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MSc Thesis

Stochastic Calculus for Fractional Brownian Motion

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Abstract

In this thesis we describe the Malliavin calculus based approach to stochastic integration with fractional Brownian motion (fBm). This approach is an extension of Itô calculus to non-semimartingales and was developed in two separate streams in the early 2000s. While other approaches exist, it is the most common way to interpret a stochastic integral due to the fact that it is centered, a property which other approaches do not share.

Next to the definition of the stochastic integral, the thesis contains an elaborate introduction into Malliavin calculus, the Skorohod integral, white noise theory and the properties of the fractional Brownian motion itself. Furthermore, we introduce a new numerical method to approximate fractional Brownian motions and linear SDEs. This method can also be applied to simulate paths of the regular Brownian motion.

Keywords: *Fractional Brownian Motion, Stochastic Integration, Malliavin Calculus, White Noise Theory*

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Notation

W_t	regular Brownian motion
W_t^H	fractional Brownian motion with Hurst-index H
s $\wedge t$	minimum of s and t
s $\vee t$	maximum of s and t
$\perp$	orthogonal on a Hilbert space
$\langle x, y\rangle_{\mathscr{H}}$	inner product on Hilbert space $\mathscr{H}$
$\langle x, x^*\rangle$	linear functional x^* evaluated at x
$\|x\|_p$	the L_p norm of x
$C(X)$	continuous functions on X
$C^\gamma_{\text{Hölder}}(X)$	hölder continuous functions with Hölder constant γ.
$C^p(X)$	p-times continuously differentiable functions on X, where $p \in \mathbb{N} \cup \infty$
$\partial_i f(x)$	partial derivative of f with respect to the i-th coordinate, evaluated at x
$\phi(x), \Phi(x)$	PDF/CDF for standard normal random variable
X^*	topological Dual space of X
$X_t \stackrel{d}{=} Y_t$	X_t and Y_t are equivalent in distribution
$\stackrel{\mathbb{P}}{\to}$	convergence in probability
$\stackrel{d}{\to}$	convergence in distribution
$\stackrel{a.s}{\to}$	almost sure convergence
$\stackrel{L^p}{\to}$	L^p convergence
$\langle X\rangle_t$	quadratic variation process
$X \sim N(\mu, \sigma^2)$	the process X follows a normal distribution with mean μ and variance σ^2
$\delta_{n,m}$	Dirac delta function

1 Introduction

Classical stochastic integration as introduced by Itô is a standard tool in stochastic analysis and is fundamental to modern mathematical finance. The Itô integral is well-defined for any semimartingale and thus for a large array of processes. The fractional Brownian motion (fBm) as introduced by Mandelbrot and van Ness [1], however, does not belong to the class of semimartingales. Itô's formulation is thus not well-defined for stochastic integrals with respect to a fractional Brownian motion.

The fractional Brownian motion is a generalization of the standard Brownian motion and a popular stochastic process to model additive white noise in weather patterns, electricity prices, certain financial assets and more [2],[3]. The process allows for positively or negatively auto-correlated increments, which have been found to be realistic for such problems. In order to use models based on fractional Brownian motions, a stochastic calculus framework is required. It is thus a relevant question to interpret the meaning of

$$\mathrm{d}X_t = Y_t \mathrm{d}W_t^H, \tag{1.1}$$

where Y_t is a suitable possibly stochastic process and W_t^H a fractional Brownian motion.

Over the years, different interpretations of a fractional stochastic integral as in Eq. (1.1) have been formed using different tools from stochastic analysis. One of the first approaches was by Lin [4] to consider the integral as a limit of Riemann sums

$$\int_0^t Y_s \mathrm{d}W_s^H = \lim_{|\Delta|\to 0} \sum Y_{t_{i-1}}(W_{t_i}^H - W_{t_{i-1}}^H).$$

While this approach was certainly intuitive, this integral does not satisfy

$$\mathbb{E}\left[\int_0^t Y_s \mathrm{d}W_s^H\right] = 0,$$

and is thus not centered. This revealed to be problematic for practical purposes, as the integral loses its meaning as "additive noise" [5]. Other notable approaches such as the path-wise integral [6] followed shorty after. However, none of these approaches managed to solve the problem with the expected value.

In an attempt to remedy the non-centered issue, Duncan et al. [7] introduced a similar idea to Lin's utilizing the properties of Wick calculus. While this approach was promising, it only worked for a Hurst-index greater than $\frac{1}{2}$. In subsequent years, Elliott, van der Hoek, Øksendal and Biagini [8], [9] expanded on Duncan's idea in several papers. They showed how the integral can be constructed for any H, using classical white noise theory by Hida and so-called fractional operators. This approach will be referred to as the Wick-Skorohod-Itô (WIS) integral in this thesis.

At about the same time, Decreusefond and Üstünel [10] introduced a fractional stochastic integral also using fractional operators. In contrast to the WIS integral, their approach is based on the divergence operator from classical Malliavin calculus and does not use white noise theory. This approach was further driven by Alòs, Mazet and Nualart [11], [12], [13] as well as Cheridito and Nualart [14], showing several interesting properties. This approach will be referred to as the divergence integral.

In current literature, the WIS integral and the divergence integral are most commonly used to interpret Eq. (1.1). While the two approaches seem to be different at first sight, it is known that the two approaches are equivalent [15]. The reason for the equivalence is that they are based on the same tools from stochastic analysis, namely Malliavin calculus. The approaches are thus referred to as the Malliavin calculus approaches.

In a recent paper, Xia et al. [16] introduced two new formulations, which they call the forward and symmetric Wick-Itô integrals. Their formulation is intuitive and similar to Duncan's formulation, but works for any H. Xia also showed that both these integrals coincide with the divergence integral and thus also fall within the class of Malliavin-calculus approaches.

The purpose of this thesis is threefold. Firstly, it serves as a complete and self-contained introduction to the Malliavin approaches of fractional stochastic integrals. We will introduce the necessary concepts from Malliavin calculus and white noise theory, which are crucial for the formulation of a fractional stochastic integral. We will then introduce the three Malliavin-based approaches (WIS, Divergence and Xia) and show the advantages and disadvantages of each approach.

The second purpose of this thesis is to show the equivalence of WIS integral and Xia integral. While this equivalence is implied by an equivalence of both approaches to the divergence integral, a direct proof of the equivalence has not yet been provided.

Thirdly, we will provide a new numerical method to simulate paths of regular Brownian motions, fractional Brownian motions as well as stochastic differential equations containing fractional stochastic integrals. The approximation method is based on the results of the WIS integral.

The thesis is structured in the following way: In Section 2, we will introduce the fractional Brownian motion and its properties. In Section 3, we will then give an introduction to both classical Malliavin calculus as well as white noise theory. This section will lay the foundation for Section 4, where we will discuss three different ways to formulate the fractional stochastic integral. In Section 5 we will consider the numerical aspects of (fractional) Brownian motions.

2 Fractional Brownian Motion

2.1 Definition and Basic Properties

The fractional Brownian motion is a stochastic process which generalizes the regular Brownian motion. Its properties depend on a parameter H, called the *Hurst*-index. Suppose that $(\mathbb{P}, \Omega, \mathcal{F})$ is a probability space.

Definition 2.1. *Let $H \in (0,1)$ be a constant and $W_t^H = W^H(t,\omega)\colon [0,\infty) \times \Omega \to \mathbb{R}$ be a continuous and centered Gaussian process, such that*

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.1}$$

We call W_t^H a fractional Brownian motion *(fBm) with* Hurst index *H.*

Note that if $H = \frac{1}{2}$, the covariance from Eq. (2.1) reduces to $t \wedge s$, which shows that in that case W_t^H is a standard Brownian motion. From the definition, we can derive the three important properties of the fractional Brownian motion, called similarity (i), stationary increments (ii) and time inversion (iii) .

Proposition 2.1. *Let W_t^H be a fBm. Then:*

i) For any $a > 0$, we have

$$(a^{-H}) W_{at}^H \stackrel{d}{=} W_t^H.$$

ii) For any $h > 0$, we have

$$W_{t+h}^H - W_h^H \stackrel{d}{=} W_t^H.$$

iii) We have

$$t^{2H} W^H_{\frac{1}{t}} \stackrel{d}{=} W^H_t.$$

Proof. As Gaussian processes are uniquely determined by their mean and covariance [17, Lemma 14.1], it suffices to show that the left-hand side is a centered Gaussian process with covariance as in Definition 2.1. In all cases it is trivial to see that the process is centered and Gaussian as W^H_t is centered and Gaussian. Furthermore, for i), we have

$$\mathbb{E}[(a^{-H})W^H_{at}(a^{-H})W^H_{as}] = a^{-2H}\mathbb{E}[W^H_{at}W^H_{as}] = \frac{a^{-2H}}{2}(|at|^{2H} + |as|^{2H}) - |at - as|^{2H}$$
$$= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

For ii), we have

$$\mathbb{E}[(W^H_{t+h} - W^H_h)(W^H_{s+h} - W^H_h)] = \mathbb{E}[W^H_{t+h}W^H_{s+h}] - \mathbb{E}[W^H_{t+h}W^H_h]$$
$$- \mathbb{E}[W^H_{s+h}W^H_h] + \mathbb{E}[(W^H_h)^2] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

Similarly, for iii),

$$\mathbb{E}[t^{2H}W^H_{\frac{1}{t}}s^{2H}W^H_{\frac{1}{s}}] = t^{2H}s^{2H}\mathbb{E}[W^H_{\frac{1}{t}}W^H_{\frac{1}{s}}] = \frac{t^{2H}s^{2H}}{2}(|\frac{1}{t}|^{2H} + |\frac{1}{s}|^{2H} - |\frac{1}{t} - \frac{1}{s}|^{2H}),$$

from which we derive

$$= \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H} = \frac{1}{2}(|t|^{2H} + |s|^{2H}) - |t-s|^{2H}).$$

This concludes the proof for all cases. □

The stationary increments property implies that the sequence

$$\{W^H_{n+1} - W^H_n : n \in \mathbb{N}\}$$

is stationary. Unlike for the regular Brownian motion, this sequences exhibits auto-correlation. If $H > \frac{1}{2}$, the fBm exhibits positive auto-correlation, and if $H < \frac{1}{2}$, we have negative auto-correlation. This can be seen by simple evaluation of the covariance of the increments.

Due to the difference in positive versus negative autocorrelation, one often has to distinguish the cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$. In fact, the process can behave very differently in the two cases. For instance, if $H > \frac{1}{2}$, the fBm exhibits so-called long-range dependence, meaning that

$$\sum_{n=1}^{\infty} \mathbb{E}\left[W^H_1\left(W^H_{n+1} - W^H_n\right)\right] = \infty,$$

which heuristically can be understood as that the increments significantly depend on the history of the process [15, Section 1.4].

It is commonly known that the fractional Brownian motion has so-called *stochastic representations*, meaning that one can express the process as a stochastic integral (see for instance [18, Section 2.3]).

2.2 Semimartingale Property

In order to verify the premise of this thesis, we want to confirm that the fBm is indeed not a semimartingale if $H \neq \frac{1}{2}$ and thus that Itô calculus is not sufficient. A common approach to show this is due to Rogers [19]. The idea is to consider the p-variations of the fractional Brownian motion

$$V_p(W^H;\Pi^n) := \sum_{k=1}^{n} |W^H_{t_k} - W^H_{t_{k-1}}|^p,$$

where Π^n is a finite partition of the interval $[0,\infty)$. The reason this quantity is relevant is the following: If W^H_t is a semimartingale, many properties about the convergence of $V_p(W^H_t;\Pi)$ as $|\Pi| \to 0$ are known. For instance, if $p=2$, it will converge uniformly on compacts in probability (UCP) to its quadratic variation process [20, Proposition 7.4].

Proposition 2.2. *For the p-variation of W^H_t, we have*

$$V_p(W^H;\Pi^n\cap[0,1]) = \sum_{k=1}^{n} |W^H_{t_k\wedge 1} - W^H_{t_{k-1}\wedge 1}|^p \xrightarrow{\mathbb{P}} \begin{cases} 0, & \text{if } pH>1, \\ \infty, & \text{if } pH<1. \end{cases}$$

Proof. Consider the expression

$$Y_{n,p} = 2^{n(pH-1)} \sum_{j=1}^{2^n} |W^H_{j2^{-n}} - W^H_{(j-1)2^{-n}}|^p,$$

which due to Proposition 2.1 i) is equal in distribution to

$$\tilde{Y}_{n,p} = 2^{-n} \sum_{j=1}^{2^n} |W^H_j - W^H_{j-1}|^p,$$

Now, note that by Proposition 2.1 ii), the $(W^H_k - W^H_{k-1})_{k\in\mathbb{N}}$ is an i.i.d sequence and thus ergodic. By the Ergodic Theorem [17, Theorem 25.6], we have

$$\tilde{Y}_{n,p} \to \mathbb{E}[|W^H_1 - W^H_0|^p] = \mathbb{E}[|W^H_1|^p]$$

in $L^2(\Omega)$. This also implies that

$$Y_{n,p} \to \mathbb{E}[|W^H_1|^p]$$

in distribution. However, by definition,

$$Y_{n,p} = 2^{n(pH-1)} V_p(W^H;\Pi^n\cap[0,1]).$$

This proves the proposition. □

The fact that the fBm is not a semimartingale follows directly from this proposition.

Theorem 2.1. *The fBm W^H_t is not a semimartingale for any $H \neq \frac{1}{2}$.*

Proof. Suppose that W^H_t is a semimartingale. Then, we have

$$\sup_{t\le 1} |V_2(W^H;\Pi^n\cap[0,t]) - \langle W^H\rangle_t| \xrightarrow{\mathbb{P}} 0,$$

by the UCP convergence. Suppose that $H < \frac{1}{2}$. Then,

$$|V_2(W^H;\Pi^n\cap[0,1]) - \langle W^H\rangle_1| \xrightarrow{\mathbb{P}} 0.$$

However, since $V_2(W^H;\Pi^n\cap[0,1]) \xrightarrow{\mathbb{P}} \infty$ by Proposition 2.2 this is a direct contradiction since $\langle W^H\rangle_1 < \infty$ almost surely.

Suppose now that $H > \frac{1}{2}$. With the same reasoning, we have $\langle W^H \rangle_t = 0$ almost surely and thus W_t^H has bounded variation. This contradicts the fact that $V_1(W^H; \Pi^n \cap [0,1]) \xrightarrow{\mathbb{P}} \infty$, which is given by Proposition 2.2.

We conclude that, if $H < \frac{1}{2}$ of $H > \frac{1}{2}$, the process W_t^H is not a semimartingale. $\square$

It turns out that the semimartingale property is not the only property that is lost for fractional Brownian motions. In a similar fashion, one can show that the fBm is not a Markovian process if $H \neq \frac{1}{2}$ [18, Theorem 2.3]. This is an indication for the difficulties which can occur when working with the fractional Brownian motion.

2.3 Applications of Fractional Brownian Motion

The intrinsic properties of fractional Brownian motions, such as autocorrelation and long-range dependence make it suitable for the modelling of various naturally occurring random problems. Here, we list a few examples of such models.

In financial modelling, the famous Black-Scholes model uses a geometric Brownian motion to model stock returns. However, critics of this model argue that a process with long-range dependence is more appropriate to model stock returns [21],[22]. An extension of the Black-Scholes model to allow for fBm is thus considered a logical next step. However, various papers have already shown that such an extension is not trivially possible without violating a fundamental principal of financial markets. Rogers [19] showed that due to the fact that the fBm is not a semimartingale, theoretical arbitrage exists and thus violates the no-arbitrage hypothesis. Several authors tried to circumvent this issue by restricting the set of admissible trading strategies, under the premise that such strategies are not realistic due to transaction costs, et cetera. Hu and Øksendal [23] showed that an arbitrage-free fractional Black-Scholes model can exist if one restricts the admissible strategies to so-called *WIS-admissible* strategies. However, it turned out that such a restriction has no economic interpretation and is not realistic, as simple easy-to-implement strategies were deemed not admissible. A fractional Black-Scholes model which does not violate the principles of financial markets remains to be found. A more extensive discussion on this topic can be found in [5].

Next to a Black-Scholes extension, researchers saw the potential of the fractional Brownian motion in other financial applications. A promising, relatively new development in financial modelling are so-called *rough volatility models*. A rough volatility model is a specific case of a stochastic volatility model, where the volatility process is driven by a rough fractional Brownian motion ($H < \frac{1}{2}$). An often applied model is the so-called rough Bergomi model, where the volatility process is given by

$$\sigma_t^2 = \exp Z_t,$$

where Z_t is a fractional Ornstein-Uhlenbeck process [24]. It has been shown empirically that rough volatility models are remarkably close to reality [25]. Due to the fact that the price process is still driven by a regular Brownian motion, the models do not violate the non-arbitrage condition.

We have seen that the no-arbitrage condition causes problems in modelling stock prices with fBm and thus cannot be directly applied. There are, however, financial derivatives whose underlying does not need to satisfy the no-arbitrage condition, due to the fact that it is not a tradable asset. A prime example are weather derivatives, where the underlying depends on future weather patterns, such as amount of sunny days versus rainy days. Furthermore, the weather is suitable to be modelled by a fractional Brownian motion, due to its obvious autocorrelation. The most notable work in pricing weather derivative is a paper by Benth [2], who applies Øksendal's approach to fractional stochastic calculus to price weather derivatives.

The fractional Brownian motion is not only applicable to financial problems. As Mandelbrot and van Ness [1] suggest, thickness of tree rings can be modelled by a fBm. In fact, the earliest ideas for a fractional Brownian motion developed when Hurst attempted to model the flow of water through the Nile [26]. We see that applications to the fractional Brownian motion are versatile.

In this section we have seen that the fractional Brownian motion is a centered Gaussian process, which is characterised by the covariance function Eq. (2.1). Next to a few properties, we have seen that the fBm is not a semimartingale. In the last part we showed a few examples of practical applications of the fBm.

3 Stochastic Integration with Malliavin Calculus

We will now introduce and discuss the necessary background in Malliavin calculus and white noise theory needed to construct the Malliavin-based approaches. The theory of this chapter is more general and is not only applicable to fractional Brownian motions.

The main part of this chapter is based on Malliavin calculus, which is the idea to define a stochastic derivative operator on a Gaussian probability space. Next to such a derivative, one obtains an operator called the *Skorohod* integral. This operator, as we will see, can be derived in various ways. As the name suggests, it is a stochastic integral and it generalizes Itô's integral to a larger domain of integrands.

We start the section by introducing the classic definitions of the Malliavin derivative as described in [13]. From this, we obtain our first definition of the Skorohod integral as the adjoint of the Malliavin derivative. Afterwards, we consider iterated stochastic integrals, which give us a more intuitive definition of the Skorohod integral.

Lastly, we introduce classical white noise theory as described in [27]. From this framework, we obtain the Skorohod integral from the Pettis integral on the Hida space and the white noise process.

3.1 Isonormal Gaussian Processes

We begin with the notion of isonormal Gaussian processes, which form the foundation of classical Malliavin calculus.

Definition 3.1. *Let $\mathscr{H}$ be a real-valued separable Hilbert space and let $W\colon \mathscr{H} \to L^2(\Omega)$ be a mapping onto the space of square integrable functions for some probability space $(\mathbb{P}, \Omega, \mathcal{F})$. We call the family of random variables $\{W(h) : h \in \mathscr{H}\}$* an $\mathscr{H}$-isonormal Gaussian process *if*

i) $W(h)$ is a Gaussian random variable for all $h \in \mathscr{H}$,

ii) $\mathbb{E}[W(h)W(g)] = \langle h, g\rangle_{\mathscr{H}}$ for all $h, g \in \mathscr{H}$.

Note that the second property implies that the map W is an isometry between $\mathscr{H}$ and $L^2(\Omega)$. For simplicity, we will mostly drop the "Gaussian" and simply call the process W an $\mathscr{H}$-isonormal process, or isonormal process if it is clear which Hilbert space is meant. The following proposition shows that such maps can be found for suitable Hilbert space, and that $W(h)$ constitutes a Gaussian process in the usual sense.

Proposition 3.1. *Let $\mathscr{H}$ be any real-valued separable Hilbert space. Then, there is a map W, such that $\{W(h) : h \in \mathscr{H}\}$ is an isonormal Gaussian process. Furthermore, the process $W(h)$ is a Gaussian stochastic process.*

Proof. We will first show the existence of the map W. Denote $(h_k)_{k\in\mathbb{N}}$ as an orthonormal basis of $\mathscr{H}$ and let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables. Then, the map $W\colon \mathscr{H} \to L^2(\Omega)$ given by

$$h \mapsto W(h) = \sum_{k=1}^{\infty} \gamma_k \langle h, h_k\rangle_{\mathscr{H}}$$

is well-defined by Bessel's inequality and the fact that, for all k, we have that

$$\gamma_k \langle h, h_k\rangle_{\mathscr{H}} \sim N(0, \langle h, h_k\rangle^2_{\mathscr{H}}).$$

It follows directly that $W(h)$ is a Gaussian random variable for all $h \in \mathscr{H}$. It remains to show the second property of Definition 3.1. We have that

$$\mathbb{E}[\sum_{k=1}^{\infty} \gamma_k \langle h, h_k\rangle_{\mathscr{H}} \sum_{l=1}^{\infty} \gamma_l \langle g, h_l\rangle_{\mathscr{H}}] = \sum_{k,l=1}^{\infty} \mathbb{E}[\gamma_k\gamma_l \langle h, h_k\rangle_{\mathscr{H}} \langle g, h_l\rangle_{\mathscr{H}}] = \sum_{k}^{\infty} \langle h, h_k\rangle_{\mathscr{H}} \langle g, h_k\rangle_{\mathscr{H}},$$

since $\mathbb{E}[\gamma_k\gamma_l] = \delta_{k,l}$. By [28, Theorem 4.13], it follows that

$$\sum_{k}^{\infty} \langle h, h_k\rangle_{\mathscr{H}} \langle g, h_k\rangle_{\mathscr{H}} = \langle h, g\rangle_{\mathscr{H}},$$

and thus that W is indeed an isonormal process.

To show that $W(h)$ is a Gaussian process, we will first show that it is a linear map. Let $\mu, \lambda \in \mathbb{R}$ and see that

$$\begin{aligned}
&\mathbb{E}[\big(W(\lambda h + \mu g) - \lambda W(h) - \mu W(g)\big)^2] \\
&= \mathbb{E}[W(\lambda h + \mu g)^2 + \lambda^2 W(h)^2 + \mu^2 W(g)^2 \\
&- 2\lambda W(\lambda h + \mu g)W(h) - 2\mu W(\lambda h + \mu g)W(g) + \lambda\mu W(h)W(g)].
\end{aligned}$$

Using the isometry property, we find

$$\begin{aligned}
&= \langle \lambda h + \mu g, \lambda h + \mu g\rangle_{\mathscr{H}} + \lambda^2 \langle h, h\rangle_{\mathscr{H}} + \mu^2 \langle g, g\rangle_{\mathscr{H}} - 2\lambda \langle \lambda h + \mu g, h\rangle_{\mathscr{H}} \\
&- 2\mu \langle \lambda h + \mu g, g\rangle_{\mathscr{H}} + \lambda\mu \langle h, g\rangle_{\mathscr{H}} \\
&= 2\langle \lambda h + \mu g, \lambda h + \mu g\rangle_{\mathscr{H}} - 2\langle \lambda h + \mu g, \lambda h + \mu g\rangle_{\mathscr{H}} = 0.
\end{aligned}$$

Clearly, the expression $W(\lambda h + \mu g) - \lambda W(h) - \mu W(g)$ is the zero element in $L^2(\Omega)$. This shows that W is a linear map.

The fact that $\big(W(h)\big)_{h\in\mathscr{H}}$ is a Gaussian process follows directly by the linearity. Indeed, for any $h_1, h_2 \dots, h_n \in \mathscr{H}$ and $\alpha_1, \alpha_2 \dots \alpha_n \in \mathbb{R}$, we have

$$\sum_{k=1}^{n} \alpha_k W(h_k) = W(\sum_{k=1}^{n} \alpha_k h_k),$$

which is a Gaussian random variable by the definition of an isonormal process. □

A classic example of an isonormal process is the so-called Wiener integral as an isonormal process of square-integrable functions over a finite interval.

Example 3.1. Fix $T > 0$ and consider the set

$$\mathcal{E} = \mathrm{Span}\big(\{I_{[0,t]} : t \in [0, T]\}\big),$$

which is the linear space of indicator functions on [0,T]. We define the inner product for any $s, t \in [0,T]$ as

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{E}} = s \wedge t.$$

The closure of $\mathcal{E}$ with respect to this inner product is a Hilbert space. In fact, this Hilbert space is equal to $L^2(\gamma, [0,T], \mathcal{B}(0,T))$, where γ is the Lebesgue measure and $\mathcal{B}(0,T)$ the Borel σ-algebra of $[0,T]$. We can thus find an isonormal process W, such that $W(I_{[0,t]})$ is a Gaussian random variable for all t and

$$\mathbb{E}[W(I_{[0,t]})^2] = t \wedge t = t.$$

Furthermore, for any $0 \leq t_1 < t_2 \leq t_3 < t_4 \leq T$, we have that

$$\begin{aligned}
&\mathbb{E}\left[(W(I_{[0,t_2]}) - W(I_{[0,t_1]}))(W(I_{[0,t_4]}) - W(I_{[0,t_3]}))\right] \\
&= \mathbb{E}[W(I_{[0,t_2]})W(I_{[0,t_4]}) + W(I_{[0,t_1]})W(I_{[0,t_3]}) \\
&- W(I_{[0,t_2]})W(I_{[0,t_3]}) - W(I_{[0,t_1]})W(I_{[0,t_4]})] \\
&= (t_2 \wedge t_4) + (t_1 \wedge t_3) - (t_2 \wedge t_3) - (t_1 \wedge t_4) \\
&= 0.
\end{aligned}$$

The process $W_t := W(I_{[0,t]})$ therefore satisfies almost all the properties of a Brownian motion, only missing pathwise continuity. However, by Kolmogorov's continuity theorem [17, Theorem 4.23], we can find a continuous modification of W_t. This shows that W_t is a Brownian motion. The random variable

$$W(h) \in L^2(\Omega)$$

is called the *Wiener integral* of $h \in L^2([0,T])$. △

3.2 The Malliavin Derivative and its Adjoint

It is well-known that paths of stochastic processes, such as the Brownian motion, are not differentiable in the classical sense. However, with the help of isonormal processes, it is possible to define such a derivative operator in a generalized sense. This is called the *Malliavin derivative*, which we will introduce here.

Throughout Section 3.2, we assume that W is an $\mathscr{H}$-isonormal Gaussian process on a probability space $(\mathbb{P}, \Omega, \mathcal{F})$ for a real-valued separable Hilbert space $\mathscr{H}$.

3.2.1 Smooth Random Variables

The definition of a classical derivative for a function requires the function to be sufficiently smooth. Let us first define different subsets of such smooth functions. We denote the set of completely smooth functions, i.e. the set of all infinitely continuously differentiable, by $C^\infty(\mathbb{R}^n)$. Furthermore, we can define the subset $C_b^\infty(\mathbb{R}^n)$ as the set of all smooth functions which are bounded and for which all partial derivatives are bounded.

$$C_b^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \exists M : |f(x)| \leq M, |\partial_i f(x)| \leq M \text{ for all } x \in \mathbb{R}^n, i \leq n\}$$

One can now define "smooth" random variables with the following idea.

Definition 3.2. *Let $h_1, h_2, \ldots, h_n \in \mathscr{H}$ and let $f \in C^\infty(\mathbb{R}^n)$. The random variable X of the form*

$$X = f(W(h_1), W(h_2), \ldots, W(h_n)) \tag{3.1}$$

is smooth *in the sense that*

$$\partial_i f(W(h_1), W(h_2), \ldots, W(h_n))$$

exists. We denote by S the set of all random variables of this form.

$$S = \{f(W(h_1), W(h_2), \ldots, W(h_n)) : f \in C^\infty(\mathbb{R}^n), h_1, h_2, \ldots, h_n \in \mathscr{H}, n \in \mathbb{N}\}$$

Furthermore, we define S_b as the random variables of the form Eq. (3.1)*, where we restrict f to $C_b^\infty(\mathbb{R}^n)$.*

Remark 3.1. Note that since bounded functions are square-integrable, we can consider S_b as a linear subspace of $L^2(\Omega)$. Notice also that the representation of X in S and S_b is by no means unique. In fact, using that W is a linear map we can apply a Gram-Schmidt procedure to find orthonomal elements $e_1, e_2, \ldots, e_n$, with

$$X = f(W(h_1), W(h_2), \ldots, W(h_n)) = g(W(e_1), W(e_2), \ldots, W(e_n)),$$

for some map $g \in C^\infty(\mathbb{R}^n)$.

Example 3.2. Suppose that $W(h)$ is the isonormal process from Example 3.1 with $W_t = W(I_{[0,t]})$. The function $f(x_1, x_2) = \sin(x_1)\cos(x_2)$ is in C_b^∞ and thus the expression

$$f(W_t, W_s) = \sin(W_t)\cos(W_s)$$

is a smooth random variable in S_b. △

Since the functions $f \in C_b(\mathbb{R})$ are completely smooth, they have a well-defined derivative everywhere. This enables us to define a derivative for smooth random variables.

Definition 3.3. *Let $X \in S_b$ be a random variable of the form*

$$X = f(W(h_1), W(h_2), \ldots, W(h_n)).$$

We define the Malliavin derivative *as the operator $D\colon S_b \to L^2(\Omega; \mathscr{H})$, such that*

$$D(X) = \sum_{i=1}^{n} \partial_i f\left(W(h_1), W(h_2), \ldots, W(h_n)\right)\ h_i.$$

Remark 3.2. Since we choose f as well as the partial derivatives to be bounded, the expression $D(X)$ is certainly square-integrable as $W(h_i)$ are square-integrable random variables.

Example 3.3. The smooth random variable

$$f(W_t, W_s) = \sin(W_t)\cos(W_s)$$

has Malliavin derivative

$$D[f(W_t, W_s)] = \cos(W_t)\cos(W_s)I_{[0,t]} - \sin(W_t)\sin(W_s)I_{[0,s]} \in L^2(\Omega; \mathscr{H}).$$

△

Remember from Example 3.1 that we considered $W(h)$ as the "Wiener integral" of h. The Malliavin derivative then, heuristically, coincides with the chain rule, if we consider h_i to be the derivative of $W(h_i)$. We now see that this coincides with what we expect from ordinary Itô calculus.

Example 3.4. We can now consider the Malliavin derivative of the Brownian motion. We have $f(x) = x$ and thus

$$DW(I_{[0,t]}) = f'(W(I_{[0,t]})) \cdot I_{[0,t]} = I_{[0,t]}.$$

We see that h_i is indeed the derivative of $W(h_i)$ in the Malliavin sense. Furthermore, this agrees with what we expect from classical Itô calculus, as we have

$$W_t = \int_0^T I_{[0,t]}(s)\mathrm{d}W_s,$$

for an Itô integral with respect to the Brownian motion W(t). △

Remark 3.3. The Malliavin derivative DX of a random variable X is an $\mathscr{H}$-valued random variable. This means that

$$\langle DX, h\rangle_{\mathscr{H}} \in L^2(\Omega)$$

is well-defined as a random variable. Due to the definition of a derivative as a limit for regular smooth function, it follows that for any $h \in \mathscr{H}$, this expression can be interpreted as a *directional derivative*. We have that

$$\begin{aligned}\langle DF, h\rangle_{\mathscr{H}} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[f\left(W(h_1)+\epsilon\langle h_1, h\rangle_{\mathscr{H}}, \dots, W(h_n)+\epsilon\langle h_n, h\rangle_{\mathscr{H}}\right)\\ - f\left(W(h_1),\dots,W(h_n)\right)].\end{aligned} \tag{3.2}$$

The Malliavin derivative is therefore to be interpreted as the change in the random variable X due to a change in randomness, which is induced by h through $W(h)$.

Just as in ordinary calculus, the Malliavin derivative has many of the standard properties. For instance, the Malliavin product rule follows directly from the ordinary product rule and the definition of the Malliavin derivative. Hence, for any two smooth random variables X, Y, we have

$$D(XY) = D(X)Y + D(Y)X.$$

Similarly, a stochastic integration-by-parts formula exists if we again consider $W(h)$ to be the integral of h. To proof it, we first need to prove the following relation:

Proposition 3.2. *Let X be a smooth random variable. Then, we have that*

$$\mathbb{E}[\langle DX, h\rangle_{\mathscr{H}}] = \mathbb{E}[XW(h)],$$

for all $h \in \mathscr{H}$.

The proof is based on the classic integration-by-parts formula.

Proof. Let $h \in \mathscr{H}$. Since W is a linear operator, we may assume WLOG that $\|h\| = 1$. We now construct a function $g \in C_b^\infty(\mathbb{R}^n)$ and orthonormal elements $e_1, e_2, \dots, e_n$, such that

$$X = g(W(e_1), W(e_2), \dots, W(e_n)),$$

where $e_1 = h$. The random variables $W(e_1), W(e_2), \dots$ follow a standard normal distribution and are jointly Gaussian. Therefore,

$$\begin{aligned}\mathbb{E}[\langle DX, h\rangle_{\mathscr{H}}] &= \int_\Omega \partial_1 g(W(e_1), W(e_2), \dots, W(e_n))\\ &= \int_{\mathbb{R}^n} \partial_1 g(x)\phi(x)\mathrm{d}x = -\int_{\mathbb{R}^n} g(x)\partial_1\phi(x)\mathrm{d}x,\end{aligned}$$

where $\phi(x)$ is the Gaussian kernel. Note that we've used that $\phi(x)$ vanishes at infinity. Furthermore,

$$-\int_{\mathbb{R}^n} g(x)\partial_1\phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} x_1 g(x)\phi(x)\mathrm{d}x = E[W(e_1)g(W(e_1), W(e_2), \dots, W(e_n))],$$

which confirms the identity. □

This proposition is important because it tells us that the expectation in the integrated state (i.e. X and $W(h)$) is the same as in the "derivative state" DX and h. The integration-by-parts formula follows from this property.

Proposition 3.3. *Let $X, Y \in S_b$ and $h \in \mathscr{H}$. Then,*

$$\mathbb{E}[Y\langle DX, h\rangle_{\mathscr{H}}] = \mathbb{E}[XYW(h) - X\langle DY, h\rangle_{\mathscr{H}}].$$

Proof. We may assume that

$$X = f(W(h_1), W(h_2), \ldots, W(h_n)),$$

and also

$$Y = g(W(h_1), W(h_2), \ldots, W(h_n)).$$

Note that since f and g are bounded, so is fg. From the regular product rule for real-valued functions, it follows that

$$D(XY) = D(X)Y + D(Y)X,$$

and thus from the previous proposition, it follows that

$$\mathbb{E}[XYW(h)] = \mathbb{E}[\langle D(XY), h\rangle_{\mathscr{H}}] = \mathbb{E}[X\langle D(Y), h\rangle_{\mathscr{H}}] + \mathbb{E}[Y\langle D(X), h\rangle_{\mathscr{H}}].$$

□

In classical multivariable calculus, derivative operators are usually unbounded. The same is true for the Malliavin derivative operator D. However, we can show that it is at least closable, meaning that we can extend it to a closed operator.

Proposition 3.4. *The derivative operator is closable.*

Proof. To show that it is closable, it suffices to show that if $x_n \to 0$ and $Dx_n \to y$, then $y = 0$. Let $G \in S_b$ be any random variable. Then,

$$\mathbb{E}[\langle y, h\rangle_{\mathscr{H}} G] = \mathbb{E}[\langle \lim_{x_n \to 0} Dx_n, h\rangle_{\mathscr{H}} G].$$

□

We can thus expand the Malliavin derivative from Definition 3.3 to a closed operator which is defined on a larger space. From now on, we define the operator D as the closed operator on $\mathbb{D}^1$, which we define as the closure of S_b in $L^2(\Omega; \mathscr{H})$ with respect to the norm

$$\|X\|_{\mathbb{D}^1} := \sqrt{\mathbb{E}[X^2] + \mathbb{E}[\|DX\|^2_{\mathscr{H}}]}.$$

A further question to consider is the existence of higher order derivatives. This requires a slight generalization of the Malliavin derivative, as the image DX of the derivative operator is in $L^2(\Omega, \mathscr{H})$ rather than $L^2(\Omega, \mathbb{R})$. The following example shows the generalization, where the second-order derivative operator exists as a map into $L^2(\Omega, \mathscr{H} \otimes \mathscr{H})$.

Example 3.5. Suppose that $X = W(h)^3$ is a smooth random variable, where $h \in \mathscr{H}$. Then, the first Malliavin derivative of X is given by

$$DX = 3W(h)^2\, h \in L^2(\Omega; \mathscr{H}).$$

For the second Malliavin derivative, we write

$$D^2(X) = 6W(h)(h, h) \in L^2(\Omega; \mathscr{H} \otimes \mathscr{H}),$$

where $(h, g) \in \mathscr{H} \otimes \mathscr{H}$ for some $h, g \in \mathscr{H}$. △

3.2.2 The Divergence Operator

While the study of derivatives of random variables is fruitful in general stochastic analysis, we are mainly interested in the adjoint operator of the Malliavin derivative, which is known as the *divergence operator*.

Theorem 3.1. *Let $X \in L^2(\Omega; \mathscr{H})$ be an $\mathscr{H}$-valued random variable, such that for all $F \in \mathbb{D}^1$, we have that*

$$|\mathbb{E}[\langle DF, X\rangle_{\mathscr{H}}]| \le c\|F\|_2, \tag{3.3}$$

where c is some constant $c \ge 0$. Then, there exists a unique random variable $\delta(X) \in L^2(\Omega)$, such that

$$\mathbb{E}[F\delta(X)] = \mathbb{E}[\langle DF, X\rangle_{\mathscr{H}}],$$

for all $F \in \mathbb{D}^1$.

Proof. In that case, the map $L\colon \mathbb{D}^1 \to \mathbb{R}$ given by

$$L(F) = \mathbb{E}[\langle DF, X\rangle_{\mathscr{H}}]$$

is a bounded linear functional on the subspace $\mathbb{D}^1$ of $L^2(\Omega)$. Using the Hahn-Banach theorem, we can extend the linear functional to all of $L^2(\Omega)$ and by the Riesz representation theorem, there exists a unique element $\delta(X) \in L^2(\Omega)$, such that

$$L(F) = \langle F, \delta(X)\rangle = \mathbb{E}[F\delta(X)].$$

□

Definition 3.4. *We define the* divergence operator *as the operator*

$$\delta\colon \mathit{Dom}\,\delta \to L^2(\Omega),$$

which is the map $X \mapsto \delta(X)$ in the sense of Theorem 3.1 and $\mathit{Dom}\,\delta$ is given by the square-integrable random variables such that Eq. (3.3) *holds.*

The random variable $\delta(X)$ is often called the *Skorohod integral* of X. As we will later see, the operator δ can indeed be interpreted as an integral operator.

Example 3.6. By Proposition 3.2, we have that if $X = h$ is a deterministic element in $\mathscr{H}$, we have that

$$\delta(h) = W(h).$$

Now suppose that

$$X = \sum_{k=1}^{n} X_k h_k,$$

where $X_k \in S_b$ for all k. Then, for any F, we have

$$\mathbb{E}[\langle DF, \sum_{k=1}^{n} X_k h_k\rangle_{\mathscr{H}}] = \sum_{k=1}^{n} \mathbb{E}[X_k \langle DF, h_k\rangle_{\mathscr{H}}].$$

We can thus apply Proposition 3.3 to obtain

$$\mathbb{E}[\langle DF, X\rangle_{\mathscr{H}}] = \mathbb{E}\left[F \sum_{k=1}^{n} X_k W(h_k) - \langle DX_k, h_k\rangle_{\mathscr{H}}\right],$$

and therefore

$$\delta(X) = \sum_{k=1}^{n} X_k W(h_k) - \langle DX_k, h_k\rangle_{\mathscr{H}}. \tag{3.4}$$

△

An interesting question is to consider what would happen when composing the derivative and the integral operator. Note that since $\mathrm{ran}(\delta) \subset L^2(\Omega)$ is not necessarily contained in $\mathbb{D}^1$, we cannot freely compose the operators. However, for sufficiently regular elements, there exists a handy relation showing the "commutation relation", which follows directly from the result in the previous example.

Proposition 3.5. *Let $h \in \mathscr{H}$ and let X be of the form $X = \sum_{k=1}^n X_k h_k$ where for all k we have $X_k \in S_b$. Then,*

$$\langle D\delta(X), h\rangle_{\mathscr{H}} = \langle X, h\rangle_{\mathscr{H}} + \delta(\langle DX, h\rangle_{\mathscr{H}}).$$

Proof. We remember that the product rule is also true for Malliavin derivatives. From Eq. (3.4), we have that

$$\delta(X) = \sum_{k=1}^n X_k W(h_k) - \langle DX_k, h_k\rangle_{\mathscr{H}},$$

and thus

$$\langle D\delta(X), h\rangle_{\mathscr{H}} = \sum_{k=1}^n \langle X_k h_k, h\rangle_{\mathscr{H}} + \langle D(X_k)W(h), h\rangle_{\mathscr{H}} - \langle D(\langle DX_k, h_k\rangle_{\mathscr{H}}), h\rangle_{\mathscr{H}}.$$

The first term on the right-hand side is simply $\langle X, h\rangle_{\mathscr{H}}$. It thus suffices to prove that

$$\sum_{k=1}^n \langle D(X_k)W(h), h\rangle_{\mathscr{H}} - \langle D(\langle DX_k, h_k\rangle_{\mathscr{H}}), h\rangle_{\mathscr{H}} = \delta(\langle DX, h\rangle_{\mathscr{H}}).$$

We first see that

$$\langle DX, h\rangle_{\mathscr{H}} = \sum_{k=1}^n \langle D(X_k h_k), h\rangle_{\mathscr{H}} = \sum_{k=1}^n \langle D(X_k), h\rangle_{\mathscr{H}} h_k \in L^2(\Omega, \mathscr{H}).$$

According to Eq. (3.4) again, we find

$$\delta\left(\sum_{k=1}^n \langle D(X_k), h\rangle_{\mathscr{H}} h_k\right) = \sum_{k=1}^n \langle D(X_k), h\rangle_{\mathscr{H}} W(h_k) - \langle D(\langle DX_k, h\rangle_{\mathscr{H}}), h_k\rangle_{\mathscr{H}}.$$

It remains to show that

$$\langle D(\langle DX_k, h\rangle_{\mathscr{H}}), h_k\rangle_{\mathscr{H}} = \langle D(\langle DX_k, h_k\rangle_{\mathscr{H}}), h\rangle_{\mathscr{H}}.$$

By the definition of a smooth random variable as well as Remark 3.1, we can now find an orthonormal set $\{e_1, e_2, \ldots, e_N\}$ such that $X_k = f_k(W(e_1), W(e_2), \ldots, W(e_N))$ for some smooth functions f_k.

$$
\begin{aligned}
\langle D(\langle DX_k, h\rangle_{\mathscr{H}}), h_k\rangle_{\mathscr{H}} &= \sum_{i=1}^{N} \langle D(\partial_i f_k \langle e_i, h\rangle_{\mathscr{H}}), h_k\rangle_{\mathscr{H}} \\
&= \sum_{i=1}^{N} \langle e_i, h\rangle_{\mathscr{H}} \langle D(\partial_i f_k), h_k\rangle_{\mathscr{H}} \\
&= \sum_{i,j=1}^{N} \langle e_i, h\rangle_{\mathscr{H}} \langle h_k, e_j\rangle_{\mathscr{H}} \langle D(\partial_i f_k), e_j\rangle_{\mathscr{H}} \\
&= \sum_{i,j=1}^{N} \langle e_i, h\rangle_{\mathscr{H}} \langle h_k, e_j\rangle_{\mathscr{H}} \langle D(\partial_j f_k), e_i\rangle_{\mathscr{H}} \\
&= \sum_{j=1}^{N} \langle h_k, e_j\rangle_{\mathscr{H}} \langle D(\partial_j f_k), h\rangle_{\mathscr{H}} \\
&= \langle D(\langle DX_k, h_k\rangle_{\mathscr{H}}), h\rangle_{\mathscr{H}},
\end{aligned}
$$

where we have used the expansion of h as well as the symmetry of partial derivatives. This completes the proof. □

3.3 Malliavin Calculus with Iterated Integrals

The derivative operator and its adjoint are the cornerstones of Malliavin calculus and its applications. We will now consider a specific example of an isonormal process and analyse the derivative/divergence operators in terms of so-called iterated Itô integrals. We will thus define these operators using the regular tools from Itô calculus. This will allow us to see why the divergence operator is an "integral". Throughout this section, we will work on a complete probability space $(\mathbb{P}, \Omega, \mathcal{F})$ with $W_t = W(t, \omega), t \geq 0$ being a classical Brownian motion.

3.3.1 The iterated Itô integral

We start by introducing a special subset of square-integrable functions. Let $T > 0$ be finite and let $L^2\big([0,T]^n\big)$ be the space of real-valued square-integrable functions on $[0,T]^n$, i.e. $f\colon [0,T]^n \to \mathbb{R}$, such that

$$
\|f\|^2_{L^2([0,T]^n)} := \int_{[0,T]^n} \big(f(t_1, t_2, \dots, t_n)\big)^2 \mathrm{d}t_1 \mathrm{d}t_2 \dots \mathrm{d}t_n < \infty.
$$

Definition 3.5. *We call a function $f\colon [0,T]^n \to \mathbb{R}$* symmetric, *if*

$$
f(t_1, t_2, \dots, t_n) = f(t_{\sigma_1}, t_{\sigma_2}, \dots, t_{\sigma_n}),
$$

for any permutation $(\sigma_1, \sigma_2, \dots, \sigma_n)$ of $(1, 2 \dots, n)$.

We denote the set of all symmetric square-integrable functions as $\tilde{L}^2\big([0,T]^n\big)$. For any non-symmetric function $f \in L^2\big([0,T]^n\big)$, we can define a *symmetrization* $\tilde{f} \in \tilde{L}^2\big([0,T]^n\big)$ which is given by

$$
\tilde{f} = \frac{1}{n!} \sum_{\sigma \in \Sigma} f(t_{\sigma_1}, t_{\sigma_2}, \dots, t_{\sigma_n}).
$$

Example 3.7. Let us consider some trivial examples of symmetric functions.

- Any single-variate $f\colon [0,T] \to \mathbb{R}$ function is symmetric.

- The function
$$f(x_1, x_2) = x_1 x_2$$
is clearly symmetric due to the fact that $f(x_1, x_2) = f(x_2, x_1)$.

- The function
$$g(x_1, x_2) = \frac{x_1}{x_2 + 1}$$
is not symmetric. The symmetrization of g is given by
$$\tilde{g} = \frac{1}{2}\left(\frac{x_1}{x_2 + 1} + \frac{x_2}{x_1 + 1}\right),$$
which is a symmetric function.

$\triangle$

Note that for an already symmetric function $f \in \tilde{L}^2\big([0,T]^n\big)$, we have that $\tilde{f} = f$. Also note that
$$\|f\|_{L^2([0,T]^n)} = \|\tilde{f}\|_{L^2([0,T]^n)},$$
meaning that the operator which sends f to $\tilde{f}$ is an isometry. Symmetric functions are particularly handy for so-called *iterated Itô-integrals.*

Definition 3.6. *Let $f_n \in \tilde{L}^2\big([0,T]^n\big)$ be symmetric. We define the* n-fold iterated Itô integral *as*
$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, t_2, \ldots, t_n) \mathrm{d}W_{t_1} \mathrm{d}W_{t_2} \ldots \mathrm{d}W_{t_n}.$$
Since f is square-integrable, we have that $I_n(f_n) \in L^2(\Omega)$.

The reason that iterated integrals are interesting to consider is that they form an orthogonal structure on $L^2(\Omega)$. This follows from the regular Itô isometry for stochastic integrals.

Proposition 3.6. *Let f_n, g_m be symmetric square-integrable functions. Then,*
$$\mathbb{E}[I_n(f_n) I_m(g_m)] = \begin{cases} 0 & \text{if } m \neq n, \\ n! \langle f_n, g_n \rangle_{L^2([0,T]^n)} & \text{if } m = n. \end{cases}$$

Proof. Assume, without loss of generality, that $n > m$ and denote, for any $1 \leq i \leq n$, the square-integrable random variables
$$F_i = n! \int_0^{t_{i+1}} \int_0^{t_i} \cdots \int_0^{t_2} f(t_1, t_2, \ldots, t_n) \mathrm{d}W_{t_1} \mathrm{d}W_{t_2} \ldots \mathrm{d}W_{t_i},$$
and, for any $1 \leq j \leq m$,
$$G_j = m! \int_0^{t_{j+1}} \int_0^{t_j} \cdots \int_0^{t_2} g(t_1, t_2, \ldots, t_m) \mathrm{d}W_{t_1} \mathrm{d}W_{t_2} \ldots \mathrm{d}W_{t_j}).$$
The values t_{n+1} and t_{m+1} correspond to T, such that $F_n = I_n(f_n)$ and $G_n = I_m(g_m)$. Now, we use the Itô isometry repeatedly to find
$$\mathbb{E}[I_n(f_n) I_m(g_m)] = \mathbb{E}[F_n G_m] = \int_0^T \mathbb{E}[F_{n-1} G_{n-1}] \mathrm{d}t_n = \int_0^T \int_0^{t_n} \mathbb{E}[F_{n-2} G_{n-2}] \mathrm{d}t_{n-1} \mathrm{d}t_n.$$

We can continue this for m steps until we have

$$\begin{aligned} &= m! \int_0^T \int_0^{t_n} \cdots \int_0^{t_{n-m+2}} \mathbb{E}[F_{n-m}] g(t_{n-m+1}, t_{n-m+2} \dots, t_n) \mathrm{d}t_{n-m+1} \dots \mathrm{d}t_{n-1} \mathrm{d}t_n \\ &= 0, \end{aligned} \tag{3.5}$$

since $\mathbb{E}[F_{n-m}] = 0$. If $m = n$, we see from Eq. (3.5) that

$$\begin{aligned} \mathbb{E}[I_n(f_n)^2] &= (n!)^2 \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, t_2, \dots, t_n) g(t_1, t_2 \dots, t_n) \mathrm{d}t_1 \mathrm{d}t_2 \dots \mathrm{d}t_n \\ &= n! \langle f, g \rangle_{L^2([0,T]^n)}. \end{aligned}$$

For the last step, note that the integral is exactly over a region of size $\frac{1}{n!}$ of the whole n-dimensional box $[0,T]^n$. Using a symmetry argument, we conclude that the integral above is the same as $\frac{1}{n!}$ times the integral over the whole box. □

The next theorem, which is attributed to Itô [29], shows why iterated Itô integrals are useful in the study of stochastic analysis. The orthogonal structure on $L^2(\Omega)$ provides a decomposition in iterated integrals.

Theorem 3.2 (Wiener Itô Expansion I)**.** *Let $X \in L^2(\Omega)$ be an $\mathcal{F}_T$-measurable random variable. Then, there is a unique sequence $(f_n)_{n\in\mathbb{N}}$ of symmetric functions, $f_n \in \tilde{L}^2\big([0,T]^n\big)$, such that*

$$\|X - \sum_{i=0}^{n} I_i(f_i)\|_{L^2(\Omega)} \to 0,$$

where $I_0(f_0) = \mathbb{E}[X]$. We write $X = \sum_{n=0}^{\infty} I_n(f_n)$. Furthermore, the L^2-norm of X is given by

$$\|X\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2([0,T]^2)}.$$

Proof. See [27, Theorem 1.10] □

The decomposition of random variables leads directly to the definition of the Skorohod integral for iterated Itô integrals. Let us consider a square-integrable random process X_t. Using the expansion theorem, we find unique sequences $(f_{n,t})_{n\in\mathbb{N}}$ of symmetric functions such that

$$X_t = \sum_{n=0}^{\infty} I_n(f_{n,t}).$$

Since the functions $f_{n,t}$ depend on t, we can view f_n as a function of $n+1$ variables such that

$$\begin{aligned} f_n \colon [0,T]^{n+1} &\to \mathbb{R} \\ (t_1, t_2, \dots, t_n, t) &\mapsto f_{n,t}(t_1, t_2 \dots, t_n). \end{aligned} \tag{3.6}$$

Definition 3.7. *Let $X_t \in L^2(\Omega)$, for all $t \in [0,T]$, and let $(f_{n,t})_{n\in\mathbb{N}}$ be the sequences of symmetric functions of the Wiener Itô Chaos Expansion (Theorem 3.2). Furthermore, let $\tilde{f}_n$ be the symmetrization of f_n as in Eq.* (3.6)*.*

We define the Skorohod integral *of X_t as the random variable*

$$\delta^I\big(X_t\big) := \int_0^T X_s \delta W_s := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n),$$

if the sum converges in $L^2(\Omega)$. In that case, we call the process X_t Skorohod-integrable *and write $X_t \in Dom\,\delta$.*

The Skorohod integral is thus an integral for a stochastic process, where the integrand does not have to be adapted. How does the integral compare to the Itô integral if X_t is adapted? It turns out that in this case the Skorohod integral coincides with the Itô integral with respect to the Brownian motion.

Proposition 3.7. *If X_t is an adapted square-integrable stochastic process, then the integrals $\int_0^T X_s \mathrm{d}W_s$ and $\int_0^T X_s \delta W_s$ coincide.*

Proof. See [27, Theorem 2.9]. □

Let us consider a simple example of a Skorohod integral.

Example 3.8. Consider the integral

$$\int_0^T W_T \delta W_t.$$

First of all, note that this cannot be evaluated using ordinary stochastic calculus due to the fact that W_T is not adapted. The Wiener-decomposition of W_T is given by

$$W_T = I_1(1),$$

according to Definition 3.6. Therefore, we find that $f_1(t_1) = 1$. Since this function is not time-dependent, we find

$$\int_0^T W_T \delta W_t = I_2(1) = 2! \int_0^T \int_0^{t_2} 1 \mathrm{d}W_{t_1} \mathrm{d}W_{t_2} = 2! \int_0^T W_{t_2} \mathrm{d}W_{t_2} = W^2(T) - T,$$

where the last equality follows from ordinary Itô calculus. △

3.3.2 The white noise case

In the previous two sections, we have introduced two different approaches to the Skorohod integral, leading to Definition 3.4 and Definition 3.7. We will now show that iterated Itô integrals can be constructed using an isonormal process and that these definitions of the Skorohod integral agree with the divergence operator. This particular construction is known in the literature as the *white noise case*. Since the construction is somewhat technical, we will only show results here and refer to Appendix B for the rigorous construction.

The idea is to construct a Hilbert space which governs the time direction of a stochastic process. We let

$$\mathscr{H} = L^2(\mu, [0,T], \mathcal{B}[0,T]),$$

where μ is a σ-finite atomless measure. In that case, the $\mathscr{H}$-isonormal process $W\colon \mathscr{H} \to L^2(\Omega)$ is characterized by the mapping on the indicator functions, as their linear span is dense in $\mathscr{H}$. We denote

$$W(A) = W(I_A),$$

where A is a measurable set such that $\mu(A) < \infty$. We can now analogously define an iterated stochastic integral.

Definition 3.8. *Let f be an indicator function on $L^2(\mu^n, [0,T]^n)$, such that*

$$f(t_1, t_2, \ldots, t_n) = I_{A_1 \times \cdots \times A_n}(t_1, t_2, \ldots t_n), \tag{3.7}$$

where $A_1 \times \cdots \times A_n$ are pairwise disjoint. Then, we define

$$\hat{I}_n(f) \coloneqq W(A_1) \dots W(A_n).$$

We expand this definition to elementary functions and then to their closure. This means that for any function $f \in L^2([0,T]^n)$, the expression

$$\hat{I}_n(f) \in L^2(\Omega)$$

is well-defined.

We use the hat for the operator $\hat{I}_n$ to distinguish it from Definition 3.6, as we have yet to establish how the definitions are related (see Theorem 3.4 below).

Remark 3.4. Note that, unlike Definition 3.6, this operator is defined on all of $L^2([0,T]^n)$ rather than just symmetric functions. However, by linearity, it is immediate that

$$\hat{I}_n(f) = \hat{I}_n(\tilde{f}),$$

where $\tilde{f}$ is the symmetrization of f.

Analogously to Theorem 3.2, we can orthogonally decompose any random variable $X \in L^2(\Omega)$.

Theorem 3.3. *Let $X \in L^2(\Omega)$ be a square-integrable random variable. Then, there is a unique sequence $(f_n)_{n\in\mathbb{N}}$ of symmetric functions $f_n \in L^2\big([0,T]^n\big)$, such that*

$$X = \sum_{n=0}^{\infty} \hat{I}_n(f_n).$$

Since W is an isonormal process, the derivative operator is defined as an operator $D\colon \mathbb{D}^1 \subset L^2(\Omega) \to L^2(\Omega; \mathscr{H})$. We now identify the space $L^2(\Omega; \mathscr{H})$ with $L^2(\Omega \times T)$ in the following way. For the random variable $X \in L^2(\Omega; \mathscr{H})$ and a given $\omega \in \Omega$, we have that

$$X(\omega) \in L^2([0,T])$$

and thus is a function on $[0,T]$. We can thus consider

$$X(\omega, t) \in L^2(\Omega \times T).$$

Proposition 3.8. *Let $X \in \mathbb{D}^1$ be given by the expansion*

$$X = \sum_{n=0}^{\infty} \hat{I}_n(f_n)$$

for a sequence of symmetric functions. Then, its Malliavin derivative is given by

$$D(X) = \sum_{n=1}^{\infty} n\hat{I}_{n-1}(f_n(\cdot, t)) \in L^2(\Omega \times T).$$

In a similar fashion, we can now express the divergence operator for a random variable $X \in L^2(\Omega \times T)$. First, notice that for any $t \in [0,T]$, the expression X_t is a random variable in $L^2(\Omega)$ and thus we can write

$$X_t = \sum_{n=0}^{\infty} \hat{I}_n(f_{n,t}).$$

Proposition 3.9. *Let $X_t \in Dom\,\delta \subset L^2(\Omega \times T)$ be given by*

$$X_t = \sum_{n=0}^{\infty} \hat{I}_n(f_{n,t}).$$

Then, we have

$$\delta(X_t) = \sum_{n=0}^{\infty} \hat{I}_{n+1}(\tilde{f}_n), \tag{3.8}$$

where $\tilde{f}_n$ is given as in Eq. (3.6).

We thus see that Eq. (3.8) is equal to Definition 3.7 if we can show that $\hat{I}_n(f) = I_n(f)$ for all f. Let $L^2(\mu, [0,T], \mathcal{B}[0,T]) = L^2(\gamma, [0,T], \mathcal{B}[0,T])$, where γ is the usual Lebesgue measure, which is known to be atomless. Remember from Example 3.1 that in this case,

$$W_t = W(I_{[0,t]})$$

is a Brownian motion.

Theorem 3.4. *Let $f \in \tilde{L}^2([0,T]^n) = \tilde{L}^2(\gamma^n, [0,T]^n, \mathcal{B}([0,T]^n)$ be a symmetric function. Then, the iterated stochastic integrals coincide, i.e.*

$$I_n(f) = \hat{I}_n(f).$$

Proof. Since we defined $\hat{I}$ on the span of elementary functions, it suffices to show that this is true for elementary functions. Note that these functions are not necessarily symmetric. Due to Remark 3.4, this does not matter as we can simply consider symmetrizations.

Let f be an indicator function in $L^2([0,T]^n)$, such that

$$f(t_1, t_2, \ldots, t_n) = I_{A_1 \times \cdots \times A_n}(t_1, t_2, \ldots t_n),$$

where $A_1 \times \cdots \times A_n$ are pairwise disjoint and let $\tilde{f}$ be its symmetrization.

From the definition it is clear that

$$\hat{I}_n(\tilde{f}) = \hat{I}_n(f) = \prod_{k=1}^{n} W(A_k).$$

To compute $I_n(f)$, we will use Proposition 1.8 from [27]. The function f can be written as $f = I_{A_1} \otimes I_{A_2} \cdots \otimes I_{A_n}$, and therefore

$$I_n(\tilde{f}) = \prod_{k=1}^{n} \int_0^T h_1(I_{A_k}(t))\mathrm{d}W_t = \prod_{k=1}^{n} \int_0^T I_{A_k}(t)\mathrm{d}W_t.$$

Since $W(A_k) = \int_0^T I_{A_k}(t)\mathrm{d}W_t$, the proof is complete. □

3.4 Expansion of White Noise Theory

So far, we have defined the Skorohod integral by means of the divergence operator as well as by iterated Itô integrals. In this section, we will introduce the third approach, which is based on the grounds of the so-called *white noise probability space*. This chapter is based on the theory of *Hida's white noise theory* as well as on the *Wick product*. As the original theory due to Hida is too general for the purpose of this thesis, we will follow the approach as introduced by Holden et al. [30]. However, in order to compare this approach to the previous sections, we will slightly adapt the approach to be able to consider a finite interval $[0,T]$.

We begin this chapter with the construction of the white noise probability space.

Definition 3.9. *Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the* Schwartz space*, which is defined as the locally convex vector space*

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall \alpha, \beta, \|f\|_{\alpha,\beta} < \infty\},$$

where

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha f^{(\beta)}(x)|.$$

Remark 3.5. A function $f \in \mathcal{S}$ is called *rapidly decreasing*, if the function and all of its derivatives go to 0 faster than at a polynomial rate as $|x|$ increases.

The topological dual space $\mathcal{S}^*$ of the Schwartz space is called the *space of tempered distributions*. We write $\langle f, f^* \rangle = f^*(f)$ for any $f \in \mathcal{S}, f^* \in \mathcal{S}^*$. Furthermore, for every tempered distribution $f^* \in \mathcal{S}^*$, we can identify real-valued functions as kernels and obtain the following inclusions:

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^*(\mathbb{R})$$

For a more detailed overview of Schwartz spaces and their dual, we refer to Chapter 3 and 4 of [31]. We continue with the construction of a probability measure on $\mathcal{S}^*(\mathbb{R})$.

Definition 3.10. *Let $\mathcal{B} = \mathcal{B}(\mathcal{S}^*)$ be the Borel σ-algebra of $\mathcal{S}^*$ with respect to the weak-star topology. The probability measure $\mathbb{P} \colon \mathcal{B} \to [0, 1]$, such that*

$$\int_{\mathcal{S}^*} \exp(i\langle f, \omega\rangle) d\mathbb{P}(\omega) = \exp\big(-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}\big), \tag{3.9}$$

for all $f \in \mathcal{S}$ is called the white noise probability measure*.*

The existence of such a probability measure is guaranteed by the Bochner Minlos theorem [30, Theorem 2.1.1]. It is often referred to as a *Gaussian measure* on $\mathcal{S}^*$.

The next lemma shows that in fact Gaussian properties are easily obtained with this measure.

Lemma 3.5. *For any $f \in \mathcal{S}$, the map $\omega \mapsto \langle f, \omega\rangle$ is a Gaussian random variable on the probability space $(\mathbb{P}, \mathcal{S}^*, \mathcal{B})$. Furthermore, we have that*

$$\mathbb{E}[\langle f, \cdot\rangle] = 0,$$

and

$$\mathbb{E}[\langle f, \cdot\rangle^2] = \|f\|^2_{L^2(\mathbb{R})}. \tag{3.10}$$

Proof. To see that $\langle f, \omega\rangle$ is a random variable, note that the map

$$\omega \mapsto \langle \omega, f^{**}\rangle$$

is continuous on the weak-star topology of $\mathcal{S}^*$, where f^{**} is naturally embedded in the double dual of $\mathcal{S}$. Let $t \in \mathbb{R}$ be any scalar. For the map $t \cdot f(x) \in \mathcal{S}$, we have

$$\int_{\mathcal{S}^*} \exp(i\langle tf, \omega\rangle) \mathrm{d}\mathbb{P}(\omega) = \int_{\mathcal{S}^*} \exp(ti\langle f, \omega\rangle) \mathrm{d}\mathbb{P}(\omega) = \varphi_{\langle f,\cdot\rangle}(t),$$

where $\varphi_{\langle f,\cdot\rangle}(t)$ is the characteristic function of $\langle f, \cdot\rangle$ showing that it is $N(0, \|f\|^2_{L^2})$ distributed. $\square$

The lemma shows that Gaussian random variables are easily constructed. Moreover, this result can be extended to Gaussian vectors.

Lemma 3.6. *Let $\{e_k : 1 \leq k \leq n\}$ be functions in $\mathcal{S}$ which are orthonormal. Then, the random vector*

$$\omega \mapsto (\langle e_1, \omega\rangle, \langle e_2, \omega\rangle, \dots, \langle e_n, \omega\rangle)$$

is a standard normal random vector.

Proof. Let $t \in \mathbb{R}^n$ be given and consider the characteristic function of the random vector

$$\varphi_{(\langle e_1,\cdot\rangle,\dots,\langle e_n,\cdot\rangle)}(t) = \int_{\mathcal{S}^*} \exp\left(i\sum_{k=1}^n t_k\langle e_k, \omega\rangle\right) \mathrm{d}\mathbb{P}(\omega) = \int_{\mathcal{S}^*} \exp\left(i\langle\sum_{k=1}^n t_k e_k, \omega\rangle\right) \mathrm{d}\mathbb{P}(\omega).$$

By the definition of $\mathbb{P}$, it follows that

$$\varphi_{(\langle e_1,\cdot\rangle,\dots,\langle e_n,\cdot\rangle)}(t) = \exp\left(-\frac{1}{2}\|\sum_{k=1}^n t_k e_k\|^2_{L^2(\mathbb{R})}\right) = \exp\left(-\frac{1}{2}\sum_{k=1}^n t_k^2\right),$$

since e_k are orthonormal in $L^2(\mathbb{R})$. The right-hand side is the characteristic function of a standard normal random vector. □

Remark 3.6. This result implies in particular that we can calculate the expectation

$$\mathbb{E}[f(\langle e_1, \omega\rangle, \langle e_2, \omega\rangle, \dots, \langle e_n, \omega\rangle)]$$

for any suitable f as the distribution of the random vector is given by

$$d\lambda_n(x) = (2\pi)^{-\frac{n}{2}} \exp(-\frac{1}{2}|x|^2)\mathrm{d}x_1\mathrm{d}x_2\dots\mathrm{d}x_n.$$

The objective now is to construct a Brownian motion $W_t, t \geq 0$ on the probability space $(\mathbb{P}, \mathcal{S}^*, \mathcal{B})$. To do so, we want to study the properties of the random variable

$$\omega \mapsto \langle I_{[0,t]}, \omega\rangle.$$

However, notice that as $I_{[0,t]}$ is not rapidly decreasing, the expression $\langle I_{[0,t]}, \cdot\rangle$ is not well-defined.

Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$[1], we can extend the definition to square-integrable functions. We can define for any $f \in L^2(\mathbb{R})$

$$\langle f, \omega\rangle = \lim_{n\to\infty} \langle f_n, \omega\rangle,$$

where the limit converges in $L^2(\Omega)$ due to the isometry of Eq. (3.10). We can thus define the random process

$$\tilde{W}_t(\omega) = \langle I_{[0,t]}, \omega\rangle,$$

where, according to Lemma 3.5, for all t, we have that $\tilde{W}_t \sim N(0, \|I_{[0,t]}\|^2_{L^2(\mathbb{R})}) = N(0,t)$.

Using Kolmogorov's continuity theorem [17, Theorem 4.23], there exists a continuous modification of $\tilde{W}_t$ which we will denote as W_t as it is in fact a Brownian motion.

Definition 3.11. *The map $w\colon L^2(\mathbb{R}) \times S' \to \mathbb{R}$ given by*

$$w(f, \omega) = \langle f, \omega\rangle = w_f$$

is called the white noise process *defined on the white noise probability space.*

[1]Note that the set of test functions C_0^∞ is contained in $\mathcal{S}$.

Remark 3.7. Note that if supp$f \in [0, \infty)$, then w_f coincides with the Itô integral. This can be seen as the integrals agree on indicator functions, which span a dense subset of functions with support on $[0, \infty)$. Therefore, we also write

$$w_f = \int_{\mathbb{R}} f(s) \mathrm{d}W_s.$$

Similar to the first Wiener Itô chaos expansion (Theorem 3.2), we want to decompose random variables as sums of white noise processes. This is achieved using the natural structure of Hermitian polynomials for Gaussian random variables, as described in Appendix A.

For the remainder of the section, we will fix a finite $T > 0$ and consider the interval $[0, T]$ as a finite time horizon. Furthermore, we fix an orthonormal basis $\{e_k\}_k$ of $L^2([0, T])$ and let α be a multi-index of length m. Generally, the choice of basis does not matter. It can be chosen for instance as

$$e_k(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{k\pi t}{T}\right). \tag{3.11}$$

Consider the random variable, $H_\alpha \colon \Omega \to \mathbb{R}$, given by

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(w_{e_j}),$$

where h_{α_j} is the α_j-th Hermitian polynomial. The functions e_k are embedded into $L^2(\mathbb{R})$ in a natural way. It turns out that the random variables of this type indexed over the set of multi-indices $\mathcal{J}$ form an orthogonal basis of the Hilbert space $L^2(\Omega)$.

Theorem 3.7 (Wiener Itô Chaos Expansion II)**.** *The family of random variables $\{H_\alpha : \alpha \in \mathcal{J}\}$ form an orthogonal basis of $L^2(\Omega)$. I.e, for any $X \in L^2(\Omega)$ there exist unique elements $c_\alpha \in \mathbb{R}$, such that*

$$X = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha$$

in $L^2(\Omega)$. Furthermore, we have

$$\|X\|^2 = \mathbb{E}[X^2] = \sum_{\alpha \in J} \alpha! c_\alpha^2.$$

Proof. Let us first show that the random variables are orthogonal. Let $\alpha, \beta \in \mathcal{J}$ be multi-indices. We may assume that they have the same length as one can extend the length by adding zeros. Then,

$$\mathbb{E}[H_\alpha H_\beta] = \mathbb{E}\left[\prod_{j=1}^{m} h_{\alpha_j}(w_{e_j}) h_{\beta_j}(w_{e_j})\right].$$

Consider the function $f \colon \mathbb{R}^m \to \mathbb{R}$, such that

$$x \mapsto \prod_{j=1}^{m} h_{\alpha_j}(x_j) h_{b_j}(x_j).$$

By Lemma 3.6 and in particular Remark 3.6, it thus follows that

$$\begin{aligned}
\mathbb{E}[H_\alpha H_\beta] &= \mathbb{E}\left[f(w_{e_1}, w_{e_2} \dots w_{e_m})\right] \\
&= \int_{\mathbb{R}^m} \prod_{j=1}^{m} h_{\alpha_j}(x_j) h_{\beta_j}(x_j) \mathrm{d}\lambda_m \\
&= \prod_{j=1}^{m} \int_{\mathbb{R}} h_{\alpha_j}(x_j) h_{\beta_j}(x_j) \mathrm{d}\lambda = \prod_{j=1}^{m} \delta_{\alpha_j, \beta_j},
\end{aligned}$$

which shows the orthogonality. To show that the system is a basis, consider the set of random variables of the form

$$p(w_{e_1}, w_{e_2}, \dots, w_{e_n}),$$

where p is a polynomial in n variables. Such random variables are called *stochastic polynomials*. It is well-known that the set of stochastic polynomials is dense in $L^2(\Omega)$ [17]. Since any polynomial can be written as a linear combination of products of Hermite polynomials, the proof is complete. □

Remark 3.8. This expansion is an analog to Theorem 3.2. One can show that if

$$X = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha,$$

we can write

$$X = \sum_{n=0}^{\infty} I_n(f_n),$$

where

$$f_n = \sum_{\alpha \in \mathcal{J}, |\alpha| = n} c_\alpha e_1^{\otimes \alpha_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \alpha_k}.$$

Here, the function $f \hat{\otimes} g$ is the symmetrization of

$$f \otimes g(x, y) = f(x) g(y).$$

We will use the notation

$$e_1^{\otimes \alpha_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \alpha_k} := e^{\hat{\otimes} \alpha}.$$

It thus also follows that

$$H_\alpha = I_{|\alpha|}(e^{\hat{\otimes} \alpha}).$$

As an example for the second Wiener Itô expansion, let us consider white noise process w_f for some $f \in L^2([0, T])$. We make use of the following notation. We write $\epsilon^{(k)}$ to indicate the multi-index of type

$$\epsilon^{(k)} = (0, 0, \dots, 0, 1, 0 \dots, 0),$$

where the 1 is on the k-th position.

Example 3.9. Consider a function $f \in L^2([0, T])$. Then, since e_k is an orthonormal basis of $L^2([0, T])$, we can write

$$f = \sum_{k=1} \langle f, e_k \rangle_{L^2([0,T])} e_k,$$

and thus

$$w_f = \langle \sum_k \langle f, e_k \rangle_{L^2([0,T])} e_k, \omega \rangle = \sum_k \int_0^T f(a) e_k(a) \mathrm{d}a \, \langle e_k, \omega \rangle = \sum_k \left(\int_0^T f(a) e_k(a) \mathrm{d}a \right) H_{\epsilon^{(k)}},$$

since the white noise process is a linear functional. Therefore,

$$\int_0^T f \mathrm{d}W(s) = \sum_k c_k H_{\epsilon^{(k)}},$$

where

$$c_k = \int_0^T f(a) e_k(a) \mathrm{d}a.$$

Of particular interest is the decomposition where $f = I_{[0,t]}$, since we have seen that the white noise process is the Brownian motion in this case. We see that for any $t \in [0, T]$, we have

$$W_t = \sum_k \left(\int_0^t e_k(s) \mathrm{d}s \right) H_{\epsilon^{(k)}}.$$

$\triangle$

Using the Wiener Chaos expansion, we can characterize an important subset of $L^2(\Omega)$, the so-called stochastic test functions. They form a vector space, which we call the Hida test function space. The precise definition of such random variables is complex and beyond the scope of this thesis. It suffices for us to have a characterization of stochastic test functions and its topological dual space according to [32].

Definition 3.12. *Let $X = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha$ be a random variable on the white noise probability space. We call X a* stochastic test function *if*

$$\sup_{\alpha \in \mathcal{J}} \alpha! a_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty, \text{ for all } k \in \mathbb{N},$$

where

$$(2\mathbb{N})^\gamma := \prod_{a=1}^m (2a)^{\gamma_a}.$$

The set of all stochastic test functions $(\mathcal{S})$[2] *is a vector space which we call the* Hida test function space.

Definition 3.13. *The dual space $(\mathcal{S})^*$ of $(\mathcal{S})$ can be identified as expressions $f = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha$ such that*

$$\sup_{\alpha \in \mathcal{J}} \alpha! b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty, \text{ for some } q \in \mathbb{N}.$$

The action of $f = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha \in (\mathcal{S})^*$ on $X = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha$ is given by

$$\langle X, f \rangle = f(X) = \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha b_\alpha.$$

The topology on $(\mathcal{S})^*$ is induced by the norms

$$\|f\|_q^2 := \sum_{\alpha \in \mathcal{J}} \alpha! b_\alpha^2 (2\mathbb{N})^{-q\alpha}, \tag{3.12}$$

i.e. $f_n \to f$ if and only if $\|f_n - f\|_q \to 0$ for some $q \in \mathbb{N}$.

We can thus express $(\mathcal{S})$ and $(\mathcal{S})^*$ in terms of (possibly infinite) linear combinations over the orthogonal set $\{H_\alpha : \alpha \in \mathcal{J}\}$. The spaces can be ordered in the following way:

$$(\mathcal{S}) \subset L^2(\Omega) \subset (\mathcal{S})^*$$

The reason to introduce stochastic test function spaces is to define another integral. While the integral is defined on $(\mathcal{S})^*$, the inclusion above shows that it can be interpreted as a stochastic integral. This integral is defined in a weak sense, and hence is called the *Pettis* integral.

Definition 3.14. *Let $Y \colon [0, T] \to (\mathcal{S})^*$ be such that*

$$\langle X, Y_t \rangle \in L^1([0, T]) \tag{3.13}$$

[2]The space $(\mathcal{S})$ should not be confused with $\mathcal{S}$, which is the space of the Schwartz functions.

for all $X \in (\mathcal{S})$. Then, we denote the Pettis *integral* $\int_0^T Y_t \mathrm{d}t$ *of* Y_t *as the unique element in* $(\mathcal{S})^*$ *such that*

$$\langle X, \int_0^T Y_t \mathrm{d}t\rangle = \int_0^T \langle X, Y_t\rangle \mathrm{d}t,$$

for all $X \in (\mathcal{S})$. In this case, we call Y_t Pettis-integrable.

Before considering examples of such integrals, we want to consider the characterization for a stochastic process to be *Pettis-integrable*. The following proposition shows that it is not necessary to check Eq. (3.13) for all X.

Proposition 3.10. *Let $Y_t = \sum_\alpha c_\alpha(t) H_\alpha$ be such that*

$$\sum_{\alpha\in\mathcal{J}} \alpha! \|c_\alpha\|^2_{L^1([0,T])} (2\mathbb{N})^{-\alpha p} < \infty, \tag{3.14}$$

for some $p < \infty$. Then, Y_t is Pettis-integrable.

Proof. Let $X = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha(\mathcal{S})$ be any stochastic test function. Then, for any p, we have,

$$\begin{aligned}\int_0^T \left|\sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha(t) b_\alpha\right| \mathrm{d}t &\leq \sum_{\alpha\in\mathcal{J}} \alpha! |b_\alpha| \|c_\alpha\|_{L^1([0,T])} \\ &= \sum_{\alpha\in\mathcal{J}} \sqrt{\alpha!}(2\mathbb{N})^{\frac{\alpha p}{2}} |b_\alpha| \sqrt{\alpha!}(2\mathbb{N})^{\frac{-\alpha p}{2}} \|c_\alpha\|_{L^1([0,T])},\end{aligned}$$

which can be split into

$$= \left(\sum_{\alpha\in\mathcal{J}} \alpha!(2\mathbb{N})^{\alpha p} b_\alpha^2\right)^{\frac{1}{2}} \left(\sum_{\alpha\in\mathcal{J}} \alpha!(2\mathbb{N})^{-\alpha p} \|c_\alpha\|^2_{L^1([0,T])}\right)^{\frac{1}{2}}.$$

Since X is a stochastic test function, the first term is bounded for all p. This concludes the proof. □

While the definition of the Pettis integral is in a weak sense, we will see with the next example that it is obtained easily by integrating the coefficients, under the condition that Eq. (3.13) is fulfilled.

Example 3.10. Suppose that $Y_t = \sum_\alpha c_\alpha(t) H_\alpha$ is Pettis-integrable and let $X = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})$. To find the integral of Y_t, we consider

$$\int_0^T \langle X, Y_t\rangle \mathrm{d}t = \int_0^T \sum_\alpha \alpha! c_\alpha(t) b_\alpha \mathrm{d}t = \sum_\alpha \alpha! b_\alpha \int_0^T c_\alpha(t)\mathrm{d}t.$$

If we denote the coefficients of $\int_0^T Y(t)\mathrm{d}t$ as a_α, we see that by the definition of the Pettis integral that

$$\sum_\alpha \alpha! a_\alpha b_\alpha = \sum_\alpha \alpha! b_\alpha \int_0^T c_\alpha(t)\mathrm{d}t,$$

and thus

$$a_\alpha = \int_0^T c_\alpha(t)\mathrm{d}t.$$

△

The Pettis integral is thus another way to integrate a stochastic process. Given the story line of this thesis, we would expect it to coincide with the Skorohod integral and thus also with the Itô integral. However, before comparing it to these integrals, we want to first elaborate more on the structure of $(\mathcal{S})^*$, by introducing a natural product on this space, called the Wick product.

3.5 Wick Products and Integration

In this section, we will introduce the *Wick product*, which is a natural operator on $(\mathcal{S})^*$. This will allow us to derive the Skorohod integral in terms of a Pettis integral in Theorem 3.10.

The Wick product originated from physics in the context of quantum field theory. Mainly due to Hida, it made a significant impact in stochastic analysis as a product for random variables, particularly in the context of the white noise probability space. There are many ways to define the Wick product. For this thesis, it makes sense to use the definition as follows:

Definition 3.15. *Let $X, Y \in (\mathcal{S})^*$ be given by $\sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha, \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha$, respectively. The* Wick product *of X and Y is given as*

$$X \diamond Y = \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta H_{\alpha+\beta} = \sum_{\gamma\in\mathcal{J}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) H_\gamma.$$

It is clear that the Wick product is associative just as a regular product. Importantly, the spaces $(\mathcal{S})$ and $(\mathcal{S})^*$ are closed under the Wick product.

Proposition 3.11. *Let $X, Y \in (\mathcal{S})$. Then,*

$$X \diamond Y \in (\mathcal{S}).$$

Alternatively, let $X, Y \in (\mathcal{S})^*$. Then,

$$X \diamond Y \in (\mathcal{S})^*.$$

Proof. Let X, Y be given by the usual decomposition (i.e Definition 3.15). The coefficients of $X \diamond Y$ are thus

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta.$$

We have

$$\begin{aligned}
\sup_{\gamma\in\mathcal{J}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right)^2 (2\mathbb{N})^{k\gamma} &\leq \sup_{\gamma\in\mathcal{J}} \sum_{\alpha+\beta=\gamma} a_\alpha^2 \sum_{\alpha+\beta=\gamma} b_\beta^2 (2\mathbb{N})^{k\gamma} \\
&= \sup_{\gamma\in\mathcal{J}} \sum_{\alpha+\beta=\gamma} a_\alpha^2 (2\mathbb{N})^{k\alpha} \sum_{\alpha+\beta=\gamma} b_\beta^2 (2\mathbb{N})^{k\beta} \\
&\leq \sup_{\alpha,\beta\in\mathcal{J}} \sum_\alpha a_\alpha^2 (2\mathbb{N})^{k\alpha} \sum_\alpha b_\beta^2 (2\mathbb{N})^{k\beta},
\end{aligned}$$

for all $k \in \mathbb{N}$. Since $X, Y \in (\mathcal{S})$, the claim follows directly. The proof for $(\mathcal{S})^*$ works in a similar fashion. $\square$

In some cases, the Wick product coincides with the regular product, or is at least comparable. First of all, the products coincide if one of the factors is deterministic. We have that

$$X \diamond Y = X \cdot Y,$$

if X or Y is deterministic. Furthermore, the next Lemma shows that the Wick product for white noise processes is connected to their regular product.

Lemma 3.8. *Let $f, g \in L^2([0,T])$ be square-integrable functions. Then, we have*

$$w_f \diamond w_g = w_f \cdot w_g - \langle f, g \rangle_{L^2([0,T])}.$$

Proof. Since e_k is an orthonormal basis, we can write

$$w_f = \langle \sum_k \langle f, e_k \rangle_{L^2([0,T])} e_k, \cdot \rangle = \sum_k \langle f, e_k \rangle_{L^2([0,T])} \langle e_k, \cdot \rangle = \sum_k \langle f, e_k \rangle_{L^2([0,T])} H_{\epsilon^{(k)}},$$

and thus

$$\begin{aligned} w_f \diamond w_g &= \sum_{k,l} \langle f, e_k \rangle_{L^2([0,T])} \langle g, e_l \rangle_{L^2([0,T])} H_{\epsilon^{(k)}+\epsilon^{(l)}} \\ &= \sum_k \langle f, e_k \rangle_{L^2([0,T])} H_{\epsilon^{(k)}} \sum_l \langle g, e_l \rangle_{L^2([0,T])} H_{\epsilon^{(l)}} - \sum_i \langle f, g \rangle_{L^2([0,T])}. \end{aligned}$$

The last term appears as $H_{\epsilon^{(k)}} H_{\epsilon^{(l)}} = H_{\epsilon^{(k)}+\epsilon^{(l)}}$ if $k \neq l$, but

$$H^2_{\epsilon^{(k)}} - 1 = H_{2\epsilon^{(k)}}.$$

This proves the lemma. □

By Remark 3.8, one can derive the expansion of a random variable in terms of iterated integrals. This implies that we can derive the Wick product in terms of iterated Itô integrals.

Proposition 3.12. *Let $X, Y \in L^2(\Omega)$ be given by the expansions $\sum_{n=0}^{\infty} I_n(f_n)$ and $\sum_{n=0}^{\infty} I_n(g_n)$, respectively. Then, we have that*

$$X \diamond Y = \sum_{n,m=0}^{\infty} I_{n+m}(f_n \hat{\otimes} g_m) = \sum_{i=0}^{\infty} I_i(\sum_{n+m=i} f_n \hat{\otimes} g_m).$$

Proof. Suppose that X, Y are given by the usual decomposition. The Wick product $X \diamond Y$ has coefficients

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta.$$

Therefore, we can write

$$X \diamond Y = \sum_{i=0} I_i(h_i),$$

with

$$h_i = \sum_{|\gamma|=i} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) e_1^{\otimes \gamma_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \gamma_k}.$$

For any n, m, let f_n, g_m be given by

$$f_n = \sum_{|\alpha|=n} a_\alpha e_1^{\otimes \alpha_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \alpha_k},$$

and

$$g_m = \sum_{|\beta|=m} b_\beta e_1^{\otimes \beta_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \beta_k}.$$

Then, we can compute

$$\sum_{n+m=i} f_n \hat{\otimes} g_m = \sum_{n+m=i} \sum_{|\alpha|=n, |\beta|=m} a_\alpha b_\beta e_1^{\otimes \alpha_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \alpha_k} \hat{\otimes} e_1^{\otimes \beta_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \beta_k},$$

which reduces to

$$\sum_{|\gamma|=i} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) e_1^{\otimes \gamma_1} \hat{\otimes} \dots \hat{\otimes} e_k^{\otimes \gamma_k},$$

and is thus equal to h_i. □

Let us now see how the Pettis integral, introduced in Definition 3.14 is related to the Skorohod integral. In order to do so, we need to introduce what is referred to as the *singular white noise.*

Remember the construction of the Brownian motion $W_t = \langle I_{[0,t]}, \omega \rangle$ on the white noise probability space. By Example 3.9, the decomposition of W_t is given by

$$W_t = \sum_k \left(\int_0^t e_k(s) \mathrm{d}s \right) H_{\epsilon^{(k)}}.$$

This can be considered as a map $W_t \colon [0,T] \to (\mathcal{S})^*$, given by

$$t \mapsto W_t.$$

A question worth considering is: Is this function differentiable? In other words, does the limit

$$\lim_{h \to 0} \frac{W_{t+h} - W_t}{h}$$

exist for any $t \in [0,T]$?

Proposition 3.13 (and Definition)**.** *The function $t \to W_t$ is differentiable in $(\mathcal{S})^*$ for all $t \in [0,T]$ and its derivative is given by the process $\dot{W} \colon [0,T] \to (\mathcal{S})^*$, where*

$$\dot{W}_t = \sum_k e_k(t) H_{\epsilon^{(k)}}.$$

This process is called the singular white noise*.*

Proof. Remember that the topology on $(\mathcal{S})^*$ is induced by the norms given in Eq. (3.12). For any h, t, we have

$$\frac{W_{t+h} - W_t}{h} = \sum_k \frac{1}{h} \int_t^{t+h} e_k(s) \mathrm{d}s \; H_{\epsilon^{(k)}},$$

and thus

$$\| \frac{W_{t+h} - W_t}{h} - \dot{W}_t \|_q^2 = \sum_k \left(\frac{1}{h} \int_t^{t+h} e_k(s) \mathrm{d}s - e_k(t) \right)^2 (2\mathbb{N})^{-q\epsilon^{(k)}}.$$

Since $\left(\frac{1}{h} \int_t^{t+h} e_k(s) \mathrm{d}s - e_k(t) \right)^2 \to 0$ as $h \to 0$, and

$$\sum_k (2\mathbb{N})^{-q\epsilon^{(k)}} = \sum_k 2k^{-q} < \infty,$$

for any $q \geq 2$, we see that $\| \frac{W_{t+h} - W_t}{h} - \dot{W}_t \|_q^2 \to 0$ for any $q \geq 2$, which proves the proposition. $\square$

We can now derive an equivalent Skorohod integral using a Pettis integral and the singular white noise. The integral we want to consider is of the form

$$\int_0^T X_s \diamond \dot{W}_s \mathrm{d}s.$$

Let us first consider what the domain of such an integral would be.

Lemma 3.9. *Let* $X_t = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha$ *be such that there is a value* $q \in \mathbb{R}_+$, *such that*

$$K \coloneqq \sup_{\alpha\in\mathcal{J}} \{\alpha! \|c_a\|_{L^1([0,T])} (2\mathbb{N})^{-q\alpha}\} < \infty.$$

Then, $X_t \diamond \dot{W}_t \in (\mathcal{S})^*$ *is integrable.*

Proof. The Wick product is given by

$$X_t \diamond \dot{W}_t = \sum_{\alpha\in\mathcal{J},k} c_\alpha(t) e_k(t) H_{\alpha+\epsilon^{(k)}} = \sum_{\gamma\in\mathcal{J}} \sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} c_\alpha(t) e_k(t) H_\gamma.$$

According to Proposition 3.10, it suffices if there is some p with

$$\sum_{\gamma\in\mathcal{J}} \gamma! \, \| \sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} c_\alpha(t) e_k(t) \|^2_{L^1([0,T])} (2\mathbb{N})^{-p\gamma} < \infty.$$

We first evaluate the norm of the coefficients:

$$\| \sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} c_\alpha e_k \|^2_{L^1([0,T])} \le \left(\sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} \|c_\alpha e_k\|_{L^1([0,T])} \right)^2 \le C^2 \left(\sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} \|c_\alpha\|_{L^1([0,T])} \right)^2,$$

where $C^2 = \frac{2}{T}$ is obtained by Eq. (3.11).

The number of terms in the sum is determined by $l(\gamma)$, which is the number of non-zero elements in γ. Thus, we can simplify to

$$\| \sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} c_\alpha e_k \|^2_{L^1([0,T])} \le C^2 l(\gamma)^2 \sum_{\substack{\alpha\in\mathcal{J}\\ \exists k: \alpha+\epsilon^{(k)}=\gamma}} \|c_\alpha\|^2_{L^1([0,T])}.$$

Now, let $p = 2q$. Then,

$$\begin{aligned}
&\sum_{\gamma\in\mathcal{J}} \gamma! \, \| \sum_{\substack{\alpha\in\mathcal{J},k\\ \alpha+\epsilon^{(k)}=\gamma}} c_\alpha(t) e_k(t) \|^2_{L^1([0,T])} (2\mathbb{N})^{-2q\gamma} \\
&\le C^2 \sum_{\alpha\in\mathcal{J},k} (\alpha+\epsilon^{(k)})! l(\alpha+\epsilon^{(k)})^2 \|c_\alpha\|^2_{L^1([0,T])} (2\mathbb{N})^{-2q(\alpha+\epsilon^{(k)})} \\
&\le C^2 K \sum_{\alpha\in\mathcal{J},k} \frac{(\alpha+\epsilon^{(k)})!}{\alpha!} l(\alpha+\epsilon^{(k)})^2 (2\mathbb{N})^{-\epsilon^{(k)}} (2\mathbb{N})^{-q\alpha} < \infty,
\end{aligned}$$

for any $q > \frac{1}{2}$. This concludes the proof. □

Theorem 3.10. *Let* X_t *be a stochastic process which is Skorohod integrable (in an iterated integral sense) and such that*

$$\int_0^T \mathbb{E}[X_t^2] d\mathrm{d}t < \infty.$$

Then, $X_t \diamond \dot{W}_t$ *is Pettis-integrable for all* t, *and*

$$\int_0^T X_s \delta W_s = \int_0^T X_s \diamond \dot{W}_s \mathrm{d}s.$$

Remark 3.9. Combining this result with Example 3.10 shows that if we know the Wiener-Itô decomposition of a square-integrable process Y_t, the Skorohod integral is easily found.

Proof. Since X_t is square-integrable, it follows by Lemma 3.9 that $X_t \diamond \dot{W}$ is integrable. This follows because

$$\sum_{\alpha\in\mathcal{J}} \alpha! \int_0^T a_\alpha(t)^2 \mathrm{d}t = \int_0^T \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha(t)^2 \mathrm{d}t = \int_0^T \mathbb{E}[X_t^2]\mathrm{d}t < \infty$$

and therefore

$$\alpha! \|a_\alpha\|^2_{L^1[0,T]} \leq \alpha! \int_0^T a_\alpha(t)^2 \mathrm{d}t$$

is certainly bounded. Let $X_t = \sum_{\alpha\in\mathcal{J}} a_\alpha(t) H_\alpha$ be the composition of X_t. The right-hand side is thus given by

$$\int_0^T \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} a_\alpha(s) e_k(s) H_{\alpha+\epsilon^{(k)}} \mathrm{d}s = \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} \int_0^T a_\alpha(s) e_k(s) \mathrm{d}s H_{\alpha+\epsilon^{(k)}}.$$

The process X_t is given in the first Wiener expansion by

$$X_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$$

with $f_n = \sum_{|\alpha|=n} a_\alpha(t) e^{\hat{\otimes}\alpha}$. Therefore,

$$\delta^I(X_t) = \delta^I(\sum_{n=0}^{\infty} I_n(f_n(\cdot,t))) = \sum_{n=0}^{\infty} \delta^I(I_n(f_n(\cdot,t))) = \sum_{n=0}^{\infty} \delta^I(I_n(\sum_{|\alpha|=n} a_\alpha(t) e^{\hat{\otimes}\alpha}),$$

as the operator δ^I is linear. Since e_k is an orthonormal basis, we can write

$$= \sum_{n=0}^{\infty} \delta^I(I_n(\sum_{|\alpha|=n} \sum_k \langle a_\alpha, e_k\rangle_{L^2[0,T]} e_k e^{\hat{\otimes}\alpha})) = \sum_{n=0}^{\infty} \delta^I(\sum_{|\alpha|=n} I_n(\sum_k \langle a_\alpha, e_k\rangle_{L^2[0,T]} e_k e^{\hat{\otimes}\alpha}))$$

and thus using the definition of the Skorohod integral, we have

$$= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} I_{n+1}(\sum_k \langle a_\alpha, e_k\rangle_{L^2[0,T]} e^{\hat{\otimes}\alpha} \hat{\otimes} e_k) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \sum_k \langle a_\alpha, e_k\rangle_{L^2[0,T]} I_{n+1}(e^{\hat{\otimes}\alpha+\epsilon^{(k)}}).$$

This simplifies to

$$\sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} \int_0^T a_\alpha(s) e_k(s) \mathrm{d}s H_{\alpha+\epsilon^{(k)}},$$

which shows that the two expressions are equal. □

This shows that, in direct combination with Example 3.10, the Skorohod integral is easily obtained if the second Wiener Itô expansion is known.

Corollary 3.1. *Let $X_t = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha$ be a square-integrable process which is Skorohod integrable. Then, we have*

$$\int_0^T X_s \delta W_s = \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} \langle c_\alpha, e_k\rangle_{L^2([0,T])} H_{\alpha+\epsilon^{(k)}} \tag{3.15}$$

and

$$\mathbb{E}[\int_0^T Y_s \delta W_s] = 0.$$

Proof. Using Theorem 3.10 it suffices to evaluate

$$\int_0^T X_s \diamond \dot{W}_s \mathrm{d}s = \int_0^T \sum_{\alpha \in \mathcal{J}, k \in \mathbb{N}} c_\alpha(s) e_k(s) H_{\alpha+\epsilon^{(k)}} \mathrm{d}s.$$

by the definition of the Wick product. But as seen in Example 3.10, this is equal to

$$\sum_{\alpha \in \mathcal{J}, k \in \mathbb{N}} \int_0^T c_\alpha(s) e_k(s) \mathrm{d}s H_{\alpha+\epsilon^{(k)}},$$

which concludes the proof. □

In this section, we introduced the Skorohod integral as a stochastic integral. We showed three different formulation for it. First, we defined it as the adjoint operator of the Malliavin derivative. Then, we defined with iterated Itô integrals. Lastly, we introduced the Wick calculus to define the Skorohod integral as a Pettis integral with the singular white noise process.

4 Stochastic Integrals with Fractional Brownian Motion

The Skorohod integral from the previous section will lay the foundation for the formulation of a stochastic fractional integral, which we will introduce here. In order to move from the Skorohod integral to a stochastic fractional integral, we will make use of so-called fractional operators, which originate from fractional calculus. These operators will "render" either the integrand or the white noise in a way that we obtain the desired properties of the stochastic integral.

As in the previous section, we will introduce the integral in two ways, as a divergence integral and as a Pettis integral, which is how they are described in the literature. Then, we will show that they coincide and thus describe the same integral.

4.1 The Divergence Integral

Here, we will introduce a fractional stochastic integral as the divergence operator of a Gaussian isonormal process.

In Example 3.1 it has been shown how a Brownian motion can be constructed using an isonormal process. This construction works due to the fact that we choose the inner product on the Hilbert space such that it coincides with what we expect from the covariance function of a Brownian motion.

This indicates that we can generalize this approach to represent a fractional Brownian motion as an isonormal process. In fact, let $\mathscr{H}$ be the closure of the set

$$\mathcal{E} = \mathrm{Span}\big(\{I_{[0,t]} : t \in [0,T]\}\big),$$

with respect to the inner product

$$\langle I_{[0,s]}, I_{[0,t]} \rangle_{\mathscr{H}} = \frac{1}{2}(t^{2h} + s^{2H} - |t-s|^{2H}) := R_H(s,t).$$

Clearly, the process

$$W_t^H = W^H(I_{[0,t]})$$

is a centered Gaussian process with covariance function

$$\mathbb{E}[W_s^H W_t^H] = \langle I_{[0,s]}, I_{[0,t]} \rangle_{\mathscr{H}} = R_H(s,t).$$

Using Kolmogorov's theorem, this process has a continuous modification. Therefore, we consider W_t^H as a fractional Brownian motion with Hurst-index H. The definition of the divergence type integral follows directly from the isonormal process W^H.

Definition 4.1. *Let X be an $\mathscr{H}$-valued square-integrable random variable such that for all $F \in \mathbb{D}_1$,*

$$|\mathbb{E}[\langle DF, X \rangle_{\mathscr{H}}]| \leq c\|F\|_2,$$

where D is the derivative operator and c some constant. We define the divergence stochastic integral of X as the random variable

$$\delta^H(X),$$

where $\delta^H \colon Dom\, \delta^H \to L^2(\Omega)$ is the divergence operator as defined in Definition 3.4.

Here, $\mathscr{H}$-valued square-integrable random variables can again be considered as square-integrable stochastic processes by identifying $L^2(\Omega, \mathscr{H})$ with $L^2(\Omega \times T)$.

First of all, let us confirm that the divergence integral is indeed centered.

Proposition 4.1. *Let $X_t \in L^2(\Omega \times T)$ be a stochastic process, which is in $Dom\, \delta^H$. Then, we have that*

$$\mathbb{E}[\delta^H(X_t)] = 0.$$

Proof. The (deterministic) process $F = 1$ has Malliavin derivative $DF = 0$. Therefore,

$$\mathbb{E}[\delta^H(X_t)] = \mathbb{E}[\delta^H(X_t) \cdot 1] = \mathbb{E}[\langle X_t, 0 \rangle_{\mathscr{H}}] = \mathbb{E}[0] = 0,$$

which follows by the definition of the divergence operator. □

While the definition of the divergence integral is straightforward, it can be very difficult to evaluate an integral due to the fact that the divergence operator is defined in a weak sense. This was less of a problem for the divergence operator δ in the white noise case, due to the fact that the Skorohod integral could be calculated using iterated integrals. We thus want to develop the connection between the fractional divergence operator δ^H and the regular divergence operator or Skorohod integral from the previous section. This link is established with the help of fractional operators. These operators play an important role for the construction of fractional stochastic integrals. We will, however, mainly consider their properties and refer for a full derivation to the existing literature, where they are discussed in full detail.

The idea of fractional operators is to establish an isometry between the spaces $\mathscr{H}$ and $L^2([0,T])$, which are the Hilbert spaces associated to δ^H and δ, respectively. It can be shown that there exists a kernel $K_H(t,s), 0 < s < t < T$, depending on the Hurst-index H, such that

$$\int_0^{t \wedge s} K_H(s,u) K_H(t,u) \mathrm{d}u = R_H(s,t).$$

An important property of the kernel is that it is differentiable in the first variable. Furthermore, it has different properties for $H > \frac{1}{2}$ and $H < \frac{1}{2}$, which are discussed in [13].

We use the kernel to define the linear operator $\mathcal{K}_H \colon \mathscr{H} \to L^2([0,T])$, which maps

$$\mathcal{K}_H(I_{[0,t]})(s) = K_H(t,s) I_{[0,t]}(s) \tag{4.1}$$

and is therefore an isometry.

Proposition 4.2 (and Definition)**.** *The linear operator*

$$\mathcal{K}_H\phi(s) = \mathcal{K}\phi(s) := \phi(s)K_H(T,s) + \int_s^T [\phi(t) - \phi(s)]\partial_1 K_H(t,s)\mathrm{d}t$$

satisfies Eq. (4.1) *and is an isometric isomorphism between* $\mathscr{H}$ *and* $L^2([0,T])$*.*

Proof. See for instance [13]. □

Theorem 4.1. *Let* $X \in \mathbb{D}^{H^1}$ *be a random variable. Then,*

$$D^H X = \mathcal{K}^{-1}(DX) \in L^2(\Omega; \mathscr{H})$$

Furthermore, if $Y \in$ *Dom* δ^H *is a* $\mathscr{H}$*-valued random variable, then,*

$$\delta^H(Y) = \delta(\mathcal{K}Y) \in L^2(\Omega).$$

Proof. It suffices to show that the equality is true for all smooth random variables. Let $f \in C_b(\mathbb{R})$ be given and let

$$X = f(W^H(h_1), W^H(h_2), \ldots W^H(h_n)).$$

The derivative of X is given by

$$\begin{aligned} D^H X &= \sum_{k=1}^n \partial_k f(W^H(h_1), W^H(h_2), \ldots W^H(h_n))h_k \\ &= \sum_{k=1}^n \partial_k f(W(\mathcal{K}h_1), W(\mathcal{K}h_2), \ldots W(\mathcal{K}h_n))h_k \\ &= \mathcal{K}^{-1}\sum_{k=1}^n \partial_k f(W(\mathcal{K}h_1), W(\mathcal{K}h_2), \ldots W(\mathcal{K}h_n))\mathcal{K}h_k \\ &= \mathcal{K}^{-1}(DX), \end{aligned}$$

which shows the first equality.

To prove the second equality, we need to show that

$$\mathbb{E}[X\delta(\mathcal{K}Y)] = \mathbb{E}[\langle DX, Y\rangle_{\mathscr{H}}],$$

for any $X \in \mathbb{D}^{H^1}$. As K is an isometry, we have that

$$\mathbb{E}[\langle D^H X, Y\rangle_{\mathscr{H}}] = \mathbb{E}[\langle \mathcal{K}(D^H X), \mathcal{K}Y\rangle_{L^2([0,T])}] = \mathbb{E}[\langle \mathcal{K}\mathcal{K}^{-1}(DX), \mathcal{K}Y\rangle_{L^2([0,T])}],$$

and thus

$$\mathbb{E}[\langle D^H X, Y\rangle_{\mathscr{H}}] = \mathbb{E}[\langle DX, \mathcal{K}Y\rangle_{L^2([0,T])}].$$

By the definition of the divergence operator, the right-hand side is equal to

$$\mathbb{E}[X\delta(\mathcal{K}Y)].$$

□

This shows that one can interpret the fractional stochastic integral $\delta^H(X)$ as a regular Skorohod integral, where the kernel $\mathcal{K}$ is applied to the integrand.

4.2 The Regularity of the Fractional Operator

The fractional operator $\mathcal{K}$ establishes the connection between the regular white noise setting and the fractional white noise setting. It is well-known that in the regular white noise setting, and thus for the Itô integral, a relatively large class of processes is integrable. For instance, an adapted process X_t is Itô integrable, if

$$\mathbb{E}[\int_0^T X_t^2 \mathrm{d}t] < \infty, \tag{4.2}$$

which in the case of deterministic integrals reduces to being square-integrable.

Whether a process X_t is thus integrable with respect to the fractional Brownian motion roughly depends on whether $\mathcal{K}(X_t)$ fulfills Eq. (4.2). It will thus be the properties of this operator which will answer the question of integrability.

This is where difficulties with the divergence integral arise, since the operator consists of integrals with singularities, meaning that the domain of integration is severely impacted. To see this issue, let us first consider a deterministic function $f \in L^2([0,T])$, and ask ourselves the question: "Under which conditions for f does

$$\delta^H(f) = \delta(\mathcal{K}f) \tag{4.3}$$

exist?" Since the properties of $\mathcal{K}$ heavily depend on the Hurst index, we will make a distinction between $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

Case $H > \frac{1}{2}$

In this case the paths of the fractional Brownian motion are trending and more regular than for the regular Brownian motion. This leads to the fact that the operator $\mathcal{K}$ has certain regularity. We can show that Eq. (4.3) exists for any $f \in L^2([0,T])$ by showing that

$$L^2([0,T]) \subset \mathscr{H}. \tag{4.4}$$

The idea of the proof is to consider a linear subspace of $|\mathscr{H}|$ of $\mathscr{H}$ induced by a metric $\|\cdot\|_{|\mathscr{H}|}$. The special property this metric has is that we have that

$$\|f\|_{|\mathscr{H}|} \leq C\|f\|_{L^{\frac{1}{H}}[0,T]},$$

for some constant C. This in particular implies that

$$L^{\frac{1}{H}}([0,T]) \subset \|f\|_{|\mathscr{H}|},$$

and since $H > \frac{1}{2}$, we have $\frac{1}{H} < 2$. This shows that

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathscr{H}| \subset \mathscr{H},$$

and thus that Eq. (4.4) holds. An analytic expression for $\|\cdot\|_{|\mathscr{H}|}$ and a more detailed proof can be found in [33].

Case $H < \frac{1}{2}$

The previous proof does not work for this case anymore, since $H < \frac{1}{2}$ implies that $\frac{1}{H} > 2$ and thus the inclusion

$$L^{\frac{1}{H}}(0,T) \subset L^2([0,T])$$

is not true anymore. In fact, it can be shown that for $H < \frac{1}{2}$ the space $L^2[(0,T)]$ is strictly larger than $\mathscr{H}$ [10].

The reason for this is that for $H < \frac{1}{2}$, the operator $\mathcal{K}$ has problematic singularities, which require the function $f \in \mathscr{H}$ to have a certain Hölder-continuity. One can show that

$$C^{\gamma}_{\text{Hölder}} \subset \mathcal{H},$$

for all $\gamma > \frac{1}{2} - H$. Here, $C^{\gamma}_{\text{Hölder}}$ denotes all Hölder-continuous functions with Hölder constant γ.

As expected, these problems with integration are not exclusive to deterministic integrals. Consider the equation

$$\mathrm{d}X_t = W^H_t \mathrm{d}W^H_t. \tag{4.5}$$

This equation can be considered elementary, since it is certainly linear and in general we would expect a solution to exist. However, as Cheridito and Nualart [14] pointed out, for $H < \frac{1}{4}$, this integral does not exist due to the fact that the paths become too irregular for $H < \frac{1}{4}$. In the same paper, they propose an extension of the divergence integral δ^H to a larger space using limit properties. However, while the proposed extension allows for a solution of Eq. (4.5), it is not enough to remedy all the issues occurring for $H < \frac{1}{2}$.

4.3 Wick-Itô-Skorohod Integral

We will now show the construction of the Malliavin-based fractional stochastic integral using the foundations of classical white noise theory. This construction is originally attributed to Elliott and van der Hoek and further described mainly by Øksendal and Biagini. Note that in these original papers, the authors usually work on $\mathbb{R}$. We use an approach adjusted to a fixed interval $[0, T]$ such that the formulation coincides with the approach from the previous section.

For the construction of this integral, we will again use the fractional operator to render the integral. However, instead of applying the operator on the integrand, as in the previous section, we will apply the operator to the white noise process. In fact, we want to find a "fractional" white noise process $\dot{W}^H_t$, such that the fractional integral still takes the form of the right-hand side of Theorem 3.10, but the white noise process $\dot{W}_t$ replaced by $\dot{W}^H_t$.

Note that due to Theorem 4.1, we have that for a fBm,

$$W^H_t = \delta^H(I_{[0,t]}) = \delta(\mathcal{K}I_{[0,t]}),$$

where the right-hand side is the usual Skorohod integral. By Theorem 3.10, we can thus write

$$W^H_t = \int_0^T \mathcal{K}I_{[0,t]}(s) \diamond \dot{W}_s \mathrm{d}s = \int_0^T \mathcal{K}I_{[0,t]}(s) \diamond \dot{W}_s \mathrm{d}s = \sum_{k=0}^{\infty} \langle \mathcal{K}I_{[0,t]}, e_k \rangle_{L^2} H_{\epsilon^{(k)}}. \tag{4.6}$$

Now, consider again the integral operator from Proposition 4.2. The operator is well defined for all e_k

Lemma 4.2. *The operator $\mathcal{K}$ has an adjoint operator $\mathcal{K}^*$ on $L^2([0,T])$, which is given by*

$$\mathcal{K}^*(\phi)(s) = \int_0^s \phi(u) \partial_1 K(s,u) \mathrm{d}u.$$

Proof. It is easy to see that $\mathcal{K}$ is well-defined on e_k for all k, as they are continuous functions.

Since $\{e_k : k \in \mathbb{N}\}$ is densely defined in $L^2([0,T])$, the operator $\mathcal{K}$ is densely defined and thus there exists an adjoint operator by the Riesz representation theorem. To show that it is equal to the expression above, let f, g be functions. By the definition of the adjoint, we then have

$$\begin{aligned}\int_0^T \mathcal{K}f(s)g(s)\mathrm{d}s &= \int_0^T \int_s^T f(u)\partial_1 K(u,s)g(s)\mathrm{d}u\mathrm{d}s \\ &= \int_0^T \int_0^u f(u)\partial_1 K(u,s)g(s)\mathrm{d}s\mathrm{d}u \\ &= \int_0^T f(u)\int_0^u g(s)\partial_1 K(u,s)\mathrm{d}s\mathrm{d}u,\end{aligned}$$

where we have changed the order of integration. □

By Eq. (4.6), we thus have

$$W_t^H = \sum_{k=0}^{\infty} \langle I_{[0,t]}, \mathcal{K}^* e_k \rangle_{L^2([0,T])} H_{\epsilon^{(k)}} = \sum_{k=0}^{\infty} \int_0^t \mathcal{K}^* e_k(s)\mathrm{d}s H_{\epsilon^{(k)}}.$$

This calculation indicates what the choice for the fractional white noise should be.

Definition 4.2. *Let T, H be fixed. The process*

$$\dot{W}_t^H = \mathcal{K}^* e_k(t) H_{\epsilon^{(k)}}$$

is called the fractional white noise *on $[0,T]$. Furthermore, for a function $Y_t \colon [0,T] \to (\mathcal{S})^*$, we define the* Wick Itô Skorohod (WIS) integral *as*

$$\delta_{\mathrm{WIS}}^H(Y_t) := \int_0^T Y_t \diamond \dot{W}_t^H \mathrm{d}t,$$

where the right-hand side is the Pettis integral according to Definition 3.14, assuming the integral exists. In that case, we call Y_t WIS-integrable.

Remark 4.1. We will use the notation

$$e_k^H(t) := \mathcal{K}^* e_k(t).$$

Example 4.1. We will consider the fractional stochastic integral with W_t^H as the integrand. We know that

$$W_t^H = \sum_k \int_0^t e_k^H(s)\mathrm{d}s H_{\epsilon^{(k)}}.$$

Therefore, the Wick product is given by

$$W_t^H \diamond \dot{W}_t^H = \sum_{k,l} \int_0^t e_k^H(s)\mathrm{d}s\, e_l^H(t) H_{\epsilon^{(k)}+\epsilon^{(l)}}.$$

This implies that

$$\begin{aligned}\delta_{\mathrm{WIS}}^H(W_t^H) = \int_0^T W_t^H \diamond \dot{W}_t^H \mathrm{d}t &= \sum_{k,l} \int_0^T \int_0^t e_k^H(s)\mathrm{d}s\, e_l^H(t)\mathrm{d}t\, H_{\epsilon^{(k)}+\epsilon^{(l)}} \\ &= \frac{1}{2}\sum_{k,l} \int_0^T e_k^H(s)\mathrm{d}s \int_0^T e_l^H(s)\mathrm{d}s\, H_{\epsilon^{(k)}+\epsilon^{(l)}},\end{aligned}$$

where the last equality follows from integration by parts .

Note that this is exactly equal to $W_T^H \diamond W_T^H$. △

Let us now examine the link between the divergence integral from Section 4.1 and the WIS integral as introduced above. The connection between the integrals is due to the direct link to their respective definitions of the Skorohod integral.

Theorem 4.3. *Let Y_t be a square-integrable stochastic process which is in Dom δ^H. Then, the divergence integral and WIS integral coincide.*

$$\delta^H_{\mathrm{WIS}}(Y_t) = \delta^H(Y_t). \tag{4.7}$$

Proof. Let $Y_t = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha$ be the Wiener-Itô expansion of Y_t. Then, the left-hand side is given by

$$\begin{aligned}\int_0^T \sum_{k\in\mathbb{N},\alpha\in\mathcal{J}} c_\alpha(t) e^H_k(t) H_{\alpha+\epsilon^{(k)}} \mathrm{d}t &= \sum_{k\in\mathbb{N},\alpha\in\mathcal{J}} \int_0^T c_\alpha(t)\mathcal{K}^* e_k(t)\mathrm{d}t H_{\alpha+\epsilon^{(k)}} \\ &= \sum_{k\in\mathbb{N},\alpha\in\mathcal{J}} \int_0^T \mathcal{K} c_\alpha(t) e_k(t)\mathrm{d}t H_{\alpha+\epsilon^{(k)}} \\ &= \int_0^T \mathcal{K} Y_t \diamond \dot{W}_t \mathrm{d}t\end{aligned}$$

On the other hand, the right-hand side of Eq. (4.7) is equal to

$$\delta^H(Y_t) = \delta(\mathcal{K}Y_t),$$

according to Theorem 4.1. Lastly, by Theorem 3.10, we see that the two sides are equal. □

The WIS-integral provides us thus with a straightforward way to calculate stochastic fractional integrals. Furthermore, thanks to Proposition 3.10, we have a classification for the domain of the WIS-integral. However, as for the divergence integral, the domain might be restricted, in particular for integrals with Hurst index $< \frac{1}{2}$.

4.4 Xia Integral

We will now analyse the stochastic fractional integral developed by Xia et al. [16]. This approach is similar to formulations by for instance Duncan [7], which form the basis for the WIS integral. Xia et al. also show that their formulation is equivalent to the divergence integral from Section 4.1. Due to Eq. (4.7), a direct link between the WIS-integral and the Xia integral must exist. Showing this link is the main result of this section.

The idea of Xia's formulation is intuitive and does not require any Malliavin calculus besides the Wick product.

Definition 4.3. *Let Y_t be a stochastic process, such that the limit*

$$\lim_{\epsilon\to 0+} \frac{1}{\epsilon} \int_0^T Y_t \diamond (W^H_{t+\epsilon} - W^H_t)\mathrm{d}t$$

converges in probability to a random variable $\delta^H_{\mathrm{Xia}}(Y_t)$. In this case, we call Y_t Xia-integrable *and call $\delta^H_{\mathrm{Xia}}(Y_t)$ the* Xia integral *of Y_t.*

Remark 4.2. Since the Wick product can be defined in various ways, the Xia integral can also be defined in different ways and not necessarily on the white noise probability space. Here, however, we will consider it as a random variable on the white noise probability space.

Theorem 4.4. *Let $Y_t \in [0,T] \times (\mathcal{S})^*$ be a Xia-integrable random process which is also WIS-integrable in $(\mathcal{S})^*$ Then, we have that*

$$\delta^H_{\mathrm{Xia}}(Y_t) = \delta^H_{\mathrm{WIS}}(Y_t)$$

in $(\mathcal{S})^*$.

Proof. Let $Y_t = \sum_{\alpha\in\mathcal{J}} c_\alpha(t)H_\alpha$ be the Wiener-Itô expansion of Y_t. For any $t \in [0,T], \epsilon > 0$, the fractional Brownian increment is given by

$$W^H_{t+\epsilon} - W^H_t = \sum_k \int_t^{t+\epsilon} e^H_k(s)\mathrm{d}s\ H_{\epsilon^{(k)}}.$$

We can write the limit of the Xia integral as

$$\begin{aligned}
&\lim_{\epsilon\to 0+} \frac{1}{\epsilon} \int_0^T \sum_{\alpha\in\mathcal{J}} c_\alpha(t)H_\alpha \diamond (W^H_{t+\epsilon} - W^H_t)\mathrm{d}t \\
&= \lim_{\epsilon\to 0+} \frac{1}{\epsilon} \int_0^T \sum_{\alpha\in\mathcal{J}} c_\alpha(t)H_\alpha \diamond \sum_k \int_t^{t+\epsilon} e^H_k(s)\mathrm{d}s\ H_{\epsilon^{(k)}}\mathrm{d}t \\
&= \lim_{\epsilon\to 0+} \frac{1}{\epsilon} \int_0^T \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} c_\alpha(t) \int_t^{t+\epsilon} e^H_k(s)\mathrm{d}s\ H_{\alpha+\epsilon^{(k)}}\mathrm{d}t \\
&= \int_0^T \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} c_\alpha(t) \lim_{\epsilon\to 0+} \frac{1}{\epsilon} \int_t^{t+\epsilon} e^H_k(s)\mathrm{d}s\ H_{\alpha+\epsilon^{(k)}}\mathrm{d}t \\
&= \int_0^T \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} c_\alpha(t)e^H_k(t)\ H_{\alpha+\epsilon^{(k)}}\mathrm{d}t \\
&= \int_0^T Y_t \diamond \dot{W}^H_t \mathrm{d}t.
\end{aligned}$$

This proves the equality. □

In this section, we introduced three formulations to a Malliavin-based fractional stochastic integral. We have shown that the methods are equivalent if the integrals exists. Furthermore, we have shown their connection to fractional operators and the Skorohod integral from Section 3.

5 Numerical Solutions to Stochastic Differential Equation

The previous sections built the theoretical foundation for stochastic integration with a fractional Brownian motion. We will now apply these results to simulate numerical solutions to stochastic differential equations (SDEs). In particular, the goal of this section is to find solutions to

$$\mathrm{d}Y_t = \mu(t)Y_t\mathrm{d}t + \sigma(t)Y_t\mathrm{d}W^H_t, \tag{5.1}$$

where the second term is a stochastic fractional integral and μ, σ are deterministic functions. We will interpret the stochastic fractional integral as a WIS integral, because the numerical scheme is mainly based on the results of Section 4.3.

As with the theoretical part, we will first show the result for the non-fractional case, i.e. when $H = \frac{1}{2}$ and $W^H_t = W_t$ is a regular Brownian motion. Note that for this case, many other numerical methods exist to find solutions to Eq. (5.1)

5.1 A Malliavin-Calculus Approach for a Brownian Motion Simulation

We start by finding solutions to the easiest type of SDEs. We want to find paths for

$$\mathrm{d}Y_t = \mathrm{d}W_t^H,$$

which means that Y_t is a (fractional) Brownian motion.

Regular Brownian Motion

The common way to simulate paths of a Brownian motion is via a random walk approximation. A path is set on a finite grid where each increment is drawn independently from a distribution. In this section, we will introduce another numerical scheme, based on white noise theory and the Wiener-Itô expansion from Theorem 3.7. The resulting paths can be evaluated on a continuous spectrum, which is different to the random walk approximation.

Remember from Example 3.9 that the Wiener-Itô expansion for a Brownian motion is given by

$$W_t = \sum_k \int_0^t e_k(s)\mathrm{d}s \; H_{\epsilon^{(k)}}.$$

The idea is to approximate the sum with a finite sum

$$W_t(\omega) \approx W_t^N := \sum_{k=1}^N \int_0^t e_k(s)\mathrm{d}s \; H_{\epsilon^{(k)}}(\omega). \tag{5.2}$$

for some $\omega \in \mathcal{S}^*$. One can then evaluate the sum at any $t \in [0, T]$ for any ω. Two questions arise which need to be addressed:

i) How does one draw samples from $\mathcal{S}^*$, the dual space of the Schwartz functions?

ii) What are the modes of convergence?

To restate i), in order to evaluate Eq. (5.2), one needs to sample the random vector

$$H_\epsilon^N(\omega) = \begin{bmatrix} H_{\epsilon^{(1)}}(\omega) \\ H_{\epsilon^{(2)}}(\omega) \\ \dots \\ H_{\epsilon^{(N)}}(\omega) \end{bmatrix} = \begin{bmatrix} \langle e_1, \omega \rangle \\ \langle e_2, \omega \rangle \\ \dots \\ \langle e_N, \omega \rangle \end{bmatrix}.$$

Due to the fact that $\langle e_k, \cdot \rangle$ are Gaussian random variables, drawing an ω from $\mathcal{S}^*$ is not necessary. Note that since $\{e_k : k \in \mathbb{N}\}$ forms an orthonomal basis of $L^2([0, T])$, the random vector H_ϵ^N follows a multivariate normal distribution with covariance matrix

$$\Sigma = \begin{pmatrix} \langle e_1, e_1 \rangle_2 & \langle e_1, e_2 \rangle_2 & \dots & \langle e_1, e_N \rangle_2 \\ \langle e_2, e_1 \rangle_2 & \langle e_2, e_2 \rangle_2 & \dots & \langle e_2, e_N \rangle_2 \\ & \dots & & \\ \langle e_N, e_1 \rangle_2 & \langle e_N, e_2 \rangle_2 & \dots & \langle e_N, e_N \rangle_2 \end{pmatrix} = I_N.$$

The random variables $\langle e_k, \cdot \rangle$ are thus independent standard normals and we can simply sample N standard normal samples to generate a realization for H_ϵ^N. After evaluating

$$\int_0^t e_k(s)\mathrm{d}s$$

for each $k \leq N$, we can evaluate Eq. (5.2) and obtain sample paths for W_t. Fig. 1 shows the simulation of 10 paths of a Brownian motion with $N = 100$. An example python code is listed in Appendix C.2

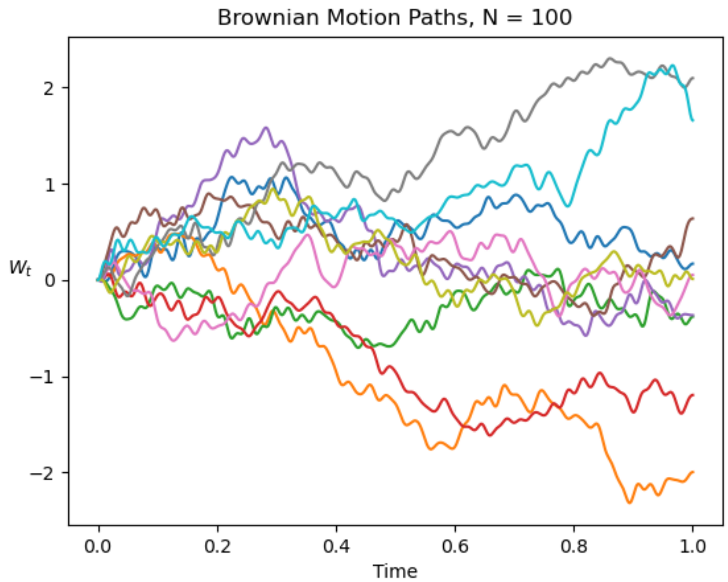

Figure 1: Simulation of 10 paths using the white noise method

Let us now address the second question. Suppose we fix some time t and consider the sequence of random variables

$$\{W_t^n : n \in \mathbb{N}\}.$$

Proposition 5.1. *For all $t \in [0, T]$, the sequence $\{W_t^n : n \in \mathbb{N}\}$ converges, both, almost surely and in $L^2(\Omega)$, to the random variable W_t.*

Proof. Note that L^2-convergence is implied by the fact that W_t is well-defined. For completeness, we will show it nevertheless. We can directly evaluate the integral of the orthonormal vectors e_k and find

$$\left|\int_0^t e_k(s)\mathrm{d}s\right| = \left|\frac{\sqrt{2T}}{\pi k}\left(1 - \cos\left(\frac{k\pi}{T}\right)\right)\right| \leq \frac{\sqrt{8T}}{\pi k}.$$

Then, we have

$$\begin{aligned}
\mathbb{E}[(W_t - W_t^n)^2] &= \mathbb{E}\left[\left(\sum_{k=n+1}^{\infty}\int_0^t e_k(s)\mathrm{d}s H_{\epsilon^{(k)}}\right)^2\right] \\
&\leq \sum_{k=n+1}^{\infty}\frac{8T}{\pi^2k^2}\mathbb{E}[H^2_{\epsilon^{(k)}}] \\
&= \sum_{k=n+1}^{\infty}\frac{8T}{\pi^2k^2} \\
&\to 0
\end{aligned}$$

as $n \to \infty$ due to the fact that $\sum_{k=1}^{\infty} \frac{8T}{\pi^2 k^2}$ is a convergent sum. This shows that W_t^n converges in L^2 and therefore also in probability.

It remains to prove that the sequence converges almost surely. We will do so by applying Kolmogorov's Three-Series theorem. The convergence in L^2 implies that the variance of the limit is bounded. Furthermore, as each $H_{\epsilon(k)}$ is centered, we also have that $\mathbb{E}[W_t^n] \to 0$. It remains to show that there exists some $A > 0$, such that

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\left|\int_0^t e_k(s)\mathrm{d}s H_{\epsilon(k)}\right| > A\right) < \infty. \tag{5.3}$$

We see that

$$\mathbb{P}\left(\left|\int_0^t e_k(s)\mathrm{d}s H_{\epsilon(k)}\right| > A\right) \leq \mathbb{P}\left(H_{\epsilon(k)} > \frac{A\pi k}{\sqrt{2T}}\right).$$

Choosing $A = \frac{\sqrt{2T}}{\pi}$, the equation Eq. (5.3) is certainly true as $H_{\epsilon(k)}$ is standard normal. □

Let us further analyze efficiency of this method. In the previous proposition we showed an upper bound for the variance of the error term $W_t - W_t^N$. We now create a sample of 100'000 paths for $t = 10$, for different values of N and compare the run time with the error variance. Table 1 gives a comparison of this upper bound to the sample variance.

N	Theoretical upper bound	Sample variance	Run time
10	0.771388	0.397591	0.03s
100	0.077139	0.033971	2.99s
1000	0.007714	0.003401	12.54s

Table 1: Error variance analysis

Note that compared to Brownian motion approximations from symmetric random walks, this simulation has two advantages:

- One can calculate $W_t^N(\omega)$ for any discrete time point $t \in \{t_1, t_2, \ldots, t_n\} \subset [0, T]$ without having to simulate any other time steps.
- The error of the approximation is controlled by N and does not depend on a time discretization.
- Once the values for $\int_0^t e_k(s)\mathrm{d}s$ are calculated, the computational cost to evaluate a new path is small, and only a single standard normal random sample needs to be drawn.

The simulation is thus especially effective for cases where the path only needs to be simulated at a few time steps, rather than over $[0, T]$.

Fractional Brownian Motion

We can now use the same procedure to simulate paths of a fractional Brownian motion. We approximate

$$W_t^H(\omega) \approx W_t^{H^N} := \sum_{k=1}^{N} \int_0^t e_k^H(s)\mathrm{d}s \ H_{\epsilon(k)}(\omega). \tag{5.4}$$

For $H > \frac{1}{2}$, the value for $e_k^H(s) = \mathcal{K}^* e_k(s)$ can be computed by computing the analytical expression from Lemma 4.2. Since we do not know an analytical expression for $\mathcal{K}^*$ if $H < \frac{1}{2}$, this is not possible. Nevertheless, it can be used that

$$\int_0^t e_k^H(s)\mathrm{d}s = \int_0^T \mathcal{K}I_{[0,t]}(s)e_k(s)\mathrm{d}s = \int_0^T K_H(t,s)e_k(s)\mathrm{d}s,$$

which can be evaluated. We simulate 10 paths of a fractional Brownian motion using this approximation, which are shown in Fig. 2, for $H = 0.7$.

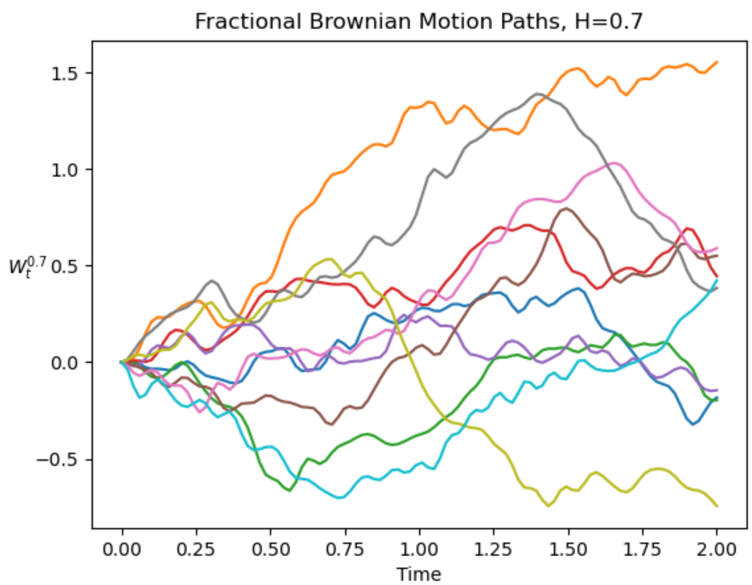

Figure 2: Simulation of 10 paths

Let us again compare the error in variance of the method for different values of N. We set $t = 2$ and compare the sample variance of W_t^H to the variance of the fBm, which is given by

$$\mathbb{E}[W_2^H] = 2^{2H}.$$

In Table 2, the results are presented for a chosen Hurst index of $H = 0.7$. We can see that the sample variance is approaching the actual variance from below. This shows the convergence of the method. From $N = 200$ on, however, the difference increases again. The cause for this divergence is that the numerical integration becomes more difficult as N increases, due to the fact that $e_n(s)$ oscillates more for high values of n. This suggests that a different choice for $e_n(s)$ would be more appropriate, such as a basis based on indicator functions.

This issue is especially noticeable for values of $H < 0.5$, which is why we will center our analysis on values $H > 0.5$ from now on.

Remark 5.1. Note that the divergence did not occur in Table 1. This is due to the fact that there exists an analytical expression for

$$\int_0^t e_k(s)\mathrm{d}s,$$

meaning no numerical integration is necessary.

$H = 0.7$				
N	Variance	Sample variance	Δ	Run time
10	2.639016	2.502646	0.136370	0.03s
50	2.639016	2.582831	0.056185	0.46s
100	2.639016	2.612176	0.02684	0.77s
200	2.639016	2.528869	0.110147	2.25s
500	2.639016	5.161344	-2.522328	7.5s

Table 2: Sample variance $H = 0.7$

Besides analyzing the sample variance, we also want to test the samples for autocorrelation. We do this with a Durbin-Watson statistic, which is mainly used to detect autocorrelation in residuals of regressions. Suppose that the sequence $\Delta W_t^H = W_t^H - W_{t-1}^H$ is given for a grid of time points. The Durbin-Watson test then evaluates

$$d = \frac{\sum_{t=2}^{T}(\Delta W_t^H - \Delta W_{t-1}^H)^2}{\sum_{t=1}^{T}(\Delta W_t^H)^2}$$

The value $d \in [0, 4]$ thus indicates the amount of autocorrelation, with a lower score indicating positive autocorrelation and $d = 2$ in case of no autocorrelation.

To test for autocorrelation in our method, we sample 1000 paths with $N = 50$ and 100 time points. The mean and variance of d for different values of H are shown in Table 3.

$H = 0.7$		
H	Mean d	Variance d
0.6	0.596609	0.013117
0.7	0.491339	0.0134853
0.8	0.376435	0.013240
0.9	0.259012	0.012240

Table 3: Various Durbin-Watson statistics

The decreasing values of d clearly show the increase in positive autocorrelation, which is what we expect from a fBm.

5.2 Stochastic Integrals with Deterministic Integrals

Let us now consider a slightly more complex SDE. In this step, we will find paths to solve

$$\mathrm{d}Y_t = \sigma(t)\mathrm{d}W_t^H,$$

where $\sigma(t)$ is a suitable deterministic function.

The process Y_t is a Wiener-integral which is written in integral form as

$$Y_t = \int_0^t \sigma(t)\, \mathrm{d}W_t^H,$$

where $t \leq T$.

In the non-fractional case, as a result of Theorem 3.10, we have that Y can be written as

$$Y_t = \sum_k \int_0^T \sigma(s)e_k(s)\mathrm{d}s\ H_{\epsilon(k)}.$$

We can thus approximate this integral with

$$Y_t \approx \sum_{k=1}^{N} \int_0^t \sigma(s) e_k(s) \mathrm{d}s \ H_{\epsilon^{(k)}},$$

and we only need to calculate k deterministic integrals at any time t to find the coefficients. These coefficients can be calculated numerically or, in special cases, analytically.

Let us now consider the fractional case, where $H \neq \frac{1}{2}$. Due to the Definition of the WIS-integral (Definition 4.2), we can write

$$Y_t = \sum_k \int_0^t \sigma(s) e_k^H(s) \mathrm{d}s \ H_{\epsilon^{(k)}} = \sum_k \int_0^t \sigma(s) \mathcal{K}^* e_k(s) \mathrm{d}s \ H_{\epsilon^{(k)}},$$

which we again approximate as a finite sum with terms including up until N. Here, the coefficients can be calculated in two separate ways, either by directly using the adjoint $\mathcal{K}^*$ and thus evaluating

$$\int_0^t \sigma(s) \mathcal{K}^* e_k(s) \mathrm{d}s,$$

or using $\mathcal{K}$ by noting that

$$\int_0^t \sigma(s) \mathcal{K}^* e_k(s) \mathrm{d}s = \int_0^T I_{[0,t]}(s) \sigma(s) \mathcal{K}^* e_k(s) \mathrm{d}s = \int_0^T \mathcal{K}(I_{[0,t]}\sigma)(s) e_k(s) \mathrm{d}s.$$

As $\mathcal{K}^*$ does not have an analytical expression for $H < \frac{1}{2}$, the first method does not work for this case.

Remark 5.2. In order to properly implement this method, an efficient calculation of $\mathcal{K}^* e_k$ or $\mathcal{K}\sigma$ is necessary. Although one possible implementation is given in Appendix C.2, we will focus for now on the theoretical aspects and leave the efficient implementation for further studies.

5.3 Exponential Wick Calculus

Note that SDEs of the type Eq. (5.1) have a stochastic integrand, as Y_t itself is stochastic. In this section, we will show that it is possible to express solutions to Eq. (5.1) using only deterministic integrals. This requires us to first extend the results for the Wick calculus of Section 3.5.

5.3.1 Non-fractional Case

We will here consider solutions to the differential equation

$$\mathrm{d}Y_t = Y_t \mathrm{d}W_t,$$

with initial value $Y_0 = y_0$, almost surely. The solution to this equations is well-known as the Doléans-exponential $y_0\mathcal{E}(W_t)$, where [20, Section 7.4],

$$Y_t = \mathcal{E}(W_t) := \exp\left(W_t - \frac{1}{2}t\right).$$

Using the equivalence of the Skorohod integral and the Pettis integral, we can write the equation in the Wick product form as

$$\mathrm{d}Y_t = Y_t \diamond \dot{W}_t \ \mathrm{d}t. \tag{5.5}$$

This suggests that a solution with Wick products, which is equivalent to the Doléans-exponential, should be available.

Consider the k-th Wick power, which we write as

$$X^{\diamond k} = \underbrace{X \diamond X \diamond \cdots \diamond X}_{k \text{ times}}.$$

One can find $X^{\diamond k}$ quite easily for any X of the type

$$X = \int_0^T f(t) \mathrm{d}W_t,$$

due to the fact that

$$X^{\diamond k} = I_1(f)^{\diamond k} = I_k(f^{\otimes \cdot k}).$$

This, on the other hand, can be written in terms of Hermitian polynomials as [27, p. 1.19]

$$I_k(f^{\otimes \cdot k}) = \|f\|^k h_k \left(\frac{X}{\|f\|} \right).$$

Hence, we have that

$$W_t^{\diamond k} = \sqrt{t^k} h_k \left(\frac{X}{\sqrt{t}^k} \right).$$

Remark 5.3. Note how the various definitions of the Skorohod integral are still important, as the different Wiener-Itô expansions have different properties.

Combining these facts, we can find a solution to Eq. (5.5) the following way.

Proposition 5.2. *Let the* Wick-exponential *be defined as*

$$\exp^{\diamond}(X) := \sum_{k=0}^{\infty} \frac{1}{k!} X^{\diamond k},$$

whenever the sum converges in $(\mathcal{S})^*$. *The Wick-exponential of the Brownian motion* $Y_t = y_0 \exp^{\diamond}(W_t)$ *is a solution to the initial-value problem of Eq.* (5.5).

Proof. The Wick-exponential of the Brownian motion is given by

$$\exp^{\diamond}(W_t) := \sum_{k=0}^{\infty} \frac{1}{k!} W_t^{\diamond k} = \sum_{k=0}^{\infty} \frac{\sqrt{t^k}}{k!} h_k \left(\frac{W_t}{\sqrt{t}} \right).$$

By the generating function for Hermitian polynomials, this is exactly equal to the Doléans-exponential,

$$\exp^{\diamond}(W_t) = \mathcal{E}(W_t).$$

Therefore, we conclude that $Y_t = y_0 \exp^{\diamond}(W_t)$ is a solution to Eq. (5.5) and thus to the differential equation

$$\mathrm{d}Y_t = Y_t \mathrm{d}W_t,$$

with initial value y_0. □

Remark 5.4. The Wick exponential in $(\mathcal{S})^*$ has the same properties as the regular exponential due to the fact that the Wick product is associative. For example, one can show that, [27, p. 5.34],

$$\exp^{\diamond}(X + Y) = \exp^{\diamond}(X) \diamond \exp^{\diamond}(Y). \tag{5.6}$$

We can further confirm that $\exp^{\diamond}(W_t)$ is a solution to Eq. (5.5) by utilizing Wick calculus. Opposed to regular SDEs, which are always to be understood as integral equations, the Wick SDEs of the form Eq. (5.5) can be considered in their differential form, where we write

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = Y_t \diamond \dot{W}_t.$$

The left-hand side is then the Malliavin derivative in $(\mathcal{S})^*$. A comprehensive overview of Wick calculus can be found in Appendix A.3 of [15]. By the Wick chain rule [15, Theorem A.3.4], we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} W_t^{\diamond k} = k W_t^{\diamond k-1} \diamond \dot{W}_t$$

and therefore

$$\frac{d}{dt} \exp^{\diamond}(W_t) = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{1}{k!} W_t^{\diamond k} = \exp^{\diamond}(W_t) \diamond \dot{W}_t. \tag{5.7}$$

Remark 5.5. Despite the notation, the expression $\frac{d}{dt} \exp^{\diamond}(W_t)$ is not to be interpreted as a derivative with respect to time t, but rather a derivative with respect to $\omega \in \mathcal{S}^*$, as it is a Malliavin derivative.

A generalization to an SDE with a drift and diffusion coefficient follows directly. Consider the equation

$$\mathrm{d}Y_t = \mu(t) Y_t \mathrm{d}t + \sigma(t) Y_t \mathrm{d}W_t.$$

The solution to this SDE is again given by the Doléans exponential $\mathcal{E}(X_t)$, where $dX_t = \mu(t)dt + \sigma(t)dW_t$ is a semimartingale.

The equation can again be written in terms of a Wick equation as

$$dY_t = \mu(t) \diamond Y_t dt + \sigma(t) \diamond Y_t \diamond \dot{W}_t dt = Y_t \diamond (\mu(t) + \sigma(t)\dot{W}_t) dt,$$

A natural guess for a solution is thus

$$Y_t = \exp^{\diamond}(X_t),$$

with $X_t = \int_0^t (\mu(s) + \sigma(s)\dot{W}_s) \mathrm{d}s$. By the Wick chain rule, we again find

$$\frac{d}{dt} X_t^{\diamond k} = k X_t^{\diamond k-1} \diamond (\mu(t) + \sigma(t)\dot{W}_t),$$

and thus also

$$\frac{d}{dt} \exp^{\diamond}(X_t) = \exp^{\diamond}(X_t) \diamond (\mu(t) + \sigma(t)\dot{W}_t),$$

which satisfies the SDE.

5.3.2 Fractional Case

The Wick product approach to SDEs is an alternative way to look at SDEs, such as to the equations in $(\mathcal{S})^*$. While it may not seem as interesting for the non-fractional case, it is the most efficient way to consider SDEs for fractional Brownian motions. Let us now consider the Wick SDE, d

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = Y_t \diamond (\mu(t) + \sigma(t)\dot{W}_t^H), \tag{5.8}$$

for an arbitrary Hurst coefficient $H \in (0, 1)$. A solution is again given by

$$\exp^{\diamond}(X_t) = \sum_{k=0}^{\infty} \frac{1}{k!} X_t^{\diamond k}, \tag{5.9}$$

with $X_t = \int_0^t (\mu(s) + \sigma(s)\dot{W}_s^H)\mathrm{d}s$. Since Eq. (5.9) contains the Wick power, however, it is not easily approximated for a numerical scheme. Therefore, we first need to rewrite the expression.

Proposition 5.3. *For $X_t = \int_0^t (\mu(s) + \sigma(s)\dot{W}_s^H)\mathrm{d}s$, we have that*

$$\exp^{\diamond}(X_t) = \exp\left(X_t - \frac{1}{2}\|\mathcal{K}\sigma 1_{[0,t]}\|_{L^2[0,T]}^2\right),$$

and thus the solution to Eq. (5.8) *is given by*

$$Y_t = \exp\left(\int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\dot{W}_s^H \mathrm{d}s - \frac{1}{2}\|\mathcal{K}\sigma 1_{[0,t]}\|_{L^2[0,T]}^2\right).$$

This shows that simulating paths, as in Section 5.2, suffices to simulate paths of solutions to Eq. (5.8) without having to compute the stochastic integral

$$\int_0^t \sigma(s) Y_s \mathrm{d}W_t^H.$$

Proof. The proof follows from the property for Wick exponentials of Eq. (5.6) and the general properties of the Wick product. First, we see that

$$\begin{aligned}\exp^{\diamond}(\int_0^t (\mu(s) + \sigma(s)\dot{W}_s^H)\mathrm{d}s) &= \exp^{\diamond}\left(\int_0^t \mu(s)\mathrm{d}s\right) \diamond \exp^{\diamond}\left(\int_0^t \sigma(s)\dot{W}_s^H \mathrm{d}s\right)\\ &= \exp\left(\int_0^t \mu(s)\mathrm{d}s\right) \exp^{\diamond}\left(\int_0^t \sigma(s)\dot{W}_s^H \mathrm{d}s\right),\end{aligned}$$

since Wick products coincide with regular products if one factor is deterministic. We will thus only need to work out the term $\exp^{\diamond}\left(\int_0^t \sigma(s)\dot{W}_s^H \mathrm{d}s\right)$. By Eq. (5.6), we have

$$\exp^{\diamond}\left(\int_0^t \sigma(s)\dot{W}_s^H \mathrm{d}s\right) = \exp^{\diamond}\left(\sum_k \int_0^t \sigma(s)e_k^H(s)\mathrm{d}s H_{\epsilon^{(k)}}\right) = \prod_k^{\diamond} \exp^{\diamond}\left(\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s H_{\epsilon^{(k)}}\right).$$

Since the random variables $\{H_{\epsilon^{(k)}} k \in \mathbb{N}\}$ are independent, the terms

$$\{\exp^{\diamond}\left(\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s H_{\epsilon^{(k)}}\right) : k \in \mathbb{N}\}$$

are also independent, which means that the Wick product will be equal to the regular product. We thus have

$$\exp^{\diamond}\left(\int_0^t \sigma(s)\dot{W}_s^H ds\right) = \prod_k \exp^{\diamond}\left(\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s H_{\epsilon^{(k)}}\right).$$

The individual factors are again decomposed using the generating function for Hermitian polynomials. For any k, we have

$$\begin{aligned}\exp^{\diamond}\left(\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s H_{\epsilon^{(k)}}\right) &= \sum_{n=0}^{\infty} \frac{\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s}{n!} H_{n\epsilon^{(k)}}\\ &= \sum_{n=0}^{\infty} \frac{\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s}{n!} h_n(w_{e_k})\\ &= \exp\left(w_{e_k}\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s - \frac{1}{2}\left(\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s\right)^2\right).\end{aligned}$$

We can now use this term to find

$$
\begin{aligned}
&\exp^{\diamond}\left(\int_0^t \sigma(s)\dot{W}_s^H \mathrm{d}s\right) \\
&= \exp\left(\sum_k w_{e_k}\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s - \sum_k \frac{1}{2}\left(\int_0^t \sigma(s)e_k^H(s)\mathrm{d}s\right)^2\right) \\
&= \exp\left(\int_0^t \sigma(s)\mathrm{d}W^H(s) - \sum_k \frac{1}{2}\left(\int_0^T \mathcal{K}\sigma 1_{[0,t]}(s)e_k(s)\mathrm{d}s\right)^2\right) \\
&= \exp\left(\int_0^t \sigma(s)\mathrm{d}W^H(s) - \frac{1}{2}\|\mathcal{K}\sigma 1_{[0,t]}\|^2_{L^2[0,T]}\right).
\end{aligned}
$$

The last equality follows from Parseval's identity. This completes the proof. $\square$

With this method, we generate a simulation of 10 paths of the SDE

$$\mathrm{d}Y_t = \mu(t)Y_t\mathrm{d}t + \sigma(t)Y_t\mathrm{d}W_t^H,$$

with

$$
\begin{aligned}
\mu(t) &= t,\, t \in [0,T] \\
\sigma(t) &= t,\, t \in [0,T].
\end{aligned}
$$

The results are presented in Fig. 3 and the code for the simulation can be found in Appendix C.3. Note, that, as in Remark 5.2, further work is required for an efficient implementation.

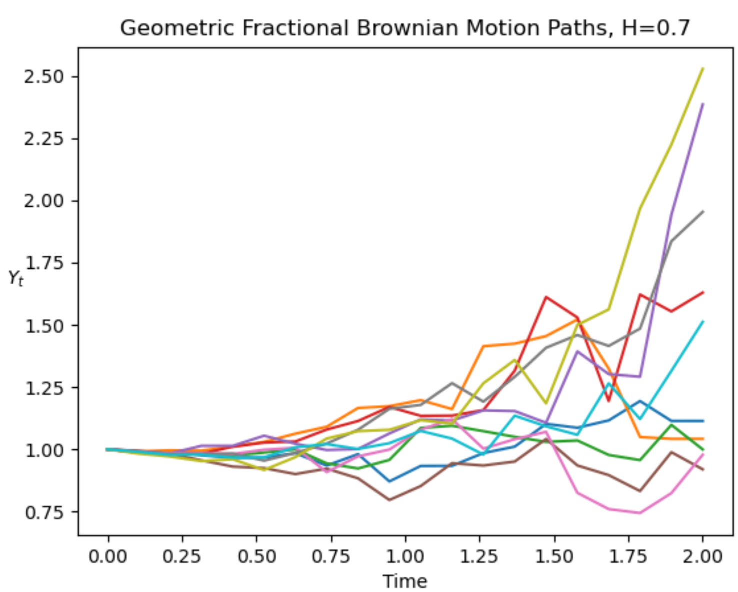

Figure 3: Simulation of 10 paths

6 Conclusions

6.1 Stochastic Integration

In this thesis we introduced the Malliavin-based approach to stochastic integration with respect to a fractional Brownian motion. We presented three ways to evaluate the integral

$$\int_0^T X_t \mathrm{d}W_t^H.$$

The first approach is to consider the integral as a divergence operator of a particularly chosen isonormal Gaussian process, an approach developed by authors including Alos, Nualart, Decreusefond and Üstünel. The second approach is based on the Wick product on the white noise probability space, which is originally attributed to Elliott and van der Hoek, but then followed up by Øksendal, Nunno and Biagini. The third, more recent approach, is by Xia et al. The most important property of these integrals is that they are all centered, i.e. that

$$\mathbb{E}[\int_0^T X_t \mathrm{d}W_t^H] = 0,$$

for any process X_t. Furthermore, they all generalize the Itô integral. This means that if $H = \frac{1}{2}$, the stochastic integrals coincide with the Itô integral.

Over the course of the thesis, we have proven that the integrals coincide due to their common relation to the Skorohod integral and fractional operators. It is, however, unclear how the respective domains of integration behave. It has been shown that, especially for small values of H ($H < \frac{1}{2}$), the rough paths of the fBm require a certain regularity on the integrand; much more than for the regular Itô integral. For instance, the simple integral

$$\int_0^T W_t^H \mathrm{d}W_t^H$$

is not well-defined in the divergence case if $H < \frac{1}{4}$, and a limit extension is necessary to define the integral.

Each integral formulation has positive and negative aspects. The divergence integral is defined in a weak way. It is thus difficult to find an analytic expression for the integral. Conversely, the integral is a direct extension of the white noise Malliavin calculus case (i.e Example 3.1) and requires the least amount of prior knowledge.

The WIS-integral can be computed in a straightforward way if the Wiener-Itô expansion of the process to be integrated is known. Furthermore, Proposition 3.10 gives a simple characterization for Pettis-integrable processes, thus further specifying the domain of integration. However, this integral does require one to explicitly work with the fractional operator $\mathcal{K}$ and its adjoint operator $\mathcal{K}^*$, which, due to the irregular properties, can be challenging.

The Xia integral is the most intuitive integral, as it follows as a limit of some sort of Riemann-sum approximation. This makes the integral especially convenient for time-discrete processes. Furthermore, an integral of a continuous process can be approximated by such an integral. Nevertheless, the integral contains a Wick product, which makes numerical approximation more challenging.

6.2 Numerical Scheme

Continuing with the integral formulation, we introduced a new numerical scheme to simulate paths on the white noise probability space in the last section of the thesis. The method includes sampling the Gaussian bases $H_{\epsilon^{(k)}}$ and then approximating paths by truncation of the Wiener-Itô expansion, which by definition, converges quadratically. We have showed numerical simulations for

i) regular Brownian motions,

ii) fractional Brownian motions,

iii) fractional stochastic integrals with respect to the fBm,

iv) fractional linear stochastic differential equations.

There are several practical applications of the method. The method can be used as an alternative to Euler-approximations of regular Itô integrals and SDEs. The method can particularly efficient in applications where paths need to be evaluated at arbitrary discrete time points. Furthermore, while the initial computational cost is higher than with an Euler-scheme, any additional path only requires sampling of one random variable as opposed to the large amount of time points in the Euler-case.

This method also allows for a simulation of paths of the solution to SDEs of the type

$$\mathrm{d}Y_t = \mu(t)Y_t\mathrm{d}t + \sigma(t)Y_t\mathrm{d}W_t^H, \tag{6.1}$$

where $\mu(t), \sigma(t)$ are suitable deterministic functions. Such SDEs are common in applications related to financial modelling, where for instance Y_t could be interpreted as the price process of some asset.

It is important to note that the content of Section 5 is from a theoretical point of view and that there are several challenges when implementing the method. The method requires the evaluation of $\mathcal{K}(\sigma)(\cdot)$ of $e_k^H(t)$, which means that a numerical integration scheme is necessary. As we have pointed out, this can lead to high computational costs or even inaccurate solutions, due to the evaluation of improper integrals. Additionally, the choice of the basis vectors $\{e_k : k \in \mathbb{N}\}$ can influence the convergence of these integrals.

6.3 Further Considerations

The fractional Brownian motion has been shown to be a very suitable model for problems occurring in various practical fields. Therefore, it has been a popular topic among mathematicians, especially around the year 2000. However, the difficulties occurring due to irregular paths have been challenging for applications. In financial mathematics, for instance, a solution to the arbitrage problem, induced by the fact that fBm is not a semimartingale, remains to be found.

Nevertheless, models with fractional Brownian motions are gaining popularity, such as the modelling of rough volatility. In order to properly apply such models, a solid understanding of the fBm, stochastic integration, and the difficulties around it is crucial. Furthermore, it is important to define a standard way to interpret such a stochastic integral. We believe that the standard way should be the WIS-integral, as introduced by Elliott and van der Hoek. Although the integral requires a solid understanding of the white noise probability space, it allows for simple computations and additionally allows for numerical approximations.

Further research should be done into the domain of integration for the WIS integral. The irregularities of the fractional operator need to be properly analyzed and compared with the issues which have already been determined for the divergence integral. Additionally, in order to speed up the simulations of paths, a more suitable expression for the fractional operator is needed. The double integral occurring in the computation of

$$\int_0^t e_k^H(s)\mathrm{d}s$$

could possibly be approximated in a more efficient way than currently proposed in this thesis.

In the end, the fractional Brownian motion remains a challenging topic compared to the regular Brownian motion, due to the martingale and Markov properties of the latter. Therefore, the benefits of using fractional models versus models with a regular Brownian motion as noise term should be carefully compared to the additional computational challenges.

Bibliography

[1] Benoit B. Mandelbrot and John W. Van Ness. "Fractional Brownian Motions, Fractional Noises and Applications". In: *SIAM Review* 10.4 (1968), pp. 422–437. DOI: 10.1137/1010093.

[2] Fred Espen Benth. "On arbitrage-free pricing of weather derivatives based on fractional Brownian motion". In: *Applied Mathematical Finance* 10.4 (2003), pp. 303–324. DOI: 10.1080/1350486032000174628.

[3] B.L.S. Prakasa Rao. "Pricing geometric Asian power options under mixed fractional Brownian motion environment". In: *Physica A: Statistical Mechanics and its Applications* 446 (2016), pp. 92–99. DOI: 10.1016/j.physa.2015.11.013.

[4] S.J. Lin. "Stochastic analysis of Fractional Brownian motions". In: *Stochastics and Stochastic Reports* 55.1-2 (1995), pp. 121–140. DOI: 10.1080/17442509508834021.

[5] Sottinen T. Bender C. and E Valkeila. "Arbitrage with fractional brownian motion?" In: *Theory Stoch. Process.* 13.1 (2007), pp. 23–34.

[6] M. Zähle. "Integration with respect to fractal functions and stochastic calculus. I". In: *Probability Theory and Related Fields* 111.3 (1998), pp. 333–374. DOI: 10.1007/s004400050171.

[7] T.e. Duncan, Y.z. Hu, and B. Pasik-Duncan. "Stochastic calculus for fractional Brownian motion. I. Theory". In: *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No.00CH37187)* (2000). DOI: 10.1109/cdc.2000.912761.

[8] Robert J. Elliott and John Van Der Hoek. "A General Fractional White Noise Theory And Applications To Finance". In: *Mathematical Finance* 13.2 (2003), pp. 301–330. DOI: 10.1111/1467-9965.00018.

[9] Francesca Biagini et al. "An introduction to white–noise theory and Malliavin calculus for fractional Brownian motion". In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 460.2041 (2004), pp. 347–372. DOI: 10.1098/rspa.2003.1246.

[10] Laurent Decreusefond and Ali Suleyman Üstünel. "Fractional brownian motion: Theory and applications". In: *ESAIM: Proceedings* 5 (1998), pp. 75–86. DOI: 10.1051/proc:1998014.

[11] Elisa Alòs, Olivier Mazet, and David Nualart. "Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than 1/2". In: *Stochastic Processes and their Applications* 86.1 (2000), pp. 121–139. DOI: 10.1016/s0304-4149(99)00089-7.

[12] David Nualart. "Stochastic integration with respect to fractional Brownian motion and applications". In: *Contemporary Mathematics Stochastic Models* (2003), pp. 3–39. DOI: 10.1090/conm/336/06025.

[13] David Nualart. *The Malliavin calculus and related topics.* Springer, 2006.

[14] P Cheridito and D Nualart. "Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H$(0,\frac{1}{2})$". In: *Annales de l'Institut Henri Poincare (B) Probability and Statistics* 41.6 (2005), pp. 1049–1081. DOI: 10.1016/j.anihpb.2004.09.004.

[15] Francesca Biagini. *Stochastic calculus for fractional Brownian motion and applications*. Springer, 2008.

[16] Fuquan Xia, Litan Yan, and Jianhui Zhu. "Forward and symmetric Wick-Itô integrals with respect to fractional Brownian motion". In: *Frontiers of Mathematics in China* 16.2 (2021), pp. 623–645. DOI: 10.1007/s11464-021-0923-8.

[17] Olav Kallenberg. *Foundations of modern probability*. Springer, 2021.

[18] Ivan Nourdin. *Selected aspects of fractional Brownian motion*. Springer Verlag, 2012.

[19] L. C. G. Rogers. "Arbitrage with Fractional Brownian Motion". In: *Mathematical Finance* 7.1 (1997), pp. 95–105. DOI: 10.1111/1467-9965.00025.

[20] Peter J.C Spreij. *Lecture Notes: Stochastic integration*. Tech. rep. University of Amsterdam, 2021.

[21] Myron T. Greene and Bruce D. Fielitz. "Long-term dependence in common stock returns". In: *Journal of Financial Economics* 4.3 (1977), pp. 339–349. DOI: 10.1016/0304-405x(77)90006-x.

[22] Koji Kuroda and Joshin Murai. "Long memory in finance and Fractional Brownian motion". In: *Progress of Theoretical Physics Supplement* 179 (2009), pp. 26–37. DOI: 10.1143/ptps.179.26.

[23] Yaozhong Hu and Bernt Øksendal. "Fractional white noise calculus and applications to finance". In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 06.01 (2003), pp. 1–32. DOI: 10.1142/s0219025703001110.

[24] Christian Bayer, Peter Friz, and Jim Gatheral. "Pricing under rough volatility". In: *Quantitative Finance* 16.6 (2015), pp. 887–904. DOI: 10.1080/14697688.2015.1099717.

[25] Jim Gatheral, Thibault Jaisson, and Mathieu Rosenbaum. "Volatility is rough". In: *SSRN Electronic Journal* (2014). DOI: 10.2139/ssrn.2509457.

[26] H. E. Hurst. "Long-term storage capacity of reservoirs". In: *Transactions of the American Society of Civil Engineers* 116.1 (1951), pp. 770–799. DOI: 10.1061/taceat.0006518.

[27] Giulia Di Nunno, B. K. Øksendal, and Frank Proske. *Malliavin calculus for levy processes with applications to finance*. Springer, 2009.

[28] John B. Conway. *A course in functional analysis*. Springer, 2000.

[29] Kiyosi Ito. "Multiple Wiener Integral". In: *Journal of the Mathematical Society of Japan* 3.1 (1951). DOI: 10.2969/jmsj/00310157.

[30] Helge Holden. *Stochastic partial differential equations: A modeling, White Noise Functional Approach user*. Birkhäuser, 1996.

[31] Dorina Mitrea. *Distributions, Partial Differential Equations, and Harmonic Analysis*. Springer International Publishing, 2019.

[32] Tu-sheng Zhang. "Characterizations of white noise test functions and Hida distributions". In: *Stochastics and Stochastic Reports* 41 (1992).

[33] Elisa Alòs and David Nualart. "Stochastic integration with respect to the fractional Brownian motion". In: *Stochastics and Stochastic Reports* 75.3 (2003), pp. 129–152. DOI: 10.1080/1045112031000078917.

A Hermitian Polynomials

Hermitian polynomials form a family of polynomials which occur naturally in the relation to Gaussian random variables.

Definition A.1. *Let $n \geq 0$. We denote the n-th* Hermitian polynomial *as*

$$h_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{\frac{-x^2}{2}} \right), \ n \geq 1,$$

and $h_0(x) \equiv 1$.

The polynomials have various properties which follow directly from their definition. For instance, it can be shown that they follow the recurrence relation

$$h_{n+1}(x) = x h_n - h_n'(x).$$

Furthermore, for $n \geq 1$, we have that

1. $h_n'(x) = h_{n-1}(x)$,
2. $(n+1)h_{n+1}(x) = x h_n(x) - h_{n-1}(x)$,
3. $h_n(-x) = (-1)^n h_n(x)$.

Lastly, they can also be characterized via the so-called (polynomial) generating function. It can be shown that

$$e^{tx - \frac{t^2}{2}} = \sum_{k=0}^{\infty} h_n(x) t^n. \tag{A.1}$$

The reason why Hermitian polynomials are connected to Gaussian random variables is that they form an orthogonal structure for normal random variables. It can easily be shown that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(x) h_m(x) e^{\frac{-x^2}{2}} \mathrm{d}x = \sqrt{n!}\sqrt{m!}\delta_{nm}, \tag{A.2}$$

for any $n, m \in \mathbb{N}$. Since $e^{\frac{-x^2}{2}}$ is the Gaussian kernel, this indicates the orthonormal structure for Gaussian random variables. In fact, we have the following lemma.

Lemma A.1. *Let X, Y be standard Gaussian random variables which are jointly Gaussian distributed. Then, for any $m, n \geq 0$, we have*

$$\mathbb{E}[h_n(X) h_m(Y)] = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{n!}\mathbb{E}[XY]^n & \text{if } n = m. \end{cases}$$

Proof. Since X and Y are jointly Gaussian with mean zero, we have that $aX + bY \sim N(0, a^2 + b^2 + 2ab\rho)$, where $\rho = Cov(X, Y)$. The moment generating function of $aX + bY$, evaluated at $t = 1$, is thus given by

$$\mathbb{E}[e^{aX+bY}] = e^{a^2+b^2+2ab\rho}.$$

Rearranging the terms, we find that

$$\mathbb{E}[e^{aX - \frac{a^2}{2} + bY - \frac{b^2}{2}}] = e^{ab\rho}.$$

By the generating function Eq. (A.1), the left-hand side can be written as

$$\mathbb{E}[\sum_{k=0}^{\infty} h_k(X) a^k \sum_{l=0}^{\infty} h_l(Y) b^l] = e^{ab\rho}.$$

The solution can be obtained by differentiating n times with respect to a, m times with respect to b, and a subsequent evaluation at $a = b = 0$. Assuming, without loss of generality, that $m \leq n$, we find

$$\mathbb{E}[n!m!h_n(X)h_m(Y)] = m!\rho^n b^{n-m} e^{ab\rho} = \delta_{nm}\rho^n m!.$$

Since $\rho = \mathbb{E}[XY]$, the equality is proven. $\square$

B Iterated Itô Integrals with Isonormal Processes

Here, we will discuss the rigorous construction of the iterated integral $\hat{I}_n(f)$, according to Definition 3.8, where $f \in L^2([0,T])$. For simplicity, we will write $I_n(f)$ instead of $\hat{I}_n(f)$.

Let $\mathscr{H}$ be the Hilbert space

$$\mathscr{H} = L^2(\mu, [0,T], \mathcal{B}[0,T]),$$

where μ is a σ-finite atomless measure on $[0,T]$ and $\mathcal{B}[0,T] = \mathcal{B}$ the set of all measurable subsets. Since $\mathscr{H}$ is a Hilbert space, there exists an isonormal Gaussian process $W\colon \mathscr{H} \to L^2(\Omega)$. In particular, if $A \in \mathcal{B}$ is a measurable set of $[0,T]$, the function I_A is in $\mathscr{H}$ and we denote

$$W(A) = W(I_A). \tag{B.1}$$

We now notice that random variables of the type Eq. (B.1) determine the whole isonormal process W. This is the case since $\{I_A : A \in \mathcal{B}, \mu(A) < \infty\}$ is dense in $L^2([0,T])$ and thus for any $h \in \mathscr{H}$, we can write

$$W(h) = W\left(\lim_{n\to\infty} \sum_{k=1}^{N} a_k A_k\right) = \lim_{n\to\infty} W\left(\sum_{k=1}^{N} a_k A_k\right),$$

for some sequences $\{a_n \in \mathbb{R} : n \in \mathbb{N}\}$ and $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$. The last equality follows from the fact that W is an isometry. Denoting $\mathcal{B}_0 = \{A \in \mathcal{B} : \mu(A) < \infty\}$, the set

$$\{W(A) : A \in \mathcal{B}_0\}$$

is thus dense in $L^2(\Omega)$.

Now, let $n \in \mathbb{N}$ be fixed and consider a sequence $\{A_1, A_2, \ldots, A_n\}$ in $\mathcal{B}_0$ with the property that $A_k \cap A_l = \emptyset$ for all $k \neq l$. The sets are thus pairwise disjoint. Furthermore, for some fixed $m \in \mathbb{N}$, we define the sequence of coefficients

$$\{a_{i_1 \ldots i_m} : i_k \leq n, i \leq m\}$$

with the property that if $i_k = i_l$ for any $k, l \leq m$, we have that $a_{i_1 \ldots i_m} = 0$. Now, consider functions of the form

$$f(t_1, t_2, \ldots, t_m) = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1\ldots i_m} I_{A_{i_1}\times\cdots\times A_{i_m}}. \tag{B.2}$$

We call functions of the type Eq. (B.2) *elementary functions* and denote the space of all elementary functions as $\mathcal{E}_m$, where m is fixed and n is chosen freely. One can interpret elementary functions the following way. Let us consider the case when $m = 2$ and the subsets A_1, A_2, as drawn in Fig. 4.

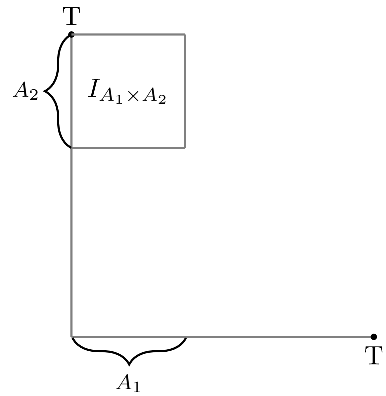

Figure 4: Graphical Representation of $I_{A_1\times A_2}$

An elementary function is for instance given by

$$f(t_1,t_2) = 2I_{A_1\times A_2} + I_{A_2\times A_1},$$

which is represented in Fig. 5. As we can see, due to the fact that A_1 and A_2 are pairwise disjoint and $a_{1,1} = a_{2,2} = 0$, the diagonal line $t_1 = t_2$ remains at 0.

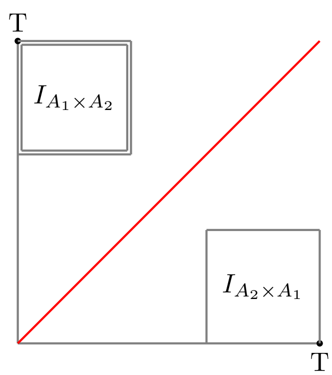

Figure 5: An elementary function in $\mathcal{E}_2$

The space of elementary functions will build the basis for a limit argument to define $I_m(f)$ for any square-integrable function f. To do so, we have to first prove that $\mathcal{E}_m$ is dense in $L^2([0,T]^m)$

Proposition B.1. *For any $m \in \mathbb{N}$, the space $\mathcal{E}_n$ is dense in $L^2(\mu^m, [0,T]^m, \mathcal{B}^m)$.*

Proof. Since the regular simple functions are dense in L^2, it suffices to show that for any $A \in \mathcal{B}_0^m$, the function I_A is contained in the closure of $\mathcal{E}_m$. The set $A \in \mathcal{B}_0[0,T]^m$ in turn can be written as

$$A_1 \times \cdots \times A_m,$$

for some $A_1,\ldots,A_m \in \mathcal{B}$. Let us now fix some $\epsilon > 0$. Since μ has no atoms, we can find a collection of pairwise disjoint sets $\{E_1,\ldots,E_n\}$, such that $\mu(E_k) \leq \epsilon$ for all $k \leq n$ and such that A_j can be written as a union of some of the sets E_k. The indicator function I_A can thus be written as

$$I_A = \sum_{i_1,\ldots,i_m=1} a_{i_1,\ldots,i_m} I_{E_1\times\cdots\times E_n},$$

for some $a_{i_1,\dots,i_m} \in \{0,1\}$. Note that the right-hand side is almost an elementary function, only missing that $a_{i_1,\dots,i_m}$ is not necessarily 0 if $i_k = i_l$ for some $k \neq l$. We thus define I_B as the elementary function

$$I_B = \sum_{i_1,\dots,i_m=1} b_{i_1,\dots,i_m} I_{E_1\times\cdots\times E_n},$$

where $b_{i_1,\dots,i_m} = a_{i_1,\dots,i_m}$ if $i_1 \neq \ldots \neq i_m$ and 0 otherwise. The set B is thus a subset of A and we find

$$I_A - I_B = \sum_{i_1,\dots,i_m\in J} c_{i_1,\dots,i_m} I_{E_1\times\cdots\times E_n},$$

where J is the set of indices for which we put the coefficients to 0. Note that $|J|$ can be at most $\binom{m}{2}$. Therefore,

$$\|I_A - I_B\|_2^2 \leq \epsilon\mu(A)^{m-1}\binom{m}{2} \to 0,$$

as $\epsilon \to 0$. The function I_A is thus a limit point of I_B. This concludes the proof. □

Remark B.1. We thus see why it is important to assume that μ is atomless, as the choice of $E_1,\dots,E_n$ is not necessarily given for a measure with atoms.

The definition of the iterated stochastic integral, according to Definition 3.8, now follows as a limit of Eq. (3.7).

C Python Code

C.1 Brownian motion paths

The code below generates and plots paths of a regular Brownian motion.

```
import numpy as np
import matplotlib.pyplot as plt
import math


def E(k: int, t: float, T: float = 1.0) -> float:
    """
    Returns: \int_0^t e_k(s)ds, where {e_k : k \in N} forms
             an orthonormal basis of [0,T]
    """
    return (
        math.sqrt(2 / T)
        * (T)
        / (k * math.pi)
        * (1 - math.cos(k * math.pi * t / T))
    )


def SampleHermitianBase(K: int, NoOfPaths: int) -> np.array:
    """
    Returns: NoOfPaths samples for <e_k,w>, which are standard normal.
    """
    return np.random.normal(0.0, 1.0, [K, NoOfPaths])


```

```
def BrownianMotionPaths(
    T: float, NoOfPaths: int, NoOfSteps: int, N: int
) -> np.array:
    """
    Returns: [NoOfPaths, NoOfSteps] array of Brownian motion sample paths
    """
    base = SampleHermitianBase(N, NoOfPaths)
    result = np.zeros([NoOfPaths, NoOfSteps + 1])
    time = np.zeros([NoOfSteps + 2])
    dt = T / float(NoOfSteps)
    for i in range(NoOfSteps + 1):
        for k in range(1, N + 1):
            result[:, i] += E(k, time[i], T) * base[k - 1, :]
        time[i + 1] = time[i] + dt
    return result


def Plot(paths: np.array, T) -> None:
    _NoOfSteps = len(paths[0])
    plt.plot(np.linspace(0, T, _NoOfSteps), np.transpose(paths))
    plt.show()


N = 100
T = 10
NoOfPaths = 10000
NoOfSteps = 100
Plot(BrownianMotionPaths(T, NoOfPaths, NoOfSteps, N), T)
```

C.2 Fractional Brownian motion paths

The code below generates and plots paths of a fractional Brownian motion.

```
import numpy as np
import matplotlib.pyplot as plt
import math
from scipy.special import gamma, hyp2f1
import scipy.integrate as integrate

# Determine Hurst-Index
H_set = 0.7


def _e(k: int, t: float, T=1) -> float:
    """
    Basis vectors of [0,T]
    """
    return math.sqrt(2 / T) * math.sin(k * math.pi * t / (T))


def _K_H(t: float, s: float, H=H_set) -> float:
    """
    Fractional operator K_H(t,s)
    """
```

```
    return (
        (t - s) ** (H - 1 / 2)
        / gamma(H + 1 / 2)
        * hyp2f1(H - 1 / 2, 1 / 2 - H, H + 1 / 2, 1 - t / s)
    )


def _KInt(start: float, end: float, k: int, T: float, H=H_set) -> float:
    """
    Calculates int_0^t K_H(t,s)e_k(s)ds
    """
    return integrate.quad(
        lambda x: _K_H(end, x, H) * _e(k, x, T), start, end
    )


def SampleHermitianBase(K: int, NoOfPaths: int) -> np.array:
    """
    Returns: NoOfPaths samples for <e_k,w>, which are standard normal.
    """
    return np.random.normal(0.0, 1.0, [K, NoOfPaths])


def FBM(T: float, NoOfPaths: int, NoOfSteps: int, N: int) -> np.array:
    """
    Returns: [NoOfPaths, NoOfSteps] array of fBm paths
    """
    base = SampleHermitianBase(N, NoOfPaths)
    result = np.zeros([NoOfPaths, NoOfSteps + 1])
    time = np.zeros([NoOfSteps + 2])
    dt = T / float(NoOfSteps)
    for i in range(NoOfSteps + 1):
        for k in range(1, N + 1):
            # Make sure no singularity at 0
            if time[i] == 0:
                time[i] = 0.00000001
            result[:, i] += (
                _KInt(0, time[i], k, T, H_set)[0] * base[k - 1, :]
            )
        time[i + 1] = time[i] + dt
    return result


def Plot(paths: np.array, T: float) -> None:
    _NoOfSteps = len(paths[0])
    plt.plot(np.linspace(0, T, _NoOfSteps), np.transpose(paths))
    plt.show()


N = 100
T = 2.0
NoOfPaths = 100
NoOfSteps = 100
paths = FBM(T, NoOfPaths, NoOfSteps, N)
```

```
Plot(paths, T)
```

C.3 Fractional SDE

The code below generates and plots paths of the SDE Eq. (6.1).

```
import numpy as np
import matplotlib.pyplot as plt
import math
from scipy.special import beta
import scipy.integrate as integrate
import matplotlib.pyplot as plt

H_set = 0.7
c_H = (H_set * (2 * H_set - 1) / (beta(2 - 2 * H_set, H_set - 0.5))) ** (
    1 / 2
)


def _e(k: int, t: float, T=1) -> float:
    """
    Basis vectors of [0,T]
    """
    return math.sqrt(2 / T) * math.sin(k * math.pi * t / (T))


def dK_H(t: float, s: float) -> float:
    """
    Derivative of K_H(t,s) with respect to t
    """
    if t == s:
        t += 0.0000000001
    return c_H * (t / s) ** (H_set - 1 / 2) * (t - s) ** (H_set - 3 / 2)


def SampleHermitianBase(K: int, NoOfPaths: int) -> np.array:
    """
    Returns: NoOfPaths samples for <e_k,w>, which are standard normal.
    """
    return np.random.normal(0.0, 1.0, [K, NoOfPaths])


def fbmIntegral(
    NoOfPaths: int, NoOfSteps: int, sigma, T=1, K=10
) -> np.array:
    """
    Returns: [NoOfSteps, NoOfPaths] array of int_0^t sigma(s)dW^H_s
    """
    HBSamples = SampleHermitianBase(K, NoOfPaths)
    time = np.zeros([NoOfSteps + 2])
    dt = 1 / float(NoOfSteps)
    sum = np.zeros([NoOfSteps, NoOfPaths])
    coeff = np.zeros(K)
    E = lambda k, start, end: integrate.dblquad(
```

```
        lambda s, u: dK_H(s, u) * _e(k + 1, u, T) * sigma(s),
        start,
        end,
        lambda s: 0,
        lambda s: s,
    )[0]
    for i in range(1, NoOfSteps):
        time[i] = time[i - 1] + dt
        for k in range(K):
            coeff[k] += E(k, time[i - 1], time[i])
            sum[i] += coeff[k] * HBSamples[k]
    return sum


def drift_term(mu, NoOfPaths: int, NoOfSteps: int) -> np.array:
    """
    Returns: [NoOfSteps, NoOfPaths] array of int_0^t mu(s)ds
    """
    result = np.zeros([NoOfSteps, NoOfPaths])
    time = np.zeros([NoOfSteps + 2])
    dt = 1 / float(NoOfSteps)
    for i in range(1, NoOfSteps):
        time[i] = time[i - 1] + dt
        result[i] = np.repeat(integrate.quad(mu, 0, time[i])[0], NoOfPaths)
    return result


def correction_term(sigma, NoOfPaths: int, NoOfSteps: int) -> np.array:
    """
    Returns: [NoOfSteps, NoOfPaths] array of ||K*sigma||_L^2
    """
    result = np.zeros([NoOfSteps, NoOfPaths])
    time = np.zeros([NoOfSteps + 2])
    dt = 1 / float(NoOfSteps)
    for i in range(1, NoOfSteps):
        time[i] = time[i - 1] + dt
        result[i] = np.repeat(
            integrate.dblquad(
                lambda t, s: (dK_H(t, s) * sigma(t)) ** 2,
                0,
                time[i],
                lambda s: s,
                lambda s: T,
            )[0],
            NoOfPaths,
        )
    return result


def Plot(paths: np.array, T: float) -> None:
    _NoOfSteps = len(paths)
    plt.plot(np.linspace(0, T, _NoOfSteps), paths)
    plt.show()
```

```
K = 50
NoOfSteps = 50
NoOfPaths = 1000
T = 2
correction = correction_term(lambda x: x, NoOfPaths, NoOfSteps)
drift = drift_term(lambda x: x, NoOfPaths, NoOfSteps)
diff = fbmIntegral(NoOfPaths, NoOfSteps, lambda x: x, T, K)

paths = np.exp(diff) * np.exp(drift) / (np.exp(1 / 2 * correction))
Plot(paths, T)
```