MASTER THESIS

Constraints on moduli masses in Type IIB orientifold compactifications

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Abstract

Strings generally exist in more dimensions than we are able to observe. To get back to four-dimensional spacetime, we have to compactify the other dimensions. However, these compactifications typically lead to extra massless fields, the moduli, and we have to give them masses to get a phenomenological effective field theory. This is called moduli stabilization. In this thesis we will focus specifically on type IIB string theory compactified on so-called Calabi-Yau orientifolds. For this theory, the moduli can acquire masses by turning on extra background fields or fluxes. We will calculate the trace of these mass values corresponding to two of these moduli, the axio-dilaton and the complex structure moduli and try to find constraints for this trace. We then find that this can be written in terms of a special metric, called the Hodge metric. We also discuss further constraints on these moduli masses by relating them to the so-called flux number and looking at three special cases for the complex structure moduli space: the case of one complex structure modulus, the case of the complex structure moduli space as an Kähler-Einstein manifold and the case where we look at the boundary of the complex structure moduli space.

Contents

Chapter 1

Introduction

Quantum field theory was the result of a long-during attempt to combine special relativity and quantum mechanics. From this theory the standard model spawned, which allowed to understand the behaviour of many elementary particles. An important condition to make quantum field theory work is renormalizability. In quite an other region of theoretical physics, general relativity was the product of Einstein's attempt to combine Newton's theory of gravity and special relativity. However, general relativity is non-renormalizable and therefore the possibility to combine two very successful physical theories seemed to be tiny. Then string theory came in. It was discovered that for string theory, which was an extension of quantum field theory, quickly a field shows up that gives rise to a gravitational force. Thus, a quantum gravity theory was born and the interest for string theory ignited.

The main idea behind string theory is very simple: Instead of looking at point particles moving through spacetime, we now look at lines or strings moving through spacetime. These strings do not follow a worldline as the point particles do, but rather sweep out a surface, called the worldsheet. Worldlines are parametrized by the proper time τ, but to describe worldsheets we need another quantity, given by σ which is periodic and stays within the range $\sigma \in [0, l)$. These coordinates are usually written as $\sigma^\alpha = (\tau, \sigma)$ [1]. We can then distinguish between closed strings that have their ends connected and open strings for which the ends hang freely in spacetime. Also, the strings can have an orientation, i.e. they can be right-moving or left-moving, or can just be unoriented.

Before going more into detail, we should make a side note. The masses of strings are all expected to be close to the Planck mass and this is of the order of 10^{18} GeV. The maximum order of magnitude for energy that can be measured at the Large Hadron Collider right now is 10^3 GeV [2]. There is thus a gap of 10^{15} GeV that has to bridged before it will be possible to measure any string theory phenomena, but in the meanwhile we can learn a lot about what a quantum gravity theory would look like by studying string theory.

1.1 Bosonic String Theory

We will start our discussion of string theory with a short description of the most basic form of string theory, which is called bosonic string theory. As the name suggests, this is a theory that only possesses bosons in the spectrum and it is therefore a very intuitive theory since we do not have to deal with the more difficult spinors yet. The main starting point of bosonic string theory is an action that is given by [1]

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{1.1}$$

This is called the Polyakov action. It is just an action for a point particle following a worldline, generalized to a string sweeping out a worldsheet, with the addition of a dynamical metric $g_{\alpha\beta}$ on the worldsheet. In this action X^μ is a field that maps the worldsheet coordinates (σ, τ) to D-dimensional Minkowski spacetime. Also, T is the tension of the string that is often written in the form $T = \frac{1}{2\pi\alpha'}$ with α' being called the Regge slope and being related to the string length scale by $\alpha' = (\frac{l_s}{2\pi})^2$. The Polyakov action has many different symmetries, and using these we can gauge fix the worldsheet metric as

$$g_{\alpha\beta} = \eta_{\alpha\beta}. \tag{1.2}$$

This means that locally we can choose the worldsheet metric to be flat. The action then simplifies to

$$S = -\frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = \frac{T}{2} \int d^2\sigma \big\{ (\partial_\tau X)^2 - (\partial_\sigma X)^2 \big\} \tag{1.3}$$

Varying this action in order to find the equations of motion, i.e. imposing $\delta X(\tau_0) = \delta X(\tau_1)$, we get

$$\delta S = T \int d^2\sigma \partial_\alpha \partial^\alpha X^\mu \delta X_\mu - T \int_{\tau_0}^{\tau_1} d\tau \partial_\sigma X^\mu \delta X_\mu \Big|_{\sigma=0}^{\sigma=l}. \tag{1.4}$$

Because of the principle of least action, these terms have to be zero and for the first term this means that

$$\partial_\alpha \partial^\alpha X^\mu = 0, \tag{1.5}$$

which is just an ordinary wave equation. To achieve the dissappearance of the second term, it is important to distinguish between closed and open strings. For closed strings naturally $X^\mu(\sigma + l) = X^\mu(\sigma)$ and therefore the second term vanishes automatically.
For open strings we see that

$$\partial_\sigma X^\mu \delta X_\mu \Big|_{\sigma=0}^{\sigma=l} = 0 \tag{1.6}$$

and this leads to the two different conditions that are called the Neumann and Dirichlet boundary conditions:

$$\partial_\sigma X^\mu \Big|_{\sigma=0}^{\sigma=l} = 0 \ \text{(Neumann)} \quad \text{or} \quad \delta X^\mu \Big|_{\sigma=0}^{\sigma=l} = 0 \ \text{(Dirichlet)}. \tag{1.7}$$

These boundary conditions are of special interest, because they define objects called D-branes. To see this, we look at the case where some of the D coordinates satisfy the Neumann condition and some the Dirichlet condition.Then we get

$$\begin{aligned} &\partial_\sigma X^m \Big|_{\sigma=0}^{\sigma=l} = 0 \quad \text{for} \quad m = 0, ..., p \\ &X^M = c^M \quad \text{for} \quad M = p+1, ..., D-1, \end{aligned} \tag{1.8}$$

where we rewrote the Dirichlet boundary conditions [2]. These Dirichlet boundary conditions thus fix the end points of the string in a $(p+1)$-dimensional hypersurface and this hypersurface is called a D-brane (D from Dirichlet). It is thus the case that open strings never end or start in free space, but they are always stuck to a D-brane, although they can move along the D-brane because of the Neumann boundary conditions. [1]
Up to this point we have only looked at a classical string, but if we also want to accommodate quantum physics, we have to quantize it. Doing this, we can derive the mass spectrum for the strings. For closed strings this is

$$M^2 = \frac{4}{\alpha'}(N - \frac{D-2}{24}) = \frac{4}{\alpha'}(\tilde{N} - \frac{D-2}{24}), \tag{1.9}$$

where N and $\tilde{N}$ are the levels of the right- and left-moving strings respectively. Since there is no difference between the right- and left-moving strings besides their orientation, these are equal to one another. Using this we can slowly build up the spectrum of the closed bosonic string, and logically we start with $N = \tilde{N} = 0$. Then:

$$M^2 = -\frac{D-2}{6\alpha'}. \tag{1.10}$$

Interestingly enough, this means that whenever $D > 2$ (which it certainly is) we get a ground state with a negative mass. The counter-intuitive particles that are related to these states are called tachyons. Ignoring this quite unphysical result for now, we proceed to the first excited state. To obtain a quantum theory with Lorentz symmetry, all the states have to transform under a representation of the group of the different Lorentz transformations. This results in the fact that all massive states have to transform under a representation of the group $SO(D-1)$ and all massless states under a representation of the group $SO(D-2)$ [3]. Since the first excited state actually belongs to the vector representation of the group $SO(D-2)$, it immediately follows that it should be massless. Looking at the mass-value

$$M^2 = \frac{4}{\alpha'}(1 - \frac{D-2}{24}) \tag{1.11}$$

this quickly leads to the condition $D = 26$. In fact, working out the mass spectrum further it follows that if we want to preserve Lorentz invariance for all the higher excited states, the same dimensional condition holds. Moreover, even the open strings have to satisfy the same condition to preserve Lorentz symmetry. This dimensional condition is a unique feature of string theory.

Let us return to the massless states where we have one left- and one right-moving level. These states divide themselves between three different parts. One, the traceless and symmetric part denoted by $g_{\mu\nu}$ leads to a massless spin-two particle called the graviton. We also have an antisymmetric part, denoted by $B_{\mu\nu}$. Lastly, we have a trace term denoted by ϕ which corresponds to a massless scalar that is called the dilaton, which will often appear in later chapters of the thesis [3]. The first term implies something quite remarkable. The appearance of the massless spin two particle, the graviton, means that Einstein gravity (whether accompanied by terms of higher derivatives or not) has appeared in our theory. We thus obtained a quantized and Lorentz-invariant theory that naturally leads to a force-carrying field for gravity. This result mainly led to the astonishing interest in string theory, because it gave insight of how a theory that combines quantum field theory and general relativity would look.

We will now reevaluate our very short description of bosonic string theory.
The great property we found was:

- Quantum gravity: As explained bosonic string theory spontaneously leads to a field that can be related to gravity. Since it is also a quantized and Lorentz invariant theory, we obtain a theory that combines both general relativity and quantum field theory.

We also found some strange properties that certainly need to be solved in order to get a theory that describes reality. These include:

- Tachyons: We observed from the mass spectrum that we end up with particles that possess a negative squared mass. This is certainly not something that we observe in nature.
- Dimension: The spacetime dimension of bosonic string theory is $D = 26$. Since we seem to live in a four-dimensional world, we have a big amount of dimensions that we need to lose before getting a reality-proof theory.
- No fermions: This is quite obvious, but bosonic string theory only describes bosonic particles. In order to get a theory that also describes electrons, muons, quarks and many others that have been observed, we need a version of string theory that allows for these particles.

We will see that for all these obstacles solutions have been devised, albeit most solutions come with problems of their own [2].

1.2 Superstring Theory

Let us start with the obvious question: How do we get fermions? One way of approaching this problem, is by just adding fermionic fields to the Polyakov action and hoping that this will also lead to fermionic particles in the spectrum. This is called the Ramond-Neveu-Schwarz (RNS) formalism and will lead to manifest supersymmetry on the worldsheet. Fermionic fields are represented by spinors, which are quite different objects compared to the bosonic fields X^μ. Denoting these fermionic fields by ψ^μ (we do not write the spinor indices) and gauge fixing the different supersymmetries, the action in superconformal gauge becomes [1]

$$S = -\frac{1}{8\pi}\int d^2\sigma\Big(\frac{2}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X_\mu + 2i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\Big), \tag{1.12}$$

which is just the action of the bosonic string theory supplemented with a Dirac action. The matrices ρ_α therefore also represent the Dirac matrices. Furthermore, note that ψ^μ are fermionic fields on the worldsheet and not in spacetime (then it has normal vector indices). So why do we call the theory that is related to this action Superstring Theory? This is due to the fact that the action is invariant under the following infinitesimal transformations:

$$\delta_\epsilon X^\mu = i\bar{\epsilon}\sqrt{\frac{\alpha'}{2}}\psi^\mu; \quad \delta\psi^\mu = \frac{1}{\sqrt{2\alpha'}}\rho^\alpha \partial_\alpha X^\mu \epsilon, \tag{1.13}$$

where ϵ is a spinor that parametrizes the transformations. As can be seen, these are symmetries that relate the bosonic fields to the fermionic fields and vice versa. These are what we call supersymmetry transformations and this theory is therefore supersymmetric. Hence the name superstring theory. Just as in in the bosonic case we would like to look at the boundary conditions for this action. For the bosonic fields these do not change and we therefore focus on the action for the fermionic fields. Deriving these boundary conditions is most easily done in lightcone coordinates, defined by $\sigma_\pm = \tau \pm \sigma$. In these coordinates the action for the fermionic fields becomes

$$S_F = \frac{i}{2\pi}\int d^2\sigma(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-). \tag{1.14}$$

Varying this action just like for the bosonic case, leads to the equation of motion

$$\partial_- \psi_+^\mu = \partial_+ \psi_-^\mu = 0. \tag{1.15}$$

The variation term for the boundaries that has to vanish this time is

$$\delta S_F = \frac{1}{2\pi} \int_{\tau_0}^{\tau_1} d\tau \left(\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-\right)\Big|_{\sigma=0}^{\sigma=l}. \tag{1.16}$$

Using that we imposed $\delta\psi^\mu(\tau_0) = \delta\psi^\mu(\tau_1) = 0$ by deriving the equations, there are different ways to get this variation to vanish which depends on whether we look at closed or open strings.
For closed strings the boundary conditions become

$$\psi_\pm^\mu(\tau, \sigma + l) = \pm\psi_\pm^\mu(\tau, \sigma) \tag{1.17}$$

and we can thus choose periodicity and anti-periodicity for the fermionic fields. The periodic boundary conditions are named Ramond (R) boundary conditions and the anti-periodic boundary conditions Neveu-Schwarz (NS) boundary conditions. We can choose the conditions for ψ_+^μ independently from the conditions for ψ_-^μ and we can thus identify four sectors for the closed string: the (R,R), (R,NS), (NS,R) and (NS,NS) sectors.
For the open string we have to impose $\psi_+^\mu = \pm\psi_-^\mu$ at every end of the string. Since the overall sign is just convention, we can without loss of generality set [3]

$$\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0). \tag{1.18}$$

Doing this, the conditions at the other end become

$$\psi_+^\mu(\tau, l) = \pm\psi_-^\mu(\tau, l), \tag{1.19}$$

where again the periodic boundary condition is called the Ramond boundary condition and the anti-periodic boundary condition the Neveu-Schwarz boundary condition. These conditions are important since they will show up in the spectrum later on. For closed strings the states of this spectrum that are in the (R,R) and (NS,NS) sectors will correspond to spacetime bosons and the states in the (R,NS) and (NS,R) sectors to spacetime fermions. For the open string the states in the R sector will correspond to spacetime fermions and the states in the NS sector to spacetime bosons.
Following the path of bosonic string theory, we should again quantize this theory and thereafter we can derive the mass spectrum of the strings. For open strings we then have to look separately at the NS and R sectors. For the NS sector the spectrum will turn out to be

$$M_{NS}^2 = \frac{1}{\alpha'}(N_B + N_F + (\frac{\Delta X}{2\pi\alpha'})^2 - \frac{D-2}{16}), \tag{1.20}$$

where ΔX is the distance between the (possibly different) D-branes that the ends of the string are attached to. Also N_B and N_F are the integer bosonic and half-integer fermionic levels respectively. Additionally, the spectrum in the R sector is given by

$$M_R^2 = \frac{1}{\alpha'}(N_B + N_F + (\frac{\Delta X}{2\pi\alpha'})^2), \tag{1.21}$$

where this time the N_F has integer values. Again, in the NS sector the first excited state is a vector representation of the group $SO(D-2)$ and thus for the same reasons as mentioned before should be massless. Since for the fermionic string the first excited state is given by $N_B = 0$ and $N_F = \frac{1}{2}$ the mass for the first excited state in the NS sector obeys

$$M_{NS}^2 = \frac{1}{\alpha'}(\frac{1}{2} - \frac{D-2}{16}) = 0, \tag{1.22}$$

which then leads to the dimensional condition $D = 10$. Just as with the bosonic string it follows that this condition does not only hold for the first excited states, but for all of them. So by adding the fermionic fields we at least lost more than half of the 26 dimensions, but a four-dimensional theory is unfortunately still not close at hand.

There are also two additional problems with this spectrum. Firstly we can still end up with tachyons in the NS sector and secondly it is not spacetime supersymmetric. We would like to have this spacetime supersymmetry, because supersymmetric theories are promising candidates for an extension of the standard model. In this extension every particle has a supersymmetric partner or superpartner. However, we want to introduce the minimal amount of these new unobserved particles that are needed for a theory consistent with phenomenology. Therefore this model is called the Minimal Supersymmetric Standard Model(MSSM). This model predicts the unification of all three gauge couplings and thus suggests a Grand Unified Theory and it could also provide an explanation for the mysterious dark matter [4].
To achieve this spacetime supersymmetry in the RNS formalism, Gliozzi, Scherk and Olive came up with a limiting operation on the spectrum, therefore called the GSO projection [5]. It is based on the operator $(-1)^F$, with F the worldsheet fermion number. The projection is then defined as demanding that either $(-1)^F = 1$ or $(-1)^F = -1$ for all states. Using this projection, we obtain a tachyon-free and spacetime supersymmetric spectrum for the open string and after this we can look at the spectrum of a closed string. For the spectrum of the closed string we can just take the tensor product of two open string spectra, where one is for the left-moving strings and the other for the right-moving ones. Ofcourse, we then also need to apply the GSO projection on both the left-moving and right-moving string spectra, which again amounts to two possibilities, namely

$$(-1)^F = (-1)^{\tilde{F}} \quad \text{or} \quad (-1)^F = -(-1)^{\tilde{F}}, \tag{1.23}$$

where the tilde refers to the left-moving strings once more. Both of these spectra, now tachyon-free, contains two gravitinos (fermions with spin 3/2) and this means that the theory is $N = 2$ supersymmetric. These gravitinos are the supersymmetric partners of the graviton. Since we have two choices in the GSO projection, these choices also correspond to two different superstring theories. For $(-1)^F = (-1)^{\tilde{F}}$ the theory is called type IIB superstring theory and for $(-1)^F = -(-1)^{\tilde{F}}$ type IIA superstring theory. The massless spectra of these theories are those of the corresponding supergravity theory in ten dimensions, i.e. a supersymmetric gravity theory. [1] We define the massless fields for the closed string at each sector below.

- NS-NS sector: This sector contains a scalar, an anti-symmetric two-tensor and a symmetric two-tensor. This may sound very familiar and indeed, these are just the massless fields that we found for the bosonic string: the graviton $g_{\mu\nu}$ the anti-symmetric field $B_{\mu\nu}$ and the dilaton ϕ.
- R-R sector: For the R-R sector we have to distinguish between type IIA and IIB. For type IIA we have a one-form gauge field C_1 and a three-form gauge field C_3. For type IIB we have three gauge fields, namely a scalar (zero-form) gauge field C_0, a two-form gauge field C_2 and a self-dual four-form gauge field C_4. The field C_0 is often called an axion. Here the forms are the ordinary differential forms of which a simple definition is given in [6].

- NS-R and R-NS sectors: These sectors contain the fermions and in both sectors there is one gravitino and one dilatino. For type IIA the gravitini have opposite chirality, while for type IIB they have the same chirality.

1.3 Kaluza-Klein Compactification

A problem that showed up for both bosonic string theory and superstring theory is that we end up with more dimensions than we actually observe. A manner to make these superfluous dimensions small enough such that we can not observe them by any modern-day experiments is called compactification. We will try to convey some general aspects of compactification by discussing Kaluza-Klein compactification on a circle.
Let us consider a free massless scalar field $\phi(x^M)$ that lives in five space-time dimensions. With the coordinates $x^M = (x^\mu, y)$, we can compactify the last coordinate on a circle by using the equivalence $y \sim y + 2\pi n R$, with $n \in \mathbb{Z}$ and R the radius of the circle. What we basically do here is mathematically turning a line into a circle. What does this change in the fifth coordinate mean for the scalar field? The equation of motion for ϕ is the very famous wave equation given by

$$\partial_\mu \partial^\mu \phi + \partial_y^2 \phi = 0. \tag{1.24}$$

Since $\phi(x, y + 2\pi R) = \phi(x, y)$, we can perform a Fourier expansion that looks like

$$\phi(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iny/R}. \tag{1.25}$$

When we fill this into the wave equation, we obtain

$$\partial_\mu \partial^\mu \phi_n - \frac{n^2}{R^2} \phi = 0 \Rightarrow \partial_\mu \partial^\mu \phi_n = \frac{n^2}{R^2} \phi_n. \tag{1.26}$$

Remembering the Klein-Gordon equation (not coincidently the same Oskar Klein) from Quantum Field Theory [7], we see that $\phi_n(x)$ are now four-dimensional scalar fields with masses $\frac{n}{R}$. Since we do not want to observe the compactified dimension, at a certain moment we have to make the circle on which we compactify very small, i.e. set $R \to 0$. Then we see that the only mode for which the masses do not get overwhelmingly big, is the zero mode at $n = 0$ and we can safely ignore the others if we look at energies that are much smaller than $1/R$. Thus, we get an extra massless scalar. This emergence of extra massless fields is actually a general aspect of Kaluza-Klein compactifications.
As we saw above, the superstring generally has ten dimensions and we thus need to lose six of them. It is therefore natural to ask how we can compactify more than one dimension. The first thing we can do is compactifying on more circles. Since a circle is a 1-dimensional torus, one can generalize this for higher dimensions n by compactifying on a n-dimensional torus. Ofcourse, now we would like to know how we can apply these compactifications to string theory. It will turn out that for the type IIB superstring, the one that we will focus on in this thesis, it is not a torus that we have to compactify on but a quite unusual mathematical object that will be introduced shortly. It is called a Calabi-Yau manifold. [1]

1.4 The five superstring theories

We naturally came across two different superstring theories, type IIA and IIB, which raises the question: Can we define more superstring theories? Indeed there turn out to be five different consistent 10-dimensional superstring theories and I will give a short description of them below [3].

- Type IIA/IIB: As discussed above, the type II theories are $N = 2$ supersymmetric and they are theories of closed oriented strings, since we distinguish between left- and right-moving strings. Since type IIB has two gravitini of the same handedness it is a chiral theory and therefore contains chiral matter. Type IIA has two gravitini of the opposite handedness and is therefore a non-chiral theory. Since the fermions in the standard model are clearly chiral, one could assume that type IIA is not important at all. However, when applying a certain projection on this theory it also becomes chiral.
- Type I: Type I superstring theory is the theory of unoriented strings, both closed and open. This theory can be obtained from type IIB superstring theory by using that is symmetric under the world-sheet parity operator Ω_p that changes the orientation of the string. Gauging this symmetry then relates the left- and -rightmoving strings and thus leads to unoriented strings [8]. However, this resulting theory has one-point functions that cause divergencies in the interaction amplitudes. It can be made consistent by adding D-branes, which means that the type I theory besides having closed strings also contains open strings.
- Heterotic string theory: Heterotic string theory is a theory that only contains closed strings and is a combination of the left-moving sector of the 26-dimensional bosonic string and the right-moving sector of the 10-dimensional superstring. This is possible since these sectors are totally decoupled. There are two types of these that differ by their gauge group in ten dimensions. The 16 extra dimensions in the right-moving sector are then compactified on an even self-dual lattice for which there are two options: $E_8 \times E_8$ or Spin$(32)/\mathbb{Z}$. These are related to the gauge groups $E_8 \times E_8$ or $SO(32)$ respectively and we can thus define the $E_8 \times E_8$ heterotic string and the $SO(32)$ heterotic string.

1.5 Dualities and M-theory

We already mentioned that type I string theory can be obtained from type IIB string theory by applying a projection that makes a string unoriented. It turns out that there are more relations between the five different string theories, and a very important form of these relations are called dualities. An example of a duality that is very famous, is the particle-wave duality in quantum mechanics: In some circumstances light is described more easily as colliding particles, in others as propagating waves. In string theory dualities work the same way. The most important dualities in string theory are called the T-duality and the S-duality. With T-duality a circular dimension with radius R is related to a circular dimension with radius $\frac{\alpha'}{R}$. S-duality relates a theory with coupling constant g_s to a theory with coupling constant $\frac{1}{g_s}$. The nice attribute of these dualities is that some string theories appeared to be equivalent when applying them, just as light can be described equivalently as particle or as wave. For instance, it was discovered that the T-duality relates the type IIA to the type IIB string theory and that the S-duality relates the type I string theory to the $SO(32)$ heterotic string theory. [3]
These different relations between all the superstring theories led to the question if there is some overlapping theory of which the five superstring theories are just some limit. This theory was first proposed by Edward Witten and is called M-theory. (The M is chosen very randomly, and according to Witten stands for "Magic, "Mystery" or "Membrane" [9].) M-theory is a 11-dimensional theory for which the low-energy limit is $D = 11$ supergravity. Different 10-dimensional superstring theories can be obtained from the 11-dimensional theory. For example, M-theory with one dimension compactified

on a circle corresponds to type IIA string theory when it has a certain coupling constant for the strings. Similarly, if we compactify M-theory on the orbifold $S^1/\mathbb{Z}$, this corresponds to the $E_8 \times E_8$ heterotic string theory. [3]

1.6 Outline

As is probably clear by now, the landscape of string theory is very vast and it is therefore impossible to give a complete overview. To make this thesis still somewhat approachable for physicists who are not familiar with string theory, we have tried to introduce most of the concepts needed for the main focus of this thesis. For the central and more detailed part of the thesis, we will give a short outline below.

We will start with discussing the mathematical language that will be used throughout the thesis. After that we will introduce Calabi-Yau manifolds and discuss what its moduli spaces look like. Thereafter, we look at type IIB string theory, the theory that we will be focusing on in this thesis, and discuss its compactification on a Calabi-Yau manifold. The next thing we do is looking at type IIB compactified on Calabi-Yau orientifolds accompanied by fluxes. These fluxes then lead us to moduli stabilization, and we will discuss it in the KKLT scenario. After moduli stabilization, we will derive an expression for the moduli mass matrix, take the trace of this and relate it to a special metric that is called the Hodge metric. We will then try to find different constraints on this mass value. Lastly, we will look at this relation in three specific cases for the complex structure moduli space: the case of one complex structure modulus, the case of a complex structure moduli space that is Kähler-Einstein, and the case where we look at the boundary of complex structure moduli space.

Chapter 2

Mathematical background

To describe the effective field theory of type IIB, we have to introduce concepts as differential forms and cohomology. Also, since a Calabi-Yau manifold is a complex manifold, we will establish some complex geometry that will be used throughout the thesis and that paves the way for a complete definition of a Calabi-Yau manifold. Since we will need the calculus of both real manifolds and complex manifolds, we will divide this chapter in these separate sections. However, we will see that many expressions on a real manifold that we discuss have their counterpart on a complex manifold. We will assume that a metric g is always defined on the manifold. For this discussion, we will also assume that the reader is familiar with manifolds in general and real differential forms (see [10] for a complete overview) and we will just define the mathematical objects that are needed for the rest of the dissertation.

2.1 Real manifolds

2.1.1 Real differential forms

We will start with defining the anti-symmetric product between differential forms. This exterior product or wedge product $\wedge$ of an r-form is given by

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} = \sum_{P \in S_r} \mathrm{sgn}(P) dx^{\mu_{P(1)}} \wedge \cdots \wedge dx^{\mu_{P(r)}}, \tag{2.1}$$

with P a permutation from the symmetric group S_r and $\mathrm{sgn}(P) = 1$ for even permutations and $\mathrm{sgn}(P) = -1$ for odd permutations. Also, dx^μ is the differential of x^μ and defines a one-form. Thus for two forms $dx \wedge dy = -dy \wedge dx$. Using this product, a general r-form can be written as

$$A = \frac{1}{r!} A_{\mu_1 \cdots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \tag{2.2}$$

We can also define an operator $*$ called the Hodge star, which works on a differential form as

$$*A = \frac{\sqrt{g}}{r!(n-r)!} A_{\mu_1 \mu_2 \ldots \mu_r} \epsilon^{\mu_1 \mu_2 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_n} \, \mathrm{d}x^{\nu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_n}, \tag{2.3}$$

where n is the dimension of the manifold and $\epsilon^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_{r+1}\ldots\nu_n}$ is the totally anti-symmetric tensor. Also, g is the determinant of the metric on the manifold [10]. The exterior derivative of a differential form is a $(r+1)$-form given by

$$dA = \frac{1}{r!}\partial_\mu A_{\mu_1\cdots\mu_r} dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \tag{2.4}$$

A differentiable form is called closed if $dA = 0$ and it is called exact if there exists a $(r-1)$-form B such that $B = dA$. Using the Hodge star and the exterior derivative we can define the following new operator

$$d^* = - * d* \tag{2.5}$$

and with that the Laplacian

$$\Delta = dd^* + d^*d. \tag{2.6}$$

The r-forms A_r are then called harmonic whenever

$$\Delta A_r = 0. \tag{2.7}$$

2.1.2 De Rham cohomology and Hodge theorem

We can now put these closed and exact r-forms into separate groups. Given a real manifold M we can denote the set of all closed differential r-forms by $Z^r(M)$. Similarly, we can denote the set of exact r-forms by $B^r(M)$. The de Rham cohomology group is then defined by the quotient of these sets, i.e.

$$H^r(M) = Z^r(M)/B^r(M) \tag{2.8}$$

and thus consists of the closed but non-exact forms. This is called the cohomology group since it is dual to the homology group H_r. If we take an r-form $\omega_r \in Z^r$ that is thus closed, its cohomology class $[\omega_r] \in H^r$ is then given by $\{\omega'_r \in Z^r | \omega'_r = \omega_r + d\psi_{r-1}\}$, i.e. each closed form that differs by an exact form. More details of homology, next to cohomology, can also be found in [10].
The Hodge decomposition theorem states that on any orientable compact Riemannian manifold any r-form A_r can be uniquely decomposed as

$$A_r = h_r + d\alpha_{r-1} + d^*\beta_{r+1}, \tag{2.9}$$

where h_r is an harmonic r-form, i.e it obeys $\Delta h_r = 0$. However, for closed forms $d^*\beta_{r+1} = 0$. Thus if we take an element $\omega_r \in H^r$, which are by definition closed, the expansion just becomes $\omega_r = h_r + d\alpha_{r-1}$. Since α_{r-1} is an exact form, it follows that each cohomology class has an unique harmonic representative. This property will be put to use later on [1].

2.2 Complex manifolds

Just as real manifolds with dimension m locally look like $\mathbb{R}^m$, complex manifolds of dimension n locally look like $\mathbb{C}^n$. Also, a complex manifold of dimension n can be seen as a manifold of $2n$ real dimensions. Returning to superstring theory, we saw that we need to lose six real dimensions. Since we want to look at compactification on Calabi-Yau manifolds and these are complex manifolds, we will look at complex manifolds of dimension $n = 3$ in particular. Calabi-Yau manifolds are not only complex but also Kähler manifolds, and therefore Hermitian. We will also discuss the properties of these kinds of manifolds below.

2.2.1 Complex differentiable forms

A differentiable form of complex dimension k, can be split up in terms of p holomorphic coordinates and q anti-holomorphic coordinates such that $k = p + q$. A complex differentiable (p, q)-form ψ can then be written as

$$\psi = \frac{1}{p!q!}\psi_{i_1\cdots i_p j_1\cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge d\bar{x}^{j_1} \wedge \cdots \wedge d\bar{x}^{j_q}, \tag{2.10}$$

where the wedge product $\wedge$ works in the same way as before. As one can see, we have now defined two sets of coordinates: x^i and its complex conjugate $\bar{x}^i$ with $i = 1, 2, 3$.
The corresponding Hodge star operation in three complex dimensions is given by

$$\begin{aligned} *\psi = {} & \frac{i(-1)^{3p}}{p!q!(3-p)!(3-q)!} g\epsilon^{m_1\ldots m_p}{}_{j_1\ldots \bar{j}_{3-p}}\epsilon^{\bar{n}_1\ldots\bar{n}_q}{}_{i_1\ldots i_{3-q}} \\ & \cdot \psi_{m_1\ldots m_p\bar{n}_1\ldots\bar{n}_q} dx^{i_1} \wedge \cdots \wedge dx^{i_{3-q}} \wedge d\bar{x}^{j1} \wedge \cdots \wedge d\bar{x}^{j_{3-p}}. \end{aligned} \tag{2.11}$$

$*\psi$ is then called the Hodge dual of the form ψ. Furthermore ψ is called self-dual if these are equal. The exterior derivative of a (p, q)-form ψ given by $d\psi$, is a $(p + 1, q + 1)$-form and it can be split up in the operators ∂ and $\bar{\partial}$ that act like

$$\begin{aligned} \partial\psi &= \frac{1}{p!q!}\partial_i\psi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dx^i \wedge dx^{i_1} \wedge \cdots \wedge d\bar{x}^{j_q} \\ \bar{\partial}\psi &= \frac{1}{p!q!}\partial_{\bar{j}}\psi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} d\bar{x}^j \wedge dx^{i_1} \wedge \cdots \wedge d\bar{x}^{j_q}, \end{aligned} \tag{2.12}$$

with $\partial_{\bar{j}} = \frac{\partial}{\partial\bar{x}^j}$. The total exterior derivative is then just defined as $d = \partial + \bar{\partial}$.

2.2.2 Hermitian manifolds

A Hermitian manifold is a complex manifold that is paired with a metric g that obeys $g_{j\bar{\imath}} = \overline{g_{i\bar{\jmath}}}$ and that is positive definite matrix, i.e. $g_{i\bar{\jmath}}v^i v^{\bar{j}} \geq 0$ for any $v^i \in \mathbb{C}^n$ and with equality if and only if $v^i = 0$. Because of these two conditions $g_{i\bar{\jmath}}$ are also the components of a positive-definite Hermitian matrix. An Hermitian metric satisfies $g_{ij} = -g_{ij}$ and therefore $g_{ij} = 0$. Using this metric, we can define a special $(1, 1)$-form by

$$J = ig_{i\bar{\jmath}}dx^i \wedge d\bar{x}^j, \tag{2.13}$$

that satisfies $\bar{J} = J$ and is thus real. This is called the fundamental form of g and we will use it again later. Using the Hermitian metric defined above, we can define a covariant derivative. The covariant derivative along a real basis vector e_μ is given by

$$\nabla_\mu e_\nu = \Gamma^\lambda_{\mu\nu}e^\lambda, \tag{2.14}$$

with $\Gamma^\lambda_{\mu\nu}$ the connection coefficients. So for the bases $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial\bar{x}^j}$ of a complex manifold this means that

$$\nabla_i\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}, \quad \nabla_{\bar{\imath}}\frac{\partial}{\partial\bar{x}^j} = \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}}\frac{\partial}{\partial\bar{x}^k}, \tag{2.15}$$

where $\Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = \overline{\Gamma}^{k}_{ij}$. These two components are the only non-vanishing connection coefficients since $\nabla_i \frac{\partial}{\partial \bar{x}^j} = \nabla_{\bar{\imath}} \frac{\partial}{\partial x^j} = 0$. On the dual bases or the one forms dx^i and $d\bar{x}^j$ the covariant derivative then behaves as

$$\nabla_i \mathrm{d}x^j = -\Gamma^j_{il} dx^l \quad \nabla_{\bar{\imath}} \mathrm{d}\bar{x}^j = -\Gamma^{\bar{\jmath}}_{\bar{\imath}\bar{l}} d\bar{x}^l. \tag{2.16}$$

We also demand the connection to be metric compatible, i.e. we make sure that the inner product between vectors stays the same along any curve. This condition amounts to

$$\nabla_i g_{j\bar{l}} = \partial_i g_{j\bar{l}} - \Gamma^k_{ij} g_{k\bar{l}} = 0 \quad \text{and} \quad \nabla_{\bar{\imath}} g_{l\bar{\jmath}} = \partial_{\bar{\imath}} g_{l\bar{\jmath}} - \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} g_{l\bar{k}} = 0. \tag{2.17}$$

Using these equations, the non-vanishing components of the Christoffel symbols that describe this connection are easily shown to be

$$\Gamma^k_{ij} = g^{k\bar{l}} \partial_i g_{j\bar{l}}, \quad \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = g^{l\bar{k}} \partial_{\bar{\imath}} g_{l\bar{\jmath}}. \tag{2.18}$$

With g the Hermitian metric, this connection is fittingly called a Hermitian connection. The connection for which the mixed components vanish next to being Hermitian is unique, and is called the Chern connection [10].

Going further, we can also find the non-vanishing components of the Riemann tensor which are given by

$$R_{i\bar{\jmath}k\bar{l}} = \partial_i \partial_{\bar{\jmath}} g_{k\bar{l}} - g^{m\bar{n}} \left(\partial_i g_{k\bar{n}}\right) \left(\partial_{\bar{\jmath}} g_{m\bar{l}}\right) \tag{2.19}$$

and the other components that are related to this one by the symmetries of the Riemann tensor. Contracting this with the inverse metric then leads to the Ricci tensor, that can be written as [1]

$$R_{i\bar{\jmath}} = -\partial_i \partial_{\bar{\jmath}} \log(\det g). \tag{2.20}$$

Furthermore, the Ricci-form is defined as

$$\mathcal{R} = i R_{i\bar{\jmath}} dx^i \wedge d\bar{x}^j = -i \partial\bar{\partial} \log(\det g). \tag{2.21}$$

And lastly, using this Ricci-form, we can define the first Chern class of the manifold, which is

$$c_1(M) = [\mathcal{R}/2\pi], \tag{2.22}$$

where the equivalence class to which $\mathcal{R}/2\pi$ belongs is denoted by [...] [10].

2.2.3 Kähler manifolds

A Kähler manifold is a Hermitian manifold with the additional property that it possesses a fundamental form J such that $dJ = 0$, i.e a closed fundamental form. The fundamental form is then called a Kähler form and the metric on the manifold a Kähler metric. What does this closedness imply? If we split up the exterior derivative as $d = \partial + \bar{\partial}$ the closed-condition leads to two separate conditions, namely $\partial J = \bar{\partial} J = 0$. Using the equations (2.12), we find that

$$\begin{aligned} \partial J &= i \partial_k g_{i\bar{\jmath}} dx^k \wedge dx^i \wedge d\bar{x}^j = -i \partial_k g_{i\bar{\jmath}} dx^i \wedge dx^k \wedge d\bar{x}^j = -i \partial_i g_{k\bar{\jmath}} dx^k \wedge dx^i \wedge d\bar{x}^j = 0 \\ &\Rightarrow \partial_k g_{i\bar{\jmath}} = \partial_i g_{k\bar{\jmath}}, \end{aligned} \tag{2.23}$$

where in the first step we used the anti-symmetric property of the wedge product and in the second we interchanged indices. Doing the same thing for $\bar{\partial}$, we obtain

$$\begin{aligned} &\bar{\partial}J = i\partial_{\bar{k}}g_{i\bar{j}}d\bar{x}^k \wedge dx^i \wedge d\bar{x}^j = -i\partial_{\bar{j}}g_{i\bar{k}}d\bar{x}^k \wedge dx^i \wedge d\bar{x}^j = 0 \\ &\quad \Rightarrow \partial_{\bar{k}}g_{i\bar{j}} = \partial_{\bar{j}}g_{i\bar{k}} \end{aligned} \tag{2.24}$$

From these conditions it can then be derived that for any Kähler metric there exists a real scalar K such that locally

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K. \tag{2.25}$$

The quantity K is called the Kähler potential [10]. For a Kähler manifold, the Chern connection for this metric is symmetric and it coincides with the well-known Riemannian or Levi-Civita connection defined above [1].

2.2.4 Dolbeault cohomology and Hodge theorem

For complex differential forms we can define similar expressions as for the real forms. This time we have the two differential operators ∂ and $\bar{\partial}$ and we can thus define things like closed and exact using one of these operators. Choosing $\bar{\partial}$, the Dolbeault operator, we can define the $\bar{\partial}$-cohomology group or the Dolbeault cohomology group on a complex manifold M as [10]

$$H^{p,q}_{\bar{\partial}}(M) = Z^{p,q}_{\bar{\partial}}(M)/B^{p,q}_{\bar{\partial}}(M), \tag{2.26}$$

where $Z^{p,q}_{\bar{\partial}}(M)$ is the set of $\bar{\partial}$-closed (p,q)-forms and $B^{p,q}_{\bar{\partial}}(M)$ the set of $\bar{\partial}$-exact (p,q)-forms. The dimensions of this group are denoted by

$$h^{p,q} = \dim(H^{p,q}_{\bar{\partial}}(M)) \tag{2.27}$$

and are called Hodge numbers. For three-dimensional Kähler manifolds they satisfy

- $h^{p,q} = h^{q,p}$
- $h^{p,q} = h^{3-p,3-q}$.

Since we focus on $\bar{\partial}$ we can similarly to before define

$$\bar{\partial}^* = - * \partial *, \tag{2.28}$$

which satisfies $\bar{\partial}^{*2} = 0$ and the corresponding Laplacian is

$$\Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{2.29}$$

A (p,q)-form ψ is then called harmonic whenever $\Delta_{\bar{\partial}}\psi = 0$. There is also a Hodge decomposition theorem for forms on an Hermitian manifold. This then states that on a compact Hermitian manifold any (p,q)-form ψ can be decomposed as [11]

$$\psi_{p,q} = h_{p,q} + \bar{\partial}\varphi_{p,q-1} + \bar{\partial}^*\eta_{p,q+1}, \tag{2.30}$$

where $h_{p,q}$ is a harmonic (p,q)-form. Again, if $\psi_{p,q}$ is closed ($\bar{\partial}\psi_{p,q} = 0$), then $\bar{\partial}^*\eta_{p,q+1} = 0$ and it follows that each element of $H^{p,q}$ has an harmonic representative up to an $\bar{\partial}$-exact form. Thus, every $\bar{\partial}$-cohomology class of (p,q)-forms has a unique harmonic representative and every harmonic form defines a cohomology class [1]. A wonderful property of a Kähler manifold is that the different Laplacians align, i.e.

$$\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta. \tag{2.31}$$

This simultaneously means that all the ∂, $\bar{\partial}$ and d-harmonic forms coincide. This is a very useful property that directly relates the elements of the Dolbeault cohomology group to real differential forms. Here we will end our discussion of complex and Kähler manifolds, and turn to the main mathematical object of interest instead: Calabi-Yau manifolds.

Chapter 3

Calabi-Yau threefolds

We experience the world around us as four-dimensional and therefore a theory with 10 or even 26 space-time dimensions does not seem to be compatible with this experience. However, using compactification these superfluous dimensions can be made so small such that they become unnoticeable. The simplest model to consider here is one with only a background metric, and demanding that the D-dimensional manifold satisfies

$$M_D = M_d \times M_n, \tag{3.1}$$

where M_n is the n-dimensional internal manifold which should have a size that is much smaller than any length scales that we have measured yet. We thus consider a perturbative approach around a geometric supergravity background at large radius through which the strings propagate [1]. Another requirement of the theory is that we have spacetime supersymmetry. This comes from the fact that the standard model is incomplete, and supersymmetry breaking could solve this incompleteness. For type IIB string theory the compactification manifold M_n that preserves spacetime supersymmetry will turn out to be a Calabi-Yau manifold and we will motivate this somewhat more in the next section. Ofcourse, since we want to compacify six dimensions, and a Calabi-Yau manifold is a complex manifold, we are interested in the manifolds of complex dimension $n = 3$ or Calabi-Yau threefolds.

3.1 General properties

A condition for spacetime supersymmetry is that we need covariantly constant spinors. This means that for a random spinor ϵ

$$\bar{\nabla}_\mu \epsilon = 0; \quad \bar{\nabla}_m \epsilon = 0, \tag{3.2}$$

where the Greek indices refer to spacetime and the Roman indices to the internal manifold. This condition then also affects the choice of internal manifolds that we can make. It can be derived from Eq. (3.2) that the Ricci tensor of the internal manifold has to obey

$$R_{mn} = 0, \tag{3.3}$$

i.e the internal manifold has to be Ricci-flat [1].

The mathematical definition of a Calabi-Yau manifold is the following: A Calabi-Yau manifold is a compact Kahler manifold with a vanishing first Chern class. It was suggested by Calabi and later proved by Yau that this means that the manifold admits a Kähler metric that has zero Ricci curvature and that is precisely the condition that we derived above. For Calabi-Yau threefolds this also corresponds to having the holonomy group $SU(3)$. This means that the set of transformations that are possible for this vector when we parallel transport it along a closed curve on this manifold, is the group $SU(3)$. It turns out that the admittance of covariantly constant spinors also corresponds to the condition of the internal manifold having the holonomy group $SU(3)$. A Calabi-Yau manifold is thus the Ricci-flat manifold that we are looking for. [10] [1].

The properties of a Calabi-Yau threefold lead to the notion that there exists a, up to a constant, unique nowhere-vanishing harmonic three-form Ω, which we will use extensively later on. For Calabi-Yau threefolds some more conditions for the Hodge numbers as defined in Eq. (2.27) hold:

- $h^{3,0} = h^{0,3} = 1$
- $h^{p,0} = h^{0,p} = h^{0,3-p} = h^{3-p,0}$
- $h^{p,0} = h^{0,p} = 0$ for $0 < p < 3$

It is customary to show the Hodge number diagrammatically in the form of a diamond. In three dimensions and with the conditions stated above, the Hodge diamond becomes [1]:

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

3.2 Moduli of Calabi-Yau threefolds

In this section we will largely follow the arcticle by Philip Candelas and Xenia de la Ossa [12]. The condition of Ricci-flatness for Calabi-Yau manifolds is given by

$$R_{mn}(g) = 0, \tag{3.4}$$

where g is the Kähler metric. What we want to find out is which deformations of this metric are allowed without losing the Ricci-flatness, i.e. while retaining a Calabi-Yau manifold, which comes down to the equation

$$R_{mn}(g + \delta g) = 0. \tag{3.5}$$

Working this out, and using the coordinate condition $\nabla^n \delta g_{mn} = 0$, we obtain the Lichnerowicz equation, which is given by

$$\nabla^k \nabla_k \delta g_{mn} + 2R_{m\ n}^{\ k\ l} \delta g_{kl} = 0. \tag{3.6}$$

The components of this equation with mixed and pure indices are independent and we can therefore treat them separately. To the mixed deformations we can associate the $(1,1)$-form

$$i\delta g_{i\bar{j}} dx^i \wedge d\bar{x}^j. \tag{3.7}$$

Using the Hodge theorem the deformation $\delta g_{i\bar{j}}$ can be expanded into harmonic forms, since then they will satisfy the Lichnerowicz equation and then we have

$$\delta g_{i\bar{j}} = -iv^\alpha (\omega_\alpha)_{i\bar{j}}, \tag{3.8}$$

where v^α are $h^{1,1}$ real scalars and ω_α is harmonic. Also, in three dimensions there exists, up to constant, a unique holomorphic three-form Ω (see the previous section) given by

$$\Omega = \frac{1}{3!} \Omega_{ijk} dx^i \wedge dx^j \wedge dx^k \tag{3.9}$$

and we can associate this form to the deformations of the pure type by defining the $(2,1)$-form

$$\Omega_{ij}^{\ \ \bar{k}} \delta g_{\bar{k}\bar{l}} dx^i \wedge dx^j \wedge d\bar{x}^l. \tag{3.10}$$

When the Hodge star from Eq. (2.11) is applied on this holomophic threeform, we get [1]

$$*\Omega = i\Omega. \tag{3.11}$$

The zero modes of the Lichnerowicz equation clearly divide between the two cohomology groups $H^{2,1}(M)$ and $H^{1,1}(M)$ and they respectively give rise to $h^{2,1}$ complex parameters and $h^{1,1}$ real parameters. Also, modes of the first type are variations of the complex structure, modes of the second type variations of the Kähler class (why this is, is explained in [12] too). There also occurs a two-form B on the internal metric which can be related to the metric by supersymmetry.

The most general metric decomposed into pure and mixed parts that can be defined in terms of these quantities is given by

$$ds^2 = \frac{1}{2V} \int g^{i\bar{j}} g^{k\bar{l}} [\delta g_{ik} \delta g_{\bar{j}\bar{l}} + (\delta g_{i\bar{l}} \delta g_{k\bar{j}} + \delta B_{i\bar{l}} \delta B_{\bar{j}k})] g^{1/2} d^6 x, \tag{3.12}$$

which is a block-diagonal metric and in which $V = \frac{1}{3!} \int J \wedge J \wedge J$ with J the Kähler form. We shall now discuss the variations of the metric related to the complex strucure and the Kähler class separately.

3.2.1 The complex structure moduli space

We look at the deformations connected to the (2,1)-forms first. We can define

$$\chi_{aij\bar{k}} = -\frac{1}{2}\Omega_{ij}{}^{\bar{l}}\frac{\partial g_{\bar{l}\bar{k}}}{\partial z^a}, \quad \chi_a = \frac{1}{2}\chi_{aij\bar{k}}dx^i \wedge dx^j \wedge d\bar{x}^{\bar{k}}, \tag{3.13}$$

where the z^a are the $h^{2,1}$ parameters of the complex structure. The (2,1)-form χ_a is also imaginary selfdual, i.e.

$$*\chi_a = i\chi_a. \tag{3.14}$$

The inverse of this relation is then given by

$$\delta g_{\bar{\imath}\bar{\jmath}} = -\frac{1}{||\Omega||^2}\overline{\Omega}_{\bar{\imath}}{}^{kl}\chi_{akl\bar{\jmath}}\delta z^a, \tag{3.15}$$

which are the metric deformations of the pure type. Let us now look at the first part of the metric and fill in this relation

$$\begin{aligned} ds^2_{(2,1)} &= \frac{1}{2V}\int g^{i\bar{\jmath}}g^{k\bar{l}}\delta g_{ik}\delta g_{\bar{\jmath}\bar{l}}g^{1/2}d^6x \\ &= \frac{1}{2V||\Omega||^4}\int g^{i\bar{\jmath}}g^{k\bar{l}}\Omega_i{}^{\bar{m}\bar{n}}\overline{\Omega}_{\bar{l}}{}^{mn}\overline{\chi}_{b\bar{m}\bar{n}k}\chi_{amn\bar{\jmath}}\delta\bar{z}^b\delta z^a g^{1/2}d^6x \\ &= \frac{1}{2V||\Omega||^4}\int \Omega^{\bar{\jmath}\bar{m}\bar{n}}\overline{\Omega}^{kmn}\overline{\chi}_{b\bar{m}\bar{n}k}\chi_{amn\bar{\jmath}}\delta z^a\delta\bar{z}^b g^{1/2}d^6x \\ &= -\frac{2i}{V||\Omega||^2}\int \chi_a \wedge \overline{\chi}_b\delta z^a\delta\bar{z}^b, \end{aligned} \tag{3.16}$$

where we used the definition of $||\Omega||^2$ that can be found in [1] and also the definition of the wedge product.
Writing this as $2G_{ab}\delta z^a\delta\bar{z}^b$ we see that the metric is

$$G_{a\bar{b}} = -\frac{\int \chi_a \wedge \chi_{\bar{b}}}{\int \Omega \wedge \overline{\Omega}}, \tag{3.17}$$

which is symmetric in its indices, i.e, $G_{a\bar{b}} = G_{\bar{b}a}$ [10]. A very useful equation for the derivative of Ω (found by Kodaira, see [13]) is given by

$$\frac{\partial\Omega}{\partial z^a} = k_a\Omega + \chi_a, \tag{3.18}$$

where the k_a may depend on the moduli z^a but not on the coordinates of the manifold M.
Using this, we see that

$$\int \Omega \wedge \frac{\partial\Omega}{\partial z^a} = \int \Omega \wedge (k_a\Omega + \chi_a) = 0, \tag{3.19}$$

since the Calabi-Yau manifold does not contain a (6,0)-form or a (5,1)-form or any (p,q)-form with $p > 3$ or $q > 3$. One can similarly show that

$$\int \Omega \wedge \frac{\partial^2\Omega}{\partial z^a\partial z^b} = \int \frac{\partial\Omega}{\partial z^a} \wedge \frac{\partial\Omega}{\partial z^b} = 0 \tag{3.20}$$

We can define an inner product on this manifold in the following manner

$$\langle \alpha, \beta \rangle = -i \int \alpha \wedge \beta. \tag{3.21}$$

Applying this to Eq. (3.18), and taking the inner product with respect to $\overline{\Omega}$, we obtain

$$\langle \frac{\partial \Omega}{\partial z^a}, \overline{\Omega} \rangle = \frac{\partial}{\partial z^a} \langle \Omega, \overline{\Omega} \rangle = k_a \langle \Omega, \overline{\Omega} \rangle + \langle \chi_a, \overline{\Omega} \rangle = k_a \langle \Omega, \overline{\Omega} \rangle, \tag{3.22}$$

where we could take the derivative out of the inner product since $\Omega(\overline{\Omega})$ is (anti)-holomorphic. Going back to the metric, we can also write it as

$$G_{a\bar{b}} = -\frac{\partial}{\partial z^a} \frac{\partial}{\partial \bar{z}^b} \ln[i \int \Omega \wedge \overline{\Omega}], \tag{3.23}$$

which defines a Kähler potential of the complex structure parameter space by

$$K^{\mathrm{cs}} = -\ln[i \int \Omega \wedge \overline{\Omega}] = -\ln[-\langle \Omega, \overline{\Omega} \rangle] \tag{3.24}$$

and thus $\langle \Omega, \overline{\Omega} \rangle = -e^{-K^{\mathrm{cs}}}$. We thus see that the metric $G_{a\bar{b}}$ is also a Kähler metric. Then Eq. (3.22) becomes

$$-\frac{\partial}{\partial z^a} e^{-K^{\mathrm{cs}}} = (\frac{\partial K^{\mathrm{cs}}}{\partial z^a}) e^{-K^{\mathrm{cs}}} = -k_a e^{-K^{\mathrm{cs}}} \tag{3.25}$$

and thus

$$k_a = -\frac{\partial K^{\mathrm{cs}}}{\partial z^a}. \tag{3.26}$$

We can now rewrite Eq. (3.18) in a more simple form:

$$\chi_a = \partial_a \Omega + (\partial_a K^{\mathrm{cs}}) \Omega = D_a \Omega, \tag{3.27}$$

in which D_a is a gauge covariant derivative that was defined by Strominger [14].
To summarize, we saw that the complex structure moduli space or the space that results from varying the pure part of the metric of the Calabi-Yau manifold has its own symmetric metric, that additionally can be written in terms of a Kähler potential. Therefore the complex structure moduli space is a Kähler manifold by itself.
Finally, we can also define a so-called triple intersection number between (2,1)-forms given in terms of the third derivative of Ω by

$$\mathcal{K}_{abc} = -\int \Omega \wedge \frac{\partial^3 \Omega}{\partial z^a \partial z^b \partial z^c} \tag{3.28}$$

which will be used later on [12]. Since the indices show up in partial derivatives the intersection number $\mathcal{K}_{abc}$ is a totally symmetric tensor.

3.2.2 Curvature in complex structure moduli space

We now would like to discuss the curvature of the complex structure moduli space, or what the Ricci tensor looks like. This naturally involves covariant derivatives, and we will therefore also discuss those and how they act on the different objects that we found in the moduli space.

It was explained by Strominger [14] that the three-form Ω is not a scalar over the parameter space, but takes values in a line bundle L, which means that the three-form is undefined up to a multiplication by a complex number. A covariant derivative that depends on in which line bundle the object on which it acts takes values, is defined in the following way. Given an object $\Psi^{(c,\bar{c})}$ which takes values in line bundle $L^c \otimes \bar{L}^{\bar{c}}$, then its charge is called $(c, \bar{c})$ and the covariant derivatives are given by

$$\begin{aligned} D_a \Psi^{(c,\bar{c})} &= (\partial_a + c K_a^{\text{cs}}) \Psi^{(c,\bar{c})} \\ D_{\bar{b}} \Psi^{(c,\bar{c})} &= (\partial_{\bar{b}} + \bar{c} K_{\bar{b}}^{\text{cs}}) \Psi^{(c,\bar{c})}. \end{aligned} \tag{3.29}$$

Both Ω and χ_a have charge $(1, 0)$. We can combine this gauge covariant derivative above and the Levi-Civita connection to define how the covariant derivative acts on χ_a and $\chi_{\bar{a}}$. Because χ_a and $\chi_{\bar{a}}$ have an extra index, we need to take into account Christoffel symbols and the Levi-Civita works in the standard way on differential forms as can be found in [10]. We now go back to Eq. (3.27)

$$\chi_a = D_a \Omega. \tag{3.30}$$

We then see from Eq. (3.29) that

$$\begin{aligned} &D_{\bar{b}} \Omega = \partial_{\bar{b}} \Omega = 0; \\ &D_{\bar{b}} \chi_a = \partial_{\bar{b}} \chi_a - \Gamma^{\bar{k}}_{\bar{b}a} \chi_{\bar{k}} = \partial_{\bar{b}} (\partial_a \Omega + \partial_a K^{\text{cs}} \Omega) = (\partial_a + \partial_a K^{\text{cs}}) \partial_{\bar{b}} \Omega + \partial_a \partial_{\bar{b}} K^{\text{cs}} \Omega = G_{a\bar{b}} \Omega, \end{aligned} \tag{3.31}$$

where we used in the second equation that on a Kähler manifold, which the complex structure moduli space is, the mixed components of the Christoffel symbols vanish. Using these equations, we can now also calculate the commutator of this, given by

$$[D_a, D_{\bar{b}}] \Omega = D_a D_{\bar{b}} \Omega - D_{\bar{b}} \chi_a = -G_{a\bar{b}} \Omega. \tag{3.32}$$

Acting the covariant derivative on χ_a gives us

$$D_a \chi_b = \partial_a \chi_b + (\partial_a K) \chi_b - \Gamma^k_{ab} \chi_k. \tag{3.33}$$

where we used the standard operation of the Levi-Civita connection on forms [15]. Another way of writing the covariant derivative $D_a \chi_b$ is [12]

$$D_a \chi_b = -i e^{K^{\text{cs}}} \mathcal{K}_{ab}{}^{\bar{k}} \chi_{\bar{k}}. \tag{3.34}$$

One last useful expression that relates the Riemann tensor to the intersection number is given by [12]

$$R_{a\bar{b}k\bar{l}} = G_{a\bar{b}} G_{k\bar{l}} + G_{a\bar{l}} G_{k\bar{b}} - e^{2K^{\text{cs}}} \mathcal{K}_{ak}{}^{\bar{m}} \mathcal{K}_{\bar{b}\bar{l}\bar{m}} = G_{a\bar{b}} G_{k\bar{l}} + G_{a\bar{l}} G_{k\bar{b}} - e^{2K^{\text{cs}}} G^{m\bar{n}} \mathcal{K}_{akm} \mathcal{K}_{\bar{b}\bar{l}\bar{n}}. \tag{3.35}$$

Using that the Ricci tensor is given by $R_{a\bar{b}} = -R_{a\bar{b}k}{}^{k} = -G^{k\bar{l}} R_{a\bar{b}k\bar{l}}$ [1], we obtain

$$\begin{aligned} R_{a\bar{b}} &= -G^{k\bar{l}} (G_{a\bar{b}} G_{k\bar{l}} + G_{a\bar{l}} G_{k\bar{b}}) + e^{2K^{\text{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kmi} \mathcal{K}_{\bar{l}\bar{n}\bar{j}} \\ &= -(h^{2,1} + 1) G_{a\bar{b}} + e^{2K^{\text{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kma} \mathcal{K}_{\bar{l}\bar{n}\bar{b}}, \end{aligned} \tag{3.36}$$

where we used that $G^{a\bar{b}} G_{a\bar{b}} = h^{2,1}$ and that $G_{a\bar{b}}$ lowers indices. This formula for the Ricci tensor is also given in [14].

3.2.3 The Kähler moduli space

We know turn to the mixed deformations of the Kähler metric, the Kähler deformations. These can be written as [16] [17]

$$\delta g_{i\bar{j}} = -iv^{\alpha}(\omega_{\alpha})_{i\bar{j}}, \tag{3.37}$$

where v^{α} are $h^{1,1}$ real scalars and ω_{α} is harmonic and therefore satisfies the Lichnerowicz equation. Now we show that also the parameters of the Kähler class make up a Kähler manifold. We will first introduce some expressions. The inner product for real (1,1)-forms ρ and σ is given by

$$G(\rho,\sigma) = \frac{1}{2V}\int \rho \wedge *\sigma = \frac{1}{2V}\int \rho_{\mu\bar{\nu}}\sigma_{\rho\bar{\sigma}}g^{\mu\bar{\sigma}}g^{\rho\bar{\nu}}g^{1/2}d^{6}x. \tag{3.38}$$

We can additionally define a cubic form by

$$\mathcal{K}(\rho,\sigma,\tau) = \int \rho \wedge \sigma \wedge \tau. \tag{3.39}$$

In [14] it was found that the inner product $G(\rho,\sigma)$ can be written in terms of only this cubic form. If we then evaluate this inner product at the basis of the Kähler moduli, we get the metric of the Kähler moduli space:

$$G_{\alpha\beta} = -\frac{3}{2\mathcal{K}}(\mathcal{K}_{\alpha\beta} - \frac{3}{2\mathcal{K}}\mathcal{K}_{\alpha}\mathcal{K}_{\beta}), \tag{3.40}$$

where (see [18])

$$\mathcal{K}_{\alpha\beta} = \int \omega_{\alpha}\wedge\omega_{\beta}\wedge J; \quad \mathcal{K}_{\alpha} = \int \omega_{\alpha}\wedge J \wedge J; \quad \mathcal{K} = \int J\wedge J\wedge J. \tag{3.41}$$

Here J is the familiar fundamental (1,1)-form which therefore can be expanded in the basis ω_{α} as

$$J = v^{\alpha}\omega_{\alpha}. \tag{3.42}$$

If we complexify the Kähler coordinates, the metric above can be written as [1]

$$G_{\alpha\bar{\beta}} = \partial_{\alpha}\partial_{\bar{\beta}}(-\ln\mathcal{K}) \tag{3.43}$$

and the Kähler moduli space is thus also a Kähler manifold by itself.
Lastly, the full intersection number for the Kähler moduli is given by

$$\mathcal{K}_{\alpha\beta\gamma} = \int \omega_{\alpha}\wedge\omega_{\beta}\wedge\gamma. \tag{3.44}$$

3.2.4 Special geometry

To describe the complex structure moduli space in more detail, we can make a choice in parameters and we will do that by describing the moduli space in terms of the periods of the 3-form Ω. To do this we define the canonical homology basis for $H_3(M)$ by (A^I, B_J) with $I, J = 0, ..., h^{2,1}$ and its dual basis (α_I, β^J) for which

$$\int_{A^J}\alpha_I = \int \alpha_I\wedge\beta^J = -\int_{B_I}\beta^J = \delta_I^J, \quad \int \alpha_I\wedge\alpha_J = \int \beta^I\wedge\beta^J = 0, \tag{3.45}$$

where δ_I^J is Kronecker delta. The periods of Ω then are

$$Z^I = \int_{A^I} \Omega, \;\; \mathcal{F}_I = \int_{B_I} \Omega. \tag{3.46}$$

In the moduli space, on a local level, the complex structure is completely determined by Z^I . This directly implies that $\mathcal{F}_I = \mathcal{F}_I(Z)$. The Z^I are actually projective coordinates for the complex structure since the transformation $Z^I \to \lambda Z^I$ leads only to the replacement of Ω by $\lambda\Omega$ and therefore, in these coordinates, Ω is homogeneous of degree one. Therefore, we can define the special coordinates $Z^I = (Z^0, Z^a)$ with $z^a = \frac{Z^a}{Z^0}$, where z^a are just the complex structure coordinates. Expanding Ω in this basis, we get

$$\Omega(Z) = Z^I \alpha_I - \mathcal{F}_I(Z)\beta^I. \tag{3.47}$$

Applying the new notation, we get for the Kähler potential [14] [1] [12]

$$\begin{aligned} K^{\mathrm{cs}} = -\ln\big[i\int \Omega \wedge \overline{\Omega}\big] &= -\ln\big[i\int (Z^I\alpha_I - \mathcal{F}_I(Z)\beta^I) \wedge (\bar{Z}^J\alpha_J - \bar{\mathcal{F}}_J(Z)\beta^J)\big] \\ &= -\ln\big[i(\bar{Z}^I\mathcal{F}_I - Z^I\bar{\mathcal{F}}_I)\big]. \end{aligned} \tag{3.48}$$

Using the triple intersection number from Eq. (3.41), the prepotential can be expanded in terms of the coordinates Z^I in the following way [19]:

$$\mathcal{F} = -\frac{1}{3!}\mathcal{K}_{abc}\frac{Z^aZ^bZ^c}{Z^0} + \frac{1}{2}\alpha_{ab}Z^aZ^b + \beta_a Z^a Z^0 + \frac{1}{2}\gamma(Z^0)^2. \tag{3.49}$$

Replacing the ordinary complex structure coordinates again, we get

$$\begin{aligned} \mathcal{F} &= -\frac{1}{3!}\mathcal{K}_{abc}(Z^0)^2z^az^bz^c + \frac{1}{2}\alpha_{ab}(Z^0)^2z^az^b + \beta_a(Z^0)^2z^a + \frac{1}{2}\gamma(Z^0)^2 \\ &= (Z^0)^2z^az^bz^c(-\frac{1}{3!}\mathcal{K}_{abc} + \frac{1}{2}\frac{\alpha_{ab}}{z^c} + \frac{\beta_a}{z^bz^c} + \frac{1}{2}\frac{\gamma}{z^az^bz^c}). \end{aligned} \tag{3.50}$$

If we then take the limit $z^a \gg 1$, this becomes

$$\mathcal{F} = -\frac{1}{3!}\mathcal{K}_{abc}(Z^0)^2z^az^bz^c = -\frac{1}{3!}\mathcal{K}_{abc}\frac{Z^aZ^bZ^c}{Z^0}, \tag{3.51}$$

which is a form of the prepotential that is often used.
We need to remark here that we also defined a triple intersection number for the Kähler moduli space. So why did we not expand the prepotential in terms of that intersection number instead of the one from the complex structure moduli space? The answer is that we could have. A remarkable property is that the complex structure and Kähler moduli space are directly related and this relation is called mirror symmetry. This is a purely mathematical property that was first discovered by string physicists. For instance, it was discovered among other things that the type IIA and type IIB string theory are related to one another by this mirror symmetry [1]. Mirror symmetry is a topic that came from string theory and spawned a completely new research topic in mathematics. More can be learned about it in [19].

Chapter 4

Type IIB compactification on a Calabi-Yau threefold

4.1 Type IIB supergravity action

Now that we have defined the Calabi-Yau manifold and its moduli spaces, we will return to the specific string theory that we will focus on: the type IIB string theory. To study type IIB string theory we will only look at its bosonic massless modes and set up the low energy effective action with those. We do not study the fermionic modes, because with the use of supersymmetry they can be constructed from the bosonic modes. As discussed before, the massless spectrum of type IIB consists of the tensor product of two open string spectra. Then there are four distinguishable sectors in the massless spectrum depending on the boundary conditions in the left- and right-moving sectors, of which two lead to bosonic fields: The NS-NS sector and the R-R sector.
In the NS-NS sector, we have the familiar bosonic massless fields

$$\{g_{MN}, (B_2)_{MN}, \phi\}, \tag{4.1}$$

with $M, N = 0, ..., 10$. We gave the anti-symmetric tensor B_{MN} an extra indice to emphasize that it is a two-form. In the R-R sector, we obtain

$$\{C_0, (C_2)_{MN}, (C_4)_{MNRS}\}, \tag{4.2}$$

where C_4 satisfies the self-duality condition $dC_4 = *dC_4$. From now on we will ignore the indices of the fields and just give them a hat whenever they are defined in 10 dimensions. The low energy effective action for the bosonic fields of type IIB supergravity in the Einstein frame is then given by [16] [18] [8]

$$\begin{aligned} S_{\text{IIB}} = \int & -\frac{1}{2}\hat{R} * 1 - \frac{1}{4}d\hat{\phi} \wedge *d\hat{\phi} - \frac{1}{4}e^{-\hat{\phi}}\hat{H}_3 \wedge *\hat{H}_3 \\ & - \frac{1}{4}e^{2\hat{\phi}}\hat{F}_1 \wedge *\hat{F}_1 - \frac{1}{4}e^{\hat{\phi}}\hat{F}_3 \wedge *\hat{F}_3 - \frac{1}{8}\hat{F}_5 \wedge *\hat{F}_5 - \frac{1}{4}\hat{C}_4 \wedge \hat{H}_3 \wedge \hat{F}_3, \end{aligned} \tag{4.3}$$

where the field strengths are $\hat{H}_3 = d\hat{B}_2$, $\hat{F}_1 = d\hat{C}_0$ $\hat{F}_3 = d\hat{C}_2 - \hat{C}_0 d\hat{B}_2$ and $\hat{F}_5 = d\hat{C}_4 - d\hat{B}_2 \wedge \hat{C}_2$ and where $\hat{R}$ is the Ricci scalar. We have to impose the self-duality condition of $\hat{C}_4$ by hand, which for the field strength becomes

$$\hat{F}_5 = *\hat{F}_5. \tag{4.4}$$

This equation does not show up naturally because the variation of the action above does not lead to it. If we demand this self-duality condition to come from the action, we would get the wrong equations of motion and we should therefore only apply it to the solutions of the action [16] [8] [1].

4.2 Calabi-Yau compactification

How do we compactify this effective theory of type IIB supergravity onto a three-dimensional Calabi-Yau manifold? We will give a brief outline of the procedure that will be far from complete. For the more detailed treatment we refer to [16]. We will first look at the Ricci scalar and then turn to the other fields.

4.2.1 The Ricci scalar

The part of the action with the Ricci scalar for both type IIA and type IIB is given in the Einstein frame by [16]

$$S = \int -\frac{1}{2}\hat{R} * \mathbf{1}. \tag{4.5}$$

Now we have to expand the Ricci scalar into its four-dimensional counterpart. Since the Ricci scalar is composed of the metric we will look at this first. The main ansatz for the metric is that it is block-diagonal in the following way

$$g_{MN}(x,y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \tag{4.6}$$

where (x, y) consist of the coordinates x^μ in spacetime and the coordinates y^m on the Calabi-Yau threefold. We have now separated the graviton field into a four-dimensional part and the Calabi-Yau part, but since we also consider deformations to the background metric on the Calabi-Yau threefold, we have to add them to the background metric. For the separate components, the mixed and pure parts, this then becomes

$$\begin{aligned} g_{ij} &= 0 + \delta g_{ij} = i\bar{z}^a \frac{1}{||\Omega||^2} \chi_{ai\bar{k}\bar{l}} \Omega^{\bar{k}\bar{l}}{}_j; \\ g_{i\bar{j}} &= g^0_{i\bar{j}} + \delta g_{i\bar{j}} = g^0_{i\bar{j}} - iv^\alpha (\omega_\alpha)_{i\bar{j}}. \end{aligned} \tag{4.7}$$

Here we use the equations that we gave in section 3.2 for the deformations and we could set $\delta z^a = z^a$ since the background value of z^a is just a constant that we can set to zero by shifting the field with that same constant. We also note that here the metric field is expanded in terms of elements of Dolbeault cohomology groups from section 2.2.4. The ten-dimensional Ricci scalar can then be expanded in the following way

$$R_{10} = R_4 + g^{\mu\nu} R_{\mu i \nu}{}^i + g^{ij} \left(R_{i\mu j}{}^\mu + R_{ikj}{}^k + R_{i\bar{k}j}{}^{\bar{k}} \right) \tag{4.8}$$

$$+ g^{i\bar{j}} \left(R_{i\mu\bar{j}}{}^\mu + R_{ik\bar{j}}{}^k + R_{i\bar{k}\bar{j}}{}^{\bar{k}} \right) + \text{ complex conjugate terms,} \tag{4.9}$$

where g_{ij} and $g_{i\bar{j}}$ are now the components of the deformed metric on the Calabi-Yau manifold. If we work out this expansion explicitly in terms of the moduli fields, and throw some additional manipulations in the mix, we finally obtain for the part of the action with the Ricci scalar

$$\int -\frac{1}{2}\hat{R} * \mathbf{1} = \int -\frac{1}{2}R * \mathbf{1} - G_{a\bar{b}}dz^a \wedge *dz^b - G_{\alpha\beta}dv^\alpha \wedge *dv^\beta, \tag{4.10}$$

where remarkably $G_{\alpha\beta}$ and $G_{a\bar{b}}$ are the Kähler metrics of the complex structure moduli and the Kähler moduli space respectively from Eq. (3.17) and Eq. (3.40)
We are now left with the rest of the fields and we will discuss their compactification below.

4.2.2 The other fields

To describe how we can compactify the other fields, we return to our description of Kaluza-Klein compactification. For clarity we will again look at a scalar field, but this time massive, and then move on to the more general r-forms. The equation of motion for a ten-dimensional scalar field Φ is given by [16]

$$\Delta\Phi = m_{10}^2\Phi = \Delta_6\Phi + \Delta_4\Phi, \tag{4.11}$$

where Δ is the Laplacian as given in (2.6). Expanding this field according to the Kaluza-Klein procedure, we get

$$\Phi(x^M) = \phi(x^\mu)h_i(y^m), \tag{4.12}$$

where y^m are the compactified coordinates and h_i are yet to be determined functions. Filling this in the equation of motion, we get

$$m_{10}^2\phi(x^\mu)h_i(y^m) = m_4^2\phi(x^\mu)h_i(y^m) + \phi(x^\mu)\Delta_6 h_i(y^m) \tag{4.13}$$

which means that

$$\Delta_6 h_i(y^m) = (m_{10}^2 - m_4^2)h_i(y^m). \tag{4.14}$$

We saw that for compactification on a circle the mass of the scalar field was equal to an integer n divided by the radius. In general it will be true that the eigenvalues of Δ_6 will be equal to n/S, where S is the size of the manifold. In analogy to the circle compactification case, we then neglect the $n \neq 0$ cases and we get

$$\Delta_6 h_i(y^m) = 0. \tag{4.15}$$

This is precisely the equation for harmonic functions, and if we want to compactify fields using the Kaluza-Klein method, we thus have to expand them into a real field in spacetime and a harmonic field on the internal manifold. The compactification of the dilaton and $\hat{C}_0$ then turns out to be easy. We can just expand them in terms of their four-dimensional counterparts as

$$\hat{\phi} = \phi, \quad \hat{C}_0 = C_0, \tag{4.16}$$

For general r-forms α_r Kaluza-Klein compactification works the same way. The r-forms are then expanded in terms of all harmonic s-forms h_{i_s} for which $0 \leq s \leq r$ in the following manner

$$\alpha_r = \alpha_r^{i_0}h_{i_0} + \alpha_{r-1}^{i_1}h_{i_1} + \ldots + \alpha_0^{i_r}h_{i_r}. \tag{4.17}$$

Ofcourse, using the Hodge theorem of Eq. (2.9), each of the harmonic forms will be a unique representative of a cohomology class and will therefore be an element from the corresponding cohomology group. Using this insight the expansions of the other fields turn out to be

$$\begin{aligned}
&\hat{B}_2 = B_2 + b^\alpha \wedge \omega_\alpha \\
&\hat{C}_2 = C_2 + c^\alpha \wedge \omega_\alpha, \quad \alpha = 1, ..., h^{1,1} \\
&\hat{C}_4 = D_2^\alpha \wedge \omega_\alpha + \rho_\alpha \wedge \tilde{\omega}^\alpha + V^I \wedge \alpha_I - U_I \wedge \beta^I, \quad I = 0, ..., h^{2,1}
\end{aligned} \tag{4.18}$$

where B_2, C_2 and D_2^α are 2-forms, V^I and U_I are 1-forms and b^α, c^α and ρ_α are scalars in spacetime. Here ω_α is the basis of harmonic (1,1)-forms, whereas $\tilde{\omega}^\alpha$ is the basis of the harmonic (2,2)-forms, which are dual to eachother. Similarly, α_I and β^I are the bases of the harmonic 3-forms as discussed in section 3.2.4.
The field strengths of these fields are then given by

$$\begin{aligned}
\hat{F}_1 &= dC_0 \\
\hat{H}_3 &= d\hat{B}_2 = dB_2 + db^\alpha \wedge \omega_\alpha \\
\hat{F}_3 &= d\hat{C}_2 - \hat{C}_0 d\hat{B}_2 = dC_2 + dc^\alpha \wedge \omega_\alpha - C_0(dB_2 + db^\alpha \wedge \omega_\alpha) \\
\hat{F}_5 &= d\hat{C}_4 - d\hat{B}_2 \wedge \hat{C}_2 \\
&= dD_2^\alpha \wedge \omega_\alpha + d\rho_\alpha \wedge \tilde{\omega}^\alpha + dV^I \wedge \alpha_I - dU_I \wedge \beta^I - (dB_2 + db^\alpha \wedge \omega_\alpha) \wedge (C_2 + c^\beta \wedge \omega_\beta) \\
&= dV^I \wedge \alpha_I - dU_I \wedge \beta^I + (dD_2^\alpha - db^\alpha \wedge C_2 - c^\alpha dB_2) \wedge \omega_\alpha \\
&+ d\rho_\alpha \wedge \tilde{\omega}^\alpha + c^\alpha db^\beta \wedge \omega_\alpha \wedge \omega_\beta.
\end{aligned} \tag{4.19}$$

Now that we have expanded all the fields in terms of their four-dimensional counterparts, we "only" have to fill them in the effective action. As we already implied, this is quite an involved calculation and we will therefore just give the resulting action after filling in the new fields. There is one simplification though, which is caused by the self-duality condition. Because of this condition only half of the fields in $\hat{C}_4$ are independent and we choose to eliminate D_2^α and U_I and keep ρ_α and V^I [17]. Still, there are a lot of steps needed to get a good action of which the details are again discussed in [16]. After all these steps, the final four-dimensional action becomes (using the notation in [18]):

$$\begin{aligned}
S_{IIB}^{(4)} = \int & -\frac{1}{2}R * 1 - \frac{1}{4}d\phi \wedge *d\phi - G_{a\bar{b}}dz^a \wedge *d\bar{z}^b - G_{\alpha\beta}dv^\alpha \wedge *dv^\beta \\
& - \frac{1}{4}d\ln\mathcal{K} \wedge *d\ln\mathcal{K} - \frac{\mathcal{K}^2}{144}e^{-\phi}dB_2 \wedge *dB_2 - e^{-\phi}G_{\alpha\beta}db^\alpha \wedge *db^\beta \\
& - \frac{1}{4}e^{2\phi}dC_0 \wedge *dC_0 - \frac{\mathcal{K}^2}{144}e^{\phi}\left(dC_2 - C_0 dB_2\right) \wedge *\left(dC_2 - C_0 dB_2\right) \\
& - e^{\phi}G_{\alpha\beta}\left(dc^\alpha - C_0 db^\beta\right) \wedge *\left(dc^\alpha - C_0 db^\beta\right) \\
& - \frac{9G^{\alpha\delta}}{4\mathcal{K}^2}\left(d\rho_\alpha - \frac{1}{2}\mathcal{K}_{\alpha\beta\gamma}\left(c^\beta db^\gamma - b^\beta dc^\gamma\right)\right) \wedge *\left(d\rho_\delta - \frac{1}{2}\mathcal{K}_{\delta\kappa\lambda}\left(c^\kappa db^\lambda - b^\kappa dc^\lambda\right)\right) \\
& + \frac{1}{2}\left(db^\alpha \wedge C_2 + c^\alpha dB_2\right) \wedge \left(d\rho_\alpha - \mathcal{K}_{\alpha\beta\gamma}c^\beta db^\gamma\right) + \frac{1}{4}\mathcal{K}_{\alpha\beta\gamma}c^\alpha c^\beta dB_2 \wedge db^\gamma \\
& + \frac{1}{4}Re\mathcal{M}_{IJ}dV^I \wedge dV^J + \frac{1}{4}Im\mathcal{M}_{IJ}dV^I \wedge *dV^J
\end{aligned} \tag{4.20}$$

where the matrices $\mathcal{M}_{IJ}$ can be expressed in special geometrical language as

$$\mathcal{M}_{IJ} = \overline{\mathcal{F}}_{IJ} + 2i\frac{(\mathrm{Im}\,\mathcal{F})_{IK}Z^K(\mathrm{Im}\,\mathcal{F})_{JL}Z^L}{Z^L(\mathrm{Im}\,\mathcal{F})_{LK}Z^K}, \tag{4.21}$$

where $\mathcal{F}_{IJ} = \frac{\partial \mathcal{F}_I}{\partial Z^J}$. It is shown in [17] that this action can be written in the form of the standard $N = 2$ supersymmetric action in four dimensions by assigning the given fields to the appropriate representations of the supersymmetry algebra.

Chapter 5

Type IIB compactification on Calabi-Yau orientifolds

By compactifying type IIB on a Calabi-Yau threefold, we found an effective action (an action of the massless modes) in spacetime that can be written in the form of a standard $N = 2$ supergravity action. There are now three things we would like to do. Firstly, since the standard model is proposed to be coming from a minimal supersymmetric theory, i.e. a theory with the least amount of new particle states, we need to lose one of the two gravitinos to satisfy this. Secondly, we get a lot of extra massless fields because of the compactification. One way to lose these is by turning on quantities in the theory that are called fluxes. After doing this, we again want to write the resulting theory into the form of a standard supergravity action.

5.1 Orientifold projection

The projection that truncates the spectrum of the $N = 2$ supergravity action in such a way that we obtain a $N = 1$ supergravity action, is given by [17]:

$$\mathcal{O} = (-1)^{\tilde{F}} \Omega_p \sigma^* \tag{5.1}$$

and we will discuss its different parts consecutively. This projection is called an orientifold projection. We already encountered the first part $(-1)^{\tilde{F}}$ with $\tilde{F}$ the fermion number in the left-moving sector. Under this operator the R-NS and R-R fields are odd and the NS-R and NS-NS fields are even. Thus the bosonic fields, i.e. the metric, the anti-symmetric field and the dilaton, do not change, and the gauge fields get a minus sign under this operator. The second part Ω_p is called the world-sheet parity operator. It reverses the orientation of the worldsheet and in doing so exchanges the left- and right-moving modes of the string [3]. Just as $(-1)^{\tilde{F}}$ it is a symmetry of type IIB. For this operation $\hat{g}$, $\hat{\phi}$ and $\hat{C}_2$ are even, while $\hat{B}_2$, $\hat{C}_0$ and $\hat{C}_4$ are odd [17] . The last part, σ^*, is the pullback of a geometric symmetry operator on the Calabi-Yau manifold, since it leaves the metric and the complex structure invariant. Therefore it works on the Kähler form and the holomorphic three-form as

$$\sigma^* J = J; \quad \sigma^* \Omega = \pm\Omega. \tag{5.2}$$

We can thus choose Ω to be symmetric or anti-symmetric under σ^* and different properties result from the choice we make. We will focus on the theory where $\sigma^*\Omega = -\Omega$. The set of points on the entire ten-dimensional manifold that are fixed under the orientifold projection in Eq. (5.1) are fittingly called orientifold planes. These orientifold planes fill up the entire spacetime, but are not dynamical objects such as D-branes are [3]. With the condition $\sigma^*\Omega = -\Omega$ the spatial dimensions of these orientifold planes will be three and seven, and then they are also called O3- and O7-planes [20]. Since only the fields that are invariant under the full orientifold projection will remain, using the properties of $(-1)^{\tilde{F}}$ and Ω_p we see that the massless fields of type IIB have to obey the following conditions:

$$\begin{aligned} \sigma^*\hat{g} &= \hat{g}, & \sigma^*\hat{C}_0 &= \hat{C}_0, \\ \sigma^*\hat{B}_2 &= -\hat{B}_2, & \sigma^*\hat{C}_2 &= -\hat{C}_2 \\ \sigma^*\hat{\phi} &= \hat{\phi}, & \sigma^*\hat{C}_4 &= \hat{C}_4 \end{aligned} \tag{5.3}$$

The cohomology groups $H^{p,q}$ also split into even and odd subgroups under σ^* as

$$H^{p,q} = H^{p,q}_+ \oplus H^{p,q}_-. \tag{5.4}$$

Of course, this leads to some differences in the Hodge diamond. Firstly, from (5.2) it follows immediately that $h^{3,0}_+ = h^{0,3}_+ = 0$ and $h^{3,0}_- = h^{0,3}_- = 1$. Also, since the involution σ^* leaves the Kähler form J invariant, only the $h^{1,1}_+$ Kähler deformations are left. Since the volume form is proportional to $\Omega\wedge\overline{\Omega}$ and this needs to be invariant under σ^* also $h^{0,0}_- = h^{3,3}_- = 0$ and $h^{0,0}_+ = h^{3,3}_+ = 0$. Furthermore, since the metric is even and Ω is odd under the involution, the complex structure deformations that are kept are the ones from $H^{2,1}_-$ and the complex structure deformations thus become

$$\delta g_{ij} = \frac{i}{||\Omega||^2}\bar{z}^a(\overline{\chi_a})_{i\bar{\imath}\bar{\jmath}}\Omega^{\bar{\imath}\bar{\jmath}}_j, \;\; a = 1, ..., h^{2,1}_-. \tag{5.5}$$

In the same way, the Kähler deformations are given by

$$\delta g_{i\bar{\jmath}} = -iv^\alpha(\omega_\alpha)_{i\bar{\jmath}}, \;\; \alpha = 1, ..., h^{1,1}_+. \tag{5.6}$$

From the relations above , the expansion (4.18) changes to

$$\begin{aligned} &\hat{B}_2 = b^\kappa\omega_\kappa, \;, \\ &\hat{C}_2 = c^\kappa\omega_\kappa, \quad \kappa = 1, ..., h^{1,1}_- \\ &\hat{C}_4 = D^\alpha_2\wedge\omega_\alpha + V^k\wedge\alpha_k + U_k\wedge\beta^k + \rho_\alpha\tilde{\omega}^\alpha, \; k = 1, ..., h^{2,1}_+, \; \alpha = 1, ..., h^{1,1}_+, \end{aligned} \tag{5.7}$$

where we still had to expand $\hat{B}_2$ and $\hat{C}_2$ in terms of forms from cohomology group $H^{1,1}_-$, since these fields are odd under σ^*. Similarly, because $\hat{C}_4$ is even under σ^* it also has to be expanded in bases from the even cohomology groups.
The basis (α_I, β^J) must self-evidently also be divided among the even and odd cohomology groups. The one element that corresponds to $I = 0$ is the same for both odd and even groups and we will therefore just put it in the odd basis (α_K, β^L), where $K = 0, ..., h^{2,1}_-$. After again getting rid of U_k and D^α_2 because of the self-duality condition for $\hat{F}_5$, we finally get the following massless spectrum for $N = 1$ supergravity from Type IIB with the orientifold projection [16]:

Gravity multiplet	1	$g_{\mu\nu}$
Vector multiplets	$h_+^{2,1}$	V^k
Chiral multiplets	$h_-^{2,1}$	z^a
Chiral multiplets	1	(ϕ, C_0)
Chiral multiplets	$h_-^{1,1}$	(b^κ, c^κ)
Chiral/linear multiplets	$h_+^{1,1}$	(v^α, ρ_α)

Table 5.1: N=1 supergravity multiplets from type IIB compactified on a Calabi-Yau orientifold.

Here the different massless fields are arranged in the representations of the supersymmetry algebra, called multiplets. We will discuss the effective action for type IIB on orientifolds after turning on fluxes. The effective action for type IIB on orientifolds without fluxes can be found in [17].

5.2 Fluxes

Even after the orientifold projection, not all problems are solved in order to obtain a phenomenological string theory. We still have a lot of extra massless fields, the moduli, and as massless fields typically lead to long-range forces (such as the graviton leads to gravity), many of these would not be very probable for a physical theory. Thus we need to find out how we can give some if not all of these moduli masses, such that they will not show up anymore in the effective field theory. However, we must be careful, because if the masses get to big the effective field theory would break down. The process of making moduli massive, but also not too massive is called moduli stabilization.
In type IIB string theory it is allowed to turn on extra background fields $H_3 = \langle dB_2 \rangle$ and $F_3 = \langle dC_2 \rangle$ and with these background fields or so-called fluxes it becomes possible (although further assumptions are needed) to stabilise the moduli. The solutions for the resulting theory are then called flux vacua. As is discussed in [21] for most Calabi-Yau threefolds the number of vacua will be gigantic. Since we do not know a lot about which of the vacua are preferred for describing our universe (although some findings indicate to this, such as the tadpole conjecture), we have a huge amount of solutions to choose from. This collection of the large amount of possible string vacua is often called the string landscape.

With our newly enabled background fields H_3 and F_3 and the expansion (5.7), the field strengths become

$$\begin{aligned}
\hat{H}_3 &= db^\kappa \wedge \omega_\kappa + H_3, \\
\hat{F}_3 &= dc^\kappa \wedge \omega_\kappa - C_0 db^\kappa \wedge \omega_\kappa + F_3 - C_0 H_3, \\
\hat{F}_5 &= d\hat{C}_4 - \frac{1}{2}\hat{H}_3 \wedge \hat{C}_2 + \frac{1}{2}\hat{B}_2 \wedge d\hat{C}_2 = dD_2^\alpha \wedge \omega_\alpha + dV^k \wedge \alpha_k + dU_k \wedge \beta^k \\
&\quad + d\rho^\alpha \wedge \tilde{\omega^\alpha} - \frac{1}{2}(db^\kappa \wedge \omega_\kappa + H_3) \wedge c^\lambda \omega_\lambda + \frac{1}{2}(b^\kappa \wedge \omega_\kappa \wedge dc^\lambda \omega_\lambda) \\
&\quad = dD_2^\alpha \wedge \omega_\alpha + dV^k \wedge \alpha_k + dU_k \wedge \beta^k + d\rho^\alpha \wedge \tilde{\omega^\alpha} - \frac{1}{2}(db^\kappa c^\lambda - b^\kappa dc^\lambda) \wedge \omega_\kappa \wedge \omega_\lambda,
\end{aligned} \tag{5.8}$$

where since the orientifold projection projected out the fields C_2 and B_2, F_3 and H_3 do not affect the $\hat{F}_5$ term. Using all this, the effective action with fluxes was found to be [18]

$$\begin{aligned} S^{(4)}_{O3/O7} = \int & -\frac{1}{2}R * \mathbf{1} - \frac{1}{4}d\phi \wedge * d\phi - G_{a\bar{b}}dz^a \wedge * d\bar{z}^b - G_{\alpha\beta}dv^\alpha \wedge * dv^\beta \\ & - \frac{1}{4}d\ln\mathcal{K} \wedge * d\ln\mathcal{K} - e^{-\phi}G_{\kappa\lambda}db^\kappa \wedge * db^\lambda - \frac{1}{4}e^{2\phi}dC_0 \wedge * dC_0 \\ & - e^{\phi}G_{\kappa\lambda}\left(dc^\kappa - C_0 db^\kappa\right) \wedge * \left(dc^\lambda - C_0 db^\lambda\right) \\ & - \frac{9G^{\alpha\beta}}{4\mathcal{K}^2}\left(d\rho_\alpha - \frac{1}{2}\mathcal{K}_{\alpha\kappa\lambda}\left(c^\kappa db^\lambda - b^\kappa dc^\lambda\right)\right) \wedge * \left(d\rho_\beta - \frac{1}{2}\mathcal{K}_{\beta\rho\sigma d}\left(c^\rho db^\sigma - b^\rho dc^\sigma\right)\right) \\ & + \frac{1}{4}\operatorname{Im}\mathcal{M}_{kl}dV^k \wedge * dV^l + \frac{1}{4}\operatorname{Re}\mathcal{M}_{kl}dV^k \wedge dV^l - V * 1, \end{aligned} \tag{5.9}$$

where again $\rho, \sigma = 1, ..., h^{1,1}_-$. Looking at the action and comparing it with the ordinary Type IIB effective action, we see that the addition of fluxes causes the appearance of a certain scalar potential V that is given by [17] [22]

$$V = \frac{18ie^\phi}{\mathcal{K}^2\int\Omega\wedge\overline{\Omega}}\Big(\int\Omega\wedge\overline{G}_3\int\overline{\Omega}\wedge G_3 + G^{a\bar{b}}\int\chi_a\wedge G_3\int\chi_{\bar{b}}\wedge\overline{G}_3\Big), \tag{5.10}$$

where we have combined the two fluxes into one by

$$G_3 = F_3 - \tau H_3, \quad \tau = C_0 + ie^{-\phi}, \tag{5.11}$$

for which we also introduced a new field τ that, since it combines the axion and the dilaton, is called the axio-dilaton. It will emerge again later on. Ofcourse, in quantum field theory masses are calculated by the second derivative of a potential evaluated at the minimum [23]. The scalar potential that has appeared in the effective action above is therefore also precisely the object with which we can calculate the masses of the moduli.

5.3 N=1 supergravity effective action

We now want to write the effective action of type IIB with O3- and O7-planes into the form of the standard effective action for $N = 1$ supergravity. General $N = 1$ supergravity theories in four dimensions consist of three main components: a Kähler potential K, a superpotential W and gauge kinetic functions f [24]. The standard $N = 1$ supergravity action in four dimensions is then given by

$$S^{(4)} = -\int\frac{1}{2}R * \mathbf{1} + G_{A\bar{B}}DM^A D\bar{M}^{\bar{B}} + \frac{1}{2}\text{Re}f_{kl}dV^k \wedge * dV^l + \frac{1}{2}\text{Im}f_{\kappa\lambda}dV^k \wedge dV^l + V * \mathbf{1}, \tag{5.12}$$

where the M^A collectively denote all the moduli [18]. This scalar potential V is expressed as

$$V = e^K(G^{A\bar{B}}D_A W D_{\bar{B}}\overline{W} - 3|W|^2) + \frac{1}{2}(\text{Re}f)^{-1\ kl}D_k D_l, \tag{5.13}$$

where the Kähler metric $G_{A\bar{B}} = \partial_A\partial_{\bar{B}}K$ and the covariant derivative on W is given by $F_A \equiv D_A W = \partial_A W + (\partial_A K)W$, that we will call an F-term. The scalar potential thus consists of two parts denoted by V_F (because of the F-terms) and V_D that are given by

$$V_F = e^K (G^{A\bar{B}} F_A \overline{F}_{\bar{B}} - 3|W|^2); \quad V_D = \frac{1}{2}(\mathrm{Re}f)^{-1\ \kappa\lambda} D_\kappa D_\lambda. \tag{5.14}$$

We now have to define the massless fields that came from the orientifold action with fluxes in such a way that we can write the action in this form, which means that we need to find a Kähler potential, a superpotential and gauge kinetic functions for all the moduli.
For the axion and the dilaton it follows that we can find a Kähler potential if we combine them as in the axio-dilaton from before given by

$$\tau = C_0 + i e^{-\phi}. \tag{5.15}$$

Also, we already know from section 3.2.1 that we can set up a Kähler potential for the complex structure moduli z^a. For the other massless scalars we have to define two different moduli, namely the moduli G^κ and the Kähler moduli T_α as

$$G^\kappa = c^\kappa - \tau b^\kappa, \quad T_\alpha = \frac{3i}{2}\rho_\alpha + \frac{3}{4}\kappa_\alpha(v) - \frac{3}{2}\zeta_\alpha(\tau, \bar{\tau}, G, \bar{G}), \tag{5.16}$$

where $\mathcal{K}_{\alpha(v)} = \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma$ and $\zeta_\alpha = -\frac{i}{2(\tau-\bar{\tau}})\mathcal{K}_{\alpha\kappa\lambda} G^\kappa (G-\bar{G})^\lambda$. The total Kähler potential, separated in a complex structure part and the rest, then turns out to be

$$K = K^{cs}(z, \bar{z}) + K^k(\tau, \bar{\tau}, G, T), \tag{5.17}$$

where

$$K^{cs}(z,\bar{z}) = -\ln\Big[i \int \Omega(z) \wedge \overline{\Omega}(\bar{z})\Big];$$
$$K^k(\tau, \bar{\tau}, G, T) = -\ln\big[-i(\tau - \bar{\tau})\big] - 2\ln\Big[\mathcal{V}(\tau, G, T)\Big], \tag{5.18}$$

with $\mathcal{V} = \frac{1}{6}\mathcal{K} = \frac{1}{6}\mathcal{K}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma$ the volume of the compactification manifold. $\mathcal{K}$ is known only implicitly via $v^\alpha(\tau, G, T)$ [18]. (Here we use the convention for the Kähler potential of the complex structure from [12] and [25].) The total moduli space then gets the form of a product

$$\mathcal{M} = \mathcal{M}_{\mathrm{cs}}^{h_-^{2,1}} \times \mathcal{M}_k^{h^{1,1}+1}, \tag{5.19}$$

where each of the two parts is a Kähler manifold. Actually, it follows that the complex structure moduli space is even a special Kähler manifold [18].
If we set $h_-^{1,1} = 0$, which we will do later on, we see that the G^a-moduli vanish and that the volume $\mathcal{V}$ will not depend on the G^a-moduli and the axio-dilaton anymore. This last dependency vanishes because G^a depends itself on τ and the dependency of the Kähler moduli on τ also dissappears [18] [20].
When looking at orientifolds with $b^\kappa = c^\kappa = 0$, the superpotential was derived to be [22]

$$W = \int G_3 \wedge \Omega, \tag{5.20}$$

which is called the Glukov-Vafa-Witten superpotential. Using this Kähler potential and the Glukov-Vafa-Witten superpotential and assuming that $V_D = 0$, we can indeed transform the action (5.9) into the supergravity action when we choose $M^I = (z^a, T_\alpha, G^\kappa, \tau)$. Indeed, starting from Eq. (5.10) one can show that these choices of the Kähler potential and the superpotential lead to Eq. (5.14). The massless spectrum of the $N = 1$ supersymmetric theory then becomes

Gravity multiplet	1	$g_{\mu\nu}$
Vector multiplets	$h_+^{2,1}$	V^k
Chiral multiplets	$h_-^{2,1}$	z^a
Chiral multiplets	1	τ
Chiral multiplets	$h_-^{1,1}$	G^κ
Chiral/linear multiplets	$h_+^{1,1}$	T^α

Table 5.2: N=1 supergravity multiplets from type IIB compactified on a Calabi-Yau orientifold with fluxes.

It is good to note that we now have obtained a $N = 1$ supergravity theory in which we have unoriented closed and open strings. This may sound a lot like type I superstring theory and actually, it is. The low energy type I theory can be obtained by the orientifold projection of the low energy type IIB theory.

5.4 Tadpole cancellation conditions

We have been ignoring some problem thus far, which we now would like to adress. For type I string theories we have some consistency conditions that have to be obeyed. This is due to the fact that the divergencies of the one-loop amplitudes or tadpoles should vanish. These divergencies arise since the orientifold planes are charged by an R-R field and these charged orientifold planes lead to a violation of the equation of motion. The solution to this problem actually sounds quite simple. Since the appearance of charged objects leads to the divergencies, we simply need to add more charged objects, but of opposite charge, such that the total charge sums up to zero. Since type I string theory also contains open strings, it also contains D-branes and these are precisely the objects that can cancel the charges of the orientifold planes. Since we look at type IIB with O3/O7-planes in this thesis, we also need D3- and D7-branes to satisfy the tadpole cancellation condition. These conditions mainly come from the modified Bianchi identities for F_5 due to the introduction of the fluxes

$$dF_5 = H_3 \wedge F_3 + \rho_{\text{local}}, \tag{5.21}$$

where ρ_{local} is the charge density of all the charged local objects in the theory [26]. If we integrate this, we end up with

$$N_{\text{flux}} + Q_3^{\text{total}} = 0, \tag{5.22}$$

where Q_3^{total} is the total contribution of all the charged objects on the compact manifold, i.e. the $D3$- and $D7$-branes and the $O3$- and $O7$-planes. More detailed versions of the tadpole cancellation conditions can be found in [27] and [28]. Also the number N_{flux} is defined by

$$N_{\text{flux}} = \int F_3 \wedge H_3 \tag{5.23}$$

and is called the flux number. We haven chosen it in such a way that it is positive [25]. This number is thus bounded by the tadpole conditions, and this is one of the reasons why it is interesting to study. We will therefore use it later on.

Chapter 6

Moduli stabilization in type IIB Calabi-Yau orientifolds

Now that we have established the theory of type IIB in four dimensions with O3-O7 orientifold planes, we will fully turn to the stabilization of the moduli. We turned on fluxes in order to be able to stabilize moduli in the previous section, but did not explain how to achieve this. The method that we will consider was proposed by Kachru, Kallosh, Linde and Trivedi and is therefore called the KKLT-scenario. For the more detailed description we refer to their article [29]. We should add here that the KKLT scenario is far from undisputed, as the recent paper [30] illustrates.

6.1 The KKLT scenario

Before describing the KKLT scenario, we impose a couple conditions to simplify matters:

- We will not consider any α'-corrections, so no terms with higher order of α'.
- We assume that there is an orientifold and a Calabi-Yau manifold such that $h^{1,1}_{-} = 0$ which means that (as can be seen in Table 5.2) there are no G^a-moduli and that the volume of the internal manifold $\mathcal{V}$ will not depend on the axio-dilaton τ.
- We will not discuss the open string sector of type IIB or the effects of D-branes.

In [20] the difficulties are discussed that arise when these conditions are not imposed. The KKLT method to stabilize the remaining moduli then consists of two main steps. The first step is to stabilize all complex structure moduli and the axio-dilaton first. The second step then consists of adding non-perturbative corrections that also depend on the Kähler moduli to the superpotential and calculating the new minima of this with the complex structure moduli and the axio-dilaton fixed at their respective stabilized values [20]. In this thesis, we will however focus on the first step, i.e. stabilizing the complex structure moduli and the axio-dilaton, and assume that the Kähler moduli will be stabilized later on.

Applying the first two conditions above, the total Kähler metric becomes

$$G_{A\bar{B}} = \begin{pmatrix} G_{a\bar{b}} & 0 & 0 \\ 0 & G_{\alpha\bar{\beta}} & 0 \\ 0 & 0 & G_{\tau\bar{\tau}} \end{pmatrix}, \tag{6.1}$$

where the Latin indices correspond to the complex structure moduli z^a and the Greek indices to the Kähler moduli T^α. The scalar potential V_F from Eq. (5.14) is then given by

$$V_F = e^K (G^{a\bar{b}} F_a \overline{F}_{\bar{b}} + G^{\alpha\bar{\beta}} F_\alpha \overline{F}_{\bar{\beta}} + G^{\tau\bar{\tau}} F_\tau \overline{F}_{\bar{\tau}} - 3|W|^2). \tag{6.2}$$

We can now look at the F-terms of the separate moduli more closely. Starting with the Kähler moduli, first we note that $W = W(\tau, z^a)$ and thus does not depend on the Kähler moduli. Therefore it holds that

$$F_\alpha = (\partial_\alpha K) W \tag{6.3}$$

and similarly for its complex conjugate. An important result that only holds when we ignore the α'-corrections is given by the so-called no-scale condition [20]

$$\partial_\alpha K G^{\alpha\bar{\beta}} \partial_{\bar{\beta}} K = 3. \tag{6.4}$$

Filling this in, the scalar potential becomes quite simple:

$$V_F = e^K (G^{a\bar{b}} F_a \overline{F}_{\bar{b}} + G^{\alpha\bar{\beta}} (\partial_\alpha K)(\partial_{\bar{\beta}} K)|W|^2 + G^{\tau\bar{\tau}} F_\tau \overline{F}_{\bar{\tau}} - 3|W|^2) \tag{6.5}$$

$$= e^K (G^{a\bar{b}} F_a \overline{F}_{\bar{b}} + 3|W|^2 + G^{\tau\bar{\tau}} F_\tau \overline{F}_{\bar{\tau}} - 3|W|^2) = e^K (G^{a\bar{b}} F_a \overline{F}_{\bar{b}} + G^{\tau\bar{\tau}} F_\tau \overline{F}_{\bar{\tau}})$$

$$= e^K (G^{I\bar{J}} F_I \overline{F}_{\bar{J}}), \tag{6.6}$$

where now the indices I and $\bar{J}$ only go over the axio-dilaton and the complex structure moduli. To calculate the flux vacua, we have to calculate the supersymmetric minimum of this potential.

6.2 The F-terms

Now that we have obtained a scalar potential that does not depend on the Kähler moduli, we will compute the F-terms of the axio-dilaton and the complex structure moduli respectively. To do this we first need some additional quantities by using the Kähler potential from Eq. (5.17).
Firstly, the derivative of the Kähler potential with respect to $\bar{\tau}$ is

$$\partial_{\bar{\tau}} K = -\partial_{\bar{\tau}} \ln(-i(\tau - \bar{\tau})) = \frac{1}{\tau - \bar{\tau}}, \tag{6.7}$$

where we used that for $h^{1,1}_- = 0$ the volume does not depend on τ anymore. The metric component $G_{\tau\bar{\tau}}$ then is

$$G_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} K = -\partial_\tau \frac{1}{(\tau - \bar{\tau})} = -\frac{1}{(\tau - \bar{\tau})^2} \tag{6.8}$$

and since this is a scalar its inverse is given by $G^{\tau\bar{\tau}} = \frac{1}{G_{\tau\bar{\tau}}} = -(\tau - \bar{\tau})^2$. We already know the metric of the complex structure moduli, since that is just the metric in Eq. (3.17). Furthermore, we note that

$$\partial_\tau W = \int \partial_\tau (F_3 - \tau H_3) \wedge \Omega = -\int H_3 \wedge \Omega \tag{6.9}$$

and also that

$$\partial_i W = \int G_3 \wedge \partial_i \Omega = \int G_3 \wedge (\chi_i - (\partial_i K^{\rm cs})\Omega) = \int G_3 \wedge \chi_i - (\partial_i K^{\rm cs})W, \tag{6.10}$$

where we used (3.27).
Using the found expressions, we get

$$\begin{aligned} F_\tau = D_\tau W = \partial_\tau W + (\partial_\tau K)W &= -\int H_3 \wedge \Omega - \frac{1}{\tau - \bar\tau}\int G_3 \wedge \Omega \\ &= -\frac{1}{\tau-\bar\tau}\int (\tau - \bar\tau) H_3 \wedge \Omega - \frac{1}{\tau - \bar\tau}\int (F_3 - \tau H_3)\wedge \Omega \\ &= -\frac{1}{\tau-\bar\tau}\int (F_3 - \bar\tau H_3)\wedge\Omega = -\frac{1}{\tau-\bar\tau}\int \overline{G}_3 \wedge \Omega \end{aligned} \tag{6.11}$$

and for the complex structure moduli

$$F_i = \partial_i W + (\partial_i K)W = \partial_i W + (\partial_i K^{\rm cs})W = \int G_3 \wedge \chi_i, \tag{6.12}$$

where we applied that $\partial_i K = \partial_i K^{\rm cs}$ since the only term in the total Kähler potential that depends on the complex structure moduli is $K^{\rm cs}$. All the complex conjugated counterpart of these terms are then easily found by taking the complex conjugate of both sides of the equation, i.e.

$$\overline{F}_{\bar\tau} = -\frac{1}{\bar\tau - \tau}\int G_3 \wedge \overline{\Omega}; \quad \overline{F}_{\bar\imath} = \int \overline{G}_3 \wedge \chi_{\bar\imath}. \tag{6.13}$$

6.3 Minimum of the scalar potential

Masses get stabilized in a minimum of a potential and a minimum of a potential is calculated by setting its first derivative to zero. Let us look at that first derivative:

$$\begin{aligned} \partial_C V_F &= \partial_C (e^K G^{I\bar J} F_I \overline{F}_{\bar J}) \\ &= (\partial_C e^K) G^{I\bar J} F_I \overline{F}_{\bar J} + e^K (\partial_C G^{I\bar J}) F_I \overline{F}_{\bar J} + e^K G^{I\bar J} (\partial_C F_I)\overline{F}_{\bar J} + e^K G^{I\bar J} F_I (\partial_C \overline{F}_{\bar J}) \\ &= e^K \left[(\partial_C K) G^{I\bar J} F_I \overline{F}_{\bar J} + (\partial_C G^{I\bar J}) F_I \overline{F}_{\bar J} + G^{I\bar J}(\partial_C F_I)\overline{F}_{\bar J} + G^{I\bar J} F_I (\partial_C \overline{F}_{\bar J})\right]. \end{aligned} \tag{6.14}$$

We could go further with our calculation, but there already appeared a very easy solution to the equation $\partial_C V_F = 0$. Since all the terms contain F_I or its complex conjugate, we can simply set $F_I = 0$ and we have found a global minimum of the potential. Therefore, instead of doing more involved calculations, we will focus on this minimum for the moduli stabilization and the mass terms. Let us consider, using the expressions that we obtained in the previous section, what happens when these F-terms go to zero. Starting generally (for both the axio-dilaton and the complex structure moduli), we get that

$$F_I = D_I W = \partial_I W + (\partial_I K)W = 0 \Rightarrow \partial_I W = -(\partial_I K)W. \tag{6.15}$$

More specifically, for the axio-dilaton this means that

$$\begin{aligned} F_\tau = -\frac{1}{\tau - \bar\tau}\int \overline{G}_3 \wedge \Omega = -\frac{1}{\tau-\bar\tau}\int (F_3 - \bar\tau H_3)\wedge \Omega = 0 \Rightarrow \int F_3 \wedge \Omega = \bar\tau \int H_3 \wedge \Omega \\ \Rightarrow W = \int G_3 \wedge \Omega = -(\tau - \bar\tau)\int H_3 \wedge \Omega \end{aligned} \tag{6.16}$$

and for the complex structure moduli it means that

$$F_i = \int G_3 \wedge \chi_i = \int (F_3 - \tau H_3) \wedge \chi_i = 0 \Rightarrow \int F_3 \wedge \chi_i = \tau \int H_3 \wedge \chi_i$$
$$\Rightarrow \int \overline{G}_3 \wedge \chi_i = (\tau - \bar{\tau}) \int H_3 \wedge \chi_i. \tag{6.17}$$

By setting the F-terms to zero we can also derive another relation for which we expand the combined flux G_3 in cohomology bases as

$$G_3 = F_3 - \tau H_3 = g^0 \Omega + g^a \chi_a + g^{\bar{a}} \chi_{\bar{a}} + g^{\bar{0}} \overline{\Omega}. \tag{6.18}$$

This expansion can be done since G_3 is a three-form and can thus be written as a sum over the cohomology bases as the cohomology group H^3 can be written as

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}. \tag{6.19}$$

Since we applied the orientifold projection, it still holds that $\Omega \in H^{3,0}_-$ and $\chi_a \in H^{2,1}_-$. In this form, the complex conjugate of G_3 is

$$\overline{G}_3 = g^{\bar{0}} \Omega + g^a \chi_{\bar{a}} + g^{\bar{a}} \chi_a + g^0 \overline{\Omega}. \tag{6.20}$$

Inserting this in the equations for the F-terms we get for the axio-dilaton that

$$F_\tau = 0 \Rightarrow \int (g^{\bar{0}} \Omega + g^a \chi_{\bar{a}} + g^{\bar{a}} \chi_a + g^0 \overline{\Omega}) \wedge \Omega = \int g^0 \overline{\Omega} \wedge \Omega = 0 \Rightarrow g^0 = 0, \tag{6.21}$$

where only the $\overline{\Omega}$ part of $\overline{G}_3$ remains, since we can not have (p,q)-forms with $p > 3$ or $q > 3$. Furthermore, we used that $\Omega \wedge \overline{\Omega} > 0$ [1]. Similarly, for the complex structure moduli, we get

$$F_i = 0 \Rightarrow \int (g^0 \Omega + g^a \chi_a + g^{\bar{a}} \chi_{\bar{a}} + g^{\bar{0}} \overline{\Omega}) \wedge \chi_i = \int g^{\bar{a}} \chi_{\bar{a}} \wedge \chi_i = 0 \Rightarrow g^{\bar{a}} = 0 \tag{6.22}$$

and when satisfying the F-term equations G_3 thus becomes

$$G_3 = g^a \chi_a + g^{\bar{0}} \overline{\Omega}. \tag{6.23}$$

Applying the Hodge star to this, we get

$$*G_3 = g^a * \chi_a + g^{\bar{0}} * \overline{\Omega} = i g^a \chi_a + i g^{\bar{0}} \overline{\Omega} = i G_3, \tag{6.24}$$

where we used the imaginary self-duality of both Ω and χ_a and that $\overline{*\psi} = *\overline{\psi}$ for any form ψ. Thus G_3 is also imaginary self-dual and in this way we can express the conditions of vanishing F-terms in just one equation.

6.4 The tadpole conjecture

When we have a large number of complex structure moduli there is a proposed bound on the flux number from Eq. (5.23). This is called the tadpole conjecture and it states that

$$N_{\text{flux}} > 2\alpha(h^{2,1} + 1) \quad \text{for} \quad h^{2,1} \gg 1, \tag{6.25}$$

where the constant α is expected to be $1/3$. In the case that this conjecture is really true, this means that at this regime, all the contributions from the fluxes can not be cancelled by the orientifold charges. This means that not all the moduli can be stabilized for this case, which is an important assumption for the theory to work [27] [31]. We would like to emphasize that the flux number showed up in both the tadpole conjecture and conditions. It therefore seems to be an important number in the theory. This is why we would like to know more about it.

Chapter 7

Constraints on the moduli masses

The ultimate goal of this thesis is to calculate the mass terms of the axio-dilaton and the complex structure moduli that appeared due to the introduction of the fluxes. Then we want to know how these masses behave or in other words the constraints on these masses but it turns out that the separate mass values are not easily related to quantities that we know in the theory. An easy way to get one value out of all the separate mass values, is by calculating the sum of all the moduli masses. In other words, the trace of the matrix that contains all the different moduli mass values.

7.1 Trace of the moduli mass matrix

Before we determine the mass formulas of the moduli, we will first consider something very simple, scalar field theory. The Lagrangian for a complex scalar field without interactions is given by

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(\phi, \phi^*), \tag{7.1}$$

where $V(\phi, \phi^*) = m^2 \phi \phi^*$ [23]. The mass of this scalar field is then obviously given by

$$m^2 = \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi^*} V(\phi, \phi^*) \big|_{\phi = \phi^* = 0}. \tag{7.2}$$

Here ϕ and ϕ^* have to be set to zero to make sure that we are evaluating at the global minimum of the potential.
In Type IIB string theory with orientifolds and fluxes, the calculation of the mass values for the axio-dilaton and the complex structure moduli works in a similar way. Looking at the four-dimensional supergravity action (the bosonic part without interactions) again, the corresponding Lagrangian, that relates to the moduli fields is given by

$$\mathcal{L}^{(4)}_{\text{moduli}} = - G_{A\bar{B}}(\phi, \bar{\phi}) \partial \phi^A \partial \phi^{\bar{B}} - V_F(\phi, \bar{\phi}), \tag{7.3}$$

where ofcourse $G_{A\bar{B}} = \frac{\partial^2 K(\phi,\bar{\phi})}{\partial\phi^A \partial\phi^{\bar{B}}}$ and with ϕ^A we denote the axio-dilaton and the complex structure moduli. To get the mass values, we can expand the scalar potential around the minimum as

$$\begin{aligned}
V_F &= V_{\min} + \partial_A V_F(\phi,\bar{\phi})\big|_{F=0}(\phi^A - \phi^A_{\min}) + \partial_{\bar{B}} V_F(\phi,\bar{\phi})\big|_{F_A=0}(\bar{\phi}^B - \bar{\phi}^B_{\min}) \\
&+ \partial_A \partial_{\bar{B}} V_F(\phi,\bar{\phi})\big|_{F_A=0}(\phi^A - \phi^A_{\min})(\bar{\phi}^B - \bar{\phi}^B_{\min}) \\
&+ \frac{1}{2}\partial_A \partial_B V_F(\phi,\bar{\phi})\big|_{F_A=0}(\phi^A - \phi^A_{\min})(\phi^B - \phi^B_{\min}) \\
&+ \frac{1}{2}\partial_{\bar{A}} \partial_{\bar{B}} V_F(\phi,\bar{\phi})\big|_{F_A=0}(\bar{\phi}^A - \bar{\phi}^A_{\min})(\bar{\phi}^B - \bar{\phi}^B_{\min}) + \cdots \\
&= V_{\min} + \partial_A V_F(\phi,\bar{\phi})\big|_{F_A=0}(\phi^A - \phi^A_{\min}) + \partial_{\bar{B}} V_F(\phi,\bar{\phi})\big|_{F_A=0}(\bar{\phi}^B - \bar{\phi}^B_{\min}) \\
&+ M^2_{A\bar{B}}(\phi^A - \phi^A_{\min})(\bar{\phi}^B - \bar{\phi}^B_{\min}) + \frac{1}{2}M^2_{AB}(\phi^A - \phi^A_{\min})(\phi^B - \phi^B_{\min}) \\
&+ \frac{1}{2}M^2_{\bar{A}\bar{B}}(\bar{\phi}^A - \bar{\phi}^A_{\min})(\bar{\phi}^B - \bar{\phi}^B_{\min}) + \cdots, \qquad (7.4)
\end{aligned}$$

where we used that the second derivative of the potential, evaluated at the minimum, corresponds to the mass values of the scalar fields. We can thus set up a matrix for the mass values as

$$M^2 = \begin{pmatrix} M^2_{A\bar{B}} & M^2_{AB} \\ M^2_{\bar{A}\bar{B}} & M^2_{\bar{A}B} \end{pmatrix}, \qquad (7.5)$$

where $M^2_{A\bar{B}} = M^2_{\bar{A}B}$ because of the commuting partial derivatives. We are most interested in the value $M^2_{A\bar{B}}$ since this will show up in the trace [26]. The corresponding part of the Lagrangian is given by

$$\mathcal{L}^{(4)}_{\text{mass}} = -\, G_{A\bar{B}} \partial\phi^A \partial\phi^{\bar{B}} - M^2_{A\bar{B}}\phi^A\phi^{\bar{B}}, \qquad (7.6)$$

where we shifted $\phi^A \to \phi^A + \phi^A_{\min}$. To get a similar Lagrangian as Eq. (7.1), we need to redefine the scalar fields. The so-called canonical adjustment can be made by writing the metric as

$$G_{A\bar{B}} = (T^\dagger)_A{}^C \delta_{C\bar{D}} T^{\bar{D}}{}_{\bar{B}}, \qquad (7.7)$$

where T is a lower triangular matrix with nonnegative diagonal numbers. This decomposition is called a Cholesky decomposition and it can be applied to Hermitian positive definite matrices, which $G_{A\bar{B}}$, since it is a Kähler metric, is [32]. Filling this in the Lagrangian term, we get

$$\begin{aligned}
\mathcal{L}^{(4)}_{\text{mass}} &= -\,(T^\dagger)_A{}^C \delta_{C\bar{D}} T^{\bar{D}}{}_{\bar{B}} \partial\phi^A \partial\phi^{\bar{B}} - M^2_{A\bar{B}}\phi^A\phi^{\bar{B}} \\
&= -\delta_{C\bar{D}} \partial\phi^A (T^\dagger)_A{}^C T^{\bar{D}}{}_{\bar{B}} \partial\phi^{\bar{B}} - M^2_{A\bar{B}}\phi^A\phi^{\bar{B}}. \qquad (7.8)
\end{aligned}$$

Using that

$$\partial\phi^A (T^\dagger)_A{}^C = \partial[\phi^A (T^\dagger)_A{}^C] - \phi^A \partial[(T^\dagger)_A{}^C)] = \partial[\phi^A (T^\dagger)_A{}^C]. \qquad (7.9)$$

we can define the canonically normalized scalar field $\hat{\phi}^C = \phi^A (T^\dagger)_A{}^C$ from which it also follows that $\phi^A = \hat{\phi}^C (T^\dagger)^{-1A}_C$. Filling these in, the Lagrangian then becomes

$$\begin{aligned}
\mathcal{L}^{(4)}_{\text{mass}} &= -\delta_{C\bar{D}} \partial\hat{\phi}^C \partial\hat{\phi}^{\bar{D}} - M^2_{A\bar{B}} (T^\dagger)^{-1A}_C \hat{\phi}^C (T^{-1})^{\bar{B}}{}_{\bar{D}} \hat{\phi}^{\bar{D}} \\
&= -\delta_{C\bar{D}} \partial\hat{\phi}^C \partial\hat{\phi}^{\bar{D}} - (T^\dagger)^{-1A}_C M^2_{A\bar{B}} (T^{-1})^{\bar{B}}{}_{\bar{D}} \hat{\phi}^C \hat{\phi}^{\bar{D}} \\
&= -\delta_{C\bar{D}} \partial\hat{\phi}^C \partial\hat{\phi}^{\bar{D}} - M^2_{C\bar{D}} \hat{\phi}^C \hat{\phi}^{\bar{D}} \qquad (7.10)
\end{aligned}$$

and this way a similar Lagrangian as in (7.1) is obtained. Due to the normalization the mass values change somewhat, and if we take the trace of these new mass values $M^2_{C\bar{D}}$, we see that

$$\begin{aligned}\mathrm{Tr}[M^2_{C\bar{D}}] &= \delta^{C\bar{D}} M^2_{C\bar{D}} = \mathrm{Tr}[(T^\dagger)^{-1A}_{C} M^2_{A\bar{B}} (T^{-1})^{\bar{B}}{}_{\bar{D}}] = \mathrm{Tr}[(T^\dagger)^{-1A}_{C} (T^{-1})^{\bar{B}}{}_{\bar{D}} M^2_{A\bar{B}}] \\ &= \delta^{C\bar{D}} (T^\dagger)^{-1A}_{C} (T^{-1})^{\bar{B}}{}_{\bar{D}} M^2_{A\bar{B}} = (T^\dagger)^{-1A}_{C} \delta^{C\bar{D}} (T^{-1})^{\bar{B}}{}_{\bar{D}} M^2_{A\bar{B}}.\end{aligned} \tag{7.11}$$

Noticing from Eq. (7.7) that the inverse Kähler metric is given by

$$G^{A\bar{B}} = (T^\dagger)^{-1A}_{C} \delta^{C\bar{D}} (T^{-1})^{\bar{B}}{}_{\bar{D}}, \tag{7.12}$$

the trace of the canonically normalized masses $M^2_{C\bar{D}}$ becomes

$$\mathrm{Tr}[M^2_{C\bar{D}}] = G^{A\bar{B}} M^2_{A\bar{B}}. \tag{7.13}$$

Using that the new mass matrix is given by

$$\hat{M}^2 = \begin{pmatrix} M^2_{C\bar{D}} & M^2_{CD} \\ M^2_{\bar{C}\bar{D}} & M^2_{\bar{C}D} \end{pmatrix} \tag{7.14}$$

and using that still $M^2_{C\bar{D}} = M^2_{\bar{C}D}$ we finally obtain for the trace of the canonically normalized mass matrix that

$$m^2 \equiv \mathrm{Tr}[\hat{M}^2] = 2\,\mathrm{Tr}[M^2_{C\bar{D}}] = 2G^{A\bar{B}} M^2_{A\bar{B}}. \tag{7.15}$$

Thus, we need to calculate $M^2_{A\bar{B}}$ and for the scalar potential from Eq. (6.5), this is

$$\begin{aligned}M^2_{A\bar{B}} &= \partial_A \partial_{\bar{B}} V_F\big|_{F_I=0} = \partial_A \partial_{\bar{B}} (e^K F_I G^{I\bar{J}} \overline{F}_{\bar{J}})|_{F_I=0} \\ &= e^K (\partial_{\bar{B}} F_I) G^{I\bar{J}} (\partial_A \overline{F}_{\bar{J}}) + e^K (\partial_A F_I) G^{I\bar{J}} (\partial_{\bar{B}} \overline{F}_{\bar{J}})|_{F_I=0},\end{aligned} \tag{7.16}$$

where in the last step we utilized that the other derivative terms vanish, because of their direct dependence on F_I or $\overline{F}_{\bar{J}}$. Starting with the first term of the equation above, for the derivative we get

$$\begin{aligned}\partial_{\bar{B}} F_I &= \partial_{\bar{B}} (\partial_I W + (\partial_I K) W) = \partial_{\bar{B}} \partial_I W + (\partial_{\bar{B}} \partial_I K) W + (\partial_I K) \partial_{\bar{B}} W \\ &= (\partial_{\bar{B}} \partial_I K) W = G_{\bar{B}I} W,\end{aligned} \tag{7.17}$$

where we used that W depends only on holomorphic coordinates, and its partial derivative with respect to anti-holomorphic coordinates is zero by definition. This relation also directly implies that

$$\partial_A \overline{F}_{\bar{J}} = G_{\bar{J}A} \overline{W}. \tag{7.18}$$

Filling these in, the first term of Eq. (7.16) becomes quite simple:

$$\begin{aligned}e^K (\partial_{\bar{B}} F_I) G^{I\bar{J}} (\partial_A \overline{F}_{\bar{J}})|_{F_I=0} &= e^K (G_{\bar{B}I} W G^{I\bar{J}} G_{\bar{J}A} \overline{W}) \\ &= e^K G_{A\bar{B}} |W|^2.\end{aligned} \tag{7.19}$$

Unfortunately, the second term will be not be so concise, and we will start with calculating the derivatives of the F-terms for the different moduli. Subsequently, we apply the conditions coming from the vanishing of the F-terms. Using the expressions for the F-terms of the axio-dilaton and the complex structure moduli the first three terms of $\partial_A F_I$ in the mass matrix become

$$\begin{aligned}
\partial_\tau F_\tau|_{F_I=0} &= \partial_\tau(-\frac{1}{\tau-\bar\tau}\int \overline{G}_3\wedge\Omega)|_{F_I=0} = \partial_\tau(-\frac{1}{\tau-\bar\tau})\int \overline{G}_3\wedge\Omega)|_{F_I=0}\\
&= \frac{1}{(\tau-\bar\tau)^2}\int \overline{G}_3\wedge\Omega)|_{F_I=0} = -\frac{1}{(\tau-\bar\tau}F_\tau|_{F_I}=0;\\
\partial_a F_\tau|_{F_I=0} &= -\frac{1}{\tau-\bar\tau}\int \overline{G}_3\wedge\partial_a\Omega|_{F_I=0} = -\frac{1}{\tau-\bar\tau}\int \overline{G}_3\wedge\chi_a + \frac{1}{\tau-\bar\tau}(\partial_a K^{\text{cs}})\int \overline{G}_3\wedge\Omega|_{F_I=0}\\
&= -\frac{1}{\tau-\bar\tau}\int \overline{G}_3\wedge\chi_a - (\partial_a K^{\text{cs}})F_\tau|_{F_I=0} = -\frac{1}{\tau-\bar\tau}\int \overline{G}_3\wedge\chi_a;\\
\partial_\tau F_i|_{F_I=0} &= \partial_\tau(\int G_3\wedge\chi_i) = -\int H_3\wedge\chi_i|_{F_I=0} = -\frac{1}{\tau-\bar\tau}\int \overline{G}_3\wedge\chi_i, \qquad (7.20)
\end{aligned}$$

where we used Eq. (3.27) in the second term and for the last term we applied the condition $F_I = 0$ from Eq. (6.17). We will treat the last derivative separately. Using Eq. (6.12), we get

$$\begin{aligned}
\partial_a F_i|_{F_I=0} &= \int G_3\wedge\partial_a\chi_i|_{F_I=0} = \int G_3\wedge D_a\chi_i - (\partial_a K^{\text{cs}})\int G_3\wedge\chi_i + \Gamma^b_{ai}\int G_3\wedge\chi_b|_{F_I=0}\\
&= \int G_3\wedge D_a\chi_i - (\partial_a K^{\text{cs}})F_i + \Gamma^b_{ai}F_b|_{F_I=0} = \int G_3\wedge D_a\chi_i \qquad (7.21)
\end{aligned}$$

where in the second step we also used Eq. (3.33). This can then again be rewritten in terms of the intersection number $\mathcal{K}$ that we gave in Eq.(3.34) as

$$\partial_a F_i|_{F_i=0} = \int G_3\wedge D_i\chi_a = -ie^{K^{\text{cs}}}\mathcal{K}_{ia}{}^{\bar k}\int G_3\wedge\chi_{\bar k}, \qquad (7.22)$$

Again, the complex conjugate counterpart $\partial_{\bar B}\overline{F}_{\bar J}$ is just given by the complex conjugates of the equations above. We can fill in these terms in the second part of Eq. (7.16) and multiply it by the inverse Kähler metric to calculate the trace. We then obtain

$$\begin{aligned}
\text{Tr}[M^2_{C\bar D}] &= G^{A\bar B}M^2_{A\bar B} = e^K G^{A\bar B}G_{A\bar B}|W|^2 + e^K G^{\tau\bar\tau}(\partial_\tau F_\tau G^{\tau\bar\tau}\partial_{\bar\tau}\overline{F}_{\bar\tau} + \partial_\tau F_i G^{i\bar\jmath}\partial_{\bar\tau}\overline{F}_{\bar\jmath})\\
&+ e^K G^{a\bar b}(\partial_a F_\tau G^{\tau\bar\tau}\partial_{\bar b}\overline{F}_{\bar\tau} + \partial_a F_i G^{i\bar\jmath}\partial_{\bar b}\overline{F}_{\bar\jmath})\\
&= e^K(h^{2,1}+1)|W|^2 + e^K(G^{i\bar\jmath}\int \overline{G}_3\wedge\chi_i\int G_3\wedge\chi_{\bar\jmath})\\
&+ e^K(G^{a\bar b}\int \overline{G}_3\wedge\chi_a\int G_3\wedge\chi_{\bar b} + e^{2K^{\text{cs}}}G^{a\bar b}G^{i\bar\jmath}\mathcal{K}_{ia}{}^{\bar k}\mathcal{K}_{\bar\jmath\bar b}{}^{k}\int \overline{G}_3\wedge\chi_k\int G_3\wedge\chi_{\bar k})\\
&= e^K(h^{2,1}+1)|W|^2 + 2e^K G^{i\bar\jmath}\int \overline{G}_3\wedge\chi_i\int G_3\wedge\chi_{\bar\jmath}\\
&+ e^K e^{2K^{\text{cs}}}G^{a\bar b}G^{i\bar\jmath}\mathcal{K}_{ia}{}^{\bar k}\mathcal{K}_{\bar\jmath\bar b}{}^{k}\int \overline{G}_3\wedge\chi_k\int G_3\wedge\chi_{\bar k}, \qquad (7.23)
\end{aligned}$$

where we used that the trace of the metric is equal to the dimension of the moduli space which in this case is $h^{2,1}+1$, where the 1 comes from the axio-dilaton. Also, in the last step, we relabeled some indices. Grouping terms together, the trace of the canonically normalized moduli mass matrix becomes

$$\begin{aligned}
m^2 &= 2G^{A\bar B}M^2_{A\bar B}\\
&= 2e^K(h^{2,1}+1)|W|^2 + 2e^K[2G^{i\bar\jmath} + e^{2K^{\text{cs}}}G^{k\bar l}G^{m\bar n}\mathcal{K}_{km}{}^{\bar\jmath}\mathcal{K}_{\bar l\bar n}{}^{i}]\int \overline{G}_3\wedge\chi_i\int G_3\wedge\chi_{\bar\jmath}. \qquad (7.24)
\end{aligned}$$

We can now use the condition Eq. (6.17) that follows from $F_I = 0$ and the same sort of expansion as in (6.18) for H_3 which is given by

$$H_3 = h^0\Omega + h^a\chi_a + h^{\bar{a}}\chi_{\bar{a}} + h^{\bar{0}}\overline{\Omega}. \tag{7.25}$$

Applying these, the product of the integrals in the second term of Eq. (7.24) can be rewritten as

$$\begin{aligned}\int \overline{G}_3 \wedge \chi_i \int G_3 \wedge \chi_{\bar{\jmath}} &= -(\tau-\bar{\tau})^2 \int H_3 \wedge \chi_i \int H_3 \wedge \chi_{\bar{\jmath}} \\ &= (\tau-\bar{\tau})^2 h^a h^{\bar{b}} \int \chi_i \wedge \chi_{\bar{b}} \int \chi_a \wedge \chi_{\bar{\jmath}} \\ &= (\tau-\bar{\tau})^2 h^a h^{\bar{b}} G_{i\bar{b}} G_{a\bar{\jmath}} \Big(\int \Omega \wedge \overline{\Omega}\Big)^2, \end{aligned} \tag{7.26}$$

where in the last step we used the equation for the complex structure Kähler metric, given in Eq. (3.17). Filling this in the mass value and leaving the term dependent on W intact, we obtain

$$\begin{aligned} m^2 &= 2e^K(h^{2,1}+1)|W|^2 \\ &\quad + 2e^K[2G^{i\bar{\jmath}} + e^{2K^{\text{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{km}{}^{\bar{\jmath}} \mathcal{K}_{\bar{l}\bar{n}}{}^{i}](\tau-\bar{\tau})^2 h^a h^{\bar{b}} G_{i\bar{b}} G_{a\bar{\jmath}} \Big(\int \Omega\wedge\overline{\Omega}\Big)^2 \\ &= 2e^K(h^{2,1}+1)|W|^2 + 2e^K[2G_{a\bar{b}} + e^{2K^{\text{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kma}\mathcal{K}_{\bar{l}\bar{n}\bar{b}}] h^a h^{\bar{b}} (\tau-\bar{\tau})^2 \Big(\int \Omega\wedge\overline{\Omega}\Big)^2, \end{aligned} \tag{7.27}$$

and here we used the index-lowering effect of the Kähler metric. Next, since the total Kähler potential is given by $K = -\ln\big[-i(\tau-\bar{\tau})\big] - \ln\Big[i\int\Omega\wedge\overline{\Omega}\Big] - 2\ln(\mathcal{V})$, we see that

$$e^K = \frac{1}{\mathcal{V}^2}\frac{1}{-i(\tau-\bar{\tau})}\frac{1}{i\int\Omega\wedge\overline{\Omega}} = \frac{1}{\mathcal{V}^2}\frac{1}{(\tau-\bar{\tau})}\frac{1}{\int\Omega\wedge\overline{\Omega}} \tag{7.28}$$

which means that

$$(\tau-\bar{\tau})\int\Omega\wedge\overline{\Omega} = \frac{e^{-K}}{\mathcal{V}^2}. \tag{7.29}$$

Filling this in two times, and after relabeling some indices, we get for the trace

$$m^2 = 2e^K(h^{2,1}+1)|W|^2 + \frac{2}{\mathcal{V}^4}e^{-K}[2G_{a\bar{b}} + e^{2K^{\text{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kma}\mathcal{K}_{\bar{l}\bar{n}\bar{b}}] h^a h^{\bar{b}}. \tag{7.30}$$

The second term in front of $h^a h^{\bar{b}}$ looks quite involved, but it can actually be written in a simpler form. We can just fill in the equation for the Ricci tensor, given by Eq. (3.36) and then we obtain

$$m^2 = 2e^K(h^{2,1}+1)|W|^2 + \frac{2}{\mathcal{V}^4}e^{-K}[(h^{2,1}+3)G_{a\bar{b}} + R_{a\bar{b}}] h^a h^{\bar{b}}. \tag{7.31}$$

However, the Ricci tensor of the complex structure moduli space is generally not a quantity that we know a lot of. In general we do not know if its values are positive, zero or negative. Luckily, there is a way to overcome this problem.

7.2 The Hodge metric

It was discovered that the complex structure moduli space of Calabi-Yau manifolds with background fluxes allows for another Kähler metric of which the properties were studied in [33–38] among others. This metric is called the Hodge metric. The Hodge metric is defined in [38], where for Calabi-Yau threefolds it is given by

$$H_{a\bar{b}} = (h^{2,1} + 3)G_{a\bar{b}} + R_{a\bar{b}}, \tag{7.32}$$

where $G_{a\bar{b}}$ is just the Kähler metric of the complex structure moduli space that we defined in Eq. (3.17), which in this context is often called a Weil-Petersson metric. We will give some properties of the Hodge metric before we relate it to the expression for the mass value above. First, we note that since the Weil-Petersson metric is a Kähler metric, it is positive definite, i.e. $G_{a\bar{b}} \geq 0$ [35]. Also, using the Ricci tensor from Eq. (3.36) the Hodge metric can be written as

$$\begin{aligned} H_{a\bar{b}} &= (h^{2,1} + 3)G_{a\bar{b}} - (h^{2,1} + 1)G_{a\bar{b}} + e^{2K^{\mathrm{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kma} \mathcal{K}_{\bar{l}\bar{n}\bar{b}} \\ &= 2G_{a\bar{b}} + e^{2K^{\mathrm{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kma} \mathcal{K}_{\bar{l}\bar{n}\bar{b}}. \end{aligned} \tag{7.33}$$

The following inequalities now hold for the Hodge metric:

- $H_{a\bar{b}} > 0$ [35]
- $G_{a\bar{b}} \leq 2G_{a\bar{b}} \leq H_{a\bar{b}}$ [34]
- $H_{a\bar{b}} = (h^{2,1} + 3)G_{a\bar{b}} + R_{a\bar{b}} \geq R_{a\bar{b}}$

We will use these inequalities later on, when we try to find constraints on the value m^2.

As can be seen from Eq. (7.31), the mass term can easily be written in terms of the Hodge metric as

$$m^2 = 2e^K (h^{2,1} + 1)|W|^2 + \frac{2}{\mathcal{V}^4} e^{-K} H_{a\bar{b}} h^a h^{\bar{b}}. \tag{7.34}$$

We thus found an expression for the trace of the moduli masses that consists of just two terms: a term dependent on the superpotential and a term that depends on the Hodge metric.

7.3 The flux number revisited

We will now return to the flux number

$$N_{\mathrm{flux}} = \int F_3 \wedge H_3 \tag{7.35}$$

and try to rewrite it such that we can more easily relate it to the mass term obtained above. This is a useful quantity to get in our equation since it is bounded by physical conditions on the manifold, the tadpole cancellation conditions. To achieve this, we expand the 3-form H_3 again as

$$H_3 = h^0 \Omega + h^a \chi_a + h^{\bar{b}} \chi_{\bar{b}} + h^{\bar{0}} \overline{\Omega}. \tag{7.36}$$

Filling this H_3 into N_{flux}, we can then additionally apply the conditions from $F_I = 0$ that relate F_3 to H_3 as stated in Eq. (6.16) and Eq. (6.17) in the following way:

$$\begin{aligned}
N_{\text{flux}} &= \int F_3 \wedge H_3 = \int F_3 \wedge (h^0\Omega + h^a\chi_a + h^{\bar b}\chi_{\bar b} + h^{\bar 0}\overline{\Omega}) \\
&= h^0 \int F_3 \wedge \Omega + h^a \int F_3 \wedge \chi_a + h^{\bar b} \int F_3 \wedge \chi_{\bar b} + h^{\bar 0} \int F_3 \wedge \overline{\Omega} \\
&= \bar\tau h^0 \int H_3 \wedge \Omega + \tau h^a \int H_3 \wedge \chi_a + \bar\tau h^{\bar b} \int H_3 \wedge \chi_{\bar b} + \tau h^{\bar 0} \int H_3 \wedge \overline{\Omega} \\
&= -\bar\tau h^0 h^{\bar 0} \int \Omega \wedge \overline{\Omega} - \tau h^a h^{\bar b} \int \chi_a \wedge \chi_{\bar b} + \bar\tau h^{\bar b} h^a \int \chi_a \wedge \chi_{\bar b} + \tau h^{\bar 0} h^0 \int \Omega \wedge \overline{\Omega} \\
&= (\tau - \bar\tau) h^0 h^{\bar 0} \int \Omega \wedge \overline{\Omega} - (\tau - \bar\tau) h^a h^{\bar b} \int \chi_a \wedge \chi_{\bar b} \\
&= (\tau - \bar\tau) h^0 h^{\bar 0} \int \Omega \wedge \overline{\Omega} - (\tau - \bar\tau) h^a h^{\bar b} G_{a\bar b} \int \Omega \wedge \overline{\Omega} \\
&= \mathcal{V}^2 e^K |W|^2 + G_{a\bar b} h^a h^{\bar b} (\tau - \bar\tau) \int \Omega \wedge \overline{\Omega} \\
&= \mathcal{V}^2 e^K |W|^2 + \frac{1}{\mathcal{V}^2} e^{-K} G_{a\bar b} h^a h^{\bar b}
\end{aligned} \tag{7.37}$$

where we again applied Eq. (3.17) for the Kähler metric of the complex structure moduli and that at the minimum

$$\begin{aligned}
|W|^2 &= (\bar\tau - \tau) \int H_3 \wedge \Omega (\tau - \bar\tau) \int H_3 \wedge \overline{\Omega} = (\tau - \bar\tau)^2 h^0 h^{\bar 0} (\int \Omega \wedge \overline{\Omega})^2 \\
&= \frac{1}{\mathcal{V}^2} e^{-K} h^0 h^{\bar 0} (\tau - \bar\tau) \int \Omega \wedge \overline{\Omega},
\end{aligned} \tag{7.38}$$

which means that

$$h^0 h^{\bar 0} (\tau - \bar\tau) \int \Omega \wedge \overline{\Omega} = \mathcal{V}^2 e^K |W|^2. \tag{7.39}$$

Using this definition of the flux number, we can write the mass value in multiple forms. Starting with Eq. (7.30) and applying the expression for the flux number, we get

$$\begin{aligned}
m^2 &= \frac{2(h^{2,1}+1)}{\mathcal{V}^2} N_{\text{flux}} - \frac{2}{\mathcal{V}^4} e^{-K} (h^{2,1}+1) G_{a\bar b} h^a h^{\bar b} + \frac{2}{\mathcal{V}^4} e^{-K} [(h^{2,1}+3) G_{a\bar b} + R_{a\bar b}] h^a h^{\bar b} \\
&= \frac{2(h^{2,1}+1)}{\mathcal{V}^2} N_{\text{flux}} + \frac{2}{\mathcal{V}^4} e^{-K} (2G_{a\bar b} + R_{a\bar b}) h^a h^{\bar b}.
\end{aligned} \tag{7.40}$$

This is a way of expressing the trace that we will use later on when we start discussing specific cases where we can actually calculate the Ricci tensor. Proceeding, we can also write the mass term in terms of the superpotential and the flux number by

$$\begin{aligned}
m^2 &= \frac{2(h^{2,1}+1)}{\mathcal{V}^2} N_{\text{flux}} + \frac{4}{\mathcal{V}^2} N_{\text{flux}} - 4e^K |W|^2 + \frac{2}{\mathcal{V}^4} e^{-K} R_{a\bar b} h^a h^{\bar b} \\
&= \frac{2(h^{2,1}+3)}{\mathcal{V}^2} N_{\text{flux}} - 4e^K |W|^2 + \frac{2}{\mathcal{V}^4} e^{-K} R_{a\bar b} h^a h^{\bar b},
\end{aligned} \tag{7.41}$$

where we again used (7.37). Lastly, the expression in terms of the Hodge metric given by Eq. (7.34) can be written in the following way

$$m^2 = \frac{2}{\mathcal{V}^2}(h^{2,1}+1)N_{\text{flux}} + \frac{2}{\mathcal{V}^4}e^{-K}\left[H_{a\bar{b}} - (h^{2,1}+1)G_{a\bar{b}}\right]h^a h^{\bar{b}}. \tag{7.42}$$

We have given three different forms of the trace of the mass matrix in terms of the flux number and we will use them all below when trying to find constraints on these mass values.

7.4 Constraints on the moduli masses

As was shown in the previous sections, it seems to be quite hard if not impossible to get a direct proportionality relation between the mass values and other well-known expressions in the theory since there will always be extra terms that show up. Therefore, the next thing we will do is try to find out if any of the objects in our equation possess some bound that we could use. In that way it may be possible to still say something about the size of the mass values. Let us start from the expression given above for the trace of the squared mass in terms of the flux number

$$m^2 = \frac{2}{\mathcal{V}^2}(h^{2,1}+1)N_{\text{flux}} + \frac{2}{\mathcal{V}^4}e^{-K}\left[H_{a\bar{b}} - (h^{2,1}+1)G_{a\bar{b}}\right]h^a h^{\bar{b}}. \tag{7.43}$$

Using that $H_{a\bar{b}} - 2G_{a\bar{b}} \geq 0$, we get

$$\begin{aligned} m^2 &\geq \frac{2}{\mathcal{V}^2}(h^{2,1}+1)N_{\text{flux}} - \frac{2}{\mathcal{V}^4}e^{-K}(h^{2,1}-1)G_{a\bar{b}}h^a h^{\bar{b}} = \frac{4}{\mathcal{V}^2}N_{\text{flux}} + 2(h^{2,1}-1)e^K|W|^2 \\ &\geq \frac{4}{\mathcal{V}^2}N_{\text{flux}}, \end{aligned} \tag{7.44}$$

and using that $|W|^2 \geq 0$, we have thus found a boundary from below on the mass value with respect to the flux number. Since the trace of the mass matrix is the sum of all the eigenvalues, we can calculate the mean mass eigenvalue by dividing the trace with the number of eigenvalues from both $M^2_{A\bar{B}}$ and $M^2_{\bar{A}B}$. For the inequality this means that

$$\langle m^2 \rangle = \frac{m^2}{2(h^{2,1}+1)} \geq \frac{2N_{\text{flux}}}{(h^{2,1}+1)\mathcal{V}^2}, \tag{7.45}$$

from which it follows that the maximum mass eigenvalue should be bigger than this. Using the expression Eq. (7.41) and the third inequality for the Hodge metric, we also get

$$\begin{aligned} m^2 &= \frac{2(h^{2,1}+3)}{\mathcal{V}^2}N_{\text{flux}} - 4e^K|W|^2 + \frac{2}{\mathcal{V}^4}e^{-K}R_{a\bar{b}}h^a h^{\bar{b}} \\ &\leq \frac{2(h^{2,1}+3)}{\mathcal{V}^2}N_{\text{flux}} - 4e^K|W|^2 + \frac{2}{\mathcal{V}^4}e^{-K}H_{a\bar{b}}h^a h^{\bar{b}} \\ &\leq \frac{2(h^{2,1}+3)}{\mathcal{V}^2}N_{\text{flux}} + \frac{2}{\mathcal{V}^4}e^{-K}H_{a\bar{b}}h^a h^{\bar{b}}, \end{aligned} \tag{7.46}$$

with which we unfortunately can not proceed any further, since there is not much we can do with the term with the Hodge metric in this inequality. Trying to still get some more information about the mass value with respect to the flux number, we will now look at specific cases for the complex structure moduli space.

7.4.1 One complex structure modulus

What happens if we have one complex structure modulus, i.e. if we set $h^{2,1} = 1$? To describe this system most easily, we will use the complex structure Kähler potential in projective coordinates

$$K^{\rm cs} = -\ln[i(\bar{Z}^I \mathcal{F}_I - \bar{\mathcal{F}}_I Z^I)], \tag{7.47}$$

with $I = 0, 1$. Furthermore the prepotential from Eq. (3.51) in the large complex structure limit is given by

$$\mathcal{F} = -\frac{1}{3!}\mathcal{K}\frac{(Z^1)^3}{Z^0} \tag{7.48}$$

and thus

$$\mathcal{F}_0 = \partial_0 \mathcal{F} = \frac{1}{3!}\mathcal{K}\frac{(Z^1)^3}{(Z^0)^2}, \quad \mathcal{F}_1 = \partial_1 \mathcal{F} = -\frac{1}{2}\mathcal{K}\frac{(Z^1)^2}{Z^0}. \tag{7.49}$$

Applying homogeneous coordinates as used in [26](adjusted to to our conventions) by $Z^I = (1, U)$ this becomes

$$\mathcal{F}_0 = \partial_0 \mathcal{F} = \frac{1}{3!}\mathcal{K}U^3, \quad \mathcal{F}_1 = \partial_1 \mathcal{F} = -\frac{1}{2}\mathcal{K}U^2 \tag{7.50}$$

and the Kähler potential is

$$\begin{aligned} K^{\rm cs} &= -\ln[\frac{i}{6}\mathcal{K}(U^3 - \bar{U}^3) + \frac{i}{2}\mathcal{K}(U\bar{U}^2 - \bar{U}U^2)] = -\ln[\frac{i}{6}\mathcal{K}(U^3 - 3\bar{U}U^2 + 3U\bar{U}^2 - \bar{U}^3)] \\ &= -\ln[\frac{i}{3!}\mathcal{K}(U - \bar{U})^3]. \end{aligned} \tag{7.51}$$

Then, the only component of the Kähler metric becomes

$$G_{U\bar{U}} = \partial_U \partial_{\bar{U}} K^{\rm cs} = -\frac{3}{(U - \bar{U})^2}. \tag{7.52}$$

and the Ricci tensor from Eq. (3.36) changes to

$$\begin{aligned} R_{U\bar{U}} &= -(1+1)G_{U\bar{U}} + e^{2K^{\rm cs}}(G^{U\bar{U}})^2\mathcal{K}^2 = -2G_{U\bar{U}} - \frac{36}{\mathcal{K}^2(U-\bar{U})^6}\frac{1}{9}(U-\bar{U})^4\mathcal{K}^2 \\ &= -2G_{U\bar{U}} - \frac{4}{(U-\bar{U})^2} = (-2 + \frac{4}{3})G_{U\bar{U}} = -\frac{2}{3}G_{U\bar{U}}. \end{aligned} \tag{7.53}$$

Substituting this in the formula for the trace of the mass values of Eq. (7.40), we obtain

$$\begin{aligned} m^2 &= \frac{4}{\mathcal{V}^2}N_{\rm flux} + \frac{2}{\mathcal{V}^4}e^{-K}(2G_{U\bar{U}} + R_{U\bar{U}})h^U h^{\bar{U}} \\ &= \frac{4}{\mathcal{V}^2}N_{\rm flux} + \frac{8}{3\mathcal{V}^4}e^{-K}G_{U\bar{U}}h^U h^{\bar{U}} = \frac{20}{3\mathcal{V}^2}N_{\rm flux} - \frac{8}{3}e^K|W|^2 \\ &\le \frac{20}{3\mathcal{V}^2}N_{\rm flux}, \end{aligned} \tag{7.54}$$

and we thus found an upper bound of the trace of the moduli masses. We know that the average moduli mass is given by the trace divided by two times the total number of moduli, and for this case it means that

$$\langle m^2 \rangle = \frac{m^2}{2(h^{2,1}+1)} \leq \frac{20}{12\mathcal{V}^2} N_{\text{flux}} = \frac{5}{3\mathcal{V}^2} N_{\text{flux}}. \tag{7.55}$$

This especially is an interesting bound since then we now that the masses of both the moduli are smaller than $\frac{5}{3\mathcal{V}^2} N_{\text{flux}}$. Since the flux number is also bounded by the tadpole cancellation conditions, this gives a finite maximum mass of the moduli.

7.4.2 Kähler-Einstein moduli space

A Kähler metric on a complex manifold is called Kähler-Einstein if its Ricci tensor satisfies [11]

$$R_{a\bar{b}} = \lambda G_{a\bar{b}} \tag{7.56}$$

with $\lambda \in \mathbb{R}$. When we assume that the Ricci tensor of the complex structure moduli space satisfies this equation, we get for the mass term of Eq. (7.30):

$$m^2 = \frac{2(h^{2,1}+1)}{\mathcal{V}^2} N_{\text{flux}} + \frac{2}{\mathcal{V}^4} e^{-K} (2+\lambda) G_{a\bar{b}} h^a h^{\bar{b}}. \tag{7.57}$$

Actually, using the known boundaries of the Hodge metric, we can restrict the range of the constant λ. We know that the Hodge metric satisfies $0 \leq G_{a\bar{b}} < 2G_{a\bar{b}} \leq H_{a\bar{b}}$ and therefore

$$R_{a\bar{b}} \geq 2G_{a\bar{b}} - (h^{2,1}+3)G_{a\bar{b}} = -(h^{2,1}+1)G_{a\bar{b}}. \tag{7.58}$$

Thus for a Kähler-Einstein metric $\lambda \geq -(h^{2,1}+1)$. Filling this into the mass term, we obtain

$$\begin{aligned} m^2 &\geq \frac{2(h^{2,1}+1)}{\mathcal{V}^2} N_{\text{flux}} + \frac{2}{\mathcal{V}^4} e^{-K} (2-(h^{2,1}+1)) G_{a\bar{b}} h^a h^{\bar{b}} \\ &= \frac{2(h^{2,1}+1)}{\mathcal{V}^2} N_{\text{flux}} - \frac{2}{\mathcal{V}^4} e^{-K} (h^{2,1}-1) G_{a\bar{b}} h^a h^{\bar{b}} \\ &= \frac{4}{\mathcal{V}^2} N_{\text{flux}} + 2(h^{2,1}-1) e^{K} |W|^2 \geq \frac{4}{\mathcal{V}^2} N_{\text{flux}}, \end{aligned} \tag{7.59}$$

which unfortunately is a bound that we already found. We thus have to conclude that looking at a general Kähler-Einstein metric does not learn us something new. We can however consider a specific case where the Calabi-Yau orientifold is given by the quotient of a six-torus with $\mathbb{Z}_2 \times \mathbb{Z}_2$ i.e. [39]

$$M_3 = \frac{T^6}{\mathbb{Z}_2 \times \mathbb{Z}_2}. \tag{7.60}$$

This is a well known example of a Calabi-Yau orientifold and is mentioned in (among others) [39], [26], [40] and [41]. It turns out that its complex structure moduli space is Kähler-Einstein. As can be seen in [39], the complex structure moduli space of the torus T^6 contains three complex moduli and the only non-vanishing components of the intersection number are the six components $\mathcal{K}_{123} = \mathcal{K}_{132} = \mathcal{K}_{213} = \mathcal{K}_{231} = \mathcal{K}_{312} = \mathcal{K}_{321} = 1$. Filling these in, the prepotential of Eq. (3.51) becomes

$$\mathcal{F} = -\frac{1}{3!}(6\mathcal{K}_{123}) \frac{Z^1 Z^2 Z^3}{Z^0} = -\frac{Z^1 Z^2 Z^3}{Z^0}. \tag{7.61}$$

Again, using the projective coordinates Z^I and choosing homogenuous coordinates as in [39] the four coordinates become

$$Z^0 = 1, Z^1 = U^1, Z^2 = U^2, Z^3 = U^3. \tag{7.62}$$

Then we get for the derivatives of the prepotential

$$\begin{aligned} \partial_0 \mathcal{F} &= U^1 U^2 U^3; \quad \partial_1 \mathcal{F} = -U^2 U^3 \\ \partial_2 \mathcal{F} &= -U^1 U^3; \quad \partial_3 \mathcal{F} = -U^1 U^2. \end{aligned} \tag{7.63}$$

Using this, the term in the Kähler potential becomes

$$\begin{aligned} \bar{Z}^I \mathcal{F}_I - \bar{\mathcal{F}}_I Z^I = (&U^1 U^2 U^3 - \bar{U}^1 \bar{U}^2 \bar{U}^3 - \bar{U}^1 U^2 U^3 + U^1 \bar{U}^2 \bar{U}^3 \\ &- \bar{U}^2 U^1 U^3 + \bar{U}^1 \bar{U}^3 U^2 - \bar{U}^3 U^1 U^2 + \bar{U}^1 \bar{U}^2 U^3) \end{aligned} \tag{7.64}$$

which can be rewritten as

$$\bar{Z}^I \mathcal{F}_I - \bar{\mathcal{F}}_I Z^I = (U^1 - \bar{U}^1)(U^2 - \bar{U}^2)(U^3 - \bar{U}^3). \tag{7.65}$$

Filling this into the Kähler potential, we get

$$\begin{aligned} K &= -\ln[i(\bar{Z}^I \mathcal{F}_I - \bar{\mathcal{F}}_I Z^I)] \\ &= -\ln[i(U^1 - \bar{U}^1)(U^2 - \bar{U}^2)(U^3 - \bar{U}^3)] \\ &= -\ln[-i(U^1 - \bar{U}^1)] - \ln[-i(U^2 - \bar{U}^2)] - \ln[-i(U^3 - \bar{U}^3)]. \end{aligned} \tag{7.66}$$

Since we want the Kähler metric for this potential we have to take derivatives. Since the potential can be separated into the parts depending on U^1, U^2 and U^3 respectively, only the diagonal components of the Kähler metric will be non-zero. The first derivative we calculate is

$$\partial_{\bar{1}} K = \frac{1}{U^1 - \bar{U}^1} \tag{7.67}$$

and therefore every mixed component of $G_{a\bar{b}}$ with $\bar{U}^1$ is zero and

$$G_{1\bar{1}} = \partial_1 \partial_{\bar{1}} K = -\frac{1}{(U^1 - \bar{U}^1)^2}. \tag{7.68}$$

Since the parts of the Kähler potential that depend on U^2 and U^3 are the same as for U^1, we obtain

$$G_{a\bar{a}} = \partial_a \partial_{\bar{a}} K = -\frac{1}{(U^a - \bar{U}^a)^2}. \tag{7.69}$$

We can now calculate the Ricci-tensor as given in Eq. (3.36), which is

$$R_{a\bar{a}} = -(h^{2,1} + 1) G_{a\bar{a}} + e^{2K^{\mathrm{cs}}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{kma} \mathcal{K}_{\bar{l}\bar{n}\bar{a}}, \tag{7.70}$$

where the only non-zero components of the Ricci tensor are also the diagonal components, since any non-diagonal parts involve intersection numbers that are zero. Using that we only have three complex structure moduli, and therefore $h^{2,1} = 3$, we can calculate the Ricci tensor for the first coordinate:

$$\begin{aligned} R_{1\bar{1}} &= -4G_{1\bar{1}} + e^{2K^{cs}} G^{k\bar{l}} G^{m\bar{n}} \mathcal{K}_{km1} \mathcal{K}_{\bar{l}\bar{n}\bar{1}} \\ &= -4G_{1\bar{1}} + e^{2K^{cs}} (G^{2\bar{2}} G^{3\bar{3}} \mathcal{K}_{231} \mathcal{K}_{\bar{2}\bar{3}\bar{1}} + G^{3\bar{3}} G^{2\bar{2}} \mathcal{K}_{321} \mathcal{K}_{\bar{3}\bar{2}\bar{1}}) \\ &= -4G_{1\bar{1}} + 2 \frac{1}{-(U^1 - \bar{U}^1)^2 (U^2 - \bar{U}^2)^2 (U^3 - \bar{U}^3)^2} G^{2\bar{2}} G^{3\bar{3}} \\ &= -4G_{1\bar{1}} + 2G_{1\bar{1}} G_{2\bar{2}} G_{3\bar{3}} G^{2\bar{2}} G^{3\bar{3}} \\ &= -4G_{1\bar{1}} + 2G_{1\bar{1}} = -2G_{1\bar{1}}. \end{aligned} \tag{7.71}$$

Since all the terms are identical, the full Ricci tensor becomes

$$R_{a\bar{b}} = -2G_{a\bar{b}}, \tag{7.72}$$

by which we have indeed shown that the complex structure moduli space of the six-torus is Kähler-Einstein with $\lambda = -2$. When we fill this into Eq. (7.57), we just get

$$m^2 = \frac{8}{\mathcal{V}^2} N_{\text{flux}}. \tag{7.73}$$

This is quite interesting since then the mean moduli mass is equal to

$$\langle m^2 \rangle = \frac{m^2}{8} = \frac{N_{\text{flux}}}{\mathcal{V}^2} \tag{7.74}$$

and for the six-torus the mean moduli mass of the axio-dilaton and the three complex structure moduli is thus directly related to the flux number.

7.4.3 Boundary of complex structure moduli space

We will conclude our treatment of specific cases for the complex structure moduli space, by looking at the boundary of this space. At the boundary of the complex structure moduli space, it was found that the Hodge metric becomes a Poincaré metric. The Poincaré metric in the Poincaré half-plane model on the upper half-plane is a hyperbolic metric and can be written as [42]

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \tag{7.75}$$

where x and y are two real coordinates.
We can see this appearance of a Poincaré metric by parametrizing the complex structure moduli as is done in [43]. To do this, we choose local coordinates $t^a = e^{2\pi z^a}$, where z^a are the complex structure moduli and for which we approach the boundary in the limit $t^a \to 0$, i.e $z^a = x^a + iy^a \to x_0^a + i\infty$. For simplicity we will again look at one complex structure coordinate $z = x + iy$. In the limit $y \to \infty$ the Kähler (Weil-Peterson) line element becomes [43]

$$ds^2_{WP} = \frac{1}{y^2} (\hat{d} + \gamma(y))(dx^2 + dy^2), \tag{7.76}$$

where $\gamma(y)$ goes to zero as $y \to \infty$ and $\hat{d} \in \{0, ..., n\}$, but only for $\hat{d} > 0$ does the Kähler metric go to the Poincaré metric.
Also the Hodge line element becomes [43]

$$ds^2_H = \frac{1}{y^2} (c^{(0)} + \hat{\gamma}(y))(dx^2 + dy^2), \tag{7.77}$$

where again the $\hat{\gamma}(y)$ vanishes for $y \to \infty$ and now $c^{(0)} > 0$. In the limit $y \to \infty$ this is then also precisely the Poincaré metric multiplied by a constant. Filling in these new forms for the Hodge and the Kähler metric, the trace of the mass values for one complex structure modulus

$$m^2 = \frac{2}{\mathcal{V}^2}(h^{2,1}+1)N_{\text{flux}} + \frac{2}{\mathcal{V}^4}e^{-K}\big[H-(h^{2,1}+1)G\big]h\bar{h} \tag{7.78}$$

becomes

$$m^2 = \frac{4}{\mathcal{V}^2}N_{\text{flux}} + \frac{2}{\mathcal{V}^4}e^{-K}\frac{c^{(0)}-2\hat{d}}{y^2}h\bar{h}. \tag{7.79}$$

As is argued in [27] the flux number diverges at the boundary of the complex structure space. Therefore, if we want to know if the trace of the mass also diverges at this boundary, we have to look at the second term in the equation above. Since $c^{(0)} > 0$, $\hat{d} \geq 0$ and $\frac{e^{-K}}{\mathcal{V}^4}$ and $h\bar{h}$ are all non-negative when $y \to \infty$, we expect the second term in the equation above to go to zero. When this would be the case, it would indeed mean that at least one of the masses would diverge at the boundary of complex structure moduli space, and thus that the effective field theory breaks down. This would then correspond with the Swampland conjecture [44], since this also proposes that the effective field theory would break down at a point in the moduli space that is infinitely far away. However, the terms e^{-K} and $h\bar{h}$ still depend on the complex structure moduli, and we have to investigate this dependency to know if it does not cancel $\frac{1}{y^2}$. Due to lack of time we were unfortunately not able to make this more explicit and it would thus be a good topic for further research.

Chapter 8

Summary and Outlook

The landscape of string theory is very wide, and one map is not enough to describe it. With our thesis, we hope that we conveyed the vastness but also the richness of the area and we provided some new results on a small and specific location within the theory. Our study of string theory itself often felt like an adventure where at every step of the way new and unexpected things started to show up. We will briefly review the path we traveled below.
It all starts with a simple idea: replacing a point particle with a string. But then it follows, if want bosonic and fermionic fields, that we end up with ten dimensions, of which we are only able to observe four. To retain supersymmetry, which is interesting to us since it can provide a supersymmetric extension of the standard model, one can then either do an GSO projection or combine the bosonic string in the right-moving sector and the superstring in the left-moving sector. This then leads to the five different and consistent supersymmetrical string theories.

In our thesis we focused on type IIB string theory, a theory of closed strings and $N = 2$ supersymmetry and to lose the six dimensions but still keep the supersymmetry, it has to be compactified on a Calabi-Yau threefold. This Calabi-Yau threefold is a complex manifold and this then leads to complex geometry with concepts as Kähler potentials, cohomology groups and moduli spaces. These moduli spaces are of especial interest because after the compactification, the fields that span theses spaces as well as other massless fields start to show up in the effective action of the theory. However, this theory still has $N = 2$ supersymmetry and to get an accurate extension of the standard model we need minimal supersymmetry, i.e. $N = 1$ supersymmetry. In order to achieve this, an orientifold projection is applied by which the compactification manifold changes to an Calabi-Yau orientifold. Still, an additional complication is that we have more massless fields in the spectrum than the amount of long-range forces, to which they relate, that we measure. We thus have to give these fields masses but also have to make sure that these do not diverge, since then the effective description of the theory would break down. One method to do this, as we saw, was by turning on background fields in the theory. These then produce a scalar potential, and, as is generally the case in field theory, this scalar potential provides masses for the moduli.

We then come to the main terrain of the thesis, which is calculating the moduli masses corresponding to the axio-dilaton and the complex structure moduli for the global minimum of the scalar potential and trying to find some constraints on these values, such as relations between other objects in the theory or certain bounds. In this thesis we found that the trace of the mass matrix or the sum of all the mass eigenvalues can be related to another Kähler metric on the complex structure moduli space

of the Calabi-Yau threefold: the Hodge metric. This Hodge metric obeys certain constraints and using these in the general case we could find a lower bound for the trace. This was the only constraint that we could derive in the general case. However, to try to learn more about the behaviour of the moduli masses, we also looked at three specific cases for the complex structure moduli space.
Firstly, we investigated the case of one complex structure modulus, for which we could find an upper bound on the sum of the mass values. This then means that the largest mass eigenvalue must be smaller than this number divided by the total number of moduli, which was in this case $\frac{5}{3\mathcal{V}^2}N_{\text{flux}}$. Since the flux number is bounded too by the tadpole conditions, this means that when we have just one complex structure, there is a finite maximal moduli mass.
Secondly, we looked at the case of the complex structure moduli space as a Kähler-Einstein space, i.e. a moduli space for which the curvature is proportional to the Kähler metric. This did not lead to new constraints, but one particular compactification manifold, the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold was proven to lead to a Kähler-Einstein metric in the complex structure moduli space. For this specific manifold, a direct relation was found between the mean moduli mass and the flux number.
Lastly, we looked at the boundary of complex structure moduli space and what happens to the sum of the mass eigenvalues there. Here we used the result from [43] that the Hodge and Kähler metric (under certain conditions) change to a Poincaré metric. Filling this in, we expected the total sum of the moduli masses to diverge, which would mean that at least one moduli mass diverges, but we were unfortunately not able to show this explicitly. This hypothetical result would however agree with the idea of the swampland conjecture that the effective field theory breaks down at an infinitely far away point in the moduli space [44].

There are many more topics that would be interesting for further research. Firstly, we looked in our thesis at the global minimum of the scalar potential, but one could also investigate what the local minima of the scalar potential look like, and how the moduli masses behave there.
Additionally, one could also investigate the behaviour of the moduli masses in the Large Volume Scenario (see [20]), which is another method of moduli stabilization. Also, we found several constraints on the moduli masses that were related to the flux number and since the magnitude of the flux number is restricted by the tadpole conjecture and the tadpole cancellation conditions, it would be interesting to relate this conjecture and these conditions directly to the moduli masses.
Finally, our discussion of the boundary of the complex structure moduli space is far from complete. For instance, due to lack of time we were not able to show explicitly how the moduli masses behave there, and it would be interesting if this could be shown explicitly. Furthermore, in studying boundaries of the complex structure moduli space, a framework that is often used nowadays is asymptotic Hodge theory (see [45], [43] or [46]) and it would also be interesting to see if any of these tools can be applied to learn more about the behaviour of the moduli masses at these boundaries.

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