UTRECHT UNIVERSITY

THEORETICAL PHYSICS

MASTER THESIS

Aspects of black hole scattering and the two-body problem in General Relativity

First Examiner: Dr. Umut Gürsoy
Second Examiner: Dr. Tanja Hinderer
Daily Supervisor: Nava Gaddam

Guillermo Cortina Fernández

Theoretical Physics Master

Academic course 2020-22

Abstract:

The merging of binary black hole systems is an event that can be observed by gravitational wave observatories. It is of high importance to understand these events and to be able to model them from a theoretical perspective. Observations of the inspiral phase of these mergers are nicely explained by known theoretical tools. In this thesis, we will study and develop tools to understand the post-merger phase. These tools emerge from studying quantum gravitational dynamics of black holes via scattering amplitudes.

Contents

1 Introduction

Gravitational wave (GW) [1] astronomy is a new and exciting branch of physics capable of teaching us an extraordinary diversity of areas of nature. Unlike traditional astronomy, which is based on observing electromagnetic radiation (visible light, X-rays, radio waves, microwaves, etc.), GW astronomy is capable of measuring a completely different kind of radiation called gravitational waves. These waves are perturbations of space-time that deform the space and time itself as it propagates. Currently, experiments like LIGO [2] are capable of observing GW that come from violent events involving very massive objects like systems of binary black holes or neutron stars merging into each other. By observing these mergers we can learn not only about the macroscopic physics of these events, but also about the quantum mechanical behavior that governs, for example, the interior of neutron stars [3]. As theoretical physicists, our goal is to be able to predict and model the signals of these GW.

In this thesis, we will focus on the merger of a binary system of black holes. In this event we can distinguish three different phases. First we have the inspiral phase, which consists on the two black holes spiraling each other and getting closer to each other. Eventually the black holes collide forming only one black hole, this corresponds with the second phase called merger. Finally, we enter into the post-merger phase, on which this black hole emits the last GW until it settles from this violent interaction.

There are many different approaches to study the relativistic two-body problem in general relativity (GR). We have the work done with numerical relativity (NR) simulations [4–7], the self-force formalism [8, 9], the effective one-body (EOB) formalism [10, 11], post-Newtonian (PN) expansion [12–20] and post-Minkowskian (PM) expansion [21–30] just to name a few. PM expansions consists on making expansions in orders of G (Newton's gravitational constant). In contrast with the more familiar PN expansion, which expands in orders of the relative velocity between the two objects with respect to the speed of light (obtaining classical mechanics in the lower orders), PM expansions allows for any relative velocity. This fact is essential to study binary systems of black holes because during the inspiral phase these objects can reach relative velocity close to the speed of light.

PM expansion makes sense specially in the inspiral phase, which is dominated by large angular momentum J. The natural way of proceeding is defining the dimensionless quantity $j = cJ/Gm_1m_2$ proportional to the angular momentum and inversely proportional to G (where c is the speed of light, and m_1, m_2 are the masses of the inspiraling black holes). Therefore, for a phase dominated by large j, it makes sense to expand in orders of 1/j, which is proportional to orders of G, hence the PM expansion. Observations of the inspiral phase of these mergers are nicely explained by known theoretical tools involving scattering amplitudes [31]. These are calculations that are normally used when studying the scattering of particles, but there are clever ways of relating the study of particle physics with the study of black hole mergers. The main idea of this technique is mapping the two black holes into two scalar particles that interact gravitationally. This scattering of the two scalar particles can be calculated in a flat space-time background because during the first moments of the inspiral phase we can assume that the two black holes are far away from each other so that the space-time in between them can be treated as flat. The leading order calculation of the scattering amplitude as well as the PM expansion in this phase are known results.

In a similar way, we can propose that this mapping between the merger and scattering amplitude of scalar particles can be done also in the less understood post-merger phase. There are many differences between the inspiral and the post-merger phases. Now we can no longer assume a flat background space-time because we are working near the horizon of the black hole that remains after the merger. Therefore we can compute scattering amplitudes of these particles in a curved background space-time in order to model the perturbations that this final black hole emits in the form of GW as it settles after the merger. Also, even though the final black hole is expected to be a Kerr black hole and have a large angular momentum, working with Schwarzschild black holes (with a low angular momentum) simplifies calculations a lot and therefore we assume that the remaining black hole is of this kind. This means that now this phase is dominated by a low angular momentum, in contrast with the inspiral phase. The leading order calculations of the scattering amplitude in this regime is also known to be $1/\tilde{j} = sR^2/(\ell^2+\ell+2)$, where s is the total energy of the scattering, R is the Schwarzschild radius of the final black hole and the angular momentum is now represented by the ℓ. This result comes from the black hole eikonal approximation ($\sqrt{s} \gg \gamma M_{PL}$), where $\gamma = \kappa/R \sim M_{PL}/M_{BH}$ is an emergent dimensionless parameter, M_{PL} is Planck's mass, M_{BH} is the mass of the black hole and $\kappa = \sqrt{8\pi G}$. And the main result from [32], which says that, within this approximation, for every partial wave ℓ, the four point function is given by

$$< \phi\phi\phi\phi >= 4s \left[exp\left(i\frac{\gamma^2}{\hbar}\frac{1}{\tilde{j}} \right) - 1 \right] \tag{1}$$

This is a result of a resummation of an infinite number of ladder diagrams on the horizon. Therefore it is non-perturbative in γ and $\hbar$.

At the classical level, the physics of the post-merger phase consist on the classical ringing of black hole quasinormal modes with decreasing amplitude of oscillations, this process is called ringdown. On top of this, we suggest that black holes exhibit more than just ringdown physics and include, for example, effects of scattering near the horizon. We then propose that an expansion in orders of $\tilde{j}$ exists, which we denote as post-black hole (PBH) expansion [32–40].

The main goal of this thesis is to understand the expansion for the post-merger phase. To do so, we'll study the inspiral case with a different perspective, in particular using harmonic expansions. This is the method that was used when studying the leading order of the post-merger case. Then, we'll study the second order of the expansion of the scattering amplitude for the post-merger phase.

This thesis is divided into the following chapters. This first chapter corresponds to the introduction, on which we have set the motivation and the goals behind this work. In chapter 2 we review how to work with perturbations on a background space-time and we study the expansions of many geometric elements as well as the Einstein-Hilbert action and the scalar matter action. Then, in chapter 3 we study the flat space-time case corresponding to the inspiral phase of the merger with a different perspective that normally is used for the post-merger phase. In chapter 4 we move into studying scattering in curved space-time, related to the post-merger phase of the merger. We start working with the matter action modelled by a scalar field and we derive some Feynman rules. With these Feynman rules in hand, then we draw some Feynman diagrams and compute some scattering amplitudes. Finally, in chapter 5 we go over the most important conclusions and some future work that could be done in order to answer some open questions.

2 Linearized gravity

First of all, the convention followed by this work is the $-,+,+,+$ sign convention, the space-time four-dimensional indices are Greek letters and we use natural unites. In the case on which we explicitly write c or $\hbar$ is because in that specific scenario it is beneficial to the reader to make a comparison or for any other reason.

In a theory that describes gravitons like the work done in this thesis, the common practice to model them is as perturbations in a background metric. Because of that, the very first thing we will do is study these perturbations and how to work with them mathematically.

The perturbed space-time metric $\bar{g}_{\mu\nu}$ consists of a background metric $g_{\mu\nu}$ on which we add a perturbation metric $\delta g_{\mu\nu} = \kappa h_{\mu\nu}$. The second way of writing this perturbation metric including the constant κ is just so that we can keep count of the order of the expansion. Also, it is common practice to use this constant in particular because it simplifies the notation in some expressions. Therefore we have the following perturbed space-time metric:

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} = g_{\mu\nu} + \kappa\, h_{\mu\nu} \tag{2}$$

This metric is a symmetric 4x4 rank-two tensor with no torsion, so the symmetries from GR are allowed.

Any tensor that depends on the metric can be expanded as:

$$\bar{T}(\bar{g}_{\mu\nu}) = \bar{T}(\bar{g}_{\mu\nu})\big|_{h_{\alpha\beta}=0} + \left.\frac{\delta \bar{T}}{\delta \bar{g}_{\alpha\beta}}\right|_{h_{\alpha\beta}=0} \delta g_{\alpha\beta} + \frac{1}{2}\left.\frac{\delta^2 \bar{T}}{\delta g_{\alpha\beta}\delta g_{\rho\sigma}}\right|_{h_{\alpha\beta}=0, h_{\rho\sigma}=0} \delta g_{\alpha\beta}\delta g_{\rho\sigma} + \mathcal{O}\big(\kappa^3\big) \tag{3}$$

We can rename the terms from the expansion as:

$$\bar{T} = T + \kappa\, T^{(1)} + \kappa^2\, T^{(2)} + \mathcal{O}\big(\kappa^3\big) \tag{4}$$

Note that with our notation, any object with a bar will contain the perturbation of space-time. And every time we rise or lower the indices of a tensor with a bar we will use the perturbed metric $\bar{g}_{\mu\nu}$.

2.1 Metric elements

The inverse of $\bar{g}_{\mu\nu}$ is define such that $\bar{g}_{\mu\alpha}\, \bar{g}^{\alpha\nu} = \delta^{\nu}_{\mu}$. Therefore, it is written as:

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2\, h^{\mu}{}_{\alpha} h^{\alpha\nu} + \mathcal{O}\big(\kappa^3\big) \tag{5}$$

Also, the element $\sqrt{-\bar{g}_{\mu\nu}}$ that will appear in future calculations can be expanded as:

$$\sqrt{-\bar{g}_{\mu\nu}} = \sqrt{-g}\left(1 + \frac{1}{2}\kappa h - \frac{1}{4}\kappa^2 h^{\mu\nu} h_{\mu\nu} + \frac{1}{8}(\kappa h)^2 + \mathcal{O}\big(\kappa^3\big)\right) \tag{6}$$

where we define $g \coloneqq det(g_{\mu\nu})$.

2.2 Christoffel symbols

Christoffel symbols are defined as:

$$\Gamma^{\alpha}{}_{\mu\nu} \coloneqq \frac{1}{2}\, g^{\alpha\beta}(\partial_\mu\, g_{\beta\nu} + \partial_\nu\, g_{\beta\mu} - \partial_\beta\, g_{\mu\nu}) \tag{7}$$

And since it depends on the metric it can be expanded in different orders of κ:

$$\begin{aligned}\bar{\Gamma}^{\alpha}{}_{\mu\nu} &= \frac{1}{2}\,\bar{g}^{\alpha\beta}(\partial_\mu\,\bar{g}_{\beta\nu} + \partial_\nu\,\bar{g}_{\beta\mu} - \partial_\beta\,\bar{g}_{\mu\nu}) \\ &= \frac{1}{2}\,g^{\alpha\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu}) \\ &\quad + \kappa\,\frac{1}{2}\,\Big[g^{\alpha\beta}(\partial_\mu\,h_{\beta\nu} + \partial_\nu\,h_{\beta\mu} - \partial_\beta\,h_{\mu\nu}) - h^{\alpha\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu})\Big] \\ &\quad + \kappa^2\,\frac{1}{2}\,\Big[h^{\alpha}{}_{\rho}h^{\rho\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu}) - h^{\alpha\beta}(\partial_\mu\,h_{\beta\nu} + \partial_\nu\,h_{\beta\mu} - \partial_\beta\,h_{\mu\nu})\Big] \\ &\quad + \mathcal{O}\big(\kappa^3\big)\end{aligned} \tag{8}$$

As a summary we can write the following:

$$\begin{cases}\bar{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} + \kappa\,\Gamma^{\alpha}{}_{\mu\nu}{}^{(1)} + \kappa^2\,\Gamma^{\alpha}{}_{\mu\nu}{}^{(2)} + \mathcal{O}\big(\kappa^3\big) \\ \Gamma^{\alpha}{}_{\mu\nu} := \frac{1}{2}\,g^{\alpha\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu}) \\ \Gamma^{\alpha}{}_{\mu\nu}{}^{(1)} := \frac{1}{2}\,\big[g^{\alpha\beta}(\partial_\mu\,h_{\beta\nu} + \partial_\nu\,h_{\beta\mu} - \partial_\beta\,h_{\mu\nu}) - h^{\alpha\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu})\big] \\ \Gamma^{\alpha}{}_{\mu\nu}{}^{(2)} := \frac{1}{2}\,\big[h^{\alpha}{}_{\rho}h^{\rho\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu}) - h^{\alpha\beta}(\partial_\mu\,h_{\beta\nu} + \partial_\nu\,h_{\beta\mu} - \partial_\beta\,h_{\mu\nu})\big]\end{cases} \tag{9}$$

We can simplify the notation using covariant derivatives:

$$\begin{cases}\bar{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} + \kappa\,\Gamma^{\alpha}{}_{\mu\nu}{}^{(1)} + \kappa^2\,\Gamma^{\alpha}{}_{\mu\nu}{}^{(2)} + \mathcal{O}\big(\kappa^3\big) \\ \Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\,g^{\alpha\beta}(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\beta\mu} - \partial_\beta\,g_{\mu\nu}) \\ \Gamma^{\alpha}{}_{\mu\nu}{}^{(1)} = \frac{1}{2}\,g^{\alpha\beta}(\nabla_\mu\,h_{\beta\nu} + \nabla_\nu\,h_{\beta\mu} - \nabla_\beta\,h_{\mu\nu}) \\ \Gamma^{\alpha}{}_{\mu\nu}{}^{(2)} = -h^{\alpha}{}_{\rho}\,\Gamma^{\rho}{}_{\mu\nu}{}^{(1)}\end{cases} \tag{10}$$

The covariant derivative acting on a tensor is defined as:

$$\begin{aligned}\nabla_\sigma\,T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} := \partial_\sigma\,T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ + \Gamma^{\mu_1}{}_{\sigma\lambda}\,T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}{}_{\sigma\lambda}\,T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ - \Gamma^{\lambda}{}_{\sigma\nu_1}\,T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}{}_{\sigma\nu_2}\,T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - \ldots\end{aligned} \tag{11}$$

2.3 Riemann tensor

After having reviewed the expansion of the Christoffel symbols we can start with the Riemann tensor. This tensor is defines as:

$$R^{\alpha}{}_{\mu\beta\nu} := \partial_\beta\,\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\,\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\beta\rho}\,\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\lambda}\,\Gamma^{\lambda}{}_{\mu\beta} \tag{12}$$

Before writing what its expansion looks like we can simplify our work by using the Palatini identity (130). The derivation of this identity is shown in appendix A.

Using this identity we can write the following expressions for the expansion of the Riemann tensor:

$$\begin{cases}\bar{R}^{\alpha}{}_{\mu\beta\nu} = R^{\alpha}{}_{\mu\beta\nu} + \kappa\,R^{\alpha}{}_{\mu\beta\nu}{}^{(1)} + \kappa^2\,R^{\alpha}{}_{\mu\beta\nu}{}^{(2)} + \mathcal{O}\big(\kappa^3\big) \\ R^{\alpha}{}_{\mu\beta\nu} := \partial_\beta\,\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\,\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\nu\lambda}\,\Gamma^{\lambda}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\beta\rho}\,\Gamma^{\rho}{}_{\mu\nu} \\ R^{\alpha}{}_{\mu\beta\nu}{}^{(1)} := \nabla_\beta\,\big(\Gamma^{\alpha}{}_{\mu\nu}{}^{(1)}\big) - \nabla_\nu\,\Big(\Gamma^{\alpha}{}_{\mu\beta}{}^{(1)}\Big) \\ R^{\alpha}{}_{\mu\beta\nu}{}^{(2)} := \nabla_\beta\,\big(\Gamma^{\alpha}{}_{\mu\nu}{}^{(2)}\big) - \nabla_\nu\,\Big(\Gamma^{\alpha}{}_{\mu\beta}{}^{(2)}\Big)\end{cases} \tag{13}$$

2.4 Ricci tensor

From the Riemann tensor, the Ricci tensor is defined as:

$$R_{\mu\nu} \coloneqq R^{\alpha}{}_{\mu\alpha\nu} \tag{14}$$

Therefore, from the expansion of the Riemann tensor is easy to find the expansion of the Ricci tensor to be:

$$\begin{cases} \bar{R}_{\mu\nu} = R_{\mu\nu} + \kappa\, R_{\mu\nu}{}^{(1)} + \kappa^2\, R_{\mu\nu}{}^{(2)} + \mathcal{O}\big(\kappa^3\big) \\ R_{\mu\nu} = \partial_\alpha\, \Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\, \Gamma^{\alpha}{}_{\mu\alpha} + \Gamma^{\alpha}{}_{\nu\lambda}\, \Gamma^{\lambda}{}_{\mu\alpha} - \Gamma^{\alpha}{}_{\alpha\rho}\, \Gamma^{\rho}{}_{\mu\nu} \\ R_{\mu\nu}{}^{(1)} \coloneqq \nabla_\alpha \Gamma^{\alpha}_{\mu\nu}{}^{(1)} - \nabla_\nu \Gamma^{\alpha}_{\alpha\mu}{}^{(1)} \\ R_{\mu\nu}{}^{(2)} \coloneqq \nabla_\alpha \Gamma^{\alpha}_{\mu\nu}{}^{(2)} - \nabla_\nu \Gamma^{\alpha}_{\alpha\mu}{}^{(2)} \end{cases} \tag{15}$$

2.5 Scalar curvature

Finally, contracting the indices from the Ricci tensor we obtain the scalar curvature:

$$R \coloneqq R^{\mu}{}_{\mu} = g^{\mu\nu}\, R_{\mu\nu} \tag{16}$$

The expansion of the scalar curvature is:

$$\begin{aligned} \bar{R} = \bar{g}^{\mu\nu}\, \bar{R}_{\mu\nu} &= \big[g^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2\, h^{\mu}{}_{\alpha} h^{\alpha\nu} + \mathcal{O}\big(\kappa^3\big)\big]\, \Big[R_{\mu\nu} + \kappa\, R_{\mu\nu}{}^{(1)} + \kappa^2\, R_{\mu\nu}{}^{(2)} + \mathcal{O}\big(\kappa^3\big)\Big] = \\ &= R + \kappa\, R^{(1)} - \kappa\, h^{\mu\nu}\, R_{\mu\nu} + \kappa^2\, R^{(2)} - \kappa^2\, h^{\mu\nu}\, R^{(1)}_{\mu\nu} + \kappa^2\, h^{\mu}{}_{\alpha} h^{\alpha\nu}\, R_{\mu\nu} + \mathcal{O}\big(\kappa^3\big) \end{aligned} \tag{17}$$

Every order of this expansion is written in this summary:

$$\begin{cases} \bar{R} & = R + \kappa\, R^{(1)} + \kappa^2\, R^{(2)} + \mathcal{O}\big(\kappa^3\big) \\ R & \coloneqq g^{\mu\nu}\, R_{\mu\nu} \\ R^{(1)} & \coloneqq g^{\mu\nu}\, R_{\mu\nu}{}^{(1)} - h^{\mu\nu}\, R_{\mu\nu} = \nabla_\mu \nabla_\nu\, h^{\mu\nu} - \Box\, h - h^{\mu\nu}\, R_{\mu\nu} \\ R^{(2)} & \coloneqq g^{\mu\nu}\, R_{\mu\nu}{}^{(2)} - h^{\mu\nu}\, R_{\mu\nu}{}^{(1)} + h^{\mu}{}_{\alpha}\, h^{\alpha\nu}\, R_{\mu\nu} \end{cases} \tag{18}$$

where the d'Alembertian is the operator defined as: $\Box \coloneqq \nabla^{\mu}\nabla_{\mu}$.

2.6 Perturbed actions

After having reviewed the expansions due to the perturbation on the background metric of the elements that describe the geometry of space-time we can study the expansions of the actions used to describe the space-time and matter field dynamics.

The action that describes the dynamics of scalar particles scattering with respect to each other in a curved space-time is the sum of the Einstein-Hilbert action and the scalar field action:

$$\bar{S} = \bar{S}_{EH} + \bar{S}_{M} \tag{19}$$

Of course, we are taking into account the perturbations on the background space-time metric and therefore we write the bar on the actions. Because of this, we can proceed with the expansion on different orders of κ as we did before.

Expanding the Einstein-Hilbert action we will obtain information about gravity itself. The different orders of the expansion will describe the propagation of the graviton and even interactions among them. On the other hand, expanding the matter action will teach us about how the scalar field interacts with space-time. This means the description of the propagator of the scalar particle and interaction between gravitons and scalar particles.

We start by expanding the Einstein-Hilbert action:

$$\bar{S}_{EH} = \frac{1}{2\kappa^2} \int d^4x \, \sqrt{-\bar{g}} \, \bar{R} \tag{20}$$

By expanding $\sqrt{-\bar{g}} \, \bar{R}$ to second order in κ, we can write the expansion of the action:

$$\begin{aligned}
\sqrt{-\bar{g}} \, \bar{R} &= \sqrt{-\bar{g}} \, \bar{g}^{\mu\nu} \, \bar{R}_{\mu\nu} \\
&= \sqrt{-g} \left(1 + \frac{1}{2}\kappa h - \frac{1}{4}\kappa^2 h^{\mu\nu} h_{\mu\nu} + \frac{1}{8}(\kappa h)^2 + \mathcal{O}(\kappa^3)\right) \\
&\quad \left(g^{\mu\nu} - \kappa \, h^{\mu\nu} + \kappa^2 \, h^{\mu}{}_{\alpha} h^{\alpha\nu} + \mathcal{O}(\kappa^3)\right) \left(R_{\mu\nu} + \kappa \, {R_{\mu\nu}}^{(1)} + \kappa^2 \, {R_{\mu\nu}}^{(2)} + \mathcal{O}(\kappa^3)\right) \\
&= \sqrt{-g} \, \left(g^{\mu\nu} + \frac{1}{2}\kappa h g^{\mu\nu} - \kappa h^{\mu\nu} + \mathcal{O}(\kappa^2)\right) \, \left(R_{\mu\nu} + \kappa \, {R_{\mu\nu}}^{(1)} + \kappa^2 \, {R_{\mu\nu}}^{(2)} + \mathcal{O}(\kappa^3)\right) \\
&= \sqrt{-g} \, \left\{\kappa g^{\mu\nu} \, {R_{\mu\nu}}^{(1)} + \kappa^2 \left[\left(\frac{1}{2} h g^{\mu\nu} - h\right) {R_{\mu\nu}}^{(1)} + g^{\mu\nu} {R_{\mu\nu}}^{(2)} + \mathcal{O}(\kappa^3)\right]\right\}
\end{aligned} \tag{21}$$

In the case of the Minkowskian background metric corresponding to a flat space-time as well as for the Schwarzschild background metric corresponding to a non-spinning black hole, both metrics are vacuum solutions. This means that both the Ricci tensor and the Scalar curvature vanish at the zeroth order.

The Einstein-Hilbert action ends up like:

$$\begin{aligned}
\bar{S}_{EH} &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} \bar{R} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} \\
&= \frac{1}{2\kappa^2} \Big[\cancelto{0}{\int d^4x \sqrt{-g} g^{\mu\nu} \left(\kappa {R_{\mu\nu}}^{(1)} + \kappa^2 {R_{\mu\nu}}^{(2)}\right)} + \kappa^2 \int d^4x \sqrt{-g} \left(\frac{1}{2} h g^{\mu\nu} - h\right) \, {R_{\mu\nu}}^{(1)} + \mathcal{O}(\kappa^3) \Big] \\
&= \frac{1}{2} \int d^4x \sqrt{-g} \left(\frac{1}{2} h g^{\mu\nu} - h\right) {R_{\mu\nu}}^{(1)} + \mathcal{O}(\kappa) = \int d^4x \, \sqrt{-g} \, \mathcal{L}_{EH} + \mathcal{O}(\kappa)
\end{aligned} \tag{22}$$

Here we used the Palatini identity in combination with Stokes' theorem to make the terms involving the first and second order Ricci tensors vanish.

Therefore, the Einstein-Hilbert Lagrangian is:

$$\begin{aligned}
\mathcal{L}_{EH} &= \frac{1}{4} \left(\frac{1}{2} h g^{\mu\nu} - h^{\mu\nu}\right) \, \left(\nabla^\sigma \nabla_\nu h_{\mu\sigma} + \nabla^\sigma \nabla_\mu h_{\nu\sigma} - \nabla_\mu \nabla_\nu \, h - \Box \, h_{\mu\nu}\right) \\
&= \frac{1}{4} h_{\mu\nu} \left[(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta})\Box + g^{\mu\nu} \nabla^\alpha \nabla^\beta - g^{\mu\alpha} \nabla^\nu \nabla^\beta + g^{\alpha\beta} \nabla^\nu \nabla^\mu - g^{\alpha\nu} \nabla^\beta \nabla^\mu\right] h_{\alpha\beta} \\
&= \frac{1}{4} h_{\mu\nu} P^{\mu\nu\alpha\beta} h_{\alpha\beta}
\end{aligned} \tag{23}$$

This Lagrangian describes how the gravitons propagate. The next order in the expansion describes the interaction between three gravitons, but we have not worked out its expression.

Now, in order to study the interaction between gravity and matter we'll focus on $\bar{S}_M$:

$$\bar{S}_M = -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} \bar{\nabla}_\mu \phi \bar{\nabla}^\mu \phi = -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} \bar{g}^{\mu\nu} \bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi \tag{24}$$

We expand to second order every term from the matter action to obtain different interactions. Since we are working with a scalar field, we have that $\bar{\nabla}_\mu \phi = \partial_\mu \phi$. This means that we only have to expand $\sqrt{-\bar{g}}$ and $\bar{g}^{\mu\nu}$.

$$\begin{aligned}
\bar{S}_M &= -\frac{1}{2}\int d^4x\sqrt{-g}\left(1+\frac{1}{2}\kappa h-\frac{1}{4}\kappa^2 h^{\mu\nu}h_{\mu\nu}+\frac{1}{8}(\kappa h)^2+\mathcal{O}\big(\kappa^3\big)\right) \\
&\qquad \big(g^{\mu\nu}-\kappa\, h^{\mu\nu}+\kappa^2\, h^{\mu}{}_{\alpha}h^{\alpha\nu}+\mathcal{O}\big(\kappa^3\big)\big)\,\partial_\mu\phi\partial_\nu\phi \\
&= -\frac{1}{2}\int d^4x\sqrt{-g}\Big[\partial_\mu\phi\partial^\mu\phi+\kappa h^{\mu\nu}\left(\frac{1}{2}g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\nu\beta}\right)\partial^\alpha\phi\partial^\beta\phi \\
&\qquad +\kappa^2 h^{\mu\nu}h^{\rho\sigma}\left(g_{\rho\nu}g_{\mu\alpha}g_{\sigma\beta}-\frac{1}{2}g_{\rho\sigma}g_{\mu\alpha}g_{\nu\beta}-\frac{1}{4}g_{\rho\mu}g_{\sigma\nu}g_{\alpha\beta}+\frac{1}{8}g_{\rho\sigma}g_{\mu\nu}g_{\alpha\beta}\right)\partial^\alpha\phi\partial^\beta\phi \\
&\qquad +\mathcal{O}\big(\kappa^3\big)\Big] \\
&= S_M+\kappa S_M^{(1)}+\kappa^2 S_M^{(2)}+\mathcal{O}\big(\kappa^3\big)
\end{aligned} \tag{25}$$

S_M studies the propagation of the scalar field ϕ, $S_M^{(1)}$ studies the interaction between two matter particles and one graviton $< \phi\phi h >$ and $S_M^{(2)}$ studies the interaction between two matter particles and two gravitons $< \phi\phi hh >$. The interaction $< \phi\phi h >$ has been studied in previous work [32]. Therefore we will expand on it by studying the second order interaction.

3 Harmonic expansion in flat space-time

As we explained in the introduction, scattering in flat space-time is related to the inspiral phase of mergers, phase which is well understood. However, we are going to study this phase using harmonic expansions, which is a method that is normally used when working with the post-merger phase [42, 43]. By doing so, our hope is to gain some insight into this phase.

3.1 Spherical symmetry. Regge-Wheeler gauge

Both space-time background metrics where the scattering occurs (Minkowski for the inspiral phase and Schwarzschild for the post-merger phase) posses spherical symmetry, which we will exploit in our work. The most general metric with a spherical symmetry is:

$$ds^2 = -2A(x,y)dxdy + r^2(x,y)(d\theta^2 + sin^2\theta d\phi^2) \tag{26}$$

Where A(x,y) is a generic function that depends on the Kruskal-Szekeres light-cone coordinates (x,y) defined as: x = t + r; y = t - r. These coordinates span the (t,r) plane, and the coordinates (θ, ϕ) span the 2-sphere S^2 of x and y constant.

Following our metric convention of $-,+,+,+$, the background metric and its inverse look like:

$$g_{\mu\nu} = \begin{pmatrix} 0 & -A & 0 & 0 \\ -A & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -A^{-1} & 0 & 0 \\ -A^{-1} & 0 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} sin^{-2}\theta \end{pmatrix} \tag{27}$$

There is a clear decomposition of coordinates (light-cone and spherical) and we will follow the convention of: $x^a = [x, y]$ and $x^A = [\theta, \phi]$. So, the lowercase indices run over the light-cone coordinates while the uppercase indices span the sphere.

Spoiling the spherical symmetry of the background metric we can perform a spherical decomposition of the perturbation metric as worked by Regge and Wheeler [41] into odd and even components:

$$h_{\mu\nu} = \sum_{\ell,m} h^-_{\ell m,\mu\nu} + \sum_{\ell,m} h^+_{\ell m,\mu\nu} \tag{28}$$

These modes $h^-_{\ell m,\mu\nu}$ and $h^+_{\ell m,\mu\nu}$ can be written as a symmetric 4x4 matrix acting on a spherical harmonic function $Y_{\ell m}(\theta\phi)$. The ten degrees of freedom of the graviton get turned into ten functions in the matrix that depend on (x,y) and carry the ℓ, m indices.

There are four redundant gauge degrees of freedom that can be deleted by fixing a gauge. In our case we'll work with the Regge-Wheeler gauge. Fixing this gauge leaves six physical off-shell degrees of freedom left, corresponding with six functions. Therefore the odd and even modes of the perturbation matrix after fixing the Regge-Wheeler gauge are:

$$h^-_{\ell m,\mu\nu} = \begin{pmatrix} 0 & 0 & -h^{\ell m}_x(x,y)csc\theta\partial_\phi & h^{\ell m}_x(x,y)sin\theta\partial_\theta \\ 0 & 0 & -h^{\ell m}_y(x,y)csc\theta\partial_\phi & h^{\ell m}_y(x,y)sin\theta\partial_\theta \\ -h^{\ell m}_x(x,y)csc\theta\partial_\phi & -h^{\ell m}_y(x,y)csc\theta\partial_\phi & 0 & 0 \\ h^{\ell m}_x(x,y)sin\theta\partial_\theta & h^{\ell m}_y(x,y)sin\theta\partial_\theta & 0 & 0 \end{pmatrix} Y_{\ell m}(\theta,\phi) \tag{29}$$

$$h^+_{\ell m,\mu\nu} = \begin{pmatrix} H^{\ell m}_{xx}(x,y) & H^{\ell m}_{xy}(x,y) & 0 & 0 \\ H^{\ell m}_{xy}(x,y) & H^{\ell m}_{yy}(x,y) & 0 & 0 \\ 0 & 0 & r^2(x,y)K^{\ell m}(x,y) & 0 \\ 0 & 0 & 0 & r^2(x,y)K^{\ell m}(x,y)sin^2\theta \end{pmatrix} Y_{\ell m}(\theta,\phi) \tag{30}$$

We can write these expressions in a covariant way as:

$$h^{-}_{\ell m,aA} = -h^{\ell m}_{a}\epsilon_{A}{}^{B}\partial_B Y_{\ell m} \qquad h^{+}_{\ell m,ab} = H^{\ell m}_{ab}Y_{\ell m} \qquad h^{+}_{\ell m,AB} = K^{\ell m}g_{AB}Y_{\ell m} \tag{31}$$

where we define the metric g_{AB} and the asymmetric tensor ϵ_{AB} on S^2:

$$g_{AB} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 sin^2\theta \end{pmatrix} \qquad \epsilon_{AB} = r^2 sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{32}$$

Therefore $\epsilon_{A}{}^{B} = \begin{pmatrix} 0 & sin\theta \\ -csc\theta & 0 \end{pmatrix}$.

The spherical harmonics decomposition is coordinate independent for diffeomorphisms that act on the light-cone and angular coordinates separately. Therefore we define H_{ab} to be a 2-tensor on the light-cone, h_a to be a vector on the light-cone, and K to be a scalar on the light-cone. These fields transform under coordinate transformations on the light-cone but not under diffeomorphisms on the angular coordinates. The opposite happens for quantities with just capital letter indices. Spherical harmonics decomposition persists under any transformation that keeps the light-cone and two-sphere separate.

3.2 Decoupling of the odd and even modes

Due to the spherical symmetry we expect the coupling between the odd and the even modes of the graviton to disappear. To check this we have to take a look at the Einstein-Hilbert Lagrangian (23).

$$\mathcal{L}_{EH} = \mathcal{L}_{no\,parity-coupling} + \mathcal{L}_{parity-coupling} \tag{33}$$

From this Lagrangian we can take a look at the term that couples the different parities of the graviton. That is, the terms of the Lagrangian that contract two perturbation metric modes $h^{\pm}_{\mu\nu}$ with different parity ($\propto h^{\pm}_{\mu\nu}h^{\mp,\mu\nu}$). The way to write this parity-coupling term is:

$$\mathcal{L}_{parity-coupling} = -\frac{1}{4}h^{\mu\nu}_{+}G^{(1),-}_{\mu\nu} - \frac{1}{4}h^{\mu\nu}_{-}G^{(1),+}_{\mu\nu} \tag{34}$$

with:

$$G^{(1),\pm}_{\mu\nu} := g^{\rho\sigma}\left(\nabla_\rho\nabla_\mu h^{\pm}_{\nu\sigma} + \nabla_\rho\nabla_\nu h^{\pm}_{\mu\sigma} - \nabla_\rho\nabla_\sigma h^{\pm}_{\mu\nu} - \nabla_\mu\nabla_\nu h^{\pm}_{\rho\sigma}\right) - g_{\mu\nu}\left(\nabla^\rho\nabla^\sigma h^{\pm}_{\rho\sigma} - \Box h^{\pm}\right) \tag{35}$$

When working with the Regge-Wheeler gauge, we can work out the following results:

$$G^{(1),-}_{ab} = 0 \tag{36}$$

$$G^{(1),-}_{AB}g^{AB} = 0 \tag{37}$$

$$G^{(1),+}_{aA} = \sum_{\ell m}\left(\frac{1}{2}\partial^b H_{ab} + \frac{1}{A}\partial_a H_{xy} - \frac{1}{2A}\left(\partial_a log(Ar^2)\right)H_{xy} - \frac{1}{2}\partial_a K\right)\partial_A Y_{\ell m} = \sum_{\ell m}F^{\ell m}_{a}\partial_A Y_{\ell m} \tag{38}$$

where, in order to simplify, we define the function:

$$F^{\ell m}_{a} := \frac{1}{2}\partial^b H_{ab} + \frac{1}{A}\partial_a H_{xy} - \frac{1}{2A}\left(\partial_a log(Ar^2)\right)H_{xy} - \frac{1}{2}\partial_a K \tag{39}$$

Therefore, $\mathcal{L}_{parity-coupling}$ gets simplified to:

$$\mathcal{L}_{parity-coupling} = -\frac{1}{2}h^{aA}_{-}G^{(1),+}_{aA} = \frac{1}{2}\sum_{\ell m,\ell' m'} h^{\ell m}_{a}F^{a}_{\ell' m'}\epsilon^{AB}\partial_B Y_{\ell m}\partial_A \bar{Y}_{\ell' m'} \tag{40}$$

notice that since the Lagrangians are real, we take the complex conjugate of one of the spherical harmonics.

The action corresponding to this Lagrangian is:

$$\begin{aligned} S_{parity-coupling} &= \frac{1}{2}\sum_{\ell m,\ell' m'}\int d^4x\sqrt{-g}h_a^{\ell m}F^a_{\ell' m'}\epsilon^{AB}\partial_B Y_{\ell m}\partial_A \bar{Y}_{\ell' m'} \\ &= \frac{1}{2}\sum_{\ell m,\ell' m'}\int d^2x A(x) h_a^{\ell m}F^a_{\ell' m'}\int d\theta d\phi\sqrt{g_{S^2}}\epsilon^{AB}\partial_B Y_{\ell m}\partial_A \bar{Y}_{\ell' m'} \end{aligned} \tag{41}$$

were we use the following definitions:

$$g = det(g_{\mu\nu}) = -A^2(x)r^4 sin^2\theta \qquad \sqrt{-g} = A(x)r^2 sin\theta = A(x)\sqrt{g_{S^2}} \tag{42}$$

where $\sqrt{g_{S^2}}$ is the volume element on S^2 of radius r.

Taking a look at the action we notice that the second integral only depends on θ, ϕ and contains functions that are known. After some calculations relying on orthogonality properties of the spherical harmonics we can find that this second integral vanishes. Therefore the whole integral vanishes too.

$$S_{parity-coupling} = 0 \tag{43}$$

finishing the proof that the odd and even modes of the graviton decouple.

In this proof we have shown that the choice of light-cone coordinates play no role, and the same holds for the choice of spherical coordinates. This is because the light-cone coordinates and the angular coordinates are summed over separately. Also, this result holds for any spherical symmetric background metric.

Therefore we are left with an Einstein-Hilbert Lagrangian without coupling between the odd and even modes of the graviton.

$$\mathcal{L}_{EH} = \mathcal{L}_{no\,parity-coupling} = \mathcal{L}^+ + \mathcal{L}^- \tag{44}$$

where:

$$\mathcal{L}^+ = \frac{1}{4}h^+_{\mu\nu}\left(P^{\mu\nu\alpha\beta}h^{\pm}_{\alpha\beta}\right)^+ \qquad \mathcal{L}^- = \frac{1}{4}h^-_{\mu\nu}\left(P^{\mu\nu\alpha\beta}h^{\pm}_{\alpha\beta}\right)^- \tag{45}$$

Note that the parenthesis with the $+$ and $-$ signs cover the perturbation metric on the right as well as the operator $P^{\mu\nu\alpha\beta}$ that acts on it. This is because this operator containing covariant derivatives can act on an odd term and result into terms containing also even modes of the perturbation metric and vice versa. Therefore, as we will see later in this chapter, we have to consider this operator acting on odd modes to contribute to the even Lagrangian as well as the operator acting on even modes to contribute to the odd Lagrangian.

The odd parity mode has a subleading contribution to the scattering processes compared to the even one. This is because transverse momentum transfer is a Planckian effect [44], and it is unlikely that particles scatter at impact parameters comparable to Planck length [45, 46]. Therefore we can safely assume small transverse momentum transfer ($p_A = 0 \sim \partial_A\phi = 0$). Because of that, we will now focus on the even parity Lagrangian $\mathcal{L}^+$.

3.3 Effective two-dimensional theory

Up until now we have worked with a generic spherically symmetric background metric. For the rest of this chapter we will now work in a flat background metric, which means that now we specify the function A(x,y)=1, giving us the following metric:

$$ds^2 = -2dxdy + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{46}$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \tag{47}$$

Even though we are working in a flat background metric we have non-vanishing Christoffel symbols because of the choice of using spherical coordinates.

$$\Gamma^a_{AB} = -\frac{1}{2}g^{ab}\partial_b g_{AB} = -g^{ab}g_{AB}\partial_b log\, r = -\frac{1}{2}g^{ab}g_{AB}V_b = -\frac{1}{2}g_{AB}V^a \tag{48}$$

$$\Gamma^A_{aC} = \frac{1}{2}g^{AB}\partial_a g_{BC} = \delta^A_C \partial_a log\, r = \frac{1}{2}\delta^A_C V_a \tag{49}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = -sin^{-2}\theta\Gamma^\theta_{\phi\phi} = \frac{1}{sin\,\theta} \tag{50}$$

where we defined the potential $V_a \coloneqq 2\partial_a log\, r$.

Now we can work on finding an expression for the even Lagrangian specifically with a Minkowskian background metric and a perturbation metric with the Regge-Wheeler gauge.

$$\mathcal{L}^+ = \frac{1}{4}\sum_{\ell m,\ell' m'} h^+_{\ell m,\mu\nu}\left(P^{\mu\nu\alpha\beta}h^{\pm}_{\ell' m',\alpha\beta}\right)^+ \tag{51}$$

To do so, first let's see how the different combinations of covariant derivatives that appear in the Lagrangian act on the different components of the perturbation metric h_{cd}, h_{CD} and h_{cD}. Starting with h_{cd}, we have:

$$\begin{aligned}
\nabla^a\nabla^b h_{cd} &= \nabla^a g^{bf}\left(\partial_f h_{cd} - \cancelto{0}{\Gamma^\mu_{fc}}h_{\mu d} - \cancelto{0}{\Gamma^\mu_{fd}}h_{c\mu}\right) = \nabla^a\partial^b h_{cd} \\
&= g^{ae}g^{bf}\left(\partial_e\partial_f h_{cd} - \cancelto{0}{\Gamma^\mu_{ef}}\partial_\mu h_{cd} - \cancelto{0}{\Gamma^\mu_{ec}}\partial_f h_{\mu d} - \cancelto{0}{\Gamma^\mu_{ed}}\partial_f h_{c\mu}\right) \\
&= \partial^a\partial^b h_{cd} = \left(\partial^a\partial^b H_{cd}\right)Y_{\ell m}
\end{aligned} \tag{52}$$

$$\begin{aligned}
\nabla^A\nabla^B h_{cd} &= \partial^A\nabla^B h_{cd} - g^{AE}\left(-\Gamma^B_{E\mu}\nabla^\mu h_{cd} + \Gamma^\mu_{Ec}\nabla^B h_{\mu d} + \Gamma^\mu_{Ed}\nabla^B h_{c\mu}\right) \\
&= \partial^A\partial^B h_{cd} - \partial^A\left(g^{BE}\Gamma^\mu_{Ec}h_{\mu d} + g^{BE}\Gamma^\mu_{Ed}h_{c\mu}\right) \\
&\quad + \frac{1}{2}g^{AB}V^a\left(\partial_a h_{cd} - \cancelto{0}{\Gamma^\mu_{bc}}h_{\mu d} - \cancelto{0}{\Gamma^\mu_{bd}}h_{c\mu}\right) + g^{AE}\Gamma^B_{ED}g^{DF}\left(\partial_F h_{cd} - \cancelto{0}{\Gamma^\mu_{Fc}}h_{\mu d} - \cancelto{0}{\Gamma^\mu_{Fd}}h_{c\mu}\right) \\
&\quad - \frac{1}{2}g^{AD}V_c g^{BE}\left(\partial_E \cancelto{0}{h_{Dd}} - \Gamma^\mu_{ED}h_{\mu d} - \Gamma^\mu_{Ed}hD\mu\right) - \frac{1}{2}g^{AD}V_d g^{BE}\left(\partial_E \cancelto{0}{h_{cD}} - \Gamma^\mu_{Ec}h_{\mu D} - \Gamma^\mu_{ED}hc\mu\right) \\
&= \left(\partial^A\partial^B + g^{AE}\Gamma^B_{ED}g^{DF}\partial_F\right)h_{cd} + \frac{1}{2}g^{AB}V_a\partial^a h_{cd} + \frac{1}{4}g^{AB}\left(2V_cV_dKY_{\ell m} - V_cV^a h_{ad} - V_dV^a h_{ac}\right) \\
&= \left[\left(\partial^A\partial^B + g^{AE}\Gamma^B_{ED}g^{DF}\partial_F\right)H_{cd} + \frac{1}{2}g^{AB}\left(V_a\partial^a H_{cd} - V^aV_{(c}H_{d)a} + V_cV_dK\right)\right]Y_{\ell m}
\end{aligned} \tag{53}$$

$$\begin{aligned}
\Box h_{cd} &= \nabla^a\nabla_a h_{cd} + \nabla^A\nabla_A h_{cd} = \tilde{\Box}h_{cd} + \left(\partial^A\partial_A - g^{AB}\Gamma^C_{AB}\partial_C\right)h_{cd} + V_a\partial^a h_{cd} - V^aV_{(c}h_{d)a} + V_cV_dK \\
&= \left[\left(\tilde{\Box} - \frac{\ell(\ell+1)}{r^2} + V_a\partial^a\right)H_{cd} - V^aV_{(c}H_{d)a} + V_cV_dK\right]Y_{\ell m}
\end{aligned} \tag{54}$$

note that in this calculation we used the following property: $\partial^a g_{BD} = V^a g_{BD}$.

Now we continue with all possible combinations of covariant derivatives acting on h_{CD} that appear in the Lagrangian:

$$\nabla^a \nabla^b h_{CD} = g_{CD} \left(\partial^a \partial^b K\right) Y_{\ell m} \tag{55}$$

$$\nabla^A \nabla^B h_{CD} = g_{CD} K \left(\tilde{\nabla}^A \tilde{\nabla}^B Y_{\ell m}\right) + \frac{1}{2}\left(g^{AB} g_{CD} V^a \partial_a K - \delta^A_{(C}\delta^B_{D)}\left(K V^a V_a - V^a V^b H_{ab}\right)\right) Y_{\ell m} \tag{56}$$

$$\Box h_{CD} = g_{CD}\left[\left(\tilde{\Box} - \frac{\ell(\ell+1)}{r^2} + V^a \partial_a - \frac{1}{2} V^a V_a\right) K + \frac{1}{2} V^a V^b H_{ab}\right] Y_{\ell m} \tag{57}$$

Finally, we take a look at the covariant derivatives that act on an odd mode of the perturbation action h_{cD} but result in some terms with even modes that contribute to the even Lagrangian:

$$\begin{aligned}
\nabla^a \nabla^A h_{cD} &= g^{ab} g^{AB}\left[\partial_b\left(\nabla_B h_{cD}\right) - \Gamma^\mu_{bB}\nabla_\mu h_{cD} - \underbrace{\Gamma^\mu_{bc}}_{\to 0}\nabla_B h_{\mu D} - \Gamma^\mu_{bD}\nabla_B h_{c\mu}\right] \\
&= g^{ab} g^{AB}\Big[\partial_b\left(\partial_B h_{cD} - \Gamma^\mu_{Bc} h_{\mu D} - \Gamma^\mu_{cD} h_{c\mu}\right) - \Gamma^\mu_{bB}\left(\partial_\mu h_{cD} - \Gamma^\nu_{\mu c} h_{\nu D} - \Gamma^\nu_{\mu D} h_{c\nu}\right) \\
&\qquad - \Gamma^\mu_{bD}\left(\partial_B h_{c\mu} - \Gamma^\nu_{Bc} h_{\nu\mu} - \Gamma^\nu_{B\mu} h_{c\nu}\right)\Big] \\
&= g^{ab} g^{AB}\Bigg[\partial_b \partial_B h^-_{cD} - \frac{1}{2}\partial_b\left(V_c h^+_{BD}\right) - \partial_b\left(\Gamma^E_{BD} h^-_{cE}\right) + \frac{1}{2}\partial_b\left(g_{BD} V^d h^+_{cd}\right) \\
&\qquad - V_b\left(\partial_B h^-_{cD} - \Gamma^F_{BD} h^-_{cF} - \frac{1}{2} V_c h^+_{BD} + \frac{1}{2} g_{BD} V^d h^+_{cd}\right)\Bigg] \\
&= g^{AB}\partial^a\underbrace{\left(\partial_B h^-_{cD} - \Gamma^E_{BD} h^-_{cE}\right)}_{\to 0} - V^a\underbrace{\left(\partial^A h^-_{cD} - g^{AB}\Gamma^F_{BD} h^-_{cF}\right)}_{\to 0} \\
&\quad + \frac{1}{2}\left[g^{AB}\partial^a\left(g_{BD} V^d h^+_{cd} - V_c h^+_{BD}\right) + V^a\left(g^{AB} V_c h^+_{BD} - \delta^A_D V^d h^+_{cd}\right)\right] \\
&= \frac{1}{2}\left[g^{AB}\partial^a\left(g_{BD} V^d H_{cd} Y_{\ell m} - g_{BD} V_c K Y_{\ell m}\right) - V^a \delta^A_D\left(V^d H_{cd} Y_{\ell m} - V_c K Y_{\ell m}\right)\right] \\
&= \frac{1}{2}\delta^A_D \partial^a\left(V^d H_{cd} - V_c K\right) Y_{\ell m} = \frac{1}{2}\delta^A_D\left(\partial^a V^d H_{cd} + V^d \partial^a H_{cd} - \partial^a V_c K - V_c \partial^a K\right) Y_{\ell m}
\end{aligned} \tag{58}$$

Where the d'Alembertian on the light-cone is defined as: $\tilde{\Box} := \partial^a \partial_a$, and the Laplacian operator on the sphere is defined as: $\Box_{S^2} := \partial^A \partial_A - g^{AB}\Gamma^C_{AB}\partial_C$, which the harmonic function is an eigenfunction of, giving the following eigenvalue: $\Box_{S^2} Y_{\ell m} = -\ell(\ell+1)/r^2\, Y_{\ell m}$.

Now that we have done so, we can compute every term of the even Lagrangian:

$$\begin{aligned}
\mathcal{L}^+ = \frac{1}{4}\sum_{\ell m, \ell' m'}\Bigg[& h^+_{\ell m, ab}\left(P^{abcd} h^+_{\ell' m', cd}\right)^+ && + h^+_{\ell m, ab}\left(P^{abCD} h^+_{\ell' m', CD}\right)^+ \\
& + h^+_{\ell m, AB}\left(P^{ABcd} h^+_{\ell' m', cd}\right)^+ && + h^+_{\ell m, AB}\left(P^{ABCD} h^+_{\ell' m', CD}\right)^+ \\
& + h^+_{\ell m, ab}\left(P^{abcD} h^-_{\ell' m', cD}\right)^+ && + h^+_{\ell m, ab}\left(P^{abCd} h^-_{\ell' m', Cd}\right)^+ \\
& + h^+_{\ell m, AB}\left(P^{ABcD} h^-_{\ell' m', cD}\right)^+ && + h^+_{\ell m, AB}\left(P^{ABCd} h^-_{\ell' m', Cd}\right)^+\Bigg]
\end{aligned} \tag{59}$$

Now let's compute every term and see how the spherical harmonic function can be pulled out from the rest of the expressions. This will allow us to integrate out the sphere when studying the action corresponding to this Lagrangian:

$$
\begin{aligned}
&h^{+}_{\ell m,ab}\left(P^{abcd}h^{+}_{\ell' m',cd}\right)^{+}\\
&= h^{+}_{\ell m,ab}\left[(g^{ac}g^{bd}-g^{ab}g^{cd})\Box + g^{ab}\nabla^{c}\nabla^{d} - g^{ac}\nabla^{b}\nabla^{d} + g^{cd}\nabla^{b}\nabla^{a} - g^{cb}\nabla^{d}\nabla^{a}\right] h^{+}_{\ell' m',cd}\\
&= -2\bar{Y}_{\ell m}H_{ab}\left[g^{a[b}g^{c]d}\Box - g^{a[b}\partial^{c]}\partial^{d} - g^{c[d}\partial^{b]}\partial^{a}\right]H_{cd}Y_{\ell' m'}\\
&= -2\bar{Y}_{\ell m}\Bigg\{H_{ab}\left[g^{a[b}g^{c]d}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{r^{2}} + V^{e}\partial_{e}\right) - g^{a[b}\partial^{c]}\partial^{d} - g^{c[d}\partial^{b]}\partial^{a} - \frac{1}{2}g^{ab}V^{c}V^{d} + \frac{1}{2}V^{d}V^{(a}g^{b)c}\right]H_{cd}\\
&\qquad + H_{ab}\frac{1}{2}\left[g^{ab}V_{e}V^{e} - V^{a}V^{b}\right]K\Bigg\}Y_{\ell' m'} \qquad (60)
\end{aligned}
$$

$$
\begin{aligned}
&h^{+}_{\ell m,ab}\left(P^{abCD}h^{+}_{\ell' m',CD}\right)^{+}\\
&= h^{+}_{\ell m,ab}\left[(\cancelto{0}{g^{aC}g^{bD}} - g^{ab}g^{CD})\Box + g^{ab}\nabla^{C}\nabla^{D} - \cancelto{0}{g^{aC}}\nabla^{b}\nabla^{D} + g^{CD}\nabla^{b}\nabla^{a} - \cancelto{0}{g^{Cb}}\nabla^{D}\nabla^{a}\right] h^{+}_{\ell' m',CD}\\
&= -\bar{Y}_{\ell m}H_{ab}\left[g^{ab}\left(g^{CD}\Box - \nabla^{C}\nabla^{D}\right) - g^{CD}\nabla^{a}\nabla^{b}\right]h^{+}_{\ell' m',CD}\\
&= -2\bar{Y}_{\ell m}\Bigg\{H_{ab}\left[g^{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^{2}} + \frac{1}{2}V^{c}\partial_{c} + \frac{1}{4}V^{c}V_{c}\right) - \partial^{a}\partial^{b}\right]K + H_{ab}\left[-\frac{1}{4}g^{ab}V^{c}V^{d}\right]H_{cd}\Bigg\}Y_{\ell' m'} \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
&h^{+}_{\ell m,AB}\left(P^{ABcd}h^{+}_{\ell' m',cd}\right)^{+}\\
&= h^{+}_{\ell m,AB}\left[(\cancelto{0}{g^{Ac}g^{Bd}} - g^{AB}g^{cd})\Box + g^{AB}\nabla^{c}\nabla^{d} - \cancelto{0}{g^{Ac}}\nabla^{B}\nabla^{d} + g^{cd}\nabla^{B}\nabla^{A} - \cancelto{0}{g^{cB}}\nabla^{d}\nabla^{A}\right] h^{+}_{\ell' m',cd}\\
&= -2\bar{Y}_{\ell m}K\left[g^{cd}\left(\Box - \frac{1}{2}\nabla^{A}\nabla_{A}\right) - \nabla^{c}\nabla^{d}\right]h^{+}_{\ell' m',cd}\\
&= -2\bar{Y}_{\ell m}\Bigg\{K\left[g^{cd}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^{2}} + \frac{1}{2}V_{a}\partial^{a}\right) - \partial^{c}\partial^{d} - \frac{1}{2}V^{c}V^{d}\right]H_{cd} + K\left[\frac{1}{2}V_{a}V^{a}\right]K\Bigg\}Y_{\ell' m'} \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
&h^{+}_{\ell m,AB}\left(P^{ABCD}h^{+}_{\ell' m',CD}\right)^{+}\\
&= h^{+}_{\ell m,AB}\left[(g^{AC}g^{BD} - g^{AB}g^{CD})\Box + g^{AB}\nabla^{C}\nabla^{D} - g^{AC}\nabla^{B}\nabla^{D} + g^{CD}\nabla^{B}\nabla^{A} - g^{CB}\nabla^{D}\nabla^{A}\right] h^{+}_{\ell' m',CD}\\
&= -\bar{Y}_{\ell m}K\left[g^{CD}\left(\Box - \nabla^{A}\nabla_{A}\right)\right]h^{+}_{\ell' m',CD} = -2\bar{Y}_{\ell m}\left\{K\tilde{\Box}K\right\}Y_{\ell' m'} \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
&h^{+}_{\ell m,ab}\left(P^{abcD}h^{-}_{\ell' m',cD}\right)^{+} + h^{+}_{\ell m,ab}\left(P^{abCd}h^{-}_{\ell' m',Cd}\right)^{+}\\
&= h^{+}_{\ell m,ab}\left[(g^{ac}\cancelto{0}{g^{bD}} - g^{ab}\cancelto{0}{g^{cD}})\Box + g^{ab}\nabla^{c}\nabla^{D} + g^{ab}\nabla^{D}\nabla^{c} - g^{ac}\nabla^{b}\nabla^{D} + \cancelto{0}{g^{cD}}\nabla^{b}\nabla^{a} - g^{cb}\nabla^{D}\nabla^{a}\right] h^{-}_{\ell' m',cD}\\
&= \bar{Y}_{\ell m}H_{ab}\left[2g^{ab}\nabla^{c}\nabla^{D} - g^{ac}\nabla^{b}\nabla^{D} - g^{cb}\nabla^{a}\nabla^{D}\right]h^{-}_{\ell' m',cD}\\
&= -2\bar{Y}_{\ell m}\Bigg\{H_{ab}\Big[-g^{ab}\left(\partial^{c}V^{d} + V^{d}\partial^{c}\right) + \frac{1}{2}g^{ac}\left(\partial^{b}V^{d} + V^{d}\partial^{b}\right) + \frac{1}{2}g^{cb}\left(\partial^{a}V^{d} + V^{d}\partial^{a}\right)\Big]H_{cd}\\
&\qquad + H_{ab}\Big[g^{ab}\left(\partial^{c}V_{c} + V_{c}\partial^{c}\right) - \left(g^{c(a}\partial^{b)}V_{c} + V^{(a}\partial^{b)}\right)\Big]K\Bigg\}Y_{\ell' m'} \qquad (64)
\end{aligned}
$$

$$
\begin{aligned}
&h^{+}_{\ell m,AB}\left(P^{ABcD}h^{-}_{\ell' m',cD}\right)^{+} + h^{+}_{\ell m,AB}\left(P^{ABCd}h^{-}_{\ell' m',Cd}\right)^{+} \\
&= h^{+}_{\ell m,AB}\left[(\cancelto{0}{g^{Ac}}g^{BD} - g^{AB}\cancelto{0}{g^{cD}})\Box + 2g^{AB}\nabla^{c}\nabla^{D} - g^{AD}\nabla^{B}\nabla^{c} + \cancelto{0}{g^{cD}}\nabla^{B}\nabla^{A} - g^{DB}\nabla^{c}\nabla^{A}\right]h^{-}_{\ell' m',cD} \\
&= 2\bar{Y}_{\ell m}K\nabla^{c}\nabla^{D}h^{-}_{\ell' m',cD} = -2\bar{Y}_{\ell m}\Big\{K\left[\partial^{c}V_{c} + V_{c}\partial^{c}\right]K + K\left[-\left(\partial^{c}V^{d} + V^{d}\partial^{c}\right)\right]H_{cd}\Big\}Y_{\ell' m'} \qquad (65)
\end{aligned}
$$

Therefore the action corresponding to the even Lagrangian is:

$$
\begin{aligned}
S^{+} &= \int d^{4}x\sqrt{-g}\mathcal{L}^{+} = \int d^{2}x d\Omega r^{2}\,\mathcal{L}^{+} \\
&= -\frac{1}{2}\sum_{\ell m,\ell' m'}\int d\Omega\bar{Y}_{\ell' m'}Y_{\ell m}\int d^{2}x\, r^{2}(H^{ab}\Delta^{-1}_{abcd}H^{cd} + H^{ab}\Delta^{-1}_{L,ab}K + K\Delta^{-1}_{R,ab}H^{ab} + K\Delta^{-1}K) \\
&= -\frac{1}{2}\sum_{\ell m}\int d^{2}x\, r^{2}(H^{ab}\Delta^{-1}_{abcd}H^{cd} + H^{ab}\Delta^{-1}_{L,ab}K + K\Delta^{-1}_{R,ab}H^{ab} + K\Delta^{-1}K) \qquad (66)
\end{aligned}
$$

In the last step we make use of the orthogonality relations for the spherical harmonics:

$$
\int d\Omega\bar{Y}_{\ell' m'}Y_{\ell m} = \delta_{\ell\ell'}\delta_{mm'} \qquad (67)
$$

This is the key property behind the choice of working with the Regge-Wheeler gauge. This allows us to simplify the theory into a two-dimensional theory that only depends on the light-cone coordinates and the ℓ and m modes. The price we pay to have separation of light-cone and spherical coordinates on top of maintaining covariance on the light-cone is that the tensor H_{ab} and K will remain coupled.

With the definitions of the following terms:

$$
\begin{aligned}
\Delta^{-1}_{abcd} &:= g_{a[b}g_{c]d}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{r^{2}} + V_{e}\partial^{e}\right) - g_{a[b}\partial_{c]}\partial_{d} - g_{c[d}\partial_{b]}\partial_{a} - \frac{1}{2}g_{ab}V_{c}V_{d} + \frac{1}{2}V_{d}V_{(a}g_{b)c} \\
&\quad - \frac{1}{4}g_{ab}V_{c}V_{d} - g_{ab}\left(\partial_{c}V_{d} + V_{d}\partial_{c}\right) + \frac{1}{2}g_{ac}\left(\partial_{b}V_{d} + V_{d}\partial_{b}\right) + \frac{1}{2}g_{cb}\left(\partial_{a}V_{d} + V_{d}\partial_{a}\right) \qquad (68) \\
\Delta^{-1}_{L,ab} &:= \frac{1}{2}\left(g_{ab}V_{c}V^{c} - V_{a}V_{b}\right) + g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^{2}} + \frac{1}{2}V_{c}\partial^{c} + \frac{1}{4}V_{c}V^{c}\right) - \partial_{a}\partial_{b} \\
&\quad + g_{ab}\left(\partial_{c}V^{c} + V_{c}\partial^{c}\right) - \left(g_{c(a}\partial_{b)}V^{c} + V_{(a}\partial_{b)}\right) \qquad (69) \\
\Delta^{-1}_{R,ab} &:= g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^{2}} + \frac{1}{2}V_{c}\partial^{c}\right) - \partial_{a}\partial_{b} - \frac{1}{2}V_{a}V_{b} - \left(\partial_{a}V_{b} + V_{b}\partial_{a}\right) \qquad (70) \\
\Delta^{-1} &:= \tilde{\Box} + \frac{1}{2}V_{a}V^{a} + \left(\partial_{a}V^{a} + V_{a}\partial^{a}\right) \qquad (71)
\end{aligned}
$$

As we can see, the spherical contribution drops out and we are left with a two-dimensional theory only dependant on the light-cone coordinates (x,y). Moreover, we can do some simplifications taking into account that $\partial_{a}V_{b} = -\frac{1}{2}V_{a}V_{b}$ and the symmetry (a,b) and (c,d):

$$
\begin{aligned}
\Delta^{-1}_{abcd} &:= g_{a[b}g_{c]d}\left(\tilde{\Box} + V_{e}\partial^{e} - \frac{\ell(\ell+1)}{r^{2}}\right) - g_{a[b}\partial_{c]}\partial_{d} - g_{c[d}\partial_{b]}\partial_{a} \\
&\quad - g_{ab}\left(\frac{1}{4}V_{c}V_{d} + V_{d}\partial_{c}\right) + \frac{1}{2}g_{ac}V_{d}\partial_{b} + \frac{1}{2}g_{bc}V_{d}\partial_{a} \qquad (72) \\
\Delta^{-1}_{L,ab} &:= g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^{2}} + \frac{3}{2}V_{c}\partial^{c} + \frac{1}{4}V_{c}V^{c}\right) - \partial_{a}\partial_{b} - V_{a}\partial_{b} \qquad (73) \\
\Delta^{-1}_{R,ab} &:= g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^{2}} + \frac{1}{2}V_{c}\partial^{c}\right) - \partial_{a}\partial_{b} - V_{a}\partial_{b} \qquad (74) \\
\Delta^{-1} &:= \tilde{\Box} + V_{a}\partial^{a} \qquad (75)
\end{aligned}
$$

Based on the work from [32] we can further simplify these expressions to:

$$\Delta^{-1}_{abcd} = \frac{1}{2}\left(g_{ac}V_{[d}\partial_{b]} + g_{bd}V_{[c}\partial_{a]} - g_{ab}V_{(c}\partial_{d)} + g_{cd}V_{(a}\partial_{b)}\right) - g_{a[b}g_{c]d}\frac{\ell(\ell+1)}{r^2} - \frac{1}{4}g_{ab}V_cV_d \tag{76}$$

$$\Delta^{-1}_{L,ab} = g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^2} + \frac{3}{2}V_c\partial^c + \frac{1}{4}V_cV^c\right) - \partial_{(a}\partial_{b)} - V_{(a}\partial_{b)} \tag{77}$$

$$\Delta^{-1}_{R,ab} = g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^2} + \frac{1}{2}V_c\partial^c\right) - \partial_{(a}\partial_{b)} - V_{(a}\partial_{b)} \tag{78}$$

$$\Delta^{-1} = \tilde{\Box} + V_a\partial^a \tag{79}$$

We now want to absorb the r^2 of the action into the fields H_{ab} and K. To do so we redefine the fields as:

$$\tilde{H}_{ab} \coloneqq rH_{ab} \qquad\qquad \tilde{K} \coloneqq rK \tag{80}$$

We also have to redefine the operators that act on the new fields. To do so, we'll use a new derivative:

$$D_a \coloneqq \partial_a + \frac{1}{2}V_a \tag{81}$$

This has the property of introducing an r that multiplies whatever it acts on ($D_a() = \partial_a() + \frac{1}{2}V_a() = \partial_a[r()]$). Therefore we replace every ∂_a with $D_a - \frac{1}{2}V_a$. For example we have that $\tilde{\Box} = D^2 - V_a\partial^a$, where $D^2 = D_aD^a$ and $\partial_{(a}\partial_{b)} = D_{(a}D_{b)} - V_{(a}\partial_{b)}$. The operators from the action now take the following form:

$$\Delta^{-1}_{abcd} = \frac{1}{2}\left(g_{ac}V_{[d}D_{b]} + g_{bd}V_{[c}D_{a]} - g_{ab}V_{(c}D_{d)} + g_{cd}V_{(a}D_{b)}\right) - g_{a[b}g_{c]d}\frac{\ell(\ell+1)}{r^2} - \frac{1}{4}g_{cd}V_aV_b \tag{82}$$

$$\Delta^{-1}_{L,ab} = g_{ab}\left(D^2 - \frac{\ell(\ell+1)}{2r^2} + \frac{1}{2}V_cD^c\right) - D_{(a}D_{b)} \tag{83}$$

$$\Delta^{-1}_{R,ab} = g_{ab}\left(D^2 - \frac{\ell(\ell+1)}{2r^2} - \frac{1}{2}V_cD^c + \frac{1}{4}V_cV^c\right) - D_{(a}D_{b)} \tag{84}$$

$$\Delta^{-1} = D^2 \tag{85}$$

and since $D_a() = \frac{1}{r}\partial_a[r()]$, we can make the final replacement and definition:

$$\tilde{\Delta}^{-1}_{abcd} \coloneqq \frac{1}{2}\left(g_{ac}V_{[d}\partial_{b]} + g_{bd}V_{[c}\partial_{a]} - g_{ab}V_{(c}\partial_{d)} + g_{cd}V_{(a}\partial_{b)}\right) - g_{a[b}g_{c]d}\frac{\ell(\ell+1)}{r^2} - \frac{1}{4}g_{cd}V_aV_b \tag{86}$$

$$\tilde{\Delta}^{-1}_{L,ab} \coloneqq g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^2} + \frac{1}{2}V_c\partial^c\right) - \partial_{(a}\partial_{b)} \tag{87}$$

$$\tilde{\Delta}^{-1}_{R,ab} \coloneqq g_{ab}\left(\tilde{\Box} - \frac{\ell(\ell+1)}{2r^2} - \frac{1}{2}V_c\partial^c + \frac{1}{4}V_cV^c\right) - \partial_{(a}\partial_{b)} \tag{88}$$

$$\tilde{\Delta}^{-1} \coloneqq \tilde{\Box} \tag{89}$$

Finally, we end up with the following two-dimensional action:

$$S^+ = -\frac{1}{2}\sum_{\ell m}\int d^2x\,(\tilde{H}^{ab}\tilde{\Delta}^{-1}_{abcd}\tilde{H}^{cd} + \tilde{H}^{ab}\tilde{\Delta}^{-1}_{L,ab}\tilde{K} + \tilde{K}\tilde{\Delta}^{-1}_{R,ab}\tilde{H}^{ab} + \tilde{K}\tilde{\Delta}^{-1}\tilde{K}) \tag{90}$$

From this point, the propagators of the different fields can be obtained by inverting the operators. These propagators correspond to the propagator of the fields $\tilde{H}^{ab}$, $\tilde{K}$, as well as the propagator of the field $\tilde{H}^{ab}$ turning into the field $\tilde{K}$ and vice versa.

3.4 Discussion of results

The propagator of the $\tilde{K}$ field corresponds to the usual propagator of a scalar field. The other propagators are the less usual ones. We can see that these ones have a term with the angular momentum contribution ($\ell(\ell+1)$) in the numerator. This means that, when inverting the operators, the angular momentum dependency will be in the denominator. From here we can expect that, after discussing the Feynman rules and computing the scattering amplitudes of some Feynman diagrams, the leading order contribution will probably be dominant for low angular momentum (in contrast with the work done of the inspiral phase). Therefore, it seems like the Clebsch–Gordan coefficients also could play a role in this regime following the harmonic expansion method. We should note that a more careful analysis should be done, but as a first approximation we can see that this harmonic expansion approach will present some problems when working in flat space-time.

On a different note, during this thesis we worked on an analysis of an expansion of the metric perturbation in a partial wave basis. The basic idea behind this is method is to work in Fourier space and write plane waves in a partial wave basis like so:

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell} j_{\ell}(kr) Y_{\ell m}(\theta, \phi) Y^{*}_{\ell,m}(k_{\theta}, k_{\phi}) \tag{91}$$

where $\theta,\ \phi$ are the angular variables for $\vec{x}$ and $k_{\theta},\ k_{\phi}$ for $\vec{k}$; the radial functions j_l are the spherical Bessel functions; and $Y_{\ell,m}$ are the spherical harmonics.

With this expansion in hand, the scalar field $\phi(t, \vec{x})$ can now be expanded as

$$\begin{aligned}
\phi(x) &= \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.\vec{x}} \phi_k(\omega, \vec{k}) \\
&= \int \frac{d\omega}{2\pi} e^{-i\omega t} \sum_{\ell,m} i^{\ell} Y_{\ell m}(\theta, \phi) \int \frac{dk}{2\pi^2} k^2 d\Omega_k j_{\ell}(kr) Y^{*}_{\ell,m}(k_{\theta}, k_{\phi})\, \phi_k(\omega, \vec{k}) \\
&= \int \frac{d\omega}{2\pi} e^{-i\omega t} \sum_{\ell,m} i^{\ell} Y_{\ell m}(\theta, \phi) \int \frac{dk}{2\pi^2} k^2 d\Omega_k j_{\ell}(kr) Y^{*}_{\ell,m}(k_{\theta}, k_{\phi}) \sum_{\ell',m'} \phi_{k,\ell',m'}(\omega, k) Y_{\ell',m'}(k_{\theta}, k_{\phi}) \\
&= \sum_{\ell,m} i^{\ell} \int \frac{d\omega dk}{4\pi^3} e^{-i\omega t}\, k^2 j_{\ell}(kr) \phi_{k,\ell,m}(\omega, k) Y_{\ell m}(\theta, \phi)
\end{aligned} \tag{92}$$

where, $k = |\vec{k}|$.

From the second to the third line, we expanded $\phi_k(\omega, \vec{k})$ in spherical harmonics.

Now, the d'Alembertian acting on the scalar field gives us:

$$\begin{aligned}
\Box\phi(x) &= \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.\vec{x}} (k^{\mu} k_{\mu}) \phi_k(\omega, \vec{k}) \\
&= \sum_{\ell,m} i^{\ell} \int \frac{d\omega dk}{4\pi^3} e^{-i\omega t} k^2 j_{\ell}(kr)(-\omega^2 + k^2) \phi_{k,\ell,m}(\omega, k) Y_{\ell m}(\theta, \phi)\,.
\end{aligned} \tag{93}$$

And from here, we can recognize the propagator of the scalar field in spherical wave basis as:

$$\mathcal{P}_{\phi} = \frac{-i}{-\omega^2 + k^2 - i\epsilon} \tag{94}$$

We ran into problems when trying to work out an analogous result for the components of the perturbation metric. The tensor nature of its components makes every calculation very complex. We tried to expand the fields in partial wave basis and then act with the operators in the Lagrangian and vice versa without any luck. Also, working with (t,r) coordinates gave us problems since $r \in [0, \infty)$ and not $(-\infty, \infty)$. This makes us run into problems when Fourier transforming the results into momentum space in order to find the Feynman rules because we cannot integrate the coordinate r in the domain $(-\infty, \infty)$. Therefore we stack with the light-cone variables (x,y) because in this case $x \times y \in (-\infty, \infty) \times (-\infty, \infty)$ and we continued to study a different topic.

4 Matter action, Feynman rules and scattering amplitudes on the horizon

Now we change the subject of study and we focus on scattering in a curved background spacetime metric. Therefore we will now work on the post-merger phase. In the introduction we reviewed what each term of the expansion of the matter action contributes to the scattering amplitude calculations. And as we anticipated, we will expand on the work done in [32], on which they focused on the first order of the expansion. In this thesis we will study the second order of this expansion and its contribution to the scattering amplitude. As we explained previously, this order corresponds to the four-vertex interaction between two scalar particles and two graviton particles which we can represent like $< \phi\phi hh >$.

4.1 Matter action and Feynman rules

The second order from the expansion of the matter action is:

$$S_M^{(2)} = -\frac{\kappa^2}{2}\int d^4x\sqrt{-g}h^{\mu\nu}h^{\rho\sigma}\left[g_{\rho\nu}g_{\mu\alpha}g_{\sigma\beta} - \frac{1}{2}g_{\rho\sigma}g_{\mu\alpha}g_{\nu\beta} - \frac{1}{4}g_{\rho\mu}g_{\sigma\nu}g_{\alpha\beta} + \frac{1}{8}g_{\rho\sigma}g_{\mu\nu}g_{\alpha\beta}\right]\partial^\alpha\phi\partial^\beta\phi \tag{95}$$

Applying the Regge-Wheeler gauge on the metric perturbation and assuming no transverse momentum transfer (as explained in the previous chapter). We end up with:

$$S_M^{(2)} = -\frac{\kappa^2}{2}\sum_{\substack{\ell,m\\ \ell',m'\\ \ell_1,m_1\\ \ell_2,m_2}}\int d^2x d\Omega A(r) Y_\ell^m \bar{Y}_{\ell'}^{m'} Y_{\ell_1}^{m_1}\bar{Y}_{\ell_2}^{m_2}\left[\tilde{H}^{ab}\tilde{H}^{cd}\tilde{P}_{abcdef} + \tilde{H}^{ab}\tilde{K}\tilde{P}_{abef}\right]\partial^e\left(\frac{1}{r}\tilde{\phi}\right)\partial^f\left(\frac{1}{r}\tilde{\phi}\right) \tag{96}$$

where the following definitions of new fields were made:

$$\tilde{H}_{ab} \coloneqq rH_{ab} \qquad \tilde{K} \coloneqq rK \qquad \tilde{\phi} \coloneqq r\phi \tag{97}$$

as well as the operators:

$$\tilde{P}_{abcdef} \coloneqq g_{cb}g_{ae}g_{df} - \frac{1}{2}g_{cd}g_{ae}g_{bf} - \frac{1}{4}g_{ca}g_{db}g_{ef} + \frac{1}{8}g_{cd}g_{ab}g_{ef} \tag{98}$$

$$\tilde{P}_{abef} \coloneqq -g_{ae}g_{bf} + \frac{1}{2}g_{ab}g_{ef} \tag{99}$$

To continue with the simplifications, we will assume that the scalar particles have fixed angular momentum in this interaction vertex, let's say $\ell_1 = \ell_2 = 0$. By doing so, we do not have to keep count of the many Clebsch–Gordan coefficients that come from considering all the different combinations of angular momenta modes of each particle in this interaction. Therefore this assumption simplifies the action and eliminates the angular contribution so that now we only work on the light-cone ending up with a two-dimensional theory.

$$S_M^{(2)} = -\kappa^2\sum_{\ell,m}\int d^2x A(r)\left[2\tilde{H}_\ell^{ab}\tilde{H}_\ell^{cd}\tilde{P}_{abcdef} + \tilde{H}_\ell^{ab}\tilde{K}_\ell\tilde{P}_{abef}\right]\partial^e\left(\frac{1}{r}\tilde{\phi}_0\right)\partial^f\left(\frac{1}{r}\tilde{\phi}_0\right) \tag{100}$$

We denote this choice of angular momentum by adding a subscript ℓ and 0 to the fields with and without angular momentum respectively. Also, because of the different ways of fixing the external legs of the fields $\tilde{\phi}_0$ and $\tilde{H}^{ab}_\ell$, we add a symmetry factor of four and two to the first and second terms respectively.

Note that this choice is arbitrary and we could have chosen any other two particles in this vertex to not have angular momentum, ending up for example with: $< \phi_\ell \phi_\ell h_0 h_>$ or $< \phi_\ell \phi_0 h_\ell h_0 >$. We chose to work with $< \phi_0 \phi_0 h_\ell h_\ell >$ because we thought that gravitons are the relevant particles in this interaction and they should be considered to have any angular momentum. Also, based on the work from [32], they studied the vertex $< \phi\phi h >$ and because of the same simplification idea they chose to give angular momentum to the graviton and one of the scalar particles. This is because the various partial waves decouple at quadratic order, owing to the spherical symmetry of the background. The different partial waves are then expected to only be coupled by large transverse momentum transfers [47]. This is subdominant for scattering processes on the horizon when the black hole is greater than Planck size. Therefore, we do not expect that angular momentum is distributed among external legs. This implies that one of the external scalar legs is kept with some fixed partial wave, say ϕ_0.

It is also relevant to note that external legs have angular momentum. Scattering amplitude calculations originally comes from quantum field theory (QFT), in which external legs have four-momenta but not angular momentum. The change here comes from the spherical harmonics expansion. We go from four-momenta to two-momenta in the light-cone and the modes ℓ and m.

Now we notice that we are working on a background metric that is conformal to flat space, since $g_{ab} = A(r)\eta_{ab}$. This means that we can apply a Weyl transformation to the fields:

$$\tilde{H}_{ab} \to A(r)\mathfrak{h}_{ab} \qquad \tilde{K} \to \mathcal{K} \qquad \tilde{\phi} \to \phi \tag{101}$$

which means that: $\tilde{H}^{ab} \to A^{-1}(r)\mathfrak{h}^{ab}$ and $\partial^e\phi \to A^{-1}(r)\partial^e\phi$.

After making this Weyl rescaling we end up with the following action:

$$S^{(2)}_M = -\kappa^2 \sum_{\ell,m} \int d^2x \left[2\mathfrak{h}^{ab}_\ell \mathfrak{h}^{cd}_\ell P_{abcdef} \partial^e \left(\frac{1}{r}\phi_0\right) \partial^f \left(\frac{1}{r}\phi_0\right) + \mathfrak{h}^{ab}_\ell \mathcal{K}_\ell P_{abef} \partial^e \left(\frac{1}{r}\phi_0\right) \partial^f \left(\frac{1}{r}\phi_0\right) \right] \tag{102}$$

with the following new definitions of the operators:

$$P_{abcdef} := \eta_{cb}\eta_{ae}\eta_{df} - \frac{1}{2}\eta_{cd}\eta_{ae}\eta_{bf} - \frac{1}{4}\eta_{ca}\eta_{db}\eta_{ef} + \frac{1}{8}\eta_{cd}\eta_{ab}\eta_{ef} = A^{-3}(r)\tilde{P}_{abcdef} \tag{103}$$

$$P_{abef} := -\eta_{ae}\eta_{bf} + \frac{1}{2}\eta_{ab}\eta_{ef} = A^{-2}(r)\tilde{P}_{abef} \tag{104}$$

Now, like we announced before, we will focus on the case where the scattering takes place near a Schwarzschild black hole horizon. This means that we now specify the function $A(r)$ to be:

$$A(r) = \frac{R}{r}e^{1-\frac{r}{R}} \qquad xy = 2R^2\left(1 - \frac{r}{R}\right)e^{\frac{r}{R}-1} \tag{105}$$

where R is the Schwarzschild radius of the black hole.

Near the horizon we can take the following approximation:

$$x, y \ll R \implies r = R\left(1 + \mathcal{O}\Big(\frac{xy}{R^2}\Big)\right) \tag{106}$$

This means that the action reduces to:

$$S^{(2)}_M = \gamma^2 \sum_{\ell,m} \int d^2x \left[2\mathfrak{h}^{ab}_\ell \mathfrak{h}^{cd}_\ell P_{abcdef} \partial^e\phi_0 \partial^f\phi_0 + \mathfrak{h}^{ab}_\ell \mathcal{K}_\ell P_{abef} \partial^e\phi_0 \partial^f\phi_0 \right] \tag{107}$$

where $\gamma \coloneqq \kappa/R$.

The last step to obtain the Feynman rules is to Fourier transform the action into momentum space.

$$\phi(x) = \frac{1}{(2\pi)^2}\int d^2p\, e^{ip_a x^a}\phi(p) \qquad \phi(p) = \int d^2x\, e^{-ip_a x^a}\phi(x) \tag{108}$$

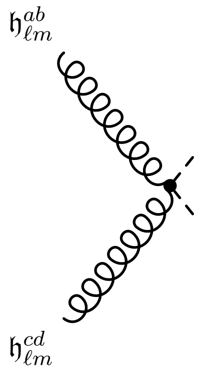
$$\mathcal{K}(x) = \frac{1}{(2\pi)^2}\int d^2p\, e^{ip_a x^a}\mathcal{K}(p) \qquad \mathcal{K}(p) = \int d^2x\, e^{-ip_a x^a}\mathcal{K}(x) \tag{109}$$

$$\mathfrak{h}^{ab}(x) = \frac{1}{(2\pi)^2}\int d^2p\, e^{ip_a x^a}\mathfrak{h}^{ab}(p) \qquad \mathfrak{h}^{ab}(p) = \int d^2x\, e^{-ip_a x^a}\mathfrak{h}^{ab}(x) \tag{110}$$

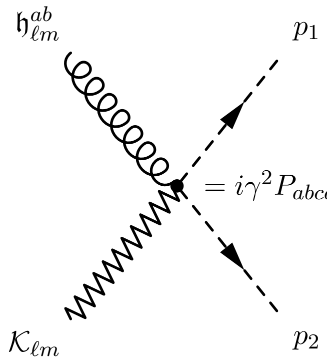
Also, we have that $\partial^e \to ip^e$. With all this, we end up withe the action in momentum space:

$$S_M^{(2)} = \gamma^2 \sum_{\ell,m}\int d^2k d^2k' d^2p_1 d^2p_2 \delta^{(2)}(k+k'+p_1+p_2)$$
$$\left[2\mathfrak{h}_\ell^{ab}(k)\mathfrak{h}_\ell^{cd}(k')P_{abcdef}p^e\phi_0(p_1)p^f\phi_0(p_2) + \mathfrak{h}_\ell^{ab}(k)\mathcal{K}_\ell(k')P_{abef}p^e\phi_0(p_1)p^f\phi_0(p_2)\right] \tag{111}$$

Which gives us the following Feynman rules for the vertices:

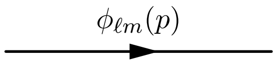


$= 2i\gamma^2 P_{abcdef}\ p_1^e p_2^f$

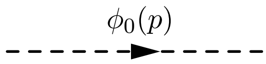


$= i\gamma^2 P_{abcd}\ p_1^c p_2^d$

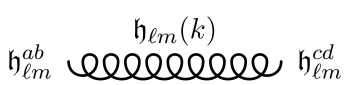
while the relevant propagators that will appear in the Feynman diagrams can be obtained from $S_M^{(0)}$ for the scalar field and $S_{EH}^{(2)}$ for the graviton fields [32]:



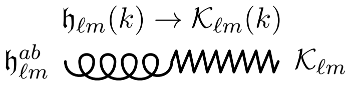
$= \dfrac{-i}{p^2+\lambda\mu^2-i\epsilon}$



$= \dfrac{-i}{p^2+\mu^2-i\epsilon}$



$= 2i\mathcal{P}^{abcd}(k)$



$= 2i\mathcal{P}^{ab}(k)$

with the following definitions:

$$\lambda \coloneqq \ell^2+\ell+1 \qquad \mu \coloneqq \frac{1}{R} \tag{112}$$

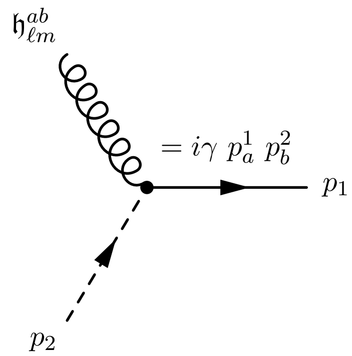
$$\mathcal{P}^{abcd}(k) \coloneqq \Delta^{abcd} + \mathcal{P}_{\mathcal{K}}(k)P^{ab}(k)P^{cd}(k) \qquad \mathcal{P}^{ab}(k) \coloneqq -\mathcal{P}_{\mathcal{K}}(k)P^{ab} \tag{113}$$

$$\Delta^{abcd} := \frac{1}{\mu^2(\lambda+1)} \left(2\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}\right) \tag{114}$$

$$\mathcal{P}_{\mathcal{K}}(k) := \frac{-(\lambda+1)}{(\lambda-3)} \frac{1}{k^2+\mu^2\lambda} \qquad\qquad P^{ab}(k) := \frac{\lambda-1}{\lambda+1}\eta^{ab} + \frac{2k^a k^b}{\mu^2(\lambda+1)} \tag{115}$$

$\mathcal{P}_{\mathcal{K}}(k)$ is the propagator of the scalar field $\mathcal{K}_{\ell m}$, but since it doesn't contribute to the scattering amplitudes that we will consider later on, we did not write it with the other propagators. As we expected for a scalar particle, it resembles the Klein-Gordon propagator with an effective mass $\mu\sqrt{\lambda}$. Note that the four-dimensional graviton is still massless, it is only in this two-dimensional reduction of the theory that the graviton gains this effective mass. This results into a natural infrared regulator. We should note that it is not well defined for the case $\ell = 1$ and that there is a sign flip for $\ell = 0$. These special cases are well known [48, 49]. Thankfully, as mentioned before, this problematic propagator won't contribute to the scattering amplitudes that we will encounter.

Finally, as discussed in [32], the interaction vertex $< \phi_0\phi_\ell h_\ell >$ has the following Feynman rule:



As we can see, only the $\mathfrak{h}^{ab}_{\ell m}$ field is consider and this is because the field $\mathcal{K}_{\ell m}$ has a subleading contribution to the scattering amplitude.

4.2 Feynman diagrams on the horizon

Now we will discuss all possible Feynman diagrams near the horizon. As we discussed in (4.1), we assume that the two scalar particles in the four-vertex have zero angular momentum while the two gravitons have non-zero angular momentum. Nevertheless, for completeness, we discuss and consider the Feynman diagrams that come from the other considerations. Therefore we can divide the diagrams in three categories:

First, we have all diagrams on which all scalar particles have unfixed angular momentum $\ell \neq 0$, which are composed by the following vertices: $< \phi_\ell\phi_\ell h_0 >, < \phi_\ell\phi_\ell h_0 h_0 >$. Second, we have all diagrams on which at least one ϕ and one h have unfixed angular momentum $\ell \neq 0$, containing the following vertex: $< \phi_0\phi_\ell h_0 h_\ell >$. Finally, we have all diagrams on which all gravitons have unfixed angular momentum $\ell \neq 0$, which are made up by the following vertices: $< \phi_0\phi_\ell h_\ell >, < \phi_0\phi_0 h_\ell h_\ell >$.

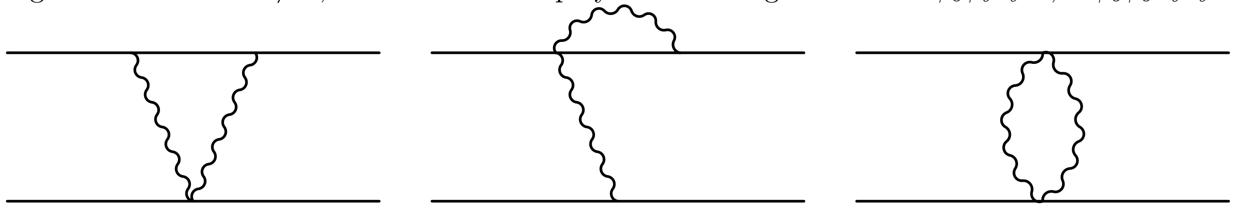


Figure 1: All scalars have $\ell \neq 0$

These are all possible diagrams considering vertices in which every scalar has an unfixed angular momentum ℓ as we can see in the diagrams with all scalar particles represented with solid lines. In this case all graviton propagators h have a fixed angular momentum $\ell = 0$.

Next, we collect the diagrams from the second family:

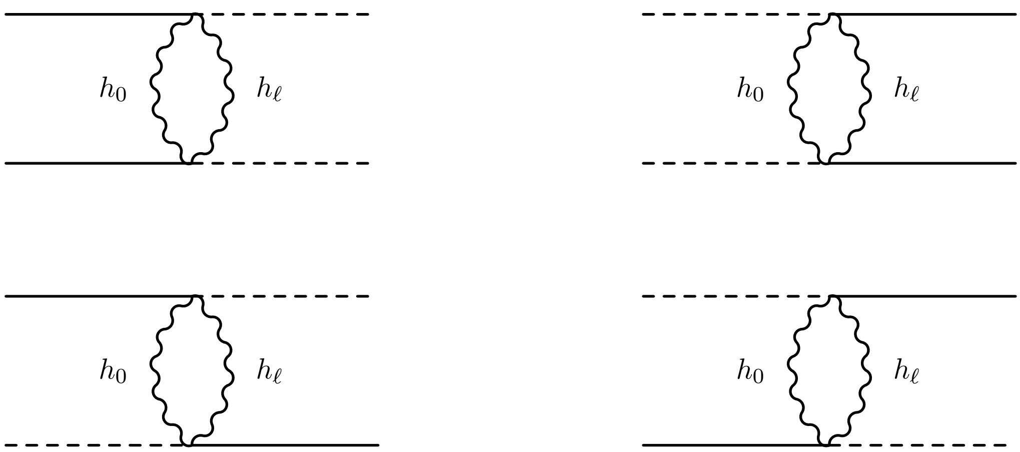


Figure 2: At least one scalar and one graviton have $\ell \neq 0$

As we can see in these diagrams there are no three-vertices. There are always two scalar particles with unfixed and two with fixed angular momentum. Also there is always one graviton with fixed and another one with unfixed angular momentum that we distinguish by writing h_0 and h_ℓ accordingly.

Finally let's take a look at the family of diagrams that we have considered studying further:

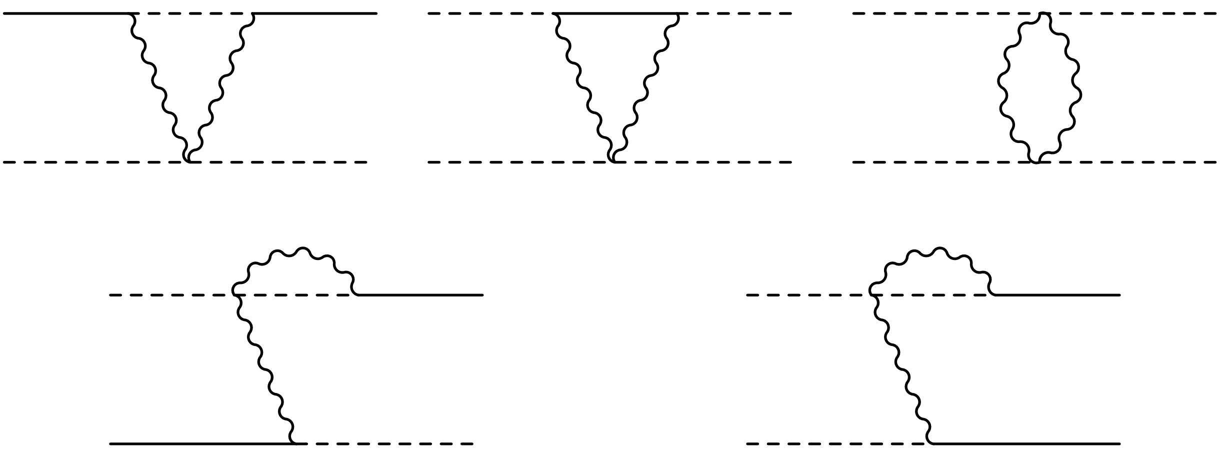


Figure 3: All gravitons have $\ell \neq 0$

These are the diagrams that we chose to study further, but note that we could have chosen any of the three families here exposed.

Moreover, from the five diagrams included in this family we computed the scattering amplitude of the diagrams with one incoming and one outgoing unfixed angular momentum $\ell \neq 0$. These correspond with the two diagrams on the left side in Fig. 3. This is because we considered that those processes were more general than the scatterings with two incoming particles or outgoing particles with fixed angular momentum. Therefore we are left with the following two diagrams:

Figure 4: All gravitons and one incoming and one outgoing scalar particle have $\ell \neq 0$

Finally, if we specify the different graviton fields that act on the different scatterings (either $\mathfrak{h}$ or $\mathcal{K}$) we obtain the six Feynman diagrams that we consider to compute its scattering amplitudes:

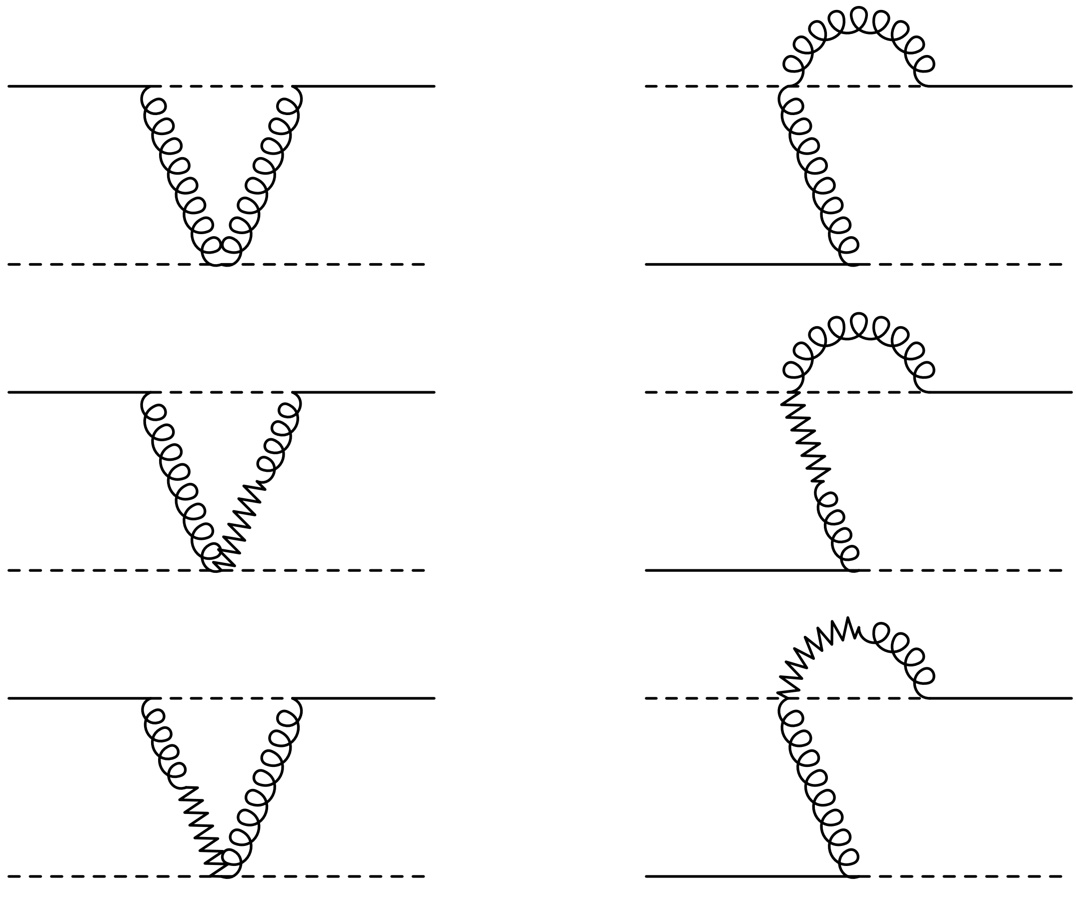


Figure 5: Specifying $\mathfrak{h}$ or $\mathcal{K}$ fields

4.3 Scattering amplitudes on the horizon

With the set of Feynman rules in hand and after having decided what diagrams we are interested on, we are now ready to compute its scattering amplitudes. These chosen diagrams are the six diagrams from Fig. 5. The only thing missing is labeling the momenta of each particle. We will do so by labeling with p the momenta of external scalar propagators and with k the internal momenta:

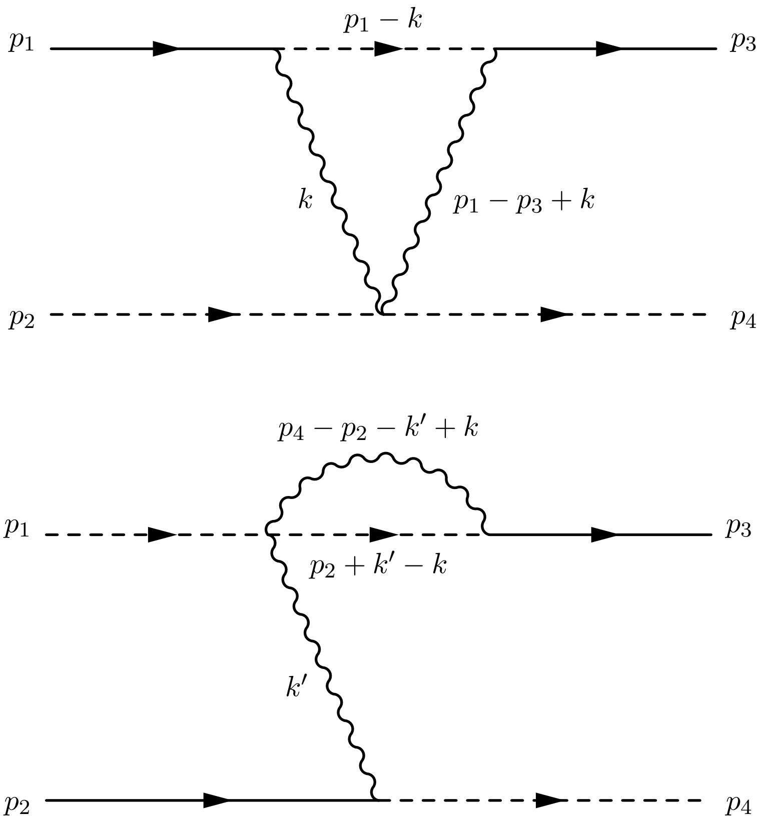


Figure 6: Momenta of each particle

We are approaching this problem for the first time, therefore we take the most simple assumptions for the momenta. Based on [21] we assume a large impact parameter parameter in these scatterings. This implies that $p_3 \approx p_1$ and $p_4 \approx p_2$. From this approximation we can safely assume that $k' \approx 0$. Now, based on [32] we assume that the momenta from the external legs dominate over the internal ones. This means that $p \pm k \approx p$ and $(p \pm k)^2 \approx p^2 \pm 2pk$. With these simplistic but rational approximations in hand we compute the scattering amplitudes of the six diagrams from 5. We will show the derivation of one of these diagrams and the rest can be seen in the appendix B.

Let's take the first diagram from Fig. 5:

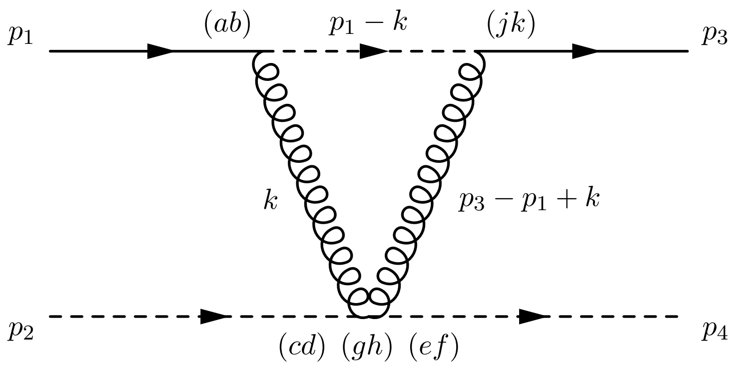


Figure 7: Scattering amplitude $i\mathcal{M}_1$

We labeled the momenta as well as the light-cone indices on each vertex that appear in the Feynman rules. Now we can compute the scattering amplitude of this diagram:

$$
\begin{aligned}
i\mathcal{M}_1 = &\int \frac{d^2k}{(2\pi)^2} \left(i\gamma p_a^1(p^1-k)_b\right)\left(\frac{-i}{(p_1-k)^2+\mu^2-i\epsilon}\right)\left(i\gamma p_j^3(p^1-k)_k\right) \\
&\qquad 2i\mathcal{P}^{abcd}(k) 2i\mathcal{P}^{efjk}(p_3-p_1+k)\left(i\gamma^2 P_{cdefgh}p_2^g p_4^h\right) \\
\approx &\frac{\gamma^4}{\pi^2}\int \frac{d^2k}{-2p_1k-i\epsilon} p_a^1 p_b^1 p_j^1 p_k^1 p_2^g p_2^h \left(\Delta^{abcd}+\mathcal{P}_{\mathcal{K}}(k)P^{ab}(k)P^{cd}(k)\right) \\
&\qquad \left(\Delta^{abcd}+\mathcal{P}_{\mathcal{K}}(k)P^{ab}(k)P^{cd}(k)\right) P_{cdefgh}
\end{aligned} \tag{116}
$$

So far, we only applied the momenta approximations. Note that the scalar propagator in between vertices has fixed angular momentum $\ell = 0$, which means that $\lambda = 1$ and therefore $(p_1)^2 = -\mu^2$. Next, we expand every term and start computing the index contractions:

$$
\begin{aligned}
i\mathcal{M}_1 = &\frac{\gamma^4}{\pi^2}\int \frac{d^2k}{-2p_1k-i\epsilon}\Bigg[\frac{2}{\mu^2(\lambda+1)}\left((p_1)^2\eta^{cd}-p_1^c p_1^d\right) \\
&- \frac{\lambda+1}{\lambda-3}\frac{1}{k^2+\mu^2\lambda}\left(\frac{\lambda-1}{\lambda+1}(p_1)^2+\frac{2(p_1k)^2}{\mu^2(\lambda+1)}\right)\left(\frac{\lambda-1}{\lambda+1}\eta^{cd}+\frac{2k^ck^d}{\mu^2(\lambda+1)}\right)\Bigg] \\
&\Bigg[\frac{2}{\mu^2(\lambda+1)}\left((p_1)^2\eta^{ef}-p_1^e p_1^f\right) \\
&- \frac{\lambda+1}{\lambda-3}\frac{1}{k^2+\mu^2\lambda}\left(\frac{\lambda-1}{\lambda+1}(p_1)^2+\frac{2(p_1k)^2}{\mu^2(\lambda+1)}\right)\left(\frac{\lambda-1}{\lambda+1}\eta^{ef}+\frac{2k^ek^f}{\mu^2(\lambda+1)}\right)\Bigg] \\
&\Bigg[\eta_{ed}p_c^2p_f^2-\frac{1}{2}\eta_{ef}p_c^2p_d^2-\frac{1}{4}\eta_{ec}\eta_{fd}(p_2)^2+\frac{1}{8}\eta_{ef}\eta_{cd}(p_2)^2\Bigg] \\
= &\frac{\gamma^4}{\pi^2}\int \frac{d^2k}{-2p_1k-i\epsilon}\Bigg[\frac{2}{\mu^2(\lambda+1)}\left((p_1)^2\eta^{cd}-p_1^c p_1^d\right) \\
&- \frac{\lambda+1}{\lambda-3}\frac{1}{k^2+\mu^2\lambda}\left(\frac{\lambda-1}{\lambda+1}(p_1)^2+\frac{2(p_1k)^2}{\mu^2(\lambda+1)}\right)\left(\frac{\lambda-1}{\lambda+1}\eta^{cd}+\frac{2k^ck^d}{\mu^2(\lambda+1)}\right)\Bigg] \\
&\qquad \left((p_2k)^2p_c^2k_d-\frac{1}{2}k^2p_c^2p_d^2-\frac{1}{4}(p_2)^2k_ck_d+\frac{1}{8}k^2(p_2)^2\eta_{cd}\right)\Bigg)\Bigg] \\
&\Bigg[\frac{-2}{\mu^2(\lambda+1)}\Bigg((p_1p_2)^2p_c^1p_d^1-\frac{1}{2}(p_1)^2p_c^2p_d^2-\frac{1}{4}(p_2)^2p_c^1p_d^1+\frac{1}{8}(p_1)^2(p_2)^2\eta_{cd} \\
&\qquad + \frac{(\lambda-1)\mu^2(p_1)^2+2(p_1k)^2}{(\lambda-3)\mu^2(k^2+\mu^2\lambda)}
\end{aligned} \tag{117}
$$

Finally, after contracting the last two indices we end up with a very long expression inside the integral. Thankfully we can simplify this integral with the help of the following identities:

Any even-dimensional integral satisfies:

$$
\int d^2k\, I(k) = \frac{1}{2}\int d^2k\, \left(I(k)+I(-k)\right) \tag{118}
$$

This, together with the scalar propagator inside the integral can be turned into the following delta identity:

$$
\frac{1}{2pk-i\epsilon}+\frac{1}{-2pk-i\epsilon} = 2\pi i\delta(2pk) \tag{119}
$$

Now, the scattering amplitude of interest has the external particles to be approximately null. Which means that one particle, say p_1, has momentum exclusively going into the black hole while the other particle, say p_2, has its momentum exclusively exiting the horizon. This is called the rest frame. This means that we have: $p_1 = (p_{1x}, p_{1y}) = (p_{1x}, 0)$ and $p_2 = (p_{2x}, p_{2y}) = (0, p_{2y})$. This means that we have the following Mandelstam variable:

$$s = -\frac{1}{2}(p_1 + p_2)^2 = -p_1 p_2 = p_{1x} p_{2y} \tag{120}$$

Also, the delta $\delta(2pk)$ can be written like:

$$\delta(2p_1 k) = \frac{1}{2|p_1|}\delta(k^y) \tag{121}$$

And if $k^y = 0$, then $k^2 = 0$ and $(p_1 k)^2 = 0$.

Putting everything together we end up with this expression for the scattering amplitude:

$$\begin{aligned} i\mathcal{M}_1 = \frac{i\gamma^4}{\pi\mu^4(\lambda+1)^2|p_1|}\int dk_x \Bigg[& \left(\frac{3}{4} - \frac{1}{2}\frac{(\lambda-1)^2}{(\lambda-3)\lambda}\right)(p_1)^4(p_2)^2 - \left(1 - \frac{(\lambda-1)^2}{(\lambda-3)\lambda}\right)(p_1)^2(p_1 p_2)^2 \\ & - \left(2 - \frac{(\lambda-1)}{(\lambda-3)\lambda}\right)\frac{(\lambda-1)}{(\lambda-3)\lambda\mu^2}(p_1)^4(p_{2y}k_x)^2\Bigg] \end{aligned} \tag{122}$$

This expression is clearly divergent. So, to make sense of it we need to use the theory of dimensional regularization [50]. From this theory we can use the following formula:

$$I(n,\alpha) := \int \frac{d^n q}{(q^2 + m^2 - i\epsilon)^\alpha} = i\pi^{n/2}\frac{\Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)}\left(m^2\right)^{\frac{1}{2}n-\alpha} \qquad for\ \alpha > \frac{n}{2} \tag{123}$$

For divergent integrals ($\alpha \leq 2$), this result still gives a representation of the integral as long as $\alpha - \frac{n}{2}$ is not equal to a non-positive integer. This means that we can use this expression for the integral in our scattering amplitude.

Therefore, after using this formula, we end up with the result of the scattering amplitude to be

$$i\mathcal{M}_1 = 0 \tag{124}$$

And, as a matter of fact, following similar calculations we end up with the result that all scattering amplitudes from the six Feynman diagrams that we consider vanish:

$$i\mathcal{M}_n = 0 \qquad for\ n \in [1,6] \tag{125}$$

5 Conclusions

In this thesis we have studied mergers of systems of binary black holes from a very interesting perspective. This unconventional idea of studying the dynamics of very massive objects with tools normally used to describe interactions among the most elementary particles that conform our universe is promising and could bring very interesting results in the near future. The main goal of the thesis has been to understand how this tool works in the post-merger phase of these collisions between black holes. We tried so by studying from a different perspective the physics of the better understood inspiral phase. The conclusion that we obtained from this is that harmonic expansion does not work as good in this phase as in the post-merger phase. This approach can be used in the inspiral phase but, the results that we obtained from it doesn't match previous known results. Nevertheless, this requires further study because more careful approximations could have been taken. Our work consisted in studying a new proposition, therefore it made sense to approach it by implementing simplistic assumptions in order to reduce the complexity of the problem. We have done so throughout the whole thesis and this is the area that requires more attention in future work. Nevertheless, after concluding this study we expect a post black hole expansion to exist in orders of $\tilde{j} = (\ell^2 + \ell + 2)/sR^2$. Also, we showed that within our approximations several diagrams do not contribute to the scattering of two scalar particles near the horizon.

As we already mentioned, the mayor area of improvement of this work is taking a more systematic analysis to the problem. As a first approach, our simplistic assumptions made sense, but now that we have come to the results from this thesis we can proceed more carefully in future work. For example, in the post-black hole expansion we could have taken into account the subleading odd parity action S^-. Also, there is interesting future work that could be done regarding the angular momentum of the particles in the vertices. As we have seen in this theses, angular momentum is very important to these results. Therefore, instead of fixing the angular momentum of some particles in these vertices, it remains as an interesting open question to complete a study of this problem without fixing any angular momentum. This will require a lot of work taking into account all the Clebsch–Gordan coefficients that appear, but this could bring interesting new results, relevant to our open problem. Moreover, we ignore due to lack of time the contribution from the three graviton vertex $< hhh >$. This is also an interesting open question that is left for the future that could give us compelling results.

Regarding the scattering amplitudes computed in this thesis, we took the rest frame assuming that the external particles are approximately null. Nevertheless, taking a look at the scalar propagator, we can see that these scalar particles are massive in the effective two-dimensional theory. But, since we are working with large black holes, we can ignore this mass and treat momenta as null [32]. Therefore, even though we obtained vanishing scattering amplitudes, perhaps taking into account non-null particles we could have obtained non-vanishing amplitudes. In the case of having obtained non-vanishing results for the post-merger phase we could have done a study regarding the angular momentum contribution. By doing so we could have checked if, as we expected, it followed an expansion in orders of $\tilde{j}$.

Acknowledgements

I would like to thank my first and second examiners Dr. Umut Gürsoy and Dr. Tanja Hinderer, my supervisor Nava Gaddam, as well as Nico Groenenboom for the great help that they have provided me during the completion of my thesis. Also, I would like to thank my fellow master students and friends with which I have spent so much time studying and working out problems, but also collecting great memories and growing as a person. Last but not least I would like to thank my family, which made it possible for me to attend this master and gave me my drive for science.

A Palatini Identity

The Riemann tensor is defined as:

$$R^{\alpha}{}_{\mu\beta\nu} \coloneqq \partial_{\beta}\,\Gamma^{\alpha}{}_{\mu\nu} - \partial_{\nu}\,\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\beta\rho}\,\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\lambda}\,\Gamma^{\lambda}{}_{\mu\beta} \tag{126}$$

If we take the variation of the Riemann tensor we end up with:

$$\begin{aligned} \delta R^{\alpha}{}_{\mu\beta\nu} &= \partial_{\beta}\,\delta\left(\Gamma^{\alpha}{}_{\mu\nu}\right) - \partial_{\nu}\delta\left(\Gamma^{\alpha}{}_{\mu\beta}\right) \\ &+ \delta\left(\Gamma^{\alpha}{}_{\beta\rho}\right)\,\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\beta\rho}\,\delta\left(\Gamma^{\rho}{}_{\mu\nu}\right) \\ &- \delta\left(\Gamma^{\alpha}{}_{\nu\lambda}\right)\,\Gamma^{\lambda}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\,\delta\left(\Gamma^{\lambda}{}_{\mu\beta}\right) \end{aligned} \tag{127}$$

where we applied the product rule for the terms with two Christoffel symbols and also the fact that the variation commutes with the partial derivative.

Christoffel symbols are not tensors; however, the variation of the Christoffel symbols are. This means that we can take the covariant derivative of the variation of a Christoffel symbol:

$$\nabla_{\alpha}\,\delta\left(\Gamma^{\beta}{}_{\mu\nu}\right) = \partial_{\alpha}\,\delta\left(\Gamma^{\beta}{}_{\mu\nu}\right) + \Gamma^{\beta}{}_{\alpha\lambda}\,\delta\left(\Gamma^{\lambda}{}_{\mu\nu}\right) - \Gamma^{\lambda}{}_{\alpha\mu}\,\delta\left(\Gamma^{\beta}{}_{\lambda\nu}\right) - \Gamma^{\lambda}{}_{\alpha\nu}\,\delta\left(\Gamma^{\beta}{}_{\mu\lambda}\right) \tag{128}$$

From this result we can see that the variation of the Riemann tensor can be written in terms of covariant derivatives acting on variations of Christoffel symbols:

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \nabla_{\beta}\,\delta\left(\Gamma^{\alpha}{}_{\mu\nu}\right) - \nabla_{\nu}\,\delta\left(\Gamma^{\alpha}{}_{\mu\beta}\right) \tag{129}$$

This is called the Palatini identity.

This identity holds to perturbations up to any order, therefore we can rewrite it as:

$$R^{\alpha}{}_{\mu\beta\nu}{}^{(n)} = \nabla_{\beta}\,\left(\Gamma^{\alpha}{}_{\mu\nu}{}^{(n)}\right) - \nabla_{\nu}\,\left(\Gamma^{\alpha}{}_{\mu\beta}{}^{(n)}\right) \tag{130}$$

This identity is very powerful because it turns variations of the Ricci tensor into terms consisting of covariant derivatives, with which we can use Stokes' theorem when they appear on an integral.

B Scattering amplitudes

In chapter 4 we computed the scattering amplitude of one Feynman diagram ($i\mathcal{M}_1$). In this appendix we will compute the other five. We will consider the same momenta approximations and follow the same procedure (obtaining the delta function and using dimensional regularization techniques) as discussed in chapter 4:

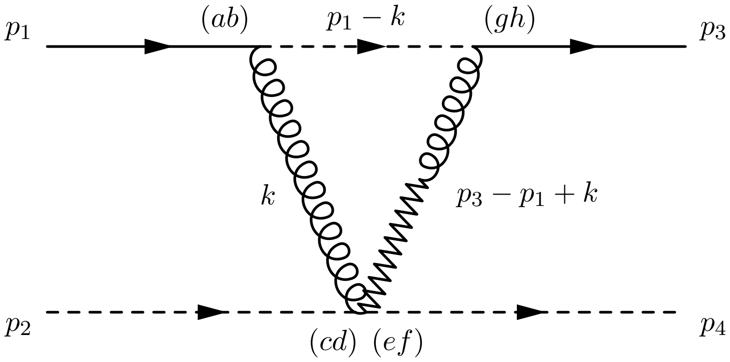


Figure 8: Scattering amplitude $i\mathcal{M}_2$

$$
\begin{aligned}
i\mathcal{M}_2 = &\int \frac{d^2k}{(2\pi)^2} \left(i\gamma p_a^1 (p^1 - k)_b\right) \left(\frac{-i}{(p_1-k)^2 + \mu^2 - i\epsilon}\right) \left(i\gamma p_g^3 (p^1 - k)_h\right) \\
&\qquad 2i\mathcal{P}^{abcd}(k) 2i\mathcal{P}^{gh}(p_3 - p_1 + k) \left(i\gamma^2 P_{cdef} p_2^e p_4^f\right) \\
\approx &\frac{\gamma^4}{\pi^2} \int \frac{d^2k}{-2p_1k - i\epsilon} p_a^1 p_b^1 p_g^1 p_h^1 p_2^e p_2^f \left(\Delta^{abcd} + \mathcal{P}_{\mathcal{K}}(k) P^{ab}(k) P^{cd}(k)\right) \left(-\mathcal{P}_{\mathcal{K}}(k) P^{gh}(k)\right) P_{cdef} \\
= &\frac{\gamma^4}{\pi^2} \int \frac{d^2k}{-2p_1k - i\epsilon} \Bigg[\frac{2}{\mu^2(\lambda+1)} \left((p_1)^2 \eta^{cd} - p_1^c p_1^d\right) \\
&\qquad - \frac{\lambda+1}{\lambda-3} \frac{1}{k^2 + \mu^2\lambda} \left(\frac{\lambda-1}{\lambda+1}(p_1)^2 + \frac{2(p_1k)^2}{\mu^2(\lambda+1)}\right) \left(\frac{\lambda-1}{\lambda+1}\eta^{cd} + \frac{2k^c k^d}{\mu^2(\lambda+1)}\right)\Bigg] \\
&\qquad \left[-p_c^2 p_d^2 + \frac{1}{2}(p_2)^2 \eta_{cd}\right] \left[\frac{1}{k^2 + \mu^2\lambda} \left(\frac{\lambda-1}{\lambda-3}(p_1)^2 + \frac{2(p_1k)^2}{\mu^2(\lambda-3)}\right)\right] \\
= &\frac{\gamma^4}{\pi^2} \int \frac{d^2k}{-2p_1k - i\epsilon} \frac{\left(\lambda-1)(p_1)^2 + \frac{2}{\mu^2}(p_1k)^2\right)}{(k^2 + \mu^2\lambda)(\lambda-3)} \\
&\qquad \left[\frac{4(p_1p_2)^2}{\mu^2(\lambda+1)} + \frac{(-k^2(p_2)^2 + 2(p_2k)^2)}{(k^2+\mu^2\lambda)(\lambda-3)\mu^2} \left(\frac{\lambda-1}{\lambda+1}(p_1)^2 + \frac{2(p_1k)^2}{\mu^2(\lambda+1)}\right)\right] \\
= &\frac{i\gamma^4}{\pi\mu^2(\lambda-3)(\lambda+1)} \int \frac{\delta(2p_1k)}{k^2+\mu^2\lambda} \left[(\lambda-1)(p_1)^2 + \frac{2}{\mu^2}(p_1k)^2\right] \\
&\qquad \left[4(p_1p_2)^2 + \frac{(2(p_2k)^2 - k^2(p_2)^2)}{(k^2+\mu^2\lambda)} \left((\lambda-1)(p_1)^2 + \frac{2(p_1k)^2}{\mu^2}\right)\right] \\
= &\frac{2i\gamma^4}{\pi\mu^2 2|p_1|(\lambda-3)(\lambda+1)} \int dk_x \frac{(\lambda-1)(p_1)^2}{\mu^2\lambda} \left(2(p_1p_2)^2 + \frac{(\lambda-1)(p_1)^2}{\mu^2\lambda}(p_2k)^2\right) \overset{Dim.reg.}{=} 0
\end{aligned}
\tag{131}
$$

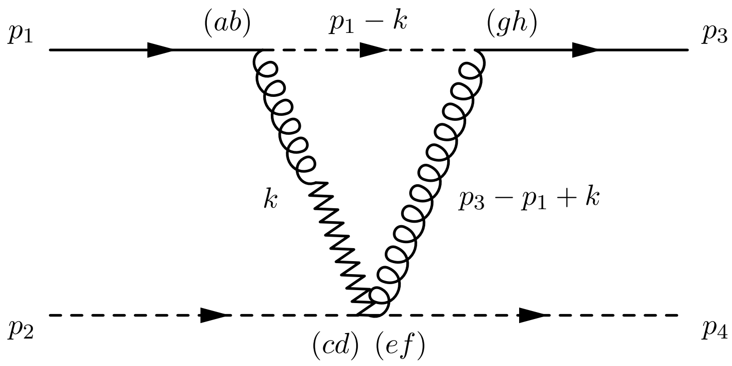


Figure 9: Scattering amplitude $i\mathcal{M}_3$

$$
\begin{aligned}
i\mathcal{M}_3 = &\int \frac{d^2k}{(2\pi)^2} \left(i\gamma p_a^1 (p^1 - k)_b\right) \left(\frac{-i}{(p_1 - k)^2 + \mu^2 - i\epsilon}\right) \left(i\gamma p_g^3 (p^1 - k)_h\right) \\
&\qquad 2i\mathcal{P}^{ab}(k) 2i\mathcal{P}^{cdgh}(p_3 - p_1 + k) \left(i\gamma^2 P_{cdef} p_2^e p_4^f\right) \\
\approx &\frac{\gamma^4}{\pi^2} \int \frac{d^2k}{-2p_1 k - i\epsilon} p_a^1 p_b^1 p_g^1 p_h^1 p_2^e p_2^f \left(-\mathcal{P}_{\mathcal{K}}(k) P^{ab}(k)\right) \left(\Delta^{cdgh} + \mathcal{P}_{\mathcal{K}}(k) P^{cd}(k) P^{gh}(k)\right) P_{cdef} \\
= &i\mathcal{M}_2 \, (ab \leftrightarrow gh) = 0
\end{aligned} \tag{132}
$$

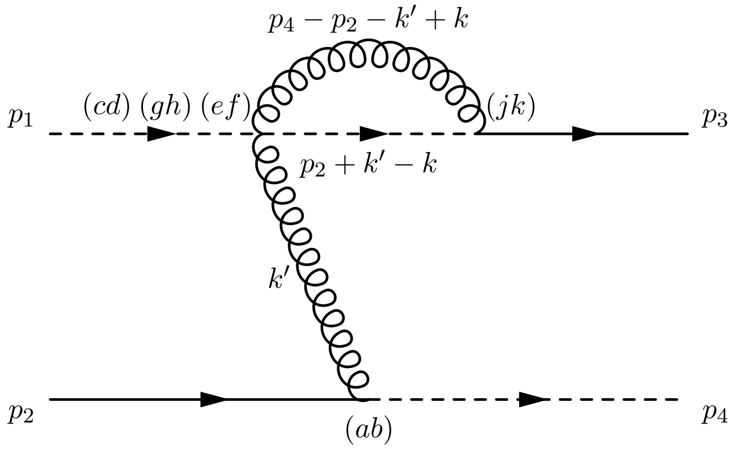


Figure 10: Scattering amplitude $i\mathcal{M}_4$

$$
\begin{aligned}
i\mathcal{M}_4 = &\int \frac{d^2k}{(2\pi)^2} \left(i\gamma p_a^1 p_b^3\right) 2i\mathcal{P}^{abcd}(k') \left(2i\gamma^2 P_{cdefgh} p_2^g (p_2 + k' - k)^h\right) 2i\mathcal{P}^{efjh}(p_4 - p_2 + k - k') \\
&\qquad \left(\frac{-i}{(p_2 - k + k')^2 + \mu^2 - i\epsilon}\right) \left(i\gamma p_j^4 (p_2 - k + k')_k\right) \\
\approx &\frac{2\gamma^4}{\pi^2} \int \frac{d^2k}{-2p_2 k - i\epsilon} p_a^1 p_b^1 p_j^2 p_k^2 p_2^g p_2^h \left(\Delta^{abcd} + \mathcal{P}_{\mathcal{K}}(0) P^{ab}(0) P^{cd}(0)\right) \\
&\qquad P_{cdefgh} \left(\Delta^{efjk} + \mathcal{P}_{\mathcal{K}}(k) P^{ef}(k) P^{jk}(k)\right)
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}_4 = \frac{2\gamma^4}{\pi^2}\int \frac{d^2k}{-2p_2k - i\epsilon}&\left[\frac{2}{\mu^2(\lambda+1)}((p_1)^2\eta^{cd} - p_1^c p_1^d) - \frac{(\lambda-1)^2(p_1)^2\eta^{cd}}{\mu^2\lambda(\lambda-3)(\lambda+1)}\right] \\
&\left[\eta_{ed}p_c^2p_f^2 - \frac{1}{2}\eta_{ef}p_c^2p_d^2 - \frac{1}{4}\eta_{ec}\eta_{fd}(p_2)^2 + \frac{1}{8}\eta_{ef}\eta_{cd}(p_2)^2\right] \\
&\left[\frac{2}{\mu^2(\lambda+1)}\left((p_2)^2\eta^{ef} - p_2^e p_2^f\right)\right. \\
&\left. - \frac{\lambda+1}{\lambda-3}\frac{1}{k^2+\mu^2\lambda}\left(\frac{\lambda-1}{\lambda+1}(p_2)^2 + \frac{2(p_2k)^2}{\mu^2(\lambda+1)}\right)\left(\frac{\lambda-1}{\lambda+1}\eta^{ef} + \frac{2k^ek^f}{\mu^2(\lambda+1)}\right)\right] \\
= \frac{i\gamma^4}{\pi\mu^4(\lambda+1)^2|p_2|}\int dk_y&\Bigg[(p_1p_2)^2(p_2)^2 + \left(\frac{1}{2} - \frac{(\lambda^2-4\lambda-1)}{\lambda(\lambda-3)}\right)(p_1)^2(p_2)^4 \\
&- \frac{(\lambda-1)}{\mu^2\lambda(\lambda-3)}(p_1^xk_y)^2\Bigg] \overset{Dim.reg.}{=} 0
\end{aligned} \tag{133}
$$

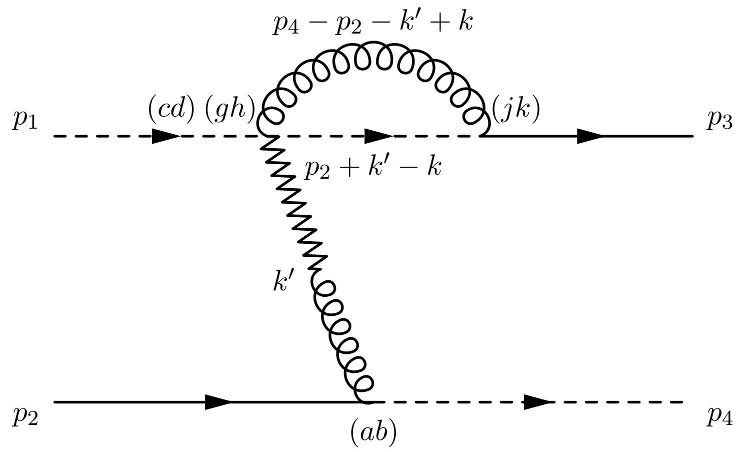


Figure 11: Scattering amplitude $i\mathcal{M}_5$

$$
\begin{aligned}
i\mathcal{M}_5 &= \int \frac{d^2k}{(2\pi)^2}\left(i\gamma p_a^1 p_b^3\right) 2i\mathcal{P}^{ab}(k')\left(i\gamma^2 P_{cdgh}p_2^g(p_2+k'-k)^h\right) 2i\mathcal{P}^{cdjk}(p_4-p_2+k-k') \\
&\qquad\left(\frac{-i}{(p_2-k+k')^2+\mu^2-i\epsilon}\right)\left(i\gamma p_j^4(p_2-k+k')_k\right) \\
&\approx \frac{\gamma^4}{\pi^2}\int \frac{d^2k}{-2p_2k-i\epsilon}p_a^1p_b^1p_j^2p_k^2p_2^gp_2^h\left(-\mathcal{P}_{\mathcal{K}}(0)P^{ab}(0)\right)P_{cdgh}\left(\Delta^{cdjk} + \mathcal{P}_{\mathcal{K}}(k)P^{cd}(k)P^{jk}(k)\right) \\
&= \frac{\gamma^4(\lambda-1)(p_1)^2}{\pi^2\mu^6\lambda(\lambda+1)(\lambda-3)}\int \frac{d^2k}{-2p_2k-i\epsilon}\Bigg[(p_2)^4 + \frac{4(p_2k)^4}{(\lambda-3)(k^2+\lambda\mu^2)} \\
&\qquad - \frac{2(p_2)^2(p_2k)^2}{(\lambda-3)(k^2+\lambda\mu^2)}\left(k^2 + \frac{(\lambda-1)\mu^2}{(p_2k)^2}\left((p_2)^2k^2 - \mu^2(\lambda-1)\right)\right)\Bigg] \\
&= \frac{i\gamma^4(\lambda-1)(p_1)^2}{2\pi\mu^6\lambda(\lambda+1)(\lambda-3)|p_2|}\int dk_y(p_2)^4 \overset{Dim.reg.}{=} 0
\end{aligned} \tag{134}
$$

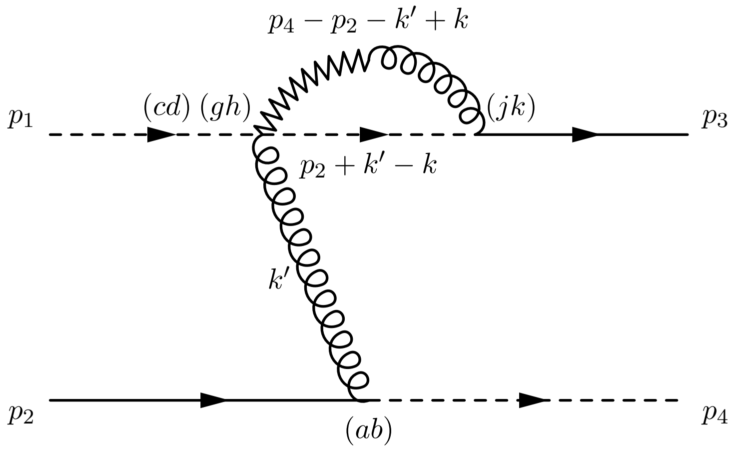


Figure 12: Scattering amplitude $i\mathcal{M}_6$

$$
\begin{aligned}
i\mathcal{M}_6 = &\int \frac{d^2k}{(2\pi)^2} \left(i\gamma p_a^1 p_b^3\right) 2i\mathcal{P}^{abcd}(k') \left(i\gamma^2 P_{cdgh} p_2^g (p_2 + k' - k)^h\right) 2i\mathcal{P}^{jk}(p_4 - p_2 + k - k') \\
&\left(\frac{-i}{(p_2 - k + k')^2 + \mu^2 - i\epsilon}\right) (i\gamma p_j^4 (p_2 - k + k')_k) \\
\approx\ &\frac{\gamma^4}{\pi^2} \int \frac{d^2k}{-2p_2k - i\epsilon} p_a^1 p_b^1 p_j^2 p_k^2 p_2^g p_2^h \left(\Delta^{abcd} + \mathcal{P}_{\mathcal{K}}(0) P^{ab}(0) P^{cd}(0)\right) P_{cdgh} \left(-\mathcal{P}_{\mathcal{K}}(k) P^{jk}(k)\right) \\
=\ &\frac{2\gamma^4}{\pi^2\mu^2(\lambda+1)} \int \frac{d^2k}{-2p_2k - i\epsilon} \left((p_1p_2)^2 - \frac{1}{2}(p_1)^2(p_2)^2\right) \frac{(\lambda-1)(p_2)^2 + \frac{2}{\mu^2}(kp_2)^2}{(k^2 + \mu^2\lambda)(\lambda-3)} \\
=\ &\frac{i\gamma^4}{\pi\mu^4\lambda(\lambda+1)(\lambda-3)|p_2|} \int dk_y (p_2)^2 \left((p_1p_2)^2 - \frac{1}{2}(p_1)^2(p_2)^2\right) \overset{Dim.reg.}{=} 0
\end{aligned}
\tag{135}
$$

C References

[1] A. Einstein and N. Rosen. On gravitational waves. *Journal of the Franklin Institute*, 223(1):43–54, 1937.

[2] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116(6):061102, 2016.

[3] Éanna É. Flanagan and Tanja Hinderer. Constraining neutron-star tidal love numbers with gravitational-wave detectors. *Phys. Rev. D*, 77:021502, Jan 2008.

[4] Frans Pretorius. Evolution of binary black-hole spacetimes. *Physical review letters*, 95(12):121101, 2005.

[5] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower. Accurate evolutions of orbiting black-hole binaries without excision. *Phys. Rev. Lett.*, 96:111101, Mar 2006.

[6] John G. Baker, Joan Centrella, Dae-Il Choi, Michael Koppitz, and James van Meter. Gravitational-wave extraction from an inspiraling configuration of merging black holes. *Phys. Rev. Lett.*, 96:111102, Mar 2006.

[7] Francois Foucart, Pablo Laguna, Geoffrey Lovelace, David Radice, and Helvi Witek. Snowmass2021 cosmic frontier white paper: Numerical relativity for next-generation gravitational-wave probes of fundamental physics, 2022.

[8] Yasushi Mino, Misao Sasaki, and Takahiro Tanaka. Gravitational radiation reaction to a particle motion. *Phys. Rev. D*, 55:3457–3476, Mar 1997.

[9] Theodore C. Quinn and Robert M. Wald. Axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved spacetime. *Phys. Rev. D*, 56:3381–3394, Sep 1997.

[10] A. Buonanno and T. Damour. Effective one-body approach to general relativistic two-body dynamics. *Phys. Rev. D*, 59:084006, Mar 1999.

[11] Alessandra Buonanno and Thibault Damour. Transition from inspiral to plunge in binary black hole coalescences. *Phys. Rev. D*, 62:064015, Aug 2000.

[12] A. Einstein, L. Infeld, and B. Hoffmann. The gravitational equations and the problem of motion. *Annals of Mathematics*, 39(1):65–100, 1938.

[13] A. Einstein and L. Infeld. The gravitational equations and the problem of motion. ii. *Annals of Mathematics*, 41(2):455–464, 1940.

[14] Tadayuki Ohta, Hiroshi Okamura, Toshiei Kimura, and Kichiro Hiida. Physically Acceptable Solution of Einstein's Equation for Many-Body System. *Progress of Theoretical Physics*, 50(2):492–514, 08 1973.

[15] Piotr Jaranowski and Gerhard Schäfer. Third post-newtonian higher order adm hamilton dynamics for two-body point-mass systems. *Phys. Rev. D*, 57:7274–7291, Jun 1998.

[16] Thibault Damour, Piotr Jaranowski, and Gerhard Schäfer. Dynamical invariants for general relativistic two-body systems at the third post-newtonian approximation. *Phys. Rev. D*, 62:044024, Jul 2000.

[17] Luc Blanchet and Guillaume Faye. On the equations of motion of point-particle binaries at the third post-newtonian order. *Physics Letters A*, 271(1):58–64, 2000.

[18] Thibault Damour, Piotr Jaranowski, and Gerhard Schäfer. Dimensional regularization of the gravitational interaction of point masses. *Physics Letters B*, 513(1):147–155, 2001.

[19] Thibault Damour, Piotr Jaranowski, and Gerhard Schäfer. Nonlocal-in-time action for the fourth post-newtonian conservative dynamics of two-body systems. *Phys. Rev. D*, 89:064058, Mar 2014.

[20] Piotr Jaranowski and Gerhard Schäfer. Derivation of local-in-time fourth post-newtonian adm hamiltonian for spinless compact binaries. *Phys. Rev. D*, 92:124043, Dec 2015.

[21] Thibault Damour. High-energy gravitational scattering and the general relativistic two-body problem. *Phys. Rev. D*, 97:044038, Feb 2018.

[22] Thibault Damour. Gravitational scattering, post-minkowskian approximation, and effective-one-body theory. *Phys. Rev. D*, 94:104015, Nov 2016.

[23] B. Bertotti. On gravitational motion. *Il Nuovo Cimento*, 4(4):898–906, October 1956.

[24] R. P. Kerr. The lorentz-covariant approximation method in general relativity. — i. *Il Nuovo Cimento*, 13(3):469–491, August 1959.

[25] B Bertotti and J Plebanski. Theory of gravitational perturbations in the fast motion approximation. *Annals of Physics*, 11(2):169–200, 1960.

[26] M Portilla. Momentum and angular momentum of two gravitating particles. *Journal of Physics A: Mathematical and General*, 12(7):1075–1090, jul 1979.

[27] K. Westpfahl and M. Goller. Gravitational scattering of two relativistic particles in post-linear approximation. *Lettere al Nuovo Cimento*, 26(17):573–576, December 1979.

[28] M Portilla. Scattering of two gravitating particles: Classical approach. *Journal of Physics A: Mathematical and General*, 13(12):3677–3683, dec 1980.

[29] Luis Bel, Thibaut Damour, Nathalie Deruelle, Jesus Ibanez, and Jesus Martin. Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity. *General Relativity and Gravitation*, 13(10):963–1004, October 1981.

[30] Konradin Westpfahl. High-speed scattering of charged and uncharged particles in general relativity. *Fortschritte der Physik/Progress of Physics*, 33(8):417–493, 1985.

[31] Henriette Elvang and Yu tin Huang. Scattering amplitudes, 2013.

[32] Nava Gaddam and Nico Groenenboom. Soft graviton exchange and the information paradox, 2020.

[33] Gerard 't Hooft. The Scattering matrix approach for the quantum black hole: An Overview. *Int. J. Mod. Phys. A*, 11:4623–4688, 1996.

[34] Gerard 't Hooft. Diagonalizing the Black Hole Information Retrieval Process. 9 2015.

[35] Gerard 't Hooft. Black hole unitarity and antipodal entanglement. *Found. Phys.*, 46(9):1185–1198, 2016.

[36] Panagiotis Betzios, Nava Gaddam, and Olga Papadoulaki. The Black Hole S-Matrix from Quantum Mechanics. *JHEP*, 11:131, 2016.

[37] Panagiotis Betzios, Nava Gaddam, and Olga Papadoulaki. Black holes, quantum chaos, and the Riemann hypothesis. *SciPost Phys. Core*, 4:032, 2021.

[38] Nava Gaddam, Nico Groenenboom, and Gerard 't Hooft. Quantum gravity on the black hole horizon. *JHEP*, 01:023, 2022.

[39] Panos Betzios, Nava Gaddam, and Olga Papadoulaki. Black hole S-matrix for a scalar field. *JHEP*, 07:017, 2021.

[40] Nava Gaddam and Nico Groenenboom. $2 \to 2N$ scattering: Eikonalisation and the Page curve. *JHEP*, 01:146, 2022.

[41] Tullio Regge and John A. Wheeler. Stability of a schwarzschild singularity. *Phys. Rev.*, 108:1063–1069, Nov 1957.

[42] Frank J. Zerilli. Gravitational field of a particle falling in a schwarzschild geometry analyzed in tensor harmonics. *Phys. Rev. D*, 2:2141–2160, Nov 1970.

[43] Karl Martel and Eric Poisson. Gravitational perturbations of the schwarzschild spacetime: A practical covariant and gauge-invariant formalism. *Phys. Rev. D*, 71:104003, May 2005.

[44] Tevian Dray and Gerard 't Hooft. The gravitational shock wave of a massless particle. *Nuclear Physics B*, 253:173–188, 1985.

[45] G. Dvali and C. Gomez. Black hole's quantum n-portrait. *Fortschritte der Physik*, 61(7-8):742–767, April 2013.

[46] G. Dvali, C. Gomez, R.S. Isermann, D. Lüst, and S. Stieberger. Black hole formation and classicalization in ultra-planckian $2 \to n$ scattering. *Nuclear Physics B*, 893:187–235, 2015.

[47] Herman Verlinde and Erik Verlinde. Scattering at planckian energies. *Nuclear Physics B*, 371(1):246–268, 1992.

[48] Frank J. Zerilli. Effective potential for even-parity regge-wheeler gravitational perturbation equations. *Phys. Rev. Lett.*, 24:737–738, Mar 1970.

[49] Alessandro Nagar and Luciano Rezzolla. Gauge-invariant non-spherical metric perturbations of schwarzschild black-hole spacetimes. *Classical and Quantum Gravity*, 22(16):R167–R192, jul 2005.

[50] George Leibbrandt. Introduction to the technique of dimensional regularization. *Reviews of Modern Physics*, 47(4):849, 1975.