Non-commutative Persistent Homology

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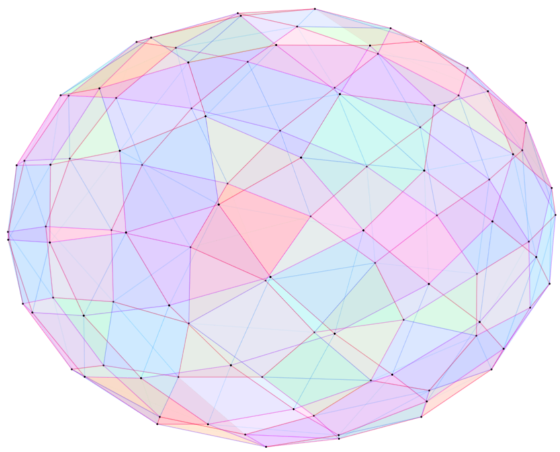

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Contents

Chapter 1. Introduction 1
1. Notation and Conventions 3
2. Acknowledgements 3

Chapter 2. Non-commutative Geometry 5
1. C*-algebras 5
2. (Non-)commutative topology 8
3. Spectral Triples 11
4. The Quantum Torus 17
5. Fuzzy spheres 19
6. Operator Spaces 21
7. Non-commutative Convexity 25

Chapter 3. Topological persistence 29
1. Persistent Homology on finite vertex sets 29
2. General Persistent Homology 33

Chapter 4. Topological Barcodes 39
1. The Barcode Space 39
2. Estimating Barcodes from Topology 42

Chapter 5. Barcodes of quantum compact topological spaces 45
1. Quantum Compact Metric Spaces 45
2. Persistent Homology on Pure States 49
3. Persistent Homology on non-commutative pure states 50
4. Are β_μ, β_ϑ and β_κ eligible? 57

Chapter 6. Discussion 59
1. Future Directions 59

Bibliography 63

CHAPTER 1

Introduction

The topology and geometry of a space X can be studied using only algebraic information. For example, the Serre-Swan theorem tells us that there is a bijective correspondence between finitely generated projective $C^\infty(X)$-modules and vector bundles over X. Another landmark result is the Gelfand-Naimark theorem, published in 1943, which states that locally compact Hausdorff spaces can be reconstructed, up to homeomorphism, from the commutative C*-algebra $C_0(X)$ of continuous functions vanishing at infinity and, vice versa, every commutative C*-algebra A is isomorphic to an algebra of functions vanishing at infinity on some locally compact space X_A. It appears that we can phrase many topological properties using only the algebraic structure of $C_0(X)$, like connectedness and dimension. If we now consider non-commutative C*-algebras instead, we can still use these algebraic formulations of topological features despite our algebra lacking an underlying topological space. We can refer to the 'virtual' topological space underlying our non-commutative algebra as a *non-commutative space*. The mathematical discipline that concerns itself with the study of these non-commutative spaces is called *non-commutative geometry*. On a more advanced level, a closed Riemannian manifold (M, g) can be described in terms of a *spectral triple* $(\mathcal{A}, H, D)$ where $\mathcal{A}$ is an algebra (typically, the algebra of smooth functions on M), H is a Hilbert space (for example: square integrable sections of the exterior bundle) with a representation $\pi : \mathcal{A} \to B(H)$ ($B(H)$ are bounded linear operators on H) and D is an unbounded operator (for example, a Dirac operator) satisfying some additional properties . Interestingly, not only M, as a topological space, can be recovered from this, but also the Riemannian metric and the differentiable structure are contained in the triple $(\mathcal{A}, H, D)$. Spectral triples and other advanced techniques in non-commutative geometry have been studied intensely and unceasingly since the 1980s, starting with the work of Alain Connes. Non-commutative geometry includes non-commutative measure theory, cyclic (co)homology, K-theory and much more. In this thesis we will explore the possibilities of computing *topological barcodes* for non-commutative spaces.

Topological barcodes are the central objects in *persistent homology* which is, arguably, the most prominent technique in the mathematical field of *topological data analysis*. Persistent homology is an algorithm that takes a point cloud (a metric space with finitely many points) and produces a *barcode* which is, roughly speaking, a multiset of intervals that contains information about the 'approximate shape' of the point cloud. Persistent homology has the beautiful property of producing similar barcodes for similar spaces and, by a simple analytical argument, persistent homology can be computed for general compact metric spaces. The measure of similarity between point clouds is given by the

Gromov-Hausdorff distance, whereas the similarity of barcodes is given by the bottleneck metric (to be introduced in Chapter 3. In technical terms, we have a Gromov-Hausdorff-to-Bottleneck Lipschitz-continuous map

$$\beta : \{\text{isometry classes of compact metric spaces}\} \longrightarrow \{\text{topological barcodes}\}.$$

We reserve the term *topological barcodes* for barcodes arising from general compact spaces, in contrast to *barcodes* that come from point clouds.

The main aim of this thesis is to find an extension of β to 'non-commutative metric spaces'. Multiple (successful) attempts to define 'non-commutative metric spaces' have been made [Rie99],[Ker03],[Wu06b],[Lat13]. Only [Lat13] does in fact make use of multiplication in the algebra, whereas the others only use the order structure of C*-algebras (or the more general *order-unit spaces*). Therefore, we will only rarely encounter the adjective 'non-commutative' in the context of metrics in this thesis. In addition to the metric structure, the mentioned articles also formulate non-commutative versions of the Gromov-Hausdorff distance which generalize the classical Gromov-Hausdorff distance. Hence, we can compactly formulate the main question of this thesis in technical terms: does there exist a non-commutative-Gromov-Hausdorff-to-Bottleneck Lipschitz-continuous map

$$\tilde{\beta} : \{\text{isometry classes of non-commutative metric spaces }\} \longrightarrow \{\text{topological barcodes}\}?$$

We will propose three *candidate non-commutative barcode maps* all of them based on the observation that the pure states $\partial^e S(A)$ of a C*-algebra A generalize the points of a topological space: $\partial^e S(C(X)) \approx X$. The simplest of these candidate maps can be shown quite easily to be inadequate, as it cannot even be continuous. The eligibility of the remaining two are harder to refute.

This thesis is structured as follows:

- In Chapter 2 we treat the most elementary theory of non-commutative topology and non-commutative geometry.[1]
- In Chapter 3 we introduce the basics of persistent homology (pictures are included!) and indicate how it fits in the framework of topological data analysis.
- In Chapter 4 we treat topological barcodes; we show that they can be countably infinite and we give two results that can be used to infer some information about the barcodes from the topology of a space.
- In Chapter 5 we expose the main contribution of this thesis: we define the non-commutative versions of metric space and Gromov-Hausdorff distance, we formulate our candidate barcode maps and discuss their eligibility.

[1]It must be noted that Section 5 is a 'ghost section'; I thought it could have been used to disqualify one of the candidate barcode maps, but I was mistaken. Yet it is included, because I find the concept is very interesting.

- In Chapter 6 we briefly summarize the results of Chapter 5 and theorize about possible future directions and the very *meaning* of hypothetical non-commutative topological barcodes.

1. Notation and Conventions

In this thesis the natural numbers $\mathbb{N}$ start at 1. We use $\approx$ for homeomorphism and $\simeq$ for (natural) isomorphism where the category is always clear from the context. The matrix algebras are denoted by M_n are over $\mathbb{R}$ unless specified otherwise (for example, $M_n(\mathbb{C})$). For any convex set K the set of extreme points of K, the extreme boundary, is denoted by $\partial^e K$. The dual space of a topological vector space V is denoted by V'.

2. Acknowledgements

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CHAPTER 2

Non-commutative Geometry

1. C*-algebras

The central objects of this chapter are *C*-algebras*, which are commonly encountered in functional analysis and operator theory. C*-algebras were originally conceived in 1947 by Irving Segal who defined them as closed *-subalgebras of $B(H)$, the algebra of operators on a Hilbert space H. The contemporary definition is seemingly more general, but it is well known that all C*-algebras can be characterized as these closed *-subalgebras for some Hilbert space H (we will come back to this in Theorem 2.17).

Definition 2.1. A **C*-algebra** is an algebra A with a norm $\|\cdot\|$ and an involutive operator $*$ such that $(A, \|\cdot\|)$ is complete and for all $a, b \in A$ we have

$$\|ab\| \leq \|a\|\|b\|, \tag{2.1}$$

$$\|a^*a\| = \|a\|^2. \tag{2.2}$$

△

Remark 2.2. If A satisfies all properties of definition 2.1 except for the completeness condition, we speak of a *pre-C*-algebra*. We refer to equation 2.2 as the *C*-property*. If A satisfies all properties in definition 2.1, but the C*-property fails, A is a *Banach-*-algebra*.

Example 2.3. Let H be some Hilbert space, then $B(H)$ (bounded operators on H) and $K(H)$ (compact operators on H) are C*-algebras. Multiplication is given by composition and involution is given by transposition. The former contains the identity operator, which is a unit for $B(H)$, whereas $K(H)$ does not. C*-algebras with a unit are referred to as **unital C*-algebras**. If H is of dimension n (that is $H \simeq \mathbb{K}^n$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), we have $B(H) = K(H) \simeq M_n(\mathbb{K})$. Because $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|T(x)\|^2$ we see that $\|T^*T\| = \|T\|^2$ for all $T \in B(H)$. ♠

Example 2.4. Let X be a compact Hausdorff space, then $C(X)$ is a C*-algebra, where multiplication and involution are pointwise and the norm is given by the supremum over X. The constant function 1 acts as the unit and for $f, g \in C(X)$ we must have $fg = gf$, because pointwise multiplication is commutative. Hence, we speak of a **commutative C*-algebra** $C(X)$. The C*-property is easily verified: $f(x)\overline{f(x)} = |f(x)|^2$ for all x, so in particular $\sup_{x\in X} f(x)\overline{f(x)} = \sup_{x\in X} |f(x)|^2$. ♠

Example 2.5. Let X be compact Hausdorff and suppose that A is a C*-algebra, then $C(X, A)$ (the continuous functions $X \to A$) is a C*-algebra. Again, multiplication and involution are pointwise, but do now take place in A. It is true that $C(X, A)$ is unital and/or commutative if and only if A is unital/commutative. ♠

Because C*-algebras are in particular algebras, they have *ideals*. It turns out however, that ideals in C*-algebras are easier to work with if they are topologically closed:

Definition 2.6. Let A be a C*-algebra. A (left, right, two-sided) **ideal** $I \subseteq A$ is a *closed* C*-subalgebra of A such that for all $a \in A$ we have $aI \subseteq I$ (left ideal), $Ia \subseteq I$ (right ideal) or $aI, Ia \subseteq I$ (two-sided ideal). If I is a two-sided ideal of A, we use the notation $I \triangleleft A$. Whenever we wish to speak of ideals that are not closed, we will refer to them as **algebraic ideals**. △

If $I \triangleleft A$, we can define $A/I := \{a + I : a \in A\}$ as usual (e.g. group theory). There is a natural norm on A/I given by $\|a + I\|_{A/I} := \inf\{\|a + b\| : b \in I\}$. We call A/I the *quotient C*-algebra* of A by I.

Example 2.7. The compact operators $K(H) \subseteq B(H)$ are a two-sided ideal of $B(H)$ and the quotient $C(H) := B(H)/K(H)$ is called the *Calkin algebra*. The C*-algebra $K(H)$ does not have any ideals, we call such C*-algebras **simple**. ♠

Example 2.8. Commutative C*-algebras usually have many ideals. Take $A = C(X)$ where X is compact Hausdorff and let $V \subseteq X$ be a closed set, then the subalgebra $I_V := \{f \in C(X) : f\big|_V = 0\}$ is a closed (two-sided) ideal. ♠

Operators on a Hilbert space have a spectrum, which is a purely algebraic concept and hence can be translated to C*-algebras as is.

Definition 2.9. Let A be a unital C*-algebra and let $a \in A$. The **spectrum** of a, denoted by $\sigma_A(a)$, consists of all $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible in A. △

We use the notation $\sigma_A(a)$ to emphasise that $\sigma_A(a)$ may depend on the ambient C*-algebra. The spectrum is a very important concept in operator theory, as it grants the ability to perform *functional calculus* and it allows us to define a notion of *positivity* (which in turn gives a C*-algebra an order structure). For this, we first need a few facts and a lemma about the intrinsicality of the spectrum.

Fact 2.10. *Let A be a unital C*-algebra and let $a \in A$. The following statements are true:*
1. *We have $\|a\| = \|a^*\|$.*
2. *If $a = a^*$ (that is, a is self-adjoint), we have $\sigma(a) \subseteq \mathbb{R}$.*
3. *The C*-subalgebra $C^*(a) \subseteq A$ which is the smallest unital C*-algebra containing a is commutative.*
4. *if $B \subseteq A$ is a C*-subalgebra with the same unit, then $\sigma_A(a) \subseteq \sigma_B(a)$ and $\partial\sigma_B(a) \subseteq \partial\sigma_A(a)$. (The latter being very much non-trivial)*

Lemma 2.11. *Suppose that B is a unital C*-algebra and that $A \subseteq B$ is a C*-subalgebra with the same unit. Let $a \in A \subseteq B$, then $\sigma_A(a) = \sigma_B(a)$. Therefore, we can henceforth write $\sigma(a)$ without ambiguity.*

Proof. First note that it is clear that $\sigma_B(a) \subseteq \sigma_A(a)$, because if $a - \lambda$ is invertible in A it must be invertible in B. We also have $\partial\sigma_A(a) \subseteq \partial\sigma_B(a)$. Now, assume that a is selfadjoint, then $\sigma_B(a), \sigma_A(a) \subseteq \mathbb{R}$ and so we have $\sigma_A(a) = \partial\sigma_A(a) \subseteq \partial_B(a) = \sigma_B(a)$. Therefore $\sigma_A(a) = \sigma_B(a)$.

Now suppose that $a = \tilde{a} - \lambda \in A$ is invertible in B, then there exists $b \in B$ such that $aa^*b^*b = 1$, but aa^*, bb^* are self-adjoint and inverses are unique, so aa^* is invertible in A which implies that $a^*(b^*b) \in A$ is an inverse for a. □

Remark 2.12. If a C*-algebra A is not unital, we can add a unit by considering $A \oplus \mathbb{C}$ with norm $\|\cdot\|_1$

$$\|(a, \lambda)\|_1 = \|a\| + |\lambda|$$

and multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a,\ \lambda\mu).$$

This allows us to define the spectrum for elements of A using this *unitization* which we will do without explicit indication. From now on A will be a general C*-algebra (unital or not), unless specified otherwise.

Definition 2.13. An element $a \in A$ is called **positive** if it is self-adjoint and $\sigma(a) \subset \mathbb{R}_{\geq 0}$. The positive elements of A induce an order on the self-adjoint elements A_{sa} of A by letting $a < b$ whenever $b - a$ is positive. A linear functional $\tau \in A'$ is called **positive** if $\tau(a) \geq 0$ whenever a is positive.

△

Note that the 0-element is also positive by convention.

Remark 2.14. It is a consequence of functional calculus that $a \in A$ is positive if and only if $a = b^*b$ for some $b \in A$.

Definition 2.15. Let A be a C*-algebra, the **state space** of A, denoted by $S(A)$ is the set of positive linear functionals with norm 1.

△

The state space $S(A)$ will play an important role in this thesis. It is important to note that it is a normed subspace of A' that is convex and if A is unital it is also weak-* compact. The latter property follows from the fact that $S(A) \subseteq B_1(A')$ (where $B_1(A')$ is the unit ball in A' and $B_1(A')$ is weak* compact by the Theorem of Banach-Alaoglu) and the fact that $S(A) = \{\psi \in A' : \psi(1_A) = 1\} \cap B_1(A')$ which is the intersection of the unit ball with a hyperplane. We are now ready for the two landmark results. The first one fully characterizes commutative C*-algebras.

Theorem 2.16 (Gelfand-Naimark)**.** *Let A be a commutative C*-algebra. The extreme points $\partial^e S(A)$ of $S(A)$ are the states τ for which $\tau(ab) = \tau(a)\tau(b)$. Equipped with the relative weak* topology on $S(A)$, the topological space $X := \partial^e S(A)$ is locally compact Hausdorff. The map*

$$\Psi : A \to C_0(X), \quad a \mapsto (\tau \mapsto \tau(a))$$

*is a *-isometric isomorphism. Hence, we can view A as the space of continuous functions vanishing at infinity on some locally compact Hausdorff space X.*

Even better, we can also concretely describe non-commutative C*-algebras.

Theorem 2.17 (Gelfand-Naimark-Segal)**.** *Every C*-algebra has a faithful *-representation. That is, there exists a Hilbert space H and an injective *-homomorphism $\pi : A \to B(H)$.*

The Gelfand-Naimark-Segal (henceforth GNS) theorem thus states that every C*-algebra can be seen as a concrete operator algebra. The mere fact that 'there exists' a representation is not always sufficient, so we will give the construction of the *universal GNS-representation* for an arbitrary C*-algebra A.

Let $\tau \in S(A)$ and define $N_\tau := \{a \in A : \tau(a^*a) = 0\}$ which is a closed left ideal in A. The map $\langle a+N_\tau, b+N_\tau\rangle_\tau = \tau(b^*a)$ defines an inner product on A/N_τ which we can use to obtain a Hilbert space H_τ which is the completion of A/N_τ with this inner product. For each $a \in A$ we can define an operator $\tilde{\varphi}_\tau(a) \in B(A/N_\tau)$ by $\varphi(a)(b) := ab + N_\tau$. It appears that $\tilde{\varphi}_\tau$ extends uniquely to a bounded operator $\varphi_\tau(a) \in B(H_\tau)$. The map $\varphi_\tau : A \to B(H_\tau)$ that sends a to $\varphi_\tau(a)$ is a *-homomorphism. Bundling these representations together for each τ, we obtain

$$\varphi : A \to \bigoplus_{\tau \in S(A)} B(H_\tau), \quad a \mapsto \bigoplus_{\tau \in S(A)} \varphi_\tau(a)$$

which is a *-representation of A. More interestingly, φ is also faithful. This follows from the fact that there exists states τ_a for every $a \in A$ such that $\tau_a(a^*a) = \|a^*a\|$. Now if $\varphi(a) = 0$ for some $a \in A$ we see that $\varphi_{\tau_a}(a) = 0$ but this implies that $a = 0$ from functional calculus. Note that the universal representation is most definitely not the most economical representation. For example, if $A = M_2(\mathbb{C})$ then A is already a concrete operator space, yet the universal representation would consist of an uncountable sum of factors.

2. (Non-)commutative topology

The Gelfand-Naimark theorem grants us an interesting perspective. Every locally compact Hausdorff space X gives us a commutative C*-algebra and every commutative C*-algebra gives us a locally compact Hausdorff space. It appears that we can describe a good number of topological properties of X in terms of algebraic properties of $C(X)$. This entices us to think of non-commutative C*-algebras as 'non-commutative topological spaces'. It turns out that this paradigm is not just a philosophical pastime, but it gives us tools to study naturally occuring non-commutative spaces using, for instance, K-theory. Some good examples are leaf spaces of foliated manifolds which behave erratically as topological spaces, but carry meaningful information in their C*-algebras [Con94].

Let us start out with a few topological properties that can be described in algebraic terms. From now on, X is a locally compact Hausdorff space.

Compactness $\iff$ unital: If $A \simeq C_0(X)$ is a commutative C*-algebra with unit 1_A, it is easy to see that 1_A must correspond to a constant function with value 1 in $C_0(X)$. This can only happen if X is compact, otherwise 1_A would not vanish at infinity. On the other hand, on compact spaces all continuous functions vanish at infinity. Hence, compactness corresponds to the presence of a unit element. Unital C*-algebras can be viewed as corresponding to non-commutative compact spaces.

Connectedness $\iff$ projectionless: If X is compact and consists of n connected components $X_1, \dots, X_n$, the indicator functions 1_{X_i} are continuous. Obviously, $1_{X_i}^2 = 1_{X_i}$ for each i, so $C(X)$ contains non-trivial projections corresponding to the connected components. On the other hand, the condition that $f \in C(X)$ satisfies $f^2 = f$ implies that f only takes values 0 and 1, hence the only projections in $C(X)$ can be indicator functions. We conclude that connected spaces X have a projectionless algebra $C(X)$, that is $C(X)$ only contains trivial projections.

Metrizability $\iff$ separable: If X is metrizable and locally compact, one can find a countable dense subset $\{x_i\}_{i\in\mathbb{N}}$ of X and we can define an algebra of compactly supported continuous functions $\{f_{i,j}\}_{i,j\in\mathbb{N}}$ supported on the neighbourhoods $U_{i,j} := B(x_i, 2^{-j})$. If we look at the linear span of these elements we have a dense subalgebra of $C_0(X)$ that separates points and vanishes nowhere, so we can use Stone-Weierstrass to establish that $C_0(X)$ is separable. Conversely, if $C_0(X)$ is separable we consider the natural embedding $X \to C_0(X)^*$ given by $x \mapsto \delta_x$ where δ_x is the corresponding pure state. The unit ball of the dual of a separable Banach space is metrizable. If we take this metric and restrict to the pure states, we obtain a metric on X.

Covering Dimension = real rank: Suppose that X is compact, so $C(X)$ is unital. We define the *real rank* of $C(X)$ as the smallest integer n such that for any $n+1$ self-adjoint elements $f_1, \dots, f_{n+1} \in C(X)$ and every $\epsilon > 0$ there exists an $(n+1)$-tuple of elements $g_1, \dots, f_{n+1}$ such that

$$\sum_{i=1}^{n+1} g_i^2 = 1, \quad \text{and} \quad \| \sum_{i=1}^{n+1} (f_i - g_i) \| < \epsilon.$$

One can identify the tuple $(f_1, \dots, f_{n+1})$ with a continuous function $f : X \to \mathbb{R}^{n+1}$. The Lebesgue covering dimension of X is the smallest n for which f can be approximated arbitrarily well by a function $g : X \to \mathbb{R}^{n+1}$ that vanishes nowhere. The fact that g vanishes nowhere implies that g_i vanishes nowhere for each component $g_i : X \to \mathbb{R}$. A function is invertible if and only if it vanishes nowhere, so we are back at our definition and we see that the real rank of $C(X)$ coincides with the Lebesgue covering dimension of X.

Remark 2.18. Sometimes it is possible to formulate multiple algebraic characterization of topological properties such that the different notions overlap for commutative C*-algebras. The covering dimension is a good example. There are many non-commutative analogues for the covering dimension: real rank, stable rank, decomposition rank, nuclear dimension[WZ09] and more. Usually, these notions coincide on commutative C*-algebras but can differ in the non-commutative setting. For instance, irrational rotation algebras $\mathcal{A}_\theta$ (also known as quantum tori, see Section 4) have real rank zero, except for $\theta = 0$, because $\mathcal{A}_0 \simeq C(\mathbb{T}^2)$ whereas the nuclear dimension of $\mathcal{A}_\theta$ is 2 whenever $\theta \in \mathbb{Q}$ and 1 whenever $\theta \notin \mathbb{Q}$ (see [Cas20] Section 5).

2.1. Non-commutative quotients. We now describe a natural occurrence of non-commutative spaces. Suppose that we have a locally compact Hausdorff space X and suppose that $\Gamma = \dot{\bigcup}_{\lambda\in\Lambda}\Gamma_\lambda$ is a partition of X into dense sets and denote by $\sim_\Gamma$ the corresponding equivalence relation (Γ is suggestive of the notation of a group action for a reason), then X/Γ is a bland space: its topology is trivial and hence $C(X/\Gamma) \simeq \mathbb{C}$ (we also have the problem that X/Γ is not even Hausdorff). Moreover, any map $f : Y \to X/\Gamma$ is continuous which makes X/Γ contractible and so there is nothing to study in terms of topology. However, it is desirable that such bizarre quotients do contain meaningful information. For this, we can introduce *non-commutative quotients*. We follow the exposition in [Ala06].

Let X and Γ be as before and fix some Borel measure on X. We have just seen that 'classical' quotients can yield very dull results, so let us look instead at the following C*-algebra:

$$C^{nc}(X/\Gamma) := \{f = (f_{\alpha\beta})_{\alpha,\beta} \in C(X\times X) : \alpha,\beta \in X, \alpha \sim_\Gamma \beta\} \subseteq B(L^2(X)).$$

The elements of $C^{nc}(X/\Gamma)$ act on $L^2(X)$ by

$$(fh)(\gamma) = \sum_{\beta\sim_\Gamma\gamma} f_{\gamma\beta}h(\beta)$$

where $f \in C^{nc}(X/\Gamma)$ and $h \in L^2(X)$.
The multiplication on $C^{nc}(X/\Gamma)$ is given by convolution over equivalent points:

$$fg(\alpha) = \sum_{\beta,\gamma\sim_\Gamma\alpha} f_{\beta\alpha}g_{\alpha\gamma}.$$

In order to get a better grasp on this construction, we give an explicit computation for X/Γ when X is finite. The construction for (uncountable) infinite sets X with dense orbits is similar.

Example 2.19. Suppose that $X = \{x_1, \dots, x_n\}$ with the discrete topology and let Γ be an equivalence relation given by the partition $X = X_1 \dot\cup X_2 \dot\cup \dots \dot\cup X_k$, then

$$\begin{aligned} C(X/\Gamma) &= \{f = (f_{\alpha\beta})_{\alpha,\beta} \in C(X\times X) : \alpha,\beta \in X, \alpha \sim_\Gamma \beta\} \\ &= \bigoplus_{i=1}^k \{f = (f_{\alpha\beta})_{\alpha,\beta} \in C(X_i \times X_i) : \alpha,\beta \in X_i\} \\ &\simeq \bigoplus_{i=1}^k M_{|X_i|}(\mathbb{C}). \end{aligned}$$

So, finite non-commutative quotients are precisely the non-commutative C*-algebras. ♠

In some cases the above approach works also for infinite topological spaces. The next example is sometimes used in differential geometry to show that locally Euclidean spaces can be non-Hausdorff.

Example 2.20. Let $X = [-1,1]\dot\cup[-1,1]$, label the points in the first and second interval by x_1, x_2 respectively and let Γ be the equivalence relation given by $x \sim y$ if $x = y$, $x = 0_1$ and $y = 0_2$ or $x = 0_2$, $y = 0_1$. That is, we glue only the midpoints of both intervals. It turns out that

$$C^{nc}(X/\Gamma) \simeq C([-1,0[) \oplus M_2 \oplus C(]0,1]).$$

So we obtain a non-commutative C*-algebra that is 'almost' commutative. ♠

As mentioned before; if we have an ordinary uncountable topological space X and we take a quotient X/Γ where Γ is a dense subset or a dense orbit, the non-commutative quotient contains information that the ordinary (dull) quotient does not. The ordinary quotient gives us a trivial topological space and as such a trivial algebra of functions, but the non-commutative quotient yields a large non-commutative C*-algebra. A prime example is given by the *Penrose tilings*[MR05] which have interesting K-theories. We do not cover K-theory in this thesis despite its importance in non-commutative geometry. Very briefly, operator K-theory is a generalization of topological K-theory, which is in turn a cohomology theory that 'counts' isomorphism classes of vector bundles over a base space. The K-theory of Penrose tilings can be computed rather easily using the fact that they are *AF algebras*: approximately finite C*-algebras or, more specifically, direct limits of finite C*-algebras.
The non-commutative torus which we will define in Section 4 can also be seen as a non-commutative quotient. See remark 2.39.

3. Spectral Triples

In the previous section we discussed how topological properties can be phrased in the language of operator algebras. If we want to generalize phenomena that are not purely topological, like a metric structure, we require more information and we enter the domain of *non-commutative geometry*. Here, we introduce the notion of a *spectral triple* which was developed by Alain Connes in the 1980s and 1990s.

Definition 2.21. A **spectral triple** is a triple $(\mathcal{A}, \pi, D)$ consisting of a *-algebra A, a faithful representation $\pi : \mathcal{A} \to B(H)$ for some Hilbert space H and a self-adjoint (unbounded) operator $D : H \to H$. Moreover, D must satisfy two additional properties:

1. The resolvent operators $(D \pm i)^{-1}$ must be compact.
2. The commutator bracket must be well-behaved with respect to the representation: $[D, \pi(a)] \in B(H)$ for all $a \in A$.

△

This definition is likely to overwhelm the unsuspecting reader, so we will give an example and indicate to what extent spectral triples capture the differential and Riemannian structure of manifolds. In the following we loosely follow chapter 5 and 9 of [Bon01].
Suppose that (M, g) is an orientable Riemannian manifold. The differential structure of M is wholly contained in the exterior bundle $\Lambda^\bullet T^* M$ that consists of all differential forms on M. Moreover, $\Lambda^\bullet T^* M$ comes with some natural operations. Most importantly, the exterior product $\wedge$ and the exterior derivative d. In order to condense the Riemannian information into our spectral triple, we will define a Hilbert space structure on the space of differential forms and then impose a *Clifford structure* onto these forms to define the adjoint of the exterior derivative. This will grant us a self-adjoint operator that can be used to recover metric and differential information of (M, g) in terms of Hilbert spaces and operators.

3.1. Integrable differentiable forms. Let $\Omega^\bullet(M) := \Gamma(\Lambda^\bullet T^*M)$ be the graded algebra of differential forms on M. The orientability of (M,g) gives us a volume form μ_g, hence we can define an inner product by

$$\langle \alpha, \beta \rangle := \int_M (\alpha, \beta) \mu_g. \tag{2.3}$$

Where $\alpha, \beta \in \Omega^\bullet(M)$ and $(\cdot,\cdot) : \Omega^\bullet(M) \times \Omega^\bullet(M) \to C(M)$ is defined as the unique extension of

$$(\alpha_1 \wedge \ldots \wedge \alpha_k,\ \beta_1 \wedge \ldots \wedge \beta_l) := \delta_{kl} \det\left[\langle \alpha_i, \beta_j \rangle\right]_{i,j}.$$

In the latter expression $\langle\cdot,\cdot\rangle$ is the point-wise inner product induced by the metric g, the determinant is well-defined because $k = l$ must hold in order for δ_{kl} to be non-zero. We can now define:

Definition 2.22. The Hilbert space of **square integrable differential forms on** M denoted by

$$L^{2,\bullet}(M)$$

is defined to be the completion of $\Omega^\bullet(M)$ under the inner product defined in 2.3. $\triangle$

It is clear from the definition that we have a decomposition $L^{2,\bullet}(M) = \bigoplus_{k=1}^{\dim M} L^{2,k}(M)$, where $L^{2,k}(M)$ is, similarly, the completion of $\Omega^k(M)$ under 2.3.

3.2. Clifford bundles. If V is a finite-dimensional vector space, and $q : V \times V \to \mathbb{R}$ is a quadratic form, there exists a unique unital associative algebra $Cl(V,q)$ and a linear map $i : V \to Cl(V,q)$ such that for each unital associative algebra A and each linear map $f : V \to A$ satisfying $f(v)^2 = q(v)1_A$ uniquely factors through $Cl(V,q)$. That is, there exists a unique $\tilde{f} : Cl(V,q) \to A$ such that $f = \tilde{f} \circ i$. The quadratic forms and inner products on V are in natural bijection. After all, a quadratic form can be used to normalize any orthogonal basis of V; to limit confusion we write $\langle\cdot,\cdot\rangle_q$ for this inner product.

Definition 2.23. The algebra $Cl(V,q)$ defined above is the **Clifford algebra** generated by (V,q). We can also *complexify* the Clifford algebra: $\mathbb{C}\mathrm{l}(V,q) := Cl(V,q) \otimes_{\mathbb{R}} \mathbb{C}$. $\triangle$

From now on, we work with $\mathbb{C}\mathrm{l}(V,q)$ and define the adjoint $(ab)^*$ for $a, b \in \mathbb{C}\mathrm{l}(V,q)$ by taking the complex conjugation of the underlying vector space and by reversing products: $(ab)^* = \overline{b}\overline{a}$. The map $-i$ (where $i : V \to \mathbb{C}\mathrm{l}(V,q)$ is the embedding) on $\mathbb{C}\mathrm{l}(V,q)$ preserves the quadratic form and therefore, by the universal property described above, this map extends to an involution $\alpha : \mathbb{C}\mathrm{l}(V,q) \to \mathbb{C}\mathrm{l}(V,q)$. Thus, we obtain a canonical algebra decomposition $\mathbb{C}\mathrm{l}(V,q) \simeq \mathbb{C}\mathrm{l}(V,q)^{\mathrm{even}} \oplus \mathbb{C}\mathrm{l}(V,q)^{\mathrm{odd}}$ where the even elements $x \in \mathbb{C}\mathrm{l}(V,q)$ are those satisfying $\alpha(x) = x$ and the odd elements satisfy $\alpha(x) = -x$. Note that $\mathbb{C}\mathrm{l}(V,q)$ is a finite-dimensional C*-algebra. The universal construction described above gives us the following useful property:

Fact 2.24. *For $v, w \in \mathbb{C}\mathrm{l}(V,q)$ we have*

$$v \cdot w - w \cdot v = -2\langle v, w \rangle_q 1.$$

We have chosen to work over complex Clifford algebras for a reason: A Clifford algebra $\mathbb{C}\mathrm{l}(V,q)$ is completely (up to isomorphism) determined by the dimension of V. This is not the case for real Clifford algebras. In fact, for $n = \dim V$ we have (henceforth omitting the quadratic form)

$$\begin{aligned} &\mathbb{C}\mathrm{l}(V) \simeq M_{2^m}(\mathbb{C}) && \text{if } n = 2m \text{ is even,} \\ &\mathbb{C}\mathrm{l}(V) \simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) && \text{if } n = 2m+1 \text{ is odd.} \end{aligned} \tag{2.4}$$

It turns out, that $\mathbb{C}\mathrm{l}(V,q)$ is linearly isomorphic to $\Lambda^\bullet T^*M$ under the identification

$$e_1 \cdots e_n \mapsto e_1 \wedge \cdots \wedge e_n. \tag{2.5}$$

Hence, the Clifford algebra $\mathbb{C}\mathrm{l}(V,q)$ equips the exterior bundle $\Omega^\bullet(M)$ with the structure of a unital associative algebra. For a Riemannian manifold (M,g) and a point $p \in M$ we have a natural quadratic form on T_pM induced by g, namely $q_p(v,w) = \langle v,w\rangle$. The same metric structure provides us with a natural linear isomorphism $T_pM \simeq T_p^*M$, so we obtain a Clifford multiplication on $\mathbb{C}\mathrm{l}(T_p^*M, q_p) \simeq \Omega^\bullet(M)$.

Definition 2.25. We define the **Clifford bundle** on a Riemannian manifold (M,g) as

$$\mathbb{C}\mathrm{l}(M) \coloneqq \dot{\bigcup}_{p\in M} \mathbb{C}\mathrm{l}(T_pM, q_p).$$

△

We can let the Clifford bundle act on the differential forms by left multiplication. Denote this action by $c : \mathbb{C}\mathrm{l}(M) \times \Omega^\bullet(M) \to \Omega^\bullet(M)$, so that for $\gamma \in \mathbb{C}\mathrm{l}(M)$ we have $c(\gamma) : \Omega^\bullet(M) \to \Omega^\bullet(M)$. This is a bounded operator and hence, it can be extended to $c(\gamma) \in B(L^{2,\bullet})$. Next, we have a grading on Clifford algebras which is given by the *chirality element* γ. At each point $p \in M$ we can define $\gamma(p) \coloneqq (-i)^n v_1 v_2 \cdots \wedge v_n$ where $\dim M = 2n$ or $\dim M = 2n+1$ and $v_1, \ldots, v_n$ form the orthonormal basis of the tangent space T_pM multiplied by i. By orientability of M we can define γ pointwise to yield a global section $\gamma \in \Gamma(\mathbb{C}\mathrm{l}(M))$. We see that $\gamma^* = (-1)^n(-1)^{n(n-1)/2} v_n \cdots v_1 = \gamma$. It follows that $\gamma^2 = \gamma^*\gamma = e_1 \cdots e_n e_n \cdots e_1 = 1$, so γ is self-adjoint en involutive.

Definition 2.26. The **Hodge star operator** $\star$ is defined as the action of the chirality element: $c(\gamma) : \Omega^\bullet(M) \to \Omega^\bullet(M)$. △

More specifically, the Hodge star operator swaps k and $n-k$ forms: $\star : L^{2,k}(M) \to L^{2,n-k}(M)$ where $n = \dim M$. We can now characterize the adjoint of the exterior derivative.

Lemma 2.27 (Lemma 9.31 from [Bon01]). *The adjoint $d^* : L^{2,\bullet}(M) \to L^{2,\bullet}(M)$ of the exterior derivative is given by*

$$d^* = (-1)^n \star d \star .$$

*We call d^* the **codifferential**.*

In fact, we can use the fact that d^* is a formal adjoint of d to perform explicit computations.

Proposition 2.28. *Let $\pi : C^\infty(M) = \Omega^0(M) \hookrightarrow \Omega^\bullet(M)$ be the canonical embedding and let $f \in C^\infty(M), \omega \in \Omega^\bullet(M)$, then*

$$d^*(f\omega) = f d^*\omega - \iota_{df}(\omega).$$

Proof. The equation

$$\begin{aligned}\langle \eta, d^*(f\omega)\rangle = \langle d\eta, f\omega\rangle &= \langle \overline{f} d\eta, \omega\rangle \\ &= \langle d(\overline{f}\eta), \omega\rangle - \langle d\overline{f} \wedge \eta, \omega\rangle \\ &= \langle \overline{f}\eta, d^*\omega\rangle - \langle \eta, \iota_{df}\omega\rangle \\ &= \langle \eta, f d^*\omega - \iota_{df}\omega\rangle\end{aligned}$$

holds for every $\eta \in \Omega^\bullet(M)$, hence we obtain the desired equality.

□

Also important is the following:

Proposition 2.29. *Let $v \in T^*M, \varphi \in \mathbb{C}\mathrm{l}(T^*M)$, then*

$$v \cdot \varphi = v \wedge \varphi - \iota_v \varphi.$$

Proof. We can express v, φ as elements in $\Lambda^* M$:

$$v = \sum_i \alpha_i e_i, \quad \varphi = \sum_j \beta_j e_{k(j)_1} \wedge \cdots \wedge e_{k(j)_{n_j}},$$

where $e_1, \ldots, e_n$ is an orthonormal basis. Next, we can use equation 2.5 to compute

$$v \cdot \varphi = \sum_i \sum_j \alpha_i \beta_j e_i \cdot e_{k(j)_1} \cdots \cdot e_{k(j)_{n_j}}.$$

If $i \notin k(j)$ we obtain an ordinary wedge product $e_i \wedge e_{k(j)_1} \cdots \wedge e_{k(j)_{n_j}}$. Otherwise, we get

$$e_i e_{k(j)_1} \cdots e_i \cdots e_{k(j)_{n_j}} = (-1)^m e_{k(j)_1} \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{k(j)_{n_j}}$$

by equation 2.24, where $k(j)_m = i$. This means precisely that

$$v \cdot \varphi = v \wedge \varphi - \iota_v \varphi.$$

□

Following proposition 2.28, we see that $d^* : L^{2,k}(M) \to L^{2,k-1}(M)$. This is precisely what one would expect from the adjoint of d. Moreover, d^* is zero when acting on 0-forms. It turns out to be the case that the inner product 2.3 on $L^{2,\bullet}(M)$ is in fact the same as

$$\langle \alpha, \beta\rangle = \int_M \alpha \wedge \star\overline{\beta}.$$

3.3. Hodge-de Rham Spectral Triple. We now have all ingredients to define a spectral triple. As in the preceding parts, let (M,g) be a compact orientable Riemannian manifold, let $L^{2,\bullet}(M)$ be the square integrable differential forms on M, let d and d^* be the exterior derivative and codifferential respectively. Let $\pi : C^\infty(M) \longrightarrow B(L^{2,\bullet}(M))$ be defined by sending $f \in C^\infty(M)$ to the multiplication operator $g \mapsto fg$ for $g \in L^{2,\bullet}(M)$. Define $D := d + d^*$, then D is a self-adjoint linear operator on $L^{2,\bullet}(M)$. It is important to note that D is not a bounded operator.

Fact 2.30. *The triple $(C^\infty(M), \pi, D)$ is a spectral triple.*

This is a non-trivial result. The triple must satisfy two properties:

- The resolvent operators $(D \pm i)^{-1}$ must be compact.
- The commutator bracket must be well-behaved with respect to the representation: $[D, \pi(a)] \in B(H)$ for all $a \in A$.

The first property implies that $(1 + D^2)^{-1}$ is compact. Employing the theory of Sobolev embeddings, we can assert that $(1 + D^2)^{-1/2}$ is also compact. Now, by direct computation

$$(D(1 + D^2)^{-1/2})^2 = D^2(1 + D^2)^{-1} = 1 - (1 - D^2)^{-1}$$

and so $D(1 + D^2)^{-1}$ is essentially invertible[1] (thus it is Fredholm). In fact, D is (unbounded) Fredholm: $D \cdot (D(1 + D^2)^{-1}) = 1 - (1 - D^2)^{-1}$, so $D(1 + D^2)^{-1}$ is an essential inverse. Because of the unboundedness of D, it is not straightforward what the index of D should be. Again, by invoking Sobolev space theory (without providing the details here) we can define the Fredholm index of D to be the Fredholm index of D in its Sobolev embeddings and because the index turns out to be the same under every choice of embedding, this is a well-defined invariant.

Now, the fact that $(d+d^* \pm i)^{-1}$ are compact follows from some deep results in the theory of *elliptic (pseudo)differential operators* which we will not discuss in depth. Because $d + d^*$ is self-adjoint, its index is 0. To circumvent this problem, we use the grading of our Hodge-de Rham spectral triple. Because $d + d^*$ swaps odd and even forms, we can write $d + d^*$ as

$$\begin{pmatrix} 0 & (d + d^*)^- \\ (d + d^*)^+ & 0 \end{pmatrix} : \Omega^{\text{even}}(M) \oplus \Omega^{\text{odd}}(M) \to \Omega^{\text{even}}(M) \oplus \Omega^{\text{odd}}(M)$$

where $(d + d^*)^-$ and $(d + d^*)^+$ are each other's adjoint.
It turns out that for the Hodge Laplacian $\Delta := (d + d^*)^2$, write $\Delta = \bigoplus_{p=1}^n \Delta_p$ where $\Delta_p := \Delta\big|_{\Omega^p(M)}$ we have

$$H^p := \operatorname{Ker} \Delta_p \simeq H^p_{dR}(M, \mathbb{R})$$

and

$$\Omega^p(M) = H^p \oplus \operatorname{Im}(d) \oplus \operatorname{Im}(d^*).$$

[1]The adjective 'essential' means 'when projected onto the Calkin algebra'.

This is known as the *Hodge Decomposition Theorem*[War83]. From this we can derive the following.

Theorem 2.31. *The Fredholm index of* $(d+d^*)^+$ *is*

$$\mathrm{Index}(d+d^*)^+ = \sum_{k=0}^{n}(-1)^k \dim H^k_{dR}(M,\mathbb{R}) = \chi(M).$$

Therefore, we obtain another *topological invariant* from analytic data only!

Remark 2.32 (Spectral Geometry)**.** At this point it is interesting to point out that the (Hodge) Laplacian Δ is a very interesting operator. Its zero-eigenvectors are precisely the harmonic forms on M and these represent the cohomology classes. Naturally, one wonders what the other eigenvalues and vectors may be. This question is fundamentally the basis of *Spectral Geometry* which is concerned with deriving geometric properties of manifolds by studying the spectra of elliptic differential operators defined on them. Whenever two manifolds M, N have Laplacians with the same spectrum (counted with multiplicity), we call M and N *isospectral.* It turns out that isometric manifolds are isospectral, but not vice-verse. Nevertheless, the spectrum of the Laplacian remains an interesting invariant.

For the second part, we have a small corollary.

Corollary 2.33. *Let* $f \in C^\infty(M)$ *and* $\omega \in L^{2,\bullet}(M)$, *then*

$$[D,f]\omega = c(df)\omega.$$

Proof. We have

$$\begin{aligned}
[D,f]a &= (d+d^*)f\omega - f(d+d^*)\omega \\
&= d(f\omega) + d^*(f\omega) - fd\omega - fd^*\omega \\
&= df\wedge\omega + fd\omega + fd^*\omega - \iota_{df}(\omega) - fd\omega - fd^*\omega \\
&= df\wedge\omega - \iota_{df}(\omega) \\
&= c(df)\omega.
\end{aligned}$$

Here we used Proposition 2.28 in the third step and proposition 2.29 in the last step. □

It follows that $[D,f]$ is a bounded operator on $B(L^{2,\bullet}(M))$. In fact, this corollary holds in more generality: we can use the Leibniz rule repeatedly to make $[D,-]$ work on general forms:

$$[D,a]\omega = c(\nabla a)\omega$$

where $a,\omega \in L^{2,\bullet}(M)$. Operators satisfying this relation are called *generalized Dirac operators* associated to ∇. Here, ∇ is the Levi-Civita connection. In fact, we can choose ∇ to be any connection satisfying some Clifford action compatibility relation (these are the *Clifford connections*), but we will not concern ourselves with this theory.

The following proposition makes our journey through the realms of abstraction worthwhile:

Proposition 2.34. *Let $(C^\infty(M), L^{2,\bullet}(M), D)$ be a spectral triple, where D is a generalized Dirac operator associated to the Clifford connection ∇ and let $p, q \in M$, then*

$$d(p,q) = \sup\{|f(p) - f(q)| : f \in C^\infty(M), \|[D,f]\| \le 1\}$$

where d is the geodesic distance on M.

Proof. We know that $[D,f] = c(df)$ (recall that c is the Clifford action). Hence

$$\begin{aligned}\|[D,f]\|^2 = \|c(df)\|^2 &\overset{!}{=} \operatorname{ess\,sup}_{x\in M} \|(\overline{df_x}, df_x)\| = \sup_{x\in M} \|(\overline{df_x}, df_x)\| \\ &= \sup_{x\in M} \|g_x^{-1}(\overline{df_x}, df_x)\| = \sup_{x\in M} \|g_x(\operatorname{grad}_x \overline{f}, \operatorname{grad}_x f)\| = \|\operatorname{grad} f\|_\infty^2\end{aligned}$$

Here '$\overset{!}{=}$' makes use of 2.3, where the inner product $(a,b) = g_x^{-1}(\overline{a}, b)$ for $a, b \in T_x^*M$ and g_x^{-1} the inner product induced by the Riemannian metric on T_xM. We have also made use of the fact that $\mathbb{C}l(T_x^*M)$ is a C*-algebra with this norm and Clifford action as multiplication. Hence, $(df_x \cdot \alpha, df_x \cdot \alpha) = \|df \cdot \alpha\|^2$ for all α.

Now, consider the function $d_p : M \to \mathbb{R}$ defined by $q \mapsto d(p,q)$. This function is continuous and has Lipschitz-constant 1, so $\|\operatorname{grad} d_p\|_\infty = 1$. Approximating d_p by smooth functions gives us $\|[D, d_p]\| = 1$. Suppose now that there exists an $f \in C(M)$ such that $|f(p) - f(q)| > d(p,q)$, then $\|\operatorname{grad} f_x\| > 1$ for some x on the minimal geodesic segment between p and q. Therefore, $\|[D,f]\| > 1$. The desired equality follows. ☐

The map $a \mapsto \|[D,a]\|$ is a semi-norm that satisfies a certain set of properties that make it a *Lip-norm*. Most importantly, it induces a metric on the state space that extends the geodesic distance on the manifold. We will introduce this concept in Chapter 5 where it will be crucial.

4. The Quantum Torus

The non-commutative torus is an example of a family of C*-algebras that represent a 'non-commutative deformation' of the ordinary torus (in our case: the 2-torus). There are multiple equivalent definitions, but in this thesis we will define it using the theory of spectral triples. We refer to the *quantum torus* as the non-commutative torus equipped with a (non-commutative) metric structure. The adjective 'quantum' replaces 'non-commutative' in Chapter 5, because we will not make use of the multiplicative structure.

The most economic description of the non-commutative torus is the following:

Definition 2.35. The **non-commutative torus** A_θ is the univeral C*-algebra generated by unitaries u, v satisfying

$$uv = e^{2\pi i\theta} vu$$

where $\theta \in [0,1]$. △

We will not give the definition of a *universal C*-algebra*, because it is very involved. It is enough for now to think of universal C*-algebras generated by some elements and constrained by some given relations as the 'smallest' such C*-algebra. Of course, in the case $\theta = 0$ the unitaries correspond to the functions $e^{2\pi ix}, e^{2\pi iy}$ where $(x,y) \in \mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$. In order to obtain a spectral triple from A_θ, we need to obtain a dense *-subalgebra that corresponds to the smooth functions for $\theta = 0$. This can be done, by taking the 'non-commutative Schwarz space'. More precisely:

Definition 2.36. We define the **Schwartz-Bruhat space** on A_θ^∞ to be the *-algebra

$$\mathcal{A}_\theta^\infty := \{ \sum_{n,m\in\mathbb{Z}} c_{n,m} u^n v^m : |c_{n,m}|(|n| + |m|)^q \text{is bounded for all } q \in \mathbb{N}\}$$

where u, v are unitaries corresponding to the functions $e^{2\pi ix}, e^{2\pi iy}$ respectively, and $uv = e^{2\pi i\theta} vu$. △

The functions in the Schwartz-Bruhat space are smooth and, if we were to define them on Euclidean space, *rapidly decreasing* (in which case we would just call them *Schwartz* functions). The Fourier transform is an isomorphism of the Schwartz-Bruhat spaces on $\mathbb{T}^2$ and $\mathbb{Z}^2$ (its dual). In particular, $\mathcal{A}_\theta^\infty$ is a dense *-subalgebra of A_θ.

We will now construct a spectral triple that represents the quantum torus. First, we need a Hilbert space on which we can represent $\mathcal{A}_\theta$. For this, we assert that

$$\tau : \mathcal{A}_\theta \to \mathbb{C}, \quad \tau(\sum_{n,m} c_{n,m} u^n v^m) = c_{0,0}$$

is a *faithful tracial state*, meaning that $\tau(ab) = \tau(ba)$ and $\tau(a^* a) = 0$ if and only if $a = 0$. This means that we can produce an inner product $\langle \cdot, \cdot \rangle_\tau$ on $\mathcal{A}_\theta$ defined by $\langle a, b\rangle_\tau := \tau(b^* a)$, just as we did in the GNS-representation. Denote by H_θ^τ the completion of $\mathcal{A}_\theta$ under this inner product. Define $H := H_\theta^\tau \oplus H_\theta^\tau$ and let $\mathcal{A}_\theta$ act on both summands by multiplication, then this representation is faithful. Finally, we must specify a self-adjoint operator that has bounded commutator and compact resolvents. We assert (consult [CPR11] for proof) that

$$D := \begin{pmatrix} 0 & \delta_1 + \kappa\delta 2 \\ -(\delta_2 + \overline{\kappa}\delta_1) & 0 \end{pmatrix}$$

is such an operator. Where $\kappa \in \mathbb{C}$ and δ_1, δ_2 derivations (linear maps satisfying the Leibniz rule) such that

$$\delta_1(u) = 2\pi iu, \delta_1(v) = 0,$$
$$\delta_2(u) = 0, \delta_2(v) = 2\pi iv.$$

Note that for $\theta = 0$, δ_1 and δ_2 correspond to the vector fields $\operatorname{grad} u, \operatorname{grad} v$ respectively. We now have our spectral triple $(\mathcal{A}_\theta, H, D)$.

Definition 2.37. The **quantum torus** is the tuple (A_θ, L_θ), where

$$A_\theta := \overline{\mathcal{A}_\theta^\infty} \subseteq B(H)$$

and

$$L_\theta(a) := \|[D, a]\|.$$

△

Remark 2.38. We have just defined a spectral triple using a *Dirac operator*. Dirac operators appear on manifolds that allow a *spin structure* and therefore they are less general than the Hodge-de Rham spectral triple that we defined in Section 3.3. However, later in this thesis we will require the geodesic structure on A_0 (the commutative case) and these coincide for both triples in the case $\theta = 0$, because both the Dirac and Hodge-de Rham operators are of *Dirac-type*.

Remark 2.39. The quantum torus for irrational θ can also be seen as a non-commutative quotient,albeit with some consideration. Consider the flat torus $\mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$ and consider the vector field $(dx, \theta dy)$ where $\theta \in [0,1] \setminus \mathbb{Q}$. Then this vector field gives us the *Kronecker foliation* which is the collection of maximal connected submanifolds V_α which we call *leaves* such that $T_{(x,y)} V_\alpha = (dx, \theta dy)$. It appears that every leaf is non-compact and lies dense in $\mathbb{T}^2$, as each leaf corresponds to an orbit of the irrational rotation.

5. Fuzzy spheres

Fuzzy spheres are families of matrix algebras that in some way 'converge' to ordinary spheres. We will only cover the fuzzy 2-sphere: there is no general recipe for defining fuzzy spheres. The motivation behind approximating ordinary spheres using matrix algebras is rooted in physics, where 'quantization' of the algebra of observables for some manifold M, that is $C(M)$, is an important theoretical tool in studying quantum mechanics.

Let us start with the definition right away. Denote by x_1, x_2, x_3 the coordinate maps of $S^2 \subseteq \mathbb{R}^3$ so that $x_1^2 + x_2^2 + x_3^2 = r^2$. Let us consider the algebra $\mathcal{A} \subseteq C(S^2)$ consisting of functions f that can be written as polynomial expansions:

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} f_\alpha x_\alpha$$

where α is a multiset, so $x_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ and f_α is a constant in $\mathbb{C}$ for each α. Because $\mathcal{A}$ separates points, contains the identity and is $*$-closed we must have that $\mathcal{A}$ lies dense in $C(S^2)$.

Now, we denote by $\mathcal{A}_k$ the vector subspaces of $\mathcal{A}$ that consist of truncated power series up to power k. So $\dim \mathcal{A}_0 = 1, \dim \mathcal{A}_1 = 4, \dim \mathcal{A}_2 = 9, \ldots$. It is not possible to multiply two elements of $\mathcal{A}_k$ as we would in $\mathcal{A}$, because we might get powers higher than k. We would still like the $\mathcal{A}_k$ to be C*-algebras, so we define a new multiplication on each $\mathcal{A}_k$. We could go ahead and multiply pointwise for each basis function, but then we would obtain a (boring) commutative C*-algebra corresponding to k^2 points, which breaks the symmetry of the $SO(3)$-action on S^2. Instead, we opt for a 'maximally' non-commutative multiplication; turning $\mathcal{A}_k$ into the full matrix algebra $M_k(\mathbb{C})$.

For $k = 1$ we can only define $f \cdot g := fg = f_0 g_0$.

For $k = 2$ we replace the coordinates x_i by the *Pauli sigma matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

times some constant κ. So, if $\tilde{x}_i = \kappa\sigma_i$, then we choose κ such that $\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 = r^2$ which means that $\kappa^2 = \frac{r^2}{3}$.
These matrices together generate $M_2(\mathbb{C})$ and we can write

$$f = \frac{\kappa}{2}(f_0 + f_3)\sigma_0 + \frac{\kappa}{2}(f_1 + f_2)\sigma_1 + \frac{i\kappa}{2}(f_1 - f_2)\sigma_2 + \frac{\kappa}{2}(f_0 - f_3)\sigma_3$$

where σ_0 is the identity matrix. We can multiply two truncated functions f, g by multiplying the Pauli-matrices. It is obvious that this turns $\mathcal{A}_2$ into M_2.

For $k \geq 3$ we must first note that the Pauli matrices span the Lie algebra $\mathfrak{su}(2)$ linearly: $\{i\sigma_i\}_{i=1}^3$ is an $\mathbb{R}$-basis for $\mathfrak{su}(2)$. Observe that the identity is not contained in $\mathfrak{su}(2)$, this follows from the definition in terms of one-parameter families:

$$\mathfrak{su}(2) \coloneqq \{X \in M_2(\mathbb{C}) : (e^{(tX)})^* = (e^{tX})^{-1} = e^{-tX}, \det(e^{tX}) = 1, \text{ for all } t \in \mathbb{R}\}.$$

We will now generalize to dimension $k = 3$ and higher by taking not the Pauli matrices, but the generators of an irreducible k-dimensional representation of $\mathfrak{su}(2)$. After all, the Pauli matrices span the irreducible representation on $\mathbb{C}^2$. In the following we will make use of sections 4.3 and 4.4 of [Hal15].

Let us construct irreducible representations for the lie algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$ and use the fact that the real Lie algebra $\mathfrak{su}(2)$ has the same representations as its complexification $\mathfrak{sl}(2,\mathbb{C})$. Because any two irreducible representations $\mathfrak{sl}(2,\mathbb{C}) \to GL(V)$ are in fact isomorphic, we will be content with a single specific representation. Denote by V_m the linear space of homogeneous polynomials in 2 complex variables of degree m, then $\dim V_m = m + 1$. Let us define a representation $\Pi_m : SU(2) \to GL(V_m)$ by

$$(\Pi_m(U)f)(z) = f(U^{-1}z).$$

It is clear that Π_m is a representation. As it turns out, we have the following...

Fact 2.40 (Propositions 5.1 and 5.3 in [BD85])**.** *The representation Π_m is irreducible and any V_m-representation of $SU(2)$ is isomorphic to Π_m.*

We define the Lie algebra representation $\pi_m : \mathfrak{sl}(2,\mathbb{C}) \to GL(V_m)$ by

$$\pi_m(X) \coloneqq \frac{\mathrm{d}}{\mathrm{d}x} e^{tX}\Big|_{t=0}$$

where $e^{tX} = \alpha_X^t(e)$, α_t^X the flow at time t of the left-invariant vector field X and $e \in SU(2)$ the identity. Let us make this representation slightly more concrete. The space $\mathfrak{sl}(2,\mathbb{C})$ has $\mathbb{C}$-basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying commutation relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Now, π_m acts on monomials $z_1^k z_2^{m-k}$ as follows:

$$\begin{aligned}
\pi_m(H) z_1^k z_2^{m-k} &= (m - 2k) z_1^k z_2^{m-k}, \\
\pi_m(X) z_1^k z_2^{m-k} &= -k z_1^{k-1} z_2^{m-k+1}, \\
\pi_m(Y) z_1^k z_2^{m-k} &= (k - m) z_1^{k+1} z_2^{m-k-1}.
\end{aligned}$$

These relations fully describe π_m, as the commutation relations are preserved by Lie algebra homomorphisms.

If we let $k = m + 1$, and let $\tilde{x}_1 = \kappa\pi_m(H), \tilde{x}_2 = \kappa\pi_m(X), \tilde{x}_3 = \kappa\pi_m(Y)$ we find that normalization constraint gives us $\kappa^2 = \frac{r^2}{k^2-1} = \frac{r^2}{m^2+2m}$. We can now allow m-fold products of the $\tilde{x}_i$ to describe our functions $f \in \mathcal{A}_m$:

$$f = f_0 + \sum_{i \in I_1}^{3} f_i \tilde{x}_i + \sum_{i_1, i_2 \in I_2} f_{i_1 i_2} \tilde{x}_{i_1} \tilde{x}_{i_2} + \ldots + \sum_{i_1, i_2 \ldots, i_m \in I_m} f_{i_1 \ldots i_m} \tilde{x}_{i_1} \cdots \tilde{x}_{i_m}$$

where I_k is the set of ordered symmetric tuples satisfying the normalization constraint. This implies that $|I_k| = |\mathrm{Sym}_k - \mathrm{Sym}_{k-2}|$ where Sym_k are ordered symmetric tuples of numbers $1, \ldots, k$. So $|I_k| = 2k - 1$ and we are left with an algebra that has linear dimension $1 + 3 + (2 \cdot 3 - 1) + \ldots + (2 \cdot m - 1) = k^2$. Because our $\tilde{x}_i$ live in $M_k(\mathbb{C})$ we have a natural product on $\mathcal{A}_k$ that ensures that $\mathcal{A}_k \simeq M_k(\mathbb{C})$.

Finally, note that for $k \to \infty$ we have $\kappa \to 0$. This tempts us to say that $\mathcal{A}_k$ 'becomes commutative at infinity'. Combined with the fact that the (normalized) eigenvectors of the $\tilde{x}_i \in \mathcal{A}_k$ eventually exhaust all *spherical harmonics*[2], we allow ourselves to claim (with some necessary chutzpah) that

$$\lim_{k \to \infty} \mathcal{A}_k \text{ “ = ” } C(S^2).$$

Where $C(S^2)$ must be interpreted as the multiplication algebra inside $B(L^2(S^2)))$. This limit is made precise using the quantum Gromov-Haudorff distance for which the matrix algebras converge to the sphere[Rie02].

6. Operator Spaces

When studying operator algebras, sooner or later one will encounter operator-valued matrices. For instance, this happens very prominently when studying operator K-theory; an extraordinary cohomology theory for C*-algebras that extends topological K-theory, which is the study of isomorphism classes of vector bundles on manifolds. Operator-valued matrices are themselves operators working on direct sums of Hilbert spaces. As such, spaces of matrices of operators are C*-algebras and they inherit the norm, ordering on self-adjoint elements and multiplication. For a given C*-algebra A, one can consider

[2]The spherical harmonics are an orthonormal basis for $L^2(S^2)$, they consist of harmonic, homogeneous polynomials in $\mathbb{R}^3$ restricted to the 2-sphere.

all matrix algebras $M_n(A) = A \otimes M_n$ with their inherited structure (as we do in K-theory), but one can also generalize this and take a linear subspace V of A to study $M_n(V)$. These $M_n(V)$ will not admit multiplications (rather unsatisfactory for a set of matrices), but they will have induced norms and order-structures. The family $M_n(V)$ with these norm and order structures are what we call an *operator space*. By definition, operator spaces lie inbetween normed spaces and C*-algebras. During this section we will follow [EG 00].

Let's start right away with the main definition of this section.

Definition 2.41. An **operator space** is a linear subspace V of some concrete C*-algebra $B(H)$ where H is a Hilbert space. $\triangle$

Given an operator space V, we can consider the V-valued matrices $M_n(V) \subseteq B(H^n)$ which inherit norm and order from $B(H^n)$ where $n \in \mathbb{N}$. These 'explicit' subspaces of operator algebras are also referred to as *concrete operator spaces*. Operator spaces can be characterized abstractly as follows:

Definition 2.42. An **abstract operator space** is a linear space V with distance matrix norms $\|\cdot\|_n$ defined on $M_n(V)$, such that

- $\|v \oplus w\|_{n+m} = \max\{\|v\|_m, \|w\|_n\}$.
- $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_m \|\beta\|$.

where $v \in M_m(V), w \in M_n(W), \alpha \in M_{m,n}, \beta \in M_{n,m}$. $\triangle$

We also want to study maps between operator spaces.

Definition 2.43. Let V, W be (abstract or concrete) operator spaces. A linear map $f : V \to W$ induces maps $f_n : M_n(V) \to M_n(W)$ by

$$[v_{ij}]_{1\leq i,j\leq n} \mapsto [f(v_{ij})]_{1\leq i,j\leq n}.$$

We define the **complete norm** by $\|f\|_{cb} := \sup_{n\in\mathbb{N}} \|f_n\|_n$ where $\|\cdot\|_n$ is the norm on $M_n(W)$. We call f...

- **completely bounded** if $\|f\|_{cb} < \infty$.
- **completely contractive** (or a complete contraction) if $\|f\|_{cb} \leq 1$.
- **completely isometric** if every f_n is isometric.
- **completely positive** if for all $a \in M_n(V)^+$ we have $f_n(a) \in M_n(W)^+$.

We denote by $\mathrm{CB}(V, W)$ the set of completely bounded maps from V to W. $\triangle$

It is a consequence of the *representation theorem for operator spaces* that concrete and abstract operator spaces are *the same* up to complete isomorphism. That is, if V is an abstract operator space, then there is a completely bounded bijection $\varphi : V \to V'$ where V' is a concrete operator space and φ^{-1} is also completely bounded. We will therefore dispense with the distinction between abstract and concrete operator spaces.

Definition 2.44. An **operator system** is a closed, unital, self-adjoint linear subspace $S \subseteq B(H)$. $\triangle$

Again, there is an abstract formulation for operator systems as well, but we will not bother exploring it as we will only get to work with concrete operator systems. An important feature of operator systems is that they carry a natural ordering. One considers the cone $M_n(S)^+ := M_n(S) \cap B(H^n)^+$ for each n and this induces an ordering on $M_n(S)_{sa}$. In ordinary operator spaces, the natural maps are the completely bounded maps. In operator systems, however, we impose a stricter condition.

Definition 2.45. Let S, S' be operator systems, then $f : S \to S'$ is called a **completely positive, unital map** or simply a **morphism** if f is completely bounded, positive and $f(1_S) = 1_{S'}$. △

It appears that the f is automatically bounded if f is unital and completely positive. We therefore refer to the collection of morphisms from S to S' as unital completely positive maps, denoted by $\mathrm{UCP}(S, S')$. Due to the additional structure on operator systems, we are given an extension of the classical Hahn-Banach theorem:

Theorem 2.46 (Arveson-Hahn-Banach)**.** *Given Hilbert spaces H, K and operator systems $V \subseteq W \subseteq B(H)$. If $\varphi : V \to B(K)$ is completely positive, then there exists a completely positive extension $\tilde{\varphi} : W \to B(K)$.*

There is also a generalization of the Gelfand-Naimark-Segal theorem for completely positive contractions instead of homomorphisms:

Theorem 2.47 (Stinespring's Decomposition)**.** *Let A be a unital C^*-algebra, let H be a Hilbert space and suppose that $\varphi : A \to B(H)$ is completely positive and completely contractive. Then, there exists a Hilbert space K, a contraction $T : H \to K$ and a unital $*$-representation $\pi : A \to B(K)$ such that*

$$\varphi(a) = T^* \pi(a) T.$$

Moreover, if φ is a morphism, T is isometric.

A famous corollary of this result is:

Corollary 2.48. *Let A, B be unital C^*-algebras and suppose that $\varphi : A \to B$ is a morphism with φ^{-1} also a morphism. Then $\varphi : A \to B$ is a $*$-isomorphism.*

6.1. Operator Space Topologies. We start by defining two important operations on operator-valued matrices

Definition 2.49. Let V be an operator space and suppose that $[m_{ij}] \in M_{k,l}(V), [m'_{ij}] \in M_{k,l}(V')$, we define the **scalar pairing** of $\langle m, m \rangle \in \mathbb{C}$ to be

$$\langle m, m' \rangle = \sum_{i,j} m'_{ij}(m_{ij}).$$

Now assume, more generally, that we have $[m_{ij}] \in M_{p,q}(V')$, then the **matrix pairing** $\langle\!\langle\, m, n \,\rangle\!\rangle \in M_{kp,lq}$ is defined as

$$\langle\!\langle\, m, m' \,\rangle\!\rangle = [\langle\!\langle\, m, m' \,\rangle\!\rangle_{ik,jl}] = [m'_{ij}(m_{kl})].$$

△

If we let $V = \mathbb{C}$, then the scalar pairing corresponds to the trace operator $\mathrm{tr}(\alpha\beta^*)$.

Definition 2.50. Let V' be the dual operator space (embedded in some $B(H)$) of V, suppose that $\{v'_\lambda\}_{\lambda\in\Lambda} \subseteq M_n(V')$ is a net and let $v' \in M_n(V')$, then

- We say that v'_λ converges to v' in the **point-norm** topology, if for all $x \in H^n$ we have $\lim_\lambda \|v'_\lambda(x) - v'(x)\| = 0$.
- We say that v'_λ converges to v' in the **point-weak*** topology, if for all $v \in M_v(V)$ we have $\lim_\lambda \| \langle\!\langle\, v'_\lambda - v', v \,\rangle\!\rangle \| = 0$.

△

Clearly, if $n = 1$ and V is any normed space the point-norm topology coincides with the classical weak* topology.

Remark 2.51 (Equivalent criterion for point-weak* convergence)**.** The name *point-weak** might be a bit confusing: it seems more reminiscent of a weak* topology. Indeed, v'_λ converges to v' if and only if each matrix entry $[(v'_{ij})_\lambda] \in V'$ converges to $[v'_{ij}]$ in the weak* topology. However, because all the M_n (including M_∞) are von Neumann algebras, they are dual spaces of spaces ${}_nM$ of trace class operators. The weak* topology on $M_n = {}_nM^*$ is given by the ultraweak topology. It turns out that v'_λ converges to v' in $\mathrm{CB}(X, M_n)$ if and only if for all $x \in X$ we have $v'_\lambda(x) \to v'(x)$ in the ultraweak topology on M_n. In the literature the point-weak* topology is often referred to as the *BW-topology* (bounded weak topology).

We now state a lemma that will be of importance in Chapter 5.

Lemma 2.52 (Compactness lemma)**.** *Let X be an operator space that is separable as a normed space, then:*

(1) For every $n \in \mathbb{N}$ the unit ball of $\mathrm{CB}(X, M_n)$ endowed with the point-norm topology is compact.
(2) For every $n \in \mathbb{N}$ the space $\mathrm{UCP}_n(X)$ endowed with the point-norm topology is compact.
(3) For every $n \in \overline{\mathbb{N}}$ the space $\mathrm{UCP}_n(X)$ endowed with the point-weak topology is compact.*

Proof. (1) Fix $n \in \mathbb{N}$ and define $\Phi : \mathrm{CB}(X, M_N) \to B(M_n(X), M_n(M_n)))$ by

$$\varphi \mapsto [\varphi]_{i,j=1}^n =: \varphi_n$$

This map is point-norm to weak* continuous and injective. If we restrict Φ to the unit ball, its image must be in the unit ball of $B(M_n(X), M_n(M_n)))$. This follows from the fact that $\|\varphi\| = \|\varphi_n\|_n$ (proposition 2.2.2 in [EG 00]). Hence, Φ is a homeomorphism onto its image and so the unit ball of $\mathrm{CB}(X, M_n)$ must be compact.

(2) Because $\mathrm{UCP}_n(X) \subseteq \mathrm{CB}(X, M_n)$ is a closed subset in the point-norm topology, it follows by the previous point that $\mathrm{UCP}_n(X)$ is point-norm closed for $n \in \mathbb{N}$.

(3) This is essentially Lemma 7.1 in [Arv69]: the space $M_n(X^*) \subseteq B(X, M_n)$ is isometrically isomorphic to the dual space of some normed space, where the

latter is endowed with the usual weak* topology. Hence the result follows by Banach-Alaoglu.

□

7. Non-commutative Convexity

In 1984 Gerd Wittstock introduced the concept of *matrix convexity* which generalized convex combinations, which classically have scalar coordinates, to matrix combinations[Wit84]. Matrix convex sets do not live in a single vector space, but rather in a countable collection of vector spaces; namely, an operator space. In this section we will expose the theory of *non-commutative convexity* which is a slight but highly impactful extension of matrix convexity. It was developed in [DK19] and [KS19] and we will use these articles as our primary references. Despite both articles focusing heavily on Choquet simplices, we will only concern ourselves with the theory of non-commutative extreme points.

Definition 2.53. An **nc convex set** over an operator space E is a graded subset $K = \dot{\bigcup}_{n\in\overline{\mathbb{N}}} K_n$ where $K_n \subseteq M_n(E)$ satisfying:

- $\sum \alpha_i x_i \alpha_i^* \in K_n$ for every collection $\{x_i \in K_{n_i}\}$ and every family of isometries $\{\alpha_i \in M_{n,n_i}\}$.
- $\beta^* x \beta \in K_m$ for every $x \in K_n$ and every isometry $\beta \in M_{n,m}$.

The first condition saying that nc convex sets are *closed under direct sums*, the second saying that *nc convex sets are closed under compressions*. We call K *compact* if E is a dual operator space and each K_n is compact in the point-weak* topology on $M_n(E)$. △

We have to point out that we range over all $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and in order to allow for $n = \infty$ (which is relevant for Section 3 in Chapter 5) we set the convention that $M_n(S^*) := M_n \otimes_{\min} S^*$ where $\otimes_{\min}$ is the so-called *spatial tensor product*. The condition that E be a dual operator space in order for K to be compact is derived from the central idea of *how we actually define convex sets in the classical sense*. A compact convex set F in a vector space V is precisely the convex hull of its extreme boundary $\partial^e F$. Because $\partial^e F$ is also compact, we can view F as the state space of the C*-algebra $C(\partial^e F)$ by Gelfand-Naimark, hence $F \subseteq C(\partial^e F)'$. In fact, the state space consists of only positive functionals, so $C(\partial^e F)$ lies in the operator space of non-negative linear functionals, which is a dual operator space. One can proceed to define the nc-convex analogue K_C of C to consists of the smallest nc convex set containing $K_0 = F$.

Definition 2.54. Let X be an operator system. The **nc state space** of X is defined to be the set $S^{nc}(X) := \dot{\bigcup}_{n\in\overline{\mathbb{N}}} K_n$ where

$$K_n = \mathrm{UCP}_n(X).$$

△

Remark 2.55. The term *generalized state space* is reserved for the collection $\dot{\bigcup}_{n\in\mathbb{N}} \mathrm{UCP}_n(X)$, so only taking $n < \infty$.

As discussed in Section 6 we have a family of point-weak* compact sets $\mathrm{UCP}_n(S) \subseteq \mathrm{CB}(S, M_n) \simeq M_n(S^*)$, so $\dot{\bigcup}_{n\in\overline{\mathbb{N}}} \mathrm{UCP}_n(X)$ is an nc compact convex set. Next, we define what *nc convex combinations* and *nc extreme points* are.

Definition 2.56. If K is an nc convex set, an **nc convex combination** of elements in K is an expression of the form $\sum \alpha_i^* x_i \alpha_i$ for a bounded collection of points $\{x_i \in K_{n_i}\}$ and a collection of operators $\{\alpha_i \in M_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$ for some $n \in \overline{\mathbb{N}}$. △

Definition 2.57. Let K be an nc convex set, an element $x \in K$ is called **nc extreme** if whenever we write x as a convex combination: $x = \sum \alpha_i x_i \alpha_i^*$ as in definition 2.56, we must have that...

(1) each α_i is a scalar multiple of an isometry β_i satisfying $\beta_i x \beta_i^* = x$, and
(2) each x_i decomposes as $x_i = y_i \oplus z_i$ where $y_i, z_i \in K$ and y_i unitarily equivalent to x.

The set of all nc extreme points of K is denoted by $\partial_{nc}^e(K)$

△

If K is nc convex, each K_n is compact convex in the classical sense. However, if we have a classical extreme point $x \in K_n$ it need not be an nc extreme point. Rather, it is a *pure* point.

Definition 2.58. Let K be an nc convex set, then a point $x \in K_n$ is called **pure** if whenever we write x as a convex combination: $x = \sum \alpha_i x_i \alpha_i^*$, then each α_i is a scalar multiple of an isometry β_i satisfying $\beta_i x \beta_i^* = x$. △

We also need the notion of *maximality*.

Definition 2.59. Let K be an nc convex set, we say that $y \in K_n$ is a **dilation** of $x \in K_m$ if there is an isometry $\alpha \in M_{n,m}$ such that $x = \alpha^* y \alpha$. We call y a **trivial dilation** if $y = x \oplus z$ for some $z \in K$. We call x **maximal** if all its dilations are trivial. △

Purity and maximality are equivalent to nc extremity:

Proposition 2.60 (Pure and maximal points are nc extreme (Proposition 6.1.4 in [DK19])**.** *Let K be an nc convex set. Then $x \in K$ is nc extreme if and only if it is both pure and maximal.*

Example 2.61. Let X be an operator space. The nc extreme points of the nc state space of X are called the **nc pure states** of X and they are denoted by

$$S^e_{\mathrm{nc}}(X) := \partial^e_{nc}(S^{nc}(X)).$$

♠

For this thesis, the most important fact concerning nc pure states is the following result, which is a combination of Theorem 3.2.3 and Example 6.1.8 in [DK19]:

Theorem 2.62. *Let A be a unital and separable C*-algebra, then $S^e_{nc}(A)$ consists precisely of the irreducible representations.*

Note that $\mathrm{UCP}_n(X) = \mathrm{UCP}_n(X_{sa})$, so we are allowed to restrict ourselves to the operator system of self-adjoint elements. At this point it is important to note that this implies in particular that for commutative unital C*-algebras $C(X)$ the irreducible representations coincide with the pure states and hence with X. This means that the nc pure states can be considered a full generalization of the points of a topological space, which is precisely what we are after.

We have skipped the lion's share of the theory presented in [DK19] and [KS19], but we must emphasize an important message that these articles convey: the nc extreme points can rightfully be called *extreme points*, because they form a minimal subset $\partial^e_{nc}K$of points of an nc convex set K that generates K by taking its *(closed) nc convex hull.* This was phrased in the non-commutative Krein-Milman theorem (Theorem 6.4.2 in [DK19]). This is in stark contrast to prior notions of matrix convexity, where the existence of matrix convex sets without extreme points could be demonstrated [Eve18]. Another elegant feature of the theory described in [KS19] is that irreducible representations can be seen as 'nc point evaluations', just like classical point evaluations correspond to pure states of commutative C*-algebras.

We finish this section by showing that for an operator system X the union $\dot{\bigcup}_{n\in\mathbb{N}}\mathrm{UCP}_n(X)$ lies point-weak* dense in $\mathrm{UCP}_\infty(X)$. This is a result we will need in Chapter 5.

Lemma 2.63 (Point-weak* density)**.** *Let X be an operator system, then*

$$\overline{\dot{\bigcup}_{n\in\mathbb{N}}\mathrm{UCP}_n(X)}^{w*} = \mathrm{UCP}_\infty(X).$$

Proof. Fix $\varphi\in\mathrm{UCP}_\infty(X)=\mathrm{UCP}(X,B(\ell^2))$, let $p_n\in B(\ell^2)$ denote the projection onto an n-dimensional subspace $\ell^2_n\subseteq\ell^2$ such that $\lim_{n\to\infty}p_n(x)=x$ for all $x\in\ell^2$. Define $\varphi_n := p_n\varphi p_n+(1-p_n)$, then clearly φ_n is a sequence of bounded operators in $\mathrm{UCP}_n(X)$. Let $x\in\ell^2$ and write $x=x_n\oplus y_n$ where $x_n:=p_n(x)$ and $p_n(y_n)=0$, then $\varphi_n(x)=p_n\varphi(x_n)+y_n$. However, $\|y_n\|\to 0$ so $\varphi_n(x)\to\varphi(x)$. We conclude that $\varphi_n\xrightarrow{wk}\varphi$ so by Remark 2.51 φ is a limit point of $\dot{\bigcup}_{n\in\mathbb{N}}\mathrm{UCP}_n(X)$.

Conversely, if φ_n is a point-weak* convergent sequence its limit must be unital, because $\varphi_n(1_n)=1_n$ for all $n\in\mathbb{N}$. And if $x\in X$ is positive, each $\varphi_n(x)$ must be positive. ☐

CHAPTER 3

Topological persistence

Algebraic topology is the mathematical field concerned with studying algebraic invariants of topological spaces. One can compute homology, cohomology and homotopy groups of manifolds and infer geometric properties from these structures. Notably, these structures do not naturally admit an analytic structure: one can not speak unambiguously about a Cauchy-sequence of cohomology rings for example. This is hardly a suprise, as by definition topological *invariants* are unperturbed by homotopy equivalences, and homotopy equivalences in turn allow spaces to be stretched, scaled and deformed in many other ways. As such, it is impossible to speak of a space possessing 'almost' a homology group. However, in computational mathematics it *does* make sense to speak of a point cloud that resembles a circle or a torus. This stems from the fact that we instinctively perceive a point cloud as representing an underlying topological shape of which we can compute topological invariants. The technique that allows for explicitly computing the generating cycles for homology groups of discrete spaces is called *persistent homology*. The broader mathematical field that deals with 'topological resemblance' of finite data is called *topological data analysis*. This field encompasses techniques such as clustering (ToMATo), creating graphs from point clouds (Mapper) and many other tools.
In section 1 we introduce the ingredients and inner workings of persistent homology. Here we consider point clouds (which are by definition finite) and Vietoris-Rips complexes. In section 2 we treat the more general setting of topological persistence. Here we no longer confine ourselves to point clouds and simplicial complexes.

1. Persistent Homology on finite vertex sets

Persistent Homology is a tool in computational topology that grants the ability to 'approximate' the homology of a topological space X given a finite subset of X. It works by computing the generators for the homology group over a finite field for growing simplicical complexes.

(1) Start with a point cloud X, a finite metric space.
(2) Choose in which degree q we want to compute the Betti numbers: let $q \in \mathbb{N}_0$.
(3) Additionally, we have to choose a finite field of coefficients $\mathbb{Z}_p$ with p prime. (Finite for computational reasons)
(4) Starting at $r = 0$ we let r increase and do the following for each r (an example is given in figure 1):
    (a) Construct an abstract simplicial complex $\tilde{\mathrm{VR}}_r(X)$ that contains the k-simplex Δ_k containing elements $x_1, \dots, x_k \in X$ whenever $\operatorname{diam} \Delta_k \leq r$. This is a so-called *Vietoris-Rips complex*.

(b) Our abstract simplicial complex $\tilde{\mathrm{VR}}_r(X)$ is finite and hence can be embedded in $\mathbb{R}^N$ for some N. Thus, we obtain a geometric realization $\mathrm{VR}_r(X)$ of our simplicial complex.
(c) For $\mathrm{VR}_r(X)$ we can compute homology classes using a boundary matrix and elementary matrix operations. We denote the generators of these homology classes by H_r.
(d) For each $r' < r$, we have a natural embedding $\mathrm{VR}_r(X) \hookrightarrow \mathrm{VR}_{r'}(X)$. This, in turn, induces a map $H_{r'} \to H_r$.
(e) Using the latter map, we can keep track of the 'birth' and 'death' of homology classes. Each homology is encoded by an interval $[a,\ b]$[1], where a is the value of r at which the homology class is born, b is the point at which the class is extinguished.
(f) When $r' = \operatorname{diam} X$ the algorithm terminates.

(5) We are left with a collection of intervals (possibly with duplicates). We refer to this as the *(finite) barcode* of X.

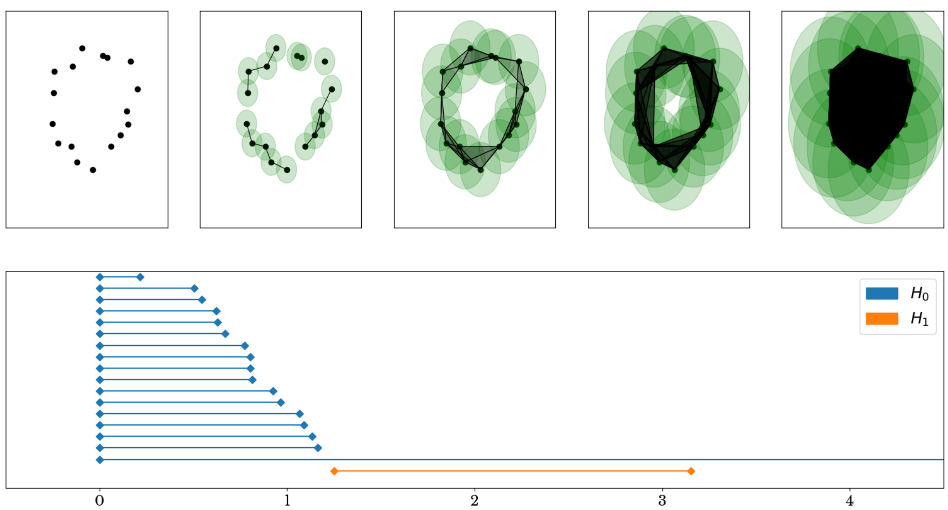

Figure 1. A visualisation of a growing Vietoris-Rips complex (top), together with the birth and death of its 0- and 1-cycles (bottom). The radii of the circles in the top image are equal to $\frac{r}{2}$. If a number circles have non-empty intersection a new simplex is added (only 0-, 1- and 2-simplicies are shown).

Definition 3.1. A **multiset** is a set that allows for multiplicity. For example $\{1,1,1,2\}$ is distinct from $\{1,2\}$ when considered as multisets.

[1]Strictly speaking, these intervals should be half-open, as homology classes *have* vanished at b. But closed intervals allow us to include single points $[a,a]$ which makes formalities in the theory easier.

△

Definition 3.2. A **(finite) barcode** is a multiset consisting of finitely many intervals with endpoints $a_i,\ b_i \in \mathbb{R}_\infty$ with $i \in I$ a finite index set and where $b_i \geq a_i \geq 0$. The set of finite barcodes is denoted by $\mathrm{Bar}_{\mathrm{fin}}$. Note that the endpoint of an interval can be infinite; $[0,\ \infty]$ is a legitimate interval.

△

A finite barcode B can be represented by its non-trivial intervals (that is the intervals $[a_i, b_i]$ for which $b_i > a_i$ and $[c_i,\ \infty]$). We denote these intervals as points in $\mathbb{R}^2$ and $\mathbb{R}_{\geq 0}$:

$$B = \{(a_1, b_1), \ldots, (a_m, b_m), (c_1, \infty), \ldots (c_n, \infty)\}.$$

We call two barcodes B_1, B_2 equivalent if they consist of the same intervals. We will also make use of the following notation for barcodes: we denote by $[a, b]_i$ the interval $[a, b]$ in the i-th homology diagram.

Definition 3.3. If $B_1 = \{(a_1,\ b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$ and $B_2 = \{(c_1, d_1), \ldots, (c_k, d_k)\}$ are two barcodes. A **matching** between B_1 and B_2 is a subset $\chi \subseteq B_1 \times B_2$ such that $(a, b), (a', b') \in \chi$ implies $a \neq a'$ and $b \neq b'$.
The **matching cost** of χ is defined by

$$c(\chi) = \max\left\{ \sup_{((a,b),(c,d))\in\chi} \|(a,b) - (c,d)\|,\ \sup_{(a,b)\in\chi^c} \frac{|b-a|}{2} \right\}.$$

In the above we let $\|(a, \infty) - (b, \infty)\| = |a - b|$ and $\|(a, b) - (c, \infty)\| = \infty$ if $b < \infty$. △

Remark 3.4. Alternatively, one can define barcodes as consisting of a finite number of non-trivial intervals together with all points on the diagonal (that is, all point intervals) counted with infinite multiplicity. One can then proceed to define a matching as a bijection between barcodes. The cost of a matching then simply reduces to taking the supremum of differences between intervals, because the intervals that would be unmatched in our definition would be matched to the closest diagonal point. We will mostly refer to barcodes as the multisets, because the finiteness can make proofs easier.

Definition 3.5. Let B_1, B_2 be two barcodes. Then

$$d_B(B_1,\ B_2) := \sup\{c(\chi) : \chi \text{ matching between } B_1 \text{ and } B_2\}$$

defines a metric, which we will call the **bottleneck distance**.

△

Remark 3.6. It follows that two barcodes B_1, B_2 are finitely close to each other if and only if they have a matching number of infinite intervals.

Example 3.7 (A dense sampling of the square)**.** In Figure 2 the persistence diagrams of two samplings of the unit square are shown. It makes clear that persistent homology does not just compute Betti numbers, but that it also encodes metric information, like the size of homological features.

♠

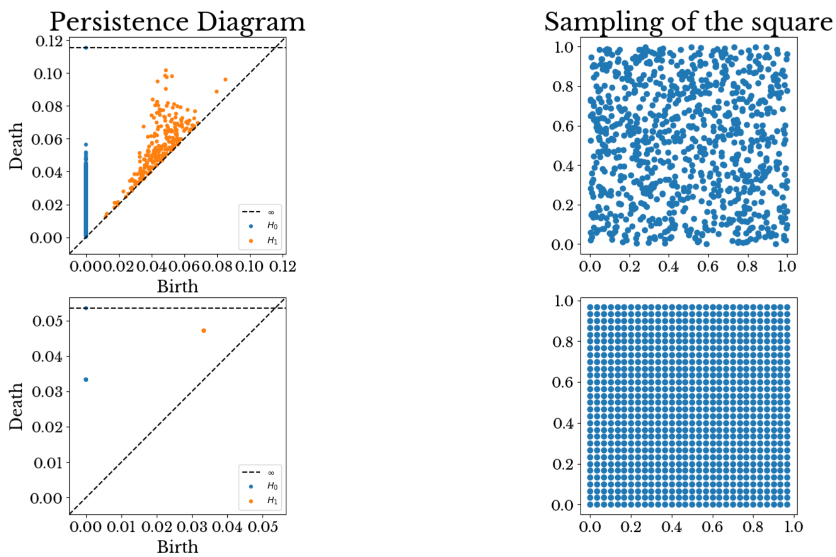

Figure 2. The plot and barcode of a random dense sampling of the circle (top) and of a uniform sampling (bottom) of the unit square. The barcode of the uniform sampling seems to have only 2 0-cycles and a single 1-cycle, but this is not true: because of the uniform distance between points, all connected components (except the last one dying at infinity) vanish at once. The same holds for the 1-cycle: every block of four points gives birth to a 1-cycle at the same time and these 1-cycles all die at the same time. Lastly, it is important to pay attention to they y-axis. The 1-cycle in the uniform sampling dies at $r \approx 0.048$, whereas the last 1-cycle in the random sampling dies at $r \approx 0.105$.

Example 3.8 (Torsion)**.** As mentioned before, one has to pick a finite field $\mathbb{Z}_p$ of coefficients in order to compute persistent homology. In general, a different choice of p can yield a different barcode. In 'ordinary' homology, this happens whenever a topological space exhibits torsion. Whenever all homology groups are free, one can pick p without changing the Betti numbers. However, the naive assumption that freeness of $H_d(C_r)$ for each $r \geq 0$ implies that the persistence diagrams are the same for each choice of p is wrong! [OY20] In Figure 3 an example is given.

♠

Arguably, the most important property of persistent homology is given by the following result. This result depends on the more general theory developed in the next section.

Theorem 3.9 (Stability of Persistent Homology[CSO12])**.** *Let $d \in \overline{\mathbb{N}}$. Let $P,\ Q$ be two point clouds and let $B(P), B(Q)$ be the corresponding barcodes for homology of degree d, then*

$$d_B(B(P)),\ B(Q)) \leq d_{GH}(P,\ Q)$$

where d_{GH} is the Gromov-Hausdorff distance.

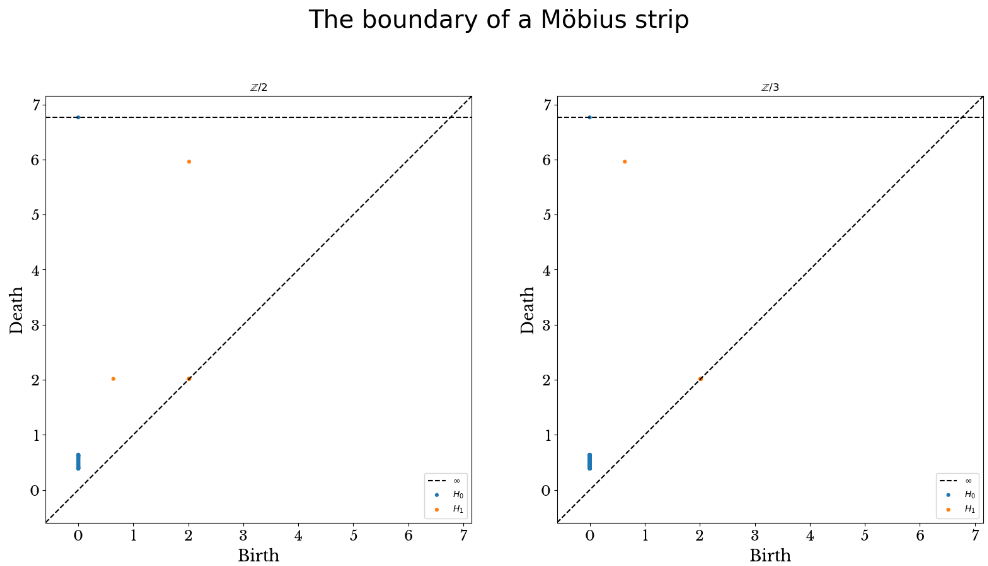

Figure 3. The persistence diagrams of the boundary of a Möbius strip, uniformly sampled with coefficients in $\mathbb{Z}_2$ (left) and coefficients in $\mathbb{Z}_3$ (right). In the $\mathbb{Z}_2$-diagram, a 1-cycle dies around $r = 2$ and another 1-cycle is born at the same time. Whereas, in $\mathbb{Z}_3$, we obtain a single persistent 1-cycle. Arguably, the field $\mathbb{Z}_3$ is superior in this case, as the boundary of a Möbius strip *is* a circle and one would expect a single persistent 1-cycle.

2. General Persistent Homology

Topological persistence generalizes persistent homology by considering not just point clouds, but more general topological spaces. Although a general definition for topological persistence does not exist, the field of Morse theory (founded by Marston Morse in the late 20s [Mor29]) can be seen as an early example. The torus and its height function provide a platitudinous (but no less illuminating) example; instead of starting with a point cloud, one starts with a single point (the bottom of the torus) and at each critical point a n-cell is added, effectively changing the homology as we climb through the torus. Therefore, we also obtain a barcode in which the intervals have end points given by the height of the torus.

To make this more precise, let us start with an alternative way to describe topological barcodes.

Definition 3.10. A **persistence module** $\mathbb{V}$ is a functor $\mathbb{V} : (P, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$. Here $(P, \leq)$ is a poset. $\triangle$

A key observation here is that $(P, \leq)$ can be considered as a category where the points in P are the objects and there is precisely one arrow $p \to q$ whenever $p \leq q$. This arrow is denoted by ι_p^q.

Example 3.11 (Point clouds)**.** The barcodes of finite point clouds from the previous section can now be described in terms of persistence modules. Let the point cloud be P, let our poset be $\mathbb{R}$ and consider vector spaces over $\mathbb{Z}_p$ (where p is the field of coefficients for our algorithm) and fix a homology degree d. At each value for $r \geq 0$ we have H_r-dimensional vector space V_r, corresponding to the number of generators for the homology group of $\mathrm{VR}_r(P)$. If $r \leq r'$, we have the inclusion map $\iota_r^{r'} : \mathrm{VR}_r(P) \hookrightarrow \mathrm{VR}_{r'}(P)$ which induces a map in homology: $\iota_r^{r'*} : V_r \longrightarrow V_{r'}$. ♠

Example 3.12. Let M be a manifold and $f : M \to \mathbb{R}$ be a Morse function, fix a homology degree d and fix a field $\mathbb{K}$. For each $r \in \mathbb{R}$, we let $V_r := H_d(f^{-1}(]-\infty, r]),\ \mathbb{K})$ and the maps between $V_r, V_{r'}$ are again induced by the inclusions $V_r \hookrightarrow V_{r'}$. ♠

Example 3.13. Let $[a,\ b[\ \subseteq \mathbb{R}$ be an interval, let $\mathbb{K}$ be a field, the persistence module $I_{[a,\ b[}$ is defined by the family of vector spaces

$$V_t = \begin{cases} \mathbb{K} & \text{if } t \in [a,\ b[, \\ \{0\} & \text{elsewhere} \end{cases}$$

together with maps

$$\iota_t^{t'} = \begin{cases} \mathrm{id} & \text{if } t, t' \in [a,\ b[, \\ \{\mathbb{K} \longrightarrow 0\} & \text{elsewhere.} \end{cases}$$

We call this persistence module an **interval module**. ♠

Later, we will need to decompose persistence modules into interval modules. This requires the notion of a direct sum.

Definition 3.14. Let $\mathbb{V} : (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$, $\mathbb{U} : (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ be persistence modules given by the family of vector spaces V_t, U_s and maps $\iota_t^{t'}, \kappa_s^{s'}$. The **direct sum of persistence modules** $\mathbb{U} \oplus \mathbb{V}$ is given by the family of vector spaces $V_t \oplus U_t$ and maps $\begin{pmatrix} \iota_t^{t'} \\ \kappa_t^{t'} \end{pmatrix} : U_t \oplus V_t \longrightarrow U_{t'} \oplus V_{t'}$. △

Just as barcodes are alternatively described by persistence modules, we need an alternative way to describe the bottleneck metric. This is achieved by using ϵ*–interleavings* which we will define for vector spaces indexed by the real numbers.

Definition 3.15. Let $\mathbb{V} : (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ be a persistence module. The **ε-shift** of $\mathbb{V}$ is the persistence module $\mathbb{V}^\epsilon$ given by vector spaces $V_t^\epsilon = V_{t+\epsilon}$. Likewise, for $s \geq r$ the maps $\eta_r^\epsilon : V_r^\epsilon \longrightarrow V_s^\epsilon$ are given by $V_{r+\epsilon} \longrightarrow V_{s+\epsilon}$. △

Definition 3.16. Let $\mathbb{V} : (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$, $\mathbb{U} : (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ be persistence modules. An **ε-interleaving** between $\mathbb{V}$ and $\mathbb{U}$ consists of two natural transformations $\varphi : \mathbb{V} \longrightarrow \mathbb{U}^\epsilon$, $\psi : \mathbb{U} \longrightarrow \mathbb{V}^\epsilon$ given by a collection of maps $\varphi_r : V_r \longrightarrow U^\epsilon_{\varphi(r)}$, $\psi_s : U_s \longrightarrow V^\epsilon_{\psi(s)}$ such that

$$\psi_{\varphi(r)} \circ \varphi_r = \eta_r^{2\epsilon}, \quad \varphi_{\psi(s)} \circ \psi_s = \eta_s^{2\epsilon}.$$

△

This seemingly abstract notion gives rise to a metric between persistence modules.

Definition 3.17. If $\mathbb{V}, \mathbb{U} : (\mathbb{R}, \leq) \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ are two persistence modules. Their **interleaving-distance** is defined as

$$d_I(\mathbb{V},\ \mathbb{U}) \coloneqq \inf\{\epsilon \mid \text{there exists an } \varepsilon\text{-interleaving between } \mathbb{V} \text{ and } \mathbb{U}\}.$$

△

Because we only consider persistence modules and barcodes that arise from topological spaces, we will suggestively denote persistence modules by $\mathbb{X}_f$ whenever they belong to a topological space X and a function $f : X \to \mathbb{R}$ (the codomain of f needs not be $\mathbb{R}$, but a more general setting is not required at this moment). That is, $\mathbb{X}_f$ consists of the family of vector spaces $\mathbb{X}_f^t \coloneqq H_d(X_t)$ where $X_t \coloneqq f^{-1}(]-\infty,\ t])$ with the straightforward maps induced by inclusion. In order to exclude pathological or unwieldy persistence modules, we will restrict ourselves to 'decent' persistent modules $\mathbb{X}_f$.

Definition 3.18. Let X be a topological space and $f : X \to \mathbb{R}$. The function f is referred to as the **filter function** on X. The persistence module $\mathbb{X}_f$ is of **finite type** if $\mathbb{X}^t$ is finite dimensional for all $t \in \mathbb{R}$. We call f **tame** if $\mathbb{X}_f$ is of finite type and f is called **q-tame** whenever $\iota_r^{r'} : \mathbb{X}_r \to \mathbb{X}_{r'}$ is of finite rank for each $r, r' \in \mathbb{R}$. △

The latter notion of *q-tameness* will not be mentioned further in this section, but we will require it in 4 where we consider persistent homology of general (possibly infinite) compact metric spaces. Now, we are ready to draw an equivalence between barcodes and persistence modules.

Theorem 3.19 ([BC19])**.** *Let X be a topological space, let $\mathbb{X}_f$ be the corresponding persistence module where $f : X \to \mathbb{R}$ is tame*[2]*. Then, there exists a decomposition into finitely many interval modules*

$$\mathbb{X} = \bigoplus_{[a,\ b[\in B(X)} I_{[a,b[}$$

where $B(X)$ is a multiset of intervals.

In fact, the bottleneck distance on barcodes and the interleaving distance on persistence modules are in a sense *the same*:

Theorem 3.20 ([CEH05])**.** *Let $\mathbb{X}, \mathbb{Y}$ be persistence modules of finite type and let $B(X), B(Y)$ be the corresponding multisets of intervals as in theorem 3.19, then we have*

$$d_I(\mathbb{X},\ \mathbb{Y}) = d_B(B(X),\ B(Y)).$$

This results allows us to prove Theorem 3.9.

Proof of theorem 3.9. Suppose P, Q are point clouds. Because P and Q are finite, they can be isometrically embedded in a common $\mathbb{R}^n$ for some n. Let $\delta = d_{GH}(P, Q)$ and define ϵ-neighbourhoods of P and Q:

$$P_\epsilon \coloneqq \{x \in \mathbb{R}^n \mid d(x, P) \leq \epsilon\}, \quad Q_\epsilon \coloneqq \{x \in \mathbb{R}^n \mid d(x, Q) \leq \epsilon\}.$$

[2]Once again, this result holds in more generality: $\mathbb{R}$ can be replaced with any totally ordered set.

Then, we must have $P \subseteq Q_\delta$ and $Q \subseteq P_\delta$. This implies that for the Vietoris-Rips complexes we have

$$\mathrm{VR}_r(P) \subseteq \mathrm{VR}_{r+\epsilon}(Q) \subseteq \mathrm{VR}_{r+2\epsilon}(P),$$
$$\mathrm{VR}_r(Q) \subseteq \mathrm{VR}_{r+\epsilon}(P) \subseteq \mathrm{VR}_{r+2\epsilon}(Q)$$

for all r. The corresponding inclusion maps form an ϵ-interleaving. Hence, by Theorem 3.20 we must have $d_I(\mathbb{P},\ \mathbb{Q}) = d_B(B(P), B(Q)) \leq \delta$. □

Now, we can state the most important theorem of this section, which is the analogue of Theorem 3.9 for filter functions.

Theorem 3.21 (Combination of [Les15] and [Bje16]). *Let $\mathbb{X}_f, \mathbb{X}_g$ be persistence modules with $f,\ g$ tame functions. Then*

$$d_I(\mathbb{X}_f, \mathbb{X}_g) \leq \|f - g\|_\infty.$$

Remark 3.22. This extension of the theory beyond mere Vietoris-Rips complexes allows us to apply persistent homology to more general settings. A few examples are listed below.

(1) To recapitulate: the persistent homology of a growing Vietoris-Rips complex fits in the framework of general topological persistence as follows: let X be the convex hull of a point cloud P and let $f : X \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \inf\{r \in \mathbb{R} \mid x \in \mathrm{VR}_r(P)\}.$$

This function is tame (but it is most definitely not continuous) and yields a persistence module with a decomposition that corresponds to the finite barcode from the previous section. We denote the this filtered complex by $\mathbb{R}\mathrm{ips}(P)$ and its persistence barcode in i-th homology by $H_i(\mathbb{R}\mathrm{ips}(P))$, we will later see that P can be replaced by a general compact metric space.

(2) One can also compute *Čech complexes* of point clouds: instead of adding a simplex Δ_r at r whenever the diameter of $\Delta_r = r$, we can consider balls of radius $\frac{r}{2}$ around each vertex and add a simplex between points $x_1, \ldots, x_n$, whenever the $r-$balls around the x_i have non-empty intersection. In terms of topological persistence, the only difference with a Vietoris-Rips complex is the choice of filter function $f : X \longrightarrow \mathbb{R}$.

(3) Another example of topological persistence is the *lower star filtration*. This is a technique that is employed in image classifaction and analysis of graphs. Given a finite graph $\Gamma \subseteq \mathbb{R}^n$ with vertices $V := \{v_i\}_{i=1}^n$ and a function $g : V \to \mathbb{R}$. We define a filter function $f : \Gamma \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in V \\ \max\{g(y),\ g(z)\} & \text{if } x \notin V \text{ and } x \text{ lies on the edge between } y \text{ and } z. \end{cases}$$

An example is given by a square grayscale image of $n \times n$ pixels that is given by a function $g : V \to [0,\ 1]$ that represents the brightness where $V = \{0, \ldots, n\} \times \{0, \ldots, n\}$. The resulting complex contains information about the size of dark and bright areas in the picture.

(4) As mentioned before, Morse functions fit seamlessly in the theory of topological persistence. Given a compact manifold X and a Morse function $f : X \to \mathbb{R}$ we immediately obtain a barcode. Classical Morse theory is non-discrete and hence cannot be used directly for computational purposes. *Discrete Morse theory* however can be used to compute persistent homology in a very efficient manner[MN13].

CHAPTER 4

Topological Barcodes

1. The Barcode Space

In Chapter 3 we introduced the notion of a *barcode* produced by a finite metric space X. Throughout this section X denotes a compact metric space, $\mathcal{P}_{\text{fin}}(X)$ denotes the set of finite subsets of X endowed with the Hausdorff distance and more generally $\mathcal{P}_{\text{fin}}$ denotes the space of finite metric spaces endowed with the Gromov-Hausdorff distance, $\mathcal{P}_{\text{c}}(X)$ denotes the closed subsets of X and by Bar_{fin} we will denote the space of finite barcodes. Because barcodes can be produced for each homology functor, we denote by $\beta_k : \mathcal{P}_{\text{fin}} \to \text{Bar}_{\text{fin}}$ the function that maps a finite point cloud to the barcode that represents its k-th homology (so $k \in \mathbb{N}_0$). Because the barcode maps β_k are 1-Lipschitz continuous we they can be uniquely extended to a Lipschitz map with domain $\overline{\mathcal{P}_{\text{fin}}}$. It is well established that $\overline{\mathcal{P}_{\text{fin}}(X)} = \mathcal{P}_{\text{c}}(X)$ and that $\mathcal{P}_{\text{c}}(X)$ is compact precisely when X is, but in order to do analysis on the space of barcodes we need to obtain a completion of Bar_{fin} that allows us to take limits. The following proposition characterizes this completion.

Proposition 4.1. *The completion* $\text{Bar}_\infty \coloneqq \overline{\text{Bar}_{\text{fin}}}$ *of* Bar_{fin} *consists of countable multisets* $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ *of intervals in* $\mathbb{R}^2$ *with* $b_i \geq a_i$ *such that for all* $\epsilon > 0$ *there are only finitely many intervals* $(a_{j_1}, b_{j_1}), (a_{j_2}, b_{j_2}), \ldots, (a_{j_n}, b_{j_n})$ *with* $|b_{j_i} - a_{j_i}| > \epsilon$ *for all* $i = 0, \ldots, n$.

Proof. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \text{Bar}_{\text{fin}}$ be a Cauchy sequence and let $\{B_k\}_{k\in\mathbb{N}} \subseteq \{A_n\}_{n\in\mathbb{N}}$ be a Cauchy subsequence such that $d(B_k, B_j) < 2^{-k}$ for all $j \geq k \geq 1$. Note that the barcodes B_i consist of intervals, say $B_i = ((a^i_1, b^i_1), (a^i_2, b^i_2), \ldots, (a^i_{n(i)}, b^i_{n(i)}))$. We will embed $\{B_n\}_{n\in\mathbb{N}}$ in $c_0(\mathbb{R}^2)$ by induction:
Firstly, define

$$\tilde{B}_1 = ((a^1_1, b^1_1), (a^1_2, b^1_2), \ldots, (a^1_{n(1)}, b^1_{n(1)}), (0,0), (0,0), \ldots) \in c_{00}(\mathbb{R}^2).$$

Now, suppose that $\tilde{B}_k$ is given by the sequence

$$((\tilde{a}^k_1, \tilde{b}^k_1), (\tilde{a}^k_2, \tilde{b}^k_2), \ldots, (\tilde{a}^k_{\tilde{n}(k)}, \tilde{b}^k_{\tilde{n}(1)}), (0,0), (0,0), \ldots) \in c_{00}(\mathbb{R}^2)$$

(where the tildes indicate that the multiset of intervals do not necessarily come from B_k). By definition of the bottleneck metric, there exists an optimal matching χ_k between B_k and B_{k+1}. As we will see, this induces a matching of $\tilde{B}_k$ with B_{k+1}. Therefore, we define for $1 \leq i \leq \tilde{n}(k)$ the intervals

$$(\tilde{a}^{k+1}_i, \tilde{b}^{k+1}_i) = \begin{cases} (a^{k+1}_\chi(i), b^{k+1}_\chi(i)) & \text{if } \chi \text{ matches the } i\text{-th interval,} \\ (0,0) & \text{otherwise.} \end{cases}$$

For $i > n(k)$, the pairs $(\tilde{a}_i^{k+1}, \tilde{b}_i^{k+1})$ are defined to be the unmatched intervals in B_{k+1} followed by $(0,0)$ after all intervals have been exhausted. It turns out that $\tilde{B}_{k+1}$ consists of intervals from B_{k+1} together with copies of the trivial interval at the origin.

We claim that $\{\tilde{B}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $c_{00}(\mathbb{R}^2)$ and hence, it has a limit $\tilde{B}$ in $c_0(\mathbb{R}^2)$.

For $j > i$, let us compute

$$\begin{aligned}\|\tilde{B}_i - \tilde{B}_j\|_\infty &\le \sum_{k=i}^{j-1} \|B_i - B_{i+1}\|_\infty = \sum_{i=i}^{j-1} \sup_{m\in\mathbb{N}} \sqrt{|\tilde{a}_m^k - a_m^{k+1}|^2 + |\tilde{b}_m^k - b_m^{k+1}|^2} \\ &\le \sum_{k=i}^{j-1} d_B(B_k, B_{k+1}) \quad (*) \\ &= \sum_{k=i}^{j-1} 2^{-k}.\end{aligned}$$

The equality after $(*)$ follows precisely from how $\tilde{B}_{k+1}$ is constructed from $\tilde{B}_k$: matched intervals differ at most 2^{-k} in $\mathbb{R}^2$ and unmatched intervals have length at most 2^{-k}. Hence, we have a Cauchy sequence.

If $\tilde{B} \in c_0(\mathbb{R}^2)$ is its limit, and if we write

$$\tilde{B} = ((a_1, b_1), (a_2, b_2), \dots) \in c_0(\mathbb{R}^2)$$

then $b_i \ge a_i$ for each $i \in \mathbb{N}$, because $(a_i^k, b_i^k)_{k\in\mathbb{N}}$ is also a Cauchy sequence for each i. If we consider $\tilde{B}$ as a barcode which we call B, we claim that $\lim_{k\to\infty} d(B_k, B) = 0$.
To prove this, we fix $k \in \mathbb{N}$ and construct a matching χ between B_k and B as follows: for each bar $[a_i^k, b_i^k] \in B_k$ track the matched bars $[a_{\chi_k(i)}^{k+1}, b_{\chi_k(i)}^{k+1}], [a_{\chi_{k+1}(\chi_k(i))}^{k+2}, b_{\chi_{k+1}(\chi_k(i))}^{k+2}], \dots$ using optimal matchings $\chi_k, \chi_{k+1}, \dots$. If at some point χ_{k+j} leaves our bar unmatched, let χ leave $[a_i^k, b_i^k]$ unmatched either; the cost of this bar is at most $\sum_{l=1}^{j} 2^{-k-l}$. On the other hand, if χ_{k+j} matches the bar for all $j \in \mathbb{N}$, the bar has a limit which is represented in $\tilde{B}$ as an element of $\mathbb{R}^2$ and so χ can match it as such. The cost of the latter matching is $\sum_{j=1}^{\infty} 2^{-2-k-j}$. We conclude that the cost of χ is smaller than 2^{-k-2} and so $\lim_{k\to\infty} B_k = B$. □

Corollary 4.2. *The barcode maps β_k extend naturally to 1-Lipschitz continuous maps*

$$\beta_k : \mathcal{P}_{\mathrm{c}} \to \mathrm{Bar}_\infty$$

PROOF. Each β_k is Lipschitz continuous and in particular uniformly continuous, so β_k can be extended to the closure of $\mathcal{P}_{\mathrm{fin}}$ which is $\mathcal{P}_{\mathrm{c}}$. As such $\beta_k(\mathcal{P}_{\mathrm{c}}) \subseteq \mathrm{Bar}_\infty$. □

There is an attractive description of topological barcodes from general compact metric spaces, which follows directly from the stability of persistent homology beyond point clouds.

Proposition 4.3. *Let X be a compact metric space, the topological barcodes $\beta_n(X)$ correspond to the filtered complex $H_n(\mathbb{R}\mathrm{ips}(X), \mathbb{Z}_p)$*

PROOF. Because X is in particular totally bounded, Proposition 5.1 from [CSO12] affirms that $H_n(\mathbb{R}\mathrm{ips}(X), \mathbb{Z}_p)$ is a q-tame persistence module. Theorem 3.9 holds in more generality for q-tame persistence modules (Theorem 5.2 from [CSO12]) and the result follows. □

Remark 4.4. The results used in the previous proposition make it clear that we do not need compactness, but that we can compute topological barcodes of totally bounded metric spaces instead by simply taking the closure. Henceforth we let the domain of β consist of all totally bounded metric spaces instead.

Remark 4.5. When we pass from point clouds to infinite compact metric spaces, a technical complication arises. Suppose X is an infinite compact metric space, the simplicial complex X_0 is precisely X as a vertex set, hence it is a totally disconnected simplex (although its geometric realisation is homeomorphic to X) and so it only contains infinitely many 0-(co)cycles. However, for each $t > 0$ the simplicial complex X_t contains X in a natural way. This means that for infinite compact metric spaces, we may have to deal with intervals that are open on the left. For example, if X is the unit circle, we would have a single representing 1-cocyle which is represented by the bar $]0,1]_1$. For the sake of simplicity, however, we rudely violate these rigorous considerations and, once again, we take the closure of these bars to make our lives easier. After all, the bottleneck metric does not discriminate between closed, open and half-open intervals.

Definition 4.6. We call Bar_∞ the *barcode space*, for each $k \in \mathbb{N}_0$ we label $\mathrm{Bar}_k := \mathrm{Bar}_\infty$ and call this the k-*th barcode space*. We call

$$\mathrm{Bar} := \prod_{k \in \mathbb{N}_0} \mathrm{Bar}_k$$

the *full barcode space* and the map

$$\beta : \mathcal{P}_{\mathrm{c}}(X) \to \mathrm{Bar}, \; P \mapsto (\beta_0(P), \beta_1(P), \dots)$$

is called the *barcode map* (or *full barcode map*). △

To get a clearer understanding of the kind of barcodes that different compact metric spaces can yield, we present a few examples.

Example 4.7. Let $X := \{0\} \cup \{2^{-n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ be endowed with the metric induced by $\mathbb{R}$. The only possible barcodes of X are supported in Bar_0, that is the $\beta_k : \mathcal{P}_{\mathrm{c}}(X) \to \mathrm{Bar}$ map to the trivial barcode. More interestingly, if $P, Q \in \mathcal{P}_{\mathrm{c}}(X)$ are distinct sets, we must have $\beta(P) \neq \beta(Q)$:the barcode map is injective. First note that $\beta_0(X)$ consists of all bars $[0, \frac{1}{2^n}]_0$ with $n \in \mathbb{N}$. If we have a closed subset P of X we remove all the intervals that correspond to the points in $X \setminus P$ and this is precisely the barcode of P. To see this, observe that the distances between any two points in X correspond precisely to a partial sum of the geometric series. Hence, any combination of points that is removed from X yields a different barcode

♠

Example 4.8 (The Baboushka space)**.** For each $n \in \mathbb{N}$ embed S^n in $\ell^2 := \{(a_n)_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |a_n|^2 < \infty\}$ by $S^n \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \ell^2$ where S^n embeds as the unit sphere, $\mathbb{R}^{n+1}$ embeds into the first $n+1$ coordinates. and scale each embedded S^n with a factor $1/n$. Consider the disjoint union of these embedded spheres:

$$X = \dot{\bigcup}_{n\in\mathbb{N}} S^n \subseteq \ell^2.$$

Then X is compact with the metric induced by $\|..\|_2$. The barcode of this space consists of the bars $[0,\ 1/n]_0$ for all $n \in \mathbb{N}$ in Bar_0 and a single bar $[0,\ 1/n]_n$ in Bar_n for each $n \in \mathbb{N}$. Therefore, this is in example of a space that has a barcode with infinite support.

♠

2. Estimating Barcodes from Topology

It is clear that the persistent homology of a compact space X cannot exists without metric information on X. However, in some cases we can infer a bit of information about the topological barcodes from its topology only.

Theorem 4.9 (Vietoris-Rips approximation of closed Riemannian manifolds [Lat01])**.** *Suppose that (X, d) is a closed Riemannian manifold, then there exists an $\epsilon > 0$ such that for a ϵ-dense sampling X^ϵ of X, the geometric Vietoris-Rips complex $\mathrm{VR}_\epsilon(X^\epsilon)$ is homotopy equivalent to X.*

Lemma 4.10 (Barcode approximation lemma)**.** *Suppose that X is a is a compact CW-complex with metric. Fix a field $\mathbb{K}$ (for example: $\mathbb{Z}_p$). Then there exists a constant $r > 0$ such that for every generator $[\alpha] \in H^\bullet(X, \mathbb{K})$ of the vector space[1] $H^\bullet(X, \mathbb{K})$ there exists a bar $[0, l_\alpha]_{k_\alpha} \in \beta(X)$ with $l_\alpha \geq r$.*

Proof. We employ the theory of *metric cohomology* introduced by [Hau95]. It states that for a compact metric space X and a ring R the functor

$$\mathcal{H}^\bullet(X, R) := \lim_{\epsilon\to 0} H^\bullet(\mathrm{VR}_\epsilon(X), R),$$

where $\lim_{\epsilon\to 0} H^\bullet(\mathrm{VR}_\epsilon(X), R)$ is a direct limit, is naturally isomorphic to $\check{H}^\bullet(X, R)$ (Čech cohomology). Because X is a finite CW-complex, it must now hold that $\mathcal{H}^\bullet(X, R) \simeq H^\bullet(X, R)$, where the latter indicates simplicial cohomology. If we now take $R = \mathbb{K}$ and ignore the cup product, we obtain a direct limit of vector spaces.

Let $V := \lim_{\epsilon\to 0} H^\bullet(\mathrm{VR}_\epsilon(X), \mathbb{K})$ as a vector space. Of course, $V \simeq H^\bullet(X, \mathbb{K})$ as vector spaces. By definition, every generator $[v] \in V$ corresponds to some $[\alpha_v]$ which is represented in some $H^k(VR_{\epsilon_v}(X), \mathbb{K})$ and hence in every $H^k(VR_{\epsilon'}(X), \mathbb{K})$ for $0 < \epsilon' \leq \epsilon_v$. Let $r' := \min_{[v]\in V} \epsilon_v$.

Now, for each cohomology class $[\omega] \in H^\bullet(X, \mathbb{K})$ there exists a barcode $[a_\omega, b_\omega]_{k_\omega}$ of length at least r'.

□

[1] we are only interested in the vector space structure. If we were to say 'generator' instead, we could end up with less elements

The following lemma is in fact a corollary of Lemma 4.10 and Theorem 17 in [ALS19].

Lemma 4.11 (Convex spaces give trivial barcodes)**.** *If X is a compact, convex metric space. Its barcode $\beta(X)$ contains precisely one bar $[0,\infty]_0$ that represents the connected component. Hence it is contained in* Bar_0.

Proof. Let $\epsilon > 0$, let X^ϵ be a finite ϵ-dense sampling of X. Embed X^ϵ isometrically into $\mathbb{R}^k$ for some k. By Theorem 17 in [ALS19] it follows that the VR complex $\mathrm{VR}_{\epsilon'}(X^\epsilon)$ is contractible for all $\epsilon' \geq \epsilon(2-\sqrt{3})$. Hence, except for the barcode in $H_0(\mathbb{R}\mathrm{ips}(X^\epsilon))$ corresponding to the main connected component, all barcodes must have died before $\epsilon(2-\sqrt{3})$. By Gromov-Hausdorff-bottleneck continuity of the barcode map, we see that $\beta(X^\epsilon) \to \beta(X)$ in the bottleneck-distance, hence $\beta(X)$ only contains the barcode $[0,\infty]_0$ in homology degree 0.

$\square$

CHAPTER 5

Barcodes of quantum compact topological spaces

In this chapter we will formulate the theory needed to answer the question: *can non-commutative persistent homology exist?* To this end, we will propose a candidate barcode map for *quantum compact metric spaces* in Section 1. This approach is based on the principle that points of a topological space X coincide with the pure states on $C(X)$: $\partial^e S(C(X))$. Instead of letting our barcode map β use X as a vertex set, we will plug $\partial^e S(A)$ into β for a C*-algebra (or, more generally: an order-unit space) A. We will show that this cannot work due to the erratic behaviour of the pure state space under the *quantum Gromov-Hausdorff distance.*
Next, in Section 3 we propose three further (similar) candidate barcode maps. This time, they are defined on *matricial* and *quantized compact metric spaces* and instead of looking at the pure states, we consider *nc pure states* (defined in Chapter 2, Section 7) which hold much more information than the pure states; the nc pure states $S^e_{\mathrm{nc}}(A)$ correspond to all concrete irreducible representations of a unital C*-algebra A. We show that for the matricial version our candidates cannot work, due to ill-definedness. Unfortunately, we are not able to disprove (nor, more optimistically, prove) the eligibility of the candidate barcode maps for the quantized version.

1. Quantum Compact Metric Spaces

During this section we follow [Rie99] and [Rie03]. Suppose that (X, d) is a compact metric space. We can define a seminorm[1] $L_d : C(X) \to [0, \infty]$ by

$$L_d(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \neq y \right\}.$$

This is the *Lipschitz-seminorm*: it assigns to a Lipschitz continuous function its Lipschitz constant. Non-Lipschitz functions are understood to have Lipschitz-constant ∞. From L_d we can now recover our metric d by taking

$$\tilde{d}(x,\ y) = \sup \{ |f(x) - f(y)| : L_d(f) \leq 1 \}.$$

Note that this principle is very reminiscent of what we did in Section 3.3 of Chapter 2. Indeed, the seminorm L_d is a generalization of the seminorm $\|[D, f]\|$ where D is a generalized Dirac operator. However, whereas earlier we made use of algebraic properties, it turns out that we can robustly generalize the concept of a metric to normed spaces that only have a specified order structure without endowing them with a multiplication.

[1]We admit seminorms that take values at infinity.

Definition 5.1. An **order-unit space** is a partially ordered vector space A with a distinguished element e (the unit) that satisfies:

(1) If $a \in A$ and $a \leq re$ for all $r \in \mathbb{R}$, then $a \leq 0$.
(2) For every $a \in A$, there exists an $r \in \mathbb{R}$ such that $a \leq re$. Moreover, on A we have a norm defined by

$$\|a\| := \inf\{r \in \mathbb{R} : -re \leq a \leq re\}.$$

△

The above definition encompasses all real unital C*-algebras or the self-adjoint elements of unital C*-algebras.

Definition 5.2. Let A be an order-unit space and let $L : A \to \mathbb{R} \cup \{\infty\}$ be a seminorm. Then the state space $S(A)$ has a natural induced metric ρ_L defined by

$$\rho_L(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| : L(f) \leq 1\}$$

which we refer to as the **Monge-Kantorovich metric**.

△

Definition 5.3. Let A be an order unit-space. A **Lip-norm** on A is a seminorm $L : A \to [0, \infty]$ satisfying the following properties:

(1) The set $L^{-1}([0, \infty[)$ is dense in A.
(2) We have $L(a) = 0$ if and only if $a \in \mathbb{R}\, e$.
(3) The topology on $S(A)$ induced by ρ_L coincides with the weak* topology.

△

Proposition 5.4. *Let (X, d) be a compact metric space and let L_d be the Lipschitz seminorm, then $(C(X), L_d)$ is a quantum compact metric space.*

PROOF. The seminorm L_d, is a Lip-norm if we set $L_d(f) = \infty$ whenever f is not Lipschitz-continuous. As Lipschitz functions on X, denoted by $\mathrm{Lip}(X)$ lie dense in $C(X)$ this fixes the density property. Clearly, the only functions with Lipschitz-constant 0 are the constant functions. For the last point, we define the quotient space $C(X)/\sim$ by letting $f \sim g$ with $f, g \in C(X)$ whenever $g = f + \lambda$ for some $\lambda \in \mathbb{R}$. The corresponding quotient supremum norm is denoted $\|\cdot\|_\sim$ and the quotient Lip-norm by $\tilde{L}$. By Theorem 1.8 in [Rie98] we must have that ρ_{L_d} induced the weak-* topology, precisely if $\{f \in C(X)/\sim : \tilde{L}(f) \leq 1\}$ is totally bounded. But this follows from Arzelá-Ascoli, where dividing out the constant functions is of fundamental importance. We see that $(C(X), L_d)$ is a quantum compact metric space. □

Definition 5.5. A **quantum compact metric space** is a pair (A, L) where A is an order-unit space and L is a Lip-norm. △

Definition 5.6. A **morphism** $\pi : A \to B$ is a unital positive linear map between order-unit spaces A and B. △

If we have a surjective morphism $\pi : A \to B$, we have an induced dual morphism $\pi' : B' \to A'$. This map is surjective: take $\mu, \nu \in S(B)$ and assume $\pi'(\mu) = \pi'(\nu)$, then $\pi'(\mu)(a) = \mu(\pi(a)) = \nu(\pi(a)) = \pi'(\nu)(a)$. But π is surjective, so for each $b = \pi(a)$ we must have $\pi'(\mu)(b) = \pi'(\nu)(b)$ and hence $\pi'(\mu) = \pi'(\nu)$. Moreover, because $\pi(e_A) = e_B$ we see that $\pi'\big|_{S(B)}$ maps to $S(A)$. Henceforth, we denote by $S(\pi)$ the map $\pi'\big|_{S(B)} : S(B) \to S(A)$.

Example 5.7. Let X, Y be operator systems, then X and Y are order-unit spaces, because they can be realized as closed self-adjoint linear subspaces in a C*-algebra. Let L_X, L_Y be their respective Lip-norms. So, unital positive maps are examples of morphisms of Lip-normed operator systems. Of particular interest are the unital completely positive maps $\mathrm{UCP}_n(X)$ (the *generalized state space*) which will become very important in Section 3. ♠

We now need to generalize the notion of a closed subset of X. Suppose (X, d) is compact metric and $Z \subseteq X$ is a closed subset. Consider the restriction morphism $\pi : C(X) \to C(Z)$. Let L_Z be the Lip-norm induced by the restricted metric $d\big|_Z$, then $L_Z(\pi(f)) \le L(f)$ simply because the supremum in L_Z is taken over a subset. On the other hand, if $g \in C(Z)$ we can extend g to a function $\tilde{g} \in C(X)$ with the same Lipschitz-constant such that $g = \tilde{g}\big|_Z$ by McShane's extension theorem[McS34]. The fact that g is real-valued is paramount, as the complex-valued analogue of this extension theorem introduces a factor $\sqrt{2}$ [Wea18] which we cannot afford. Therefore, there exists a function $g \in C(X)$ such that $L_Z(\pi(g)) = L(g)$ and we see that L_Z is the *quotient seminorm* of on $C(Z)$.

This tells us that an appropriate generalization of a closed subset is given by a surjective morphism $\pi : A \to B$. We have:

Proposition 5.8 (Proposition 3.1 in [Rie03])**.** *Let A, B be order-unit spaces and let $\pi : A \to B$ be a surjective morphism, such that $S(\pi)$ is injective. If L is a Lip-norm on A, we define the quotient seminorm*

$$L_B(b) := \inf\{L(a) : \pi(a) = b\}.$$

Then $S(\pi) : S(A) \to S(B)$ is an isometry for ρ_L and ρ_{L_B}.

Convention 1. If A is a complex unital C*-algebra, it cannot be an order-unit space, because non-self-adjoint elements are not comparable. As such, we will implicitly assume that for a C*-algebra A and Lip-norm $L : A^{sa} \to [0, \infty]$, the quantum compact metric space (A, L) is in fact (A^{sa}, L). So, if $A = C(X)$ we implicitly take $A = C(X, \mathbb{R})$ for example.

Remark 5.9. In fact, throughout this chapter we will assume that all order-unit spaces and operator systems consist of self-adjoint elements. This is clearly less general than assuming only that our spaces are closed under taking the adjoint, but it doesn't matter as the (generalized) pure states are fully determined by the values they take on self-adjoint elements. We may assume without peril that our Lip-norms satisfy $L(a) = L(a^*)$ ([Rie03],[Ker03],[Wu06b]) if we want to define them on complex C*-algebras.

Let us also introduce some notation that we will use throughout this chapter.

Notation 1. Denote by $\mathcal{D}(L)$ the domain of the Lip-norm L. Denote by $\mathcal{D}_r(L)$ the set $\{a \in A : L(a) \leq r\}$ such that $\mathcal{D}(L) = \cup_{r\in\mathbb{R}} \mathcal{D}_r(L)$. Denote by $\mathcal{B}_r$ the set $\{a \in A : L(a) \leq r, \|a\| \leq r\}$.

Definition 5.10. The **quantum Gromov-Hausdorff metric** between order-unit spaces $(A, L_A), (B, L_B)$ is defined by

$$\mathrm{dist_q}(A, B) := \sup\{\mathrm{dist}_H^{\rho_L}(S(A), S(B)) : L \in \mathcal{L}(A, B)\}$$

where $\mathcal{L}(A, B)$ is the set of Lip-norms L such that L restricted to A and B gives L_A and L_B respectively.

△

The next result is very important.

Proposition 5.11. *For compact metric spaces $(X, d_X), (Y, d_Y)$ we have*

$$\mathrm{dist_q}(C(X), C(Y)) \leq \mathrm{dist}_{GH}(X, Y).$$

Proof. Let $(X, d_X), (Y, d_Y)$ be compact metric spaces. Then $(C(X), L_X), (C(Y), L_Y)$ are the corresponding order-unit spaces (where $L_X := L_{d_X}, L_Y := L_{d_Y}$). By definition

$$\mathrm{dist}_{GH}(X, Y) = \sup\{d_H^\rho(\partial^e S(C(X)), \partial^e S(C(Y))) : \rho \in \mathcal{M}(X, Y)\}$$

where $\mathcal{M}(X, Y)$ is the set of metrics ρ on $X \coprod Y$ such that $\rho\big|_X = d_X, \rho\big|_Y = d_Y$. Fix a metric ρ and let $d_\rho = d_H^\rho(S(C(X)), S(C(Y)))$. We use the natural identifications $X \approx \partial^e S(C(X)), Y \approx \partial^e S(C(Y))$ and note that for each $x \in X$ there exists a $y \in Y$ such that $\rho(x, y) \leq d_\rho$. Any $x' \in S(C(X))$ can be written as a convex combination $x' = \sum t_i x_i$ where $x_i \in X$ (denote the corresponding distance minimizers in Y by y_i). Convexity of ρ now gives us that there exists a $y \in Y$ such that

$$\rho(x', y) \leq \sup_i \rho(x_i, y_i) \leq d_\rho.$$

The same argument holds for the roles of X and Y reversed. We obtain:

$$d_H^\rho(S(C(X)), S(C(Y))) \leq d_H^\rho(\partial^e S(C(X)), \partial^e S(C(Y)))$$

for any convex metric ρ. Taking suprema gives us

$$\mathrm{dist_q}(C(X), C(Y)) \leq d_{GH}(X, Y).$$

□

Proposition 5.11 essentially tells us that the quantum Gromov-Hausdorff distance need not be a full generalization of the classical Gromov-Hausdorff distance. Indeed, Hanfeng Li has described a set of examples in [Li01] where we have a strict inequality. To my knowledge it is still unknown whether the classical and quantum Gromov-Hausdorff distances are Lipschitz-equivalent when restricted to classical compact metric spaces.

A major result in [Rie03] is that the collection of isometry classes of quantum compact metric spaces under the quantum Gromov-Hausdorff distance is itself a complete metric space.

Theorem 5.12 (Completeness Theorem (Theorem 13.15 in [Rie03]))**.** *The set of isometry classes of quantum compact metric spaces, denoted by* $(\mathrm{QCM}, \mathrm{dist_q})$*, is a complete metric space.*

Lastly, we cite Theorem 4.5 from [Rie03], because it is important and interesting in its own right.

Theorem 5.13. *Let L be a seminorm on an order unit space A such that $L(a) = 0$ if and only if $a \in \mathbb{R}e$. Then ρ_L induces the weak* topology on $S(A)$ if and only if*

- *$S(A)$ has finite radius, and*
- *$\mathcal{B}_1$ is totally bounded for $\|\cdot\|_A$.*

2. Persistent Homology on Pure States

As we can naturally identify a compact Hausdorff space X with the pure states in $S(C(X))$ (which is precisely the extreme boundary $\partial^e S(C(X))$) and quantum compact metric spaces (A, L) come with a norm ρ_L on their state space $S(A)$, it is natural to ask whether we can obtain a reasonable metric space by restricting ρ_L to the pure states on $S(A)$. Naively, we propose a candidate barcode map

$$\beta_{\mathrm{q}} : (\mathrm{QCM}, \mathrm{dist_q}) \longrightarrow \mathrm{Bar}$$

defined by $\beta_{\mathrm{q}}(A) = \beta(\partial^e S(A))$. Note that $\partial^e S(A)$ may not be compact, but because $S(A)$ is compact, $\partial^e S(A)$ must be totally bounded, which is enough by Remark 4.4. Next, we require that β_{q} is Lipschitz-continuous with respect to the quantum Gromov-Hausdorff distance. This, however, is impossible as we will demonstrate with a counterexample. In Section 9 of [Rie03] Rieffel shows that the quantum tori (A_θ, L_θ), indexed by skew-symmetric matrices θ constitute a continuous family with respect to the quantum Gromov-Hausdorff distance. We only need the quantum 2-tori from Section 4 indexed by $\theta \in [0,1]$:

Lemma 5.14. *For $\theta \in [0,1]$ let $(\mathcal{A}, H, D) = (\mathcal{A}_\theta, H_\theta \oplus H_\theta, D)$ be the spectral triple for the non-commutative torus from Section 4. Denote $A := \overline{\mathcal{A}}^{sa}$ and let $L := \|[D,a]\|$ for $a \in A$, then (A, L) is a quantum compact metric space.*

Proof. Because $\mathcal{A}$ is unital, A is an order-unit space. We have to verify that L defines a Lip-norm.

(1) By definition of a spectral triple, $[D,a]$ is bounded for all $a \in \mathcal{A}$, hence $\|[D,a]\| < \infty$ for the dense *-subalgebra $\mathcal{A}^{sa}$ of A.

(2) Suppose that $a = \sum_{n,m\in\mathbb{Z}} c_{n,m} u^n v^m \in \mathcal{A}^{sa}$, then direct computation shows that

$$Da - aD = \begin{pmatrix} 0 & 2\pi i \sum_{n,m\in\mathbb{Z}} c_{n,m} u^n v^m (n + m\kappa) \\ -2\pi i \sum_{n,m\in\mathbb{Z}} c_{n,m} u^n v^m (n + m\overline{\kappa}) & 0 \end{pmatrix}.$$

Because $\mathrm{im}(\kappa) \neq 0$, we see that $[D,a] = 0$ implies that $c_{n,m} = 0$ whenever $n \neq 0$ and $m \neq 0$. However, it is clear that $[D,1] = 0$. Hence $L(a) = 0$ if and only if $a \in \mathbb{R}$.

(3) This is the hardest part, it is proven (specifically for Dirac operators) in [Rie98] in Theorem 4.2.

□

If $\theta = 0$, then $(A_0, L_0) \simeq (C(\mathbb{T}^2), L_{d_{\mathrm{arc}}})$ where d_{arc} is the arc-length metric induced by the Riemannian metric on $\mathbb{T}^2$, which is a flat metric determined by the constant κ.

If θ is any irrational number, A_θ is a simple, unital and separable C*-algebra. These irrational non-commutative tori therefore satisfy the conditions for the following lemma:

Lemma 5.15 (Glimm's lemma [BO08])**.** *Let $A \subseteq B(H)$ be a separable C*-algebra and H infinite-dimensional. If A contains no non-zero compact operators, then for each state $\tau \in S(A)$ there exists an orthonormal sequence of unit vectors $(\xi_n)_{n\in\mathbb{N}}$ such that $\tau_{\xi_n} \to \tau$ in the weak* topology. Where $\tau_\xi(a) := \langle a(\xi), \xi\rangle$.*

Because the states τ_{ξ_n} are pure (they correspond to rank-1 projections in $B(H)$), it follows that $\overline{S^e(A_\theta)} = S(A_\theta)$.

Now, we can use Proposition 4.11 to conclude that $\beta_{\mathrm{q}}(A_\theta)$ contains only a single barcode in homology degree 0 whenever θ is irrational. However, theorem 4.9 implies that A_0 must have at least two non-trivial barcodes in homology degree 1 and at least one non-trivial barcode in degree 2. It follows that $\theta \mapsto A_\theta$ does not map to a continuous family of barcodes under β_{q}, so our proposed barcode map cannot even be continuous. We conclude:

Theorem 5.16. *The candidate barcode map β_{q} is discontinuous.*

In fact, any non-commutative Gromov-Hausdorff analogue for which the quantum tori form a continuous family makes our proposed barcode map β_{q} discontinuous. Therefore, also the *matricial quantum Gromov-Hausdorff distance*[Ker03] and the *Gromov-Hausdorff propinquity*[Lat13] fail to make β_{q} a suitable candidate.

3. Persistent Homology on non-commutative pure states

The 'naive' approach of identifying topological spaces with pure states and using the latter as a vertex set for persistent homology does not work. In this section we develop two similar alternative barcode maps. The central idea is that we must not look at pure states, but rather at the nc pure states defined in Section 7, Chapter 2. The nc pure states contain more information than pure states, because they correspond to concrete irreducible representations. First we will briefly introduce the matricial and quantized Gromov-Hausdorff distances in parallel and combine them into the framework of non-commutative convexity. After we have formulated two new candidates for the barcode map we will show that both maps are ill-defined for the matricial Gromov-Hausdorff distance. At most one[2] of the barcode maps might theoretically be eligible for the quantized Gromov-Hausdorff distance, but this is an open problem.

Recall from Chapter 2, Section 7 the definitions of the *nc state space* and the *nc pure states*. Theorem 2.62 tells us that the nc pure states of a unital, separable C*-algebra A

[2] the eligibility of the one forces ill-definedness on the other and vice versa.

are precisely the irreducible representations $A \to M_n$ where $n \le \infty$. Let us start with an example.

Lemma 5.17 (Nc pure states of matrix algebras are projective unitaries)**.** *We have a homeomorphism*

$$S^e_{nc}(B_n) \approx PU(n)$$

where $PU(n)$ is the projective unitary group in n dimensions.

Proof. By Theorem 2.62 the nc pure states are precisely the irreducible representations $B_n \to M_k$. Because $B_n \simeq M_n$ as an algebra, B_n is simple, hence we can only have non-trivial representations $B_n \to M_n$. Any irreducible representation must be unitarily equivalent to the identity representation. Hence, we have a surjective map

$$\Phi : U(n) \to S^{\mathrm{e}}_{\mathrm{nc}}(B_n), \quad U \mapsto \varphi_U$$

where $\varphi_U : B_n \to M_n$ defined by $A \mapsto UAU^*$. Clearly, $\Phi(U) = \Phi(V)$ exactly when $UAU^* = VAV^*$ for all A. This can only happen when $V = \lambda U$ for some $\lambda \in \mathbb{R}$. Hence, Φ factors uniquely through $PU(n)$: $\Phi : U(n) \xrightarrow{\pi} PU(n) \xrightarrow{\tilde{\Phi}} S^{\mathrm{e}}_{\mathrm{nc}}(B_n)$. It follows that $\tilde{\Phi}$ is bijective. Because the space $\mathrm{UCP}_n(M_n)$ is finite-dimensional, any two vector space topologies are equivalent. The map $\tilde{\Phi}$ can easily be shown to be continuous with respect to the point-norm topology: fix $A \in M_n$ and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a net in $U(n)$ converging to $U \in U(n)$, then

$$\begin{aligned}
\|U_\lambda A U_\lambda^* - UAU^*\| &= \|U_\lambda A U_\lambda^* - U_\lambda A U^* + U_\lambda A U^* - UAU^*\| \\
&\le \|U_n A U_\lambda^* - U_\lambda A U^*\| + \|U_\lambda A U^* - UAU^*\| \\
&= \|U_\lambda A\| \|U_\lambda^* - U^*\| + \|U_\lambda - U\| \|AU^*\|.
\end{aligned}$$

Because $U_\lambda \to U$ in the Lie group topology is equivalent with convergence in operator norm, we see that $\lim_\lambda \tilde{\Phi}(U_\lambda) = \tilde{\Phi}(U)$ and hence, $\tilde{\Phi}$ is a homeomorphism. ☐

We can now formulate an imprecise version of the candidate barcode map:

Question 5.18. *Is the map $\tilde{\beta} : (\mathscr{C}, \mathrm{dist}) \to \mathrm{Bar}$ that assigns*

$$(X, L) \mapsto \beta(S^e_{nc}(X))$$

a well-defined, Lipschitz continuous map for some metric space $\mathscr{C}$ of isometry classes of operator systems endowed with a Lip-norm and appropriate non-commutative Gromov-Hausdorff distance dist *between these isometry classes?*

Clearly, the map $\tilde{\beta}$ doesn't really make sense yet. We will give sound formulations in Question 5.27 and Question 5.37. The remainder of this section is devoted to the formulation of these questions and to finding examples that answer these questions negatively.

3.1. The Matricial Gromov-Hausdorff distance. Let (X, L) be a Lip-normed operator system throughout. The Lip-norm L defines a metric on each $\mathrm{UCP}_n(X)$ for $n \in \mathbb{N}$. We will, however, also require a metric on $\mathrm{UCP}_\infty(X)$ (recall that these are the unital completely positive maps $X \to B(\ell^2)$). Luckily, it turns out that this is not a problem (Proposition 5.23).

First, we require a 'niceness' condition for Lip-norms.

Definition 5.19. A Lip-norm L is called a **closed Lip-norm**, whenever $\mathcal{D}_1(L)$ is closed in X. △

From now on, we impose upon our Lip-normed operator systems (X, L) that L be closed. We also need a definition for morphisms between operator systems that respect the Lip-norm:

Definition 5.20. If $\varphi : (X, L_X) \to (Y, L_Y)$ is a unital positive map. We call φ **Lip-isometric** whenever $\varphi(\mathcal{D}(L_X)) \subseteq \mathcal{D}(L_Y)$ and $L_Y(\varphi(x)) = L_X(x)$. Likewise, if φ is an isomorphism (a unital completely order isomorphic map) and both φ and φ^{-1} are Lip-isometric, we call φ **bi-Lip-isometric**. For brevity we refer to bi-Lip-isometric unital complete order isomorphisms as **isometries**.

△

Next, we introduce the Lip-norm induced metrics on the generalized state space.

Definition 5.21. For each $n \in \overline{\mathbb{N}}$, we define

$$\mu_{L,n}(\varphi, \psi) := \sup_{x \in \mathcal{D}_1(L)} \{\|\varphi(x) - \psi(x)\|\}$$

for $\varphi, \psi \in \mathrm{UCP}_n(X)$ and $\mathcal{D}_1(L) := \{x \in X : L(x) \leq 1\}$. We refer to the family $\{\mu_{L,n}\}_{n \in \overline{\mathbb{N}}}$ as the **matricial metrics for** L △

We include the proofs of the next two results (Proposition 2.9 and 2.12 in [Ker03] for the case $n \in \mathbb{N}$) because they are important and are also valid for $n = \infty$.

Proposition 5.22 (Matricial diameters coincide)**.** *The diameters $D_n := diam(\mathrm{UCP}_n(X))$ under $\mu_{L,n}$ is finite for each $n \in \overline{\mathbb{N}}$ and $D_n = D_m$ for every $n, m \in \overline{\mathbb{N}}$.*

Proof. The restriction map $S(X) \to S(\mathcal{D}(L))$ is a weak-* homeomorphism, because $\mathcal{D}(L)$ is dense in X. If ρ_L is the Monge-Kantorovich metric (definition 5.2) on $S(X)$, then this restriction map is $\rho_L - \mu_{L,1}$-isometric. And, the diameter of $(S(X), \rho_L)$ is finite, so $\mathrm{UCP}_1(X)$ has a finite diameter. Now, fix $n \in \overline{\mathbb{N}}$, let $x \in X$ and let $\varphi, \psi \in \mathrm{UCP}_n(X)$. There exists a (pure) state $\tau \in S(M_n(X))$ such that

$$\|\varphi(x) - \psi(x)\| = |\tau \circ (\varphi - \psi)(x)|.$$

Hence, the diameter of $\mathrm{UCP}_n(X)$ is bounded above by the diameter of $\mathrm{UCP}_1(X)$. Conversely, we can isometrically embed $S(X) \hookrightarrow \mathrm{UCP}_n(X)$ by letting $\sigma \mapsto \sigma 1_n$. This implies that $\mathrm{UCP}_n(X)$ has diameter bounded below by $S(X) \simeq \mathrm{UCP}_1(X)$. We conclude that all diameters $\mathrm{diam}(\mathrm{UCP}_n(X))$ are finite and they coincide. □

Proposition 5.23 (Matricial metrics give point-norm topologies)**.** *The matricial metrics $\{\mu_{L,n}\}_{n \in \overline{\mathbb{N}}}$ induce the point-norm topology on $\mathrm{UCP}_n(X)$ for every $n \in \overline{\mathbb{N}}$.*

Proof. Fix $n \in \overline{\mathbb{N}}$. Let

$$U_{\varphi,\Omega,\epsilon} := \{\psi \in \mathrm{UCP}_n(X) : \|\varphi(x) - \psi(x)\| < \epsilon \text{ for all } x \in \Omega\},$$

where $\varphi \in \mathrm{UCP}_n(X)$, $\epsilon > 0$ and Ω a finite subset of $\mathcal{D}_1(L)$. If we choose for each $x \in \Omega$ an element $y_x \in \mathcal{D}(L)$ such that $\|x - y_x\| < \frac{\epsilon}{2}$ and let $M > \max_{x \in \Omega} L(y_x)$, then the $\frac{\epsilon}{2M}$-ball of

$\mu_{L,n}$ is contained in $U_{\varphi,\Omega,\epsilon}$. Hence, the $\mu_{L,n}$-topology is finer than the norm topology. Let now $B(\varphi,\epsilon)$ be an ϵ-ball in the point-norm topology. Because we forced $\mathcal{D}_1(L)$ to be closed and $\mathrm{Bar}_r(X)$ is totally bounded with r is the diameter of $S(X)$, $\mathcal{D}_1(L)\cap\mathrm{Bar}_r(X)$ is totally bounded as well. Hence, we can find a finite ϵ-dense set Ω in $\mathcal{D}_1(L)\cap\mathrm{Bar}_r(X)$. Therefore, the open set

$$\{\psi\in\mathrm{UCP}_n(X):\|\varphi(x)-\psi(x)\|<\epsilon \text{ for all } x\in\Omega\}$$

is contained in $B(\varphi,\epsilon)$. We conclude that the metric and point-norm topologies on $\mathrm{UCP}_n(X)$ are the same. □

Recall from Lemma 2.52 that for $n\in\mathbb{N}$ the space $\mathrm{UCP}_n(X)$ is compact when endowed with the point-norm topology. This cannot be said of $\mathrm{UCP}_\infty(X)$, which can be a problem. In particular, we do not know whether $\mathrm{UCP}_\infty(X)$ is even totally bounded. As we are going to feed a subset of $\mathrm{UCP}_\infty(X)$ into the classical barcode map, this may put a spanner in the works. For now, however, we introduce the matricial Gromov-Hausdorff distance:

Definition 5.24. Let $(X,L_X),(Y,L_Y)$ be Lip-normed operator systems. For $n\in\mathbb{N}$ we define the **matricial n-distance** between X and Y to be

$$\mathrm{dist}_n^\mu(X,Y):=\inf_{L\in\mathcal{M}(L_X,L_Y)}\mathrm{dist}_H^{\mu_{L,n}}\{\mathrm{UCP}_n(X),\mathrm{UCP}_n(Y)\}$$

where $\mathcal{M}(L_X,L_Y)$ is the set of closed Lip-norms on $X\oplus Y$ that restrict to L_X and L_Y on the respective subspaces and $\mathrm{dist}_H^{\mu_{L,n}}$ is the Hausdorff distance with respect to the matricial n-metric on $\mathrm{UCP}_n(X\oplus Y)$.
We define the **complete matricial distance** between X and Y to be

$$\mathrm{dist}_\mu(X,Y):=\inf_{L\in\mathcal{M}(L_X,L_Y)}\sup_{n\in\mathbb{N}}\mathrm{dist}_H^{\mu_{L,n}}\{\mathrm{UCP}_n(X),\mathrm{UCP}_n(Y)\}.$$

△

This definition has the nice feature that distance zero implies that the Lip-normed operator systems are isomorphic and bi-Lip-isometric, that is: the isomorphism is a metric isometry:

Theorem 5.25 (Matricial distance zero (theorem 4.11 in [Ker03]))**.** *Let $(X,L_X),(Y,L_Y)$ be Lip-normed operator systems.*
We have $\mathrm{dist}_\mu(X,Y)=0$ if and only if there exists an isometry between X and Y.

Note that this raises another issue: we do not make use of $\mathrm{UCP}_\infty(X\oplus Y)$ in the definition of the matricial Gromov-Hausdorff distance, so whatever happens in $\mathrm{UCP}_\infty(X_k)$ for a matricial Gromov-Hausdorff convergent sequence $\{(X_k,L_k)\}_{n\in\mathbb{N}}$ is at mercy of $\mathrm{UCP}_n(X_k)$ for $n\in\mathbb{N}$. On the other hand, if there is a bi-Lip-isometric unital complete order isomorphism between X and Y, we must have $\mathrm{UCP}_\infty(X)\simeq\mathrm{UCP}_\infty(Y)$.

Just like in the case of quantum compact metric spaces, the collection of isometry classes is itself a metric space:

Theorem 5.26 (Combination of Theorems 3.7 and 4.1 in [KL04])**.** *The collection of isometry classes of Lip-normed operator spaces with the matricial Gromov-Hausdorff metric* $(\mathrm{OM}, \mathrm{dist}_\mu)$ *is a complete metric space.*

Finally, we are able to propose a candidate barcode map on $(\mathrm{OM}, \mathrm{dist}_\mu)$:

Question 5.27. *Is the map* $\beta_\mu : (\mathrm{OM}, \mathrm{dist}_\mu) \to \mathrm{Bar}$ *defined by*

$$\beta_\mu : (X, L) \mapsto \beta(S^e_{nc}(X))$$

Lipschitz continuous with respect to the matricial Gromov-Hausdorff distance?

We will examine this question in Section 4.

3.2. The Quantized Gromov-Hausdorff distance. There is an inadequacy in endowing the nc state space with the matricial Gromov-Hausdorff metrics, namely: they induce point-norm topologies. For finite n this need not be a problem, because the UCP_n are still compact, but this is not the case for UCP_∞. If our metrics on UCP_n were to yield the weak* topology instead (take $\mathrm{UCP}_n(X) \subseteq M_n(X^*)$), this problem disappears. And we are in luck, because this is precisely what the quantized Gromov-Hausdorff distance relies on. However, the quantized Gromov-Hausdorff requires a rescaling of the metric on each UCP_n with a factor n^{-2} which is likely to cause trouble.

Firstly, the quantized distance does not use a single Lip-norm on an operator space X, but rather a family of Lip-norms defined on each $M_n \otimes X$.

Definition 5.28. Let X be an operator system, a **matrix Lipschitz-seminorm** L is a family of seminorms $L_n : M_n(X) \to [0, \infty]$ such that

(1) $L_n^{-1}([0, \infty[)$ lies dense in $M_n(X)$,
(2) $L_n(a) = 0$ if and only if $a \in M_n = M_n(\mathbb{C}\, 1_X)$,
(3) $L_{m+n}(v \oplus w) = \max\{L_m(v), L_n(w)\}$,
(4) $L_n(\alpha v \beta) \leq \|a\|\beta\|\|L_n(v)$ and
(5) $L_m(v^*) = L_m(v)$

where $v \in M_m(V), w \in M_n(V), \alpha \in M_{n,m}, \beta \in M_{m,n}$. △

In Definition 5.28 we can replace $\mathbb{C}$ with $\mathbb{R}$ in condition 2 and remove the last condition that the L_m be adjoint-invariant if we consider operator systems consisting purely of self-adjoint elements.

Remark 5.29. We will not follow the exposition of Wu in [Wu06b] very strictly. This is mostly for the reason of brevity, but also to emphasise the similarity between the quantized Gromov-Hausdorff and its quantum and matricial counterparts. Moreover, we will not adopt the exact same names and notation in order to avoid confusion with the matricial Gromov-Hausdorff distance.

Again, we need metrics on the matrix state spaces:

Definition 5.30. Let X be an operator system with matrix Lipschitz-seminorm L, we define the **quantized n-metric** on $\mathrm{UCP}_n(X)$ to be

$$\kappa_{L,n}(\varphi, \psi) := \sup\left\{\|\langle\!\langle \varphi,\ a \rangle\!\rangle - \langle\!\langle \psi,\ a \rangle\!\rangle\| : a \in M_r(V), L_r(a) \leq 1, r \in \mathbb{N}\right\}.$$

Recall that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the *matrix pairing* introduced in Definition 2.49. △

Definition 5.31. A matrix Lipschitz-seminorm L on an operator system X is called a **matrix Lip-norm** whenever $\kappa_{L,n}$ induces the point-weak* topology. In this case (X, L) is called a **quantized metric space**. △

In a similar vein to the quantum and matricial setting, we need our quantized metrics to induce the point-weak* topology on each $\mathrm{UCP}_n(X)$. The point-weak* topology on $\mathrm{UCP}_1(X) = S(X)$ coicides with the classical weak* topology, so we have a generalization of the quantum Gromov-Hausdorff distance. A nice property of the point-weak* topology, is that $\bigcup_{n\in\mathbb{N}} \mathrm{UCP}_n(X)$ lies dense in $\mathrm{UCP}_\infty(X)$ with respect to the point-weak* topology, so we do not need to worry about metrizing $\mathrm{UCP}_\infty(X)$; the natural inclusions $\mathrm{UCP}_n(X) \hookrightarrow \mathrm{UCP}_{n+1}(X)$ are isometric by property 3 in Definition 5.28 and the fact that

$$\| \langle\!\langle \varphi \oplus 1, a \rangle\!\rangle - \langle\!\langle \psi \oplus 1, a \rangle\!\rangle \| = \| \langle\!\langle \varphi, a \rangle\!\rangle - \langle\!\langle \psi, a \rangle\!\rangle \|$$

for any $\varphi, \psi \in M_n(X^*)$ and $a \in M_n(X)^k$ for all $k \in \mathbb{N}$, so we can take the completion to obtain a metric on $\mathrm{UCP}_\infty(X)$ that induces the point-weak* topology.

Without further ado we introduce the distance of interest:

Definition 5.32. Let $(X, L_X), (Y, L_Y)$ be two quantized metric spaces. The **quantized Gromov-Hausdorff distance** between X, Y is defined to be

$$\mathrm{dist}_\kappa(X, Y) := \inf \left\{ \sup_{n\in\mathbb{N}} \left\{ \frac{\mathrm{dist}_H^{\kappa_{L,n}}(\mathrm{UCP}_n(X), \mathrm{UCP}_n(Y))}{n^2} \right\} : L \in \mathcal{M}(L_X, L_Y) \right\}$$

where $\mathcal{M}(L_X, L_Y))$ is the set of matrix Lip-norms on $X \oplus Y$ that restrict to L_X, L_Y respectively. △

This definition has a glaring characteristic: for each quantized n-metric we require multiplication by a factor n^{-2} for the Gromov-Hausdorff distance to work. This can turn out to be a disadvantage when we will feed $\mathrm{UCP}(X)$ into the barcode map.

Remark 5.33. Given a quantum compact metric space (A, L) one can look at the Lipschitz-continuous functions on $S(A)$ which is an operator space $\tilde{A}$. Let K be the set of Lipschitz functions on $S(A)$ with norm bounded by 1 and let $\hat{K}$ be the graded set that is the minimal matrix convex set such that $\hat{K}_1 = K$. Then the Minkowski functionals $\hat{L}_n$ for each $\hat{K}_n$ defined on each $M_n(A^*)$ give us a matrix Lipschitz seminorm $\hat{L} = \{\hat{L}_n\}$ and these satisfy all the properties that makes $(A, \hat{L})$ into a quantized metric space. It turns out that for $(A, L_A), (B, L_B)$ quantum compact metric spaces and $(\hat{A}, \hat{L}_A), (\hat{B}, \hat{L}_B)$ the corresponding quantized metric spaces we have

$$\mathrm{dist}_{\mathrm{q}}(A, B) \le \mathrm{dist}_\kappa(\hat{A}, \hat{B})$$

by Proposition 4.9 in [Wu06b].

Remark 5.34. There is another important source of quantized metric spaces; just like quantum compact metric spaces, spectral triples $(\mathcal{A}, H, D)$ give rise to quantized metric spaces. To see this, define a matrix Lipschitz seminorm L on $\mathcal{A}$ by letting

$$L_n([a_{ij}]) = \|([D, a_{ij}])_{ij}\|$$

where $[a_{ij}] \in M_n(\mathcal{A})$. The details are in [Wu06a]. In order to make this work for operator systems, take $\overline{\mathcal{A}}$ and let $L_n([a_{ij}]) = \infty$ whenever $a_{ij} \notin \mathcal{A}$ for some i, j. In particular, the quantum torus is a family of quantized metric spaces, but it does not readily follow that the quantum tori are continuous with respect to the quantized Gromov-Hausdorff distance.

And, again, distance zero between quantized metric spaces implies a complete isometry.

Theorem 5.35 (Distance zero if and only if isometric)**.** *Let $(X, L_X), (Y, L_Y)$ be quantized metric spaces, then*

$$\mathrm{dist}_\kappa(X, Y) = 0$$

implies that there exists a complete isometry $\Phi : X \to Y$, that is: a unital complete order isomorphism such that $(L_X)_n = (L_Y)_n \circ \Phi_n$ for all $n \in \mathbb{N}$.

Lastly, like in the previous settings, we have a completeness result:

Theorem 5.36 (Completeness of the quantized metric spaces (Theorem 6.5 in [Wu06b]))**.** *The collection of isometry classes of quantized metric spaces with the quantized Gromov-Hausdorff distance $(\mathrm{QM}, \mathrm{dist}_\kappa)$ is a complete metric space.*

Now, we can propose candidate barcode maps. Firstly, we remark that the map β_μ in Question 5.27 is a candidate, because the quantized distance provides us with metrics on $\mathrm{UCP}_n(X)$ for $n \in \mathbb{N}$, but we have to fix the domain. So, let us define

$$\beta_\vartheta : (\mathrm{QM}, \mathrm{dist}_\kappa) \longrightarrow \mathrm{Bar}$$

by $(X, L) \mapsto \beta(S^e_{\mathrm{nc}}(X))$. However, given the factor n^{-2} in Definition 5.32, we are tempted to present another candidate barcode map. For this, we first need to specify the metric space that we would like to 'extract' from our quantized metric space (X, L).

Let (X, L) be a quantized metric space, for $\mu \in \mathrm{UCP}_n(X) \setminus \mathrm{UCP}_{n-1}(X), \nu \in \mathrm{UCP}_k(X) \setminus \mathrm{UCP}_{k-1}(X)$ – where $k \geq n$ without loss of generality – we can define

$$d_q(\mu, \nu) := k^{-2}\, \kappa_{L,k}(\mu, \nu)$$

by κ-isometrically embedding $\mathrm{UCP}_n(X) \hookrightarrow \mathrm{UCP}_k(X)$. This gives us a metric on $\bigcup_{n\in\mathbb{N}} \mathrm{UCP}_n(X)$. Now, suppose that $\mu, \nu \in \mathrm{UCP}_\infty(X)$ and either μ or ν is not contained in any $\mathrm{UCP}_n(X)$ for $n \in \mathbb{N}$, then we can take sequences $\mu_k \to \mu, \nu_k \to \nu$ in the point-weak* topology and let

$$d_q(\mu, \nu) := \lim_{k\to\infty} n_k^{-2}\, \kappa_{L,n_k}(\mu_k, \nu_k)$$

where n_k is the smallest $n \in \mathbb{N}$ such that $\mu_k, \nu_k \in \mathrm{UCP}_{n_k}(X)$. Now, we restrict d_q to the nc pure states: $S^e_{\mathrm{nc}}(X) \subseteq \mathrm{UCP}_\infty(X)$.

Question. *Is $d_q : S^e_{nc}(X) \times S^e_{nc}(X) \to [0, \infty]$ a metric?*

If this question can be answered affirmatively, we may propose another candidate barcode map.

Question 5.37. *Is the map $\beta_\kappa : (QM, \mathrm{dist}_\kappa) \to \mathrm{Bar}$ defined by*

$$(X, L) \mapsto \beta(S^e_{nc}(X))),$$

where $\mathrm{UCP}_\infty(X)$ *is metrized by d_q, a Lipschitz-continuous map?*

Note that β_κ cannot be redefined to have domain $(\mathrm{OM}, \mathrm{dist}_\mu)$ in a straightforward manner, because $\dot{\bigcup}_{n\in\mathbb{N}} \mathrm{UCP}_n(X)$ is not point-norm dense in $\mathrm{UCP}_\infty(X)$.

4. Are β_μ, β_ϑ and β_κ eligible?

Now that we have developed the theory for both the matricial and quantized Gromov-Hausdorff distance, it is time to check whether the proposed barcode maps *can* in fact be Lipschitz-continuous with their respective distances.

4.1. The Matricial barcode map. We start with β_μ. We have already mentioned the fact that the difficulty with this approach lies in the fact that for a Lip-normed operator system (X, L) the induced metrics on $\mathrm{UCP}_n(X)$ give rise to the point-norm topologies. The point-norm topologies are decent for $\mathrm{UCP}_n(X)$ with $n \in \mathbb{N}$, but for $\mathrm{UCP}_\infty(X)$ this appears not to be the case. In fact, $\mathrm{UCP}_\infty(X)$ endowed with $\mu_{L,\infty}$ *need not even be totally bounded* as we will demonstrate in the following proposition:

Proposition 5.38. *Let $\theta \in [0,1] \setminus \mathbb{Q}$, let $(A_\theta,\ L_\theta)$ be the corresponding non-commutative torus. The nc state space for A_θ given by $(\mathrm{UCP}_\infty(A_\theta), \mu_{L_\theta,\infty})$ is not totally bounded.*

Proof. We will construct a sequence of irreducible representations in $S^e_{\mathrm{nc}}(X)$ without accumulation point. It will follow that $\mathrm{UCP}_\infty(A_\theta)$ cannot be totally bounded.

Denote by $\pi : A_\theta \to B(L^2(S^1))$ the irreducible representation determined by

$$u \mapsto (f(x) \mapsto f(x)e^{2\pi i x}),$$
$$v \mapsto (f(x) \mapsto f(x+\theta)).$$

Let $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ be the orthonormal basis for $L^2(S^1)$ and denote by u the unitary operator that sends $e^{2\pi i n x} \mapsto e^{2\pi i (n+1)x}$. We can conjugate π with powers of u to obtain new irreducible representations $\pi_k := u^k \pi u^{-k}$. Because $v \in A_\theta$ satisfies $\|[D, v]\| < \infty$ we can define $\tilde{v} := \lambda v$ for some $\lambda \in \mathbb{R}$ such that $\|[D, \tilde{v}]\| = 1$. If we let $(\pi_l - \pi_k)(v) \in B(L^2(S^1))$ act on $1 \in L^2(S^1)$ we get

$$\begin{aligned}\|(\pi_l(v) - \pi_k(v))(1)\|_2 &= \|u^l(e^{-2\pi i l(x+\theta)}) - u^k(e^{-2\pi i k(x+\theta)})\|_2 \\ &= \|e^{-2\pi i l\theta} - e^{-2\pi i k\theta}\|_2 = \sqrt{\int_{[0,1]} |e^{2\pi i l\theta} - e^{2\pi i k\theta}|^2 d\mu}.\end{aligned}$$

If we now fix $\epsilon > 0$, then we can find infinitely many distinct natural numbers $k_1, k_2, \dots$ such that $k_i\theta$ is ϵ-close to a whole number, but for whole numbers $m \neq m'$ we know that $\|e^{2\pi i m'} - e^{2\pi i m}\|_2 = \sqrt{2}$. Hence, there is a constant K dependent only on ϵ such that

$$\|\pi_{k_j}(v) - \pi_{k_i}(v)\| \geq K\sqrt{2}$$

for every distinct pair $i, j \in \mathbb{N}$. We conclude that the irreducible representations π_{k_i} form a sequence in $\mathrm{UCP}_\infty(X)$ without accumulation point for the point-norm topology. $\square$

Because the barcode map β is only defined for totally bounded metric spaces, the map $\tilde{\beta}_\mu$ is not well-defined.

4.2. The Quantized Barcode Map. This is not an easy question. To my knowledge there is only one example in the literature of a convergent sequence of quantized metric spaces, namely: the matrix algebras converge to the sphere when endowed with suitable metrics [Wu06b]. It is not unlikely that the quantum tori and other common quantum Gromov-Hausdorff-continuous families of quantum compact metric spaces will also converge for suitable quantized adaptations, but this needs to be proven first. We also know that *only one* of the distances β_ϑ and β_κ can be valid; if β_ϑ makes $\mathrm{UCP}_\infty(X)$ totally bounded, β_κ can at most be a pseudo-metric and if β_κ is a metric, β_ϑ makes $\mathrm{UCP}_\infty(X)$ unbounded.

CHAPTER 6

Discussion

We have reviewed four candidate barcode maps for non-commutative metric spaces without success. The results have been summarized in Figure 1 below.

Barcode map	Gromov-Hausdorff variant	Does it work?
β_{q}	quantum	No, discontinuous
β_{μ}	matricial	No, ill-defined
β_{ϑ}	quantzied	Unknown
β_{κ}	quantized	Unknown

Figure 1. The different candidate barcode maps proposed in this thesis.

The latter barcode maps β_{ϑ} and β_{κ} are unlikely to be suitable candidates (β_{κ} might even be ill-defined). But, providing a refutation for either of these maps is non-trivial. Examples of convergent sequences for the quantized Gromov-Hausdorff distance are scarce in the literature, so this has to be established from scratch.

1. Future Directions

In the last part of this thesis we discuss possible future leads for developing non-commutative persistent homology. We also entertain the possibility that for the non-commutative Gromov-Hausdorff distances that have been invented thus far there cannot exist a Lipschitz-continuous persistent homology map for non-commutative metric spaces.

1.1. Issues.

1.1.1. *Existence is not guaranteed.* The existence of a Lipschitz-continuous barcode map defined on non-commutative metric spaces is endangered by the fact that the classical Gromov-Hausdorff distance dominates both the quantum Gromov-Hausdorff distance $\mathrm{dist}_{\mathrm{q}}$ and the Gromov-Hausdorff propinquity dist_{Λ}(introduced in [Lat13]) when we restrict ourselves to classical compact metric spaces. In particular, $\mathrm{dist}_q, \mathrm{dist}_{\Lambda}$ may not even be Lipschitz-equivalent to dist_{GH} and so there might exist a sequence of compact metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ that converge to (X, d) for $\mathrm{dist}_{\mathrm{q}}$ or dist_{Λ}, but for which $\beta(X_n)$ does not converge to $\beta(X)$ in a Lipschitz-continuous way. If we waive Lipschitz-continuity and aim for normal continuity instead, this is not a problem: see Theorem 13.16 in [Rie03].

1.1.2. *Persistent homology is flat.* Persistent homology has been developed for point clouds, which are to be seen as finite subsets of Euclidean space. And, while we can compute topological barcodes for non-flat Riemannian manifolds, persistent homology assumes implicitly that our spaces can be embedded in a flat vector space (most notably: Theorem 3.9). This means that for a non-flat Riemannian manifold, the persistent topological barcodes are not *intrinsic* as the filtration of simplicial complexes live outside the manifold itself. Many non-commutative spaces are obtained as deformations of manifolds (most prominently: the quantum torus) and their non-commutative metrics are hence derived from an 'intrinsic' distance. Defining persistent-homology for Riemannian manifolds in an instrinsic manner is not very straightforward, however (if at all possible).

1.1.3. *Are topological barcodes enough?* Thus far, we have assumed that our candidate barcodes map into the space of full barcodes Bar, but maybe this is too inflexible a restriction. Perhaps non-commutativity calls for an extension of the definition of barcodes.

1.2. *Actual* non-commutative persistent homology. On a very fundamental level, the approach taken in this thesis violates the principle of non-commutative geometry. After all, we consider order-unit spaces, operator systems or C*-algebras and then we *extract* from these algebraic structures a topological space in the form of (nc) pure states that serves as a vertex set which we feed directly into the classical barcode map. That is, we never really left point-set topology. Instead, a purely non-commutative geometric approach would involve generalizing the whole process of persistent homology: producing a filtration of simplicial complexes from a 'non-commutative point cloud' and computing persistent homology classes. Let us examine each step in this method.

1.2.1. *Non-commutative point clouds.* The very first question is obviously: *what is a non-commutative point cloud?* Since ordinary point clouds are finite metric spaces, it does not seem far fetched to define non-commutative point clouds as finite dimensional C*-algebras (or finite dimensional operator systems) that have a Lip-norm (or a related seminorm). This definition is further motivated by the theory of *truncated spectral triples* developed in [Sui20] which are finite-dimensional truncations (X_n, L_n) of a Lip-normed operator system (X, L) such that the corresponding state spaces converge in the classical Gromov-Hausdorff distance.

1.2.2. *Non-commutative filtrations.* To formulate a non-commutative version for the filtrations of simplicial complexes, we need to ask ourselves two questions:

(1) What are non-commutative simplicial complexes?
(2) How does a non-commutative point cloud determine a filtration of non-commutative simplicial complexes?

For the first question it seems tempting to implement the *non-commutative simplicial complexes* as defined by Cuntz in [Cun02]. Despite the elegance of this theory there are two problems:

(1) For each abstract simplicial complex there is exactly *one* commutative (classical) simplicial complex and *one* non-commutative simplicial complex.
(2) Vertex sets (0-skeletons) are the same for commutative and non-commutative simplicial complexes, making it impossible to have strictly non-commutative point clouds as vertex sets.

The second question heavily depends on the first one and as such it is ill-posed. In the (likely) case that the maps in our filtration are $*$-homomorphisms we require our simplicial complex to have a sufficiently rich ideal structure.

1.2.3. *Non-commutative homology.* In the classical setting, for each simplicial complex in a filtration we can compute homology classes and between two simplicial complexes we have nicely behaved inclusions that induce maps in homology. How we would compute non-commutative simplicial (co)homology depends heavily on what our definition of non-commutative simplicial complexes. If each non-commutative simplicial complex is a C*-algebra, we may employ *cyclic (co)homology* [Con94] which is a cohomology theory for *-algebras that corresponds to the de Rham cohomology for commutative algebras of smooth functions on manifolds. In addition, we may compute K-theory which has the nice property of generalizing topological K-theory relatively intuitively. Moreoever, in some cases (such as graph C*-algebras) K-theory can be computed in terms of finite algebraic data.

1.3. Spectral Geometry. A completely different approach to non-commutative persistent homology forks off from this thesis in Chapter 2 Section 3 where we mentioned that the kernel of the (Hodge) Laplacian Δ corresponds to the cohomology vector space. In fact, this observation is used in a theoretical implementation of persistent homology on quantum computers[LGZ16]. This is a very convenient setting, because for (non-commutative) spectral triples we already have Laplacians at our disposal. Moreover, *persistent geometry*, the study of point clouds using a filtration of simplicial complexes and computing the eigenvalues of the combinatorial Laplacian [MWW20], is a very recent subject of study that seeks to employ spectral geometry in the study of point clouds.

1.4. What does non-commutative persistent homology *mean*? Classical persistent homology is widely applicable in data analysis, because we have a very clear understanding of what these barcodes mean: they signify the existence of holes and cavities (possibly including torsion) in point clouds. If we want to define a suitable persistent homology theory for non-commutative point clouds, it is essential to formulate 'what non-commutative holes and cavities should be'. If we only have the algebra of observables at our disposal how can we determine the 'non-commutative shape' of our underlying non-commutative space? An interesting toy model is the fuzzy sphere (Chapter 2, Section 5) which can reasonably be seen as a family of non-commutative point clouds (matrix algebras) converging to a sphere (a commutative C*-algebra). In what way do these matrix algebras have a hole, like the sphere does?

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