C*-Actions and Takai Duality

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In dedication to my parents
J and E
who supported me unconditionally throughout my journey.

Abstract

The study of classical dynamical systems (G, Ω) deals with actions of locally compact Hausdorff groups G on locally compact Hausdorff spaces Ω. It is long recognized that classical dynamical systems can be studied successfully via C*-algebras. A C*-action $\alpha\colon G \curvearrowright A$ is a topological group homomorphism α from a locally compact Hausdorff group G to $\mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the automorphism group of a C*-algebra A endowed with the topology of pointwise convergence. The C*-actions involving the commutative C*-algebras precisely model the classical dynamical systems.

Every C*-action $\alpha\colon G \curvearrowright A$ gives rise to a C*-algebra $G \ltimes_\alpha A$, called the crossed product, which encodes a lot of information about the C*-action. Using crossed products and duality theory for abelian locally compact Hausdorff groups one can develop a certain duality theory for C*-actions. Just as Pontryagin duality describes the second dual of G, Takai duality describes the second dual of α. The idea is that one recovers a C*-action from its crossed product up to tensoring with another C*-action.

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Contents

Introduction

The theory of C*-actions provides a deep connection between the study of dynamical systems and the study of C*-algebras. In physics C*-actions are used to exploit symmetry in quantum mechanics. A nice survey of the physical relevance of C*-actions can be found in [Rae99].

To cut to the chase, a C*-action is a topological group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$, where G is a locally compact Hausdorff group and $\mathrm{Aut}(A)$ is the automorphism group of a C*-algebra A endowed with the topology of pointwise convergence.

The famous results by Gelfand, Naimark and Segal show that C*-algebras can be studied effectively via their representations on Hilbert spaces. Likewise, we can study C*-actions by looking at their so-called covariant representations on Hilbert spaces. Covariant representations express the structure of a C*-action in terms of compositions of Hilbert space operators in a certain prescribed way.

A fundamental result states that there exists a C*-algebra $G \ltimes_\alpha A$, called the crossed product (Definition 2.3.4), whose representations are in one-to-one correspondence with the covariant representations of $\alpha\colon G \to \mathrm{Aut}(A)$. Thus the information of a C*-action is somewhat encoded in its crossed product. The main result of this thesis, Takai duality, gives a partial positive answer to the following question.

Can we recover the C-action* $\alpha\colon G \to \mathrm{Aut}(A)$ *from its crossed product* $G \ltimes_\alpha A$*?*

Before we elaborate on the main result we say a few words about how C*-actions arise naturally from classical dynamical systems.

Suppose we are given a locally compact Hausdorff space Ω and a homeomorphism $T\colon \Omega \to \Omega$. The iterates of the map T yield an action of the discrete group $\mathbb{Z}$ on Ω via $j \cdot \omega := T^j(\omega)$ for all $j \in \mathbb{Z}$ and $\omega \in \Omega$. If we consider the change of observables $\varphi \in C_0(\Omega)$, then we obtain a special case of a C*-action $\tau\colon \mathbb{Z} \to \mathrm{Aut}(C_0(\Omega))$ by defining[1,2] $\tau_j(\varphi)(\omega) := \varphi(j^{-1} \cdot \omega)$ for all $j \in \mathbb{Z}$, $\varphi \in C_0(\Omega)$ and $\omega \in \Omega$. The crossed product $\mathbb{Z} \ltimes_\tau C_0(\Omega)$ is well-studied and contains a lot of information about T. For example[3], if Ω is a compact Hausdorff space with infinite cardinality, then the irreducibility of T (in the form of 'minimality') is reflected by the irreducibility of $\mathbb{Z} \ltimes_\tau C(\Omega)$ (in the form of 'simplicity').

The pair $(\mathbb{Z}, \Omega)$ together with the action of $\mathbb{Z}$ on Ω induced by T is an example of a dynamical system. In general, a dynamical system is a pair (G, Ω), where G is a locally compact Hausdorff group and Ω a locally compact Hausdorff space, together with an action of G on Ω. Every dynamical system (G, Ω) gives rise to a C*-action $\tau\colon G \to \mathrm{Aut}(C_0(\Omega))$. A C*-action of this form is called classical. We shall see that every C*-action $\alpha\colon G \to \mathrm{Aut}(A)$ with A commutative is conjugate to a classical C*-action $\tau\colon G \to \mathrm{Aut}(C_0(\Omega))$. Due to this fact, we sometimes think of the study of C*-actions as the study of non-commutative dynamical systems.

An important example that we shall discuss in great detail is the dynamical system (G, G), where the action is given by left-translation, that is, $s \cdot t := st$ for all $s, t \in G$. The classical C*-action induced by (G, G) is denoted by $\mathrm{lt}\colon G \to \mathrm{Aut}(C_0(G))$. A reformulation of a module

[1]The presence of the inverse is just bookkeeping to ensure that we indeed have a group homomorphism.

[2]In literature the map τ_1 is commonly known as the Koopman operator of T. See for example [EFHN15].

[3]See [Dav96, Theorem VIII.3.9] for the details.

theoretic result by Rieffel says that every covariant representation of the left-translation is unitarily equivalent to a direct sum of (possibly infinitely many) copies of the so-called Schrödinger covariant representation of the left-translation. We give a modern proof of this result based on ideas from Rieffel himself. After that we use Rieffel's result to prove the Stone-Neumann theorem, which states that the crossed product $G \ltimes_{\mathrm{lt}} C_0(G)$ is isomorphic to a C*-algebra K of compact operators on a Hilbert space. The Stone-Neumann theorem plays a crucial role in the proof of the main result.

As mentioned earlier the aim of the main result is to recover a C*-action $\alpha\colon G \to \mathrm{Aut}(A)$ from its crossed product $G \ltimes_\alpha A$. It turns out that this is possible up to tensoring with another C*-action.

If G is abelian, then this can be accomplished by using the dual group $G^\sharp$ to develop a certain duality theory for C*-actions. More precisely, there is a canonical C*-action $\alpha^\sharp\colon G^\sharp \to \mathrm{Aut}(G \ltimes_\alpha A)$, which we call the dual C*-action. By Pontryagin duality we can identify $G^{\sharp\sharp} = G$ and therefore we fully recover the group G from its dual group $G^\sharp$. Hence, motivated by Pontryagin duality, we hope to recover the C*-action $\alpha\colon G \to \mathrm{Aut}(A)$ from its dual C*-action $\alpha^\sharp\colon G^\sharp \to \mathrm{Aut}(G \ltimes_\alpha A)$ and therefore from its crossed product $G \ltimes_\alpha A$.

This is where our main result comes into play; Takai duality states that the double dual C*-action $\alpha^{\sharp\sharp}\colon G \to \mathrm{Aut}(G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A))$ is conjugate to $\alpha\colon G \to \mathrm{Aut}(A)$ up to tensoring with another C*-action. More precisely, there is a C*-algebra[4] K of compact operators on a Hilbert space and a C*-action $\kappa\colon G \to \mathrm{Aut}(K)$ such that the following statement holds.

> *There is an isomorphism from the iterated crossed product $G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$ to the maximal tensor product $K \otimes_{\max} A$ under which the C*-actions $\alpha^{\sharp\sharp}\colon G \to \mathrm{Aut}(G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A))$ and $\kappa \otimes \alpha\colon G \to \mathrm{Aut}(K \otimes_{\max} A)$ are conjugate.*

In this thesis we attempt to provide a more accessible proof of this fact.

There is also a version of Takai duality for non-abelian G, where one uses the concept of C*-coactions to work around the absence of the dual group $G^\sharp$. However, in this thesis we only consider Takai duality for abelian G and refer to [Rae87] for non-abelian G.

Organization

Integration on topological groups will play a key role in thesis. Since we deviate from the more conventional measure theoretic approach, we have devoted Chapter 1 to the development of this theory. Emphasis will lie on vector-valued integration on topological groups. Anyone who is comfortable with these concepts may want to skip or postpone this chapter.

In Chapter 2 we introduce C*-actions and their covariant representations. After that we construct the crossed product C*-algebra. We prove that there is a one-to-one correspondence between the covariant representations of a C*-action and the representations of its associated crossed product. We end this chapter with the Stone-Neumann theorem (Theorem 2.6.10).

In Chapter 3 we commit to a detailed treatment of the main result. First we undertake some preparatory work and then we state and prove the Takai duality theorem (Theorem 3.5.2).

The main contribution of this thesis is contained in Section 2.6 and Section 3.5. The two diagrams below describe how sections in both Chapter 2 and Chapter 3 depend on each other.

$$\begin{tikzcd}
2.1 \arrow[d] & & 2.3 \arrow[r] \arrow[d] & 2.5 \arrow[d] \\
2.2 \arrow[urr] & & 2.4 & 2.6
\end{tikzcd}
\qquad
\begin{tikzcd}
3.1 \arrow[dr] & 3.3 \arrow[r] & 3.5 \\
3.2 \arrow[r] & 3.4 \arrow[ur] &
\end{tikzcd}$$

[4] The reuse of the symbol K is not a coincidence; the C*-algebra K in the Takai duality theorem is the same C*-algebra as the C*-algebra K coming from the Stone-Neumann theorem.

Prerequisites

Basic knowledge of topology and functional analysis is required. This includes the theory of nets, which is discussed in the appendix of Conway's book [Con90]. Usage of important functional analytic tools such as the Stone-Weierstrass theorem and the Hahn-Banach theorem, which can also be found in Conway's book, should not raise questions.

We assume that the reader has had the equivalent of a basic course in C*-algebras; the first few chapters of Murphy's book [Mur90] should be enough. However, for some less elementary results we shall refer to Murphy's book explicitly.

Familiarity with abstract harmonic analysis is beneficial, but not required as the definitions and results we need will be paraphrased from Hewitt and Ross' books [HR63] and [HR70].

Other miscellaneous prerequisites include completions of normed linear spaces, algebraic tensor products and Zorn's lemma.

Conventions, Notation and Terminology

The symbols $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$ refer, respectively, to the sets of integers, real numbers, complex numbers and complex numbers of modulus one.

Linear spaces are always assumed to be complex. If E is a normed linear space, then its dual linear space consisting of bounded linear functionals from E to $\mathbb{C}$ is denoted by $E^\sharp$. Inner products are always assumed to be linear in the second variable. We use juxtaposition to denote compositions and evaluations of Hilbert space operators.

An algebra is a linear space together with a multiplication and involution. A homomorphism between two algebras is a linear operator that preserves the multiplication and involution.

Chapter 1

Integration on Topological Groups

Many fundamental constructions in this thesis will rely on vector-valued Haar integration. The primary purpose of this chapter is to introduce this type of integration.

In Section 1.1 we discuss some properties of scalar-valued compactly supported continuous functions on topological spaces Ω. Without proofs we state Urysohn's lemma and the existence of partitions of unity. After that we introduce the notion of a Radon integral on Ω and prove a version of Fubini's theorem.

In Section 1.2 we restrict our attention to topological groups G. Without proof we mention the existence of the Haar integral on G, which is a non-zero left-shift invariant Radon integral. In Section 1.3 we extend the definition of the Haar integral to vector-valued compactly supported continuous functions on G.

Finally, in section 1.4 we return to scalar-valued Haar integration and investigate the extent to which the Haar integral fails to be right-shift invariant.

1.1 Topological Spaces and Radon Integrals

In this section we cover some topological preliminaries. In particular, we introduce the class of scalar-valued functions we intend to integrate. The main references for this section are [Fol84], [Rud87] and [Bou04].

Definition 1.1.1. Suppose that Ω is a topological space. The *support* of a function $\varphi\colon \Omega \to \mathbb{C}$ is defined as the set

$$\sigma(\varphi) := \overline{\{\omega \in \Omega : \varphi(\omega) \neq 0\}}.$$

We say that φ is *compactly supported* if $\sigma(\varphi)$ is compact. The linear space of all compactly supported continuous functions from Ω to $\mathbb{C}$ is denoted by $C_{\mathrm{cpt}}(\Omega)$.

Often we require that our topological space Ω is locally compact and Hausdorff. The following version of Urysohn's lemma for locally compact Hausdorff spaces will be used extensively throughout this thesis.

Lemma 1.1.2 (cf. [Rud87, Lemma 2.12])**.** *Let Ω be a locally compact Hausdorff space. For every open $U \subset \Omega$ and compact $K \subset U$ there exists a function $\eta \in C_{\mathrm{cpt}}(\Omega)$ such that $\sigma(\eta) \subset U$ and $\eta(\Omega) \subset [0,1]$ and, furthermore, $\eta(\omega) = 1$ for all $\omega \in K$.*

By $C_0(\Omega)$ we denote the prototypical commutative C*-algebra consisting of all continuous functions $\varphi\colon \Omega \to \mathbb{C}$ for which $\{\omega \in G : |\varphi(\omega)| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$. The operations are defined pointwise and the supremum norm of an element $\varphi \in C_0(\Omega)$ is given by

$$\|\varphi\|_\infty := \sup_{\omega \in \Omega} |\varphi(\omega)|.$$

Lemma 1.1.3. *If Ω is a locally compact Hausdorff space, then $C_{\mathrm{cpt}}(\Omega)$ is a dense subalgebra of the C^*-algebra $C_0(\Omega)$.*

Proof. We leave it to the reader to check that $C_{\mathrm{cpt}}(\Omega)$ is a subalgebra of $C_0(\Omega)$.

Using Lemma 1.1.2 one readily verifies that the subalgebra $C_{\mathrm{cpt}}(G)$ of $C_0(G)$ satisfies the criteria of the Stone-Weierstrass theorem, which implies the density statement. ■

Locally compact Hausdorff spaces admit partitions of unity, which are a powerful technical tool allowing us to pass from local to global.

Definition 1.1.4. Let Ω be a topological space. Suppose that $K \subset \Omega$ is compact and that $U_1, \ldots, U_n \subset \Omega$ is an open cover of K. A *partition of unity* on K subordinate to $U_1, \ldots, U_n$ is a collection $\eta_1, \ldots, \eta_n \in C_{\mathrm{cpt}}(\Omega)$ such that $\sigma(\eta_j) \subset U_j$ and $\eta_j(\Omega) \subset [0,1]$ and, furthermore,

$$\sum_{j=1}^{n} \eta_j(\omega) \leq 1$$

for all $\omega \in \Omega$ with equality whenever $\omega \in K$.

Lemma 1.1.5 (cf. [Rud87, Theorem 2.13])**.** *Let Ω be a locally compact Hausdorff space. If $K \subset \Omega$ is compact, then for any open cover $U_1, \ldots, U_n \subset \Omega$ of K there exists a partition of unity on K subordinate to $U_1, \ldots, U_n$.*

Inspired by [Bou04] we shall develop the integration theory on locally compact Hausdorff spaces from a rather functional analytic approach. For the link with measure theoretic integrals we refer to [Fol84, Theorem 7.2].

Definition 1.1.6. Let Ω be a locally compact Hausdorff space. A *Radon integral* on Ω is a linear functional $\mu \colon C_{\mathrm{cpt}}(\Omega) \to \mathbb{C}$ such that $\mu(\varphi) \geq 0$ whenever $\varphi \geq 0$. Given $\varphi \in C_{\mathrm{cpt}}(\Omega)$, we call $\mu(\varphi)$ the *integral* of φ and we shall sometimes write

$$\int_{\Omega} \varphi(\omega)\, \mathrm{d}_{\mu}\omega := \mu(\varphi).$$

If $\varphi \in C_{\mathrm{cpt}}(\Omega)$ satisfies $\varphi(\Omega) \subset \mathbb{R}$, then is not hard to check that $\mu(\varphi) \in \mathbb{R}$. In particular, the real and imaginary part of any function in $C_{\mathrm{cpt}}(\Omega)$ always have real integrals. We shall now use this fact to prove the following lemma.

Lemma 1.1.7. *Let μ be a Radon integral on a locally compact Hausdorff space Ω.*

(i) The equality $\mu(\varphi^*) = \mu(\varphi)^*$ holds for all $\varphi \in C_{\mathrm{cpt}}(\Omega)$.

(ii) The inequality $\mu(|\varphi|) \geq |\mu(\varphi)|$ holds for all $\varphi \in C_{\mathrm{cpt}}(\Omega)$.

Proof. For part (i) we directly compute

$$\mu(\varphi^*) = \mu(\mathrm{re}(\varphi)) - i\mu(\mathrm{im}(\varphi)) = (\mu(\mathrm{re}(\varphi)) + i\mu(\mathrm{im}(\varphi)))^* = \mu(\varphi)^*$$

and obtain the desired equality.

For part (ii) we pick $\lambda \in \mathbb{T}$ such that $\lambda\mu(\varphi) \in \mathbb{R}$. Note that $\mu(\mathrm{im}(\lambda\varphi)) = 0$. So, because $|\lambda\varphi| - |\mathrm{re}(\lambda\varphi)| \geq 0$ and $|\mathrm{re}(\lambda\varphi)| \pm \mathrm{re}(\lambda\varphi) \geq 0$, we estimate

$$\mu(|\varphi|) = \mu(|\lambda\varphi|) \geq \mu(|\mathrm{re}(\lambda\varphi)|) \geq |\mu(\mathrm{re}(\lambda\varphi))| = |\mu(\lambda\varphi)| = |\mu(\varphi)|$$

and obtain the desired inequality. ■

The next definition is justified by Lemma 1.1.2.

Definition 1.1.8. Let μ be a Radon integral on a locally compact Hausdorff space Ω. If $K \subset \Omega$ is compact, then the value

$$\mu(K) := \inf_{\eta} \mu(\eta),$$

where the infimum runs over all functions $\eta \in C_{\mathrm{cpt}}(\Omega)$ for which $\eta(\Omega) \subset [0,1]$ and $\eta(\omega) = 1$ for all $\omega \in K$, is called the *volume* of K.

The volume is a very useful tool for estimating integrals.

Lemma 1.1.9. *Let μ be a Radon integral on a locally compact Hausdorff space Ω. If $K \subset \Omega$ is compact, then $\mu(|\varphi|) \leq \mu(K)\|\varphi\|_\infty$ for all $\varphi \in C_{\mathrm{cpt}}(\Omega)$ with $\sigma(\varphi) \subset K$.*

Proof. Assume that $\varphi \in C_{\mathrm{cpt}}(\Omega)$ satisfies $\sigma(\varphi) \subset K$. For any $\eta \in C_{\mathrm{cpt}}(\Omega)$ with $\eta(\Omega) \subset [0,1]$ and $\eta(\omega) = 1$ for all $\omega \in K$ we have $|\varphi| \leq \eta\|\varphi\|_\infty$ and therefore $\mu(|\varphi|) \leq \mu(\eta)\|\varphi\|_\infty$. ■

The proofs of the following two results nicely illustrate how Lemma 1.1.9 can be used.

Lemma 1.1.10. *Assume Ω and Ω' are locally compact Hausdorff spaces with Radon integrals μ and μ', respectively. For any $f \in C_{\mathrm{cpt}}(\Omega \times \Omega')$ the functions*

$$\Omega \ni \omega \mapsto \int_{\Omega'} f(\omega, \omega') \, \mathrm{d}_{\mu'}\omega' \in \mathbb{C}, \qquad \Omega' \ni \omega' \mapsto \int_{\Omega} f(\omega, \omega') \, \mathrm{d}_{\mu}\omega \in \mathbb{C}$$

belong to $C_{\mathrm{cpt}}(\Omega)$ and $C_{\mathrm{cpt}}(\Omega')$, respectively.

Proof. We only prove the assertion for the function on the left-hand side. Define $\varphi\colon \Omega \to \mathbb{C}$ by

$$\varphi(\omega) := \int_{\Omega'} f(\omega, \omega') \, \mathrm{d}_{\mu'}\omega'$$

for all $\omega \in \Omega$. Given a net $(\omega_\lambda)_{\lambda\in\Lambda}$ that converges to ω_∞ in Ω, we must show that $(\varphi(\omega_\lambda))_{\lambda\in\Lambda}$ converges to $\varphi(\omega_\infty)$ in $\mathbb{C}$. Let $\varepsilon > 0$ be given. By compactness of $\sigma(f)$ there are compact $K \subset \Omega$ and $K' \subset \Omega'$ such that $\sigma(f) \subset K \times K'$. Define $r' := \mu'(K') + 1$. Since f is continuous, there are for each $\omega' \in K'$ open neighbourhoods $U_{\omega'} \subset \Omega$ of ω_∞ and $U'_{\omega'} \subset \Omega'$ of ω' such that $f(U_{\omega'} \times U'_{\omega'})$ is contained in the open ball of radius $\varepsilon/2r'$ centered at $f(\omega_\infty, \omega')$ in $\mathbb{C}$. Choose $\omega'_1, \dots, \omega'_n \in K'$ such that $U'_{\omega_1}, \dots, U'_{\omega_n}$ cover K'. Now consider

$$U := \bigcap_{j=1}^{n} U_{\omega'_j}$$

and choose $\lambda_0 \in \Lambda$ such that $\omega_\lambda \in U$ for all $\lambda \geq \lambda_0$. We claim that $|\varphi(\omega_\lambda) - \varphi(\omega_\infty)| < \varepsilon$ for all $\lambda \geq \lambda_0$. One the one hand, if $\omega' \notin K'$, then $|f(\omega_\lambda, \omega') - f(\omega_\infty, \omega')| = 0$. On the other hand, if $\omega' \in K'$, then $\omega' \in U'_{\omega'_j}$ for some $j = 1, \dots, n$ and therefore

$$|f(\omega_\lambda, \omega') - f(\omega_\infty, \omega)| \leq |f(\omega_\lambda, \omega') - f(\omega_\infty, \omega'_j)| + |f(\omega_\infty, \omega'_j) - f(\omega_\infty, \omega')| < \frac{\varepsilon}{2r'} + \frac{\varepsilon}{2r'} = \frac{\varepsilon}{r'}.$$

Overall, Lemma 1.1.9 implies that

$$|\varphi(\omega_\lambda) - \varphi(\omega_\infty)| \leq \int_{\Omega'} |f(\omega_\lambda, \omega') - f(\omega_\infty, \omega')| \, \mathrm{d}_{\mu'}\omega' \leq \mu'(K')\frac{\varepsilon}{r'} < \varepsilon,$$

thereby proving the claim. Finally, φ is compactly supported as $\sigma(\varphi) \subset K$. ■

Importantly, Lemma 1.1.10 allows us to formulate the following version of Fubini's theorem.

Theorem 1.1.11. *Assume Ω and Ω' are locally compact Hausdorff spaces with Radon integrals μ and μ', respectively. For any $f \in C_{\mathrm{cpt}}(\Omega \times \Omega')$ we have*

$$\int_\Omega \int_{\Omega'} f(\omega, \omega') \, \mathrm{d}_{\mu'}\omega' \, \mathrm{d}_\mu \omega = \int_{\Omega'} \int_\Omega f(\omega, \omega') \, \mathrm{d}_\mu \omega \, \mathrm{d}_{\mu'}\omega'.$$

Proof. For the sake of notation we define

$$\lambda := \int_\Omega \int_{\Omega'} f(\omega, \omega') \, \mathrm{d}_{\mu'}\omega' \, \mathrm{d}_\mu \omega, \qquad \lambda' := \int_{\Omega'} \int_\Omega f(\omega, \omega') \, \mathrm{d}_\mu \omega \, \mathrm{d}_{\mu'}\omega'.$$

Let $\varepsilon > 0$ be arbitrary. By compactness of $\sigma(f)$ there are compact $K \subset \Omega$ and $K' \subset \Omega'$ such that $\sigma(f) \subset K \times K'$. Put $r := \mu(K) + 1$ and $r' := \mu'(K') + 1$. Since f is continuous, there is for each $\omega \in K$ an open neighbourhood $U_\omega \subset \Omega$ of ω such that for each $\omega' \in K'$ the set $f(U_\omega \times \{\omega'\})$ is contained in the open ball of radius $\varepsilon/2rr'$ centered at $f(\omega, \omega')$ in $\mathbb{C}$. Choose $\omega_1, \ldots, \omega_n \in K$ such that $U_{\omega_1}, \ldots, U_{\omega_n}$ cover K. Using Lemma 1.1.5 we find a partition of unity $\eta_1, \ldots, \eta_n \in C_{\mathrm{cpt}}(\Omega)$ on K subordinate to $U_{\omega_1}, \ldots, U_{\omega_n}$. Pick any $\eta \in C_{\mathrm{cpt}}(\Omega)$ for which $\eta(\Omega) \subset [0, 1]$ and $\eta(\omega) = 1$ for all $\omega \in K$. The function $g \colon \Omega \times \Omega' \to \mathbb{C}$ given by

$$g(\omega, \omega') := \eta(\omega) \sum_{j=1}^n \eta_j(\omega) f(\omega_j, \omega')$$

for all $(\omega, \omega') \in \Omega \times \Omega'$ belongs to $C_{\mathrm{cpt}}(\Omega \times \Omega')$ and satisfies

$$\int_\Omega \int_{\Omega'} g(\omega, \omega') \, \mathrm{d}_{\mu'}\omega' \, \mathrm{d}_\mu \omega = \mu(\eta) \sum_{j=1}^n \mu(\eta_j) \int_{\Omega'} f(\omega_j, \omega') \, \mathrm{d}_{\mu'}\omega' = \int_{\Omega'} \int_\Omega g(\omega, \omega') \, \mathrm{d}_\mu \omega \, \mathrm{d}_{\mu'}\omega'.$$

By Lemma 1.1.9 we have

$$\begin{aligned}
\int_\Omega \int_{\Omega'} |f(\omega, \omega') - g(\omega, \omega')| \, \mathrm{d}_{\mu'}\omega' \, \mathrm{d}_\mu \omega &= \int_\Omega \int_{\Omega'} \eta(\omega) \sum_{j=1}^n \eta_j(\omega) |f(\omega, \omega') - f(\omega_j, \omega')| \, \mathrm{d}_{\mu'}\omega' \, \mathrm{d}_\mu \omega \\
&\leq \int_\Omega \mu'(K') \eta(\omega) \frac{\varepsilon}{2rr'} \, \mathrm{d}_\mu \omega = \mu(\eta) \mu(K') \frac{\varepsilon}{2rr'}
\end{aligned}$$

and, likewise,

$$\begin{aligned}
\int_{\Omega'} \int_\Omega |g(\omega, \omega') - f(\omega, \omega')| \, \mathrm{d}_\mu \omega \, \mathrm{d}_{\mu'}\omega' &= \int_{\Omega'} \int_\Omega \eta(\omega) \sum_{j=1}^n \eta_j(\omega) |f(\omega_j, \omega') - f(\omega, \omega')| \, \mathrm{d}_\mu \omega \, \mathrm{d}_{\mu'}\omega' \\
&\leq \mu(K') \int_\Omega \eta(\omega) \frac{\varepsilon}{2rr'} \, \mathrm{d}_\mu \omega = \mu(\eta) \mu(K') \frac{\varepsilon}{2rr'}.
\end{aligned}$$

So we deduce that

$$\begin{aligned}
|\lambda - \lambda'| &\leq \int_\Omega \int_{\Omega'} |f(\omega, \omega') - g(\omega, \omega')| \, \mathrm{d}_{\mu'}\omega' \, \mathrm{d}_\mu \omega + \int_{\Omega'} \int_\Omega |g(\omega, \omega') - f(\omega, \omega')| \, \mathrm{d}_\mu \omega \, \mathrm{d}_{\mu'}\omega' \\
&\leq \mu(\eta) \mu(K') \frac{\varepsilon}{2rr'} + \mu(\eta) \mu(K') \frac{\varepsilon}{2rr'} = \mu(\eta) \mu'(K') \frac{\varepsilon}{rr'}.
\end{aligned}$$

Since this is true for all $\eta \in C_{\mathrm{cpt}}(\Omega)$ with $\eta(\Omega) \subset [0, 1]$ and $\eta(\omega) = 1$ for all $\omega \in K$, it follows by definition of the volume that

$$|\lambda - \lambda'| \leq \mu(K) \mu'(K') \frac{\varepsilon}{rr'} < \varepsilon.$$

But $\varepsilon > 0$ was chosen arbitrarily and therefore we must have $\lambda = \lambda'$. ∎

1.2 Topological Groups and Haar Integrals

Topological groups are used to study continuous symmetries. This section is devoted to providing a quick overview of topological groups and integration thereon in the locally compact Hausdorff case. A more thorough treatment of this material can be found in [HR63] and [HR70]. A look at [Fol95] and [DE14] is also certainly recommended.

Definition 1.2.1. A *topological group* is a group G together with a topology for which the multiplication from $G \times G$ to G and inversion from G to G are continuous maps. A *locally compact Hausdorff group* is a topological group G for which the underlying topological space is locally compact and Hausdorff.

Definition 1.2.2. Let G and G' be two topological groups. A group homomorphism from G to G' is called a *topological group homomorphism* if it is continuous and a *topological group isomorphism* if it is also a homeomorphism.

Given a group G, the symbol e will be used to denote its neutral element, absent express mention to the contrary. Recall from group theory that a subset of a group is called symmetric if it is invariant under inversion.

Now let G be a topological group. For each element $s \in G$ and subset $U \subset G$ we have that sU is open if and only if U is open. Consequently, the topology of G is fully determined by the neighbourhoods of e. Thus practicality of the following lemma should not come as a surprise.

Lemma 1.2.3. *Let G be a topological group. For any open neighbourhood $V \subset G$ of e there is a symmetric open neighbourhood $U \subset G$ of e such that $UU \subset V$.*

Proof. Since the multiplication on G is continuous, there are two open neighbourhoods $U_1, U_2 \subset G$ of e such that $U_1 U_2 \subset V$. The sets U_1^{-1} and U_2^{-1} are also open neighbourhoods of e by continuity of the inversion on G. Hence $U := U_1 \cap U_1^{-1} \cap U_2 \cap U_2^{-1}$ serves. ■

Given $\theta \in G$, we define linear operators $L_\theta \colon C_{\mathrm{cpt}}(G) \to C_{\mathrm{cpt}}(G)$ and $R_\theta \colon C_{\mathrm{cpt}}(G) \to C_{\mathrm{cpt}}(G)$ by $L_\theta(\varphi)(s) := \varphi(\theta^{-1}s)$ and $R_\theta(\varphi)(s) := \varphi(s\theta)$ for $\varphi \in C_{\mathrm{cpt}}(G)$ and $s \in G$. Observe that the equalities $L_{\theta_1\theta_2} = L_{\theta_1} \circ L_{\theta_2}$ and $R_{\theta_1\theta_2} = R_{\theta_1} \circ R_{\theta_2}$ hold for all $\theta_1, \theta_2 \in G$.

Definition 1.2.4. Let G be a topological group. We call L_θ (resp. R_θ) the *left-shift* (resp. *right-shift*) by $\theta \in G$. A linear functional $\mu \colon C_{\mathrm{cpt}}(G) \to \mathbb{C}$ is said to be *left-shift invariant* (resp. *right-shift invariant*) if $\mu \circ L_\theta = \mu$ (resp. $\mu \circ R_\theta = \mu$) for all group elements $\theta \in G$.

We are now ready to define Haar integrals.

Definition 1.2.5. Let G be a locally compact Hausdorff group. A *Haar integral* on G is a non-zero left-shift invariant Radon integral μ on G.

So a Haar integral μ on a locally compact Hausdorff group G satisfies

$$\int_G \varphi(\theta^{-1}s) \, \mathrm{d}_\mu s = \int_G \varphi(s) \, \mathrm{d}_\mu s$$

for all $\varphi \in C_{\mathrm{cpt}}(G)$ and $\theta \in G$. The following theorem is fundamental and guarantees that Haar integrals exist.

Theorem 1.2.6 (cf. [HR63, Theorem 15.5]). *If G is a locally compact Hausdorff group, then there exists a Haar integral μ on G. Moreover, if ν is another Haar integral on H, then μ and ν are proportial by some constant $r > 0$.*

The following innocent-looking lemma has important applications. For instance, in the next section we shall see how it provides a method to generalize the definition of the Haar integral to vector-valued compactly supported continuous functions.

Lemma 1.2.7. *Let G be a locally compact Hausdorff group with Haar integral μ. If $\varphi \in C_{\mathrm{cpt}}(G)$ satisfies $\varphi \geq 0$ and $\mu(\varphi) = 0$, then $\varphi = 0$.*

Proof. Suppose that $\varphi \in C_{\mathrm{cpt}}(G)$ satisfies $\varphi \geq 0$ and $\mu(\varphi) = 0$, but assume to the contrary that $\varphi \neq 0$. By continuity of φ we find a constant $r > 0$ and a non-empty open subset $U \subset G$ such that $\varphi(s) \geq r$ for all $s \in U$. Now let $\eta \in C_{\mathrm{cpt}}(G)$ be such that $\eta \geq 0$. By compactness of $\sigma(\eta)$ there are $\theta_1, \dots, \theta_n \in G$ such that $\theta_1 U, \dots, \theta_n U$ cover $\sigma(\eta)$. It follows that

$$0 \leq \eta(s) \leq \frac{1}{r} \|\eta\|_\infty \sum_{j=1}^{n} \varphi(\theta_j^{-1} s)$$

for all $s \in \sigma(\eta)$. So we obtain

$$0 \leq \int_G \eta(s) \ \mathrm{d}_\mu s \leq \frac{1}{r} \|\eta\|_\infty \sum_{j=1}^{n} \int_G \varphi(\theta_j^{-1} s) \ \mathrm{d}_\mu s = \frac{n}{r} \|\eta\|_\infty \int_G \varphi(s) \ \mathrm{d}_\mu s = 0$$

and therefore $\mu(\eta) = 0$. Hence $\mu = 0$ as the set $\{\eta \in C_{\mathrm{cpt}}(G) : \eta \geq 0\}$ spans $C_{\mathrm{cpt}}(G)$. This is a contradiction as a Haar integral is non-zero by definition. ■

To every locally compact Hausdorff group G there is an associated Hilbert space H^G. Indeed, if G is a locally compact Hausdorff group with Haar integral μ, then Lemma 1.2.7 implies that the value $\langle \chi, \psi \rangle := \mu(\chi^* \psi)$ for $\chi, \psi \in C_{\mathrm{cpt}}(G)$ defines an inner product on $C_{\mathrm{cpt}}(G)$ and therefore, after completing, we obtain a Hilbert space H^G.

From now onwards we implicitly assume that any locally compact Hausdorff group G is equipped with a fixed Haar integral μ.

1.3 Vector-Valued Haar Integration

In this section we shall extend the definition of the Haar integral to functions that take values in an arbitrary Banach space.

Definition 1.3.1. Let G be a topological group and E a normed linear space. The *support* of a function $f\colon G \to E$ is defined as the set

$$\sigma(f) := \overline{\{s \in G : f(s) \neq 0\}}.$$

We say that f is *compactly supported* if $\sigma(f)$ is compact. The linear space of all compactly supported continuous functions from G to E is denoted by $C_{\mathrm{cpt}}(G, E)$.

It turns out that functions in $C_{\mathrm{cpt}}(G, E)$ satisfy a stronger notion of continuity.

Definition 1.3.2. Let G be a topological group and let E be a normed linear space. A function $f\colon G \to E$ is called *left-uniformly continuous* (resp. *right-uniformly continuous*) if for every $\varepsilon > 0$ there exists an open neighbourhood $U \subset G$ of e such that $\|f(s) - f(t)\| < \varepsilon$ for all $s, t \in G$ with $s^{-1}t \in U$ (resp. $st^{-1} \in U$).

Lemma 1.3.3. *If G is a topological group and E a normed linear space, then any $f \in C_{\mathrm{cpt}}(G, E)$ is left-uniformly continuous and right-uniformly continuous.*

Proof. We only prove left-uniform continuity. Let $\varepsilon > 0$ be given. Since f is continuous, there is for each $\theta \in \sigma(f)$ an open neighbourhood $V_\theta \subset G$ of e such that $f(\theta V_\theta)$ is contained in the open ball of radius $\varepsilon/2$ centered at $f(\theta)$ in E. By Lemma 1.2.3 there is a symmetric open neighbourhood U_θ of e such that $U_\theta U_\theta \subset V_\theta$. Since $\sigma(f)$ is compact, there are $\theta_1, \dots, \theta_n \in \sigma(f)$ such that $\theta_1 U_{\theta_1}, \dots, \theta_n U_{\theta_n}$ cover $\sigma(f)$. Note that the set

$$U := \bigcap_{j=1}^{n} U_{\theta_j}$$

is a symmetric open neighbourhood of e. We claim that $\|f(s) - f(t)\| < \varepsilon$ whenever $s, t \in G$ satisfy $s^{-1}t \in U$. If neither s nor t lies in $\sigma(f)$, then $\|f(s) - f(t)\| = 0$ and we are done. Now suppose that $s \in \sigma(f)$ or $t \in \sigma(f)$. Note that $s^{-1}t \in U$ if and only if $t^{-1}s \in U$ by symmetry of U. So without loss of generality we may assume that $s \in \sigma(f)$. Thus $s \in \theta_j U_{\theta_j}$ for some $j = 1, \dots, n$ and therefore $t \in \theta_j U_{\theta_j} U_{\theta_j}$ as $t = ss^{-1}t$. Hence $s, t \in \theta_j V_{\theta_j}$ and for that reason

$$\|f(s) - f(t)\| \leq \|f(s) - f(\theta_j)\| + \|f(\theta_j) - f(t)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves the claim. ■

Lemma 1.2.7 implies that the following definition presents a well-defined norm on $C_{\text{cpt}}(G, E)$.

Definition 1.3.4. Let G be a locally compact Hausdorff group and suppose that E is a normed linear space. The value

$$\|f\|_{\text{int}} := \int_G \|f(s)\| \, \mathrm{d}_\mu s$$

is called the *integral norm* of $f \in C_{\text{cpt}}(G, E)$.

Let G be a locally compact Hausdorff group and E a normed linear space. Now that we have the integral norm at our disposal, we shall first prove an approximation result, which ultimately allows us to integrate functions in $C_{\text{cpt}}(G, E)$ for complete E. The tensor product $C_{\text{cpt}}(G) \otimes E$ can be viewed as a linear subspace of $C_{\text{cpt}}(G, E)$. Indeed, for any $\varphi \in C_{\text{cpt}}(G)$ and $x \in E$ we can identify the pure tensor $\varphi \otimes x$ in $C_{\text{cpt}}(G) \otimes E$ with the function

$$G \ni s \mapsto \varphi(s)x \in E,$$

which is easily seen to be an element of $C_{\text{cpt}}(G, E)$.

Lemma 1.3.5. *If G is a locally compact Hausdorff group and E a normed linear space, then $C_{\text{cpt}}(G) \otimes E$ is dense in $C_{\text{cpt}}(G, E)$ with respect to the integral norm.*

Proof. Let $f \in C_{\text{cpt}}(G, E)$ and $\varepsilon > 0$ be given. Fix a compact neighbourhood $K \subset G$ of e and define $r := \mu(\sigma(f)K) + 1$. By Lemma 1.3.3 there is an open neighbourhood $U \subset G$ of e such that $\|f(s) - f(t)\| < \varepsilon/r$ whenever $s, t \in G$ satisfy $s^{-1}t \in U$. By shrinking U if necessary we may assume that U is symmetric and contained in K. Choose $t_1, \dots, t_n \in \sigma(f)$ such that $t_1 U, \dots, t_n U$ cover $\sigma(f)$. Using Lemma 1.1.5 we find a partition of unity $\eta_1, \dots, \eta_n \in C_{\text{cpt}}(G)$ on $\sigma(f)$ subordinate to $t_1 U, \dots, t_n U$. We claim that

$$g := \sum_{j=1}^{n} \eta_j \otimes f(t_j)$$

approximates f within the margin of error ε. Indeed, by symmetry of U we have $s^{-1}t_j \in U$ whenever $s \in \sigma(\eta_j)$ and therefore

$$\begin{aligned}\|f-g\|_{\mathrm{int}} &= \int_G \left\| f(s) - \sum_{j=1}^n \eta_j(s) f(t_j) \right\| \mathrm{d}_\mu s = \int_G \left\| \sum_{j=1}^n \eta_j(s)(f(s)-f(t_j)) \right\| \mathrm{d}_\mu s \\ &\leq \int_G \sum_{j=1}^n \eta_j(s) \|f(s)-f(t_j)\| \, \mathrm{d}_\mu s \leq \mu(\sigma(f)K)\frac{\varepsilon}{r} < \varepsilon\end{aligned}$$

by Lemma 1.1.9. ■

Assume that E is a Banach space. Let $I \colon E \to E$ be the identity operator. We now have all the ingredients to construct the integral of an arbitrary function in $C_{\mathrm{cpt}}(G,E)$. By the universal property of the tensor product there is a unique linear operator $\mu \times I \colon C_{\mathrm{cpt}}(G) \otimes E \to E$ that acts on pure tensors via

$$(\mu \times I)(\varphi \otimes x) = \left(\int_G \varphi(s) \, \mathrm{d}_\mu s \right) x$$

for all $\varphi \in C_{\mathrm{cpt}}(G)$ and $x \in E$. In fact, $\mu \times I$ is norm-decreasing with respect to the integral norm. Indeed, using that the natural embedding of E into $E^{\sharp\sharp}$ is isometric, we see that

$$\begin{aligned}\left\| (\mu \times I)\left(\sum_{j=1}^n \varphi_j \otimes x_j \right) \right\| &= \sup_{\substack{x^\sharp \in E^\sharp \\ \|x^\sharp\| \leq 1}} \left| x^\sharp \left(\sum_{j=1}^n \left(\int_G \varphi_j(s) \, \mathrm{d}_\mu s \right) x_j \right) \right| \\ &= \sup_{\substack{x^\sharp \in E^\sharp \\ \|x^\sharp\| \leq 1}} \left| \int_G x^\sharp \left(\sum_{j=1}^n \varphi_j(s) x_j \right) \mathrm{d}_\mu s \right| \\ &\leq \int_G \sup_{\substack{x^\sharp \in E^\sharp \\ \|x^\sharp\| \leq 1}} \left| x^\sharp \left(\sum_{j=1}^n \varphi_j(s) x_j \right) \right| \mathrm{d}_\mu s \\ &= \int_G \left\| \sum_{j=1}^n \varphi_j(s) x_j \right\| \mathrm{d}_\mu s = \left\| \sum_{j=1}^n \varphi_j \otimes x_j \right\|_{\mathrm{int}}\end{aligned}$$

for all $\varphi_1, \dots, \varphi_n \in C_{\mathrm{cpt}}(G)$ and $x_1, \dots, x_n \in E$. In the third step we used Lemma 1.1.7. So by completeness of E it follows that $\mu \times I$ can be uniquely extended to a norm-decreasing linear operator $\mu \times I \colon C_{\mathrm{cpt}}(G,E) \to E$. This leads us to the following definition.

Definition 1.3.6. Let G be a locally compact Hausdorff group and E a Banach space. Given a function $f \in C_{\mathrm{cpt}}(G,E)$, we call $(\mu \times I)(f)$ the *integral* of f and we write

$$\int_G f(s) \, \mathrm{d}_\mu s := (\mu \times I)(f).$$

Let $\theta \in G$ be arbitrary. Consider the linear operators $L_\theta \times I \colon C_{\mathrm{cpt}}(G,E) \to C_{\mathrm{cpt}}(G,E)$ and $R_\theta \times I \colon C_{\mathrm{cpt}}(G,E) \to C_{\mathrm{cpt}}(G,E)$ given by $(L_\theta \times I)(f)(s) = f(\theta^{-1}s)$ and $(R_\theta \times I)(f)(s) = f(s\theta)$ for $f \in C_{\mathrm{cpt}}(G,E)$. Since $(\mu \times I) \circ (L_\theta \times I) = \mu \times I$, it follows that $L_\theta \times I$ is isometric. Contrarily, the map $R_\theta \times I$ may fail to be isometric.

Vector-valued integrals also interact nicely with bounded antilinear and linear operators.

Theorem 1.3.7. *Let G be a locally compact Hausdorff group. Let $\Phi \colon E \to E'$ is a bounded antilinear or linear operator between Banach spaces E and E'. For any $f \in C_{\mathrm{cpt}}(G,E)$ we have*

$$\int_G \Phi(f(s)) \, \mathrm{d}_\mu s = \Phi\left(\int_G f(s) \, \mathrm{d}_\mu s \right).$$

Proof. We only prove the antilinear case; the same content serves as a proof for the linear case verbatim after omitting the adjoints. By Lemma 1.3.5 it suffices to prove that the desired equality holds for pure tensors in $C_{\mathrm{cpt}}(G) \otimes E$. It follows from Lemma 1.1.7 that

$$\begin{aligned}\int_G \Phi(\varphi(s)x) \, \mathrm{d}_\mu s &= \int_G \varphi^*(s)\Phi(x) \, \mathrm{d}_\mu s = \Big(\int_G \varphi^*(s) \, \mathrm{d}_\mu s\Big)\Phi(x) = \Big(\int_G \varphi(s) \, \mathrm{d}_\mu s\Big)^*\Phi(x) \\ &= \Phi\bigg(\Big(\int_G \varphi(s) \, \mathrm{d}_\mu s\Big)x\bigg) = \Phi\Big(\int_G \varphi(s)x \, \mathrm{d}_\mu s\Big)\end{aligned}$$

for all $\varphi \in C_{\mathrm{cpt}}(G)$ and $x \in E$. ∎

The next two results are variations of Lemma 1.1.10 and Theorem 1.1.11 for vector-valued Haar integration.

Lemma 1.3.8. *Assume G and G' are locally compact Hausdorff groups with Haar integrals μ and μ', respectively. Let E be any Banach space. For any $f \in C_{\mathrm{cpt}}(G \times G', E)$ the functions*

$$G \ni s \mapsto \int_{G'} f(s, s') \, \mathrm{d}_{\mu'} s' \in E, \qquad G' \ni s' \mapsto \int_G f(s, s') \, \mathrm{d}_\mu s \in E$$

belong to $C_{\mathrm{cpt}}(G, E)$ and $C_{\mathrm{cpt}}(G', E)$, respectively.

Proof. The proof of Lemma 1.1.10 almost works verbatim; the details are left to the reader. ∎

Importantly, Lemma 1.3.8 allows us to formulate the following version of Fubini's theorem.

Theorem 1.3.9. *Assume G and G' are locally compact Hausdorff spaces with Haar integrals μ and μ', respectively. Let E be any Banach space. For any $f \in C_{\mathrm{cpt}}(G \times G')$ we have*

$$\int_G \int_{G'} f(s, s') \, \mathrm{d}_{\mu'} s' \, \mathrm{d}_\mu s = \int_{G'} \int_G f(s, s') \, \mathrm{d}_\mu s \, \mathrm{d}_{\mu'} s'.$$

Proof. By Theorem 1.3.7 and Theorem 1.1.11 we have

$$\begin{aligned}x^\sharp\Big(\int_G \int_{G'} f(s, s') \, \mathrm{d}_{\mu'} s' \, \mathrm{d}_\mu s\Big) &= \int_G \int_{G'} x^\sharp(f(s, s')) \, \mathrm{d}_{\mu'} s' \, \mathrm{d}_\mu s \\ &= \int_{G'} \int_G x^\sharp(f(s, s')) \, \mathrm{d}_\mu s \, \mathrm{d}_{\mu'} s' \\ &= x^\sharp\Big(\int_{G'} \int_G f(s, s') \, \mathrm{d}_\mu s \, \mathrm{d}_{\mu'} s'\Big)\end{aligned}$$

for all $x^\sharp \in E^\sharp$. Hence the result follows from the Hahn-Banach theorem. ∎

Due to their elementary nature, we shall often use Theorem 1.3.7 and Theorem 1.3.9 without further reference.

We end this section with a continuity result for shifts.

Lemma 1.3.10. *Let G be a locally compact Hausdorff space and E a Banach space. Let $I\colon E \to E$ be the identity operator. For each $f \in C_{\mathrm{cpt}}(G, E)$ the maps*

$$G \ni \theta \mapsto (L_\theta \times I)(f) \in C_{\mathrm{cpt}}(G, E), \qquad G \ni \theta \mapsto (R_\theta \times I)(f) \in C_{\mathrm{cpt}}(G, E)$$

are continuous with respect to the integral norm.

Proof. We only prove the assertion for the function on the left-hand side. Given a net $(\theta_\lambda)_{\lambda\in\Lambda}$ that converges to θ_∞ in G, we must show that $((L_{\theta_\lambda} \times I)(f))_{\lambda\in\Lambda}$ converges to $(L_{\theta_\infty} \times I)(f)$ in $C_{\mathrm{cpt}}(G, E)$ for each $f \in C_{\mathrm{cpt}}(G, E)$. Let $\varepsilon > 0$ be given. Fix a compact neighbourhood $K \subset G$ of e and define $r := \mu(\theta_\infty K^{-1}\sigma(f)) + 1$. By right-uniform continuity of f there is an open neighbourhood $U \subset G$ of e such that $\|f(s) - f(t)\| < \varepsilon/r$ whenever $s, t \in G$ satisfy $st^{-1} \in U$. We may assume $U \subset K$. Pick $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $\theta_\lambda \in \theta_\infty U^{-1}$. We claim that $\|(L_{\theta_\lambda} \times I)(f) - (L_{\theta_\infty} \times I)(f)\|_{\mathrm{int}} < \varepsilon$ for $\lambda \geq \lambda_0$. Since $\theta_\lambda, \theta_\infty \in \theta_\infty K^{-1}$, we have $\theta_\lambda^{-1}s, \theta_\infty^{-1}s \notin \sigma(f)$ whenever $s \notin \theta_\infty K^{-1}\sigma(f)$. We also clearly have $\theta_\lambda^{-1}s(\theta_\infty^{-1}s)^{-1} \in U$ for all $s \in G$. Combining these last two observations yields

$$\|(L_{\theta_\lambda} \times I)(f) - (L_{\theta_\infty} \times I)(f)\|_{\mathrm{int}} = \int_G \|f(\theta_\lambda^{-1}s) - f(\theta_\infty^{-1}s)\| \, \mathrm{d}_\mu s \leq \mu(\theta_\infty K^{-1}\sigma(f))\frac{\varepsilon}{r} < \varepsilon.$$

This completes the proof of the claim and continuity follows. ■

1.4 The Modular Function

We return to scalar-valued Haar integration. Let G be a locally compact Hausdorff group. A Haar integral μ on G is left-invariant by definition. However, it may happen that μ is not right-invariant. In other words, there may exist $\theta \in G$ such that $\mu \circ R_\theta \neq \mu$. In this section we investigate the extent to which μ fails to be right-invariant. The main reference for this section is [Fol95].

For each $\theta \in G$ one readily verifies that $\mu \circ R_\theta$ is a Haar integral on G. So by Theorem 1.2.6 there exists a unique map $\Delta\colon G \to \,]0, \infty[$ such that $\Delta(\theta)\mu(R_\theta(\varphi)) = \mu(\varphi)$ for all $\theta \in G$ and $\varphi \in C_{\mathrm{cpt}}(G)$. Note that Δ is independent of the choice of Haar integral μ on G.

Definition 1.4.1. Let G be a locally compact Hausdorff group. The map Δ is called the *modular function* of G. We say that G is *unimodular* if $\Delta(\theta) = 1$ for all $\theta \in G$.

Lemma 1.4.2. *If G is a locally compact Hausdorff group, then its modular function Δ is a topological group homomorphism.*

Proof. Fix $\eta \in C_{\mathrm{cpt}}(G)$ such that $\mu(\eta) \neq 0$. By definition of Δ we have $\Delta(\theta) = \mu(\eta)/\mu(R_\theta(\eta))$ for all $\theta \in G$. So continuity of Δ is a consequence of Lemma 1.1.7 and Lemma 1.3.10. We also see that Δ respects the group structure as

$$\Delta(\theta_1\theta_2) = \frac{\mu(\eta)}{\mu(R_{\theta_1\theta_2}(\eta))} = \frac{\mu(\eta)}{\mu(R_{\theta_1}(R_{\theta_2}(\eta)))} = \frac{\Delta(\theta_1)\mu(\eta)}{\mu(R_{\theta_2}(\eta))} = \frac{\Delta(\theta_1)\Delta(\theta_2)\mu(\eta)}{\mu(\eta)} = \Delta(\theta_1)\Delta(\theta_2)$$

for all $\theta_1, \theta_2 \in G$. ■

It is clear from the definition that a Haar integral μ on G is right-invariant if and only if G is unimodular. Also, if G is abelian, then G is clearly unimodular.

We end this short section with the following lemma.

Lemma 1.4.3. *Let G be a locally compact Hausdorff group. The equality*

$$\int_G \Delta(s^{-1})\varphi(s^{-1}) \, \mathrm{d}_\mu s = \int_G \varphi(s) \, \mathrm{d}_\mu s.$$

holds for all functions $\varphi \in C_{\mathrm{cpt}}(G)$.

Proof. Consider the map $\nu\colon C_{\mathrm{cpt}}(G) \to \mathbb{C}$ defined by

$$\nu(\varphi) := \int_G \Delta(s^{-1})\varphi(s^{-1}) \,\mathrm{d}_\mu s$$

for $\varphi \in C_{\mathrm{cpt}}(G)$. It is clear that ν is a Radon integral on G. In fact, because

$$\begin{aligned}\int_G \varphi(\theta^{-1}s) \,\mathrm{d}_\nu s &= \int_G \Delta(s^{-1})\varphi((s\theta)^{-1}) \,\mathrm{d}_\mu s = \Delta(\theta) \int_G \Delta((s\theta)^{-1})\varphi((s\theta)^{-1}) \,\mathrm{d}_\mu s \\ &= \int_G \Delta(s^{-1})\varphi(s^{-1}) \,\mathrm{d}_\mu s = \int_G \varphi(s) \,\mathrm{d}_\nu s.\end{aligned}$$

for all $\varphi \in C_{\mathrm{cpt}}(G)$ and $\theta \in G$, it follows that ν is a Haar integral on G. So Theorem 1.2.6 implies that μ and ν are proportional by some constant $r > 0$. Now choose $\eta \in C_{\mathrm{cpt}}(G)$ such that $\mu(\eta) \neq 0$. Both μ and ν send the function

$$G \ni s \mapsto \eta(s) + \Delta(s^{-1})\eta(s^{-1}) \in \mathbb{C}$$

in $C_{\mathrm{cpt}}(G)$ to the scalar $\mu(\eta) + \nu(\eta) \neq 0$ in $\mathbb{C}$ and therefore we infer that $r = 1$. ■

Chapter 2

C*-Actions

The main objects that we shall study in this thesis are called C*-actions. Loosely speaking, a C*-action comprizes a locally compact Hausdorff group G parametrizing a family of automorphisms of a C*-algebra A.

Every commutative C*-algebra A is isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω. For this reason, albeit abstract jargon, the study of general C*-algebras can be thought of as non-commutative topology. In line with this philosophy we shall see in Section 2.1 that C*-actions establish a non-commutative generalization of dynamical systems.

In Section 2.2 we discuss the notion of covariant representations of a C*-action on a Hilbert space. In Section 2.3 we introduce the crossed product, which is a C*-algebra built out of a C*-action. In Section 2.4 we prove that there is a one-to-one correspondence between covariant representation of a C*-action and the representations of its associated crossed product.

After having defined crossed products, we concentrate in Section 2.5 on those associated to classical C*-actions, that is, C*-actions induced by dynamical systems. Section 2.6 is basically an in-depth case study of its predecessor. Here we prove the Stone-Neumann theorem, which states that the crossed product associated to left-translation on a locally compact Hausdorff group is isomorphic to a C*-algebra of compact operators on a Hilbert space.

2.1 From Dynamical Systems to C*-Actions

In this section we explain how C*-actions naturally arise from dynamical systems. For a good reference with plenty examples accompanying this section we suggest [GKPT18].

Let us first recall the definition of a dynamical system.

Definition 2.1.1. A *dynamical system* is a pair (G, Ω), where G is a locally compact Hausdorff group and Ω a locally compact Hausdorff space, together with a continuous map

$$G \times \Omega \ni (s, \omega) \mapsto s \cdot \omega \in \Omega,$$

called the *action* of G on Ω, such that $e \cdot \omega = \omega$ and $(st) \cdot \omega = s \cdot (t \cdot \omega)$ for $s, t \in G$ and $\omega \in \Omega$.

Note that there is a canonical one-to-one correspondence between dynamical systems of the form $(\mathbb{Z}, \Omega)$ and homeomorphisms from Ω to Ω. Namely, the action of $\mathbb{Z}$ on Ω is fully determined by the homeomorphism

$$\Omega \ni \omega \mapsto 1 \cdot \omega \in \Omega$$

as $\mathbb{Z}$ is generated by 1. We think of $(\mathbb{Z}, \Omega)$ as a discrete time evolution of Ω.

Now suppose that (G, Ω) is any dynamical system. We claim that G parametrizes a family of automorphisms of the C*-algebra $C_0(\Omega)$.

Definition 2.1.2. Let A be a C*-algebra. An *automorphism* of A is an isomorphism from A to A. The collection of all automorphisms of A, denoted by $\mathrm{Aut}(A)$, is called the *automorphism group* of A. We equip $\mathrm{Aut}(A)$ with the topology of pointwise convergence.

Lemma 2.1.3. *The automorphism group* $\mathrm{Aut}(A)$ *of a C*-algebra* A *is a topological group.*

Proof. We omit the trivial proof. ■

Let us shed light on the aforementioned claim. For each $s \in G$ we have an automorphism $\tau_s \colon C_0(\Omega) \to C_0(\Omega)$ determined by $\tau_s(\varphi)(\omega) := \varphi(s^{-1} \cdot \omega)$ for $\varphi \in C_0(\Omega)$ and $\omega \in \Omega$. Hence we obtain a well-defined map $\tau \colon G \to \mathrm{Aut}(C_0(\Omega))$.

Lemma 2.1.4. *Let* (G, Ω) *be a dynamical system. The map* τ *is a topological group homomorphism from* G *to* $\mathrm{Aut}(C_0(\Omega))$.

Proof. We begin with continuity of τ. Given $t \in G$ and $\varphi \in C_0(\Omega)$, we must find for every $\varepsilon > 0$ a neighbourhood $N \subset G$ of t such that $s \in N$ implies $\|\tau_s(\varphi) - \tau_t(\varphi)\|_\infty < \varepsilon$. Fix a compact neighbourhood $K \subset G$ of t and consider the compact set $L := \{\omega \in \Omega : |\varphi(\omega)| \geq \varepsilon/4\}$. Define $\dot{\varphi} \colon G \times \Omega \to \mathbb{C}$ via $\dot{\varphi}(s, \omega) := \varphi(s^{-1} \cdot \omega)$ for all $(s, \omega) \in G \times \Omega$. Since $\dot{\varphi}$ is continuous, there are for each $\omega \in K \cdot L$ open neighbourhoods $U_\omega \subset G$ of t and $V_\omega \subset \Omega$ of ω such that $\dot{\varphi}(U_\omega \times V_\omega)$ is contained in the open ball of radius $\varepsilon/4$ centered at $\dot{\varphi}(t, \omega)$ in $\mathbb{C}$. By compactness of $K \cdot L$ there are $\omega_1, \dots, \omega_n \in K \cdot L$ such that $V_{\omega_1}, \dots, V_{\omega_n}$ cover $K \cdot L$. We claim that

$$N := K \cap \bigcap_{j=1}^{n} U_{\omega_j}$$

is the desired neighbourhood. We divide the proof in two cases. Pick any point $s \in N$. On the one hand, if $\omega \in K \cdot L$, then $\omega \in V_{\omega_j}$ for some $j = 1, \dots, n$. Since also $s, t \in U_{\omega_j}$ and $\omega_j \in V_{\omega_j}$, it follows that

$$|\dot{\varphi}(s, \omega) - \dot{\varphi}(t, \omega)| \leq |\dot{\varphi}(s, \omega) - \dot{\varphi}(t, \omega_j)| + |\dot{\varphi}(t, \omega_j) - \dot{\varphi}(t, \omega)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

On the other hand, if $\omega \notin K \cdot L$, then $K^{-1} \cdot \omega$ does not intersect L. Since $s \in K$, we must have $s^{-1} \cdot \omega \notin L$ and therefore

$$|\dot{\varphi}(s, \omega) - \dot{\varphi}(t, \omega)| \leq |\dot{\varphi}(s, \omega)| + |\dot{\varphi}(t, \omega)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

by definition of L. Hence $\|\tau_s(\varphi) - \tau_t(\varphi)\|_\infty < \varepsilon$ and the claim follows.

It is easy to check that τ is a group homomorphism. Indeed, for $s, t \in G$ we have

$$\tau_{st}(\varphi)(\omega) = \varphi((st)^{-1} \cdot \omega) = \varphi(t^{-1} \cdot (s^{-1} \cdot \omega)) = \tau_t(\varphi)(s^{-1} \cdot \omega) = (\tau_s \tau_t)(\varphi)(\omega)$$

for all $\varphi \in C_0(\Omega)$ and $\omega \in \Omega$. ■

This motivates the following definition.

Definition 2.1.5. A *C*-action* is a topological group homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, where G is a locally compact Hausdorff group and A a C*-algebra. We shall use the notation $\alpha \colon G \curvearrowright A$ rather than $\alpha \colon G \to \mathrm{Aut}(A)$. Moreover, we agree to write $\alpha_s := \alpha(s)$ for $s \in G$. A C*-action is called *classical* if it is of the form $\tau \colon G \curvearrowright C_0(\Omega)$ for some dynamical system (G, Ω).

Actually, any C*-action $\alpha \colon G \curvearrowright A$ with A commutative is classical in the following sense.

Definition 2.1.6. Let $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$ be two C*-actions with the same underlying locally compact Hausdorff group G. A homomorphism $\Phi\colon A \to B$ is called *equivariant* for α and β if $\beta_s \circ \Phi = \Phi \circ \alpha_s$ for all $s \in G$. We say that α and β are *conjugate* if there exists an equivariant isomorphism between A and B.

The proof of the following theorem heavily relies on the classicification theory of commutative C*-algebras. We briefly recall the basic notions. The spectrum Ω of A is the collection of all non-zero homomorphisms from A to $\mathbb{C}$. Endowed with the topology of pointwise convergence, Ω becomes a locally compact Hausdorff space by [Mur90, Theorem 1.3.5]. The Gelfand transform $\Gamma\colon A \to C_0(\Omega)$ given by $\Gamma(a)(\omega) := \omega(a)$ for all $a \in A$ and $\omega \in \Omega$ is a well-defined isomorphism by [Mur90, Theorem 2.1.10].

Theorem 2.1.7. *Every C*-action $\alpha\colon G \curvearrowright A$ with A commutative is conjugate to some classical C*-action $\tau\colon G \curvearrowright C_0(\Omega)$.*

Proof. Let Ω be the spectrum of A. It turns out that the assignment $s \cdot \omega := \omega \circ \alpha_s^{-1}$ for $s \in G$ and $\omega \in \Omega$ defines an action of G on Ω. We prove that $s \cdot \omega \in \Omega$ depends continuously on $(s, \omega) \in G \times \Omega$. Suppose that $(s_\lambda, \omega_\lambda)_{\lambda \in \Lambda}$ is a net that converges to $(s_\infty, \omega_\infty)$ in $G \times \Omega$. Let $a \in A$ and $\varepsilon > 0$ be given. Choose open neighbourhoods $U \subset G$ of s_∞ and $V \subset \Omega$ of ω_∞ such that $s \in U$ implies $\|\alpha_{s_\lambda}^{-1}(a) - \alpha_{s_\infty}^{-1}(a)\| < \varepsilon/2$ and $\omega \in V$ implies $|\omega_\lambda(\alpha_{s_\infty}^{-1}(a)) - \omega_\infty(\alpha_{s_\infty}^{-1}(a))| < \varepsilon/2$. Now find $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $(s_\lambda, \omega_\lambda) \in U \times V$. So for $\lambda \geq \lambda_0$ we have

$$\begin{aligned} |(s_\lambda \cdot \omega_\lambda)(a) - (s_\infty \cdot \omega_\infty)(a)| &= |\omega_\lambda(\alpha_{s_\lambda}^{-1}(a)) - \omega_\infty(\alpha_{s_\infty}^{-1}(a))| \\ &\leq |\omega_\lambda(\alpha_{s_\lambda}^{-1}(a)) - \omega_\lambda(\alpha_{s_\infty}^{-1}(a))| + |\omega_\lambda(\alpha_{s_\infty}^{-1}(a)) - \omega_\infty(\alpha_{s_\infty}^{-1}(a))| \\ &\leq \|\alpha_{s_\lambda}^{-1}(a) - \alpha_{s_\infty}^{-1}(a)\| + |\omega_\lambda(\alpha_{s_\infty}^{-1}(a)) - \omega_\infty(\alpha_{s_\infty}^{-1}(a))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and continuity follows. It is also straightforward to check that $e \cdot \omega = \omega$ and $(st) \cdot \omega = s \cdot (t \cdot \omega)$ for all $s, t \in G$ and $\omega \in \Omega$. Hence we obtain a dynamical system (G, Ω). Let $\tau\colon G \curvearrowright C_0(\Omega)$ be the induced classical C*-action. We claim that the Gelfand transform Γ is equivariant for α and τ. Indeed, for each $s \in G$ the equalities

$$\tau_s(\Gamma(a))(\omega) = \Gamma(a)(s^{-1} \cdot \omega) = (s^{-1} \cdot \omega)(a) = \omega(\alpha_s(a)) = \Gamma(\alpha_s(a))(\omega)$$

hold for arbitrary $a \in A$ and $\omega \in \Omega$. ■

Let $\Phi\colon A \to B$ be an isomorphism between two C*-algebras A and B. If $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$ are C*-actions, then we can define C*-actions $\Phi\alpha\colon G \curvearrowright B$ and $\Phi^{-1}\beta\colon G \curvearrowright A$ via $(\Phi\alpha)_s := \Phi \circ \alpha_s \circ \Phi^{-1}$ and $(\Phi^{-1}\beta)_s := \Phi^{-1} \circ \beta_s \circ \Phi$ for all $s \in G$, respectively. One readily verifies that Φ is equivariant for α and β if and only if $\Phi\alpha = \beta$ if and only if $\alpha = \Phi^{-1}\beta$.

Definition 2.1.8. Let $\Phi\colon A \to B$ be an isomorphism between two C*-algebras A and B. If $\alpha\colon G \curvearrowright A$ is a C*-action, then $\Phi\alpha\colon G \curvearrowright B$ is called a *pushforward C*-action*. If $\beta\colon G \curvearrowright B$ is a C*-action, then $\Phi^{-1}\beta\colon G \curvearrowright A$ is called a *pullback C*-action*.

2.2 Covariant Representations

In this section we study how C*-actions can be represented on Hilbert spaces. A presentation of the material in this section can also be found in [Wil07]. For more information specifically on unitary representations of locally compact groups and representations of C*-algebras we refer to [Fol95] and [Mur90], respectively.

Definition 2.2.1. A *unitary representation* of a locally compact Hausdorff group G on a Hilbert space H is a *strongly continuous* group homomorphism $u\colon G \to U(H)$, that is, for each $x \in H$ the function

$$G \ni \theta \mapsto u(\theta)x \in H$$

is continuous. If u' is another unitary representation of G on H', then u and u' are called *unitarily equivalent* if there exists a unitary linear operator $Z\colon H \to H'$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "u(\theta)"] \arrow[d, "Z"'] & H \arrow[d, "Z"] \\ H' \arrow[r, "u'(\theta)"'] & H' \end{tikzcd}$$

commutes for all $\theta \in G$. A unitary representation is said to be *faithful* when it is injective.

Definition 2.2.2. A *representation* of a C*-algebra A on a Hilbert space H is a *non-degenerate* homomorphism $\rho\colon A \to B(H)$, that is, $\rho(A)H$ is dense in H. If ρ' is another representation of A on H', then ρ and ρ' are called *unitarily equivalent* if there exists a unitary linear operator $Z\colon H \to H'$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "\rho(a)"] \arrow[d, "Z"'] & H \arrow[d, "Z"] \\ H' \arrow[r, "\rho'(a)"'] & H' \end{tikzcd}$$

commutes for all $a \in A$. A representation is said to be *faithful* when it is injective.

It follows from [Mur90, Theorem 3.4.1] that every C*-algebra admits at least one faithful representation. In other words, every C*-algebra can be realized as a closed subalgebra of the C*-algebra of bounded linear operators on some Hilbert space.

The following definition gives a reasonable way to represent a C*-action on a Hilbert space.

Definition 2.2.3. A *covariant representation* of a C*-action $\alpha\colon G \curvearrowright A$ on a Hilbert space H is a pair (u, ρ), where u is a unitary representation of G on H and ρ a representation of A on H such that the diagram

$$\begin{tikzcd} H \arrow[r, "\rho(a)"] \arrow[d, "u(\theta)"'] & H \arrow[d, "u(\theta)"] \\ H \arrow[r, "\rho(\alpha_\theta(a))"'] & H \end{tikzcd}$$

commutes for all $\theta \in G$ and $a \in A$. If (u', ρ') is another covariant representation of α on H', then (u, ρ) and (u', ρ') are called *unitarily equivalent* if there exists a unitary linear operator $Z\colon H \to H'$ for which the diagrams

$$\begin{tikzcd} H \arrow[r, "u(\theta)"] \arrow[d, "Z"'] & H \arrow[d, "Z"] \\ H' \arrow[r, "u'(\theta)"'] & H' \end{tikzcd}, \qquad \begin{tikzcd} H \arrow[r, "\rho(a)"] \arrow[d, "Z"'] & H \arrow[d, "Z"] \\ H' \arrow[r, "\rho'(a)"'] & H' \end{tikzcd}$$

commute for all $\theta \in G$ and $a \in A$.

Let $\alpha\colon G \curvearrowright A$ be a C*-action. We show that every representation of A induces a covariant representation of α. In particular, this implies that covariant representations of α always exist. Given a representation π of A on some Hilbert space Ξ, let H^π be the completion of $C_{\mathrm{cpt}}(G, \Xi)$ with respect to the inner product given by

$$\langle x, y\rangle := \int_G \langle x(s), y(s)\rangle \ \mathrm{d}_\mu s$$

for all $x, y \in C_{\mathrm{cpt}}(G, \Xi)$.

- For each $\theta \in G$ we consider the linear operator $u^\pi(\theta)\colon C_{\mathrm{cpt}}(G, \Xi) \to H^\pi$ determined by $(u^\pi(\theta)x)(s) := x(\theta^{-1}s)$ for all $x \in C_{\mathrm{cpt}}(G, \Xi)$ and $s \in G$. Note that

$$\|u^\pi(\theta)x\|^2 = \int_G \|x(\theta^{-1}s)\|^2 \ \mathrm{d}_\mu s = \int_G \|x(s)\|^2 \ \mathrm{d}_\mu s = \|x\|^2$$

  for all $x \in C_{\mathrm{cpt}}(G, \Xi)$. Consequently, $u^\pi(\theta)$ can be uniquely extended to an isometric linear operator $u^\pi(\theta)\colon H^\pi \to H^\pi$ and a little more effort shows that this extension is also surjective. Thus we obtain a well-defined map $u^\pi\colon G \to U(H^\pi)$.

- For each $a \in A$ we now define a linear operator $\rho^\pi(a)\colon C_{\mathrm{cpt}}(G, \Xi) \to H^\pi$ by $(\rho^\pi(a)x)(s) := \pi(\alpha_s^{-1}(a))x(s)$ for all $x \in C_{\mathrm{cpt}}(G, \Xi)$ and $s \in G$. Note that

$$\|\rho^\pi(a)x\|^2 = \int_G \|\pi(\alpha_s^{-1}(a))x(s)\|^2 \ \mathrm{d}_\mu s \leq \int_G \|a\|^2\|x(s)\|^2 \ \mathrm{d}_\mu s = \|a\|^2\|x\|^2$$

  for all $x \in C_{\mathrm{cpt}}(G, \Xi)$. It follows that $\rho^\pi(a)$ uniquely extends to a bounded linear operator $\rho^\pi(a)\colon H^\pi \to H^\pi$. Hence we discover a well-defined linear operator $\rho^\pi\colon A \to B(H^\pi)$.

Lemma 2.2.4. *Let $\alpha\colon G \curvearrowright A$ be a C*-action. If π is a representation of A on Ξ, then (u^π, ρ^π) is a covariant representation of α on H^π.*

Proof. There are several things that need to be checked. We leave it to the reader to verify that u^π is a group homomorphism and that ρ^π is a homomorphism.

Let us prove that u^π is strongly continuous. A modification of the proof of Lemma 1.3.10, which we leave to the reader, shows that for each $x \in C_{\mathrm{cpt}}(G, \Xi)$ the function

$$G \ni \theta \mapsto u^\pi(\theta)x \in C_{\mathrm{cpt}}(G, \Xi)$$

is continuous with respect to our norm on H^π. This suffices. It follows that u^π is a unitary representation of G on H^π.

Let us now prove that ρ^π is non-degenerate, that is, $\rho^\pi(A)H^\pi$ is dense in H^π. A modification of the proof of Lemma 1.3.5, which we also leave to the reader, shows that $C_{\mathrm{cpt}}(G) \otimes \Xi$ is dense in $C_{\mathrm{cpt}}(G, \Xi)$ with respect to our norm on H^π. Thus $C_{\mathrm{cpt}}(G) \otimes \Xi$ is dense in H^π and therefore it suffices to approximate $x \in C_{\mathrm{cpt}}(G) \otimes \Xi$ by elements in $\rho^\pi(A)H^\pi$. Choose $x \in C_{\mathrm{cpt}}(G) \otimes \Xi$ and $\varepsilon > 0$ randomly. Pick $\varphi_1, \ldots, \varphi_n \in C_{\mathrm{cpt}}(G)$ and $\xi_1, \ldots, \xi_n \in \Xi$ such that the sum of the pure tensors $\varphi_1 \otimes \xi_1, \ldots, \varphi_n \otimes \xi_n$ equals x. Consider the scalar

$$r := \Big(\int_G \Big(\sum_{j=1}^n |\varphi_j(s)| \Big)^2 \ \mathrm{d}_\mu s + 1 \Big)^{1/2}.$$

By non-degeneracy of π there are $\tilde{a}_j \in A$ and $\tilde{\xi}_j \in \Xi$ such that $\|\pi(\tilde{a}_j)\tilde{\xi}_j - \xi_j\| < \varepsilon/2r$. By absorbing a scalar into $\tilde{a}_j$ if necessary we may assume that $\|\tilde{\xi}_j\| \leq 1$. Choose an approximate unit $(1_\lambda)_{\lambda\in\Lambda}$ for A. Since $\{\alpha_s(\tilde{a}_j) : s \in \sigma(\varphi_j)\}$ is compact, we can find a large index $\lambda \in \Lambda$, independent of $j = 1, \dots, n$ as Λ is directed, such that $\|1_\lambda \alpha_s(\tilde a_j) - \alpha_s(\tilde a_j)\| < \varepsilon/2r$ for all $s \in \sigma(\varphi_j)$. We now consider the elements $\tilde a := 1_\lambda$ and

$$\tilde x := \sum_{j=1}^{n} \varphi_j \otimes \pi(\tilde a_j)\tilde\xi_j$$

in A and H^π, respectively, and we claim that $\rho^\pi(\tilde a)\tilde x$ approximates x within tolerance ε. Indeed, the inequalities

$$\begin{aligned}
\|\rho^\pi(\tilde a)\tilde x - x\|^2 &= \int_G \left\| \sum_{j=1}^{n} \pi(\alpha_s^{-1}(1_\lambda))(\varphi_j(s)\pi(\tilde a_j)\tilde\xi_j) - \varphi_j(s)\xi_j \right\|^2 \mathrm{d}_\mu s \\
&\le \int_G \left(\sum_{j=1}^{n} |\varphi_j(s)|(\|\pi(\alpha_s^{-1}(1_\lambda)\tilde a_j - \tilde a_j)\tilde\xi_j\| + \|\pi(\tilde a_j)\tilde\xi_j - \xi_j\|) \right)^2 \mathrm{d}_\mu s \\
&\le \int_G \left(\sum_{j=1}^{n} |\varphi_j(s)|(\|1_\lambda\alpha_s(\tilde a_j) - \alpha_s(\tilde a_j)\| + \|\pi(\tilde a_j)\tilde\xi_j - \xi_j\|) \right)^2 \mathrm{d}_\mu s \\
&\le \left(\frac{\varepsilon}{2r} + \frac{\varepsilon}{2r}\right)^2 \int_G \left(\sum_{j=1}^{n} |\varphi_j(s)| \right)^2 \mathrm{d}_\mu s < \varepsilon^2
\end{aligned}$$

hold. Thus ρ^π is a representation of A on H^π.

It remains to check that $\rho^\pi(\alpha_\theta(a))u^\pi(\theta) = u^\pi(\theta)\rho^\pi(a)$ for $\theta \in G$ and $a \in A$. We have

$$\begin{aligned}
(\rho^\pi(\alpha_\theta(a))u^\pi(\theta)x)(s) &= \pi(\alpha_{\theta^{-1}t}^{-1}(a))(u^\pi(\theta)x)(s) = \pi(\alpha_{\theta^{-1}t}^{-1}(a))x(\theta^{-1}t) \\
&= (\rho^\pi(\theta)x)(\theta^{-1}s) = (u^\pi(\theta)\rho^\pi(a)x)(s)
\end{aligned}$$

for $x \in C_{\mathrm{cpt}}(G, \Xi)$ and $s \in G$. So we infer that the bounded linear operators $\rho^\pi(\alpha_\theta(a))u^\pi(\theta)$ and $u^\pi(\theta)\rho^\pi(a)$ are equal as they agree on the dense linear subspace $C_{\mathrm{cpt}}(G, \Xi)$ of H^π. ■

Definition 2.2.5. Let $\alpha\colon G \curvearrowright A$ be a C*-action and let π be a representation of A. The pair (u^π, ρ^π) is called the *regular covariant representation* of α associated to π.

Let G be a locally compact Hausdorff group, A a C*-algebra and $(H_\lambda)_{\lambda\in\Lambda}$ an orthogonal family of closed linear subspaces of some ambient Hilbert space H'. Let $(u_\lambda)_{\lambda\in\Lambda}$ be a family of unitary representation of G and $(\rho_\lambda)_{\lambda\in\Lambda}$ a family of representations of A, both on the Hilbert spaces $(H_\lambda)_{\lambda\in\Lambda}$. Routine arguments show that the natural maps

$$\bigoplus_{\lambda\in\Lambda} u_\lambda\colon G \to U\Big(\bigoplus_{\lambda\in\Lambda} H_\lambda\Big), \qquad \bigoplus_{\lambda\in\Lambda} \rho_\lambda\colon A \to B\Big(\bigoplus_{\lambda\in\Lambda} H_\lambda\Big)$$

are, respectively, a unitary representation and a representation. Moreover, if the pairs $(u_\lambda, \rho_\lambda)_{\lambda\in\Lambda}$ are covariant for some given C*-action, then their direct sum

$$\bigoplus_{\lambda\in\Lambda} (u_\lambda, \rho_\lambda) := \Big(\bigoplus_{\lambda\in\Lambda} u_\lambda, \bigoplus_{\lambda\in\Lambda} \rho_\lambda\Big),$$

is covariant representation for the same C*-action.

In what follows u' is a unitary representation of G on H', ρ' is a representation of A on H' and H is a closed linear subspace of H'.

Definition 2.2.6. Suppose that u' is a unitary representation of a locally compact Hausdorff group G on a Hilbert space H'. A closed linear subspace H of H' is called *invariant* under u' if $u'(G)H \subset H$. We say that u' is *irreducible* if $\{0\}$ and H' are the only closed linear subspaces of H' that are invariant under u'.

If H is invariant under u', then the restriction $u'|H\colon G \to U(H)$ is a unitary representation. In this case $H^\perp$ is automatically invariant under u' and, furthermore, $u' = u'|H \oplus u'|H^\perp$. If the Hilbert spaces $(H_\lambda)_{\lambda\in\Lambda}$ are invariant under u', then their direct sum is invariant under u' and

$$u'|\bigoplus_{\lambda\in\Lambda} H_\lambda = \bigoplus_{\lambda\in\Lambda} u'|H_\lambda.$$

Definition 2.2.7. Suppose that ρ' is a representation of a C*-algebra A on a Hilbert space H'. A closed linear subspace H of H' is called *invariant* under ρ' if $\rho'(A)H \subset H$. We say that ρ' is *irreducible* if $\{0\}$ and H' are the only closed linear subspaces of H' that are invariant under ρ'.

If H is invariant under ρ', then the restriction $\rho'|H\colon A \to B(H)$ is a representation. In this case $H^\perp$ is automatically invariant under ρ' and, furthermore, $\rho' = \rho'|H \oplus \rho'|H^\perp$. If the Hilbert spaces $(H_\lambda)_{\lambda\in\Lambda}$ are invariant under ρ', then their direct sum is invariant under ρ' and

$$\rho'|\bigoplus_{\lambda\in\Lambda} H_\lambda = \bigoplus_{\lambda\in\Lambda} \rho'|H_\lambda.$$

Definition 2.2.8. Let (u', ρ') be a covariant representation of a C*-action $\alpha\colon G \curvearrowright A$ on a Hilbert space H'. A closed linear subspace H of H' is said to be invariant under (u', ρ') when it is invariant under both u' and ρ'.

2.3 Crossed Products and Integrated Forms

Suppose that $\alpha\colon G \curvearrowright A$ is a C*-action. In this section we construct a C*-algebra $G \ltimes_\alpha A$, called the crossed product, from which we can recover exactly the covariant representations of α. The construction basically consists of two steps. First we turn $C_{\mathrm{cpt}}(G, A)$ into an algebra and then we complete $C_{\mathrm{cpt}}(G, A)$ with respect to a certain C*-norm. Of course, there is nothing in this section that cannot be found in [Wil07].

- Let us begin with the definition of the multiplication on $C_{\mathrm{cpt}}(G, A)$. For arbitrary $f, g \in C_{\mathrm{cpt}}(G, A)$ we see that the continuous function

  $$G \times G \ni (s,t) \mapsto f(s)\alpha_s(g(s^{-1}t)) \in A$$

  is compactly supported as its support is contained in $\sigma(f) \times \sigma(f)^{-1}\sigma(g)$. So by Lemma 1.3.8 we can integrate over the first variable and the resulting function, depending only on the second variable, belongs to $C_{\mathrm{cpt}}(G, A)$. Now define the product $fg\colon G \to A$ of two functions $f, g \in C_{\mathrm{cpt}}(G, A)$ by the assignment

  $$(fg)(t) := \int_G f(s)\alpha_s(g(s^{-1}t))\, \mathrm{d}_\mu s$$

  for all $t \in G$.

- Moving on to the involution on $C_{\mathrm{cpt}}(G, A)$, we define the adjoint $h^*\colon G \to A$ of a function $h \in C_{\mathrm{cpt}}(G, A)$ by the formula $h^*(t) := \Delta(t^{-1})\alpha_t(h(t^{-1})^*)$ for all $t \in G$.

Lemma 2.3.1. *If $\alpha\colon G \curvearrowright A$ is a C*-action, then the linear space $C_{\mathrm{cpt}}(G, A)$ is an algebra.*

Proof. It is clear that the product $fg \in C_{\mathrm{cpt}}(G, A)$ depends linearly on both $f, g \in C_{\mathrm{cpt}}(G, A)$ and that the adjoint $h^* \in C_{\mathrm{cpt}}(G, A)$ depends antilinearly on $h \in C_{\mathrm{cpt}}(G, A)$. To see that $(fg)h = f(gh)$ for $f, g, h \in C_{\mathrm{cpt}}(G, A)$, we calculate

$$\begin{aligned}
((fg)h)(t) &= \int_G (fg)(s_1)\alpha_{s_1}(h(s_1^{-1}t)) \, \mathrm{d}_\mu s_1 \\
&= \int_G \bigg(\int_G f(s_2)\alpha_{s_2}(g(s_2^{-1}s_1)) \, \mathrm{d}_\mu s_2 \bigg) \alpha_{s_1}(h(s_1^{-1}t)) \, \mathrm{d}_\mu s_1 \\
&= \int_G f(s_2) \bigg(\int_G \alpha_{s_2}(g(s_2^{-1}s_1))\alpha_{s_1}(h(s_1^{-1}t)) \, \mathrm{d}_\mu s_1 \bigg) \, \mathrm{d}_\mu s_2 \\
&= \int_G f(s_2)\alpha_{s_2} \bigg(\int_G g(s_2^{-1}s_1)\alpha_{s_2^{-1}s_1}(h(s_1^{-1}t)) \, \mathrm{d}_\mu s_1 \bigg) \, \mathrm{d}_\mu s_2 \\
&= \int_G f(s_2)\alpha_{s_2} \bigg(\int_G g(s_1)\alpha_{s_1}(h(s_1^{-1}s_2^{-1}t)) \, \mathrm{d}_\mu s_1 \bigg) \, \mathrm{d}_\mu s_2 \\
&= \int_G f(s_2)\alpha_{s_2}((gh)(s_2^{-1}t)) \, \mathrm{d}_\mu s_2 \\
&= (f(gh))(t)
\end{aligned}$$

for all $t \in G$. In the fifth step we used left-invariance of the integral. To see that $(fg)^* = g^* f^*$ for $f, g \in C_{\mathrm{cpt}}(G, A)$, we compute

$$\begin{aligned}
(fg)^*(t) &= \Delta(t^{-1})\alpha_t((fg)(t^{-1})^*) \\
&= \Delta(t^{-1})\alpha_t \bigg(\bigg(\int_G f(s)\alpha_s(g(s^{-1}t^{-1})) \, \mathrm{d}_\mu s \bigg)^* \bigg) \\
&= \Delta(t^{-1}) \int_G \alpha_{ts}(g(s^{-1}t^{-1})^*)\alpha_t(f(s)^*) \, \mathrm{d}_\mu s \\
&= \Delta(t^{-1}) \int_G \alpha_s(g(s^{-1})^*)\alpha_t(f((s^{-1}t)^{-1})^*) \, \mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_s(g(s^{-1})^*)\alpha_s(\Delta((s^{-1}t)^{-1})\alpha_{s^{-1}t}(f((s^{-1}t)^{-1})^*)) \, \mathrm{d}_\mu s \\
&= \int_G g^*(s)\alpha_s(f^*(s^{-1}t)) \, \mathrm{d}_\mu s \\
&= (g^* f^*)(t)
\end{aligned}$$

for all $t \in G$. In the fourth step we used left-invariance of the integral. Finally, it is clear that $(h^*)^* = h$ for all $h \in C_{\mathrm{cpt}}(G, A)$. ∎

We shall now construct a C*-norm on the algebra $C_{\mathrm{cpt}}(G, A)$. The idea is that for every covariant representation (u, ρ) of α on H we can produce a homomorphism $u \ltimes_\alpha \rho$ from $C_{\mathrm{cpt}}(G, A)$ to $B(H)$, which allows us to utilize the C*-norm on $B(H)$ for various Hilbert spaces H.

For $f \in C_{\mathrm{cpt}}(G, A)$ and $x \in H$ the continuous function

$$G \ni s \mapsto \rho(f(s))u(s)x \in H$$

is compactly supported as its support is contained in $\sigma(f)$. So by Lemma 1.3.8 we can integrate and estimate

$$\bigg\| \int_G \rho(f(s))u(s)x \, \mathrm{d}_\mu s \bigg\| \leq \int_G \|\rho(f(s))u(s)x\| \, \mathrm{d}_\mu s \leq \int_G \|f(s)\|\|x\| \, \mathrm{d}_\mu s = \|f\|_{\mathrm{int}}\|x\|,$$

which implies that the linear operator $u \ltimes_\alpha \rho \colon C_{\mathrm{cpt}}(G, A) \to B(H)$ given by

$$(u \ltimes_\alpha \rho)(f)x := \int_G \rho(f(s))u(s)x \, \mathrm{d}_\mu s$$

for $f \in C_{\mathrm{cpt}}(G, A)$ and $x \in H$ is well-defined. Moreover, $u \ltimes_\alpha \rho$ is norm-decreasing with respect to the integral norm, that is, $\|(u \ltimes_\alpha \rho)(f)\| \leq \|f\|_{\mathrm{int}}$ for all $f \in C_{\mathrm{cpt}}(G, A)$.

Lemma 2.3.2. *Let $\alpha \colon G \curvearrowright A$ be a C^*-action. If (u, ρ) is a covariant representation of α on H, then $u \ltimes_\alpha \rho$ is a homomorphism from $C_{\mathrm{cpt}}(G, A)$ to $B(H)$.*

Proof. On the one hand, for $f, g \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}
(u \ltimes_\alpha \rho)(fg)x &= \int_G \rho((fg)(s_1))u(s_1)x \, \mathrm{d}_\mu s_1 \\
&= \int_G \rho\bigg(\int_G f(s_2)\alpha_{s_2}(g(s_2^{-1}s_1)) \, \mathrm{d}_\mu s_2 \bigg) u(s_1)x \, \mathrm{d}_\mu s_1 \\
&= \int_G \int_G \rho(f(s_2))\rho(\alpha_{s_2}(g(s_2^{-1}s_1)))u(s_1)x \, \mathrm{d}_\mu s_2 \, \mathrm{d}_\mu s_1 \\
&= \int_G \int_G \rho(f(s_2))u(s_2)\rho(g(s_2^{-1}s_1))u(s_2^{-1}t)x \, \mathrm{d}_\mu s_1 \, \mathrm{d}_\mu s_2 \\
&= \int_G \rho(f(s_2))u(s_2)\bigg(\int_G \rho(g(s_1))u(s_1)x \, \mathrm{d}_\mu s_1 \bigg) \, \mathrm{d}_\mu s_2 \\
&= (u \ltimes_\alpha \rho)(f)(u \ltimes_\alpha \rho)(g)x
\end{aligned}$$

for all $x \in H$. In the fourth step we used that (u, ρ) is covariant for α. On the other hand, for $h \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}
\langle (u \ltimes_\alpha \rho)(h^*)x_1, x_2 \rangle &= \bigg\langle \int_G \rho(h^*(s))u(s)x_1 \, \mathrm{d}_\mu s, x_2 \bigg\rangle \\
&= \int_G \langle \rho(h^*(s))u(s)x_1, x_2 \rangle \, \mathrm{d}_\mu s \\
&= \int_G \langle \rho(\Delta(s^{-1})\alpha_s(h(s^{-1})^*))u(s)x_1, x_2 \rangle \, \mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\langle u(s)\rho(h(s^{-1})^*)x_1, x_2 \rangle \, \mathrm{d}_\mu s \\
&= \int_G \langle u(s)^*\rho(h(s))^*x_1, x_2 \rangle \, \mathrm{d}_\mu s \\
&= \int_G \langle x_1, \rho(h(s))u(s)x_2 \rangle \, \mathrm{d}_\mu s \\
&= \bigg\langle x_1, \int_G \rho(h(s))u(s)x_2 \, \mathrm{d}_\mu s \bigg\rangle \\
&= \langle x_1, (u \ltimes_\alpha \rho)(h)x_2 \rangle
\end{aligned}$$

for all $x_1, x_2 \in H$. In the fourth step we used that (u, ρ) is covariant for α and in the fifth step we used Lemma 1.4.3. ∎

Lemma 2.3.3. *Let $\alpha \colon G \curvearrowright A$ be a C^*-action. Suppose that (u^π, ρ^π) is the regular covariant representation of α associated to some faithful representation π of A on Ξ. If $f \in C_{\mathrm{cpt}}(G, A)$ satisfies $(u^\pi \ltimes_\alpha \rho^\pi)(f) = 0$, then $f = 0$.*

Proof. We give a proof by contraposition; we show that $(u^\pi \ltimes_\alpha \rho^\pi)(f) \neq 0$ whenever $f(\theta) \neq 0$ for some $\theta \in G$. Since $\pi \circ \alpha_\theta^{-1}$ is injective, there are vectors $\xi_1, \xi_2 \in \Xi$ with $\|\xi_1\| \leq 1$ and $\|\xi_2\| \leq 1$ such that the value $\lambda := \langle \xi_1, \pi(\alpha_\theta^{-1}(f(\theta)))\xi_2 \rangle$ is non-zero. Pick an open neighbourhood $V \subset G$ of e such that $s, t \in \theta V$ implies $\|\alpha_t^{-1}(f(s)) - \alpha_\theta^{-1}(f(\theta))\| < |\lambda|/4$. First use Lemma 1.2.3 to find a symmetric open neighbourhood $U \subset G$ of e for which $UU \subset V$ and then use Lemma 1.1.2 to find a non-zero $\eta \in C_{\text{cpt}}(G)$ for which $\sigma(\eta) \subset U$ and $\eta \geq 0$. By rescaling η we may assume that

$$\int_G \int_G \eta(\theta^{-1}t)\eta(s^{-1}t) \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s = 1.$$

If we put $x_1 := L_\theta(\eta) \otimes \xi_1$ and $x_2 := \eta \otimes \xi_2$, then

$$\begin{aligned}
\langle x_1, (u^\pi \ltimes_\alpha \rho^\pi)(f)x_2 \rangle &= \left\langle x_1, \int_G \rho^\pi(f(s))u^\pi(s)x_2 \; \mathrm{d}_\mu s \right\rangle \\
&= \int_G \langle x_1, \rho^\pi(f(s))u^\pi(s)x_2 \rangle \; \mathrm{d}_\mu s \\
&= \int_G \int_G \langle x_1(t), (\rho^\pi(f(s))u^\pi(s)x_2)(t) \rangle \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s \\
&= \int_G \int_G \langle x_1(t), \pi(\alpha_t^{-1}(f(s)))x_2(s^{-1}t) \rangle \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s \\
&= \int_G \int_G \eta(\theta^{-1}t)\eta(s^{-1}t)\langle \xi_1, \pi(\alpha_t^{-1}(f(s)))\xi_2 \rangle \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s.
\end{aligned}$$

So, because $s, t \in \theta V$ whenever $s, t \in G$ satisfy $\theta^{-1}t, s^{-1}t \in \sigma(\eta)$, we get

$$\begin{aligned}
|\langle x_1, (u^\pi \ltimes_\alpha \rho^\pi)(f)x_2 \rangle - \lambda| &= \left| \int_G \int_G \eta(\theta^{-1}t)\eta(s^{-1}t)\langle \xi_1, \pi(\alpha_t^{-1}(f(s)) - \alpha_\theta^{-1}(f(\theta)))\xi_2 \rangle \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s \right| \\
&\leq \int_G \int_G \eta(\theta^{-1}t)\eta(s^{-1}t)\|\alpha_t^{-1}(f(s)) - \alpha_\theta^{-1}(f(\theta))\| \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s \\
&\leq \frac{|\lambda|}{4} \int_G \int_G \eta(\theta^{-1}t)\eta(s^{-1}t) \; \mathrm{d}_\mu t \; \mathrm{d}_\mu s < \frac{|\lambda|}{2}.
\end{aligned}$$

Thus $\langle x_1, (u^\pi \ltimes_\alpha \rho^\pi)(f)x_2 \rangle \neq 0$ and therefore $(u^\pi \ltimes_\alpha \rho^\pi)(f) \neq 0$. ■

We are now ready to define a C*-norm on $C_{\text{cpt}}(G, A)$. Since A always has a faithful representation, it follows from Lemma 2.3.3 that the following definition indeed presents a well-defined norm on $C_{\text{cpt}}(G, A)$.

Definition 2.3.4. Let $\alpha \colon G \curvearrowright A$ be a C*-action. The value

$$\|f\|_{\text{env}} := \sup_{(u,\rho)} \|(u \ltimes_\alpha \rho)(f)\|,$$

where the supremum runs over all covariant representations (u, ρ) of α, is called the *enveloping norm* of $f \in C_{\text{cpt}}(G, A)$. The *crossed product*, denoted by $G \ltimes_\alpha A$, is the completion of $C_{\text{cpt}}(G, A)$ with respect to the enveloping norm.

Observe that $\|f\|_{\text{env}} \leq \|f\|_{\text{int}}$ for all $f \in C_{\text{cpt}}(G, A)$.

Theorem 2.3.5. *If $\alpha \colon G \curvearrowright A$ is a C*-action, then the crossed product $G \ltimes_\alpha A$ is a C*-algebra.*

Proof. This is a consequence of Lemma 2.3.1 and Lemma 2.3.2. Indeed, we know that $C_{\text{cpt}}(G, A)$ is an algebra and, using that $u \ltimes_\alpha \rho$ is a homomorphism for any covariant representation (u, ρ) of α and that $B(H)$ is a C*-algebra for any Hilbert space H, it is easy to check that the enveloping norm is a C*-norm on $C_{\text{cpt}}(G, A)$. Hence the completion $G \ltimes_\alpha A$ is a C*-algebra. ■

Given a covariant representation (u, ρ) of α on a Hilbert space H, we see that $\|(u \ltimes_\alpha \rho)(f)\| \leq \|f\|_{\mathrm{env}}$ for all $f \in C_{\mathrm{cpt}}(G, A)$ by definition of the enveloping norm. Hence $u \ltimes_\alpha \rho$ can be uniquely extended to a homomorphism $u \ltimes_\alpha \rho \colon G \ltimes_\alpha A \to B(H)$.

Definition 2.3.6. Let $\alpha\colon G \curvearrowright A$ be a C*-action. If (u, ρ) is a covariant representation of α, then we call $u \ltimes_\alpha \rho$ the *integrated form* of (u, ρ).

Theorem 2.3.7. *Let $\alpha\colon G \curvearrowright A$ be a C*-action. If (u, ρ) is a covariant representation of α on H, then its integrated form $u \ltimes_\alpha \rho$ is a representation of $G \ltimes_\alpha A$ on H.*

Proof. Since we already know that $u \ltimes_\alpha \rho$ is a homomorphism, we only need to show that $u \ltimes_\alpha \rho$ is non-degenerate, that is, $(u \ltimes_\alpha \rho)(G \ltimes_\alpha A)H$ is dense in H. Let $x \in H$ and $\varepsilon > 0$ be given. By non-degeneracy of ρ there are $\tilde{a} \in A$ and $\tilde{x} \in H$ such that $\|\rho(\tilde{a})\tilde{x} - x\| < \varepsilon/4$. Let $U \subset G$ be a neighbourhood of e such that $\|\tilde{a}\|\|u(s)\tilde{x} - \tilde{x}\| < \varepsilon/4$ whenever $s \in U$. By Lemma 1.1.2 there is a non-zero $\eta \in C_{\mathrm{cpt}}(G)$ such that $\sigma(\eta) \subset U$ and $\eta \geq 0$. By rescaling η we may assume that $\mu(\eta) = 1$. Note that $\tilde{f} := \eta \otimes \tilde{a}$ is an element of $G \ltimes_\alpha A$. We claim that $(u \ltimes_\alpha \rho)(\tilde{f})\tilde{x}$ approximates x within the margin of error ε. Indeed, we estimate

$$\begin{aligned}
\|(u \ltimes_\alpha \rho)(\tilde{f})\tilde{x} - x\| &= \left\| \int_G \eta(s)(\rho(\tilde{a})u(s)\tilde{x} - x) \,\mathrm{d}_\mu s \right\| \\
&\leq \int_G \eta(s)\|\rho(\tilde{a})u(s)\tilde{x} - x\| \,\mathrm{d}_\mu s \\
&\leq \int_G \eta(s)(\|\rho(\tilde{a})(u(s)\tilde{x} - \tilde{x})\| + \|\rho(\tilde{a})\tilde{x} - x\|) \,\mathrm{d}_\mu s \\
&\leq \int_G \eta(s)(\|\tilde{a}\|\|u(s)\tilde{x} - \tilde{x}\| + \|\rho(\tilde{a})\tilde{x} - x\|) \,\mathrm{d}_\mu s \\
&\leq \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) \int_G \eta(s) \,\mathrm{d}_\mu s < \varepsilon
\end{aligned}$$

and conclude that $u \ltimes_\alpha \rho$ is non-degenerate. ■

In fact, the integrated form implements a one-to-one correspondence between covariant representations of α and representations of $G \ltimes_\alpha A$. In the next section we shall develop the machinery required for the proof of this statement.

The following lemma states two useful properties of integrated forms.

Lemma 2.3.8. *Let $\alpha\colon G \curvearrowright A$ be a C*-action.*

(i) If $(u_\lambda, \rho_\lambda)_{\lambda\in\Lambda}$ is a family of covariant representations of α, then

$$\left(\bigoplus_{\lambda\in\Lambda} u_\lambda\right) \ltimes_\alpha \left(\bigoplus_{\lambda\in\Lambda} \rho_\lambda\right) = \bigoplus_{\lambda\in\Lambda} u_\lambda \ltimes_\alpha \rho_\lambda.$$

(ii) If (u, ρ) and (u', ρ') are unitarily equivalent covariant representations of α, then $u \ltimes_\alpha \rho$ and $u' \ltimes_\alpha \rho'$ are unitarily equivalent representations of $G \ltimes_\alpha A$.

Proof. We omit the trivial proof. ■

It turns out that conjugate C*-actions lead to isomorphic crossed products. Let $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$ be two C*-actions. Let $\Phi\colon A \to B$ be an equivariant isomorphism and define a linear operator $G \ltimes \Phi\colon C_{\mathrm{cpt}}(G, A) \to G \ltimes_\beta B$ via $(G \ltimes \Phi)(f) := \Phi \circ f$ for $f \in C_{\mathrm{cpt}}(G, A)$.

If (u, ρ) is a covariant representation of α, then equivariance of Φ implies that $(u, \rho \circ \Phi^{-1})$ is a covariant representation of β and, furthermore, $(u \ltimes_\beta (\rho \circ \Phi^{-1})) \circ (G \ltimes \Phi) = u \ltimes_\alpha \rho$. In fact, every covariant representation of β is of the form $(u, \rho \circ \Phi^{-1})$ for a unique covariant representation (u, ρ) of α. We conclude that $\|(G \ltimes \Phi)(f)\|_{\mathrm{env}} = \|f\|_{\mathrm{env}}$ for all $f \in C_{\mathrm{cpt}}(G, A)$. Hence $G \ltimes \Phi$ uniquely extends to an isometric linear operator $G \ltimes \Phi\colon G \ltimes_\alpha A \to G \ltimes_\beta B$.

Lemma 2.3.9. *Let $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$ be two C*-actions. Suppose that $\Phi\colon A \to B$ is an equivariant isomorphism. The map $G \ltimes \Phi$ is an isomorphism from $G \ltimes_\alpha A$ to $G \ltimes_\beta B$.*

Proof. On the one hand, for $f, g \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}(G \ltimes \Phi)(fg)(t) &= \Phi((fg)(t)) \\ &= \int_G \Phi(f(s)\alpha_s(g(s^{-1}t)))\ \mathrm{d}_\mu s \\ &= \int_G \Phi(f(s))\beta_s(\Phi(g(s^{-1}t)))\ \mathrm{d}_\mu s \\ &= \int_G (G \ltimes \Phi)(f)(s)\beta_s((G \ltimes \Phi)(g)(s^{-1}t))\ \mathrm{d}_\mu s \\ &= ((G \ltimes \Phi)(f)(G \ltimes \Phi)(g))(t)\end{aligned}$$

for all $t \in G$. In the third step we used equivariance of Φ. On the other hand, for $h \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}(G \ltimes \Phi)(h^*)(t) &= \Phi(h^*(t)) \\ &= \Delta(t^{-1})\Phi(\alpha_t(h(t^{-1})^*)) \\ &= \Delta(t^{-1})\beta_t(\Phi(h(t^{-1}))^*) \\ &= \Delta(t^{-1})\beta_t((G \ltimes \Phi)(h)(t^{-1})^*) \\ &= (G \ltimes \Phi)(h)^*(t)\end{aligned}$$

for all $t \in G$. In the third step we used equivariance of Φ.

It remains to prove that $G \ltimes \Phi$ is surjective. But this immediately follows from the fact that $G \ltimes \Phi$ is an isometric homomorphism mapping the dense subalgebra $C_{\mathrm{cpt}}(G, A)$ of $G \ltimes_\alpha A$ onto the dense subalgebra $C_{\mathrm{cpt}}(G, B)$ of $G \ltimes_\beta B$. ∎

2.4 Representations of Crossed Products

Let $\alpha\colon G \curvearrowright A$ be a C*-action. In the previous section we have seen that every covariant representation (u, ρ) of α induces a representation $u \ltimes_\alpha \rho$ of $G \ltimes_\alpha A$. As promised we shall now work towards a proof of the fact that every representation of $G \ltimes_\alpha A$ is the integrated form of a unique covariant representation of α. Our proof strategy is roughly based on [Wil07].

For reasons that become clear later we need to generalize the notion of a covariant representation. For this we need multiplier algebras. We briefly cover the basic material on multiplier algebras; for more information we refer to [Lan95].

Definition 2.4.1. Let C be a C*-algebra. A bounded linear operator $m\colon C \to C$ is said to be *adjointable* if there exists a (necessarily unique) bounded linear operator $m^*\colon C \to C$ such that $m(c_1)^*c_2 = c_1^* m^*(c_2)$ for all $c_1, c_2 \in C$. In this case m^* is called the *adjoint* of m. The linear space $M^B(C)$ of all adjointable bounded linear operators on C is called the *multiplier algebra* of C and the elements in $M^B(C)$ are called *multipliers*.

Suppose that C is a C*-algebra. One readily verifies that $m_1, m_2 \in M^B(C)$ implies $m_1 \circ m_2 \in M^B(C)$ with $(m_1 \circ m_2)^* = m_2^* \circ m_1^*$ and that $m \in M^B(C)$ implies $m^* \in M^B(C)$ with $m^{**} = m$.

If we view $c \in C$ as the bounded linear operator $c\colon C \to C$ that multiplies from the left by c, then c is clearly adjointable. Moreover, the operator norm of c equals the norm of c. Hence we obtain an inclusion of C into $M^B(C)$.

The following lemma serves as preparation for the subsequent theorem.

Lemma 2.4.2. *Let C be a C*-algebra. If $m \in M^B(C)$ and $c \in C$, then the composition $m \circ c$ and the evaluation $m(c)$ are equal in $M^B(C)$.*

Proof. Using that $m^{**} = m$ we find that

$$c_1^* m(cc_2) = m^*(c_1)^* cc_2 = c_1^* m(c) c_2$$

for all $c_1, c_2 \in C$ and therefore $m \circ c = m(c)$. ■

From now onwards we use juxtaposition for composition in $M^B(C)$, that is, we write $m_1 m_2$ instead of $m_1 \circ m_2$. In particular, the notation mc refers to both the composition $m \circ c$ and the evaluation $m(c)$ by Lemma 2.4.2.

Theorem 2.4.3. *Let C be a C*-algebra. The multiplier algebra $M^B(C)$ of C is a unital C*-algebra and contains C as a closed ideal. If C is unital, then $M^B(C) = C$.*

Proof. We leave it to the reader to check that $M^B(C)$ is a C*-algebra with respect to the operator norm. The identity operator $I\colon C \to C$ is a unit for $M^B(C)$. Now let $m \in M^B(C)$ and $c \in C$ be arbitrary. It follows from Lemma 2.4.2 that $mc \in C$. Likewise, we have $m^* c^* \in C$ by Lemma 2.4.2 and therefore $cm \in C$ as $cm = (m^* c^*)^*$. So C is an ideal in $M^B(C)$. Moreover, C is closed in $M^B(C)$ as C is complete and the inclusion of C into $M^B(C)$ is isometric.

If C has a unit, then this unit is precisely the unit I of $M^B(C)$. So C is an ideal in $M^B(C)$ containing the unit of $M^B(C)$ and therefore $M^B(C) = C$. ■

Definition 2.4.4. Let C be a C*-algebra. We denote by $M^U(C)$ the group of all unitary elements in the C*-algebra $M^B(C)$.

Often we would like to extend the domain of a homomorphism to its multiplier algebra.

Definition 2.4.5. Let C and A be C*-algebras. A *multiplier extension* of a homomorphism $\Phi\colon C \to A$ is a homomorphism $\Phi^+\colon M^B(C) \to A$ such that $\Phi^+(c) = \Phi(c)$ for all $c \in C$.

Lemma 2.4.6. *If P is a representation of a C*-algebra C on a Hilbert space H, then P has a unique multiplier extension P^+. In fact, P^+ is a representation of $M^B(C)$ on H.*

Proof. Let E denote the span of $P(C)H$ in H. If P^+ exists, then it is necessarily unique by non-degeneracy of P. Indeed, one readily verifies that for each $m \in M^B(C)$ the bounded linear operator $P^+(m)$ is fully determined by P on the dense linear subspace E of H. So we only have to worry about existence of P^+.

For each $m \in M^B(C)$ we shall construct a bounded linear operator $P^+(m)\colon H \to H$. Let $(1_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for C. If $c_j \in C$ and $x_j \in H$, then for each $\varepsilon > 0$ there is a sufficiently large $\lambda \in \Lambda$ such that

$$\left\| \sum_j P(mc_j)x_j \right\| - \varepsilon < \left\| \sum_j P(m1_\lambda c_j)x_j \right\| = \left\| P(m1_\lambda) \sum_j P(c_j)x_j \right\| \leq \|m\| \left\| \sum_j P(c_j)x_j \right\|.$$

So the linear operator $P^+(m)\colon E \to H$ given by

$$P^+(m) \sum_j P(c_j)x_j := \sum_j P(mc_j)x_j$$

for $c_j \in C$ and $x_j \in H$ is well-defined and bounded. In particular, since E is a dense linear subspace of H, it follows that $P^+(m)$ uniquely extends to a bounded linear operator $P^+(m)\colon H \to H$. Routine computations show that we obtain a homomorphism $P^+\colon M^B(C) \to B(H)$ satisfying $P^+(c) = P(c)$ for all $c \in C$.

Finally, the homomorphism P^+ is a representation of $M^B(C)$ on H as it is clearly unital and therefore non-degenerate. ■

Let us discuss an important example. Let H be a Hilbert space and consider the C*-algebra $K(H)$ of compact operators on H. For $x, y \in H$ we define a linear operator $x \oslash y \colon H \to H$ via $(x \oslash y)z := \langle x, z\rangle y$ for all $z \in H$. The finite-rank operators on H, which are spanned by $\{x \oslash y : x, y \in H\}$, are dense in $K(H)$ by [Mur90, Theorem 2.4.5]. Observe that $b_1(x \oslash y)b_2^* = b_2 x \oslash b_1 y$ for all $b_1, b_2 \in B(H)$ and that $\|x \oslash y\| = \|x\|\|y\|$. Moreover, $x \oslash y \in K(H)$ depends antilinearly on $x \in H$ and linearly on $y \in H$.

Lemma 2.4.7. *Let H be a Hilbert space. For each $b \in B(H)$ we have*

$$\sup_{\substack{c \in K(H) \\ \|c\| \leq 1}} \|bc\| = \|b\|.$$

Proof. We may assume that $H \neq \{0\}$, otherwise we are done. Fix a vector $x \in H$ with $\|x\| = 1$. Since $\|x \oslash y\| = \|y\|$ for all $y \in H$, it follows that

$$\|b\| \geq \sup_{\substack{c \in K(H) \\ \|c\| \leq 1}} \|bc\| \geq \sup_{\substack{y \in H \\ \|y\| \leq 1}} \|b(x \oslash y)\| = \sup_{\substack{y \in H \\ \|y\| \leq 1}} \|x\|\|by\| = \|b\|$$

as desired. ∎

Note that $K(H)$ is an ideal in $B(H)$. If we view $b \in B(H)$ as the bounded linear operator $b \colon K(H) \to K(H)$ that multiplies from the left by b, then b is clearly adjointable. So by Lemma 2.4.7 we obtain an inclusion of $B(H)$ into $M^B(K(H))$. In fact, this inclusion is an isomorphism. Indeed, Lemma 2.4.6 shows that the inclusion of $K(H)$ into $B(H)$ extends to a homomorphism from $M^B(K(H))$ to $B(H)$, which is easily seen to be the inverse of the inclusion of $B(H)$ into $M^B(K(H))$. Hence we can identify $B(H) = M^B(K(H))$ and $U(H) = M^U(K(H))$.

We are now ready to generalize the notion of covariant representations.

Definition 2.4.8. A *unitary multiplier representation* of a locally compact Hausdorff group G on a C*-algebra C is a *strongly continuous* group homomorphism $u \colon G \to M^U(C)$, that is, for each $c \in C$ the function

$$G \ni \theta \mapsto u(\theta)c \in C$$

is continuous.

Using that the span of $\{x \oslash y : x, y \in H\}$ is dense in $K(H)$ one readily verifies that a group homomorphism from a locally compact Hausdorff group G to $U(H) = M^U(K(H))$ is a unitary representation of G on H if and only if it is unitary multiplier representation of G on $K(H)$.

Definition 2.4.9. A *multiplier representation* of a C*-algebra A on another C*-algebra C is a *non-degenerate* homomorphism $\rho \colon A \to M^B(C)$, that is, $\rho(A)C$ is dense in C.

The next lemma shows that a homomorphism from a C*-algebra A to $B(H) = M^B(K(H))$ is a representation of A on H if and only if it is a multiplier representation of A on $K(H)$.

Lemma 2.4.10. *Let A be a C*-algebra and H a Hilbert space. If $\rho \colon A \to B(H)$ is a homomorphism, then $\rho(A)H$ is dense in H if and only $\rho(A)K(H)$ is dense in $K(H)$.*

Proof. To prove the forward implication, assume that $\rho(A)H$ is dense in H. It suffices to prove that the closure of $\rho(A)K(H)$ contains the finite-rank operators on H. Let $c \in K(H)$ be any finite-rank operator and pick any $\varepsilon > 0$. Choose $x_1, \ldots, x_n \in H$ and $y_1, \ldots, y_n \in H$ such that c equals the sum of $x_1 \oslash y_1, \ldots, x_n \oslash y_n$. By assumption there exist $\tilde{a}_j \in A$ and $\tilde{y}_j \in H$ such that $\|x_j\|\|\rho(\tilde{a}_j)\tilde{y}_j - y_j\| < \varepsilon/2n$. Now let $(1_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. Choose $\lambda \in \Lambda$ such that $\|x_j\|\|1_\lambda \tilde{a}_j - \tilde{a}_j\|\|\tilde{y}_j\| < \varepsilon/2n$ for all $j = 1, \dots, n$. Define $\tilde{a} := 1_\lambda$ and

$$\tilde{c} := \sum_{j=1}^{n} x_j \oslash \rho(\tilde{a}_j)\tilde{y}_j.$$

We discover that

$$\begin{aligned}
\|\rho(\tilde{a})\tilde{c} - c\| &= \left\| \sum_{j=1}^{n} x_j \oslash (\rho(1_\lambda \tilde{a}_j)\tilde{y}_j - y_j) \right\| \\
&\leq \sum_{j=1}^{n} \|x_j\|(\|\rho(1_\lambda \tilde{a}_j)\tilde{y}_j - \rho(\tilde{a}_j)\tilde{y}_j\| + \|\rho(\tilde{a}_j)\tilde{y}_j - y_j\|) \\
&\leq \sum_{j=1}^{n} \|x_j\|(\|1_\lambda \tilde{a}_j - \tilde{a}_j\|\|\tilde{y}_j\| + \|\rho(\tilde{a}_j)\tilde{y}_j - y_j\|) < \varepsilon,
\end{aligned}$$

which shows that c belongs to the closure of $\rho(A)K(H)$.

To prove the converse, assume that $\rho(A)K(H)$ is dense in $K(H)$. Let $x \in H$ and $\varepsilon > 0$ be given. Choose $c \in K(H)$ such that $cx = x$. By hypothesis there are $\tilde{a} \in A$ and $\tilde{c} \in K(H)$ such that $\|\rho(\tilde{a})\tilde{c} - c\|\|x\| < \varepsilon$. Put $\tilde{x} := \tilde{c}x$ and observe that

$$\|\rho(\tilde{a})\tilde{x} - x\| = \|\rho(\tilde{a})\tilde{c}x - cx\| \leq \|\rho(\tilde{a})\tilde{c} - c\|\|x\| < \varepsilon.$$

Thus $\rho(A)H$ is dense in H. ■

Definition 2.4.11. A *covariant multiplier representation* of a C*-action $\alpha\colon G \curvearrowright A$ on a C*-algebra C is a pair (u, ρ), where u is a unitary multiplier representation on C and ρ a multiplier representation of A on C such that the diagram

$$\begin{tikzcd}
C \arrow[r, "\rho(a)"] \arrow[d, "u(\theta)"'] & C \arrow[d, "u(\theta)"] \\
C \arrow[r, "\rho(\alpha_\theta(a))"'] & C
\end{tikzcd}$$

commutes for all $s \in G$ and $a \in A$.

Now let $\alpha\colon G \curvearrowright A$ be a C*-action. It goes without saying that the covariant representations of α on H are precisely the covariant multiplier representations of α on $K(H)$.

We claim that α always has a covariant multiplier representation on its associated crossed product of G and A.

- For each $\theta \in G$ we consider the linear operator $j_G(\theta)\colon C_{\mathrm{cpt}}(G, A) \to G \ltimes_\alpha A$ determined by $(j_G(\theta)f)(s) := \alpha_\theta(f(\theta^{-1}s))$ for all $f \in C_{\mathrm{cpt}}(G, A)$ and $s \in G$. If (u, ρ) is a covariant representation of α on H, then for $\theta \in G$ and $f \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}
(u \ltimes_\alpha \rho)(j_G(\theta)f)x &= \int_G \rho(\alpha_\theta(f(\theta^{-1}s)))u(s)x \; \mathrm{d}_\mu s = \int_G \rho(\alpha_\theta(f(s)))u(\theta s)x \; \mathrm{d}_\mu s \\
&= u(\theta) \int_G \rho(f(s))u(s)x \; \mathrm{d}_\mu t = u(\theta)(u \ltimes_\alpha \rho)(f)x
\end{aligned}$$

for all $x \in H$ and therefore $\|j_G(\theta)f\|_{\text{env}} = \|f\|_{\text{env}}$. It follows that $j_G(\theta)$ can be uniquely extended to an isometric linear operator $j_G(\theta)\colon G \ltimes_\alpha A \to G \ltimes_\alpha A$. For $f, g \in C_{\text{cpt}}(G, A)$ and $t \in G$ we calculate

$$\begin{aligned}
((j_G(\theta)f)^* g)(t) &= \int_G \Delta(s^{-1})\alpha_s((j_G(\theta)f)(s^{-1})^*)\alpha_s(g(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_{s\theta}(f(\theta^{-1}s^{-1})^*)\alpha_s(g(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\alpha_{s\theta^{-1}}(g(\theta s^{-1}t)) \,\mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\alpha_s((j_G(\theta^{-1})g)(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= (f^* j_G(\theta^{-1})g)(t),
\end{aligned}$$

which proves that $j_G(\theta)$ is adjointable with adjoint $j_G(\theta)^* = j_G(\theta^{-1})$. One can also check that $j_G(\theta)$ is invertible with inverse $j_G(\theta)^{-1} = j_G(\theta^{-1})$. Hence we obtain a well-defined map $j_G\colon G \to M^U(G \ltimes_\alpha A)$.

- For each $a \in A$ we now define a linear operator $j_A(a)\colon C_{\text{cpt}}(G, A) \to G \ltimes_\alpha A$ by $(j_A(a)f)(s) := af(s)$ for all $f \in C_{\text{cpt}}(G, A)$ and $s \in G$. If (u, ρ) is a covariant representation of α on H, then for $a \in A$ and $f \in C_{\text{cpt}}(G, A)$ we have

$$\begin{aligned}
(u \ltimes_\alpha \rho)(j_A(a)f)x &= \int_G \rho((j_A(a)f)(s))u(s)x \,\mathrm{d}_\mu s = \int_G \rho(af(s))u(s)x \,\mathrm{d}_\mu s \\
&= \rho(a) \int_G \rho(f(s))u(s)x \,\mathrm{d}_\mu s = \rho(a)(u \ltimes_\alpha \rho)(f)x
\end{aligned}$$

for all $x \in H$ and therefore $\|j_A(a)f\|_{\text{env}} \leq \|a\|\|f\|_{\text{env}}$. It follows that $j_A(a)$ uniquely extends to a bounded linear operator $j_A(a)\colon G \ltimes_\alpha A \to G \ltimes_\alpha A$. For $f, g \in C_{\text{cpt}}(G, A)$ and $t \in G$ we compute

$$\begin{aligned}
((j_A(a)f)^* g)(t) &= \int_G \Delta(s^{-1})\alpha_s((j_A(a)f)(s^{-1})^*)\alpha_s(g(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_s(f(s^{-1})^* a^*)\alpha_s(g(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\alpha_s(a^* g(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\alpha_s((j_A(a^*)g)(s^{-1}t)) \,\mathrm{d}_\mu s \\
&= (f^* j_A(a^*)g)(t),
\end{aligned}$$

which shows that $j_A(a)$ is adjointable with adjoint $j_A(a^*)$. So we find a well-defined linear operator $j_A\colon A \to M^B(G \ltimes_\alpha A)$.

Lemma 2.4.12. *If $\alpha\colon G \curvearrowright A$ is a C^*-action, then the pair (j_G, j_A) is a covariant multiplier representation of α on $G \ltimes_\alpha A$.*

Proof. It is clear that j_G is a group homomorphism and that j_A is a homomorphism.

Let us prove that j_G is strongly continuous. Given a net $(\theta_\lambda)_{\lambda\in\Lambda}$ that converges to θ_∞ in G, it suffices to show that $(j_G(\theta_\lambda)f)_{\lambda\in\Lambda}$ converges to $j_G(\theta_\infty)f$ in $G \ltimes_\alpha A$ for each $f \in C_{\text{cpt}}(G, A)$.

Let $f \in C_{\mathrm{cpt}}(G, A)$ and $\varepsilon > 0$ be given. By Lemma 1.3.10 there is an open neighbourhood $U_1 \subset G$ of θ_∞ such that $\theta \in U_1$ implies

$$\int_G \|f(\theta^{-1}s) - f(\theta_\infty^{-1}s)\| \,\mathrm{d}_\mu s < \frac{\varepsilon}{4}.$$

Put $r := \mu(\sigma(f)) + 1$ and choose an open neighbourhood $U_2 \subset G$ of θ_∞ such that $\theta \in U_2$ implies $\|\alpha_\theta(f(s)) - \alpha_{\theta_\infty}(f(s))\| < \varepsilon/4r$ for all $s \in \sigma(f)$. Define $U := U_1 \cap U_2$ and choose $\lambda_0 \in \Lambda$ such that $\theta_\lambda \in U$ whenever $\lambda \geq \lambda_0$. So for $\lambda \geq \lambda_0$ we have

$$\begin{aligned}
\|j_G(\theta_\lambda)f - j_G(\theta_\infty)f\|_{\mathrm{int}} &= \int_G \|\alpha_{\theta_\lambda}(f(\theta_\lambda^{-1}s)) - \alpha_{\theta_\infty}(f(\theta_\infty^{-1}s))\| \,\mathrm{d}_\mu s \\
&\leq \int_G \|\alpha_{\theta_\lambda}(f(\theta_\lambda^{-1}s)) - \alpha_{\theta_\lambda}(f(\theta_\infty^{-1}s))\| \,\mathrm{d}_\mu s \\
&\qquad + \int_G \|\alpha_{\theta_\lambda}(f(\theta_\infty^{-1}s)) - \alpha_{\theta_\infty}(f(\theta_\infty^{-1}s))\| \,\mathrm{d}_\mu s \\
&= \int_G \|f(\theta_\lambda^{-1}s) - f(\theta_\infty^{-1}s)\| \,\mathrm{d}_\mu s + \int_G \|\alpha_{\theta_\lambda}(f(s)) - \alpha_{\theta_\infty}(f(s))\| \,\mathrm{d}_\mu s \\
&\leq \frac{\varepsilon}{4} + \mu(\sigma(f))\frac{\varepsilon}{4r} < \varepsilon
\end{aligned}$$

and therefore $\|j_G(\theta_\lambda)f - j_G(\theta_\infty)f\|_{\mathrm{env}} < \varepsilon$ as the integral norm dominates the enveloping norm on $C_{\mathrm{cpt}}(G, A)$. So j_G is a unitary multiplier representation of G on $G \ltimes_\alpha A$.

Let us now prove that j_A is non-degenerate. Let $f \in C_{\mathrm{cpt}}(G, A)$ and $\varepsilon > 0$ be given. Fix an approximate unit $(1_\lambda)_{\lambda \in \Lambda}$ for A. Put $r := \mu(\sigma(f)) + 1$ and find a sufficiently large index $\lambda \in \Lambda$ such that $\|1_\lambda f(s) - f(s)\| < \varepsilon/2r$ for all $s \in \sigma(f)$. If $\tilde{a} := 1_\lambda$ and $\tilde{f} := f$, then

$$\|j_A(\tilde{a})\tilde{f} - f\|_{\mathrm{int}} = \int_G \|1_\lambda f(s) - f(s)\| \,\mathrm{d}_\mu s \leq \mu(\sigma(f))\frac{\varepsilon}{2r} < \varepsilon.$$

and therefore $\|j_A(\tilde{a})\tilde{f} - f\|_{\mathrm{env}} < \varepsilon$. Thus j_A is a multiplier representation of A on $G \ltimes_\alpha A$.

It remains to check that $j_A(\alpha_\theta(a))j_G(\theta) = j_G(\theta)j_A(a)$ for $\theta \in G$ and $a \in A$. So we compute

$$\begin{aligned}
(j_A(\alpha_\theta(a))j_G(\theta)f)(s) &= \alpha_\theta(a)(j_G(\theta)f)(s) = \alpha_\theta(af(\theta^{-1}s)) \\
&= \alpha_\theta((j_A(a)f)(\theta^{-1}s)) = (j_G(\theta)j_A(a)f)(s)
\end{aligned}$$

for $f \in C_{\mathrm{cpt}}(G, A)$ and $s \in G$. This suffices. ∎

Definition 2.4.13. Let $\alpha\colon G \curvearrowright A$ be a C*-action. The pair (j_G, j_A) is called the *canonical covariant multiplier representation* of α.

Every covariant representation (u, ρ) of α factors through the canonical covariant multiplier representation (j_G, j_A) of α in the following sense.

Lemma 2.4.14. *Let $\alpha\colon G \curvearrowright A$ be a C*-action. If (u, ρ) is a covariant representation of α on H, then the equalities $(u \ltimes_\alpha \rho)^+(j_G(\theta)) = u(\theta)$ and $(u \ltimes_\alpha \rho)^+(j_A(a)) = \rho(a)$ hold for all $\theta \in G$ and $a \in A$.*

Proof. Let $f \in G \ltimes_\alpha A$ and $x \in H$ be arbitrary. We have $(u \ltimes_\alpha \rho)^+(j_G(\theta))(u \ltimes_\alpha \rho)(f)x = u(\theta)(u \ltimes_\alpha \rho)(f)x$ for $\theta \in G$ and $(u \ltimes_\alpha \rho)^+(j_A(a))(u \ltimes_\alpha \rho)(f)x = \rho(a)(u \ltimes_\alpha \rho)(f)x$ for $a \in A$. So the desired equalities hold by non-degeneracy of $u \ltimes_\alpha \rho$. ∎

The next technical lemma is a consequence of Lemma 2.4.14.

Lemma 2.4.15. *Let $\alpha\colon G \curvearrowright A$ be a C^*-action. The equality*

$$\int_G j_A(f(s))j_G(s)g \,\mathrm{d}_\mu s = fg$$

holds for all $f \in C_{\mathrm{cpt}}(G, A)$ and $g \in G \ltimes_\alpha A$.

Proof. If (u, ρ) is a covariant representation of α on H, then Lemma 2.4.14 implies that

$$\begin{aligned}(u \ltimes_\alpha \rho)\bigg(\int_G j_A(f(s))j_G(s)g \,\mathrm{d}_\mu s\bigg)x &= \int_G (u \ltimes_\alpha \rho)(j_A(f(s))j_G(s)g)x \,\mathrm{d}_\mu s \\ &= \int_G \rho(f(s))u(s)(u \ltimes_\alpha \rho)(g)x \,\mathrm{d}_\mu s \\ &= (u \ltimes_\alpha \rho)(f)(u \ltimes_\alpha \rho)(g)x \\ &= (u \ltimes_\alpha \rho)(fg)x\end{aligned}$$

for all $x \in H$. This suffices. ■

We are now ready to prove that the integrated form implements a one-to-one correspondence between the covariant representations of α and the representations of $G \ltimes_\alpha A$.

Theorem 2.4.16. *Let $\alpha\colon G \curvearrowright A$ be a C^*-action. For every representation P of $G \ltimes_\alpha A$ on H there exists a unique covariant representation (u, ρ) of α on H such that $u \ltimes_\alpha \rho = P$.*

Proof. Note that $P^+(M^U(G \ltimes_\alpha A)) \subset U(H)$ as P^+ is non-degenerate and therefore unital. So we can define $u\colon G \to U(H)$ and $\rho\colon A \to B(H)$ via $u(\theta) := P^+(j_G(\theta))$ and $\rho(a) := P^+(j_A(a))$ for $\theta \in G$ and $a \in A$, respectively. It is clear that u is a group homomorphism and that ρ is a homomorphism. First we shall use Lemma 2.4.12 to show that (u, ρ) is a covariant representation of α on H.

Let us first prove that u is strongly continuous. Suppose that $(\theta_\lambda)_{\lambda\in\Lambda}$ is a net that converges to θ_∞ in G. By non-degeneracy of P it suffices to show that $(u(\theta_\lambda)x)_{\lambda\in\Lambda}$ converges to $u(\theta_\infty)x$ in H for each $x \in P(G \ltimes_\alpha A)H$. Choose $g \in G \ltimes_\alpha A$ and $y \in H$ such that $x = P(g)y$. By strong continuity of j_G there exists a $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $\|j_G(\theta_\lambda)g - j_G(\theta_\infty)g\|_{\mathrm{env}}\|y\| < \varepsilon$. So for $\lambda \geq \lambda_0$ we have

$$\begin{aligned}\|u(\theta_\lambda)x - u(\theta_\infty)x\| &= \|P^+(j_G(\theta_\lambda))P(g)y - P^+(j_G(\theta_\infty))P(g)y\| \\ &= \|P(j_G(\theta_\lambda)g)y - P(j_G(\theta_\infty)g)y\| \\ &\leq \|j_G(\theta_\lambda)g - j_G(\theta_\infty)g\|_{\mathrm{env}}\|y\| < \varepsilon.\end{aligned}$$

It follows that u is a unitary multiplier representation of G on H.

We now prove that ρ is non-degenerate. By non-degeneracy of P it suffices to approximate $x \in P(G \ltimes_\alpha A)H$ by elements in $\rho(A)H$. Pick $g \in G \ltimes_\alpha A$ and $y \in H$ such that $x = P(g)y$. Let $\varepsilon > 0$ be arbitrary. By non-degeneracy of j_A there are $\tilde{a} \in A$ and $\tilde{g} \in G \ltimes_\alpha A$ such that $\|j_A(\tilde{a})\tilde{g} - g\|_{\mathrm{env}}\|y\| < \varepsilon$. Define $\tilde{x} := P(\tilde{g})y$ and observe that

$$\begin{aligned}\|\rho(\tilde{a})\tilde{x} - x\| &= \|P^+(j_A(\tilde{a}))P(\tilde{g})y - P(g)y\| \\ &= \|P(j_A(\tilde{a})\tilde{g})y - P(g)y\| \\ &\leq \|j_A(\tilde{a})\tilde{g} - g\|_{\mathrm{env}}\|y\| < \varepsilon.\end{aligned}$$

We conclude that ρ is a representation of A on H.

Finally, the equalities

$$\rho(\alpha_\theta(a))u(\theta) = P^+(j_A(\alpha_\theta(a))j_G(\theta)) = P^+(j_G(\theta)j_A(a)) = u(\theta)\rho(a)$$

for $\theta \in G$ and $a \in A$ show that (u, ρ) is a covariant representation of α on H.

To see the integrated form of (u, ρ) agrees with P, it suffices to prove that $(u \ltimes_\alpha \rho)(f)x = P(f)x$ for all $f \in C_{\text{cpt}}(G, A)$ and $x \in P(G \ltimes_\alpha A)H$ by non-degeneracy of P. Choose $g \in G \ltimes_\alpha A$ and $y \in H$ such that $x = P(g)y$. By Lemma 2.4.15 we have

$$\begin{aligned}(u \ltimes_\alpha \rho)(f)x &= \int_G \rho(f(s))u(s)P(g)y \, \mathrm{d}_\mu s = \int_G P(j_A(f(s))j_G(s)g)y \, \mathrm{d}_\mu s \\ &= P\biggl(\int_G j_A(f(s))j_G(s)g \, \mathrm{d}_\mu s\biggr)y = P(fg)y = P(f)x.\end{aligned}$$

This implies that $u \ltimes_\alpha \rho = P$

Uniqueness is clear by virtue of Lemma 2.4.14. ■

2.5 Classical Crossed Products

Consider a classical C*-action $\tau\colon G \curvearrowright C_0(\Omega)$. In this section we analyze the corresponding crossed product $G \ltimes_\tau C_0(\Omega)$. For instance, it is possible to describe the structure of $G \ltimes_\tau C_0(\Omega)$ in terms of functions in $C_{\text{cpt}}(G \times \Omega)$. To see this, note that there is a canonical one-to-one correspondence between functions $f\colon G \to C_0(\Omega)$ and functions $f\colon G \times \Omega \to \mathbb{C}$ for which

$$\Omega \ni \omega \mapsto f(s, \omega) \in \mathbb{C}$$

is contained in $C_0(\Omega)$ for each $s \in G$. Indeed, for $s \in G$ and $\omega \in \Omega$ we identify $f(s)(\omega) = f(s, \omega)$. Under this identification the notation $\sigma(f)$ becomes ambiguous. So we agree to reserve this notation for the support of f viewed as a function from $G \times \Omega$ to $\mathbb{C}$.

This perspective allows us to prove the following approximation lemma.

Lemma 2.5.1. *If G is a locally compact Hausdorff group and Ω a locally compact Hausdorff space, then $C_{\text{cpt}}(G \times \Omega)$ is a linear subspace of $C_{\text{cpt}}(G, C_0(\Omega))$ and, furthermore, it is dense with respect to the integral norm.*

Proof. Let $f \in C_{\text{cpt}}(G \times \Omega)$ be given and note that f can be viewed as a function from G to $C_0(\Omega)$. We show that $f \in C_{\text{cpt}}(G, C_0(\Omega))$. Suppose that $(s_\lambda)_{\lambda \in \Lambda}$ is a net that converges to s_∞ in G. Let $K \subset G$ and $L \subset \Omega$ be compact such that $\sigma(f) \subset K \times L$. Choose for each $\omega \in L$ open neighbourhoods $U_\omega \subset G$ of s_∞ and $V_\omega \subset \Omega$ of ω such that $f(U_\omega \times V_\omega)$ is contained in the open ball of radius $\varepsilon/4$ centered at $f(s_\infty, \omega)$ in $\mathbb{C}$. By compactness of L there are $\omega_1, \ldots, \omega_n \in L$ such that $V_{\omega_1}, \ldots, V_{\omega_n}$ cover L. Consider the open neighbourhood

$$U := \bigcap_{j=1}^n U_{\omega_j}$$

of s_∞ and pick $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $s_\lambda \in U$. We claim that

$$\|f(s_\lambda) - f(s_\infty)\|_\infty = \sup_{\omega \in L} |f(s_\lambda, \omega) - f(s_\infty, \omega)| < \varepsilon$$

for all $\lambda \geq \lambda_0$. So assume that $\lambda \geq \lambda_0$ and let $\omega \in L$ be given. By construction there is some $j = 1, \ldots, n$ such that $\omega \in V_{\omega_j}$. Since we also have $s_\lambda \in U_{\omega_j}$, it follows that

$$|f(s_\lambda, \omega) - f(s_\infty, \omega)| \leq |f(s_\lambda, \omega) - f(s_\infty, \omega_j)| + |f(s_\infty, \omega_j) - f(s_\infty, \omega)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

This proves the claim and continuity of f as a function from G to $C_0(\Omega)$ follows. Moreover, it is also compactly supported as it vanishes outside of K. Hence $f \in C_{\text{cpt}}(G, C_0(\Omega))$ as desired.

Density is a straightforward application of Lemma 1.1.3 and Lemma 1.3.5; the details are left to the reader. ■

Let us argue that $C_{\rm cpt}(G \times \Omega)$ is a subalgebra of the algebra $C_{\rm cpt}(G, C_0(\Omega))$.

- For each $\omega \in \Omega$ let $x^\sharp_\omega \colon C_0(\Omega) \to \mathbb{C}$ be the bounded linear functional given by evaluation at the point ω, that is, $x^\sharp_\omega(\varphi) := \varphi(\omega)$ for all $\varphi \in C_0(\Omega)$. For any $\omega \in \Omega$ the product $fg \colon G \to C_0(\Omega)$ of two elements $f, g \in C_{\rm cpt}(G \times \Omega)$ satisfies
$$\begin{aligned} x^\sharp_\omega((fg)(t)), &= x^\sharp_\omega\left(\int_G f(s)\tau_s(g(s^{-1}t))\, \mathrm{d}_\mu s\right) \\ &= \int_G x^\sharp_\omega(f(s)\tau_s(g(s^{-1}t)))\, \mathrm{d}_\mu s \\ &= \int_G f(s,\omega)g(s^{-1}t, s^{-1}\cdot\omega)\, \mathrm{d}_\mu s \end{aligned}$$
for all $t \in G$. Since the function
$$G \times G \times \Omega \ni (s,t,\omega) \mapsto f(s,\omega)g(s^{-1}t, s^{-1}\cdot\omega) \in \mathbb{C}.$$
belongs to $C_{\rm cpt}(G \times G \times \Omega)$, we can integrate over the first variable and use Lemma 1.1.10 to conclude that $fg \in C_{\rm cpt}(G \times \Omega)$ with formula
$$(fg)(t,\omega) = \int_G f(s,\omega)g(s^{-1}t, s^{-1}\cdot\omega)\, \mathrm{d}_\mu s$$
for all $(t,\omega) \in G \times \Omega$.
- If $h^* \colon G \to C_0(\Omega)$ is the adjoint of an element $h \in C_{\rm cpt}(G \times \Omega)$, then $h^* \in C_{\rm cpt}(G \times \Omega)$ with formula $h^*(t,\omega) = \Delta(t^{-1})h(t^{-1}, t^{-1}\cdot\omega)^*$ for all $(t,\omega) \in G \times \Omega$.

Lemma 2.5.2. *If $\tau \colon G \curvearrowright C_0(\Omega)$ is a classical C*-action, then $C_{\rm cpt}(G\times\Omega)$ is a dense subalgebra of the C*-algebra $G \ltimes_\tau C_0(\Omega)$.*

Proof. This is an immediate consequence of Lemma 2.5.1 and the fact that the integral norm dominates the enveloping norm on $C_{\rm cpt}(G, C_0(\Omega))$. ■

Caution is advised when viewing $C_{\rm cpt}(G \times \Omega)$ as a subalgebra of $G \ltimes_\tau C_0(\Omega)$; the multiplication and involution are different from the standard operations coming from $C_0(G \times \Omega)$.

2.6 The Stone-Neumann Theorem

This section presents an important case study of a classical crossed product. The main reference for this section is [Rie72].

Definition 2.6.1. Let G be a locally compact Hausdorff group. The dynamical system (G, G), whose action is given by $s \cdot t := st$ (resp. $s \cdot t := ts^{-1}$) for $s, t \in G$, is called the *left-translation* (resp. *right-translation*) on G. The classical C*-action induced by (G, G) is denoted by $\mathrm{lt} \colon G \curvearrowright C_0(G)$ (resp. $\mathrm{rt} \colon G \curvearrowright C_0(G)$) and, with a slight abuse of language, will also be called the *left-translation* (resp. *right-translation*) on G.

Given a locally compact Hausdorff group G, we shall study the covariant representations of its left-translation $\mathrm{lt} \colon G \curvearrowright C_0(G)$ and show that the associated crossed product $G \ltimes_{\rm lt} C_0(G)$ is isomorphic to the C*-algebra $K(H^G)$ of compact operators on H^G. By Lemma 2.5.2 we can view $C_{\rm cpt}(G \times G)$ as a dense subalgebra of $G \ltimes_{\rm lt} C_0(G)$.

Our analysis starts with the introduction of certain functions in $C_{\rm cpt}(G \times G)$. For $\chi, \psi \in C_{\rm cpt}(G)$ the continuous function $f_{\chi,\psi} \colon G \times G \to \mathbb{C}$ given by $f_{\chi,\psi}(s,t) := \Delta(s^{-1}t)\chi^*(s^{-1}t)\psi(t)$ for all $(s,t) \in G \times G$ is compactly supported as $\sigma(f_{\chi,\psi}) \subset \sigma(\psi)\sigma(\chi)^{-1} \times \sigma(\psi)$. Moreover, $f_{\chi,\psi} \in C_{\rm cpt}(G \times G)$ depends antilinearly on $\chi \in C_{\rm cpt}(G)$ and linearly on $\psi \in C_{\rm cpt}(G)$.

Lemma 2.6.2. *Let G be a locally compact Hausdorff group.*

(i) The equality $f_{\chi_1,\psi_1} f_{\chi_2,\psi_2} = \langle \chi_1, \psi_2 \rangle f_{\chi_2,\psi_1}$ holds for all $\chi_1, \psi_1, \chi_2, \psi_2 \in C_{\mathrm{cpt}}(G)$.

(ii) The equality $f^*_{\chi,\psi} = f_{\psi,\chi}$ holds for all $\chi, \psi \in C_{\mathrm{cpt}}(G)$.

Proof. We only prove part (i); part (ii) is left to the reader. We directly compute

$$\begin{aligned}
(f_{\chi_1,\psi_1} f_{\chi_2,\psi_2})(t_1,t_2) &= \int_G f_{\chi_1,\psi_1}(s,t_2) f_{\chi_2,\psi_2}(s^{-1}t_1, s^{-1}t_2)\ \mathrm{d}_\mu s \\
&= \int_G \Delta(s^{-1}t_2)\chi_1^*(s^{-1}t_2)\psi_1(t_2)\Delta(t_1^{-1}ss^{-1}t_2)\chi_2^*(t_1^{-1}ss^{-1}t_2)\psi_2(s^{-1}t_2)\ \mathrm{d}_\mu s \\
&= \left(\int_G \Delta(s^{-1}t_2)\chi_1^*(s^{-1}t_2)\psi_2(s^{-1}t_2)\ \mathrm{d}_\mu s \right) \Delta(t_1^{-1}t_2)\chi_2^*(t_1^{-1}t_2)\psi_1(t_2) \\
&= \langle \chi_1, \psi_2 \rangle f_{\chi_2,\psi_1}(t_1,t_2)
\end{aligned}$$

for $(t_1,t_2) \in G \times G$. In the final step we used left-invariance of the integral and Lemma 1.4.3. ∎

Lemma 2.6.3. *Let G be a locally compact Hausdorff group. The span of $\{f_{\chi,\psi} : \chi, \psi \in C_{\mathrm{cpt}}(G)\}$ is dense in $C_{\mathrm{cpt}}(G \times G)$ with respect to the integral norm on $C_{\mathrm{cpt}}(G, C_0(G))$.*

Proof. Let E denote the span of $\{f_{\chi,\psi} : \chi, \psi \in C_{\mathrm{cpt}}(G)\}$ in $C_{\mathrm{cpt}}(G \times G)$. Let $f \in C_{\mathrm{cpt}}(G \times G)$ and $\varepsilon > 0$ be given. Find compact subsets $K \subset G$ and $L \subset G$ such that $\sigma(f) \subset K \times L$ and use Lemma 1.1.2 to find $\eta \in C_{\mathrm{cpt}}(G)$ such that $\eta(G) \subset [0,1]$ and $\eta(t) = 1$ for all $t \in K^{-1}L \cup L$. Put $r := \mu(\sigma(\eta)^{-1}\sigma(\eta)) + 1$. If we endow E with pointwise multiplication and pointwise involution, then a moment's thought reveals that the subalgebra E of $C_0(G \times G)$ satisfies the criteria of the Stone-Weierstrass theorem. Hence there exists an $h \in E$ such that $\|f - h\|_\infty < \varepsilon / r$. Consider the function $g\colon G \times G \to \mathbb{C}$ given by $g(s,t) := \eta(s^{-1}t)\eta(t)h(s,t)$ for all $(s,t) \in G \times G$. It is clear that $g \in E$. If $s \notin \sigma(\eta)^{-1}\sigma(\eta)$, then $\eta(s^{-1}t) = 0$ for all $t \in \sigma(\eta)$ and, furthermore, $\eta(t) = 0$ for all $t \notin \sigma(\eta)$. So we deduce that

$$\|f - g\|_{\mathrm{int}} = \int_G \sup_{t \in G} \eta(s^{-1}t)\eta(t)|f(s,t) - h(s,t)|\ \mathrm{d}_\mu s \leq \mu(\sigma(\eta)^{-1}\sigma(\eta)) \frac{\varepsilon}{r} < \varepsilon,$$

thereby completing the proof. ∎

We shall now define two covariant representation; one for the the left-translation and one for the right-translation. Recall that the Hilbert space H^G is the completion of $C_{\mathrm{cpt}}(G)$ with respect to the inner product given by $\langle \chi, \psi \rangle = \mu(\chi^* \psi)$ for $\chi, \psi \in C_{\mathrm{cpt}}(G)$.

- Given $\theta \in G$, we define linear operators $u^L(\theta)\colon C_{\mathrm{cpt}}(G) \to H^G$ and $u^R(\theta)\colon C_{\mathrm{cpt}}(G) \to H^G$ by $u^L(\theta)\chi := L_\theta(\chi)$ and $u^R(\theta)\chi := \Delta(\theta)^{1/2} R_\theta(\chi)$ for all $\chi \in C_{\mathrm{cpt}}(G)$. We have

$$\|u^L(\theta)\chi\|^2 = \int_G |\chi(\theta^{-1}s)|^2\ \mathrm{d}_\mu s = \int_G |\chi(s)|^2\ \mathrm{d}_\mu s = \|\chi\|^2$$

and by definition of the modular function we have

$$\|u^R(\theta)\chi\|^2 = \int_G \Delta(\theta)|\chi(s\theta)|^2\ \mathrm{d}_\mu s = \int_G |\chi(s)|^2\ \mathrm{d}_\mu s = \|\chi\|^2$$

for all $\chi \in C_{\mathrm{cpt}}(G)$. Consequently, $u^L(\theta)$ and $u^R(\theta)$ can be uniquely extended to isometric linear operators $u^L(\theta)\colon H^G \to H^G$ and $u^R(\theta)\colon H^G \to H^G$. Also note that $u^L(\theta)$ and $u^R(\theta)$ are surjective. Thus we obtain well-defined maps $u^L\colon G \to U(H^G)$ and $u^R\colon G \to U(H^G)$.

- For each $\varphi \in C_0(G)$ we define a linear operator $m(\varphi)\colon C_{\mathrm{cpt}}(G) \to H^G$ via $m(\varphi)\chi := \varphi\chi$ for all $\chi \in C_{\mathrm{cpt}}(G)$. The estimation

$$\|m(\varphi)\chi\|^2 = \int_G |\varphi(s)\chi(s)|^2 \, \mathrm{d}_\mu s \leq \int_G \|\varphi\|_\infty^2 |\chi(s)|^2 \, \mathrm{d}_\mu s = \|\varphi\|_\infty^2 \|\chi\|^2$$

  shows that $m(\varphi)$ can be uniquely extended to a bounded linear operator $m(\varphi)\colon H^G \to H^G$. Hence $m\colon C_0(G) \to B(H^G)$ is a well-defined linear operator.

Lemma 2.6.4. *If G is a locally compact Hausdorff group, then the pair (u^L, m) (resp. (u^R, m)) is a covariant representation of* lt *(resp.* rt*) on H^G.*

Proof. We only prove the statement that (u^L, m) is a covariant representation of lt on H^G. One readily verifies that u^L is a group homomorphism and that m is a homomorphism. Strong continuity of u^L follows from a modification of the proof of Lemma 1.3.10 and non-degeneracy of m follows from an application of Lemma 1.1.2.

We still need to show that $m(\mathrm{lt}_\theta(\varphi))u^L(\theta) = u^L(\theta)m(\varphi)$ for $\theta \in G$ and $\varphi \in C_0(G)$. We have

$$\begin{aligned}(m(\mathrm{lt}_\theta(\varphi))u^L(\theta)\chi)(s) &= \mathrm{lt}_\theta(\varphi)(s)(u^L(\theta)\chi)(s) = \varphi(\theta^{-1}s)\chi(\theta^{-1}s) \\ &= (m(\varphi)\chi)(\theta^{-1}s) = (u^L(\theta)m(\varphi)\chi)(s)\end{aligned}$$

for $\chi \in C_{\mathrm{cpt}}(G)$ and $s \in G$. Hence the bounded linear operators $m(\mathrm{lt}_\theta(\varphi))u^L(\theta)$ and $u^L(\theta)m(\varphi)$ are equal as they agree on the dense linear subspace $C_{\mathrm{cpt}}(G)$ of H^G. ■

We shall now restrict our attention to the left-translation and the covariant representations thereof.

Definition 2.6.5. If G is locally compact Hausdorff group, then the pair (u^L, m) is called the *Schrödinger covariant representation* of lt.

It turns out that every covariant representation the left-translation is unitarily equivalent to a direct sum of (possibly infinitely many) copies of the Schrödinger covariant representation of the left-translation. This is in essence a reformulation of a module theoretic result by Rieffel [Rie72]. We shall give a modern proof of this result borrowing some ideas from Rieffel himself. The following two lemmas serve as preparation.

Lemma 2.6.6. *Let G be a locally compact Hausdorff group. If (u, ρ) is a covariant representation of* lt *on a Hilbert space H, then for every $\chi \in C_{\mathrm{cpt}}(G)$ and $x \in H$ there exists a vector $q \in H$ such that*

$$(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})x = \int_G \rho(\psi)u(s)q \, \mathrm{d}_\mu s$$

for all $\psi \in C_{\mathrm{cpt}}(G)$.

Proof. The continuous function $\varphi_\chi\colon G \to \mathbb{C}$ defined by $\varphi_\chi(t) := \Delta(t)\chi^*(t)$ for all $t \in G$ clearly belongs to $C_0(G)$ as it is compactly supported. Moreover, if we view the function $f_{\chi,\psi}$ as an element of $C_{\mathrm{cpt}}(G, C_0(G))$, then

$$f_{\chi,\psi}(s)(t) = \Delta(s^{-1}t)\chi^*(s^{-1}t)\psi(t) = \varphi_\chi(s^{-1}t)\psi(t) = (\mathrm{lt}_s(\varphi_\chi)\psi)(t)$$

for all $t \in G$ and therefore $f_{\chi,\psi}(s) = \mathrm{lt}_s(\varphi_\chi)\psi$ for all $\psi \in C_{\mathrm{cpt}}(G)$ and $s \in G$. Now we have

$$\begin{aligned}(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})x &= \int_G \rho(\mathrm{lt}_s(\varphi_\chi)\psi)u(s)x \, \mathrm{d}_\mu s \\ &= \int_G \rho(\psi)\rho(\mathrm{lt}_s(\varphi_\chi))u(s)x \, \mathrm{d}_\mu s \\ &= \int_G \rho(\psi)u(s)\rho(\varphi_\chi)x \, \mathrm{d}_\mu s\end{aligned}$$

for all $\psi \in C_{\mathrm{cpt}}(G)$. Hence the vector $q := \rho(\varphi_\chi)x$ serves. ■

Lemma 2.6.7. *Let G be a locally compact Hausdorff group and let (u^L, m) be the Schrödinger covariant representation of* lt. *If (u, ρ) is another covariant representation of* lt *on some Hilbert space $H \neq \{0\}$, then there is an isometric linear operator $Q\colon H^G \to H$ for which the diagrams*

$$\begin{array}{ccc} H^G & \xrightarrow{u^L(\theta)} & H^G \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ H & \xrightarrow[u(\theta)]{} & H \end{array}, \qquad \begin{array}{ccc} H^G & \xrightarrow{m(\varphi)} & H^G \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ H & \xrightarrow[\rho(\varphi)]{} & H \end{array}$$

commute for all $\theta \in G$ and $\varphi \in C_0(G)$.

Proof. There must exist $\chi \in C_{\mathrm{cpt}}(G)$ and $y \in H$ such that $(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\chi})y \neq 0$, otherwise

$$\|\chi\|^2 (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})y = (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi} f_{\chi,\chi})y = (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\chi})y = 0$$

for all $\chi, \psi \in C_{\mathrm{cpt}}(G)$ and $y \in H$, which would contradict the non-degeneracy of $u \ltimes_{\mathrm{lt}} \rho$. After rescaling y, we may assume that $\|\chi\| = 1$. Put

$$x := \frac{1}{\|(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\chi})y\|}(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\chi})y$$

and note that $(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\chi})x = x$. Now consider the linear operator $Q\colon C_{\mathrm{cpt}}(G) \to H$ given by $Q\psi := (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})x$ for all $\psi \in C_{\mathrm{cpt}}(G)$. To see that Q is isometric, we compute

$$\begin{aligned} \|Q\psi\|^2 &= \langle (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})x, (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})x \rangle = \langle x, (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})^*(u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi})x \rangle \\ &= \langle x, (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\psi}^* f_{\chi,\psi})x \rangle = \|\psi\|^2 \langle x, (u \ltimes_{\mathrm{lt}} \rho)(f_{\chi,\chi})x \rangle = \|\psi\|^2 \langle x, x \rangle = \|\psi\|^2 \end{aligned}$$

for all $\psi \in C_{\mathrm{cpt}}(G)$. It follows that Q can be uniquely extended to an isometric linear operator $Q\colon H^G \to H$. It will be convenient to invoke Lemma 2.6.6 and find a vector $q \in H$ such that

$$Q\psi = \int_G \rho(\psi)u(s)q \; \mathrm{d}_\mu s$$

for all $\psi \in C_{\mathrm{cpt}}(G)$.

To prove that the diagrams commute, we use density of $C_{\mathrm{cpt}}(G)$ in H^G. On the one hand, the diagram on the left-hand side commutes as

$$\begin{aligned} u(\theta)Q\psi &= \int_G u(\theta)\rho(\psi)u(s)q \; \mathrm{d}_\mu s \\ &= \int_G \rho(\mathrm{lt}_\theta(\psi))u(\theta s)q \; \mathrm{d}_\mu s \\ &= \int_G \rho(u^L(\theta)\psi)u(s)q \; \mathrm{d}_\mu s \\ &= Qu^L(\theta)\psi \end{aligned}$$

for all $\psi \in C_{\mathrm{cpt}}(G)$. On the other hand, the diagram on the right-hand side commutes as

$$\begin{aligned} \rho(\varphi)Q\psi &= \int_G \rho(\varphi)\rho(\psi)u(s)q \; \mathrm{d}_\mu s \\ &= \int_G \rho(\varphi\psi)u(s)q \; \mathrm{d}_\mu s \\ &= \int_G \rho(m(\varphi)\psi)u(s)q \; \mathrm{d}_\mu s \\ &= Qm(\varphi)\psi \end{aligned}$$

for all $\psi \in C_{\mathrm{cpt}}(G)$. ∎

We are now ready to prove Rieffel's result.

Theorem 2.6.8. *Let G be a locally compact Hausdorff group. Every covariant representation of* lt *is unitarily equivalent to a direct sum of (possibly infinitely many) copies of the Schrödinger covariant representation of* lt.

Proof. Let (u', ρ') be any covariant representation of lt on H'. By Zorn's lemma there is a maximal orthogonal family $(H_\lambda)_{\lambda\in\Lambda}$ of closed linear subspaces of H' such that H_λ is invariant under (u', ρ') and $(u'|H_\lambda, \rho'|H_\lambda)$ is unitarily equivalent to the Schrödinger representation (u^L, m). It suffices to prove that

$$H' = \bigoplus_{\lambda\in\Lambda} H_\lambda.$$

Assume the contrary. If H denotes the orthogonal complement of the direct sum of $(H_\lambda)_{\lambda\in\Lambda}$ in H', then $H \neq \{0\}$ and H is invariant under (u', ρ'). Now consider the covariant representation $(u, \rho) := (u'|H, \rho'|H)$. Using Lemma 2.6.7 we find an isometric linear operator $Q\colon H^G \to H$ such that $u(\theta)Q = Qu^L(\theta)$ for $\theta \in G$ and $\rho(\varphi)Q = Qm(\varphi)$ for $\varphi \in C_0(G)$. The closed linear subspace QH^G of H' is orthogonal to H_λ as it is contained in H. Moreover, the properties of Q guarantee that QH^G is invariant under (u', ρ') and that $(u'|QH^G, \rho'|QH^G)$ is unitarily equivalent to (u^L, m). Thus we have $QH^G = \{0\}$ by maximality of $(H_\lambda)_{\lambda\in\Lambda}$, which is impossible for obvious reasons. ∎

The rest of this section is devoted to the integrated form $u^L \ltimes_{\mathrm{lt}} m$ of the Schrödinger covariant representation (u^L, m) of lt. Let us inspect how $u^L \ltimes_{\mathrm{lt}} m$ represents elements of $C_{\mathrm{cpt}}(G, C_0(G))$ on H^G. For any $f \in C_{\mathrm{cpt}}(G, C_0(G))$ we have

$$\begin{aligned}
\langle \xi_1, (u^L \ltimes_{\mathrm{lt}} m)(f)\xi_2\rangle &= \left\langle \xi_1, \int_G m(f(s))u^L(s)\xi_2 \,\mathrm{d}_\mu s\right\rangle \\
&= \int_G \langle \xi_1, m(f(s))u^L(s)\xi_2\rangle \,\mathrm{d}_\mu s \\
&= \int_G \int_G \xi_1^*(t) f(s)(t)\xi_2(s^{-1}t) \,\mathrm{d}_\mu t \,\mathrm{d}_\mu s \\
&= \int_G \xi_1^*(t)\bigg(\int_G f(s)(t)\xi_2(s^{-1}t)\,\mathrm{d}_\mu s\bigg)\,\mathrm{d}_\mu t
\end{aligned}$$

for all $\xi_1, \xi_2 \in C_{\mathrm{cpt}}(G)$ and therefore

$$((u^L \ltimes_{\mathrm{lt}} m)(f)\xi)(t) = \int_G f(s)(t)\xi(s^{-1}t)\,\mathrm{d}_\mu s$$

for all $\xi \in C_{\mathrm{cpt}}(G)$ and $t \in G$.

For $\chi, \psi \in C_{\mathrm{cpt}}(G)$ we now consider the finite-rank operator $\chi \oslash \psi\colon H^G \to H^G$.

Lemma 2.6.9. *Let G be a locally compact Hausdorff group and let (u^L, m) be the Schrödinger covariant representation of* lt*. The equality $(u^L \ltimes_{\mathrm{lt}} m)(f_{\chi,\psi}) = \chi\oslash\psi$ holds for all $\chi, \psi \in C_{\mathrm{cpt}}(G)$.*

Proof. If $\xi \in C_{\mathrm{cpt}}(G)$, then

$$\begin{aligned}
((u^L \ltimes_{\mathrm{lt}} m)(f_{\chi,\psi})\xi)(t) &= \int_G f_{\chi,\psi}(s,t)\xi(s^{-1}t)\,\mathrm{d}_\mu s = \int_G \Delta(s^{-1}t)\chi^*(s^{-1}t)\psi(t)\xi(s^{-1}t)\,\mathrm{d}_\mu s \\
&= \int_G \chi^*(s)\psi(t)\xi(s)\,\mathrm{d}_\mu s = \langle \chi, \xi\rangle\psi(t) = ((\chi \oslash \psi)\xi)(t)
\end{aligned}$$

for all $t \in G$. It follows that the bounded linear operators $(u^L \ltimes_{\mathrm{lt}} m)(f_{\chi,\psi})$ and $\chi \oslash \psi$ are equal as they agree the dense linear subspace $C_{\mathrm{cpt}}(G)$ of H^G. ∎

The following classical result is known as the Stone-Neumann theorem. In particular, it states that $G \ltimes_{\mathrm{lt}} C_0(G)$ is isomorphic to $K(H^G)$. As indicated by Sims, Szavó and Williams [SSW20], one can use Rieffel's result to prove the Stone-Neumann theorem.

Theorem 2.6.10. *Let G a locally compact Hausdorff group and let (u^L, m) be the Schrödinger covariant representation of* lt. *The representation $u^L \ltimes_{\mathrm{lt}} m$ of $G \ltimes_{\mathrm{lt}} C_0(G)$ on H^G is faithful and has range $K(H^G)$.*

Proof. Let us first prove injectivity. Suppose that $f \in G \ltimes_{\mathrm{lt}} C_0(G)$ satisfies $(u^L \ltimes_{\mathrm{lt}} m)(f) = 0$. Lemma 2.3.8 and Theorem 2.6.8 imply that $\|(u \ltimes_{\mathrm{lt}} \rho)(f)\| = 0$ for every covariant representation (u, ρ) of lt. Thus $\|f\|_{\mathrm{env}} = 0$ and therefore $f = 0$ as desired.

It remains to prove the claim $(u^L \ltimes_{\mathrm{lt}} m)(G \ltimes_{\mathrm{lt}} C_0(G)) = K(H^G)$. If E denotes the span of $\{f_{\chi,\psi} : \chi, \psi \in C_{\mathrm{cpt}}(G)\}$ in $G \ltimes_{\mathrm{lt}} C_0(G)$, then E is dense in $G \ltimes_{\mathrm{lt}} C_0(G)$ by Lemma 2.6.3 and $(u^L \ltimes_{\mathrm{lt}} m)(E)$ is dense in $K(H^G)$ by Lemma 2.6.9. So the claim follows as $u^L \ltimes_{\mathrm{lt}} m$ is isometric by injectivity. ■

Since left-translation and right-translation commute, it follows from Lemma 2.3.9 that for each $\theta \in G$ we have an automorphism $G \ltimes \mathrm{rt}_\theta$ of the crossed product $G \ltimes_{\mathrm{lt}} C_0(G)$. The following lemma will be important later.

Lemma 2.6.11. *If G is a locally compact Hausdorff group, then the diagram*

$$\begin{tikzcd} H^G \arrow[r, "(u^L \ltimes_{\mathrm{lt}} m)(f)"] \arrow[d, "u^R(\theta)"'] & H^G \arrow[d, "u^R(\theta)"] \\ H^G \arrow[r, "(u^L \ltimes_{\mathrm{lt}} m)((G \ltimes \mathrm{rt}_\theta)(f))"'] & H^G \end{tikzcd}$$

commutes for all $\theta \in G$ and $f \in G \ltimes_{\mathrm{lt}} C_0(G)$.

Proof. It suffices to prove the statement for $f \in C_{\mathrm{cpt}}(G, C_0(G))$. We directly compute

$$\begin{aligned} ((u^L \ltimes_{\mathrm{lt}} m)(\mathrm{rt}_\theta \circ f) u^R(\theta)\xi)(t) &= \int_G \mathrm{rt}_\theta(f(s))(t)(u^R(\theta)\xi)(s^{-1}t)\ \mathrm{d}_\mu s \\ &= \Delta(\theta)^{1/2} \int_G f(s)(t\theta)\xi(s^{-1}t\theta)\ \mathrm{d}_\mu s \\ &= \Delta(\theta)^{1/2}((u^L \ltimes_{\mathrm{lt}} m)(f)\xi)(t\theta) \\ &= (u^R(\theta)(u^L \ltimes_{\mathrm{lt}} m)(f)\xi)(t) \end{aligned}$$

for all $\xi \in C_{\mathrm{cpt}}(G)$ and $t \in G$. ■

Chapter 3

Takai Duality

In this chapter our goal is to recover a C*-action $\alpha\colon G \curvearrowright A$ from its crossed product $G \ltimes_\alpha A$ in the case where G is abelian. The main result of this thesis, Takai duality, tells us that this can be accomplished up to tensoring with another C*-action. In essence, the idea is to study the so-called dual C*-action $\alpha^\sharp\colon G^\sharp \curvearrowright G \ltimes_\alpha A$, where $G^\sharp$ is the dual group.

In Section 3.1 we cover some basic material on abstract harmonic analysis on abelian locally compact Hausdorff groups G. We give the definition of the dual group $G^\sharp$ and without proof we state the celebrated Pontryagin duality theorem, which basically says that $G^{\sharp\sharp} = G$. After that we discuss the Fourier transform F.

The primary purpose of Section 3.2 and Section 3.3 is to discuss some miscellanuous topics that we need for the (proof of the) main result. In the former we analyze the structure of an iterated crossed product, that is, a C*-algebra of the form $G' \ltimes_{\alpha'} (G \ltimes_\alpha A)$ for C*-actions $\alpha\colon G \curvearrowright A$ and $\alpha'\colon G' \curvearrowright (G \ltimes_\alpha A)$. In the latter we introduce the maximal tensor product $A \otimes_{\max} B$ of two C*-algebras A and B and explain what it means to tensor two C*-actions $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$.

In Section 3.4 we formally introduce the dual C*-action $\alpha^\sharp\colon G^\sharp \curvearrowright G \ltimes_\alpha A$ and talk through some of its properties. Finally, in Section 3.5 we state and prove the Takai duality theorem, which says that the double dual C*-action $\alpha^{\sharp\sharp}\colon G \curvearrowright G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$ is conjugate to $\alpha\colon G \curvearrowright A$ up to tensoring with another C*-action.

3.1 Pontryagin Duality and the Fourier Transform

In this section we discuss some important results from abstract harmonic analysis. The main references for this section are [HR63] and [HR70]. A look at [Fol95] and [DE14] is also certainly recommended. Throughout this section G denotes an abelian locally compact Hausdorff group.

Definition 3.1.1. Let G be an abelian locally compact Hausdorff group. The *dual group* of G, denoted by $G^\sharp$, is the set of all topological group homomorphisms from G to the unit circle $\mathbb{T}$. We endow $G^\sharp$ with topology of uniform convergence on compact subsets.

The neutral element of $G^\sharp$ will be denoted by $e^\sharp$.

Lemma 3.1.2 (cf. [HR63, Theorem 23.15]). *If G is an abelian locally compact Hausdorff group, then its dual group $G^\sharp$ is also an abelian locally compact Hausdorff group.*

In particular, Lemma 3.1.2 implies that $G^\sharp$ admits a Haar integral, which we denote by $\mu^\sharp$.

The following celebrated result, known as the Pontryagin duality theorem, states that the double dual group $G^{\sharp\sharp}$ can be identified with G in a natural way.

Theorem 3.1.3 (cf. [HR63, Theorem 24.8]). *Let G be an abelian locally compact Hausdorff group. The natural map from G to $G^{\sharp\sharp}$, assigning to every point in G its corresponding evaluation map from $G^\sharp$ to $\mathbb{T}$, is a topological group isomorphism.*

In notation we shall never differentiate between an element $s \in G$ and its associated evaluation map $s\colon G^\sharp \to \mathbb{T}$. In particular, since a group homomorphism from $G^\sharp$ to $U(\mathbb{C}) = \mathbb{T}$ is strongly continuous if and only if it is continuous, we obtain a canonical one-to-one correspondence between unitary representations of $G^\sharp$ on $\mathbb{C}$ and elements of G.

We shall write $C_{\mathrm{cpt}}(G^\sharp, \mathbb{C})$ for $C_{\mathrm{cpt}}(G^\sharp)$ to stress that we view it as a subalgebra of $G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$, where $\mathrm{id}\colon G^\sharp \curvearrowright \mathbb{C}$ is the trivial C*-action. Given any $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, \mathbb{C})$, we can define a function $F(f^\sharp)\colon G \to \mathbb{C}$ via

$$F(f^\sharp)(s) = \int_{G^\sharp} s^\sharp(s) f^\sharp(s^\sharp)\ \mathrm{d}_{\mu^\sharp} s^\sharp$$

for $s \in G$. By the next lemma we obtain a well-defined linear operator $F\colon C_{\mathrm{cpt}}(G^\sharp, \mathbb{C}) \to C_0(G)$.

Lemma 3.1.4 (cf. [HR70, Theorem 31.5]). *Let G be an abelian locally compact Hausdorff group. For each $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, \mathbb{C})$ one has $F(f^\sharp) \in C_0(G)$.*

Let us show that that F is isometric. Let $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, \mathbb{C})$ be arbitrary. Let 1 denote the unique representation of $\mathbb{C}$ on $\mathbb{C}$. Note that for every $s \in G$ the pair $(s, 1)$ is a covariant representation of id on $\mathbb{C}$. By definition of F we have $F(f^\sharp)(s) = (s \ltimes_{\mathrm{id}} 1)(f^\sharp)$ and therefore

$$\|F(f^\sharp)\|_\infty = \sup_{s\in G} |(s \ltimes_{\mathrm{id}} 1)(f^\sharp)| \leq \|f^\sharp\|_{\mathrm{env}}.$$

Hence it suffices to show that $|(s \ltimes_{\mathrm{id}} 1)(f^\sharp)| = \|f^\sharp\|_{\mathrm{env}}$ for some $s \in G$. For this we may assume that $f^\sharp \neq 0$, otherwise we are done. By [Mur90, Theorem 5.1.12] there exists an irreducible representation P of $G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$ on some Hilbert space $H \neq \{0\}$ such that $\|P(f^\sharp)\| = \|f^\sharp\|_{\mathrm{env}}$. Since $G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$ is commutative, [Mur90, Theorem 5.1.5] implies that $H = \mathbb{C}$. So by Theorem 2.4.16 there is an $s \in G$ such that $s \ltimes_{\mathrm{id}} 1 = P$. We conclude that $\|F(f^\sharp)\|_\infty = \|f^\sharp\|_{\mathrm{env}}$. Consequently, F uniquely extends to an isometric linear operator $F\colon G^\sharp \ltimes_{\mathrm{id}} \mathbb{C} \to C_0(G)$.

Definition 3.1.5. If G is an abelian locally compact Hausdorff group, then we call F the *Fourier transform.*

Theorem 3.1.6. *Let G be an abelian locally compact Hausdorff group. The Fourier transform F is an isomorphism from $G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$ to $C_0(G)$.*

Proof. Recall that for every element $s \in G$ the integrated form $s \ltimes_{\mathrm{id}} 1$ is a homomorphism. This immediately implies that F is a homomorphism as $F(f^\sharp)(s) = (s \ltimes_{\mathrm{id}} 1)(f^\sharp)$ for $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, \mathbb{C})$ and $s \in G$.

It remains to show that F is surjective. The subalgebra $F(C_{\mathrm{cpt}}(G^\sharp, \mathbb{C}))$ of $C_0(G)$ clearly satisfies the criteria of the Stone-Weierstrass theorem. Thus $F(C_{\mathrm{cpt}}(G^\sharp, \mathbb{C}))$ is dense in $C_0(G)$ and therefore $F(G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) = C_0(G)$ as F is isometric. ∎

A corollary of Theorem 3.1.6 is that there is a one-to-one correspondence between unitary representations of $G^\sharp$ and representations of $C_0(G)$.

Another interesting consequence of Theorem 3.1.6 is that we obtain pullback C*-actions $F^{-1}\mathrm{lt}\colon G \curvearrowright G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$ and $F^{-1}\mathrm{rt}\colon G \curvearrowright G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$, where $\mathrm{lt}\colon G \curvearrowright C_0(G)$ and $\mathrm{rt}\colon G \curvearrowright C_0(G)$ denote the left-translation and right-translation, respectively.

Lemma 3.1.7. *Let G be an abelian locally compact Hausdorff group. Evaluated at $s \in G$, the pullback C*-action $F^{-1}\mathrm{lt}$ (resp. $F^{-1}\mathrm{rt}$) of the left-transation lt (resp. right-translation rt) under the Fourier transform F is given by the formula $(F^{-1}\mathrm{lt})_s(f^\sharp)(s^\sharp) = s^{\sharp-1}(s) f^\sharp(s^\sharp)$ (resp. $(F^{-1}\mathrm{rt})_s(f^\sharp)(s^\sharp) = s^\sharp(s) f^\sharp(s^\sharp)$) for all $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, \mathbb{C})$ and $s^\sharp \in G^\sharp$.*

Proof. We only prove the statement for the left-translation. For this it suffices to observe that

$$\mathrm{lt}_s(F(f^\sharp))(t) = F(f^\sharp)(s^{-1}t) = \int_{G^\sharp} s^\sharp(s^{-1}t) f^\sharp(s^\sharp)\ \mathrm{d}_{\mu^\sharp} s^\sharp = \int_{G^\sharp} s^\sharp(t) s^{\sharp -1}(s) f^\sharp(s^\sharp)\ \mathrm{d}_{\mu^\sharp} s^\sharp$$

for all $t \in G$. ■

3.2 Iterated Crossed Products

Suppose that $\alpha\colon G \curvearrowright A$ and $\alpha'\colon G' \curvearrowright G \ltimes_\alpha A$ are C*-actions. In this section we study the structure of the iterated crossed product $G' \ltimes_{\alpha'} (G \ltimes_\alpha A)$. This material is taken from [Wil07].

There is a canonical one-to-one correspondence between functions $f'\colon G' \to G \ltimes_\alpha A$ with $f'(G') \subset C_{\mathrm{cpt}}(G, A)$ and functions $f'\colon G' \times G \to A$ for which

$$G \ni s \mapsto f'(s', s) \in A$$

is contained in $C_{\mathrm{cpt}}(G, A)$ for each $s' \in G'$. Indeed, we identify $f'(s')(s) = f'(s', s)$ for all $s' \in G'$ and $s \in G$. Throughout this section we use the notation $\sigma(f')$ for the support of f' viewed as a function from $G' \times G$ to A.

Lemma 3.2.1. *If $\alpha\colon G \curvearrowright A$ is a C*-action and G' a locally compact Hausdorff group, then $C_{\mathrm{cpt}}(G' \times G, A)$ is a linear subspace of $C_{\mathrm{cpt}}(G', G \ltimes_\alpha A)$ and, furthermore, it is dense with respect to the integral norm.*

Proof. It is clear that $f' \in C_{\mathrm{cpt}}(G' \times G, A)$ can be viewed as a function $f'\colon G' \to G \ltimes_\alpha A$ with $f'(G') \subset C_{\mathrm{cpt}}(G, A)$. We need to show that $f' \in C_{\mathrm{cpt}}(G', G \ltimes_\alpha A)$. Let $\varepsilon > 0$ be arbitrary. Fix compact subsets $K' \subset G'$ and $K \subset G$ such that $\sigma(f') \subset K' \times K$. Define $r := \mu(K) + 1$ and use left-uniform continuity of f' as a function from $G' \times G$ to A to find open neighbourhoods $U' \subset G'$ and $U \subset G$ of the neutral elements e' and e of G' and G, respectively, such that $(s', s)^{-1}(t', t) \in U' \times U$ implies $\|f'(s', s) - f'(t', t)\| < \varepsilon/2r$ for all $(s', s), (t', t) \in G' \times G$. Hence $s'^{-1}t' \in U$ implies

$$\|f'(s') - f'(t')\|_{\mathrm{int}} = \int_G \|f'(s', s) - f'(t', s)\|\ \mathrm{d}_\mu s \le \mu(K)\frac{\varepsilon}{2r} < \varepsilon$$

and therefore $\|f'(s') - f'(t')\|_{\mathrm{env}} < \varepsilon$ for all $s', t' \in G'$. Thus f' as a function from G' to $G \ltimes_\alpha A$ is left-uniformly continuous and therefore continuous. Moreover, it is compactly supported as it vanishes outside of K'.

Density is a straightforward application of Lemma 1.3.5; the details are left to the reader. ■

For further analysis of $G \ltimes_{\alpha'} (G \ltimes_\alpha A)$ we need that α' is related to α in a certain way.

Definition 3.2.2. Let $\alpha\colon G \curvearrowright A$ and $\alpha'\colon G' \curvearrowright G \ltimes_\alpha A$ be two C*-actions. We say that α' is *compatible* with α if for each $s' \in G'$ the algebra $C_{\mathrm{cpt}}(G, A)$ is invariant under $\alpha'_{s'}$ and for each $g' \in C_{\mathrm{cpt}}(G' \times G, A)$ the function

$$G' \times G \times G' \ni (t', t, s') \mapsto \alpha'_{s'}(g'(t'))(t) \in A$$

is continuous and vanishes outside of $L' \times L \times G'$ for some compact subsets $L' \subset G'$ and $L \subset G$.

Let us argue that $C_{\mathrm{cpt}}(G' \times G, A)$ is a subalgebra of the algebra $C_{\mathrm{cpt}}(G', G \ltimes_\alpha A)$ in the special case where α' is compatible with α.

- For any covariant representation (u, ρ) of α on H the product $f'g'\colon G' \to G \ltimes_\alpha A$ of two elements $f', g' \in C_{\mathrm{cpt}}(G' \times G, A)$ satisfies

$$\begin{aligned}(u \ltimes_\alpha \rho)((f'g')(t'))x &= (u \ltimes_\alpha \rho)\bigg(\int_{G'} f'(s')\alpha'_{s'}(g'(s'^{-1}t'))\, \mathrm{d}_{\mu'}s'\bigg)x \\ &= \int_{G'} (u \ltimes_\alpha \rho)(f'(s')\alpha'_{s'}(g'(s'^{-1}t')))x\, \mathrm{d}_{\mu'}s' \\ &= \int_{G'}\int_G \rho((f'(s')\alpha'_{s'}(g'(s'^{-1}t')))(t))u(t)x\, \mathrm{d}_\mu t\, \mathrm{d}_{\mu'}s' \\ &= \int_G \rho\bigg(\int_{G'} (f'(s')\alpha'_{s'}(g'(s'^{-1}t')))(t)\, \mathrm{d}_{\mu'}s'\bigg)u(t)x\, \mathrm{d}_\mu t \\ &= \int_G \rho\bigg(\int_{G'}\int_G f'(s',s)\alpha_s(\alpha'_{s'}(g'(s'^{-1}t'))(s^{-1}t))\, \mathrm{d}_\mu s\, \mathrm{d}_{\mu'}s'\bigg)u(t)x\, \mathrm{d}_\mu t\end{aligned}$$

  for all $t' \in G'$ and $x \in H$ by compatibility. Since the function

$$G' \times G \times G' \times G \ni (t', t, s', s) \mapsto f'(s', s)\alpha_s(\alpha'_{s'}(g'(s'^{-1}t'))(s^{-1}t)) \in A$$

  belongs to $C_{\mathrm{cpt}}(G' \times G \times G' \times G, A)$ by compatibility, we can use Lemma 1.3.8 and integrate over the third and fourth variable to conclude that $f'g' \in C_{\mathrm{cpt}}(G' \times G, A)$ with formula

$$(f'g')(t', t) = \int_{G'}\int_G f'(s', s)\alpha_s(\alpha'_{s'}(g'(s'^{-1}t'))(s^{-1}t))\, \mathrm{d}_\mu s\, \mathrm{d}_{\mu'}s'$$

  for all $(t', t) \in G' \times G$.

- If $h'^*\colon G' \to G \ltimes_\alpha A$ is the adjoint of an element $h' \in C_{\mathrm{cpt}}(G' \times G, A)$, then compatibility implies that $h'^* \in C_{\mathrm{cpt}}(G' \times G, A)$ with formula $h'^*(t', t) = \Delta'(t'^{-1})\alpha'_{t'}(h'(t'^{-1})^*)(t)$ for all $(t', t) \in G' \times G$.

Lemma 3.2.3. *Let $\alpha\colon G \curvearrowright A$ and $\alpha'\colon G' \curvearrowright G \ltimes_\alpha A$ be C*-actions. If α' is compatible with α, then $C_{\mathrm{cpt}}(G' \times G, A)$ is a dense subalgebra of the C*-algebra $G' \ltimes_{\alpha'} (G \ltimes_\alpha A)$.*

Proof. This is an immediate consequence of Lemma 3.2.1 and the fact that the integral norm dominates the enveloping norm on $C_{\mathrm{cpt}}(G', G \ltimes_\alpha A)$. ∎

3.3 Maximal Tensor Products

Given two C*-algebras A and B, their tensor product $A \otimes B$ forms an algebra in the natural way. There is always at least one C*-norm on $A \otimes B$ by [Mur90, Theorem 6.3.3]. However, it is a fact of life that there can be more than one C*-norm on $A \otimes B$. In other words, the algebra $A \otimes B$ may have more than one C*-completion, that is, a C*-algebra that contains $A \otimes B$ as a dense subalgebra.

In this section we introduce the maximal tensor product $A \otimes_{\max} B$, which is a specific C*-completion of $A \otimes B$. It is maximal in the sense that its C*-norm dominates all other C*-norms on $A \otimes B$. After that we discuss that any two C*-actions $\alpha\colon G \curvearrowright A$ and $\beta\colon G \curvearrowright B$ give rise to another C*-action $\alpha \otimes \beta\colon G \curvearrowright A \otimes_{\max} B$. We focus on the case where $\beta = \mathrm{id}$ is trivial and prove that the C*-algebras $G \ltimes_{\alpha\otimes\mathrm{id}} (A \otimes_{\max} B)$ and $(G \ltimes_\alpha A) \otimes_{\max} B$ can be identified.

The following definition will be essential.

Definition 3.3.1. Let A and B be two C*-algebras. Suppose that ρ and σ are representations of A and B, respectively. We say that ρ and σ *commute* if they represent on the same Hilbert space and have commuting ranges.

If W is any C*-completion of $A \otimes B$, then it follows from [Mur90, Lemma 6.3.4] that the pure tensor $a \otimes b \in W$ depends continuously on $a \in A$ for fixed $b \in B$ and continuously on $b \in B$ for fixed $a \in A$. This fact will be used in the proof of the following lemma.

Lemma 3.3.2. *Let A and B be two C*-algebras. If W is a C*-completion of $A \otimes B$, then for every representation π of W on H there exist unique commuting representations ρ and σ of A and B on H such that $\rho(a)\sigma(b) = \pi(a \otimes b)$ for all $a \in A$ and $b \in B$.*

Proof. Let E denote the span of $\pi(A \otimes B)H$ in H. If ρ and σ exist, then they are necessarily unique by non-degeneracy of π. Indeed, one readily verifies that for each $a \in A$ and $b \in B$ the bounded linear operators $\rho(a)$ and $\sigma(b)$ are fully determined by π on the dense linear subspace E of H. So only have to worry about existence of ρ and σ.

For each $b \in B$ we shall construct a bounded linear operator $\sigma(b)\colon H \to H$. Let $(1_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. Choose $r > 0$ such that $\|a \otimes b\| \leq r\|a\|$ for all $a \in A$. If $a_{k,l} \in A$, $b_{k,l} \in B$ and $x_k \in H$, then for each $\varepsilon > 0$ there is a sufficiently large $\lambda \in \Lambda$ such that

$$\begin{aligned}\Big\|\sum_k \pi\Big(\sum_l a_{k,l} \otimes bb_{k,l}\Big)x_k\Big\| - \varepsilon &< \Big\|\sum_k \pi\Big(\sum_l 1_\lambda a_{k,l} \otimes bb_{k,l}\Big)x_k\Big\| \\ &= \Big\|\pi(1_\lambda \otimes b)\sum_k \pi\Big(\sum_l a_{k,l} \otimes b_{k,l}\Big)x_k\Big\| \\ &\leq r\Big\|\sum_k \pi\Big(\sum_l a_{k,l} \otimes b_{k,l}\Big)x_k\Big\|.\end{aligned}$$

So the linear operator $\sigma(b)\colon E \to H$ given by

$$\sigma(b)\sum_k \pi\Big(\sum_l a_{k,l} \otimes b_{k,l}\Big)x_k := \sum_k \pi\Big(\sum_l a_{k,l} \otimes bb_{k,l}\Big)x_k$$

for $a_{k,l} \in A$, $b_{k,l} \in B$ and $x_k \in H$ is well-defined and bounded. In particular, since E is a dense linear subspace of H, it follows that $\sigma(b)$ uniquely extends to a bounded linear operator $\sigma(b)\colon H \to H$. Elementary calculations show that we obtain a homomorphism $\sigma\colon B \to B(H)$. Non-degeneracy of σ quickly follows from non-degeneracy of π. Thus σ is a representation of B on H. Of course the representation ρ of A on H is defined similarly. It is clear that ρ and σ commute and that $\rho(a)\sigma(b) = \pi(a \otimes b)$ for all $a \in A$ and $b \in B$. ■

Let ρ and σ be representations of A and B on the same Hilbert space H. If ρ and σ have commuting ranges, then the unique linear operator $\rho \times \sigma\colon A \otimes B \to B(H)$ that acts on pure tensors via $(\rho \times \sigma)(a \otimes b) = \rho(a)\sigma(b)$ for all $a \in A$ and $b \in B$ is a homomorphism.

We can always find ρ and σ such that $\rho \times \sigma$ maps non-zero tensors in $A \otimes B$ to non-zero operators in $B(H)$. Indeed, choose a faithful representation π of some C*-completion W of $A \otimes B$ and apply Lemma 3.3.2. In particular, the following definition presents a well-defined norm on $A \otimes B$.

Definition 3.3.3. Let A and B be two C*-algebras. The value

$$\|w\|_{\max} := \sup_{(\rho,\sigma)} \|(\rho \times \sigma)(w)\|,$$

where the supremum runs over all pairs (ρ, σ) of commuting representations ρ and σ of A and B, is called the *maximal norm* of $w \in A \otimes B$. The *maximal tensor product*, denoted by $A \otimes_{\max} B$, is the completion of $A \otimes B$ with respect to the maximal norm.

Observe that $\|a \otimes b\|_{\max} \leq \|a\|\|b\|$ for all $a \in A$ and $b \in B$. Moreover, the maximal norm dominates any other C*-norm on $A \otimes B$, thereby justifying the terminology.

Theorem 3.3.4. *If A and B are two C*-algebras, then their maximal tensor product $A \otimes_{\max} B$ is a C*-completion of $A \otimes B$.*

Proof. We omit the trivial proof. ■

It turns out that there is a one-to-one correspondence between commuting representations of A and B and representations of $A \otimes_{\max} B$. Given two commuting representations ρ and σ of A and B, we see that $\|(\rho \times \sigma)(w)\| \leq \|w\|_{\max}$ for all $w \in A \otimes B$ by definition of the maximal norm. Hence $\rho \times \sigma$ can be uniquely extended to a homomorphism $\rho \times \sigma \colon A \otimes_{\max} B \to B(H)$.

Theorem 3.3.5. *Let A and B be two C*-algebras. If ρ and σ are commuting representations of A and B on H, then $\rho \times \sigma$ is a representation of $A \otimes_{\max} B$ on H. Conversely, for every representation π of $A \otimes_{\max} B$ on H there are unique commuting representations ρ and σ of A and B on H such that $\rho \times \sigma = \pi$.*

Proof. To prove that $\rho \times \sigma$ is a representation of $A \otimes_{\max} B$ on H, we only have to show that $(\rho \times \sigma)(A \otimes_{\max} B)H$ is dense in H. Let $z \in H$ and $\varepsilon > 0$ be given. First use non-degeneracy of σ to find $\tilde{b} \in B$ and $\tilde{y} \in H$ such that $\|\sigma(\tilde{b})\tilde{y} - z\| < \varepsilon/2$ and then use non-degeneracy of ρ to find $\tilde{a} \in A$ and $\tilde{x} \in H$ such that $\|\tilde{b}\|\|\rho(\tilde{a})\tilde{x} - \tilde{y}\| < \varepsilon/2$. For $\tilde{w} := \tilde{a} \otimes \tilde{b}$ and $\tilde{z} := \tilde{x}$ we now have

$$\|(\rho \times \sigma)(\tilde{w})\tilde{z} - z\| = \|\sigma(\tilde{b})\rho(\tilde{a})\tilde{x} - z\| \leq \|\tilde{b}\|\|\rho(\tilde{a})\tilde{x} - \tilde{y}\| + \|\sigma(\tilde{b})\tilde{y} - z\| < \varepsilon$$

and density follows.

Finally, an application of Lemma 3.3.2 tells us that for every representation π of $A \otimes_{\max} B$ on H there are unique commuting representations ρ and σ of A and B on H such that $\rho \times \sigma = \pi$. ■

Let $\alpha \colon G \curvearrowright A$ and $\beta \colon G \curvearrowright B$ be two C*-actions. For each $s \in G$ the universal property of the tensor product implies the existence of a unique linear operator $(\alpha \otimes \beta)_s \colon A \otimes B \to A \otimes_{\max} B$ that acts on pure tensors via $(\alpha \otimes \beta)_s(a \otimes b) = \alpha_s(a) \otimes \beta_s(b)$ for all $a \in A$ and $b \in B$.

Let us briefly show that $(\alpha \otimes \beta)_s$ is isometric. If ρ and σ are commuting representations of A and B on H, then the same is true for $\rho \circ \alpha_s^{-1}$ and $\sigma \circ \beta_s^{-1}$. Note that $((\rho \circ \alpha_s^{-1}) \times (\sigma \circ \beta_s^{-1})) \circ (\alpha \otimes \beta)_s = \rho \times \sigma$. From this it clearly follows that $\|(\alpha \otimes \beta)_s(w)\|_{\max} = \|w\|_{\max}$ for all $w \in A \otimes B$. Thus $(\alpha \otimes \beta)_s$ extends to an isometric linear operator $(\alpha \otimes \beta)_s \colon A \otimes_{\max} B \to A \otimes_{\max} B$ in a unique way.

In fact, $(\alpha \otimes \beta)_s$ is an automorphism of $A \otimes_{\max} B$. Indeed, $(\alpha \otimes \beta)_s$ is clearly an isometric homomorphism mapping the dense subalgebra $A \otimes B$ of $A \otimes_{\max} B$ onto itself. Hence we obtain a well-defined map $\alpha \otimes \beta \colon G \to \mathrm{Aut}(A \otimes_{\max} B)$.

Lemma 3.3.6. *Let $\alpha \colon G \curvearrowright A$ and $\beta \colon G \curvearrowright B$ be two C*-actions. The map $\alpha \otimes \beta$ is a topological group homomorphism from G to $\mathrm{Aut}(A \otimes_{\max} B)$.*

Proof. We begin with continuity of $\alpha \otimes \beta$. Given $t \in G$ and $w \in A \otimes_{\max} B$, we must find for every $\varepsilon > 0$ a neighbourhood $N \subset G$ of t such that $s \in N$ implies $\|(\alpha \otimes \beta)_s(w) - (\alpha \otimes \beta)_t(w)\|_{\max} < \varepsilon$. We may assume that $w \in A \otimes B$. Pick $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ such that the sum of $a_1 \otimes b_1, \ldots, a_n \otimes b_n$ equals w. Now choose a neighbourhood $N \subset G$ of t such that $s \in N$ implies $\|\alpha_s(a_j) - \alpha_t(a_j)\|\|b_j\| < \varepsilon/2n$ and $\|a_j\|\|\beta_s(b_j) - \beta_t(b_j)\| < \varepsilon/2n$ for all $j = 1, \ldots, n$. So for $s \in N$ one has

$$\begin{aligned}
\|(\alpha \otimes \beta)_s(w) - (\alpha \otimes \beta)_t(w)\|_{\max} &\leq \sum_{j=1}^{n} \|\alpha_s(a_j) \otimes \beta_s(b_j) - \alpha_t(a_j) \otimes \beta_t(b_j)\|_{\max} \\
&\leq \sum_{j=1}^{n} \|(\alpha_s(a_j) - \alpha_t(a_j)) \otimes \beta_s(b_j)\|_{\max} \\
&\qquad + \sum_{j=1}^{n} \|\alpha_t(a_j) \otimes (\beta_s(b_j) - \beta_t(b_j))\|_{\max} \\
&\leq \sum_{j=1}^{n} \|\alpha_s(a_j) - \alpha_t(a_j)\|\|b_j\| + \sum_{j=1}^{n} \|a_j\|\|\beta_s(b_j) - \beta_t(b_j)\| < \varepsilon
\end{aligned}$$

as desired.

It is easy to check that $\alpha \otimes \beta$ is a group homomorphism. Indeed, this is a consequence of the equalities $\alpha_{st} = \alpha_s\alpha_t$ and $\beta_{st} = \beta_s\beta_t$ for all $s, t \in G$. ■

Now assume that $\beta = \mathrm{id}$ is trivial, that is, $\mathrm{id}_s(b) = b$ for $s \in G$ and $b \in B$. It turns out that we can identify the C*-algebras $G \ltimes_{\alpha\otimes\mathrm{id}} (A \otimes_{\max} B)$ and $(G \ltimes_\alpha A) \otimes_{\max} B$. To see that this is true, we need the following key lemma.

Lemma 3.3.7. *Let $\alpha\colon G \curvearrowright A$ be any C*-action and $\mathrm{id}\colon G \curvearrowright B$ a trivial C*-action. Let H be a Hilbert space, u a unitary representation of G on H, ρ a representation of A on H and σ a representation of B on H.*

(i) The maps ρ and σ have commuting ranges and $(u, \rho \times \sigma)$ is covariant for $\alpha \otimes \mathrm{id}$ if and only if (u, ρ) is covariant for α and the maps $u \ltimes_\alpha \rho$ and σ have commuting ranges.

(ii) If one of the equivalent conditions in part (i) is satisfied, then the maps $u \ltimes_{\alpha\otimes\mathrm{id}} (\rho \times \sigma)$ and $(u \ltimes_\alpha \rho) \times \sigma$ agree on $C_{\mathrm{cpt}}(G) \otimes A \otimes B$.

Proof. We begin with part (i). To prove the forward implication, assume that ρ and σ have commuting ranges and that $(u, \rho \times \sigma)$ is covariant for $\alpha \otimes \mathrm{id}$. Note that

$$\begin{aligned}
\rho(\alpha_\theta(a_1))u(\theta)(\rho \times \sigma)(a_2 \otimes b)x &= \rho(\alpha_\theta(a_1))(\rho \times \sigma)((\alpha \otimes \mathrm{id})_\theta(a_2 \otimes b))u(\theta)x \\
&= (\rho \times \sigma)((\alpha \otimes \mathrm{id})_\theta(a_1a_2 \otimes b))u(\theta)x \\
&= u(\theta)(\rho \times \sigma)(a_1a_2 \otimes b)x \\
&= u(\theta)\rho(a_1)(\rho \times \sigma)(a_2 \otimes b)x
\end{aligned}$$

for all $\theta \in G$, $a_1, a_2 \in A$, $b \in B$ and $x \in H$. Also note that

$$\begin{aligned}
(u \ltimes_\alpha \rho)(f)\sigma(b_1)(\rho \times \sigma)(a \otimes b_2)x &= \int_G \rho(f(s))u(s)\sigma(b_1)(\rho \times \sigma)(a \otimes b_2)x \, \mathrm{d}_\mu s \\
&= \int_G \rho(f(s))u(s)(\rho \times \sigma)(a \otimes b_1b_2)x \, \mathrm{d}_\mu s \\
&= \int_G \rho(f(s))(\rho \times \sigma)((\alpha \otimes \mathrm{id})_s(a \otimes b_1b_2))u(s)x \, \mathrm{d}_\mu s \\
&= \sigma(b_1) \int_G \rho(f(s))(\rho \times \sigma)((\alpha \otimes \mathrm{id})_s(a \otimes b_2))u(s)x \, \mathrm{d}_\mu s \\
&= \sigma(b_1) \int_G \rho(f(s))u(s)(\rho \times \sigma)(a \otimes b_2)x \, \mathrm{d}_\mu s \\
&= \sigma(b_1)(u \ltimes_\alpha \rho)(f)(\rho \times \sigma)(a \otimes b_2)x
\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, A)$, $a \in A$, $b_1, b_2 \in B$ and $x \in H$. Hence non-degeneracy of $\rho \times \sigma$ implies that (u, ρ) is covariant for α and that $u \ltimes_\alpha \rho$ and σ have commuting ranges.

To prove the converse, assume that (u, ρ) is covariant for α and that $u \ltimes_\alpha \rho$ and σ have commuting ranges. Let (j_G, j_A) be the canonical covariant multiplier representation of α. Recall from Lemma 2.4.14 that (u, ρ) factors through (j_G, j_A) via $(u \ltimes_\alpha \rho)^+$. So we have

$$\begin{aligned}
\rho(a)\sigma(b)(u \ltimes_\alpha \rho)(f)x &= \rho(a)(u \ltimes_\alpha \rho)(f)\sigma(b)x \\
&= (u \ltimes_\alpha \rho)(j_A(a)f)\sigma(b)x \\
&= \sigma(b)(u \ltimes_\alpha \rho)(j_A(a)f)x \\
&= \sigma(b)\rho(a)(u \ltimes_\alpha \rho)(f)x
\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, A)$, $a \in A$, $b \in B$ and $x \in H$. In a similar fashion we find

$$\begin{aligned}
(\rho \times \sigma)((\alpha \otimes \mathrm{id})_\theta(a \otimes b))u(\theta)(u \ltimes_\alpha \rho)(f)x &= \rho(\alpha_\theta(a))\sigma(b)(u \ltimes_\alpha \rho)(j_G(\theta)f)x \\
&= \rho(\alpha_\theta(a))(u \ltimes_\alpha \rho)(j_G(\theta)f)\sigma(b)x \\
&= \rho(\alpha_\theta(a))u(\theta)(u \ltimes_\alpha \rho)(f)\sigma(b)x \\
&= u(\theta)\rho(a)\sigma(b)(u \ltimes_\alpha \rho)(f)x \\
&= u(\theta)(\rho \times \sigma)(a \otimes b)(u \ltimes_\alpha \rho)(f)x
\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, A)$, $\theta \in G$, $a \in A$, $b \in B$ and $x \in H$. Hence non-degeneracy of $u \ltimes_\alpha \rho$ implies that ρ and σ have commuting ranges and that $(u, \rho \times \sigma)$ is covariant for $\alpha \otimes \mathrm{id}$.

Part (ii) follows from the computation

$$\begin{aligned}
((u \ltimes_\alpha \rho) \times \sigma)(\varphi \otimes a \otimes b)x &= \sigma(b)(u \ltimes_\alpha \rho)(\varphi \otimes a)x \\
&= \int_G \varphi(s)\sigma(b)\rho(a)u(s)x \, \mathrm{d}_\mu s \\
&= \int_G \varphi(s)(\rho \times \sigma)(a \otimes b)u(s)x \, \mathrm{d}_\mu s \\
&= (u \ltimes_{\alpha \otimes \mathrm{id}} (\rho \times \sigma))(\varphi \otimes a \otimes b)x
\end{aligned}$$

for all $\varphi \in C_{\mathrm{cpt}}(G)$, $a \in A$, $b \in B$ and $x \in H$. ■

It follows from Lemma 3.3.7 that the inclusion of $C_{\mathrm{cpt}}(G) \otimes A \otimes B$ into $(G \ltimes_\alpha A) \otimes_{\max} B$ extends to an isometric homomorphism from $G \ltimes_{\alpha \otimes \mathrm{id}} (A \otimes_{\max} B)$ to $(G \ltimes_\alpha A) \otimes_{\max} B$. So, because $C_{\mathrm{cpt}}(G) \otimes A \otimes B$ is dense in both $G \ltimes_{\alpha \otimes \mathrm{id}} (A \otimes_{\max} B)$ and $(G \ltimes_\alpha A) \otimes_{\max} B$ by Lemma 1.3.5, we can identify $G \ltimes_{\alpha \otimes \mathrm{id}} (A \otimes_{\max} B) = (G \ltimes_\alpha A) \otimes_{\max} B$.

3.4 Dual C*-Actions

Let $\alpha \colon G \curvearrowright A$ be a C*-action and suppose that G is abelian. It turns out that the dual group $G^\sharp$ parametrizes a family of automorphisms of the crossed product $G \ltimes_\alpha A$. Let $s^\sharp \in G^\sharp$ be arbitrary. Consider the linear operator $\alpha^\sharp_{s^\sharp} \colon C_{\mathrm{cpt}}(G, A) \to G \ltimes_\alpha A$ given by $\alpha^\sharp_{s^\sharp}(f)(s) := s^\sharp(s)f(s)$ for all $f \in C_{\mathrm{cpt}}(G, A)$ and $s \in G$.

Let us briefly show that $\alpha^\sharp_{s^\sharp}$ is isometric with respect to the enveloping norm. If $u \colon G \to U(H)$ is a unitary representation, then every $\theta^\sharp \in G^\sharp$ induces another unitary representation $u_{\theta^\sharp} \colon G \to U(H)$ via $u_{\theta^\sharp}(\theta) := \theta^\sharp(\theta)u(\theta)$ for all $\theta \in G$. It is trivial that $u = u_{e^\sharp}$. Moreover, if (u, ρ) is a covariant representation of α, then $(u_{\theta^\sharp}, \rho)$ is also a covariant representation of α and, furthermore, $(u_{\theta^\sharp} \ltimes_\alpha \rho) \circ \alpha^\sharp_{s^\sharp} = u_{s^\sharp\theta^\sharp} \ltimes_\alpha \rho$. So for $f \in C_{\mathrm{cpt}}(G, A)$ one finds $\|\alpha^\sharp_{s^\sharp}(f)\|_{\mathrm{env}} = \|f\|_{\mathrm{env}}$. Thus $\alpha^\sharp_{s^\sharp}$ extends to an isometric linear operator $\alpha^\sharp_{s^\sharp} \colon G \ltimes_\alpha A \to G \ltimes_\alpha A$ in a unique way.

Lemma 3.4.1. *Let $\alpha\colon G \curvearrowright A$ be a C*-action and assume that G is abelian. For each $s^\sharp \in G^\sharp$ the map $\alpha^\sharp_{s^\sharp}$ is an automorphism of $G \ltimes_\alpha A$.*

Proof. One the one hand, for $f, g \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}
\alpha^\sharp_{s^\sharp}(fg)(t) &= s^\sharp(t)(fg)(t) \\
&= s^\sharp(t) \int_G f(s)\alpha_s(g(s^{-1}t)) \, \mathrm{d}_\mu s \\
&= \int_G s^\sharp(s) f(s) \alpha_s(s^\sharp(s^{-1}t) g(s^{-1}t)) \, \mathrm{d}_\mu s \\
&= \int_G \alpha^\sharp_{s^\sharp}(f)(s) \alpha_s(\alpha^\sharp_{s^\sharp}(g)(s^{-1}t)) \, \mathrm{d}_\mu s \\
&= (\alpha^\sharp_{s^\sharp}(f)\alpha^\sharp_{s^\sharp}(g))(t)
\end{aligned}$$

for all $t \in G$. On the other hand, for $h \in C_{\mathrm{cpt}}(G, A)$ we have

$$\begin{aligned}
\alpha^\sharp_{s^\sharp}(h^*)(t) &= s^\sharp(t) h^*(t) \\
&= s^\sharp(t) \alpha_t(h(t^{-1})^*) \\
&= \alpha_t((s^\sharp(t^{-1}) h(t^{-1}))^*) \\
&= \alpha_t(\alpha^\sharp_{s^\sharp}(h)(t^{-1})^*) \\
&= \alpha^\sharp_{s^\sharp}(h)^*(t)
\end{aligned}$$

for all $t \in G$.

Thus $\alpha^\sharp_{s^\sharp}$ is an automorphism as it an isometric homomorphism mapping the dense subalgebra $C_{\mathrm{cpt}}(G, A)$ of $G \ltimes_\alpha A$ onto itself. ■

By Lemma 3.4.1 we obtain a well-defined map $\alpha^\sharp\colon G^\sharp \to \mathrm{Aut}(G \ltimes_\alpha A)$.

Lemma 3.4.2. *Let $\alpha\colon G \curvearrowright A$ be a C*-action and assume that G is abelian. The map $\alpha^\sharp$ is a topological group homomorphism from $G^\sharp$ to $\mathrm{Aut}(G \ltimes_\alpha A)$.*

Proof. We begin with continuity of $\alpha^\sharp$. Given $t^\sharp \in G^\sharp$ and $f \in G \ltimes_\alpha A$, we must find for every $\varepsilon > 0$ a neighbourhood $N^\sharp \subset G^\sharp$ of $t^\sharp$ such that $s^\sharp \in N^\sharp$ implies $\|\alpha^\sharp_{s^\sharp}(f) - \alpha^\sharp_{t^\sharp}(f)\|_{\mathrm{env}} < \varepsilon$. It does no harm to assume that $f \in C_{\mathrm{cpt}}(G, A)$ and $\|f\|_{\mathrm{int}} \leq 1$. By definition of the topology on $G^\sharp$ there is a neighbourhood $N^\sharp \subset G^\sharp$ of $t^\sharp$ such that every $s^\sharp \in N^\sharp$ uniformly approximates $t^\sharp$ on $\sigma(f)$ within the margin of error ε. So for $s^\sharp \in N^\sharp$ one has

$$\|\alpha^\sharp_{s^\sharp}(f) - \alpha^\sharp_{t^\sharp}(f)\|_{\mathrm{int}} = \int_G |s^\sharp(s) - t^\sharp(s)| \|f(s)\| \, \mathrm{d}_\mu s \leq \sup_{s \in \sigma(f)} |s^\sharp(s) - t^\sharp(s)| < \varepsilon$$

and therefore $\|\alpha^\sharp_{s^\sharp}(f) - \alpha^\sharp_{t^\sharp}(f)\|_{\mathrm{env}} < \varepsilon$ as desired.

It is easy to check that τ is a group homomorphism. Indeed, for $s^\sharp, t^\sharp \in G^\sharp$ we have

$$\alpha^\sharp_{s^\sharp t^\sharp}(f)(s) = (s^\sharp t^\sharp)(s) f(s) = s^\sharp(s) t^\sharp(s) f(s) = s^\sharp(s) \alpha^\sharp_{t^\sharp}(f)(s) = \alpha^\sharp_{s^\sharp}(\alpha^\sharp_{t^\sharp}(f))(s) = (\alpha^\sharp_{s^\sharp}\alpha^\sharp_{t^\sharp})(f)(s)$$

for all $f \in C_{\mathrm{cpt}}(G, A)$ and $s \in G$. ■

Definition 3.4.3. If $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian, then $\alpha^\sharp\colon G^\sharp \curvearrowright G \ltimes_\alpha A$ is called its *dual C*-action*.

Due to Lemma 3.2.3, the following lemma guarantuees that we can view $C_{\mathrm{cpt}}(G^\sharp \times G, A)$ as a subalgebra of the iterated crossed product $G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$.

Lemma 3.4.4. *If $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian and $\alpha^\sharp\colon G^\sharp \curvearrowright G \ltimes_\alpha A$ is the dual C*-action, then $\alpha^\sharp$ is compatible with α.*

Proof. The C*-action $\alpha^\sharp$ leaves $C_{\mathrm{cpt}}(G, A)$ invariant by definition. The rest follows directly from the fact that $\alpha^\sharp_{s^\sharp}(g^\sharp(t^\sharp))(t) = s^\sharp(t)g^\sharp(t^\sharp, t)$ for $g^\sharp \in C_{\mathrm{cpt}}(G^\sharp \times G, A)$ and $(t^\sharp, t, s^\sharp) \in G^\sharp \times G \times G^\sharp$. ■

The following lemma shows how $\alpha^\sharp$ interacts with the canonical covariant multiplier representation (j_G, j_A) of α.

Lemma 3.4.5. *Let $\alpha\colon G \curvearrowright A$ be a C*-action with G abelian and let (j_G, j_A) be the canonical covariant multiplier representation of α.*

(i) The equality $\alpha^\sharp_{\theta^\sharp}(j_G(\theta)f) = \theta^\sharp(\theta)j_G(\theta)\alpha^\sharp_{\theta^\sharp}(f)$ holds for $\theta^\sharp \in G^\sharp$, $\theta \in G$ and $f \in G \ltimes_\alpha A$.

(ii) The equality $\alpha^\sharp_{\theta^\sharp}(j_A(a)f) = j_A(a)\alpha^\sharp_{\theta^\sharp}(f)$ holds for $\theta^\sharp \in G^\sharp$, $a \in A$ and $f \in G \ltimes_\alpha A$.

Proof. To prove the formula in part (i), we assume that $f \in C_{\mathrm{cpt}}(G, A)$ and compute

$$\begin{aligned}
\alpha^\sharp_{\theta^\sharp}(j_G(\theta)f)(s) &= \theta^\sharp(s)(j_G(\theta)f)(s) \\
&= \theta^\sharp(s)\alpha_\theta(f(\theta^{-1}s)) \\
&= \theta^\sharp(\theta)\alpha_\theta(\theta^\sharp(\theta^{-1}s)f(\theta^{-1}s)) \\
&= \theta^\sharp(\theta)\alpha_\theta(\alpha^\sharp_{\theta^\sharp}(f)(\theta^{-1}s)) \\
&= \theta^\sharp(\theta)(j_G(\theta)\alpha^\sharp_{\theta^\sharp}(f))(s)
\end{aligned}$$

for all $s \in G$.

Similarly, for part (ii) we assume again that $f \in C_{\mathrm{cpt}}(G, A)$ and compute

$$\begin{aligned}
\alpha^\sharp_{\theta^\sharp}(j_A(a)f)(s) &= \theta^\sharp(s)(j_A(a)f)(s) \\
&= \theta^\sharp(s)af(s) \\
&= a\theta^\sharp(s)f(s) \\
&= a\alpha^\sharp_{\theta^\sharp}(f)(s) \\
&= (j_A(a)\alpha^\sharp_{\theta^\sharp}(f))(s)
\end{aligned}$$

for all $s \in G$. ■

Suppose that (U, P) is a covariant representation of $\alpha^\sharp$ on a Hilbert space H. In particular, P is a representation of $G \ltimes_\alpha A$ on H. Thus Theorem 2.4.16 tells us that there is a unique covariant representation (u, ρ) of α on H such that $u \ltimes_\alpha \rho = P$. Given $\theta^\sharp \in G^\sharp$, the following lemma describes how the unitary linear operator $U(\theta^\sharp)$ affects the behaviour of $u(\theta)$ and $\rho(a)$ for all $\theta \in G$ and $a \in A$.

Lemma 3.4.6. *Let $\alpha\colon G \curvearrowright A$ be a C*-action with G abelian. Suppose that (U, P) is a covariant representation of $\alpha^\sharp$ on a Hilbert space H and (u, ρ) a covariant representation of α on H such that $u \ltimes_\alpha \rho = P$. For each $\theta^\sharp \in G^\sharp$ the diagrams*

$$\begin{tikzcd}
H \arrow[r, "u(\theta)"] \arrow[d, "U(\theta^\sharp)"'] & H \arrow[d, "U(\theta^\sharp)"] \\
H \arrow[r, "\theta^\sharp(\theta)u(\theta)"'] & H
\end{tikzcd}, \qquad
\begin{tikzcd}
H \arrow[r, "\rho(a)"] \arrow[d, "U(\theta^\sharp)"'] & H \arrow[d, "U(\theta^\sharp)"] \\
H \arrow[r, "\rho(a)"'] & H
\end{tikzcd}$$

commute for all $\theta \in G$ and $a \in A$.

Proof. To prove that the diagrams commute, we use density of $P(G \ltimes_\alpha A)H$ in H. On the one hand, the diagram on the left-hand side commutes as Lemma 3.4.5 implies that

$$\begin{aligned}\theta^\sharp(\theta)u(\theta)U(\theta^\sharp)P(f)x &= \theta^\sharp(\theta)P^+(j_G(\theta))U(\theta^\sharp)P(f)x\\ &= P(\theta^\sharp(\theta)j_G(\theta)\alpha^\sharp_{\theta^\sharp}(f))U(\theta^\sharp)x\\ &= P(\alpha^\sharp_{\theta^\sharp}(j_G(\theta)f))U(\theta^\sharp)x\\ &= U(\theta^\sharp)P^+(j_G(\theta))P(f)x\\ &= U(\theta^\sharp)u(\theta)P(f)x\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, A)$ and $x \in H$. On the other hand, the diagram on the right-hand side commutes as Lemma 3.4.5 implies that

$$\begin{aligned}\rho(a)U(\theta^\sharp)P(f)x &= P^+(j_A(a))U(\theta^\sharp)P(f)x\\ &= P(j_A(a)\alpha^\sharp_{\theta^\sharp}(f))U(\theta^\sharp)x\\ &= P(\alpha^\sharp_{\theta^\sharp}(j_A(a)f))U(\theta^\sharp)x\\ &= U(\theta^\sharp)P^+(j_A(a))P(f)x\\ &= U(\theta^\sharp)\rho(a)P(f)x\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, A)$ and $x \in H$. ■

3.5 The Takai Duality Theorem

Suppose that $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian. Since one has $G^{\sharp\sharp} = G$ by Theorem 3.1.3, it is natural to ask about the structure of the double dual C*-action $\alpha^{\sharp\sharp}\colon G \curvearrowright G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$.

In order to describe the structure of $\alpha^{\sharp\sharp}$, we need to introduce another C*-action. Suppose that u is a unitary representation of a (possibly non-abelian) locally compact Hausdorff group G on some Hilbert space H. Since $K(H)$ is an ideal in $B(H)$, we have for each $s \in G$ an automorphism $\kappa(u)_s\colon K(H) \to K(H)$ given by $\kappa(u)_s(c) := u(s)cu(s)^*$ for all $c \in K(H)$. Hence we obtain a well-defined map $\kappa(u)\colon G \to \mathrm{Aut}(K(H))$.

Lemma 3.5.1. *Let u be a unitary representation of a locally compact Hausdorff group G on a Hilbert space H. The map $\kappa(u)$ is a topological group homomorphism from G to* $\mathrm{Aut}(K(H))$.

Proof. We begin with continuity of $\kappa(u)$. Given $t \in G$ and $c \in K(H)$, we must find for every $\varepsilon > 0$ a neighbourhood $N \subset G$ of t such that $s \in N$ implies $\|\kappa(u)_s(c) - \kappa(u)_t(c)\| < \varepsilon$. Assume that c is a finite-rank operator. Pick $x_1, \ldots, x_n \in H$ and $y_1, \ldots, y_n \in H$ such that the sum of $x_1 \oslash y_1, \ldots, x_n \oslash y_n$ equals c. Now use strong continuity of u to find a neighbourhood $N \subset G$ of t such that $s \in N$ implies $\|u(s)x_j - u(t)x_j\|\|y_j\| < \varepsilon/2n$ and $\|x_j\|\|u(s)y_j - u(t)y_j\| < \varepsilon/2n$ for all $j = 1, \ldots, n$. So for $s \in N$ one has

$$\begin{aligned}\|\kappa(u)_s(c) - \kappa(u)_t(c)\| &\le \sum_{j=1}^n \|u(s)x_j \oslash u(s)y_j - u(t)x_j \oslash u(t)y_j\|\\ &\le \sum_{j=1}^n \|(u(s)x_j - u(t)x_j) \oslash u(s)y_j\| + \sum_{j=1}^n \|u(t)x_j \oslash (u(s)y_j - u(t)y_j)\|\\ &= \sum_{j=1}^n \|u(s)x_j - u(t)x_j\|\|y_j\| + \sum_{j=1}^n \|x_j\|\|u(s)y_j - u(t)y_j\| < \varepsilon\end{aligned}$$

as desired.

It is easy to check that $\kappa(u)$ is a group homomorphism. Indeed, this follows from the fact that u is a homomorphism. ■

Recall that the Hilbert space H^G is the completion of $C_{\mathrm{cpt}}(G)$ with respect to the inner product given by $\langle \chi, \psi \rangle = \mu(\chi^* \psi)$ for $\chi, \psi \in C_{\mathrm{cpt}}(G)$ and that the unitary representation u^R of G on H^G is given by $u^R(\theta)\chi = \Delta(\theta)^{1/2} R_\theta(\chi)$ for $\theta \in G$ and $\chi \in C_{\mathrm{cpt}}(G)$. So by the preceding discussion we obtain a C*-action $\kappa(u^R)\colon G \curvearrowright K(H^G)$.

We are now ready to formulate the Takai duality theorem.

Theorem 3.5.2. *If $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian, then there exists an isomorphism T from the iterated crossed product $G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$ to the maximal tensor product $K(H^G) \otimes_{\max} A$, which is equivariant for $\alpha^{\sharp\sharp}$ and $\kappa(u^R) \otimes \alpha$.*

The original proof of Takai duality due to Takai [Tak75] has been simplified by Raeburn [Rae88]. Takai's proof heavily relies on representation theory and is based on a duality theorem for Neumann algebra crossed products by Takesaki [Tak73], while Raeburn's proof exploits the universal properties of crossed products.

The philosphy behind Takai duality is that one recovers the C*-action α from its crossed product up to tensoring with the C*-action $\kappa(u^R)$.

Sketch of the Proof

As indicated by Williams [Wil07], Reaburn's arguments can be altered in such manner that the proof becomes a verification of a chain of four isomorphisms, which is displayed in the figure below. If an arrow is dashed, then its domain and codomain can be identified by Lemma 3.3.7.

$$
\begin{tikzcd}
 & G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} ((G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A) \arrow[dd, "T_2"] & \\
G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A) \arrow[dd, "T_1"] & & (G \ltimes_{\mathrm{lt}} C_0(G)) \otimes_{\max} A \arrow[dd, "T_4"] \\
 & G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A) \arrow[dd, "T_3"] & \\
G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A) \arrow[uuur, dashed] & & K(H^G) \otimes_{\max} A \\
 & G \ltimes_{\mathrm{lt}\otimes\mathrm{id}} (C_0(G) \otimes_{\max} A) \arrow[uuur, dashed] &
\end{tikzcd}
$$

So the desired isomorphism for Takai duality will be the composition $T := T_4 \circ T_3 \circ T_2 \circ T_1$. Of course, we also need to keep track of how the four isomorphisms affect the behaviour of the double dual C*-action $\alpha^{\sharp\sharp}$. In other words, at each step we need to compute the pushforward C*-action. In the end we shall recover $T\alpha^{\sharp\sharp} = \kappa(u^R) \otimes \alpha$.

The First Isomorphism T_1

Let F be the Fourier transform and consider the C*-action $F^{-1}\mathrm{lt} \otimes \alpha\colon G \curvearrowright (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$. Use Lemma 3.3.7 to identify $G^\sharp \ltimes_{\mathrm{id}} A = (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$ and note that for $s \in G$ we have $(F^{-1}\mathrm{lt} \otimes \alpha)_s(f^\sharp)(s^\sharp) := s^{\sharp -1}(s)\alpha_s(f^\sharp(s^\sharp))$ for all $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, A)$ and $s^\sharp \in G^\sharp$ by Lemma 3.1.7.

Due to Lemma 3.2.3, the following lemma guarantees that we can regard $C_{\mathrm{cpt}}(G \times G^\sharp, A)$ as a subalgebra of the iterated crossed product $G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A)$.

Lemma 3.5.3. *If $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian and* id$\colon G^\sharp \curvearrowright A$ *is the trivial C*-action, then $F^{-1}\mathrm{lt} \otimes \alpha$ is compatible with* id.

Proof. It is clear that $F^{-1}\mathrm{lt}\otimes\alpha$ leaves $C_{\mathrm{cpt}}(G^\sharp, A)$ invariant. The rest follows from the fact that $(F^{-1}\mathrm{lt}\otimes\alpha)_s(g(t))(t^\sharp) = t^{\sharp-1}(s)\alpha_s(g(t,t^\sharp))$ for $g \in C_{\mathrm{cpt}}(G\times G^\sharp, A)$ and $(t, t^\sharp, s) \in G\times G^\sharp\times G$. ∎

The following lemma shows how $F^{-1}\mathrm{lt}\otimes\alpha$ interacts with the canonical covariant multiplier representation $(j_{G^\sharp}, j_A)$ of id.

Lemma 3.5.4. *Let $\alpha\colon G \curvearrowright A$ be a C^*-action with G abelian and let $(j_{G^\sharp}, j_A)$ be the canonical covariant multiplier representation of* id.

(i) The equality $(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(j_{G^\sharp}(\theta^\sharp)f^\sharp) = \theta^{\sharp-1}(\theta)j_{G^\sharp}(\theta^\sharp)(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(f^\sharp)$ holds for $\theta \in G$, $\theta^\sharp \in G^\sharp$ and $f^\sharp \in G^\sharp \ltimes_{\mathrm{id}} A$.

(ii) The equality $(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(j_A(a)f^\sharp) = j_A(\alpha_\theta(a))(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(f^\sharp)$ holds for $\theta \in G$, $a \in A$ and $f^\sharp \in G^\sharp \ltimes_{\mathrm{id}} A$.

Proof. To prove the formula in part (i), we assume that $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, A)$ and compute

$$\begin{aligned}
(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(j_{G^\sharp}(\theta^\sharp)f^\sharp)(s^\sharp) &= s^{\sharp-1}(\theta)\alpha_\theta((j_{G^\sharp}(\theta^\sharp)f^\sharp)(s^\sharp))\\
&= s^{\sharp-1}(\theta)\alpha_\theta(f^\sharp(\theta^{\sharp-1}s^\sharp))\\
&= \theta^{\sharp-1}(\theta)\alpha_\theta((\theta^{\sharp-1}s^\sharp)^{-1}(\theta)f^\sharp(\theta^{\sharp-1}s^\sharp))\\
&= \theta^{\sharp-1}(\theta)(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(f^\sharp)(\theta^{\sharp-1}s^\sharp)\\
&= \theta^{\sharp-1}(\theta)(j_{G^\sharp}(\theta^\sharp)(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(f^\sharp))(s^\sharp)
\end{aligned}$$

for all $s^\sharp \in G^\sharp$.

Similarly, for part (ii) we assume again that $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, A)$ and compute

$$\begin{aligned}
(F^{-1}\mathrm{lt}\otimes\alpha)_\theta(j_A(a)f^\sharp)(s^\sharp) &= s^\sharp(\theta)\alpha_\theta((j_A(a)f^\sharp)(s^\sharp))\\
&= s^\sharp(\theta)\alpha_\theta(af^\sharp(s^\sharp))\\
&= \alpha_\theta(a)\alpha_\theta(s^\sharp(\theta)f^\sharp(s^\sharp))\\
&= \alpha_\theta(a)\alpha_\theta((F^{-1}\mathrm{lt}\otimes\alpha)_\theta(f^\sharp)(s^\sharp))\\
&= (j_A(\alpha_\theta(a))\alpha_\theta((F^{-1}\mathrm{lt}\otimes\alpha)_\theta(f^\sharp)))(s^\sharp)
\end{aligned}$$

for all $s^\sharp \in G^\sharp$. ∎

We define a linear operator $T_1\colon C_{\mathrm{cpt}}(G^\sharp\times G, A) \to G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A)$ by $T_1(f^\sharp)(s, s^\sharp) := s^{\sharp-1}(s)f^\sharp(s^\sharp, s)$ for $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp\times G, A)$ and $(s^\sharp, s) \in G^\sharp \times G$. Our first goal is to prove that T_1 is isometric. For this we have to analyze covariant representations. The following two lemmas relate the covariant representations of $F^{-1}\mathrm{lt}\otimes\alpha$ to those of $\alpha^\sharp$.

Lemma 3.5.5. *Let $\alpha\colon G \curvearrowright A$ be a C^*-action with G abelian. Suppose that (U, P) is a covariant representation of $\alpha^\sharp$ on a Hilbert space H and (u, ρ) a covariant representation of α on H such that $u \ltimes_\alpha \rho = P$.*

(i) The pair (U, ρ) is a covariant representation of id *on H.*

(ii) The pair $(u, U \ltimes_{\mathrm{id}} \rho)$ is a covariant representation of $F^{-1}\mathrm{lt}\otimes\alpha$ on H.

Proof. Part (i) is an immediate consequence of Lemma 3.4.6.

To see that part (ii) is true, pick an arbitrary $\theta \in G$ and use Lemma 3.4.6 again to find

$$\begin{aligned}(U \ltimes_{\mathrm{id}} \rho)((F^{-1}\mathrm{lt} \otimes \alpha)_\theta(f^\sharp))u(\theta)x &= \int_{G^\sharp} \rho((F^{-1}\mathrm{lt} \otimes \alpha)_\theta(f^\sharp)(s^\sharp))U(s^\sharp)u(\theta)x \ \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= \int_{G^\sharp} \rho(\alpha_\theta(f^\sharp(s^\sharp)))u(\theta)U(s^\sharp)x \ \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= \int_{G^\sharp} u(\theta)\rho(f^\sharp(s^\sharp))U(s^\sharp)x \ \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= u(\theta)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x\end{aligned}$$

for all $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp, A)$ and $x \in H$. ■

Lemma 3.5.6. *Let $\alpha\colon G \curvearrowright A$ be a C^*-action with G abelian. Every covariant representation of $F^{-1}\mathrm{lt} \otimes \alpha$ is of the form $(u, U \ltimes_{\mathrm{id}} \rho)$ for a unique covariant representation (U, P) of $\alpha^\sharp$ and a unique covariant representation (u, ρ) of α such that $u \ltimes_\alpha \rho = P$.*

Proof. By Theorem 2.4.16 we know that every covariant representation of $F^{-1}\mathrm{lt} \otimes \alpha$ is of the form $(u, U \ltimes_{\mathrm{id}} \rho)$ for a unique covariant representation (U, ρ) of id on some Hilbert space H. Hence it suffices to show that (u, ρ) is covariant for α and (U, P) with $P := u \ltimes_\alpha \rho$ is covariant for $\alpha^\sharp$. Let $(j_{G^\sharp}, j_A)$ be the canonical covariant multiplier representation of id.

First we show that $\rho(\alpha_\theta(a))u(\theta) = u(\theta)\rho(a)$ for $\theta \in G$ and $a \in A$. By Lemma 3.5.4 we have

$$\begin{aligned}\rho(\alpha_\theta(a))u(\theta)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x &= (U \ltimes_{\mathrm{id}} \rho)^+(j_A(\alpha_\theta(a)))u(\theta)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \\ &= (U \ltimes_{\mathrm{id}} \rho)(j_A(\alpha_\theta(a))(F^{-1}\mathrm{lt} \otimes \alpha)_\theta(f^\sharp))u(\theta)x \\ &= (U \ltimes_{\mathrm{id}} \rho)((F^{-1}\mathrm{lt} \otimes \alpha)_\theta(j_A(a)f^\sharp))u(\theta)x \\ &= u(\theta)(U \ltimes_{\mathrm{id}} \rho)^+(j_A(a))(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \\ &= u(\theta)\rho(a)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x\end{aligned}$$

for all $f^\sharp \in G^\sharp \ltimes_{\mathrm{id}} A$ and $x \in H$. Non-degeneracy of $U \ltimes_{\mathrm{id}} \rho$ yields the desired equality.

We now show that $P(\alpha^\sharp_{\theta^\sharp}(f))U(\theta^\sharp) = U(\theta^\sharp)P(f)$ for $\theta^\sharp \in G^\sharp$ and $f \in G \ltimes_\alpha A$. Of course it suffices to prove this equality for $f \in C_{\mathrm{cpt}}(G, A)$. By Lemma 3.5.4 we have

$$\begin{aligned}P(\alpha^\sharp_{\theta^\sharp}(f))U(\theta^\sharp)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x &= \int_G \rho(\alpha^\sharp_{\theta^\sharp}(f)(s))u(s)U(\theta^\sharp)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \ \mathrm{d}_\mu s \\ &= \int_G \rho(\alpha^\sharp_{\theta^\sharp}(f)(s))u(s)(U \ltimes_{\mathrm{id}} \rho)^+(j_{G^\sharp}(\theta^\sharp))(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \ \mathrm{d}_\mu s \\ &= \int_G \rho(\alpha^\sharp_{\theta^\sharp}(f)(s))(U \ltimes_{\mathrm{id}} \rho)((F^{-1}\mathrm{lt} \otimes \alpha)_s(j_{G^\sharp}(\theta^\sharp)f^\sharp))u(s)x \ \mathrm{d}_\mu s \\ &= \int_G \rho(f(s))(U \ltimes_{\mathrm{id}} \rho)(j_{G^\sharp}(\theta^\sharp)(F^{-1}\mathrm{lt} \otimes \alpha)_s(f^\sharp))u(s)x \ \mathrm{d}_\mu s \\ &= \int_G \rho(f(s))(U \ltimes_{\mathrm{id}} \rho)^+(j_{G^\sharp}(\theta^\sharp))u(s)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \ \mathrm{d}_\mu s \\ &= \int_G \rho(f(s))U(\theta^\sharp)u(s)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \ \mathrm{d}_\mu s \\ &= \int_G U(\theta^\sharp)\rho(f(s))u(s)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x \ \mathrm{d}_\mu s \\ &= U(\theta^\sharp)P(f)(U \ltimes_{\mathrm{id}} \rho)(f^\sharp)x\end{aligned}$$

for all $f^\sharp \in G^\sharp \ltimes_{\mathrm{id}} A$ and $x \in H$. Again, by non-degeneracy of $U \ltimes_{\mathrm{id}} \rho$ we are done. ■

We are now ready to show that T_1 is isometric. Indeed, if (U, P) is a covariant representation of $\alpha^\sharp$ on H and (u, ρ) is a covariant representation of α on H with $u \ltimes_\alpha \rho = P$, then for each $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp \times G, A)$ one has

$$\begin{aligned}(u \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (U \ltimes_{\mathrm{id}} \rho))(T_1(f^\sharp))x &= \int_G (U \ltimes_{\mathrm{id}} \rho)(T_1(f^\sharp)(s))u(s)x \, \mathrm{d}_\mu s \\ &= \int_G \int_{G^\sharp} \rho(T_1(f^\sharp)(s, s^\sharp))U(s^\sharp)u(s)x \, \mathrm{d}_{\mu^\sharp} s^\sharp \, \mathrm{d}_\mu s \\ &= \int_G \int_{G^\sharp} \rho(f^\sharp(s^\sharp, s))u(s)U(s^\sharp)x \, \mathrm{d}_{\mu^\sharp} s^\sharp \, \mathrm{d}_\mu s \\ &= \int_{G^\sharp} \int_G \rho(f^\sharp(s^\sharp, s))u(s)U(s^\sharp)x \, \mathrm{d}_\mu s \, \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= \int_{G^\sharp} P(f^\sharp(s^\sharp))U(s^\sharp)x \, \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= (U \ltimes_{\alpha^\sharp} P)(f^\sharp)x\end{aligned}$$

for $x \in H$ by Lemma 3.4.6 and therefore $\|T_1(f^\sharp)\|_{\mathrm{env}} = \|f^\sharp\|_{\mathrm{env}}$ by Lemma 3.5.6. Hence T_1 uniquely extends to an isometric linear operator $T_1 \colon G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A) \to G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A)$.

Lemma 3.5.7. *Let $\alpha \colon G \curvearrowright A$ be a C^*-action with G abelian. The map T_1 is an isomorphism from $G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$ to $G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A)$.*

Proof. On the one hand, for $f^\sharp, g^\sharp \in C_{\mathrm{cpt}}(G^\sharp \times G, A)$ we have

$$\begin{aligned}T_1(f^\sharp g^\sharp)(t, t^\sharp) &= t^{\sharp-1}(t)(f^\sharp g^\sharp)(t^\sharp, t) \\ &= t^{\sharp-1}(t) \int_{G^\sharp} \int_G f^\sharp(s^\sharp, s)\alpha_s(\alpha^\sharp_{s^\sharp}(g^\sharp(s^{\sharp-1}t^\sharp))(s^{-1}t)) \, \mathrm{d}_\mu s \, \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= t^{\sharp-1}(t) \int_{G^\sharp} \int_G f^\sharp(s^\sharp, s)\alpha_s(s^\sharp(s^{-1}t)g^\sharp(s^{\sharp-1}t^\sharp, s^{-1}t)) \, \mathrm{d}_\mu s \, \mathrm{d}_{\mu^\sharp} s^\sharp \\ &= \int_G \int_{G^\sharp} s^{\sharp-1}(s)f^\sharp(s^\sharp, s)(s^{\sharp-1}t^\sharp)^{-1}(s)\alpha_s((s^{\sharp-1}t^\sharp)^{-1}(s^{-1}t)g^\sharp(s^{\sharp-1}t^\sharp, s^{-1}t)) \, \mathrm{d}_{\mu^\sharp} s^\sharp \, \mathrm{d}_\mu s \\ &= \int_G \int_{G^\sharp} T_1(f^\sharp)(s, s^\sharp)(s^{\sharp-1}t^\sharp)^{-1}(s)\alpha_s(T_1(g^\sharp)(s^{-1}t, s^{\sharp-1}t^\sharp)) \, \mathrm{d}_{\mu^\sharp} s^\sharp \, \mathrm{d}_\mu s \\ &= \int_G \int_{G^\sharp} T_1(f^\sharp)(s, s^\sharp)\mathrm{id}_{s^\sharp}((F^{-1}\mathrm{lt} \otimes \alpha)_s(T_1(g^\sharp)(s^{-1}t))(s^{\sharp-1}t^\sharp))) \, \mathrm{d}_{\mu^\sharp} s^\sharp \, \mathrm{d}_\mu s \\ &= (T_1(f^\sharp)T_1(g^\sharp))(t, t^\sharp)\end{aligned}$$

for all $(t^\sharp, t) \in G^\sharp \times G$. On the other hand, for $h^\sharp \in C_{\mathrm{cpt}}(G^\sharp \times G, A)$ we have

$$\begin{aligned}T_1(h^{\sharp*})(t^\sharp, t) &= t^{\sharp-1}(t)h^{\sharp*}(t^\sharp, t) \\ &= t^{\sharp-1}(t)\alpha^\sharp_{t^\sharp}(h^\sharp(t^{\sharp-1})^*)(t) \\ &= h^\sharp(t^{\sharp-1})^*(t) \\ &= \alpha_t(h^\sharp(t^{\sharp-1}, t^{-1})^*) \\ &= t^{\sharp-1}(t)\alpha_t(\mathrm{id}_{t^\sharp}(T_1(h^\sharp)(t^{-1}, t^{\sharp-1})^*)) \\ &= t^{\sharp-1}(t)\alpha_t(T_1(h^\sharp)(t^{-1})^*(t^\sharp)) \\ &= (F^{-1}\mathrm{lt} \otimes \alpha)_t(T_1(h^\sharp)(t^{-1})^*)(t^\sharp) \\ &= T_1(h^\sharp)^*(t, t^\sharp)\end{aligned}$$

for all $(t^\sharp, t) \in G^\sharp \times G$.

Surjectivity of T_1 follows from the fact that T_1 is an isometric homomorphism mapping the dense subalgebra $C_{\mathrm{cpt}}(G^\sharp \times G, A)$ of $G^\sharp \ltimes_{\alpha^\sharp} (G \ltimes_\alpha A)$ onto the dense subalgebra $C_{\mathrm{cpt}}(G \times G^\sharp, A)$ of $G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A)$. ■

It remains to study the pushforward C*-action $T_1\alpha^{\sharp\sharp}\colon G \curvearrowright G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} (G^\sharp \ltimes_{\mathrm{id}} A)$. For $s \in G$ the C*-action $F^{-1}\mathrm{rt}\otimes\mathrm{id}\colon G \curvearrowright (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$ gives rise to an automorphism $(F^{-1}\mathrm{rt}\otimes\mathrm{id})_s$ of the maximal tensor product $(G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$. Using Lemma 2.3.9 one readily verifies that we have an automorphism $G \ltimes (F^{-1}\mathrm{rt}\otimes\mathrm{id})_s$ of the crossed product $G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} ((G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A)$.

As before we identify $G^\sharp \ltimes_{\mathrm{id}} A = (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$ with Lemma 3.3.7. For $f \in C_{\mathrm{cpt}}(G \times G^\sharp, A)$ we have $(G \ltimes (F^{-1}\mathrm{rt}\otimes\mathrm{id})_s)(f)(t, t^\sharp) = t^\sharp(s) f(t, t^\sharp)$ for all $(t, t^\sharp) \in G \times G^\sharp$ by Lemma 3.1.7.

Lemma 3.5.8. *Let $\alpha\colon G \curvearrowright A$ be a C*-action with G abelian. The pushforward C*-action $T_1\alpha^{\sharp\sharp}$ satisfies $(T_1\alpha^{\sharp\sharp})_s = G \ltimes (F^{-1}\mathrm{rt}\otimes\mathrm{id})_s$ for all $s \in G$.*

Proof. We need to show that $(G \ltimes (F^{-1}\mathrm{rt}\otimes\mathrm{id})_s) \circ T_1 = T_1 \circ \alpha_s^{\sharp\sharp}$. We have

$$\begin{aligned}(G \ltimes (F^{-1}\mathrm{rt}\otimes\mathrm{id})_s)(T_1(f^\sharp))(t, t^\sharp) &= t^\sharp(s) T_1(f^\sharp)(t, t^\sharp)\\ &= t^{\sharp-1}(s^{-1}t) f^\sharp(t^\sharp, t)\\ &= t^{\sharp-1}(t) \alpha_s^{\sharp\sharp}(f^\sharp)(t, t^\sharp)\\ &= T_1(\alpha_s^{\sharp\sharp}(f^\sharp))(t, t^\sharp)\end{aligned}$$

for all $f^\sharp \in C_{\mathrm{cpt}}(G^\sharp \times G, A)$ and $(t, t^\sharp) \in G \times G^\sharp$. ■

The Second Isomorphism T_2

For the map T_2 we first construct an isomorphism Φ_2 from $(G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$ to $C_0(G) \otimes_{\max} A$ and then show that Φ_2 is equivariant for the C*-actions $F^{-1}\mathrm{lt} \otimes \alpha\colon G \curvearrowright (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A$ and $\mathrm{lt}\otimes\alpha\colon G \curvearrowright C_0(G) \otimes_{\max} A$. The induced isomorphism $G \ltimes \Phi_2$ from $G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} ((G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A)$ to $G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A)$ will be our desired isomorphism T_2.

Since F is an isomorphism, one readily verifies that there is a unique isometric homomorphism $\Phi_2\colon (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A \to C_0(G) \otimes_{\max} A$ such that $\Phi_2(f^\sharp \otimes a) = F(f^\sharp) \otimes a$ for $f^\sharp \in G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}$ and $a \in A$. In fact, the map Φ_2 is even an isomorphism.

Lemma 3.5.9. *Suppose that $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian. The isomorphism Φ_2 is equivariant for the C*-actions $F^{-1}\mathrm{lt} \otimes \alpha$ and $\mathrm{lt} \otimes \alpha$.*

Proof. This immediately follows from the fact that F is equivariant for $F^{-1}\mathrm{lt}$ and lt. ■

By Lemma 2.3.9 and Lemma 3.5.9 we obtain an isomorphism $G \ltimes \Phi_2\colon G \ltimes_{F^{-1}\mathrm{lt}\otimes\alpha} ((G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A) \to G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A)$. We put $T_2 := G \ltimes \Phi_2$.

Let us inspect the pushforward C*-action $T_2T_1\alpha^{\sharp\sharp}\colon G \curvearrowright G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A)$. The C*-action $\mathrm{rt} \otimes \mathrm{id}\colon G \curvearrowright C_0(G) \otimes_{\max} A$ produces for each $s \in G$ an automorphism $(\mathrm{rt} \otimes \mathrm{id})_s$ of the maximal tensor product $C_0(G) \otimes_{\max} A$. Using Lemma 2.3.9 one readily verifies that we have an automorphism $G \ltimes (\mathrm{rt} \otimes \mathrm{id})_s$ of the crossed product $G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A)$.

Lemma 3.5.10. *Let $\alpha\colon G \curvearrowright A$ be a C*-action with G abelian. The pushforward C*-action $T_2T_1\alpha^{\sharp\sharp}$ satisfies $(T_2T_1\alpha^{\sharp\sharp})_s = G \ltimes (\mathrm{rt} \otimes \mathrm{id})_s$ for all $s \in G$.*

Proof. We need to show that $(G \ltimes (\mathrm{rt} \otimes \mathrm{id})_s) \circ T_2 = T_2 \circ (T_1\alpha^{\sharp\sharp})_s$. Note that Φ_2 is equivariant for $F^{-1}\mathrm{rt} \otimes \mathrm{id}$ and $\mathrm{rt} \otimes \mathrm{id}$ as F is equivariant for $F^{-1}\mathrm{rt}$ and rt. So by Lemma 3.5.8 we have

$$\begin{aligned}(G \ltimes (\mathrm{rt} \otimes \mathrm{id})_s)(T_2(f))(t) &= (\mathrm{rt} \otimes \mathrm{id})_s(\Phi_2(f(t)))\\ &= \Phi_2((F^{-1}\mathrm{rt} \otimes \mathrm{id})_s(f(t)))\\ &= T_2((G \ltimes (F^{-1}\mathrm{rt} \otimes \mathrm{id})_s)(f))(t)\\ &= T_2((T_1\alpha^{\sharp\sharp})_s(f))(t)\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, (G^\sharp \ltimes_{\mathrm{id}} \mathbb{C}) \otimes_{\max} A)$ and $t \in G$ as desired. ■

The Third Isomorphism T_3

For the map T_3 we first construct an isomorphism Φ_3 from $C_0(G) \otimes_{\max} A$ to $C_0(G) \otimes_{\max} A$ and then show that Φ_3 is equivariant for the C*-actions $\mathrm{lt} \otimes \alpha\colon G \curvearrowright C_0(G) \otimes_{\max} A$ and $\mathrm{lt} \otimes \mathrm{id}\colon G \curvearrowright C_0(G) \otimes_{\max} A$. The induced isomorphism $G \ltimes \Phi_3$ from $G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A)$ to $G \ltimes_{\mathrm{lt}\otimes\mathrm{id}} (C_0(G) \otimes_{\max} A)$ will be our desired isomorphism T_3.

Since $C_0(G) \otimes_{\max} A$ is defined as the completion of $C_0(G) \otimes A$ with respect to the maximal norm, it may occur that not every element in $C_0(G) \otimes_{\max} A$ can be expressed as a sum of pure tensors. This is an obstacle; explicit descriptions of Φ_3 require concrete realizations of arbitrary elements in $C_0(G) \otimes_{\max} A$ as Φ_3 will not leave $C_0(G) \otimes A$ invariant.

Consider the C*-algebra $C_0(G, A)$ consisting of all continuous functions $w\colon G \to A$ for which $\{t \in G : \|w(t)\| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$. The operations are defined pointwise and the supremum norm of an element $w \in C_0(G, A)$ is given by

$$\|w\|_\infty := \sup_{t\in G} \|w(t)\|.$$

The tensor product $C_0(G) \otimes A$ can be viewed as a subalgebra of $C_0(G, A)$. Indeed, for any $\varphi \in C_0(G)$ and $a \in A$ we can identify the pure tensor $\varphi \otimes a$ in $C_0(G) \otimes A$ with the function

$$G \ni s \mapsto \varphi(s)a \in A,$$

which is easily seen to be an element of $C_0(G, A)$. The inclusion of $C_0(G) \otimes A$ into $C_0(G, A)$ extends to an isomorphism from $C_0(G) \otimes_{\max} A$ to $C_0(G, A)$ by [Mur90, Theorem 6.4.17]. Hence we can identify $C_0(G) \otimes_{\max} A = C_0(G, A)$.

Define $\Phi_3\colon C_0(G, A) \to C_0(G, A)$ by $\Phi_3(w)(t) := \alpha_t^{-1}(w(t))$ for $w \in C_0(G, A)$ and $t \in G$. It is clear that Φ_3 is an isomorphism as α_s is an automorphism for every $s \in G$.

Lemma 3.5.11. *Suppose that $\alpha\colon G \curvearrowright A$ is a C*-action with G abelian. The isomorphism Φ_3 is equivariant for the C*-actions $\mathrm{lt} \otimes \alpha$ and $\mathrm{lt} \otimes \mathrm{id}$.*

Proof. For each $s \in G$ we directly compute

$$\begin{aligned}
(\mathrm{lt} \otimes \mathrm{id})_s(\Phi_3(w))(t) &= \Phi_3(w)(s^{-1}t) \\
&= \alpha_{s^{-1}t}^{-1}(w(s^{-1}t)) \\
&= \alpha_t^{-1}(\alpha_s(w(s^{-1}t))) \\
&= \alpha_t^{-1}((\mathrm{lt} \otimes \alpha)_s(w)(t)) \\
&= \Phi_3((\mathrm{lt} \otimes \alpha)_s(w))(t)
\end{aligned}$$

for all $w \in C_0(G, A)$ and $t \in G$. ∎

By Lemma 2.3.9 and Lemma 3.5.11 we obtain an isomorphism $G \ltimes \Phi_3\colon G \ltimes_{\mathrm{lt}\otimes\alpha} (C_0(G) \otimes_{\max} A) \to G \ltimes_{\mathrm{lt}\otimes\mathrm{id}} (C_0(G) \otimes_{\max} A)$. We put $T_3 := G \ltimes \Phi_3$.

We now examine the pushforward C*-action $T_3T_2T_1\alpha^{\sharp\sharp}\colon G \curvearrowright G \ltimes_{\mathrm{lt}\otimes\mathrm{id}} (C_0(G) \otimes_{\max} A)$. The C*-action $\mathrm{rt} \otimes \alpha\colon G \curvearrowright C_0(G) \otimes_{\max} A$ yields for each $s \in G$ an automorphism $(\mathrm{rt} \otimes \alpha)_s$ of the maximal tensor product $C_0(G) \otimes_{\max} A$. Using Lemma 2.3.9 one readily verifies that we have an automorphism $G \ltimes (\mathrm{rt} \otimes \alpha)_s$ of the crossed product $G \ltimes_{\mathrm{lt}\otimes\mathrm{id}} (C_0(G) \otimes_{\max} A)$.

Lemma 3.5.12. *Let $\alpha\colon G \curvearrowright A$ be a C*-action with G abelian. The pushforward C*-action $T_3T_2T_1\alpha^{\sharp\sharp}$ satisfies $(T_3T_2T_1\alpha^{\sharp\sharp})_s = G \ltimes (\mathrm{rt} \otimes \alpha)_s$ for all $s \in G$.*

Proof. We need to show that $(G \ltimes (\mathrm{rt} \otimes \alpha)_s) \circ T_3 = T_3 \circ (T_2 T_1 \alpha^{\sharp\sharp})_s$. By Lemma 3.5.10 we have

$$\begin{aligned}
(G \ltimes (\mathrm{rt} \otimes \alpha)_s)(T_3(f))(t_1)(t_2) &= (\mathrm{rt} \otimes \alpha)_s(\Phi_3(f(t_1)))(t_2) \\
&= \alpha_s(\Phi_3(f(t_1))(t_2 s)) \\
&= \alpha_s(\alpha_{t_2 s}^{-1}(f(t_1)(t_2 s))) \\
&= \alpha_{t_2}^{-1}(f(t_1)(t_2 s)) \\
&= \alpha_{t_2}^{-1}((\mathrm{rt} \otimes \mathrm{id})_s(f(t_1))(t_2)) \\
&= \Phi_3((\mathrm{rt} \otimes \mathrm{id})_s(f(t_1)))(t_2) \\
&= T_3((G \ltimes (\mathrm{rt} \otimes \mathrm{id})_s)(f))(t_1)(t_2) \\
&= T_3((T_2 T_1 \alpha^{\sharp\sharp})_s(f))(t_1)(t_2)
\end{aligned}$$

for all $f \in C_{\mathrm{cpt}}(G, C_0(G) \otimes_{\max} A)$ and $t_1, t_2 \in G$. ■

The Fourth Isomorphism T_4

The construction of T_4 basically boils down to an application of the Stone-Neumann theorem. Since $u^L \ltimes_{\mathrm{lt}} m$ is an isometric homomorphism with range $K(H^G)$, one readily verifies that there is a unique isometric homomorphism $T_4\colon (G \ltimes_{\mathrm{lt}} C_0(G)) \otimes_{\max} A \to K(H^G) \otimes_{\max} A$ such that $T_4(f \otimes a) = (u^L \ltimes_{\mathrm{lt}} m)(f) \otimes a$ for $f \in G \ltimes_{\mathrm{lt}} C_0(G)$ and $a \in A$. In fact, the map T_4 is even an isomorphism.

Lemma 3.5.13. *Let $\alpha\colon G \curvearrowright A$ be a C^*-action with G abelian. The pushforward C^*-action $T_4 T_3 T_2 T_1 \alpha^{\sharp\sharp}$ satisfies $(T_4 T_3 T_2 T_1 \alpha^{\sharp\sharp})_s = (\kappa(u^R) \otimes \alpha)_s$ for all $s \in G$.*

Proof. We need to show that $(\kappa(u^R) \otimes \alpha)_s \circ T_4 = T_4 \circ (T_3 T_2 T_1 \alpha^{\sharp\sharp})_s$. By Lemma 3.5.12 we have $(T_3 T_2 T_1 \alpha^{\sharp\sharp})_s(f \otimes a) = (G \ltimes \mathrm{rt}_s)(f) \otimes \alpha_s(a)$ for $f \in C_{\mathrm{cpt}}(G, C_0(G))$ and $a \in A$. So it even suffices to show that $\kappa(u^R)_s((u^L \ltimes_{\mathrm{lt}} m)(f)) = (u^L \ltimes_{\mathrm{lt}} m)((G \ltimes \mathrm{rt}_s)(f))$ for $f \in C_{\mathrm{cpt}}(G, C_0(G))$. But this is precisely the content of Lemma 2.6.11. ■

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