Anisotropic deformation on contact three-manifolds

from subRiemannian geodesics to spectral properties of metric-dependent operators

A thesis presented for the
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Abstract

Anisotropic deformation, sometimes generically called adiabatic deformation, is a metric blow-up that is natural to construct on contact manifolds. Such manifolds admit a Riemannian metric which is adapted to the contact distribution in the following sense: the metric g consists of a horizontal part g_h, that comes from a subRiemannian structure on the tangent distribution, and another one g_v on the vertical direction. The deformation then rescales the metric by a factor of $1/\epsilon^2$ on the vertical component. Every variation of the value of ϵ, especially in the limit $\epsilon \to 0$, massively affects almost every metric-dependent attribute of the manifold. Gromov proved that the induced metric space converges in the Gromov-Hausdorff sense to the subRiemannian one.

This thesis is a collection of recent results and possible future developments in the study of anisotropic limits. It starts with a description of some geometrical features of the anisotropic limit, namely the convergence of some Riemannian geodesics to Reeb orbits through *spiraling* subRiemannian geodesics. Then the discussion continues with the study of the perturbation of some Laplace and Dirac-type operators. The purpose is to analyze the convergence of such operators in the resolvent sense.

Acknowledgments

First of all, I would like to express my gratitude to my supervisor Alvaro del Pino, who guided me through this multidisciplinary project. Over the last year, he has always been present to assist me with great enthusiasm. His passion motivated me, and his intuitions and insights were extremely helpful for understanding such delicate topics.

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Contents

Introduction

The notion of *Anisotropy* arises in contrast to the term *Isotropy*. Roughly speaking, an isotropic manifold is a Riemannian manifold whose geometry does not depend on *directions*. In more precise terms, a (pseudo)Riemannian manifold is (locally) isotropic if, for any point $p \in M$ and any pair of non-zero tangent vectors $v, w \in T_pM$ of same length, there exists a (local) isometry φ that fixes p and whose pushforward sends one vector to the other, that is $\varphi(p) = p$ and $\varphi_*(v) = w$. Riemannian manifolds of constant sectional curvature are locally isotropic ([37]), and in general, isotropic manifolds are often related to spaces with constant curvature ([36]). On the contrary, an *anisotropic manifold* is a Riemannian manifold with geometrical properties that differ according to the direction of observation. In physics, the term anisotropic is used to describe the same concept. Crystals, for example, can be anisotropic because they might exhibit properties, such as strength or light refraction, with different values depending on the direction of the measurement. In mathematics, we could say that an *anisotropic deformation* is a process that makes a Riemannian manifold depend on directions. This is still a very vague interpretation. A more precise description of the process is the following: suppose that there exists a splitting $TM = \xi \oplus \xi^\perp$, where both ξ and $\xi^\perp$ possess a smooth fiberwise inner product g_h and g_v, respectively. We may want to endow TM with a (pseudo)Riemannian metric that derives from the subRiemannian structures on ξ and $\xi^\perp$. By anisotropic deformation we mean the process of blowing up the metric $g = g_h + g_v$ along the vertical directions. For any $\epsilon > 0$, we define

$$g_\epsilon = g_h + \frac{1}{\epsilon^2} g_v.$$

If g_v can be canonically chosen, the sequence of metrics $\{g_\epsilon\}_\epsilon$ is sometimes called *canonical family of Riemannian metrics* associated to g_h. Anisotropic deformation has been largely utilized over the last decades ([15],[27],[13],[19],[21],[3]), especially because it is a very natural approach to the study of subRiemannian structures: in principle, the goal of a subRiemannian geometer is to extend the results from Riemannian Geometry to this general setting. Thanks to anisotropic deformation, one could think of the subRiemannian structure (ξ, g_h) as the limit of the Riemannian structures (TM, g_ϵ) as ϵ tends to zero.

One of the classical frameworks on which this is a useful resource is the study of subRiemannian dynamics. SubRiemannian geodesics are length minimizing curves, whose velocity at each point belongs to the *horizontal* distribution ξ. What is interesting to study, within this setting, is the convergence of Riemannian geodesics of (M, g_ϵ) under anisotropic deformation. One of the typical questions is whether it is possible to create a sequence of Riemannian geodesics that converges to a fixed subRiemannian geodesic of (ξ, g_h). The answer to this question depends on the properties of the subRiemannian structure itself. In Riemannian Geometry, every geodesic is determined by the projection to the manifold of a Hamiltonian trajectory. This may not be the case for subRiemannian geodesics. For example, *non-fat* distributions may have ([20]) subRiemannian geodesics that do not derive from solutions to the Hamiltonian equations, but rather from never vanishing characteristics, which are curves on $Ann(\xi) - \{0\}$ with derivatives that preserve the tautological Liouville form on the cotangent bundle. The existence of such singular geodesics complicates the study of the convergence of Riemannian geodesics, and it is sometimes convenient to restrict the framework to fat distributions. Contact manifolds are a typical example of this setup, and will be the main focus of this thesis.

The setup that we will consider is the following: let (M, ξ) be a closed coorientable contact three-manifold. The tangent distribution ξ is completely determined as the kernel of a nowhere vanishing 1-form α, which is unique up to rescaling by smooth positive functions, and it is maximally non-degenerate, meaning that

$\alpha \wedge d\alpha$ is a volume form. Each of these contact forms defines a vector field r, called *Reeb vector field*, that is pointwise transverse to the contact distribution, and preserves the contact form α. For any subRiemannian metric g_h on the two-plane field ξ, we can construct a Riemannian metric g as follows:

$$g := g_h + \alpha \otimes \alpha.$$

We will generally ask g to be adapted to the contact structure, meaning that we want the area form of g to be a multiple of $\alpha \wedge d\alpha$ by a positive constant λ. This is a condition on the subRiemannian metric g_h, but we can always construct one that satisfies it (Theorem 1.4.13). The anisotropic deformation in this setting generates the following 1-parametric family of Riemannian metrics:

$$g_\epsilon := g_h + \frac{1}{\epsilon^2}\alpha \otimes \alpha.$$

Each metric of this family becomes adapted to the contact structure as soon as we rescale the contact form by a factor of ϵ^{-1}. In turn, the Reeb vector field is rescaled by ϵ.

There are two classical approaches to subRiemannian limits on contact manifolds: The first one, studied in [21], wants to take the limit of the perturbed metric for $\epsilon \to \infty$. This construction reduces the length of the Reeb orbits, which will eventually tend to zero. It does not change the topology of the manifold, but it radically affects the dynamics and the induced metric structure of the manifold. In particular, points within the manifold that are connected by a Reeb orbit will have zero distance. This construction can be quite problematic, especially when there are dense Reeb orbits. In this scenario, the induced distance on the manifold collapses in the limit. This subRiemannian limit on contact 3-manifolds is generally considered when the Reeb vector field is Killing, meaning that the Reeb flow generates isometries. Topologically, compact three-manifolds that admit such a Reeb vector field are diffeomorphic to *Seifert manifolds* ([21, Proposition 1.3]). It means that there exists a compact two-orbifold over which M is an S^1-bundle. A typical and insightful example of this setting is the so called *Boothby-Wang fibration.* ([5]) The base manifold of the fibration is a compact symplectic surface (B, ω), whose symplectic structure pulls back to $d\alpha$ via the fiberwise projection. In this construction, the contact form α is also chosen so that the Reeb vector field coincides with the infinitesimal action of S^1. Any metric h on B, with a multiple of ω as area form, can be pulled back to ξ to obtain a subRiemannian metric g_h on ξ that generates an adapted metric on (M, ξ). Consider, as in Figure 1, the splitting of the tangent bundle into the contact

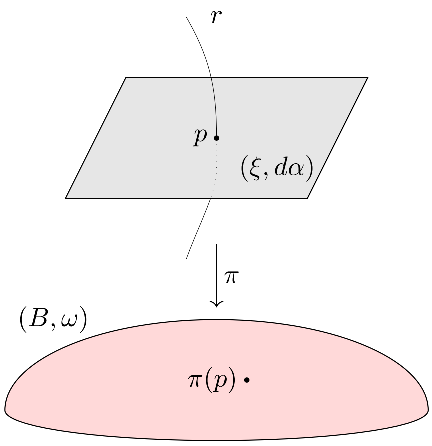

Figure 1: Example of Seifert manifold: Boothby-Wang fibration.

plane ξ and the Reeb direction r. Taking the limit with $\epsilon \to \infty$, makes the S^1-fibers very short, which in turn makes the projection $\pi : M \to B$ an isometry. In this setting, the metric space (M, D_ϵ) with distance induced by g_ϵ converges in the Gromov-Hausdorff sense to (B, D_h) ([15]). The metric spaces (M, D_ϵ) and (B, D_h) are induced by Riemannian metrics. Therefore their Hausdorff dimension coincides with the topological one. In particular, both the topological and Hausdorff dimension of the sequence (M, D_ϵ) changes in the limit from three to two.

The second approach wants to take the limit of the perturbed metric when $\epsilon \to 0$. It is the most common construction in the literature (see [27],[3], [14], [15]), because it can be be geometrically meaningful, regardless of any special requirements on the contact manifold. This construction makes the length of every Reeb orbits diverge, and, in turn, it punishes motion in the vertical direction. In this setting, we expect some kind of convergence of Riemennian geodesics towards subRiemannian ones. We know that the geodesics converge at least in length. In fact, Gromov ([15]) proved that the metric space (M, D_ϵ) converges in the Gromov-Hausdorff sense to (M, D_h), where D_h is the Carnot-Caratheodory distance induced by the subRiemannian metric g_h. The limiting metric space has the same topological distance of any element in the sequence (M, D_ϵ). What changes is the Hausdorff dimension, which in this case is four for (M, D_h) and three for (M, D_ϵ).

In this project, we focus on the second approach towards anisotropic deformation. One of our research questions concerns precisely the uniform convergence of Riemannian geodesics under anisotropic deformation. In particular, in three-dimensional contact dynamics, it is known ([34]) that subRiemannian geodesics with increasingly high initial momenta converge to a Reeb orbit. We want to study the existence of a sequence of Riemannian geodesics of the canonical family of Riemannian metrics $\{g_\epsilon\}_\epsilon$ that converges to a Reeb orbit in the limit $\epsilon \to 0$. In [32], Colin de Verdiére proved that a non-degenerate closed Reeb orbit admits a sequence of closed subRiemannian geodesics converging to it, and he asks if the set of periods of the Reeb flow is a spectral invariant of the subRiemannian contact Laplacian. We believe that proving the existence of a sequence of closed geodesics converging to a non-degenerate closed Reeb orbit under anisotropic deformation would be useful for proving such a conjecture. Richard Melrose ([18]) showed that the set of lengths of closed geodesics is, generically, a spectral invariant of the subRiemannian contact Laplacian. One could hope to recover the Reeb periods from the lengths of the closed geodesics converging to them.

The contact Laplacian is an operator defined on the so called Rumin's complex [26]. This complex is natural constructed on contact manifolds of any (odd) dimension, and its construction relies on the splitting between horizontal and vertical forms. Since the restriction of the exterior derivation to horizontal forms does not square to zero, the Rumin complex is an attempt to get a cochain complex of horizontal forms by forcing the equalities $d_Q^2 = 0$. In the three dimensional case, it takes the form

$$0 \longrightarrow \mathbb{R} \hookrightarrow \mathcal{C}^\infty(M) \xrightarrow{d_Q} \Omega^1\xi \xrightarrow{D} \alpha \wedge \Omega^1\xi \xrightarrow{d_Q} \Omega^3 M \longrightarrow 0,$$

where D is a second order operator connecting horizontal 1-forms with vertical 2-forms. It was Zhong Ge ([13]) who first noted some relations between the Rumin complex and the spectrum of operators of Laplace-type. He proved that the parts of the spectrum of the scalar Laplacian that have finite limits (for $\epsilon \to 0$) tend to concentrate on the spectrum of their counterparts in the Rumin complex. Few years later, Michel Rumin proved ([27]) an analogous result for the Hodge-Laplacian operator. In particular, he showed that the spectrum of the family of Hodge-Laplacians $\{\Delta_\epsilon\}_\epsilon$ can be divided in three classes:

- an *exploding part*, which consists of every eigenvalue that diverges to infinity for $\epsilon \to 0$;
- a *collapsing part*, which consists of all the eigenvalues that tend to zero in the limit;
- a *converging part*, which consists of all the remaining eigenvalues of Δ_ϵ.

The last class of eigenvalues is precisely the part of the spectrum that converges to the spectrum of Δ_Q. It is our goal to follow the work of Rumin in the proof of this classification in the three-dimensional setting and try to observe how would be possible to relate the previous convergence of (closed) geodesics to closed Reeb orbits with this class of converging eigenvalues. We think that a precise description of this relation would involve the generalized Duistermaat-Guillemin trace formula developed by Melrose ([18]).

This thesis is structured in two somewhat separate discussions. The first one, contained in Chapter 1, 2 and 3, treats the geometrical aspects of anisotropic deformation. In the first chapter, we provide the basic background material to understand this thesis, including a brief introduction to subRiemannian and Contact Geometry. We still expect the reader to have some familiarity with Riemannian Geometry and Symplectic Topology, for which we refer to [9], [4] and [17]. Starting from Chapter 2, we focus on three dimensional contact manifolds. In particular, we provide some important results about this three-dimensional setting that cannot be generalized to higher dimensions (see Theorem 2.1.1 and 2.1.3). Then, in Section 2.3.1, we analyze the structural equations of the Levi-Civita connection associated to an adapted metric, and we describe how they are affected by anisotropic deformation. The third is the most relevant chapter of this first part. We first give the intuition behind the circular behaviour of subRiemannian geodesics around Reeb orbits in the Heisenberg group (Section 3.1), and then we present a Theorem (3.2.7), proved in [34], which ensures that subRiemannian geodesics with high initial momenta spiral around a Reeb orbit. We conclude the chapter with some considerations on the uniform convergence of the Riemannian Hamiltonian to the subRiemannian Hamiltonian under anisotropic deformation. Furthermore, we state a result around the convergence of Riemannian geodesics towards closed non-degenerate Reeb orbits.

The second part, consisting of Chapters 4 and 5, has a more analytical flavour: we consider two metric dependent differential operators and look at how they react to the anisotropic deformation. The fourth is the most technically demanding chapter, and it requires some knowledge of functional analysis and estimates with delicate Sobolev norms. We will only state or lightly sketch this type of results. Nevertheless, Chapter 4 gives a complete description of the resolvent convergence of the Hodge-Laplacian on contact three-manifolds, following the work of Rumin ([27]). Chapter 5 treats the behaviour of the Dirac operator under anisotropic deformation. It is somewhat inconclusive in terms of convergence results, but it still gives the idea of what problems could arise in the study of the resolvent convergence of the Dirac operator on contact three-manifolds.

Chapter 1

Tangent distributions and metric perturbations

Tangent distributions are mathematical objects that have been a center of intensive studies over the last decades. They lay the foundations to important areas of mathematics such as Contact Topology, Control Theory and subRiemannian Geometry. Sections 1.2 and 1.3 provide the typical background of these subjects. The final Section introduces the notion of adapted Riemannian metrics on contact manifolds, and describes the general properties of anisotropic deformation of adapted metrics.

For a thorough description of tangent distributions, we suggest [24], while for subRiemannian Geometry and Control Theory, we refer to [19],[16] and [12].

1.1 Tangent distributions

A tangent distribution is a smooth collection of vector subspaces distributed inside the tangent bundle of a given manifold. Let M be a smooth manifold of dimension n. Each of these subspaces is situated inside a fiber of TM. In mathematical language, this object is called *vector subbundle* of TM.

Definition 1.1.1. Let M be a smooth manifold. A *tangent distribution* is a vector subbundle ξ of the tangent bundle TM. The *rank* of ξ is the dimension of the fibers of ξ.

Example 1.1.2. Here a handful of simple examples of tangent distributions:

1. *Trivial line:* Every nowhere vanishing vector field spans a tangent distribution of rank 1.
2. *Trivial distribution:* Every collection of k non-vanishing linearly independent vector fields spans a tangent distribution of rank k.
3. *Kernel of differential 1-forms:* Let $\alpha \in \Omega^1 M$ be a non-vanishing differential 1-form in M. The fiberwise kernel of α defines a codimension 1 tangent distribution on M.
4. *Martinet:* The Martinet distribution is the distribution on $\mathbb{R}^3$ defined as $\ker(dy - z^2 dx)$.
5. *Cartan:* The Cartan distribution is the distribution on $\mathbb{R}^{2n+1}$ defined by $\ker(dz - \sum_{i=1}^n y_i dx_i)$, for the set of coordinates $\{x_1, y_1, ..., x_n, y_n, z\}$.
6. *Carnot:* The Carnot distribution is the distribution on $\mathbb{R}^3$ defined by $\ker\left(dz - \frac{1}{2}(ydx - xdy)\right)$.

Tangent distributions arise naturally in the study of the dynamics of a given system when the solutions are only allowed to travel along specific tangent directions. These type of systems are called *control systems* and their solutions are usually called *admissable curves*. Think for example of the movement of a monocycle on a plane. This vehicle is only allowed to undertake two precise movements: the first one is to follow a straight line in the plane and the second one is to rotate and change direction. Together, they define a tangent distribution on the manifold $\mathbb{R}^2 \times S^1$. Let us see this example in full mathematical terms.

Example 1.1.3. Consider the 3-dimensional configuration space $\Sigma = \mathbb{R}^2 \times S^1$, parametrized by (x, y, θ). Let

$$X_1 := \cos(\theta)\partial_x + \sin(\theta)\partial_y, \qquad X_2 := \partial_\theta$$

be the two possible tangent directions. Geometrically, the first tangent vector corresponds to infinitesimally following a straight line, and X_2 corresponds to changing angle and thus direction as shown in Figure 1.1. The subbundle $\xi := \langle X_1, X_2 \rangle$ becomes the tangent distribution of the system. In particular, a solution can always move in S^1, but for a fixed θ, it cannot move in all directions in $\mathbb{R}^2$. Precisely, it cannot move outside of the red line of Figure 1.1. However, as the line is dependent to the angle θ, it could change direction and reach any point in the configuration space. We will see that this property derives from a curvature condition on the tangent distribution.

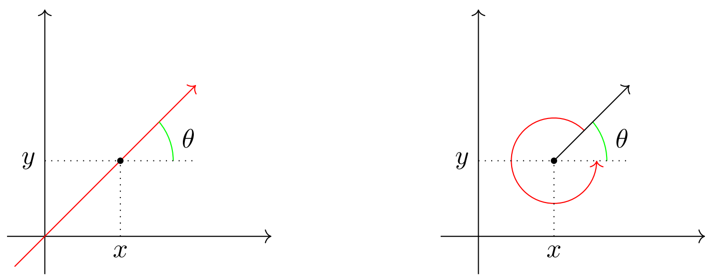

Figure 1.1: Allowed movements of a monocycle on a plane (in red).

Let us formally define the admissable curves of a distribution ξ.

Definition 1.1.4. A curve $\gamma : I \to M$ is called *admissable* or *horizontal* if $\dot{\gamma}(t) \in \xi$ for all $t \in I$.

We do not require γ to be smooth but simply to be absolutely continuous, i.e. to admit one derivative at almost every point and to satisfy the Fundamental Theorem of Calculus on every local representation. The reasons to these assumptions are the following: most times, horizontal curves will be constructed as concatenations of smooth curves, and so, they could be non smooth at finitely many points. Secondly, the integrability condition is fundamental for computing the *length* of an horizontal curve. It is a metric depend attribute of the curve. To define and compute it, we will first introduce a metric on ξ and integrate the norm of the velocity of the horizontal curve over its domain. This is carried out in detail in Section 1.2.

1.1.1 Lie Flag and curvature

To better understand how to construct horizontal curves, we need to study some typical features of the distributions, namely its Lie flag and curvature, both of which depend on where the Lie bracket of the vectors of the distribution takes values.

Definition 1.1.5. We call the *Lie Flag* of a distribution $\xi \subseteq TM$, the monotonically incresing sequence of $\mathcal{C}^\infty(M)$-modules $\{\xi^k\}_{k\in\mathbb{N}}$ defined by

$$\xi^{k+1} := \left\langle [X, Y] : X \in \xi^k, Y \in \xi^1 \right\rangle,$$

where $\xi^1 := \Gamma(\xi)$ is the space of sections of the distribution.

The Lie flag of a general distribution does not necessarily consist of vector bundles, since the Lie bracket is computed pointwise and so it might belong or not to the distribution depending on the fiber on which is computed. The Martinet distribution is an example of such irregularity.

Example 1.1.6. Let ξ be the Martinet distribution introduced in Example 1.1.2. A simple computation shows that ξ is spanned by $\partial_x + z^2\partial_y$ and ∂_z. The Lie bracket of this frame is

$$[\partial_x + z^2\partial_y, \partial_z] = -2z\partial_y,$$

which vanishes on the plane $\{z = 0\}$.

To avoid these cases, we will only consider distributions whose Lie flag consists of vector bundles. These are called *regular* distributions.

Assumption 1.1.7. If not otherwise stated, all distributions from now on will be regular.

The motivation for this assumption is the following.

Lemma 1.1.8. *If ξ is regular, then there exists $m \leq dim(M)$ such that $\xi^k = \xi^m$ for all $k \geq m$. We call this bundle, limiting bundle of the Lie flag of ξ.*

Proof. It follows directly from the fact that M has dimension $n < \infty$ and that, as subbundles, ξ^{k+1} either has strictly greater rank than ξ^k or it is equal to ξ^k. As the tangent bundle has finite rank, ξ^k must stabilize at some point. □

Now that convergence is assured, what can it say about the controllability of the associated system?

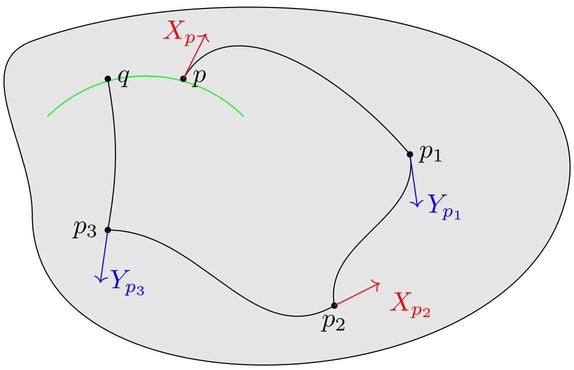

Figure 1.2: Geometrical procedure to reach a point in the integral curve of the Lie bracket (in green) of two horizontal vector fields X, Y, through a point p.

Remark 1.1.9. Consider two vector fields $X, Y \in \xi^1$ in a neighbourhood of a point $p \in M$. As in Figure 1.2, follow the integral curve of X through p for a time t sufficiently small to remain in the neighbourhood of p. Then, at the end point p_1, do the same with the integral curve of Y. Continue this procedure for X and Y with the exception of going backward instead of forward along the integral curves. The end point of this linear combination of smooth horizontal curves is a point q that does not necessarily coincide with the starting point p. In fact, it can be proven that, for

t sufficiently small, $p = q$ if and only if $[X, Y] = 0$ in the neighbourhood. In general, the point q belongs, up to an error of order t^3, to the integral curve of the Lie bracket $[X, Y]$ through p. Therefore, from a point p, this procedure allows us to reach any point q in the integral curves of the elements of ξ^2 through p, via a concatenation of horizontal curves. We can extend this reasoning also to multiple Lie brackets and reach any point in the integral curve of the elements of limiting bundle ξ^n. In particular, the closer ξ^n is to TM, the greater is the space of reachable points in the configuration space and in turn, the controllability of the system.

The monocycle in the plane is an example of setup where the Lie flag of the distribution *converges* to the whole tangent bundle. It is a first example of what we will call *bracket-generating distributions.*

Example 1.1.10. In the setting of Example 1.1.3, we have that the Lie flag of ξ converges to TM already for $k = 2$, because $[X_1, X_2] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$ is pointwise linearly independent to X_1, X_2. This implies that ξ is regular and that for any pair of points (x_1, y_1, θ_1) and (x_2, y_2, θ_2), there exists a piecewise smoth admissable curve connecting the two. To construct it, let θ_0 be the direction of the line containing (x_1, y_2) and (x_2, y_2). Then a possible solution is the combination of the following allowed movements:

$$(x_1, y_1, \theta_1) \xrightarrow{rotating} (x_1, y_1, \theta_0) \xrightarrow{straight} (x_2, y_2, \theta_0) \xrightarrow{rotating} (x_2, y_2, \theta_2).$$

This is the mathematical explanation of what was previously mentioned at the end of Example 1.1.3: simply by a combination of rotating the wheel and following straight lines, the monocycle can reach any point.

It is time to introduce also the notion of *curvature* of a tangent distribution ξ, a differential invariant that measures the Lie bracket properties of the elements of each bundle ξ^k.

Definition 1.1.11. The *curvature* of ξ is the collection of morphisms

$$c_{i,j} : \xi^i/\xi^{i-1} \times \xi^j/\xi^{j-1} \to \xi^{i+j+1}/\xi^{i+j}$$

defined, for each $i, j \geq 1$, as

$$c_{i,j}(X, Y) = [X, Y] \mod \xi^{i+j}.$$

The quotient ξ^k/ξ^{k-1} consists of all the new directions added by the Lie bracket of the vector fields in ξ^k. Clearly, if $\xi^k = \xi^{k-1}$ for all k, then all the $c_{i,j}$ are trivial. For the purpose of the thesis, we are interested in the tangent distributions whose curvature is as non trivial as possible.

We are ready to classify tangent distributions in terms of their curvature.

1.1.2 Foliations

Let us define the first important type of tangent distribution.

Definition 1.1.12. Let M be a smooth manifold of dimension n and $\xi \subseteq TM$ be a (regular) tangent distribution. The distribution ξ is called *foliation* or *flat* if the first curvature $c_{1,1}$ is trivial.

By definition, a foliation is a degenerate distribution, meaning that $\xi^k = \xi^{k-1}$ for all $k \geq 1$. Foliations are generally known for their property of inducing a partition of the manifold. This property is a result of the Frobenius Theorem.

Theorem 1.1.13 (Frobenius). *A distribution ξ of rank k is a foliation if and only if, at every point $p \in M$, there exists an embedding $\psi : [0,1]^n \to M$, called foliation chart, satisfying $p \in Im(\psi)$ and*

$$\psi^* \xi = \ker(dx_{k+1}) \cap ... \cap \ker(dx_n)$$

where $(x_1, ..., x_n)$ are the coordinates in $[0,1]^n$.

This is a celebrated theorem that gives a natural local form to distributions satisfying an infinitesimal property and the other way around.

Remark 1.1.14. Recall that, in the Euclidean space $\mathbb{R}^n$ with standard coordinates $(x_1, ..., x_n)$, the Lie bracket $[\partial_{x_i}, \partial_{x_j}] = 0$ vanishes for all i, j. When it happens to a local frame of a distribution ξ of rank k, Theorem 1.1.13 asserts that there exists an isomorphism that sends those local vector fields to the standard Euclidean frame $(\partial_{x_1}, ..., \partial_{x_k})$, giving a flat local form to the distribution.

The intersections of the level sets of the coordinates $x_{k+1}, ..., x_n$ gives us a partition into k-dimensional submanifolds of $[0,1]^n$ and, in turn, of $\psi([0,1]^n)$. These local partitions are compatible with one another in the sense that they create a global partition into submanifolds $\{L_i\}_{i \in I}$, called *leaves*, which have the distribution ξ as tangent bundle.

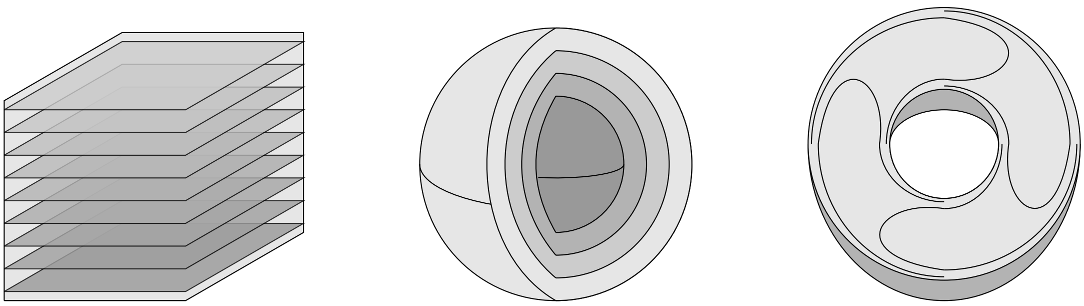

Figure 1.3: Examples of codimension 1 foliations on three-dimensional manifolds. From the left, flat foliation on $\mathbb{R}^3$, sphere foliation on a three-dimensional ball and Reeb foliation on the solid torus.

Example 1.1.15. In Figure 1.3, three examples of codimension one foliations on three-dimensional manifolds are shown. On the left, we have the simplest possible scenario, where flat horizontal planes give a partition of $\mathbb{R}^3$. At least locally, every other codimension one foliation on a three manifold looks like this. In the center, we have a partition of a three dimensional ball B (without the center) by two dimensional spheres of increasingly small radius. For every point $p \in B$, consider the neighbourhood $B - \ell$, where ℓ is a line connecting the center to the boundary of the ball and not containing p. Then, within that open set, each leaf of the partition can be straightened up to make it look like a horizontal plane. On the right, we have a more subtle foliation on a solid torus, called *Reeb foliation*. We can think of each element of this partition as a disk whose center has been pulled along an inside circle. Also in this case, we could cut the solid torus and straighten the leaves to go back to the flat case.

Any non vanishing vector field $X \in \mathfrak{X}(M)$ spans a distribution ξ of rank 1. As all the horizontal vector fields in ξ^1 are just multiples of X, the first curvature of ξ is trivial. Therefore, ξ is a foliation. We can apply the Frobenius Theorem to this setting. This Corollary is usually known as *Flow-box Theorem*. We will rephrase it in a different, but equivalent form.

Corollary 1.1.16 (Flow-box)**.** *Let $X \in \mathfrak{X}(M)$ be non-vanishing vector field. For every $p \in M$, there exist $\epsilon > 0$ and a set of local coordinates $(x_1, ..., x_n)$ around p such that in the box*

$$C := \{(x_1, ..., x_n) \in \mathbb{R}^n : |x_i| < \epsilon, i = 1, ..., n\}$$

the point $p \in M$ corresponds to the origin 0 and X is given by ∂_{x_1}.

An important feature to study about foliations is the compactness of the generated leaves. Let ξ be a foliation on a n-dimensional manifold M and $\{L_i\}_{i \in I}$ be the set of leaves. When M is compact, we have the following equivalent condition.

Theorem 1.1.17. *If M is compact, then each L_i is compact if and only if L_i is covered by finitely many foliation charts in which L_i is a single plaque.*

By *single plaque*, it is meant that the leaf L_i intersects a foliation chart only once. For example, in Figure

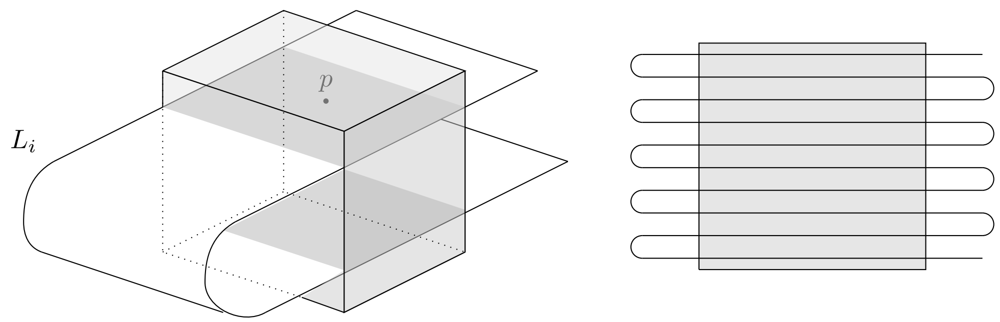

Figure 1.4: Example of foliation chart around a point p where the leaf L_i containing p intersects the foliation chart twice (on the left) and multiple times (on the right).

1.4, the leaf L_i consists of two plaques inside a foliation chart around p. We want to prevent this from happening; or at least, if it happens, we want to consider only the cases where it is possible to cut the cube in two, so that each of them contains only one plaque. This is not always possible because the leaf might be dense in the foliation chart.

Proof. Recall that, for every $p \in L_i$, there exists a foliation chart $\psi : [0, 1]^n \to M$ whose image is closed and it containes p. If we suppose L_i to be compact, then we just need to prove that there exists such a foliation chart at each point $p \in L_i$. By assumption, L_i is not dense in any neighbourhood of p, so we can shrink any foliation chart around p until it contains only one plaque of the leaf L_i.
Conversely, by compactness of M, we just need to prove that M is closed. However, by hypothesis, L_i consists of finitely many (closed) single plaques. □

This Theorem will be extremely useful for checking the compactness of the integral curves of a nowhere vanishing vector field. These are just the leaves of the foliation that the vector field spans.

1.1.3 Bracket-generating distributions

Bracket generating distributions are the central focus of this thesis.

Definition 1.1.18. Let M be a smooth manifold of dimension n and $\xi \subseteq TM$ be a (regular) tangent distribution. The distribution ξ is called *bracket-generating* if the limiting bundle ξ^n coincides with the tangent bundle TM.

The reason to their importance is the following result which assures that the associated system of such distributions is fully controllable.

Theorem 1.1.19 (Chow)**.** *Let ξ be a bracket generating distribution on M and $p \in M$ be a point on the manifold. Then, the set of points that can be connected to p using a horizontal curve is the connected component of M containing p.*

Proof. [19, Theorem 1.6.2]. □

This Theorem confirms the intuition of Remark 1.1.9: the larger ξ^n is, the greater is the controllability of the associated system. In particular, it ensures that if the manifold is connected, then it is horizontally path connected.

Remark 1.1.20. Consider any non bracket-generating distribution ξ. Under the regularity assumption, its Lie flag stabilizes to a subbundle ξ^n that is not equal to $\mathfrak{X}(M)$. This subbundle defines a foliation on M. It can be shown that the leaves of this foliation coincide with the subspaces of points that can be connected through horizontal curves of ξ.

In the next section we will consider only bracket-generating distributions, and we will introduce a metric on them so that it is possible to compute the length of the horizontal curves.

1.2 SubRiemannian geometry

SubRiemannian geometry is the study of bracket generating tangent distributions of a given manifold, when endowed with smooth inner products on the fibers. In the case of a maximal rank subbundle, we recover Riemannian geometry. However, when the rank of the subbundle is smaller than the dimension of the manifold, many major differences to the Riemannian setting arise, as we shall see later on. The purpose of subRiemannian geometry is to overcome these differencies and try to generalize the properties of the Riemannian setting to the subRiemannian one.

The central focus of the area consists of the study of the dynamics of horizontal curves on the distributions ξ. The metric on the distribution can be seen as $(0, 2)$-tensor on the vector bundle ξ. Let us briefly recall some notation and language of tensor algebras. Let M be a smooth n-dimensional manifold and $\pi : V \to M$ be a vector bundle on M.

Definition 1.2.1. A (p, q)-tensor of V is a section τ of the bundle

$$\tau \in \Gamma(\underbrace{V \otimes ... \otimes V}_{\#p} \otimes \underbrace{V^* \otimes ... \otimes V^*}_{\#q}),$$

where V^* is the dual vector bundle.

Example 1.2.2. here a handful of examples

1. A function $f \in \mathcal{C}^\infty(M)$ is a $(0, 0)$-tensor.
2. A vector field $X \in \mathfrak{X}(M)$ is a $(1, 0)$-tensor of the tangent bundle TM or any tangent distribution ξ that contains the vector field. Each $(1, 0)$ tensor X on ξ gives rise to the so called *momentum function* $P_X : TM^* \to \mathbb{R}$ defined by $P_X(p, w) = w(X_p)$.
3. Any differential form $\alpha \in \Omega^q M$ is a $(0, q)$-tensor of TM or either a $(q, 0)-$tensor of the cotangent bundle TM^* when TM is endowed with a metric.

4. By the universal property of the tensor product, any bilinear map on a vector bundle V gives rise naturally to a section of the dual of the tensor product bundle $(V \otimes V)^*$. In the finite dimensional setting, $(V \otimes V)^* \cong V^* \otimes V^*$, meaning that any such bilinear map corresponds to a $(0,2)$-tensor.

5. By the previous point, any metric g on a vector bundle V defines a $(0,2)$-tensor of V. Furthermore, they define a particular type of $(0,2)-$tensor.

The subRiemannian setting is the natural generalization of the Riemannian one.

Definition 1.2.3. A *subRiemannian structure* on M is a pair (ξ, g_h) where ξ is a tangent distribution and g_h is a metric tensor on ξ, often called *horizontal metric.*

Remark 1.2.4. We use the subscript h on the metric to stress the subRiemmanian setting. Shortly, we will be talking about Riemannian metrics g deriving from subRiemannian ones g_h, and we want to stay consistent to that notation.

Clearly, a Riemannian structure (TM, g) is also subRiemannian. Moreover, by the isomorphism $TM \cong TM^*$, Riemannian structures correspond uniquely to fiber inner products of TM^*, thus to positive definite symmetric $(2,0)$ tensors $g^* \in \Gamma(M, Sym^2 TM)$.
We can say something similar in the subRiemannian case.

Proposition 1.2.5. *SubRiemannian structures are in 1:1 correspondence with positive semi-definite symmetric* $(2,0)$*-tensors* $g^* \in \Gamma(M, Sym^2 TM)$ *of constant rank.*

Proof. Given a subRiemannian structure (ξ, g_h), consider the bundle map $\flat : \xi \to \xi^*$ given by $\flat(X) = g_h(-, X)$. By non-degeneracy of g_h, this is an isomorphism of vector bundles, so let $\#$ be its inverse. Denote by $Ann(\xi) \subset TM^*$ the *annihilator* of ξ, i.e. the bundle of forms $\alpha \in TM^*$ satisfying $\alpha \mid_\xi = 0$. Then the dual of ξ, ξ^* is isomorphic to the quotient $TM/Ann(\xi)$. If we let $q : TM^* \to \xi^*$ be the quotient map, then we can say that the subRiemannian structure g_h defines a positive $(2,0)$ tensor g_h^* by

$$g_h^*(\alpha, \beta) := g_h(\#^q \alpha, \#^q \beta) \qquad \forall \alpha, \beta \in TM^*$$

where $\#^q = \# \circ q$. Clearly, g^* vanishes in $Ann(\xi)$ and it is positive elsewhere, so the tensor g^* is a cometric if and only if $Ann(\xi) = \{0\}$, i.e. the Riemannian case.
Conversely, let $g^* \in \Gamma(M, Sym^2 TM)$. Define $\xi \subset TM$ as the image of the map $\#^q : TM^* \to TM$ sending $\alpha \in TM^*$ to $g^*(-, \alpha) \in TM^{**} \cong TM$. Then, endow H with the fiber inner product

$$g_h(\#^q \alpha, \#^q \beta) := g^*(\alpha, \beta) \qquad \forall \alpha, \beta \in TM^*$$

□

Positive semi-definite symmetric $(2,0)$-tensors of constant rank are usually called *subRiemannian cometrics*, in view of the clear correspondence to the Riemannian case. Note that a subRiemannian cometric g_h^*, when restricted to the dual of the corresponding distribution (ξ, g_h), is exactly the cometric g_h^* on ξ^*. Therefore, the abuse of notation is motivated.

There is another classification of subRiemannian structures in terms of the Hamiltonian of the system.

Definition 1.2.6. Given a subRiemannian cometric g_h^*, we define the subRiemannian Hamiltonian $H : TM^* \to \mathbb{R}$ as $H(p, w) = g_h^*(w, w)$ where we specified the dependency on the fiber $T_p^* M$ for later convenience.

Any subRiemannian structure is completely described by its Hamiltonian: each non-negative fiber quadratic function $H : TM^* \to \mathbb{R}$ of constant fiber rank defines a unique subRiemannian structure. This is a clear consequence of Proposition 1.2.5.

1.2.1 Hamiltonian dynamics

Generally speaking, a *Hamiltonian* is simply a function on a phase space X, i.e. a smooth manifold with a symplectic structure. When we want a general smooth manifold M to be the configuration space, the usual phase space becomes the cotangent bundle TM^*. This is due to the fact that it has a canonical symplectic structure.

Intermezzo: canonical symplectic structure on the cotangent bundle

A way to define the canonical symplectic structure on TM^* is through one of its primitives, namely the *Liouville tautological 1-form*.

Definition 1.2.7. The*Liouville tautological 1-form* is a form $\lambda_{can} \in \Omega^1(TM^*)$ satisfying

$$(\lambda_{can})_{(p,w)}(v) = w((d\pi)_p(v)), \qquad \forall (p,w) \in TM^*,$$

where $\pi : TM^* \to M$ is the standard projection.

It is called tautological because of its tautological behaviour under pullback of 1-forms in M. To clarify this statement, let $\alpha \in \Omega^1(M)$ be a 1-form on M. By definition, α is a section in $\Gamma(TM^*)$, so in particular, it is a smooth map $\alpha : M \to TM^*$. Therefore, its pullback α^* takes forms on TM^* to forms in M. We claim that $\lambda_{can} \in \Omega^1(TM^*)$ is the unique 1-form satisfying $\alpha^*\lambda_{can} = \alpha$.

Proposition 1.2.8. *The Liouville tautological* 1*-form* λ_{can} *is the unique form satisfying* $\alpha^*\lambda_{can} = \alpha$ *for every* $\alpha \in \Omega^1 M$

Proof. Let $\alpha \in \Omega^1 M$ and $(p, v) \in TM$. Then

$$(\alpha^*\lambda_{can})_p(v) = (\lambda_{can})_{\alpha(p)}((d\alpha)_p(v)) = \alpha_p((d(\pi \circ \alpha))_p(v)) = \alpha_p(v),$$

because α is a section of the bundle $\pi : TM^* \to M$. For the uniqueness, consider a second form λ'_{can}. Then, $\alpha^*(\lambda_{can} - \lambda'_{can}) = 0$ for every $\alpha \in \Omega^1 M$, meaning that $\lambda_{can} = \lambda'_{can}$. □

We want to prove that the exterior derivative of λ_{can} defines a symplectic structure on TM^*. To do so, we first need to describe λ_{can} in local terms. Let $\{x_i\}_i$ be local coordinates on the base manifold and $\{P_i\}_i$ be the *momentum functions* induced on the fibers of TM^*. We can think of each P_i as a smooth function $P_i : TM^* \to \mathbb{R}$ given by the vector field ∂_{x_i} when seen as a $(1, 0)$-tensor as in Example 1.2.2. Explicitly, for $(p, w) \in TM^*$, we write

$$P_i(p, w) := w(\partial_{x_i}).$$

Thus, $\{x_1, ..., x_n, P_i, ..., P_n\}$ becomes a set of local coordinates on the total space TM^*. In local terms, the defining property of λ_{can} is the following:

$$(\lambda_{can})_{(p,w)}\left(\sum_{i=1}^n (a_i\partial_{x_i} + b_i\partial_{P_i})\right) = \sum_{i=1}^n w(a_i\partial_{x_i}) = \sum_{i=1}^n a_i P_i(p, w).$$

Therefore, $\lambda_{can} = -\sum_{i=1}^n P_i dx_i$. Its differential corresponds to

$$d\lambda_{can} = \sum_{i=1}^n dP_i \wedge dx_i.$$

Lemma 1.2.9. *The 2-form $\omega_{can} := d\lambda_{can}$ is a symplectic structure on TM^*. It is called canonical symplectic form.*

Proof. Clearly, it is a closed form. To see that it is non degenerate, note that, in local terms,

$$\sum_{i=1}^{n} \iota_{(a_i\partial_{x_i}+b_i\partial_{P_i})}\omega_{can} = \sum_{i=1}^{n} b_i dx_i - a_i dP_i = 0 \qquad \iff \qquad a_i = b_i = 0.$$

□

Another important object in this setup is the *Liouville vector field* $X \in \mathfrak{X}(TM^*)$. By definition, it satisfies $\iota_X(\omega_{can}) = \lambda_{can}$. Thus, in local terms, it is given by

$$X := \sum_{i=1}^{n} P_i \partial_{P_i}. \tag{1.1}$$

We will see how this vector field relates to the Hamiltonian flow. An interesting feature of this vector field is that it is always transverse to the unit sphere bundle STM^* constructed from any subRiemannian cometric g^*. The unit sphere bundle STM^* is the locus of points (p, w) with unit norm, i.e. $g_p^*(w, w) = 1$. Note that, unless g^* is positive definite, STM^* does not actually look like a sphere, but rather a cylinder. In any case, the integral curves of X are not tangent to it. In fact, they are tangent to the fibers of TM^* and each of them can be written as

$$\gamma(t) = (p, e^t w) \in T_p^*M.$$

In particular, $g^*(\gamma(t), \gamma(t)) = e^{2t} g_p^*(w, w)$. So, for $(p, w) \notin Ann(\xi)$ we have that

$$\frac{d}{dt} g^*(\gamma(t), \gamma(t)) > 0,$$

which means that X is transverse to the positive level sets of g_p^*.

Hamiltonian flow

Let H be a Hamiltonian on (TM^*, ω_{can}). Associated to H, there is the vector field $X_H \in \mathfrak{X}(TM^*)$ defined on the tangent bundle of TM^*.

Definition 1.2.10. We call *Hamiltonian flow* of H the vector field X_H satisfying

$$dH = -\iota_{X_H} \omega_{can} \tag{1.2}$$

This is unique by non-degeneracy of ω_{can}.

The Hamiltonian flow will be the vector field that characterizes the *geodesics* of the system, defined in Section 1.2.2. Recall that each vector field has some associated integral curves, which are the curves on the manifold whose velocity at each point coincides with the value of the vector field there.

Remark 1.2.11. We call *Hamiltonian equations* of H the ordinary differential equations for the integral curves of X_H. In the local coordinates $\{x_i, P_i\}_i$, these are given by

$$\dot{x}_i = \frac{\partial H}{\partial P_i} \qquad\qquad \dot{P}_i = -\frac{\partial H}{\partial x_i}. \tag{1.3}$$

A curve $\zeta(t) = (\gamma(t), \alpha(t))$, with $\gamma(t)$ being a curve on M and $\alpha(t)$ a curve on the cotangent bundle, is a solution of (1.3) if and only if, in local coordinates,

$$\dot{\zeta}(t) = \left(\frac{\partial H}{\partial P_1}\bigg|_{\zeta(t)}, \ldots, \frac{\partial H}{\partial P_n}\bigg|_{\zeta(t)}, -\frac{\partial H}{\partial x_1}\bigg|_{\zeta(t)}, \ldots, -\frac{\partial H}{\partial x_n}\bigg|_{\zeta(t)} \right).$$

This is the same as saying

$$\omega(\dot\zeta(t),-) = \sum_{i=1}^{n} dP_i(\dot\zeta(t))dx_i - dx_i(\dot\zeta(t))dP_i = -\sum_{i=1} \frac{\partial H}{\partial P_i}\bigg|_{\zeta(t)} dP_i - \frac{\partial H}{\partial x_i}\bigg|_{\zeta(t)} dx_i = -dH,$$

which means that $\dot\zeta(t) = (X_H)_{\zeta(t)}$ by non degeneracy of ω.

One immediate example of Hamiltonian could be the momentum function $H = P_X : TM^* \to \mathbb{R}$ of a vector field X. In this setting, there is an interesting relation between the vector fields X and X_H: if we lift the flow of X to a flow in TM^* through pushforward of forms, then it turns out that the lifted flow is generated by the Hamiltonian flow X_H of H.

SubRiemannian Hamiltonian

We will mostly focus on the subRiemannian Hamiltonian, defined in Definition 1.2.6, due to its correspondence to horizontal geodesics, which will be introduced in the next section.

An useful alternative (local) description of the subRiemannian Hamiltonian is contained in the following Lemma.

Lemma 1.2.12. *Let $\{X_i\}_i$ be a local orthonormal frame of the horizontal distribution ξ and $\{P_{X_i}\}_i$ be their momentum functions. The subRiemannian Hamiltonian takes the form*

$$H = \sum_i P_{X_i}^2.$$

Proof. On $Ann(\xi)$, the cometric g^* is trivial, and so are the momentum functions P_{X_j}, for every element X_j of the local frame of ξ. Since the local frame is orthogonal, then it is trivial to prove to see that $g^* = \sum_i P_{X_i}^2$. □

1.2.2 Horizontal geodesics

Horizontal geodesics are the length minimizing horizontal curves of a given subRiemannian structure. This means that if γ is a horizontal geodesic connecting two points p and q on the manifold M, then there is no other horizontal curve between p, q with smaller length than γ. In some sense, geodesics are the *optimal* way to move around the manifold. As such, they define a metric on M, where the distance between 2 points is precisely given by the length of the horizontal geodesic connecting them, if it exists. Let (ξ, g_h) be a subRiemannian structure on M.

Definition 1.2.13. We set the *length* and the *energy* of a horizontal curve $\gamma : I \to M$ to be

$$\ell(\gamma) := \int_I \|\dot\gamma(t)\|_h dt = \int_I \sqrt{g_h(\dot\gamma(t), \dot\gamma(t))}dt, \qquad E(\gamma) := \int_I \|\dot\gamma(t)\|_h^2 dt. \tag{1.4}$$

The existence of horizontal curves connecting any two points in each connected component of a manifold is assured by the Chow Theorem in the case of bracket generating distrbutions. We will usually assume ξ to be bracket generating and M to be connected, so that every pair of points in M can be connected by a horizontal curves.

Assumption 1.2.14. From now on, unless otherwise specified, any subRiemannian structure (ξ, g_h) will consist of a bracket-generating distribution.

Remark 1.2.15. The quantities in Equation (1.4) satisfy an interesting relation. Let $\gamma : I \to M$ be a horizontal curve. Then, by Cauchy-Schwartz,

$$\ell(\gamma) = \int_I \|\dot{\gamma}(t)\|_h dt \leq \sqrt{\int_I \|\dot{\gamma}(t)\|_h^2 dt}\sqrt{\int_I 1 dt} = \sqrt{2E(\gamma)}\sqrt{|I|}.$$

In particular, there is an equality only for curves with constant speed $\|\dot{\gamma}(t)\| = c \in \mathbb{R}$. Therefore, γ minimizes the energy if and only if it minimizes also the length and it can be reparameterized to have constant speed. This is always the case for geodesics because they have non zero velocity at every point.

Definition 1.2.16. We define the *Carnot-Caratheodory distance* in M as

$$d(p, q) = \inf_{\gamma \text{ hor. curves } p \to q} \ell(\gamma).$$

A *subRiemannian geodesic* $\gamma : I \to M$ between p and q is a horizontal curve that realizes the distance between those points.

Under our assumptions, local existence of subRiemannian geodesics (but not uniqueness) is ensured.

Theorem 1.2.17. *Let (ξ, g_h) be a subRiemannian structure on M such that ξ is a bracket generating distribution and let $p \in M$ be a point on the manifold. There exists a neighbourhood $U \subset M$ of p such that each point in U can be connected to p through geodesics.*

Proof. [19, Theorem 1.6.3] □

We can actually do more that this, if we require the Carnot-Charatheodory distance to be *complete*.

Theorem 1.2.18. *Let (ξ, g_h) be a subRiemannian structure with bracket-generating distribution on a connected manifold M that is complete with respect to the Carnot-Caratheodory distance. Then, any two points can be connected by a subRiemannian geodesic.*

Proof. [19, Theorem 1.6.4]. □

Proving that a horizontal curve is a subRiemannian geodesic often relies on the following process. Let $\gamma : I \to M$ be a horizontal curve with velocity contained in the sphere bundle $S\xi$, i.e. γ is a horizontal curve of constant speed 1. We can always assume this to be true up to reparametrization. In turn, we can write $I = [0, T]$ where T is the length of the curve. Let $p := \gamma(0)$ and $q := \gamma(T)$ be the points connected by γ and let

$$\Omega_{p,T} := \{\zeta : [0, T] \to M : \zeta' \in S\xi, \zeta(0) = p\}$$

be the set of all horizontal curves of constant speed 1 starting at p. We can endow $\Omega_{p,T} \cap L^2$ with a smooth structure deriving from M ([19, Theorem E.0.1]), so that it becomes an infinite dimensional Hilbert manifold. Define also the *end-point map*

$$end_p^T : \Omega_{p,T} \to M,$$

sending a subRiemannian geodesic ζ to its endpoint $\zeta(T)$, and

$$R_{p,T} := \{\zeta(T) \in M : \zeta \in \Omega_{p,T}\}$$

as the set of *reachable* points by curves of length T. Note that $R_{p,T}$ is just the image of the endpoint map. The space $R_{p,T}$ coincides with the ball $B_T(p)$ of radius T with center at p induced by the Carnot-Caratheodory metric. Moreover, it is precisely the image of the endpoint map defined on the space of subRiemannian geodesics with length at most T, which is a subset of $\Omega_{p,T}$. The shape and topological properties of $R_{p,T}$ are still subject to heavy investigation and remain an open question.
By construction, the point q is clearly inside of $R_{p,T}$ and proving that it has precisely distance T from p is equivalent to proving that γ is a subRiemannian geodesic.

> *Remark* 1.2.19. By definition, γ is a subRiemannian geodesic if and only if $q \in \partial R_{p,T}$, i.e. if q is contained in the boundary of the reachable points. A boundary point is not easily reachable, so it must be reached in an optimal way, through geodesics.

By construction, end_p^T is a smooth map between manifolds and therefore, we can study its critical points. Its differential at γ, $d_\gamma end_p^T : T_\gamma \Omega_{p,T} \to T_{\gamma(T)}M$, can be seen as the linearization of the map

$$d_\gamma end_p^T : \Gamma_\gamma(TS\xi) \to T_{\gamma(T)}M \tag{1.5}$$

where $\Gamma_\gamma(TS\xi)$ corresponds to the variations of γ along $S\xi$. An element of $\Gamma_\gamma(TS\xi)$ is a smooth section of the tangent space of the sphere bundle $S\xi$, restricted to the image of γ inside M. In Figure 1.5, we

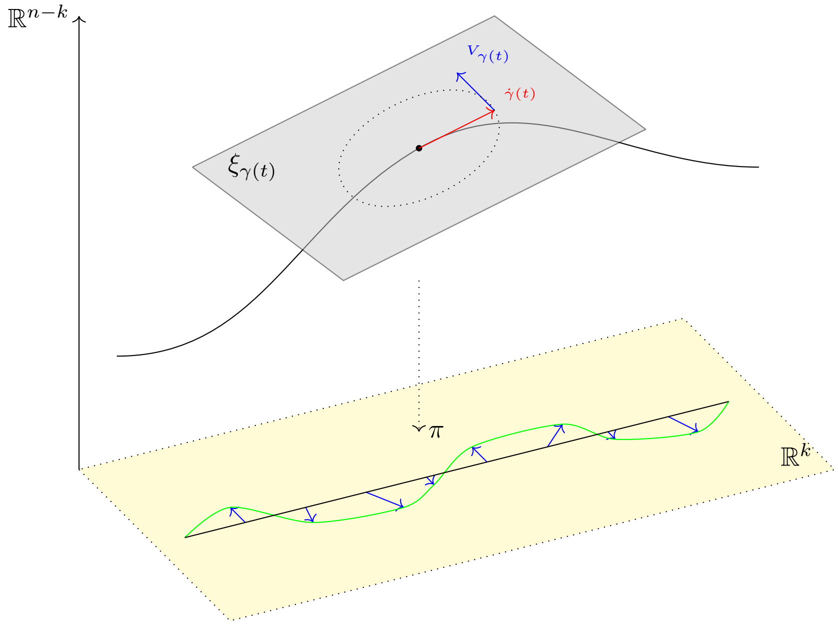

Figure 1.5: Projection of a neighbourhood of the curve γ to $\mathbb{R}^k$. Along the curve it is shown how to construct a vector $V_\gamma(t)$ and how it projects down to $\mathbb{R}^k$. The green curve is the perturbed one.

highlight the meaning of this statement. Under appropriate conditions on the curve, we can find a tubular neighbourhood of γ and a map π defined on the neighbourhood, which sends γ to the first coordinate axis of $\mathbb{R}^n$ and makes the tangent distribution ξ flat and spanned by the first k coordinates at each point. Then, a section $V \in \Gamma_\gamma(TS\xi)$ projects to an actual variation in $\mathbb{R}^k$ through $d\pi$ and therefore, we can integrate and obtain the curve ζ of the perturbation $d\pi(V)$ in $\mathbb{R}^k$. This curve is drawn in green in Figure 1.5. By lifting ζ to the original neighbourhood of γ, we create a new curve $\widetilde{\zeta} \in \Omega_{p,T}$. Due to the lifting process, $\widetilde{\gamma}$ differ from the original γ by a section $\widetilde{V} \in \Gamma_\gamma(TS\xi)$ that satisfies

$$d\pi(\widetilde{V} - V) = 0,$$

or equivalently, that projects to the same euclidean variation. Therefore, it might be possible that γ and $\widetilde{\zeta}$ do not even share the same ending point. The map in Equation (1.5), sends V to the variation between the two endpoints of γ and $\widetilde{\zeta}$. From this description, we see that if q is a boundary point of $R_{p,T}$, then the map in Equation (1.5) cannot be surjective, because, for any variation of γ, the variation of the endpoints can only point in a direction inside the ball, and not outside. This concept is graphically shown in Figure

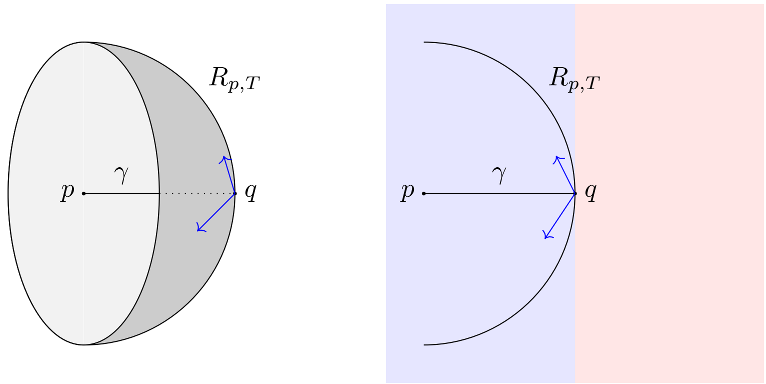

Figure 1.6: Right hemisphere of $R_{p,T}$ assuming $q \in \partial R_{p,T}$

1.6. In particular, it implies that subRiemannian geodesics are critical points of the endpoint map end_p^T. The point of doing this process locally is that if γ is a geodesic, then it must be a critical point of the endpoint map for every $t \in [0, T]$. In fact, to be optimal, a curve needs to be optimal at every previous time.

Lemma 1.2.20. *If $\gamma \in \Omega_{p,T}$ is a geodesic, then it must be a critical point of the endpoint map.*

A careful analysis of the critical points of the endpoint map is the typical procedure to prove (or disprove) that a curve is a subRiemannian geodesic.

1.2.3 Cogeodesic flow

The previous reasoning can be used to prove the following result. The idea is that a curve γ is a critical point of the endpoint map if and only if there exists a lift $\zeta : [0, T] \to TM^*$ of the curve γ to the cotangent bundle which is either an Hamiltonian orbit of the cometric or belongs to the annihilator bundle $Ann(\xi)$. In the first case, γ is called *normal* geodesic, and this is what the following Theorem claims.

Theorem 1.2.21. *Let (ξ, g_h) be a subRiemannian structure and H be the subRiemannian Hamiltonian. If $\zeta(t) = (\gamma(t), P(t))$ is a solution of Equation (1.3), then every sufficiently short arc of γ is a horizontal geodesic. Moreover, γ is the unique geodesic joining its endpoints.*

Proof. [19, Theorem 1.5.7]. □

This theorem is a general result, but it has some limitations. In particular, the Hamiltonian equations do not always describe every geodesic of the subRiemannian structure: there exist horizontal geodesics that are not solutions to the Hamiltonian equations, these are called *singular* geodesics.

Example 1.2.22. Consider once again the Martinet distribution $\xi := \ker(dy - z^2 dx)$. We saw that it is not a regular distribution. Independently of the subRiemannian metric on ξ, it also admits singular geodesics, which are precisely the straight lines on the plane $\{y = 0\}$, parallel to the x-axis.

The singular geodesics, even if they are not solutions to the Hamiltonian equations, are still geodesics, and thus they are critical points of the endpoint map. In particular, they are the critical points which admit a particular lift to the contangent bundle with values in $Ann(\xi)$. Let ω_{can}, be the standard symplectic structure on TM^* and ω be the restriction of ω_{can} to the annihilator bundle. It is not necessarily true that $(Ann(\xi) - \{0\}, \omega)$ is a symplectic manifold. When this is the case, ξ is called *fat*. For the moment, we do not require that.

Definition 1.2.23. A *characteristic* for $Ann(\xi)$ is an absolutely continuous curve in $Ann(\xi)$, which never intersects the zero section and whose derivatives lie in the kernel of ω almost everywhere.

Therefore, an absolute continuous curve $\zeta : I \to Ann(\xi)$ is a characteristic if $\zeta(t)$ never intersects the zero covector and

$$\zeta^*\omega = 0.$$

Lucas Hsu proved the following Theorem:

Theorem 1.2.24 (Hsu)**.** *As horizontal curve is a singular geodesic if and only if there exists a never vanishing characteristic for $Ann(\xi)$ which projects to it.*

Proof. [8, Proposition 1]. □

It turns out that if the distribution ξ is fat, then there are no nowhere vanishing characteristics for $Ann(\xi)$, and therefore, no singular geodesics.

Proposition 1.2.25. *If ξ is fat, then all horizontal geodesics are normal.*

Therefore, the existence of singular geodesics is restricted to non-fat distributions. In the rest of the thesis, we will focus on contact distribution, which are indeed fat.

We saw in Remark 1.2.11 that the solutions of the Hamiltonian equations coincide with the integral curves of the Hamiltonian flow X_H. Since X_H is nowhere vanishing, we can always reparametrize the

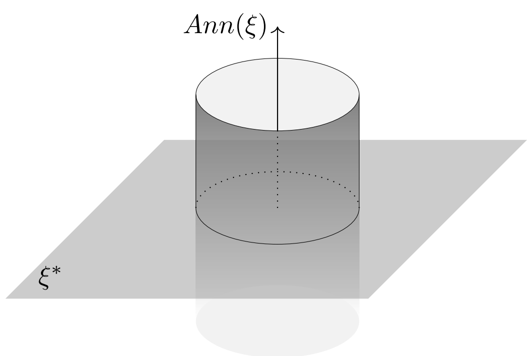

Figure 1.7: Identify the cotangent bundle at one point with $\mathbb{R}^3$. The cylinder represents the sphere bundle STM^* for the subRiemannian cometric g^*.

solutions to have constant speed and in particular, to be curves inside the sphere bundle STM^*. This space, as shown in Figure 1.7, is parallel to the annihilator bundle and so, it consists of a regular levelset of the cometric g^*. Moreover, we showed that it is transverse to the Liouville vector field of the canonical symplectic structure on the cotangent bundle. We will explain many more interesting features of this space in the contact setting.

1.3 Contact geometry

Contact structures are a particular type of (fat) bracket-generating distributions that appear only on odd-dimensional manifolds. For many reasons, Contact Geometry is often viewed as the odd-dimensional counterpart of Symplectic Geometry. These areas relate to each other in interesting ways. For example, some *contact manifolds*, which are pairs (M, ξ) where ξ is a contact distribution on M, can be naturally constructed as boundaries of symplectic ones. This relation allows one to move from the symplectic setting to the contact one. Moreover, as a fat distribution, its annihilator bundle admits a symplectic structure given by the restriction of the canonical one. We will see other interesting relations between the two areas in Section 1.3.5, once we have established some notation for contact structures.

First, let us see the definition of contact distributions and why they appear only on odd-dimensional manifolds. Let M be a smooth $n-$dimensional manifold.

Definition 1.3.1. A *contact structure* on M is a codimension 1 distribution ξ on M whose first curvature $c_{1,1}$ is a non degenerate two-form on ξ with values in the *dual bundle* TM/ξ.

Asking for the first curvature to be non degenerate is generally stronger than asking for ξ to be bracket-generating. In the first case, we have that, for any horizontal vector field X, there exists $Y \in \Gamma(\xi)$, such that $c_{1,1}(X, Y) \neq 0$. In the second case, we just have that there exists $X, Y \in \Gamma(\xi)$ such that $c_{1,1}(X, Y) \neq 0$. On a 3-dimensional manifold, these two conditions are the same but it is generally not true in higher dimensions.

Lemma 1.3.2. *Let ξ be a contact structure on M. Then, the rank of ξ is even.*

Proof. Let $p \in M$ be a point in M. Identify $T_pM/\xi_p \cong \mathbb{R}$ by choosing a vector in the fiber. On that fiber, the first curvature $c_{1,1}$ corresponds to a skew-symmetric non degenerate bilinear form $(c_{1,1})_p : \xi_p \times \xi_p \to \mathbb{R}$ on a $n-1$ dimensional vector space. It is commonly known in linear algebra that this type of maps only exist on even dimensional vector spaces. To give an idea of the proof, let V be a vector space of dimension k and proceed by induction on k. If $k = 1$, a simple computation shows that there are no non-degenerate bilinear forms on V. In higher odd dimensions, assume by contradiction the existence of such form. Then, we can split V into two vector spaces V_1 and V_2 where one of them has dimension 2. The contradiction comes from the fact that the restriction of the form to each of these vector spaces is still non degenerate. $\square$

Since the rank of a contact structure is even, the dimension of a manifold that admits such a structure must be odd. Let us see some interesting features of contact structures.

1.3.1 Orientation and coorientation

Let us briefly talk about the orientability of the manifold and the contact distribution. It is an important topic because, whenever the two orientations behave *nicely* together, it is possible to define a global 1-form that characterizes the contact structure. This is an anticipation of Theorem 1.3.7.
Recall that an orientation on a vector bundle $V \to M$ of rank r, is just a choice of a sign for the volume forms on the bundle, which gives, in turn, a global trivialization of the determinant line bundle $\det(V) := \Lambda^r V$. In the case of TM and ξ, we have the following vector bundle isomorphism

$$TM \cong \xi \oplus TM/\xi,$$

that splits the tangent bundle into the distribution ξ of rank $2n$ and the line bundle TM/ξ. We can compute the determinant line bundle of TM in terms of this splitting: we get

$$\det(TM) \cong \det(\xi \oplus TM/\xi) \cong \det(\xi) \otimes TM/\xi.$$

Moreover, by the fact that $c_{1,1}$ is a non degenerate 2-form on TM/ξ, its n-th power provides an isomorphism of $\det(\xi)$ with $\otimes^n TM/\xi$. All together,

$$\det(TM) \cong \otimes^{n+1} TM/\xi.$$

For even powers, the tensor product of a line bundle is canonically trivial. Indeed, every real line bundle L is isomorphic to its dual L^*, and therefore $L \otimes L \cong L \otimes L^* \cong Hom(L, L)$, which is a line bundle that admits a nowhere vanishing section, namely Id. We can generalize this to any even power.
Thus, depending on the parity of n, either $\det(\xi)$ or $\det(TM)$ is trivial which means that we obtain canonical orientations either on ξ or TM.

- if n is even, then $\det(\xi)$ is trivial and $\det(M) \cong TM/\xi$. Thus, there is a canonical orientation of ξ.
- if n is odd, then $\det(\xi) \cong TM/\xi$ and $\det(TM)$ is trivial. Thus, there is a canonical orientation on TM.

From this description we obtain that TM and ξ are both orientable if and only if the dual bundle TM/ξ is trivial.

Definition 1.3.3. We say that M and ξ are *coorientable* if the dual bundle TM/ξ is trivial.

When this is the case, both bundles have a canonical orientation in every dimension and we can globally define a $1-$form that only vanishes on the contact distribution.

1.3.2 Contact form

Let M be a $(2n+1)$-dimensional manifold and let ξ be a contact distribution on M.

Definition 1.3.4. A *contact form* of the contact manifold (M, ξ) is a 1-form $\alpha \in \Omega^1(M)$ satisfying $\xi = \ker(\alpha)$.

We mentioned (and we will prove) that the existence of such a form depends on the coorientability of M and ξ. At least locally, such a form always exists. This is due to the fact that, for every point $p \in M$, there exists a neighbourhood U on which TM/ξ is trivial. In turn, the contact manifold $(U, \xi\mid_U)$ becomes coorientable. This is a sufficient condition for the existence of a contact form for $(U, \xi\mid_U)$.

Lemma 1.3.5. *For every point $p \in M$, there exist a neighbourhood U of p and a 1-form $\alpha_U \in \Omega^1(U)$ such that $\xi\mid_U = \ker(\alpha_U)$ and $\alpha_U \wedge (d\alpha_U)^n \neq 0$.*

Proof. Locally, $\xi\mid_U$ is just a smooth collection of hyperplanes of $\mathbb{R}^{2n+1}$. It is always possible to construct a $1-$form α_U that vanishes pointwise on those hyperplanes. Moreover, the composition of α_U with the zero curvature gives the following relation: for $X, Y \in \Gamma(\xi\mid_U)$,

$$\alpha_U \circ c_{1,1}(X, Y) = \alpha_U([X, Y]) = -d\alpha_U([X, Y]).$$

As α_U does not vanish on TM/ξ, the non-degeneracy of $c_{1,1}$ implies the non-degeneracy of $d\alpha_U$ on $\xi\mid_U$. In turn, $(d\alpha_U)^n$ becomes a volume form in $\xi\mid_U$ and $\alpha_U \wedge (d\alpha_U)^n \neq 0$. □

The previous Lemma assures that, at least locally, we can always find a $1-$form that defines the distribution and whose differential is non degenerate on the distribution. The natural question to ask at this point is whether these local forms can be patched together to obtain a global 1-form that characterizes the distribution.

Remark 1.3.6. Let U and V be two open subsets of M such that $U \cap V \neq \emptyset$ and let α_U and α_V be the two local 1-forms satisfying the properties of Lemma 1.3.5. As they are non vanishing only on the line bundle TM/ξ, they are multiple of each other on $U \cap V$. Thus, there exists a smooth function $f \in \mathcal{C}^\infty(U \cap V)$ such that $\alpha_U = f\alpha_V$. The function f has is sign preserving only if the bundle TM/ξ is oriented, but we know that any real line bundle is orientable if only if it is trivial. Thus, the coorientability condition.
It is interesting to prove again, also in this setting, that there are canonical orientations on ξ and M, independently of the coorientability condition. Note that,

$$\alpha_U \wedge (d\alpha_U)^n = f^{n+1}\alpha_V \wedge (d\alpha_V), \qquad (d\alpha_U)^n = f^n(d\alpha_V)^n.$$

If n is odd, then f^{n+1} is a positive function, and the two volume forms in TM have the same sign, meaning that the contact structure defines an orientation of M. When M and ξ are coorientable, the orientation of M given by the local forms coincide with the already existing one.
If n is even, then f^n is a positive function, and the two volume forms in ξ have the same sign, meaning that the local forms give ξ a canonical orientation.

Theorem 1.3.7. *M and ξ be coorientable if and only if there exists a contact form $\alpha \in \Omega^1 M$ of (M, ξ).*

Proof. If M and ξ are coorientable, then the line bundle TM/ξ is trivial. In the setting of Remark 1.3.6, we can choose a common sign for all the local forms, so that they can patch together.
If, on the other hand, there exists a contact form of (M, ξ), we can construct a global non vanishing section ζ of the line bundle TM/ξ, meaning that TM/ξ is trivial. Indeed, α is non vanishing outside of ξ and thus, we can define $r \in \mathfrak{X}(M)M$ pointwise as the vector ζ_p such that $\alpha_p(r_p) = 1$, which projects to a non vanishing section of TM/ξ. We will see this construction in detail in the follow section. □

Every contact form is uniquely associated to a coorientable contact structure defined by $\xi = \ker \alpha$. The converse is also true, up to rescaling, i.e. every coorientable contact structure ξ determines a contact form up to a scalar multiple.

A contact form α not only characterize the contact structure but also singles out a particular type of vector fields in $\mathfrak{X}(M)$ called *duals* of α. These are all the vector fields $X \in X(M)$ satisfying $\iota_X \alpha = 1$ and will be of the main focus of the next section.

1.3.3 Reeb vector field

The Reeb vector field is a non horizontal and non vanishing vector field $r \in \mathfrak{X}(M)$ that satisfies specific Lie bracket conditions and it is uniquely related to a given contact form of a contact manifold (M, ξ). It is an element of a space of vector fields called *contact fields*, which are infinitesimal symmetries of the horizontal distribution ξ.

Definition 1.3.8. Let (M, ξ) be a contact manifold. A vector field $X \in \mathfrak{X}(M)$ is called *contact* if $[X, Y] \in \xi^1$ for all $Y \in \xi^1$. We denote by $\mathfrak{C}(M, \xi)$ the space of such vector fields.

Note that $\mathfrak{C}(M, \xi)$ is a vector space but it is not the space of sections of some vector bundle as it is not a $C^\infty(M)$-module. In fact, for $X \in \mathfrak{C}(M, \xi)$, $Y \in \xi^1$ and $f \in \mathcal{C}^\infty(M)$

$$[fX, Y] = f[X, Y] - \iota_Y df X$$

is not necessarily horizontal, because X could be non horizontal.

Consider the projection $\pi : \mathfrak{X}(M) \to \Gamma(TM/\xi)$. The restriction of this map to $\mathfrak{C}(M,\xi)$ has an interesting property.

Lemma 1.3.9. *The restriction $\pi : \mathfrak{C}(M,\xi) \to \Gamma(TM/\xi)$ is an isomorphism of vector spaces.*

Proof. Let us prove the injectivity of π by checking elements in its kernel. If $X \in \ker(\pi)$, it is a horizontal vector field. Since it is also contact, $c_{1,1}(X,-) = 0$. By non degeneracy of the first curvature, it must be the zero vector field.
To prove surjectivity, pick a section $\zeta \in \Gamma(TM/\xi)$ and a vector field $Z \in \mathfrak{X}(M)$ whose projection to TM/ξ coincide with ζ. This Z is not necessarily contact but we want to prove that there exists $Y \in \xi^1$ such that $X := Z - Y \in \mathfrak{C}(M,\xi)$. For $Y \in \xi^1$, X is contact if and only if the projection $\pi([X,-]) = 0$ is zero. Therefore, we have to find a $Y \in \xi^1$ such that $\iota_Y c_{1,1} = \pi([Z,-])$. The existence of such Y is again given by the non-degeneracy of $c_{1,1}$. □

In the presence of a contact form α of the contact manifold (M,ξ), contact fields are described by the following equivalent condition.

Lemma 1.3.10. *Let $\alpha \in \Omega^1 M$ be a contact form of (M,ξ). Then, $X \in \mathfrak{X}(M)$ is a contact field if and only if there exists a smooth function $f \in \mathcal{C}^\infty(M)$ such that $\mathcal{L}_X\alpha = f\alpha$.*

Proof. Let $X \in \mathfrak{C}(M,\xi)$ and $Y \in \xi^1$ be a horizontal vector field. Then, $\alpha(Y) = 0$ and

$$0 = \mathcal{L}_X(\alpha(Y)) = (\mathcal{L}_X\alpha)(Y) + \alpha([X,Y]) = (\mathcal{L}_X\alpha)(Y).$$

Therefore, $\ker(\mathcal{L}_X\alpha) \subset \ker\alpha$, and thus there exists a smooth function f such that $\mathcal{L}_X\alpha = f\alpha$.
For the converse, suppose $\mathcal{L}_X(\alpha) = f\alpha$. By analogous arguments, we can pick any horizontal field $Y \in \xi^1$ and prove that $[X,Y] \in \ker(\alpha)$. □

A corollary of this lemma states that the correspondence between a contact field X and the function f for which $L_X\alpha = f\alpha$ is unique.

Corollary 1.3.11. *Let $\alpha \in \Omega^1 M$ be a contact form of (M,ξ). Then the space $\mathfrak{C}(M,\xi)$ is in 1:1 correspondence with $\mathcal{C}^\infty(M)$.*

Proof. How to associate a contact field $X \in \mathfrak{C}(M,\xi)$ to a function $f \in \mathcal{C}^\infty(M)$ and viceversa derives from Lemma 1.3.10. The uniqueness follows from the fact that α never vanishes outside of ξ. □

Under the assumption of coorientability of the contact structure ξ and M, the dual bundle TM/ξ is a trivial line bundle. Therefore, there exists a non vanishing and globally defined section that spans it. We saw in the proof of Theorem 1.3.7 that every contact form α gives a natural choice for this spanning section. Its preimage, under the isomorphism π, will be the so called *Reeb vector field*.

Definition 1.3.12. Let $\alpha \in \Omega^1 M$ be a contact form of (M,ξ). We call *Reeb vector field*, associated to α, the contact field $r \in \mathfrak{C}(M,\xi)$ whose image under the isomorphism π is the dual of α in TM/ξ.

In other words, given a contact form α, the Reeb vector field is the unique dual of α whose Lie bracket with any horizontal section is still horizontal. By construction it is non vanishing.

Remark 1.3.13. Let α be a contact form and r be its associated Reeb vector field. If we rescale the contact form α by any non-vanishing smooth function f, then its associated Reeb vector field is not just the rescaling of r by $1/f$. Depending on f, there is a horizontal vector field X that is added to $\frac{1}{f}r$. It can be proven that the horizontal vector field X derives from the Hamiltonian flow of $\log f$. We will come back to this in Lemma 1.3.17

There is an alternative definition of the Reeb vector field, and it is also the most common definition of the object in the literature. Many authors define the Reeb vector field r as a dual of the contact form α in $\mathfrak{X}(M)$ along which the contraction of the differential of α is trivial.

Definition 1.3.14. Let $\alpha \in \Omega^1 M$ be a contact form of (M, ξ). The *Reeb vector field*, associated to α, is the unique vector field $r \in \mathfrak{X}(M)$ satisfying

$$\iota_r \alpha = 1, \qquad \iota_r d\alpha = 0.$$

We shall see how these two definitions are equivalent.

Lemma 1.3.15. *Definition 1.3.12 and Definition 1.3.14 are equivalent.*

Proof. In both definitions the Reeb vector field is a dual of the contact form. Let us prove the equivalence of the other conditions.
Suppose r to be the unique contact dual of α. Then, by Lemma 1.3.10, there exists $f \in \mathcal{C}^\infty(M)$ such that $\mathcal{L}_r \alpha = f\alpha$. By Cartan's formula,

$$\iota_r d\alpha = \mathcal{L}_r(\alpha) - d\iota_r \alpha = f\alpha,$$

which means that $\xi \subset \ker(\iota_r d\alpha)$. Let $\zeta \in \Gamma(TM/\xi)$ be the dual of α in TM/ξ. Any section in $\Gamma(TM/\xi)$ is a multiple of ζ. Moreover, by construction, there exists $Y \in \xi^1$ such that $r = \zeta + Y$. Hence, for any $g \in \mathcal{C}^\infty(M)$,

$$\iota_r d\alpha(g\zeta) = -\iota_r d\alpha(gY) = -g\iota_r d\alpha(Y) = 0,$$

because $\alpha(r, r) = 0$ and $\xi \subset \ker(\iota_r d\alpha)$. Therefore, $\iota_r d\alpha$ is the zero 1-form.
Conversely, if $\iota_r d\alpha = 0$, then, by Cartan's formula, $\mathcal{L}_r \alpha = 0$, meaning that r is a contact field. Therefore, r is both contact and dual of α. □

Remark 1.3.16. In the proof of the previous Theorem, we saw that the Reeb vector field r preserves the associated contact form α in the sense that $\mathcal{L}_r \alpha = 0$. This means that the pullback of the flow of r sends α to α. In general, we saw in Lemma 1.3.10 that the covariant derivative of α along a contact field is just a multiple of α.

Lemma 1.3.17. *If we rescale α by a positive function $f : \mathcal{C}^\infty(M)$, then the new Reeb vector field is given by $r_f := \frac{1}{f}(r - X_{\log f})$.*

Proof. Let $\alpha_f := f\alpha$ be the rescaled contact form. Then, by the normalizing condition of Definition 1.3.14, r_f must satisfy $r_f = \frac{1}{f}(r + X)$ for some horizontal vector field $X \in \xi^1$. The second condition requires $\iota_{r_f} d\alpha_f = 0$. If we compute explicitly this contraction, we obtain

$$\begin{aligned} 0 = \iota_{r_f} d\alpha_f &= \frac{1}{f} df(r + X)\alpha - \frac{1}{f}\alpha(r + X) df + \iota_{(r+X)} d\alpha. \\ &= \frac{1}{f}\left(df(r) + df(X)\right)\alpha - \frac{1}{f} df + \iota_X d\alpha \end{aligned}$$

If we contract again with r, we obtain

$$0 = \iota_r \iota_{r_f} d\alpha_f = \frac{1}{f} df(r) + \frac{1}{f} df(X) - \frac{1}{f} df(r) = \frac{1}{f} df(X).$$

Therefore, we can explicitly describe $\iota_X d\alpha$ as

$$\iota_X d\alpha = \frac{1}{f}(df - df(r)\alpha).$$

The term $df - df(r)\alpha$ is the projection of df to the horizontal plane ξ. Since $d\alpha$ is symplectic on ξ, it uniquely defines X as $-X_{\log f}$. □

The previous Lemma has problematic consequences, especially in Reeb dynamics, which is the study of the integral curves of the given Reeb vector field. As soon as we rescale the contact form of the system by f, the integral curves are affected by the rescaling and may not be integral curves of the rescaled Reeb vector field.

Reeb Dynamics

The integral curves of the Reeb vector field of a contact manifold (M, α) are very important objects that have been immensively studied over the years. They are always transverse to the contact plane and so, they are the focal point in the study of the vertical dynamic.

Definition 1.3.18. We call *Reeb orbits* the integral curves of the Reeb vector field r.

As already mentioned, a nowhere vanishing vector fields spans a foliation. This is obviously the case also for the Reeb vector field. In particular, the leaves generated by this foliation coincide with its orbits. One of the main question about Reeb orbits is whether they are closed/periodic.

Definition 1.3.19. We say that a curve $\gamma : \mathbb{R} \to M$ is *closed/periodic* if there exists $T > 0$, such that $\gamma(t + T) = \gamma(t)$ for all t. The first $T > 0$ satisfying this property is called *period* of γ.

A periodic curve is compact with respect to the topology of the manifold. If also the manifold is compact, all periodic curves become embedded circles in the manifold.

Definition 1.3.20. If all the Reeb orbits are embedded circles of the manifold M, then the Reeb vector field r is called *regular*.

In particular, the Reeb vector fields is regular if and only if the foliation that it spans has compact leaves. If the base manifold is compact, then Theorem 1.1.17 already gives us an equivalent condition for that.

Theorem 1.3.21. *Let $(M, \ker\alpha)$ be a closed contact manifold. The Reeb vector field r is regular if and only if, around every point $p \in M$, there exists a flowbox for the Reeb vector field r that is pierced at most once by any Reeb orbit.*

This is a strong condition that is not always satisfied, and it is often difficult to check whether it is holds or not. There is a special framework in which, even without a compact manifold, we always have periodic Reeb orbits. This contact manifold is called *Boothby-Wang fibration* and it will be the object of study of Section 1.3.7.

Generally, regular Reeb vector fields are very rare. In fact, there are exist contact manifolds, such as open contact manifolds, which do not admit any closed Reeb orbits. A famous conjecture, conceived by Weinstein [35, pg. 358], claims that simply connected closed contact manifolds have at least one closed Reeb orbit. The conjecture was then extended to any closed contact manifold. It has been proven by

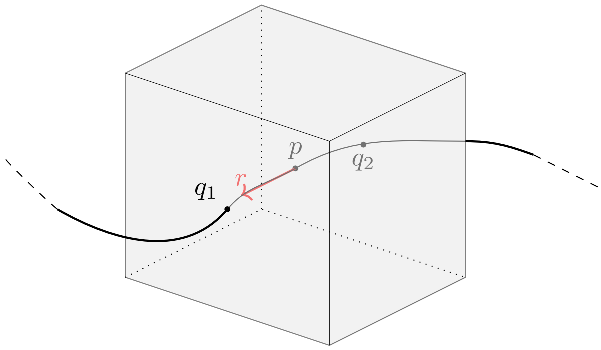

Figure 1.8: Reeb orbits interacting with a flowbox at a point p.

Taubes [31] for the three dimensional case, but the statement in full generality is still unsolved.

Even when closed Reeb orbits exist, there is still a distinction between *degenerate* and *non-degenerate* ones. This distinction depends on whether, around the curve, the restriction to ξ of the linearized Reeb flow has 1 as eigenvalue. Let γ be a closed Reeb orbit of period T, and $\phi_r^t : M \to M$ be the flow of r at time t. Recall that this map sends a point p to $\gamma_p(t)$ of the integral curve passing through p at time 0.

Definition 1.3.22. We call *linearized return map* of γ at $p \in \gamma$, the linear map $\Psi_\gamma : \xi_p \to \xi_p$ given by the restriction to $(d\phi_r^T)_p$ to ξ_p.

It is well defined by the fact that γ has period T, and $\mathcal{L}_r\alpha = 0$. We are looking at the linearization of the flow of r when it first comes back at the same point p. This is a symplectic map because the Reeb flow preserves $d\alpha$. In turn, the study of its eigenvalues become very insightful. For example, if 1 is an eigenvalue of Ψ_γ, it means that there is a direction in ξ_p on which $(d\phi_r^T)_p$ is the identity. Thus, every vector on that direction is fixed. This is something that we want to avoid because, at a linear level, the flow seems to have a 1-parametric family of fixed points, which is an unstable scenario. Indeed, some perturbations to the system causes the loss of all the fixed points at once.

Definition 1.3.23. We say that the closed Reeb orbit γ is *non-degenerate* if 1 is not an eigenvalue of Ψ_γ.

Non-degenerate Reeb orbits are *stable* in the sense that any perturbation of the contact form does not decrease drastically the fixed points of Ψ_γ. It turns out that this is something that we can always ask up to rescaling of the contact form.

Lemma 1.3.24. *For a given (coorientable) contact structure (M, ξ), there exists a non degenerate contact form α, i.e. a contact form whose closed Reeb orbits are all non-degenerate.*

Proof. Rather technical proof contained in [7, Lemma 2]. $\square$

Therefore, there is the notion of *stable* and *unstable* Reeb dynamic, but it is always possible to pass from one to the other by rescaling the contact form. We will come back to non degenerate contact forms in Chapter 3.

The study of Reeb orbits might appear similar to the previous discussion about Hamiltonian equations. There are cases in which we can naturally construct a symplectic manifold $C(M)$ from a contact one M, and relate the Hamiltonian flow of Hamiltonians on $C(M)$ with the Reeb vector field on the other. We will see some of these cases in Section 1.3.5.

1.3.4 Heisenberg calculus

The Heisenberg calculus is an alternative to the usual symbolic calculus for differential operators on the so called *Heisenberg manifolds*. The important difference between the two approaches is the notion of order of an operator, that we will shortly explain. First, let us give some context.

Definition 1.3.25. An *Heisenberg manifold* is a pair (M, ξ) where M is a smooth manifold and ξ is a codimension 1 tangent distribution.

Clearly, contact manifolds are also Heisenberg manifolds. Thus, for the purpose of this thesis, it is general enough to talk about Heisenberg calculus on contact manifolds. Let (M, α) be a coorientable contact manifold with contact form α and Reeb vector field r.

Definition 1.3.26. We say that that a positive local frame $\{r_0, r_1, ..., r_{2n}\}$ is a $\xi-$frame if r_0 is the restriction of the Reeb vector field and $\{r_1, ...r_{2n}\}$ is a local frame for ξ. We will often avoid the first index to stress the fact that r restricts to r_0.

Let $E \to M$ be a vector bundle of rank k over M. To define the *Heisenberg order* of a linear operator $P : \Gamma(E) \to \Gamma(E)$, we will use the following multi-index notation: For $I = (i_0, ..., i_{2n}) \in \mathbb{Z}^{2n+1}$ and $R = \{r, ..., r_{2n}\}$ an $\xi-$frame, we will write

$$R^I := r^{i_0} r_1^{i_1} \cdots r_{2n}^{i_{2n}}, \qquad \langle I \rangle := 2i_0 + i_1 + ... + i_{2n},$$

where the r_i terms are seen as differential operators. The factor 2 in front of the first value of I generates the difference in computing the order of an operator between the Heisenberg calculus and the standard symbolic calculus.

Definition 1.3.27. We say that $P : \Gamma(E) \to \Gamma(E)$ is a *differential operator of Heisenberg order* m if in any local trivialization (U, ϕ) of E, and for any ξ-frame R on U, there exists local sections $b_I \in \mathcal{C}^\infty(U, \mathbb{R}^{k \times k})$ such that P has the form

$$\phi \circ P \circ \phi^{-1} = \sum_{\langle I \rangle \leq m} b_I R^I.$$

The motivation for the factor 2 in front of i_0 is the following observation, which follows directly from the previous Definition.

Lemma 1.3.28. *The Lie derivative $\mathcal{L}_r : \Omega^* M \to \Omega^* M$ is a differential operator of Heisenberg order* 2*.*

While $\mathcal{L}_r$ has Heisenberg order 2, the covariant derivative along any horizontal direction has Heisenberg order 1. This difference of order between derivatives along vertical and horizontal directions want to point out which operator has just horizontal components and which does not. In fact, in the Heisenberg calculus, the horizontal vector fields have order 1, and this automatically makes the Reeb vector field have order 2 because it is a commutator of horizontal vector fields plus lower order terms.

We will see some interesting application of the Heisenberg calculus in Chapter 4.

1.3.5 Symplectization and return

As mentioned at the beginning of this section, contact and symplectic structures are very much related, and we can sometimes move from one to the other. We will see few cases and constructions. The process of creating a symplectic manifold from a contact one is called *symplectization*. The inverse process is called *contactization*.

Let $(M, \ker \alpha)$ be a contact manifold of dimension $2n+1$.

Lemma 1.3.29. *On the cone $C(M) := M \times \mathbb{R}^+$, we can define a symplectic form ω as $\omega := d(t\widetilde{\alpha})$, where $t : C(M) \to \mathbb{R}^+$, $\pi : C(M) \to M$ are the canonical projections and $\widetilde{\alpha} := \pi^*\alpha$.*

Proof. It is not really difficult to see that the symplectization of $(M, \ker \alpha)$ is, indeed, a symplectic manifold. The only thing to prove is the non-degeracy of ω. We have that

$$\omega^{n+1} = (d(t\widetilde{\alpha}))^{n+1} = (dt \wedge \widetilde{\alpha} + td\widetilde{\alpha})^{n+1} = t^{2n} dt \wedge \widetilde{\alpha} \wedge (d\widetilde{\alpha})^n. \tag{1.6}$$

which means that ω is non degerate, outside of $t = 0$, by the defining properties of α. □

Remark 1.3.30. Under the condition of coorientability, the dual bundle TM/ξ is trivial. If we endow TM with a Riemannian metric, we have an isomorphism between TM/ξ and the annihilator bundle $Ann(\xi) \subset TM^*$. All together, we have that the positive annihilator bundle $Ann(\xi)^+$ is isomorphic to the cone $C(M)$ and we can endow it with a symplectic structure deriving from Lemma 1.3.29. This is an important construction that will be essential in Chapter 3.

It turns out that ω being a symplectic form on the cone $C(M)$ is also a sufficient condition for α being a contact form.

Theorem 1.3.31. *$\alpha \in \Omega^1 M$ is a contact form if and only if $\omega = d(t\alpha)$ is a symplectic form on $C(M)$.*

Proof. The first implication is precisely the definition. Let us see the other one. If $\omega = d(t\alpha)$ is a symplectic form on $C(M)$, then $t\widetilde{\alpha} \in \Omega^1(C(M))$, when restricted to $M \times \{1\}$, is a non-degenerate one form that corresponds to α. Moreover, by Equation (1.6), we conclude that, on $M \times \{1\}$, $\alpha \wedge (d\alpha)^n \neq 0$, meaning that α defines a contact form on M. □

Remark 1.3.32. The previous construction works for any power of t. Indeed, for any $k > 1$, $d(t^k\widetilde{\alpha})$ is a symplectic structure on $C(M)$ if and only if α is a contact form. Moreover, in the setting of Remark 1.3.30, the symplectic form $d(t^2\widetilde{\alpha})$ is precisely the restriction of the standard symplectic structure on TM^* to $Ann^+(\Sigma)$.

We can also construct a contact manifold from a symplectic one in an analogous manner to the one in Lemma 1.3.29.

Lemma 1.3.33. *Let (B, ω) be a $2n$-dimensional manifold with an exact symplectic form, and let $\lambda \in \Omega^1 M$ be a primitive. Then, on $M := B \times \mathbb{R}$ we can define a contact form α as*

$$\alpha = \pi^*\lambda + dt,$$

where $\pi : B \times \mathbb{R} \to B$ and $t : B \times \mathbb{R} \to \mathbb{R}$ are the canonical projections.

Proof. If we take the differential of α, we get

$$d\alpha = d\pi^*\lambda + d^2 t = \pi^* d\lambda = \pi^*\omega,$$

meaning that $d\alpha$ is non degenerate. Thus $\alpha \wedge (d\alpha)^n$ is a non vanishing volume form. □

There are contactizations that, from a symplectic manifold, create a contact manifold of lower dimension. An important one, particularly used in classical mechanics, considers the levelsets of a Hamiltonian on a phase space. Let (B, ω) be a $2n+2$-dimensional symplectic manifold and $X \in \mathfrak{X}(B)$ be a *Liouville field*. Recall that a Liouville field is a direction that preserves the symplectic form, i.e. satisfying $\mathcal{L}_X \omega = \omega$.

Definition 1.3.34. Any hypersurface M of (B, ω) transverse to X is called of *contact type*.

By transverse to X, we just mean that $X_p \notin T_p M$ for all $p \in M$. The previous definition is motivated by the following Lemma.

Lemma 1.3.35. *The contraction $\alpha := \iota_X \omega$ is a contact form on any hypersurfaces M of contact type.*

Proof. Let M be an hypersurface transverse to X. Then X is never a direction in TM. In particular, the restriction of α to $\mathfrak{X}(M)$ is non-zero at all points, by non degeneracy of ω. Moreover, by Cartan's formula, $d\alpha = d\iota_X \omega = \mathcal{L}_X \omega = \omega$. Therefore, $d\alpha$ is a non degenerate 2-form, and

$$\alpha \wedge (d\alpha)^n = \iota_X \omega \wedge \omega^n = \frac{1}{n+1} \iota_X (\omega^{n+1}).$$

Since ω^{n+1} is a volume form on B, we get that $\iota_X(\omega^{n+1})$ is a volume form on M. □

Remark 1.3.36. Consider the symplectization of a contact manifold $(M, \ker \alpha)$ as in Lemma 1.3.29. In this setup, it is not difficult to prove that ∂_t is a Liouville field of $(C(M), \omega)$, implying that any hypersurface $M \times \{t_0\} \cong M$ is of contact type. Moreover, on each $M \times \{t_0\}$ we define the contact form $\widetilde{\alpha} := \iota_{\partial_t} d(t^2 \alpha) = 2t_0 \alpha$.

Corollary 1.3.37. *If the symplectic form ω admits a primitive λ, then λ is a contact form for any hypersurface of contact type.*

Proof. The symplectic form is exact, and so $\omega = d\lambda$. To be a Liouville vector field, X must satisfy $\mathcal{L}_X \omega = \omega$, which, by Cartan's formula, is equivalent to $\iota_X \omega = \lambda$. □

If we define an Hamiltonian on (B, ω) whose levelsets are hypersurfaces of contact type, we obtain an interesting relation between the Hamiltonian flow and the Reeb vector field of the generated contact structure.

Lemma 1.3.38. *Let H be an Hamiltonian on (B, ω). If a levelset $M \subset B$ of H is also an hypersurface of contact type, then the Hamiltonian flow X_H of H and the Reeb vector field r on $(M, \ker \iota_X \omega)$ coincide up to scaling.*

Proof. As before, for $\alpha := \iota_X \omega$, we have that $d\alpha = \omega$. The associated Reeb vector field r satisfies

$$0 = \iota_r d\alpha = \iota_r \omega,$$

precisely as the Hamiltonian flow X_H on the levelsets of H, since $dH(\mathfrak{X}(M)) = 0$. By non-degeneracy of ω, r and X_H are equal up to scaling. □

What happens when the symplectic space (B, ω) is the cotangent bundle of some manifold M with standard symplectic structure? Can we apply the previous Lemma to the subRiemannian Hamiltonian?

Remark 1.3.39. We saw in Section 1.2.1, that TM^* is endowed with an exact symplectic form $\omega_{can} = d\lambda_{can}$, where λ_{can} is the tautoligical Liouville form. In this setting, we can retrieve a Liouville field $X \in \mathfrak{X}(M)$ simply by asking $\iota_X\omega_{can} = \lambda_{can}$. In local terms, it is given by $X = \sum_{i=1}^n P_i\partial_{P_i}$. We also saw that this Liouville field is transverse to the unit sphere bundle STM^* constructed from any subRiemannian structure (ξ, g_h) with associated cometric g^*. This means that STM^* is an hypersurface of contact type. By Corollary 1.3.37, λ_{can} is a contact form on STM^*. The associated Reeb vector field r is

$$r = \sum_{i=1}^k P_i\partial_{x_i},$$

where k is the codimension of tangent distribution ξ. If we consider also the subRiemannian Hamiltonian H, then we have that $H^{-1}(1) = STM^*$, which implies that the Hamiltonian flow X_H and the Reeb r differ by scaling. We will continue this discussion in Chapter 2.

Another interesting interaction between contact and symplectic geometry is contained in the Section 1.3.7, where we construct contact structures on principal S^1-bundles over symplectic manifolds. This construction is known as *Boothby-Wang construction.*

1.3.6 Local model

Just like in the foliation case, a local model for a contact structure is an *equivalence* between a neighbourhood of the manifold and a *standard structure* in $\mathbb{R}^{2n+1}$. For contact manifolds, the equivalences are called *contactomorphisms* and they are defined as follows.

Definition 1.3.40. Let (M_1, ξ_1) and (M_2, ξ_2) be two contact manifolds. A *contactomorphism* between M_1 and M_2 is a diffeomorphism $\varphi : M_1 \to M_2$ satisfying $\varphi_*(\xi_1) = \xi_2$. If there exists such map, we say that (M_1, ξ_1) and (M_2, ξ) are *contactomorphic.*

A contactomorphism is just a smooth isomorphism, whose push-forward sends one contact distribution to the other one. Darboux proved that contact manifolds of the same dimension are always locally contactomorphic.

Theorem 1.3.41 (Darboux)**.** *Let (M_i, ξ_i) be a contact manifold of dimension $2n+1$ for $i = 1, 2$. Given any pair points $p_i \in M_i$, there exist neighbourhoods $U_i \subseteq M_i$ of p_i and a contactomorphism $\varphi : (U_1, \xi_1 |_{U_1}) \to (U_2, \xi_2 |_{U_2})$.*

Therefore, we can always pass from one contact manifold to another though local contactomorphisms. What is the easiest setting to work on? As always, we take a look at structures on $\mathbb{R}^{2n+1}$ first. The Cartan's distribution of Example 1.1.2 seems a good candidate.

Lemma 1.3.42. *Let $(x_1, y_1, ..., x_n, y_n, z)$ be coordinates on $\mathbb{R}^{2n+1}$ and define*

$$\alpha_{std} := dz - \sum_{i=1}^n y_i dx_i \in \Omega^1(\mathbb{R}^{2n+1}).$$

Then $(\mathbb{R}^{2n+1}, \ker \alpha_{std})$ is a contact manifold.

Proof. We need to prove the non-degeneracy of $d\alpha_{can}$. A simple computation shows that

$$d\alpha_{std} = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Therefore, $d\alpha_{std}$ vanishes only vertically for ∂_z. $\square$

The pair $(\mathbb{R}^{2n+1}, \xi_{std:} = \ker \alpha_{std})$ will be the canonical local setting for any contact manifold (M, ξ). Let us see how the Reeb vector field looks like. A simple computation shows that $r_{std} = \partial_z$. In fact,

$$\iota_{\partial_z} dx_i \wedge dy_i = 0 \qquad \iota_{\partial_z} \alpha_{std} = 1.$$

Moreover, a possible global frame for the distribution is

$$\{\partial_{y_1}, ..., \partial_{y_n}, \partial_{x_1} + y_1\partial_z, ..., \partial_{x_n} + y_n\partial_z\}.$$

In particular, every y_i-direction is tangent to the distribution and, on a fixed plane $\{y = 0\}$, also the x_i-direction. The picture to have in mind is Figure 1.9. It is an example of standard contact structure

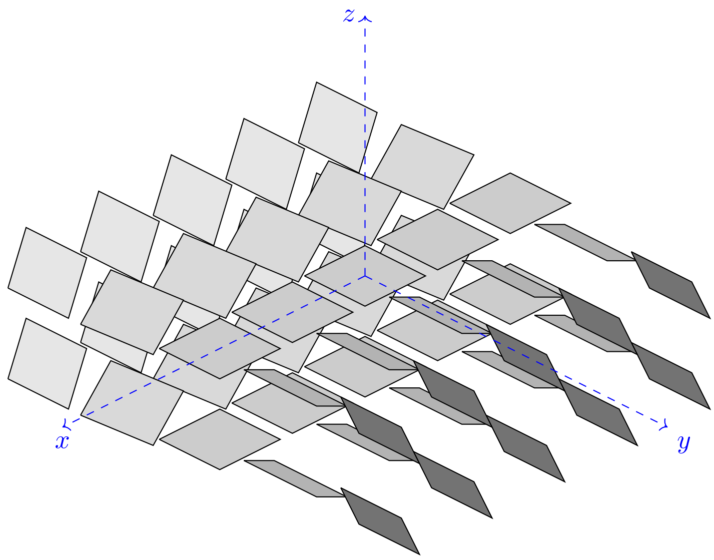

Figure 1.9: Standard contact structure in $\mathbb{R}^3$.

in 3-dimensions. It shows two layers of contact planes for two values of z. On each layer, a grid of 25 different configurations of the contact plane around $(x, y) = 0$ are shown. It is evident from the picture that ξ_{std} does not change along the x-direction while it spins along the y-direction. This is coherent with the mathematical description. In this setting, the contact form is simply given by $\alpha_{std} = dz - ydx$ and the Reeb vector field is still the vertical direction. Therefore, on a fixed plane $\{z = k\}$, ξ_{std} changes inclination as long as we move along the y-coordinate, but always keeping the vertical direction transverse.

We will see other important features of this contact structure when we will endow it with a subRiemannian metric.

1.3.7 Boothby-Wang fibrations

A particularly important type of contact manifolds are the total spaces of principal S^1-bundles over symplectic manifolds. The contact structure is given by a principal connection on the bundle, and the S^1-fibers are the orbits of a Reeb vector field. These spaces have some interesting properties, and their simple construction allow us to see features of contact manifolds that would be much more complicated to see in a general setting. We can construct contact forms on such manifolds. The procedure to obtain them is commonly known as *Boothby-Wang construction* and it relies heavily on the fact that the base manifold is symplectic. To understand why, let assume at first that B is just an even-dimensional manifold. The interesting thing about the Boothby-Wang construction is that it defines the Reeb vector field first, and then the associated contact form. Let $S^1 \hookrightarrow M \xrightarrow{\pi} B$ be a principal S^1-bundle.

Lemma 1.3.43. *There exists a vector field $r \in \mathfrak{X}(M)$ such that the S^1-action on M is generated by the flow of r.*

Proof. By definition, there is a free, proper right action of S^1 on M. It is a map $P : M \times S^1 \to M$ denoted by $p \cdot e^{it} := P(p, e^{it})$ whose orbits $p \cdot S^1$ are smooth. Its infinitesimal action gives a choice of the vector field r. We construct r pointwise as the velocity of the orbits, that is

$$r_p := \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{it}.$$

Since P is smooth, so is r. Note that the integral curves $\gamma : \mathbb{R} \to M$ of r are precisely given by the orbits of the S^1-action, i.e. $\gamma(t) = p \cdot e^{it}$ for some $p \in M$. $\square$

Therefore, the flow ϕ_r^t of the Reeb vector field corresponds to the right multiplication $R_{e^{it}} : M \to M$ given by $R_{e^{it}}(p) = p \cdot e^{it}$, for all $t \in \mathbb{R}$.
Now the question is: how do we define a coorientable contact structure ξ on M with r as Reeb vector field? We shall recall the definition of connections on principal bundles first. The most geometrical definition is the following.

Definition 1.3.44. A *connection* on M is a tangent distribution $\xi \subset TM$ that is isomorphic to π^*TB and it is compatible with the S^1-action, meaning that it satisfies $(\phi_r^t)_*\xi = \xi$ for all $t \in \mathbb{R}$.

Therefore, a connection on M is a codimension 1 distribution that is transverse to the S^1-fiber at each point. Such distributions are equivalently described by the 1-forms that vanish on them. Precisely, we have the following result.

Proposition 1.3.45. *Let ξ be a connection on $S^1 \hookrightarrow M \xrightarrow{\pi} B$. Then there exists a unique form $\alpha \in \Omega^1(M)$ such that*

$$\ker(\alpha) = \xi, \qquad \alpha(r) = 1, \qquad \mathcal{L}_r\alpha = 0.$$

Proof. At each point $p \in M$, we have a splitting $T_pM = \langle r_p \rangle \oplus \xi_p$. We define α pointwise as

$$\alpha_p(kr_p + w) = k,$$

for every $k \in \mathbb{R}$ and $w \in \xi_p$. Therefore, α vanishes on ξ and measures the magnitude of the vector field along the S^1-orbits. Moreover, since ξ is smooth, then so is α. Note that the first two conditions are already satisfied by this construction. The third comes from the fact that ξ is S^1-equivariant, meaning that it satisfies $(\phi_r^t)_*\xi = \xi$. In particular, by definition of Lie derivative,

$$\mathcal{L}_r\alpha = \left.\frac{d}{dt}\right|_{t=0} (\phi_r^t)^*\alpha = 0,$$

because α vanishes on ξ and the pushforward of ϕ_r^t sends horizontal vectors to horizontal vectors. The uniqueness of the form is by construction (normalization). $\square$

The connection 1-form of Proposition 1.3.45 looks like a contact form on M. However, this is not enough to prove that a connection ξ is a contact structure on M. We need that either $d\alpha$ or the first curvature $c_{1,1}$ is non-degenerate on ξ. To get that, we need to put some extra conditions on the base manifold.

Lemma 1.3.46. *The differential 2-form $d\alpha \in \Omega^2 M$ induces a closed 2-form ω on the base manifold.*

Proof. The connection form α satisfies $\alpha(r) = 1$ and $\mathcal{L}_r \alpha = 0$. Therefore, it is easy to prove that $\iota_r d\alpha = 0$ and $\mathcal{L}_r d\alpha = 0$. As the Reeb direction coincide with the S^1-fibers, we can project $d\alpha$ to a 2-form $\omega \in \Omega^2 B$ satisfying $\pi^*\omega = d\alpha$. □

Remark 1.3.47. It is clear that $d\alpha$ is non degenerate on ξ if and only if ω is non degenerate on B. Therefore, if ω is a symplectic form on the base manifold B, then the principal connection $\xi := \ker \alpha$ is a contact distribution.

Definition 1.3.48. A contact manifold (M, α) is called *Boothby-Wang fibration* if there exists a symplectic manifold (B, ω) over which M is the total space of a principal S^1-bundle, and $d\alpha = \pi^*\omega$.

We can do more that this: as ω is closed, it also defines a cohomology class that, up to rescaling by $1/2\pi$, is integral ([17, Theorem 2.72]). Moreover, it is an invariant for the fiber bundle, and it determines it up to isomorphism ([6, Section 6]).

Theorem 1.3.49 (Boothby-Wang)**.** *Let (B, ω) be a closed symplectic manifold and $S^1 \hookrightarrow M \xrightarrow{\pi} B$ be the principal S^1 bundle with Euler class of $[\omega/2\pi]$. Then, there exists a contact form α on M whose differential is $\pi^*\omega$ and whose Reeb vector field generates the S^1-action.*

Proof. [14, Theorem 7.2.4]. □

1.4 (Adapted) Riemannian geometry

The previous section described the general geometric setting of contact geometry. Now the questions are: what happens if we add a horizontal metric to the contact structure? Can we extend it to a Riemannian metric that is somewhat *adapted* to the contact structure? This section answers all these questions, but first, we need to understand what adapted metric means.

Definition 1.4.1. Let $(M, \xi = \ker \alpha)$ be a contact manifold. An *adapted metric structure* is a pair (λ, g), where $\lambda \in \mathbb{R}^+$ is a positive constant and g is a Riemannian metric on M satisfying $\|\alpha\|^2 = 1$ and $(d\alpha)^n = \lambda^n \star \alpha$ where $\star$ corresponds to the Hodge dual of g.

Remark 1.4.2. In the literature, adapted metrics are usually defined for a fixed $\lambda = 2$. This is a convention that simplifies some computations. For our purpose, it is useful to let λ vary since we will often *perturb* a adapted metric structure (λ, g) and obtain another adapted metric structure (λ', g') with a different coefficient. For further details, check Section 1.4.2.
If we say that a Riemannian metric g is adapted to ξ without referring to any λ, we mean that $(1, g)$ is an adapted metric structure.

Let $(M, \xi = \ker \alpha)$ be the usual $(2n + 1)$-dimensional contact manifold. If (λ, g) is an metric adapted to ξ, it follows that the volume form $\alpha \wedge (d\alpha)^n$ corresponds to a multiple of the area form induced by the Riemannian metric g. Indeed, by definition of the Hodge dual,

$$\alpha \wedge (d\alpha)^n = \lambda^n(\alpha \wedge \star\alpha) = \lambda^n \|\alpha\|^2 vol_g = \lambda^n vol_g.$$

A natural way to construct adapted metrics on the contact manifold (M,ξ) is the following.

Remark 1.4.3. Given a subRiemannian metric g_h on the contact manifold (M,ξ), we can extend it to a Riemannian metric on M simply by adding a metric on $\langle r\rangle$. Precisely, we extend g_h to the whole tangent bundle by making it vanish on r. Then we define

$$g := g_h + \alpha \otimes \alpha.$$

In this setup, the Reeb vector field r becomes the unitary vector field orthogonal to ξ. Moreover, if the area form of g_h corresponds to $\frac{1}{\lambda^n}(d\alpha)^n$, for some $\lambda \in \mathbb{R}^+$, then (λ, g) becomes adapted to the contact structure.

Assumption 1.4.4. We will always assume that a contact manifold (M,ξ) is coorientable. Therefore there always exists a contact form α.

There are many choices for the horizontal metric, some more natural than others. For example, the next Section describes the construction of a special subRiemannian metric that is naturally recovered from the differential of the contact form. This is called *contact metric structure* and it is the most natural choice. Another interesting choice could be a metric that is preserved by the Reeb orbits. In that case we say that the Reeb vector field r is *Killing*.

Definition 1.4.5. Let (ξ, g_h) be a subRiemannian contact structure on M. The Reeb vector field r is Killing if $\mathcal{L}_r g_h = 0$.

As $\mathcal{L}_r\alpha = 0$ by definition of contact structure, we have that $\mathcal{L}_r g_h = 0$ if and only if $\mathcal{L}_r g = 0$. An example of Killing Reeb vector field appears in the Boothby-Wang construction.

Example 1.4.6. In line with Section 1.3.7, let $S^1 \hookrightarrow (M,\xi) \xrightarrow{\pi} B$ be an S^1-principal bundle over a symplectic manifold (B,ω). We can lift any metric h of TB to obtain a subRiemannian metric g_h on ξ. The resulting metric g_h makes the Reeb vector field Killing by construction. In fact,

$$\mathcal{L}_r g_h = \mathcal{L}_r \pi^* h = 0 = \left.\frac{d}{dt}\right|_{t=0} (\phi_r^t)^* \pi^* h = \left.\frac{d}{dt}\right|_{t=0} (\pi \circ \phi_r^t)^* h = 0,$$

because $\pi \circ \phi_r^t = \mathrm{Id}$ by construction.

In the presence of an adapted Riemannian metric (λ, g), the Reeb vector field becomes the unitary vector field orthogonal to the contact distribution. This observation induces the following definition:

Definition 1.4.7. An *adapted local frame* is an orthonormal ξ-frame $\{r, r_1, ..., r_{2n}\}$.

Given an adapted frame $\{r, r_1, ..., r_{2n}\}$, its dual coframe can be written as $\{\alpha, \alpha_1, ..., \alpha_{2n}\}$, where α is the unitary contact form of (M,ξ), with respect to g. This dual coframe is also said to *adapted* when the first entry is the contact form.

1.4.1 Contact-metric structure

A *contact metric structure* is a very natural type of adapted Riemannian metric, in the sense of Remark 1.4.3. In this setting, the horizontal metric is recovered from the differential of the contact form $d\alpha$ in a Kahler geometry style. To construct it, we will need, in particular, an almost complex structure on the contact distribution ξ. Let (M,ξ) be a $(2n+1)$-dimensional contact manifold.

Definition 1.4.8. An *almost complex structure* on (M, ξ) is a $(1,1)-$tensor J on M satisfying

$$J^2 = -\operatorname{Id} + \alpha \otimes r.$$

If restricted to ξ, J is an almost complex structure in the usual sense: it is a endomorphism $J : \xi^1 \to \xi^1$ satisfying $J^2 = -\operatorname{Id}$. This claim is a result of the following Lemma.

Lemma 1.4.9. *Let J be a almost complex structure on (M, ξ). Then J has rank $2n$ and $Jr = 0$.*

Proof. Let us prove first that J vanishes in the vertical direction. By definition, we have that

$$J^2 r = -r + \alpha(r) r = r - r = 0.$$

Therefore, either $Jr = 0$ or Jr is a non-trivial eigenvector of J of the zero eigenvalue. In both cases we have that

$$0 = J(J^2 r) = J^2(Jr) = -Jr + \alpha(Jr) r \qquad \Longrightarrow \qquad Jr = \alpha(Jr) r.$$

By applying J to the last equality, we get

$$0 = J^2 r = J(Jr) = J(\alpha(Jr) r) = \alpha(Jr) Jr,$$

which implies that $Jr = 0$. To prove that it vanishes only there, take another vector field $X \in \mathfrak{X}(M)$ satisfying $JX = 0$. Then,

$$0 = J(JX) = J^2 X = -X + \alpha(X) r,$$

which means that X is a multiple of r. □

Corollary 1.4.10. *Let J be a almost complex structure on (M, ξ). Then $Im(J) = \xi^1$.*

Proof. From $\alpha(r) = 0$, it is simple to see that $\alpha \circ J = 0$, resulting in $Im(J) \subseteq \ker \alpha$. Moreover, as J has rank $2n$, then $Im(J) = \xi^1$. □

Therefore, J is precisely an almost complex structure on the bundle ξ. As $d\alpha$ is a closed non degenerate form on ξ, we may suspect that $d\alpha(-, J-)$ defines a subRiemannian metric on ξ. When this is the case, we say that J is *compatible* with α, which motivates the following definition.

Definition 1.4.11. A *contact metric structure* on a contact manifold (M, ξ) is a triple (λ, J, g) where J is an almost complex structure on (M, ξ), $\lambda \in \mathbb{R}^+$ and g is Riemannian metric on M satisfying

- $d\alpha(X, Y) = \lambda g(JX, Y)$;
- $g(X, Y) = g(JX, JY) + \alpha^2(X, Y)$.

If there exists a contact-metric structure on (M, ξ), then the Riemannian metric is given by

$$g = \frac{1}{\lambda} d\alpha(-, J-) + \alpha \otimes \alpha.$$

Once again the coefficient λ is usually assumed to be 1. However, it will change under anisotropic deformation. It is related to the previous λ by the following Lemma.

Lemma 1.4.12. *The contact metric g is indeed adapted to ξ with coefficient λ.*

Proof. By construction, we can create a positive local orthonormal frame of ξ given by $\{r_1, Jr_1..., r_n, Jr_n\}$, where $r_i \in \xi$. We can extend the frame to TM by adding the Reeb vector field r as a first entry. Indeed, r is orthonormal to ξ because

$$\|r\|^2 = d\alpha(r, Jr) + (\alpha(r))^2 = 1.$$

Since $\{r_1, Jr_1, ..., r_n, Jr_n\}$ is orthonormal in ξ, we have that $g(r_i, r_i) = 1$ and $g(Jr_1, Jr_i) = 1$, and the rest is 0. Therefore,

$$d\alpha(r_i, Jr_i) = \lambda, \qquad d\alpha(r_i, Jr_j) = 0,$$

for every $i \neq j$. This is sufficient to show that the volume form $\alpha \wedge (d\alpha)^n$ sends the frame to λ^n. Therefore, the area form of g is exactly $(1/\lambda^n)\alpha \wedge (d\alpha)^n$, which implies $\lambda^n \star \alpha = (d\alpha)^n$.

□

The only problem now is the existence of such structure. The next Theorem will assure that contact metric structures always exist. The proof relies on the algebraic process of *polarization*. It allows to decompose a non singular matrix into the product of an orthogonal one and a positive definite symmetric one. For further reference, check [4, Section 4.2].

Theorem 1.4.13. *Let $(M, \xi = \ker \alpha)$ be a coorientable contact manifold. There exists a contact-metric structure on (M, ξ).*

Proof. Let h be any Riemannian metric on M. Define the map $T : \mathfrak{X}(M) \to \mathfrak{X}(M)$ as $T = -\operatorname{Id} + \alpha \otimes r$ and a new metric on M as

$$h'(X, Y) = h(T(X), T(Y)) + \alpha(X)\alpha(Y).$$

By construction, we have that $h'(X, r) = \alpha(X)$. Let $X_1, ..., X_{2n}$ be a local orthonormal basis of ξ with respect to h'. Denote by A the skew symmetric non singular matrix whose component are given by $d\alpha(X_i, X_j)$. Polarize A into the product FG, where F is some orthogonal matrix and G is positive definite and symmetric. Now define

$$g'(X_i, X_j) := G_{ij}, \qquad J'X_i := F_{ij}X_j.$$

The new metric g and almost complex structure J are independent of the choice of the frame. In fact, given another orthonormal frame $\{Y_1, ..., Y_{2n}\}$, there exists an orthogonal matrix P such that

$$d\alpha(Y_i, Y_j) = (PAP^{-1})_{ij}.$$

Therefore g' and J' are globally defined on ξ. Moreover, $(J')^2 = -\operatorname{Id}$ by skewsymmetry of F. Extend g' to a metric g on M that agrees with h' in the Reeb direction. Do the same for J' and construct a morphism J by setting $J(r) = 0$. The pair (g, J) becomes a contact-metric structure of (M, ξ). □

Example 1.4.14. In the setting of the Boothby-Wang construction, it is possible to create a contact-metric structure on the S^1-bundle $M \xrightarrow{\pi} (B, \omega)$ in a natural way. As a symplectic manifold, B admits an almost complex structure $j : TB \to TB$ that makes $h := \omega(-, j-)$ a Riemannian metric on B. The pair $(\pi^* h + \alpha^2, \pi^* j)$ is a contact-metric structure of (M, ξ).

1.4.2 Anisotropic deformation

A typical approach to subRiemannian geometry on contact manifolds consists of blowing up the adapted Riemannian metric transversally to the tangent distribution. This process is commonly known as *anisotropic deformation.* In this setup, it is insightful to study the convergence of the induced metric on the manifold. Specifically, let (M, ξ) be a contact manifold and $(\lambda, g := g_h + \alpha \otimes \alpha)$ be an adapted metric structure in the sense of Remark 1.4.3. Then, we perturb the metric in the following way: for $\epsilon > 0$, let g_ϵ be a new Riemannian metric on M given by

$$g_\epsilon := g_h + \frac{1}{\epsilon^2}\alpha \otimes \alpha.$$

This new Riemannian metric is adapted to the contact structure as long as we rescale the contact form α.

Lemma 1.4.15. *If we rescale the contact form by* $1/\epsilon$*, then* $(\lambda/\epsilon, g_\epsilon)$ *becomes an adapted metric structure on* (M, ξ)*.*

Proof. Let α_ϵ be the rescaled contact form given by $\alpha_\epsilon := \frac{1}{\epsilon}\alpha$. Then, we have that

$$\|\alpha_\epsilon\|_\epsilon^2 = \epsilon^2\|\alpha_\epsilon\|^2 = \|\alpha\|^2 = 1.$$

Moreover, the area form induced by g_ϵ satisfies $\lambda^n vol_{g_\epsilon} = \alpha_\epsilon \wedge (d\alpha)^n$, from which we derive the relation $vol_g = \epsilon vol_{g_\epsilon}$. Hence,

$$\alpha_\epsilon \wedge (d\alpha_\epsilon)^n = \frac{1}{\epsilon^{n+1}}\alpha \wedge (d\alpha)^n = \frac{\lambda^n}{\epsilon^{n+1}} vol_g = \frac{\lambda^n}{\epsilon^n} vol_{g_\epsilon}.$$

□

In turn, the associated Reeb vector field becomes

$$r_\epsilon := \epsilon r.$$

The vector field r is the unitary vector field of the metric g orthogonal to the contact distribution. Anisotropic deformation does not change the geometrical shape of the manifold, but it changes quite radically the metric, and thus the intrinsic geometry of the manifold. Clearly if $\epsilon = 1$, the two metrics coincide and everything remains the same. However, depending on whether $\epsilon \to \infty$ or $\epsilon \to 0$, the perturbation affects the dynamic of the system in two opposite ways: it reduces or increases the length and energy of the integral curves of the Reeb vector field. To see this, consider a Reeb orbit $\gamma : I \to M$. Then, there exists $f \in \mathcal{C}^\infty(M)$ such that $\dot\gamma(t) = f(\gamma(t))r = \frac{f(\gamma(t))}{\epsilon} r_\epsilon$. In turn, we obtain the following relations:

$$\ell_\epsilon(\gamma) = \int_I \frac{|f(\gamma(t))|}{\epsilon} dt = \frac{1}{\epsilon}\ell(\gamma), \qquad E_\epsilon(\gamma) = \frac{1}{\epsilon^2}E(\gamma).$$

Therefore, it influences the dynamic of the system in the sense that the geodesics of the Riemannian manifold (M, g_ϵ) will either prefer or avoid the vertical direction. It turn, it affects the metric D_ϵ of the manifold. In the limiting cases $\epsilon \to \infty$ or $\epsilon \to 0$, we want to talk about the convergence of the metric space (M, D_ϵ) to some other metric space, and to do so, we need the notion of *Gromov-Hausdorff distance*.

Definition 1.4.16. Let $(X, D_X), (Y, D_Y)$ be two compact metric spaces. The *Gromov-Hausdorff distance* between X, Y is set to be

$$d_{GH}(X, Y) := \inf_{Z,f,g} d_H(f(X), g(Y)),$$

where $f : X \to Z$ and $g : Y \to Z$ are isometric embeddings and d_H is the Hausdorff distance d_H in Z.

In particular, a sequence of compact metric spaces (N_ϵ, d_ϵ) converges to some other compact metric space (N, d) in the Gromov-Hausdorff sense if the Gromov-Hausdorff distance between the spaces tends to zero.

First scenario

In the first case, for a large ϵ, the new Reeb vector field is longer than the previous one. This means that the cost of traveling along the Reeb orbits, in terms of energy, drastically reduces. The greater ϵ is, the easier to move along the vertical direction. In the limit $\epsilon \to \infty$, the Reeb orbits will have zero length. If the contact manifold (M, ξ) has dense Reeb orbits, every pair of points in the manifold will have zero distance in the limit $\epsilon \to \infty$. Therefore, the induced distance D_ϵ collapses in the limit.

What happens instead when the contact manifold has a regular Reeb vector field? As usual, let us describe this setting for the Boothby-Wang fibration.

> *Example* 1.4.17. Let $S^1 \hookrightarrow (M, \alpha) \xrightarrow{\pi} (B, \omega)$ be a a Boothby-Wang fibration. Consider a metric h on B such that $\lambda^n vol_h = \omega^n$ for some $\lambda \in \mathbb{R}^+$, and lift it to the adapted metric $g = \pi^* h + \alpha \otimes \alpha$. Under anisotropic deformation, the new metric becomes
>
> $$g_\epsilon = \pi^* h + \frac{1}{\epsilon^2} \alpha \otimes \alpha.$$
>
> In the limit $\epsilon \to \infty$, the S^1-fibers become very short, making the projection π an isometry of metric spaces. We say that (M, D_ϵ) converges to (B, D_h) in the Gromov-Hausdorff sense.

Since the metric collapses on contact manifolds with dense Reeb orbits, the study of the convergence for $\epsilon \to \infty$ is often restricted to contact manifolds with regular Reeb vector field, such as Boothby-Wang fibration. We will focus on the other scenario, because of its more general setup.

Second scenario

Whenever ϵ is very small, the new Reeb vector field becomes shorter and the Reeb orbits diverge in length. This construction punishes motion in the vertical direction, which means that the cost, in terms of energy, of traveling along the Reeb orbits will tend to infinity in the limit. In this sense, we expect that the induced metric D_ϵ on M will eventually coincide with the horizontal one, given by the length of horizontal geodesics. Clearly, if two points in the manifold are not connected by an horizontal geodesics, then, for any fixed ϵ, the distance D_ϵ will consider geodesics with vertical components. This will happen also in the limit $\epsilon \to 0$, but in this case, those curves will have infinite length and so, the distance between the two points will coincide with the subRiemannian one. This is precisely what Gromov proved for compact contact manifolds.

> **Proposition 1.4.18** (Gromov)**.** *Let (M, ξ) be a compact contact manifold under anisotropic deformation. Moreover, let D_ϵ and D_h be the distances induced by g_ϵ and g_h, respectively. Then, for $\epsilon \to 0$, the metric space (M, D_ϵ) converges to (M, D_h) in the Gromov-Hausdorff sense.*

This result is the reason why anisotropic deformation is such a useful tool for subRiemannian geometers. We will see in the next chapters, other features of this limit.

Chapter 2

The tame world of closed 3-manifolds

Three-manifolds are the lowest dimensional spaces that admit contact structures. Together with their natural visual interpretation, they provide a somewhat typical framework for studying contact manifolds. Among the features that make it a simpler setup, we encounter the existence of contact structures and the existence of closed Reeb orbits, claimed in Theorem 2.1.2 and 2.1.3, respectively.

In this chapter, we provide some relevant results about closed oriented three-manifolds. The most important part of this chapter is contained in Section 2.3.3, where we carefully study the structural equations of the Levi-Civita connection associated to the $1-$parametric family of Riemannian metrics $\{g_\epsilon\}_\epsilon$.

2.1 Relevant results

As mentioned before, three-manifolds are the easiest possible framework for studying contact structures. To make it even easier, we assume the manifolds to be closed, i.e. compact without boundary. In this setup, we have many remarkable results, some of which require particularly involved proofs and techniques that we are likely to withhold. The following statement, for example, assures a property of oriented three-manifolds that is obvious for line bundles but not in higher dimensions.

Theorem 2.1.1. *Closed oriented three-manifolds have trivializable tangent bundle.*

Proof. [14, Theorem 4.2.1]. □

This Theorem is a first step towards the following result, first proved by Martinet in 1971.

Theorem 2.1.2 (Martinet)**.** *Closed oriented three-manifolds always admit a contact distribution.*

Thanks to Martinet's Theorem, we can always construct a contact structure on a closed oriented three-manifold M. But we can do more than this. It has been proven that there are many contact structures, at least one for each element of $H^2(M,\mathbb{Z})$. Indeed, contact distributions on M are in surjection onto $H^2(M,\mathbb{Z})$ by associating their Euler classes as vector bundles. Another important result, also attributed to Martinet, is that every oriented codimension 2 distribution on M can be homotopically deformed to a contact distribution ([14, Theorem 4.3.1]).

What about Reeb orbits? Are there any closed Reeb orbits for a given Reeb vector field?

Let (M,ξ) be a coorientable closed contact three-manifold. There are many examples in literature of non-vanishing vector fields on M without periodic orbits. However, it has been proven by Taubes [31, Theorem 1] that the Reeb vector field of (M, α) has at least one closed orbit.

Theorem 2.1.3 (Weinstein). *(M, α) admits at least one closed Reeb orbit.*

Together with Lemma 1.3.24, this Theorem is a great result for studying closed Reeb orbits. The purpose of the following chapter will be to describe the behaviour of the horizontal geodesics around the closed Reeb orbit. One of the open question about this is whether infinitesimally close geodesics are periodic too.

2.2 Local model

As we said in Section 1.3.6, the local model of a general contact manifold (M, ξ) is usually given by $(\mathbb{R}^{2n+1}, \xi_{std})$ where ξ_{std} is the Cartan's distribution. Nevertheless, in the three-dimensional case, we choose as a local model a distribution in $\mathbb{R}^3$ that arises naturally from the so called *Heisenberg group*, of which we are about to give a brief description.

Intro: the Heisenberg group

The *Heisenberg group*, or *first Carnot group* is a subgroup of general linear group in $\mathbb{R}^3$ given by the upper triangular matrices.

Definition 2.2.1. The *Heisenberg group* $H \subset GL_3(\mathbb{R})$ is defined by

$$H := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R}) : a, b, c \in \mathbb{R} \right\}.$$

It is a Lie subgroup of $GL_3(\mathbb{R})$ whose background manifold is $\mathbb{R}^3$. In this sense, it is convenient to consider the set of coordinates $(a, b, c) : H \to \mathbb{R}^3$. Its Lie algebra $\mathfrak{h}$ is the tangent space at $(0, 0, 0)$ and its canonical basis is given by the velocity at 0 of the following curves: $\gamma_1(t) = (t, 0, 0)$, $\gamma_2(t) = (0, t, 0)$ and $\gamma_3(t) = (0, 0, t)$. For any $p = (a, b, c) \in H$, these curves generate a frame of T_pH given the vectors $(r_i)_p := \frac{d}{dt}\big|_{t=0} \gamma_i(t) \cdot p$. Precisely, we have that

$$(r_1)_p = \begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (r_2)_p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad (r_3)_p = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

is a basis of the tangent space at $(a, b, c) \in H$. The exponential map $exp : \mathfrak{h} \to H$ of H is explicitly given by, for $X \in \mathfrak{h}$,

$$exp(X) = \mathrm{Id} + X + \frac{1}{2}X^2.$$

A simple computation shows indeed that $X^k = 0$ for all $k \geq 3$ and $X \in \mathfrak{h}$. Moreover, we can also show that the *lower central series* of the Lie brackets inside $\mathfrak{h}$ terminates, meaning that the Lie algebra $\mathfrak{h}$ is nilpotent. As a consequence, by general Lie group theory, we obtain that the exponential map is an diffeomorphism. For any $X = xr_1 + yr_2 + zr_3$, the exponential map is given by

$$exp(X) = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

With a change of coordinates, let (x, y, z) be the element in H corresponding to $xr_1 + yr_2 + zt_3 \in \mathfrak{h}$ through the exponential map as in Equation (2.2). In other words, let $exp^{-1} : H \to \mathfrak{h} \cong \mathbb{R}^3$ be the new set of coordinates in H. With respect to this change of coordinates, the old frame of equation (2.1) takes the form

$$\begin{aligned} r_1 &= \partial_x + \frac{1}{2} y \partial_z \\ r_2 &= \partial_y - \frac{1}{2} x \partial_z \\ r_3 &= \partial_z. \end{aligned}$$

The first 2 vector fields r_1, r_2 will create a global frame for the distribution defining the so-called *canonical contact structure* in $\mathbb{R}^3$.

Canonical contact structure

Definition 2.2.2. The *canonical contact structure* in $\mathbb{R}^3$ is (ξ_{can}, g_h) where ξ_{can} is the distribution spanned by $\{r_1, r_2\}$ and g_h is the metric making those vector fields orthonormal. Precisely,

$$\xi_{can} = \left\langle \partial_x + \frac{1}{2} y \partial_z, \partial_y - \frac{1}{2} x \partial_z \right\rangle, \qquad g_h = dx \otimes dx + dy \otimes dy.$$

Consider the contact form

$$\alpha_{can} := dz + \frac{1}{2}(xdy - ydx).$$

It is easy to see that $\ker \alpha_{can} = \xi_{can}$ and that its differential is precisely $dx \wedge dy$. In particular, if $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection onto the first two coordinates, then g_h is the pullback of the Euclidean metric in $\mathbb{R}^2$ and $d\alpha$ is the pullback of the area form in $\mathbb{R}^2$ through π. This proves that $(1, g := g_h + \alpha^2)$ is an adapted metric structure on $(\mathbb{R}^3, \xi_{can})$. Moreover, if we set $r := \partial_z$, then $\{r, r_1, r_2\}$ becomes an adapted frame, which will be the canonical one.

With respect to the local model of Figure 1.9, this new local form rotates also in the $y-$direction. In fact, ξ_{can} is also usually defined in polar coordinates. For $\{\rho, \theta, z\}$, we have that

$$\alpha_{can} = dz + \rho^2 d\theta.$$

This local form is useful to observe the spiraling behavior of horizontal geodesics around the Reeb orbits. We will see this in the next Chapter.

2.3 Structural equations

In the previous chapter, we proved that coorientable contact manifolds always admit an adapted metric. In this section, we explore some properties of such metrics, both in the general and specific cases. In particular, we will describe the *structural equations* of some adapted metric on the contact structure.

2.3.1 Adapted metrics

Let (λ, g) be an adapted metric on (M, ξ). The associated Levi-Civita connection ∇ is the unique connection on the tangent bundle TM that is torsion free and compatible with the metric, i.e. satisfying for any $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} &[X, Y] = \nabla_X Y - \nabla_Y X \\ &X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \end{aligned}$$

Since M is oriented and endowed with a metric, the structure group of its tangent bundle reduces to $SO(3)$. This means that, by compatibility with the metric g, every local description of ∇ takes value in $\mathfrak{so}(3)$. In other words, for any local orthonormal frame $\{r_0, r_1, r_2\}$ on an open subset $U \subseteq M$, there exists $\omega \in \Omega^1(U, \mathfrak{so}(3))$ such that

$$\nabla \mid_U = d + \omega.$$

Explicitly, ω is an antisymmetric matrix of 1-forms on U defined by

$$\omega = \begin{pmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{pmatrix} \qquad a, b, c \in \Omega^1(U). \tag{2.3}$$

Consequently, the action of the Levi-Civita connection on the local frame is given by

$$\begin{cases} \nabla(r_0) = -a \otimes r_1 + b \otimes r_2; \\ \nabla(r_1) = a \otimes r_0 - c \otimes r_2; \\ \nabla(r_2) = -b \otimes r_0 + c \otimes r_1. \end{cases} \tag{2.4}$$

In terms of the dual coframe, $\{\alpha_0, \alpha_1, \alpha_2\}$, we have that

$$d \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{pmatrix} \wedge \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{2.5}$$

The adapted metric g induces some particular constraints on the local 1-forms a, b, c whenever the local frame is adapted. Let $\{r, r_1, r_2\}$ be an adapted local frame in U and $\{\alpha, \alpha_1, \alpha_2\}$ be the dual adapted coframe. By definition of adapted metric, we have that α is unitary and

$$d\alpha = \lambda \alpha_1 \wedge \alpha_2. \tag{2.6}$$

With respect to this frame, we can decompose the 1-forms of Equation (2.3) in $a = a_0\alpha + a_1\alpha_1 + a_2\alpha_2$ and $b = b_0\alpha + b_1\alpha_1 + b_2\alpha_2$ for some smooth functions $a_i, b_i \in \mathcal{C}^\infty(U)$.

Lemma 2.3.1. *The following relations hold:*

$$a_0 = b_0 = 0, \qquad a_2 + b_1 = \lambda, \qquad a_1 = b_2. \tag{2.7}$$

Proof. By Equation (2.5), we have that

$$\begin{aligned} d\alpha &= -a_0\alpha \wedge \alpha_1 - a_2\alpha_2 \wedge \alpha_1 + b_0\alpha \wedge \alpha_2 + b_1\alpha_1 \wedge \alpha_2 \\ &= a_0\alpha \wedge \alpha_1 + b_0\alpha \wedge \alpha_2 + (b_1 + a_2)\alpha_1 \wedge \alpha_2. \end{aligned}$$

Together with Equation (2.6), we obtain the following costraints on a_i, b_i:

$$a_0 = b_0 = 0, \qquad a_2 + b_1 = \lambda.$$

Moreover, by $d^2\alpha = 0$, we obtain that

$$\begin{aligned} 0 = d(d\alpha) &= \lambda d(\alpha_1 \wedge \alpha_2) = \lambda(d\alpha_1 \wedge \alpha_2 - \alpha_1 \wedge d\alpha_2) \\ &= \lambda(a \wedge \alpha \wedge \alpha_2 + \alpha_1 \wedge b \wedge \alpha) \\ &= \lambda(b_2 - a_1) vol_g \end{aligned}$$

which implies the constraint $a_1 = b_2$. □

Note also that the structural equations in (2.5) and evaluations in (2.7) give a nice interpretation of the covariant derivative along the Reeb direction. Indeed,

$$\begin{cases} \nabla_r r_1 = a_0 r - c_0 r_2 = -c_0 r_2 \\ \nabla_r r_2 = -b_0 r + c_0 r_1 = c_0 r_1. \end{cases} \tag{2.8}$$

The next Lemma gives an idea of what is the geometrical meaning of the previous equations.

Lemma 2.3.2. *the function c_0 defines the infinitesimal rotation of ξ produced by the parallel transport along r.*

Proof. Let $p \in M$ be a point, and $\gamma : [0, T] \to M$ be a Reeb orbit starting at p and ending at $q := \gamma(T)$. For every vector $v \in T_pM$, there exists a unique lift $\widetilde{\gamma}_v : [0, T] \to TM$ such that

$$\pi \circ \widetilde{\gamma}_v = \gamma, \qquad \widetilde{\gamma}_v(0) = v, \qquad (\gamma^*\nabla)_{\frac{d}{dt}}(\widetilde{\gamma}_v) = 0.$$

The parallel transport $T^{\gamma}_{\nabla} : T_pM \to T_qM$ of ∇ along γ is defined by $T^{\gamma}_{\nabla}(v) = \widetilde{\gamma}_v(T)$. In particular, by Equation (2.8), we obtain that

$$\begin{cases} T^{\gamma}_{\nabla}((r_1)_p) = -c_0(q)(r_2)_q \\ T^{\gamma}_{\nabla}((r_2)_p) = c_0(q)(r_1)_q. \end{cases}$$

Therefore, c_0 describe the rotation of the frame vectors r_1, r_2 between ξ_p and ξ_q. This concept is shown in Figure 2.1. □

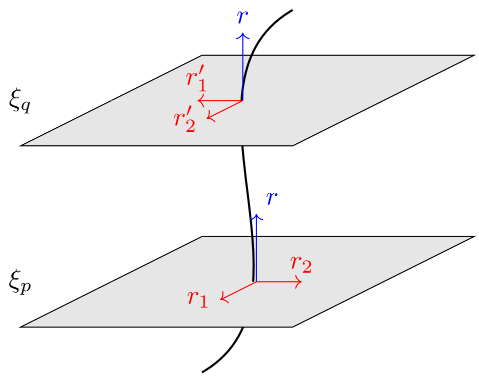

Figure 2.1: Rotation of the frame of ξ_p induced by the parallel transport of ∇ along a Reeb orbit, where $r_i' = T^{\gamma}_{\nabla}((r_i)_p)$.

We can then project ∇ to the contact bundle ξ, to obtain a connection on ξ.

Definition 2.3.3. Let $\nabla^{\perp}$ be the connection on ξ defined by $\nabla^{\perp} := \Pi_{\xi} \circ \nabla\mid_{\xi}$ where $\Pi_{\xi} : TM \to \xi$ is the fiberwise orthogonal projection. $\nabla^{\perp}$ is called *horizontal Levi-Civita connection.*

Remark 2.3.4. By linearity of the projection, it is a well defined connection. Indeed, for all $X, Y \in \mathfrak{X}(\xi)$ and $f \in \mathcal{C}^{\infty}(M)$,

$$\nabla^{\perp}_{fX}Y = \Pi_{\xi}(f\nabla_X Y) = f\Pi_{\xi}(\nabla_X Y) = f\nabla^{\perp}_X Y$$
$$\nabla^{\perp}_X(fY) = \Pi_{\xi}(X(f)Y) + \Pi_{\xi}(f\nabla_X Y) = X(f)Y + f\nabla^{\perp}_X Y.$$

Moreover, equation (2.4) gives us the structural equations of $\nabla^{\perp}$ as

$$\begin{cases} \nabla^{\perp} r_1 = \Pi_{\xi}(\nabla r_1) = \Pi_{\xi}(a \otimes r - c \otimes r_2) = -c \otimes r_2 \\ \nabla^{\perp} r_2 = \Pi_{\xi}(\nabla r_2) = \Pi_{\xi}(-b \otimes r + c \otimes r_1) = c \otimes r_1. \end{cases} \tag{2.9}$$

This connection will defined the canonical Spin$^{\mathbb{C}}$-connection on the complexified bundle $\xi \otimes \mathbb{C}$ as we will see in Section 5.2.3.

2.3.2 Killing scenario

Let (λ, g) be an adapted metric structure on (M, ξ), for which the Reeb vector field is Killing. In this framework the structural equations of (λ, g) are even simpler than before. Let ∇ be the Levi-Civita connections on TM associated to g. We need a preliminary Lemma on the Lie derivative of a metric tensor.

Lemma 2.3.5. *For any $X, Y, Z \in \mathfrak{X}(M)$, we have the following relation:*

$$\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).$$

Proof. The Lie derivative along a vector field is a derivation, so it satisfies

$$\mathcal{L}_X g(Y, Z) = \mathcal{L}_X(g(Y, Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z)$$

which is the Leibniz rule with respect to tensor contraction. Moreover, by the definition of Lie bracket ($[X, Y] := \mathcal{L}_X(Y)$) and the defining properties of the Levi-Civita connection, we can add

$$\begin{aligned}
\mathcal{L}_X g(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z]) \\
&= g(\nabla_X Y - [X, Y], Z) + g(Y, \nabla_X Z - [X, Z]) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X)
\end{aligned}$$

□

Thanks to this Lemma, we can observe the effects of the Killing condition on the structural coefficients of g. Let $\{r, r_1, r_2\}$ be a local adapted frame with dual coframe $\{\alpha, \alpha_1, \alpha_2\}$. Moreover, let a, b, c be the local forms for the representation of ∇. By the Killing condition, we have that $L_r(Y, Z) = 0$ for every $Y, Z \in \mathfrak{X}(M)$. If we substitute Y, Z with pairs of frame vectors $\{r, r_1, r_2\}$ and use the structural equations in (2.4), we obtain

$$a_0 = a_1 = b_0 = b_2 = 0, \qquad b_1 = a_2 \tag{2.10}$$

Together with the constraints on the structural coefficients for general adapted structures, we obtain the following Lemma.

Lemma 2.3.6. *For Killing adapted metric structures (λ, g), we have the following relations:*

$$a = \frac{\lambda}{2}\alpha_2, \qquad b = \frac{\lambda}{2}\alpha_1.$$

If we compute the Lie brackets $[r_1, r]$ and $[r_2, r]$ in this setting we obtain that

$$\begin{cases} [r_1, r] = (b_1 + c_0) r_2 = \left(\frac{\lambda}{2} + c_0\right) r_2 \\ [r_2, r] = -(a_2 + c_0) r_1 = -\left(\frac{\lambda}{2} + c_0\right) r_1. \end{cases}$$

Therefore, the function $\varphi := \frac{\lambda}{2} + c_0$ defines the magnitude of the Lie bracket between the Reeb vector field and the horizontal basis vectors.

> *Example* 2.3.7. In line with Section 1.3.7, let (M, α) be a Boothby-Wang fibration over a symplectic 2-manifold (B, ω). Consider a metric h on B whose area form is $\frac{1}{\lambda}\omega$, for some $\lambda \in \mathbb{R}^+$. Then, we know that
>
> $$(\lambda, g := \pi^* h + \alpha^2)$$
>
> is an adapted metric structure on (M, ξ), and that the Reeb vector field is Killing. Let ∇^B be the Levi-Civita connection on B. Then $\nabla = \pi^* \nabla^B$. Consider an adapted local frame $\{r, r_1, r_2\}$ of M. By construction, there exists a local orthonormal frame $\{b_1, b_2\}$ of B satisfying $\pi^* b_i = r_i$. In this setting, the structural equations of ∇ derive from the ones of ∇^B.

2.3.3 Anisotropic deformation

Let $(\lambda, g := g_h + \alpha^2)$ be an adapted metric structure of (M, ξ) in the sense of Remark 1.4.3. Then, by Lemma 1.4.15, the anisotropic deformation of (λ, g) by $\epsilon > 0$ is an adapted metric structure of type $(\lambda/\epsilon, g_\epsilon)$ for the rescaled contact form $\alpha_\epsilon = \frac{1}{\epsilon}\alpha$. Can we relate the structural equations of (λ, g) with the ones of $(\lambda/\epsilon, g_\epsilon)$?

Denote by ∇, ∇^ϵ the Levi Civita connections associated to g and g_ϵ, respectively. Consider the local adapted frames $\{r, r_1, r_2\}$ and $\{r_\epsilon, r_1, r_2\}$ and denote by $\{a, b, c\}$ and $\{a^\epsilon, b^\epsilon, c^\epsilon\}$ the structural 1-forms of the local representations of ∇ and ∇^ϵ, as in Equation (2.3). The structural equations of ∇^ϵ in terms of an adapted frame $\{r_\epsilon, r_1, r_2\}$ rely on the constraints

$$a_0^\epsilon = b_0^\epsilon = 0, \qquad a_2^\epsilon + b_1^\epsilon = \frac{\lambda}{\epsilon}, \qquad a_1^\epsilon = b_2^\epsilon.$$

We can state this as a Lemma to clarify the relations with the previous structural coefficients.

Lemma 2.3.8. *We have the following relations:*

$$a_0^\epsilon = b_0^\epsilon = a_0 = b_0 = 0, \qquad \epsilon(a_2^\epsilon + b_1^\epsilon) = (a_2 + b_1), \qquad a_1^\epsilon = b_2^\epsilon = \epsilon a_1 = \epsilon b_2.$$

Proof. The only thing to prove is the last relation. The Lie bracket is metric-independent. Thus,

$$\begin{aligned} [r_\epsilon, r_1] &= \nabla^\epsilon_{r_\epsilon} r_1 - \nabla^\epsilon_{r_1} r_\epsilon = a_1^\epsilon r_1 - (b_1^\epsilon + c_0^\epsilon) r_2 \\ &= \epsilon(\nabla_r r_1 - \nabla_{r_1} r) = \epsilon a_1 r_1 - \epsilon(b_1 + c_0) r_2. \end{aligned}$$

This implies that $a_1^\epsilon = \epsilon a_1$. We can do the same for $[r, r_2]$ and obtain $b_2^\epsilon = \epsilon b_2$. □

What about the structural form c^ϵ?

Lemma 2.3.9. *We have the following relations:*

$$c_0^\epsilon = \epsilon c_0 + \frac{\lambda}{2}\left(\epsilon - \frac{1}{\epsilon}\right), \qquad c_1^\epsilon = c_1, \qquad c_2^\epsilon = c_2.$$

Proof. First, we prove that $c_i^\epsilon = c_i$ for $i = 1, 2$. To see this, compute the Lie bracket of $[r_1, r_2]$ for both ∇^ϵ and ∇. The result is the equality

$$-\lambda r + c_1 r_1 + c_2 r_2 = [r_1, r_2] = -\frac{\lambda}{\epsilon} r_\epsilon + c_1^\epsilon r_1 + c_2^\epsilon r_2. \tag{2.11}$$

Therefore, $c_i^\epsilon = c_i$ for $i = 1, 2$. Let us compute the relation between rotational coefficients c_0 and c_0^ϵ. By the Koszul formula, we get that

$$-2c_0^\epsilon = 2g_\epsilon(\nabla^\epsilon_{r_\epsilon} r_1, r_2) = g_\epsilon([r_\epsilon, r_1], r_2) - g_\epsilon([r_\epsilon, r_2], r_1) - g_\epsilon([r_1, r_2], r_\epsilon)$$

Note that the Lie bracket is independent of the metric, and therefore it is not affected by the deformation. Recall that r_ϵ is a contact field, which means that, by Definition 1.3.8, $[r_\epsilon, X]$ is horizontal for any horizontal vector field X. Therefore,

$$-2c_0^\epsilon = \epsilon g([r, r_1], r_2) - \epsilon g([r, r_2], r_1) - g_\epsilon([r_1, r_2], r_\epsilon)$$

We expect an exploding term for $g_\epsilon([r_1, r_2], r)$ because $[r_1, r_2]$ has a nowhere vanishing vertical component, as shown in Equation (2.11). Therefore,

$$g_\epsilon([r_1, r_2], r_\epsilon) = -\frac{\lambda}{\epsilon} g_\epsilon(r_\epsilon, r_\epsilon) = -\frac{\lambda}{\epsilon}.$$

Always by the Koszul formula, but this time on g, we obtain that

$$-2c_0^\epsilon = \underbrace{\epsilon g([r,r_1],r_2) - \epsilon g([r,r_2],r_1 - \epsilon g([r_1,r_2],r)}_{2\epsilon g(\nabla_r r_1, r_2)} + \epsilon g([r_1,r_2],r) - g_\epsilon([r_1,r_2],r_\epsilon)$$

Finally, we obtain that

$$-2c_0^\epsilon = 2\epsilon g(\nabla_r r_1, r_2) - \lambda\left(\epsilon - \frac{1}{\epsilon}\right) = -2\epsilon c_0 - \lambda\left(\epsilon - \frac{1}{\epsilon}\right),$$

which proves the claim.

□

The previous Lemma gives a further description of the horizontal Levi Civita connection $\nabla^{\epsilon,\perp}$.

Corollary 2.3.10. *The two horizontal Levi-Civita connections are related by*

$$\begin{cases} \nabla^{\epsilon,\perp} r_1 = \epsilon\nabla^\perp r_1 - \frac{\lambda}{2\epsilon}\left(\epsilon - \frac{1}{\epsilon}\right)\alpha\otimes r_2 \\ \nabla^{\epsilon,\perp} r_2 = \epsilon\nabla^\perp r_1 + \frac{\lambda}{2\epsilon}\left(\epsilon - \frac{1}{\epsilon}\right)\alpha\otimes r_1. \end{cases} \tag{2.12}$$

Remark 2.3.11. Lemma 2.3.9 asserts that the the coefficients c_0 and c_0^ϵ are related by

$$c_0^\epsilon = \epsilon c_0 + \frac{\lambda}{2}\left(\epsilon - \frac{1}{\epsilon}\right).$$

What is the geometrical meaning of this transformation? By Lemma 2.3.2, c_0 and c_0^ϵ define the infinitesimal rotation of ξ produced by the parallel transport along the Reeb orbits of ∇ and ∇^ϵ, respectively. We described in the previous chapter how Reeb orbits are metrically affected by the anisotropic deformation. For $\epsilon > 1$, the orbits are cheaper to travel on, and, in the limit $\epsilon \to \infty$, they are basically free of expenses. On the contrary, if $\epsilon < 1$, the Reeb orbits are more expensive, and practically impossible to undertake in the limit $\epsilon \to 0$. In both cases, the rotational coefficient c_0^ϵ seems to explode.

We could state a similar result for the other structural coefficients b_1^ϵ and a_2^ϵ.

Corollary 2.3.12. *The coefficients a_2 and b_1 satisfy:*

$$b_1^\epsilon = \epsilon b_1 - \frac{\lambda}{2}\left(\epsilon - \frac{1}{\epsilon}\right), \qquad a_2^\epsilon = \epsilon a_2 - \frac{\lambda}{2}\left(\epsilon - \frac{1}{\epsilon}\right).$$

Proof. Consider again the Lie bracket $[r_\epsilon, r_1]$. When computed with respect to ∇ and ∇^ϵ, it leads to the the equality

$$\epsilon a_1 r_1 - \epsilon(b_1 + c_0) r_2 = a_1^\epsilon r_1 - (b_1^\epsilon + c_0^\epsilon) r_2.$$

Specifically, we obtain that

$$b_1^\epsilon = \epsilon(b_1 + c_0) - c_0^\epsilon = \epsilon b_1 - \frac{\lambda}{2}\left(\epsilon - \frac{1}{\epsilon}\right).$$

The Lie bracket $[r_\epsilon, r_2]$ gives the other equality. Moreover, we see that

$$b_1^\epsilon + \alpha_2^\epsilon = \epsilon(b_1 + a_2) - \lambda\left(\epsilon - \frac{1}{\epsilon}\right) = \frac{\lambda}{\epsilon},$$

which is coherent with the previous description. □

Chapter 3

Geodesic flow in the 3-dimensional contact case

SubRiemannian geodesics can often be very delicate to study. In the literature, there are many examples of subRiemannian geodesics with pathological behaviour ([20],[29]). Surely, the existence of singular geodesics does not simplify the matter. Thus, it is slightly less challenging to work with contact manifolds. Furthermore, in the three dimensional case, we saw that closed oriented manifolds have also other interesting features. As shown in the previous chapter, they always admit a contact structure, and the generalized Weinstein conjecture holds, meaning that there exists at least one closed Reeb orbit for each contact form.

The goal of this chapter is to study the behaviour of horizontal geodesics that are very close to Reeb orbits. We will see that such curves spiral up around the Reeb orbits. Precisely, we will first consider the canonical contact structure in $\mathbb{R}^3$. In that setting, the spiraling behaviour will be evident. Afterwards, we will study the work of C. de Verdière, L. Hillairet and E. Trelat contained in [34], where they proved that subRiemannian geodesics with increasingly high initial momenta converge to a Reeb orbit. In the final section, we will relate this result to geodesics of the 1-parametric family g_ϵ.

For a thorough discussion on subRiemannian Geometry of contact manifolds we refer to [1] and [12].

3.1 Canonical contact structure

Consider the canonical contact structure $(\mathbb{R}^3, \xi_{can})$ of Definition 2.2.2. In the coordinates $\{x, y, z\}$, it is characterized by the contact form and Reeb vector field

$$\alpha_{can} = dz - \frac{1}{2}(ydx - xdy), \qquad r_{can} = \partial_z.$$

We can endowed ξ_{can} with the Euclidean metric $g_h = dx \otimes dx + dy \otimes dy$. Then $(1, g = g_h + \alpha_{can}^2)$ becomes an adapted metric structure on $(\mathbb{R}^3, \xi_{can})$. Consider the following horizontal vector fields:

$$r_1 := \partial_x + \frac{1}{2}y\partial_z, \qquad r_2 := \partial_y - \frac{1}{2}x\partial_z.$$

It is not difficult to see that $\{r_{can}, r_1, r_2\}$ is an adapted frame on $(\mathbb{R}^3, \xi_{can})$.

When we first introduced horizontal curves, we explained that one can make a choice on the regularity of the curves. As mentioned in Section 1.2, we usually assume that all horizontal curves are absolutely continuous and square integrable. In this canonical setting, it means that, for every horizontal curve γ, there exist unique square integral functions $u_1, u_2 \in L^2(I)$, defined on the domain of γ, such that

$$\dot{\gamma} = u_1 r_1 + u_2 r_2.$$

By the fundamental Theorem of ODEs, the converse is also true where the starting point is fixed. Let $p \in \mathbb{R}^3$ and $u \in L^2(I, \mathbb{R}^2)$, then there is only one solution γ of the ODE $\dot{\gamma} = u$ with initial condition $\gamma(0) = p$. This 1:1 correspondence will allow us to identify the space of all horizontal curves starting at a point p with $L^2(I, \mathbb{R}^2)$.

Let us take a look at the geometric behaviour of these horizontal curves. Consider the projections $\pi : T^*\mathbb{R}^3 \to \mathbb{R}^3$ and $\widetilde{\pi} : \mathbb{R}^3 \to \mathbb{R}^2$ onto the first two variables. Since the Reeb vector field r_{can} coincides with the $z-$direction ∂_z, it spans the kernel of the projection $\widetilde{\pi}$. The following Lemma ensures that horizontal geodesics undertake a spiraling behaviour in this setting.

Lemma 3.1.1. *A horizontal curve γ is a subRiemannian geodesic of (ξ_{can}, g_h) if and only if its projection $\widetilde{\pi} \circ \gamma$ is a circular segment in the plane.*

Proof. Let $\lambda_{can} = P_x dx + P_y dy + P_z dz$ be the tautological Liouville 1-form of the standard symplectic structure on $T^*\mathbb{R}^3$. Let a, b, c be functions on the cotangent bundle such that $\lambda_{can} = a\alpha_{can} + bdx + cdy$. Then, $\{a, b, c\}$ is a new set of coordinates on the fibers of TM^* and we will write $\{\partial_a, \partial_b, \partial_c\}$ as the induced frame of $T(T^*\mathbb{R}^3)$. So in particular, a full frame of $T(T^*\mathbb{R}^3)$ is given by $\{r_{can}, r_1, r_2, \partial_a, \partial_b, \partial_c\}$. The canonical symplectic structure ω_{can} in this coordinates is given by

$$\omega_{can} = d\lambda_{can} = da \wedge \alpha_{can} + db \wedge dx + dc \wedge dy + ad\alpha_{can}.$$

The contractions of ω_{can} with respect to the vectors in the frame are

$$\begin{cases} \iota_{r_{can}}\omega_{can} = -da\iota_r\alpha_{can} + a\iota_r d\alpha_{can} = -da \\ \iota_{r_1}\omega_{can} = -db\iota_{r_1}dx + a\iota_{r_1}d\alpha_{can} = -db + ady \\ \iota_{r_2}\omega_{can} = -dc\iota_{r_1}dy + a\iota_{r_2}d\alpha_{can} = -dc + adx \\ \iota_{\partial_a}\omega_{can} = \alpha_{can} \\ \iota_{\partial_b}\omega_{can} = dx \\ \iota_{\partial_c}\omega_{can} = dy. \end{cases} \tag{3.1}$$

In this coordinates, the subRiemannian Hamiltonian is $H(x, y, z, a, b, c) = b^2 + c^2$. Thus,

$$dH = 2bdb + 2cdc.$$

Together with the previous observations on ω_{can}, we conclude that the Hamiltonian flow takes the following form:

$$X_H = 2br_1 + 2cr_2 - 2a(b\partial_c - c\partial_b)$$

Indeed,

$$\begin{aligned} \iota_{X_H}\omega_{can} &= 2b\iota_{r_1}\omega_{can} + 2c\iota_{r_2}\omega_{can} - 2ab\iota_{\partial_c}\omega_{can} + 2ac\iota_{\partial_b}\omega_{can} \\ &= -2bdb + 2abdy - 2cdc + 2acdx - 2abdy + 2acdx = -dH. \end{aligned}$$

This implies that, if $\zeta(t) = (\gamma(t), P(t))$ is Hamiltonian orbit of X_H, the vector $2b\partial_x + 2c\partial_y$ is the velocity of the projection $\widetilde{\pi} \circ \gamma$. To see this, note that the direction of the velocity of the curve $\widetilde{\pi} \circ \gamma$ is given by

$$d(\widetilde{\pi} \circ \pi)(X_H) = 2bd\widetilde{\pi}(r_1) + 2cd\widetilde{\pi}(r_2) = 2b\partial_x + 2c\partial_y.$$

In particular, γ is a piece of circumference of radius $\frac{4(b^2+c^2)}{a}$ when $a \neq 0$ and a straight line otherwise. $\square$

Now that we know that horizontal geodesics move in a circular manner, let us observe in particular how geodesics close to the Reeb orbits behave. To do so, we will pick the origin $p = (0, 0, 0)$ as starting point for the curves, and a sufficiently close points $q = (0, 0, \epsilon)$, as the end one. Denote by $\Omega_{p,q}(\xi)$, the set of horizontal curves connecting p, q. As ξ is bracket generating, Chow Theorem (1.1.19) ensures that $\Omega_{p,q}(\xi)$ is not empty. For simplicity, we assume that all curves in $\Omega_{p,q}(\xi)$ are defined on the interval $I = [0, 2\pi]$. This is possible up to reparametrization of the curve. Let $\Omega \subseteq L^2(I, \mathbb{R}^2)$ be the subspace of square integrable functions in 1 : 1 correspondence to $\Omega_{p,q}(\xi)$. This correspondence works nicely when computing the energy of a curve γ_u whose associated function is $u \in \Omega$. Indeed, we can write

$$E(\gamma_u) = \frac{1}{2}\int_I \|\dot{\gamma}(t)\|^2 dt = \frac{1}{2}\|u\|_0^2$$

for the usual L^2-norm $\|\cdot\|_0$.

Corollary 3.1.2. *The critical points of E are all the curves $\gamma \in \Omega_{p,q}(\xi)$ whose projection $\widetilde{\pi} \circ \gamma$ is a circle in $\mathbb{R}^2$.*

Proof. The critical points of the energy functional are the horizontal geodesics. Therefore, by Lemma 3.1.1, we have the claim. □

Denote by $\Omega^L_{p,q}(\xi_{can})$ the space of horizontal curves of length at most $L > 0$. For $L < L'$, it is clear that

$$\Omega^L_{p,q}(\xi_{can}) \subseteq \Omega^{L'}_{p,q}(\xi_{can}) \subseteq \Omega_{p,q}(\xi_{can}).$$

Therefore, for any monotonically increasing sequence $\{L_i\}_i \subset \mathbb{R}$, we obtain a monotonically increasing sequence of spaces

$$\Omega^{L_1}_{p,q}(\xi_{can}) \hookrightarrow \Omega^{L_2}_{p,q}(\xi_{can}) \hookrightarrow \Omega^{L_3}_{p,q}(\xi_{can}) \hookrightarrow \Omega^{L_4}_{p,q}(\xi_{can}) \hookrightarrow \cdots \hookrightarrow \Omega_{p,q}(\xi_{can}).$$

As $\Omega_{p,q}(\xi_{can})$ is not empty, there exists an $L > 0$ such that $\Omega^L_{p,q}(\xi)$ is not empty as well. It seems reasonable to expect that the first number L_1 to satisfy this property would be the length of a horizontal geodesic γ connecting p and q.

By Lemma 3.1.1, γ projects to an arc of a circle in $\mathbb{R}^2$. As $\widetilde{\pi}(p) = \widetilde{\pi}(q)$, $\widetilde{\pi} \circ \gamma$ consists of the full circle. In particular, γ spirals up from p to q like in Figure 3.1. We can relate the energy and the length L_1 of

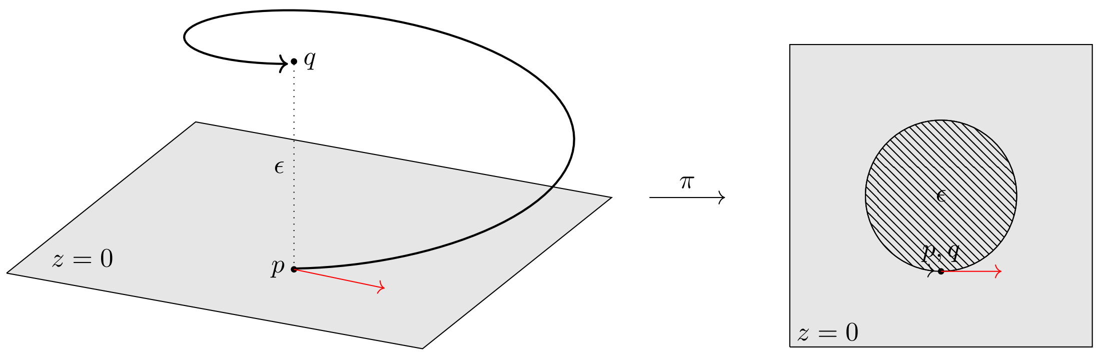

Figure 3.1: Horizontal geodesic connecting p and q: three-dimensional representation on the left and projection onto the plane $\{z = 0\}$ on the right.

the horizontal geodesic γ with the $z-$coordinate of the point q. To see this, note that the height of q is exactly the m-times the area of the disk whose boundary is the projection of γ onto $\{z = 0\}$, where m is the number of intersections (excluding p) between the curve γ and the z-axis. Without of loss of generalities, we assume that $m = 1$. We can parameterize γ as $\gamma(t) = (\rho(\cos(t) - 1), \rho\sin(t), \epsilon t/2\pi)$ for some $\rho \in \mathbb{R}$ that corresponds to the ray of the disk. To make sure that it is horizontal, compute its velocity. This is precisely

$$\dot{\gamma}(t) = -\rho\sin(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + \rho\cos(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} + \frac{\epsilon}{2\pi}\left(\frac{\partial}{\partial z}\right)_{\gamma(t)}.$$

Then, if we apply it to the standard contact form α_{can}, we obtain the following condition

$$\alpha_{can}(\dot{\gamma}(t)) = \frac{\epsilon}{2\pi} - \frac{1}{2}\rho^2 = 0 \qquad \Longleftrightarrow \qquad \epsilon = \pi\rho^2.$$

Therefore, the area of the disk in $\{z = 0\}$ is precisely the height of q. In turn, the length and the energy of γ are easily computable and they correspond to $L_1 = 2\sqrt{\pi\epsilon}$ and

$$E(\gamma) = \frac{1}{2}\int_I \|\dot{\gamma}(t)\|_h^2 dt = \pi\rho^2 = \epsilon.$$

The space $\Omega_{p,q}^{L_1}(\xi)$ is very interesting. It contains infinitely many horizontal curves but, as the maximum length is L_1, it contains only horizontal geodesics. In particular, it is homeomorphic to S^1. We can see this by projecting all the curves to the plane $\mathbb{R}^2$ via $\widetilde{\pi}$, and associating to each curve the angular coefficient of their velocity at time $t = 0$. Figure 3.2 clarifies the claim. Each curve in $\Omega_{p,q}^{L_1}(\xi)$ projects to

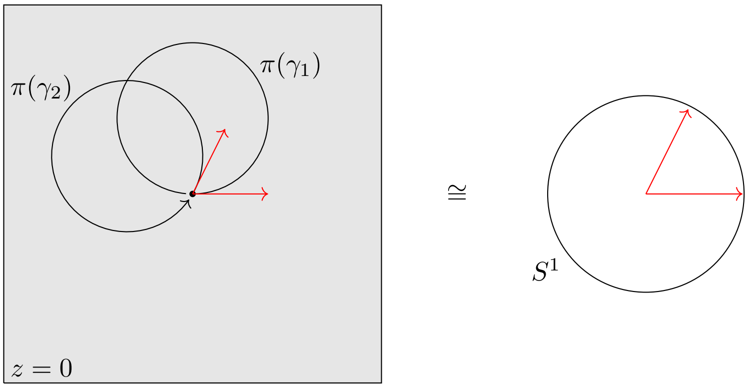

Figure 3.2: Projection of two curves $\gamma_1, \gamma_2 \in \Omega_{p,q}^{2\sqrt{\pi\epsilon}}$ to the plane $z = 0$ and their correspondence to elements in S^1.

a circle of area ϵ. Each of these circles is uniquely associated to the (positive) tangent vector at p, which corresponds to the initial velocity of the associated curve.

For $k \in \mathbb{N}$, let $L_k = 2\sqrt{\pi k\epsilon}$. Similarly to the case $k = 1$, we have that in each $\Omega_{p,q}^{L_k}(\xi)$, there are S^1 critical points of the energy E that spiral up k times and each one projects to the boundary of a disk of area ϵ/k. The energy of those curves γ_k is precisely $E(\gamma_k) = \epsilon k^2$. Now consider the sequence of spaces $\Omega_{p,q}^{L_k}(\xi)$ evaluated on the Energy functional E in a Morse Theory style

$$\begin{array}{ccccccccc}
\Omega_{p,q}^{L_1}(\xi) & \hookrightarrow & \Omega_{p,q}^{L_2}(\xi) & \hookrightarrow & \Omega_{p,q}^{L_3}(\xi) & \hookrightarrow & \cdots & \hookrightarrow & \Omega_{p,q}(\xi) \\
\downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} & & & & \downarrow{\scriptstyle E} \\
\{\epsilon\} & & [\epsilon, 4\epsilon] & & [\epsilon, 9\epsilon] & & \cdots & & [\epsilon, \infty)
\end{array} \tag{3.2}$$

Then, we can say that

$$\{\text{critical points of } E\} \cap \Omega_{p,q}^{L_k}(\xi) \cong \underbrace{S^1 \cup \cdots \cup S^1}_{\#k}.$$

For a further discussion on this construction, we refer to [2].

3.2 SubRiemannian geodesics around Reeb orbits

Let (λ, g) be an adapted structure on the closed three manifold (M, α), of the type $g = g_h + \alpha^2$. In this setting, any covector $(p, w) \in TM^*$ splits into horizontal and vertical components, i.e $w = w_h + h_w\alpha_p$. Consider the cometric g_h^*, uniquely associated to (ξ, g_h) by Proposition 1.2.5. Since the cometric vanishes on $Ann(\xi)$, we have that, for any $w, w' \in T_p^*M$,

$$g_h^*(w_h + h_w\alpha_p, w'_h + f_{w'}\alpha_p) = g_h^*(w_h, w'_h), \tag{3.3}$$

which is just the horizontal cometric applied to the projection onto ξ^*. The Hamiltonian of the system becomes

$$H_0(p,w) = g_h^*(w_h + h_w\alpha_p, w_h + h_w\alpha_p) = g_h^*(w_h, w_h).$$

The Hamiltonian associated to g is something of the form $H = H_0 + P_r^2$, where P_r is the momentum function of the Reeb vector field of (M,α).

The purpose of the upcoming Section is to construct a symplectic space (normal form) with a Hamiltonian $\widetilde{H}$, and a local symplectomorphism that pulls back $\widetilde{H}$ to H_0. In this way, we can study the subRiemanian geodesics from the Hamiltonian trajectories of $\widetilde{H}$.

3.2.1 Normal form

We will usually denote an element of $Ann(\xi)$ as $(p, h\alpha)$ for some $p \in M$ and $h \in \mathbb{R}$. Depending on the sign of h, we can split $Ann(\xi)$ into positive and negative bundles. Let $\Sigma := Ann^+(\xi)$ be the positive annihilator. By Remark 1.3.30, we can endow Σ with a symplectic structure given by $\omega_\Sigma := d(h\widetilde{\alpha})$, where $\widetilde{\alpha} := \pi^*\alpha$ is the pull back of the contact form under the projection $\pi : TM^* \to M$. The restriction of the Riemannian Hamiltonian to $Ann(\xi)$, $H_{Ann(\xi)} : \Sigma \to \mathbb{R}$, is given by

$$P_r^2(p, h\alpha) = h^2.$$

We consider its square root as a Hamiltonian on Σ. Therefore $H_\Sigma(p, h\alpha) := h$. The motivation to this choice is the following Lemma. Recall that the Hamiltonian flow $X_\Sigma \in \mathfrak{X}(\Sigma)$ satisfies

$$\omega_\Sigma(X_\Sigma, -) = -dH_\Sigma.$$

Lemma 3.2.1. *The projection onto M of the Hamiltonian flow X_Σ at a point $(p, h\alpha)$ is exactly the Reeb vector field r.*

Proof. Explicitly, we have

$$H_\Sigma(p,h) = h, \qquad \omega_\Sigma = dh \wedge \widetilde{\alpha} + hd\widetilde{\alpha}.$$

It is clear that, to satisfy the defining property, $\pi_*(X_\Sigma) = r$. Indeed, we just need that

$$\iota_{X_\Sigma}\widetilde{\alpha} = 1, \qquad \iota_{X_\Sigma}dh = \iota_{X_\Sigma}d\widetilde{\alpha} = 0.$$

Since r is the only vector field satisfying $\iota_r\alpha = 1$ and $\iota_r d\alpha = 0$, X_Σ must project to it. □

Consider the manifold $\widetilde{\Sigma} := \Sigma \times (\mathbb{R}^2 - \{0\})$ with symplectic structure given by $\widetilde{\omega} := \omega_\Sigma + du \wedge dv$ and conic structure

$$\lambda \cdot (p, h\alpha, u, v) := (p, \lambda h\alpha, \sqrt{\lambda}u, \sqrt{\lambda}v),$$

for any $\lambda > 0$. We define an Hamiltonian on $\widetilde{\Sigma}$ as

$$\widetilde{H}_\Sigma(p, h\alpha, u, v) = \sqrt{h(u^2+v^2)}. \tag{3.4}$$

Note that $\widetilde{H}_\Sigma$ vanishes on the annihilator bundle, just like the subRiemannian Hamiltonian H_0. The next Theorem ensures that there exists a symplectomorphism that sends one Hamiltonian to the other, up to a small error.

If two functions $f_1, f_2 : TM^* \to \mathbb{R}$ coincide up to order k on Σ, we will write $f_1 = f_2 + O_\Sigma(k)$.

Theorem 3.2.2 (Birkhoff)**.** *Let $p \in M$ be a point. There exists a conic neighbourhood $C \subseteq TM^* - \{0\}$ of $(p,\alpha) \in \Sigma$ and a smooth homogeneous symplectomorphism $\chi : C \to \widetilde{\Sigma}$ satisfying*

$$\chi\mid_{\Sigma^+\cap C} = Id_{\Sigma^+\cap C} \times \{0\} \qquad \& \qquad (\chi^{-1})^* H_0 = \widetilde{H}_\Sigma^2 + O_\Sigma(\infty),$$

where H_0 is the subRiemannian Hamiltonian.

Proof. Technical proof contained in [33]. □

The space $\widetilde{\Sigma}$ is therefore our normal form. For each point in the manifold, we have a neighbourhood of (p, α) and a symplectomorphism, sending the subRiemannian Hamiltonian to the (squared) Hamiltonian of a well constructed symplectic manifold. By expanding the term in $\widetilde{H}_\Sigma$, we have that

$$H_0 \circ \chi^{-1}(p, h\alpha, u, v) = (u^2 + v^2)h + O_\Sigma(\infty).$$

The term $(u^2 + v^2)$ implies that the level sets of the Hamiltonian outside of the annihilator bundle are paraboloids. It stimulates the idea that there might be a circular behaviour of the critical points of the Hamiltonian outside of Σ. It is precisely what we are hoping for, and the central focus of the next Section.

The next Theorem generalizes the Birkhoff normal form for any closed arc of Reeb orbit. We will denote by Σ_U the local positive bundle on a neighbourhood $U \subseteq M$.

Theorem 3.2.3 (Melrose)**.** *Let Γ_r be a closed arc of a Reeb orbit in M parametrized by $[0, 1]$. There exist*

- *a neighbourhood $U \subseteq M$ of Γ_r;*
- *two conical neighbourhoods $C \subseteq TU^*$ of Σ_U and $C' \subseteq \widetilde{\Sigma}_U$ of $\Sigma_U \times \{0\}$;*
- *a homogeneous symplectic diffeomorphism $\chi : C \to C'$ satisfying*

$$\chi\,|_{\Sigma_U} = Id_{\Sigma_U} \times \{0\}, \qquad \chi^*(\widetilde{H}_\Sigma^2) = H_0.$$

Proof. Proof contained in [34]. □

Normal forms generally serve a specific purpose, by making some objects and particular features much easier to observe. In the case of spiraling of geodesics around the Reeb orbits, the normal form is given by the previous Theorem, mostly because it exists only for neighbourhoods around the Reeb orbits. The next section uses this Theorem to prove the spiraling of horizontal geodesics. To do so, it first studies the geodesics in the normal form $\widetilde{\Sigma}$.

3.2.2 Spiraling Theorem

Consider the normal form $\widetilde{\Sigma}$ with Hamiltonian $\widetilde{H}_\Sigma$ defined in Equation (3.4) and symplectic structure given by $\widetilde{\omega} := d(h\widetilde{\alpha}) + du \wedge dv$. Endow $\mathbb{R}^2_{u,v}$ with coordinates (τ, θ) where θ is the usual angle and

$$\tau(p, h\alpha, u, v) := \sqrt{(u^2 + v^2)/H_\Sigma(p, h\alpha)} = \sqrt{\frac{u^2 + v^2}{h}},$$

which is the distance from $0 \in \mathbb{R}^2$ scaled by the value of the Hamiltonian, which corresponds to the height in Σ.

Lemma 3.2.4. *The Hamiltonian flow $\widetilde{X}_\Sigma$ of $\widetilde{H}_\Sigma$ in the new coordinates is given by*

$$\widetilde{X}_\Sigma = \frac{\tau}{2} X_\Sigma + \frac{1}{\tau}\partial_\theta.$$

Proof. Let us start by writing $\widetilde{H}_\Sigma$ and $\widetilde{\omega}$ in terms of the new coordinates. It is not difficult to see that

$$\widetilde{H}_\Sigma(p, h\alpha, \tau, \theta) = h\tau.$$

As $u = \sqrt{h}\tau\cos\theta$ and $v = \sqrt{h}\tau\sin\theta$, we have that

$$\begin{cases} du = \frac{\tau}{2\sqrt{h}}\cos\theta dh + \sqrt{h}\cos\theta d\tau - \sqrt{h}\tau\sin\theta d\theta; \\ dv = \frac{\tau}{2\sqrt{h}}\sin\theta dh + \sqrt{h}\sin\theta d\tau + \sqrt{h}\tau\cos\theta d\theta. \end{cases}$$

A simple computation shows that

$$du \wedge dv = \frac{\tau^2}{2} dh \wedge d\theta + h\tau d\tau \wedge d\theta.$$

Therefore, the symplectic form $\widetilde{\omega}_\Sigma$ in terms of the new coordinates is given by

$$\widetilde{\omega}_\Sigma = \omega_\Sigma + \frac{\tau^2}{2} dh \wedge d\theta + h\tau d\tau \wedge d\theta.$$

Compute the contraction of $\widetilde{\omega}_\Sigma$ by $\widetilde{X}_\Sigma$ and obtain

$$\iota_{\widetilde{X}_\Sigma}\widetilde{\omega}_\Sigma = -\frac{\tau}{2}dh - \frac{\tau^2}{2\tau}dh - \frac{h\tau}{\tau}d\tau = -\tau dh - hd\tau = -d\widetilde{H}_\Sigma$$

Therefore, $\widetilde{X}_\Sigma$ must be the Hamiltonian flow by non degeneracy of $\widetilde{\omega}_\Sigma$. □

The integral curves of this vector field do not vary in the τ direction. Therefore, for each integral curve $\zeta : I \to \widetilde{\Sigma}$, there exists $c \in \mathbb{R}^+$ such that $\tau(\zeta(t)) = c$ for all $t \in I$. The level set $\tau^{-1}(c)$ is not a cylinder, as you may expect from normal polar coordinates. In fact, since τ is scaled by the height on Σ, each level set is fiberwise a quadratic cone of radius $\sqrt{h}c$. In Figure 3.3, a fiber of a level set of τ is shown. Denote

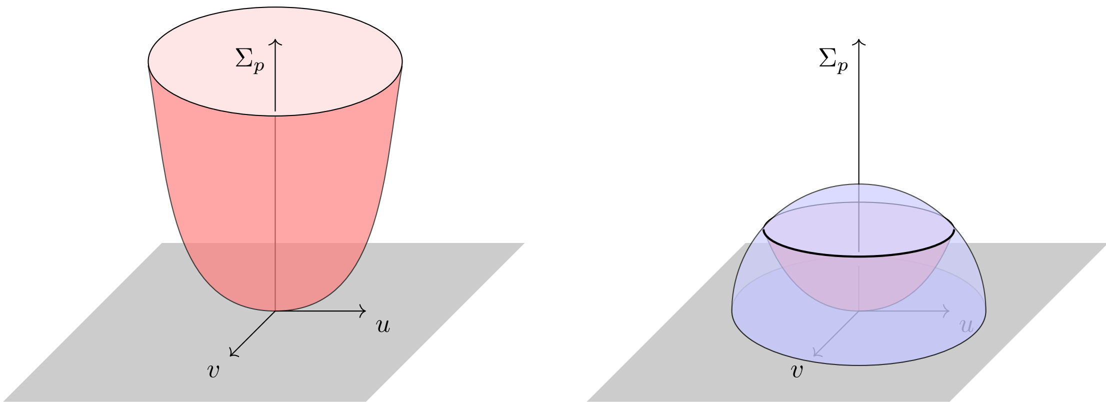

Figure 3.3: The picture on the left displays a fiber of a level set of the coordinate τ. The picture on the right depicts the intersection between the sphere bundle and the level set of τ.

by $S\widetilde{\Sigma}$ the sphere bundle on $\widetilde{\Sigma}$, whose fibers, in terms of the old coordinates, satisfy

$$h^2 + u^2 + v^2 = 1.$$

In terms of the new coordinates, the previous condition is equivalent to

$$h^2 + h\tau^2 - 1 = 0.$$

We will denote a point $(p, h\alpha, \tau, \theta) \in S\widetilde{\Sigma}$ simply by (p, τ, θ), because τ already possesses information about the height of the point. Indeed, for a fixed τ, there is a unique positive real number for which h satisfies the previous relation. Graphically, this value corresponds to the height at which the level set of τ intersects the sphere as in Figure 3.3. We are interested in the integral curves of $\widetilde{X}_\Sigma$ of unit speed, thus the integral curves fiberwise in the intersection between a level set of τ and $S\widetilde{\Sigma}$. The following Lemma gives a proper description.

Lemma 3.2.5. *An integral curve $\zeta : I \to \widetilde{\Sigma}$ of $\widetilde{X}_\Sigma$ in $S\widetilde{\Sigma}$ starting at (p, τ_0, θ_0) is parameterized by*

$$\zeta(t) = \left(\Gamma_r\left(\frac{\tau_0 t}{2}\right), \tau_0, \theta_0 + \frac{t}{\tau_0}\right),$$

where $\Gamma_r(t)$ is a Reeb orbit satsfying $\gamma(0) = p$.

Proof. The proof of this Lemma follows directly from putting together Lemma 3.2.1 and 3.2.4. □

At this point we want to use Theorem 3.2.3 to send the geodesics of (M, g) to the integral curves of the previous Lemma, via the symplectomorphism χ. We can do so only for geodesics with high initial momenta.

Definition 3.2.6. Let $\zeta(t) = (\gamma(t), P(t))$ be a solution of the Hamiltonian equations with starting point (p, w). We say that γ has *high initial momentum* if the lift ζ is contained in the conical neighbourhood $C \subseteq TU^*$ of Theorem 3.2.3.

It is even simpler to describe the integral curves of $\widetilde{X}_\Sigma$ in a complex setting. In $\Sigma \times (\mathbb{C} - \{0\})$, the integral curve ζ of $\widetilde{X}_\Sigma$ is parameterized by

$$\zeta(t) = \left(\Gamma_r\left(\frac{\tau_0 t}{2}\right), \tau_0 e^{\theta_0 + it/\tau_0}\right).$$

We can also endow ξ with the usual almost complex structure J given by rotation by $\pi/2$. Then, we can state the fundametal result of this Section.

Theorem 3.2.7. *Let $(p, w) \in TM^*$ be a point with large momentum, and γ be the subRiemannian geodesic starting at p with unit speed $X_0 \in S\xi$. Then, there exist a point p' close to p and $Y \in \xi_{p'}$ such that*

$$\begin{cases} \gamma(t) = \Gamma_r\left(\frac{\tau_0 t}{2}\right) - \tau_0 e^{it/\tau_0} Y\left(\frac{\tau_0 t}{2}\right) + O(\tau_0^2) \\ \dot{\gamma}(t) = e^{it/\tau_0} Y\left(\frac{\tau_0 t}{2}\right) + O(\tau_0) \end{cases}$$

where $Y(t)$ is the parallel transport of Y along Γ_r.

Proof. To prove this Theorem, one picks a covering of the Reeb trajectory Γ_r starting at p. Restricted to an element of the covering, the Reeb trajectory is a closed arc parametrized by $[0, 1]$. Then, by Theorem 3.2.3, there exist neighbourhoods $U \subseteq M$, $C \subseteq TU^*$ and $C' \subseteq \widetilde{\Sigma}_U$ and a symplectomorphism $\chi : C \to C'$ that sends the Hamiltonian $\widetilde{H}^2_\Sigma$ to H_0. In turn, it also sends the integral curves of $\widetilde{X}_\Sigma$ to solutions of the Hamiltonian equations in TU^*. An explicit description of χ, together with Lemma 3.2.5, gives the rest. □

This is an incredible result that proves the spiraling of horizontal geodesics around a Reeb orbit. It also says that the period of the spiraling depends on the initial momenta. Indeed, $T = \frac{1}{h_0}$, which means that the higher the initial momentum is, the faster it spirals. It also says that the higher the momentum is, the closer the curves are, and the smaller is the error. In some sense, we could say that, if an Hamiltonian trajectory of H_0 is very high in the unit cylinder, then its whole behaviour is determined by how high it is, because the curve will be almost periodic with period depending on the height.

It is nonetheless not so trivial to see if there is any closed subRiemannian geodesic γ of Theorem 3.2.7 around a closed Reeb orbits. The Spiraling Theorem describes its behaviour, but what can we say about the existence? In a very recent article [32, Theorem 0.1], C. de Verdière proved there exists a sequence of closed horizontal geodesics that converges to a closed non-degenerate Reeb orbit.

One further research question now could be: what happens in the degenerate scenario? What is commonly known in this setting, is that the dynamics complicate incredibly. For example, in [29], the author shows that pseudoholomorphic curves approach a degenerate Reeb orbit in a very unnatural way. It would be interesting to see an example of this also for subRiemannian geodesics.

The purpose of the following discussion is to relate these results with the convergence of Riemannian geodesics to spiraling subRiemannian ones, under anisotropic deformation.

3.3 Anisotropic deformation

In the previous section, we studied some properties of subRiemannian geodesics approaching a Reeb orbit. In this section, we want to build on these results by blowing the metric through anisotropic deformation.

Let (M, ξ) be a closed contact three-manifold with contact form α. Let $g = g_h + \alpha^2$ be an adapted metric on (M, ξ). On the cotangent bundle, there are two induced cometrics. The first one is subRiemannian, and it describes the subRiemannian structure (ξ, g_h). We will denote it by g_h^*. Its sphere bundle S_hTM^* is the subRiemannian cylinder depicted in the second picture of Figure 3.4. The second cometric is the

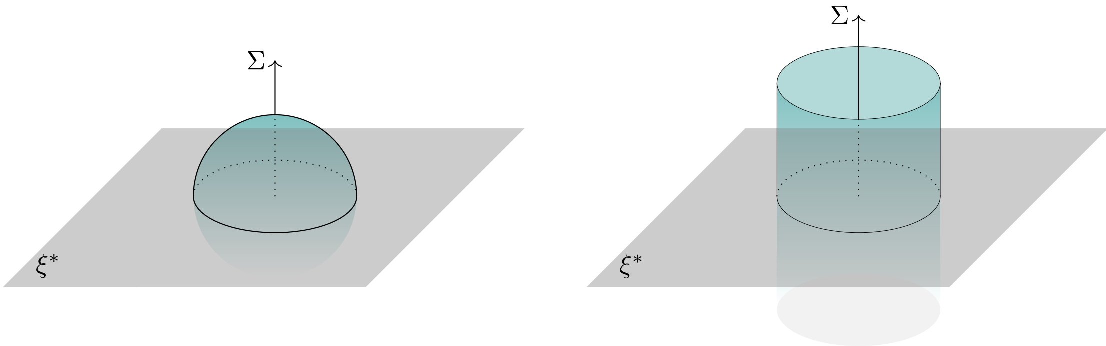

Figure 3.4: Identify a fiber of the cotangent bundle at one point with $\mathbb{R}^3$. Vertically, we have the positive annihilator bundle. On the left, the sphere bundle STM^* is plotted and, on the right, the cylinder of the subRiemannian structure.

Riemannian one, and it corresponds to

$$g^* = g_h^* + P_r^2,$$

where P_r is the momentum function of the Reeb vector field. The sphere bundle STM^* generated by this cometric is precisely a sphere, as depicted in Figure 3.4. Under anisotropic deformation, this cometric is deformed by

$$g_\epsilon^* = g_h^* + P_{r_\epsilon}^2 = g_h^* + \epsilon^2 P_r^2.$$

In particular, it converges to the subRiemannian cometric when $\epsilon \to 0$. What happens to $S_\epsilon TM^*$? A point $(p, h\alpha_\epsilon) \in \Sigma$ corresponds to $(p, h/\epsilon\alpha_\epsilon)$ with respect to the old coordinates. In particular, a fiber of the bundle $S_\epsilon TM$ is a vertically stretched sphere as displayed in Figure 3.5. When the poles are removed, it converges to the cylinder. Both bundles are defined as the 1 level set of the Hamiltonian. At each $\epsilon > 0$, the preimage of H_ϵ is compact, meaning that the Hamiltonian is a proper map. However, H_ϵ converges to the subRiemannian Hamiltonian H_0 which is not proper. If we restrict the domain of the two functions to horizontal bands, then we can talk about convergence on compacts, which is the main focus of the next Section.

3.3.1 Compact Convergence

In this section, we will focus on the convergence of the stretched sphere $S_\epsilon TM^*$ to the cylinder S_hTM^* on compacts. We recall the definition of this type of convergence.

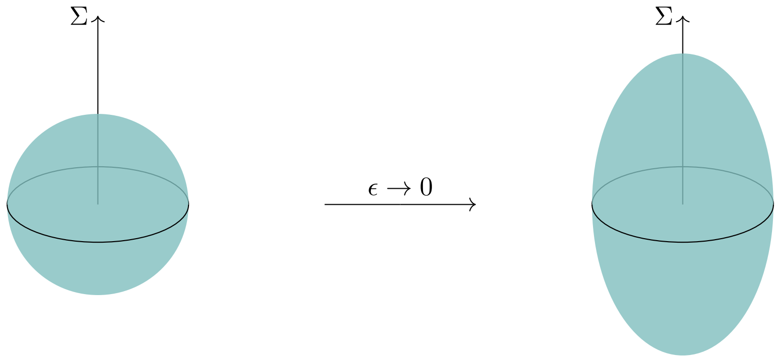

Figure 3.5: Anisotropic deformation of the unit sphere in the cotangent bundle.

Definition 3.3.1. Let (X, τ) be a topological space and $\{f_k : X \to \mathbb{R}\}_k$ be a sequence of functions. We say that f_k converges to some $f : X \to \mathbb{R}$ *on compacts* if, for every compact set $K \subseteq X$, $f_k \mid_K$ converges uniformly to f_K.

In the previous setup, consider the fiber bundle

$$\mathcal{B}_\lambda := \{(p, w) \in TM^* : P_r(p, w) \leq \lambda\},$$

for some $\lambda \in \mathbb{R}^+$. At each fiber, it consists of an horizontal band of size λ. The intersections $\mathcal{B}_\lambda \cap S_\epsilon TM^*$ and $\mathcal{B}_\lambda \cap S_h TM^*$ are fiberwise compact for every λ. Graphically, the two intersections are shown in Figure 3.6. In particular, for every λ, there exists an $\epsilon > 0$ such that $\frac{1}{\epsilon} > \lambda$, meaning that the stretched

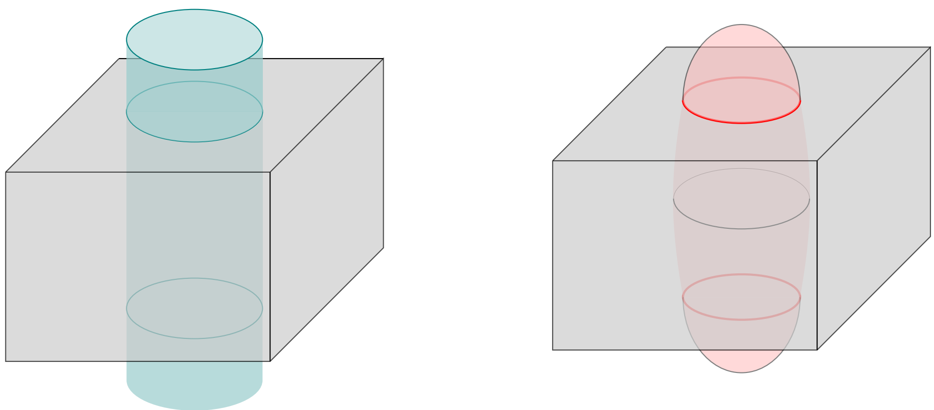

Figure 3.6: Fiber Intersection between $S_h TM^*, S_\epsilon TM^*$ and a band $\mathcal{B}_\lambda$.

sphere will eventually pop out from the band $\mathcal{B}_\lambda$, as soon as ϵ becomes very small. It is not difficult to see that the intersections converge to each other in the limit $\epsilon \to 0$. When this happens, it means that the two Hamiltonian converge to each other on compacts.

Lemma 3.3.2. *The Hamiltonian H_ϵ converges to H_0 on compacts in the $\mathcal{C}^\infty$-topology.*

A classical application of such convergence relies on basic ODE theory, and precisely on Gronwall's Lemma. In fact, we could pick any closed arc of Hamiltonian trajectory of H_0 and a compact cover for it. Then, by the uniform convergence of H_ϵ to H_0 on that cover, we could choose a distance $\delta > 0$ to the closed arc, and find a sequence of Hamiltonian trajectories of H_ϵ that are all δ-close to the initial trajectory. The following Corollary is a precise mathematical description of this claim.

Corollary 3.3.3. *If ζ is a closed arc of Hamiltonian trajectory of H_0, then there exists a sequence of closed arcs of Hamiltonian trajectories $\{\zeta_\epsilon\}$ that converges to ζ.*

Proof. Consider $(p, w) \in S_hTM^*$, and let $\zeta : I \to TM^*$ be a closed arc of subRiemannian Hamiltonian trajectory starting at (p, w). Choose $\lambda \in \mathbb{R}^+$ such that ζ is completely contained in the band $\mathcal{B}_\lambda$. By Lemma 3.3.2, there exists a sequence of points $\{(p, w_\epsilon)\}_\epsilon \in S_\epsilon TM^*$ approaching (p, w). Since, H_ϵ converges uniformly to H_0, so do the differentials $dH_\epsilon \to dH_0$ and Hamiltonian flows $X_{H_\epsilon} \to X_{H_0}$. If we consider, for every point (p, w_ϵ), a closed arc of Riemannian trajectory ζ_ϵ that is contained in the band, then, by Gronwall's Lemma, ζ_ϵ converges to ζ in norm. This means that for any desired distance $\delta > 0$ to ζ, there exists $\epsilon_0 > 0$ such that, for all $\epsilon < \epsilon_0$,

$$\|\zeta - \zeta_\epsilon\| < \delta.$$

□

With an analogous procedure, we can also prove the following Corollary.

Corollary 3.3.4. *Let $\{\zeta_\epsilon\}$ be a sequence of Hamiltonian trajectories of H_ϵ. If every element of the sequence has bounded initial momenta, then $\{\zeta_\epsilon\}_\epsilon$ admits a subsequence that converges to a subRiemannian trajectory.*

In other words, if there exists λ such that $\mathcal{B}_\lambda$ containes every element of the sequence, then the sequence converges (up to subsequences) to a lift of an horizontal geodesic.

3.3.2 Convergence of geodesics

The projection to M of an Hamiltonian trajectory ζ of H_0 with high initial momenta, is a curve that is relatively close to a Reeb orbit. In view of this observation, we would like to relate Theorem 3.2.7 with Corollary 3.3.3. We have two separate statements:

1. The spiraling Theorem (3.2.7) ensures that any subRiemannian geodesic (with sufficiently high initial momentum) spirals up around the Reeb orbit. Moreover, it also proved that the higher their momentum is, the closer it gets to the Reeb orbit. In some sense, it proved the uniform convergence of Hamiltonian trajectories of H_0 to the ones of the vertical Hamiltonian H_Σ outside of a band $\mathcal{B}_\lambda$.
2. Corollary 3.3.3 proved that for any Hamiltonian trajectory ζ of H_0, there exists a sequence of Riemannian trajectories of H_ϵ that converges uniformly to ζ, when restricted to any band $\mathcal{B}_\lambda$.

Therefore, we have uniform convergence at infinity and another one inside any compact. We do not know anything about what happens in the middle, and precisely around the north pole of the sphere $S_\epsilon TM^*$. This is somewhat counter intuitive: one would expect that, around the vertical geodesic, curves would be easier to control in the limit, because they should reflect a similar behaviour to the vertical one. However, it seems that Hamiltonian trajectories that are close, but not too close, to the vertical geodesic are the only ones whose behaviour is fully understood. In principle, there could be a region around the pole, where curves have an unpredictable behaviour. We expect that this is not the case for non-degenerate Reeb orbits. We think that the stability of the orbit would require a coherent behaviour of the nearby geodesics. This should follow from the same reasoning used in the recent article [32] by C. de Verdière.

Theorem 3.3.5. *Let Γ_r be a non degenerate closed Reeb orbit. Then, any closed arc of Γ_r can be approximated by Riemannian geodesics under anisotropic deformation.*

Proof. Since the Reeb orbit is non degenerate, there exists a sequence $\{\gamma_k\}_k$ of closed spiraling subRiemannian geodesics that converges to the Reeb orbit. For each k, we could consider a lift consisting of a Hamiltonian trajectories $\{\zeta_k\}$ of γ_k. These are trajectories with progressively higher initial momenta (p_k, w_k). By Corollary 3.3.3, for any closed arc of ζ_k, there exists a sequence of Riemannian trajectories $\{\zeta_{k,\epsilon}\}_\epsilon$ that converges to ζ_k when restricted to a band $\mathcal{B}_{\lambda_k}$ that contains the closed arc ζ_k. In particular, the higher the initial momenta of ζ_k, the bigger the band $\mathcal{B}_{\lambda_k}$ and the closer γ_k to Γ_r. For a sufficiently large k and small ϵ, the projection to M of the closed arc of $\zeta_{\epsilon,k}$ inside of $\mathcal{B}_{\lambda_k}$, approximate a closed arc of Γ_r. The error of this approximation depends on k and ϵ. $\square$

One question that remains unsolved is whether we can choose closed geodesics to approximate the Reeb orbit. Since non degenerate Reeb orbits admit a sequence of closed subRiemannian geodesic converging to it, it seems very plausible that this is the case also for Riemannian geodesics under anisotropic defomation. We think that stability in this setting would play a pivotal role.

Chapter 4

Subriemannian limit of the Hodge-Laplacian

The Laplace-type operators are a family of second order differential operators that have been immensely studied due to their natural construction and geometrical properties. They are all metric dependent operators, meaning that they recover information about the Riemannian metric of the manifold, such as curvatures, and they are affected by any perturbation of the metric. The *Hodge-Laplacian*, or *Laplace-De Rham operator*, is an operator of Laplace type that is constructed by combining suitably the exterior derivative d with the Hodge star operator $\star$, thus the name. Following the work [27] of M. Rumin, we will focus on the deformation of the Hodge-Laplacian under anisotropic deformation. We are particularly interested in the consequences of this perturbation for the spectrum of the operator.

The first two Sections of this chapter treat the so called *Rumin's complex*. It is a cochain complex, which is only possible to construct on coorientable contact manifolds, and which computes the singular cohomology of the manifold. It is a complex that can also be defined in higher dimensional contact manifold. We decided to focus on the three-dimensional case to observe more clearly its features, and try to relate the results with the previous discussion on dynamics.

The goal of the rest of the Chapter is to prove that the spectrum of the family of Hodge-Laplacians can be divided in three classes:

- an *exploding part*, which consists of every eigenvalue that diverges to infinity for $\epsilon \to 0$;
- a *collapsing part*, which consists of all the eigenvalues that tend to zero in the limit;
- a *converging part*, which consists of all the remaining eigenvalues of Δ_ϵ.

For each of these classes, we want to study where they concentrate. In more precise terms, we want to predict where the eigenspace of an eigenvalue λ lies, based only on what type of eigenvalue λ is. It turns out that the exploding part of the spectrum concentrates outside of the Rumin complex, which means that the remaining part of the spectrum concentrates inside of the Rumin complex.

The most technical part of this Chapter, especially of Sections 4.5.3 and 4.5.4, consists of tedious procedures. We choose to withhold some very involved proofs, but we hope to explain sufficiently well the key aspects of the approach of [27].

4.1 Rumin's decomposition

Let (M, ξ) be a coorientable contact 3-manifold with contact form α. As we know, there is a clear splitting of $\mathfrak{X}(M)$ into horizontal and vertical fields, i.e.

$$\mathfrak{X}(M) = \mathfrak{X}(\xi) \oplus \langle r \rangle ,$$

where r is the Reeb vector field associated to α. The same happens in the ring of forms Ω^*M. Let $\Omega^*\xi$ be the set of forms on the contact plane. We define $\Omega^*\xi$ as the quotient $\Omega^*M/\mathcal{J}^*M$, where $\mathcal{J}^*M$ is the following two-sided ideal:

$$\mathcal{J}^*M := \alpha \wedge \Omega^*M.$$

The elements of the quotient $\Omega^*\xi$ are classes of the form $[\beta] = \beta + \mathcal{J}^*M$.

Lemma 4.1.1. *We have the following splitting: $\Omega^*M \cong \ker(\iota_r) \oplus \mathcal{J}^*M$, where ι_r is the contraction by the Reeb vector field.*

Proof. The two spaces clearly intersect each other only at the zero form. Moreover, for each form $\omega \in \Omega^*M$, there exists $\omega_h \in \ker \iota_r$ such that $\alpha \wedge \omega = \alpha \wedge \omega_h$. Moreover, there exists $\alpha \wedge \omega_v \in \mathcal{J}^*M$ such that $\iota_r \omega = \omega_v$. □

An immediate consequence of this statement is that $\Omega^*\xi \cong \ker(\iota_r)$. Thanks to this isomorphism, we will always think of $\Omega^*\xi$ as the kernel of the contraction by the Reeb vector field. In turn, if we set $\mathcal{J}^*\xi := \alpha \wedge \ker(\iota_r)$, we obtain the isomorphism $\mathcal{J}^*\xi \cong \mathcal{J}^*M$. All together, we have a splitting of Ω^*M into horizontal and vertical components

$$\Omega^*M = \Omega^*M/\mathcal{J}^*M \oplus \mathcal{J}^*M \cong \Omega^*\xi \oplus \mathcal{J}^*\xi.$$

Clearly, $\Omega^*\xi \cap \mathcal{J}^*\xi = \{0\}$ and therefore, every differential form $\omega \in \Omega^*M$ can be split into

$$\omega = \omega_h + \alpha \wedge \omega_v$$

for unique $\omega_h, \omega_v \in \Omega^*\xi$. How is the exterior differentiation affected by this splitting? we need to define a few operators first.

Definition 4.1.2. The operator $L : \Omega^*M \to \Omega^*M$, called *Lefschetz operator*, takes the wedge product of forms with $d\alpha$.

Another important operator is the horizontal derivation d_h.

Definition 4.1.3. We define $d_h : \Omega^*\xi \to \Omega^*\xi$ as the projection to $\Omega^*\xi$ of the exterior derivation, i.e. $d_h\beta := (d\beta)_h$.

We are now ready to study the block decomposition of the exterior derivation in terms of the Rumin's splitting.

Lemma 4.1.4. *The block decomposition of the exterior derivative is*

$$d = \begin{pmatrix} d_h & L \\ \mathcal{L}_r & -d_h \end{pmatrix}, \tag{4.1}$$

Proof. A simple computation shows that

$$\begin{aligned} d\omega &= d(\omega_h + \alpha \wedge \omega_v) \\ &= d\omega_h + d\alpha \wedge \omega_v - \alpha \wedge d\omega_v \\ &= (d\omega_h)_h + \alpha \wedge (d\omega_h)_v + d\alpha \wedge \omega_v - \alpha \wedge (d\omega_v)_h \\ &= d_h\omega_h + d\alpha \wedge \omega_v + \alpha \wedge ((d\omega_h)_v - d_h\omega_v). \end{aligned}$$

By Cartan's magic formula, the vertical component of the differential of a horizontal form coincides with the Lie derivative of the form along the Reeb direction. Precisely, we see that, for $\beta \in \Omega^*\xi$,

$$\mathcal{L}_r\beta = \iota_r(d\beta) = \iota_r(d_h\beta + \alpha \wedge (d\beta)_v) = (d\beta)_v.$$

All together,
$$d\omega = d_h\omega_h + L(\omega_v) + \alpha \wedge (\mathcal{L}_r\omega_h - d_h\omega_v).$$
□

Note that both L and $\mathcal{L}_r$ preserve the horizontal and vertical components. Indeed,
$$L(\omega) = d\alpha \wedge \omega_h + \alpha \wedge d\alpha \wedge \omega_v = L(\omega_h) + \alpha \wedge L(\omega_v)$$
$$\mathcal{L}_r(\omega) = \mathcal{L}_r(\omega_h) + \mathcal{L}_r(\alpha \wedge \omega_h) = \mathcal{L}_r(\omega_h) + \alpha \wedge \mathcal{L}_r(\omega_v).$$
It is important to mention another immediate property of the Lefschetz operator

Lemma 4.1.5. *The restriction of the Lefschetz operator to the horizontal forms is always trivial except for the case $L : \mathcal{C}^\infty(M) \to \Omega^2\xi$ where it is an isomorphism.*

Proof. This follows easily from the fact that $d\alpha$ is a volume form in ξ. □

We will denote by $L^{-1} : \Omega^*\xi \to \Omega^*\xi$ the linear operator which is the inverse of the Lefschetz operator at degree two and zero elsewhere.

Consider the following sequence of spaces:
$$0 \longrightarrow \mathbb{R} \hookrightarrow \mathcal{C}^\infty(M) \xrightarrow{d_h} \Omega^1\xi \xrightarrow{d_h} \Omega^2\xi \longrightarrow 0 \tag{4.2}$$
An interesting question to ask is whether this sequence is locally exact or not. The answer to this question is no, because d_h^2 is not necessarily trivial.

Lemma 4.1.6. *The following relations hold:*
$$d_h^2 = -L\mathcal{L}_r, \qquad [L, \mathcal{L}_r] = [d_h, L] = [\mathcal{L}_r, d_h] = 0.$$

Proof. By the trivial relation $d^2 = 0$ we conclude that
$$0 = d^2 = \begin{pmatrix} d_h & L \\ \mathcal{L}_r & -d_h \end{pmatrix}\begin{pmatrix} d_h & L \\ \mathcal{L}_r & -d_h \end{pmatrix} = \begin{pmatrix} d_h^2 + L\mathcal{L}_r & [d_h, L] \\ [\mathcal{L}_r, d_h] & d_h^2 + L\mathcal{L}_r \end{pmatrix}$$
which implies that $d_h^2 = -L\mathcal{L}_r = -\mathcal{L}_r L$ and $[d_h, L] = [\mathcal{L}_r, d_h] = 0$. In turn,
$$[L, \mathcal{L}_r] = L\mathcal{L}_r - \mathcal{L}_r L = -d_h^2 + d_h^2 = 0.$$
□

Remark 4.1.7. Since L is non trivial only in degree 0, so is d_h^2. In particular, for $f \in \mathcal{C}^\infty(M)$, then by the previous Lemma
$$d_h^2 f = -\mathcal{L}_r(f d\alpha) = -\mathcal{L}_r(f)d\alpha.$$
In the Heisenberg calculus, we constructed $\mathcal{L}_r$ to have order 2. One of the reasons for this choice is precisely this relation with the squaring of the horizontal derivation.

There are contact manifolds on which d_h^2 can be trivial. The Boothby-Wang fibration is an example of this, as we will see in Section 4.1.1.

4.1.1 Boothby-Wang fibration

Let $S^1 \hookrightarrow (M, \alpha) \xrightarrow{\pi} (B, \omega)$ be a Boothby-Wang fibration of dimension three. Recall that, in this construction, the symplectic form pulls back to the differential $d\alpha$ of the contact form α of (M, ξ), and almost every other horizontal object comes from the base manifold B in an analogous way.

Lemma 4.1.8. *The pullback π^* has values in $\Omega^*\xi$.*

Proof. It is easy to see that the contraction $\iota_r \pi^*\beta$ vanishes for all $\beta \in \Omega^* B$. It comes directly from the fact that $d\pi(r) = 0$. $\square$

By the fact that $d\pi : \xi \to TB$ is fiberwise an isometry, one may think that π^* defines an isomorphism between $\Omega^* B$ and $\Omega^*\xi$. This is not the case, and it is already observable at degree 0. Indeed, the image of $\pi^* : \mathcal{C}^\infty(B) \to \mathcal{C}^\infty(M)$ consists of fiberwise constant functions, which means that it is not even surjective.

Consider the de Rham complex of the base manifold B. Together with the sequence in Equation (4.2), we construct the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, hook] \arrow[d] & \mathcal{C}^\infty(B) \arrow[r, "d"] \arrow[d, hook, "\pi^*"] & \Omega^1 B \arrow[r, "d"] \arrow[d, hook, "\pi^*"] & \Omega^2 B \arrow[r] \arrow[d, hook, "\pi^*"] & 0 \\ 0 \arrow[r] & \mathbb{R} \arrow[r, hook] & \mathcal{C}^\infty(M) \arrow[r, "d_h"] & \Omega^1\xi \arrow[r, "d_h"] & \Omega^2\xi \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

Lemma 4.1.9. *The diagram in Equation (4.3) commutes.*

Proof. The pullback π^* commutes with the exterior derivations. Therefore, for each $\beta \in \Omega^* B$, $\pi^* d\beta = d\pi^*\beta$. By Lemma 4.1.8, $\pi^*\beta \in \Omega^*\xi$. Then, by definition of d in $\Omega^* M$,

$$(\pi^* \circ d)\beta = (d \circ \pi^*)\beta = (d_h \circ \pi^*)\beta + \alpha \wedge \mathcal{L}_r \pi^*\beta = (d_h \circ \pi^*)\beta.$$

Therefore, we have commutativity in the second and third square. The same holds for the first square as $\pi^* f = f \circ \pi$ for every $f \in \mathcal{C}^\infty(B)$. If the function is constant, then the pre-composing makes it constant also in the fibers. $\square$

Corollary 4.1.10. *The horizontal derivation d_h squares to zero when restricted to forms that come from the base manifold.*

Proof. By the commutativity of the diagram, $\pi^* \circ d = d_h \circ \pi^*$. Therefore,

$$0 = \pi^* \circ d^2 = (\pi^* \circ d) \circ d = d_h \circ (pi^* \circ d) = d_h^2 \circ \pi^*.$$

Therefore, $d_h^2 = 0$. $\square$

This Corollary follows also directly from 4.1.6 because function coming from B are invariant under the Reeb flow.

> *Remark* 4.1.11. This result fails to hold when we rescale the contact form α, since the deriving Reeb orbits would not coincide the S^1 fibers.

We will continue this discussion about Boothby-Wang fibrations as soon as we have introduced the Rumin complex.

4.2 Rumin's complex

The Rumin complex is a famous subcomplex of the de Rham complex for contact manifolds, first introduced by M. Rumin in [26]. It has the property of computing the singular cohomology of M ([26, p. 401]). Moreover, as first suggested by Ge in [13] for the scalar Laplacian, and developed by Rumin in [27] for the Hodge-Laplacian, the converging part of their spectrum concentrate on the spectrum of their restriction to the Rumin complex.

Definition 4.2.1. In the three dimensional case, Rumin's complex consists of the following sequence

$$0 \longrightarrow \mathbb{R} \hookrightarrow \mathcal{C}^\infty(M) \xrightarrow{d_h} \Omega^1\xi \xrightarrow{D} \alpha \wedge \Omega^1\xi \xrightarrow{d} \Omega^3 M \longrightarrow 0 \tag{4.4}$$

The construction of the central morphism D relies on the following Lemma. We will denote the sequence of spaces as $F^* \subset \Omega^* M$.

For any $\eta \in \Omega^*\xi$, we call *lift of* β any form $\widetilde{\beta} \in \Omega^* M$ whose horizontal part coincides with β.

Lemma 4.2.2. *For all $\beta \in \Omega^1\xi$ there exists a unique lift $\widetilde{\beta} \in \Omega^1 M$ such that $d\widetilde{\beta}$ is a vertical 2 form.*

Proof. Any lift of β is given by $\widetilde{\beta} = \beta + f\alpha$ for some $f \in \mathcal{C}^\infty(M)$. As the Lefschetz operator $L : \mathcal{C}^\infty(M) \to \Omega^2\xi$ is an isomorphism, let f be the unique smooth function with $L(f) = fd\alpha = -d_h\beta$. In this way

$$\alpha \wedge d\widetilde{\beta} = \alpha \wedge d\beta + \alpha \wedge fd\alpha = 0$$

which means that $d\widetilde{\beta} \in \alpha \wedge \Omega^1\xi$. □

We can now define the operator $D : \Omega^1\xi \to \alpha \wedge \Omega^1\xi$ as the map $D(\beta) = d\widetilde{\beta}$ given by the previous Lemma.

Proposition 4.2.3. *The operator $D : \Omega^1\xi \to \alpha \wedge \Omega^1\xi$ is described by*

$$\iota_r D = \mathcal{L}_r + d_h L^{-1} d_h.$$

Proof. First note that $\iota_r D(\beta) = (D\beta)_v$. By definition, $D\beta = d(\beta + f\alpha)$ with f being the unique map satisfying $fd\alpha = -d_h\beta$. Equivalently, we can apply L^{-1} and obtain $f = L^{-1}(fd\alpha) = -L^{-1}d_h\beta$. By Cartan's magic formula

$$\begin{aligned}\iota_r D\beta &= \mathcal{L}_r(\beta + f\alpha) - d\iota_r(\beta + f\alpha) = \mathcal{L}_r(\beta) + \mathcal{L}_r(f)\alpha - df \\ &= \mathcal{L}_r\beta - d_h f = \mathcal{L}_r\beta + d_h L^{-1} d_h\beta.\end{aligned}$$

□

Remark 4.2.4. It is interesting to observe that D bear some similarities with the so called *horizontal Hodge-Laplacian* Δ_h. When M is endowed with an adapted metric, the map $L^{-1} : \Omega^2\xi \to \mathcal{C}^\infty(M)$ is the restriction to the horizontal forms of the Hodge star $\star$ (Lemma 4.3.12). Thus, the term $d_h L^{-1} d_h$ corresponds to $d_h \star_h d_h$, which is very related to the terms in the horizontal Laplacian. The Lie derivative $\mathcal{L}_r$ along the Reeb direction is the extra term in this correspondence. As already mentioned in Lemma 1.3.28, it is a first order operator in the usual differential calculus and second order in the Heisenberg calculus. We will return to this observation as soon as we introduce the horizontal Hodge-Laplacian Δ_h, but also already in Remark 4.2.8 for the Boothby-Wang construction.

Let us prove that the complex in Equation (4.4) is exact.

Proposition 4.2.5. *The Rumin complex (see Equation (4.4)) is locally exact.*

Proof. First we see that the image of any map in the sequence is included in the kernel of the next one. It is clear that $d_h \circ i : \mathbb{R} \to \Omega^1\xi = 0$ so let us consider $D \circ d_h$. By Proposition 4.2.3 and Lemma 4.1.6, we get that

$$\iota_r D \circ d_h = \mathcal{L}_r d_h + d_h L^{-1} d_h^2 = \mathcal{L}_r d_h - d_h L^{-1} L \mathcal{L}_r = \mathcal{L}_r d_h - d_h \mathcal{L}_r = [\mathcal{L}_r, d_h] = 0.$$

Analogously, using the block decomposition of d in Equation (4.1), we get that

$$d \circ D = L\iota_r D - \alpha \wedge d_h \iota_r D = -\alpha \wedge \left(d_h \mathcal{L}_r + d_h^2 L^{-1} d_h\right) = \alpha \wedge [\mathcal{L}_r, d_h] = 0.$$

We shall prove the other inclusion. Note that it is trivially satisfied in the $\mathbb{R}$ case. In the case of $\mathcal{C}^\infty(M)$, it is not as easy as one may think. In fact, if $d_h f = 0$, it does not automatically mean that $df = 0$. We would just know that f is constant along horizontal curves. However, by Chow Theorem, any pair of points in M can be connected by a horizontal curve, implying that f is constant everywhere. To prove that the sequence is locally exact in $\Omega^1\xi$, consider $\beta \in \ker(D)$. This means that there exists a closed lift $\widetilde{\beta}$ of β. In turn, by Poincarè lemma, the restriction of $\widetilde{\beta}$ to any contractible open subset $U \subset M$ is exact. Therefore, there exists a local smooth map $f \in \mathcal{C}^\infty(U)$ such that $df = \widetilde{\beta}\mid_U$. By construction, we obtain that

$$d_h f = \beta \mid_U .$$

To prove the local exactness in $\alpha \wedge \Omega^1\xi$, consider a vertical closed 2 form $\omega \in \mathcal{J}^2\xi$. Once again, since every closed form is locally exact, there exists a 1-form $\beta \in \Omega^1 U$ with $d\beta = \omega\mid_U \in F^2$. Obviously, β is a lift of its horizontal part $\beta_h \in \Omega^1\xi\mid_U$. Hence, by construction of D, we have that $D(\beta_h) = \omega\mid_U$.
Finally, for $f\alpha \wedge d\alpha \in \Omega^3 M$, we use the local exactness of the deRham complex, meaning that there exists $\omega \in \Omega^2 U$ such that $d\omega = f\alpha \wedge d\alpha$. We have to check that ω is not just horizontal. The form ω splits into $\omega_h + \alpha \wedge \omega_v$ for ω_h horizontally closed. By the decomposition of the derivative, also the form $\widetilde{\omega} = \alpha \wedge \omega_v$ is a local primitive of $f\alpha \wedge d\alpha$. □

Following this Proposition, the Rumin complex becomes a resolution of the sheaf $\mathbb{R}$. We have a major result for the induced cohomology ([26, p.401]).

Theorem 4.2.6. *The cohomology induced by the contact complex is isomorphic to the singular cohomology of M.*

4.2.1 Boothby-Wang fibration

It is interesting to see how the Rumin complex relates to the de Rham complex of the underlying surface in the Boothby-Wang construction. Let $S^1 \hookrightarrow (M, \xi) \xrightarrow{\pi} (B, \omega)$ be a Boothby-Wang fibration. Similarly to what we have seen in Section 4.1.1, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, hook] \arrow[d, hook] & \mathcal{C}^\infty(B) \arrow[r, "d"] \arrow[d, hook, "\pi^*"] & \Omega^1 B \arrow[r, "d\Lambda_B d"] \arrow[d, hook, "\pi^*"] & \Omega^1 B \arrow[r, "-d"] \arrow[d, hook, "\alpha\wedge\pi^*"] & \Omega^2 B \arrow[r] \arrow[d, "\alpha\wedge\pi^*"] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, hook] & \mathcal{C}^\infty(M) \arrow[r, "d_h"] & \Omega^1\xi \arrow[r, "D"] & \alpha \wedge \Omega^1\xi \arrow[r, "d"] & \Omega^3 M \arrow[r] & 0
\end{tikzcd} \tag{4.5}$$

where $\Lambda_B : \Omega^2 B \to \mathcal{C}^\infty(M)$ is defined by $\Lambda_B(f\omega) = f$.

Lemma 4.2.7. *The diagram in Equation (4.5) commutes.*

Proof. Most of the work has already been done in Lemma 4.1.9. We just need to prove the commutativity of the third square. Note that, by Proposition 4.2.3,

$$\iota_r D \circ \pi^* = \mathcal{L}_r \pi^* + d_h L^{-1} d_h \pi^* = \pi^* d\Lambda_B d.$$

Indeed, $\mathcal{L}_r\pi^* = 0$, $\pi^*\Lambda_B = L^{-1}\pi^*$ and $\pi^* d = d_h\pi^*$. □

Remark 4.2.8. In this setup, the observation made in Remark 4.2.4 becomes even clearer. The horizontal part of D comes directly from a Laplace-type operator in B, since

$$(\iota_r D) \circ \pi^* = \pi^* \circ (d\Lambda_B d).$$

If g_B is a metric on TB whose area form coincides with ω, then it is clear that $\Lambda_B = \star_B$. Thus,

$$(\iota_r D) \circ \pi^* = \pi^* \circ (d \star_B d).$$

We may be interested in computing the cohomology of the first row in Equation (4.5). Let us look at the first $\Omega^1 B$ term. A closed 1-form $\beta \in \Omega^1 B$, is still in the kernel of $d\Lambda_B d$. Therefore, $\ker(d) \subseteq \ker(d\Lambda_B d)$. We expect also the converse to hold. In fact, since Λ_B is an isomorphism, $\ker(d) = \ker(\Lambda_B d)$. Therefore, a part from closed 1-forms, $\ker(d\Lambda d)$ consists of constant multiples of primitives of ω. However, such primitives do not exist because ω is not exact. As for the second $\Omega^1 B$ term, note that $\mathrm{Im}(\Lambda_B d) = \ker(\delta L)^\perp = \ker(\delta)^\perp$. Therefore, $\mathrm{Im}(\Delta_B d)$ is the orthogonal part of constant functions. In this way, we see that $\mathrm{Im}(d\Delta_B d) = \mathrm{Im}(d)$. In particular, the cohomology of the first row consists of the singular cohomology of B with a copy of $H^1(B, \mathbb{R})$.

4.3 Hodge operator and L2 product

In this section, we are going to introduce the products with which we will compute the adjoint of the operators. We will start with a general Riemannian metric, and then we will describe the adapted case.

Let g be a Riemannian metric on M. We know that g induces a cometric g^* on the cotangent bundle. We can generalize this construction and say that g induces a fiber inner product g^* on the space of $(0,q)$-tensors $\mathcal{T}_0^q(M)$. Consider a positive orthonormal frame $\{r_0, r_1, r_2\}$, and its coframe $\{\alpha_0, \alpha_1, \alpha_2\}$. These objects consist of $(1,0)$ and $(0,1)$ tensors, respectively. Then, we can locally describe an element $\tau \in \mathcal{T}_0^q(M)$ as

$$\tau = \sum_{I,J} f_{I,J}(\alpha_{j_0}^{i_0} \otimes \cdots \otimes \alpha_{j_k}^{i_k}). \tag{4.6}$$

where $f_{I,J}$ is a smooth function, $I = (i_0, ..., i_k)$ is any multi-index that satisfies $i_0 + \cdots + i_k = q$, and $J = \{j_0, ..., j_k\}$ has values in $\{0,1,2\}$. The metric g induces a fiber inner product on $\mathcal{T}_0^q(M)$ given by

$$g^*(\tau, \rho) = \sum_{I,J} f_{I,J} f'_{I,J},$$

where $f_{I,J}, f'_{I,J}$ represents τ and ρ, respectively. It can be proven that this definition does not depend on the choice of the frame. Moreover, when applied to q-forms, it coincide with the usual coframe g^*.

Definition 4.3.1. On $\mathcal{T}^q(M)$, we have the global inner product

$$\langle \tau, \rho \rangle_0 = \int_M g^*(\tau, \rho) vol_g.$$

Moreover, we say that τ is *square-integrable* (or in L^2) if

$$\|\tau\|_0^2 < \infty.$$

Clearly, $q-$forms are $(0,q)$-tensors. In particular, when restricted to $\Omega^q M$, there is an alternative description of the L^2-product, which relies on the *Hodge operator* $\star$.

Definition 4.3.2. We define $\star : \Omega^q M \to \Omega^{3-q} M$ as the unique operator satisfying the relation

$$\omega \wedge \star\eta = g^*(\omega, \eta) vol_g$$

for all $\omega, \eta \in \Omega^q(M)$.

In terms of this operator, the L^2-product of Definition 4.3.1, when restricted to forms, is

$$\langle \omega, \eta \rangle_0 = \int_M \omega \wedge \star\eta.$$

Note that, by Linear Algebra arguments, it can be proven that $\star^2 = \mathrm{Id} : \Omega^q M \to \Omega^q M$. This depends on the fact that the dimension of the manifold is odd. We can now define the k-Sobolev space of forms.

Definition 4.3.3. The *Sobolev space* $\mathcal{H}^k(M)$ is the the space of differential forms on M that are square-integrable together with their covariant/weak derivatives up to order k, i.e.

$$\|\omega\|_0^2 = \int_M \omega \wedge \star\omega < \infty, \qquad \|\nabla^i \omega\|_0^2 < \infty \quad i = 1, ..., k.$$

We also define the *Sobolev inner product* of $\alpha \in \mathcal{H}^k(M)$ as

$$\langle \omega, \eta \rangle_k = \sum_{i=0}^{k} \left\langle \nabla^i \omega, \nabla^i \eta \right\rangle_0 .$$

Note that the covariant derivative of a $(0, q)-$tensor τ is a $(0, q+1)$-tensor given by

$$\nabla\tau(X_1, ..., X_q, Y) := \tau(\nabla_Y X_1, ..., \nabla_Y X_q).$$

In general, $\nabla^\ell \tau$ is a $(0, q+\ell)$-tensor defined as the $\ell-$th power of ∇ on each component of τ.

Remark 4.3.4. The pair $(\mathcal{H}^k(M), \| \cdot \|_k)$ is an Hilbert space with bounded norm. In general, the completion of $\Omega^* M$ with respect to the norm $\| \cdot \|_k$ is a Hilbert space.

Lemma 4.3.5. *Any differential operator $P : \mathcal{H}^k(M) \to \mathcal{H}^0(M)$ of order k is bounded.*

Proof. This comes from the fact that all the first k-weak derivatives of any element of $\mathcal{H}^k$ are square-integrable, and so is the image of any element through P. □

4.3.1 Adapted metric

Let (λ, g) be an adapted metric structure on (M, α) of the usual form $g = g_h + \alpha^2$. In an analogous way to what we have seen in Definition 4.3.1, the horizontal metric g_h defines a positive semi-definite inner product on $\mathcal{T}_0^q(M)$. Indeed, if the orthonoraml frame $\{r_0, r_1, r_2\}$ is adapted, then, in the local representation of τ in Equation (4.6), we can decompose τ in horizontal and vertical component, based on what terms have a non-zero power of α_0. We have

$$g^*(\tau, \rho) = g_h^*(\tau, \rho) + g_v^*(\tau, \rho),$$

where $g_h^*(-, -)$ is the sum of the products $f_{I,J} f'_{I,J}$ for all I, J such that, if $j_k = 0$, then $i_k = 0$ (for $k = 0, 1, 2$). Both $g_h^*(-, -)$ and $g_v^*(-, -)$ are positive semi-definite fiber inner products.

Definition 4.3.6. We will denote by $\langle -,-\rangle_{0,h}$ or $\langle -,-\rangle_{0,v}$, the L^2-products of Definition 4.3.1 induced by g^*_h and g^*_v, respectively.

When restricted to forms, the horizontal inner product of tensors g^*_h corresponds to the subRiemannian cometric g^*_h. Moreover, by construction, g^*_v satisfies

$$g^*_v(\alpha \wedge \beta) = g^*_h(\beta).$$

Moreover, we have the following relations.

Lemma 4.3.7. *For any $\omega \in \Omega^k M$, it holds*

$$\|\omega\|^2_{0,h} = \|\omega_h\|^2_0, \qquad \|\omega\|^2_{0,v} = \|\omega_v\|^2_0.$$

Proof. It is immediate from the construction of the norms. □

Corollary 4.3.8. *For any $\omega \in \Omega^k M$, it holds*

$$\|\omega\|^2_0 = \|\omega\|^2_{0,h} + \|\omega\|^2_{0,v}.$$

Proof. It follows from the previous Lemma. □

Horizontal Hodge-operator

The horizontal L^2-product $\langle -,-\rangle_{0,h}$ also have a description in terms of the Hodge dual, when restricted to forms. Since the metric is adapted, we have the relation $\lambda vol_g = \alpha \wedge d\alpha$. In turn, we can easily restrict the dual operator $\star$ to the horizontal components.

Definition 4.3.9. We define the *horizontal $\star$-operator* as the unique operator $\star_h : \Omega^*\xi \to \Omega^*\xi$ satisfying

$$\beta \wedge \star_h \theta = \frac{1}{\lambda} g^*_h(\beta,\theta) d\alpha \qquad \forall \beta,\theta \in \Omega^k\xi.$$

In terms of this operator, the horizontal L^2-products on forms can be computed as

$$\langle \omega,\eta\rangle_{0,h} = \int_M \alpha \wedge \omega_h \wedge \star_h \eta_h.$$

We will conclude this section with a formula that describes how the $\star-$operator behaves in terms of the splitting in horizontal and vertical forms.

Lemma 4.3.10. *The matrix description of $\star : \Omega^k M \to \Omega^{3-k} M$ is*

$$\star = \begin{pmatrix} 0 & \star_h \\ (-1)^k \star_h & 0 \end{pmatrix}. \tag{4.7}$$

Proof. For $\omega, \eta \in \Omega^k M$,

$$\begin{aligned}\omega \wedge \star\eta &= \frac{1}{\lambda} g^*(\omega,\eta)\alpha \wedge d\alpha \\ &= \frac{1}{\lambda} g_h^*(\omega_h,\eta_h)\alpha \wedge d\alpha + \frac{1}{\lambda} g_h^*(\omega_v,\eta_v)\alpha \wedge d\alpha \\ &= \alpha \wedge \omega_h \wedge \star_h \eta_h + \alpha \wedge \omega_v \wedge \star_h \eta_v \\ &= \omega_h \wedge (\alpha \wedge (-1)^k \star_h \eta_h) + (\alpha \wedge \omega_v) \wedge \star_h \eta_v.\end{aligned}$$

The claim follows from this description of the product $\omega \wedge \star\eta$. □

Remark 4.3.11. The fact that $\star^2 = \mathrm{Id}$ on $\Omega^k M$ implies that $\star_h^2 = (-1)^k \mathrm{Id}$ on $\Omega^k \xi$.

It is also important to note the relation between the Lefschetz operator L and the horizontal Hodge operator.

Lemma 4.3.12. *At degree 0, the operator $L : \mathcal{C}^\infty(M) \to \Omega^2\xi$ corresponds to $\lambda\star_h$.*

Proof. By definition, for $f \in \mathcal{C}^\infty(M)$, we have that $L(f) = f d\alpha = \lambda \star_h f$. □

Perturbed metric

Let (λ, g) be an adapted structure on (M,ξ) and g_ϵ be the perturbed metric. If we rescale the contact form, then $(g_\epsilon, \lambda/\epsilon)$ is an adapted metric structure on (M, α_ϵ). Formally, as soon as we do this, we are moving from the metric space $(\Omega^k M, g_\epsilon^*)$, where the metric changes with ϵ, to the unperturbed $(\Omega^k M, g^*)$ through an isometry that sends α_ϵ to α.

Lemma 4.3.13. *The map $C_\epsilon : (\Omega^k M, g_\epsilon^*) \to (\Omega^k M, g^*)$ of metric spaces given by*

$$C_\epsilon(\omega_h + \alpha \wedge \omega_v) := \omega_h + \epsilon\alpha \wedge \omega_v.$$

is an isometry.

Proof. This is a trivial statement because

$$g^*(C_\epsilon(\omega), C_\epsilon(\eta)) = g_h^*(\omega_h,\eta_h) + \epsilon^2 g_h^*(\omega_v,\eta_v) = g_\epsilon^*(\omega,\eta),$$

meaning that C_ϵ is an isometry. □

What happens to the L^2-norms? Denote by $\langle -,-\rangle_{0,\epsilon}$ the L^2-product induced by the adapted metric g_ϵ.

Lemma 4.3.14. *For $\omega, \eta \in \Omega^* M$, we have the following equality*

$$\langle \omega, \eta\rangle_{0,\epsilon} = \frac{1}{\epsilon} \langle C_\epsilon(\omega), C_\epsilon(\eta)\rangle_0 .$$

Proof. A straighforward computation shows

$$\langle \omega, \eta\rangle_{0,\epsilon} = \int_M g_\epsilon^*(\omega,\eta)\alpha_\epsilon \wedge d\alpha = \frac{1}{\epsilon}\int_M g^*(C_\epsilon(\omega), C_\epsilon(\eta))\alpha \wedge d\alpha = \frac{1}{\epsilon} \langle C_\epsilon(\omega), C_\epsilon(\eta)\rangle_0 ,$$

which is the claim. □

In terms of the norm $\|\cdot\|_{0,\epsilon}$, we have the following result.

Corollary 4.3.15. *For $\omega \in \Omega^* M$, it holds*

$$\|\omega\|_{0,\epsilon}^2 = \frac{1}{\epsilon}\|\omega\|_{0,h}^2 + \epsilon\|\omega\|_{0,v}^2.$$

Proof. By Lemma 4.3.14, $\|\omega\|_{0,\epsilon}^2 = \frac{1}{\epsilon}\|C_\epsilon(\omega)\|_0^2$. Together with Lemma 4.3.7, we obtain,

$$\|\omega\|_{1,\epsilon}^2 = \frac{1}{\epsilon}\|\omega_h + \epsilon\alpha\wedge\omega_v\|_0^2 = \frac{1}{\epsilon}\|\omega\|_{0,h}^2 + \epsilon\|\omega\|_{0,v}^2 + 2\left\langle \omega_h, \alpha\wedge\omega_v\right\rangle_0,$$

which is the claim. □

Corollary 4.3.16. *For $f \in \mathcal{C}^\infty(M)$, it holds*

$$\|f\|_{1,\epsilon}^2 = \frac{1}{\epsilon}\|f\|_{1,h}^2 + \epsilon\|f\|_{1,v}^2.$$

Proof. It follows from the previous Corollary and the fact that the covariant derivative of f is a 1-form. □

4.4 The Hodge-Laplacian operator

In this section, we are going to define the Hodge-Laplacian Δ in terms of the Rumin's splitting. Let g be a Riemannian metric structure on M. Consider the *codifferential* δ, which is defined as the (formal) adjoint of the exterior derivative d with respect to the L^2-product $\langle -,-\rangle_0$.

Lemma 4.4.1. *The codifferential $\delta : \Omega^k M \to \Omega^{k-1}M$ corresponds to $\delta = (-1)^k \star d\star$.*

Proof. By Stokes, we can see that, for $\omega \in \Omega^{k-1}M$ and $\eta \in \Omega^k M$,

$$0 = \int_M d(\omega\wedge\star\eta) = \int_M d\omega\wedge\star\eta + (-1)^{k-1}\omega\wedge d\star\eta = \langle d\omega,\eta\rangle_0 - \left\langle \omega, (-1)^k \star^{-1} d\star\eta\right\rangle_0.$$

Therefore $\delta = (-1)^k \star^{-1} d\star$. In particular, we know that $\star^{-1} = \star$ and thus, the claim. □

Definition 4.4.2. We define the *Hodge-Laplacian* $\Delta : \Omega^k M \to \Omega^k M$ as $\Delta := (d+\delta)^2 = d\delta + \delta d$.

We want to study the decomposition of Δ in terms of the splitting in horizontal and vertical components. We will do so in the next Subsection.

4.4.1 Block decomposition

In this section we will define the horizontal counterparts of δ and Δ. We assume that the adapted metric structure (λ, g) is of the form $g = g_h + \alpha^2$. In this way, we can consider the horizontal L^2-product, and talk about adjoints.

Codifferential

Just like the codifferential δ, we want to the define the horizontal codifferential δ_h as the adjoint of d_h with respect to the horizontal L^2-product induced by g_h on $\Omega^*\xi$. As the two L^2-products coincide on horizontal forms, δ_h is also the adjoint of d_h, with respect to the Riemannian L^2-norm on $\Omega^*\xi$. We also expect a nice description of δ_h in terms of the horizontal $\star_h$ operator.

Lemma 4.4.3. *The horizontal codifferential δ_h takes the form*

$$\delta_h = - \star_h d_h \star_h : \Omega^k\xi \to \Omega^{k-1}\xi.$$

Proof. Note that, using Equation (4.7), we obtain that, for any $\beta \in \Omega^{k-1}\xi$ and $\theta \in \Omega^k\xi$,

$$\beta \wedge \star\theta = (-1)^k \alpha \wedge (\beta \wedge \star_h \theta) \in \mathcal{J}^2 M.$$

By Equation (4.1), its exterior derivative becomes

$$\begin{aligned} d(\beta \wedge \star\theta) &= (-1)^k L(\beta \wedge \star_h\theta) - (-1)^k \alpha \wedge d_h(\beta \wedge \star_h \theta) \\ &= (-1)^{k-1} \alpha \wedge d_h(\beta \wedge \star_h\theta), \end{aligned}$$

because L is trivial on $\Omega^1\xi$. It is easy to check that d_h also satisfies the Leibniz rule and thus,

$$d_h(\beta \wedge \star_h\theta) = d_h\beta \wedge \star_h\theta + (-1)^k \beta \wedge d_h \star_h \theta.$$

All together,

$$\begin{aligned} d(\beta \wedge \star\theta) &= (-1)^{k-1} \alpha \wedge d_h\beta \wedge \star_h\theta - \alpha \wedge \beta \wedge d_h \star_h \theta \\ &= (-1)^{k-1} d_h\beta \wedge ((-1)^k \alpha \wedge \star_h \theta) - (-1)^{k-1} \beta \wedge \alpha \wedge d_h \star_h \theta \\ &= (-1)^{k-1} d_h\beta \wedge \star\theta + \beta \wedge \alpha \wedge \star_h^2 d_h \star_h \theta \\ &= (-1)^{k-1} d_h\beta \wedge \star\theta + (-1)^{k-1} \beta \wedge \star(\star_h d_h \star_h \theta). \end{aligned}$$

Finally, by Stokes we obtain that

$$0 = \int_M d(\beta \wedge \star\theta) = (-1)^{k-1} \int_M d_h \wedge \star\theta + (-1)^{k-1} \int_M \beta \wedge \star(\star_h d_h \star_h \theta),$$

which means that $\langle d_h\beta, \theta\rangle_0 = \langle \beta, - \star_h d_h \star_h \theta\rangle_0$ as we wanted. □

Lemma 4.4.4. *The adjoint of the Lefschetz operator L is $\Lambda : \Omega^k\xi \to \Omega^{k-2}\xi$ described by*

$$\Lambda = (-1)^k \star_h L \star_h .$$

Proof. We prove this statement in a general setting even if L is non trivial only at degree 0. A direct computation shows that, for $\beta \in \Omega^k\xi$ and $\theta \in \Omega^{k+2}\xi$,

$$\begin{aligned} \langle L\beta, \theta\rangle_0 &= \int_M \beta \wedge d\alpha \wedge \star\theta = (-1)^{k+2} \int_M \beta \wedge \alpha \wedge L \star_h \theta \\ &= (-1)^{k+2} \int_M \beta \wedge \star(\star_h L \star_h \theta) = \langle \beta, \Lambda\theta\rangle_0 , \end{aligned}$$

which proves the claim. □

We can use the fact that L is non zero only at degree 0, where it corresponds to $\lambda\star_h$.

Corollary 4.4.5. *The operator $\Lambda : \Omega^2\xi \to \mathcal{C}^\infty(M)$ corresponds to $\lambda\star_h$.*

Proof. A quick computation shows that, at degree 2, $\Lambda = \lambda\star_h^3 = \lambda\star_h$. □

Lemma 4.4.6. *The adjoint of $\mathcal{L}_r$ on horizontal form is $\mathcal{L}_r^\star : \Omega^k\xi \to \Omega^k\xi$, and it is described by*

$$\mathcal{L}_r^\star = (-1)^{k+1} \star_h \mathcal{L}_r \star_h .$$

Proof. For $\beta, \theta \in \Omega^k\xi$,

$$\langle \mathcal{L}_r\beta, \theta\rangle_0 = \int_M \mathcal{L}_r\beta \wedge \star\theta = \int_M \alpha \wedge \mathcal{L}_r\beta \wedge \star_h\theta.$$

and, analogously,

$$\langle \beta, \mathcal{L}_r^*\theta\rangle_0 = (-1)^{k+1} \int_M \beta \wedge \star(\star_h \mathcal{L}_r \star_h \theta) = -\int_M \alpha \wedge \beta \wedge \mathcal{L}_r \star_h \theta.$$

All together,

$$\langle \mathcal{L}_r\beta, \theta\rangle_0 - \langle \beta, \mathcal{L}_r^\star\theta\rangle_0 = \int_M \alpha \wedge \mathcal{L}_r(\beta \wedge \star_h\theta) = \int_M \mathcal{L}_r(\alpha \wedge \beta \wedge \star_h\theta) = \int_M \mathcal{L}_r(\beta \wedge \star\theta).$$

which is zero always by Stokes. □

Now we can decompose δ with respect to the Rumin's splitting.

Proposition 4.4.7. *The matrix description of $\delta : \Omega^k M \to \Omega^{k-1} M$ is*

$$\delta = \begin{pmatrix} \delta_h & \mathcal{L}_r^* \\ \Lambda & -\delta_h \end{pmatrix}, \tag{4.8}$$

Proof. By putting together Equation (4.1) and (4.7) with the relation $\delta = (-1)^k \star d \star$, we obtain

$$\delta = \begin{pmatrix} \delta_h & (-1)^k \star_h \mathcal{L}_r \star_h \\ (-1)^k \star_h L \star_h & -\delta_h \end{pmatrix},$$

which is the claim by Lemma 4.4.4 and 4.4.6. □

In the same way as we did for the horizontal differential, we can observe if the horizontal codifferential squares to 0.

Lemma 4.4.8. *The following relations hold:*

$$\delta_h^2 = -\mathcal{L}_r^\star\Lambda, \qquad [\mathcal{L}_r^\star, \Lambda] = [\delta_h, \mathcal{L}_r^\star] = [\Lambda, \delta_h] = 0$$

Proof. Since δ squares to zero, we obtain

$$0 = \delta^2 = \begin{pmatrix} \delta_h & \mathcal{L}_r^\star \\ \Lambda & -\delta_h \end{pmatrix} \begin{pmatrix} \delta_h & \mathcal{L}_r^\star \\ \Lambda & -\delta_h \end{pmatrix} = \begin{pmatrix} \delta_h^2 + \mathcal{L}_r^\star\Lambda & [\delta_h, \mathcal{L}_r^\star] \\ [\Lambda, \delta_h] & \delta_h^2 + \Lambda\mathcal{L}_r^\star \end{pmatrix},$$

which implies $\delta_h^2 = -\mathcal{L}_r^\star\Lambda = -\Lambda\mathcal{L}_r^\star$ and $[\delta_h, \mathcal{L}_r^\star] = [\Lambda, \delta_h] = 0$. In turn,

$$[\mathcal{L}_r^\star, \Lambda] = \mathcal{L}_r^\star\Lambda - \Lambda\mathcal{L}_r^\star = \delta_h^2 - \delta_h^2 = 0.$$

□

Hodge-Laplacian

Just like the Hodge-Laplacian Δ, the horizontal Hodge-Laplacian is $\Delta_h := (d_h + \delta_h)^2$. However, we know that, by Lemma 4.1.6 and 4.4.8, both d_h and δ_h do not square to zero, so it means that the horizontal Laplacian has some the extra terms. Precisely,

$$\Delta_h = d_h\delta_h + \delta_h d_h - L\mathcal{L}_r - \mathcal{L}_r^\star\Lambda. \tag{4.9}$$

In terms of the usual decomposition, we have the following result

Proposition 4.4.9. *The block decomposition of Δ is*

$$\Delta = \begin{pmatrix} \Delta_h + (L + \mathcal{L}_r^\star)(\Lambda + \mathcal{L}_r) & [d_h, \mathcal{L}_r^\star] + [\delta_h, L] \\ [\mathcal{L}_r, \delta_h] + [\Lambda, d_h] & \Delta_h + (\Lambda + \mathcal{L}_r)(L + \mathcal{L}_r^\star) \end{pmatrix}.$$

Proof. A straightforward computation using Equation (4.1), (4.8) and (4.9) proves the statement. □

In view of Equation (4.9), the terms in the diagonal can be rewritten as

$$\begin{cases} \Delta_h + (L + \mathcal{L}_r^\star)(\Lambda + \mathcal{L}_r) = d_h\delta_h + \delta_h d_h + L\Lambda + \mathcal{L}_r^\star\mathcal{L}_r. \\ \Delta_h + (\Lambda + \mathcal{L}_r)(L + \mathcal{L}_r^\star) = d_h\delta_h + \delta_h d_h + \Lambda L + \mathcal{L}_r\mathcal{L}_r^\star. \end{cases}$$

We will sometimes denote by $\widetilde{\Delta}_h := d_h\delta_h + \delta_h d_h$ the Laplacian that maintains the degree.

> *Example* 4.4.10. Let us compute the Laplacian of α. Clearly, $\alpha_h = 0$ and $\alpha_v = 1$. From the fact that the metric is adapted we know that $d\alpha = \star_h\lambda$. Therefore,
>
> $$\Delta\alpha = \begin{pmatrix} [d_h, \mathcal{L}_r^\star]1 + [\delta_h, L]1 \\ \widetilde{\Delta}_h 1 + \Lambda L 1 + \mathcal{L}_r\mathcal{L}_r^\star 1 \end{pmatrix} = \begin{pmatrix} -\lambda^{-1} d_h \star_h \mathcal{L}_r(d\alpha) - \star_h d_h(\lambda) \\ (\Lambda - \mathcal{L}_r)d\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^2 \end{pmatrix} = \lambda^2\alpha$$
>
> is an eigenform. The same happens for $d\alpha$ because the Laplacian commutes with the exterior derivative, i.e.
>
> $$\Delta(d\alpha) = d(\Delta\alpha) = \lambda^2 d\alpha.$$

The Laplacian is usually studied in term of its square root $P : \Omega^* M \to \Omega^* M$ defined as

$$P := d + \delta.$$

In a block decomposition, this is given by

$$P = \begin{pmatrix} P_h & L + \mathcal{L}_r^\star \\ \Lambda + \mathcal{L}_r & -P_h \end{pmatrix}, \tag{4.10}$$

where $P_h = d_h + \delta_h$.

Connecting operator

Consider the connecting operator D of the Rumin complex. For practical matters, we extend D to be an operator from $\Omega^1 M \to \Omega^2 M$, which is zero on the orthogonal space of its previous domain.

Lemma 4.4.11. *The block decomposition of $D : \Omega^1 M \to \Omega^2 M$ is*

$$D = \begin{pmatrix} 0 & 0 \\ \mathcal{L}_r + \frac{1}{\lambda^2} d_h \Lambda d_h & 0 \end{pmatrix}. \tag{4.11}$$

Proof. By Proposition 4.2.3, we have that $\iota_r D = \mathcal{L}_r + d_h L^{-1} d_h$. Then, by Lemma 4.3.12 and Corollary 4.4.5, we obtain that $L^{-1} = \frac{1}{\lambda^2}\Lambda$, which gives the result. □

What can we say about its adjoint?

Lemma 4.4.12. *The adjoint of D, $D^\star : \Omega^2 M \to \Omega^1 M$ has the block decomposition*

$$D^\star = \begin{pmatrix} 0 & \mathcal{L}_r^\star + \frac{1}{\lambda^2}\delta_h L \delta_h \\ 0 & 0 \end{pmatrix}.$$

Proof. For $\beta, \theta \in \Omega^1\xi$, it is easy to see that

$$\langle D\beta, \alpha \wedge \theta\rangle_0 = \langle \mathcal{L}_r\beta, \theta\rangle_0 + \frac{1}{\lambda^2}\langle d_h \Lambda d_h \beta, \theta\rangle_0 = \langle \beta, \mathcal{L}_r^* \theta\rangle_0 + \frac{1}{\lambda^2}\langle \beta, \delta_h \Lambda \delta_h \theta\rangle_0 = \langle \beta, D^\star(\alpha \wedge \theta)\rangle_0,$$

giving the desired result. □

Note that both D and $D^\star$ square to zero. We may define a Laplacian from these operators.

Definition 4.4.13. We call *connecting Laplacian* the operator

$$\Delta_D := DD^\star + D^\star D : \Omega^1 M \to \Omega^1 M.$$

We denote its square root by $P_D := D + D^\star$.

The Laplacian Δ_D will central in the study of the collapsing part of the spectrum of Δ, when we perturb the metric under anisotropic. We refer to Corollary 4.5.25 for the full statement.

Q-Laplacian

Another operator that will be taken under cosideration when studying the spectrum of the Laplacian is

$$P_Q := d_Q + \delta_Q.$$

The operator d_Q is defined as what is left from the Rumin complex when we eliminate D. Degree by degree, it is defined by

$$0 \longrightarrow \mathcal{C}^\infty(M) \xrightarrow{d_h} \Omega^1\xi \xrightarrow{0} \alpha \wedge \Omega^1\xi \xrightarrow{d} \Omega^3 M \longrightarrow 0. \tag{4.12}$$

It is defined to be trivial outside of the Rumin complex. The operator δ_Q is the adjoint of d_Q with respect to the L^2-product.

Definition 4.4.14. We define the Q-Laplacian Δ_Q as $\Delta_Q := d_Q\delta_Q + \delta_Q d_Q$.

Note that its square root is given by P_Q, because both d_Q and δ_Q square to zero. The Laplacian Δ_Q will be the limiting operator for the converging part of the spectrum of the deformed Hodge-Laplacian Δ_ϵ under anisotropic deformation. Moreover, its kernel $K := (\ker \Delta_Q)$ will be the subspace of forms where the collapsing part of the spectrum of Δ_ϵ concentrates. Once again, check Section 4.5.5. We state now a technical result that will be useful to prove the previous claim.

Proposition 4.4.15. *For a purely imaginary $\lambda \in \mathbb{C}$, the operator $(\lambda - P_Q)$ is a bicontinuous bijection, when seen as an operator from $\mathcal{H}^1(M)$ to $L^2(M)$.*

Proof. [27, Proposition 6.2] □

4.4.2 Contact-metric structure

In this Section, we simplify the setting even more by assuming that the subRiemannian metric g_h comes from $d\alpha$ in the Khaler-manifold style. It will be clear the motivation for this assumption, once we prove Proposition 4.4.20. Let (J, λ, g) be a contact-metric structure, as in Definition 1.4.11. By Lemma 1.4.12, the pair (λ, g) in this setting is an adapted structure, where the metric is defined by

$$g = \frac{1}{\lambda} d\alpha(-, J-) + \alpha^2.$$

J is an almost complex structure on (M, ξ), compatible with α. Recall, from Section 1.4.1, that J is a $(1,1)$-tensor on TM that satisfies

$$J^2 = -\operatorname{Id} + \alpha \otimes r.$$

In other words, J kills the Reeb direction and fixes ξ, where it acts as an actual almost complex structure.

Remark 4.4.16. J extends to a dual map $J^* : TM^* \to TM^*$ by precomposition, i.e. for $(p, w) \in TM^*$ and $v \in T_pM$,

$$J_p^*(w)(v) := w(J_p(v)).$$

In turn, it extends to any form $\Omega^* M$ by precomposition on each component.

When restricted to ξ, J is an endomorphism satisfying $J^2 = -\operatorname{Id}$. In particular, its inverse in ξ is $-J$. The same happens in the dual ξ^*, where J^* is an endomorphism with inverse $-J^*$. In view of this observation, we define a derivation given by pre and post-componing J with d_h. This operator describes an important commutator of the matrix description of the Hodge-Laplacian, as we will show in Proposition 4.4.20

Definition 4.4.17. On $\Omega^*\xi$, we define $d_h^J : \Omega^k\xi \to \Omega^{k+1}\xi$ as

$$d_h^J := -J^* d_h J^*.$$

It is an interesting operator that simplifies some computations. We want to talk about its adjoint, but to do so, we first need to see how J^* relates with the Hodge dual.

Lemma 4.4.18. *The commutator $[\star_h, J^*]$ is always trivial in $\Omega^k\xi$. Moreover, in $\Omega^1\xi$, $\star_h$ coincides with $-J^*$.*

Proof. Consider a local non vanishing vector field $r_1 \in \xi$. Then $\{r, r_1, Jr_1\}$ is an adapted local frame of TM. It is easy to see that it dual frame in TM^* is $\{\alpha, \alpha_1, -J^*\alpha_1\}$, where $\alpha_1 \in \Omega^1\xi$ is the dual of r_1. Indeed,

$$-J^*\alpha_1(Jr_1) = \alpha_1(-J^2 r_1) = \alpha_1(r_1) = 1.$$

We can write $d\alpha$ in this local terms as

$$d\alpha = -\alpha_1 \wedge J^*\alpha_1 = J^*\alpha_1 \wedge \alpha_1.$$

Therefore, $\star_h \alpha_1 = -J^*\alpha_1$, and $\star_h J^*\alpha_1 = \alpha_1$. In turn, $J^* \star_h \alpha_1 = -(J^*)^2\alpha_1 = \alpha_1$, which means that J^* is a right and left inverse of $\star_h$ in $\Omega^1\xi$. Let us see the other degrees. For $f \in \mathcal{C}^\infty(M)$, we have that $\star_h f = f d\alpha$ and $\star_h^2 f = f$. Then

$$[J^*, \star_h] f = f J^* d\alpha - f d\alpha = f d\alpha - f d\alpha = 0.$$

Analogously $[J^*, \star_h] f d\alpha = 0$. □

Thanks to the previous Lemma, we can easily describe the adjoint of d_h^J, in terms of $\star_h$ and J^*.

Lemma 4.4.19. *The adjoint of d_h^J, denoted by $\delta_h^J : \Omega^{k+1}\xi \to \Omega^k\xi$ corresponds to*

$$\delta_h^J = - \star_h d_h^J \star_h = -J^* \delta_h J^*.$$

Proof. A simple computation shows that, for $\beta \in \Omega^k\xi$ and $\theta \in \Omega^{k+1}\xi$,

$$\left\langle d_h^J \beta, \theta \right\rangle_0 = - \left\langle J^* d_h J^* \beta, \theta \right\rangle_0 = \left\langle d_h J^* \beta, J^* \theta \right\rangle_0 = \left\langle J^* \beta, \delta_h J^* \theta \right\rangle_0 = \left\langle \beta, -J^* \delta_h J^* \theta \right\rangle_0 .$$

Therefore, by the fact that J^* and $\star_h$ commute, we get that

$$\delta_h^J = -J^* \delta_h J^* = J^* \star_h d_h \star_h J^* = - \star_h d_h^J \star_h,$$

which is the claim. ☐

The following key formulas give a meaning to the operators that we just introduced. They are usually called *almost-contact Kahler identities*. In our three-dimensional setting, they are pretty easy to prove.

Proposition 4.4.20. *It holds*

$$[\Lambda, d_h] = -\lambda \delta_h^J, \qquad [\delta_h, L] = \lambda d_h^J$$

where J is the almost complex structure on ξ.

Proof. We will prove just the first equality, because the second one is completely analogous. In $\Omega^0\xi$, both δ_h^J and the commutator $[\Lambda, d_h]$ are zero, because Λ is non trivial only in $\Omega^2\xi$, when it coincides with $\lambda\star_h$. For $\beta \in \Omega^1\xi$,

$$[\Lambda, d_h]\beta = \Lambda d_h \beta = \lambda \star_h d_h \beta = \lambda \star_h d_h \star_h J^* \beta = \lambda J^* \delta_h J^* \beta = -\lambda \delta_h^J \beta.$$

For $\beta \in \Omega^2\xi$, we have

$$[\Lambda, d_h]\beta = -d_h \star_h \beta = - \star_h J^* d_h J^* \star_h \beta = -\delta_h^J \beta.$$

We used the fact that $\star_h J^* = J^* \star_h = \mathrm{Id}$ at degree 1 and $J^* = \mathrm{Id}$ at degree 0. ☐

Remark 4.4.21. Thanks to this Proposition, the Hodge-Laplacian has the following block decomposition:

$$\Delta = \begin{pmatrix} \widetilde{\Delta}_h + L\Lambda + \mathcal{L}_r^\star \mathcal{L}_r & [d_h, \mathcal{L}_r^\star] + \lambda d_h^J \\ [\mathcal{L}_r, \delta_h] - \lambda \delta_h^J & \widetilde{\Delta}_h + \Lambda L + \mathcal{L}_r \mathcal{L}_r^\star \end{pmatrix}. \tag{4.13}$$

We do not really care about the commutators $[\mathcal{L}_r, \delta_h]$ and $[d_h, \mathcal{L}_r^\star]$, because they will not be very problematic when we perturbe the metric.

We also have the following Corollary.

Corollary 4.4.22. *It holds*

$$[\Lambda, d_h^J] = \lambda \delta_h, \qquad [\delta_h^J, L] = -\lambda d_h.$$

Proof. We just give the proof for the first commutator. It is trivial at degree 0, so let $\beta \in \Omega^1\xi$. Then

$$[\Lambda, d_h^J]\beta = -\lambda \star_h J^* d_h J^* \beta = -\lambda J^* \star_h d_h (J^*)^2 \star_h \beta = \lambda \delta_h \beta.$$

Analogously, for $\beta \in \Omega^2\xi$,

$$[\Lambda, d_h^J]\beta = -\lambda J^* d_h J^* \star_h \beta = -\lambda (J^*)^2 \star_h d_h \star_h \beta = \lambda \delta_h \beta.$$

The other equality is completely analogous. ☐

We are ready to perturb the metric and see how the Hodge-Laplacian of Equation (4.13) is affected.

4.5 Anisotropic deformation

Now that we have a satisfying description of every element in the matrix decomposition of the Hodge-Laplacian, we can finally perturb the metric and see what happens to Δ. Let g_ϵ be the anisotropic deformation on a contact metric structure $(J, 1, g)$ in (M, ξ). Therefore,

$$g = d\alpha(-, J-) + \alpha^2 \qquad \rightsquigarrow \qquad g_\epsilon = d\alpha(-, J-) + \frac{1}{\epsilon^2}\alpha^2.$$

At this point, we have to make a choice: on one hand, we could rescale the contact form α by a factor of $1/\epsilon$, so that $(J, 1/\epsilon, g_\epsilon)$ becomes another contact metric structure. On the other hand, we could not rescale α, and we would have to work without an adapted metric structure. Let us explore the latter scenario.

It is well known that the exterior derivative does not depend on the metric. So, it does not change with g_ϵ, and its matrix representation remains

$$d = \begin{pmatrix} d_h & L \\ \mathcal{L}_r & -d_h \end{pmatrix}. \tag{4.14}$$

However, the Hodge-dual and the codifferential do depend on the metric. To compute their matrix representation, we use a straightforward computation analogous to Lemma 4.3.10 and Proposition 4.4.7.

Lemma 4.5.1. *The matrix decomposition of $\star_\epsilon$ and δ in $(\Omega^k M, g_\epsilon^*)$ are*

$$\star_\epsilon = \begin{pmatrix} 0 & \epsilon\star_h \\ (1/\epsilon)(-1)^k\star_h & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} \delta_h & \epsilon^2\mathcal{L}_r^\star \\ (1/\epsilon^2)\Lambda & -\delta_h \end{pmatrix}. \tag{4.15}$$

In this setup, d has no ϵ-terms while δ has only order 2 terms in ϵ. All together, we could compute the Hodge-Laplacian and we would obtain

$$\Delta = \begin{pmatrix} \Delta_h & [\delta_h, L] \\ [\mathcal{L}_r, \delta_h] & \Delta_h \end{pmatrix} + \epsilon^2 \begin{pmatrix} \mathcal{L}_r\mathcal{L}_r^\star & [d_h, \mathcal{L}_r] \\ 0 & \mathcal{L}_r^\star\mathcal{L}_r \end{pmatrix} + \frac{1}{\epsilon^2}\begin{pmatrix} L\Lambda & 0 \\ [\Lambda, d_h] & \Lambda L \end{pmatrix}. \tag{4.16}$$

In this setup, the Hodge-Laplacian has only second order terms in ϵ. We will see that this is not the case when we rescale the contact form. The point is that, without rescaling, we have to be careful with the ambient metric, because it changes in ϵ. In the other setting, we can somehow fix the metric and make everything else change. We are now about to see these subtleties of the rescaled setup.

4.5.1 Rescaled setup

When we rescale the contact form α to $\alpha_\epsilon := \frac{1}{\epsilon}\alpha$, then $(1/\epsilon, g_\epsilon)$ becomes an adapted metric structure on $(M, \ker\alpha_\epsilon)$.

Rescaled Operators

To rescale the operators with respect to the rescaling of the contact form, we use the isometry in Lemma 4.3.13. Since it does not affect the horizontal component of the forms, the horizontal operators such as $d_h, \star_h$ and δ_h are not affected by the perturbation. Many others are.

Definition 4.5.2. We define the *rescaled differential* d_ϵ and *codifferential* δ_ϵ as

$$d_\epsilon := C_\epsilon d C_\epsilon^{-1}, \qquad \delta_\epsilon := C_\epsilon \delta C_\epsilon^{-1},$$

where d and δ are the differential and codifferential in $(\Omega^k M, g_\epsilon^*)$.

The matrix description of C_ϵ and C_ϵ^{-1} is given by

$$C_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \qquad C_\epsilon^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\epsilon \end{pmatrix}.$$

Together with the block decomposition of d and δ, respectively in Equation 4.14 and 4.15, we obtain the following result.

Lemma 4.5.3. *The matrix description of d_ϵ and δ_ϵ is*

$$d_\epsilon = \begin{pmatrix} d_h & (1/\epsilon)L \\ \epsilon\mathcal{L}_r & -d_h \end{pmatrix}, \qquad \delta_\epsilon = \begin{pmatrix} \delta_h & \epsilon\mathcal{L}_r^\star \\ (1/\epsilon)\Lambda & -\delta_h \end{pmatrix}. \tag{4.17}$$

This matrix description is coherent with what we would intuitively obtain by rescaling the contact form. Indeed, for $\alpha_\epsilon = \frac{1}{\epsilon}\alpha$, the rescaled Lefschetz operator would be $L_\epsilon = \frac{1}{\epsilon}L$. Moreover, as the Reeb vector field is also rescaled to $r_\epsilon = \epsilon r$, we would have $\mathcal{L}_{r_\epsilon} = \epsilon\mathcal{L}_r$. All together, the rescaled exterior derivative d_ϵ would be

$$d_\epsilon = \begin{pmatrix} d_h & L_\epsilon \\ \mathcal{L}_{r_\epsilon} & -d_h \end{pmatrix} = \begin{pmatrix} d_h & (1/\epsilon)L \\ \epsilon\mathcal{L}_r & -d_h \end{pmatrix}.$$

Then, we would have computed the same codifferential δ_ϵ, because of the following Lemma.

Lemma 4.5.4. *The codifferential δ_ϵ is the adjoint of d_ϵ with respect to the L^2-product in $(\Omega^k M, g^*)$.*

Proof. We know that $\delta = (-1)^k \star_\epsilon d\star_\epsilon$ and that, by isometry, $\star = C_\epsilon \star_\epsilon C_\epsilon^{-1}$. Therefore,

$$\begin{aligned} \delta_\epsilon &= C_\epsilon \delta_\epsilon C_\epsilon^{-1} = (-1)^k C_\epsilon \star_\epsilon d \star_\epsilon C_\epsilon^{-1} \\ &= (-1)^k C_\epsilon \star_\epsilon C_\epsilon^{-1} d_\epsilon C_\epsilon \star_\epsilon C_\epsilon^{-1} \\ &= (-1)^k \star d_\epsilon \star . \end{aligned}$$

which is exactly what we needed, so that it holds $\langle d_\epsilon\omega, \eta\rangle_0 = \langle \omega, \delta_\epsilon\eta\rangle_0$ for all $\omega \in \Omega^{k-1}\xi$ and $\eta \in \Omega^k\xi$. $\square$

Now we could compute the rescaled Hodge-Laplacian, either through $\Delta_\epsilon := C_\epsilon \Delta C_\epsilon^{-1}$, where Δ is given by Equation 4.16, or by $\Delta_\epsilon := d_\epsilon\delta_\epsilon + \delta_\epsilon d_\epsilon$. In any case, we would obtain the following matrix decomposition.

Proposition 4.5.5. *The matrix decomposition of Δ_ϵ is*

$$\Delta_\epsilon = \begin{pmatrix} \widetilde{\Delta}_h + (1/\epsilon^2)L\Lambda & \epsilon[d_h, \mathcal{L}_r^\star] + (1/\epsilon)[\delta_h, L] \\ \epsilon[\mathcal{L}_r, \delta_h] + (1/\epsilon)[\Lambda, d_h] & \widetilde{\Delta}_h + (1/\epsilon^2)\Lambda L \end{pmatrix} + \epsilon^2 \begin{pmatrix} \mathcal{L}_r^\star\mathcal{L}_r & 0 \\ 0 & \mathcal{L}_r\mathcal{L}_r^\star \end{pmatrix}. \tag{4.18}$$

The Hodge-Laplacian in this rescaled scenario has a very complicated description. It has all sorts of $\epsilon-$terms of different orders. It is clearly more complicated than Δ of Equation 4.16. However, as already said, it is worth using the rescaled setting in order to avoid confusions with the ambient metric. Let us make a first Remark about the eigenforms of Δ_ϵ.

Remark 4.5.6. Outside of middle degrees, the perturbed Laplacian takes a nice form. Indeed, for $f \in \mathcal{C}^\infty(M)$, we have

$$\Delta_\epsilon f = \Delta_Q f + \epsilon^2 \mathcal{L}_r^\star \mathcal{L}_r f.$$

In this case the scalar Laplacian simply consists of the horizontal operator Δ_Q plus an operator that just involves the derivative along the Reeb direction. In particular, if we unravel the term

$\mathcal{L}_r^\star \mathcal{L}_r f$, we see that it is precisely the Lie derivative $\mathcal{L}_r$ squared. Indeed

$$\mathcal{L}_r^\star \mathcal{L}_r f = - \star_h \mathcal{L}_r \star_h \mathcal{L}_r f = -\mathcal{L}_r^2 f.$$

The same happens for volume forms $f\alpha \wedge d\alpha \in \Omega^3 M$ as

$$\Delta_\epsilon (f\alpha \wedge d\alpha) = (\Delta_Q f)\alpha \wedge d\alpha - \epsilon^2 \mathcal{L}_r^2 f\alpha \wedge d\alpha = (\Delta_\epsilon f)\alpha \wedge d\alpha.$$

In both cases the Laplacian converges to the horizontal Laplacian Δ_h when ϵ goes to 0. As will be explained next, this is enough to draw first conclusions about the spectrum of Δ_ϵ: the eigenforms of this family of operators $\{\Delta_\epsilon\}_\epsilon$ converge strongly to the eigenforms of Δ_Q.

4.5.2 Convergence at middle degrees

At middle degrees, as we will prove in Section 4.5.5, the spectrum of the Laplacian splits into a converging part, an exploding and a collapsing one. The final goal is to describe where the eigenspaces of each of these three types of eigenforms lie.

For simplicity, we will first study the spectrum of $P_\epsilon := d_\epsilon + \delta_\epsilon$ and then generalize to $P_\epsilon^2 = \Delta_\epsilon$. In a block decomposition, P_ϵ is given by

$$P_\epsilon = \begin{pmatrix} P_h & (1/\epsilon)L + \epsilon\mathcal{L}_r^\star \\ (1/\epsilon)\Lambda + \epsilon\mathcal{L}_r & P_h \end{pmatrix} = P_h + \epsilon \begin{pmatrix} 0 & \mathcal{L}_r^\star \\ \mathcal{L}_r & 0 \end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix} 0 & L \\ \Lambda & 0 \end{pmatrix}. \tag{4.19}$$

From this description, it is easy to see that it is self-adjoint, meaning that $P_\epsilon^\star = P_\epsilon$.

Lemma 4.5.7. *The operator P_ϵ is self-adjoint with respect to the L^2-product induced by g.*

By pushing ϵ to 0, it appears clear that the exploding terms in Equation (4.19) are the ones that include the zero-order operators L or Λ. Recall that L and Λ, as horizontal operators, are trivial outside of degree 0 and 2, respectively. In turn, we may imagine that converging problems arise on $1-$forms with non-zero vertical component and on $2-$forms with non-zero horizontal component.

Lemma 4.5.8. *The elements of the Rumin complex correspond to*

$$F = \ker \begin{pmatrix} 0 & L \\ \Lambda & 0 \end{pmatrix}.$$

Proof. A straightforward analysis of the kernel shows that

$$F^0 = \mathcal{C}^\infty(M) \qquad F^1 = \Omega^1\xi \qquad F^2 = \alpha \wedge \Omega^1\xi \qquad F^3 = \Omega^3 M.$$

This follows easily from the fact that the only non trivial L and Λ are on $\langle\alpha\rangle$ and $\Omega^2\xi$, respectively. ☐

In view of this Lemma, we can assert that the eigenspaces of the non exploding part of the spectrum of P_ϵ lie inside the Rumin complex.

When generalizing to the Laplacian, things complicate a bit. By Equation (4.18), we can write

$$\Delta_\epsilon = \frac{1}{\epsilon^2}\begin{pmatrix} L\Lambda & 0 \\ 0 & \Lambda L \end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix} 0 & [\delta_h, L] \\ [\Lambda, d_h] & 0 \end{pmatrix} + \widetilde{\Delta}_h + \mathcal{O}(\epsilon).$$

In this case we have exploding terms of order 1 and 2. This is due to the fact that $\mathrm{Im}(P_\epsilon \mid_F)$ has components both in F and $F^\perp$.

Remark 4.5.9. By Lemma 4.5.8, the second order term has kernel in F, i.e.

$$F = \ker \begin{pmatrix} L\Lambda & 0 \\ 0 & \Lambda L \end{pmatrix}. \tag{4.20}$$

Thus, it is non-zero only outside of the Rumin complex.

The question to ask is whether the space F also controls the operator

$$T := \begin{pmatrix} 0 & [\delta_h, L] \\ [\Lambda, d_h] & 0 \end{pmatrix}.$$

Recall that the commutator $[\delta_h, L] : \mathcal{C}^\infty(M) \to \Omega^1\xi$ is precisely defined as $\delta_h L f = \delta_h(f d\alpha)$, while $[\Lambda, d_h] : \Omega^1\xi \to \mathcal{C}^\infty(M)$ corresponds to $\Lambda d_h \beta$. In particular, we see that the Rumin complex is not in the kernel of T because, for $\beta \in F^1$ and $\omega \in F^2$, we get that β is horizontal and $\omega = \alpha \wedge \omega_v$ and in turn,

$$[\Lambda, d_h]\beta = \Lambda d_h \beta = 0 \qquad \iff \qquad d_h\beta = 0 \tag{4.21}$$

$$[L, \delta_h]\omega_v = L\delta_h\omega_v = 0 \qquad \iff \qquad \delta_h\omega_v = 0. \tag{4.22}$$

We need to study the intersection between the kernel of T and the Rumin complex. In this space, the Hodge-Laplacian has no exploding terms in ϵ. The relations in Equation (4.21) and (4.22) characterize this intersection.

Lemma 4.5.10. *Degree by degree, the intersection* $C := \ker(T) \cap F$ *is*

$$C^0 = \mathcal{C}^\infty(M), \qquad C^1 = \ker(d_h), \qquad C^2 = \ker(\delta_h \iota_r), \qquad C^3 = \Omega^3 M.$$

We want to relate this space C with another space coming from the $Q-$Laplacian. Consider the chain of operators d_Q on F defined in Equation (4.12) and define

$$K := \ker(d_Q + \delta_Q).$$

The operators d_Q and δ_Q are not defined outside of F. Therefore, K is a subset of the Rumin complex, and more importantly, it is also a subset of the kernel of the Q-Laplacian Δ_Q. These operators are explicitly given by

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^\infty(M) \arrow[r, bend left, "d_h"] & \Omega^1\xi \arrow[l, bend left, "\delta_h"] \arrow[r, bend left, "0"] & \alpha\wedge\Omega^1\xi \arrow[l, bend left, "0"] \arrow[r, bend left, "-\alpha\wedge d_h\iota_r"] & \Omega^3 M \arrow[l, bend left, "-\alpha\wedge\delta_h\iota_r"] \arrow[r] & 0. \end{tikzcd} \tag{4.23}$$

Since the exploding terms are in middle degrees, we will restrict F to $\Omega^1 M \oplus \Omega^2 M$ and we will call it K_{mid}, i.e.

$$K_{mid} := K \cap (\Omega^1 M \oplus \Omega^2 M). \tag{4.24}$$

Lemma 4.5.11. *Degree by degree, the space* K_{mid} *is*

$$K^0_{mid} = 0, \qquad K^1_{mid} = \ker(\delta_h), \qquad K^2_{mid} = \ker(d_h\iota_r), \qquad K^3_{mid} = 0.$$

We are interested in its orthogonal space inside the Rumin complex. We want to see if it is the suitable space on which the operator T is trivial.

Lemma 4.5.12. *Degree by degree, the space* $E := F \cap K_{mid}^{\perp}$ *is*

$$E^0 = \mathcal{C}^\infty(M), \qquad E^1 = \mathrm{Im}(d_h), \qquad E^2 = \mathrm{Im}(\alpha \wedge \delta_h), \qquad E^3 = \Omega^3 M.$$

Proof. It follows from the relation between the kernel of an operator and the space othogonal to the image of its adjoint. Precisely, $\mathrm{Im}(P^*) = \ker(P)^\perp$, for any closed, densely defined (unbounded) operator P, by Proposition A.2.2. ☐

How can we relate the space E of the previous Lemma with the space C of Lemma 4.5.10? Outside of the middle degrees, they coincide. What about degree one? The elements of E^1 are exact horizontal forms of the form $d_h f$, while the ones of C^1 are closed horizontal forms. As long as $d_h^2 \neq 0$, we cannot relate these two spaces. Nevertheless, the space E will be pivotal in the study of the convergence of the spectrum of Δ_ϵ.

For the study of the convergence of the spectrum, we will see that the useful decomposition of $\Omega^* M$ is

$$\Omega^* M = F^\perp \oplus E \oplus K_{mid}.$$

The first one is outside of the Rumin complex. Thus, on $F^\perp$, the Laplacian has second order terms in ϵ. We will see that if an eigenvalue λ of the Laplacian Δ_ϵ tends to infinity in the limit $\epsilon \to 0$, then its eigenspace is a subset of $F^\perp$. We will say that $F^\perp$ represents the *exploding* part of the spectrum. On the other hand, both E and K_{mid} are inside the Rumin complex. We will see that the eigenvalues of the Laplacian that tend to zero in the limit have eigenspace inside K, which contains the harmonic forms. Note that K overlaps with E outside of middle degrees, and coincides with K_{mid} at middle degrees. We will say that K represents the *collapsing* part of the spectrum, while E represent the *converging* one. In particular, in $E \cap K$ we will have the part of the spectrum converging to the 0 eigenvalue of the $Q-$Laplacian.

The upcoming two Sections will be rather technical in view of Section 4.5.5, in which we prove the fundamental results about the convergence of the spectrum of Δ_ϵ for $\epsilon \to 0$. It is therefore not strictly necessary to read the sections entirely. We advice the reader to skip these two sections and go directly to Section 4.5.5, and come back if necessary.

4.5.3 Ad hoc decomposition

Let $(J, 1, g)$ be a contact metric structure on (M, α). In this Section, we decompose the operator P_ϵ in terms of the the splitting $\Omega^* M = F \oplus F^\perp$. It will be a very useful decomposition, since we want to classify the diverging and non-diverging part of the spectrum of P_ϵ in terms of this splitting. Outside of middle degrees, the splitting is trivial because $F^\perp = \{0\}$. In middle degrees, we have that,

$$\Omega^1 M = F^1 \oplus F^{1,\perp} = \Omega^1 \xi \oplus \langle \alpha \rangle, \qquad \Omega^2 M = F^2 \oplus F^{2,\perp} = (\alpha \wedge \Omega^1 \xi) \oplus \Omega^2 \xi.$$

We may sometime refer to middle degree spaces as *tough* and to the others as *tame*. Precisely, we will call $F^{tough} := F^1 \oplus F^2$, and $F^{tame} := F^0 \oplus F^3$. This notation stresses the fact that the convergence problems arise at middle degrees.

Exterior differential

We will start decomposing d_ϵ. As always, the usual matrix decomposition of d_ϵ is

$$d_\epsilon = \begin{pmatrix} d_h & 0 \\ 0 & -d_h \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ \mathcal{L}_r & 0 \end{pmatrix} + \frac{1}{\epsilon} \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} =: D_h + \epsilon D_r + \frac{1}{\epsilon} D_L.$$

We want to decompose each operator D_h, D_r and D_L in terms of the splitting $\Omega^* M = F \oplus F^\perp$. Therefore we need matrix of the form

$$D_* = \begin{pmatrix} \Pi_F D_* \Pi_F & \Pi_F D_* \Pi_{F^\perp} \\ \Pi_{F^\perp} D_* \Pi_F & \Pi_{F^\perp} D_* \Pi_{F^\perp} \end{pmatrix},$$

where Π is the orthogonal projection. We will be especially interested in the first row of each operator, since it has values in the Rumin complex.

Proposition 4.5.13. *The block decomposition of d_ϵ in terms of $F \oplus F^\perp$ is*

$$d_\epsilon = \begin{pmatrix} d_Q + \epsilon D & \Pi_{F^{tame}}(\epsilon D_r)\Pi_{F^{tough,\perp}} \\ \Pi_{F_1^{tough,\perp}}(\epsilon D_r)\Pi_{F^{tame}} & 0 \end{pmatrix} + \epsilon(\phi_\epsilon^2)^\star \phi_\epsilon^1,$$

where

$$\begin{cases} \phi_\epsilon^1 := \delta_Q^J \Pi_{F^1} - (1/\epsilon)\iota_r \Pi_{F^{1,\perp}} \\ \phi_\epsilon^2 := \delta_Q \iota_r \Pi_{F^2} - (1/\epsilon)\Lambda \Pi_{F^{2,\perp}} \end{cases}$$

Proof. We will study degree by degree, each operator D_h, D_L and D_r. For D_h, we have

$$D_h^0 = \begin{pmatrix} d_Q & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_h^1 = \begin{pmatrix} d_Q & -\alpha \wedge d_Q \iota_r \\ d_h & 0 \end{pmatrix} \qquad D_h^2 = \begin{pmatrix} d_Q & 0 \\ 0 & 0 \end{pmatrix},$$

for which, we will write, in a compact form

$$D_h = \begin{pmatrix} d_Q & 0 \\ \Pi_{F^{tough,\perp}} D_h \Pi_{F^{tough}} & 0 \end{pmatrix} - \alpha \wedge d_Q \iota_r \Pi_{F^{1,\perp}}.$$

Then, D_L is always trivial unless it acts on $F^{1,\perp} = \langle \alpha \rangle$ and, in that case, $D_L(F^{1,\perp}) \subset \Omega^2\xi = F^{2,\perp}$. Finally, for D_r, we have that, by Lemma 4.4.11 together with the Kahler identities in 4.4.20,

$$D_r(\beta) = \alpha \wedge \mathcal{L}_r \beta = \alpha \wedge \iota_r D\beta + \alpha \wedge d_h \delta_h^J \beta = D\beta + \alpha \wedge d_Q \delta_Q^J \beta.$$

for any $\beta \in \Omega^1\xi = F^1$, where D is the connecting operator of the Rumin complex. Degree by degree, we have

$$D_r^0 = \begin{pmatrix} 0 & 0 \\ \alpha \wedge \mathcal{L}_r & 0 \end{pmatrix}, \qquad D_r^1 = \begin{pmatrix} D + \alpha \wedge d_Q \delta_Q^J & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_r^2 = \begin{pmatrix} 0 & \alpha \wedge \mathcal{L}_r \\ 0 & 0 \end{pmatrix}.$$

In a compact form, we write,

$$D_r = \begin{pmatrix} D & \Pi_{F^{tame}} D_r \Pi_{F^{tough,\perp}} \\ \Pi_{F^{tough,\perp}} D_r \Pi_{F^{tame}} & 0 \end{pmatrix} + \alpha \wedge d_Q \delta_Q^J \Pi_{F^1}.$$

All together, we can write d_ϵ as

$$d_\epsilon = \begin{pmatrix} d_Q + \epsilon D & \Pi_{F^{tame}}(\epsilon D_r)\Pi_{F^{tough,\perp}} \\ \Pi_{F^{tough,\perp}} D_h \Pi_{F^{tough}} + \Pi_{F^{tough,\perp}}(\epsilon D_r)\Pi_{F^{tame}} & \epsilon^{-1} D_L \end{pmatrix} + \epsilon\alpha \wedge d_Q \phi_\epsilon^1,$$

where $\phi_\epsilon^1 := \delta_Q^J \Pi_{F^1} - (1/\epsilon)\iota_r \Pi_{F^{1,\perp}}$. Define also $\phi_\epsilon^2 := \delta_Q \iota_r \Pi_{F^2} - (1/\epsilon)\Lambda \Pi_{F^{2,\perp}}$, and note that its adjoint is a map from $\mathcal{C}^\infty(M)$ to $\Omega^2 M$ that sends a function f to $(\phi_\epsilon^2)^\star f = \alpha \wedge d_Q f - \epsilon^{-1} L f$. In turn, we could write

$$\begin{aligned} \epsilon\alpha \wedge d_Q \phi_\epsilon^1 &= \epsilon\alpha \wedge d_Q \delta_Q^J \Pi_{F^1} - \alpha \wedge d_Q \iota_r \Pi_{F^{1,\perp}} \\ &= \epsilon \left(\alpha \wedge d_Q - \frac{1}{\epsilon} L \right) \left(\delta_Q^J \Pi_{F^1} - \frac{1}{\epsilon} \iota_r \Pi_{F^{1,\perp}} \right) + L\delta_Q^J \Pi_{F^1} - \frac{1}{\epsilon} L \iota_r \Pi_{F^{1,\perp}} \\ &= \epsilon(\phi_\epsilon^2)^* \phi_\epsilon^1 + d_Q \Pi_{F^1} - \frac{1}{\epsilon} L \iota_r \Pi_{F^{1,\perp}} \\ &= \epsilon(\phi_\epsilon^2)^* \phi_\epsilon^1 + \Pi_{F^{2,\perp}} D_h \Pi_{F^1} - \frac{1}{\epsilon} \Pi_{F^{2,\perp}} D_L \Pi_{F^{1,\perp}} \end{aligned}$$

where $L\delta_Q^J = d_Q$ comes from the second Kahler identities. All together, we obtain the claim. □

Codifferential

From Proposition 4.5.13, we can take the adjoint and obtain the matrix decomposition of δ_ϵ in terms of the splitting $F \oplus F^\perp$. The result is the following Proposition.

Proposition 4.5.14. *The block decomposition of δ_ϵ in terms of $F \oplus F^\perp$ is*

$$\delta_\epsilon = \begin{pmatrix} \delta_Q + \epsilon D^\star & \Pi_{F^{tame}}(\epsilon D_r^\star)\Pi_{F^{tough,\perp}} \\ \Pi_{F_1^{tough,\perp}}(\epsilon D_r^\star)\Pi_{F^{tame}} & 0 \end{pmatrix} + \epsilon(\phi_\epsilon^1)^\star \phi_\epsilon^2,$$

where

$$\begin{cases} \phi_\epsilon^1 := \delta_Q^J \Pi_{F^1} - (1/\epsilon)\iota_r \Pi_{F^{1,\perp}} \\ \phi_\epsilon^2 := \delta_Q \iota_r \Pi_{F^2} - (1/\epsilon)\Lambda \Pi_{F^{2,\perp}} \end{cases}$$

This decomposition will be very useful to study the convergence of the eigenvalues of P_ϵ with eigenspaces inside the Rumin complex. See for example, Theorem 4.5.20.

4.5.4 A priori estimates

In this subsection we state some technical results that are nonetheless fundamental in the study of the convergence of the spectrum of the family $\{\Delta_\epsilon\}_\epsilon$. We will deal with delicate estimates on Sobolev spaces that will be introduced shortly. We define the following Sobolev norm, which is somehow related to $\| - \|_{1,\epsilon}$ defined in Definition 4.3.3.

Definition 4.5.15. For $\omega \in \Omega^p M$, we define the following Sobolev norm:

$$\|\omega\|^2_{1',\epsilon} := \begin{cases} \|\omega\|^2_{1,h} + \epsilon^2 \|\omega\|^2_{1,v} + \|\omega\|_0^2 & p = 0, 3; \\ \|\omega\|^2_{1,h} + \epsilon^2 \|\omega\|^2_{1,v} + \left(1 + \frac{1}{\epsilon^2}\right)\|\omega\|_0^2 + \|\delta_h \omega_h\|_0^2 + \|\delta_h^J \omega_h - \frac{\omega_v}{\epsilon}\|_0^2 & p = 1; \\ \| \star \omega\|^2_{1',\epsilon} & p = 2. \end{cases}$$

For example, at degree 0, we can relate $\|f\|^2_{1',\epsilon}$ with $\epsilon\|f\|^2_{1,\epsilon} + \|f\|_0^2$, in view of Corollary 4.3.16. The norm $\| - \|_{1',\epsilon}$ is specifically constructed to prove some delicate estimates for the Hodge-Laplacian. We collect some useful results about this norm.

Proposition 4.5.16. *There exists $C_1, C_2 > 0$ such that, for any $\epsilon > 0$,*

$$C_1\|\eta\|^2_{1',\epsilon} \leq \langle \Delta_\epsilon \eta, \eta \rangle_0 + \|\eta\|_0^2 \leq C\|\eta\|^2_{1',\epsilon}.$$

Proof. The first inequality comes from a SubRiemannian Bochner-type of formula that can be found in [27, Proposition 5.3]. The second inequality comes from the continuity of the differential operator P_ϵ from $\mathcal{H}^1$ to $\mathcal{H}^0$. Indeed, since P_ϵ is self adjoint, we could rephase $\langle \Delta_\epsilon \eta, \eta \rangle_0$ as $\|P_\epsilon \eta\|_0^2$. The rest follows from the continuity. □

Lemma 4.5.17. *There exists $C > 0$ such that, for all $\epsilon \in (0, 1]$ and $\eta \in \Omega^1 M$,*

$$\|\eta_v\|_0^2 \leq \epsilon C \|\eta\|^2_{1',\epsilon}.$$

Proof. [27, Lemma 5.6]. □

We can now state and prove the convergence Theorems for the Hodge-Laplacian.

4.5.5 Convergence Theorems

In this Section, we are finally going to state the results about the convergence of the spectrum of the Hodge Laplacian. Both P_ϵ and Δ_ϵ are self-adjoint unbounded operators on $\Omega^* M$. Therefore, the notion of convergence comes with complications. The statements are given in terms of their resolvent operator, which is a bounded operator, conveniently constructed to solve the problem of convergence. Since it is bounded and globally defined, we are allowed to talk about strong and weak convergence in the usual sense. Appendix A gives an introduction on the matter, and covers the required material to understand the upcoming discussion.

Diverging part of the spectrum

The first Theorem that we are going to present, describes the exploding part of the spectrum. It ensures that the restriction of the resolvent of P_ϵ to $K_{mid}^\perp$ converges in norm to the zero operator.

Theorem 4.5.18. *There exists $\lambda \in \mathbb{C}$ such that*

$$\|\Pi_{F^\perp} R(\lambda, P_\epsilon) \Pi_{K_{mid}^\perp}\| \xrightarrow{\epsilon \to 0} 0.$$

Proof. Consider the following equation

$$(\lambda - P_\epsilon)\eta_\epsilon = \omega$$

for some purely imaginary $\lambda \in \mathbb{C}$ and $\omega \in K_{mid}^\perp$. By Lemma 4.5.7, P_ϵ is self-adjoint. Thus, it does not have purely imaginary eigenvalues and so, $\lambda \in Res(P_\epsilon)$ for all ϵ. This means that $\ker(\lambda - P_\epsilon) = \{0\}$ and so, there exists $\eta_\epsilon \in \Omega^* M$ satisfying the equation. Using Lemma A.2.7, we can split $\|(\lambda - P_\epsilon)\eta_\epsilon\|_0^2$ in

$$\|\omega\|_0^2 = |\Im(\lambda)|^2 \|\eta_\epsilon\|_0^2 + \|P_\epsilon \eta_\epsilon\|_0^2,$$

where $\Im(\lambda)$ is the imaginary part of λ. Moreover, by Proposition 4.5.16, there exists $C > 0$ such that

$$\|\omega\|_0^2 \geq |\Im(\lambda)|^2 \|\eta_\epsilon\|_0^2 + \|P_\epsilon \eta_\epsilon\|_0^2 = |\Im(\lambda)|^2 \|\eta_\epsilon\|_0^2 + \langle \Delta_\epsilon \eta_\epsilon, \eta_\epsilon \rangle_0 \geq C \|\eta_\epsilon\|_{1,\epsilon}^2$$

which means that the perturbed Sobolev 1-norm of η_ϵ is controlled by the L^2-norm of ω. By construction, the orthogonal space $F^\perp$ of the Rumin complex is trivial at degree $0, 3$. Thus, we need to control $\Pi_{F^\perp}\eta_\epsilon$ only for middle degree $\eta_\epsilon \in \Omega^1 M \oplus \Omega^2 M$. By Lemma 4.5.17, if $\eta_\epsilon \in \Omega^1 M$, then, there exists $C > 0$ such that

$$C\|\eta_\epsilon\|_{1,\epsilon}^2 \geq \frac{1}{\epsilon}\|\eta_{\epsilon,v}\|_0^2 = \frac{1}{\epsilon}\|\alpha \wedge \eta_{\epsilon,v}\|_0^2 = \frac{1}{\epsilon}\|\Pi_{F^\perp}\eta_\epsilon\|_0^2.$$

By Definitio 4.5.15, for $\eta_\epsilon \in \Omega^2 M$, we have that $\|\eta_\epsilon\|_{1,\epsilon}^2 = \|\star\eta_\epsilon\|_{1,\epsilon}^2$. Moreover, by the previous arguments, there exists $C > 0$, such that

$$C\|\eta_\epsilon\|_{1,\epsilon}^2 = C\| \star \eta_\epsilon\|_{1,\epsilon}^2 \geq \frac{1}{\epsilon}\|(\star\eta_\epsilon)_v\|_0^2 = \frac{1}{\epsilon}\| \star \eta_{\epsilon,h}\|_0^2 = \frac{1}{\epsilon}\|\eta_{\epsilon,h}\|_0^2 = \frac{1}{\epsilon}\|\Pi_{F^\perp}\eta_\epsilon\|_0^2,$$

because the Hodge dual $\star$ preserves the L^2-norm induced by the same metric. All together, we can say that there exists $C > 0$ such that

$$\|\Pi_{F^\perp}(\lambda - P_\epsilon)^{-1}\omega\|_0^2 = \|\Pi_{F^\perp}\eta_\epsilon\|_0^2 \leq \epsilon C \|\omega\|_0^2.$$

Therefore, we can conclude

$$\|\Pi_{F^\perp} R(\lambda, P_\epsilon)\Pi_{K_{mid}^\perp}\| = \sup_{\|\omega\|=1} \|\Pi_{F^\perp}(\lambda - P_\epsilon)^{-1}\Pi_{F_2^\perp}\omega\|_0^2 \leq \epsilon C \tag{4.25}$$

which tends to 0 whenever ϵ does. $\square$

This Theorem is the first important result on the convergence of the spectrum of the family of the operators P_ϵ. It ensures that that the part of the spectrum of P_ϵ which concentrates in $F^\perp$ is exploding when ϵ goes to zero. Firstly, we see that this is also a strong convergence. In particular, for any sequence of L^2-bounded eigenforms $\{\omega_\epsilon\}_\epsilon \subset F^\perp$, let $\nu_\epsilon \in \mathbb{R}$ be its sequence of eigenvalues. Then

$$\|\Pi_{F^\perp} R(\lambda, P_\epsilon)\Pi_{K_{mid}^\perp}\omega_\epsilon\|_0^2 = \left\|\frac{1}{\lambda - \nu_\epsilon}\omega_\epsilon\right\|_0^2 \to 0$$

implies that ν_ϵ is diverging. Therefore, every sequence of eigenvalues $\{\nu_\epsilon\}_\epsilon$ with eigenspace in $F^\perp$ for all ϵ sufficiently small, diverges.

Example 4.5.19. The contact form α is a bounded eigenform for all P_ϵ, and it is not contained in the Rumin complex. In particular, a simple computation shows that $P_\epsilon\alpha = \frac{1}{\epsilon}\alpha$, which gives an example of sequence of eigenvalues that diverges.

Converging part of the spectrum

For the converging part of the spectrum, we have the following statement.

Theorem 4.5.20. *There exists $\lambda \in \mathbb{C}$ such that*

$$\|\Pi_E R(\lambda, P_\epsilon)\Pi_{K_{mid}^\perp} - R(\lambda, P_Q)\Pi_E\| \xrightarrow{\epsilon\to 0} 0.$$

Proof. Consider, as in the previous proof, the equation

$$(\lambda - P_\epsilon)\eta_\epsilon = \omega$$

for some purely imaginary $\lambda \in \mathbb{C}$ and $\omega \in K_{mid}^\perp$. As before, there exists $\eta_\epsilon \in \Omega^* M$ satisfying the equation. What can we say about $\Pi_F\eta_\epsilon$? We are interested in the projection to the subspace $E \subseteq F$, but for the moment, it is sufficient to study the projection to the Rumin complex. To do this, we need to use the decomposition of P_ϵ with respect to the splitting $\Omega^*M = F \oplus F^\perp$. From Proposition 4.5.13 and 4.5.14, we can rewrite the initial equation as

$$(\lambda - P_Q)\Pi_E\eta_\epsilon - \Pi_F(\epsilon P_r)\Pi_{F^\perp}\eta_\epsilon - \epsilon\Pi_E((\phi_\epsilon^2)^*\phi_\epsilon^1 + (\phi_\epsilon^1)^*\phi_\epsilon^2)\eta_\epsilon = \Pi_F\omega,$$

where $P_r = D_r + D_r^\star$, for the operator D_r defined in Section 4.5.3. Equivalently, we write,

$$(\lambda - P_Q)\Pi_E\eta_\epsilon - \Pi_F\omega = \Pi_F(\epsilon P_r)\Pi_{F^\perp}\eta_\epsilon + \epsilon\Pi_E((\phi_\epsilon^2)^*\phi_\epsilon^1 + (\phi_\epsilon^1)^*\phi_\epsilon^2)\eta_\epsilon.$$

Now, we want to compose both terms with $(\lambda - P_Q)^{-1}$. The equation becomes

$$\Pi_E\eta_\epsilon - (\lambda - P_Q)^{-1}\Pi_F\omega = \epsilon(\lambda - P_Q)^{-1}\Pi_F P_r \Pi_{F^\perp}\eta_\epsilon + \epsilon(\lambda - P_Q)^{-1}\Pi_E((\phi_\epsilon^2)^*\phi_\epsilon^1 + (\phi_\epsilon^1)^*\phi_\epsilon^2)\eta_\epsilon$$

By Proposition 4.4.15, we know that $(\lambda - P_Q)\Pi_E$ is a bicontinuous bijection between $\mathcal{H}^1(M)$ and $L^2(M)$. Moreover, P_r can be viewed as a second order differential operator, which means that is continuous between $\mathcal{H}^2$ and L^2. Also the projections Π inside the Rumin complex are bounded in L^2. In norm, we get that there exists $C > 0$ such that

$$\|\Pi_E\eta_\epsilon - (\lambda - P_Q)^{-1}\Pi_F\omega\|_0^2 \le \epsilon C(\|\eta_\epsilon\|_0^2 + \|\phi_\epsilon\eta_\epsilon\|_0^2) \le \epsilon C\|\omega\|_{1,\epsilon}^2$$

By construction, also $\|\phi_\epsilon\eta_\epsilon\|_0^2$ can be controlled. The last inequality follows from Proposition 4.5.16, and implies that $\Pi\eta_\epsilon$ converges to $(\lambda - P_Q)^{-1}\omega$ in norm. The form η_ϵ was the solution to $(\lambda - P_\epsilon)\eta_\epsilon = \omega$ for all ϵ. Therefore, we have uniform (L^2) convergence of

$$\Pi_E(\lambda - P_\epsilon)^{-1}\Pi_{K_{mid}^\perp} \xrightarrow{\epsilon\to 0} (\lambda - P_Q)^{-1}\Pi_E,$$

which is precisely the claim. □

Outside of middle degrees, F and $K_{mid}^{\perp}$ coincide with $\Omega^* M$. Therefore, the previous statement can be rephrased as convergence in the norm-resolvent sense, as in Definition A.4.1. Indeed, there exists $\lambda \in \mathbb{C} - \mathbb{R}$ such that

$$\|R(\lambda, P_\epsilon) - R(\lambda, P_Q)\| \xrightarrow{\epsilon \to 0} 0.$$

In turn, by Theorem A.4.2, we obtain that $P_\epsilon \xrightarrow{n} P_Q$. We can therefore apply all the machinery from functional analysis explained in Appendix A.4. In particular, Theorem A.4.4, ensures that for any open interval $(a, b) \subseteq \mathbb{R}$, the spectral projection $\chi_{(a,b)}(P_\epsilon)$ converges in norm to $\chi_{(a,b)}(P_Q)$. It means that that for every eigenvalue of P_Q, there exists a sequence of eigenvalues of P_ϵ converging to it. This claim was already stated in Remark 4.5.6. At middle degrees, we have a similar consequence, but one needs to be careful on where the eigenspace of an eigenvalue of P_Q lies. The full statement, which merges the previous two Theorems together, is the following.

Corollary 4.5.21. *There exists $\lambda \in \mathbb{C}$ such that*

$$\|\Pi_{K_{mid}^{\perp}} R(\lambda, P_\epsilon) \Pi_{K_{mid}^{\perp}} - R(\lambda, P_Q)\Pi_E\| \xrightarrow{\epsilon \to 0} 0.$$

Collapsing part of the spectrum

We can now state the result about the collapsing part of the spectrum. We will not just prove where the collapsing eigenvalues of P_ϵ concentrate, but we will give a description of where they would converge if P_ϵ was rescaled by a factor of $\frac{1}{\epsilon}$. An eigenvalue of P_ϵ/ϵ is just an eigenvalue of P_ϵ multiplied by $\frac{1}{\epsilon}$. Thus, every eigenvalue of P_ϵ, that was not converging to 0 in the limit $\epsilon \to 0$, is now diverging. Formally, the claim is the following.

Theorem 4.5.22. *There exists $\lambda \in \mathbb{C} - \mathbb{R}$ such that*

$$\|\Pi_{K^{\perp}} R(\lambda, P_\epsilon/\epsilon)\| \xrightarrow{\epsilon \to 0} 0.$$

Proof. Proof analogous to Theorem 4.5.18, with a different norm in $\mathcal{H}^1$. ([27, Theorem 3.6]). □

The claim of the previous Theorem is rather intuitive. It means that the part of the spectrum of P_ϵ/ϵ that concentrates in $K^{\perp}$ is diverging. Recall that $K^{\perp}$ coincides with $K_{mid}^{\perp}$ at middle degrees. This is not a surprising result because the exploding eigenvalues of P_ϵ/ϵ coincide with the non collapsing ones of P_ϵ. Corollary 4.5.21 was already suggesting this result. The interesting part relies on describing the converging eigenvalues of P_ϵ/ϵ. In particular, by describing the resolvent convergence of the rescaled operator P_ϵ/ϵ, we can therefore describe the convergence of the collapsing part of the spectrum of P_ϵ.

Theorem 4.5.23. *There exists $\lambda \in \mathbb{C} - \mathbb{R}$ such that*

$$\|\Pi_K R(\lambda, P_\epsilon/\epsilon) - R(\lambda, P_D)\Pi_K\| \xrightarrow{\epsilon \to 0} 0,$$

where $P_D = D + D^\star$, for D the connecting operator of the Rumin complex.

Proof. Proof analogous to Theorem 4.5.20, with a different norm in $\mathcal{H}^1$. ([27, Theorem 3.6]) □

This last Theorem is rather surprising and the reason is the following. The limiting operator P_D of the convergence is a second order operator in the classical calculus, which means that it is a higher order operator than P_ϵ. This is not what one would naively expect from the convergence of a differential operator. We think that this is change of order is somehow related to the change in Hausdorff dimension of the metric spaces (M, D_ϵ) converging to (M, D_h) in the Gromov-Hausdorff sense.

Hodge-Laplacian

Every previous result can be generalized to the sequence of elliptic operators $\Delta_\epsilon = P_\epsilon^2$. For the non-collapsing part of the spectrum, we have the following result.

Corollary 4.5.24. *There exists $\lambda \in \mathbb{C} - \mathbb{R}$ such that*

$$\|\Pi_{K_{mid}^\perp} R(\lambda, \Delta_\epsilon) \Pi_{K_{mid}^\perp} - R(\lambda, \Delta_Q)\Pi_E\| \xrightarrow{\epsilon \to 0} 0.$$

Proof. A simple manipulation of the resolvent gives

$$R(\lambda^2, \Delta_\epsilon) = (\lambda^2 - \Delta_\epsilon)^{-1} = \frac{1}{2\lambda}\left((\lambda - P_\epsilon)^{-1} + (\lambda + P_\epsilon)^{-1}\right).$$

In turn, we can apply Corollary 4.5.21, and obtain the claim. □

Analogously to what we did for P_ϵ after Theorem 4.5.20, we can state some consequence of the previous Corollary on the local spectrum convergence of Δ_ϵ. We already saw that, outside of middle degrees, this statement says that $\Delta_\epsilon \xrightarrow{n} \Delta_Q$. By Theorem A.4.4 and A.4.6, we will have that any compact subset of the spectrum of the Hodge-Laplacian will converge to a compact subset of the spectrum of the Rumin's Laplacian Δ_Q, i.e. for any $C > 0$,

$$(0, C] \cap \sigma(\Delta_\epsilon) \longrightarrow (0, C] \cap \sigma(\Delta_Q)$$

whenever $\epsilon \to 0$. This is an important result in terms of spectral invariances, but it was already expected by Remark 4.5.6. Inside middle degrees, the exploding part of the spectrum lies in $F^\perp$ and the one in E is in bijection with the spetrum of Δ_Q.

A further research question in this setting could be: how can we relate Corollary 4.5.24 with the period of closed Reeb orbits and the convergence of geodesics to it of Theorem 3.3.5? We believe that if one generalize the latter theorem to closed geodesics, then we should be see a relation between the lengths of such closed geodesics and the part of the spectrum that concentrates in E. This might be done through Duistermaat-Guillemin trace formulas as in [18].

For the collapsing part of the spectrum of Δ_ϵ, we generalize Theorem 4.5.22 and 4.5.23 in the following way.

Corollary 4.5.25. *There exists $\lambda \in \mathbb{C} - \mathbb{R}$ such that*

$$\|R(\lambda, \Delta_\epsilon/\epsilon^2) - R(\lambda, \Delta_D)\Pi_{K_{mid}}\| \xrightarrow{\epsilon \to 0} 0.$$

All things considered, we split the space of forms into three components:

$$\Omega^* M = F^\perp \oplus E \oplus K_{mid}.$$

In the first one lies the exploding part of the spectrum, in the second the converging one and in the last component the collapsing one. Note that K intersect E at degree 0 and 3. The eigenvalues that concentrate in that intersection are still converging to 0, but at the same time, 0 is an eigenvalue of Δ_Q with eigenspace in that intersection. If we rescale Δ_ϵ by $\frac{1}{\epsilon^2}$, the the eigenvalues of $\Delta_\epsilon/\epsilon^2$, that concentrates on $K_{mid} \subset K$, converge to the ones of Δ_D.

Chapter 5

Subriemannian limit of the Dirac

An operator of Dirac type is generally any first order differential operator acting on a vector bundle over a Riemannian manifold. The square root of the Hodge-Laplacian considered in the previous chapter is an example of Dirac-type operator. Generally, they are metric dependent operators, and so, they are affected by any metric perturbation.

By Theorem 2.1.1, we know that the second Stiefel-Whitney class of a closed oriented three-manifold is trivial. Therefore, we can always endowed it with a Spin-structure. Furthermore, if the manifold is contact (and coorientable), and the metric is a contact metric structure, then there is a natural way of constructing $\mathrm{Spin}^{\mathbb{C}}$-structures on it that are somehow *adapted* to the contact structure. In turn, the deriving Dirac operator splits into vertical and horizontal component, so that we can define the *horizontal Dirac operator*. We expect it to be the limiting operator in the subRiemannian limit of the perturbed Dirac.

This final chapter is fairly brief and it does not contain any significant result regarding the convergence of the spectrum of the Dirac. Nonetheless, we decided to include it, because it was part of our research and it gives a valuable description of the perturbed Dirac operator under anisotropic deformation. The literature on the subject is not a vast, so hopefully, the upcoming discussion will be useful for future research.

For a thorough discussion on $\mathrm{Spin}^{\mathbb{C}}$-structure on contact manifolds, we refer to ([10],[23],[30], and [22]).

5.1 Spin-C structures on contact manifolds

As usual, let (M, α) be a closed contact three-manifold. We endow TM with a contact metric structure (J, λ, g) as in Definition 1.4.11. In this setting, $g_h = \lambda^{-1} d\alpha(-, J-)$. We will first define $\mathrm{Spin}^{\mathbb{C}}$-structure on the contact plane, and only afterwards we will see how they naturally extend to $\mathrm{Spin}^{\mathbb{C}}$-structures on the full tangent bundle.

Definition 5.1.1. A $\mathrm{Spin}^{\mathbb{C}}$-structure on (ξ, g_h) is a pair (W, Γ) where $W \to M$ is a rank 2 hermitian vector bundle and $\Gamma : \xi \to End(W)$ is a bundle morphism which satisfies the Clifford relations

$$\Gamma(X)^* + \Gamma(X) = 0, \qquad \Gamma(X)^* \circ \Gamma(X) = |X|^2 \,\mathrm{Id} \qquad \forall X \in \xi. \tag{5.1}$$

An *isomorphism* of $\mathrm{Spin}^{\mathbb{C}}$-structures between (W, Γ) and (W', Γ') is a unitary bundle isomorphism $\Phi : W \to W'$ that commutes with $\Gamma(v)$ and $\Gamma'(v)$ for any $v \in \xi$.

We will now prove that the morphism Γ of a $\mathrm{Spin}^{\mathbb{C}}$-structure (W, Γ) extends to the complexified Clifford algebra of ξ, making (W, Γ) a Clifford bundle on ξ.

Lemma 5.1.2. *A* Spin$^{\mathbb{C}}$*-structure* (W,Γ) *on* (ξ, g_h) *is a* $Cl^{\mathbb{C}}(\xi)$*-module.*

Proof. Let (W,Γ) be a Spin$^{\mathbb{C}}$-structure on ξ and consider a adapted frame $\{r, r_1, r_2\}$ of TM. By construction $\{r_1, r_2\}$ is a positive oriented orthonormal basis of ξ. The Clifford relations in (5.1) allows us to extend the map Γ to the Clifford bundle $Cl^{\mathbb{C}}(\xi)$ and in particular, if we take the element $c = r_2 \cdot r_1 \in Cl(\xi)$ we get that

$$\Gamma(c)^2 = \Gamma(c^2) = \Gamma(-r_2 \cdot r_1 \cdot r_1 \cdot r_2) = \Gamma(-1) = -\operatorname{Id}.$$

This means that the map $\Gamma(c)$ has eigenvalues $\pm i$. Let

$$W^{\pm} := \{\phi \in W : \Gamma(c)(\phi) = \pm i\phi\}$$

the two hermitian line bundles that span W. Note that the map $\Gamma(X)$ swaps the bundles $W^{\pm}$ for all $X \in \xi$. This is enough to prove that (W,Γ) is a $Cl^{\mathbb{C}}(\xi)-$module. □

In particular, this statement ensures that there exists a splitting into positive and negative spinors, and that the Clifford action $\Gamma(X)$ swaps the two bundles, for every horizontal vector X.

Since the metric is adapted, we can also restrict any Spin$^{\mathbb{C}}$-structure on (M, g_h) to ξ and obtain a Spin$^{\mathbb{C}}$-structure on (ξ, g_h).

Lemma 5.1.3. *Let* (W,Γ) *be a* Spin$^{\mathbb{C}}$*-structure on* (M,g)*. Then,* $(W, \Gamma\mid_{\xi})$ *is a* Spin$^{\mathbb{C}}$*-structure on* (ξ, g_h)*.*

Proof. It follows directly from the fact that the metric is adapted, and we can split $g = g_h + \alpha^2$. □

5.1.1 Horizontal Dirac

Let (W,Γ) be a Spin$^{\mathbb{C}}$-structure on (ξ, g_h). Consider the connection $\nabla^{\perp}$ on ξ defined as the projection to ξ of the Levi-Civita connection of (M,g), as in Definition 2.3.3.

Definition 5.1.4. A hermitian connection ∇ on W is called Spin$^{\mathbb{C}}$*-connection (compatible with* ξ) if

$$\nabla_X(\Gamma(Y)\phi) = \Gamma(Y)\nabla_X\phi + \Gamma(\nabla^{\perp}_X Y)\phi$$

for all $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(\xi)$ and $\phi \in \mathcal{C}^{\infty}(M,W)$.

In terms of any Spin$^{\mathbb{C}}$-connection compactible with ξ, we can define the *horizontal Dirac operator.*

Definition 5.1.5. We define the *Horizontal Dirac operator* $\not{D}_h : \mathcal{C}^{\infty}(M,W) \to \mathcal{C}^{\infty}(M,W)$ as

$$\not{D}_h\phi = \sum_{j=1}^{2} \Gamma(r_j)\nabla_{r_j}\phi$$

for a Spin$^{\mathbb{C}}$-connection ∇ (compatible with ξ) on (W,Γ), and an adapted frame $\{r_0, r_1, r_2\}$.

We are ready to construct the canonical Spin$^{\mathbb{C}}$-structure on the contact plane (and consequently on the whole tangent bundle).

5.2 Canonical Spin-C structure

Let (J, λ, g) be a contact metric structure on (M, α). The construction of the canonical $\mathrm{Spin}^{\mathbb{C}}$-structure on the contact manifold relies heavily on the presence of the contact metric structure. In fact, the hermitian vector bundle W_{can} will be defined in terms of the eigenvalue decomposition of the complexified contact plane $\xi \otimes_{\mathbb{R}} \mathbb{C}$ induced by the almost complex structure J. So, before defining it, we need to briefly talk about this decomposition.

5.2.1 Complex decomposition

Consider the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$, the complex plane bundle $\xi_{\mathbb{C}} := \xi \otimes_{\mathbb{R}} \mathbb{C}$ and the line bundle $\langle r \rangle_{\mathbb{C}} := \langle r \rangle \otimes_{\mathbb{R}} \mathbb{C}$. Also in this complex setting, we have the decomposition of complex bundles

$$T_{\mathbb{C}}M = \xi_{\mathbb{C}} \oplus \langle r \rangle_{\mathbb{C}} .$$

We can extend the almost complex structure $J : \xi \to \xi$ to the complexified contact bundle $\xi_{\mathbb{C}}$ and obtain another decomposition in terms of eigenvalue bundles, i.e.

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \oplus \langle r \rangle_{\mathbb{C}} . \tag{5.2}$$

Precisely, $T^{1,0}M$ and $T^{0,1}M$ are the complex line bundles spanning $\xi_{\mathbb{C}}$ and generated by the eigenvalues i and $-i$ of J, respectively.

Lemma 5.2.1. *Given an adapted local frame $\{r, r_1, r_2\}$, a typical (local) element of $T^{1,0}M$ is*

$$\kappa := \frac{1}{\sqrt{2}}(r_1 + ir_2).$$

Proof. Since the frame is adapted to the contact metric structure, we have that $r_2 = Jr_1$. It is easy to see that it is an eigenvector with corresponding eigenvalue i. □

The two complex line bundles $T^{1,0}M$ and $T^{0,1}M$ are isomorphic by coniugation. In turn, $\overline{\kappa}$ becomes a typical element of $T^{0,1}M$. By duality, it is possible to extend this construction to the complexified cotangent bundle $T^*_{\mathbb{C}}M$. Let $\mathcal{K}^{\pm}$ be the eigenvalue bundles of $J : \xi^*_{\mathbb{C}} \to \xi^*_{\mathbb{C}}$ with respect to the eigenvalues $\pm i$. The dual version of the decomposition in Equation (5.2) becomes

$$T^*_{\mathbb{C}}M = \mathcal{K}^+ \oplus \mathcal{K}^- \oplus \langle \alpha \rangle_{\mathbb{C}} . \tag{5.3}$$

Locally, for an adapted coframe $\{\alpha, \alpha_1, \alpha_2\}$, the typical element of $\mathcal{K}$ is

$$\omega := \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2).$$

Lemma 5.2.2. *The elements ω and $\overline{\kappa}$ (respectively $\overline{\omega}$ and κ) are in duality.*

Proof. A simple computation shows that $\omega(\overline{\kappa}) = \frac{1}{2}(\alpha_1(r_1) + \alpha_2(r_2)) = 1$. □

In general, the dual takes $(0, 1)$ vectors to $(1, 0)$ forms and viceversa, so that

$$\mathcal{K}^+ = (T^{0,1}M)^*, \qquad \mathcal{K}^- = (T^{1,0}M)^*.$$

In terms of general complex $m-$forms, the splitting in (5.3) generalizes to

$$\Lambda^m T^*_{\mathbb{C}}M = \bigoplus_{p+q+r=m} \Lambda^p \mathcal{K}^+ \oplus \Lambda^q \mathcal{K}^- \oplus \Lambda^r \langle \alpha \rangle_{\mathbb{C}} .$$

The first two components represent the horizontal part of the forms so, similarly to what we did in the previous chapter, we will write

$$\Lambda^{p,q}\xi^* := \Lambda^p \mathcal{K}^+ \oplus \Lambda^q \mathcal{K}^-, \qquad \Omega^{p,q}\xi := \mathcal{C}^\infty(M, \Lambda^{p,q}\xi^*).$$

Lemma 5.2.3. *$\mathcal{K}^+ \otimes \mathcal{K}^-$ is the trivial line bundle.*

Proof. We know from the previous chapter that $\Lambda^2\xi^*$ is a trivial real line bundle. Its complexification is a trivial complex line bundle that corresponds to

$$(\Lambda^2\xi^*)_{\mathbb{C}} \cong \Lambda^{1,1}\xi^* = \mathcal{K}^+ \otimes \mathcal{K}^-.$$

Therefore, $\mathcal{K}^+ \otimes \mathcal{K}^-$ is the trivial line bundle. □

5.2.2 Pseudo-Dolbeault space

The goal of this section is to prove that the *pseudo Dolbeault graded vector space*

$$\Lambda^{0,*}\xi^* := \Lambda^{0,0}\xi^* \oplus \Lambda^{0,1}\xi^* \cong \mathbb{C} \oplus \mathcal{K}^{-1} \tag{5.4}$$

defines a $\mathrm{Spin}^{\mathbb{C}}$-structure on ξ. We can endow $\mathcal{K}^{-1}$ with an hermitian structure that derives from g_h in a natural way, making $\overline{\omega}$ unitary. Together with the canonical hermitian product on $\mathbb{C}$, we obtain an hermitian structure h on $\Lambda^{0,*}\xi^*$, which, in local terms, satisfies

$$h(f + g\overline{\omega}, f' + g'\overline{\omega}) = f\overline{f'} + g\overline{g'}.$$

Proposition 5.2.4. *There exists a bundle morphism $\Gamma : \xi \to End(\Lambda^{0,*}\xi^*)$ that makes $(\Lambda^{0,*}\xi^*, \Gamma)$ a $\mathrm{Spin}^{\mathbb{C}}$-structure on (ξ, g_h).*

Proof. Define the morphism Γ locally as

$$\Gamma(r_1)\phi = \overline{\omega} \otimes \phi - \iota_\kappa\phi = \overline{\omega} \otimes \phi - \sqrt{2}\iota_{r_1}\phi$$
$$\Gamma(r_2)\phi = i(\overline{\omega} \otimes \phi + \iota_\kappa\phi) = i\overline{\omega} \otimes \phi - \sqrt{2}\iota_{r_2}\phi$$

where $\{r, r_1, r_2\}$ is the adapted frame in duality with the coframe $\{\alpha, \alpha_1, \alpha_2\}$ that defines ω. Given a pair of local complex valued functions (f, g), we obtain

$$\Gamma(r_1)(f + g\overline{\omega}) = \overline{\omega} \otimes (f + g\overline{\omega}) - \iota_\kappa(f + g\overline{\omega}) = f\overline{\omega} - g\iota_\kappa(\overline{\omega}) = f\overline{\omega} - g;$$
$$\Gamma(r_2)(f + g\overline{\omega}) = i(\overline{\omega} \otimes (f + g\overline{\omega}) + \iota_\kappa(f + g\overline{\omega})) = i(f\overline{\omega} + g\iota_\kappa(\overline{\omega})) = if\overline{\omega} + ig.$$

Their adjoint operators, with respect to the hermitian L^2-product induced by h, are

$$(\Gamma(r_1))^*(f' + g'\overline{\omega}) = g' - f'\overline{\omega}$$
$$(\Gamma(r_2))^*(f' + g'\overline{\omega}) = -ig' - if'\overline{\omega}$$

A straightforward computation shows that both $\Gamma(r_1)$ and $\Gamma(r_2)$ satisfy the Clifford relations of Definition 5.1.1 □

Definition 5.2.5. The $\mathrm{Spin}^{\mathbb{C}}$-structure $(\Lambda^{0,*}\xi^*, \Gamma)$ is called *canonical* $\mathrm{Spin}^{\mathbb{C}}$*-structure* on (ξ, g_h), and it is denoted by (W_{can}, Γ_{can}). We will denote by $W^+_{can} := \mathbb{C}$ and $W^-_{can} := \mathcal{K}^{-1}$, the positive and negative spinor bundles, respectively.

If we denote a local spinor $f + g\overline{\omega} \in W_{can}$ as (f, g), we obtain the following block decompositions:

$$\Gamma_{can}(r_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_{can}(r_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

Together with $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, these three matrices resemble the Pauli spin matrices. This gives an idea of how to extend Γ_{can} to the entire tangent bundle.

Lemma 5.2.6. *The morphism Γ_{can} canonically extends to TM, making (W_{can}, Γ_{can}) a* Spin$^{\mathbb{C}}$*-structure on (M, g).*

We could say that this Spin$^{\mathbb{C}}$-structure on M is somewhat *adapted* to the contact structure, because it is naturally constructed from a Spin$^{\mathbb{C}}$-structure on ξ. With a slightly different notation, we will write

$$\Gamma_{can}(r) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \Gamma_{can}(r_1) = \begin{pmatrix} 0 & -\omega \\ \overline{\omega} & 0 \end{pmatrix}, \qquad \Gamma_{can}(r_2) = \begin{pmatrix} 0 & i\omega \\ i\overline{\omega} & 0 \end{pmatrix},$$

with the idea that $\omega \otimes \overline{\omega} = 1$ becuase of Lemma 5.2.3.

5.2.3 Canonical Spin-C connection

In the setting of the canonical Spin$^{\mathbb{C}}$ $-$structure (W_{can}, Γ), there is a natural way to construct a Spin$^{\mathbb{C}}$ $-$connection compactible with ξ. Let ∇^h be the connection on $\xi_{\mathbb{C}}$ that is complex linear and coincides with $\nabla^\perp$ on ξ. Then, by the structural equations in 2.9, we have that

$$\begin{aligned} \nabla^h(r_1 + ir_2) &= \nabla^\perp r_1 + i\nabla^\perp r_2 = -c \otimes r_2 + ic \otimes r_1 = ic(r_1 + ir_2) \\ \nabla^h(r_1 - ir_2) &= \nabla^\perp r_1 - i\nabla^\perp r_2 = -c \otimes r_2 - ic \otimes r_1 = -ic(r_1 - ir_2) \end{aligned}$$

Therefore, the connection preserves the splitting of $\xi_{\mathbb{C}}$ into the eigenspaces $T^{0,1}M$ and $T^{0,1}M$. This follows from the fact that the connection $\nabla^\perp$ preserves the metric g_h in ξ and therefore also the almost complex structure J. By diality, is also acts on the canonical and anticanonical bundles $\mathcal{K}^\pm$ as

$$\begin{aligned} \nabla^h \omega &= ic \otimes \omega \\ \nabla^h \overline{\omega} &= -ic \otimes \overline{\omega}. \end{aligned}$$

In particular, ∇^h also preserves $\mathcal{K}^\pm$. This property will make him the canonical spin$^{\mathbb{C}}$-connection on the canonical spinor bundle W_{can}.

Lemma 5.2.7. *∇^h is a* Spin$^{\mathbb{C}}$*-connection compactible with ξ.*

Proof. We need to show that ∇^h satisfies

$$\nabla^h_X(\Gamma_{can}(Y)\phi) = \Gamma_{can}(Y)(\nabla^h_X \phi) + \Gamma_{can}(\nabla^\perp_X Y)\phi,$$

where $\nabla^\perp$ is the orthogonal Levi-Civita connection. Locally, for $Y = r_1$, the first term corresponds to

$$\nabla^h_X(\Gamma_{can}(r_1)(f + g\overline{\omega})) = -\partial_X g + \partial_X f\overline{\omega} - ic(X)f\overline{\omega},$$

The second terms is given by

$$\Gamma_{can}(r_1)(\nabla^h_X(f + g\overline{\omega})) = -\partial_X g + ic(X)g + \partial_X f\overline{\omega}.$$

Moreover, by Structural equations, $\nabla_X r_1 = -c(X)r_2$ and so,

$$\Gamma_{can}(\nabla_X r_1)(f + g\overline{\omega}) = -ic(X)g - ic(X)f\overline{\omega}.$$

All together, it is clear that the compatibility condition is satisfied. The case for r_2 is analogous. ☐

5.2.4 Canonical Dirac operator

In this section, we will define the Dirac operator in the canonical setting.

Proposition 5.2.8. *In a compact form, the horizontal Dirac operator is given by*

$$\not{D}_h(f, g\overline{\omega}) = \sqrt{2}\begin{pmatrix} igc(\overline{\kappa}) - \partial_{\overline{\kappa}} g \\ \partial_\kappa f\overline{\omega} \end{pmatrix}.$$

Proof. A straightforward computation in a local setting shows that

$$\begin{aligned}
\not{D}_h(f, g\overline{\omega}) &= \begin{pmatrix} 0 & -\omega \\ \overline{\omega} & 0 \end{pmatrix}\begin{pmatrix} \partial_{r_1} f \\ \partial_{r_1} g\overline{\omega} - igc_1\overline{\omega} \end{pmatrix} + \begin{pmatrix} 0 & i\omega \\ i\overline{\omega} & 0 \end{pmatrix}\begin{pmatrix} \partial_{r_2} f \\ \partial_{r_2} g\overline{\omega} - igc_2\overline{\omega} \end{pmatrix} \\
&= \begin{pmatrix} igc_1 - \partial_{r_1} g \\ \partial_{r_1} f\overline{\omega} \end{pmatrix} + \begin{pmatrix} i\partial_{r_2} g + gc_2 \\ i\partial_{r_2} f\overline{\omega} \end{pmatrix} \\
&= \sqrt{2}\begin{pmatrix} igc(\overline{\kappa}) - \partial_{\overline{\kappa}} g \\ \partial_\kappa f\overline{\omega} \end{pmatrix}.
\end{aligned}$$

□

This description of the Dirac operator is not very insightful. Furthermore, it is not a global representation. For these reasons, in the next Section, we will give an alternative description of the Dirac operator in terms of the so called *Pseudo-Dolbeault operator*.

5.2.5 Pseudo-Dolbeault operator

Consider an hermitian connection ∇ on a vector bundle W on M. By definition,

$$\nabla : \mathcal{C}^\infty(M, W) \to \mathcal{C}^\infty(M, T^*M \otimes W).$$

By complex linearity, it can be extended to

$$\nabla : \mathcal{C}^\infty(M, W) \to \mathcal{C}^\infty(M, T^*_{\mathbb{C}}M \otimes W).$$

Using the decomposition in Equation (5.3), we obtain that ∇ can be splitted in three parts:

$$\begin{aligned}
\nabla_\omega &: \mathcal{C}^\infty(M, W) \to \mathcal{C}^\infty(M, \mathcal{K}^+ \otimes W) \\
\nabla_{\overline{\omega}} &: \mathcal{C}^\infty(M, W) \to \mathcal{C}^\infty(M, \mathcal{K}^- \otimes W) \\
\nabla_r &: \mathcal{C}^\infty(M, W) \to \mathcal{C}^\infty(M, W).
\end{aligned}$$

The local description of the first two operators give meaning to the notation.

Lemma 5.2.9. *The local description of the previous operators are*

$$\nabla_\omega\phi = \sqrt{2}\omega \otimes \nabla_{\overline{\kappa}}\phi \qquad \nabla_{\overline{\omega}}\phi = \sqrt{2}\overline{\omega} \otimes \nabla_\kappa\phi. \tag{5.5}$$

Assume $W = \mathcal{K}^-$ and $\nabla = \nabla^h$. In this case, we obtain that the operator ∇^h_ω sends negative spinors of the canonical $\mathrm{Spin}^{\mathbb{C}}$-structure to positive ones. Precisely, we have

$$\nabla^h_\omega : \mathcal{C}^\infty(M, W^-_{can}) = \mathcal{C}^\infty(M, \mathcal{K}^-) \to \mathcal{C}^\infty(M, \mathcal{K}^+ \otimes \mathcal{K}^-) = \mathcal{C}^\infty(M, \mathcal{C}) = \mathcal{C}^\infty(M, W^+_{can}).$$

If we assume instead $W = \mathbb{C}$ and $\nabla_X = \partial_X$ is the usual weak derivative of functions, then,

$$\partial_{\overline{\omega}} : \mathcal{C}^\infty(M, W^+_{can}) = \mathcal{C}^\infty(M, \mathcal{C}) \to \mathcal{C}^\infty(M, \mathcal{K}^- \otimes \mathcal{C}) = \mathcal{C}^\infty(M, \mathcal{K}^-) = \mathcal{C}^\infty(M, W^-_{can}).$$

In terms of these operators, we can define the so called *pseudo-Dolbeault operator*,

Definition 5.2.10. The *pseudo-Dolbeault operator* $\sigma : \mathcal{C}^\infty(M, W_{can}) \to \mathcal{C}^\infty(M, W_{can})$ is defined as

$$\sigma = \begin{pmatrix} -i\partial_r & -\nabla^h_\omega \\ \partial_{\overline{\omega}} & i\nabla^h_r \end{pmatrix}. \tag{5.6}$$

This operator is very much related to the Dirac operator on the canonical Spin$^{\mathbb{C}}$-structure.

Proposition 5.2.11. *The canonical horizontal Dirac is described by the elements in the antidiagonal of the pseudo-Dolbeault operator σ,*

$$\not{D}_h = \begin{pmatrix} 0 & -\nabla^h_\omega \\ \partial_{\overline{\omega}} & 0 \end{pmatrix}$$

Proof. Locally, given a spinor $f + g\overline{\omega}$,

$$\begin{aligned}
\begin{pmatrix} 0 & -\nabla^h_\omega \\ \partial_{\overline{\omega}} & 0 \end{pmatrix}\begin{pmatrix} f \\ g\overline{\omega} \end{pmatrix} &= \begin{pmatrix} -\sqrt{2}\omega \otimes \nabla^h_{\overline{\kappa}}(g\overline{\omega}) \\ \sqrt{2}\partial_\kappa f\overline{\omega} \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{2}g\omega \otimes \nabla^h_{\overline{\kappa}}(\overline{\omega}) - \sqrt{2}\partial_{\overline{\kappa}} g \\ \sqrt{2}\partial_\kappa f\overline{\omega} \end{pmatrix} \\
&= \sqrt{2}\begin{pmatrix} igc(\overline{\kappa}) - \partial_{\overline{\kappa}} g \\ \partial_\kappa f\overline{\omega} \end{pmatrix}
\end{aligned}$$

which is exactly the description of the Dirac operator in Proposition 5.2.8. □

A first consequence of this Proposition is the following Lemma.

Lemma 5.2.12. *The horizontal Dirac operator is self-adjoint.*

Proof. for $f\overline{\omega} \in \mathcal{K}^-$ and $g \in \mathcal{C}^\infty(M, \mathbb{C})$,

$$\begin{aligned}
\left\langle \nabla^h_\omega(f\overline{\omega}), g\right\rangle_0 + \langle f\overline{\omega}, \partial_{\overline{\omega}} g\rangle_0 &= \sqrt{2}(\omega \otimes \nabla^h_{\overline{\kappa}}(f\overline{\omega}), g) + \sqrt{2}(f\overline{\omega}, \partial_\kappa g\overline{\omega}) \\
&= \sqrt{2}(f\overline{\omega}\partial_{\overline{\kappa}} f - i\sqrt{2}c(\overline{\kappa})f, g) + \sqrt{2}(f\overline{\omega}, \partial_\kappa g\overline{\omega}) \\
&= \sqrt{2}\int_U (\partial_{\overline{\kappa}} f - ic(\overline{\kappa})f)\overline{g} \quad vol_g + \sqrt{2}\int_U h(f\overline{\omega}, \partial_\kappa g\overline{\omega}) \quad vol_g \\
&= \sqrt{2}\int_U \partial_{\overline{\kappa}} f\overline{g} \quad vol_g - \sqrt{2}\int_U c(\overline{\kappa})f\overline{g} \quad vol_g + \sqrt{2}\int_U f\partial_{\overline{\kappa}}\overline{g} \quad vol_g \\
&= \sqrt{2}\int_U \partial_{\overline{\kappa}}(f\overline{g}) - ic(\overline{\kappa})f\overline{g} \quad vol_g \\
&= \int_U \nabla^h_\omega(f\overline{g}\overline{\omega}) \quad vol = 0.
\end{aligned}$$

□

Another immediate consequence of Proposition 5.2.11 is that we can describe the pseudo-Dolbeault operator σ as

$$\sigma = \not{D}_h + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\nabla^h_r.$$

By construction, we could also phrase this relation as follows:

$$\sigma = \not{D}_h + \Gamma_{can}(r)\nabla^h_r,$$

which means that σ coincides with the Dirac operator whenever ∇^h is a Spin$^{\mathbb{C}}$-connection of the canonical Spin$^{\mathbb{C}}$-structure (W_{can}, Γ_{can}) on (M, ξ).

Proposition 5.2.13. *The pseudo Dolbeault operator σ is a self-adjoint operator.*

Proof. It is easy to see that both $i\partial_r$ and $i\nabla_r^h$ are self adjoint operators.
Firstly, for $f, g \in \mathcal{C}^\infty(M, \mathbb{C})$,

$$\langle i\partial_r f, g\rangle_0 - \langle f, i\partial_r g\rangle_0 = \int_M i\partial_r f\overline{g} + if\partial_r\overline{g} \quad vol_g = \int_M \partial_r(if\overline{\omega}) \quad vol_g = 0.$$

Secondly,

$$\left\langle i\nabla_r^h f\overline{\omega}, g\overline{\omega}\right\rangle_0 - \left\langle f\overline{g}, i\nabla_r^h g\overline{\omega}\right\rangle_0 = \int_M \nabla_r^h(if\overline{g\omega}) \quad vol_g = 0.$$

□

5.2.6 Anisotropic deformation

Now that we have a satisfying global description the horizontal Dirac operator, we can finally perturb the metric and see what happens to it. Let g_ϵ be the anisotropic deformation on a contact metric structure $(J, 1, g)$ in (M, ξ). Therefore,

$$g = d\alpha(-, J-) + \alpha^2 \qquad \rightsquigarrow \qquad g_\epsilon = d\alpha(-, J-) + \frac{1}{\epsilon^2}\alpha^2.$$

Just like in the study of the Hodge-Laplacian, we choose to rescale the contact form. Let α_ϵ be the rescaled contact form, and r_ϵ its Reeb vector field. Denote by ∇^ϵ the Levi Civita connection associated to the perturbed metric. The horizontal Levi civita connection $\nabla^{\epsilon,\perp}$ is described by

$$\begin{cases} \nabla^{\epsilon,\perp} r_1 = -c^\epsilon \otimes r_2 \\ \nabla^{\epsilon,\perp} r_2 = c^\epsilon \otimes r_1. \end{cases} \tag{5.7}$$

Moreover, by Lemma 2.3.9, we can describe the structural form c^ϵ in terms of the unperturbed one as follows:

$$c^\epsilon = c + \left(\epsilon - \frac{1}{\epsilon}\right)\alpha_\epsilon.$$

The deformed structural equation for $\nabla^{\epsilon,h}$ becomes

$$\nabla^{\epsilon,h}\omega = \nabla^h\omega + i\left(\epsilon - \frac{1}{\epsilon}\right)\alpha_\epsilon \otimes \omega$$
$$\nabla^{\epsilon,h}\overline{\omega} = \nabla^h\overline{\omega} - i\left(\epsilon - \frac{1}{\epsilon}\right)\alpha_\epsilon \otimes \overline{\omega}.$$

Lemma 5.2.14. *The horizontal operator $\nabla_\omega^{\epsilon,h}$ is not affected by the deformation.*

Proof. In local terms,

$$\nabla_\omega^{\epsilon,h} = \sqrt{2}\omega \otimes \nabla_{\overline{\kappa}}^{\epsilon,h} = \nabla_\omega^h - \sqrt{2}i\left(1 - \frac{1}{\epsilon^2}\right)\alpha(\overline{\kappa})\omega = \nabla_\omega^h,$$

because $\eta(\overline{\kappa})$ is clearly zero for an adapted orthonormal basis $\{r, r_1, r_2\}$. □

Proposition 5.2.15. *The perturbed pseudo-Dolbeault operator takes the form*

$$\sigma_\epsilon = \not{D}_h + \epsilon \begin{pmatrix} -i\partial_r & 0 \\ 0 & i\nabla^h_r + \mathrm{Id} \end{pmatrix} - \frac{1}{\epsilon}\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{5.8}$$

Proof. A straightforward computation in this setting shows that

$$\sigma_\epsilon = \begin{pmatrix} -i\partial_{r_\epsilon} & -\nabla^{\epsilon,h}_\omega \\ \partial_{\overline{\omega}} & i\nabla^{\epsilon,h}_{r_\epsilon} \end{pmatrix} = \begin{pmatrix} -i\epsilon\partial_r & -\nabla^h_\omega \\ \partial_{\overline{\omega}} & i\epsilon\nabla^h_r + \epsilon - \frac{1}{\epsilon} \end{pmatrix}, \tag{5.9}$$

which is precisely the claim. □

We see that the issue of convergence for $\epsilon \to 0$ is strictly related to the eigenvalue problem in the negative spinors. This setting is somewhat different from what we saw for the Hodge-Laplacian. In particular, in the latter case, it was rather intuitive to classify the spectrum of P_ϵ, simply by studying the kernel of the exploding operators in ϵ. A simple analysis showed immediately that the exploding part of the spectrum of the Laplacian might be concentrating outside of the Rumin's complex. On the other hand, in the pseudo-Dolbeault case, we have an eigenvalue problem $\frac{1}{\epsilon}\,\mathrm{Id}$ and, therefore, the exploding part of the spectrum might presumably concentrates everywhere in the negative spinors. The eigenvalue problem basically shifts the spectrum of σ. In principle, one may be able to study this shift and prove some alternative resolvent convergence of σ_ϵ. For the moment however, we do not know how to approach this shifting problem, and so it remains an open question.

Remark 5.2.16. The convergent part of the spectrum of σ_ϵ concentrate on the space of spinors with zero negative part and whose positive part belongs in the kernel of ∂_r. We are therefore looking at holomorphic sections of the contact plane.

5.3 General Spin-C structure

From the canonical $\mathrm{Spin}^{\mathbb{C}}$-structure (W_{can}, Γ_{can}) on (ξ, g_h), we easily construct others by tensoring W_{can} with any hermitian line bundle.

Lemma 5.3.1. *For any hermitian line bundle $L \to M$, the pair $(W_{can} \otimes L, \Gamma_{can} \otimes \mathrm{Id})$ becomes a $\mathrm{Spin}^{\mathbb{C}}$-structure on (M, g).*

Also the converse is true: we can completely describe a $\mathrm{Spin}^{\mathbb{C}}$-structure (W, Γ) on (ξ, g_h) by an hermitian line bundle.

Proposition 5.3.2. *Any $\mathrm{Spin}^{\mathbb{C}}$-structure (W, Γ) on (ξ, g_h) is isomorphic to*

$$(W_{can} \otimes L, \Gamma_{can} \otimes \mathrm{Id})$$

for some hermitian line bundle $L \to M$.

Proof. For a local adapted frame $\{r, r_1, r_2\}$, consider the Clifford element $c := r_2 \cdot r_1 \in Cl(\xi)$. We know that $\Gamma(c)$ split W in positive and negative spinors depending on the sign of their eigenvalue by Lemma 5.1.2. Then, for any $X \in \xi$ and $\phi \in W^-$, we have

$$\Gamma(JX)\phi = \Gamma(X)\Gamma(c)\phi = -i\Gamma(X)\phi.$$

Therefore, $\Gamma(-)\phi$ is a section of $Hom(T^{1,0}M, W^+) \cong \mathcal{K}^- \otimes W^+$. In particular, the map $T : W^- \to W_{can} \otimes W^+$, sending ϕ to $\frac{1}{\sqrt{2}}\Gamma(-)\phi$ is a unitary isomorphism. Set $L := W^+$ with hermitian structure induced by W. Then $W^+ \cong \mathbb{C} \otimes W^+ = \mathbb{C} \otimes L$ and $W \cong W_{can} \otimes L$. □

5.3.1 General Spin-C connections

We saw in the previous section that a general spinor bundle is given by tensoring the canonical bundle $C \oplus \mathcal{K}^-$ with any hermitian line bundle L. The same happens with $\mathrm{Spin}^{\mathbb{C}}$-connections. In fact, any hermitian connection ∇ on L gives rise to a $\mathrm{Spin}^{\mathbb{C}}$-connection on $W = W_{can} \otimes L$ given by

$$\nabla^h \otimes \nabla.$$

Lemma 5.3.3. *$\nabla^h \otimes \nabla$ is a $\mathrm{Spin}^{\mathbb{C}}$-connection on $(W_{can} \otimes L, \Gamma_{can} \otimes \mathrm{Id})$ compatible with ξ.*

Proof. The Clifford action is given by $\Gamma = \Gamma_{can} \otimes \mathrm{Id}$. Hence, for $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(\xi)$ and $\phi \otimes s \in \mathcal{C}^\infty(W)$,

$$\begin{aligned}
(\nabla^h \otimes \nabla)_X(\Gamma(Y)(\phi \otimes s)) &= (\nabla^h \otimes \nabla)_X((\Gamma_{can}(Y)(\phi) \otimes s)) \\
&= \nabla^h_X(\Gamma_{can}(Y)\phi) \otimes s + \Gamma_{can}(Y)(\phi) \otimes \nabla_X e \\
&= (\Gamma_{can}(Y)\nabla^h_X \phi + \Gamma_{can}(\nabla^\perp_X Y)\phi) \otimes s + \Gamma_{can}(Y)(\phi) \otimes \nabla_X e \\
&= \Gamma_{can}(Y)(\nabla^h_X \phi) \otimes s + \Gamma_{can}(Y)(\phi) \otimes \nabla_X e + \Gamma_{can}(\nabla^\perp_X Y)(\phi) \otimes s \\
&= \Gamma(Y)(\nabla^h_X \phi \otimes s + \phi \otimes \nabla_X e) + \Gamma_{can}(\nabla^\perp_X Y)(\phi) \otimes s \\
&= \Gamma(Y)((\nabla^h \otimes \nabla)_X(\phi \otimes s)) + \Gamma(\nabla^\perp_X Y)(\phi \otimes s).
\end{aligned}$$

□

5.3.2 General pseudo-Dolbeault operator

Generalize the previous construction in the case $W = \mathcal{K}^- \otimes L$ and $W = L$ for any hermitian line bundle. Then, given an hermitian connection ∇ on L, the pseudo Dolbeault operator $\sigma_L : \mathcal{C}^\infty(M, W_{can} \otimes L) \to \mathcal{C}^\infty(M, W_{can} \otimes L)$ takes the form

$$\sigma_L = \begin{pmatrix} -i\nabla_r & -(\nabla^h \otimes \nabla)_\omega \\ \nabla_{\overline{\omega}} & i(\nabla^h \otimes \nabla)_r \end{pmatrix}. \tag{5.10}$$

It is not difficult to see that σ_L remains a selfadjoint operator and still corresponds to the Dirac operator of the $\mathrm{Spin}^{\mathbb{C}}$-structure $(W_{can} \otimes L, \Gamma_{can} \otimes \mathrm{Id})$. Note that, given $f\overline{\omega} \otimes s \in \mathcal{C}^\infty(W^-)$, then

$$\begin{aligned}
(\nabla^h \otimes \nabla)_\omega(f\overline{\omega} \otimes s) &= \sqrt{2}\omega \otimes (\nabla^h \otimes \nabla)_{\overline{\kappa}}(f\overline{\omega} \otimes s) \\
&= \sqrt{2}\omega \otimes \nabla^h_{\overline{\kappa}}(f\overline{\omega}) \otimes s + \sqrt{2} f \nabla_{\overline{\kappa}} s \\
&= \nabla^h_\omega(f\overline{\omega}) \otimes s + f\overline{\omega} \otimes \nabla_\omega s \\
&= \nabla^h_\omega \otimes \nabla_\omega(f\overline{\omega} \otimes s).
\end{aligned}$$

Which means that $(\nabla^h \otimes \nabla)_\omega = \nabla^h_\omega \otimes \nabla_\omega$.

Appendix A

Unbounded operators

In this Appendix, we review some basic material on unbounded operators, in order to talk about their convergence in the resolvent sense. Let X be a separable Hilbert space, which means an Hilbert space which contains a countable dense subset. Let $T : D(T) \subseteq X \to X$ be a linear operator defined on a dense domain, and $\Gamma(T)$ be the graph of T, that is the Hilbert space defined by

$$\Gamma(T) := \{(x, Tx) \in X \times X : x \in D(T)\}$$

with inner product $\langle (x, y), (x', y') \rangle := \langle x, x' \rangle + \langle y, y' \rangle$.

Definition A.0.1. We say that T is *closed* if $\Gamma(T)$ is a closed subset of $X \times X$ with the product topology.

It is trivial to see that every continous operator is closed. The converse is true if $D(T)$ is closed, meaning that $D(T) = X$.

Theorem A.0.2 (Closed graph Theorem). *If $D(T) = X$, then T is bounded if and only if T is closed.*

Proof. [28, Theorem 1.5]. □

In order to talk about the resolvent of an operator, the following result plays a particular role.

Lemma A.0.3. *T is closed if and only if $(\lambda - T)$ is closed on X*

Proof. [28, Corollary 1.8]. □

Definition A.0.4. The *resolvent set* of an operator T is given by

$$Res(T) := \{\lambda \in \mathbb{C} : (\lambda - T) : D(T) \to X \text{ is bijective}\}.$$

For $\lambda \in Res(T)$, we define the *resolvent operator* associated to T as

$$R(\lambda, T) : (\lambda - T)^{-1} : X \to X.$$

By definition, the spectrum of T is exactly $\sigma(T) = \mathbb{C} - Res(T)$. Moreover, By the previous Lemma, we obtain that the resolvent operator is continuous.

Lemma A.0.5. *$R(\lambda, T)$ is bounded if and only if T is closed.*

Proof. The resolvent is globally defined, therefore, by the closed graph Theorem, it is bounded if and only if it's closed. In turn, by flipping the coordinates, we note that $\Gamma((\lambda - T)^{-1})$ is closed if and only if $\Gamma(\lambda - T)$. By Lemma A.0.3, we obtain the result. $\square$

A.1 Topology on bounded operators

Let X be a separable Hilbert space and $\mathcal{L}(X)$ be the space of continuous operators. By Theorem A.0.2, every operator $T \in \mathcal{L}(X)$ is closed. The space $\mathcal{L}(X)$ is a Banach space with norm

$$\|T\| := \sup_{\|x\|=1} \|Tx\|.$$

The topology on $\mathcal{L}(X)$ induced by this norm is usually called *uniform topology*. We will say that a sequence of bounded operators T_k converges *uniformly* to $T \in \mathcal{L}(X)$ if it converges in the uniform topology. There is another topology on $\mathcal{L}(X)$ that is very important for the upcoming discussion about the convergence of unbounded operators.

Definition A.1.1. We call *strong topology* the smallest topology on $\mathcal{L}(X)$ such that the map $E_x : \mathcal{L}(X) \to X$ given by $E_x(T) := Tx$ is continuous for all $x \in X$.

In this setting a sequence of bounded operators T_k converges to $T \in \mathcal{L}(X)$ if and only if $\|T_k x - Tx\| \to 0$ whenever $k \to \infty$. When this is the case, we will say that T_k converges *strongly* to T.

Lemma A.1.2. *If a sequence $T_k \in \mathcal{L}(X)$ converges uniformly to $T \in \mathcal{L}(X)$, then it is also true that it converges strongly.*

Proof. This is clear by the fact that the map $E_x : \mathcal{L}(X) \to X$ is continuous also in the uniform topology, so the strong one is a weaker topology. It can also be seen by the inequality

$$\|(T_k - T)x\| \leq \|T_k - T\| \|x\| \qquad \forall x \in X.$$

$\square$

The choice of the underlying topology on $\mathcal{L}(X)$ not only plays a significant role when talking about convergence of bounded operators, but also when checking the continuity of operators $\mathcal{L}(X) \to \mathcal{L}(X)$. It is worth mentioning the problem of the continuity of the adjoint in the strong topology.

Definition A.1.3. Given a bounded operator $T \in \mathcal{L}(X)$, the *adjoint* of T is the unique bounded operator $T^* \in \mathcal{L}(X)$ satisfying the relation

$$(Tx, y) = (x, T^* y) \qquad \forall x, y \in X.$$

An immediate consequence of this definition is that the adjoint $* : \mathcal{L}(X) \to \mathcal{L}(X)$ is conjugate linear, i.e.

$$(\lambda T)^* = \overline{\lambda} T^*.$$

A less trivial property of the adjoint is that it preserves the uniform norm.

Lemma A.1.4. *For any $T \in \mathcal{L}(X)$, we have that $\|T\| = \|T^*\|$.*

Proof. This can be seen by checking one of the two inequalities because $T^{**} = T$. To prove $\|T^*\| \leq \|T\|$, define $\phi_y \in X^*$ as the continuous functional that sends a point $x \in X$ to (Tx, y) for some $y \in X$. By the Riesz Theorem, this functional is uniquely associated to the adjoint of T and, in particular,

$$\|T^*x\| = \|\phi_y\| = \sup_{\|x=1\|} |(Tx, y)| \leq \|T\|\|y\|,$$

for all $x, y \in X$. This implies that $\|T^*\| \leq \|T\|$. □

Remark A.1.5. The adjoint $* : \mathcal{L}(X) \to \mathcal{L}(X)$ is a continuous map in the uniform topology but not in the strong one. The first statement follows easily from the fact that, given any sequence of bounded operators T_k converging uniformly to $T \in \mathcal{L}(X)$,

$$\|T_k^* - T^*\| = \|(T_k - T)^*\| = \|T_k - T\| \to 0.$$

A counter example of the continuity in the strong sense is the following: let

$$\ell_2 := \left\{ (x_k)_k \in \mathbb{R}^{\mathbb{N}} : \sum_{k \in \mathbb{N}} x_k^2 < \infty \right\}$$

and $W_n : \ell_2 \to \ell_2$ be the right shift of $n-$entries. W_n does not converge strongly to the zero operator but its adjoint does.

A.2 Self-adjoint operators

The notion of adjoint can be extended to linear operators defined on a dense domain. Let X be an Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $T : D(T) \to X$ a linear operator defined on a dense domain $D(T) \subset X$. In this general setting, the adjoint is the linear operator is defined on the domain

$$D(T^*) := \{y \in X : \exists! z \in D(T), \forall x \in D(T) \quad (Tx, y) = (x, z)\}$$

and sends y to the unique $z \in D(T)$ satisfying the relation. This means that, for all $x \in D(T)$ and $y \in D(T^*)$,

$$\langle Tx, y \rangle = \langle x, T^*y \rangle .$$

An interesting properties of adjoints is that they are closed operators.

Lemma A.2.1. *The adjoint operator $T^* : D(T^*) \subset X \to X$ is closed.*

Proof. [11, Proposition 1.6] □

The following Proposition relates the kernel of a densely defined operator to the image of its adjoint.

Proposition A.2.2. *For a linear operator $T : D(T) \to X$, it holds $\ker(T) = \mathrm{Im}(T^*)^\perp$. Moreover, if T is also closed, $\ker(T) = \mathrm{Im}(T^*)^\perp$*

Proof. [11, Proposition 1.13]. □

Definition A.2.3. T is called *self-adjoint* if $T = T^*$ (so $D(T) = D(T^*)$).

An immediate consequence of this definition is that a self-adjoint operator is always closed, and so if it is globally defined, it is also bounded. This result is usually known as the Hellinger-Toeplitz Theorem.

Theorem A.2.4 (Hellinger-Toeplitz)**.** *Let $T : X \to X$ be a self-adjoint operator. Then, T is bounded.*

What can we say about the spectrum of a self-adjoint operator?

Lemma A.2.5. *If T is self-adjoint, then its spectrum is real.*

Proof. [11, Corollary 2.9]. □

What can we say about the resolvent? Is it self-adjoint too?

Remark A.2.6. For a self-adjoint operator T, we have

$$R(\lambda, T)^* = ((\lambda - T)^{-1})^* = ((\lambda - T)^*)^{-1} = (\overline{\lambda} - T^*)^{-1} = R(\overline{\lambda}, T).$$

In particular, the resolvent is self-adjoint if and only if $\lambda \in \mathbb{R} \cap Res(T)$.

Lemma A.2.7. *Let T be self-adjoint, $x \in D(T)$ and $\lambda \in \mathbb{C}$. Then we have that $(x, Tx) \in \mathbb{R}$ and*

$$\|(\lambda - T)x\|^2 = \|(Re(\lambda) - T)x\|^2 + |\Im(\lambda)|^2 \|x\|^2.$$

Proof. By definition of inner product, $(x, y) = \overline{(y, x)}$. However, by self-adjointness

$$(Tx, x) = (x, Tx) = \overline{(Tx, x)} \in \mathbb{R}.$$

Now, let $\lambda = a + ib$, then

$$\begin{aligned}
\|\lambda x - Tx\|^2 &= \langle ax - Tx + ibx, ax - Tx + ibx \rangle = \|(a - T)x\|^2 + 2Re \langle ibx, ax - Tx \rangle + \|ibx\|^2 \\
&= \|(a - T)x\|^2 + 2Re(iab\|x\|^2 - ib \langle x, Tx \rangle) + |b|^2 \|x\|^2 \\
&= \|(a - T)x\|^2 + |b|^2 \|x\|^2.
\end{aligned}$$

□

Let T be a self-adjoint operator and $\lambda \in \mathbb{C}$ with $Re(\lambda) \in \sigma(T)$ and x eigenvector of $Re(\lambda)$. Then we have

$$\|Tx\|^2 = |Re(\lambda)|^2 \|x\|^2 \geq 0 = \|(Re(\lambda) - T)x\|^2.$$

In particular, then it is not true that

$$\|(\lambda - T)x\|^2 \geq \|Tx\|^2 + |\Im(\lambda)|^2 \|x\|^2.$$

This inequality is useful for the proof of Theorem 4.5.18.

A.3 Functional calculus

It is the purpose of this subsection to give the essential tools from spectral theory and in particular, from the continuous functional calculus. Let $\mathcal{B}(\mathbb{R})$ be the space of bounded Borel, complex valued functions and $\mathcal{L}(X)$ be the space of bounded operators on a Hilbert space X. They are both Banach spaces with norms

$$\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)| \qquad \& \qquad \|T\|_{\mathcal{L}(X)} := \sup_{\|x\|=1} \|Tx\|,$$

respectively. Consider now a self-adjoint operator $T : D(T) \subset X \to X$ defined on a dense subset $D(T)$.

Theorem A.3.1. *There exists a unique map $\phi : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(X)$ from the bounded Borel functions on $\mathbb{R}$ into the bounded operators on X satisfying*

1. ***Spectral property:*** *If $Tx = \lambda x$, then $\phi(f)x = f(\lambda)x$ for all $f \in \mathcal{B}(\mathbb{R})$;*
2. ***Linearity:*** *ϕ is an algebraic $*$-homomorphism, i.e. ...*
3. ***Norm continuity:*** *ϕ is norm-continuous, i.e. $\|\phi(f)\|_{\mathcal{L}(X)} \leq \|f\|_\infty$;*
4. ***Strong sequence continuity:*** *for all sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ converging to a function $f_k \in \mathcal{B}(\mathbb{R})$, $\phi(f_k)$ converges strongly to $\phi(f)$.;*
5. ***Naturality:*** *for all sequences $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ converging poitwise to the identity map and satisfying $|h(t)| \leq |t|$ for all $t \in \mathbb{R}$, the following holds:*
$$\lim_{k\to\infty} \phi(f_k)x = Tx \qquad \forall x \in D(T);$$
6. ***Positivity:*** *If $f \geq 0$, then $\phi(f) \geq 0$.*

Proof. [25, Theorem 8.5]. □

For any bounded function $f \in \mathcal{B}(\mathbb{R})$, we will denote by $f(T)$ the bounded operator given by $\phi(f)$. For example, given a Borel subset $B \subset \mathbb{R}$, the characteristic function χ_B is a bounded Borel function that gives rise to the so called *spectral projection* $\chi_B(T)$.

Remark A.3.2. Let T be a self-adjoint operator. Then, $\sigma(T) \subset \mathbb{R}$ is a Borel set. For any $\lambda \in \mathbb{C} - \mathbb{R}$, consider the Borel function $f_\lambda : \sigma(T) \to \mathbb{C}$ given by

$$f_\lambda(x) = \frac{1}{x - \lambda}.$$

By the spectral property, we have that $f_\lambda(T) = R(\lambda, T)$. In particular,

$$\|R(\lambda, T)\| \leq \sup_{x\in\sigma(T)} \left|\frac{1}{x - \lambda}\right| = \frac{1}{\inf_{x\in\sigma(T)} \sqrt{(x - Re(\lambda))^2 + Im(\lambda)^2}} \leq \frac{1}{|Im(\lambda)|}.$$

A.4 Convergence of unbounded self-adjoint operators

The Hellinger-Toeplitz Theorem asserts that every globally defined self-adjoint operator on an Hilbert space is bounded. In turn, it suggests that every non bounded self-adjoint operator must be defined on a dense domain $D(T) \subset X$. Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of self-adjoint unbounded operators. As each term of the sequence may have a completely different domain, the notion of convergence of T_k to a another self-adjoint operator T comes with complications. We will use the convergence of the resolvent in order to solve the problem since it is globally defined and bounded. Recall that self-adjoint operators have real spectrum, therefore any $\lambda \in \mathbb{C} - \mathbb{R}$ is contained in the resolvent set.

Definition A.4.1. We say that the sequence T_k converges to a self-adjoint operator T in the *norm-resolvent sense*, in notation $T_k \xrightarrow{n} T$, if $R(\lambda, T_k)$ converges to $R(\lambda, T)$ in norm for all $\lambda \in \mathbb{C} - \mathbb{R}$. Moreover, we say that $T_k \to T$ in the *strong-resolvent sense*, in notation $T_k \xrightarrow{s} T$, if $R(\lambda, T_k) \to R(\lambda, T)$ strongly for all $\lambda \in \mathbb{C} - \mathbb{R}$.

The following Theorem assures us that we can simply check the resolvent convergence at any fixed $\lambda \in \mathbb{C} - \mathbb{R}$.

Theorem A.4.2. *Let $\lambda_0 \in \mathbb{C} - \mathbb{R}$ be a fixed not real number.*

1. *If $\|R(\lambda_0, T_k) - R(\lambda_0, T)\| \to 0$, then $T_k \xrightarrow{n} T$;*
2. *If $\|R(\lambda_0, T_k)x - R(\lambda_0, T)x\| \to 0$ for all $x \in X$, then $T_k \xrightarrow{s} T$.*

Proof. [25, Theorem 8.19]. □

Example A.4.3. Let $f_k(x) := \frac{x}{k} \in L^2(\mathbb{R})$. We define a sequence of unbounded operators

$$T_k : D(T_k) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

as multiplication by f_k. These operators are clearly not globally defined, because not every function remains in $L^2(\mathbb{R})$ when multiplied by x. However, their domain is dense. By definition, it contains every bounded function $f \in L^2(\mathbb{R})$ such that

$$\|f_k f\|_2^2 = \int_{\mathbb{R}} |f_k f|^2 dx < \infty.$$

Indeed, it contains all the compactly supported functions $\mathcal{C}_c^\infty(\mathbb{R})$, which is a dense subset of $L^2(\mathbb{R})$. Note moreover, that T_k is self-adjoint and that $\sigma(T_k) = \mathbb{R}$ for all $k > 0$. Then, for $\lambda = i$, the resolvent of T_k is given by

$$R(i, T_k)(f)(x) = \frac{k}{ik - x} f(x) = -\frac{kx + ik^2}{x^2 + k^2} f(x)$$

Consider $T = 0$, the zero operator. Clearly, its resolvent is $R(\lambda, T)(f) = -if$. Then,

$$\begin{aligned} \|R(i, T_k)(f) - R(i, T)(f)\|_2^2 &= \int_{\mathbb{R}} \left| -\frac{kx + ik^2}{x^2 + k^2} f(x) + if(x) \right|^2 dx \\ &= \int_{\mathbb{R}} \frac{x^2}{x^2 + k^2} |f(x)|^2 dx \to 0 \qquad for \quad k \to \infty \end{aligned}$$

Therefore $T_k \xrightarrow{s} T$. From the next Theorem, it will be evident that $T_k \xrightarrow{n} T$ doesn't hold.

What are the implications of the resolvent convergence to the spectrum of the limit operator?

Theorem A.4.4. *Suppose that $T_k \xrightarrow{n} T$. If $\lambda \in Res(T)$, then there exists $k_0 \in \mathbb{N}$ such that $\lambda \in Res(T_k)$ for all $k \geq k_0$. Moreover, it is true that*

$$\|R(\lambda, T_k) - R(\lambda, T)\| \to 0.$$

Proof. [25, Theorem 8.23]. □

Corollary A.4.5. *Let $I = (a, b) \subset \mathbb{R}$ be an open interval and suppose that $T_k \xrightarrow{n} T$. If $\partial I \in Res(T)$, then the spectral projection $\chi_{(a,b)}(T_k)$ converges in norm to $\chi_{(a,b)}(T)$.*

Proof. Let $I = (a, b) \subset \mathbb{R}$ be an interval satisfying $\partial I \subset Res(T)$. Suppose that $T_k \xrightarrow{n} T$ and there exists $\lambda \in \sigma(T_k) \cap I$ for all k sufficiently large. By the spectral property of Theorem A.3.1, for any $x_k \in D(T_k)$ satisfying $T_k x_k = \lambda x_k$, then

$$\chi_I(T_k)x_k = \chi_I(\lambda)x_k = x_k.$$

Let $\{x_k\}_{k\geq k_0} \subset X$ be a sequence satisfying $T_k x_k = \lambda x_k$ and $\|x_k\| = 1$. By the second point of Theorem A.4.4,

$$\|\chi_I(T_k) - \chi_I(T)\| = \sup_{\|x\|=1} \|\chi_I(T_k)x - \chi_I(T)x)\| \to 0.$$

Therefore,

$$\|\chi_I(T_k)x_k - \chi_I(T)x_k\| = \|x_k - \chi_I(T)x_k\| \to 0$$

which means that the series $\{x_k\}_k$ converges to a point $x \in D(T)$ satisfying $Tx = \lambda x$. In particular, $\lambda \in \sigma(T)$. □

This Corollary tells us that, if $T_k \xrightarrow{n} T$ and there exists an eigenvalue $\lambda \in \sigma(T_k)$ for all k sufficiently large, then $\lambda \in \sigma(T)$. In other words, the spectrum of T cannot suddenly contract.

Theorem A.4.6. *Let $(a, b) \subset \mathbb{R}$ be an open interval and suppose that $T_k \xrightarrow{s} T$. If $(a, b) \cap \sigma(T_k) = \emptyset$ for all k, then $(a, b) \cap \sigma(T) = \emptyset$.*

Proof. [25, Theorem 8.24]. □

The previous Theorem gives a result in the opposite direction to Corollary A.4.5. In fact, the strong resolvent convergence assures us that the spectrum of the limiting operator T cannot suddenly expand. Indeed, by the previous Theorem, if there is no eigenvalues of any operator in the sequence in one open real interval, there there is no way that the interval contains an eigenvalue of T. It is possible however, that the spectrum of T contracts.

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