf{s})]+[a_{\mu\nu}(\mathbf{k}),b_\sigma(\mathbf{p})\rho_c(\mathbf{s})]\rho_b(\mathbf{r})\\ &= -a_{\mu\nu}(\mathbf{k})\Big(b_\sigma(\mathbf{p})[\rho_c(\mathbf{s}),\rho_b(\mathbf{r})]+[b_\sigma(\mathbf{p}),\rho_b(\mathbf{r})]\rho_c(\mathbf{s})\Big)\\ &= -a_{\mu\nu}(\mathbf{k})b_\sigma(\mathbf{p})\Big(c^\dagger(\mathbf{s})c(\mathbf{r})-c^\dagger(\mathbf{r})c(\vec{s})+d^\dagger(\mathbf{s})d(\mathbf{r})-d^\dagger(\mathbf{r})d(\mathbf{s})\Big)\delta^{(3)}(\mathbf{r}-\mathbf{s})\end{aligned}$$

where in the second and third equality we used that commutators of different fields commute, as well as Eq.(B.4) of Appendix B. When substituting back in the integral, the delta function will kill the s-integral and everything that contains **s** will be evaluated at $\mathbf{r}$. Therefore, the parenthesis automatically vanishes. It is very straightforward to show that the same argument holds for the rest of the commutators.

A.5 Computation of Eq.(6.48)

We have:

$$Q_{gp}(t_1,t_2) = [V^I_{as,gp}(t_1), V^I_{as,gp}(t_2)] = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}\frac{d^3\mathbf{l}d^3\mathbf{q}}{2q_0\sqrt{2l_0}} p^\mu p^\nu q^\alpha q^\beta$$
$$\times\left[\left(a_{\mu\nu}(\mathbf{k})e^{i\frac{k\cdot p}{p^0}t_1} + a^\dagger_{\mu\nu}(\mathbf{k})e^{-i\frac{k\cdot p}{p^0}t_1}\right)\rho^{ph}(\mathbf{p}), \left(a_{\alpha\beta}(\mathbf{l})e^{i\frac{l\cdot q}{q^0}t_2} + a^\dagger_{\alpha\beta}(\mathbf{l})e^{-i\frac{l\cdot q}{q^0}t_2}\right)\rho^{ph}(\mathbf{q})\right] \quad \text{(A.17)}$$

The commutators to be computed are

1. $e^{i\left(\frac{l\cdot q}{q^0}t_2 + \frac{k\cdot p}{p^0}t_1\right)}[a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})]$
2. $e^{-i\left(\frac{l\cdot q}{q^0}t_2 - \frac{k\cdot p}{p^0}t_1\right)}[a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a^\dagger_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})]$
3. $e^{i\left(\frac{l\cdot q}{q^0}t_2 - \frac{k\cdot p}{p^0}t_1\right)}[a^\dagger_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})]$
4. $e^{-i\left(\frac{l\cdot q}{q^0}t_2 + \frac{k\cdot p}{p^0}t_1\right)}[a^\dagger_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a^\dagger_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})]$

For the 1^{st} commutator we have

$$\begin{aligned}
[a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})] &= a_{\mu\nu}(\mathbf{k})[\rho^{ph}(\mathbf{p}), a_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})] + [a_{\mu\nu}(\mathbf{k}), a_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})]\rho^{ph}(\mathbf{p}) \\
&= -a_{\mu\nu}(\mathbf{k})\Big(a_{\alpha\beta}(\mathbf{l})[\rho^{ph}(\mathbf{q}), \rho^{ph}(\mathbf{p})] + [a_{\alpha\beta}(\mathbf{l}), \rho^{ph}(\mathbf{p})]\rho^{ph}(\mathbf{q})\Big) \\
&- \Big(a_{\alpha\beta}(\mathbf{l})[\rho^{ph}(\mathbf{q}), a_{\mu\nu}(\mathbf{k})] + [a_{\alpha\beta}(\mathbf{l}), a_{\mu\nu}(\mathbf{k})]\rho^{ph}\Big)\rho^{ph}(\mathbf{p}) \\
&= -a_{\mu\nu}(\mathbf{k})a_{\alpha\beta}(\mathbf{l})[\rho^{ph}(\mathbf{q}), \rho^{ph}(\mathbf{p})]
\end{aligned}$$

where we used that commutators of different fields vanish, as well as the relations given in Appendix B. However,

$$[\rho^{ph}(\mathbf{q}), \rho^{ph}(\mathbf{p})] = \Big(b^\dagger_\delta(\mathbf{q})b^\delta(\mathbf{p}) - b^\dagger_\delta(\mathbf{p})b^\delta(\mathbf{q})\Big)\delta^{(3)}(\mathbf{p}-\mathbf{q})$$

Therefore, this will vanish upon integration over p's. The same is also true for the 4^{th} commutator. For the 2^{nd} commutator we have

$$
\begin{aligned}
[a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a^{\dagger}_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})] &= a_{\mu\nu}(\mathbf{k})[\rho^{ph}(\mathbf{p}), a^{\dagger}_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})] + [a_{\mu\nu}(\mathbf{k}), a^{\dagger}_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})]\rho^{ph}(\mathbf{p}) \\
&= -a_{\mu\nu}(\mathbf{k})\Big(a^{\dagger}_{\alpha\beta}(\mathbf{l})[\rho^{ph}(\mathbf{q}), \rho^{ph}(\mathbf{p})] + [a^{\dagger}_{\alpha\beta}(\mathbf{l}), \rho^{ph}(\mathbf{p})]\rho^{ph}(\mathbf{q})\Big) \\
&\quad - \Big(a^{\dagger}_{\alpha\beta}(\mathbf{l})[\rho^{ph}(\mathbf{q}), a_{\mu\nu}(\mathbf{k})] + [a^{\dagger}_{\alpha\beta}(\mathbf{l}), a_{\mu\nu}(\mathbf{k})]\rho^{ph}(\mathbf{q})\Big)\rho^{ph}(\mathbf{p}) \\
&= [a_{\mu\nu}(\mathbf{k}), a^{\dagger}_{\alpha\beta}(\mathbf{l})]\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p}) \\
&= (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})\delta^{(3)}(\mathbf{k}-\mathbf{l})\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p})
\end{aligned}
\tag{A.18}
$$

with the time dependence $e^{-i\left(\frac{l\cdot q}{q^0}t_2 - \frac{k\cdot p}{p^0}t_1\right)}$. Similar computations for the 3^{rd} commutator give

$$
[a_{\mu\nu}(\mathbf{k})\rho^{ph}(\mathbf{p}), a^{\dagger}_{\alpha\beta}(\mathbf{l})\rho^{ph}(\mathbf{q})] = -(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})\delta^{(3)}(\mathbf{k}-\mathbf{l})\rho^{ph}(\mathbf{q})\rho^{ph}(\mathbf{p})
\tag{A.19}
$$

with the time dependence $e^{i\left(\frac{l\cdot q}{q^0}t_2 - \frac{k\cdot p}{p^0}t_1\right)}$. Replacing these two commutators in Eq.(A.17), killing the l-integral and using that

$$
(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})p^{\mu}p^{\nu}q^{\alpha}q^{\beta} = 2(p\cdot q)^2
\tag{A.20}
$$

one can easily obtain Eq.(6.48)

A.6 Vanishing contributions to $V^{I}_{as,gp}(t)$

From the most general expression given by Eq.(5.22), we can calculate the contributions to the asymptotic potential of the various terms appearing in V_{gp}. We find:

$$
-\frac{1}{2}\eta^{\mu\rho}h^{\nu\sigma}\partial_{\mu}A_{\nu}\partial_{\rho}A_{\sigma} = -\frac{1}{2(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^2\big(a^{\mu\nu}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger\ \mu\nu}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\mu\nu}(\mathbf{p})
$$

This term vanishes on-shell. We also have

$$
\frac{1}{2}\eta^{\mu\rho}h^{\nu\sigma}\partial_{\mu}A_{\nu}\partial_{\sigma}A_{\rho} = \frac{1}{2(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^{\rho}p_{\beta}\big(a^{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger\ \alpha\beta}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\alpha\rho}(\mathbf{p})
$$

and

$$\frac{1}{2}\eta^{\mu\rho}h^{\nu\sigma}\partial_\nu A_\mu \partial_\rho A_\sigma = \frac{1}{2(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p_\alpha p^\mu\big(a^{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger\ \alpha\beta}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\mu\beta}(\mathbf{p})$$

Both these terms vanish by virtue of the Lorentz condition $p^\mu \cdot \epsilon^n_\mu(\mathbf{p})$, since $b_\mu(\mathbf{p}) = \epsilon^n_\mu(\mathbf{p})b^n(\mathbf{p})$, with $\epsilon^n_\mu(\mathbf{p})$ denoting the polarization vector of the external photon. We also have the contributions:

$$\frac{1}{8}\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}h_{\alpha\beta}\partial_\mu A_\nu \partial_\rho A_\sigma = \frac{1}{8}\frac{\eta^{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^2\big(a_{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\alpha\beta}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{\nu\ ph}_{\nu}(\mathbf{p})$$

$$\frac{1}{8}\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}h_{\alpha\beta}\partial_\nu A_\mu \partial_\sigma A_\rho = \frac{1}{8}\frac{\eta^{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^2\big(a_{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t}\big)j^{\mu\ ph}_{\mu}(\mathbf{p})$$

$$-\frac{1}{8}\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}h_{\alpha\beta}\partial_\nu A_\mu \partial_\rho A_\sigma = -\frac{1}{8}\frac{\eta^{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^\sigma p^\mu\big(a_{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\alpha\beta}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\mu\sigma}(\mathbf{p})$$

$$-\frac{1}{8}\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}h_{\alpha\beta}\partial_\mu A_\nu \partial_\sigma A_\rho = -\frac{1}{8}\frac{\eta^{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^\rho p^\nu\big(a_{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\alpha\beta}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\rho\nu}(\mathbf{p})$$

which vanish for the same reasons. A last contribution comes from the term $-\frac{1}{4}h(\partial\cdot A)^2$, which, however, vanishes as well:

$$-\frac{1}{4}h(\partial\cdot A)^2 = -\frac{1}{4}\frac{\eta^{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}d^3\mathbf{p}}{2p_0\sqrt{2k_0}}p^\mu p^\nu\big(a_{\alpha\beta}(\mathbf{k})e^{i\frac{p\cdot k}{p^0}t} + a^{\dagger}_{\alpha\beta}(\mathbf{k})e^{-i\frac{p\cdot k}{p^0}t}\big)j^{ph}_{\mu\nu}(\mathbf{p})$$

A.7 Total real soft contribution

We have

$$\begin{aligned}
M_{soft} &= 2\sum_n \int d^3\mathbf{k}\big(S_f - S_i\big)^2 \\
&= 2\sum_n \int d^3\mathbf{k}\frac{1}{4(2\pi)^3 2k_0}\left(\frac{p_f^\mu p_f^\nu \epsilon^n_{\mu\nu}}{p_f\cdot k} - (\frac{p_i^\mu p_i^\nu \epsilon^n_{\mu\nu}}{p_i\cdot k}\right)^2 \\
&= \frac{1}{32\pi^3}\sum_n \int \frac{d^3\mathbf{k}}{k_0}\left(\frac{p_f^\mu p_f^\nu p_f^\rho p_f^\sigma \epsilon^{\star n}_{\mu\nu}\epsilon^n_{\rho\sigma}}{(p_f\cdot k)^2}\right. \\
&\qquad + \frac{p_i^\mu p_i^\nu p_i^\rho p_i^\sigma \epsilon^{\star n}_{\mu\nu}\epsilon^n_{\rho\sigma}}{(p_i\cdot k)^2} \\
&\qquad \left. - 2\frac{p_f^\mu p_f^\nu p_i^\rho p_i^\sigma \epsilon^{\star n}_{\mu\nu}\epsilon^n_{\rho\sigma}}{(p_i\cdot k)(p_f\cdot k)}\right)
\end{aligned}$$

Using the polarization completeness relation given by Eq.(D.1), we obtain:

$$\frac{1}{2}p_f^\mu p_f^\nu p_f^\rho p_f^\sigma I_{\mu\nu\rho\sigma} = \frac{1}{2}p_f^4$$

and similarly

$$\frac{1}{2}p_i^\mu p_i^\nu p_i^\rho p_i^\sigma I_{\mu\nu\rho\sigma} = \frac{1}{2}p_i^4$$

These terms vanish on-shell. The last term gives:

$$-2\frac{1}{2}p_f^\mu p_f^\nu p_i^\rho p_i^\sigma I_{\mu\nu\rho\sigma} = -2\big((p_i \cdot p_f)^2 - \frac{1}{2}p_i^2 p_f^2\big) = -2(p_i \cdot p_f)^2$$

Therefore

$$M_{soft} = -\frac{1}{16\pi^3}\int \frac{d^3\mathbf{k}}{k_0}\frac{(p_i \cdot p_f)^2}{(p_i \cdot k)(p_f \cdot k)}$$

A.8 Normalization & Cloud-to-cloud contribution

For the cloud-to-cloud contribution, we have:

$$\begin{aligned}
M_{cloud\text{-}cloud} &= \sum_{n,n'}\int d^3\mathbf{k}d^3\mathbf{k}' S_i^n S_f^{n'} \epsilon_{\mu\nu}^{\star n}\epsilon_{\rho\sigma}^{n'}\langle 0|a^{\rho\sigma}(\mathbf{k}')a^{\dagger\mu\nu}(\mathbf{k})|0\rangle \\
&= \sum_{n,n'}\int d^3\mathbf{k}d^3\mathbf{k}' S_i^n S_f^{n'} \epsilon_{\mu\nu}^{\star n}\epsilon_{\rho\sigma}^{n'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')I^{\rho\sigma\mu\nu} \\
&= \sum_{n,n'}\int d^3\mathbf{k} S_i^n S_f^{n'} \epsilon_{\mu\nu}^{\star n}\epsilon_{\rho\sigma}^{n'} I^{\rho\sigma\mu\nu} \\
&= \sum_{n,n'}\int d^3\mathbf{k} S_i^n S_f^{n'} 2\epsilon_n^{\star\ \rho\sigma}\epsilon_{\rho\sigma}^{n'} \\
&= \sum_{n,n'}\int d^3\mathbf{k} S_i^n S_f^{n'} 2\delta_{n,n'} \\
&= 2\sum_{n}\int d^3\mathbf{k} S_i^n S_f^{n}
\end{aligned}$$

where in the first equality we used the commutation relation for graviton creation and annihilation operators given by Eq.(B.2) and in the fourth one we the polarization tensor normalization $\epsilon_n^{\star\ \rho\sigma}\epsilon_{\rho\sigma}^{n'} = \delta_{n,n'}$ If we sum M_{norm} and $M_{cloud\text{-}cloud}$, we obtain:

$$M' = M_{norm} + M_{cloud\text{-}cloud} = -\sum_n \int d^3\mathbf{k}\big(S_f^n - S_i^n\big)^2$$

This is exactly equal to what we calculated in subsection A.7, multiplied by the factor $-\frac{1}{2}$. Thus, we directly get

$$M' = \frac{1}{32\pi^3} \int \frac{d^3\mathbf{k}}{k_0} \frac{(p_i \cdot p_f)^2}{(p_i \cdot k)(p_f \cdot k)}$$

Appendix B

Commutation Relations

The only non vanishing commutators for the complex scalar field creation and annihilation operators are

$$\begin{aligned} [c(\mathbf{r}), c^{\dagger}(\mathbf{q})] &= \delta^{(3)}(\mathbf{r}-\mathbf{q}) \\ [d(\mathbf{s}), d^{\dagger}(\mathbf{q})] &= \delta^{(3)}(\mathbf{s}-\mathbf{q}) \end{aligned} \tag{B.1}$$

The commutation relation for gravitational field operators is

$$[a_{\mu\nu}(\mathbf{k}), a^{\dagger}_{\rho\sigma}(\mathbf{l})] = I_{\mu\nu\rho\sigma}\delta^{(3)}(\mathbf{k}-\mathbf{l}) \tag{B.2}$$

while for the electomagnetic ones:

$$[b_{\mu}(\mathbf{p}), b^{\dagger}_{\nu}(\mathbf{q})] = \eta_{\mu\nu}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{B.3}$$

Other important commutation relations are

$$\begin{aligned} [\rho_b(\mathbf{s}), \rho_b(\mathbf{r})] &= \Big(c^{\dagger}(\mathbf{s})c(\mathbf{r}) - c^{\dagger}(\mathbf{r})c(\mathbf{s}) + d^{\dagger}(\mathbf{s})d(\mathbf{r}) - d^{\dagger}(\mathbf{r})d(\mathbf{s})\Big)\delta^{(3)}(\mathbf{r}-\mathbf{s}) \\ &= [\rho_c(\mathbf{s}), \rho_b(\mathbf{r})] \\ &= [\rho_c(\mathbf{s}), \rho_c(\mathbf{r})] \end{aligned} \tag{B.4}$$

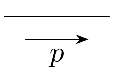
$$[\rho^{ph}(\mathbf{q}), \rho^{ph}(\mathbf{p})] = \Big(b^{\dagger}_{\delta}(\mathbf{q})b^{\delta}(\mathbf{p}) - b^{\dagger}_{\delta}(\mathbf{p})b^{\delta}(\mathbf{q})\Big)\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{B.5}$$

$$[\rho^{ph}(\mathbf{p}), b^{\dagger}_{\sigma}(\mathbf{q})] = b^{\dagger}_{\sigma}(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{B.6}$$

Appendix C

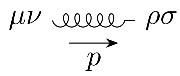
Feynman Rules

Scalar Propagator:



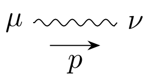
$$= -i\frac{1}{p^2 + m^2 - i\epsilon}$$

Graviton Propagator:



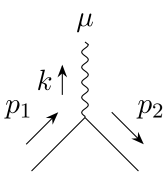
$$= -\frac{i}{2} I^{\mu\nu\rho\sigma} \frac{1}{p^2 - i\epsilon}$$

Photon Propagator:



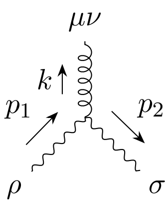
$$= -i\eta^{\mu\nu} \frac{1}{p^2 - i\epsilon}$$

Scalar-scalar-photon vertex:



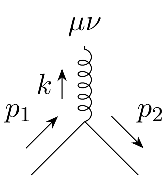
$$= ie(p_1 + p_2)_\mu$$

Photon-photon-graviton vertex:



$$= \frac{i}{2}\big(I^{\mu\nu\rho\sigma} p_2 \cdot p_1 + \Lambda^{\mu\nu\rho\sigma}\big)$$

Scalar-scalar-graviton vertex:



$$= -\frac{i}{2}\big(p_2^\rho p_1^\sigma + p_2^\sigma p_1^\rho - \eta^{\rho\sigma}(p_2 \cdot p_1 - m^2)\big)$$

Appendix D

Formulas & more

From [44]:

$$\sum_{n,n'} \epsilon^{\star n}_{\mu\nu} \epsilon^{n'}_{\rho\sigma} = \frac{1}{2}\big(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big)\delta_{n,n'} \tag{D.1}$$

$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \tag{D.2}$$

From [6]:

$$\int \frac{d^3\mathbf{k}}{2k^0} sin(k(\frac{q}{q_0}\tau_1 - \frac{p}{p_0}\tau_2)) = 2\pi^2\delta((\frac{q}{q_0}\tau_1 - \frac{p}{p_0}\tau_2)^2)\delta(\tau_1 - \tau_2) \tag{D.3}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = 1 \tag{D.4}$$

$$\delta^{(3)}(\mathbf{p} - \mathbf{q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d^3\mathbf{x} e^{-i\mathbf{x}(\mathbf{p}-\mathbf{q})} \tag{D.5}$$

From [62]:

$$\sqrt{\mathbf{p}^2 + m^2} - \sqrt{(\mathbf{p}-\mathbf{k})^2 + m^2} - k_0 \approx \frac{\mathbf{pk}}{p_0} - k_0 = \frac{p\cdot k}{p_0} \tag{D.6}$$

For two operators A,B:

$$\textit{BCH formula:} \quad e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\ldots} \tag{D.7}$$

$$\Rightarrow e^{A}Be^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \overbrace{[A, \ldots, [A, B] \ldots]}^{n} \tag{D.8}$$

Lemma D.1 (Riemann-Lebesgue lemma [63])**.** *If the Lebesgue integral of a function $|f|$ is finite, then its Fourier transform satisfies:*

$$\hat{f}(k) = \int_{\mathbf{R}^d} dx f(x) e^{-ikx} \to 0 \quad , as \quad |k| \to \infty \tag{D.9}$$

In the Feynman rules of Appendix C we have defined

$$I_{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} \tag{D.10}$$

and

$$\begin{aligned}\Lambda^{\mu\nu\rho\sigma} &= \eta^{\mu\nu}p_2p_1^{\sigma} - \eta^{\rho\nu}p_2^{\mu}p_1^{\sigma} + \eta^{\rho\sigma}p_2^{\mu}p_1^{\nu} - \eta^{\mu\sigma}p_2^{\rho}p_1^{\nu} - \eta^{\nu\sigma}p_2^{\rho}p_1^{\mu} + \eta^{\rho\sigma}p_2^{\nu}p_1^{\mu} - \eta^{\rho\mu}p_2^{\nu}p_1^{\sigma} \\ &+ \frac{1}{2}\eta^{\mu\nu}(p_1^{\rho}p_2^{\sigma} + p_1^{\sigma}p_2^{\rho}) + \frac{1}{2}\eta^{\mu\nu}(p_1^{\rho}p_1^{\sigma} + p_2^{\rho}p_2^{\sigma}) - \eta^{\nu\sigma}(p_1^{\rho}p_1^{\sigma} + p_1^{\rho}p_1^{\sigma})\end{aligned} \tag{D.11}$$

Appendix E

SFT supplement

E.1 BRST quantization of the string

The symmetries of a Lagrangian dictate the possible terms to be considered. However, if the path integral is gauge fixed, the original symmetry is not manifest. One can introduce BRST quantization either by the standard QFT process, or within the frame of CFT. Here, we will review the former, following a mixed treatment of [20, 36]. The CFT treatment will not be discussed. Only some important results will be presented. In the worldsheet path integral of Eq.(2.25), There is however a global symmetry remnant of the local symmetry. This can be recovered by adding in the path integral the gauge fixing action

$$S_{g.f.}[g,\hat{g},B] = -\frac{i}{4\pi}\int d^2\sigma B^{\alpha\beta}(\sqrt{-g}g_{\alpha\beta} - \sqrt{-\hat{g}}\hat{g}_{\alpha\beta})$$

where $B^{\alpha\beta}$ is an auxiliary field. The path integral is then written as

$$Z[g] = \int \mathcal{D}X\mathcal{D}g\mathcal{D}b\mathcal{D}Bce^{-S_P[X,g]-S_{gh}[b,c,g]-S_{g.f.}[g,\hat{g},B]} \tag{E.1}$$

One can then show that the total action

$$S = S_P[X,g] + S_{gh}[b,c,g] + S_{g.f.}[g,\hat{g},B] \tag{E.2}$$

is invariant under the infinitesimal BRST transformations:

$$\delta_B X^\mu ic\partial X^\mu, \quad \delta_B b = B_{ab}, \quad \delta_B c = ic\partial c, \quad \delta_B B_{\alpha\beta} = 0 \tag{E.3}$$

By Noether's theorem, the global symmetry of Eq.(E.3) is generated by a conserved current j_B^α, the so-called BRST current. A conserved charge, called the BRST charge[39] is associated to this current, given by

$$Q_B = \int d^2\sigma j_B^0(\sigma) \tag{E.4}$$

One can show that Q_B is nilpotent, that is $Q_B^2 = 0$ and has ghost number $N_{gh}(Q_B) = 1$. Physical states $|\psi\rangle$ are elements of the BRST cohomology and obey

$$Q_B|\psi\rangle = 0, \tag{E.5a}$$

$$|\psi\rangle \sim |\psi\rangle + Q_B|\chi\rangle \tag{E.5b}$$

As we have already mentioned, the BRST procedure can also be carried out on the complex plane. The conserved current takes the form

$$j_B(z) = c(z)T^m(z) + \frac{1}{2}c(z)T^{gh}(z) + \frac{3}{2}\partial^2 c(z) \tag{E.6}$$

where the stress-tensors for bosonic matter and ghosts are

$$T^m = -\frac{1}{\alpha'} : \partial X \partial X : \tag{E.7a}$$

$$T^{gh} = -2 : b\partial c : - : \partial bc : + \frac{1}{2}\partial^2 c(z) \tag{E.7b}$$

respectively. The BRST charge can be decomposed as following:

$$Q = c_0 L_0 - b_0 M_+ + \hat{Q} \tag{E.8}$$

In Eq.(E.8), L_0, b_0 and c_0 are.. and $M_+, \hat{Q}$ are given by

$$M_+ = \sum_{n \neq 0} n c_{-n} c_n \tag{E.9a}$$

$$\hat{Q} = \sum_{n \neq 0} c_{-n} L_n^m - \frac{1}{2} \sum_{\substack{m,n \\ m+n \neq 0}} (m-n) c_{-m} c_{-n} b_{m+n} \tag{E.9b}$$

[39] or BRST operator

E.2 Properties of the BPZ inner product

The BPZ inner product satisfies[40]

$$\langle A, B\rangle = (-1)^{|A||B|}\langle B, A\rangle \tag{E.10a}$$

$$\langle Q_B A, B\rangle = -(-1)^{|A|}\langle A, Q_B B\rangle \tag{E.10b}$$

It is non-degenerate, meaning that

$$\forall A: \quad \langle A|B\rangle = 0 \Rightarrow |B\rangle = 0 \tag{E.11}$$

Let $\{|\phi_s\rangle\}$ be an arbitrary basis in some Hilbert space $\mathcal{H}$, supplied with the BPZ inner product. The dual basis $\{|\phi_r^c\rangle\}$ defined by the BPZ inner product is:

$$\langle\phi_r^c|\phi_s\rangle = \delta_{r,s} \tag{E.12}$$

Note that

$$\langle\phi_r|\phi_s^c\rangle = (-1)^{|\phi_r|}\delta_{r,s} \tag{E.13}$$

E.3 Useful formulas for SFT

We define

$$c_0^{\pm} = \frac{1}{2}(c_0 \pm \bar{c}_0) \tag{E.14a}$$

$$b_0^{\pm} = (b_0 \pm \bar{b}_0) \tag{E.14b}$$

$$L_0^{\pm} = (L_0 \pm \bar{L}_0) \tag{E.14c}$$

The ghost numbers:

$$N_{gh}(c) = 1, \quad N_{gh}(b) = -1, \quad N_{gh}(\bar{c}) = 1, \quad N_{gh}(\bar{b}) = -1, \quad N_{gh}(Q) = 1, N_{gh}(\Psi) = 2 \tag{E.15}$$

[40] We denote with $|C|$ the Grassmann parity of the operator C. It takes values 0 or 1 and the operator is called Grassmann even or Grassmann odd, respectively. Notice that if both A,B in Eq.(E.10a) are Grassmann odd, then the inner product (viewed as a binary operation) is anti-commutative.

Euler characteristics:

$$\chi_{g,n} = 2 - 2g - n \tag{E.16}$$

Mixing matrix:

$$M_{ab}(k) := \langle \phi^c_\alpha(k) | b_0^+ b_0^- | \phi^c_\beta(-k) \rangle \tag{E.17}$$

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