Utrecht University

Cluster algebras: Positivity conjecture

A thesis presented for the degree of
Master of Mathematics

Author: Stijn Nugteren

Supervisor: Dr. J.W. van de Leur,
Department of Mathematics,
Utrecht University

Second reader: Dr. G.S.K.S. Heuts
Department of Mathematics,
Utrecht University

July 27, 2021

Abstract

A well-known fact is that the cluster variables of a cluster algebra can be expressed as Laurent polynomials in the variables of any given cluster (*The Laurent phenomenon*). Sergey Fomin and Andrei Zelevinsky conjectured in 2002 that the coefficients of these Laurent polynomials are nonnegative integer linear combinations over the coefficient group of the cluster algebra (*The Positivity conjecture*). Since then special cases of this conjecture have been proven. In this thesis we will investigate this conjecture. We will introduce *coefficient matrices*, which we will use to give a proof of a new and slightly stronger version of the Laurent phenomenon, and we will discuss these coefficient matrices in relation with the Positivity conjecture.

Contents

Introduction

In 2002 Sergey Fomin and Andrei Zelevinsky introduced a class of commutative rings called cluster algebras ([1]). They did this to create an algebraic framework for dual canonical bases and total positivity in semisimple groups. These rings can be found as coordinate rings of algebraic varieties, for instance as homogeneous coordinate rings of Grassmannians. In the last two decades a lot more applications were found in various fields, such as: Teichmüller theory, Poisson geometry and Lie theory.

Cluster algebras are constructed using a set of generators called cluster variables which are grouped into possibly overlapping sets of fixed cardinality m, called clusters (m is called the rank of the cluster algebra). Cluster variables in adjacent clusters are related to one and other using exchange relations. In their paper Fomin and Zelevinsky proved the so-called Laurent phenomenon, which states that any cluster variable, which can initially be viewed as a rational function in the variables of any given cluster, is in fact a Laurent polynomial. Moreover, they stated the positivity conjecture, which states that the coefficients of these Laurent polynomials are positive integer linear combinations over the chosen coefficient group. This conjecture has been proven in various special cases:

1. Philippe Caldero and Markus Reineke proved the positivity conjecture for acyclic cluster algebras in [2];
2. Grégoire Dupont used the result of Caldero and Reineke to prove the positivity conjecture for (coefficient-free) cluster algebras of rank 2 in [3];
3. Kyungyong Lee and Ralf Shiffler proved that the positivity conjecture holds for all skew-symmetric cluster algebras in [4].

In this thesis we examine the structure of cluster variables in arbitrary cluster algebras. We start of by giving a brief introduction to the theory of cluster algebras where we follow the exposition in [1]. In chapter 2 we use a new approach to prove the Laurent phenomenon, using what we call coefficient matrices. After that, in chapter 3, we give an introduction to quiver representation theory and describe the relation to the theory of cluster algebras. In this chapter we also state some results obtained by Caldero, Reineke and others, and we give a proof of the result of Dupont. We will use these results to deduce some interesting properties of the so-called minimal coefficient matrices in chapter 4. In this final chapter we also state some conjectures about minimal coefficient matrices which might lead to a proof of the positivity conjecture for arbitrary cluster algebras.

Notation

Throughout this thesis, we will use the following notation: For any integer $a \in \mathbb{Z}$ we write

$$[a]_+ = \max\{0, a\}.$$

For any integers $m, n \in \mathbb{Z}$ we write

$$[m, n] = \{ \ k \in \mathbb{Z} \mid \ m \leq k \leq n \ \},$$

moreover, we write

$$\mathrm{I}_{k,l} = [0, k] \times [0, l] \qquad \text{and} \qquad \mathrm{I}^{\ulcorner}_{k,l} = \mathrm{I}_{k,l} \setminus \{(k, l)\}.$$

For $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, we use the following definition for the binomial coefficient:

$$\binom{m}{n} = \frac{m_{(n)}}{n!},$$

where $m_{(n)} = \prod_{i=0}^{n-1} (m - i)$ denotes the falling factorial. If $0 \leq n \leq m$ this means we can write

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \binom{m}{m-n}.$$

Moreover, for $n \in \mathbb{Z}_{<0}$ we use the convention that $\binom{m}{n}$ is equal to 0.

1 Cluster algebras

In this chapter we give a brief introduction to cluster algebras. If the reader is already familiar with this subject, he or she may wish to skip this chapter.
This chapter is mainly derived from the paper in which Fomin and Zelevinsky introduce cluster algebras ([1]).

Definition 1.1: Let $N \in \mathbb{Z}_{>0}$, then an N-regular tree $\mathbb{T}$ is a tree containing at least one vertex, whose edges are considered to be undirected, and where each vertex has degree N. For $N = 1$ this means we have that $\mathbb{T}$ consists of two vertices connected with a single undirected edge. For $N > 1$, we have that $\mathbb{T}$ is an infinite undirected graph which can be constructed recursively as follows: We start with a single vertex t_0, and add N new vertices which we each connect with t_0 with an undirected edge. Now the tree contains N vertices of degree 1. For each vertex of degree 1 we add $N - 1$ new vertices to the tree which we each connect to this vertex with an undirected edge. This last step can be repeated endlessly to create the N-regular tree $\mathbb{T}$. We can regard the vertex t_0 as the root of $\mathbb{T}_N$, however it is important to note that any vertex of $\mathbb{T}$ can be regarded as the root due to the fact that the edges are undirected.

Let $N \in \mathbb{Z}_{>0}$ and let I be a finite set of N elements. We let $\mathbb{T}_I$ denote the N-regular tree, whose edges are labelled by the elements of I, such that the N edges emanating from each vertex have distinct labels. Slightly abusing notation, we write $t \in \mathbb{T}_I$ for a vertex t of $\mathbb{T}_I$, i.e., we regard $\mathbb{T}_I$ as the set of all vertices in $\mathbb{T}_I$. Given $t, t' \in \mathbb{T}_I$ and $i \in I$, we write $t \overset{i}{\text{---}} t'$ if the vertices t and t' are connected with an edge labelled i. Finally, if we have $I = [1, N]$ we write $\mathbb{T}_N$ for $\mathbb{T}_I$. We now take $I = [1, N]$.

Definition 1.2 ([1, Definition 2.1, Proposition 4.3])**:** Let I be a finite nonempty set of cardinality N. To each vertex $t \in \mathbb{T}_I$ we associate a cluster of N generators (called cluster variables) $\mathbf{x}(t) = (x_i(t))_{i \in I}$, moreover, we also associate an $N \times N$ integer matrix $B(t) = (b_{ij}(t))_{i,j \in I} = (b_{ij}(t))$ to the vertex t, which we will call the exchange matrix. Finally, let $\mathbb{P}$ be a torsion-free multiplicative abelian group, then for any $t \in \mathbb{T}_I$ we let $\mathbf{p}(t) = (p_i(t))_{i \in I}$ denote an N-tuple of so-called coefficients in the coefficient group $\mathbb{P}$.
Now let $\mathcal{E} = \big((\mathbf{x}(t))_{t \in \mathbb{T}_I}, (B(t))_{t \in \mathbb{T}_I}, (\mathbf{p}(t))_{t \in \mathbb{T}_I}\big)$, then $\mathcal{E}$ is called an exchange pattern on $\mathbb{T}_I$ with coefficients in $\mathbb{P}$ if the following conditions are satisfied:
For any vertex $t \in \mathbb{T}_I$, we have that

1. the matrix $B(t) = (b_{ij})$ is sign-skew-symmetric: For any $i, j \in I$ we have $b_{ij}b_{ji} < 0$ or $b_{ij} = b_{ji} = 0$. (In particular we have $b_{ii} = 0$ for all $i \in I$.)

For any edge $t \overset{k}{\text{---}} t'$ in $\mathbb{T}_I$, we have that

2. the matrix $B(t') = B' = (b'_{ij})$ is obtained from the matrix $B(t) = B = (b_{ij})$ by matrix mutation in direction k (we write $B' = \mu_k(B)$), which means that for any $i, j \in I$ we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise;} \end{cases}$$

(Note that we have $\mu_k^2(B) = B$.)

3. $x_i(t) = x_i(t')$ for all $i \neq k$;

4. $x_k(t)x_k(t') = p_k(t)M_k(t) + p_k(t')M_k(t')$ in $\mathbb{ZP}[x_i(t), x_i(t') \mid i \in I]$ where $\mathbb{ZP}$ denotes the group ring of $\mathbb{P}$ with integer coefficients, where for any $t'' \in \mathbb{T}_I$ we have

$$M_k(t'') = \prod_{i \in I} x_i(t'')^{[b_{ki}(t'')]_+}.$$

Finally, whenever $t_1 \overset{l}{\text{———}} t_2 \overset{k}{\text{———}} t_3 \overset{l}{\text{———}} t_4$ in $\mathbb{T}_I$ (with $k \neq l$), we have that

5. $\dfrac{p_l(t_1)}{p_l(t_2)} = \dfrac{p_l(t_4)}{p_l(t_3)} \cdot \dfrac{p_k(t_2)^{[b_{lk}(t_2)]_+}}{p_k(t_3)^{[b_{lk}(t_3)]_+}}.$

The equalities in conditions 3 and 4 are called exchange relations between the cluster variables of adjacent clusters.
We call an exchange pattern $\mathcal{E}$ coefficient-free if for all $t \in \mathbb{T}_I$ we have $\mathbf{p}(t) = (1)_{i \in I}$. In this case we write $\mathcal{E} = \big((\mathbf{x}(t))_{t \in \mathbb{T}_I}, (B(t))_{t \in \mathbb{T}_I}\big)$.

Remark 1.3: Note that conditions 2 and 5 correspond to the condition (2.7) from Definition 2.1 in [1], which comes down to the following statement: whenever $t_1 \overset{l}{\text{———}} t_2 \overset{k}{\text{———}} t_3 \overset{l}{\text{———}} t_4$ in $\mathbb{T}_I$ (with $k \neq l$), we have

$$\frac{p_l(t_1)M_l(t_1)}{p_l(t_2)M_l(t_2)} = \frac{p_l(t_4)M_l(t_4)}{p_l(t_3)M_l(t_3)} \cdot \left(\frac{p_k(t_2)M_k(t_2)}{x_k^2}\right)^{[b_{lk}(t_2)]_+} \cdot \left(\frac{x_k^2}{p_k(t_3)M_k(t_3)}\right)^{[b_{lk}(t_3)]_+}.$$

For the remainder of this chapter we work with $I = [1, N]$, i.e., we take $\mathbb{T}_I$ to be $\mathbb{T}_N$, however, everything also works over $\mathbb{T}_I$ for an arbitrary set I of cardinality N.

Example 1.4 ([1, Example 2.4])**:** Take $N = 1$. Note that $\mathbb{T}_1$ contains a single edge: $t \overset{1}{\text{———}} t'$. Hence an exchange pattern on $\mathbb{T}_1$ with coefficients in some coefficient group $\mathbb{P}$ must satisfy the single exchange relation

$$x_1(t)x_1(t') = p_1(t) + p_1(t'),$$

and hence is completely determined by the choice of the coefficients $p_1(t)$ and $p_1(t')$.

Example 1.5 ([1, Example 2.5])**:** Now take $N = 2$. We have that $\mathbb{T}_2$ can be written as

$$\cdots \overset{2}{\text{———}} t_0 \overset{1}{\text{———}} t_1 \overset{2}{\text{———}} t_2 \overset{1}{\text{———}} t_3 \overset{2}{\text{———}} t_4 \overset{1}{\text{———}} \cdots.$$

Now note that any coefficient-free exchange pattern on $\mathbb{T}_2$ is completely determined by our choice of $B(t_0)$. Of course we have the trivial example, where $B(t_0)$ is the zero-matrix, in which case all matrices $B(t)$ are zero, and where the cluster variables are given by

$$x_1(t_1) = \frac{2}{x_1(t_0)}, \qquad x_2(t_2) = \frac{2}{x_2(t_0)}, \qquad x_1(t_3) = x_1(t_0) \quad \text{and} \quad x_2(t_4) = x_2(t_0).$$

Note that for any $a, b \in \mathbb{Z}_{>0}$ taking

$$B(t_0) = B = \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix}$$

uniquely determines a coefficient-free exchange pattern on $\mathbb{T}_2$. This follows directly from the fact that we have

$$\mu_1(B) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} = \mu_2(B).$$

Using the exchange relations, the first few cluster variables can be written as

$$x_1(t_1) = \frac{x_2(t_0)^b + 1}{x_1(t_0)}, \qquad x_2(t_2) = \frac{x_1(t_1)^a + 1}{x_2(t_1)} = \frac{(x_2(t_0)^b + 1)^a + x_1(t_0)^a}{x_1(t_0)^a x_2(t_0)},$$

$$\begin{aligned} x_1(t_3) = \frac{x_2(t_2)^b + 1}{x_1(t_2)} &= \frac{x_1(t_0) \cdot \big((x_2(t_0)^b + 1)^a + x_1(t_0)^a\big)^b + x_1(t_0)^{ab+1} x_2(t_0)^b}{x_1(t_0)^{ab} x_2(t_0)^b \cdot \big(x_2(t_0)^b + 1\big)} \\ &= \frac{\prod_{j=1}^{b} \binom{b}{j} (x_2(t_0)^b + 1)^{ja-1} x_1(t_0)^{(b-j)a} + x_1(t_0)^{ab}}{x_1(t_0)^{ab-1} x_2(t_0)^b}. \end{aligned}$$

Next, we will give an example of a coefficient-free exchange pattern on $\mathbb{T}_N$ for general N, however to do this we need some preparation:

Definition 1.6: Let $B = (b_{ij})$ be an $N \times N$ integer matrix, then B is called skew-symmetric if $b_{ij} = -b_{ji}$ for all $i, j \in [1, N]$. We call the matrix B skew-symmetrizable if there exists some diagonal $N \times N$ integer matrix D whose diagonal entries are positive, such that DB is skew-symmetric. In this case D is called the skew-symmetrizing matrix of B.

Proposition 1.7: Let $B = (b_{ij})$ be a skew-symmetrizable $N \times N$ integer matrix, and let D be the skew-symmetrizing matrix of B whose N diagonal entries we denote with $d_1, \dots, d_n$, then for any $k \in [1, N]$ we have that the $N \times N$ integer matrix $\mu_k(B) = B' = (b'_{ij})$ obtained from B by matrix mutation in direction k is skew-symmetrizable with skew-symmetrizing matrix equal to D.

Proof. Let $i, j \in [1, N]$. Then we have by definition

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Hence if $i = k$ or $j = k$, we have

$$d_i b'_{ij} = -d_i b_{ij} = d_j b_{ji} = -d_j b'_{ji},$$

and otherwise we have

$$d_i b'_{ij} = d_i b_{ij} + \frac{|d_i b_{ik}| d_k b_{kj} + d_i b_{ik} |d_k b_{kj}|}{2 d_k} = -d_j b_{ji} - \frac{|d_k b_{ki}| d_j b_{jk} + d_k b_{ki} |d_j b_{jk}|}{2 d_k} = -d_j b'_{ji}.$$

This means B' is indeed skew-symmetrizable with skew-symmetrizing matrix D.

□

Example 1.8: This proposition gives us directly an example of a coefficient-free exchange pattern on $\mathbb{T}_N$ for general N: For any $N \times N$ skew-symmetrizable matrix B and for any $t_0 \in \mathbb{T}_N$ there exists a unique coefficient-free exchange pattern $\mathcal{E} = \big((\mathbf{x}(t))_{t\in\mathbb{T}_N}, (B(t))_{t\in\mathbb{T}_N}\big)$ on $\mathbb{T}_N$, such that $B(t_0) = B$.

Finally, we give an example of exchange pattern on $\mathbb{T}_N$ with coefficients in $\mathbb{Q}(y_1, \dots, y_M)$, the field of rational functions in M variables, for some $M \in \mathbb{Z}_{>0}$:

Example 1.9: Let $(B(t))_{t\in\mathbb{T}_N}$ be a family of $N \times N$ sign-skew-symmetric integer matrices with $B(t') = \mu_k(B(t))$ for any edge $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_N$. Now let $(C(t))_{t\in\mathbb{T}_N}$ be a family of $N \times M$ integer matrices such that for any edge $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_N$ we have that the matrices $C(t) = (c_{ij}(t)) = (c_{ij})$ and $C(t') = (c_{ij}(t')) = (c'_{ij})$ are related by

$$c'_{ij} = \begin{cases} -c_{ij} & \text{if } i = k, \\ c_{ij} + \frac{|b_{ik}(t)|c_{kj} + b_{ik}(t)|c_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Finally, for any $t \in \mathbb{T}_N$, let $\mathbf{p}(t) = (p_i(t))_{i\in I}$ be an N-tuple of nonzero rational functions in $\mathbb{Q}(y_1, \dots, y_M)$, such that for any $k \in [1, N]$ we have

$$p_k(t) = \prod_{j=1}^{M} y_j^{[c_{kj}(t)]_+}.$$

Note that for any edge $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_N$ this means we have

$$\frac{p_k(t)}{p_k(t')} = \prod_{j=1}^{M} y_j^{[c_{kj}(t)]_+} \cdot \prod_{j=1}^{M} y_j^{-[c_{kj}(t')]_+} = \prod_{j=1}^{M} y_j^{c_{kj}(t)}.$$

Now suppose we have $t_1 \overset{l}{\text{———}} t_2 \overset{k}{\text{———}} t_3 \overset{l}{\text{———}} t_4$ in $\mathbb{T}_N$ (with $k \neq l$), then, writing $B(t_2) = (b_{ij})$, $C(t_2) = (c_{ij})$ and $C(t_3) = (c'_{ij})$, we have

$$c'_{lj} = c_{lj} + \frac{|b_{lk}|c_{kj} + b_{lk}|c_{kj}|}{2} = \begin{cases} c_{lj} + b_{lk}\,[c_{kj}]_+ & \text{if } b_{lk} \geq 0, \\ c_{lj} + b_{lk}\,[-c_{kj}]_+ & \text{if } b_{lk} \leq 0, \end{cases}$$

which means we have

$$\frac{p_l(t_1)}{p_l(t_2)} = \frac{p_l(t_4)}{p_l(t_3)} \cdot \frac{p_k(t_2)^{[b_{lk}(t_2)]_+}}{p_k(t_3)^{[b_{lk}(t_3)]_+}}.$$

We conclude that $\mathcal{E} = \big((\mathbf{x}(t))_{t\in\mathbb{T}_N}, (B(t))_{t\in\mathbb{T}_N}, (\mathbf{p}(t))_{t\in\mathbb{T}_N}\big)$ is an exchange pattern on $\mathbb{T}_N$ with coefficients in $\mathbb{Q}(y_1, \dots, y_M)$, which is uniquely determined by the matrices $B(t_0)$ and $C(t_0)$ at a given vertex $t_0 \in \mathbb{T}_N$. Any exchange pattern of this form is called an exchange pattern of geometric type.

We now fix an exchange pattern $\mathcal{E} = \big((\mathbf{x}(t))_{t\in\mathbb{T}_N}, (B(t))_{t\in\mathbb{T}_N}, (\mathbf{p}(t))_{t\in\mathbb{T}_N}\big)$ on $\mathbb{T}_N$ with coefficients in some coefficient group $\mathbb{P}$. Now to any edge $t \overset{k}{\text{——}} t'$ we can associate the following binomial:

$$P = p_k(t) \prod_{i=1}^{N} x_i^{[b_{ki}(t)]_+} + p_k(t') \prod_{i=1}^{N} x_i^{[b_{ki}(t')]_+} \in \mathbb{ZP}[x_1, \ldots, x_N].$$

We now can write

$$x_k(t)x_k(t') = P(\mathbf{x}(t)) = P(\mathbf{x}(t')),$$

and we call P te exchange polynomial associated to the edge $t \overset{k}{\text{——}} t'$. Now note that since $\mathbb{P}$ is torsion-free, the ring $\mathbb{ZP}$ contains no zero divisors and neither does the polynomial ring $\mathbb{ZP}[x_1(t), \cdots, x_N(t)]$ for any vertex $t \in \mathbb{T}_N$, hence to any vertex $t \in \mathbb{T}_N$ we can associate a field $\mathcal{F}(t)$ which is the field of fractions of the polynomial ring $\mathbb{ZP}[x_1(t), \cdots, x_N(t)]$. Now note that for any edge $t \overset{k}{\text{——}} t'$ in $\mathbb{T}_N$ with associated exchange polynomial P, we have a $\mathbb{ZP}$-linear field isomorphism $R_{tt'} : \mathcal{F}(t') \longrightarrow \mathcal{F}(t)$, which is given by

$$R_{tt'}(x_i(t')) = x_i(t) \quad \text{for } i \neq k \qquad\qquad \text{and} \qquad\qquad R_{tt'}(x_k(t')) = \frac{P(\mathbf{x}(t))}{x_k(t)},$$

and the exchange relations from Definition 1.2 give us that $R_{tt'}^{-1} = R_{t't}$. We call these maps the transition maps, and these allow us to identify all fields $\mathcal{F}(t)$ with each other, hence we can regard them as a single field $\mathcal{F}$ which contains all the cluster variables $x_i(t)$ for $i \in [1, N]$ and $t \in \mathbb{T}_N$ in such a way that they satisfy the exchange relations in $\mathcal{F}$. Now we can define a cluster algebra as follows:

Definition 1.10 ([1, Definition 2.3])**:** Let $\mathbb{A}$ be a subring (with unit) in $\mathbb{ZP}$ containing all the coefficients $p_i(t)$ for $i \in [1, N]$ and $t \in \mathbb{T}_N$, then the cluster algebra $\mathcal{A} = \mathcal{A}_{\mathbb{A}}(\mathcal{E})$ of rank N over $\mathbb{A}$ associated to the exchange pattern $\mathcal{E}$ is the $\mathbb{A}$-subalgebra (with unit) in $\mathcal{F}$ generated by all cluster variables $x_i(t)$ for $i \in [1, N]$ and $t \in \mathbb{T}_N$.

For examples of cluster algebras we refer the reader to the discussion after Definition 2.3 in [1].

2 Cluster polynomials and Coefficient matrices

For the remainder of this chapter we fix some $N > 1$, some coefficient group $\mathbb{P}$ and some exchange pattern $\mathcal{E} = \big((\mathbf{x}(t))_{t\in\mathbb{T}_N}, (B(t))_{t\in\mathbb{T}_N}, (\mathbf{p}(t))_{t\in\mathbb{T}_N}\big)$ on $\mathbb{T}_N$ with coefficients in $\mathbb{P}$. Fixing a vertex $t_0 \in \mathbb{T}_N$ and for $u \in [1, N]$ writing $t_0 \overset{u}{\text{———}} t_u$ with P_u denoting the associated exchange polynomial, we will show that for any $t \in \mathbb{T}_N$ and for any $u \in [1, N]$ we can find a Laurent polynomial G in variables $x_1, \ldots, x_N$ with coefficients in $\mathbb{ZP}$ such that $x_u(t) = G(\mathbf{x}(t_0))$. We will show that such a Laurent polynomial G can be written as F/M with M a Laurent monomial

$$M = \prod_{i=1}^{N} x_i^{m_i}$$

for $m_i \in \mathbb{Z}$ and $F \in \mathbb{ZP}[x_1, \ldots, x_N]$ a polynomial not divisible by any of the variables $x_1, \ldots, x_N$. Moreover, for all $v \in [1, N]$ with $m_v \geq 0$ we can find polynomials $F_{v,0}, \ldots, F_{v,m_v} \in \mathbb{ZP}[x_1, \ldots, x_N]$ with $F_{v,i}$ not containing x_v for all $i \in [1, m_v]$, such that we have

$$F = \sum_{i=0}^{m_v} F_{v,i} \cdot x_v^{m_v - i} \cdot P_v^i.$$

A Laurent polynomial of this form we call an <u>M-cluster polynomial</u> associated to the vertex t_0. Note that it is enough to show that for any $u \in [1, N]$ substituting P_u/x_u for x_u in any M-cluster polynomial G associated t_0 gives us an M'-cluster polynomial associated to t_u, where

$$M' = \frac{M}{x_u^{m_u}} \cdot x_u^{m'_u - m_u},$$

with m'_u equal to the largest exponent of x_u in F. To be able to prove this, we will first introduce what we will call 'coefficient matrices'.

2.1 Coefficient matrices

Definition 2.1: For any commutative monoid R, written additively, we denote with $\mathrm{Mat}_2(R)$ the set containing all indexed sets of the form $M = \{m_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ whose elements, which we will call <u>entries</u>, lie in R and of which only finitely many are nonzero, we will call M a <u>matrix</u>. Given a nonzero matrix $M \in \mathrm{Mat}_2(R)$, we refer to the smallest rectangle of entries of M containing all the nonzero entries of M i.e. the set $S = \{m_{x_0+i,y_0+j}\}_{(i,j)\in\mathrm{I}_{m,n}}$ with $x_0, y_0 \in \mathbb{Z}$ maximal and $m, n \in \mathbb{Z}_{\geq 0}$ minimal such that all nonzero entries of M are contained in S, presented as an $(m+1) \times (n+1)$ matrix, as the <u>nonzero part</u> of M. The quadruple $(x_0, y_0, x_0 + m, y_0 + n)$ we call the <u>dimensions</u> of M which we denote with $\dim(M)$, and we call the tuple (x_0, y_0) the <u>origin</u> of M. Given two matrices $M, M' \in \mathrm{Mat}_2(R)$, we define $M'' = M + M' \in \mathrm{Mat}_2(R)$ to be the matrix, whose entries are given by

$$m''_{k,l} = m_{k,l} + m'_{k,l} \qquad \big((k,l) \in \mathbb{Z}^2\big).$$

Clearly $M + M' = M' + M$, and we have an obvious zero matrix: The matrix $0 \in \mathrm{Mat}_2(R)$ with no nonzero entries. This makes $\mathrm{Mat}_2(R)$ into a commutative monoid. If we moreover have a commutative multiplication on R which makes R into a multiplicative semigroup, and such that the multiplication is distributive with respect to the addition on R, then we can also define scalar multiplication on $\mathrm{Mat}_2(R)$: Given a matrix $M \in \mathrm{Mat}_2(R)$ and some $\lambda \in R$, we define $\lambda \cdot M$ to be the matrix $M' \in \mathrm{Mat}_2(R)$ whose entries are given by

$$m'_{k,l} = \lambda \cdot m_{k,l} \qquad \big((k,l) \in \mathbb{Z}^2\big).$$

We will only be interested in the case where R is equal to $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$.

Definition 2.2: Let **Seq** denote the set of all sequences of nonnegative integers $(a_i)_{i\geq 0}$ satisfying:

1. $a_0 \geq a_1$ with $a_0 = a_1$ if and only if $a_0 = 0$;

2. $a_{i+1} = [a_i - (a_{i-1} - a_i)]_+$ for all $i \in \mathbb{Z}_{>0}$.

We denote the zero-sequence in **Seq** with $\mathbf{0}$, and we will denote a sequence $(a_i)_{i\geq 0} \in \mathbf{Seq}$ with $\mathbf{a}$. We say that $\mathbf{a} \in \mathbf{Seq}$ has length $l \in \mathbb{Z}_{\geq 0}$ if $a_l = 0$ and (in case $l \neq 0$) $a_{l-1} > 0$, we will denote the length of $\mathbf{a}$ with $\ell(\mathbf{a})$. Finally, for $c \in \mathbb{Z}$ and $d \in \mathbb{Z}_{>0}$ we write $\mathbf{seq}(c,d)$ for the sequence $\big([c - id]_+\big)_{i \in \mathbb{Z}_{\geq 0}}$ (which is equal to $\mathbf{0}$ if $c \leq 0$).

Definition 2.3: Let $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$ and let $C \in \mathrm{Mat}_2(\mathbb{Z})$ be a nonzero matrix with origin in $\mathbb{Z}^2_{\geq 0}$, then we call C an $(\mathbf{m}, \mathbf{n})$-coefficient matrix if there exist nonzero matrices $D, E \in \mathrm{Mat}_2(\mathbb{Z})$ with the same origin as C, such that for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$c_{k,l} = \sum_{i=0}^{m_l} d_{k-i,l} \binom{m_l}{i} = \sum_{j=0}^{n_k} e_{k,l-j} \binom{n_k}{j}.$$

We call C moreover minimal, if we have $c_{x,y} = 1$ (and therefore $d_{x,y} = e_{x,y} = 1$), and for all $(k,l) \in \mathbb{Z}_{\geq x} \times \mathbb{Z}_{\geq y} \setminus \{(x,y)\}$ we have

$$c_{k,l} = \max\left\{ \sum_{i=x}^{k-1} d_{i,l} \binom{m_l}{k-i}, \quad \sum_{j=y}^{l-1} e_{k,j} \binom{n_k}{l-j} \right\}.$$

In which case we have that C, D and E are matrices in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

We will clarify this definition with some examples, but first we recall two basic results for binomial coefficients:

Remark 2.4: For $m, n, k \in \mathbb{Z}_{\geq 0}$, we have the following equalities:

1. $\binom{m}{k}\binom{m-k}{n-k} = \binom{m}{n}\binom{n}{k}$ if $0 \leq k \leq n \leq m$;

2. $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{m+n}{k}$.

The first equality follows from the following calculation:

$$\binom{m}{k}\binom{m-k}{n-k} = \frac{m!}{k!(m-k)!} \cdot \frac{(m-k)!}{(n-k)!(m-n)!} = \frac{m!}{n!(m-n)!} \cdot \frac{n!}{k!(n-k)!} = \binom{m}{n}\binom{n}{k}.$$

The second equality can be observed by calculating the coefficient of x^k on both sides of the following polynomial equality in $\mathbb{Z}[x]$:

$$(x+1)^n(x+1)^m = (x+1)^{m+n}.$$

Example 2.5: For $m, n, a, b \in \mathbb{Z}_{>0}$ let $\mathbf{m} = \mathbf{seq}(m, a)$ and $\mathbf{n} = \mathbf{seq}(n, b)$. Now let $C \in \mathrm{Mat}_2(\mathbb{Z})$ be a matrix with origin $(0, 0)$ which for any $(k, l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c_{k,l} = \binom{m}{k}\binom{n}{l},$$

then C is an $(\mathbf{m}, \mathbf{n})$-coefficient matrix (this follows from the second part of the remark above). If $m = 6$ and $n = 8$ then the nonzero part of C is given by

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 6 & 48 & 168 & 336 & 420 & 336 & 168 & 48 & 6 \\ 15 & 120 & 420 & 840 & 1050 & 840 & 420 & 120 & 15 \\ 20 & 160 & 560 & 1120 & 1400 & 1120 & 560 & 160 & 20 \\ 15 & 120 & 420 & 840 & 1050 & 840 & 420 & 120 & 15 \\ 6 & 48 & 168 & 336 & 420 & 336 & 168 & 48 & 6 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \end{pmatrix}.$$

Example 2.6: For $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$ let $\mathbf{m} = \mathbf{seq}(m, 1)$ and let $\mathbf{n} = \mathbf{seq}(n, 1)$. Now let $C \in \mathrm{Mat}_2(\mathbb{Z})$ be a matrix with origin $(0, 0)$, which for $(k, l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c_{k,l} = \binom{m}{k}\binom{n_k}{l},$$

then we claim that C is an $(\mathbf{m}, \mathbf{n})$-coefficient matrix. To see this, we first consider the case where we again have $m = 6$ and $n = 8$. In this case the nonzero part of C is given by

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 6 & 42 & 126 & 210 & 210 & 126 & 42 & 6 & 0 \\ 15 & 90 & 225 & 300 & 225 & 90 & 15 & 0 & 0 \\ 20 & 100 & 200 & 200 & 100 & 20 & 0 & 0 & 0 \\ 15 & 60 & 90 & 60 & 15 & 0 & 0 & 0 & 0 \\ 6 & 18 & 18 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the nonzero part of the matrix D associated to C is given by

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 0 & 2 & 14 & 42 & 70 & 70 & 42 & 6 & 0 \\ 0 & 0 & 1 & 6 & 15 & 20 & 15 & 0 & 0 \end{pmatrix}.$$

Now to prove our claim, we note that for $(k,l) \in \mathbb{Z}^2_{>0}$ with $k \leq n$ and $l \leq m$ we can write

$$\begin{aligned}
\binom{m}{k}\binom{n_k}{l} &= \binom{m}{k}\binom{n-k}{l} = \sum_{j=0}^{l}\binom{n-m}{j}\binom{m}{k}\binom{m-k}{l-j} \\
&= \sum_{j=0}^{l}\binom{n-m}{j}\binom{m}{k}\binom{m-k}{m-(l-j)-k} = \sum_{j=0}^{l}\binom{n-m}{j}\binom{m}{m-l+j}\binom{m-l+j}{k} \\
&= \sum_{i=0}^{k}\left(\sum_{j=0}^{l}\binom{n-m}{j}\binom{m}{m-l+j}\binom{j}{i}\right)\binom{m-l}{k-i}.
\end{aligned}$$

Now note that for $(k,l) \in \mathbb{Z}^2_{>0}$ with $k \leq n$ and $l \leq m$ we can write

$$\begin{aligned}
\sum_{j=0}^{l}\binom{n-m}{j}\binom{m}{m-l+j}\binom{j}{k} &= \sum_{j=k}^{l}\binom{n-m}{j}\binom{j}{k}\binom{m}{l-j} \\
&= \sum_{j=k}^{l}\binom{n-m}{k}\binom{n-m-k}{j-k}\binom{m}{l-j} \\
&= \binom{n-m}{k}\sum_{j'=0}^{l-k}\binom{n-m-k}{j'}\binom{m}{l-k-j'} \\
&= \binom{n-m}{k}\binom{n-k}{l-k}.
\end{aligned}$$

This means that we can find $D \in \mathrm{Mat}_2(\mathbb{Z})$ with origin $(0,0)$, and which for $(k,l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$d_{k,l} = \begin{cases} \binom{n-m}{k}\binom{n-k}{l-k} = \binom{n-m}{k}\binom{n-k}{n-l} & \text{if } k \leq n \text{ and } l \leq m; \\ \binom{m}{k}\binom{n_k}{l} & \text{otherwise,} \end{cases}$$

such that for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$\binom{m}{k}\binom{n_k}{l} = \sum_{i=0}^{k} d_{i,l}\binom{m_l}{k-i}.$$

We encourage the reader to verify that this is enough to prove that C is actually a minimal coefficient matrix. We make the argument needed to prove this claim precise in the following proposition:

Proposition 2.7: Let $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$, and let $C, D, E \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ be three nonzero matrices which all three have their origin $\mathbb{Z}^2_{\geq 0}$, such that for all $(k,l) \in \mathbb{Z}_{\geq 0}$ we can write

$$c_{k,l} = \sum_{i=0}^{k} d_{i,l} \binom{m_l}{k-i} = \sum_{j=0}^{l} e_{k,j} \binom{n_k}{l-j}.$$

Moreover, assume that there exists precisely one pair $(x,y) \in \mathbb{Z}^2_{\geq 0}$ such that $d_{x,y} \neq 0$ and $e_{x,y} \neq 0$. Then $d_{x,y} = e_{x,y}$ and $C = d_{x,y} \cdot C'$, where C' is a minimal $(\mathbf{m},\mathbf{n})$-coefficient matrix with origin (x,y).

Proof. We clearly have that $c_{k,l} = 0$ for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ satisfying $k < x$ and/or $l < y$, which automatically means we have $d_{x,y} = e_{x,y}$. Moreover, for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ with $k \geq x$ and $l \geq y$ such that $(k,l) \neq (x,y)$ we must have

$$c_{k,l} = \max \left\{ \sum_{i=x}^{k-1} d_{i,l} \binom{m_l}{k-i}, \quad \sum_{j=y}^{l-1} e_{k,j} \binom{n_k}{l-j} \right\}.$$

If $d_{x,y} = 1 = e_{x,y}$ this already means that C is a minimal $(\mathbf{m},\mathbf{n})$-coefficient matrix with origin (x,y). If $d_{x,y} = a = e_{x,y}$ for some $a \in \mathbb{Z}_{>1}$, then we can use the above equality to prove, using induction, that $a \mid c_{k,l}$ for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$.

□

We end with two concrete examples of coefficient matrices, one of which is zero in its origin:

Example 2.8: Let $\mathbf{m} = \mathbf{seq}(6,1)$ and $\mathbf{n} = \mathbf{seq}(8,2)$, then the matrix $C \in \mathrm{Mat}_2(\mathbb{Z})$ with origin $(0,0)$ whose non-zero part is given by

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 6 & 40 & 114 & 180 & 170 & 96 & 30 & 4 & 0 \\ 15 & 80 & 176 & 204 & 131 & 44 & 6 & 0 & 0 \\ 20 & 80 & 124 & 92 & 32 & 4 & 0 & 0 & 0 \\ 15 & 40 & 36 & 12 & 1 & 0 & 0 & 0 & 0 \\ 6 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

is an $(\mathbf{m},\mathbf{n})$-coefficient matrix, where the nonzero parts of the associated matrices $D, E \in \mathrm{Mat}_2(\mathbb{Z})$ are respectively given by

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 0 & 0 & 2 & 12 & 30 & 40 & 30 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 6 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 & 0 \\ 15 & 20 & 6 & 0 & 0 \\ 20 & 40 & 24 & 4 & 0 \\ 15 & 40 & 36 & 12 & 1 \\ 6 & 8 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly C is minimal. Another example of an $(\mathbf{m},\mathbf{n})$-coefficient matrix is given by taking the nonzero part of C equal to

$$\begin{pmatrix} 0 & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 1 & 6 & 32 & 122 & 270 & 346 & 256 & 102 & 17 & 0 \\ 6 & 24 & 48 & 198 & 582 & 804 & 516 & 126 & 0 & 0 \\ 15 & 65 & 85 & 142 & 578 & 835 & 364 & 0 & 0 & 0 \\ 20 & 105 & 220 & 38 & 210 & 307 & 0 & 0 & 0 & 0 \\ 15 & 96 & 318 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 46 & 212 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

now the nonzero parts of the associated matrices $D, E \in \mathrm{Mat}_2(\mathbb{Z})$ are respectively given by

$$\begin{pmatrix} 0 & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 1 & 1 & 0 & 38 & 158 & 276 & 256 & 102 & 17 & 0 \\ 0 & 9 & 0 & 0 & 210 & 528 & 516 & 126 & 0 & 0 \\ 0 & 0 & 53 & 0 & 0 & 307 & 364 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 17 & 0 & 0 & 0 \\ 6 & 0 & 12 & 126 & 0 & 0 \\ 15 & 35 & 0 & 107 & 364 & 0 \\ 20 & 105 & 220 & 38 & 210 & 307 \\ 15 & 96 & 318 & 0 & 0 & 0 \\ 6 & 46 & 212 & 0 & 0 & 0 \\ 1 & 9 & 53 & 0 & 0 & 0 \end{pmatrix}.$$

Note that in these examples we talk about the matrices D and E associated to C, that this unambiguous follows from the following proposition:

Proposition 2.9: For any $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$, given an $(\mathbf{m}, \mathbf{n})$-coefficient matrix C, there exist unique nonzero matrices $D, E \in \mathrm{Mat}_2(\mathbb{Z})$ with the same origin as C such that for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$c_{k,l} = \sum_{i=0}^{m_l} d_{k-i,l} \binom{m_l}{i} = \sum_{j=0}^{n_k} e_{k,l-j} \binom{n_k}{j}.$$

We will denote these matrices with $D(C)$ and $E(C)$ respectively.

Proof. Let $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$, let $C \in \mathrm{Mat}_2(\mathbb{Z})$ be an $(\mathbf{m}, \mathbf{n})$-coefficient matrix with origin $(x, y) \in \mathbb{Z}^2_{\geq 0}$, and let $D, E \in \mathrm{Mat}_2(\mathbb{Z})$ be nonzero matrices with origin (x, y) such that for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$c_{k,l} = \sum_{i=0}^{m_l} d_{k-i,l} \binom{m_l}{i} = \sum_{j=0}^{n_k} e_{k,l-j} \binom{n_k}{j}.$$

Then in particular, we have $d_{x,y} = e_{x,y} = c_{x,y}$. Now for any $(k, l) \in \mathbb{Z}_{\geq x} \times \mathbb{Z}_{\geq y} \setminus \{(x, y)\}$, we have

$$d_{k,l} = c_{k,l} - \sum_{i=1}^{m_l} d_{k-i,l} \binom{m_l}{i} \qquad \text{and} \qquad e_{k,l} = c_{k,l} - \sum_{j=1}^{n_k} e_{k,l-j} \binom{n_k}{j}.$$

Hence if we know that for all $(k', l') \in \mathrm{I}^{\ulcorner}_{k,l}$ the values of $d_{k',l'}$ and $e_{k',l'}$ are uniquely determined by C, then the values of $d_{k,l}$ and $e_{k,l}$ are also uniquely determined by C, hence by induction on k and l we have that D and E are uniquely determined by C.

□

Remark 2.10: From this proposition it follows that for any $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$ and for any $(x, y) \in \mathbb{Z}^2_{\geq 0}$ there exists a unique minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix with origin (x, y), which we therefore will call the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix with origin (x, y), moreover, if we do not specify an origin, we take it to be $(0, 0)$.

This remark gives rise to the following proposition:

Proposition 2.11: For $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$, let C be an $(\mathbf{m}, \mathbf{n})$-coefficient matrix. For any $(x, y) \in \mathbb{Z}^2_{\geq 0}$, let $C^{x,y}$ denote the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix with origin (x, y). Then there exists a unique matrix $S \in \mathrm{Mat}_2(\mathbb{Z})$ with same origin as C, such that for any $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$c_{k,l} = \sum_{(x,y) \in \mathbb{Z}^2_{\geq 0}} s_{x,y} c^{x,y}_{k,l}.$$

We will denote this matrix with $S(\mathbf{m}, \mathbf{n})(C)$.

Proof. Note that constructing such a matrix $S \in \mathrm{Mat}_2(\mathbb{Z})$ is straightforward: For any $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we take

$$s_{k,l} = c_{k,l} - \sum_{(x,y) \in \mathrm{I}^{\ulcorner}_{k,l}} s_{x,y} c^{x,y}_{k,l}.$$

We have that S lies in $\mathrm{Mat}_2(\mathbb{Z})$ because C lies in $\mathrm{Mat}_2(\mathbb{Z})$, and $m_{\ell(\mathbf{m})+l} = 0$ and $n_{\ell(\mathbf{n})+k} = 0$ for all $l, k \in \mathbb{Z}_{\geq 0}$.

Now suppose that S is not unique, then there exists some matrix $T \in \mathrm{Mat}_2(\mathbb{Z})$ with origin equal to the origin of C, not equal to S, such that for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$c_{k,l} = \sum_{(x,y) \in \mathbb{Z}^2_{\geq 0}} t_{x,y} c^{x,y}_{k,l}.$$

This means we can find $(k, l) \in \mathbb{Z}^2_{\geq 0}$ such that $s_{k,l} \neq t_{k,l}$ and such that for all $(x, y) \in \mathrm{I}^{\ulcorner}_{k,l}$ we have $s_{x,y} = t_{x,y}$. Now note that by definition of the minimal coefficient matrix we have $c^{x,y}_{k,l} = 0$ for all $(x, y) \in \mathbb{Z}^2_{\geq 0} \setminus \mathrm{I}_{k,l}$, hence we must have

$$t_{k,l} = c_{k,l} - \sum_{(x,y) \in \mathrm{I}^{\ulcorner}_{k,l}} t_{x,y} c^{x,y}_{k,l} = s_{k,l},$$

which gives us a contradiction.

□

Definition 2.12: Given $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$ and an $(\mathbf{m}, \mathbf{n})$-coefficient matrix C, then we say C is positive if $S(\mathbf{m}, \mathbf{n})(C)$ lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

We display the proposition above with an example:

Example 2.13: For $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$ let $\mathbf{m} = \mathbf{seq}(m, 1)$ and let $\mathbf{n} = \mathbf{seq}(n, 1)$. We saw in

Example 2.6 that the minimal positive $(\mathbf{m}, \mathbf{n})$-coefficient matrix C is given by

$$c_{k,l} = \binom{m}{k}\binom{n_k}{l} \qquad \big((k,l) \in \mathbb{Z}^2_{\geq 0}\big).$$

Hence we have that C in this case is also an $(\mathbf{m}', \mathbf{n})$-coefficient matrix, now for $\mathbf{m}' = \mathbf{seq}(m-1, 1)$. Taking $m = 6$ and $n = 8$ this can be seen from the fact that the nonzero part of the minimal $(\mathbf{m}', \mathbf{n})$-coefficient matrix with origin $(0, 0)$ is given by

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 5 & 35 & 105 & 175 & 175 & 105 & 35 & 5 & 0 \\ 10 & 60 & 150 & 200 & 150 & 60 & 10 & 0 & 0 \\ 10 & 50 & 100 & 100 & 50 & 10 & 0 & 0 & 0 \\ 5 & 20 & 30 & 20 & 5 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the nonzero part of the minimal $(\mathbf{m}', \mathbf{n})$-coefficient matrix with origin $(1, 0)$ is given by

$$\begin{pmatrix} 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ 5 & 30 & 75 & 100 & 75 & 30 & 5 & 0 \\ 10 & 50 & 100 & 100 & 50 & 10 & 0 & 0 \\ 10 & 40 & 60 & 40 & 10 & 0 & 0 & 0 \\ 5 & 15 & 15 & 5 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding these two matrices together indeed gives us C, whose nonzero part is equal to

$$\begin{pmatrix} 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ 6 & 42 & 126 & 210 & 210 & 126 & 42 & 6 & 0 \\ 15 & 90 & 225 & 300 & 225 & 90 & 15 & 0 & 0 \\ 20 & 100 & 200 & 200 & 100 & 20 & 0 & 0 & 0 \\ 15 & 60 & 90 & 60 & 15 & 0 & 0 & 0 & 0 \\ 6 & 18 & 18 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We end this discussion of coefficient matrices with two final remarks:

Remark 2.14: For $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$, let C be an $(\mathbf{m}, \mathbf{n})$-coefficient matrix with $\dim(C) = (x, y, K, L)$. Now let $C^T \in \mathrm{Mat}_2(\mathbb{Z})$ be the matrix which for $(k, l) \in \mathbb{Z}^2$ is given by $c^T_{k,l} = c_{l,k}$, then C^T is an $(\mathbf{n}, \mathbf{m})$-coefficient matrix with $\dim(C^T) = (y, x, L, K)$. In particular, if C is minimal, then C^T is also minimal. In general, for matrices $M, M^T \in \mathrm{Mat}_2(\mathbb{Z})$ with $m^T_{k,l} = m_{l,k}$ for all $(k, l) \in \mathbb{Z}^2$, we call the matrix M^T the transpose of M, and we have $(M^T)^T = M$.

Remark 2.15: Let C be an $(\mathbf{m}, \mathbf{n})$-coefficient matrix, and let C' be an $(\mathbf{m}', \mathbf{n}')$-coefficient matrix with origin $(x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, such that $m'_{y+i} = m_i$ and $n'_{x+i} = n_i$ for all $i \in \mathbb{Z}_{\geq 0}$, then for any $(k, l) \in \mathbb{Z}^2$ we have

$$c_{k,l} = c'_{x+k,y+l}.$$

Moreover, if C is minimal, then C' is also minimal.

2.2 Cluster polynomials and the Laurent phenomenon

We now return to our discussion about the structure of cluster variables in $\mathcal{E}$ to get an understanding of how these matrices will be used to get the desired result. Take some $t_0 \in \mathbb{T}_N$ and let $u, v \in [1, N]$ be distinct, then we will look at the structure of cluster variables belonging to vertices which can be connected to t_0 with a sequence of edges labelled u or v. To do this we define a so-called 'minimal cluster polynomial':

Definition 2.16: Given some vertex $t \in \mathbb{T}_N$, write $B(t) = (b_{ij})$, let $u, v \in [1, N]$ distinct, and write $a = |b_{vu}|$ and $b = |b_{uv}|$. Now let

$$P_u = p_{u,1}M_{u,1} + p_{u,2}M_{u,2} \qquad \text{and} \qquad P_v = p_{v,1}M_{v,1} + p_{v,2}M_{v,2}$$

be the exchange polynomials in $\mathbb{ZP}[x_1, \ldots, x_N]$ associated to the edges emanating from t labelled u and v respectively, where $p_{\_,\_}$ denotes an element in $\mathbb{P}$ and $M_{\_,\_}$ denotes a monomial in the variables $\{x_1, \ldots, x_N\}$. In what follows we will use similar notation for exchange polynomials without further explanation. If $a \neq 0$ and $b \neq 0$, we assume we have $x_v^b \mid M_{u,1}$ and $x_u^a \mid M_{v,1}$. Then for $m, n \in \mathbb{Z}$ and some polynomial $F \in \mathbb{ZP}[x_1, \ldots, x_N]$ we call the Laurent polynomial

$$G = \frac{F}{x_u^m x_v^n}$$

a <u>minimal $x_u^m x_v^n$-cluster polynomial</u> associated to t if there exists some $p \in \mathbb{P}$ such that one of the following statements holds:

1. $a = b = 0$ and $F = p \cdot P_u^{[m]_+} P_v^{[n]_+}$;
2. $a \neq 0$, $b \neq 0$ and

$$\begin{aligned} F &= \frac{p}{M} \cdot \sum_{k=0}^{K} \sum_{l=0}^{L} c_{k,l} (p_{u,1}M_{u,1})^k (p_{u,2}M_{u,2})^{K-k} (p_{v,1}M_{v,1})^l (p_{v,2}M_{v,2})^{L-l} \\ &= \frac{p \cdot (p_{u,2}M_{u,2})^K (p_{v,2}M_{v,2})^L}{M} \cdot \sum_{k=0}^{K} \sum_{l=0}^{L} c_{k,l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k \left(\frac{p_{v,1}M_{v,1}}{p_{v,2}M_{v,2}}\right)^l \end{aligned}$$

   where M is a monomial in the variables $\{x_1, \ldots, x_N\} \backslash \{x_u, x_v\}$ such that non of these variables divide F, and where $C \in \mathrm{Mat}_2(\mathbb{Z})$ is the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix with $\dim(C) = (0, 0, K, L)$, for $\mathbf{m} = \mathbf{seq}(m, a)$ and $\mathbf{n} = \mathbf{seq}(n, b)$. Writing $D = D(C)$ and $E = E(C)$, this means we have

$$\begin{aligned} F &= \frac{p \cdot (p_{v,2}M_{v,2})^L}{M} \cdot \sum_{l=0}^{L} \sum_{k=0}^{K-m_l} d_{k,l} \left(\frac{p_{v,1}M_{v,1}}{p_{v,2}M_{v,2}}\right)^l (p_{u,1}M_{u,1})^k (p_{u,2}M_{u,2})^{K-m_l-k} \cdot P_u^{m_l} \\ &= \frac{p \cdot (p_{u,2}M_{u,2})^K}{M} \cdot \sum_{k=0}^{K} \sum_{l=0}^{L-n_k} e_{k,l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k (p_{v,1}M_{v,1})^l (p_{v,2}M_{v,2})^{L-n_k-l} \cdot P_v^{n_k}. \end{aligned}$$

We say G is a reduced $x_u^m x_v^n$-cluster polynomial associated to t if F satisfies one of the statements above, but in the second case the coefficient matrix C does not necessarily need to be a minimal. In particular, any minimal $x_u^m x_v^n$-cluster polynomial is a reduced $x_u^m x_v^n$-cluster polynomial.

The following example displays a simple class of minimal cluster polynomials:

Example 2.17: Assume $N = 2$ and $\mathbb{P} = 1$ is the trivial group. Now fix some vertex $t_0 \in \mathbb{T}_2$ and let $a, b \in \mathbb{Z}_{>0}$ such that the exchange polynomials associated to the edges $t_0 \overset{1}{\text{——}} t_1$ and $t_0 \overset{2}{\text{——}} t_2$ are respectively given by $P_1 = x_2^b + 1$ and $P_2 = x_1^a + 1$. Now take $m, n \in \mathbb{Z}_{\geq 0}$ and let C be the minimal $(\mathbf{seq}(m,a), \mathbf{seq}(n,b))$-coefficient matrix, then the Laurent polynomial

$$\frac{1}{x_1^m x_2^n} \cdot \sum_{(k,l) \in \mathbb{Z}_{\geq 0}^2} c_{k,l} x_1^{la} x_2^{kb}$$

is a (actually 'the') minimal $x_1^m x_2^n$-cluster polynomial associated to t_0.

We now recall the definition of a 'general' cluster polynomial: Let $t_0 \in \mathbb{T}_N$, and let M be a Laurent monomial in the variables $x_1, \ldots, x_N$. Writing

$$M = \prod_{i=1}^{N} x_i^{m_i} \qquad \big(m_i \in \mathbb{Z}\big),$$

we recall that an M-cluster polynomial G associated to the vertex t_0 is a Laurent polynomial in the variables $x_1, \ldots, x_N$ with coefficients in $\mathbb{ZP}$ which can be written as a fraction F/M, where $F \in \mathbb{ZP}[x_1, \ldots, x_N]$ is a polynomial not divisible by any of the variables $x_1, \ldots, x_N$. Moreover, for all $v \in [1, N]$ with $m_v \geq 0$ we can find polynomials $F_{v,0}, \ldots, F_{v,m_v} \in \mathbb{ZP}[x_1, \ldots, x_N]$ with $F_{v,i}$ not containing x_v for all $i \in [1, m_v]$, such that we have

$$F = \sum_{i=0}^{m_v} F_{v,i} \cdot x_v^{m_v - i} \cdot P_v^i.$$

Here, for $v \in [1, N]$, P_v denotes the exchange polynomial associated to the edge $t_0 \overset{v}{\text{——}} t_v$.

Any cluster polynomial associated to some vertex in $\mathbb{T}_N$ can be written by definition as a fraction of a polynomial in $\mathbb{ZP}[x_1, \ldots, x_N]$ and a Laurent monomial in the variables $x_1, \ldots, x_N$ as above. Hence, when we talk about a fraction F/M as being an M-cluster polynomial associated to some vertex in $\mathbb{T}_N$, we mean that M is a Laurent monomial in the variables $x_1, \ldots, x_N$ and that F is a polynomial in $\mathbb{ZP}[x_1, \ldots, x_N]$ satisfying the properties above.

Now consider the following two remarks which relate the general cluster polynomial with minimal/reduced cluster polynomials:

Remark 2.18: Let $u, v \in [1, N]$ be distinct, and let $t \in \mathbb{T}_N$ and let $m, n \in \mathbb{Z}$, then any reduced (and in particular minimal) $x_u^m x_v^n$-cluster polynomial associated to t is an $x_u^m x_v^n$-cluster polynomial associated to t.

Remark 2.19: Let $t \in \mathbb{T}_N$ and for a given Laurent monomial

$$M = \prod_{i=1}^{N} x_i^{m_i} \qquad (m_i \in \mathbb{Z})$$

let $G = F/M$ be an M-cluster polynomial associated to t. Then for any distinct $u, v \in [1, N]$ we can write

$$G = \frac{x_u^{m_u} x_v^{m_v}}{M} \left(\sum_{i=0}^{[m_u]_+} \sum_{j=0}^{[m_v]_+} F_{i,j} G_{i,j} \right),$$

where for $(i, j) \in [0, [m_u]_+] \times [0, [m_v]_+]$ we have that $G_{i,j}$ is a minimal $x_u^i x_v^j$-cluster polynomial, and $F_{i,j}$ is a polynomial in $\mathbb{ZP}[x_1, \ldots, x_N]$ such that it does not contain the variable x_u if $i > 0$ and it does not contain the variable x_v if $j > 0$.

In line with our discussion at the start of this chapter we have the following result:

Theorem 2.20: Let $u, v \in [1, N]$ be distinct, let $P_u \in \mathbb{ZP}[x_1, \ldots, x_N]$ be the exchange polynomial associated to a given edge $t \overset{u}{\text{——}} t'$ in $\mathbb{T}_N$, and, for $m, n \in \mathbb{Z}$ (not necessarily nonnegative), let $G = F/x_u^m x_v^n$ be a reduced $x_u^m x_v^n$-cluster polynomial associated to t, then substituting P_u/x_u for x_u in G gives us a reduced $x_u^{m'-m} x_v^n$-cluster polynomial associated to t', where m' is the largest exponent of x_u in F.

We are not yet ready to give a full proof of this theorem, however we can give an argument which reduces the theorem to a statement about coefficient matrices:

Let $u, v \in [1, N]$ be distinct, and consider the edges $t_1 \overset{v}{\text{——}} t_2 \overset{u}{\text{——}} t_3 \overset{v}{\text{——}} t_4$ in $\mathbb{T}_N$, with associated exchange polynomials P_v, P_u and P'_v in $\mathbb{ZP}[x_1, \ldots, x_N]$ respectively. Write $B(t_2) = (b_{ij})$ and $B(t_3) = (b'_{ij})$, and let $a = |b_{vu}|$ and $b = |b_{uv}|$. Now let G be some reduced $x_u^m x_v^n$-cluster polynomial associated to t_2, and let H be the Laurent polynomial obtained from G by substituting P_u/x_u for x_u. If we have $a = b = 0$, then, by definition, we have

$$G = \frac{p \cdot P_u^{[m]_+} P_v^{[n]_+}}{x_u^m x_v^n}$$

for some $p \in \mathbb{P}$. This means we have

$$H = \begin{cases} \dfrac{p \cdot x_u^m P_v^{[n]_+}}{x_v^n} & \text{if } m \geq 0; \\[2ex] \dfrac{p \cdot P_u^{|m|} P_v^{[n]_+}}{x_u^{|m|} x_v^n} & \text{if } m < 0. \end{cases}$$

Now note that $b_{vu} = 0$ implies we have $b_{vj} = b'_{vj}$ for all $j \in [1, N]$, hence we have $M_v(t_1) = M_v(t_4)$ and $M_v(t_2) = M_v(t_3)$. Using condition five from Definition 1.2 we have

$$\frac{P_v}{p_v(t_2)} = \frac{P'_v}{p_v(t_3)}.$$

This means that H is a reduced $x_u^{-m}x_v^n$-cluster polynomial associated to t_3.

Next, assume we have $a \neq 0$ (and hence $b \neq 0$). If $n \leq 0$ we directly have that H is a reduced $x_u^{-m}x_v^n$-cluster polynomial associated to t_3. Now assume we have $n > 0$ and write

$$P_v = p_{v,1}M_{v,1} + p_{v,2}M_{v,2}, \qquad P_u = p_{u,1}M_{u,1} + p_{u,2}M_{u,2} \quad \text{and} \quad P'_v = p'_{v,1}M'_{v,1} + p'_{v,2}M'_{v,2},$$

such that $x_u^a \mid M_{v,1}$, $x_v^b \mid M_{u,1}$ and $x_u^a \mid M'_{v,1}$. Using Remark 1.3 we have

$$\frac{p_{v,1}M_{v,1}}{p_{v,2}M_{v,2}} = \frac{p'_{v,2}M'_{v,2}}{p'_{v,1}M'_{v,1}} \cdot \frac{x_u^{2a}}{p_{u,2}^a M_{u,2}^a}. \tag{1}$$

In particular this means we have

$$M'_{v,1} = \frac{x_u^a \cdot M_{v,2}}{M_{\text{gcd}}} \qquad \text{and} \qquad M'_{v,2} = \frac{M_{v,1}M_{u,2}^a}{x_u^a \cdot M_{\text{gcd}}},$$

where $M_{\text{gcd}} = \gcd\left(M_{v,2}, M_{u,2}^a\right)$. Now let $\mathbf{m} = \mathbf{seq}(m,a)$, $\mathbf{n} = \mathbf{seq}(n,b)$, then for some $(\mathbf{m},\mathbf{n})$-coefficient matrix C with $\dim(C) = (0,0,K,L)$, writing $D = D(C)$, we have

$$G = \frac{p \cdot (p_{v,2}M_{v,2})^L}{M \cdot x_u^m x_v^n} \cdot \sum_{l=0}^{L} \sum_{k=0}^{K-m_l} d_{k,l} \left(\frac{p_{v,1}M_{v,1}}{p_{v,2}M_{v,2}}\right)^l (p_{u,1}M_{u,1})^k (p_{u,2}M_{u,2})^{K-m_l-k} \cdot P_u^{m_l},$$

for some $p \in \mathbb{P}$ and some monomial M in the variables $\{x_1, \dots, x_N\} \setminus \{x_u, x_v\}$. Writing $m' = La - m$, let $\mathbf{m'} = \mathbf{seq}(m', a)$. Now we have

$$\begin{aligned} H &= \frac{p \cdot (p_{v,2}M_{v,2})^L}{M \cdot x_v^n} \cdot \sum_{l=0}^{L} \sum_{k=0}^{K-m_l} d_{k,l} \left(\frac{p_{v,1}M_{v,1}}{x_u^a \cdot p_{v,2}M_{v,2}}\right)^l (p_{u,1}M_{u,1})^k (p_{u,2}M_{u,2})^{K-m_l-k} \cdot P_u^{m'_{L-l}} \cdot x_u^{m-la} \\ &= \frac{p \cdot (p_{v,2}M_{v,2})^L}{M \cdot x_u^{m'} x_v^n} \cdot \sum_{l=0}^{L} \sum_{k=0}^{K-m_l} d_{k,l} \left(\frac{p_{v,1}M_{v,1}}{x_u^a \cdot p_{v,2}M_{v,2}}\right)^l (p_{u,1}M_{u,1})^k (p_{u,2}M_{u,2})^{K-m_l-k} \cdot P_u^{m'_{L-l}} \cdot x_u^{(L-l)a}. \end{aligned}$$

Let $C' \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ which for $(k,l) \in \mathbb{Z}^2$ is given by

$$c'_{k,l} = \sum_{i=0}^{k} d_{i,L-l} \binom{m'_l}{k-i},$$

then $\dim(C) = (0,0,K',L)$ for some $K' \geq K - m$. Now, using the fact that we have

$$m'_{L-l} - m_l - la = [la - m]_+ - [m - la]_+ - la = -m, \tag{2}$$

we can write

$$H = \frac{p \cdot (p_{u,2}M_{u,2})^{K-m}(p_{v,2}M_{v,2})^L}{M \cdot x_u^{m'} x_v^n} \cdot \sum_{k=0}^{K'} \sum_{l=0}^{L} c'_{k,L-l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k \left(\frac{p_{v,1}M_{v,1}}{p_{v,2}M_{v,2}} \cdot \frac{p_{u,2}^a M_{u,2}^a}{x_u^a}\right)^l \cdot x_u^{(L-l)a}.$$

Using equality (1), we now have

$$
\begin{aligned}
H &= \frac{p \cdot (p_{u,2}M_{u,2})^{K-m}(p_{v,2}M_{v,2})^L}{M \cdot x_u^{m'} x_v^n} \cdot \sum_{k=0}^{K'} \sum_{l=0}^{L} c'_{k,L-l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k \left(\frac{x_u^a \cdot p'_{v,2}M'_{v,2}}{p'_{v,1}M'_{v,1}}\right)^l \cdot x_u^{(L-l)a} \\
&= \frac{p \cdot (p_{u,2}M_{u,2})^{K-m}(p_{v,2}M_{v,2})^L}{M \cdot x_u^{m'} x_v^n} \cdot \left(\frac{x_u^a \cdot p'_{v,2}M'_{v,2}}{p'_{v,1}M'_{v,1}}\right)^L \cdot \sum_{k=0}^{K'} \sum_{l=0}^{L} c'_{k,L-l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k \left(\frac{p'_{v,1}M'_{v,1}}{p'_{v,2}M'_{v,2}}\right)^{L-l} \\
&= \frac{p \cdot (p_{u,2}M_{u,2})^{K-m}}{M \cdot x_u^{m'} x_v^n} \cdot \left(\frac{p_{v,1}M_{v,1}p_{u,2}^a M_{u,2}^a}{x_u^a}\right)^L \cdot \sum_{k=0}^{K'} \sum_{l=0}^{L} c'_{k,l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k \left(\frac{p'_{v,1}M'_{v,1}}{p'_{v,2}M'_{v,2}}\right)^l .
\end{aligned}
$$

Now let I denote the set of indices $(k,l) \in \mathbb{Z}^2$ such that $d_{k,l} \neq 0$, then we can write

$$M = \gcd\left\{ M_{u,1}^k M_{u,2}^{K-m_l-k} M_{v,1}^l M_{v,2}^{L-l} \;\middle|\; (k,l) \in I \right\}.$$

Now let

$$M' = \gcd\left\{ M_{u,1}^k M_{u,2}^{K'-m'_{L-l}-k} (M'_{v,1})^{L-l} (M'_{v,2})^l \;\middle|\; (k,l) \in I \right\},$$

then using the identities for $M'_{v,1}$ and $M'_{v,2}$ derived from equality (1), we have

$$M' = \frac{1}{M_{\text{gcd}}^L} \cdot \gcd\left\{ M_{u,1}^k M_{u,2}^{K'-m'_{L-l}-k} (x_u^a M_{v,2})^{L-l} (M_{v,1}M_{u,2}^a / x_u^a)^l \;\middle|\; (k,l) \in I \right\}.$$

Note that x_u is not contained in $M_{u,1}, M_{u,2}$ and $M_{v,2}$, and since $\dim(C) = (0,0,K,L)$, we know there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $d_{k,0} \neq 0$, hence we can write

$$M' = \frac{1}{M_{\text{gcd}}^L} \cdot \gcd\left\{ M_{u,1}^k M_{u,2}^{K'-m'_{L-l}-k} (M_{v,2})^{L-l} (M_{v,1}M_{u,2}^a)^l \;\middle|\; (k,l) \in I \right\}.$$

Finally, using equality (2) and using the fact that $K' \geq K - m$, we have

$$M' = \frac{M_{u,2}^{K'-K+m}}{M_{\text{gcd}}^L} \cdot \gcd\left\{ M_{u,1}^k M_{u,2}^{K-m_l-k} (M_{v,2})^{L-l} (M_{v,1})^l \;\middle|\; (k,l) \in I \right\} = \frac{M_{u,2}^{K'-K+m}}{M_{\text{gcd}}^L} \cdot M.$$

This means, using the identity for $M'_{v,2}$ derived from equality (1), that we have

$$\frac{M_{u,2}^{K-m}(M_{v,1}M_{u,2}^a/x_u^a)^L}{M} = M_{u,2}^{K-m}(M_{v,1}M_{u,2}^a/x_u^a)^L \cdot \frac{M_{u,2}^{K'-K+m}}{M' \cdot M_{\text{gcd}}^L} = \frac{M_{u,2}^{K'}(M'_{v,2})^L}{M'}.$$

We conclude that we can write

$$H = \frac{p' \cdot (p_{u,2}M_{u,2})^{K'}(p'_{v,2}M'_{v,2})^L}{M' \cdot x_u^{m'} x_v^n} \cdot \sum_{k=0}^{K'} \sum_{l=0}^{L} c'_{k,l} \left(\frac{p_{u,1}M_{u,1}}{p_{u,2}M_{u,2}}\right)^k \left(\frac{p'_{v,1}M'_{v,1}}{p'_{v,2}M'_{v,2}}\right)^l,$$

where

$$p' = \frac{p \cdot p_{u,2}^{K+La-m} \cdot p_{v,1}^{L}}{p_{u,2}^{K'} \cdot (p'_{v,2})^{L}},$$

hence if we have that C' is an $(\mathbf{m}', \mathbf{n})$-coefficient matrix, then H is a reduced $x_u^{m'} x_v^{n}$-cluster polynomial. This means we have reduced the theorem to the following statement:

For any $m, n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_{>0}$, let $\mathbf{m} = \mathbf{seq}(m, a)$ and $\mathbf{n} = \mathbf{seq}(n, b)$. Moreover, let C be an $(\mathbf{m}, \mathbf{n})$-coefficient matrix with $\dim(C) = (0, 0, K, L)$ and with associated matrices $D = D(C)$ and $E = E(C)$. Writing $\mathbf{m}' = \mathbf{seq}(La - m, a)$ and $\mathbf{n}' = \mathbf{seq}(Kb - n, b)$, let $C', C'' \in \mathrm{Mat}_2(\mathbb{Z})$ be matrices with origin $(0, 0)$ and which for $(k, l) \in \mathbb{Z}^2_{\geq 0}$ are given by

$$c'_{k,l} = \sum_{i=0}^{k} d_{i,L-l} \binom{m'_l}{k-i} \qquad \text{and} \qquad c''_{k,l} = \sum_{j=0}^{l} e_{K-k,j} \binom{n'_k}{l-j},$$

then C' is an $(\mathbf{m}', \mathbf{n})$-coefficient matrix and C'' is an $(\mathbf{m}, \mathbf{n}')$-coefficient matrix.

Using Remark 2.14, we can derive this statement from the following lemma:

Lemma 2.21: For $m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}_{>0}$ let $\mathbf{m} = \mathbf{seq}(m, a)$, and take $\mathbf{n} \in \mathbf{Seq}$. Now let C be an $(\mathbf{m}, \mathbf{n})$-coefficient matrix with $\dim(C) = (x, y, K, L)$ and with associated matrices $D = D(C)$ and $E = E(C)$. Writing $\mathbf{m}' = \mathbf{seq}(La - m, a)$, we can find a matrix $E' \in \mathrm{Mat}_2(\mathbb{Z})$ such that for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$d_{k,l} = \sum_{i=0}^{k} \left(\sum_{j=0}^{l} e'_{i,j} \binom{n_i}{l-j} \right) \binom{-m'_{L-l}}{k-i}.$$

That this lemma indeed implies the result we need to prove Theorem 2.20, follows from the following result for binomial coefficients:

Remark 2.22: We recall that $(1 + x)$ is an invertible element in $\mathbb{Z}[[x]]$, the ring of formal power series in the variable x over $\mathbb{Z}$, and we have

$$(1 + x)^{-1} = \sum_{i \geq 0} (-x)^i.$$

More generally, for $m \in \mathbb{Z}$ we can write

$$(1 + x)^m = \sum_{i \geq 0} \binom{m}{i} x^i.$$

Now let $\{a_i\}_{i \geq 0}$ and be a sequence of integers, let $m \in \mathbb{Z}_{\geq 0}$ and for $k \in \mathbb{Z}_{\geq 0}$ let

$$b_k = \sum_{i=0}^{k} a_i \binom{-m}{k-i},$$

then for any $l \in \mathbb{Z}_{\geq 0}$ we have

$$a_l = \sum_{i=0}^{l} b_i \binom{m}{l-i}.$$

This follows directly from calculating the coefficient of x^k in the following formal power series in $\mathbb{Z}[[x]]$:

$$(1+x)^{-m} \sum_{i \geq 0} a_i x^i.$$

To prove Lemma 2.21 however, we first need the following result:

Proposition 2.23: For $p, q \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$, $s \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}_{\geq s}$ there exists a function

$$\Psi(p, q, r, s, t) : \mathbb{Z} \longrightarrow \mathbb{Z},$$

such that for any $u \in \mathbb{Z}$ we have $\Psi(p, q, r, s, t)(u) = 0$ if $u \notin [0, t]$, and such that for any $v \in \mathbb{Z}_{\geq 0}$ we have

$$\binom{p}{v} \binom{q - vr}{s} = \sum_{u=0}^{v} \Psi(p, q, r, s, t)(u) \binom{p-t}{v-u}.$$

For any $u \in \mathbb{Z}$, we will let $\Psi(p, q, r, s, t, u)$ denote the value of the function $\Psi(p, q, r, s, t)$ evaluated at u.

Proof. Let $p, q \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{>0}$. If we have $s \in \mathbb{Z}_{\geq 0}$ such that the function $\Psi(p, q, r, s, s)$ exists, then we have for any $t \in \mathbb{Z}_{>s}$ that the function $\Psi(p, q, r, s, t)$ exists. This follows from the fact that for any $v \in \mathbb{Z}_{\geq 0}$ we can write

$$\begin{aligned}
\binom{p}{v} \binom{q - vr}{s} &= \sum_{u=0}^{v} \Psi(p, q, r, s, s, u) \binom{p-s}{v-u} \\
&= \sum_{u=0}^{v} \Psi(p, q, r, s, s, u) \left(\sum_{i=0}^{v-u} \binom{t-s}{i} \binom{p-t}{v-u-i} \right) \\
&= \sum_{u'=0}^{v} \left(\sum_{j=0}^{u'} \binom{t-s}{j} \Psi(p, q, r, s, s, u'-j) \right) \binom{p-t}{v-u'}.
\end{aligned}$$

Which means that for any $u \in \mathbb{Z}$ we can write

$$\Psi(p, q, r, s, t, u) = \sum_{j=0}^{t-s} \binom{t-s}{j} \Psi(p, q, r, s, s, u-j). \tag{3}$$

Hence it is enough to show that $\Psi(p, q, r, s, s)$ exists for any $s \in \mathbb{Z}_{\geq 0}$.

Now note that for $s = 0$, we directly have that $\Psi(p, q, r, s, s)$ exists and for any $u \in \mathbb{Z}$ we have

$$\Psi(p, q, r, 0, 0, u) = \begin{cases} 1 & \text{if } u = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Hence without loss of generality we may assume we have $p > 0$ and $s > 0$. Now assume that the function $\Psi(p, q, r, s-1, s-1)$ exists. Let $v \in \mathbb{Z}_{\geq 0}$, then we can write

$$\binom{q-vr}{s} = \frac{(q-vr)_{(s)}}{s!} = \frac{q-vr-s+1}{s} \cdot \binom{q-vr}{s-1},$$

and hence we have

$$\begin{aligned}\binom{p}{v}\binom{q-vr}{s} &= \frac{q-vr-s+1}{s} \cdot \binom{p}{v}\binom{q-vr}{s-1} \\ &= \frac{q-vr-s+1}{s}\sum_{u=0}^{v} \Psi(p,q,r,s-1,s-1,u)\binom{p-s+1}{v-u} \\ &= \frac{1}{s}\sum_{u=0}^{v} \Psi(p,q,r,s-1,s-1,u)\Bigg((q-ur-s+1)\binom{p-s+1}{v-u} \\ &\qquad - r(p-s+1)\binom{p-s}{v-u-1}\Bigg)\end{aligned}$$

where we use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$ for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.

Now using the identity $\binom{p-s+1}{v-u} = \binom{p-s}{v-u} + \binom{p-s}{v-u-1}$, we can write

$$\begin{aligned}\binom{p}{v}\binom{q-vr}{s} = \frac{1}{s}\sum_{u=0}^{v} \Psi(p,q,r,s-1,s-1,u)\Bigg(&(q-ur-s+1)\binom{p-s}{v-u} \\ &+ \big(q-ur-s+1-r(p-s+1)\big)\binom{p-s}{v-u-1}\Bigg).\end{aligned}$$

Using the fact that for any $u \in \mathbb{Z} \setminus [0, s-1]$ we have $\Psi(p,q,r,s-1,s-1,u) = 0$, we now have

$$\begin{aligned}\binom{p}{v}\binom{q-vr}{s} &= \frac{1}{s}\sum_{u'=0}^{v} \Big(\big(q-(u'-1)r-s+1-r(p-s+1)\big)\Psi(p,q,r,s-1,s-1,u'-1) \\ &\qquad + \big(q-u'r-s+1\big)\Psi(p,q,r,s-1,s-1,u')\Big)\binom{p-s}{v-u'} \\ &= \frac{1}{s}\sum_{u'=0}^{v} \Big(\big(q-u'r-s+1-r(p-s)\big)\Psi(p,q,r,s-1,s-1,u'-1) \\ &\qquad + \big(q-u'r-s+1\big)\Psi(p,q,r,s-1,s-1,u')\Big)\binom{p-s}{v-u'}.\end{aligned}$$

Finally, using equality (3), we have

$$\Psi(p,q,r,s-1,s,u) = \Psi(p,q,r,s-1,s-1,u-1) + \Psi(p,q,r,s-1,s-1,u),$$

for all $u \in \mathbb{Z}$, which means that we have

$$\binom{p}{v}\binom{q-vr}{s} = \frac{1}{s}\sum_{u'=0}^{v}\Big(\big(q-u'r-s+1\big)\Psi(p,q,r,s-1,s,u') - r(p-s)\Psi(p,q,r,s-1,s-1,u'-1)\Big)\binom{p-s}{v-u'}.$$

Hence for all $u \in \mathbb{Z}$ we must have

$$\Psi(p,q,r,s,s,u) = \frac{1}{s}\Big(\big(q-ur-s+1\big)\Psi(p,q,r,s-1,s,u) - r(p-s)\Psi(p,q,r,s-1,s-1,u-1)\Big).$$

That $\Psi(p,q,r,s,s,u)$ lies in $\mathbb{Z}$ for all $u \in [0,s]$ follows by induction on u and using the fact that we can write

$$\Psi(p,q,r,s,s,u) = \binom{p}{u}\binom{q-ur}{s} - \sum_{u'=0}^{u-1}\Psi(p,q,r,s,s,u')\binom{p-s}{u-u'}.$$

□

We now are ready to prove Lemma 2.21:

Proof. Take $m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}_{>0}$, let $\mathbf{m} = \mathbf{seq}(m,a)$ and take $\mathbf{n} \in \mathbf{Seq}$. Moreover, let C be an $(\mathbf{m},\mathbf{n})$-coefficient matrix with $\dim(C) = (x,y,K,L)$ and with associated matrices $D = D(C)$ and $E = E(C)$, and let $\mathbf{m}' = \mathbf{seq}(La-m,a)$. Using Remark 2.15 we may assume C has origin $(0,0)$. To prove the lemma we show how E' can be constructed from C. Clearly we must have that E' has origin $(0,0)$, and for all $l \in \mathbb{Z}_{\geq 0}$ we must have $e'_{0,l} = e_{0,l}$. Moreover, if E' exists we must have $e'_{k,L-n_k+l} = 0$ for all $k,l \in \mathbb{Z}_{\geq 0}$ (otherwise it contradicts C having dimensions $(0,0,K,L)$). Now we construct E' using induction with respect to k. We fix $k \in [1,K]$ and assume that for all $k' \in [0,k-1]$ we have

$$d_{k',l} = \sum_{i=0}^{k'}\left(\sum_{j=0}^{l} e'_{i,j}\binom{n_i}{l-j}\right)\binom{-m'_{L-l}}{k'-i}$$

for all $l \in \mathbb{Z}_{\geq 0}$. If $n_k = 0$, then for $l \in \mathbb{Z}_{\geq 0}$ we can just take

$$e'_{k,l} = \sum_{i=0}^{k} d_{i,l}\binom{m'_{L-l}}{k-i}.$$

Now assume $n_k > 0$ and fix some $l \in \mathbb{Z}_{\geq 0}$. Note that we have

$$m - la = \begin{cases} m_l & \text{if } la \leq m; \\ -m'_{L-l} & \text{if } la \geq m, \end{cases}$$

hence we can write

$$\sum_{i=0}^{k-1}\left(\sum_{j=0}^{l} e'_{i,j}\binom{n_i}{l-j}\right)\binom{m-la}{k-i} = \begin{cases} \sum_{i=0}^{k-1} d_{i,l}\binom{m-la}{k-i} & \text{if } la \le m; \\ -\sum_{i=0}^{k-1} d_{i,l}\binom{la-m}{k-i} & \text{if } la \ge m. \end{cases}$$

The case where we have $la \ge m$ follows from applying Remark 2.22: For $k' \in [0, k-1]$ write $b_{k'} = d_{k',l}$ and $a_{k'} = \sum_{j=0}^{l} e'_{k',j}\binom{n_{k'}}{l-j}$, moreover, write $a_k = 0$. Now for any $k' \in [0, k-1]$ we can write

$$a_{k'} = \sum_{i=0}^{k'} b_i \binom{la-m}{k'-i} \qquad \text{and} \qquad b_{k'} = \sum_{i=0}^{k'} a_i \binom{m-la}{k'-i}.$$

Now let $b_k \in \mathbb{Z}$, such that

$$a_k = \sum_{i=0}^{k} b_i \binom{la-m}{k-i},$$

then we have

$$b_k = a_k - \sum_{i=0}^{k-1} b_i \binom{la-m}{k-i} = -\sum_{i=0}^{k-1} b_i \binom{la-m}{k-i},$$

and we also have

$$b_k = \sum_{i=0}^{k} a_i \binom{m-la}{k-i} = \sum_{i=0}^{k-1} a_i \binom{m-la}{k-i}.$$

By definition, we have that $n_i - n_k \ge k - i$, hence, using Proposition 2.23, we can now write the sum

$$\sum_{i=0}^{k-1}\left(\sum_{j=0}^{l} e'_{i,j}\binom{n_i}{l-j}\binom{m-la}{k-i}\right)$$

as

$$\sum_{i=0}^{k-1}\left(\sum_{j=0}^{l} e'_{i,j}\left(\sum_{j'=0}^{l-j} \Psi(n_i, m-ja, a, k-i, n_i-n_k, j')\binom{n_k}{l-j-j'}\right)\right).$$

Rearranging the summation we get:

$$\sum_{j_1=0}^{l}\left(\sum_{i=0}^{k-1}\sum_{j_2=0}^{j_1} e'_{i,j_2}\Psi(n_i, m-j_2a, a, k-i, n_i-n_k, j_1-j_2)\right)\binom{n_k}{l-j_1}.$$

Now, since we have

$$d_{k,l} = \sum_{j=0}^{l} e_{k,j}\binom{n_k}{l-j} - \sum_{i=0}^{k-1} d_{i,l}\binom{m_l}{k-i},$$

we can take

$$e'_{k,l} = e_{k,l} - \sum_{i=0}^{k-1} \sum_{j=0}^{l} e'_{i,j} \Psi(n_i, m - ja, a, k - i, n_i - n_k, l - j) \tag{4}$$

By the properties of the Ψ-function, we indeed have that $e'_{k,L-n_k+l} = 0$ for all $l \in \mathbb{Z}_{\geq 0}$, and by construction we have for $l \in \mathbb{Z}_{\geq 0}$ with $la \leq m$

$$d_{k,l} = \sum_{j=0}^{l} e'_{k,j} \binom{n_k}{l-j}.$$

Now note that for $l \in \mathbb{Z}_{\geq 0}$ with $la > m$, we have by definition

$$d_{k,l} = \sum_{j=0}^{l} e_{k,j} \binom{n_k}{l-j},$$

substituting equality (4) and reverting the steps above, we obtain

$$d_{k,l} = \sum_{j=0}^{l} e'_{k,j} \binom{n_k}{l-j} + \sum_{i=0}^{k-1} \left(\sum_{j=0}^{l} e'_{i,j} \binom{n_i}{l-j} \right) \binom{-m'_{L-l}}{k-i},$$

which is what we needed.

□

Remark 2.24: Note that the matrix E' constructed in the proof above is uniquely determined by C, and hence we will denote this matrix with $E'(C)$. Moreover, we denote with $D'(C)$ the matrix $(E'(C^T))^T$. Writing $D' = D'(C)$ and $\mathbf{n} = \mathbf{seq}(n, b)$ and $\mathbf{n'} = \mathbf{seq}(Kb - n, b)$ for $n \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}_{>0}$, we now, for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$, have

$$e_{k,l} = \sum_{j=0}^{l} \left(\sum_{i=0}^{k} d'_{i,j} \binom{m_j}{k-i} \right) \binom{-n'_{K-k}}{l-j}.$$

We now can prove a slightly stronger version of the Laurent phenomenon:

Theorem 2.25: Given a vertex $t_0 \in \mathbb{T}_N$ and $u \in [1, N]$, then for any $t \in \mathbb{T}_N$ we can find a cluster polynomial G associated to t_0 such that $x_u(t) = G(\mathbf{x}(t_0))$.

Proof. Fix any vertex $t \in \mathbb{T}_N$, and for any $u \in [1, N]$ let $t_0 \overset{u}{\text{———}} t_u$ be an edge in $\mathbb{T}_N$ with associated exchange polynomial P_u. Now assume we know that for $v \in [1, N]$ (not necessarily different from u) we have that $x_v(t) = G(\mathbf{x}(t_0))$ for some cluster polynomial G associated to t_0, then using Theorem 2.20 and Remark 2.19 we have that substituting P_u/x_u for x_u in G gives us some cluster polynomial H associated to t_1, and we have $x_v(t) = H(\mathbf{x}(t_u))$. If $t \neq t_0$ then for all but one choice of v we have that the length of the shortest path between t_u and t is the length of the shortest path between t_0 and t plus 1. Hence using induction on the length of the shortest path between vertices the theorem now follows by the argument above.

□

Now the Positivity conjecture can be stated as follows:

Conjecture 2.26: The cluster polynomials occurring in Theorem 2.25 have subtraction free numerators.

2.3 Coefficient matrices and the Positivity conjecture

The remainder of this thesis will be focused on establishing a relation between Conjecture 2.26 and some properties and conjectures for cluster polynomials and (minimal) coefficient matrices. Before we end this chapter we discuss some useful results following from Lemma 2.21 which will be essential for the discussion in the following chapters.

We have the following useful corollary from Lemma 2.21 concerning the dimensions of minimal coefficient matrices:

Corollary 2.27: Let $\mathbf{m}, \mathbf{n} \in \mathbf{Seq}$, then the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix C has dimensions $(0, 0, m_0, n_0)$.

Proof. Write $\mathbf{m} = \mathbf{seq}(m, a)$ for $m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}_{>0}$ and write $n = n_0$. Moreover, write $\dim(C) = (0, 0, K, L)$, then, using Remark 2.14, it is enough to show that $L = n$ (note that by definition of C we have $L \geq n$). Now let $D = D(C)$, $E = E(C)$ and $E' = E'(C)$, then it is enough to show that for any $k \in [0, K]$ we have $e_{k,l} = 0 = e'_{k,l}$ for all $l \in \mathbb{Z}_{\geq 0} \setminus [0, n - n_k]$. For $k = 0$ this follows directly from the definition of C. Now fix $k \in [1, K]$ and assume that for all $k' \in [0, k-1]$ we have that $e_{k',l} = 0$ for all $l \in \mathbb{Z}_{\geq 0} \setminus [0, n - n_{k'}]$. If we have $n_k = 0$, we have nothing to show, hence we assume we have $n_k > 0$. For $l \in \mathbb{Z}_{\geq 0}$ write

$$e^{\star}_{k,l} = \sum_{i=0}^{k-1} \sum_{j=0}^{l} e'_{i,j} \Psi(n_i, m - ja, a, k - i, n_i - n_k, l - j),$$

then, applying equality (4), we have $e'_{k,l} = e_{k,l} - e^{\star}_{k,l}$. Now fix some $l \in \mathbb{Z}_{\geq 0}$. Then, using the fact that for all $i \in [0, k-1]$ and for all $j \in [0, l]$ we have

$$\Psi(n_i, m - ja, a, k - i, n_i - n_k, l - j) = 0 \qquad \text{if} \quad l - j > n_i - n_k,$$

and $e'_{i,j} = 0$ if $j > n - n_i$, we have $e^{\star}_{k,l} = 0$ if $l > n - n_k$. This means that for $l > n - n_k$ we have $e_{k,l} = e'_{k,l}$. Since $d_{k,l}$ and $e_{k,l}$ are not both nonzero (we have $k > 0$), we have that $e_{k,l} = 0 = e'_{k,l}$ if $l \in [n - n_k, \ell(\mathbf{m})]$, and if $l \in [\ell(\mathbf{m}), n]$ we always have $e_{k,l} = 0$. We conclude that $e_{k,l} = 0 = e'_{k,l}$ if $l > n - n_k$, which is precisely what we needed to prove. Hence by induction on k we have that $L = n$.

□

This corollary gives rise to the following lemma:

Lemma 2.28: Let $\mathbf{m} = \mathbf{seq}(m, a)$ and $\mathbf{n} = \mathbf{seq}(n, b)$ for $m, n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_{>0}$. Let C be the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix and write $D = D(C)$, $E = E(C)$ and $E' = E'(C)$. Writing $\mathbf{m}' = \mathbf{seq}(na - m, a)$, let $C' \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ be the matrix with origin $(0, 0)$, which for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$

is given by

$$c'_{k,l} = \sum_{i=0}^{k} d_{i,n-l} \binom{m'_l}{k-i} = \sum_{j=0}^{n-n_k} e'_{k,n-n_k-j} \binom{n_k}{l-j}.$$

Then C' is the minimal $(\mathbf{m}', \mathbf{n})$-coefficient matrix if and only if E' lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

Proof. That for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ we have

$$\sum_{i=0}^{k} d_{i,n-l} \binom{m'_l}{k-i} = \sum_{j=0}^{n-n_k} e'_{k,n-n_k-j} \binom{n_k}{l-j}$$

follows from Lemma 2.21. Clearly if C' is the minimal $(\mathbf{m}', \mathbf{n})$-coefficient matrix this implies we have E' in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$, hence we just have to prove that E' lying in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ implies that C' is the minimal $(\mathbf{m}', \mathbf{n})$-coefficient matrix. Now assume E' lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.
Using Proposition 2.7 it is enough to prove that $d_{k,n-l} \cdot e'_{k,n-n_k-l} = 0$ for all $(k,l) \in \mathbb{Z}^2_{\geq 0} \backslash \{(0,0)\}$. By definition, we know that $d_{0,n-l} \cdot e'_{0,-l} \neq 0$ if and only if $l = 0$. Moreover, since $\dim(C) = (0,0,m,n)$, it is enough to show that for any $k \in [1,m]$ we have that $d_{k,n-l} \cdot e'_{k,n-n_k-l} = 0$ for all $l \in [0, n-n_k]$. Now fix some $k \in [1,m]$. If $n_k = 0$, then we saw in the proof of Lemma 2.21 that for all $l \in [0,n]$ we have

$$e'_{k,n-l} = \sum_{i=0}^{k} d_{i,n-l} \binom{m'_l}{k-i},$$

and by definition of the minimal coefficient matrix we have $d_{k,n-l} = 0$ in this case, hence we can assume $n_k > 0$. Now fix some $l \in [0, n-n_k]$ such that $e'_{k,n-n_k-l} \neq 0$. Note that $d_{k,n-l} > 0$ implies that $e_{k,l'} > 0$ for some $l' \in [n-n_k-l, n-l-1]$. Now suppose such an l' exists and assume it to be minimal. We must have $l'a < m$ by definition of the minimal coefficient matrix (otherwise $e_{k,l'} = 0$). However, since $e'_{k,j} \geq 0$ for all $j \in [0, n-n_k]$, this means we have

$$d_{k,l'} = \sum_{j=0}^{l'} e'_{k,j} \binom{n_k}{l'-j} > 0$$

which means we have $d_{k,l'} > 0$ and $e_{k,l'} > 0$ which gives us as contradiction (by Proposition 2.7). □

Using Remark 2.14 we have automatically the following result:

Corollary 2.29: Writing $\mathbf{n}' = \mathbf{seq}(mb-n, b)$, then the matrix $C'' \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$, with origin $(0,0)$, which for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c'_{k,l} = \sum_{j=0}^{l} e_{m-k,j} \binom{n'_k}{l-j} = \sum_{i=0}^{m-m_l} d'_{m-m_l-i,l} \binom{m_l}{k-i},$$

is the minimal $(\mathbf{m}, \mathbf{n}')$-coefficient matrix if and only if $D'(C)$ lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

This lemma is useful when dealing with minimal cluster polynomials in the setting of Theorem 2.20:

Corollary 2.30: Let $u, v \in [1, N]$ be distinct, let $P_u \in \mathbb{ZP}[x_1, \ldots, x_N]$ be the exchange polynomial associated to a given edge $t \overset{u}{\longrightarrow} t'$ in $\mathbb{T}_N$, and, for $m, n \in \mathbb{Z}$ (not necessarily nonnegative), let $G = F/x_u^m x_v^n$ be a minimal $x_u^m x_v^n$-cluster polynomial associated to t. Write $B(t) = (b_{ij})$ and let $a = |b_{vu}|$ and $b = |b_{uv}|$. Assume we have $a \cdot b \neq 0$ and write $\mathbf{m} = \mathbf{seq}(m, a)$ and $\mathbf{n} = \mathbf{seq}(n, b)$. Let C denote the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix, then substituting P_u/x_u for x_u in G gives us a minimal $x_u^{na-m}x_v^n$-cluster polynomial associated to t' if and only if $E'(C)$ lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.
Assuming $E'(C) \notin \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ (which in particular means that m and n must both be positive), let G' denote the reduced $x_u^{na-m}x_v^n$-cluster polynomial associated to t' obtained from substituting P_u/x_u for x_u in G, and let P_v be the exchange polynomial associated to the edge $t' \overset{v}{\longrightarrow} t''$. Then, writing m' for the largest exponent of x_v in the numerator of G' and writing $n' = m'b - n$, substituting P_v/x_v for x_v in G' results in a reduced $x_u^{na-m}x_v^{n'}$-cluster polynomial associated to t'' whose numerator is not subtraction free.

Proof. The first part of the corollary is a direct result from Theorem 2.20 and Lemma 2.28.
The second part follows from Theorem 2.20 and Lemma 2.21: let G'' denote the reduced $x_u^{na-m}x_v^{n'}$-cluster polynomial associated to t'', and write $\mathbf{m}' = \mathbf{seq}(na - m, a)$ and $\mathbf{n}' = \mathbf{seq}(n', b)$. Write $D = D(C)$ and $E' = E'(C)$, and let $C' \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ be the $(\mathbf{m}', \mathbf{n})$-coefficient matrix which for $(k, l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c'_{k,l} = \sum_{i=0}^{k} d_{i,n-l} \binom{m'_l}{k-i} = \sum_{j=0}^{n-n_k} e'_{k,n-n_k-j} \binom{n_k}{l-j}.$$

Now write $E'' = E(C')$, and let $C'' \in \mathrm{Mat}_2(\mathbb{Z})$ be the $(\mathbf{m}, \mathbf{n}')$-coefficient matrix which for $(k, l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c''_{k,l} = \sum_{j=0}^{l} e''_{m'-k,j} \binom{n'_k}{l-j}.$$

From the definition of a reduced cluster polynomial we can deduce that it is enough to prove that $C'' \notin \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

Note that we have $E'' \notin \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ since $E' \notin \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$. If $m'b \leq n$, then for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$ we have $c''_{k,l} = e''_{m'-k,l}$, which means $C'' \notin \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$. This means we may assume $m'b > n$.

Now fix some $k \in \mathbb{Z}_{\geq 0}$. From the proof of Lemma 2.21 we know that if $n_k = 0$ we have

$$e'_{k,l} = \sum_{i=0}^{k} d_{i,l} \binom{m'_{n-l}}{k-i} > 0 \qquad \big(l \in \mathbb{Z}_{\geq 0}\big),$$

which means that having $e'_{k,l} < 0$ for some $l \in [0, n - n_k]$ implies we have $n_k > 0$. Assume $n_k > 0$, then we have $kb < n$ and hence $(m' - k)b > m'b - n$ which implies $n'_{m'-k} = 0$. Now assume for some $l \in [0, n - n_k]$ we have $e'_{k,l} < 0$, then $e''_{k,n-n_k-l} < 0$, and hence $c_{m'-k,n-n_k-l} = e''_{k,n-n_k-l}$. We conclude that $C'' \notin \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

□

This corollary in particular illustrates why minimal coefficient matrices are very interesting objects to consider when studying the Positivity conjecture.

3 Quivers and Cluster algebras

In this chapter we introduce the necessary definitions to discuss the result regarding the Positivity conjecture obtained by Philippe Caldero and Markus Reineke in [2], and we illustrate how Grégoire Dupont in [3] deduces from this result the Positivity conjecture for coefficient-free cluster algebras of rank 2. We end this chapter with the discussion of a potential generalization of the result of Dupont.

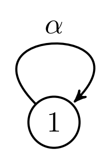
3.1 Introduction to the representation theory of quivers

Our exposition in this section is based on the lecture notes of the course 'Introduction to the representation theory of quivers' given by Claus M. Ringel ([5], [6]) and on the lecture notes of Michel Brion on this subject ([7]).

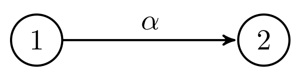
Definition 3.1: A quiver Q is a directed graph which can contain loops, and which can have multiple arrows between vertices. We write $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices of Q and Q_1 is the set of arrows of Q. We additionally have two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$, where for an arrow $\alpha : i \to j \in Q_1$ we have that $s(\alpha) = i$ is the source and $t(\alpha) = j$ is the target of α. We say that Q is finite if $\#Q_0 < \infty$ and $\#Q_1 < \infty$, and we say that Q is acyclic if Q does not contain any oriented cycles.

We consider three simple examples of quivers:

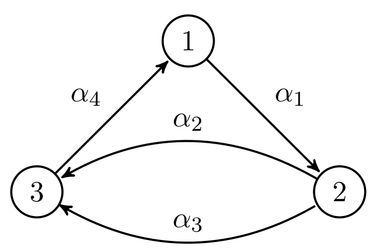
Example 3.2: The loop quiver $Q = (\{1\}, \{\alpha\})$ is given by the graph



The Kronecker quiver $Q = (\{1, 2\}, \{\alpha\})$ is given by the graph



Finally we consider the quiver $Q = (\{1, 2, 3\}, \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ given by the graph



The following remark indicates there is a relation between quivers and cluster algebras:

Remark 3.3: To a finite quiver $Q = (Q_0, Q_1)$ which does not contain loops or 2-cycles we can associate a skew-symmetric matrix $B_Q = (b_{ij})_{i,j \in Q_0}$ where for any $i, j \in Q_0$ we have

$$b_{ij} = \#\{\alpha \in Q_1 \mid s(\alpha) = i \text{ and } t(\alpha) = j\} - \#\{\alpha \in Q_1 \mid s(\alpha) = j \text{ and } t(\alpha) = i\}.$$

This gives a one-to-one correspondence between skew-symmetric matrices and quivers without loops or 2-cycles.

To expand on the relation between quivers and cluster algebras we first introduce path algebras associated to quivers.

Definition 3.4: Let $Q = (Q_0, Q_1)$ be a quiver, then a path $w = \alpha_1 \cdots \alpha_n$ in Q of length $n \geq 1$ is a sequence of arrows $\alpha_1, \dots, \alpha_n \in Q_1$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i \leq n-1$. We let $s(w) = s(\alpha_n)$ denote the source and $t(w) = t(\alpha_1)$ denote the target of w. Additionally, for vertex $i \in Q_0$ we define e_i to be a path of length 0 with source and target equal to i. Given two paths w and w' in Q with $s(w) = t(w')$, we write ww' for the concatenation of w and w', and we write $e_{t(w)}w = w = we_{s(w)}$.

Example 3.5: In the loop quiver the set of all paths is given by the set $\{e_1\} \cup \{\alpha^n \mid n \geq 1\}$ which is in particular an infinite set. In the Kronecker quiver the set of all paths is $\{e_1, e_2, \alpha\}$.
Now let $Q = (\{1,2,3\}, \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ be the quiver as given in Example 3.2, write $\gamma_1 = \alpha_4\alpha_2\alpha_1$ and $\gamma_2 = \alpha_4\alpha_3\alpha_1$. Now let $\Gamma_1 = \{\gamma_1, \gamma_2\}$, and for any $n > 1$ we define

$$\Gamma_n = \{\gamma_1 w \mid w \in \Gamma_{n-1}\} \cup \{\gamma_2 w \mid w \in \Gamma_{n-1}\},$$

and let

$$\Gamma = \bigcup_{n \in \mathbb{Z}_{>0}} \Gamma_n.$$

Moreover, we define the sets $S = \{e_1, \alpha_4, \alpha_4\alpha_2, \alpha_4\alpha_3\}$ and $T = \{e_1, \alpha_1, \alpha_2\alpha_1, \alpha_3\alpha_1\}$. Now the set of all paths in Q is

$$\{e_1, e_2, e_3\} \cup Q_1 \cup S \cup T \cup \{t\gamma s \mid t \in T, \gamma \in \Gamma, s \in S\}.$$

This example shows that simple quivers can already have infinitely many paths. The quivers which have a finite number of paths are precisely the finite acyclic quivers ([6, Corollary 4.1]).

We now fix an algebraically closed field k.

Definition 3.6: Let $Q = (Q_0, Q_1)$ be a quiver, then the path algebra over k of the quiver Q, denoted with kQ, is the k-vector space with basis the set of all paths in Q. On kQ we can define a multiplication as follows: for any two paths w, w' in Q we define the product ww' in kQ to be 0 if $s(w) \neq t(w')$ and otherwise the path obtained by the concatenation of w and w'. This induces indeed a multiplication on kQ with identity given by

$$1 = \sum_{i \in Q_0} e_i.$$

This makes kQ into an associative k-algebra.

Example 3.7: The path algebra over k of the loop quiver is naturally isomorphic to the polynomial ring $k[\alpha]$.

Next we consider quiver representations:

Definition 3.8: A representation of a quiver $Q = (Q_0, Q_1)$ is of the form $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$,

where for each $i \in Q_0$ we have that M_i is a k-vector space and for each $\alpha \in Q_1$ we have that $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a k-linear map. We say that M is finite dimensional if each M_i is a finite dimensional k-vector space.
Given Q-representations M, N we say that N is a subrepresentation of M, if for any $i \in Q_0$ we have that N_i is a linear subspace of M_i and if for any $\alpha \in Q_1$ we have that N_α is the restriction of M_α to $N_{s(\alpha)}$.
A morphism of representations $f : M \to N$, for given Q-representations M, N, is a family of k-linear maps $(f_i : M_i \to N_i)_{i \in Q_0}$ such that for all $\alpha \in Q_1$ we have the following commutative diagram:

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{M_\alpha} & M_{t(\alpha)} \\ \downarrow{\scriptstyle f_{s(\alpha)}} & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ N_{s(\alpha)} & \xrightarrow{N_\alpha} & N_{t(\alpha)} \end{array}$$

This means we have a well-defined category $\mathbf{Rep}(Q, k)$ of representations of Q, and a well-defined category $\mathbf{rep}(Q, k)$ of finite dimensional representations of Q. If there is no confusion we will write $\mathbf{Rep}(Q)$ and $\mathbf{rep}(Q)$.

We now fix a finite quiver $Q = (Q_0, Q_1)$. The following theorem shows that there is a strong resemblance between quiver representations and left kQ-modules:

Theorem 3.9 ([6, p. 5])**:** We have an equivalence of categories $\mathbf{Rep}(Q) \simeq kQ$-**Mod**, where kQ-**Mod** denotes the category of left kQ-modules.

Since we are just concerned with left kQ-modules, we will omit the prefix ‘left’ from here on.

The equivalence in the theorem above is given by the following correspondences:

- Let $(M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be a representation of Q, then let $M = \bigoplus_{i \in Q_0} M_i$ be the corresponding kQ-module, where the left multiplication by a path $w = \alpha_1 \cdots \alpha_n$ is given by the mapping

$$M \longrightarrow M, \qquad (a_i)_{i \in Q_0} \longmapsto (M_{\alpha_1} \circ \cdots \circ M_{\alpha_n})(a_{s(\alpha_n)}).$$

  Moreover, let $f : M \to N$ be a morphism of representations of Q, then f corresponds to the morphism of kQ-modules $\bigoplus_{i \in Q_0} f_i : \bigoplus_{i \in Q_0} M_i \to \bigoplus_{i \in Q_0} N_i$.

- Let M be a kQ-module, then the corresponding representation of Q is $(M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where for $i \in Q_0$ we have $M_i = e_i M$, and where for $\alpha \in Q_1$ we have

$$M_\alpha : M_{s(\alpha)} \longrightarrow M_{t(\alpha)}, \qquad x \mapsto \alpha x.$$

  Moreover, let $f : M \to N$ be a morphism of kQ-modules, then $f(e_i M) \subseteq e_i N$ for all $i \in Q_0$ (since f is a kQ-linear map). Hence for any $i \in Q_0$ let f_i denote the restriction of f to M_i, then the morphism of Q-representations corresponding to f is $(f_i)_{i \in Q_0}$.

We have the following corollary:

Corollary 3.10 ([6, p. 6])**:** We have an equivalence of categories $\mathbf{rep}(Q) \simeq kQ$-**mod**, where kQ-**mod** denotes the category of finitely generated kQ-modules.

Since the equivalence in Theorem 3.9 is natural in the sense that the correspondences given above are natural, we will think of a Q-representation as a kQ-module and vice versa. Moreover, by Theorem 3.9, we can study kQ-modules to get a better understanding of Q-representations. We will only be interested in finite dimensional Q-representations, hence when talking of a Q-representation, we assume it to be finite dimensional (and similarly, when talking of a kQ-module, we assume it to be finitely generated). Finally, (as the fact that Caldero and Reineke proved the Positivity conjecture for acyclic cluster algebras suggests) we are only interested in the case where Q is an acyclic quiver, hence we assume our quiver Q from here on to be acyclic.

We end this section with a discussion of two important classes of kQ-modules.

Definition 3.11: Let M be a nonzero kQ-module, recall that:

- M is called simple if the only kQ-submodules of M are 0 and M.
- M is called indecomposable if M cannot be written as the direct sum of two nonzero kQ-modules.
- M is called free if $M \cong kQ^n$ for some $n \in \mathbb{Z}_{\geq 0}$.
- M is called projective if there exists another kQ-module N such that $M \oplus N$ is a free kQ-module.

Definition 3.12: For any $i \in Q_0$ we define the following kQ-modules:

- $P_i = kQe_i$, which is the k-vector space generated by all paths in Q with source equal to i.
- $S_i = P_i / kQ_{\geq 1} P_i$, where $kQ_{\geq 1}$ is the kQ-ideal generated by Q_1.

For these kQ-modules we have the following results:

- Any simple kQ-module is isomorphic to S_i for some $i \in Q_0$ ([7, Proposition 1.3.1]).
- Any indecomposable projective kQ-module is isomorphic to P_i for some $i \in Q_0$, and for $i, j \in Q_0$ we have that P_i and P_j are not isomorphic if $i \neq j$ ([7, Proposition 1.3.7]).

3.2 Cluster categories

For the entirety of this section we let $k = \mathbb{C}$ and we fix a finite acyclic quiver $Q = (Q_0, Q_1)$. In this section we will introduce the cluster category of Q (introduced by Aslak Bakke Buan et al. in [8]).

We first recall some definitions of category theory, where we follow Franco Rota's lecture notes [9].

Definition 3.13: Let $\mathcal{A}$ be an abelian category (for instance the category of modules over a ring). A cochain complex $(A^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ is a sequence of objects $\ldots, A^{-1}, A^0, A^1, A^2, \ldots$ in $\mathcal{A}$ connected by morphisms $d^n : A^n \to A^{n+1}$ (called boundary operators or differentials) such that $d^{n+1} \circ d^n = 0$. We also write a cochain complex of $\mathcal{A}$ as

$$\cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^3} \cdots$$

A morphism of cochain complexes $f^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ is a family of morphisms $(f^n : A^n \to B^n)_{n \in \mathbb{Z}}$ such that $f^{n+1} \circ d_A^n = d_B^n \circ f^n$ for all $n \in \mathbb{Z}$.
This means we have a category of cochain complexes of objects in $\mathcal{A}$, which we denote with $\mathbf{Kom}(\mathcal{A})$.

We also have a category of bounded cochain complexes of objects in $\mathcal{A}$ whose objects are those cochain complexes $(A^\bullet, d^\bullet)$ in $\mathbf{Kom}(\mathcal{A})$ for which there exists an integer $N \in \mathbb{Z}_{>0}$ such that $A_n = 0$ for all $n \in \mathbb{Z}$ with $|n| \geq N$. This category is denoted with $\mathbf{Kom}^b(\mathcal{A})$ and is a full subcategory of $\mathbf{Kom}(\mathcal{A})$.
To a cochain complex $(A^\bullet, d^\bullet)$ in $\mathbf{Kom}(\mathcal{A})$ we associate the n-th cohomology group $H^n(A^\bullet)$ which is equal to $\mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$ (which is an object of $\mathcal{A}$). Note that a morphism $f^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ induces a morphism on the n-th cohomology groups $H^n(f^\bullet) : H^n(A^\bullet) \to H^n(B^\bullet)$. We call $f^\bullet$ a quasi-isomorphism if $H^n(f^\bullet)$ is an isomorphism in $\mathcal{A}$ for all $n \in \mathbb{Z}$.
Finally, the bounded derived category $D^b(\mathcal{A})$, is a category whose objects are bounded cochain complexes of objects in $\mathcal{A}$, i.e., $\mathrm{ob}(D^b(\mathcal{A})) = \mathrm{ob}(\mathbf{Kom}^b(\mathcal{A}))$, together with a functor $F : \mathbf{Kom}^b(\mathcal{A}) \to D^b(\mathcal{A})$ satisfying the following universal property: For any category $\mathcal{C}$ and functor $G : \mathbf{Kom}^b(\mathcal{A}) \to \mathcal{C}$ such that any quasi-isomorphism $f^\bullet$ in $\mathbf{Kom}^b(\mathcal{A})$ maps to an isomorphism in $\mathcal{C}$ under the functor G, we have that G factors through F.

Since kQ-**mod** is an abelian category, the bounded derived category of kQ-**mod**, which we denote as $D^b(kQ)$, is well-defined, and we have a projection $\mathrm{ob}(kQ\text{-}\mathbf{mod}) \to \mathrm{ob}(D^b(kQ))$ which maps a kQ-module M to the cochain complex

$$\cdots \xrightarrow{d^{-2}} 0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} 0 \xrightarrow{d^1} \cdots$$

We will think of a kQ-module as being an object in $D^b(kQ)$ under this projection. Moreover, we have that $D^b(kQ)$ is a triangulated category:

Definition 3.14: Let $\mathcal{D}$ be an additive category (for instance the category of modules over a ring or the category of cochain complexes over such a category). The structure of a triangulated category on $\mathcal{D}$ is given by an additive auto-equivalence $T : \mathcal{D} \to \mathcal{D}$ called the shift functor, and a set of distinguished triangles or exact triangles, where a triangle is a sequence in $\mathcal{D}$ of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A),$$

such that axioms **TR1-TR4** below are respected. In stating of these axioms we use the following notation: For any $n \in \mathbb{Z}$ and for any object A in $\mathcal{D}$, we write $A[n] = T^n(A)$, and for any morphism $f : A \to B$ we write $f[n]$ for the morphism $T^n(f) : A[n] \to B[n]$. A morphism of triangles is given by morphisms f, g and h in $\mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

Moreover, it is called an isomorphism if f, g and h all three are isomorphisms.

We now state the axioms:

TR1.

- Any triangle of the form

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished;

- Any triangle isomorphic to a distinguished triangle is distinguished;
- Any morphism $f : A \to B$ fits in a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

TR2. A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished.

TR3. Suppose we have a diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

where the rows are distinguished triangles and the leftmost square is commutative, then there exists a (not necessarily unique) morphism $h : C \to C'$, which, if added to the diagram above, makes the diagram into a morphism of distinguished triangles.

TR4. (Octahedral axiom) Let $u : A \to B$ and $v : B \to C$ be morphisms, then by **TR1** we have distinguished triangles

$$A \xrightarrow{u} B \xrightarrow{f_1} C' \xrightarrow{f_2} A[1],$$

$$B \xrightarrow{v} C \xrightarrow{g_1} A' \xrightarrow{g_2} B[1],$$

$$A \xrightarrow{v \circ u} C \xrightarrow{h_1} B' \xrightarrow{h_2} A[1].$$

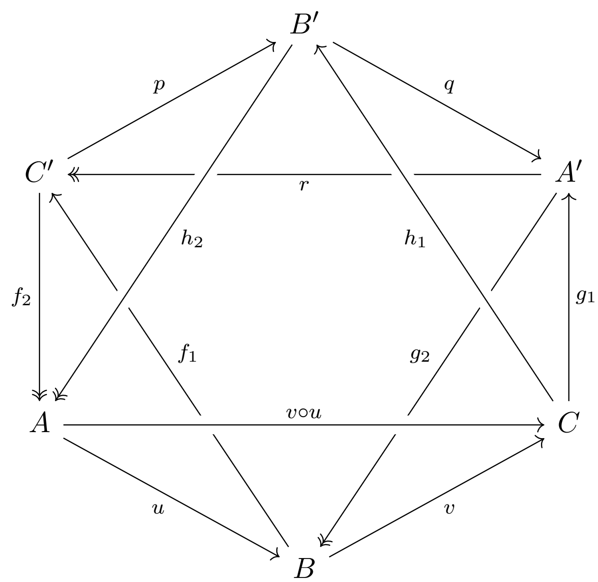
The axiom then states that there exists a distinguished triangle

$$C' \xrightarrow{p} B' \xrightarrow{q} A' \xrightarrow{r} C'[1],$$

such that

$$g_1 = q \circ h_1, \qquad f_2 = h_2 \circ p, \qquad r = f_1[1] \circ g_2, \qquad g_2 \circ q = u[1] \circ h_2, \qquad \text{and} \qquad p \circ f_1 = h_1 \circ v.$$

The name of this axiom comes from the fact that these morphisms fit in a 'commutative' diagram which gives the skeleton of an octahedron:



Where the arrows of the form $X \twoheadrightarrow Y$ mean the morphism is from X to $Y[1]$.

Given objects A, B in $\mathcal{D}$, we end this definition of triangulated categories with introducing a special notation for the set of morphisms from A to $B[n]$ for $n \in \mathbb{Z}$: we will write $\mathrm{Hom}_{\mathcal{D}}(A, B[n]) = \mathrm{Ext}^n_{\mathcal{D}}(A, B)$, and the set $\mathrm{Ext}^1_{\mathcal{D}}(A, A)$ are called the self-extensions of A.

For more information on the definitions above, we refer to [9].

The shift functor on $D^b(kQ)$ is defined as follows: Let $(A^\bullet, d^\bullet_A)$ be an object in $D^b(kQ)$, then for $n \in \mathbb{Z}$ we have that $(A^\bullet, d^\bullet_A)[n] = (B^\bullet, d^\bullet_B)$ where for any $m \in \mathbb{Z}$ we have $B^m = A^{n+m}$ and $d^m_B = (-1)^n d^{n+m}_A$. We let $\tau = D\,\mathrm{Tr}$ denote the Auslander-Reiten translation on $D^b(kQ)$ which is an auto-equivalence of $D^b(kQ)$. The exact definition of this functor is not important for our discussion in this chapter, for more information on this translation we refer to the article of Henning Krause and Jue Le on this subject ([10]).

Definition 3.15: The cluster category of Q is the orbit category $\mathcal{C}_Q = D^b(kQ)/F$, where F is the auto-equivalence $\tau^{-1}[1]$.

We state some properties for $\mathcal{C}_Q$ which can be found in [8, Section 1]. The objects of the category $\mathcal{C}_Q$ are the F-orbits of objects in $D^b(kQ)$. It can be shown that $\mathcal{C}_Q$ is a triangulated category, and that the natural functor $\pi : D^b(kQ) \to \mathcal{C}_Q$ is a triangle functor, i.e. π commutes with the shift functor on both categories and preserves distinguished triangles ([11]). The shift functor on $\mathcal{C}_Q$ is induced by the shift in $D^b(kQ)$. For objects $X, Y \in D^b(kQ)$, let $\widetilde{X}$ and $\widetilde{Y}$ denote the corresponding objects in $\mathcal{C}_Q$, then we have

$$\mathrm{Hom}_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) = \bigsqcup_{i \in \mathbb{Z}} \mathrm{Hom}_{D^b(kQ)}(F^i X, Y),$$

and we have $\mathrm{Hom}_{D^b(kQ)}(F^i X, Y) \neq 0$ for only finitely many $i \in \mathbb{Z}$.

For any category $\mathcal{A}$ we write $\mathrm{ind}(\mathcal{A})$ for the set of isomorphism classes of indecomposable objects in $\mathcal{A}$. We end this section with following important result:

Proposition 3.16 ([8, Proposition 1.6])**:** We have that any set of representatives for ind(kQ-**mod**) (seen as objects in $\mathcal{C}_Q$) together with the objects $\pi(P_i[1])$ for all $i \in Q_0$, forms a set of representatives for ind($\mathcal{C}_Q$).

3.3 The Caldero-Chapoton map

In this section we introduce the Caldero-Chapoton map (introduced by Philippe Caldero and Frédéric Chapoton in [12]), and discuss the relation it induces between the objects of the cluster category associated to a finite acyclic quiver and the cluster variables of the cluster algebra associated to this quiver.

As before, we let $k = \mathbb{C}$ and we fix a finite acyclic quiver $Q = (Q_0, Q_1)$.

Let V be a finite dimensional k-vector space, then for $d \in \mathbb{Z}_{\geq 0}$ the Grassmannian $\mathrm{Gr}_d(V, k)$ is the set of all linear subspaces of M of dimension d. It is a well-known fact that $\mathrm{Gr}_d(V, k)$ smooth projective variety. Now let M be a kQ-module then we can also consider the Grassmannian $\mathrm{Gr}_d(M, kQ)$ of kQ-submodules of M with dimension d, then $\mathrm{Gr}_d(M, kQ)$ is a closed subvariety of $\mathrm{Gr}_d(M, k)$ and hence it is a projective variety. Since every kQ-module naturally corresponds to a Q-representation, we now consider the following definition:

Definition 3.17: Let M be a Q-representation, then the dimension vector of M is the vector $\mathbf{dim}(M) = (\dim_k(M_i))_{i \in Q_0} \in \mathbb{N}^{Q_0}$.

Let M be a Q-representation, then regarding M as a kQ-module, we have

$$\mathbf{dim}(M) = (\dim_k(e_i M))_{i \in Q_0},$$

in particular, we have

$$\dim_k(M) = \sum_{i \in Q_0} \dim_k(e_i M).$$

Now let $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ and let $d = \sum_{i \in Q_0} d_i$. We define the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ to be the closed subvariety of $\mathrm{Gr}_d(M, kQ)$ of all kQ-submodules N of M with $\mathbf{dim}(N) = \mathbf{d}$ (which again is a projective variety).

Definition 3.18: The homological Euler form on kQ-**mod** is defined as the bilinear form

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}, \qquad (\mathbf{a}, \mathbf{b}) \longmapsto \sum_{i \in Q_0} a_i b_i - \sum_{\alpha \in Q_1} a_{s(\alpha)} b_{t(\alpha)}.$$

Now let $\mathbf{x} = \{x_i \mid i \in Q_0\}$ be a set of indeterminates over $\mathbb{Q}$, and let $\mathbb{Z}[\mathbf{x}^{\pm 1}]$ denote the ring of all Laurent polynomials in the variables $\{x_i \mid i \in Q_0\}$ with coefficients in $\mathbb{Z}$.

Definition 3.19: The Caldero-Chapoton map is a map $X_? : \mathrm{ob}(\mathcal{C}_Q) \to \mathbb{Z}[\mathbf{x}^{\pm 1}]$ which assigns a Laurent polynomial $X_M \in \mathbb{Z}[\mathbf{x}^{\pm 1}]$ to any object M in the category $\mathcal{C}_Q$. Let M be an object in $\mathcal{C}_Q$, then X_M is defined as follows:

1. If M is an indecomposable kQ-module (recall that we can regard kQ-modules as objects in

$D^b(kQ)$ and objects of $D^b(kQ)$ can be projected onto $\mathcal{C}_Q$), then

$$X_M = \sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_{\mathbf{d}}(M)) \prod_{i\in Q_0} x_i^{-\langle \mathbf{d},\mathbf{dim}(S_i)\rangle - \langle \mathbf{dim}(S_i),\mathbf{dim}(M)-\mathbf{d}\rangle}, \tag{5}$$

where $\chi(\mathrm{Gr}_{\mathbf{d}}(M))$ denotes the Euler-Poincaré characteristic of the projective variety $\mathrm{Gr}_{\mathbf{d}}(M)$.

2. If $M = P_i[1]$ for some $i \in Q_0$, then

$$X_M = x_i.$$

3. If $M = N_1 \oplus N_2$ for objects N_1, N_2 in $\mathcal{C}_Q$, then

$$X_M = X_{N_1} \cdot X_{N_2}.$$

By Proposition 3.16 the Caldero-Chapoton map is well-defined. Moreover, since for any two kQ-modules M and N, and for any dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ we have (by [2, Proposition 1]) that

$$\chi(\mathrm{Gr}_{\mathbf{d}}(M \oplus N)) = \sum_{\mathbf{e}+\mathbf{f}=\mathbf{d}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \cdot \chi(\mathrm{Gr}_{\mathbf{f}}(N)),$$

we have that the formula (5) for X_M also holds when M is not indecomposable.

Remark 3.20: Let M be a kQ-module, write $\mathbf{dim}(M) = \mathbf{m} = (m_i)_{i\in Q_i}$, and let $\mathbf{d} \in \mathbb{N}^{Q_0}$, then for any $i \in Q_0$ we have that $-\langle \mathbf{d}, \mathbf{dim}(S_i)\rangle - \langle \mathbf{dim}(S_i), \mathbf{dim}(M) - \mathbf{d}\rangle$ is equal to

$$-d_i + \sum_{\alpha\in Q_1 : t(\alpha)=i} d_{s(\alpha)} - (m_i - d_i) + \sum_{\alpha\in Q_1 : s(\alpha)=i} (m_{t(\alpha)} - d_{t(\alpha)}),$$

which can be written as

$$-m_i + \sum_{\alpha\in Q_1 : t(\alpha)=i} d_{s(\alpha)} + \sum_{\alpha\in Q_1 : s(\alpha)=i} (m_{t(\alpha)} - d_{t(\alpha)}).$$

This implies we have

$$X_M = \prod_{i\in Q_0} x_i^{-m_i} \left(\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_{\mathbf{d}}(M)) \prod_{\alpha\in Q_1} x_{s(\alpha)}^{m_{t(\alpha)}-d_{t(\alpha)}} x_{t(\alpha)}^{d_{s(\alpha)}} \right).$$

As we mentioned at the beginning of this section, we will discuss the relation $X_?$ induces between objects in $\mathcal{C}_Q$ and the cluster variables of the 'the cluster algebra associated to Q'. What this last part means we now make precise:

Definition 3.21: Write $\mathbb{T}_Q = \mathbb{T}_{Q_0}$, and let B_Q be the skew-symmetric matrix associated to Q as in Remark 3.3. Now take some vertex $t_Q \in \mathbb{T}_Q$ and let $\mathcal{E}_Q = \big((\mathbf{u}(t))_{t\in\mathbb{T}_Q}, (B(t))_{t\in\mathbb{T}_Q}\big)$ denote the unique coefficient-free exchange pattern on $\mathbb{T}_Q$ with $B(t_Q) = B_Q$ (as in Example 1.8). Then we let $\mathcal{A}(Q)$ denote the cluster algebra of rank $\#Q_0$ over $\mathbb{Z}$ associated to the exchange pattern $\mathcal{E}_Q$. We call t_Q the initial vertex of $\mathcal{A}(Q)$, and we call $\mathcal{A}(Q)$ the cluster algebra associated to Q (this is unambiguous since the choice of initial vertex does not change the structure of the resulting cluster algebra). We call a coefficient-free cluster algebra acyclic if it can be obtained in this way from a finite acyclic quiver.

The relation induced by $X_?$ between objects of $\mathcal{C}_Q$ and cluster variables of $\mathcal{A}(Q)$ is given by the following theorem proved by Philippe Caldero and Bernhard Keller:

Theorem 3.22 ([13, Theorem 4])**:** The map $X_?$ induces a one-to-one correspondence between the indecomposable objects without self-extensions of $\mathcal{C}_Q$ and the cluster variables of $\mathcal{A}(Q)$.

Which they deduced from the following theorem:

Theorem 3.23 ([13, Theorem 2])**:** Let M and N be indecomposable objects in $\mathcal{C}_Q$ such that $\mathrm{Ext}^1(M,N)$ is one-dimensional. Then we have

$$X_M X_N = X_B + X_{B'},$$

where B and B' are the unique objects (up to isomorphism) such that there exist non-split triangles

$$N \longrightarrow B \longrightarrow M \longrightarrow N[1], \qquad M \longrightarrow B' \longrightarrow N \longrightarrow M[1].$$

After this, Philippe Caldero and Markus Reineke proved the following theorem:

Theorem 3.24 ([2, Theorem 1])**:** For any kQ-module M without self-extensions then for any $\mathbf{d} \in \mathbb{N}^{Q_0}$ we have that the Euler-Poincaré characteristic $\chi(\mathrm{Gr}_{\mathbf{d}}(M))$ is nonnegative.

They used this theorem to deduce the Positivity conjecture for acyclic cluster algebras:

Theorem 3.25 ([2, Theorem 2])**:** The cluster variables of $\mathcal{A}(Q)$ expressed in the variables of any cluster $\mathbf{x}$ lie in $\mathbb{Z}_{\geq 0}[\mathbf{x}^{\pm 1}]$.

3.4 Positivity for coefficient-free cluster algebras of rank 2

In this section we give an overview of how Grégoire Dupont proves the Positivity conjecture for coefficient-free cluster algebras of rank 2, after which we prove this result using just the definitions of Chapter 1, the results of Chapter 2 and the fact that the Positivity conjecture holds for acyclic cluster algebras.

As we saw in Example 1.5, we can write $\mathbb{T}_2$ as

$$\cdots \overset{2}{\text{———}} t_0 \overset{1}{\text{———}} t_1 \overset{2}{\text{———}} t_2 \overset{1}{\text{———}} t_3 \overset{2}{\text{———}} t_4 \overset{1}{\text{———}} \cdots.$$

This means that the set of vertices of $\mathbb{T}_2$ is of the form $\{t_n\}_{n\in\mathbb{Z}}$ such that for any $n \in \mathbb{Z}$ we have that $\mathbb{T}_2$ contains the edges $t_{n-1} \text{ ——— } t_n$ and $t_n \text{ ——— } t_{n+1}$ (in particular, we have that these edges cannot have the same label). We therefore can assume that for any $n \in \mathbb{Z}$ we have that $\mathbb{T}_2$ contains the edges $t_{2n-1} \overset{2}{\text{———}} t_{2n}$ and $t_{2n} \overset{1}{\text{———}} t_{2n+1}$. We also saw in Example 1.5 that any coefficient-free exchange pattern $\mathcal{E}$ on $\mathbb{T}_2$ is completely determined by the sign-skew-symmetric 2×2 matrix $B(t_0) = B = (b_{ij})$. Moreover, since for any $n \in \mathbb{Z}$, we have that the exchange polynomials associated to the edges $t_{2n-1} \overset{2}{\text{———}} t_{2n}$ and $t_{2n} \overset{1}{\text{———}} t_{2n+1}$ are respectively given by $x_1^{|b_{21}|}+1$ and $x_2^{|b_{12}|}+1$, any coefficient-free exchange pattern $\mathcal{E}$ on $\mathbb{T}_2$ is completely determined by the values of $|b_{12}|$ and $|b_{21}|$. This means that any nontrivial coefficient-free cluster algebra of rank 2 is uniquely determined by a pair of positive integers a, b such that

$$B(t_0) = \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix},$$

since taking

$$B(t_0) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix},$$

induces the same cluster algebra. We denote a cluster algebra of this form with $\mathcal{A}(a, b)$.

To prove the Positivity conjecture for a cluster algebra $\mathcal{A}(a, b)$, Dupont first defines a finite acyclic quiver $K_{a,b}$ as follows: Let $\mathbf{v} = \{v_1, \ldots, v_a\}$ and $\mathbf{w} = \{w_1, \ldots, w_b\}$ be two sets, then we take

$$(K_{a,b})_0 = \mathbf{v} \sqcup \mathbf{w}$$

(where $\mathbf{v} \sqcup \mathbf{w}$ denotes the disjoint union of $\mathbf{v}$ and $\mathbf{w}$), and

$$(K_{a,b})_1 = \{ \ v_i \to w_j \mid \ i \in [1, a], \ j \in [1, b] \ \}.$$

To ease notation, we write $Q = K_{a,b}$. Now let $\mathcal{A}(Q)$ be the cluster algebra associated to the quiver Q with initial vertex $t_Q \in \mathbb{T}_Q$, and let $\mathbf{u} = \{u_i\}_{i \in Q_0}$ be a cluster in $\mathcal{A}(Q)$ associated to t_Q. To relate the cluster algebras $\mathcal{A}(Q)$ and $\mathcal{A}(a, b)$, Dupont defines a $\mathbb{Z}$-algebra homomorphism π called a <u>folding</u>:

$$\pi : \mathbb{Z}[\mathbf{u}^{\pm 1}] \longrightarrow \mathbb{Z}[x_1(t_0)^{\pm 1}, x_2(t_1)^{\pm 1}], \qquad u_i \longmapsto \begin{cases} x_1(t_0) & \text{if } i \in \mathbf{v}; \\ x_2(t_1) & \text{if } i \in \mathbf{w}. \end{cases}$$

Given $n \in \mathbb{Z}$, Dupont shows that for any $v, v' \in \mathbf{v}$ we have $\pi(X_{P_v[n]}) = \pi(X_{P_{v'}[n]})$ and $w, w' \in \mathbf{w}$ we have $\pi(X_{P_w[n]}) = \pi(X_{P_{w'}[n]})$. This allows for the following description of cluster variables in $\mathcal{A}(a, b)$:

Proposition 3.26 ([3, Proposition 7])**:** For any $n \in \mathbb{Z}$, for any $v \in \mathbf{v}$ and for any $w \in \mathbf{w}$ we have

$$x_1(t_{2n}) = \pi(X_{P_v[n+1]}) \qquad \text{and} \qquad x_2(t_{2n+1}) = \pi(X_{P_w[n+1]}).$$

For completion, we include the proof given by Dupont:

Proof. The proof goes by induction on n. We have $x_1(t_0) = \pi(u_v) = \pi(X_{P_v[1]})$ and $x_2(t_1) = \pi(u_w) = \pi(X_{P_w[1]})$.
Now fix some $n \in \mathbb{Z}$ and assume we have $\pi(X_{P_v[n]}) = x_1(t_{2n-2})$ and $\pi(X_{P_w[n]}) = x_2(t_{2n-1})$. It can be shown that $\mathrm{End}_{\mathcal{C}_Q}(P_v[n+1]) \cong k$ (see [8]), this means we have

$$\begin{aligned} k &\cong \mathrm{End}_{\mathcal{C}_Q}(P_v[n+1]) \\ &\cong \mathrm{Hom}_{\mathcal{C}_Q}(P_v[n+1], P_v[n+1]) \\ &\cong \mathrm{Hom}_{\mathcal{C}_Q}(P_v[n+1], (P_v[n])[1]) \\ &\cong \mathrm{Ext}^1_{\mathcal{C}_Q}(P_v[n+1], P_v[n]). \end{aligned}$$

Since $\mathcal{C}_Q$ is 2-Calabi-Yau (see [11]), we have an isomorphism of k-vector spaces

$$k \cong \operatorname{Ext}^1_{\mathcal{C}_Q}(P_v[n+1], P_v[n]) \cong \operatorname{Ext}^1_{\mathcal{C}_Q}(P_v[n], P_v[n+1]),$$

The associated triangles as in Theorem 3.23 are

$$P_v[n] \longrightarrow 0 \longrightarrow P_v[n+1] \longrightarrow P_v[n+1],$$
$$P_v[n+1] \longrightarrow \bigoplus_{j=1}^{b} P_{w_j}[n] \longrightarrow P_v[n] \longrightarrow P_v[n+2],$$

Hence we have

$$X_{P_v[n]} X_{P_v[n+1]} = \prod_{j=1}^{b} X_{P_{w_j}[n]} + 1.$$

Using our induction hypothesis we get

$$\pi(X_{P_v[n+1]}) = \frac{x_2(t_{2n-1})^b + 1}{x_1(t_{2n-2})} = x_1(t_{2n}).$$

The other cases are proved in a similar way.

□

From this proposition the Positivity conjecture for coefficient-free cluster algebras of rank 2 can be easily deduced:

Theorem 3.27 ([3, Theorem 8])**:** Any cluster variable of $\mathcal{A}(a,b)$ expressed in the variables of any cluster $\mathbf{x}$ lies in $\mathbb{Z}_{\geq 0}[\mathbf{x}^{\pm 1}]$.

Proposition 3.26 is the crucial ingredient for proving the Positivity conjecture for coefficient-free cluster algebras of rank 2. Our goal for the remainder of this section is to describe the relation between the cluster variables of $\mathcal{A}(a,b)$ and the cluster variables of $\mathcal{A}(Q)$ as in this proposition but without use of the Caldero-Chapoton map. To do this, we consider the following definition:

Definition 3.28: Let $t_Q \in \mathbb{T}_Q$ be the initial vertex of $\mathcal{A}(Q)$, then an $\mathcal{A}(Q)$-embedding of $\mathcal{A}(a,b)$ is a map $\varphi : \mathbb{T}_2 \to \mathbb{T}_Q$ satisfying:

1. $\varphi(t_0) = t_Q$;
2. For any edge $t \overset{1}{\longrightarrow} t'$ in $\mathbb{T}_2$ we have that there exists a path between $\varphi(t)$ and $\varphi(t')$ of length a, with the occurring edges having distinct labels, all of which are in $\mathbf{v}$;
3. For any edge $t \overset{2}{\longrightarrow} t'$ in $\mathbb{T}_2$ we have that there exists a path between $\varphi(t)$ and $\varphi(t')$ of length b, with the occurring edges having distinct labels, all of which are in $\mathbf{w}$,

together with a family of maps $(\varphi_t)_{t \in \mathbb{T}_2}$ where for $t \in \mathbb{T}_2$ we have:

$$\varphi_t : \mathbb{Z}[\mathbf{u}(\varphi(t))^{\pm 1}] \longrightarrow \mathbb{Z}[x_1(t)^{\pm 1}, x_2(t)^{\pm 1}], \qquad u_i(\varphi(t)) \longmapsto \begin{cases} x_1(t) & \text{if } i \in \mathbf{v}; \\ x_2(t) & \text{if } i \in \mathbf{w}. \end{cases}$$

We call φ_t the folding centered at t.

Note that we have the following result:

Proposition 3.29: Let $B = (b_{ij})$ be a matrix for which there exists a sequence $(k_i)_{i=1}^n$ of elements in $\mathbf{v}$ such that

$$B = (\mu_{k_1} \circ \cdots \circ \mu_{k_n})(B_Q),$$

then for any $v \in \mathbf{v}$ we have $b_{vw} = b_{vw'} \neq 0$ (and hence $b_{wv} = b_{w'v} \neq 0$) for any $w, w' \in \mathbf{w}$. Moreover, for any $v, v' \in \mathbf{v}$ we have $b_{vv'} = 0$, and we have $b_{ww'} = 0$ for any $w, w' \in \mathbf{w}$.

Proof. We proof this by induction on n. For $n = 0$, we have $B = B_Q$ in which case we have nothing to prove. Now assume that $n > 0$ and let $(k_i)_{i=1}^{n-1}$ be a sequence of elements in $\mathbf{v}$. Now let

$$B' = (b'_{ij}) = (\mu_{k_2} \circ \cdots \circ \mu_{k_n})(B_Q),$$

then, writing $v = k_1$, we have $B = \mu_v(B')$. Hence, for any $i, j \in Q_0$, we have by definition:

$$b_{ij} = \begin{cases} -b'_{ij} & \text{if } i = v \text{ or } j = v, \\ b'_{ij} + \frac{|b'_{iv}|b'_{vj} + b'_{iv}|b'_{vj}|}{2} & \text{otherwise.} \end{cases}$$

Now let $i, j \in Q_0$, then we can consider the following cases:

- $i = v$ or $j = v$: We have $b_{ij} = -b'_{ij}$. In particular, we therefore have $b_{vw} = b_{vw'} \neq 0$ for all $w, w' \in \mathbf{w}$;
- $i \in \mathbf{v}$ or $j \in \mathbf{v}$: This means we have $b'_{iv} = 0$ or $b'_{vj} = 0$, either way we have $b_{ij} = b'_{ij}$;
- $i, j \in \mathbf{w}$: Now we have b'_{iv} and b'_{vj} are both nonzero and have opposite sign, which implies we have $b_{ij} = b'_{ij}$.

From the case distinction above it is directly clear that B satisfies the necessary properties.

□

By a symmetric argument we also have:

Corollary 3.30: Let $B = (b_{ij})$ be a matrix for which there exists a sequence $(l_j)_{j=1}^n$ of elements in $\mathbf{w}$ such that

$$B = (\mu_{l_1} \circ \cdots \circ \mu_{l_n})(B_Q),$$

then for any $w \in \mathbf{w}$ we have $b_{vw} = b_{v'w} \neq 0$ (and hence $b_{wv} = b_{wv'} \neq 0$) for any $v, v' \in \mathbf{v}$. Moreover, for any $v, v' \in \mathbf{v}$ we have $b_{vv'} = 0$, and we have $b_{ww'} = 0$ for any $w, w' \in \mathbf{w}$.

From these results we can deduce the following result:

Corollary 3.31: Let φ be an $\mathcal{A}(Q)$-embedding of $\mathcal{A}(a, b)$, then the following statements hold:

1. For any $t \in \mathbb{T}_2$ we have $B(\varphi(t)) = \pm B_Q$;
2. Let $t \overset{1}{\text{———}} t'$ be an edge in $\mathbb{T}_2$. Let $\{t_i\}_{i=0}^a$ denote the vertices in $\mathbb{T}_Q$ occurring in the path from Definition 3.28.2, then for any $i, j \in [0, a]$ and for any $v \in \mathbf{v}$ the exchange polynomials associated the edges $t_i \overset{v}{\text{———}} \_$ and $t_j \overset{v}{\text{———}} \_$ are the same.
3. Let $t \overset{2}{\text{———}} t'$ be an edge in $\mathbb{T}_2$. Let $\{t_j\}_{j=0}^b$ denote the vertices in $\mathbb{T}_Q$ occurring in the path from Definition 3.28.3, then for any $i, j \in [0, b]$ and for any $w \in \mathbf{w}$ the exchange polynomials

associated the edges $t_i \xrightarrow{\ w\ } \_$ and $t_j \xrightarrow{\ w\ } \_$ are the same.

From this corollary we can deduce the desired result:

Lemma 3.32: Let φ be an $\mathcal{A}(Q)$-embedding of $\mathcal{A}(a,b)$, then for any $t, t' \in \mathbb{T}_2$, for any $v \in \mathbf{v}$ and for any $w \in \mathbf{w}$ we have

$$x_1(t') = \varphi_t(u_v(\varphi(t'))) \qquad \text{and} \qquad x_2(t') = \varphi_t(u_w(\varphi(t'))).$$

Proof. We prove this lemma with induction on the length of the shortest path between t and t'. By definition, we have $\varphi_t(u_v(\varphi(t))) = x_1(t)$ for any $v \in \mathbf{v}$ and $\varphi_t(u_w(\varphi(t))) = x_2(t)$ for any $w \in \mathbf{w}$. Now let $t' \in \mathbb{T}_2$ such that $t \neq t'$ and

$$x_1(t') = \varphi_t(u_v(\varphi(t'))) \qquad \text{and} \qquad x_2(t') = \varphi_t(u_w(\varphi(t'))).$$

For any $v \in \mathbf{v}$ and for any $w \in \mathbf{w}$ let G_v and H_v denote the cluster polynomials in $\mathbb{Z}[\mathbf{u}^{\pm}]$ associated to the vertex $\varphi(t)$, such that

$$G_v(\mathbf{u}(\varphi(t'))) = \varphi_t(u_v(\varphi(t'))) \qquad \text{and} \qquad H_w(\mathbf{u}(\varphi(t'))) = \varphi_t(u_w(\varphi(t'))).$$

Moreover, let G and H denote the cluster polynomials in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ associated to the vertex t, such that

$$G_v(\mathbf{u}(\varphi(t'))) = \varphi_t(u_v(\varphi(t'))) \qquad \text{and} \qquad H_w(\mathbf{u}(\varphi(t'))) = \varphi_t(u_w(\varphi(t'))).$$

Now we define the $\mathbb{Z}$-algebra homomorphism $\pi : \mathbb{Z}[\mathbf{u}^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ given by the mapping

$$u_i \longmapsto \begin{cases} x_1 & \text{if } i \in \mathbf{v}; \\ x_2 & \text{if } i \in \mathbf{w}. \end{cases}$$

Then we have $G = \pi(G_v)$ for all $v \in \mathbf{v}$ and $H = \pi(H_w)$ for all $w \in \mathbf{w}$. Now let $t' \xrightarrow{\ 1\ } t''$ be an edge in $\mathbb{T}_2$ with associated exchange polynomial $P_1 = x_2^b + 1$, then for any $v \in \mathbf{v}$ we have that exchange polynomial associated to the edge $\varphi(t') \xrightarrow{\ v\ } \_$ in $\mathbb{T}_Q$ is given by

$$P_v = \prod_{w \in \mathbf{w}} u_w + 1,$$

which follows from the first part of Corollary 3.31. This means that for any $v \in \mathbf{v}$ we have $P_1 = \pi(P_v)$. This means we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}[\mathbf{u}^{\pm 1}] & \xrightarrow{\ \phi_{\mathbf{v}}\ } & \mathbb{Z}[\mathbf{u}^{\pm 1}] \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}] & \xrightarrow{\ x_1 \mapsto P_1/x_1\ } & \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}], \end{array}$$

where $\phi_{\mathbf{v}} : \mathbb{Z}[\mathbf{u}^{\pm 1}] \to \mathbb{Z}[\mathbf{u}^{\pm 1}]$ denotes the $\mathbb{Z}$-algebra homomorphism given by the mapping

$$u_i \longmapsto \begin{cases} P_i/u_i & \text{if } i \in \mathbf{v}; \\ u_i & \text{if } i \in \mathbf{w}. \end{cases}$$

In particular we have

$$x_1(t'') = \varphi_t(u_v(\varphi(t''))) \qquad \text{and} \qquad x_2(t'') = \varphi_t(u_w(\varphi(t''))).$$

A similar argument can be given for the edge $t' \overset{2}{\longrightarrow} \_$, which then proves the lemma.

□

In the next chapter we will generalize this procedure of creating an embedding of a coefficient free cluster algebra into a acyclic cluster algebra to deduce some properties of coefficient matrices.

3.5 Generalization of $\mathcal{A}(Q)$-embedding

In the previous section we saw that by embedding a coefficient-free cluster algebra of rank 2 in a particular acyclic cluster algebra, the Positivity conjecture for the embedded cluster algebra could be deduced from the fact that the Positivity conjecture holds for acyclic cluster algebras (Theorem 3.27). In this section we will show a potential way to generalize this procedure for a certain class of cluster algebras of rank ≥ 3.

Definition 3.33: We say that a sign-skew-symmetric $N \times N$ matrix $B = (b_{ij})$ is acyclic if there exists a finite acyclic quiver Q, with

$$Q_0 = \bigsqcup_{i=1}^{N} \mathbf{v}^i,$$

where $\mathbf{v}^i$ denotes a finite set for each $i \in [1, N]$, and, writing $B_Q = (b'_{vw})$ for the skew-symmetric matrix associated to Q (see Remark 3.3), we have for any $i, j \in [1, N]$ and for any $v^i \in \mathbf{v}^i$ that

$$b_{ij} = \sum_{v^j \in \mathbf{v}^j} b'_{v^i v^j} \qquad \text{and} \qquad \left| \sum_{v^j \in \mathbf{v}^j} b'_{v^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |b'_{v^i v^j}|.$$

For any $i \in [1, N]$ we will denote a representative of the set $\mathbf{v}^i$ with $\mathbf{v}^i$, and when using this notation in an equation, we take every occurrence of $\mathbf{v}^i$ in that equation to be the same representative. Using this notation we can write the above equalities as

$$b_{ij} = \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} \qquad \text{and} \qquad \left| \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |b'_{\mathbf{v}^i v^j}|$$

for any $i, j \in [1, N]$.

Example 3.34: Let $a, b \in \mathbb{Z}_{>0}$, then we consider the matrix

$$B = \begin{pmatrix} 0 & b & 1 \\ -a & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

To the matrix B we associate a finite acyclic quiver K_B as follows: Let $\mathbf{v} = \{v_1, \dots, v_a\}$, $\mathbf{w} = \{w_1, \dots, w_b\}$ and $\mathbf{y} = \{y_1, \dots, y_a\}$ be three sets, then we take

$$(K_B)_0 = \mathbf{v} \sqcup \mathbf{w} \sqcup \mathbf{y},$$

and

$$(K_B)_1 = \{ \ v_i \to w_j \ | \ i \in [1,a], j \in [1,b] \ \} \cup \{ \ v_i \to y_i \ | \ i \in [1,a] \ \} \,.$$

Then it is clear that matrix associated to K_B satisfies the equalities above and we have that B is acyclic.

We now fix an acyclic sign-skew symmetric $N \times N$ matrix $B = (b_{ij})$, and we let Q be an associated quiver as in the definition above.

Remark 3.35: Let $i \in [1,N]$ and $v_1^i \in \mathbf{v}^i$, and write $\mu_{v_1^i}(B_Q) = B'_Q = (b''_{vw})$. Then for any $v_2^i \in \mathbf{v}^i$ not equal to v_1^i, we have for any $w \in Q_0$ that $b''_{v_2^i w} = b'_{v_2^i w}$ and $b''_{w v_2^i} = b'_{w v_2^i}$. This follows from the fact that $b'_{v_1^i v_2^i} = 0 = b'_{v_2^i v_1^i}$ (otherwise the equalities in the definition fail). In particular, for any $j \in [1,N]$, we have

$$|b_{ij}| = \sum_{v^j \in \mathbf{v}^j} |b''_{v_1^i v^j}| = \sum_{v^j \in \mathbf{v}^j} |b''_{\mathbf{v}^i v^j}|.$$

Since this implies that $b''_{v_1^i v_2^i} = 0 = b''_{v_2^i v_1^i}$ for any $v_2^i \in \mathbf{v}^i$, we see that consecutively mutating B_Q in any sequence of directions in $\mathbf{v}^i$ does not depend on the ordering of this sequence. This allows us to write $\mu(\mathbf{v}^i, B_Q)$ for the matrix obtained by mutating the matrix B_Q consecutively, once in each direction in $\mathbf{v}^i$, in any order.

Lemma 3.36: For any $n \in [1,N]$, let $C = (c_{ij}) = \mu_n(B)$ and let $C_Q = \mu(\mathbf{v}^n, B_Q) = (c'_{vw})$, then for any $i,j \in [1,N]$ we have

$$c_{ij} = \sum_{v^j \in \mathbf{v}^j} c'_{\mathbf{v}^i v^j} \qquad \text{and} \qquad \left| \sum_{v^j \in \mathbf{v}^j} c'_{\mathbf{v}^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |c'_{\mathbf{v}^i v^j}|$$

Proof. For $j \in [1,N]$ we have

$$c_{nj} = -b_{nj} = -\sum_{v^j \in \mathbf{v}^j} b'_{v^n v^j} = \sum_{v^j \in \mathbf{v}^j} c'_{v^n v^j},$$

and similarly for $i \in [1,N]$ we have

$$c_{in} = -b_{in} = -\sum_{v^n \in \mathbf{v}^n} b'_{v^i v^n} = \sum_{v^n \in \mathbf{v}^n} c'_{v^i v^n}.$$

That the right equality holds in these two cases is clear. Now let $i, j \in [1, N] \setminus \{n\}$, then we have

$$\begin{aligned}
\sum_{v^j \in \mathbf{v}^j} c'_{\mathbf{v}^i v^j} &= \sum_{v^j \in \mathbf{v}^j} \left(b'_{\mathbf{v}^i v^j} + \sum_{v^n \in \mathbf{v}^n} \frac{|b'_{\mathbf{v}^i v^n}| b'_{v^n v^j} + b'_{\mathbf{v}^i v^n} |b'_{v^n v^j}|}{2} \right) \\
&= \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} + \frac{1}{2} \sum_{v^n \in \mathbf{v}^n} \left(|b'_{\mathbf{v}^i v^n}| \cdot \left(\sum_{v^j \in \mathbf{v}^j} b'_{v^n v^j} \right) \right) + \frac{1}{2} \sum_{v^n \in \mathbf{v}^n} \left(b'_{\mathbf{v}^i v^n} \cdot \left(\sum_{v^j \in \mathbf{v}^j} |b'_{v^n v^j}| \right) \right) \\
&= \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} + \frac{1}{2} \sum_{v^n \in \mathbf{v}^n} \left(|b'_{\mathbf{v}^i v^n}| \cdot \left(\sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^n v^j} \right) \right) + \frac{1}{2} \sum_{v^n \in \mathbf{v}^n} \left(b'_{\mathbf{v}^i v^n} \cdot \left(\sum_{v^j \in \mathbf{v}^j} |b'_{\mathbf{v}^n v^j}| \right) \right) \\
&= b_{ij} + \frac{b_{nj}}{2} \sum_{v^n \in \mathbf{v}^n} |b'_{\mathbf{v}^i v^n}| + \frac{|b_{nj}|}{2} \sum_{v^n \in \mathbf{v}^n} b'_{\mathbf{v}^i v^n} \\
&= b_{ij} + \frac{|b_{in}| b_{nj} + b_{in} |b_{nj}|}{2} = c_{ij}.
\end{aligned}$$

Now we just have to show that for any $i \in [1, N]$ not equal to n we have

$$\left| \sum_{v^j \in \mathbf{v}^j} c'_{\mathbf{v}^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |c'_{\mathbf{v}^i v^j}|.$$

Now let $i, j \in [1, N] \setminus \{n\}$ be distinct, and let $v_i \in \mathbf{v}^i$ and $v^j \in \mathbf{v}^j$ such that $b'_{v^i v^j} \neq 0$, $b'_{v^n v^j} \neq 0$ and $b_{v^i v^n} \neq 0$. Moreover, assume that $b'_{v^n v^j}$ and $b'_{v^n v^j}$ have the same sign s (where $s = \pm 1$). Without loss of generality, we assume $s = -1$. This means there exist arrows

$$v^j \longrightarrow v^n \qquad \text{and} \qquad v^n \longrightarrow v^i$$

in Q. This implies we must have $b'_{v^i v^j} < 0$, otherwise we would have that there exists an arrow $v^i \to v^j$ in Q which would mean that Q contains an oriented cycle and that cannot happen since Q is acyclic. This means that every nonzero term in the sum

$$\sum_{v^n \in \mathbf{v}^n} \frac{|b'_{v^i v^n}| b'_{v^n v^j} + b'_{v^i v^n} |b'_{v^n v^j}|}{2}$$

has the same sign as $b'_{v^i v^j}$. We conclude that for any $i \in [1, N]$ not equal to n we have

$$\left| \sum_{v^j \in \mathbf{v}^j} c'_{\mathbf{v}^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |c'_{\mathbf{v}^i v^j}|.$$

□

Now note that the quiver associated to the matrix C_Q in the lemmma above is not necessarily acyclic, which can be seen in the following example:

Example 3.37: Consider the sign-skew-symmetric matrix

$$B = \begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then B is acyclic, and we can associate a quiver Q to B as follows: Let $\mathbf{v} = \{v_1\}$, $\mathbf{w} = \{w_1, w_2\}$ and $\mathbf{y} = \{y_1\}$, then we set

$$Q_0 = \mathbf{v} \sqcup \mathbf{w} \sqcup \mathbf{y},$$

and let

$$Q_1 = \{v_1 \to w_1, v_1 \to w_2, y_1 \to v_1\}.$$

This means that the matrix B_Q can be written as

$$B_Q = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

where the rows and columns are indexed over the set $\{v_1, w_1, w_2, y_1\}$. Now we have

$$\mu_1(B) = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix},$$

and

$$\mu_{v_1}(B_Q) = \mu(\mathbf{v}, B_Q) = \begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}.$$

The quiver associated to $\mu_{v_1}(B_Q)$ is the quiver Q' with

$$Q'_0 = \mathbf{v} \sqcup \mathbf{w} \sqcup \mathbf{y},$$

and with

$$Q'_1 = \{v_1 \to y_1, w_1 \to v_1, w_2 \to v_1, y_1 \to w_1, y_1 \to w_2\}.$$

Clearly we have the path $v_1 \to y_1 \to w_1 \to v_1$ in Q, which is an oriented cycle. We conclude that Q' is not acyclic.

This gives rise to the following definition:

Definition 3.38: Fix some vertex $t_0 \in \mathbb{T}_N$. Let $\big(B(t)\big)_{t\in\mathbb{T}_N}$ be a family of matrices such that $B(t_0) = B$ and such that for any vertex $t \overset{n}{\text{——}} t'$ in $\mathbb{T}_N$ we have $B(t') = \mu_n(B(t))$. Moreover, let $\big(B_Q(t)\big)_{t\in\mathbb{T}_N}$ be a family of matrices such that $B_Q(t_0) = B_Q$ and such that for any vertex $t \overset{n}{\text{——}} t'$ in $\mathbb{T}_N$ we have $B_Q(t')$ is a matrix obtained by mutating the matrix $B_Q(t)$ consecutively, once in each direction in $\mathbf{v}^n$ (order of applying the mutations does not matter). We call the matrix B

quiver representable (with respect to the quiver Q) if for any vertex $t \in \mathbb{T}_N$, writing $B(t) = (b_{ij})$ and $B_Q(t) = (b'_{vw})$, we have

$$b_{ij} = \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} \qquad \text{and} \qquad \left| \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |b'_{\mathbf{v}^i v^j}|$$

for any $i, j \in [1, N]$.

If B is quiver representable with respect to the quiver Q, then we have the following results:

1. For any $t \in \mathbb{T}_N$, the matrix $B(t)$ is sign-skew-symmetric;
2. To the matrix B we can associate a unique coefficient-free exchange pattern on $\mathbb{T}_N$ with $B(t_0) = B$, as in Example 1.8.
3. Writing $\mathcal{A}(B)$ for the cluster algebra of rank N over $\mathbb{Z}$ associated to this unique coefficient-free exchange pattern on $\mathbb{T}_N$ associated to B, then we can define an $\mathcal{A}(Q)$-emmbedding of $\mathcal{A}(B)$ in a similar fashion as in the previous section and we have that an analogue of Lemma 3.32 holds for such an embedding. (In particular, the Positivity conjecture holds for the cluster algebra $\mathcal{A}(B)$.)

A case when an acyclic matrix is quiver representable is given by the lemma:

Lemma 3.39: For any $a, b \in \mathbb{Z}_{>0}$, then the acyclic matrix

$$B = \begin{pmatrix} 0 & b & 1 \\ -a & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

is quiver representable with respect to the associated quiver Q equal to the quiver K_B as constructed in Example 3.34.

Proof. We claim that for any $t \in \mathbb{T}_3$ we can write

$$B_Q(t) = \begin{pmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ -B_{0,1}^T & B_{1,1} & B_{1,2} \\ -B_{0,2}^T & -B_{1,2}^T & B_{2,2} \end{pmatrix},$$

where:

1. $B_{0,0}, B_{1,1}$ and $B_{2,2}$ are respectively the $a \times a$, the $b \times b$ and the $a \times a$ zero matrix;
2. $B_{0,1}$ is an $a \times b$ matrix, all of whose entries are the same;
3. $B_{1,2}$ is an $b \times a$ matrix, all of whose entries are the same;
4. $B_{0,2}$ is an $a \times a$ matrix, with all of its diagonal entries the same and all of its non-diagonal entries the same, such that the absolute value of the difference of a diagonal entry and a non-diagonal entry is equal to 1.

We know that $B_Q(t_0)$ is of this form. Now we can prove our claim using induction on the length of the shortest path between t and t_0: Let $t \in \mathbb{T}_3$ such that $B_Q(t)$ satisfies our claim, and for

$n \in \{1,2,3\}$ let $t \overset{n}{\text{———}} t_n$ be an edge in $\mathbb{T}_3$. Write

$$B_Q(t_n) = \begin{pmatrix} C_{0,0} & C_{0,1} & C_{0,2} \\ -C_{0,1}^T & C_{1,1} & C_{1,2} \\ -C_{0,2}^T & -C_{1,2}^T & C_{2,2} \end{pmatrix}.$$

For all values of n we have $C_{0,0} = B_{0,0}$, $C_{1,1} = B_{1,1}$ and $C_{2,2} = B_{2,2}$. Let p denote the unique value of all the entries in $B_{0,1}$, let q denote the unique value of all the entries in $B_{1,2}$, and let $r, s \in \mathbb{Z}$ such that $|r - s| = 1$ and such that the diagonal entries of $B_{0,2}$ are equal to r and the non-diagonal entries of $B_{0,2}$ are equal to s. Now we distinguish three cases:

- $n = 1$: Now we have $C_{0,1} = -B_{0,1}$ and $C_{0,2} = -B_{0,2}$. If $p \neq 0$ and the sign of p is equal to the sign of $r + s$, then each entry of $C_{1,2}$ is equal to $q + |p| \cdot (r + (a-1)s)$. Otherwise, we have $C_{0,2} = B_{0,2}$.
- $n = 2$: In this case we have $C_{0,1} = -B_{0,1}$ and $C_{1,2} = -B_{1,2}$. If p and q are both nonzero with equal sign, we have that $C_{0,2} = B_{0,2} + C$ where C is the $a \times a$ integer matrix whose entries are equal to $b \cdot |p| \cdot q$. Otherwise, we have $C_{0,2} = B_{0,2}$.
- $n = 3$: We now have $C_{0,2} = -B_{0,2}$ and $C_{1,2} = -B_{1,2}$. If $q \neq 0$ and the sign of q is equal to the sign of $r + s$, then each entry of $C_{0,1}$ is equal to $p + |q| \cdot (r + (a-1)s)$. Otherwise, we have $C_{0,1} = B_{0,1}$.

In all three cases the matrix $B_Q(t_n)$ satisfies the necessary properties, hence our claim holds.

Now let $t \in \mathbb{T}_3$ and write $B(t) = (b_{ij})$ and $B_Q(t) = (b'_{vw})$. From our claim we directly have

$$\left| \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} \right| = \sum_{v^j \in \mathbf{v}^j} |b'_{\mathbf{v}^i v^j}| \qquad \big(\forall i, j \in [1,3]\big).$$

Now applying the same argument as in the first part of the proof of Lemma 3.36, we have by induction on the length of the shortest path between t and t_0 in $\mathbb{T}_3$ that

$$b_{ij} = \sum_{v^j \in \mathbf{v}^j} b'_{\mathbf{v}^i v^j} \qquad \big(\forall i, j \in [1,3]\big).$$

□

As mentioned before, we can deduce the following theorem from this lemma:

Theorem 3.40: Let $a, b \in \mathbb{Z}_{>0}$ and let

$$B = \begin{pmatrix} 0 & b & 1 \\ -a & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

then any cluster variable of $\mathcal{A}(B)$ expressed in the variables of any cluster $\mathbf{x}$ lies in $\mathbb{Z}_{\geq 0}[\mathbf{x}^{\pm 1}]$.

Remark 3.41: We expect that there exist many more acyclic matrices which are quiver representable. For instance, for $a, b \in \mathbb{Z}_{>0}$ the matrix

$$B = \begin{pmatrix} 0 & b & 1 & 0 \\ -a & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

can be proven to be quiver representable using a similar argument as in Lemma 3.39. Moreover, one can extend this process of embedding a cluster algebra into an acyclic cluster algebra to an embedding of a cluster algebra into a skew-symmeteric cluster algebra. Then using the fact that the Positivity conjecture holds for these skew-symmeteric cluster algebras ([4]), one can deduce the Positivity conjecture for the cluster algebras which can be embedded into a skew-symmeteric cluster algebra.

4 Results and Conjectures

Now that we have seen some cases where the Positivity conjecture holds, does this allow us to say more about (minimal) coefficient matrices and cluster polynomials? We already saw in Section 2.3 how there is a strong relation between the Positivity conjecture and minimal coefficient matrices (see Corollary 2.30). In this chapter we will discuss the results we can deduce from the previous chapter, and we end with a discussion of some conjectures about properties of minimal coefficient matrices.

4.1 Totally positive coefficient matrices

To study the relation of coefficient matrices with the Positivity conjecture, we introduce the notion of totally positive coefficient matrices.

Definition 4.1: For $a, b \in \mathbb{Z}_{>0}$, let $\mathbf{Cf}(a,b)$ denote the set of all triples $(m,n,C) \in \mathbb{Z}\times\mathbb{Z}\times\mathrm{Mat}_2(\mathbb{Z})$ such that C is an $(\mathbf{seq}(m,a), \mathbf{seq}(n,b))$-coefficient matrix. We identify an element (m,n,C) of $\mathbf{Cf}(a,b)$ with its matrix C and we write $m(C) = m$ and $n(C) = n$. In other words, we think of $\mathbf{Cf}(a,b)$ as the set of all $(\mathbf{m},\mathbf{n})$-coefficient matrices C with $\mathbf{m} = \mathbf{seq}(m(C),a)$ and $\mathbf{n} = \mathbf{seq}(n(C),b)$.

On $\mathbf{Cf}(a,b)$ we define two maps

$$\phi_D^{(a,b)} : \mathbf{Cf}(a,b) \longrightarrow \mathbf{Cf}(a,b) \qquad \text{and} \qquad \phi_E^{(a,b)} : \mathbf{Cf}(a,b) \longrightarrow \mathbf{Cf}(a,b),$$

which we also will denote with ϕ_D and ϕ_E respectively if there is no confusion about the domains. These maps are defined as follows: Let $C \in \mathbf{Cf}(a,b)$ be an $(\mathbf{m},\mathbf{n})$-coefficient matrix with $m = m(C)$ and $n = n(C)$, and write $D = D(C)$, $E = E(C)$ and $\dim(C) = (x,y,K,L)$. Let $\mathbf{m}' = \mathbf{seq}(La - m, a)$, then $\phi_D(C) = (La - m, n, C')$, where C' is the $(\mathbf{m}',\mathbf{n})$-coefficient matrix with origin $(x,0)$, which for $(k,l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c'_{k,l} = \sum_{i=0}^{k} d_{i,L-l} \binom{m'_l}{k-i}.$$

For $\mathbf{n}' = \mathbf{seq}(Kb - n, b)$, we have $\phi_E(C) = (m, Kb - n, C'')$, where C'' is the $(\mathbf{m},\mathbf{n}')$-coefficient matrix with origin $(0,y)$, which for $(k,l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c''_{k,l} = \sum_{j=0}^{l} e_{K-k,j} \binom{n'_k}{l-j}.$$

Note that the maps ϕ_D and ϕ_E are well-defined by Lemma 2.21. The <u>orbit</u> of a coefficient matrix $C \in \mathbf{Cf}(a,b)$ (denoted with $\mathcal{O}^{(a,b)}(C)$ or just $\mathcal{O}(C)$) is a family $\{C^n\}_{n\in\mathbb{Z}}$ of coefficient matrices in $\mathbf{Cf}(a,b)$ such that $C^0 = C$, and for all $n \in \mathbb{Z}$ we have

$$C^{2n+1} = \phi_D(C^{2n}) \qquad \text{and} \qquad C^{2n} = \phi_E(C^{2n-1}).$$

We say that C is <u>reducible</u> if there exists some $N \in \mathbb{Z}$ such that $m(C^N) \leq 0$ and $n(C^M) \leq 0$, moreover, we call C^N a <u>reduced element</u> of $\mathcal{O}(C)$. We say that C is <u>totally positive</u> if every coefficient matrix in $\mathcal{O}(C)$ lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

We give a simple example of totally positive coefficient matrices:

Example 4.2: Every minimal coefficient matrix in $\mathbf{Cf}(1,1)$ is totally positive. This follows from the fact for any $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$ we have that for the minimal $(\mathbf{seq}(m,1), \mathbf{seq}(n,1))$-coefficient matrix C the matrices $D'(C)$ and $E'(C)$ lie in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$: We saw in Example 2.6 that the matrix $D = D(C)$ is for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ given by

$$d_{k,l} = \begin{cases} \binom{n-m}{k}\binom{n-k}{n-l} & \text{if } k \leq n \text{ and } l \leq m; \\ \binom{m}{k}\binom{n_k}{l} & \text{otherwise.} \end{cases}$$

From this we can deduce that $E'(C)$ lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$. Moreover, we have that the matrix $E = E(C)$ is for all $(k,l) \in \mathbb{Z}^2_{\geq 0}$ given by

$$e_{k,l} = \begin{cases} \binom{m}{k} & \text{if } l = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This means that the matrix $D'(C)$ also lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

We conclude, using Lemma 2.28 and Corollary 2.29, that every minimal coefficient matrix in $\mathbf{Cf}(1,1)$ is totally positive.

Remark 4.3: Let $C \in \mathbf{Cf}(a,b)$, then $0 \in \mathcal{O}(C)$ if and only if $C = 0$

Remark 4.4: To any coefficient matrix $C \in \mathbf{Cf}(a,b)$ with $m = m(C)$ and $n = n(C)$ we can associate a Laurent polynomial

$$G_C = \frac{1}{x_1^m x_2^n} \cdot \sum_{(k,l) \in \mathbb{Z}^2_{\geq 0}} c_{k,l} x_1^{la} x_2^{kb}. \tag{6}$$

By definition, we now have that substituting $(x_2^b + 1)/x_1$ for x_1 in G_C results in the Laurent polynomial $G_{\phi_D(C)}$, and substituting $(x_1^a + 1)/x_2$ for x_2 in G_C results in the Laurent polynomial $G_{\phi_E(C)}$.

Lemma 4.5: Let $C \in \mathbf{Cf}(a,b)$ be a reducible coefficient matrix such that $\mathcal{O}(C)$ contains a reduced element which lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$, then C is totally positive.

Proof. Write $m = m(C)$ and $n = n(C)$. Without loss of generality assume we have $C \neq 0$ is a reduced element of $\mathcal{O}(C)$ such that $C \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$. Now consider the polynomial

$$F = x_1^{|m|} x_2^{|n|} \cdot \sum_{(k,l) \in \mathbb{Z}^2_{\geq 0}} c_{k,l} x_1^{la} x_2^{kb},$$

let $\mathcal{A}(a,b)$ be as in the previous chapter, and let $t_0 \in \mathbb{T}_2$. Then for any coefficient matrix $C' \in \mathcal{O}(C)$ there exists $t \in \mathbb{T}_2$, and G and H, cluster polynomials associated to t_0 satisfying $x_1(t) = G(\mathbf{x}(t_0))$ and $x_2(t) = H(\mathbf{x}(t_0))$, such that the Laurent polynomial $G_{C'}$ associated to C', as defined in the remark above, is equal to $F(G, H)$. By Theorem 3.27 we have that G and H both have positive coefficients, and by assumption F also has positive coefficients, hence $C' \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

□

Corollary 4.6: Let C be the matrix which has a single nonzero entry given by $c_{0,0} = 1$, then for any $n, m \in \mathbb{Z}_{\geq 0}$ we have that the coefficient matrix $(-m, -n, C) \in \mathbf{Cf}(a, b)$ is totally positive.

This means that the Positivity conjecture holds for all cluster algebras of rank 2 (also the non-coefficient-free cluster algebras):

Theorem 4.7: Let $\mathcal{E}$ be an exchange pattern on $\mathbb{T}_2$ with coefficients in some coefficient group $\mathbb{P}$, then any cluster algebra $\mathcal{A}$ associated to $\mathcal{E}$ satisfies the Positivity conjecture.

Proof. Follows directly from Corollary 2.30 and Corollary 4.6.

□

We end this section with discussing some other classes of totally positive coefficient matrices.

Proposition 4.8: Let $(m, n, C) \in \mathbf{Cf}(a, b)$ be a totally positive coefficient matrix, then for any $m', n' \in \mathbb{Z}_{\geq 0}$ we have that $(m - m', n - n', C) \in \mathbf{Cf}(a, b)$ is also a totally positive coefficient matrix.

Proof. Let $G_{(m,n,C)}$ denote the Laurent polynomial associated to (m, n, C) as in Remark 4.4. Then we have $G_{(m-m',n-n',C)} = x_1^{m'} x_2^{n'} G_{(m,n,C)}$ is the Laurent polynomial associated to $(m-m', n-n', C)$. Now we apply the same reasoning as in the proof of Lemma 4.5. Let $\mathcal{A}(a, b)$ be as in the previous chapter, and let $t_0 \in \mathbb{T}_2$. Now for any $C' \in \mathcal{O}((m - m', n - n', C))$ there exists $t \in \mathbb{T}_2$, and G and H, cluster polynomials associated to t_0 satisfying $x_1(t) = G(\mathbf{x}(t_0))$ and $x_2(t) = H(\mathbf{x}(t_0))$, such that the Laurent polynomial $G_{C'}$ associated to C', as defined in the remark above, is equal to

$$G_{(m-m',n-n',C)}(G, H) = G^{m'} H^{n'} G_{(m,n,C)}(G, H).$$

By Theorem 3.27, G and H have both positive coefficients, and by the assumption that C is totally positive, we have that $G_{(m,n,C)}(G, H)$ has positive coefficients. We conclude that $G_{C'}$ has positive coefficients, which means that $(m - m', n - n', C)$ is totally positive.

□

Proposition 4.9: Let $(m, n, C), (m', n', C') \in \mathbf{Cf}(a, b)$ be two totally positive coefficient matrices. Let $C'' \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ be the matrix which for all $(k, l) \in \mathbb{Z}^2$ is given by

$$c''_{k,l} = \sum_{(x,y) \in \mathbb{Z}^2} c_{x,y} c'_{k-x,l-y},$$

then we have that $(m + m', n + n', C'')$ is totally positive in $\mathbf{Cf}(a, b)$.

Proof. This follows directly from the fact that

$$G_{(m+m',n+n',C'')} = G_{(m,n,C)} \cdot G_{(m',n',C')}.$$

□

Corollary 4.10: For any $m, n \in \mathbb{Z}_{\geq 0}$, let $C \in \mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ be a matrix with origin $(0, 0)$ and which for $(k,l) \in \mathbb{Z}^2_{\geq 0}$ is given by

$$c_{k,l} = \binom{m}{k}\binom{n}{l},$$

then (m,n,C) is totally positive in $\mathbf{Cf}(a,b)$.

4.2 Conjectures

In the previous section we saw that any minimal coefficient matrix in $\mathbf{Cf}(a,b)$ which is reducible is totally positive. Since every minimal coefficient matrix is defined and constructed in the same way, one would expect that total positivity would be a property of every minimal coefficient matrix in $\mathbf{Cf}(a,b)$. This would be in line with the Positivity conjecture in general, since one would not expect the coefficient matrices occurring in the cluster polynomials in Theorem 2.25 to have negative summands (regarded as a sum of minimal coefficient matrices see Proposition 2.11). This gives rise to the following conjecture:

Conjecture 4.11: For any $a, b \in \mathbb{Z}_{>0}$, any minimal coefficient matrix in $\mathbf{Cf}(a,b)$ is totally positive.

We note that, as most of the classes of totally positive minimal coefficient matrices arise from the fact that the Positivity conjecture holds for acyclic cluster algebras. As we mentioned in Remark 3.41, there are more results to obtain in this direction. We expect that obtaining these results it might result in a proof of the conjecture above.

For the remainder of this section we fix some $N > 1$, some coefficient group $\mathbb{P}$ and some exchange pattern $\mathcal{E} = \big((\mathbf{x}(t))_{t\in\mathbb{T}_N}, (B(t))_{t\in\mathbb{T}_N}, (\mathbf{p}(t))_{t\in\mathbb{T}_N}\big)$ on $\mathbb{T}_N$ with coefficients in $\mathbb{P}$. That Conjecture 4.11 represents an important step in the direction of proving the Positivity conjecture in for an arbitrary cluster algebra, follows from the following discussion:

Definition 4.12: Let M be a Laurent monomial in the variables $x_1, \ldots, x_N$, write

$$M = \prod_{i=1}^{N} x_i^{m_i} \qquad \big(m_i \in \mathbb{Z}\big).$$

Let $G = F/M$ be an M-cluster polynomial associated to some vertex $t \in \mathbb{T}_N$. Then G is called <u>positive</u> if for any distinct $u, v \in [1,N]$ we have that

$$G = \frac{x_u^{m_u} x_v^{m_v}}{M} \cdot \left(\sum_{i=0}^{[m_u]_+} \sum_{j=0}^{[m_v]_+} F_{i,j} \cdot G_{i,j} \right),$$

where for any $i \in [0, [m_u]_+]$, $j \in [0, [m_v]_+]$, we have that $G_{i,j}$ is a minimal $x_u^{m_u-i} x_v^{m_v-j}$-cluster polynomial, and $F_{i,j} \in \mathbb{ZP}[x_1, \ldots, x_N]$ is a subtraction free polynomial.

Now let G be some positive M-cluster polynomial associated to some vertex $t \in \mathbb{T}_N$. Now fix distinct $u, v \in [1,N]$, and let $\mathbb{T}_{\{u,v\}}$ denote the 2-regular subtree of $\mathbb{T}_N$, containing the vertex t. Now for any $t' \in \mathbb{T}_{\{u,v\}}$ let $G_{u,t'}$ and $G_{v,t'}$ denote the cluster polynomials such that $x_u(t') = G_{u,t'}(\mathbf{x}(t))$ and $x_v(t') = G_{v,t'}(\mathbf{x}(t))$. Then, assuming Conjecture 4.11 holds, we have for any $t' \in \mathbb{T}_{\{u,v\}}$ that substituting $G_{u,t'}$ for x_u and $G_{v,t'}$ for x_v in G gives us a cluster polynomial G' associated to the vertex t' whose numerator is a subtraction free polynomial. In line with the Positivity conjecture, one would expect that G' is again a positive cluster polynomial. A result which brings us close to proving this can be stated as the following conjecture:

Conjecture 4.13: Let $m, n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{Z}_{>0}$, write $\mathbf{m} = \mathbf{seq}(m, a)$ and $\mathbf{n} = \mathbf{seq}(n, b)$, and let C be the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix. Then the matrices $S(\mathbf{m}, \mathbf{seq}(n-1, b))(C)$ and $S(\mathbf{seq}(m-1, a), \mathbf{n})(C)$ lie in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

This conjecture would give us that for any $u \in [1, N]$ and for any positive M-cluster polynomial G associated to some vertex $t \in T_N$, we have that $x_u G$ is a positive M/x_u-cluster polynomial associated to t.

A case where Conjecture 4.13 holds is given by the following proposition:

Proposition 4.14: Let $m, n \in \mathbb{Z}_{\geq 0}$ such that $m \leq n$. Write $\mathbf{m} = \mathbf{seq}(m, 1)$ and $\mathbf{n} = \mathbf{seq}(n, 1)$. Let C be the minimal $(\mathbf{m}, \mathbf{n})$-coefficient matrix, then $S(\mathbf{m}, \mathbf{seq}(n-1, 1))(C)$ and $S(\mathbf{seq}(m-1, 1), \mathbf{n})(C)$ lie in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$.

Proof. We use the results obtained in Example 2.6. Since the matrix $E = E(C)$ is for all $(k, l) \in \mathbb{Z}^2_{\geq 0}$ given by

$$e_{k,l} = \begin{cases} \binom{m}{k} & \text{if } l = 0; \\ 0 & \text{otherwise,} \end{cases}$$

we directly have that $S = S(\mathbf{seq}(m-1, 1), \mathbf{n})(C)$ is a matrix in $\mathrm{Mat}_2(\mathbb{Z})$ whose only nonzero entries are given by $s_{0,0} = 1$ and $s_{1,0} = 1$ (this follows from the fact that $m - 1 \leq n - 1$). This also proves that $S(\mathbf{m}, \mathbf{seq}(n-1, 1))(C)$ lies in $\mathrm{Mat}_2(\mathbb{Z}_{\geq 0})$ if $m = n$, hence we now assume $m < n$. Now we have that $m \leq n - 1$ and $m - 1 < n - 1$, from this we can conclude that $S' = S(\mathbf{m}, \mathbf{seq}(n-1, 1))(C)$ is a matrix in $\mathrm{Mat}_2(\mathbb{Z})$ whose only nonzero entries are given by $s'_{0,0} = 1$, $s'_{0,1} = 1$ and $s'_{1,1} = 1$.

□

We end with a final conjecture which induces the Positivity conjecture:

Conjecture 4.15: Let $G = F/M$ be a positive M-cluster polynomial associated to some vertex $t \in \mathbb{T}_N$. Let $w \in [1, N]$ and let $t \overset{w}{\text{———}} t'$ in $\mathbb{T}_N$ with associated exchange polynomial P_w. Then substituting P_w/x_w for x_w in G results in a positive M'-cluster polynomial G' associated to t', where

$$M' = \frac{x_w^{m'_w}}{x_w^{m_w}} \cdot \frac{M}{x_w^{m_w}}$$

with m'_w the largest exponent of x_w in F.

It is clear from Theorem 2.25 that this conjecture indeed induces the Positivity conjecture. We have not much ground to state this conjecture, however, assuming Conjecture 4.11 and Conjecture 4.13 hold, one might be able to prove Conjecture 4.15 in the following way:

First proving the following statement:
Let $u, v \in [1, N] \setminus \{w\}$ distinct, let $m, n \in \mathbb{Z}_{\geq 0}$ and let $M = x_u^m x_v^n$, then substituting P_w/x_w for x_w in G results in a positive $x_u^m x_v^n x_w^{m'_w}$-cluster polynomial associated to the vertex t'.

Next, let M be any Laurent monomial in the variables $x_1, \ldots, x_N$, write

$$M = \prod_{i=1}^{N} x_i^{m_i} \qquad (m_i \in \mathbb{Z}),$$

and let

$$M' = \frac{x_w^{m'_w}}{x_w^{m_w}} \cdot \frac{M}{x_w^{m_w}}$$

as in the conjecture. Using Conjecture 4.11 we know that for any $u \in [1, N]$ we can write

$$G' = \frac{x_u^{m_u} x_w^{m'_w - m_w}}{M} \cdot \left(\sum_{i=0}^{[m_u]_+} \sum_{j=0}^{[m'_w - m_w]_+} F_{i,j} \cdot G_{i,j} \right),$$

where for any $i \in [0, [m_u]_+]$, $j \in [0, [m'_w - m_w]_+]$, we have that $G_{i,j}$ is a minimal $x_u^{m_u - i} x_v^{m'_w - m_w - j}$-cluster polynomial, and $F_{i,j} \in \mathbb{ZP}[x_1, \ldots, x_N]$ is a subtraction free polynomial. Hence we just need to show that for any distinct $u, v \in [1, N] \setminus \{w\}$ we can write

$$G' = \frac{x_u^{m_u} x_v^{m_v}}{M} \cdot \left(\sum_{i=0}^{[m_u]_+} \sum_{j=0}^{[m_v]_+} F'_{i,j} \cdot G'_{i,j} \right),$$

where for any $i \in [0, [m_u]_+]$, $j \in [0, [m_v]_+]$, we have that $G'_{i,j}$ is a minimal $x_u^{m_u - i} x_v^{m_v - j}$-cluster polynomial, and $F'_{i,j} \in \mathbb{ZP}[x_1, \ldots, x_N]$ is a subtraction free polynomial.

If $m_w \leq 0$, then the result follows directly from the statement above. Now assume $m_w > 0$, and assume that for any positive cluster polynomial associated to t with denominator having exponent of x_w less than m_w we know that the substitution of P_w/x_w for x_w results in a positive cluster polynomial associated to t'. Note that $x_w G$ is a positive M/x_w-cluster polynomial associated to t by Conjecture 4.13. Hence by our assumption we have that

$$\frac{P_w}{x_w} \cdot G'$$

is a positive $x_w M'$-cluster polynomial. From this, one might be able to deduce that G' is a positive M'-cluster polynomial, which then makes it possible to prove Conjecture 4.15 with use of induction.

We conclude that although there is still much work to be done, the coefficient matrices introduced in this thesis are very important objects in studying the structure of cluster variables and possibly also (as displayed by the discussion above) in proving the Positivity conjecture.

References

[1] Sergey Fomin and Andrei Zelevinsky. Cluster algebras I: Foundations. *Journal of the American Mathematical Society*, 15(2):497–529, 2002.

[2] Philippe Caldero and Markus Reineke. On the quiver grassmannian in the acyclic case. *Journal of Pure and Applied Algebra*, 212(11):2369–2380, 2008.

[3] Grégoire Dupont. Positivity in coefficient-free rank two cluster algebras. *The Electronic Journal of Combinatorics*, 16, 2009.

[4] Kyungyong Lee and Ralf Schiffler. Positivity for cluster algebras. *Annals of Mathematics*, 182:73–125, 2015.

[5] Claus M. Ringel. Introduction to the representation theory of quivers: Chapter 1. `https://www.math.uni-bielefeld.de/~sek/kau/leit1.pdf`, 2012. [Online; accessed 13-7-2021].

[6] Claus M. Ringel. Introduction to the representation theory of quivers: Chapter 4. `https://www.math.uni-bielefeld.de/~sek/kau/leit4.pdf`, 2012. [Online; accessed 13-7-2021].

[7] Michel Brion. Representations of quivers. `https://www-fourier.ujf-grenoble.fr/~mbrion/notes_quivers_rev.pdf`, 2008. [Online; accessed 13-7-2021].

[8] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov. Tilting theory and cluster combinatorics. *Advances in mathematics*, 204(2):572–618, 2006.

[9] Franco Rota. An introduction to the derived category of cherent sheaves. `https://www.math.utah.edu/~goller/F15_SAGS/Franco_DerivedCategory.pdf`. [Online; accessed 14-7-2021].

[10] Henning Krause and Jue Le. The Auslander–Reiten formula for complexes of modules. *Advances in Mathematics*, 207(1):133–148, 2006.

[11] Bernhard Keller. On triangulated orbit categories. *Documenta Mathematica*, 10:551–581, 2005.

[12] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. *Commentarii Mathematici Helvetici*, 81(3):596–616, 2006.

[13] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras II. *Annales Scientifiques de l'Ecole Normale Supérieure*, 39(4):83–100, 2006.