# Murphy's Law on the Fixed Point Locus of the Quot-Scheme, and Classifying Continuous Constraint Satisfaction problems

Reinier F. Schmiermann

Supervisors: Martijn Kool, Tillmann Miltzow

## Abstract

This thesis consists of two independent parts.

The first part is about Murphy's law on the fixed point locus of the Quot-scheme. In this part, we analyze the singularities of the fixed point locus of the Quot-scheme on $\mathbb{A}_k^4$ under some torus action. We prove that this scheme satisfies a version of Murphy's law, meaning that it contains every singularity which it could conceivably contain.

The second part is about continuous constraint satisfaction problems (CCSPs). In an instance of such a problem, we are given variables $x_1, \ldots, x_n$ and constraints $c_1, \ldots, c_m$ on these variables, and we want to know whether we can assign real numbers to these variables such that all constraints are satisfied. We prove that a large number of these CCSPs are $\exists\mathbb{R}$-complete, which means that solving such a problem is as hard as solving a general system of equations. Finally we apply these results on CCSPs to show that the problem of packing convex polygons into a square container is also $\exists\mathbb{R}$-complete.

# Contents

# Chapter 1

# Introduction

This master thesis consists of two different parts. The first part concerns algebraic geometry, and is about the singularities of the Quot-scheme. The second part concerns theoretical computer science, and is about the complexity of continuous constraint satisfaction problems. Both these parts can be read independently of each other. In particular, no knowledge of algorithmics is needed to read the first part of the thesis, and no knowledge of algebraic geometry is needed for reading the second part of the thesis. In this introduction, a short overview of the contents of both the parts will be given, together with a short section on the shared philosophy between the two parts.

## 1.1 Murphy's Law on the Fixed Point Locus of the Quot-Scheme

Quot-schemes form an important tool within algebraic geometry. A Quot-scheme over a space $X$ is some kind of geometrical space, such that each point of this space corresponds to a quotient of a kind of algebraic structure on $X$. Unfortunately, when trying to do computations with such a Quot-scheme, it turns out that the entire Quot-scheme is too complicated to easily work with. Luckily, if the space $X$ admits a certain kind of symmetry (a "torus action"), then this symmetry carries over to the Quot-scheme. The points of the Quot-scheme which are fixed under this symmetry form the *fixed point locus*. This fixed point locus admits a convenient combinatorial description, which makes it a lot easier to work with than with the full Quot-scheme. Furthermore, a lot of information about the Quot-scheme can be obtained by just studying this fixed point locus.

It would have been nice if the fixed point locus of the Quot-scheme were smooth, since smooth spaces are easier to work with than singular spaces. Unfortunately, this turns out not to be the case. In the first part of this thesis, we study the singularities of this fixed point locus. The main result is that the fixed point locus of the Quot-scheme over the space $\mathbb{C}^4$ satisfies a version of *Murphy's law* in algebraic geometry. The term Murphy's law was introduced by Vakil [57], and it means that the space is as singular as possible: every singularity which could concievably occur on the space, does actually occur.

The main tool in this proof is Mnëv's universality theorem. This theorem basically states that any set of polynomial equations with integer coefficients can be modeled by a set of incidence relations $I$ of points and lines in the plane. Such a set $I$ consists of requirements of the form "point $i$ lies on line $j$". Now Mnëv's universality theorem says that for every system of equations we can find such an $I$, such that the set of configurations $C_I$ of points and lines satisfying the relations in $I$ looks like the set of solutions to the original system of equations. We can use this to prove Murphy's law on the fixed point locus of the Quot-scheme by showing that for any such set $C_I$, we can find a connected component of our fixed point locus which is essentially the same as $C_I$.

## 1.2 On Classifying Continuous Constraint Satisfaction Problems

In the second part of the thesis, we study continuous constraint satisfaction problems (CCSPs). In an instance of such a problem, we are given an interval $U \subseteq \mathbb{R}$, variables $x_1, \ldots, x_n$ and a set of constraints $c_1, \ldots, c_m$ in these variables (an example of such a constraint could be $x_1^2 \geq x_2$). Now we ask whether there is some way to assign a value in $U$ to every variable such that all the constraints are satisfied.

By allowing different sets of constraints, we can get different continuous constraint satisfaction problems. For example, we might allow just constraints of one of the forms $x+y=z$, $x \cdot y = z$ and $x=1$. In this thesis, we prove $\exists\mathbb{R}$-completeness for a large number of these CCSPs. A problem is called $\exists\mathbb{R}$-complete if it is as hard as solving general systems of polynomial equations and inequalities over the real numbers. In particular, proving that a problem is $\exists\mathbb{R}$-complete involves reducing any system of equations over the reals to an instance of this problem, such that this instance has a solution if and only if the system of equations has a solution.

We focus on two classes of CCSPs. The curved equality problem (CE) belonging to a function $f\colon U^2 \to \mathbb{R}$ and a positive $\delta \in \mathbb{R}$ is the CCSP where we are only allowed to use constraints of the forms

$$x + y = z, \quad f(x,y) = 0, \quad x \geq 0, \quad x = \delta.$$

We prove that this problem is $\exists\mathbb{R}$-complete for all functions $f$ such that $f(x,y)=0$ describes a smooth curve through the origin with non-zero curvature, and which furthermore satisfy some technical conditions.

The convex concave inequality problem (CCI) corresponding to functions $f, g\colon U^2 \to \mathbb{R}$ and a real number $\delta > 0$ is the CCSP where we only use constraints of the form

$$x + y = z, \quad f(x,y) \geq 0, \quad g(x,y) \geq 0, \quad x \geq 0, \quad x = \delta.$$

We prove that this problem is $\exists\mathbb{R}$-complete for all functions $f, g$ such that the equalities $f(x,y) = 0$ and $g(x,y) = 0$ both describe a smooth curve through the origin with non-zero curvature, and such that one of the inequalities $f(x, y) \geq 0$ and $g(x, y) \geq 0$ describes a convex region, and the other one describes a concave region near the origin.

Finally, we apply these results to geometric packing by answering an open question by Abrahamsen et al. [5, FOCS 2020]. We prove that the problem of packing convex polygonal pieces into a square container, using translations and rotations of the pieces, is $\exists\mathbb{R}$-complete.

This chapter is based on the paper [41].

## 1.3 Shared Philosophy

Even though the two halves of this thesis are not directly related to each other, there is a common theme in both the parts. Both topics are about encoding systems of polynomial equations in a different setting. In the first part of the thesis, it is shown that any such system of polynomial equations is encoded by a component of the fixed point locus of the Quot scheme. In the second part of the thesis, any system of equations is modeled by some other system of equations or inequalities, but this new system of equations only consists of a very limited set of equations. Furthermore, at the end of the second part, every system of equations is also encoded as a packing problem.

The kinds of reductions and encodings which are used in the different parts of the thesis are different though. When proving Murphy's law on the Quot-scheme, it is important that singularities of the space are preserved. On the other hand, during the $\exists\mathbb{R}$-reductions in the second part, we are mostly interested in preserving the existence of a solution, and we care less about preserving the other properties of the solution set. However, in this second part it is important that all reductions can be executed efficiently, while there is no such requirement when working with singularities on the Quot-scheme.

Another way of looking at the similarities between the parts, is by noting that in each part we look for objects which seem simple, but turn out to exhibit a rich, complicated structure. In the first part this simple object is the fixed point locus of the Quot scheme, which satisfied Murphy's law. In the second part we look for continuous constraint satisfaction problems which are as simple as possible, while still being $\exists\mathbb{R}$-complete.

# Chapter 2

# Murphy's Law on the Fixed Point Locus of the Quot-Scheme

## 2.1 Introduction

The Quot-scheme is an important tool in algebraic geometry. Roughly stated, the Quot-scheme $Q$ of rank $r$ over a quasi-projective scheme $X$ over a field $k$ parameterizes 0-dimensional quotients of $\mathcal{O}_X^r$. What this means will be explained in more detail later. It turns out that this full Quot-scheme is fairly complicated, and that it is quite cumbersome to work with.

In the case that $X$ admits a group action of the $d$-dimensional algebraic torus $T = (k^*)^d$, computations can be simplified. This is because such a torus action lifts to the Quot-scheme $Q$, and it is possible to look at the fixed point locus $Q^T$ under this torus action, which admits a nice combinatorial description as we will see later. Computations on $Q$ can be reduced to simpler computations on $Q^T$. In this thesis, we study the singularities of this fixed point locus $Q^T$ in the case where the space $X$ we are working over is the affine space $\mathbb{A}_k^d$. We will denote the Quot scheme in this case by $Q_{r,d}^\bullet$, and its fixed point locus by $(Q_{r,d}^\bullet)^T$.

In particular, the goal is to check whether this scheme satisfies Murphy's law in algebraic geometry. This notion was introduced by Vakil [57]. A scheme satisfies Murphy's law if it contains every singularity of finite type over $\mathbb{Z}$. Intuitively, this means that the scheme is as singular as possible. Vakil himself already proved in [57] that Murphy's law holds for a large number of schemes. Payne has also proven that Murphy's law holds on the moduli scheme of toric vector bundles on a toric variety [44], and Jelisiejew proved that Murphy's law holds up to retraction for the Hilbert scheme of points on $\mathbb{A}^{16}$ [29].

The main result of this first part of the thesis is the following:

**Theorem 2.1.1.** *Every singularity type of finite type over $\mathbb{Z}$ occurs, up to a base change to $k$, on the scheme $(Q_{3,4}^\bullet)^T$.*

Note that in this thesis we are working over an algebraically closed field $k$ of characteristic 0, while Murphy's law is about schemes over $\mathbb{Z}$. This is the reason why we do not exactly get Murphy's law on $(Q^{\bullet}_{3,4})^T$, but instead need to base change the singularities to $k$ first. It is expected that the construction described in this thesis can also be performed while working over $\mathbb{Z}$, but verifying this is beyond the scope of this thesis.

The most important tool in proving this theorem is Mnëv's universality theorem, as described in [35] and [34, Section 1.8]. Mnëv's universality theorem states that incidence schemes satisfy Murphy's law. Here an incidence scheme is a scheme which parametrizes sets of points and lines which satisfy certain incidence relations of the form "point $i$ lies on line $j$". It turns out that the fixed point locus $(Q^{\bullet}_{3,4})^T$ is always a disjoint union of such incidence schemes, to prove the theorem we need to show that every incidence scheme also occurs as some connected component of $(Q^{\bullet}_{3,4})^T$. This construction, where we find a connected component of this fixed point locus for every incidence scheme, is the main contribution of this thesis.

Before we prove this theorem, we also discuss how our approach fails when working over $\mathbb{A}^2_k$ or $\mathbb{A}^3_k$. We do this by showing that not all incidence schemes occur as connected components of $(Q^{\bullet}_{3,2})^T$ and $(Q^{\bullet}_{3,3})^T$, however, this does not yet imply that Murphy's law does not hold in these cases. It would be interesting to see whether it is possible to give an explicit singularity which does not occur on these schemes, or whether it is possible to derive Murphy's law in some other way, but this is a topic for future research.

Finally we also give a short discussion of how our result on the fixed point locus might be extended to the entire Quot-scheme. In particular, we discuss the Białynicki-Birula decomposition, which is a central tool in the proof from Jelisiejew [29] of Murphy's law on the Hilbert scheme of points on $\mathbb{A}^{16}$. In particular, we show that applying this decomposition in a somewhat naive way cannot directly work. This does however not yet exclude the possibility that, by using more advanced tools similar to those used by Jelisiejew, we can prove Murphy's law on some $Q^{\bullet}_{r,d}$ with $d$ less than 16.

In Section 2.2, we will give an overview of the already known results which are needed for the rest of the thesis. In particular we give a definition of the Quot-scheme, and an explicit description of its fixed point locus. We also introduce Murphy's law and Mnëv's universality theorem in more detail here. In Section 2.3, we specialize the description of the fixed point locus to the case $r = 3$. After discussing the Quot-scheme on $\mathbb{A}^2_k$ and $\mathbb{A}^3_k$, we prove our main result on $\mathbb{A}^4_k$. Finally in Section 2.4, we discuss how this might be extended to the whole Quot-scheme by using the Białynicki-Birula decomposition.

## 2.2 Preliminaries

Before proving the results of this thesis, first some definitions and results are needed about the Quot-scheme and Murphy's law in algebraic geometry. This section serves to introduce these topics. In this section, and also in the rest of this thesis, we will work over an algebraically closed field $k$ with characteristic 0, unless indicated otherwise.

### 2.2.1 Quot-scheme

The main object which we will study in this thesis, is the Quot-scheme $\mathrm{Quot}_X(\mathcal{O}_X^r, n)$, which we will define in this section. Here $X$ is a quasi-projective scheme over $k$, and $r$ and $n$ are nonnegative integers. The Quot-scheme is a so-called moduli space.

A moduli space is a kind of geometric space (in this case a scheme, but it might also be for example a variety or a stack), such that each point of this space represents some geometric object. An important and fairly simple example of a moduli space is that of the Grasmannian $\mathrm{Gr}(m, n)$, which is a variety which models the set of $m$-dimensional subspaces of the $k$-vector space $k^n$. For example, $\mathrm{Gr}(1, 3)$ is the set of 1-dimensional subspaces of $k^3$, which is exactly $\mathbb{P}_k^2$. Also, $\mathrm{Gr}(2, 3)$ is the set of 2-dimensional subspaces of $k^3$, which is the set of lines in $\mathbb{P}_k^2$, so $\mathrm{Gr}(2, 3) \cong \mathbb{P}_k^{2\vee}$.

The Quot-scheme $\mathrm{Quot}_X(\mathcal{O}_X^r, n)$ parametrizes the set of zero-dimensional coherent quotients $\mathcal{O}_X^r \to Q$ of length $n$ as an $\mathcal{O}_X$-module. Here "zero-dimensional" means that $Q$ has a zero-dimensional support $\mathrm{Supp}(Q)$; this support is the set of points $x \in X$ such that the stalk $Q_x$ is nontrivial. Stated differently, $Q$ should be supported on a finite set of points. The usual way to formalize such a definition of a moduli space is by first giving a functor from schemes to sets, and then showing that this functor is represented by some scheme. In the following, we will explain this a bit more carefully using chapter 2 of the book by Huybrechts and Lehn [28]. This book however only discusses the case where the scheme $X$ is projective, and a little extra care is needed to show existence of the Quot-scheme over general quasi-projective schemes.

#### Quot-functor

As mentioned before, the first step in defining the Quot-scheme is to define the Quot-functor

$$\mathcal{Q} := \underline{\mathrm{Quot}}_X(\mathcal{O}_X^r, n)\colon (Sch/k)^o \to (Sets).$$

Informally, this functor should send a certain scheme $S$ to the set of geometric objects over $S$ which we want the Quot-scheme to represent. In particular, it should send $\mathrm{Spec}(k)$ to the set of zero-dimensional coherent quotients $\mathcal{O}_X^r \to Q$ of length $n$. For some field extension $\ell$ of $k$, we instead want $\mathcal{Q}(\mathrm{Spec}(\ell))$ to be the set of zero-dimensional coherent quotients $\mathcal{O}_{X_\ell}^r \to Q$ of length $n$ as an $\mathcal{O}_{X_\ell}$-module (that is, we base-change the whole definition to be over $\ell$ instead of $k$).

For a general $k$-scheme $S$, the definition is a bit more complicated. We denote $X_S := S \times_k X$. Now we let $\mathcal{Q}(S)$ be the set of all zero-dimensional $S$-flat coherent quotient sheaves $\mathcal{O}_{X_S}^r \to Q$, where $Q$ has length $n$ as an $\mathcal{O}_{X_S}$-module. Here we identify two quotients $q_1\colon \mathcal{O}_{X_S}^r \to Q_1$ and $q_2\colon \mathcal{O}_{X_S}^r \to Q_2$ if they have the same kernel; this is equivalent to requiring that there is an isomorphism $\Phi\colon Q_1 \to Q_2$ such that $q_2 = \Phi \circ q_1$. The way to think about an element of $\mathcal{Q}(S)$ is as a family of quotient sheaves parametrized by the points of $S$: for every point $s \in S$, there is a sheaf $Q_s := Q|_{X_s}$ over $X_s := \mathrm{Spec}(\kappa(s)) \times_k X \subset X_S$. The fact that all these sheaves $Q_s$ together form an $S$-flat sheaf over $X_S$ implies that they vary in some "continuous" manner when $s$ varies over $S$, we will not discuss these conditions in more detail here.

Now that we have described what the Quot-functor does with objects, we also need to describe how it acts on morphisms between schemes. Let $g\colon S \to T$ be a morphism of $k$-schemes. Now we let $\mathcal{Q}(g)\colon \mathcal{Q}(T) \to \mathcal{Q}(S)$ be the map which sends a quotient $q\colon \mathcal{O}^r_{X_T} \to Q$ to the quotient $g_X^* q\colon \mathcal{O}^r_{X_S} = g_X^*\mathcal{O}^r_{X_T} \to g_X^* Q$. Here $g_X^*$ denotes the pullback by the map $g_X\colon X_S \to X_T$. By right exactness of this pullback, it follows that $g_X^* q$ is indeed a quotient.

In this definition of the Quot-functor, we used quotients $\mathcal{O}^r_{X_S} \to Q$. It is however also possible to identify each such quotient with its kernel, which is a subsheaf of $\mathcal{O}^r_{X_S}$. In this way, it is possible to define the sets $\mathcal{Q}(S)$ by using these subsheaves with $S$-flat cokernel, instead of quotients, and in some settings this description can be more convenient. Later in this thesis we will mostly use this alternative definition. It should however be noted that the maps $\mathcal{Q}(g)$ for $g\colon S \to T$ cannot easily be defined by using these subsheaves; this is because the pullback $g_X^*$ is not left exact, so it will not necessarily send an inclusion $E \to \mathcal{O}^r_{X_T}$ to another inclusion $g_X^* E \to \mathcal{O}^r_{X_S}$.

### Representing the Quot-functor

We define the Quot-scheme to be the $k$-scheme which represents the Quot-functor we just defined. What this means will be explained next.

Let $\mathcal{C}$ be a locally small category, and let $\mathcal{C}'$ be the category of contravariant functors $\mathcal{C}^o \to (Sets)$. Here locally small means that for any pair of objects $x, y$ of $\mathcal{C}$, the morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$ form a set. We can define a functor $\mathcal{C} \to \mathcal{C}'$ by mapping an object $x$ of $\mathcal{C}$ to the functor $\underline{x}\colon y \mapsto \mathrm{Mor}_{\mathcal{C}}(y, x)$. The Yoneda Lemma now states that this functor $\mathcal{C} \to \mathcal{C}'$ embeds $\mathcal{C}$ as a subcategory into $\mathcal{C}'$. In particular it follows that if two objects $x$ and $y$ define the same functors $\underline{x}$ and $\underline{y}$, then $x$ and $y$ are isomorphic. This gives rise to the following definition:

**Definition 2.2.1.** A functor $\mathcal{F}\colon \mathcal{C}^o \to (Sets)$ is *represented* by an object $F$ of $\mathcal{C}$ if $\mathcal{F}$ is isomorphic to $\underline{F}$ in $\mathcal{C}'$.

Note that by the Yoneda Lemma it follows that, if $\mathcal{F}$ is represented by some $F$, then this $F$ is unique up to isomorphism.

Now we apply this definition to the case where $\mathcal{C}$ is the category of schemes over $k$, and $\mathcal{F}$ is the Quot-functor $\underline{\mathrm{Quot}}_X(\mathcal{O}^r_X, n)$. We get the following theorem, originally due to Grothendieck:

**Theorem 2.2.2.** *The functor $\underline{\mathrm{Quot}}_X(\mathcal{O}^r_X, n)$ is represented by a quasi-projective $k$-scheme $\mathrm{Quot}_X(\mathcal{O}^r_X, n)$.*

We will not prove this theorem here, but it is proven as Theorem 2.2.4 in [28] for the case where $X$ is projective. The scheme $\mathrm{Quot}_X(\mathcal{O}^r_X, n)$ from this theorem is what we use as the definition of the Quot-scheme. In particular, this means that the set of $k$ points of $\mathrm{Quot}_X(\mathcal{O}^r_X, n)$, which is defined as $\mathrm{Mor}(\mathrm{Spec}(k), \mathrm{Quot}_X(\mathcal{O}^r_X, n))$, is exactly the set of zero-dimensional coherent quotients $\mathcal{O}^r_X \to Q$ of length $n$.

### The case $X = \mathbb{A}^d_k$

In the rest of the thesis, we will mostly work with the Quot-scheme over $\mathbb{A}^d_k$ for some $d$. To shorten the notation, we will write $Q^n_{r,d} := \mathrm{Quot}_{\mathbb{A}^d_k}(\mathcal{O}^r, n)$. Furthermore, we are interested in the disjoint union of these schemes over $n$, which we denote by $Q^\bullet_{r,d} := \coprod_n Q^n_{r,d}$.

In this affine case, the set of $k$-points of $Q^n_{r,d}$ can be described slightly simpler: instead of working with quotients of the $\mathcal{O}_X$-module $\mathcal{O}^n_X$, we can also work with quotients $Q$ of the $k[x_1, \ldots, x_d]$-module $k[x_1, \ldots, x_d]^n$. The length of such a quotient is exactly its dimension $\dim_k(Q)$ as a $k$-vector space. Specifying such a quotient is furthermore equivalent to specifying a submodule of $k[x_1, \ldots, x_d]$, which corresponds to the kernel of the quotient map.

Now we can also observe that the Grasmannian $\mathrm{Gr}(m, n)$ is actually a special case of a Quot-scheme; it is isomorphic to the scheme $Q^{n-m}_{n,0}$. This is because points in this Quot-scheme correspond to $(n-m)$-dimensional quotients of the vector space $k^n$. These in turn correspond to $m$-dimensional subspaces of $k^n$.

## 2.2.2 Torus action and the fixed point locus

This section serves to define a torus action on the Quot-scheme over $\mathbb{A}^d$, and to give a combinatorial description of the fixed point locus under this action. The main results of this thesis all concern this fixed point locus.

### Torus action

Before describing the torus action, we first give a definition of the torus itself:

**Definition 2.2.3.** For a nonnegative integer $m$, the *$m$-dimensional algebraic torus* is the algebraic group $(k^*)^m$.

A $d$-dimensional torus acts on $\mathbb{A}^d_k$ as follows: an element $\mathbf{t} = (t_1, \ldots, t_d) \in (k^*)^d$ sends a point $(x_1, \ldots, x_d) \in \mathbb{A}^d_k$ to $(t_1x_1, \ldots, t_dx_d) \in \mathbb{A}^d_k$. Note that this action depends on the choice of coordinates on $\mathbb{A}^d_k$. The element $\mathbf{t}$ acts on the coordinate ring $k[x_1, \ldots, x_n]$ of $\mathbb{A}^d_k$ by mapping $x_i \mapsto t_i^{-1}x_i$ for all $i$. In the rest of this thesis, we will denote the $d$-dimensional torus by $T$.

Also the projective space $\mathbb{P}^d_k$ admits a torus action by $T$, by taking $(t_1, \ldots, t_d) \cdot (x_0 : x_1 : \cdots : x_d) = (x_0 : t_1x_1 : \cdots : t_dx_d) \in \mathbb{P}^d_k$. Both affine space and projective space furthermore admit an embedding of $T$ as a dense open subvariety, by considering the subset of $\mathbb{A}^d_k$ or $\mathbb{P}^d_k$ where all coordinates are nonzero, and the torus acts on these subvarieties by the usual multiplication. Varieties which allow for such a torus embedding are called *toric varieties.* These toric varieties allow for a fully combinatorial description, and a lot of computations simplify when working with toric varieties. This makes them a useful class of varieties to study in many settings. We will however not discuss this combinatorial description in this thesis.

It turns out that the action of $T$ on a toric variety lifts to the Quot-scheme on this variety. In particular, we have an action of the $d$-dimensional torus $T$ on the Quot-scheme $Q^n_{r,d}$ for all $r$, $d$ and $n$. For a description of how this torus action lifts to the Quot-scheme, and also of the description of the fixed point locus we will give next, we refer to a paper by Kool [32]. It should be noted that this paper actually focuses on the moduli space of Gieseker stable torsion-free sheaves, but similar (and even somewhat simpler) arguments apply when working with the Quot-scheme instead. The reason for this is that, as noted before, points on the Quot-scheme can be identified with subsheaves of $\mathcal{O}_X^r$. These subsheaves are furthermore always torsion-free, which makes the study of these subsheaves very similar to the study of torsion-free sheaves.

### Fixed point locus

Now that we have introduced the action of $T$ on $Q^n_{r,d}$, and therefore on $Q^\bullet_{r,d}$, we can also consider the fixed point locus $(Q^\bullet_{r,d})^T$. We will discuss this fixed point locus by first describing its $k$-points. The final goal of this section is to give a decomposition $(Q^\bullet_{r,d})^T = \coprod_\chi Q_\chi$ into connected components, which turns out to allow for a fairly easy description.

Recall that every point of $Q^\bullet_{r,d}$ can be identified by a submodule $E \subseteq k[x_1, \ldots, x_d]^r$ by taking the kernel of the quotient map corresponding to this point. Note furthermore that $k[x_1, \ldots, x_d]^r$, being a vector space in $d$ variables, admits a $\mathbb{Z}^d$ grading, and that it splits as a $k$-vector space as $k[x_1, \ldots, x_d]^r = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} k^r \cdot x^{\mathbf{a}}$, where we denote $x^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}$. This grading also is compatible with the torus action on $\mathbb{A}^d_k$, since a $\mathbf{t} \in T$ acts on the component $k^r \cdot x^{\mathbf{a}}$ of $k[x_1, \ldots, x_d]^r$ by multiplication by $\mathbf{t}^{-\mathbf{a}} := t_1^{-a_1} \cdots t_d^{-a_d}$.

It turns out that every point of the fixed point locus $(Q^\bullet_{r,d})^T$ corresponds to a graded submodule $E \subseteq k[x_1, \ldots, x_d]^r$. Each such submodule splits into homogeneous components as $E = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} E_{\mathbf{a}} \cdot x^{\mathbf{a}}$, where every $E_{\mathbf{a}}$ is a subspace of $k^r$. To such a graded submodule $E$ we assign a characteristic function $\chi_E$, which is the map $\mathbb{Z}^d_{\geq 0} \to \mathbb{Z}_{\geq 0}$ which sends $\mathbf{a} \mapsto \dim_k E_{\mathbf{a}}$ for every $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$. We will see later that it is easier to describe $Q^n_{r,d}$ by looking at the points belonging to one such characteristic function at a time. Also note that $0 \leq \chi_E(\mathbf{a}) \leq r$ for all $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$.

We can visualize a graded submodule $E = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} E_{\mathbf{a}} \cdot x^{\mathbf{a}}$ by putting its components $E_{\mathbf{a}}$ in a $d$-dimensional grid. An example is drawn in Figure 2.1a. Similarly, we can visualize the corresponding characteristic function by instead putting the numbers $\chi(\mathbf{a})$ in the grid, as shown in Figure 2.1b.

We denote by $\mathcal{X}_{r,d}$ the set of all characteristic functions $\chi$ which are of the form $\chi_E$ for some $E$ which corresponds to a point of $(Q^\bullet_{r,d})^T$. That is,

$$\mathcal{X}_{r,d} := \left\{ \chi_E \,\middle|\, [E \hookrightarrow k[x_1, \ldots, x_d]^r] \in (Q^\bullet_{r,d})^T \right\}.$$

We can now deduce some properties of the characteristic functions $\chi$ in $\mathcal{X}_{r,d}$. The cokernel of the embedding $E \hookrightarrow k[x_1, \ldots, x_d]^r$ should have finite length as a $k[x_1, \ldots, x_d]$-module, and therefore it should have finite dimension as a $k$-vector space. This implies

$$\begin{array}{|ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
k^3 & k^3 & k^3 & k^3 & k^3 & k^3 & \cdots \\
W_1 & k^3 & k^3 & k^3 & k^3 & k^3 & \cdots \\
V_1 & V_1 & W_2 & k^3 & k^3 & k^3 & \cdots \\
0 & V_1 & W_2 & W_2 & W_2 & k^3 & \cdots \\
0 & 0 & V_2 & W_2 & W_2 & k^3 & \cdots \\
0 & 0 & 0 & 0 & V_3 & k^3 & \cdots \\
\hline
\end{array}$$

(a) A submodule $E$ of $k[x_1, x_2]^3$

$$\begin{array}{|ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
3 & 3 & 3 & 3 & 3 & 3 & \cdots \\
2 & 3 & 3 & 3 & 3 & 3 & \cdots \\
1 & 1 & 2 & 3 & 3 & 3 & \cdots \\
0 & 1 & 2 & 2 & 2 & 3 & \cdots \\
0 & 0 & 1 & 2 & 2 & 3 & \cdots \\
0 & 0 & 0 & 0 & 1 & 3 & \cdots \\
\hline
\end{array}$$

(b) The corresponding characteristic function $\chi_E$

Figure 2.1: A visualization of a graded submodule $E$ of $k[x_1, x_2]^3$, and its characteristic function $\chi_E$. Here $V_1$, $V_2$ and $V_3$ are 1-dimensional subspaces of $k^3$, and $W_1$ and $W_2$ are 2-dimensional subspaces of $k^3$, satisfying $V_1 \subseteq W_1$ and $V_1, V_2, V_3 \subseteq W_2$.

that only finitely many of the spaces $E_{\mathbf{a}}$ can be different from $k^r$, and therefore $\chi_E(\mathbf{a}) \neq r$ for only finitely many $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$.

Also for every $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$, there is an injective map $E \to E$ given by multiplication by $x^{\mathbf{a}}$. This map sends the component $E_{\mathbf{b}}$ into $E_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{b} \in \mathbb{Z}^d_{\geq 0}$, and therefore $\chi_E(\mathbf{a}+\mathbf{b}) \geq \chi_E(\mathbf{b})$. This condition can also be stated by noting that for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d_{\geq 0}$ with $\mathbf{a} \leq \mathbf{b}$, we have $\chi_E(\mathbf{a}) \leq \chi_E(\mathbf{b})$. Here, by $\mathbf{a} \leq \mathbf{b}$ we mean that for all $i$ we have $a_i \leq b_i$.

These conditions on the characteristic functions $\chi_E$ actually define the whole set $\mathcal{X}_{r,d}$.

**Lemma 2.2.4.** *The set $\mathcal{X}_{r,d}$ contains exactly those maps $\chi\colon \mathbb{Z}^d_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the following properties:*

- $0 \leq \chi(\mathbf{a}) \leq r$ *for all* $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$,
- $\chi(\mathbf{a}) = r$ *for all but finitely many* $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$,
- *if* $\mathbf{a} \leq \mathbf{b}$, *then* $\chi(\mathbf{a}) \leq \chi(\mathbf{b})$, *for all* $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d_{\geq 0}$.

*Proof.* We have already seen that every element of $\mathcal{X}_{r,d}$ satisfies the given properties. Let $\chi$ be some arbitrary map satisfying the properties from the lemma. Let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = k^r$ be a chain of inclusions of $k$-vector spaces, where every $V_i$ had dimension $i$. Now take

$$E = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} V_{\chi(\mathbf{a})} \cdot x^{\mathbf{a}}.$$

This $E$ can be seen to be a graded submodule of $k[x_1, \ldots x_d]$, and it furthermore satisfies $\chi_E = \chi$. So we see that $\chi \in \mathcal{X}_{r,d}$. $\square$

Now that we have found the set of characteristic functions $\chi \in \mathcal{X}_{r,d}$, we next focus on finding all $E$ with a given characteristic function. To describe such an $E$, it is sufficient to give the subspaces $E_{\mathbf{a}} \subseteq k^r$ for all $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$. These subspaces should furthermore be compatible, in the sense that $\mathbf{a} \leq \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d_{\geq 0}$ implies that $E_{\mathbf{a}} \subseteq E_{\mathbf{b}}$. This suggests that the following scheme might describe all possible choices of $E$:

**Definition 2.2.5.** Let $\chi \in \mathcal{X}_{r,d}$. We define the scheme $Q_\chi$ to be the closed subscheme of

$$\prod_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} \mathrm{Gr}(\chi(\mathbf{a}), r),$$

which consists of those points $([E_{\mathbf{a}}])_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} \in \prod_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} \mathrm{Gr}(\chi(\mathbf{a}), r)$ such that $\mathbf{a} \leq \mathbf{b}$ implies $E_{\mathbf{a}} \subseteq E_{\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d_{\geq 0}$.

Note that the product occurring in this definition is an infinite product, however, only finitely many of the factors are not equal to $\mathrm{Gr}(r, r)$, which is just a single point. Furthermore, each scheme $Q_\chi$ can be seen to be connected. In the remainder of this thesis, we will denote a point $([E_{\mathbf{a}}])_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} \in Q_\chi$ just by $[E]$, where $E := \bigoplus_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} E_{\mathbf{a}} x^{\mathbf{a}}$ is the corresponding graded submodule of $k[x_1, \ldots, x_d]^r$. The following theorem implies that the given definition of $Q_\chi$ is the correct one:

**Theorem 2.2.6.** *There is an isomorphism*

$$(Q^\bullet_{r,d})^T \cong \coprod_{\chi \in \mathcal{X}_{r,d}} Q_\chi.$$

We will not prove this theorem here. Instead we refer to [32], where a similar result is deduced for the moduli space of torsion free sheaves. Finally we also mention that it is possible to give a similar description of the fixed point locus of the Quot-scheme on toric varieties different from $\mathbb{A}^d_k$, but we will not discuss this in this thesis.

### 2.2.3 Murphy's law and Mnëv's universality theorem

Murphy's law in algebraic geometry was first introduced by Vakil [57]. Informally, we say that a scheme satisfies Murphy's law if it is as degenerate as possible; that is, if every possible singularity type occurs somewhere on the scheme. For stating what Murphy's law means, we should work over $\mathbb{Z}$ instead of a field $k$, which is what we will do in this section.

In order to define what a singularity type is, we first need to introduce an equivalence relation $\sim$ on pointed schemes $(X, p)$. This equivalence relation is generated by taking $(X, p) \sim (Y, q)$ if there is a smooth morphism $X \to Y$ which maps $p$ to $q$. Saying that a morphism is *smooth* means that each of its fibers is nonsingular, and that the morphism is furthermore flat. Flatness of a morphism can intuitively be interpreted as saying that the fibers of the morphism vary in a somewhat continuous manner. In this thesis, we will just use that open embeddings and projections of the form $X \times Y \to X$, where $Y$ is a smooth scheme, are always smooth.

Singluarity types are now defined as the equivalence classes of pointed schemes $(X, p)$ under this equivalence relation $\sim$. We say that a singularity type has finite type over $\mathbb{Z}$ if there is some representative $(X, p)$ of this singularity type where $X$ has finite type over $\mathbb{Z}$. Now we can define Murphy's law in algebraic geometry:

**Definition 2.2.7.** We say that a scheme $X$ satisfies *Murphy's law* if every singularity type of finite type over $\mathbb{Z}$ occurs somewhere on $X$. That is: every singularity type of finite type over $\mathbb{Z}$ has some representative $(X, p)$ with $p \in X$.

Note that this definition in particular implies the following: if $f \colon X \to Y$ is a smooth morphism of schemes, and $X$ satisfies Murphy's law, then also $Y$ satisfies Murphy's law. If instead we know that $Y$ satisfies Murphy's law, and that $f$ is surjective, then also $X$ satisfies Murphy's law. This is caused by the fact that smooth morphisms preserve singularity types.

In [57], Vakil proved for a large number of important moduli spaces that they satisfy Murphy's law. The central tool in these proofs is Mnëv's Universality Theorem, which we will introduce next.

We define a set of incidence relations on $m$ points and $n$ lines to be a subset $I \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$, where we consider two of those sets of incidence relations to be the same if one can be obtained from the other by permuting the index set of points $\{1, \ldots, m\}$ and permuting the index set of lines $\{1, \ldots, n\}$. More formally, we consider a set of incidence relations to be an element of

$$\mathcal{P}(\{1, \ldots, m\} \times \{1, \ldots, n\}) / (\mathfrak{S}_m \times \mathfrak{S}_n),$$

but we will usually refer to such an element just by some set $I \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$. To such a set $I$ we assign an incidence scheme $C_I$.

**Definition 2.2.8.** The *incidence scheme $C_I$* corresponding to a set of incidence relations $I$ is the subscheme of $(\mathbb{P}^2_{\mathbb{Z}})^m \times (\mathbb{P}^{2\vee}_{\mathbb{Z}})^n$ which parameterize $m$ points $p_1, \ldots, p_m$ and $n$ lines $\ell_1, \ldots, \ell_n$ satisfying the following conditions:

- The points $p_1, \ldots, p_m$ are pairwise distinct.
- The lines $\ell_1, \ldots, \ell_n$ are pairwise distinct.
- A points $p_i$ and a line $\ell_j$ are incident if and only if $(i, j) \in I$.

For us it will be slightly more convenient to work with a modification of this definition, which concerns what we will call closed incidence schemes:

**Definition 2.2.9.** The *closed incidence scheme $\overline{C}_I$* corresponding to a set of incidence relations $I$ is the subscheme of $(\mathbb{P}^2_{\mathbb{Z}})^m \times (\mathbb{P}^{2\vee}_{\mathbb{Z}})^n$ which parameterize $m$ points $p_1, \ldots, p_m$ and $n$ lines $\ell_1, \ldots, \ell_n$, such that $(i, j) \in I$ implies that $p_i$ and $\ell_j$ are incident.

Stated differently, we define the closed incidence scheme $\overline{C}_I$ to be like the usual incidence scheme $C_I$, except that we drop the conditions that the points and lines need to be pairwise distinct, and that a point $p_i$ and line $\ell_j$ cannot be incident if $(i, j) \notin I$. Since these conditions that we dropped are all open conditions, we see that $C_I$ is an open subscheme of $\overline{C}_I$. Furthermore, $\overline{C}_I$ is a closed subscheme of $(\mathbb{P}^2_{\mathbb{Z}})^m \times (\mathbb{P}^{2\vee}_{\mathbb{Z}})^n$. It is however not necessarily the case that $\overline{C}_I$ is the closure of $C_I$ in $(\mathbb{P}^2_{\mathbb{Z}})^m \times (\mathbb{P}^{2\vee}_{\mathbb{Z}})^n$: for example, it could be the case that $C_I$ is the empty scheme, even though $\overline{C}_I$ is always nonempty.

Now the version of Mnëv's universality theorem which we will use, states the following:

**Theorem 2.2.10** (Mnëv's univerality theorem)**.** *Every singularity type of finite type over $\mathbb{Z}$ occurs somewhere on some closed incidence scheme $\overline{C}_I$.*

*Proof.* Lafforgue proves in Section 1.8 of [34] that for every singularity type, there is some incidence scheme $C_I$ such that $\mathrm{PGL}_3$ acts freely on this incidence scheme, and such that the quotient $C_I/\mathrm{PGL}_3$ contains the given singularity type. A similar result is proven in a somewhat more detailed manner by Lee and Vakil in [35]. Since both the open embedding $C_I \hookrightarrow \overline{C}_I$ and the projection $C_I \to C_I/\mathrm{PGL}_3$ are smooth, we conclude that also $\overline{C}_I$ contains the given singularity type. □

Since we want to work over a field $k$ instead of over $\mathbb{Z}$ for the rest of this thesis, we also introduce the notation $\overline{C}_{k,I} := \overline{C}_I \times_{\mathbb{Z}} \mathrm{Spec}(k)$. Note that we could also have defined $\overline{C}_{k,I}$ as some closed subscheme of $(\mathbb{P}^2_k)^m \times_k (\mathbb{P}^{2\vee}_k)^n$ instead, this would result in the same scheme.

## 2.3 Singularities of $(Q^{\bullet}_{3,d})^T$

In this section we will prove the main result of this thesis, which is that every singularity type of finite type over $\mathbb{Z}$ occurs, up to a base change to $k$, on the scheme $(Q^{\bullet}_{3,4})^T$. First we illustrate that, for any $d$ and for $r = 3$, every connected component $Q_\chi$ of $(Q^{\bullet}_{3,d})^T$ is isomorphic to some closed incidence scheme $\overline{C}_{k,I_\chi}$. In particular this means that every singularity occurring on $\overline{C}_{k,I_\chi}$ also occurs somewhere on $Q_\chi$, and therefore on $(Q^{\bullet}_{3,d})^T$.

The remaining question is whether every closed incidence scheme $\overline{C}_{k,I_\chi}$ also occurs as some connected component $Q_\chi$. We show that for $d = 2$ or $d = 3$, the answer to this question is negative. In these cases we can write down properties of $I_\chi$ which are not satisfied by all sets of incidence relations.

Finally we will show that, for $d = 4$, every closed incidence scheme $\overline{C}_{k,I}$ is isomorphic to some connected component $Q_\chi$ of $(Q^{\bullet}_{3,4})^T$. This is done by explicitly constructing $\chi$ such that it exactly encodes the relations in $I$. Combining this with the previous step implies that every singularity which occurs on some closed incidence scheme $\overline{C}_I$, also occurs on $(Q^{\bullet}_{3,4})^T$. In particular, using Mnëv's universality theorem, this yields our main result.

### 2.3.1 $Q_\chi$ as a closed incidence scheme

As discussed in Section 2.2.2, every connected component of $(Q^{\bullet}_{r,d})^T$ is isomorphic to some closed subscheme of a product of Grassmannians. In the case where $r = 3$, these Grassmannians represent linear subspaces of $k^3$, which can be identified with points and lines in $\mathbb{P}^2_k$. In particular, $\mathrm{Gr}(0,3)$ and $\mathrm{Gr}(3,3)$ are both just a point, while $\mathrm{Gr}(1,3) \cong \mathbb{P}^2_k$ and $\mathrm{Gr}(2,3) \cong \mathbb{P}^{2\vee}_k$. Therefore, it is unsurprising that these connected components $Q_\chi$ are isomorphic to the closed incidence schemes defined in Section 2.2.3. In this section we will explicitly construct the set of incidence relations $I_\chi$ such that $\overline{C}_{k,I_\chi} \cong Q_\chi$ for all $\chi \in \mathcal{X}_{3,d}$.

Let $\chi \in \mathcal{X}_{3,d}$ be a characteristic function. Furthermore, let $[E]$ be a point in the corresponding connected component $Q_\chi$ for some graded submodule $E$ of $k[x_1, \ldots, x_d]^3$. Before defining the set of incidence relations $I_\chi$, we first define the index sets of points and lines on which we define this set. Note that our original definition of sets of incidence relations always assumes these index sets to be of the form $\{1, \ldots, n\}$. We can however also use any other pair of finite sets, since after choosing any ordering of the elements we get a set of incidence relations by our original definition anyway.

We index the points by the set

$$P_\chi := \left\{ \mathbf{a} \in \mathbb{Z}^d \,\middle|\, \chi(\mathbf{a}) = 1 \right\} / \sim,$$

where the equivalence relation $\sim$ is generated by setting $\mathbf{a} \sim \mathbf{b}$ if $\mathbf{a} \leq \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ with $\chi(\mathbf{a}) = \chi(\mathbf{b}) = 1$. This means that we add one point to our incidence relations for each 1 dimensional subspace $E_\mathbf{a}$, except that we identify points corresponding to $E_\mathbf{a}$ and $E_\mathbf{b}$ if there is some constraint $E_\mathbf{a} \subseteq E_\mathbf{b}$ (which would imply $E_\mathbf{a} = E_\mathbf{b}$, since these two subspaces of $k^3$ have the same dimension).

The index set of the lines of $I_\chi$ is defined in the same way, except that we focus on 2 dimensional subspaces of $k^3$ now:

$$L_\chi := \left\{ \mathbf{a} \in \mathbb{Z}^d \,\middle|\, \chi(\mathbf{a}) = 2 \right\} / \sim,$$

where the equivalence relation $\sim$ is generated by setting $\mathbf{a} \sim \mathbf{b}$ if $\mathbf{a} \leq \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ with $\chi(\mathbf{a}) = \chi(\mathbf{b}) = 2$.

Finally we define the actual incidence relations: for a point-index $i \in P_\chi$ and a line-index $j \in L_\chi$ we add the pair $(i, j)$ to the set of incidence relations if and only if $i$ has a representative $\mathbf{a}$ and $j$ has a representative $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b}$. In this way we get a subset of $P_\chi \times L_\chi$, we call the corresponding set of incidence relations $I_\chi$.

**Lemma 2.3.1.** *For every $\chi \in \mathcal{X}_{3,d}$, the scheme $Q_\chi$ is isomorphic to the closed incidence scheme $\overline{C}_{k,I_\chi}$.*

*Proof.* $Q_\chi$ can be seen to be exactly the closed subscheme of $\prod_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} \mathrm{Gr}(\chi(\mathbf{a}), 3)$ which corresponds to the points $[E]$ with $E_\mathbf{a} = E_\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are in the same class of $P_\chi$ or in the same class of $L_\chi$, and with $E_\mathbf{a} \subseteq E_\mathbf{b}$ if $\mathbf{a}$ is in some class $i \in P_\chi$ and $\mathbf{b}$ is in some class $j \in L_\chi$ with $(i, j) \in I_\chi$.

Using this characterization of $Q_\chi$, it is fairly straightforward to write down an isomorphism $Q_\chi \to \overline{C}_{k,I_\chi}$ together with its inverse. □

### 2.3.2 The case $d = 2$

Before we analyze the singularities of $(Q^\bullet_{3,4})^T$, we will first consider the schemes $(Q^\bullet_{3,2})^T$ and $(Q^\bullet_{3,3})^T$. As we will see, in these cases we are unable to prove that all singularity types

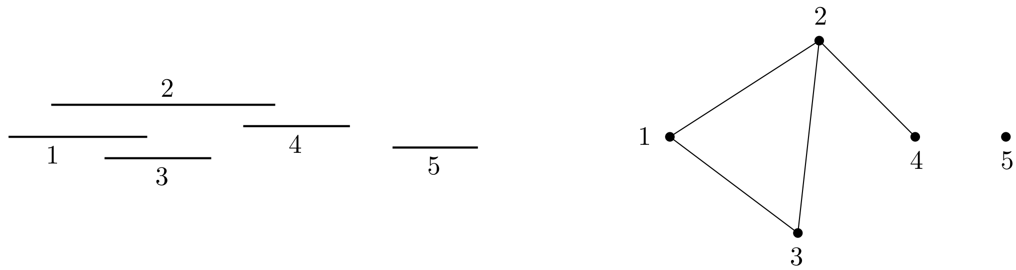



Figure 2.2: An example of a set of real intervals, and the corresponding interval graph.

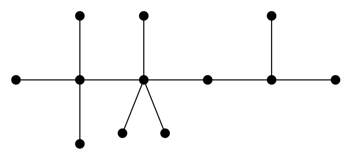


Figure 2.3: An example of a caterpillar graph.

occur on these schemes. This is because there are sets of incidence relations $I$ which are not of the form $I_\chi$ for some $\chi$ in $\mathcal{X}_{3,2}$ or $\mathcal{X}_{3,3}$.

It turns out that it is useful to identify a set of incidence relations $I$ with a bipartite graph. This graph has as vertices the point-indices and line-indices of $I$, and has an edge between a point-index $i$ and line-index $j$ exactly if $(i, j) \in I$. The following graph classes play an important role when working with the case $d = 2$:

**Definition 2.3.2.** Given a set of $m$ intervals $\{S_i \subseteq \mathbb{R} \,|\, 1 \leq i \leq m\}$, we define the intersection graph of these intervals as the graph with vertex set $\{S_i \subseteq \mathbb{R} \,|\, 1 \leq i \leq m\}$, and with an edge between two intervals $S_i$ and $S_j$ if and only if $S_i \cap S_j \neq \emptyset$.

A graph which is the intersection graph of a set of intervals, is called an *interval graph*.

**Definition 2.3.3.** A *caterpillar graph* $G$ is a tree which contains a path $P$ as a subgraph, such that every vertex in $G$ has distance at most one from a vertex in $P$.

An example of an interval graph is given in Figure 2.2, and an example of a caterpillar graph is drawn in Figure 2.3. Now we have the following result:

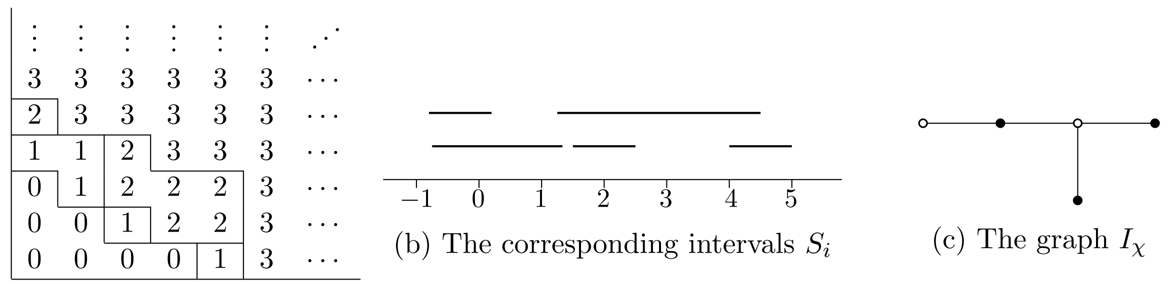
**Proposition 2.3.4.** *For any $\chi \in \mathcal{X}_{3,2}$, the set of incidence relations $I_\chi$ is, as a graph, a disjoint union of caterpillar graphs.*

*Proof.* First we will prove that $I_\chi$ is an interval graph, then we will deduce from this that $I_\chi$ is actually a disjoint union of caterpillar graphs.

To every point $(a_1, a_2) \in \mathbb{Z}^2_{\geq 0}$, we assign a real closed unit interval $S_{(a_1,a_2)} := [a_1 - 1 + 1/(a_2 + 1), a_1 + 1/(a_2 + 1)] \subseteq \mathbb{R}$. This assignment has the following two properties:

1. If $S_\mathbf{a} \cap S_\mathbf{b} \neq \emptyset$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^2_{\geq 0}$, then $\mathbf{a} \leq \mathbf{b}$ or $\mathbf{b} \leq \mathbf{a}$.

2. For all $(a_1, a_2) \in \mathbb{Z}^2_{\geq 0}$ we have that $S_{(a_1,a_2)}$ has nonempty intersection with each of the intervals $S_{(a_1+1,a_2)}$, $S_{(a_1-1,a_2)}$, $S_{(a_1,a_2+1)}$ and $S_{(a_1,a_2-1)}$, if they are defined.




(a) A characteristic function $\chi \in \mathcal{X}_{3,2}$

(b) The corresponding intervals $S_i$

(c) The graph $I_\chi$

Figure 2.4: A visualization of the proof of Proposition 2.3.4.

Now we define the interval corresponding to a point-index $i \in P_\chi$ as $S_i := \bigcup_{\mathbf{a}\in i} S_\mathbf{a}$, and similarly for $j \in L_\chi$ we define $S_j := \bigcup_{\mathbf{a}\in j} S_\mathbf{a}$. See Figure 2.4b for an example. The fact that these are indeed connected intervals follows from the second property of $S_\mathbf{a}$ from above. Furthermore it follows from the first property that for all $i_1, i_2 \in P_\chi$ with $i_1 \neq i_2$, we have $S_{i_1} \cap S_{i_2} = \emptyset$, and similarly for $j_1, j_2 \in L_\chi$ with $j_1 \neq j_2$ we have $S_{j_1} \cap S_{j_2} = \emptyset$. Finally it can be checked that for $i \in P_\chi$ and $j \in L_\chi$, we have $S_i \cap S_j \neq \emptyset$ if and only if $(i, j) \in I_\chi$. This proves that $I_\chi$ is the intersection graph belonging to the intervals $S_i$ for $i \in P_\chi$ and $S_j$ for $j \in L_\chi$.

We now know that $I_\chi$ is a bipartite interval graph, see Figure 2.4c. Next we will see that this implies that $I_\chi$ is a disjoint union of caterpillar graphs. Consider one connected component $C$ of $I_\chi$. First, we want to show that $C$ is a tree. Suppose that there is some cycle in $C$, consider the interval $S_0$ in this cycle with the right-most endpoint. This interval should intersect with two other intervals in the cycle, $S_{-1}$ and $S_1$. Because of bipartiteness, $S_{-1}$ and $S_1$ cannot intersect, without loss of generality we may assume that $S_{-1}$ is to the left of $S_1$. This however implies that $S_1$ is completely contained in $S_0$ (recall that the right-most point in $S_0$ lies to the right of $S_1$), so every interval intersecting $S_1$ also intersects $S_0$. There exists such an interval which intersects $S_1$, this follows by the fact that $S_1$ is part of a cycle in $C$ and therefore has degree at least 2. Let $S_2$ be this interval. Now $S_2$ intersects both $S_0$ and $S_1$, but this contradicts the bipartiteness of the graph. We conclude that $C$ has to be a tree.

Next let $P$ be the path in this tree from the interval with the smallest starting point to the interval with the largest end point. The union of all intervals in this path has to be connected, and it contains both the left-most point and the right-most point of any interval in $C$. It follows that this union contains all intervals in the connected component $C$. This implies that every interval in $C$ intersects some interval in the path, and therefore every vertex of $C$ has distance at most 1 to $P$. This completes the proof that $C$ is a caterpillar graph, and therefore $I_\chi$ is the disjoint union of caterpillar graphs. □

Now that we know that $I_\chi$ is always a disjoint union of caterpillar graphs for $\chi \in \mathcal{X}_{3,2}$, we in particular know that not all sets of incidence relations occur as some $I_\chi$ when $d = 2$. It seems unlikely that every singularity type of finite type over $\mathbb{Z}$ occurs on some incidence

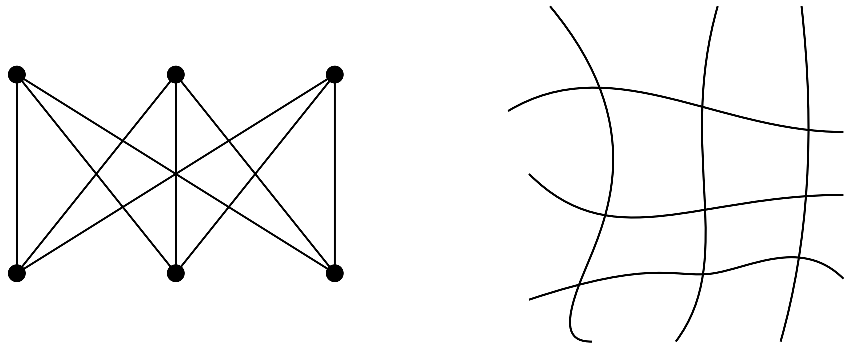


Figure 2.5: The graph $K_{3,3}$, and a set of strings which have $K_{3,3}$ as their intersection graph.

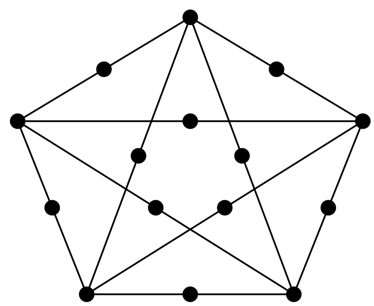


Figure 2.6: A subdivided $K_5$, which is not a string graph

scheme of such a simple set of incidence relations, so it would be interesting to investigate what kind of singularities do occur in this case.

### 2.3.3 The case $d = 3$

Also in the case where $d = 3$, it turns out that not all closed incidence schemes occur as some connected component of $(Q^{\bullet}_{3,3})^T$. The argument is similar to that of the case $d = 2$, although it is a bit more complicated.

**Definition 2.3.5.** Given a set of $m$ continuous paths in $\mathbb{R}^2$ of finite length without self-intersections $\{S_i \subseteq \mathbb{R} \,|\, 1 \leq i \leq m\}$ ("strings"), we define the intersection graph of these strings as the graph with vertex set $\{S_i \subseteq \mathbb{R} \,|\, 1 \leq i \leq m\}$, and with an edge between two intervals $S_i$ and $S_j$ if and only if $S_i \cap S_j \neq \emptyset$.

A graph which is the intersection graph of such a set of strings is called a *string graph.*

String graphs form quite a large class of graphs, for example every planar graph can be seen to be a string graph. There are also string graphs which are not planar graphs, for example the complete bipartite graph on two sets of 3 vertices, $K_{3,3}$, see Figure 2.5. An example of a graph which is not a string graph is the subdivision of $K_5$ shown in Figure 2.6, where every edge is divided into two edges by a vertex.

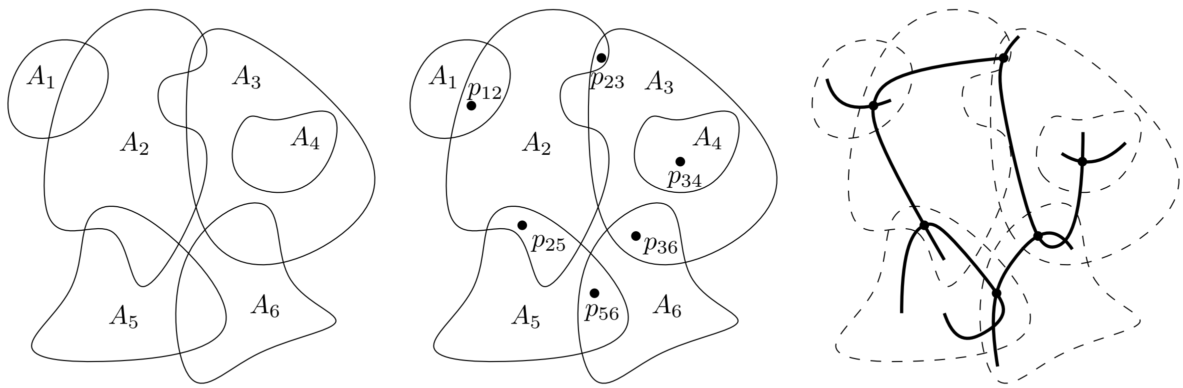



Figure 2.7: A visualization of the last half of the proof of Proposition 2.3.6.

**Proposition 2.3.6.** *For any $\chi \in \mathcal{X}_{3,3}$, the set of incidence relations $I_\chi$ is, as a graph, a bipartite string graph.*

*Proof.* To every $(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0}$ we assign the following subset of $\mathbb{R}^2$:

$$A_{(a_1,a_2,a_3)} := \left[a_1 - 1 + \frac{1}{a_3+1}, a_1 + \frac{1}{a_3+1}\right] \times \left[a_2 - 1 + \frac{1}{a_3+1}, a_2 + \frac{1}{a_3+1}\right] \setminus \left\{\left(a_1 + \frac{1}{a_3+1}, a_2 - 1 + \frac{1}{a_3+1}\right)\right\}.$$

This is a unit-square where the bottom right corner is removed. The reason for removing this corner is that we do not want the sets $A_{(a_1,a_2,a_3)}$ and $A_{(a_1+1,a_2-1,a_3)}$ to intersect. These sets now satisfy the following two properties:

1. If $A_\mathbf{a} \cap A_\mathbf{b} \neq \emptyset$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^3_{\geq 0}$, then $\mathbf{a} \leq \mathbf{b}$ or $\mathbf{b} \leq \mathbf{a}$.

2. If the distance between $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^3_{\geq 0}$ is at most 1, then $S_\mathbf{a} \cap S_\mathbf{b}$ is nonempty. Furthermore, this intersection is not just a point, but instead contains at least some line segment with positive length.

For every point-index $i \in P_\chi$ we take $A_i := \bigcup_{\mathbf{a} \in i} A_\mathbf{a} \subseteq \mathbb{R}^2$, and for $j \in L_\chi$ we take $A_j := \bigcup_{\mathbf{a} \in j} A_\mathbf{a} \subseteq \mathbb{R}^2$. Both these sets are connected by the second property, and cannot be disconnected by removing some finite number of points. Furthermore, for distinct $i_1, i_2 \in P_\chi$ we have that $A_{i_1}$ and $A_{i_2}$ are disjoint by the first property, similarly for distinct $j_1, j_2 \in L_\chi$ the sets $A_{j_1}$ and $A_{j_2}$ are disjoint. It also follows from the properties that for $i \in P_\chi$ and $j \in L_\chi$, we have $(i, j) \in I_\chi$ if and only if $A_i \cap A_j \neq \emptyset$. Stated differently, $I_\chi$ is the intersection graph belonging to the sets $A_i$ for $i \in P_\chi$ and $A_j$ for $j \in L_\chi$.

It remains to be shown that we can replace each set $A_i$ or $A_j$ by a string $S_i$ or $S_j$, such that the intersection graph does not change. This last part of the proof is visualized in Figure 2.7. We take a point $p_{ij} \in \mathbb{R}^2$ in every nonempty intersection $A_i \cap A_j$. Now for every set $A_i$ with $i \in P_\chi$, we draw a curve $S_i$ without self-intersections connecting all points $p_{ij}$

within this set, such that the curve does not leave $A_i$. This is always possible because of the connectivity of the set $A_i$. We do the same for the sets $A_j$ for $j \in L_\chi$. This way, two strings $S_i$ and $S_j$ intersect if and only if the corresponding sets $A_i$ and $A_j$ intersect. We conclude that $I_\chi$ is indeed a string graph. $\square$

This proposition implies in particular that the subdivision of $K_5$ from Figure 2.6, which is not a string graph, does not occur as a set of incidence relations $I_\chi$ with $\chi \in \mathcal{X}_{3,3}$. As a set of incidence relations, this graph corresponds to a configuration with 5 points and 10 lines, where there is one line passing through every pair of points. We note that we have not proven yet that every bipartite string graph does occur as some $I_\chi$. It is also unclear whether Murphy's law should hold for $(Q^\bullet_{3,3})^T$, this depends on whether the class of bipartite string graphs is general enough to encode any singularity type.

### 2.3.4 The case $d = 4$

The next and final case we consider is $d = 4$. We have seen in the preceding sections that for $d = 2$, the graph corresponding to $I_\chi$ with $\chi \in \mathcal{X}_{3,d}$ is always an intersection graph of intervals in $\mathbb{R}$, and that for $d = 3$ this graph is always an intersection graph of strings in $\mathbb{R}^2$. This suggests that for $d = 4$, the graph identified with $I_\chi$ must be some kind intersection graph of objects in $\mathbb{R}^3$. However, every graph can be embedded in $\mathbb{R}^3$, so this does not form an obstacle. As we will see in this section, it is indeed the case that every set of incidence relations is of the form $I_\chi$ for some $\chi \in \mathcal{X}_{3,4}$.

Let $I$ be some set of incidence relations on $m$ points and $n$ lines. We will find a characteristic function $\chi \in \mathcal{X}_{3,4}$ such that $I_\chi = I$. To do this, we should specify $\chi(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{Z}^4_{\geq 0}$.

The construction of $\chi$ will use only two hyperplanes in $\mathbb{Z}^4_{\geq 0}$: the hyperplane $H_1$ defined by $a_1 + a_2 + a_3 + a_4 = M_1 + M_2$ and the hyperplane $H_2$ defined by $a_1 + a_2 + a_3 + a_4 = M_1 + M_2 + 1$, where $M_1 = 2m - 2$ and $M_2 = 2n - 1$. The values of $M_1$ and $M_2$ are chosen in such a way to make the rest of the construction work out, there is no further significance in these values. The rest of the construction is completed by taking $\chi(a_1, a_2, a_3, a_4) = 0$ for $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4_{\geq 0}$ with $a_1 + a_2 + a_3 + a_4 < M_1 + M_2$ and taking $\chi(a_1, a_2, a_3, a_4) = 3$ for $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4_{\geq 0}$ with $a_1 + a_2 + a_3 + a_4 > M_1 + M_2 + 1$.

Furthermore, we will make sure that most $\mathbf{a} \in H_1$ satisfy $\chi(\mathbf{a}) = 0$ and most points $\mathbf{a} \in H_2$ satisfy $\chi(\mathbf{a}) = 3$. This way, the only pairs of points $\mathbf{a}, \mathbf{b}$ which actively influence the definition of $I_\chi$ are those with $\mathbf{a} \in H_1$ and $\mathbf{b} \in H_2$, and with $\chi(\mathbf{a}), \chi(\mathbf{b}) \in \{1, 2\}$.

Now we can start describing the part of the construction which actually encodes $I$. For this construction, we assume that the points from $I$ are indexed from 0 to $m - 1$, and that the lines are indexed from 0 to $n - 1$, since this makes the formulas slightly easier. We will first describe how the points are encoded, then we describe how the lines are encoded, and finally we show how incidences between points and lines are encoded.

An example of the construction is sketched in Figure 2.8. The dots in this construction correspond to coordinates $\mathbf{a} \in H_1$ with $\chi(\mathbf{a}) \neq 0$, in particular the black dots have $\chi(\mathbf{a}) = 1$ and the white dots have $\chi(\mathbf{a}) = 2$. The segments in the figure correspond to coordinates

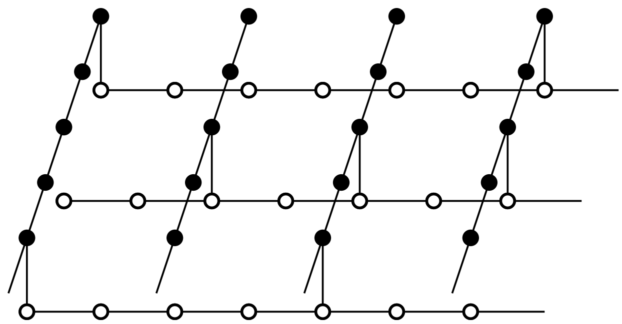


Figure 2.8: A visualization of the characteristic function $\chi$ which encodes the set of incidence relations $I = \{(0,0),(0,2),(1,1),(2,1),(2,2),(3,0),(3,1)\}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\iddots$ |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | $\cdots$ |
| 1 | 1 | 3 | 3 | 3 | 3 | 3 | $\cdots$ |
| 0 | 1 | 1 | 3 | 3 | 3 | 3 | $\cdots$ |
| 0 | 0 | 1 | 1 | 3 | 3 | 3 | $\cdots$ |
| 0 | 0 | 0 | 1 | 1 | 3 | 3 | $\cdots$ |
| 0 | 0 | 0 | 0 | 1 | 1 | 3 | $\cdots$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 3 | $\cdots$ |

Figure 2.9: $\chi$ restricted to the plane with $a_1 = 2i$ and $a_2 = M_1 - 2i$ for some $0 \le i < m$, in the case that $n = 3$. The points drawn here with $\chi(\mathbf{a}) = 1$ are exactly the points in $A_i$.

$\mathbf{b} \in H_2$ with $\chi(\mathbf{b}) \neq 3$, a segment joining the points $\mathbf{a}_1, \mathbf{a}_2 \in H_1$ corresponds to some $\mathbf{b} \in H_2$ with $\mathbf{b} \ge \mathbf{a}_1$ and $\mathbf{b} \ge \mathbf{a_2}$. Each of the 4 chains of black points in the figure corresponds to some point from the set of incidence relations, and each of the 3 chains of white points corresponds to a line.

All the points of $I$ are encoded in two parallel planes in $\mathbb{Z}^4_{\ge 0}$: the plane given by $a_1 + a_2 = M_1$ and $a_3 + a_4 = M_2$ which is contained in $H_1$, and the plane given by $a_1 + a_2 = M_1$ and $a_3 + a_4 = M_2 + 1$, which is contained in $H_2$. In particular, the point with index $i$ is represented by the following set of coordinates:

$$A_i = \{(2i, M_1 - 2i, a_3, M_2 - a_3) \mid 0 \le a_3 \le 2n - 2\} \\ \cup \{(2i, M_1 - 2i, a_3 + 1, M_2 - a_3) \mid 0 \le a_3 \le 2n - 2\}\,.$$

Note that the definitions of $M_1$ and $M_2$ were chosen in such a way to ensure that all points in this set have non-negative coordinates. For each $i$ and for each $\mathbf{a} \in A_i$ we set $\chi(\mathbf{a}) = 1$. The points in $A_i$ can be seen to form a kind of staircase in the plane with $a_1 = 2i$ and $a_2 = M_1 - 2i$, see Figure 2.9. Note that all points in $A_i$ will be in the same equivalence class of $P_\chi$. We will not add any more points $\mathbf{a}$ with $\chi(\mathbf{a}) = 1$ later in the construction, so we see that every equivalence class of $P_\chi$ will correspond to exactly one set $A_i$. We will therefore also use index $i$ for this equivalence class.

Next we encode all lines of $I$. This happens in a very similar manner. Now we use the plane given by $a_1 + a_2 = M_1 + 1$ and $a_3 + a_4 = M_2 - 1$, which is contained in $H_1$, and the plane $a_1 + a_2 = M_1 + 2$ and $a_3 + a_4 = M_2 - 1$, which is contained in $H_2$. Note that these planes are also parallel to the planes used for encoding the points. Now the line with index $j$ is represented by the following set of coordinates:

$$
\begin{aligned}
B_j = \{(a_1, M_1 + 1 - a_1, 2j, M_2 - 1 - 2j) \,|\, 0 \leq a_1 \leq 2m - 2\} \\
\cup \{(a_1 + 1, M_1 + 1 - a_1, 2j, M_2 - 1 - 2j) \,|\, 0 \leq a_1 \leq 2m - 2\}\,.
\end{aligned}
$$

For each point $\mathbf{a}$ in such a set, we take $\chi(\mathbf{a}) = 2$. Just like the points in $A_i$, we can see that the points in $B_j$ also form a staircase-like pattern in the plane with $a_3 = 2j$ and $a_4 = M_2 - 1 - 2j$. Again this adds one equivalence class to $L_\chi$ for each set $B_j$, we will refer to such an equivalence class by the index $j$ too. Note that there are no $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^4$ with $\mathbf{a} \leq \mathbf{b}$ such that one of these two points is in some $A_i$, and the other is in some $B_j$. Later in the construction, some more points $\mathbf{a}$ with $\chi(\mathbf{a}) = 2$ will be added, these points will all become part of an equivalence class corresponding to some $B_j$.

Finally we need to encode the actual incidence relations between the points and lines. To do this, we use the set of coordinates

$$
F = \{(2i, M_1 + 1 - 2i, 2j, M_2 - 2j) \,|\, (i, j) \in I\}\,.
$$

For every $\mathbf{a} \in F$ we take $\chi(\mathbf{a}) = 2$. Note that $F$ is completely contained in $H_2$. To see why we choose these points, consider one such point $(2i, M_1 - 2i, 2j, M_2 - 1 - 2j) \in F$ (so $(i, j) \in I$). Note that we have

$$
\begin{aligned}
(2i, M_1 + 1 - 2i, 2j, M_2 - 2j) &\geq (2i, M_1 - 2i, 2j, M_2 - 2j) \in A_i, \\
(2i, M_1 + 1 - 2i, 2j, M_2 - 2j) &\geq (2i, M_1 + 1 - 2i, 2j, M_2 - 1 - 2j) \in B_j.
\end{aligned}
$$

This implies that $(2i, M_1 - 2i, 2j, M_2 - 1 - 2j)$ is in equivalence class $j$ of $L_\chi$, and that $I_\chi$ will contain the incidence relation $(i, j)$. Furthermore, the point $(2i, M_1 - 2i, 2j, M_2 - 1 - 2j)$ does not dominate any other point in some $A_i$ or $B_j$. So we see that, if we take $\chi(\mathbf{a}) \in \{0, 3\}$ for all other points not in some $A_i$, $B_j$ or $F$, then the set $I_\chi$ is exactly the same as $I$ (when identifying the right indices). In particular, we get the following result:

**Lemma 2.3.7.** *For every set of incidence relations $I$, there is some $\chi \in \mathcal{X}_{3,4}$ with $I = I_\chi$.*

*Proof.* Let $I$ be some set of incidence relations on $m$ points and $n$ lines. Let $A_i$, $B_j$ and $F$ be as defined above for all $i$ and $j$. Now consider the $\chi \in \mathcal{X}_{3,4}$ which satisfies

$$
\chi(\mathbf{a}) = \begin{cases}
0 & \text{if } a_1 < 0 \vee a_2 < 0 \vee a_3 < 0 \vee a_4 < 0, \\
1 & \text{if } \mathbf{a} \in \bigcup_i A_i, \\
2 & \text{if } \mathbf{a} \in \bigcup_j B_j, \\
2 & \text{if } \mathbf{a} \in F, \\
0 & \text{otherwise if } a_1 + a_2 + a_3 + a_4 \leq M_1 + M_2, \\
3 & \text{otherwise if } a_1 + a_2 + a_3 + a_4 \geq M_1 + M_2 + 1,
\end{cases}
$$

for all $\mathbf{a} \in \mathbb{Z}^4$. To see that there is indeed such a $\chi$ in $\mathcal{X}_{3,4}$, we need to verify that for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^4$ we have that $\mathbf{a} \leq \mathbf{b}$ implies $\chi(\mathbf{a}) \leq \chi(\mathbf{b})$. This can be verified from the definition of the sets $A_i$, $B_j$ and $F$. Furthermore, by the preceding discussion, we see that $I_\chi$ is indeed exactly the same as $I$. This completes the proof of the lemma. □

From this lemma, we derive our main result:

**Theorem 2.1.1.** *Every singularity type of finite type over $\mathbb{Z}$ occurs, up to a base change to $k$, on the scheme $(Q^\bullet_{3,4})^T$.*

*Proof.* Mnëv's universality theorem (Theorem 2.2.10) yields, after a base change, that every singularity type of finite type over $\mathbb{Z}$ appears on some $\overline{C}_{k,I}$, up to base changing to $k$, for some set of incidence relations $I$. Furthermore, by Lemma 2.3.7, there is some $\chi \in \mathcal{X}_{3,4}$ such that $I = I_\chi$. Using Lemma 2.3.1 we now see that $\overline{C}_{k,I} = \overline{C}_{k,I_\chi}$ is isomorphic to the connected component $Q_\chi \subseteq (Q^\bullet_{3,4})^T$. This implies that $(Q^\bullet_{3,4})^T$ contains every singularity which occurs on some closed incidence scheme $\overline{C}_I$. We conclude that the theorem holds. □

## 2.4 Extending the result to the full Quot-scheme

In this section it will be discussed how it might be possible to extend the results from the previous section to the entire Quot-scheme, instead of just the fixed point locus. The inspiration for this section comes from a paper by Jelisiejew [29], where it is proven that the Hilbert scheme of points on $\mathbb{A}^{16}_{\mathbb{Z}}$, denoted $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$, satisfies Murphy's law up to retraction. The Hilbert scheme of points is a moduli space which parametrizes sets of points on a scheme, it is actually the special case of the Quot-scheme with $r = 1$. So in our notation, this moduli space would be denoted as $Q^\bullet_{1,16}$. The "up to retraction" part of the result furthermore means that singularity types with representatives $(X, x)$ and $(Y, y)$ are also identified if there is a retraction $(X, x) \to (Y, y)$. Here a retraction is a pair of morphisms of pointed schemes $f\colon (X, x) \to (Y, y)$ and $s\colon (Y, y) \to (X, x)$ such that $f \circ s = \mathrm{id}_Y$.

Jelisiejew derives this result by first proving Murphy's law for the fixed point locus of $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^5_{\mathbb{Z}})$ with respect to the action of some 1-dimensional torus $k^*$. Then, quite some complicated machinery is used to extend this result to $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$. Note in particular that Jelisiejew starts out with a result on a Hilbert space on $\mathbb{A}^5_{\mathbb{Z}}$, and ends up with a result over $\mathbb{A}^{16}_{\mathbb{Z}}$. In this thesis, we have proven a result on the fixed point locus of a Quot-scheme over $\mathbb{A}^4_{\mathbb{Z}}$, one might hope that by applying similar tools, we can prove a version of Murphy's law on $Q^\bullet_{3,d}$, where $d$ is strictly smaller than 16.

Modifying the tools used by Jelisiejew to the case of the Quot-scheme is however beyond the scope of this thesis. Instead, we focus on a single tool which plays an important role in [29], namely the *generalized Białynicki-Birula decomposition*, and check whether we can directly apply this tool to get some version of Murphy's law on $Q^\bullet_{3,4}$. This turns out not to be the case, as we will see in this section.

### 2.4.1 The Białynicki-Birula decomposition

Suppose we have some scheme $X$ with an action of the 1-dimensional torus $k^*$. Furthermore, suppose that its fixed point locus $X^{k^*}$ decomposes into connected components as $X^{k^*} = \coprod_{i=1}^n Y_i$. Now the *Białynicki-Birula decomposition* of $X$, if it exists, is a scheme $X^+ = \coprod_{i=1}^m X_i^+$, where every $X_i^+$ consists (informally) of the points $x \in X$ such that the limit $\lim_{t\to\infty} t \cdot x$ is contained in $Y_i$. This decomposition comes with a map $\theta_0 \colon X^+ \to X$, which is a injection on points, a retraction $\pi \colon X^+ \to X^{k^*}$ which sends each $X_i^+$ to $Y_i$, and an embedding $i \colon X^{k^*} \to X^+$ which embeds each $Y_i$ in the corresponding $X_i^+$. Furthermore, we have $\pi \circ i = \mathrm{id}$ and $\theta_0 \circ i \colon X^{k^*} \to X$ is the embedding of fixed points.

In order to apply this to the Quot-scheme $Q^n_{r,d}$, we first need to choose an action of the 1-dimensional torus $k^*$ on it. For this, we embed $k^*$ into our $d$-dimensional torus $T$ by sending $t \mapsto (t^{e_1}, \ldots, t^{e_d})$ for certain $e_1, \ldots, e_d \in \mathbb{Z}$, and let $k^*$ act on $Q^n_{r,d}$ through this embedding. It can be shown that, if we choose the numbers $e_1, \ldots e_d$ generically, then $(Q^n_{r,d})^{k^*} = (Q^n_{r,d})^T$. In particular, all our preceding results about singularities on $(Q^n_{r,d})^T$ carry over to $(Q^n_{r,d})^{k^*}$.

Now assume that a Białynicki-Birula decomposition of $Q^n_{r,d}$ with respect to the given torus action exists, denote it by $Q^{n+}_{r,d} = \coprod_\chi Q^+_\chi$, where $Q^+_\chi$ is the component of the decomposition belonging to $Q_\chi \subseteq Q^n_{r,d}$. Because the map $\pi \colon Q^{n+}_{r,d} \to (Q^n_{r,d})^{k^*}$ is a retraction, it follows that any singularity type which occurs on $(Q^n_{r,d})^{k^*} = (Q^n_{r,d})^T$ also occurs, up to retraction, on $Q^{n+}_{r,d}$. It would be nice if we could use this to show that these singularities also occur on $Q^n_{r,d}$. This would in particular be the case if the map $\theta_0 \colon Q^{n+}_{r,d} \to Q^n_{r,d}$ is an open embedding. This is the reason why we will focus next on determining for which points the map $\theta_0$ will locally be an open embedding: we hope that for every singularity type, we can find some point $[E]$ with this singularity on some $(Q^n_{r,d})^T$, such that $\theta_0$ is locally around $i([E])$ an open embedding.

### 2.4.2 Trivial negative tangents on the Quot-scheme

Let $[E]$ be a point in $(Q^n_{r,d})^T = (Q^n_{r,d})^{k^*}$, corresponding to a submodule $E$ of $k[x_1, \ldots, x_d]^r$. It holds that $\theta_0$ is an open embedding around $i([E]) \in Q^{n+}_{r,d}$, if and only if the tangent space to $i([E]) \in Q^{n+}_{r,d}$ and the tangent space to $[E] \in Q^n_{r,d}$ coincide. It turns out that the tangent space to $[E] \in Q^n_{r,d}$ is given by

$$\mathrm{Hom}_{k[x_1,\ldots,x_d]}(E, k[x_1, \ldots, x_d]^r/E).$$

This tangent space inherits a $\mathbb{Z}^d$-grading from the $\mathbb{Z}^d$-grading on $E$ and $k[x_1, \ldots, x_d]$. An element $f \colon E \to k[x_1, \ldots, x_d]^r/E$ of the tangent space is homogeneous of degree $\mathbf{a} \in \mathbb{Z}^d$ if, for any $\mathbf{b} \in \mathbb{Z}^d$, it maps $E_\mathbf{b}$ into the set of homogeneous elements in $k[x_1, \ldots, x_d]^r/E$ of degree $\mathbf{a} + \mathbf{b}$. Stated differently, a homogeneous element of degree $\mathbf{a}$ of the tangent space, is a morphism which adds $\mathbf{a}$ to the degree. In particular, the maps with degree $0 \in \mathbb{Z}^d$ are exactly those maps which preserve the grading, so these are the morphisms of graded modules.

Now consider the 1-dimensional torus $k^*$ again, which embeds into $T$ by sending $t \mapsto (t^{e_1}, \dots, t^{e_d})$. This torus induces a $\mathbb{Z}$-grading on $k[x_1, \dots, x_d]$ as follows: any element of $k[x_1, \dots, x_d]$ which is homogeneous of $\mathbb{Z}^d$-degree $\mathbf{a} \in \mathbb{Z}^d$, gets $\mathbb{Z}$-degree equal to $-e_1a_1 - e_2a_2 - \cdots - e_da_d \in \mathbb{Z}$. The reason for this grading is that a $t \in k^*$ acts on the polynomial ring by sending $x^{\mathbf{a}} \mapsto t^{-e_1a_1-\cdots-e_da_d}x^{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^d$. This $\mathbb{Z}$-grading on $E$ also yields a $\mathbb{Z}$-grading on the tangent space $\mathrm{Hom}_{k[x_1,\dots,x_d]}(E, k[x_1, \dots, x_d]^r/E)$.

The $\mathbb{Z}$-grading we just defined can be used to describe the tangent spaces to $[E]$ of both $(Q^n_{r,d})^{k^*}$ and $Q^{n+}_{r,d}$. The tangent space of $(Q^n_{r,d})^{k^*}$ is exactly the degree 0 part of the full tangent space of $Q^n_{r,d}$, so it is

$$\mathrm{Hom}_{k[x_1,\dots,x_d]}(E, k[x_1, \dots, x_d]^r/E)_0.$$

The tangent space of $Q^{n+}_{r,d}$ in $i([E])$ consists of those elements of non-negative degree:

$$\mathrm{Hom}_{k[x_1,\dots,x_d]}(E, k[x_1, \dots, x_d]^r/E)_{\geq 0}.$$

As indicated before, the map $\theta_0\colon Q^{n+}_{r,d} \to Q^n_{r,d}$ is an open embedding around $i([E])$ if and only if this tangent space to $Q^{n+}_{r,d}$ in $i([E])$ is equal to the full tangent space to $Q^n_{r,d}$. Now we see that this is equivalent to requiring that this full tangent space has no elements of negative degree, that is:

$$\mathrm{Hom}_{k[x_1,\dots,x_d]}(E, k[x_1, \dots, x_d]^r/E)_{<0} = 0.$$

If this property is satisfied, we say that $Q^n_{r,d}$ has *trivial negative tangents* in $[E]$. In this case, the singularity type of $((Q^n_{r,d})^T, [E])$ is, up to retraction, the same as that of $(Q^n_{r,d}, [E])$. Unfortunately, the following proposition shows that this is not a feasible approach for finding singularities on $Q^n_{r,d}$.

**Proposition 2.4.1.** *Let $\chi \in \mathcal{X}_{r,d}$ be a characteristic function such that there exist two different $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}^d_{\geq 0}$ with $\mathbf{a} \leq \mathbf{a}'$ and $0 < \chi(\mathbf{a}) \leq \chi(\mathbf{a}') < r$. Let $[E] \in Q_\chi$ be a point. Let $k^*$ be a 1-dimensional subtorus which embeds into $T$ as $t \mapsto (t^{e_1}, \dots, t^{e_d})$ for certain $e_1, \dots, e_d \in \mathbb{Z}$, such that $(Q^n_{r,d})^T = (Q_{r,d})^{k^*}$.*

*Assume that the Białynicki-Birula decomposition $Q^{n+}_{r,d}$ of $Q^n_{r,d}$ exists. Now $Q^n_{r,d}$ does not have trivial negative tangents in $[E]$, and therefore $\theta_0\colon Q^{n+}_{r,d} \to Q^n_{r,d}$ is not an open embedding around $i([E])$.*

*Proof.* We will construct two nontrivial elements $f, g \in \mathrm{Hom}_{k[x_1,\dots,x_d]}(E, k[x_1, \dots, x_d]^r/E)$ with opposite $\mathbb{Z}^d$-degrees, it follows from this that they also have opposite $\mathbb{Z}$-degrees. Finally we also show that their degrees cannot be 0, and therefore either $f$ or $g$ has a negative degree.

Without loss of generality we may assume that $a_1 < a'_1$. Let $\mathbf{a}'' = (a_1 + 1, a_2, \dots, a_d)$, we have $\mathbf{a} \leq \mathbf{a}'' \leq \mathbf{a}'$, so also $\chi(\mathbf{a}'') < r$. Let $\mathbf{v}$ be a nonzero vector in the vector space $E_{\mathbf{a}}$, and let $\mathbf{v}'$ be a nonzero vector in $k^r \setminus E_{\mathbf{a}''}$. Let $h$ now be some linear map $k^r \to k^r$ which sends $\mathbf{v}$ to $\mathbf{v}'$.

We define $f\colon E \to k[x_1, \dots, x_d]^r/E$ by sending $\mathbf{u}x^{\mathbf{b}} \mapsto h(\mathbf{u})x^{\mathbf{b}} \cdot x_1$ for all $\mathbf{b} \in \mathbb{Z}^d_{\geq 0}$ and $\mathbf{u} \in E_{\mathbf{b}}$. This map $f$ has $\mathbb{Z}^d$-degree $(1, 0, \dots, 0)$, and therefore $\mathbb{Z}$-degree $-e_1$. Furthermore, it sends $\mathbf{v}x^{\mathbf{a}}$ to the class of $\mathbf{v}'x^{\mathbf{a}''}$ in $k[x_1, \dots, x_n]/E$, which is nonzero. Therefore $f$ is not the trivial map.

Next we define $g$. We do this by sending $\mathbf{u}x^{\mathbf{b}}$ to the class of $\mathbf{u}x^{(b_1-1, b_2, \dots b_d)}$ if $b_1 \geq 1$, and to 0 otherwise for all $\mathbf{b} \in \mathbb{Z}^d_{\geq 0}$ and $\mathbf{u} \in E_{\mathbf{b}}$. It is clear that this $g$ respects the action of $k[x_2, \dots, x_d]$, however, it is not immediately clear why it respects the action of $x_1$. To see that this is the case, notice that for all $e \in E$ we have $g(x_1 \cdot e) = [e] = 0 \in k[x_1, \dots, x_d]/E$, and it can also be seen from the definition of $g$ that $x_1 \cdot g(e)$ will also always be 0. So $g$ is also an element of $\mathrm{Hom}_{k[x_1, \dots, x_d]}(E, k[x_1, \dots, x_d]^r/E)$, and it has degree $-e_1$.

Finally we note that $e_1$ cannot be zero, since this would imply that the torus $k^*$ does not act at all on one of the coordinates of $\mathbb{A}^d_{\mathbb{Z}}$. This would be in contradiction with the assumption that $(Q^d_{n,r})^T = (Q^d_{n,r})^{k^*}$. □

Note that almost all $\chi \in \mathcal{X}_{r,d}$ satisfy the condition from the preceding proposition. For any $\chi$ which does not satisfy this condition, we have that $Q_\chi$ is equal to the full product $\prod_{\mathbf{a} \in \mathbb{Z}^d_{\geq 0}} \mathrm{Gr}(\chi(\mathbf{a}), r)$, and in particular it is smooth. So if $[E]$ is not a smooth point of $Q_\chi$, then $\chi$ has to satisfy the condition of the proposition.

We conclude that $\theta_0$ cannot be directly used to show the existence of singularities on $Q^d_{n,r}$. However, this does not imply that the more complicated method applied by Jelisiejew [29] cannot be used in this situation. Further research would be needed in this direction.

# Chapter 3

# On Classifying Continuous Constraint Satisfaction Problems

## 3.1 Introduction

In geometric packing, we are given a set of two-dimensional pieces, a container and a set of motions. The aim is to move the pieces into the container without overlap, and while respecting the given motions. Recently, Abrahamsen, Miltzow and Seiferth showed that many geometric packing problems are $\exists\mathbb{R}$-complete (FOCS 2020) [5]. Despite the fact that the arXiv version is roughly 100 pages long, the high-level approach follows the same principle as many other hardness reductions. First, they showed that a technical intermediate problem is hard and then they reduced from this technical problem. In their case, a specific continuous constraint satisfaction problem called ETR-INV serves as this intermediate $\exists\mathbb{R}$-complete problem. Specifically, ETR-INV contains essentially only addition constraints $(x + y = z)$ and inversion constraints $(x \cdot y = 1)$. In the second step, they showed how to encode addition and inversion using geometric objects. This enabled them to show in a unified framework that various geometric packing problems are $\exists\mathbb{R}$-complete, see Figure 3.2.

The inversion constraint is particularly handy as it was shown in various other works that it is particularly easy to encode geometrically [36, 20, 21, 2]. Curiously, Abrahamsen, Miltzow and Seiferth left arguably the most interesting case of packing convex polygonal objects into a square container open. The missing puzzle piece seemed to be a gadget to encode the inversion constraint for this case. They dedicate Section 8.4 on explaining the difficulties to find such a gadget and give the following comment.

*"Gadget wanted! Most interesting to us is whether there exists a gadget encoding $x \cdot y \leq 1$, using a polygonal container, convex polygonal pieces, and rotation. [...] Despite much effort, we could not find such a gadget."*

We take an alternative approach and engage in a systematic study of continuous constraint satisfaction problems in their own respect. The aim is to fully classify all continuous constraint satisfaction problems, by their computational complexity. Polynomial time, NP-

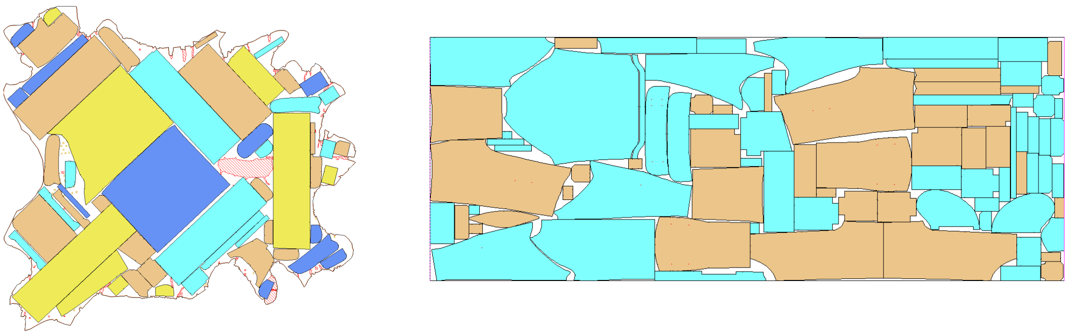


Figure 3.1: Real examples of packing a leather hide (left) and a piece of fabric (right), kindly provided by MIRISYS and produced by their software for automatic nesting, `https://www.mirisys.com/`.

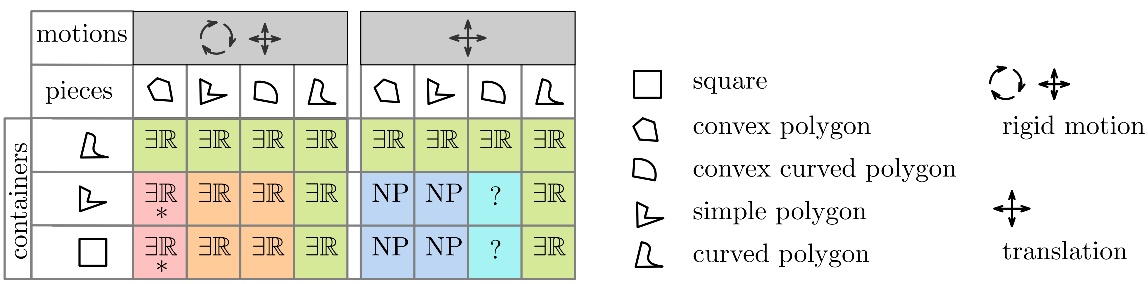



Figure 3.2: Various packing variants and their complexity status established in [5]. The results with an asterisk are new and complete the picture for packing variants that involve rotations. Image used with permission [5].

complete, and ∃ℝ-complete are some apparent complexities, but as we will see, they may not be the only ones that are relevant, see Section 3.1.4. Our first theorem arises as a combination of a small adaption of the framework by [5] and an application of our structural results.

**Theorem 3.1.1.** *Packing convex polygons into a square under rigid motions is ∃ℝ-complete.*

Interestingly, we employ our structural result only for one simple special case. Still, we would not know how to prove that case without establishing the theorem in its full generality. Before we explain our alternative approach, we give a short introduction to constraint satisfaction problems and the complexity class ∃ℝ. See Figure 3.2 for a comparison to previous ∃ℝ-completeness results for geometric packing.

### 3.1.1 Constraint Satisfaction Problems

Constraint satisfaction problems (CSPs) are a wide class of computational decision problems. In order to give a formal definition, we first introduce several other terms.

**Definition 3.1.2** (Signature)**.** A *signature* is a finite set of symbols together with *arities* $\ell \in \mathbb{N}$. Note that each symbol has exactly one arity attached to it. Often the signature distinguishes between function symbols and relational symbols. We will only use relational symbols.

Note that a signature is also called *template*, *constraint language*, or *vocabulary* in the literature.

**Definition 3.1.3** (Structure)**.** A structure consists of a set $U$, called the *domain*, a signature $\tau$ and an *interpretation* of each symbol. If $\alpha \in \tau$ is a symbol of arity $\ell$, then the interpretation is a set $\alpha \subseteq U^\ell$.

To make this more tangible, consider the following example. We define the domain as $U = \{0, 1\}$, the symbol $+_2$ of arity 3 and the symbol $\mathbf{1}$ of arity 1. We interpret $+_2$ as $\{(x, y, z) \in U^3 \mid x + y \equiv z \pmod{2}\}$, and $\mathbf{1}$ as $\{x \in U \mid x = 1\}$. This defines a structure $S_1 = \langle U = \{0, 1\}, +_2, \mathbf{1}\rangle$. Note that it is common to use a symbol and its interpretation interchangeably. Specifically, many symbols are used in the literature with their common interpretation, e.g., $\leq$ is interpreted as $\{(x, y) \in U \mid x \leq y\}$ and $+$ is interpreted as $\{(x, y, z) \in U^3 \mid x + y = z\}$. We refer to the symbols and interpretations of a structure merely as *constraints*. We will usually denote these constraints by the equation that they enforce. For example, we write $x^2 = y$ for the constraint $c = \{(x, y) \in U^2 \mid x^2 = y\}$.

**Definition 3.1.4** (Constraint satisfaction problem)**.** Given a structure $S = (U, \tau)$ we define a *constraint formula* $\Phi := \Phi(x_1, \ldots, x_n)$ to be a conjunction $c_1 \wedge \ldots \wedge c_m$ for $m \geq 0$, where each $c_i$ is of the form $c(y_1, \ldots, y_\ell)$ for some $c \in \tau$ and variables $y_1, \ldots, y_\ell \in \{x_1, \ldots, x_n\}$. We also define $V(\Phi) \subseteq U^n$ as $V(\Phi) := \{\mathbf{x} \in U^n \mid \Phi(\mathbf{x})\}$. In the *constraint satisfaction problem* (CSP) with structure $S$, we are given a constraint formula $\Phi$, and are asked whether $V(\Phi) \neq \emptyset$.

Consider the constraint formula $\Phi = (x_1 + x_2 \equiv x_4 \pmod{2}) \wedge (x_2 + x_3 \equiv x_4 \pmod{2}) \wedge (x_2 = 1)$ gives an instance of a CSP with structure $S_1$ as above. Note that $(0, 1, 0, 1) \in V(\Phi)$. It can be interesting whether the CSP with structure $S_1$ is polynomial time solvable.

In this thesis, we restrict ourselves to interval domains $U \subseteq \mathbb{R}$ and denote them as *continuous constraint satisfaction problem*s (CCSPs).

Constraint satisfaction problems (CSPs) have a long history in algorithmic studies [53, 14, 15, 59, 38, 22]. There are two application driven motivations to study them. On the one hand, it is possible to easily encode many fundamental algorithmic problems directly as a CSP. Then, given an efficient algorithm for those types of CSPs, we have immediately also solved those other algorithmic problems. On the other hand, if we can encode CSPs into algorithmic problems, then then any hardness result for the CSP immediately carries over to the algorithmic problem. Next to an application driven motivation, it is fair to say that they deserve a study in their own right as fundamental mathematical objects. CSPs are a very versatile language and often allow for a complete classification by their computational complexity. Specifically, the *tractability conjecture* states that every class of CSP with a finite domain is either NP-complete or polynomial time solvable. Schaefer showed the conjecture for domains of size two [53]. Recently Bulatov and Zhuk could confirm the conjecture independently [15, 59] for any finite domain. Note that one can also try to find a classification from the parametrized complexity perspective [38] or the approximation counting perspective [22].

In this thesis, we focus on CSPs with a interval domain $U \subset \mathbb{R}$ and we are interested in the class of CSPs that are $\exists\mathbb{R}$-complete. We want to point out that there is also a large body of research that deals with infinite domains [60, 58, 10, 12, 30]. Most relevant for us is the work by Bodirsky, Jonsson and von Oertzen [11], who also studied CSPs over the reals and showed that a host of them are NP-hard to decide. Specifically, they defined a subset $S$ of $\mathbb{R}^n$ as *essentially convex* if for all $a, b \in S$, the straight line segment intersects the complement $\overline{S}$ of $S$ in finitely many points. They show that CSPs that contain $x = 1$, $x > 0$, $x \leq y$, and at least one constraint that is *not* essentially convex are NP-hard. However, their techniques do not imply $\exists\mathbb{R}$-hardness. See also [13] for an overview of results for the real domain.

### 3.1.2 Existential Theory of the Reals

The class of the existential theory of the reals $\exists\mathbb{R}$ (pronounced as 'ER') is a complexity class which has gained a lot of interest, specifically within the computational geometry community. To define this class, we first consider the problem ETR, which also stands for Existential Theory of the Reals. In an instance of this problem we are given some sentence of the form

$$\exists x_1, \ldots, x_n \in \mathbb{R} : \Phi(x_1, \ldots, x_n),$$

where $\Phi$ is a well-formed quantifier-free formula consisting of the symbols $\{0, 1, x_1, \ldots, x_n, +, \cdot, \geq, >, \wedge, \vee, \neg\}$, the goal is to check whether this sentence is true. As an example of an ETR-instance, we could take $\Phi = x \cdot y^2 + x \geq 0 \wedge \neg(y < 2x)$. The goal of this instance would be to determine whether there exist real numbers $x$ and $y$ satisfying this formula. Now the class $\exists\mathbb{R}$ is the family of all problems that admit a polynomial-time many-one reduction to ETR. It is known that

$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}.$$

The first inclusion follows from the definition of $\exists\mathbb{R}$. Showing the second inclusion was first done by Canny in his seminal paper [16]. The CSP of the structure $\mathbf{R} = \langle \mathbb{R}, \cdot, +, \mathbf{1} \rangle$ is $\exists\mathbb{R}$-complete [39]. The reason that $\exists\mathbb{R}$ is an important complexity class is that a number of common problems in computational geometry have been shown to be complete for this class.

**Scope.** The main reason that the complexity class $\exists\mathbb{R}$ gained traction in recent years is the increasing number of important algorithmic problems that are $\exists\mathbb{R}$-complete. Marcus Schaefer established the current name and pointed out first that several known NP-hardness reductions actually implied $\exists\mathbb{R}$-completeness [48]. Note that some important reductions that establish $\exists\mathbb{R}$-completeness were done before the class was named.

Problems that have a continuous solution space and non-linear relation between partial solutions are natural candidates to be $\exists\mathbb{R}$-complete. Early examples are related to recognition of geometric structures: points in the plane [42, 56], geometric linkages [49, 1], segment graphs [33, 39], unit disk graphs [40, 31], ray intersection graphs [17], and point visibility graphs [18]. In general, the complexity class is more established in the graph drawing community [36, 20, 50, 23]. Yet, it is also relevant for studying polytopes [47, 21]. There is a series of papers related to Nash-Equilibria [7, 52, 25, 8, 9]. Another line of research studies matrix factorization problems [19, 54, 55, 51]. Other $\exists\mathbb{R}$-complete problems are the Art Gallery Problem [2] and training neural networks [3].

**Practical Implications.** It is maybe at first not entirely clear why we should care about $\exists\mathbb{R}$-completeness, when we know for most of those problems that they are NP-hard. The answer has different aspects. One reason is that we are intrinsically interested in establishing the true complexity of important algorithmic problems. Furthermore, $\exists\mathbb{R}$-completeness helps us to understand better the difficulties encountered when designing algorithms for those types of problems. While we have a myriad of techniques for NP-complete problems, most of these techniques are of limited use when we consider $\exists\mathbb{R}$-complete problems. The reason being that $\exists\mathbb{R}$-complete problems have an infinite set of possible solutions that are intertwined in a sophisticated way. Many researchers have hoped to discretize the solution space, but success was limited [27, 39]. The complexity class $\exists\mathbb{R}$ connects all of those different problems and tells us that we can either discretize all of them or none of them. To illustrate our lack of sufficient worst-case methods, note that we do not know the smallest square container to pack eleven unit squares, see Figure 3.3.

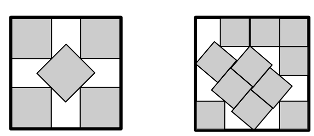


Figure 3.3: Left: Five unit squares into a minimum square container. Right: This is the best known packing of eleven unit squares into a square container [26].

**Technique.** In order to show $\exists\mathbb{R}$-completeness, usually two steps are involved. The first step is a reduction to a technical variant of ETR. The second step is a reduction from that variant to the problem at hand. Those ETR variants are typically CCSP's with only very limited types of constraints. It is common to have an addition constraint $(x + y = z)$, and a non-linear constraint, like one of the following:

$$z = x \cdot y, \quad z = x^2, \quad 1 = x \cdot y.$$

To find the right non-linear constraint is crucial for the second step, as it is often very difficult to encode non-linear constraints in geometric problems. Previous proof techniques relied on expressing multiplication indirectly using other operations. To be precise, we say that a constraint $c$ of arity $\ell$ has a *positive primitive definition* in structure $S$, if there is a constraint formula $\Phi$ in $S$ such that $c(y_1, \ldots, y_\ell)$ if and only if $\exists x_1, \ldots, x_k$ : $\Phi(y_1, \ldots, y_\ell, x_1, \ldots, x_k)$. In that case, $\Phi$ is called a *positive primitive formula*, or just *pp-formula*. For instance, we can express multiplication using squaring and addition as follows:

$$x \cdot y = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2.$$

This translates into a pp-formula as follows. $\exists A_0, A_1, A_2, B_0, B_1, B_2$ :

$$\begin{array}{rclcrclc}
A_0 &=& x + y, & \quad & x &=& y + B_0, & \\
A_0 &=& A_1 + A_1, & & B_0 &=& B_1 + B_1, & \quad A_2 = B_2 + z. \\
A_2 &=& A_1^2, & & B_2 &=& B_1^2, &
\end{array}$$

Given a pp-formula, we can reduce a CSP with constraint $c$ to a CSP with a different signature. Here, we replaced the ternary constraint $x \cdot y = z$ by the binary constraint $x^2 = y$.

Furthermore, there are often some range constraints of the form $x > 0$, $x \in [1/2, 2]$ or even $x \in [-\delta, \delta]$, for some $\delta = O(n^{-c})$, where $n$ is the number of variables. The above reduction becomes even more involved as the range constraints need to hold for every intermediate variable as well. They are important as we may only be able to encode variables in a certain limited range. Finally, it may be useful to know some structural properties of the variable constraint graph, like planarity. Those structural properties can often be imposed solely by self-reductions and repeated usage of the addition constraint.

Overall, those techniques have their limitations. As the reductions rely on an explicit way to express one non-linear constraint by another non-linear constraint and addition, we have to find those identities. To illustrate this, we encourage the reader to find a way to express multiplication using $x^2 + y^2 = 1$ and linear constraints. (See Section 3.4 for the solution.) This gets more tricky when dealing with inequality constraints. For instance, it is not clear how to express multiplication with $x \cdot y \geq 1$ and $x^2 + y^2 \geq 1$. We offer 10 euro to the first person, who is able to find a pp-formula to do so. Note that our theorems imply that those two inequalities together with linear constraints are enough to establish $\exists\mathbb{R}$-completeness, but we do not describe a pp-formula. We do not know whether the concave constraint $x \cdot y \leq 1$ is sufficient to establish $\exists\mathbb{R}$-completeness. At last, translating those identities into a reduction that respects the range constraints for every variable becomes very tedious and lengthy. Furthermore, it only establishes $\exists\mathbb{R}$-completeness for those specific constraints. See Abrahamsen and Miltzow [4] for some of those reductions.

We overcome this limitation by developing a new technique that establishes $\exists\mathbb{R}$-completeness for virtually any one non-linear equality constraint. We extend our results and show that any one convex and any one concave inequality constraint are also sufficient to establish $\exists\mathbb{R}$-completeness. See Section 3.1.3 for a formal description of our results and Section 3.1.5

for an overview of our techniques.

### 3.1.3 Results

We focus on the special case with essentially only one addition constraint and any one non-linear constraint. While this may seem like a strong limitation, note that addition constraints are commonly easy to encode. In most applications, the non-linear constraint is the crucial one.

**Definition 3.1.5** (Curved equality problem (CE))**.** We assume that we are given $\delta > 0$, a domain $[-\delta, \delta] \subseteq U$ and a function $f\colon U^2 \to \mathbb{R}$. Then we define the signature $C(f, \delta)$ as

$$C(f, \delta) = \{x + y = z,\ f(x, y) = 0,\ x \geq 0,\ x = \delta\}.$$

The *CE problem* is the CCSP given by $C(f, \delta)$. Furthermore, we are promised that $V(\Phi) \subseteq [-\delta, \delta]^n$.

By slight abuse of notation, we also allow that $\delta$ is given as part of the input. Note that although the problem is called curved equality problem, we make no assumptions on $f$ as part of the definition. We do this explicitly, as there are various technical ways to formulate those assumptions. Abrahamsen, Adamaszek and Miltzow [2, 4] essentially showed that CE is $\exists\mathbb{R}$-complete for $f = (x - 1)(y - 1) - 1$. Here, we generalize this to a wider set of functions $f$ defined as follows.

**Definition 3.1.6** (Well-behaved)**.** We say a function $f\colon U^2 \to \mathbb{R}$ is *well-behaved* around the origin if the following conditions are met.

- $f$ is a $C^2$-function, with $U \subseteq \mathbb{R}$ being a neighborhood of $(0, 0)$,
- $f(0, 0) = 0$, and all partial derivatives $f_x$, $f_y$, $f_{xx}$, $f_{xy}$ and $f_{yy}$ are rational, in $(0, 0)$.
- $f_x(0, 0) \neq 0$ or $f_y(0, 0) \neq 0$,
- $f(x, y)$ can be computed on a real RAM [24].

Note that if $p(x, y)$ is a polynomial of the form $\sum_{i,j} a_{i,j} x^i y^j$, then $p$ is well-behaved if and only if $a_{0,0} = 0$, $a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}$ are rational, and ($a_{1,0} \neq 0$ or $a_{0,1} \neq 0$).

**Definition 3.1.7** (Curved)**.** Let $f : U^2 \to \mathbb{R}$ be a function that is well-behaved around the origin. We write the curvature of $f$ at zero by

$$\kappa = \kappa(f) = \left( \frac{f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}} \right)(0, 0),$$

see Figure 3.4 for an illustration. We say $f$ is

- **curved** if $\kappa(f) \neq 0$,

- **convexly curved** if $\kappa(f) < 0$, and
- **concavely curved** if $\kappa(f) > 0$.

Note that we can define the simpler expression $\kappa' = \kappa'(f)$

$$\kappa'(f) = \left(f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}\right)(0,0),$$

and it holds that $\mathrm{sign}(\kappa) = \mathrm{sign}(\kappa')$. For this reason, we will work with $\kappa'$ instead of $\kappa$.

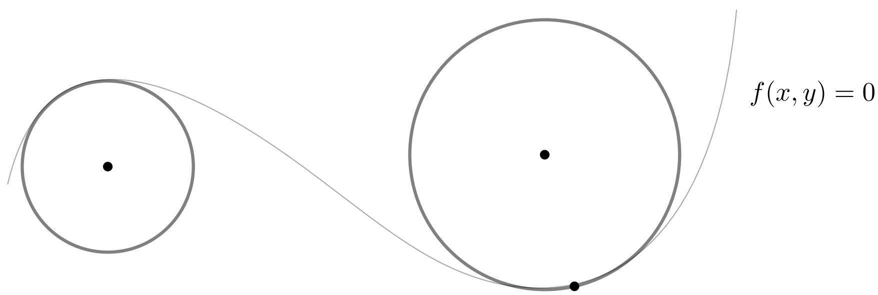



Figure 3.4: The expression $\kappa(f)$, has a geometric interpretation as the inverse of the radius of the tangent circle at $(0,0)$. The sign of $\kappa(f)$ tells us on which side of the curve $(f(x,y) = 0)$ the circle lies.

Consider a polynomial $p$ of the form $p(x,y) = \sum_{i,j} a_{ij} x^i y^j$. Then $\kappa'(p)$ equals

$$\kappa'(p) = a_{01}^2 a_{20} - a_{10} a_{01} a_{11} + a_{10}^2 a_{02}.$$

Now, we are ready to state our main theorem for equality constraints.

**Theorem 3.1.8.** *Let $f \colon U^2 \to \mathbb{R}$ be well-behaved and curved around the origin. Then CE is $\exists\mathbb{R}$-complete, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

Recall that we assume $\delta$ to be given as part of the input. Note that $\exists\mathbb{R}$-membership follows from the fact that $f$ is computable on the real RAM and the structural theorem by Erickson Hoog and Miltzow [24, Theorem 2.1]. It states that a problem is contained in $\exists\mathbb{R}$, if and only if there is an algorithm on the real RAM that can verify a given solution.

Unfortunately, we are only capable of encoding inequality constraints in geometric packing. Thus, in order to apply our techniques to geometric packing, we generalize Theorem 3.1.8 to inequality constraints. In the following we define the *convex concave inequality problem* (CCI), which is completely analogous to CE with one subtle difference. The constraint $f(x,y) = 0$ is replaced by $f(x,y) \geq 0$ and $g(x,y) \geq 0$. The curved condition is replaced by convexly curved and concavely curved conditions.

**Definition 3.1.9** (Convex concave inequality problem (CCI))**.** We assume that we are given $\delta > 0$, a domain $[-\delta, \delta] \subseteq U$ and functions $f, g : U^2 \to \mathbb{R}$. Then we define the signature $C(f, g, \delta)$ as

$$C(f,g,\delta) = \{x + y = z,\ f(x,y) \geq 0,\ g(x,y) \geq 0,\ x \geq 0,\ x = \delta\}.$$

The *CCI problem* is the CCSP given by $C(f, g, \delta)$. Furthermore, we are promised that $V(\Phi) \subseteq [-\delta, \delta]^n$.

Note that despite its name, we do not make any assumptions on $f, g$ in the problem definition of CCI. We do this explicitly, as there are various technical ways to formulate those assumptions. We will use different assumptions in our proofs culminating in the following theorem.

**Theorem 3.1.10.** *Let $f, g\colon U^2 \to \mathbb{R}$ be two well-behaved functions, one being convexly curved, and the other being concavely curved. Then the CCI problem is $\exists\mathbb{R}$-complete, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

In order to compare Theorem 3.1.8 and Theorem 3.1.10, consider the following two signatures for some given well-behaved and curved $f$:

$$C_1 = \{x + y = z,\ x \geq 0,\ x = \delta,\ f(x, y) = 0\},$$

and

$$C_2 = \{x + y = z,\ x \geq 0,\ x = \delta,\ f(x, y) \geq 0,\ -f(x, y) \geq 0\}.$$

Clearly, $C_2$ is more expressive than $C_1$. Therefore Theorem 3.1.8 implies a special case of Theorem 3.1.10. Namely, the case that $g = -f$. For general $f$ there is no further relation between the two theorems. Yet for the special case of $f = y - \bar{f}(x)$ it holds that Theorem 3.1.10 implies Theorem 3.1.8 as follows. We can encode each constraint of the form $f(x, y) = y - \bar{f}(x) \geq 0$ as follows: $f(x, z_1) = z_1 - \bar{f}(x) = 0$, $z_2 = y - z_1$, and $z_2 \geq 0$. Similarly, constraints of the form $-f(x, y) = \bar{f}(x) - y \geq 0$ can be encoded in $C_1$.

### 3.1.4 Discussion

Theorem 3.1.8 and Theorem 3.1.10 are a strong generalization of the $\exists\mathbb{R}$-completeness of ETR-INV. The problem ETR-INV played a central role both in $\exists\mathbb{R}$-completeness of the Art Gallery problem [2] and geometric packing [5]. One of the major obstacles of the $\exists\mathbb{R}$-completeness proofs of the Art Gallery problem was to find a way to encode inversion. If the authors had known Theorem 3.1.8 back then, it would have been sufficient to find essentially any well-behaved and curved constraint on two variables, which is much easier.

Let us now discuss further implications of our results and some potential future research directions.

- We want to point out that addition and convexly curved constraints alone seem not to be sufficient to establish $\exists\mathbb{R}$-completeness, as convex programs are *believed* to be polynomial time solvable. See [43] for an in-depth discussion.

- When we allow formulas in the first order theory of the reals, it is easier to describe arbitrary semi-algebraic sets, even when only allowing a single non-linear constraint. For example, if we allow a single convex constraint $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, then the following formula described the upper half of the boundary of the disk, given by $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \wedge y \geq 0\}$:

$$\varphi(x,y) = D(x,y) \wedge \forall_{z\in\mathbb{R}}(D(x,z) \Rightarrow z \leq y).$$

Using just the language of CCSPs, it is however impossible to encode such a set using only linear constraints and the constraint $D(x,y)$, as any CCSP instance of this form describes a convex set. Note in particular that we may not apply quantifier elimination to the given formula $\varphi$, since this is impossible without introducing non-linear constraints different from $D(x,y)$. For a more extensive analysis of the semi-algebraic sets which can be described using the first order theory of the reals when the set of constraints is restricted, we refer to [37, 45, 46].

- When we remove the convex constraint, but keep the concave constraint in Theorem 3.1.10 then we do not know if the problem is $\exists\mathbb{R}$-complete. It is easy though to establish NP-hardness in this case [11]. We consider the option that there is another complexity class `Concave` that characterizes such CCSPs. As with geometric packing with convex pieces, polygonal containers and translations grant the possibility to encode only linear and concave constraints. This problem is a natural candidate to be `Concave`-complete. We are curious if this intuition could be supported in some mathematically rigorous way.

- The constraint $x = \delta$ is required in order to establish the range promise. If we are aiming for a classification that is not interested in range promises, then we assume that this constraint can be replaced by $x = 1$. Note that without such a constraint, the origin is always a valid solution.

- If we remove the addition constraint, we are left only with constraints in at most two variables. This seems too weak to establish $\exists\mathbb{R}$-completeness, as setting $x$ determines $y$, up to finitely many options once we impose the constraint $f(x,y) = 0$. On the other hand, very large and irrational solutions can be enforced, which makes it unlikely for those CCSPs to be contained in NP. We wonder whether those CCSPs can be solved with an oracle to arithmetic circuits and non-determinism.

- Given the discussion above, it seems plausible that at least one ternary constraint is required to establish $\exists\mathbb{R}$-completeness. Therefore, the case of CCSPs with exactly one ternary constraint appears interesting to us. Note that the ternary multiplication constraint $x \cdot y = z$ can be transformed to the linear constraint $\log x + \log y = \log z$ [13]. Therefore it seems unlikely that the multiplication constraint by itself leads to $\exists\mathbb{R}$-completeness. (Note that taking $\exp(x)$ is an analytic function that is generally not supported on the real RAM [24].) It is plausible that this trick or similar tricks can only be applied to exceptional ternary constraints. Therefore, it may be common that one well-behaved and curved ternary equality constraint is sufficient to imply $\exists\mathbb{R}$-completeness.

- We want to point out that we do not consider arbitrary constraints. Otherwise, we could easily include constraints which enforce that variables are integers, thereby allowing us to encode arbitrary Diophantine equations. This would make the problem undecidable. As a consequence any type of classification of continuous constraints must limit the set of allowed constraints in some way.

- We have in this thesis completely neglected the variable-constraint incidence graph. Previous work showed that this graph can be restricted, by self-reduction and a clever application of the addition constraint [20, 36]. We are curious if it is possible to classify hereditary graph classes for which CCI is $\exists\mathbb{R}$-complete.

- Although $x \geq 0$ may not be necessary to imply $\exists\mathbb{R}$-completeness, our proof heavily relies on it.

- Previous reductions of $\exists\mathbb{R}$-completeness usually also imply so-called universality results. They translate topological and algebraic phenomena from one type of CSP to another type. Our methods seem not to imply these types of universality results. Specifically, if $f$ is a complicated function that is not even a polynomial, it seems implausible that $f$ can be used to construct, say, $\sqrt{2}$.

### 3.1.5 Proof Overview for CE and CCI

The proof goes into several steps which we explain in the following.

**Ball Theorem.** One of the most important tools that we employ is a lemma from real algebraic geometry. It states that for every ETR-formula $\Phi$ there is a ball $B$ whose radius only depends on the description complexity $L$ of $\Phi$, such that the following property is satisfied: if $\Phi$ has at least one solution $x$ then there must be also a solution $y$ inside the ball $B$. This theorem tells us that solutions cannot get too large. To get an intuition, consider the system of equations $x_0 = 2, x_{i+1} = x_i^2$. Clearly, $x_n = 2^{2^n}$, which is double exponentially large. The ball theorem essentially states that we cannot get much larger numbers.

**Range.** To introduce range constraints is common practice and we inherit them from a previous work [4, 5]. We repeat here the argument, for the benefit of the reader. In order to restrict the range of every variable, we first note that the ball theorem already tells us that the range of each variable is limited by some number $r$. We construct $\varepsilon = \delta/r$ and replace every variable $x$ by $[\![\varepsilon x]\!] = \varepsilon \cdot x$ and consequently we need to adopt all constraints. For instance $x \cdot y = z$ becomes $[\![\varepsilon x]\!] \cdot [\![\varepsilon y]\!] = [\![\varepsilon z]\!]\, \varepsilon$. In this way, we can easily ensure that if there is a solution at all than there is at least one solution with all variables in the range $[-\delta, \delta]$.

We will make use of this re-scaling trick to place all variables in an even smaller range close to zero. As the behavior of $f$ and $g$ is better understood close to the origin. Specifi-

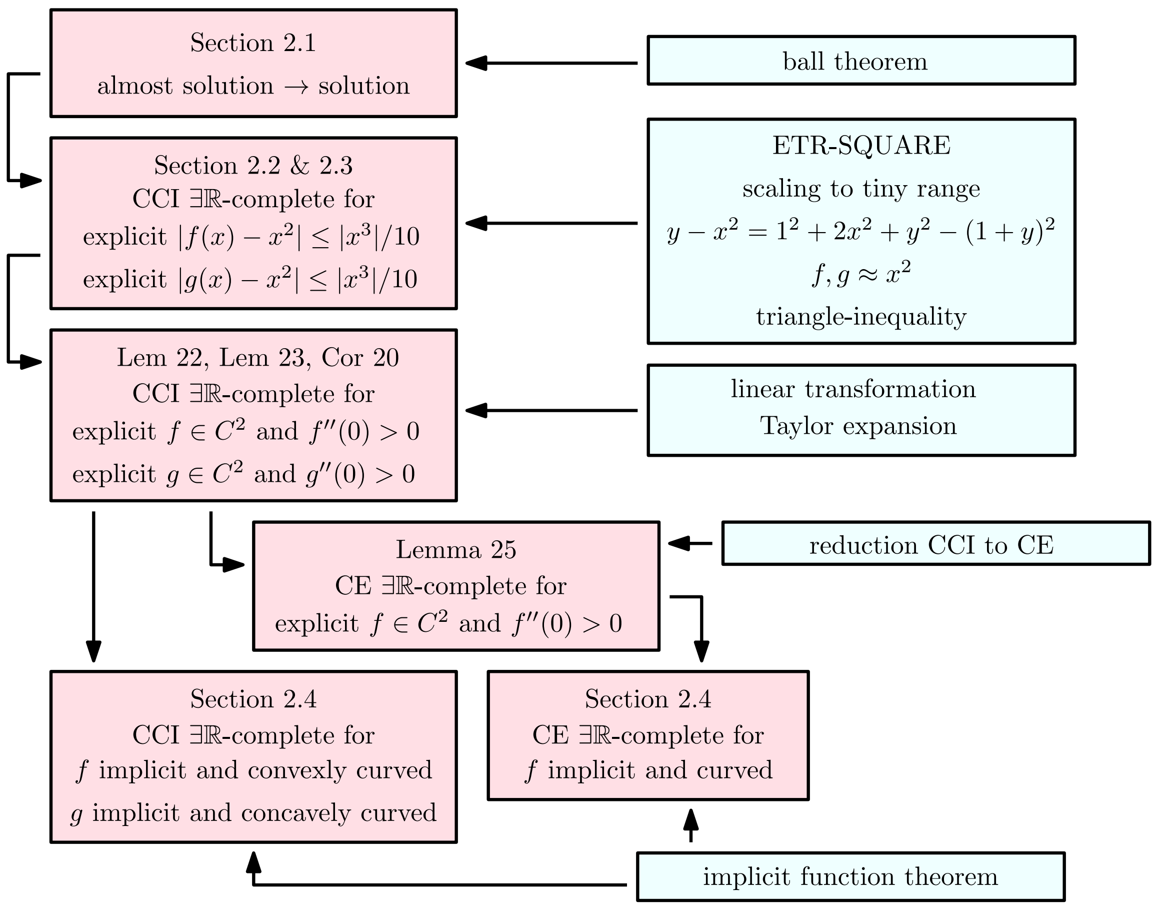



Figure 3.5: A formal overview of the different steps of the proof to Theorem 3.1.8 and Theorem 3.1.10.

cally, the error $|f(x) - x^2| \leq \varepsilon^3$ is small enough to pretend that $f$ behaves like a squaring function.

**Approximate Solution.** Using the ball theorem, we will establish that equality constraints of the form $p(x) = 0$ can be slightly weakened to $|p(x)| \leq \varepsilon$ for some sufficiently small $\varepsilon$. To get an intuition consider the following highly simplified cases.

Assume we have given a polynomial equation $p(x) = 0$, with $p \in \mathbb{Z}[X_1, \dots, X_n]$ and we are looking for a solution $x \in \mathbb{Z}^n$. Then in particular, we know that for all $x \in \mathbb{Z}^n$ that $p(x) \in \mathbb{Z}$. This readily implies that we can equivalently ask for some $x \in \mathbb{Z}^n$ that satisfies $|p(x)| \leq \frac{1}{2}$. Now, this is trivial for integers as integers have distance at least one to each other. But we can generalize the same principle also to rational and algebraic numbers.

Let $S = \{\frac{a_1}{b_1}, \dots, \frac{a_1}{b_1}\}$ be $n$ rational numbers with $|a_i|, |b_i| \leq L$. Thus it is easy to see that $q, r \in S$ have minimum distance $\frac{1}{L^2}$. This implies that if $|q - r| \leq \frac{1}{L^2}$, for some $q, r \in S$, we can infer that $q = r$. Again, this may seem almost trivial, but relies on the simple fact that rational numbers with bounded description complexity have a minimum distance to one another.

Lemma 3.2.5 generalizes the idea to algebraic numbers. Using the ball theorem, we will establish that algebraic numbers also have some minimum distance to one another, if we restrict their description complexity.

**ETR-SQUARE.** We use a theorem by Abrahamsen and Miltzow that shows that ETR-SQUARE is $\exists\mathbb{R}$-complete [4]. In this variant, we essentially have only addition $(x + y = z)$ and squaring constraints $(x^2 = y)$. Furthermore, the range of each variable is restricted to a small range around zero.

**Explicit.** Given those tools, we can show that we can replace a squaring constraint by explicit constraints $(f(x) = y)$. We start by only considering $f$ which satisfy

$$|f(x) - x^2| \leq \frac{1}{10}x^3. \tag{3.1}$$

The idea of the reduction from ETR-SQUARE is simple, but tedious. We can rewrite the constraint $x^2 = y$ as a linear combination of squares as follows

$$1^2 + 2x^2 + y^2 - (1 + y)^2 = 0.$$

Now, we can replace each square using the function $f$ to $f(1) + 2f(x) + f(y) - f(1 + y) = 0$ As $f$ is approximately squaring, this implies that we are approximately enforcing the constraint $x^2 = y$. In other words, we enforce $|x^2 - y| \leq \varepsilon$. Note that this is the technically most tedious step to make rigorous as we will later see. As we have discussed above it is sufficient to enforce each constraint approximately. The technical difficulty is many-fold. We need to work with scaled variables, instead of the original variables. Furthermore, we have to take into consideration that when we construct $\varepsilon$ that this also makes the formula longer. In particular this means that definition of $\varepsilon$ cannot depend on the newly constructed instance, but has to depend on the original instance.

Using linear transformations and Taylor expansion on $f$, we can replace Condition 3.1 relatively easily by Condition 3.2

$$f \text{ twice differentiable and, } f''(0) > 0 \tag{3.2}$$

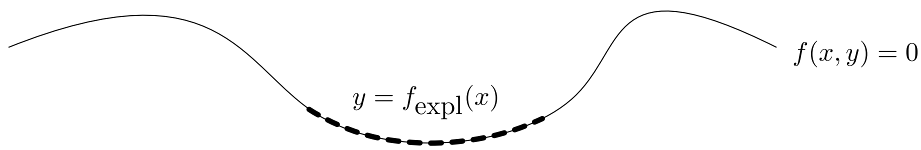



Figure 3.6: The implicit function theorem tells us that there is an function $f_{\text{expl}}$ such that the curve $y = f_{\text{expl}}(x)$ is locally identical to the curve $f(x, y) = 0$.

**Implicit.** We are now ready to handle the more general case of constraints in implicit form ($f(x, y) = 0$). The implicit function theorem tells us that there is a function $f_{\text{expl}}$ such that the curve $y = f_{\text{expl}}(x)$ is locally identical to the curve $f(x, y) = 0$, see Figure 3.6. The properties of the partial derivatives of $f$ translate to properties of the partial derivatives of $f_{\text{expl}}$. In this way, we can infer hardness of the CSP with constraint $f(x, y) = 0$ from the problem with constraint $y = f_{\text{expl}}(x)$.

**Computability of $f_{\text{expl}}$.** Interestingly, the fact that $f$ is computable does not readily imply that $f_{\text{expl}}$ is computable as well. Yet, we only require that we can test $f_{\text{expl}}(x) = y$, which is equivalent to $f(x, y) = 0$. And by definition we can compute $f(x, y)$.

**Inequalities.** The case of inequalities goes analogous to the equality case. We need one convexly curved and one concavely curved inequality. Whenever we want to upper bound an expression, we use one inequality and whenever we need to lower bound something, we use the other one. While on the surface this is not so difficult, it makes the reduction considerably more tedious. Specifically, it makes it harder to have an intuition on several technical steps and the meaning of several intermediate variables.

### 3.1.6 Proof Overview to Packing

We are following the framework by Abrahamsen et al. [5], with a few modifications. Instead of solely using the inversion constraint, we use, up to linear transformations, the following two constraints

$$x \cdot y \geq 1, \quad x^2 + y^2 \geq 1$$

Theorem 3.1.10 tells us that those two constraints are sufficient to establish $\exists\mathbb{R}$-completeness of geometric packing. While there is already a gadget for the first one, we needed to construct a new gadget for the second constraint.

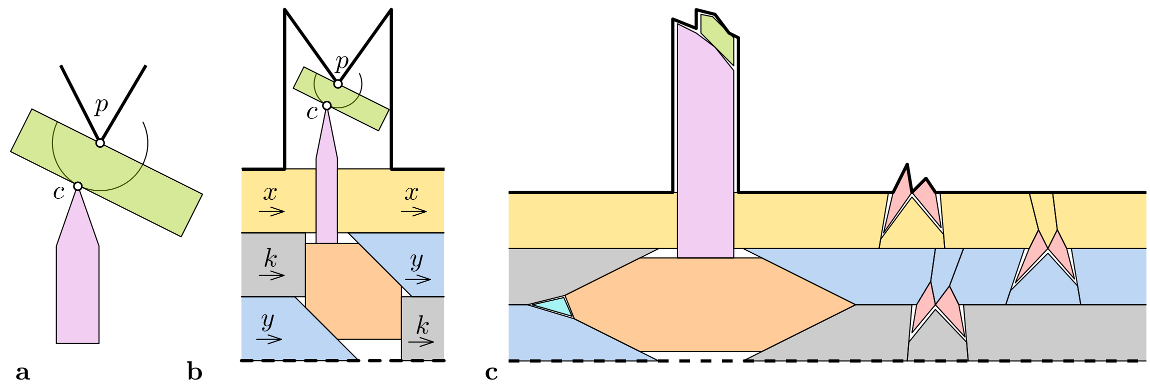



Figure 3.7: The idea of the gramophone construction. (For black and white printouts, see Figure 3.14 for color-codes.)

Consider Figure 3.7 for the following description. The core idea is to have a rectangle rotate around a pivot point $p$. Thus the tip $c$ of a second piece is constrained to lie on a circle around $p$, see Figure 3.7 **a**. To be precise $c$ has distance at least 1 from $p$. As a second step, we refine the construction such that we can read of the $x$ and $y$-coordinate of $c$ with the help of other pieces (yellow and blue), see Figure 3.7 **b**. In a third step, we need to modify this construction such that the slack is extremely low and every piece is fingerprinted. This also involves some modifications at other parts of the construction so that the fingerprinting is not interfering with the vertical edge-edge contact between the pink and yellow piece, see Figure 3.7 **c**.

## 3.2 Proof of CCSP-Theorems

In this section, we will prove Theorem 3.1.8 and Theorem 3.1.10.

### 3.2.1 Approximate Solutions

In this section, we show that if all constraints are "almost satisfied", then all constraints are also "exactly satisfied". We start by considering the following problem, which is a special case of the problem ETR-SMALL from [5, Definition 36]:

**Definition 3.2.1** (ETR-AMI-$\frac{1}{2}$)**.** We define the set of constraints $C_{\text{AMI}}$ as

$$C_{\text{AMI}} = \left\{x + y = z,\ x \cdot y = z,\ x \geq 0,\ x = \tfrac{1}{2}\right\}$$

Now we define the *ETR-AMI-*$\frac{1}{2}$ problem as the CCSP given by $C_{\text{AMI}}$. Furthermore, we are promised that $V(\Phi) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$.

We will not repeat the hardness proof of this problem here. Instead we refer to [5, Lemma D], where a full proof is given.

**Lemma 3.2.2** ([5])**.** *The problem ETR-AMI-$\frac{1}{2}$ is $\exists\mathbb{R}$-hard.*

From here, we can prove hardness of the following problem:

**Definition 3.2.3** (ETR-SQUARE-1)**.** We define the set of constraints $C_{\text{SQUARE}}$ as

$$C_{\text{SQUARE}} = \left\{x + y = z,\ y = x^2,\ x \geq 0,\ x = 1\right\}.$$

Now we define the *ETR-SQUARE-1* problem as the CCSP given by $C_{\text{SQUARE}}$. Furthermore, we are promised that $V(\Phi) \subseteq [-1, 1]$.

Before we start the proof of $\exists\mathbb{R}$-hardness of ETR-SQUARE-1, we make some remarks about the notation used in this and future proofs. In those proofs, newly introduced variables will often be denoted by using double square brackets, like this: $\llbracket x^2 \rrbracket$. In this notation, formally the whole expression including the brackets and the symbols within it should be understood as the name of the variable, without any special meaning. The symbols within the brackets will usually denote the value which is intuitively represented by the variable.

**Lemma 3.2.4.** *The problem ETR-SQUARE-1 is $\exists\mathbb{R}$-hard.*

*Proof.* We start with an ETR-AMI-$\frac{1}{2}$ instance $\Phi$. To this instance we add a variable $\llbracket 1 \rrbracket$ and a constraint $\llbracket 1 \rrbracket = 1$. Next we replace every constraint of the form $x = \frac{1}{2}$ by a constraint $x + x = \llbracket 1 \rrbracket$. Finally, for every constraint of the form $x \cdot y = z$, we introduce the following new variables:

$$\left\llbracket x^2 \right\rrbracket, \left\llbracket y^2 \right\rrbracket, \llbracket x + y \rrbracket, \left\llbracket (x+y)^2 \right\rrbracket, \left\llbracket x^2 + 2xy \right\rrbracket, \llbracket 2xy \rrbracket, \llbracket xy \rrbracket$$

and we add the following constraints:

$$\begin{aligned}
\left\llbracket x^2 \right\rrbracket &= x^2 \\
\left\llbracket y^2 \right\rrbracket &= y^2 \\
\llbracket x + y \rrbracket &= x + y \\
\left\llbracket (x+y)^2 \right\rrbracket &= \llbracket x + y \rrbracket^2 \\
\left\llbracket (x+y)^2 \right\rrbracket &= \left\llbracket x^2 + 2xy \right\rrbracket + \left\llbracket y^2 \right\rrbracket \\
\left\llbracket x^2 + 2xy \right\rrbracket &= \left\llbracket x^2 \right\rrbracket + \llbracket 2xy \rrbracket \\
\llbracket 2xy \rrbracket &= \llbracket xy \rrbracket + \llbracket xy \rrbracket \\
\llbracket xy \rrbracket &= z.
\end{aligned}$$

Every constraint of the form $x + y = z$ or $x \geq 0$ is not changed. After performing all these changes, which only needs linear time, we have an ETR-SQUARE-1 formula $\Psi$. Furthermore, it can be checked that every solution of this ETR-SQUARE-1 formula corresponds uniquely to a solution of the original ETR-AMI-$\frac{1}{2}$ formula. Also the fact that $V(\Phi) \subseteq [-1/2, 1/2]^n$ can be seen to imply that $V(\Psi) \subseteq [1, 1]^m$, where $m$ is the number of variables in $\Psi$. This proves that $\Psi$ is an ETR-SQUARE-1 instance which is equivalent to the original ETR-AMI-$\frac{1}{2}$ instance. Therefore the reduction is valid, and the problem ETR-SQUARE-1 is $\exists\mathbb{R}$-hard. $\square$

Next we will state the main lemma of this section. This lemma intuitively states the following: if an ETR-SQUARE-1 formula $\Phi$ has something which is "almost a solution", with an error of at most $2^{-2^{O(|\Phi|)}}$, then $\Phi$ also admits an actual solution.

**Lemma 3.2.5.** *Let $\Phi = \Phi(x_1, \ldots, x_n)$ be a ETR-SQUARE-1 formula such that $V(\Phi) \subseteq [-1, 1]^n$. Define $\Phi_\varepsilon$ as the formula where every constraint of the form $y = x^2$ is replaced by a constraint of the form $|y - x^2| \leq \varepsilon$, and where constraints $-1 \leq x \leq 1$ are added for every $x \in \{x_1, \ldots, x_n\}$.*

*Now there exists a constant $M \in \mathbb{Z}$ with $M = O(|\Phi|)$, such that if we take $\varepsilon = 2^{-2^M}$, then $V(\Phi_\varepsilon) \neq \emptyset$ if and only if $V(\Phi) \neq \emptyset$.*

The proof of this lemma requires a result from real algebraic geometry. The formulation here is taken from [5, Corollary 37], which is in turn based on a statement from [6].

**Corollary 3.2.6.** *If a bounded semi-algebraic set in $\mathbb{R}^n$ has complexity at most $L \geq 5n$, then all its points have distance at most $2^{2^{L+5}}$ from the origin.*

*Proof (of Lemma 3.2.5).* Consider the formula $\Psi$ which is obtained from $\Phi$ by adding constraints $-\delta \leq x \leq \delta$ for each $x \in \{x_1, \ldots, x_n\}$, replacing every constraint $y = x^2$ in $\Phi$ by a constraint $y + \eta_i = x^2$, where $\eta_i$ is a new variable (we add one such variable for each original squaring constraint), adding a new variable $u$, and finally adding a constraint

$$u(\eta_1^2 + \eta_2^2 + \cdots + \eta_r^2) = 1.$$

In this way, after choosing values for all original variables from $\Phi$, the new variable $u$ encodes the inverse of the total squared error in all squaring constraints. Now it can be seen that $V(\Psi)$ is bounded if and only if $V(\Phi) = \emptyset$. In the case where $V(\Psi)$ is bounded, we have by Corollary 3.2.6 the upper bound $u \leq 2^{2^{L+5}}$ where $L$ is the complexity of $\Psi$, which is linear in the complexity of $\Phi$.

We take $M$ to be an integer such that $r \cdot 2^{-2^{M+1}} < 2^{-2^{L+5}}$, note that this can be achieved while keeping $M$ bounded by a linear polynomial in the size of $\Phi$. Also let $\varepsilon = 2^{-2^M}$, so $r\varepsilon^2 < 2^{-2^{L+5}}$.

Suppose that $V(\Phi_\varepsilon) \neq \emptyset$, let $P \in V(\Phi_\varepsilon)$. Now we can obtain a point in $P' \in V(\Psi)$ which has the same values of each of the $x_i$'s as $P$, and which has the unique values of $\eta_i$ and $u$ that make all constraints hold. Using $P \in V(\Phi_\varepsilon)$, it can be seen that all values of the $\eta_i$ of $P'$ are at most $\varepsilon$. This implies that the value of $\eta_1^2 + \cdots + \eta_r^2$ is at most $r\varepsilon^2 < 2^{-2^{L+5}}$, and therefore the value of $u$ corresponding to $P'$ is larger than $2^{2^{L+5}}$. So $V(\Psi)$ contains a point with distance from the origin larger than $2^{2^{L+5}}$. This implies that $V(\Psi)$ is not bounded, and therefore $V(\Phi) \neq \emptyset$.

So $V(\Phi_\varepsilon) \neq \emptyset$ implies $V(\Phi) \neq \emptyset$. We also know that $V(\Phi) \neq \emptyset$ implies that $V(\Phi_\varepsilon) \neq \emptyset$, since $V(\Phi) \subseteq V(\Phi_\varepsilon)$. This completes the proof of the lemma. $\square$

### 3.2.2 Almost Square Explicit Equality Constraints

Using Lemma 3.2.5, we are able to prove that an explicit version CE is also $\exists\mathbb{R}$-complete, with some additional assumptions. Note that this subsection is technically not needed for the proof of Theorem 3.1.8 and Theorem 3.1.10. We will prove a similar lemma also for the inequality case. And the inequality case can be used to also prove the equality case. Yet, we believe that seeing the proof first for the equality case makes it much easier to read Section 3.2.4.

**Definition 3.2.7** (CE-EXPL)**.** Let $f\colon U \to \mathbb{R}$ be a function. Now we define the CE-EXPL problem to be the CE problem corresponding to the function $f^*\colon U^2 \to \mathbb{R}$ defined by $f^*(x,y) = y - f(x)$.

The goal of this section is to prove the following result:

**Lemma 3.2.8.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of $0$, and let $f\colon U \to \mathbb{R}$ be a function such that $|f(x) - x^2| \leq \frac{1}{10}|x|^3$ for all $x \in U \subseteq \mathbb{R}$. Now the problem CE-EXPL is $\exists\mathbb{R}$-complete, even if $\delta = O(n^{-c})$ for any constant $c > 0$.*

The reason that we impose these specific constraints on $f$, which enforce $f$ to be very similar to squaring, is that the proof will use $\exists\mathbb{R}$-hardness of a problem involving a squaring constraint. Furthermore, this specific case can be generalized to more general functions $f$.

*Proof.* Before giving the details of the construction, we will first give an overview of the used approach. The idea of this proof is to start with an instance of ETR-SQUARE-1, and convert this into a CE-EXPL instance by using $f$ to approximate squaring. In order to ensure that $f$ approximates squaring close enough, the whole instance is scaled by some small factor $\varepsilon$, so every variable $x$ is replaced by a variable representing $\varepsilon x$ instead.

The linear constraints and inequalities are easy to rewrite in terms of $\varepsilon x$, for example a constraint of the form $x + y = z$ can be rewritten to $\varepsilon x + \varepsilon y = \varepsilon z$.

Handling a squaring constraint $y = x^2$ is a bit more complicated. The first step is to rewrite this to a constraint involving $\varepsilon x$ and $\varepsilon y$, we get $\varepsilon \cdot \varepsilon y = (\varepsilon x)^2$. However, in the CE-EXPL problem there is no easy way to simulate the multiplication on the left-hand side of this equation. To solve this, we rewrite this equation to only use sums and differences of squares:

$$\varepsilon^2 + 2(\varepsilon x)^2 + (\varepsilon y)^2 - (\varepsilon + \varepsilon y)^2 = 0.$$

To simplify notation a bit, we will denote $t_1 = \varepsilon$, $t_2 = \varepsilon x$, $t_3 = \varepsilon y$ and $t_4 = \varepsilon + \varepsilon y$. Using this notation the equation becomes $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$. This is still not something we can directly enforce in a CE formula. However, by applying the function $f$, squaring can be approximated. Furthermore, Lemma 3.2.5 on a high level implies that such an approximation is enough to guarantee existence of a solution to the original equations. This is why in the CE formulation we enforce

$$f(t_1) + 2f(t_2) + f(t_3) - f(t_4) = O(\varepsilon^3). \tag{3.3}$$

Enforcing the $= O(\varepsilon^3)$ presents another problem: we cannot easily compute $\varepsilon^3$. To counter this, we instead bound the left-hand side of the equation in absolute value by $2(f(\varepsilon + f(\varepsilon)) - f(\varepsilon))$, which is approximately equal to $4\varepsilon^3$ (note that in the case that $f(x) = x^2$ for all $x$, this expression would actually be equal to $4\varepsilon^3 + 2\varepsilon^4$). The details of this reduction and a proof of its correctness will be worked out in the remainder of this proof.

**Reduction.** Let $\Phi$ be an ETR-SQUARE-1 formula. We pick some $\delta > 0$ such that $\delta < \frac{1}{4}$, and which furthermore satisfies the additional constraint $\delta = O(n^{-c})$ if required (how we choose $\delta$ does not matter, as long as is satisfies these constraints). We will now construct a CE formula $\Psi$ such that $V(\Phi) \neq \emptyset$ if and only if $V(\Psi) \neq \emptyset$.

Let $M$ be the constant obtained by applying Lemma 3.2.5 to $\Phi$, and let $L$ be the smallest positive integer such that $2^{-2^L} \leq \frac{1}{100} \cdot 2^{-2^M}$ and $2^{-2^L} \leq \frac{1}{100}$. We start by introducing variables $[\![\delta_i]\!]$ for $0 \leq i \leq L$. The variable $[\![\delta_0]\!]$ satisfies the constraint $[\![\delta_0]\!] = \delta$, and for each $1 \leq i \leq L$ we add a constraint

$$[\![\delta_i]\!] = f([\![\delta_{i-1}]\!]).$$

Denote $[\![\varepsilon]\!] = [\![\delta_L]\!]$. The idea behind these definitions is to simulate repeated squaring, as we will see later they force the value of $[\![\varepsilon]\!]$ to be in the interval $\left(0, 2^{-2^L}\right]$.

Next, we introduce a new variable $[\![\approx 2\varepsilon^3]\!]$ together with a (constant) number of constraints and auxiliary variables that enforce

$$\left[\!\!\left[\approx 2\varepsilon^3\right]\!\!\right] = f([\![\varepsilon]\!] + f([\![\varepsilon]\!])) - f([\![\varepsilon]\!]).$$

This can be done explicitly by introducing auxiliary variables $[\![f(\varepsilon)]\!]$, $[\![\varepsilon + f(\varepsilon)]\!]$ and $[\![f(\varepsilon + f(\varepsilon))]\!]$ and adding the following constraints:

$$\begin{aligned} [\![f(\varepsilon)]\!] &= f([\![\varepsilon]\!]) \\ [\![\varepsilon + f(\varepsilon)]\!] &= [\![\varepsilon]\!] + [\![f(\varepsilon)]\!] \\ [\![f(\varepsilon + f(\varepsilon))]\!] &= f([\![\varepsilon + f(\varepsilon)]\!]) \\ [\![f(\varepsilon + f(\varepsilon))]\!] &= \left[\!\!\left[\approx 2\varepsilon^3\right]\!\!\right] + [\![f(\varepsilon)]\!]\,. \end{aligned}$$

In the rest of this proof, and in future proofs of this thesis, we will not give such explicit constraints anymore. The variable $[\![\approx 2\varepsilon^3]\!]$ will be used to bound the error on the constraints replacing squaring constraints, as mentioned in the overview of this proof. Stated differently, it replaces the "$= O(\varepsilon^3)$" part of Equation (3.3).

Now, for each variable $x$ of $\Phi$, we add a variable $[\![\varepsilon x]\!]$ to $\Psi$, together with some constraints which enforce that $-[\![\varepsilon]\!] \leq [\![\varepsilon x]\!] \leq [\![\varepsilon]\!]$. Furthermore each constraint $x + y = z$ is replaced by $[\![\varepsilon x]\!] + [\![\varepsilon y]\!] = [\![\varepsilon z]\!]$, each constraint $x \geq 0$ is replaced by $[\![\varepsilon x]\!] \geq 0$ and each constraint $x = 1$ is replaced by $[\![\varepsilon x]\!] = [\![\varepsilon]\!]$.

For each constraint $y = x^2$, we build Equation (3.3) as in the overview. To do this, we first introduce variables $[\![t_1]\!]$, $[\![t_2]\!]$, $[\![t_3]\!]$ and $[\![t_4]\!]$ satisfying

$$\begin{aligned} [\![t_1]\!] &= [\![\varepsilon]\!] \\ [\![t_2]\!] &= [\![\varepsilon x]\!] \\ [\![t_3]\!] &= [\![\varepsilon y]\!] \\ [\![t_4]\!] &= [\![\varepsilon]\!] + [\![\varepsilon y]\!]\,. \end{aligned}$$

(Note that, even though $x$ and $y$ are suppressed in the notation here, the variables $[\![t_1]\!]$, $[\![t_2]\!]$, $[\![t_3]\!]$ and $[\![t_4]\!]$ should actually be distinct variables for each constraint $y = x^2$.) Next we introduce a new variable $[\![\eta_{x,y}]\!]$ representing the left-hand side of Equation (3.3):

$$[\![\eta_{x,y}]\!] = f([\![\varepsilon]\!]) + 2f([\![\varepsilon x]\!]) + f([\![\varepsilon y]\!]) - f([\![\varepsilon]\!] + [\![\varepsilon y]\!]).$$

The next step is to bound this variable, for this we add constraints which enforce

$$\begin{aligned} &[\![\eta_{x,y}]\!] \geq -2\left[\!\!\left[\approx 2\varepsilon^3\right]\!\!\right] \quad \text{and} \\ &[\![\eta_{x,y}]\!] \leq 2\left[\!\!\left[\approx 2\varepsilon^3\right]\!\!\right]. \end{aligned}$$

This completes the construction of $\Psi$. Note that in this construction, every constraint of $\Phi$ is replaced by a constant number of constraints in $\Psi$, and therefore we have $|\Psi| = O(|\Phi|)$. In particular this reduction can be executed in linear time.

**Calculations.** To show validity of the reduction, we first perform some side calculations. We define $\delta_0 = \delta$, for $1 \leq i \leq L$ we take $\delta_i = f(\delta_{i-1})$, and we take $\varepsilon = \delta_L$. We use the following facts:

$$|f(x) - x^2| \leq \frac{1}{10}|x|^3 \qquad \text{for } x \in [-\delta, \delta] \setminus \{0\} \tag{3.4}$$

$$0 < f(x) \leq 2x^2 \qquad \text{for } x \in [-\delta, \delta] \setminus \{0\} \tag{3.5}$$

$$\varepsilon \leq \frac{1}{100} \min\left(2^{-2^M}, \delta\right) \tag{3.6}$$

$$f(\varepsilon) < \varepsilon \tag{3.7}$$

$$f(\varepsilon + f(\varepsilon)) - f(\varepsilon) \in [\varepsilon^3, 3\varepsilon^3] \tag{3.8}$$

Inequality 3.4 is one of the assumptions, and is repeated here just for clarity. Combining this with $\delta \leq \frac{1}{4}$, Inequality 3.5 follows.

Using induction with the fact that $0 < f(x) \leq 2x^2$ for $x \in [-\delta, \delta] \setminus \{0\}$, it follows that $0 < \delta_i \leq 2^{-2^i - 1}$ for all $i$, so $0 < \varepsilon \leq \frac{1}{2} 2^{-2^L}$. Using the definition of $L$, we get Inequality 3.6. The fact $f(\varepsilon) < \varepsilon$ now follows from Inequalities 3.5 and 3.6.

For deriving Inequality 3.8, we first rewrite by adding and subtracting some terms, and applying the triangle inequality:

$$\begin{aligned} |f(\varepsilon + f(\varepsilon)) - f(\varepsilon) - 2\varepsilon^3| &= |f(\varepsilon + f(\varepsilon)) - (\varepsilon + f(\varepsilon))^2 + \varepsilon^2 - f(\varepsilon) \\ &\quad + (f(\varepsilon) + \varepsilon^2)(f(\varepsilon) - \varepsilon^2) + \varepsilon^4 + 2\varepsilon(f(\varepsilon) - \varepsilon^2)| \\ &\leq |f(\varepsilon + f(\varepsilon)) - (\varepsilon + f(\varepsilon))^2| + |\varepsilon^2 - f(\varepsilon)| \\ &\quad + (f(\varepsilon) + \varepsilon^2)|f(\varepsilon) - \varepsilon^2| + \varepsilon^4 + 2\varepsilon|f(\varepsilon) - \varepsilon^2|. \end{aligned}$$

To this we apply Inequalities 3.4, 3.6 and 3.7 to obtain the desired bound:

$$\begin{aligned}
|f(\varepsilon+f(\varepsilon))-f(\varepsilon)-2\varepsilon^3| &\leq |f(\varepsilon+f(\varepsilon))-(\varepsilon+f(\varepsilon))^2|+|\varepsilon^2-f(\varepsilon)| \\
&\quad + (f(\varepsilon)+\varepsilon^2)|f(\varepsilon)-\varepsilon^2|+\varepsilon^4+2\varepsilon|f(\varepsilon)-\varepsilon^2| \\
&\leq \frac{1}{10}(\varepsilon+f(\varepsilon))^3+\frac{1}{10}\varepsilon^3+(f(\varepsilon)+\varepsilon^2)\varepsilon^3+\varepsilon^4+\frac{1}{5}\varepsilon^4 \\
&\leq \frac{8}{10}\varepsilon^3+\frac{1}{10}\varepsilon^3+\frac{1}{10}\varepsilon^3 \\
&\leq \varepsilon^3,
\end{aligned}$$

so $f(\varepsilon+f(\varepsilon))-f(\varepsilon)\in[\varepsilon^3,3\varepsilon^3]$.

$V(\Phi)$ **nonempty implies** $V(\Psi)$ **nonempty.** Now we can start to prove the validity of the reduction. First suppose that $V(\Phi)\neq\emptyset$, so there is some $P\in V(\Phi)$. It needs to be shown that also $V(\Psi)\neq\emptyset$, to do this we construct a point $Q\in V(\Psi)$. For a variable $x$ of $\Phi$, we will use the notation $x(P)$ for the value of this variable for the point $P$. Similar notation is used for $Q$. To define $Q$, we take $[\![\varepsilon x]\!]\,(Q)=\varepsilon x(P)$ for all variables $x$ of $\Phi$, and we enforce that $Q$ satisfies all equality constraints of $\Psi$. This uniquely defines the value of $Q$ in all other variables of $\Psi$. In particular, we get that

$$\begin{aligned}
[\![\varepsilon]\!]\,(Q) &= \varepsilon \\
[\![\approx 2\varepsilon^3]\!]\,(Q) &= f(\varepsilon+f(\varepsilon))-f(\varepsilon) \\
[\![\eta_{x,y}]\!]\,(Q) &= f(\varepsilon)+2f(\varepsilon x(P))+f(\varepsilon y(P))-f(\varepsilon+\varepsilon y(P)),
\end{aligned}$$

where the last equality holds for all constraints $y=x^2$ in $\Phi$.

It is left to show that $Q$ also satisfies all inequalities of $\Psi$. There are three types of these inequalities. Firstly, we have inequalities which enforce $|\,[\![\varepsilon x]\!]\,(Q)|\leq[\![\varepsilon]\!]\,(Q)$. That these are satisfied for $Q$ follows from the fact that $|x(P)|\leq 1$ since $\Phi$ is a 1-ETR-SMALL-SQUARE formula. Secondly, for every inequality $x\geq 0$ in $\Phi$, we get a corresponding inequality $[\![\varepsilon x]\!]\geq 0$, that this is satisfied follows by combining $[\![\varepsilon x]\!]\,(Q)=\varepsilon x(P)$ and $x(P)\geq 0$.

Finally, for every constraint $y=x^2$ in $\Phi$ we get constraints enforcing $|\,[\![\eta_{x,y}]\!]\,|\leq 2\,[\![\approx 2\varepsilon^3]\!]$. To see that these are satisfied, first we shorten the notation a bit by writing $t_1=[\![t_1]\!]\,(Q)=\varepsilon$, $t_2=[\![t_2]\!]\,(Q)=\varepsilon x(P)$, $t_3=[\![t_3]\!]\,(Q)=\varepsilon y(P)$ and $t_4=[\![t_4]\!]\,(Q)=\varepsilon+\varepsilon y(P)$. Now $[\![\eta_{x,y}]\!]\,(Q)$ can be bounded, for this we first use the triangle inequality:

$$\begin{aligned}
|\,[\![\eta_{x,y}]\!]\,(Q)| &= |f(t_1)+2f(t_2)+f(t_3)-f(t_4)| \\
&= |f(t_1)-t_1^2+2(f(t_2)-t_2^2)+f(t_3)-t_3^2-(f(t_4)-t_4^2) \\
&\quad + t_1^2+2t_2^2+t_3^2-t_4^2| \\
&\leq |f(t_1)-t_1^2|+2|f(t_2)-t_2^2|+|f(t_3)-t_3^2|+|f(t_4)-t_4^2| \\
&\quad + |t_1^2+2t_2^2+t_3^2-t_4^2|
\end{aligned}$$

Note that $t_1, t_2, t_3$ and $t_4$ were chosen in such a way to ensure that, given $y(P) = x(P)^2$, we have $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$. Using this together with Inequality 3.4 and the facts that $t_1$, $t_2$ and $t_3$ are bounded in absolute value by $\varepsilon$, and $t_4$ is bounded in absolute value by $2\varepsilon$, we find

$$\begin{aligned} | \llbracket \eta_{x,y} \rrbracket (Q)| &\leq |f(t_1) - t_1^2| + 2|f(t_2) - t_2^2| + |f(t_3) - t_3^2| + |f(t_4) - t_4^2| \\ &\quad + |t_1^2 + 2t_2^2 + t_3^2 - t_4^2| \\ &\leq \frac{1}{10}t_1^3 + \frac{1}{5}t_2^3 + \frac{1}{10}t_3^3 + \frac{1}{10}t_4^3 + 0 \\ &\leq \frac{6}{5}\varepsilon^3. \end{aligned}$$

Finally using Inequality 3.8 we derive

$$\begin{aligned} | \llbracket \eta_{x,y} \rrbracket (Q)| &\leq \frac{6}{5}\varepsilon^3 \\ &\leq 2(f(\varepsilon + f(\varepsilon)) - f(\varepsilon)). \end{aligned}$$

This completes the proof that $Q \in V(\Psi)$, so $V(\Psi) \neq \emptyset$.

**$V(\Psi)$ nonempty implies $V(\Phi)$ nonempty.** Next, suppose that there is some $Q \in V(\Psi)$, now we want to show that $|x(Q)| \leq \delta$ for all variables $x$ in $\Psi$, and we want to prove that $V(\Phi) \neq \emptyset$. We start by bounding the coordinates. Note that $\llbracket \varepsilon \rrbracket (Q) = \varepsilon$, now for every variable $x$ of $\Phi$ it follows that $| \llbracket \varepsilon x \rrbracket (Q)| \leq | \llbracket \varepsilon \rrbracket (Q)| \leq \varepsilon$. Using this, it can be shown that also all values of the auxiliary variables except for the $\llbracket \delta_i \rrbracket$ are bounded by $100\varepsilon \leq \delta$. The values $\llbracket \delta_i \rrbracket (Q) = \delta_i$ are furthermore also all smaller than $\delta$, so this shows that $Q$ is contained in $[-\delta, \delta]^m$, where $m$ is the number of variables of $\Psi$.

Now we need to show that $V(\Phi) \neq \emptyset$. For this we use Lemma 3.2.5, we construct a point $P$ in $V(\Phi_{100\varepsilon})$, which implies $V(\Phi) \neq \emptyset$ since $100\varepsilon \leq 2^{-2^M}$. We define the point $P$ by taking $x(P) = \frac{\llbracket \varepsilon x \rrbracket (Q)}{\varepsilon}$ for all variables $x$ of $\Phi$. It immediately follows that $P$ satisfies all linear constraints and inequality constraints of $\Phi$, it is only left to check that it satisfies the constraints $|y - x^2| \leq 100\varepsilon$ of $\Phi_{100\varepsilon}$. To do this, first we observe that, using Inequality 3.8,

$$\begin{aligned} | \llbracket \eta_{x,y} \rrbracket (Q)| &\leq 2 \left\llbracket \approx 2\varepsilon^3 \right\rrbracket (Q) \\ &= 2(f(\varepsilon + f(\varepsilon)) - f(\varepsilon)) \\ &\leq 6\varepsilon^3. \end{aligned}$$

Now we will try to bound $|y(P) - x(P)^2|$. First we again write $t_1 = \llbracket t_1 \rrbracket (Q) = \varepsilon$, $t_2 = \llbracket t_2 \rrbracket (Q) = \varepsilon x(P)$, $t_3 = \llbracket t_3 \rrbracket (Q) = \varepsilon y(P)$ and $t_4 = \llbracket t_4 \rrbracket (Q) = \varepsilon + \varepsilon y(P)$. These choices were made such that

$$t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 2\varepsilon^2(x(P)^2 - y(P)),$$

so we see

$$2\varepsilon^2 |y(P) - x(P)^2| = \left| t_1^2 + 2t_2^2 + t_3^2 - t_4^2 \right|.$$

Next we apply the triangle inequality to get an expression to which we can apply Inequality 3.4 and the bound on $|\llbracket \eta_{x,y} \rrbracket (Q)|$:

$$\begin{aligned}
2\varepsilon^2|y(P) - x(P)^2| &= \left|t_1^2 + 2t_2^2 + t_3^2 - t_4^2\right| \\
&= |t_1^2 - f(t_1) + 2(t_2^2 - f(t_2)) + t_3^2 - f(t_3) - (t_4^2 - f(t_4)) \\
&\quad + f(t_1) + 2f(t_2) + f(t_3) - f(t_4)| \\
&\le |t_1^2 - f(t_1)| + 2|t_2^2 - f(t_2)| + |t_3^2 - f(t_3)| + |t_4^2 - f(t_4)| \\
&\quad + |f(t_1) + 2f(t_2) + f(t_3) - f(t_4)| \\
&= |t_1^2 - f(t_1)| + 2|t_2^2 - f(t_2)| + |t_3^2 - f(t_3)| + |t_4^2 - f(t_4)| \\
&\quad + |\llbracket \eta_{x,y} \rrbracket (Q)|
\end{aligned}$$

Applying Inequality 3.4 and the bound on $|\llbracket \eta_{x,y} \rrbracket (Q)|$ yields

$$\begin{aligned}
2\varepsilon^2|y(P) - x(P)^2| &\le |t_1^2 - f(t_1)| + 2|t_2^2 - f(t_2)| + |t_3^2 - f(t_3)| + |t_4^2 - f(t_4)| \\
&\quad + |\llbracket \eta_{x,y} \rrbracket (Q)| \\
&\le \frac{1}{10}t_1^3 + \frac{1}{5}t_2^3 + \frac{1}{10}t_3^3 + \frac{1}{10}t_4^3 + 6\varepsilon^3
\end{aligned}$$

Finally we use that $t_1$, $t_2$ and $t_3$ are bounded in absolute value by $\varepsilon$, and that $t_4$ is bounded by $2\varepsilon$:

$$\begin{aligned}
2\varepsilon^2|y(P) - x(P)^2| &\le \frac{1}{10}t_1^3 + \frac{1}{5}t_2^3 + \frac{1}{10}t_3^3 + \frac{1}{10}t_4^3 + 6\varepsilon^3 \\
&\le \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 + 6\varepsilon^3 \\
&< 200\varepsilon^3.
\end{aligned}$$

So $|y(P) - x(P)^2| \le 100\varepsilon$. This proves that $P \in V(\Phi_{100\varepsilon})$, and therefore $V(\Phi) \neq \emptyset$.

This finishes the proof of the validity of the reduction of ETR-SQUARE-1 to CE-EXPL. We conclude that for the given $f$, the problem CE-EXPL is $\exists\mathbb{R}$-hard. □

### 3.2.3 Almost Square Explicit Inequality Constraints

In this section, we will prove a number of hardness results about the explicit version of CCI. Before we can describe these results, we first need the following definition:

**Definition 3.2.9** (CCI-EXPL)**.** Let $f, g\colon U \to \mathbb{R}$ be two functions. Now we define the CCI-EXPL problem to be the CCI problem corresponding to the functions $f^*, g^*\colon U^2 \to \mathbb{R}$ defined by $f^*(x, y) = y - f(x)$ and $g^*(x, y) = g(x) - y$.

We will prove that CCI-EXPL is $\exists\mathbb{R}$-hard in a large number cases. In particular, we prove the following:

**Corollary 3.2.10.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of* $0$*, and let $f, g\colon U \to \mathbb{R}$ be functions which are 2 times differentiable such that $f(0) = g(0) = 0$ and $f'(0), f''(0), g'(0), g''(0) \in \mathbb{Q}$ with $f''(0), g''(0) > 0$. Now the problem CCI-EXPL is $\exists\mathbb{R}$-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

This corollary will be an important step towards proving Theorem 3.1.8. Before we can prove this corollary, we first prove another result which is similar to Lemma 3.2.8.

**Lemma 3.2.11.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of 0, and let $f, g\colon U \to \mathbb{R}$ be functions such that $|f(x) - x^2| \leq \frac{1}{10}|x|^3$ and $|g(x) - x^2| \leq \frac{1}{10}|x|^3$ for all $x \in U$. In this setting, the problem CCI-EXPL is $\exists\mathbb{R}$-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* The idea is to use almost the same construction as in Lemma 3.2.8, so we recommend the reader to first read the proof to this lemma. The the first main difference is that some extra care needs to be taken when making the constraints for the $[\![\delta_i]\!]$ variables. Also the squaring constraints need to be handled in a slightly different way. In order to do this, we replace the variables $[\![\eta_{x,y}]\!]$ by two new variables $[\![\eta_{x,y}^{\text{low}}]\!]$ and $[\![\eta_{x,y}^{\text{high}}]\!]$, which impose a lower bound, respectively upper bound, on the value of $t_1^2 + 2t_2^2 + t_3^2 - t_4^2$. Here $t_1 = \varepsilon$, $t_2 = \varepsilon x$, $t_3 = \varepsilon y$ and $t_4 = \varepsilon + \varepsilon y$, as before.

**Reduction.** Let $\Phi$ be an ETR-SQUARE-1 formula. We again pick any $\delta$ such that $\delta < \frac{1}{4}$ and which satisfies any imposed requirement of the form $\delta = O(n^{-c})$. We will construct a CCI-EXPL formula $\Psi$ such that $V(\Phi) \neq \emptyset$ if and only if $V(\Psi) \neq \emptyset$. Let $M$ be the constant obtained by applying Lemma 3.2.5 to $\Phi$, and let $L$ be a constant such that $2^{-2^L} \leq \frac{1}{100} \cdot 2^{-2^M}$ and $2^{-2^L} \leq \frac{1}{100}$, just like in the proof of Lemma 3.2.8.

We again introduce $[\![\delta_i]\!]$ for $0 \leq i \leq L$, where the variable $[\![\delta_0]\!]$ should satisfy the constraint $[\![\delta_0]\!] = \delta$. For each $1 \leq i \leq L$ we now add constraints enforcing

$$\frac{1}{2} f([\![\delta_{i-1}]\!]) \leq [\![\delta_i]\!] \leq g([\![\delta_{i-1}]\!]).$$

Denote $[\![\varepsilon]\!] = [\![\delta_L]\!]$. The constraints $[\![\delta_i]\!] \leq g([\![\delta_{i-1}]\!])$ are there to enforce that $[\![\varepsilon]\!] \leq 2^{-2^L}$, and the constraints $\frac{1}{2} f([\![\delta_{i-1}]\!]) \leq [\![\delta_i]\!]$ are there to enforce that $\varepsilon > 0$.

We continue by defining variables $[\![\leq g(\varepsilon)]\!]$ and $[\![\lesssim 2\varepsilon^3]\!]$ using constraints

$$\begin{aligned}
&[\![\leq g(\varepsilon)]\!] \leq g([\![\varepsilon]\!]),\\
&[\![\leq g(\varepsilon)]\!] \geq 0,\\
&[\![\lesssim 2\varepsilon^3]\!] \leq g([\![\varepsilon]\!] + [\![\leq g(\varepsilon)]\!]) - f([\![\varepsilon]\!]),\\
&[\![\lesssim 2\varepsilon^3]\!] \geq 0.
\end{aligned}$$

This new variable $[\![\lesssim 2\varepsilon^3]\!]$ is a replacement for the variable $[\![\approx 2\varepsilon^3]\!]$ which occurred in the proof of Lemma 3.2.8. Later, we will show that $[\![\lesssim 2\varepsilon^3]\!]$ is upper bounded by $3\varepsilon^3$.

Next for each variable $x$ of $\Phi$, we add a variable $[\![\varepsilon x]\!]$ to $\Psi$, with constraints enforcing $-[\![\varepsilon]\!] \leq [\![\varepsilon x]\!] \leq [\![\varepsilon]\!]$. Constraints of type $x + y = z$, type $x \geq 0$ or type $x = 1$ are handled by replacing them by constraints $[\![\varepsilon x]\!] + [\![\varepsilon y]\!] = [\![\varepsilon z]\!]$, $[\![\varepsilon x]\!] \geq 0$ and $[\![\varepsilon x]\!] = [\![\varepsilon]\!]$, respectively.

For each constraint $y = x^2$, we introduce variables $[\![t_1]\!]$, $[\![t_2]\!]$, $[\![t_3]\!]$ and $[\![t_4]\!]$ with constraints

$$\begin{aligned}
[\![t_1]\!] &= [\![\varepsilon]\!]\,,\\
[\![t_2]\!] &= [\![\varepsilon x]\!]\,,\\
[\![t_3]\!] &= [\![\varepsilon y]\!]\,,\\
[\![t_4]\!] &= [\![\varepsilon]\!] + [\![\varepsilon y]\!]\,.
\end{aligned}$$

Next we introduce two new variables: $[\![\eta^{\text{low}}_{x,y}]\!]$ and $[\![\eta^{\text{high}}_{x,y}]\!]$, together with constraints enforcing

$$\begin{aligned}
[\![\eta^{\text{low}}_{x,y}]\!] &\leq g([\![t_1]\!]) + 2g([\![t_2]\!]) + g([\![t_3]\!]) - f([\![t_4]\!]),\\
[\![\eta^{\text{high}}_{x,y}]\!] &\geq f([\![t_1]\!]) + 2f([\![t_2]\!]) + f([\![t_3]\!]) - g([\![t_4]\!]),\\
[\![\eta^{\text{low}}_{x,y}]\!] &\geq -2\,[\![\lesssim 2\varepsilon^3]\!]\,,\\
[\![\eta^{\text{high}}_{x,y}]\!] &\leq 2\,[\![\lesssim 2\varepsilon^3]\!]\,.
\end{aligned}$$

Note that the variables $[\![\eta^{\text{low}}_{x,y}]\!]$ and $[\![\eta^{\text{high}}_{x,y}]\!]$ are not completely necessary, and that it is also possible to use direct constraints

$$\begin{aligned}
-2\,[\![\lesssim 2\varepsilon^3]\!] &\leq g([\![t_1]\!]) + 2g([\![t_2]\!]) + g([\![t_3]\!]) - f([\![t_4]\!]),\\
2\,[\![\lesssim 2\varepsilon^3]\!] &\geq f([\![t_1]\!]) + 2f([\![t_2]\!]) + f([\![t_3]\!]) - g([\![t_4]\!])
\end{aligned}$$

instead. The two variables $[\![\eta^{\text{low}}_{x,y}]\!]$ and $[\![\eta^{\text{high}}_{x,y}]\!]$ are included since these slightly simplify the notation when proving correctness of this construction later on. This completes the construction of $\Psi$, which can be performed in linear time.

**Calculations.** Let $\varepsilon$ be any real number such that there exist reals $\delta_i$ for $0 \leq i \leq L$ satisfying $\delta_0 = \delta$, $\delta_L = \varepsilon$ and $\frac{1}{2}f(\delta_{i-1}) \leq \delta_i \leq g(\delta_{i-1})$ for all $1 \leq i \leq L$. Now we have the following facts (these facts hold in particular if $\varepsilon = [\![\varepsilon]\!]\,(Q)$ for some $Q \in V(\Psi)$):

$$\begin{aligned}
|f(x) - x^2| &\leq \frac{1}{10}|x|^3 && \text{for } x \in [-\delta, \delta], && (3.9)\\
|g(x) - x^2| &\leq \frac{1}{10}|x|^3 && \text{for } x \in [-\delta, \delta], && (3.10)\\
\frac{1}{2}f(x) &\leq g(x) && \text{for } x \in [-\delta, \delta], && (3.11)\\
\varepsilon &\leq \frac{1}{100}\min\left(2^{-2^M}, \delta\right), && && (3.12)\\
f(\varepsilon) < \varepsilon,\quad & g(\varepsilon) < \varepsilon, && && (3.13)\\
[\![\lesssim 2\varepsilon^3]\!]\,(Q) &\leq 3\,[\![\varepsilon]\!]\,(Q)^3 && \text{for } Q \in V(\Psi), && (3.14)\\
g(\varepsilon + g(\varepsilon)) - f(\varepsilon) &\geq \varepsilon^3. && && (3.15)
\end{aligned}$$

Inequalities 3.9 and 3.10 are assumptions from the statement of the lemma. Inequality 3.11 follows from this, together with the fact that $|x|$ is bounded by $\frac{1}{4}$:

$$\frac{1}{2}f(x) \leq \frac{1}{2}x^2 + \frac{1}{20}|x|^3 \leq x^2 - \frac{1}{10}|x|^3 \leq g(x).$$

Inequality 3.12 can be derived in the same way as Inequality 3.6 from Lemma 3.2.8, and now Inequality 3.13 follows from this with Inequalities 3.9 and 3.10.

In order to derive Inequality 3.14, we use the definition of the variable $[\![\lesssim 2\varepsilon^3]\!]$ and apply Inequalities 3.9 and 3.10 to this (here we take $\varepsilon = [\![\varepsilon]\!](Q)$ to simplify the notation a bit):

$$\begin{aligned}[\![\lesssim 2\varepsilon^3]\!](Q) &\leq g(\varepsilon + [\![\leq g(\varepsilon)]\!](Q)) - f(\varepsilon)\\ &\leq (\varepsilon + [\![\leq g(\varepsilon)]\!](Q))^2 + \frac{1}{10}(\varepsilon + [\![\leq g(\varepsilon)]\!](Q))^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3.\end{aligned}$$

Combining this with the constraint $[\![\leq g(\varepsilon)]\!] \leq g([\![\varepsilon]\!])$ and Inequalities 3.10 and 3.13, we get

$$\begin{aligned}[\![\lesssim 2\varepsilon^3]\!](Q) &\leq (\varepsilon + [\![\leq g(\varepsilon)]\!](Q))^2 + \frac{1}{10}(\varepsilon + [\![\leq g(\varepsilon)]\!](Q))^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3\\ &\leq (\varepsilon + g(\varepsilon))^2 + \frac{1}{10}(\varepsilon + g(\varepsilon))^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3\\ &\leq \left(\varepsilon + \varepsilon^2 + \frac{1}{10}\varepsilon^3\right)^2 + \frac{1}{10}(\varepsilon + \varepsilon)^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3\\ &= \frac{29}{10}\varepsilon^3 + \frac{6}{5}\varepsilon^4 + \frac{1}{5}\varepsilon^5 + \frac{1}{100}\varepsilon^6.\end{aligned}$$

Combining this with $\varepsilon \leq \frac{1}{100}$ (which follows from Inequality 3.12) yields that $[\![\lesssim 2\varepsilon^3]\!](Q) \leq 3\varepsilon^3$, as we wanted.

Finally Inequality 3.15 follows from Inequalities 3.9, 3.10 and 3.13 in the following manner:

$$\begin{aligned}g(\varepsilon + g(\varepsilon)) - f(\varepsilon) &\geq (\varepsilon + g(\varepsilon))^2 - \frac{1}{10}(\varepsilon + g(\varepsilon))^3 - \varepsilon^2 - \frac{1}{10}\varepsilon^3\\ &\geq \left(\varepsilon + \varepsilon^2 - \frac{1}{10}\varepsilon^3\right)^2 - \frac{1}{10}(\varepsilon + \varepsilon)^3 - \varepsilon^2 - \frac{1}{10}\varepsilon^3\\ &= \frac{11}{10}\varepsilon^3 + \frac{4}{5}\varepsilon^4 - \frac{1}{5}\varepsilon^5 + \frac{1}{100}\varepsilon^6\\ &\geq \varepsilon^3.\end{aligned}$$

**$V(\Phi)$ nonempty implies $V(\Psi)$ nonempty.** Suppose that $V(\Phi) \neq \emptyset$, so there exists some $P \in V(\Phi)$. We want to show that also $V(\Psi) \neq \emptyset$, to do this we construct a point $Q \in V(\Psi)$. We start by taking $[\![\delta_0]\!](Q) = \delta$ and $[\![\delta_i]\!](Q) = g([\![\delta_{i-1}]\!])$ for all $1 \leq i \leq L$. By Inequality 3.11, this definition satisfies all constraints on the $[\![\delta_i]\!]$. Denote $\varepsilon = [\![\varepsilon]\!](Q) =$

$[\![\delta_L]\!](Q)$. Next we take $[\![\leq g(\varepsilon)]\!](Q) = g(\varepsilon)$ and $[\![\lesssim 2\varepsilon^3]\!](Q) = g(\varepsilon + g(\varepsilon)) - f(\varepsilon)$, so by Inequality 3.15 we know that $[\![\lesssim 2\varepsilon^3]\!](Q) \geq \varepsilon^3$.

For all variables $x$ of $\Phi$, we take $[\![\varepsilon x]\!](Q) = \varepsilon x(P)$. Since $V(P) \subseteq [-1,1]^n$, it follows that all inequalities of the form $-[\![\varepsilon]\!](Q) \leq [\![\varepsilon x]\!](Q) \leq [\![\varepsilon]\!](Q)$ are satisfied in this way. Also for every constraint from $\Phi$ of one of the forms $x + y = z$, $x \geq 0$ or $x = 1$, the corresponding constraint in $\Psi$ is clearly satisfied.

Next we consider a squaring constraint $y = x^2$ from $\Phi$, for each such constraint we take

$$\begin{aligned}
[\![t_1]\!](Q) &= \varepsilon,\\
[\![t_2]\!](Q) &= [\![\varepsilon x]\!](Q),\\
[\![t_3]\!](Q) &= [\![\varepsilon y]\!](Q),\\
[\![t_4]\!](Q) &= \varepsilon + [\![\varepsilon y]\!](Q),\\
\left[\!\left[\eta^{\text{low}}_{x,y}\right]\!\right](Q) &= g([\![t_1]\!](Q)) + 2g([\![t_2]\!](Q)) + g([\![t_3]\!](Q)) - f([\![t_4]\!](Q)),\\
\left[\!\left[\eta^{\text{high}}_{x,y}\right]\!\right](Q) &= f([\![t_1]\!](Q)) + 2f([\![t_2]\!](Q)) + f([\![t_3]\!](Q)) - g([\![t_4]\!](Q)).
\end{aligned}$$

Using these definitions, the only constraints for which we still need to check whether $Q$ satisfies them, are the constraints of the form $\left[\!\left[\eta^{\text{low}}_{x,y}\right]\!\right] \geq -2\,[\![\lesssim 2\varepsilon^3]\!]$ and $\left[\!\left[\eta^{\text{high}}_{x,y}\right]\!\right] \leq 2\,[\![\lesssim 2\varepsilon^3]\!]$.

We start by checking the first of these constraints. Denote $t_1 = [\![t_1]\!](Q)$, $t_2 = [\![t_2]\!](Q)$, $t_3 = [\![t_3]\!](Q)$ and $t_4 = [\![t_4]\!](Q)$. Now we can apply Inequalities 3.9 and 3.10 to the definition of $\left[\!\left[\eta^{\text{low}}_{x,y}\right]\!\right]$:

$$\begin{aligned}
\left[\!\left[\eta^{\text{low}}_{x,y}\right]\!\right](Q) &= g(t_1) + 2g(t_2) + g(t_3) - f(t_4)\\
&\geq t_1^2 - \frac{1}{10}|t_1|^3 + 2t_2^2 - \frac{1}{5}|t_2|^3 + t_3^2 - \frac{1}{10}|t_3|^3 - t_4^2 - \frac{1}{10}|t_4|^3\\
&= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 - \left(\frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3\right).
\end{aligned}$$

Since $t_1, \dots t_4$ were chosen such that $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = \varepsilon^2 x(P)^2 - \varepsilon^2 y(P)$, and by the fact that $y(P) = x(P)^2$, it follows that $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$. Furthermore, $t_1$, $t_2$ and $t_3$ are all bounded by $\varepsilon$ in absolute value, while $|t_4|$ is bounded by $2\varepsilon$. This yields

$$\begin{aligned}
\left[\!\left[\eta^{\text{low}}_{x,y}\right]\!\right](Q) &\geq t_1^2 + 2t_2^2 + t_3^2 - t_4^2 - \left(\frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3\right)\\
&\geq 0 - \left(\frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3\right)\\
&\geq -2\varepsilon^3.
\end{aligned}$$

Combining this with $[\![\lesssim 2\varepsilon^3]\!](Q) \geq \varepsilon^3$, we find $\left[\!\left[\eta^{\text{low}}_{x,y}\right]\!\right](Q) \geq -2\,[\![\lesssim 2\varepsilon^3]\!](Q)$.

Next we consider the constraint $\left[\!\left[\eta^{\text{high}}_{x,y}\right]\!\right] \leq 2\,[\![\lesssim 2\varepsilon^3]\!]$. We apply Inequalities 3.9 and 3.10

to the definition of $\llbracket \eta_{x,y}^{\text{high}} \rrbracket$:

$$\begin{aligned}\llbracket \eta_{x,y}^{\text{high}} \rrbracket (Q) &= f(t_1) + 2f(t_2) + f(t_3) - g(t_4) \\ &\leq t_1^2 + \frac{1}{10}|t_1|^3 + 2t_2^2 + \frac{1}{5}|t_2|^3 + t_3^2 + \frac{1}{10}|t_3|^3 - t_4^2 + \frac{1}{10}|t_4|^3 \\ &= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3.\end{aligned}$$

Here we can apply that $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$, and that $|t_1|$, $|t_2|$ and $|t_3|$ are bounded by $\varepsilon$ and $|t_4|$ is bounded by $2\varepsilon$ to get

$$\begin{aligned}\llbracket \eta_{x,y}^{\text{high}} \rrbracket (Q) &\leq t_1^2 + 2t_2^2 + t_3^2 - t_4^2 + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \\ &\leq 0 + \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 \\ &\leq 2\varepsilon^3.\end{aligned}$$

So we get that $\llbracket \eta_{x,y}^{\text{high}} \rrbracket (Q) \leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket (Q)$.

We conclude that $Q$ satisfies all constraints from $\Psi$, and therefore $Q \in V(\Psi)$. This proves that $V(\Psi) \neq \emptyset$.

$V(\Psi)$ **nonempty implies** $V(\Phi)$ **nonempty.** Next, let $Q \in V(\Psi)$. Just as in the proof of Lemma 3.2.8, we want to show that $|x(Q)| \leq \delta$ for all variables $x$ of $\Psi$, and we want to prove that $V(\Phi) \neq \emptyset$. Bounding the coordinates goes in exactly the same way as in Lemma 3.2.8.

To show that $V(\Phi) \neq \emptyset$, we again use Lemma 3.2.5, we construct a point $P$ in $V(\Phi_{100\varepsilon})$, where we again denote $\varepsilon = \llbracket \varepsilon \rrbracket (Q)$. This would imply $V(\Phi) \neq \emptyset$ since $100\varepsilon \leq 2^{-2^M}$. We take $x(P) = \frac{\llbracket \varepsilon x \rrbracket (Q)}{\varepsilon}$ for all variables $x$ of $\Phi$. Now $P$ satisfies all linear constraints and inequality constraints of $\Phi$, it only remains to be checked that it satisfies the constraints $|y - x^2| \leq 100\varepsilon$ of $\Phi_{100\varepsilon}$.

We start by proving that $x(P)^2 - y(P) \leq 100\varepsilon$. Denote $t_1 = \llbracket t_1 \rrbracket (Q) = \varepsilon$, $t_2 = \llbracket t_2 \rrbracket (Q) = \varepsilon x(P)$, $t_3 = \llbracket t_3 \rrbracket (Q) = \varepsilon y(P)$ and $t_4 = \llbracket t_4 \rrbracket (Q) = \varepsilon + \varepsilon y(P)$. We have that

$$t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 2\varepsilon^2(x(P)^2 - y(P)).$$

Note that from Inequalities 3.9 and 3.10 it also follows that $x^2 \leq f(x) + \frac{1}{10}|x|^3$ and $x^2 \geq g(x) - \frac{1}{10}|x|^3$ for all $x \in [-\delta, \delta]$. Using this, we find

$$\begin{aligned}2\varepsilon^2(x(P)^2 - y(P)) &= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 \\ &\leq f(t_1) + \frac{1}{10}|t_1|^3 + 2f(t_2) + \frac{1}{5}|t_2|^3 \\ &\quad + f(t_3) + \frac{1}{10}|t_3|^3 - g(t_4) + \frac{1}{10}|t_4|^3 \\ &= f(t_1) + 2f(t_2) + f(t_3) - g(t_4) \\ &\quad + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3.\end{aligned}$$

To bound this, we use the variable $\llbracket \eta^{\text{high}}_{x,y} \rrbracket$, and the observation that $t_1$, $t_2$ and $t_3$ are bounded in absolute value by $\varepsilon$, and $|t_4|$ is bounded by $2\varepsilon$:

$$\begin{aligned}
2\varepsilon^2(x(P)^2 - y(P)) &\leq f(t_1) + 2f(t_2) + f(t_3) - g(t_4) \\
&\quad + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \\
&\leq \llbracket \eta^{\text{high}}_{x,y} \rrbracket (Q) + \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 \\
&\leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket (Q) + 2\varepsilon^3.
\end{aligned}$$

Here we can apply Inequality 3.14 to find

$$2\varepsilon^2(x(P)^2 - y(P)) \leq 8\varepsilon^3 < 200\varepsilon^3.$$

This implies $x(P)^2 - y(P) \leq 100\varepsilon$, as we wanted.

The proof that $x(P)^2 - y(P) \geq -100\varepsilon$ works in a similar manner. Leaving out some intermediate steps, it looks as follows:

$$\begin{aligned}
2\varepsilon^2(x(P)^2 - y(P)) &= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 \\
&\geq g(t_1) + 2g(t_2) + g(t_3) - f(t_4) \\
&\quad - \left( \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \right) \\
&\geq \llbracket \eta^{\text{low}}_{x,y} \rrbracket (Q) - 2\varepsilon^3 \\
&\geq -8\varepsilon^3 > -200\varepsilon^3,
\end{aligned}$$

and therefore $x(P)^2 - y(P) \geq -100\varepsilon$. This implies that $P \in V(\Phi_{100\varepsilon})$, and therefore $V(\Phi) \neq \emptyset$.

This completes the proof of the validity of the reduction of ETR-SQUARE-1 to CCI-EXPL. So for $f$ and $g$ satisfying the conditions from the lemma, the problem CCI-EXPL is $\exists\mathbb{R}$-hard. $\square$

Now that hardness of this restricted version of CCI-EXPL is proven, this result can be generalized in small steps until finally Theorem 3.1.10 is proven.

Before we do this, we first note that in any CCI-EXPL formula, constraints of the form $x = q \cdot y$, where $x, y$ are variables and $q$ is a rational constant, can be enforced using a constant number of addition constraints and new variables. To illustrate this, we will discuss the case where $q \in [0, 1]$ here. Other cases can be handled in a similar manner. Assume that $q = a/b$ for a positive integer $b$ and an integer $0 \leq a \leq b$. Now we can introduce variables $\llbracket \frac{i}{b} y \rrbracket$ for all $0 \leq i \leq b$, which satisfy constraints

$$\begin{aligned} \left\llbracket \frac{0}{b} y \right\rrbracket &= \left\llbracket \frac{0}{b} y \right\rrbracket + \left\llbracket \frac{0}{b} y \right\rrbracket, \\ \left\llbracket \frac{i+1}{b} y \right\rrbracket &= \left\llbracket \frac{i}{b} y \right\rrbracket + \left\llbracket \frac{1}{b} y \right\rrbracket && \text{for } 0 \leq i < b, \\ \left\llbracket \frac{b}{b} y \right\rrbracket &= \left\llbracket \frac{0}{b} y \right\rrbracket + y, \\ \left\llbracket \frac{a}{b} y \right\rrbracket &= \left\llbracket \frac{0}{b} y \right\rrbracket + x. \end{aligned}$$

This exactly enforces that $x = \frac{a}{b} \cdot y$.

Now the next step when working towards Theorem 3.1.10, is to slightly relax the constraints on $f$ and $g$, by allowing the difference with squaring to be any $O(x^3)$ function, instead of just functions bounded by $\frac{1}{10}|x|^3$ in absolute value.

**Lemma 3.2.12.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of 0, and let $f, g \colon U \to \mathbb{R}$ be functions such that $f(x) = x^2 + O(x^3)$ and $g(x) = x^2 + O(x^3)$ as $x \to 0$. Now the problem CCI-EXPL is $\exists\mathbb{R}$-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* Let $c$ be a constant such that $|f(x) - x^2| \leq c|x|^3$ and $|g(x) - x^2| \leq c|x|^3$ for all $x \in U'$ where $U' \subseteq U$ is a neighborhood of 0. Now let $N$ be a positive integer larger than $10c$. This implies that for all $x \in U'$

$$\begin{aligned} |N^2 f(x/N) - x^2| &\leq \frac{1}{10}|x|^3 \text{ and} \\ |N^2 g(x/N) - x^2| &\leq \frac{1}{10}|x|^3. \end{aligned}$$

If we define $f^*$ and $g^*$ by $f^*(x) = N^2 f(x/N)$ and $g^*(x) = N^2 g(x/N)$, then using Lemma 3.2.11, we get that the problem CCI-EXPL is $\exists\mathbb{R}$-hard for $f^*$ and $g^*$. For the rest of the proof of this lemma, we will denote this specific CCI-EXPL version by CCI-EXPL$^*$.

We give a reduction from CCI-EXPL$^*$ to the CCI-EXPL version with $f$ and $g$. Let $(\delta, \Phi)$ be a CCI-EXPL$^*$ instance. Now for every variable $x$ in this instance, we add extra variables $\llbracket x/N \rrbracket$, $\llbracket x/N^2 \rrbracket$ and we add constraints enforcing

$$\begin{aligned} \left\llbracket \frac{x}{N} \right\rrbracket &= \frac{\llbracket x \rrbracket}{N}, \\ \left\llbracket \frac{x}{N^2} \right\rrbracket &= \frac{\llbracket x \rrbracket}{N^2}. \end{aligned}$$

Next we replace every constraint of the form $y \geq f^*(x)$ by a constraint $\llbracket y/N^2 \rrbracket \geq f(\llbracket x/N \rrbracket)$, and we replace every constraint of the form $y \leq g^*(x)$ by a constraint $\llbracket y/N^2 \rrbracket \leq g(\llbracket x/N \rrbracket)$. This results in a CCI formula which has a solution exactly when $\Phi$ has one, and furthermore for all possible solutions all variables can be seen to be bounded in absolute value by $\delta$. This whole reduction only takes linear time. $\square$

In the next lemma, we allow even more possible $f$ and $g$.

**Lemma 3.2.13.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of $0$, and let $f, g\colon U \to \mathbb{R}$ be functions such that $f(x) = ax + bx^2 + O(x^3)$ and $g(x) = cx + dx^2 + O(x^3)$ as $x \to 0$, where $a, b, c, d \in \mathbb{Q}$ and $b, d > 0$. Now the problem CCI-EXPL is $\exists\mathbb{R}$-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* We define $f^*$ and $g^*$ as $f*(x) = (f(x) - ax)/b$ and $g*(x) = (g(x) - cx)/d$. From the constraints on $f$ and $g$ it follows that $f^*(x) = x^2 + O(x^3)$ and $g^*(x) = x^2 + O(x^3)$. Therefore we can apply the previous lemma to these functions to find that the CCI-EXPL problem with $f^*$ and $g^*$ is $\exists\mathbb{R}$-hard. We will denote this problem by CCI-EXPL$^*$, and give a reduction from CCI-EXPL$^*$ to the CCI-EXPL problem with $f$ and $g$ as defined in the lemma statement.

Let $(\delta, \Phi)$ be any instance of CCI-EXPL$^*$. We denote

$$\delta' = (1 + |a| + |b| + |c| + |d|)\delta.$$

Now we build an instance $(\delta', \Psi)$ of CCI-EXPL in the following manner: We start by adding a variable $[\![\delta]\!]$ which is meant as a replacement for the $\delta$ in conditions of the form $x = \delta$ in $\Phi$. To introduce this variable, we introduce an auxiliary variable $[\![\delta']\!]$ and enforce the following constraints:

$$\begin{aligned}
&[\![\delta']\!] = \delta', \\
&[\![\delta']\!] = (1 + |a| + |b| + |c| + |d|)\, [\![\delta]\!]\,.
\end{aligned}$$

We also add every variable of $\Phi$ and all constraints of the form $x + y = z$ or $x \geq 0$ from $\Phi$ to $\Psi$, and for every constraint $x = \delta$ in $\Phi$ we add a constraint $x = [\![\delta]\!]$ to $\Psi$.

For every constraint $y \geq f^*(x)$ of $\Phi$ we introduce a new variable $[\![ax + by]\!]$ to $\Psi$ which we force to equal $ax + by$ using some linear constraints. Furthermore we add a constraint $[\![ax + by]\!] \geq f(x)$. For constraints of the form $y \leq g^*(x)$ we do something similar.

In this way, $(\delta', \Psi)$ is a valid CCI-EXPL instance since all values of the variables in any solutions can be seen to be bounded by $\delta'$ using the triangle inequality. Furthermore, the new instance $\Psi$ differs from $\Phi$ only by new auxiliary variables, and otherwise has exactly the same constraints on the original variables. Thus $V(\Psi)$ is non-empty if and only if $V(\Phi)$ is nonempty. $\square$

The next step is to notice that any function which is 2 times differentiable with nonzero second derivative satisfies the constraints from the previous lemma. This leads to the following result:

**Corollary 3.2.10.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of $0$, and let $f, g\colon U \to \mathbb{R}$ be functions which are 2 times differentiable such that $f(0) = g(0) = 0$ and $f'(0), f''(0), g'(0), g''(0) \in \mathbb{Q}$ with $f''(0), g''(0) > 0$. Now the problem CCI-EXPL is $\exists\mathbb{R}$-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* Using Taylor's theorem, we find that

$$f(x) = f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3) \text{ and}$$
$$g(x) = g'(0)x + \frac{g''(0)}{2}x^2 + O(x^3).$$

To this we can apply the previous lemma to find that CCI-EXPL is ∃ℝ-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$. □

### 3.2.4 Implicit Constraints

Using, Corollary 3.2.10 we will show Theorem 3.1.10 and Theorem 3.1.8 in this order. The next lemma is almost equivalent to Theorem 3.1.10. The only difference is that the conditions $f_y(0,0) > 0$ and $g_y(0,0) < 0$ are added.

**Lemma 3.2.14.** *Let $f, g \colon U^2 \to \mathbb{R}$ be two functions, with $f$ well-behaved and convexly curved, and $g$ well-behaved and concavely curved. Furthermore assume that their partial derivatives satisfy $f_y(0,0) > 0$ and $g_y(0,0) < 0$. Then the CCI problem is ∃ℝ-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* Using the implicit function theorem, we find that in a neighborhood $(U')^2 \subseteq U^2$ of $(0,0)$, the curve $f(x,y) = 0$ can also be given in an explicit form $y = f_{\text{expl}}(x)$, where $f_{\text{expl}}$ is some $C^2$-function $U' \to \mathbb{R}$. So for $(x,y) \in (U')^2$ we have $f(x,y) = 0$ if and only if $y = f_{\text{expl}}(x)$. Since $f_y(0,0) > 0$, it also follows that $f(x,y) \geq 0$ if and only if $y \geq f_{\text{expl}}(x)$. Furthermore, the implicit function theorem also states that the derivative of $f_{\text{expl}}$ is given by

$$f'_{\text{expl}}(x) = \frac{f_x(x, f'_{\text{expl}}(x))}{f_y(x, f'_{\text{expl}}(x))}.$$

From this it can be computed that the second derivative in 0 is

$$f''_{\text{expl}}(0) = -\left(\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{f_y^3}\right)(0,0).$$

Note that the fact that $f$ is convexly curved exactly implies that the numerator of this expression is a positive number. Using the assumptions from the lemma statement, we conclude that $f''_{\text{expl}}(0)$ is a positive rational number.

In a analogous way, we can write the condition $g(x,y) \geq 0$ in the form $y \leq g_{\text{expl}}(x)$ in some neighborhood of $(0,0)$, where $g_{\text{expl}}$ is a $C^2$-function with rational first and second derivative in 0, and with positive second derivative.

We conclude that the problem CCI is equivalent to the problem CCI-EXPL for $f_{\text{expl}}$ and $g_{\text{expl}}$, and by the Corollary 3.2.10 this problem is ∃ℝ-hard. □

From here it is a small step to prove the main result:

**Theorem 3.1.10.** *Let $f, g\colon U^2 \to \mathbb{R}$ be two well-behaved functions, one being convexly curved, and the other being concavely curved. Then the CCI problem is $\exists\mathbb{R}$-complete, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* Without loss of generality, we may assume that $f_y(0,0) \neq 0$ and $g_y(0,0) \neq 0$. In any other case, we can just interchange the variables in one of the functions.

In the case where $f_y(0,0) > 0$ and $g_y(0,0) < 0$, we can apply the previous lemma and we are done. For the case $f_y(0,0) < 0$ and $g_y(0,0) < 0$, we can provide a reduction from CCI with functions $f^*(x,y) = f(-x,-y)$ and $g^*(x,y) = g(x,y)$. For the case $f_y(0,0) > 0$ and $g_y(0,0) > 0$ we can make a reduction from CCI with functions $f^*(x,y) = f(x,y)$ and $g^*(x,y) = g(-x,-y)$. and for the case $f_y(0,0) < 0$ and $g_y(0,0) > 0$, we can provide a reduction from CCI with functions $f^*(x,y) = f(-x,-y)$ and $g^*(x,y) = g(-x,-y)$. Note that flipping the signs of the inputs of $f$ or $g$ does not influence any second partial derivative, while it does negate the first partial derivatives; therefore the mentioned starting points for the reductions can all be seen to satisfy the conditions from Lemma 3.2.14.

As an example we discuss the case $f_y(0,0) > 0$ and $g_y(0,0) > 0$. We want to give reduction from the problem CCI with functions $f^*(x,y) = f(-x,-y)$ and $g^*(x,y) = g(x,y)$; we denote this CCI variation by CCI$^*$. So suppose that $(\delta, \Phi)$ is a CCI$^*$ instance. Now we will construct a CCI instance $(\delta, \Psi)$ (with the $f$ and $g$ from the theorem statement). We add every variable of $\Phi$ to $\Psi$, and for every such variable $x$ we also add an extra variable $[\![-x]\!]$, together with a constraint enforcing $x + [\![-x]\!] = 0$. Furthermore, we copy every constraint from $\Phi$ to $\Psi$, except for constraints of the form $f^*(x,y) \geq 0$, these are replaced by $f([\![-x]\!], [\![-y]\!]) \geq 0$. This finishes the construction. □

As a final result in this section, we prove Theorem 3.1.8 as well. To do this, we start from Corollary 3.2.10 and convert this to a result about CE-EXPL.

**Lemma 3.2.15.** *Let $U \subseteq \mathbb{R}$ be a neighborhood of $0$, and let $f\colon U \to \mathbb{R}$ be a function which is 2 times differentiable such that $f(0) = 0$ and $f'(0), f''(0) \in \mathbb{Q}$ with $f''(0) \neq 0$. Now the problem CE-EXPL is $\exists\mathbb{R}$-hard.*

*Proof.* We apply Corollary 3.2.10 to the case where $g = f$ to find that in this case CCI-EXPL is $\exists\mathbb{R}$-hard. We can reduce this problem to CE-EXPL. Let $(\delta, \Phi)$ be a CCI-EXPL instance. Now we construct an CE-EXPL formula $\Psi$. We copy all constraints of the form $x + y = z$, $x \geq 0$ and $x = \delta$ from $\Phi$. For every constraint $y \geq f(x)$ we introduce two new variables $[\![f(x)]\!]$ and $[\![y - f(x)]\!]$, which we restrict by constraints

$$\begin{aligned} [\![f(x)]\!] &= f(x), \\ y &= [\![y - f(x)]\!] + [\![f(x)]\!]\,, \text{ and} \\ [\![y - f(x)]\!] &\geq 0. \end{aligned}$$

In a similar manner we replace every constraint $y \leq f(x)$ by introducing new variables

$[\![f(x)]\!]$ and $[\![f(x)-y]\!]$ and imposing the constraints

$$\begin{aligned}[\![f(x)]\!] &= f(x),\\ [\![f(x)]\!] &= [\![f(x)-y]\!] + y, \text{ and}\\ [\![f(x)-y]\!] &\geq 0.\end{aligned}$$

This completes the construction. It can easily be checked that every solution of $\Phi$ corresponds to a solution of $\Psi$, and vice versa. $\square$

**Theorem 3.1.8.** *Let $f\colon U^2 \to \mathbb{R}$ be well-behaved and curved around the origin. Then CE is $\exists\mathbb{R}$-complete, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* This proof is very similar to that of Lemma 3.2.14. Without loss of generality, we may assume that $f_y(0,0) \neq 0$, otherwise we can swap the variables. Using the implicit function theorem, we can write the condition $f(x,y) = 0$ in some neighborhood $(U')^2 \subseteq U^2$ of $(0,0)$ as $y = f_{\text{expl}}(x)$, where $f_{\text{expl}}$ is some $C^2$-function $U' \to \mathbb{R}$. Using the fact that the curvature of $f$ is nonzero, the implicit function theorem also tells us that $f''_{\text{expl}}(0) \neq 0$.

Since $f(x,y) = 0$ if and only if $y = f_{\text{expl}}(x)$, we find that the problem CE is equivalent to the problem CE-EXPL, which we know to be $\exists\mathbb{R}$-hard by the previous lemma. We conclude that also CE is $\exists\mathbb{R}$-hard. $\square$

## 3.3 Packing

In this section we will apply the result from the previous section to prove $\exists\mathbb{R}$-completeness of packing convex polygons into a square container under rotations and translations.

**Theorem 3.1.1.** *Packing convex polygons into a square under rigid motions is $\exists\mathbb{R}$-complete.*

$\exists\mathbb{R}$-membership has been proven in Section 1 of [5]; this follows fairly easily by applying a result from Erickson, Hoog and Miltzow [24].

To prove $\exists\mathbb{R}$-hardness, we will use the framework by Abrahamsen, Miltzow and Seiferth [5]. They used this framework to prove hardness of a broad number of packing problems, however, the case of packing convex polygons in a square container was left open. This is because of the difficulties of encoding the constraint $x \cdot y \leq 1$ in this specific setting. Using Theorem 3.1.10 on the classification of constraint satisfaction problems, it is not necessary anymore to find a gadget encoding exactly $x \cdot y \leq 1$. It becomes sufficient to find a gadget which encodes any concave constraint on two variables, allowing a much greater freedom in gadget-construction.

In Section 3.3.1, an overview of the framework from [5] is given. Some small changes to this framework are necessary to allow a reduction from CCI for suitable $f$ and $g$, instead of the problem ETR-INV which was used in [5]. These changes will be discussed in Section 3.3.2. The new gadget used in the construction will be discussed in Section 3.3.3.

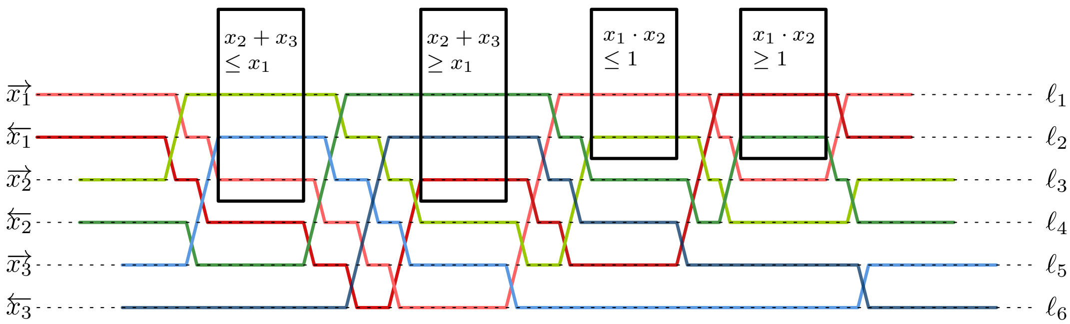



Figure 3.8: A wiring diagram corresponding to the formula $x_2 + x_3 = x_1 \wedge x_1 \cdot x_2 = 1$. This figure is taken with permission from [5].

### 3.3.1 Overview of Previous Work

**RANGE-ETR-INV.** The paper [5] gives a reduction from the $\exists\mathbb{R}$-complete problem RANGE-ETR-INV to the various packing problems; the hardness of RANGE-ETR-INV is proven in Section 5 in [5]. We repeat the definition of RANGE-ETR-INV here:

**Definition 3.3.1** (Definition 1 from [5])**.** An *ETR-INV formula* $\Phi = \Phi(x_1, \ldots, x_n)$ is a conjunction

$$\left(\bigwedge_{i=1}^{n} 1/2 \leq x_i \leq 2\right) \wedge \left(\bigwedge_{i=1}^{m} C_i\right),$$

where $m \geq 0$ and each $C_i$ is of one of the forms

$$x + y = z, \quad x \cdot y = 1$$

for $x, y, z \in \{x_1, \ldots, x_n\}$.

**Definition 3.3.2** (Definition 2 from [5])**.** An instance $\mathcal{I} = [\Phi, \delta, (I(x_1), \ldots, I(x_n))]$ of the *RANGE-ETR-INV problem* consists of an ETR-INV formula $\Phi$, a number $\delta := 2^{-l}$ for a positive integer $l$, and, for each variable $x \in \{x_1, \ldots, x_n\}$, an interval $I(x) \subseteq [1/2, 2]$ such that $|I(x)| \leq 2\delta$. For every *inversion constraint* $x \cdot y = 1$, we have either $I(x) = I(y) = [1 - \delta, 1 + \delta]$ or $I(x) = [2/3 - \delta, 2/3 + \delta]$ and $I(y) = [3/2 - \delta, 3/2 + \delta]$. We are promised that $V(\Phi) \subset I(x_1) \times \cdots \times I(x_n)$. The goal is to decide whether $V(\Phi) \neq \emptyset$.

**WIRED-INV.** In Section 5.8 of [5] an auxiliary problem WIRED-INV is introduced. An instance of this problem consists of a RANGE-ETR-INV instance together with a wiring diagram. Such a diagram is a geometric representation of the ETR-INV formula, where every variable $x$ is represented by two wires $\overrightarrow{x}$ and $\overleftarrow{x}$, and every constraint is represented by two constraint boxes intersecting some of these wires. See Figure 3.8 for a drawing of such a wiring diagram.

**Main idea construction.** Next, this wiring diagram needs to be converted into a packing instance. For this, we use an instance with $\delta = O(n^{-300})$. In the conversion, every wire is converted to a lane of variable pieces, which can move horizontally, but not vertically. The horizontal position of each of these pieces encodes the value of the corresponding variable. Every constraint box of the wiring diagram is replaced by a gadget. The idea behind the construction is described in more detail at the start of Section 2 of [5]. After this part of the construction, the container is a 4-monotone polygon. A 4-monotone polygon is a polygon that consists of two $x$-monotone and two $y$-monotone polygonal curves. See Figure 3.9 for a sketch of an instance which might result from this construction.

**Fingerprinting.** For the construction as mentioned in the previous paragraph, it is important that in every valid placement, all the pieces are placed almost as intended; it should not be allowed to place the pieces in a completely different configuration. To ensure this, a technique called fingerprinting is used, which is described in Section 6 of [5]. To apply this technique, it is important that every pieces has some corner where its angle is in $[5\pi/180, \pi/3]$, such that this angle is sufficiently different from all other angles of all other pieces. Furthermore, it is important to keep the slack $\mu$ low. The slack is the area of the container minus the total area of all the pieces. See Figure 3.10 for an illustration of this technique and its underlying idea.

**Correctness of the reduction.** The proof of correctness of the reduction happens in a number of steps, which are described in Section 2 of [5]. Most of these steps happen separately for every gadget in Sections 7 and 8. Here we will give a brief overview of the steps.

The first step is to show that every solution to the original RANGED-ETR-INV instance gives rise to a solution of the packing instance. The remaining steps are concerned with proving that any solution to the packing instance also gives a solution to the RANGED-ETR-INV instance.

First, for every gadget, a set of *canonical* placements are defined. These are the "intended" placements, where all pieces are in the right positions and have the intended edge-edge contacts. Furthermore, a *valid* placement is any placement that respects the condition that no pieces should overlap and all the pieces are inside the container. The next step is to prove using fingerprinting that any valid placement of all the pieces is *almost-canonical*. This means that each piece is moved by a distance of at most $n^{-1}$ from a canonical placement. Now it is proven that any almost-canonical placement is also an *aligned $N\mu$-placement*. In such a placement, every variable piece should be correctly aligned, and should furthermore encode a variable in the range $[m - N\mu, m + N\mu]$, where $N = O(n^4)$ is the total number of gadgets in the packing instance. The variable $\mu = O(n^{-296})$ is the total amount of slack in the instance, and $m$ is the midpoint of the interval $I(x)$ corresponding to the variable piece.

After this, it needs to be shown that all the variable pieces corresponding to a single variable actually encode the same value. For this purpose, a dependency graph $G_x$ is

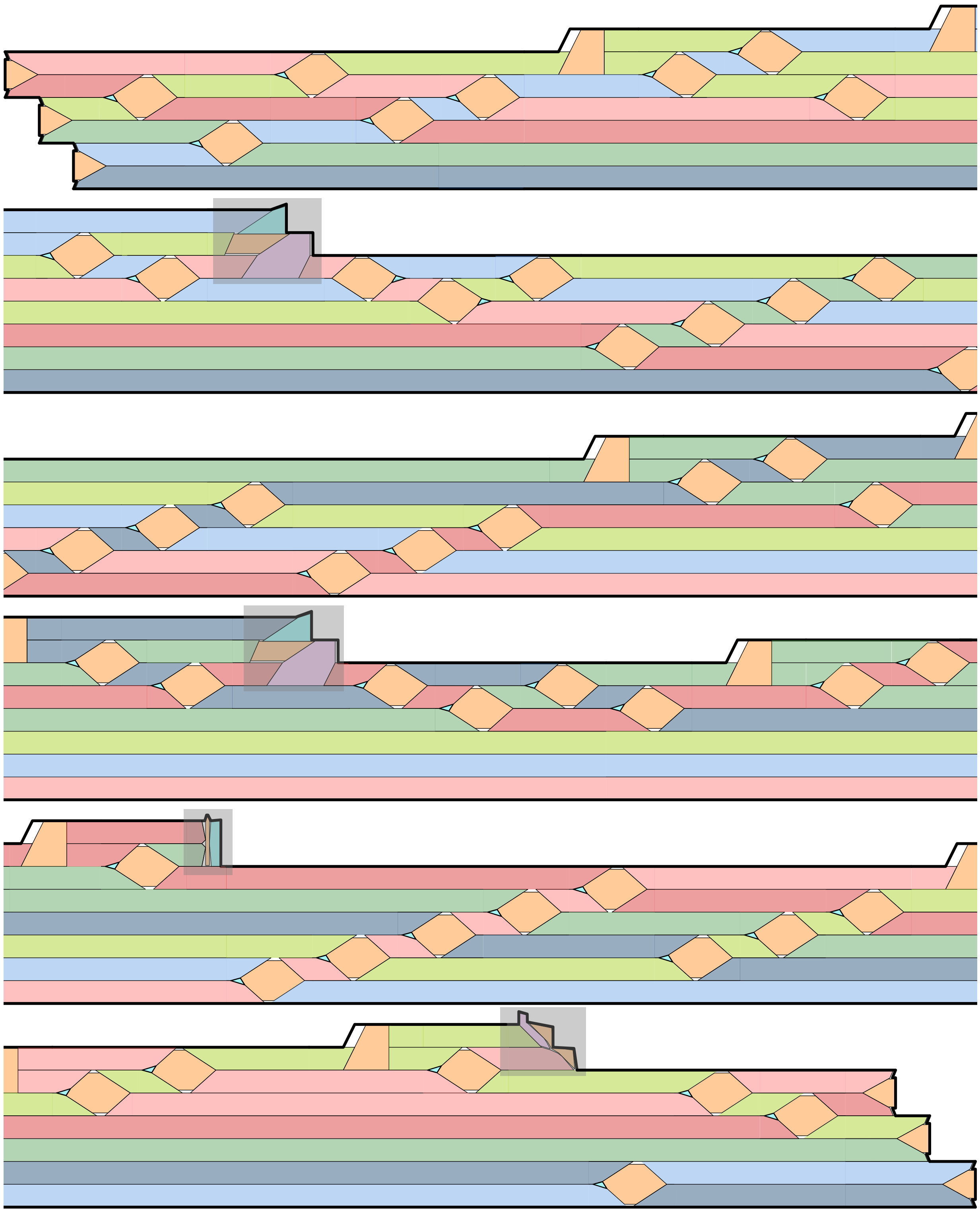


Figure 3.9: A sketch of the instance of , broken over six lines, we get from the wiring diagram in Figure 3.8 (except that the order of the inverters has been swapped to decrease the number of crossings). The adders and inverters are marked with gray boxes. This figure is taken from [5].

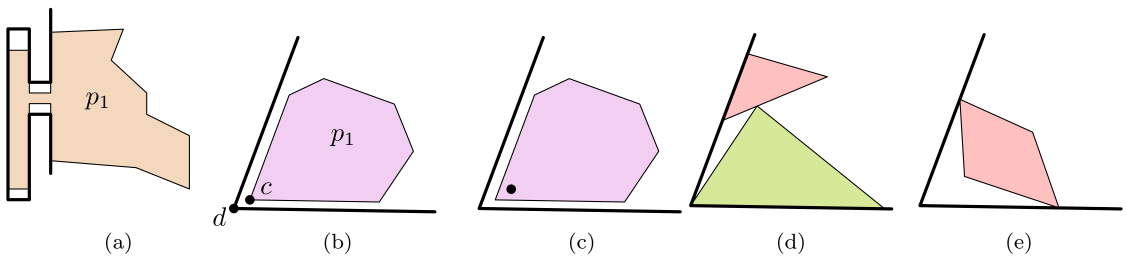



Figure 3.10: (a): A pocket and an augmentation that fit perfectly together as in a jigsaw puzzle. If non-convex pieces are allowed, this mechanism can be easily used to enforce a canonical placement of all the pieces. The idea for convex polygonal pieces is similar, but technically more intricate. (b): A wedge of the empty space and a piece which fit together. (c): The corner of the piece we are fingerprinting is marked with a dot. (d) and (e): Two examples where space is wasted because a wedge is not occupied by a piece with a matching angle. This figure is taken from [5].

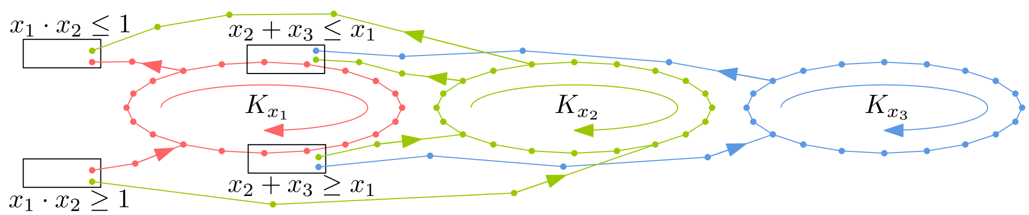



Figure 3.11: An abstract drawing of the dependency graphs of the instance we get from the wiring diagram in Figure 3.8 and how the graphs connect to the gadgets for addition and inversion constraints. The number of vertices on the cycles and paths are neither important nor correct. This figure is taken from [5].

introduced for every variable $x$. The vertices of this graph are all the variable pieces which encode this variable. Two of those vertices are connected by a directed edge if one of the corresponding pieces enters some gadget, and the other leaves the same gadget. It needs to be proven that for any such edge $(p_1, p_2)$ of $G_x$, we have $\langle p_1 \rangle \leq \langle p_2 \rangle$ where $\langle p \rangle$ indicates the value encoded by piece $p$. The packing instance was constructed in such a way to ensure that this graph $G_x$ contains a large cycle $K_x$. This implies that every piece in this cycle encodes exactly the same value $\langle K_x \rangle$. See Figure 3.11 for a visualization of the graph $G_x$.

The last step of the correctness proof consists of showing that for any addition constraint $x + y = z$ of $\Phi$, we actually have $\langle K_x \rangle + \langle K_y \rangle = \langle z \rangle$, and similarly that for every inversion constraint $x \cdot y = 1$ we have $\langle K_x \rangle \cdot \langle K_y \rangle = 1$. From this it follows that any valid packing indeed encodes a solution to $\Phi$.

**Square container.** Finally in Section 9 of [5], it is described how the packing instance with a 4-monotone container can be converted to a packing instance with a square container.

### 3.3.2 Changes to the framework

**CCI and WIRED-CCI.** As mentioned in the previous section, the proof from [5] works by reducing from RANGED-ETR-INV via the problem WIRED-INV. In the reduction in this thesis, we will instead reduce from CCI via a new problem WIRED-CCI. The specific version of CCI which is used, uses the following polynomials $f$ and $g$:

$$f(x,y) = (x-1)(y-1)-1, \text{ and}$$
$$g(x,y) = (x-1)^2 + \left(\frac{1}{4}y - 1\right)^2 - 2.$$

Note that those are just linear transformations of the polynomials $xy - 1$ and $x^2 + y^2 - 1$. After expanding the brackets in this definition, it is straightforward to check that these do indeed satisfy the necessary properties applying Theorem 3.1.10. For the rest of this section on packing, any reference to CCI or the polynomials $f$ and $g$ will refer to this specific definition.

An instance of WIRED-CCI consists of a CCI instance together with a wiring diagram. Let $\Phi$ be a CCI formula, now we define a wiring diagram of this formula $\Phi$ in almost the same way as is done in Section 5.8 from [5], see also Figure 3.8. The only difference in the definition concerns the constraint boxes which are needed in the diagram. Addition constraints are still modeled in the same way: for a constraint $x_i + x_j = x_k$, we have two constraint boxes for enforcing respectively $x_i + x_j \leq x_k$ and $x_i + x_j \geq x_k$. The first box should intersect the right-oriented wires $\overrightarrow{x_i}$, $\overrightarrow{x_j}$ and $\overrightarrow{x_k}$ in positions $\ell_1$, $\ell_2$ and $\ell_3$ respectively. The second box should intersect the left-oriented wires $\overleftarrow{x_i}$, $\overleftarrow{x_j}$ and $\overleftarrow{x_k}$ in positions $\ell_1$, $\ell_2$ and $\ell_3$ respectively.

For each constraint of the form $f(x_i, x_j) \geq 0$, we have a single constraint box which intersects the left-oriented wires $\overleftarrow{x_i}$ and $\overleftarrow{x_j}$ in positions $\ell_1$ and $\ell_2$. Similarly for each constraint of the form $g(x_i, x_j) \geq 0$ we should have a single constraint box intersecting the right-oriented wires $\overrightarrow{x_i}$ and $\overrightarrow{x_j}$ in positions $\ell_1$ and $\ell_2$.

Lastly, constraints of the form $x_i \geq 0$ or $x_1 = \delta$ do not need to be encoded in the wiring diagram in any way. The reason for this is that we will directly encode these constraints in the anchor-gadgets, so no separate constraint boxes are needed.

**Definition 3.3.3** (WIRED-CCI, modified from [5, Definition 42])**.** An instance $\mathcal{I} = [\mathcal{I}', D]$ of the *WIRED-CCI problem* consists of an instance $\mathcal{I}'$ of CCI together with a wiring diagram $D$ of the CCI formula $\Phi(\mathcal{I}')$.

**Lemma 3.3.4** (Modified from [5, Lemma 43])**.** *Given an CCI formula $\Phi$ with variables $x_1, \ldots, x_n$, we can in $O(n^4)$ time construct a wiring diagram of $\Phi$. Therefore, the problem WIRED-CCI is $\exists\mathbb{R}$-hard, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

*Proof.* The proof is completely analogous to that of Lemma 43 of [5]. The idea is that we can construct the wiring diagram from left to right. For every constraint we first swap some pairs of wires to make sure that the correct wires are on top, and then we insert a constraint box into the diagram. This reduction requires $O(n^4)$ time, where $n$ is the number of variables in $\Phi$, since there are at most $O(n^3)$ distinct constraints, and for each of those constraints at most $O(n)$ swaps need to be inserted. $\square$

Using Lemma 3.3.4, it follows that the problem WIRED-CCI is $\exists\mathbb{R}$-hard. Using the framework from Abrahamsen, Miltzow and Seiferth [5] with the modifications below, we get a reduction from WIRED-CCI to the desired packing problem. In this reduction the teeter-totter is used for encoding constraints of the form $f(x, y) \geq 0$ and the wobbly gramophone is used for encoding constraints of the form $g(x, y) \geq 0$.

**Variable pieces.** In the framework from [5], every variable $x$ of an ETR-INV formula comes with an interval $I(x) \subset [1/2, 2]$. Following this example, we also define $I(x)$ for any variable of a CCI formula $\Phi$. We take:

$$I(x) = \begin{cases} \{\delta\} & \text{if } \Phi \text{ has a constraint } x = \delta, \\ [0, \delta] & \text{else, if } \Phi \text{ has a constraint } x \geq 0, \\ [-\delta, \delta] & \text{otherwise.} \end{cases}$$

Note that the construction from [5] only uses the fact that $I(x) \subset [1/2, 2]$ when translating between positions of pieces and the corresponding variable values. This can still be done in our construction, where we know that all intervals $I(x)$ are instead contained in $[-\delta, \delta]$. For the reduction to work, it is merely necessary that every interval $I(x)$ has size at most $2\delta$, which is still the case.

**Gadgets.** We can directly reuse most of the gadgets from [5]. In particular we keep using the anchor, swap, split, adder and teeter-totter. The seesaw and gramophone are not used anymore, those are replaced by the new wobbly gramophone, which will be introduced in more detail later. The anchor is placed at both ends of every variable lane, and ensures that the left-oriented and right-oriented lanes corresponding to the same variable also encode the same value. The swap is used when two lanes need to cross. The split duplicates one of the lanes; this is necessary for embedding the adders and teeter-totters in the construction. The adder can be used for encoding both constraints of the form $x + y \leq z$ and $x + y \geq z$. Finally, the teeter-totter encodes constraints of the form $xy \geq 1$, and the wobbly gramophone will encode constraints of the form $g(x, y) \geq 0$. An overview of the gadgets is given in Figure 3.12.

The anchor, swap, split and adder gadgets can be directly used without any modification. It should be noted that the anchor always ensures that for any valid packing, the value of some variable $x$ encoded by this packing is always in the interval $I(x)$. Therefore, because of our definition of the intervals $I(x)$, the anchor automatically enforces the constraints of the form $x = \delta$ and $x \geq 0$.

Some more care is needed for using the teeter-totter. The original teeter-totter is used to encode $xy \geq 1$ in some neighborhood of either $(1, 1)$ or $(2/3, 3/2)$. For our result, we

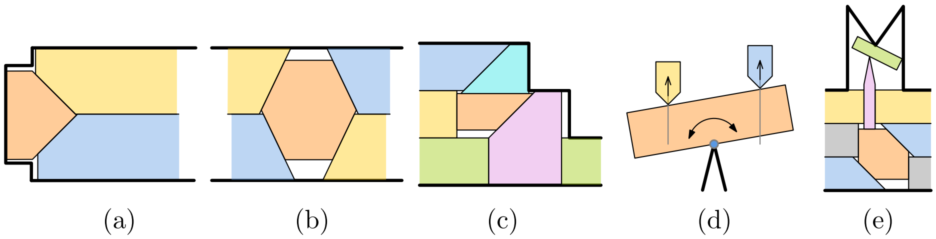



Figure 3.12: (a): anchor, (b): swap, (c): adder, (d): teeter-totter, (e): wobbly gramophone

however need a constraint in a neighborhood of $(0, 0)$. To obtain this, we use the teeter-totter for encoding $xy \geq 1$ around $(1, 1)$, but consider both the variables to be translated by $-1$. In this way we can use exactly the same gadget to enforce $(x+1)(y+1) \geq 1$ in a neighborhood of $(0, 0)$. This new constraint is equivalent to $f(x, y) \geq 0$ with the $f$ defined earlier.

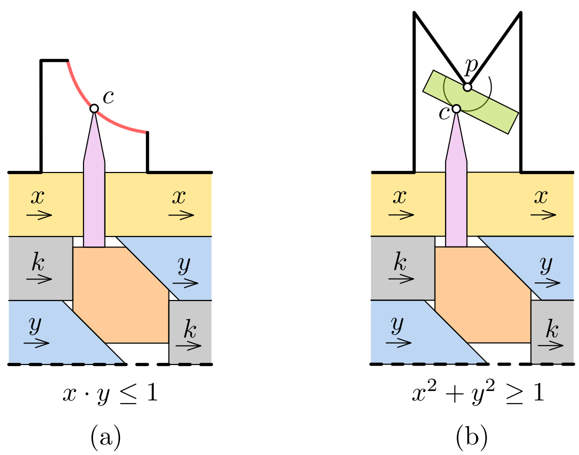



Figure 3.13: (a) The original gramophone from [5]. (b) The concept behind the new gadget. Just like in the original gramophone, the gray pieces marked $k$ cannot move and can be thought of representing a constant. Now the green piece prevents the tip $c$ of the pink piece from coming too close to the point $p$ on the container boundary, thereby enforcing a constraint of the form $x^2 + y^2 \geq 1$.

### 3.3.3 Gadget

The next step in proving $\exists\mathbb{R}$-hardness of the packing problem of convex polygons is to find a suitable gadget which encodes some concave constraint on two variables, while using only convex pieces. Here we introduce this new gadget, the wobbly gramophone, which is a modification of the gramophone gadget from [5]. The idea behind the gadget is to change the original gramophone by replacing the curved boundary by a rotating piece. See Figure 3.13. The exact concave constraint enforced by the actual form of this gadget will be $(x-1)^2 + \left(\frac{1}{4}y - 1\right)^2 \geq 2$.

For the rest of this section, we will write $\Phi$ for the CCI formula which we are reducing to a packing instance, and we let $\mathbf{p}_i$ be the set of all pieces which are part of the first $i$ gadgets of the constructed packing instance.

We will start by explaining the principle behind the simplified version of the gadget. Then the actual wobbly gramophone will be described, and it will be shown that it satisfies all the properties which are necessary for embedding it in the framework from [5].

**Principle.** Just like in the original gramophone from [5], the new wobbly gramophone contains a pink piece with a tip $c$, which encodes both the variables $x$ and $y$. Instead of having a fixed curve which bounds the corner $c$, we use a rotating rectangular piece of width 1 (the green piece) which enforces that the distance between $c$ and some corner $p$ on the outside boundary is at least 1. Therefore this exactly encodes a constraint of the form $x^2 + y^2 \geq 1$ in a neighborhood of the point $(0, -1)$. In the actual gadget, we will use a scaled and translated version of this constraint, as will be described next.

**Actual gadget.** A more accurate drawing of the actual wobbly gramophone can be seen in Figure 3.14. Here all the side-lengths of the pieces should be chosen in such a way to make sure the total amount of empty space (slack) is only $O(\delta)$. This gadget differs on quite some aspects from the simplified drawing in Figure 3.13. First we observe that the only pieces which are involved in enforcing the correct constraint, are pieces with numbers 1, 2, 3, 5, 6, 7, 10, 21 and 24; the rest of the pieces are necessary just for fingerprinting purposes.

The pieces numbered 1 and 10 should have edge-edge contacts with the pink piece. Together these pieces enforce that the pink piece has the intended orientation, and they fix the $x$-coordinate of the pink piece. The pieces 2, 3, 4, 5, 21 and 24 together form a swap gadget, as described in [5]. Furthermore, the gray pieces all have a fixed position, so they encode some constant $k$. However, instead of freely choosing the corners of pieces 2, 3, 21 and 24 for fingerprinting (as is done for the original gramophone), we always take these angles to be $\arctan(1/2)$. In this way the slope of the slanted sides of these pieces is exactly 1/2. Now if the value encoded by pieces 3 and 21 is increased by $\Delta y$, then the orange piece 5 will move up by $\frac{1}{4}\Delta y$. Stated differently, if piece 3 and 21 encode some variable $y$, then the $y$-coordinate of the orange piece encodes $\frac{1}{4}y$, and the $y$-coordinate of the pink piece encodes some variable $y'$ which satisfies $y' \geq \frac{1}{4}y$.

The green piece (6) should have distance exactly $\sqrt{2}$ between its long sides. (Note that this can be done while still ensuring all corners of this piece have rational coordinates. To see this, note that the two parallel lines $y = -x$ and $y = -x+2$ have exactly distance $\sqrt{2}$.) In this way, the green piece enforces that the distance between corner $c$ of the pink piece and corner $p$ of the outside boundary is at least $\sqrt{2}$. Since the $x$- and $y$-coordinates of the pink piece encode the variables $x$ and $y$ in some sense, we find that the gadget enforces the constraint $(x-1)^2 + \left(\frac{1}{4}y - 1\right)^2 \geq 2$ around the point $(x, y) = (0, 0)$. Note that that $(x, y) = (0, 0)$ corresponds exactly to the case where the $x$- and $y$-coordinates of corner $p$ are both 1 larger than the $x$- and $y$-coordinates of corner $c$.

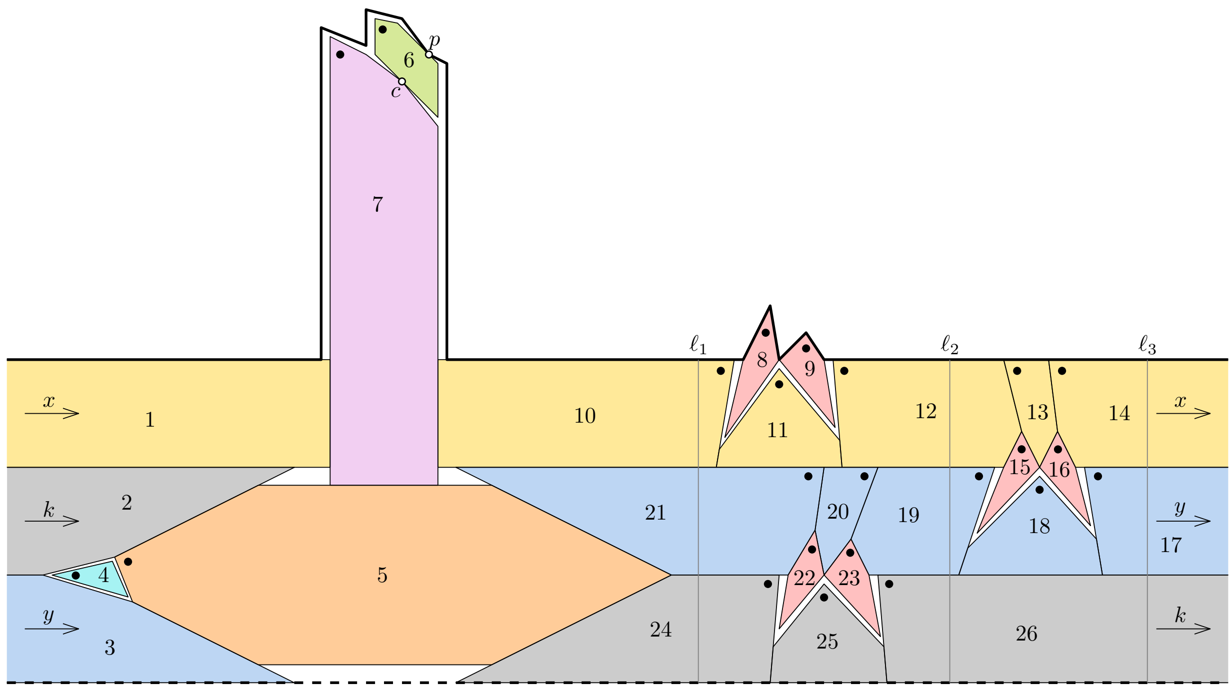



Figure 3.14: The actual wobbly gramophone. Color codes: 1 yellow, 2 gray, 3 blue, 4 turquoise, 5 orange, 6 green, 7 pink, 8, 9 red, 10, 11, 12, 13, 14 yellow, 15, 16 red, 17, 18, 19, 20, 21 blue, 22, 23 red, 24, 25, 26 gray.

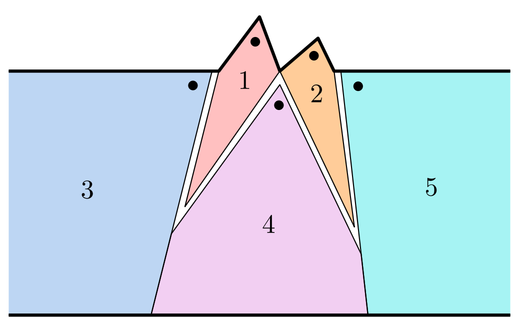



Figure 3.15: The fingerprinting subgadget. Color codes: 1 red, 2 orange, 3 blue, 4 pink, 5 turquoise.

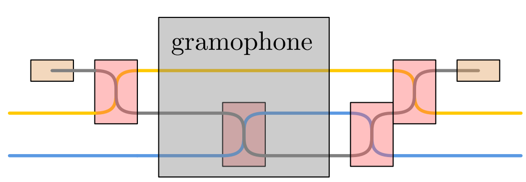



Figure 3.16: The installation manual of the wobbly gramophone, taken from [5].

The rest of the pieces, which form the right half of the gadget, have as a purpose to fingerprint the pieces 10, 21 and 24. This is necessary since these three pieces are not fingerprinted at their corners next to the orange and pink pieces, as is done in the gramophone described in [5]. The right half consists of three subgadgets, as drawn in Figure 3.15. Such a subgadget allows us to fingerprint a variable piece from the right instead of from the left.

It is important that the fingerprinting corner of piece 13 (in Figure 3.14) is strictly smaller than that of piece 14; this ensures that piece 13 attains its maximum width at the top, and therefore cannot be translated or rotated without pushing piece 14 to the right. Similarly, the fingerprinting corner of piece 20 should be smaller than the fingerprinting corner of piece 21.

Embedding this gadget into the total construction is done in exactly the same way as the embedding of the gramophone from [5]. See Figure 3.16. In particular there is a separate lane introduced for the constant $k$, which ends immediately before and after the gadget. This lane is ended in such a way to ensure that the first variable piece of this lane encodes at least 0, and the last piece encodes at most 0.

**Canonical placements and solution preservation.** The variable pieces in the wobbly gramophone are pieces 1, 3, 10, 12, 14, 17, 19 and 21. Furthermore we will also treat pieces 2, 24 and 26 as variable pieces, even though they do not encode actual variables, but they do encode a constant $k$.

We take the canonical placements to be those where the boundary and the yellow, gray, blue and orange pieces have all the edge-edge contacts between them which are drawn in Figure 3.14. The gray pieces should furthermore have exactly the same position in all canonical placements, and this should be the position as drawn in Figure 3.14. The pink piece should have edge-edge contacts with pieces 1 and 10 (but not necessarily with piece 5 below it). The turquoise and red pieces should all be enclosed by the pieces which are enclosing them in Figure 3.14, but here no edge-edge contacts are required. Finally we impose that the segment between the corner $c$ of the pink piece and corner $p$ of the boundary intersects both long sides of the green piece.

**Lemma 3.3.5** (Solution Preservation, Lemma 6 from [5])**.** *Suppose that this gadget is used as the i'th gadget in the construction of the packing instance, and that for every solution to* $\Phi$*, there is a canonical placement of the pieces* $\mathbf{p}_{i-1}$ *of the preceding gadgets which encodes this solution. Then the same holds for* $\mathbf{p}_i$*.*

*Proof.* The pieces 1 and 3 are already fixed, since these belong to the set $\mathbf{p}_{i-1}$. Now we can place all the remaining pieces except for the green piece in such a manner as to have all edge-edge contacts as drawn in Figure 3.14. Since the given placements of all the pieces encode a valid solution to $\Phi$, it follows that the distance between $c$ and $p$ is $\sqrt{(x-1)^2 + \left(\frac{1}{4}y - 1\right)^2}$. Since $\Phi$ contains some constraint $g(x, y) \geq 0$, we find that this distance is at least $\sqrt{2}$. This shows that we can place the green piece between these two corners, we do this in such a way that its long sides are perpendicular to the segment between $c$ and $p$. There should be enough slack in the construction to make sure that the green piece can be placed in this manner without intersecting the boundary. □

**Fingerprinting and almost-canonical placement.**

**Lemma 3.3.6** (Almost-canonical Placement, Lemma 10 from [5])**.** *Suppose that the wobbly gramophone is used as the i'th gadget in the construction of the packing instance. Consider a valid placement $P$ (of all the pieces) for which the pieces $\mathbf{p}_{i-1}$ have an aligned $(i-1)\mu$-placement. It then holds for $P$ that the pieces $\mathbf{p}_i$ have an almost-canonical placement.*

*Proof.* This proof goes analogous to the proof of Lemma 10 for the anchor in [5], where we fingerprint the pieces in the order indicated by the numbers in Figure 3.14. Because of how technical the proof is, we will not repeat the details here. □

**Aligned placement.** The next step is to show that the variable pieces in the gadget are correctly aligned, and encode variables in the correct ranges. Here the term $i\mu$-placement means that every variable piece is correctly aligned, and encodes a value in the interval $[-i\mu, i\mu]$.

**Lemma 3.3.7** (Aligned Placement, Lemma 11 from [5])**.** *Suppose that this gadget is used as the i'th gadget in the construction of the packing instance. Consider a valid placement $P$ (of all the pieces) for which the pieces $\mathbf{p}_{i-1}$ have an aligned $(i-1)\mu$-placement and the pieces $\mathbf{p}_i$ have an almost-canonical placement. It then holds for $P$ that the pieces 10, 12, 14, 17, 19, 21, 24 and 26 have an aligned $i\mu$-placement.*

*Proof.* From the alignment lines $\ell_1$, $\ell_2$ and $\ell_3$ in Figure 3.14 we conclude that all the mentioned pieces are indeed aligned.

By using arguments similar to the arguments used for proving Lemma 11 for the swap in [5], it can be shown that also all encoded variables are in the right ranges. We will briefly discuss the main ideas here. By the inequalities which will be proven in Lemma 3.3.8, and the fact that the values encoded by pieces 1, 2 and 3 are in $[-(i-1)\mu, (i-1)\mu]$, it follows that all variable pieces encode a value which is at least $-(i-1)\mu$. What remains to be shown, is that no variable piece can encode a value which is more than $\mu$ larger than the value encoded by piece 1, 2 or 3. For example, consider piece 19, the other pieces can be handled in a similar way. If piece 19 would encode a value which is more than $\mu$ larger than the value of piece 3, then the total area enclosed by the pieces 1, 2, 3, 12, 19 and 25 would be at least $\mu$ more than the total area of all the pieces which would be placed inside this region in a canonical placement. This implies that an area of more than $\mu$ is empty withing this region, but this is impossible (recall that $\mu$ is the amount of empty space in the whole construction). We conclude that the value encoded by piece 19 is indeed at most $\mu$ larger than the value encoded by piece 3. □

**Edge inequalities.** Next we prove the edge inequalities for this gadget. In the following lemma we denote by $\langle p_i \rangle$ the value encoded by the piece with number $i$ in Figure 3.14.

**Lemma 3.3.8** (Edge inequality, Lemma 13 from [5])**.** *Let an aligned $i\mu$-placement of the pieces in the wobbly gramophone be given. Now the values encoded by pieces 1, 2, 3, 14, 17 and 26 satisfy the following inequalities:*

$$\begin{aligned}
&\langle p_1 \rangle \leq \langle p_{10} \rangle \leq \langle p_{12} \rangle \leq \langle p_{14} \rangle \\
&\langle p_2 \rangle \leq \langle p_{21} \rangle \leq \langle p_{19} \rangle \leq \langle p_{17} \rangle \\
&\langle p_3 \rangle \leq \langle p_{24} \rangle \leq \langle p_{26} \rangle .
\end{aligned}$$

*Proof.* First consider pieces 1 and 10. In a canonical placement where both piece 1 and 10 have edge-edge contacts with piece 7, we would have $\langle p_1 \rangle = \langle p_{10} \rangle$. Note that piece 1 and 10 can never get closer to each other than this: piece 7 will in any almost-canonical placement be between them. This implies that we always have $\langle p_1 \rangle \leq \langle p_{10} \rangle$. Similarly, we can look at piece 2 and 24. These pieces are always separated by piece 5, and the horizontal distance between them is minimized exactly when both of these pieces have edge-edge contacts with piece 5, like in a canonical placement. It follows that also $\langle p_2 \rangle \leq \langle p_{21} \rangle$. Similarly we get $\langle p_3 \rangle \leq \langle p_{24} \rangle$.

Next consider piece 18 in the gadget. In a canonical placement, this piece has edge-edge contacts with the pieces 17, 19 and 26. In the current aligned placement, we know that these three pieces are all aligned. This implies that piece 18 cannot move downward. Also rotating piece 18 will always increase the horizontal distance between pieces 17 and 19. Furthermore the width of piece 18 is largest at its bottom, meaning that translating this piece upwards would also increase the distance between piece 17 and 19. We conclude that in any case we get $\langle p_{19} \rangle \leq \langle p_{17} \rangle$. By a similar argument applied to pieces 11 and 25, we find that also $\langle p_{10} \rangle \leq \langle p_{12} \rangle$ and $\langle p_{24} \rangle \leq \langle p_{26} \rangle$.

Similar arguments also apply to pieces 13 and 20; for example 20 is bounded by pieces 12, 19 and 21, and its width is maximal at its top. In this way we find the additional inequalities $\langle p_{12} \rangle \leq \langle p_{14} \rangle$ and $\langle p_{21} \rangle \leq \langle p_{19} \rangle$. This completes the proof of the lemma. $\square$

Note that the statement about pieces 3 and 26 in this lemma implies that the gray pieces actually model a constant. This works as follows: from the way the gray pieces are embedded, it follows that $\langle p_2 \rangle \geq 0$ and $\langle p_{26} \rangle \leq 0$. Combining this with the inequalities which we just proved, it indeed follows that all gray pieces encode 0.

**The gadget works.** Since all the edge-inequalities for all the gadgets used in the construction together form a cycle for every variable, it follows that for any valid placement of all the pieces and any variable $x$, all the vairable pieces in this placement corresponding to $s$ encode the same value, denoted $\langle K_x \rangle$. This is Lemma 14 from [5]. Now we get the following result:

**Lemma 3.3.9.** *For every constraint of the form* $(x-1)^2 + \left(\frac{1}{4}y - 1\right)^2 \geq 2$ *of* $\Phi$*, we have* $f(\langle K_x\rangle, \langle K_y\rangle) \geq 0$.

*Proof.* Since all the values of the variables are consistently encoded, we find in particular that piece 1 and 10 encode the same variable, and piece 3 and 21 encode the same variable. We also know that piece 2 and 24 encode the same constant. This implies that the orange piece has edge-edge contacts with pieces 2, 3, 21 and 24, and that the pink piece has edge-edge contacts with pieces 1 and 10. We can now slide the pink piece downwards until it touches the orange piece; note that this leaves the packing valid.

Consider a translated version of the standard coordinate system, in which corner $p$ of the outside boundary has coordinates $(1, 1)$. If all the pieces in this gadget would have the canonical position encoding $x = y = 0$, then the corner $c$ of the pink piece gets position $(0, 0)$ by construction of the gadget. Furthermore, if the value of $x$ would be increased by some value $\Delta x$, then the pink piece will shift $\Delta x$ to the right. If the value of $y$ were increased by some value $\Delta y$, then the pink piece will shift $\frac{1}{4}\Delta y$ upwards.

This has as a consequence that, in the current situation, the corner $c$ of the pink piece has coordinates exactly $\left(\langle K_x\rangle, \frac{1}{4}\langle K_y\rangle\right)$ with respect to this new coordinate system. Furthermore, since the placement is almost canonical, we can see that the green piece has to separate the corners $c$ and $p$; there is not enough space for the green piece to go anywhere else. We conclude that the distance between $c$ and $p$ is at least $\sqrt{2}$; this exactly implies the condition $(\langle K_x\rangle - 1)^2 + \left(\frac{1}{4}\langle K_y\rangle - 1\right)^2 \geq 2$. This is equivalent to $f(\langle K_x\rangle, \langle K_y\rangle) \geq 0$. □

**Valid placements are canonical.**

**Lemma 3.3.10** (Lemma 20 from [5])**.** *Suppose that the wobbly gramophone is used as the i'th gadget in the construction of the packing instance. Consider a valid placement* $P$ *of all the pieces, such that the pieces* $\mathbf{p}_{i-1}$ *have a canonical placement. It then holds for* $P$ *that the pieces* $\mathbf{p}_i$ *have a canonical placement.*

*Proof.* Using that the placement of all the variable pieces is consistent, it follows by the arguments given in the proof of Lemma 3.3.8 that the pieces 11, 13, 18, 20 and 25 have all the edge-edge contacts with other pieces which are desired in a canonical placement. Also by the proof of Lemma 3.3.9 it follows that there are edge-edge contacts between the pink piece and pieces 1 and 10, and between the orange piece and pieces 2, 3, 21 and 24. In this proof it was also observed that the green piece always separates the corners $c$ and $p$ in the desired manner. Since we know the placement to be almost-canonical, we furthermore know that the turquoise and red pieces all have the correct positions. This completes the proof that the placement is canonical. □

## 3.4 Appendix: Circle-Constraint

Here, we discuss the question of expressing multiplication via equations which are all of the form $x + y = z$ and the circle constraint $x^2 + y^2 = 1$. We note that for real numbers $x$ and $y$ the following equivalence holds:

There exists a real $z$ such that $z^2 + (x + y)^2 = 1$ and $(z + x - y)^2 + (z - x + y)^2 = 1$ if and only if $8xy = 1$ and $|x + y| <= 1$ This can be used in turn to express the inversion constraint $(x \cdot y = 1)$ after some scaling and imposing range constraints. Note that $x \cdot y = 1$ can be used to express squaring as follows

$$\frac{1}{\frac{1}{x} - \frac{1}{x+1}} - x = x^2.$$

And we saw already in the Section 3.1 how squaring can be used to express multiplication.

# Bibliography


[1] Zachary Abel, Erik Demaine, Martin Demaine, Sarah Eisenstat, Jayson Lynch, and Tao Schardl. Who needs crossings? Hardness of plane graph rigidity. In *32nd International Symposium on Computational Geometry (SoCG 2016)*, pages 3:1–3:15, 2016.

[2] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The art gallery problem is $\exists\mathbb{R}$-complete. In *STOC*, pages 65–73, 2018.

[3] Mikkel Abrahamsen, Linda Kleist, and Tillmann Miltzow. Training neural networks is er-complete. *arXiv preprint arXiv:2102.09798*, 2021.

[4] Mikkel Abrahamsen and Tillmann Miltzow. Dynamic Toolbox for ETRINV. *arXiv e-prints*, page arXiv:1912.08674, December 2019.

[5] Mikkel Abrahamsen, Tillmann Miltzow, and Nadja Seiferth. Framework for $\exists\mathbb{R}$-Completeness of Two-Dimensional Packing Problems. *arXiv e-prints*, page arXiv:2004.07558, April 2020.

[6] Saugata Basu and Marie-Françoise Roy. Bounding the radii of balls meeting every connected component of semi-algebraic sets. *Journal of Symbolic Computation*, 45(12):1270 – 1279, 2010. MEGA'2009.

[7] Marie Louisa Tølbøll Berthelsen and Kristoffer Arnsfelt Hansen. On the computational complexity of decision problems about multi-player nash equilibria. In *International Symposium on Algorithmic Game Theory*, pages 153–167. Springer, 2019.

[8] Vittorio Bilò and Marios Mavronicolas. A catalog of exists-r-complete decision problems about nash equilibria in multi-player games. In *33rd Symposium on Theoretical Aspects of Computer Science (STACS 2016)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016.

[9] Vittorio Bilò and Marios Mavronicolas. Existential-r-complete decision problems about symmetric nash equilibria in symmetric multi-player games. In *34th Symposium on Theoretical Aspects of Computer Science (STACS 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

[10] Manuel Bodirsky and Martin Grohe. Non-dichotomies in constraint satisfaction complexity. In *International Colloquium on Automata, Languages, and Programming*, pages 184–196. Springer, 2008.

[11] Manuel Bodirsky, Peter Jonsson, and Timo Von Oertzen. Essential convexity and complexity of semi-algebraic constraints. *arXiv preprint arXiv:1210.0420*, 2012.

[12] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. *Journal of the ACM (JACM)*, 57(2):1–41, 2010.

[13] Manuel Bodirsky and Marcello Mamino. Constraint satisfaction problems over numeric domains. In *Dagstuhl Follow-Ups*, volume 7. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

[14] Andrei A Bulatov. A dichotomy theorem for constraint satisfaction problems on a 3-element set. *Journal of the ACM (JACM)*, 53(1):66–120, 2006.

[15] Andrei A Bulatov. A dichotomy theorem for nonuniform csps. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 319–330. IEEE, 2017.

[16] John Canny. Some algebraic and geometric computations in PSPACE. In *Proceedings of the twentieth annual ACM symposium on Theory of computing (STOC 1988)*, pages 460–467. ACM, 1988.

[17] Jean Cardinal, Stefan Felsner, Tillmann Miltzow, Casey Tompkins, and Birgit Vogtenhuber. Intersection graphs of rays and grounded segments. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 153–166. Springer, 2017.

[18] Jean Cardinal and Udo Hoffmann. Recognition and complexity of point visibility graphs. *Discrete & Computational Geometry*, 57(1):164–178, 2017.

[19] Dmitry Chistikov, Stefan Kiefer, Ines Marusic, Mahsa Shirmohammadi, and James Worrell. On Restricted Nonnegative Matrix Factorization. In *ICALP*, volume 55 of *LIPIcs*, pages 103:1–103:14, 2016. ArXiv 1605.06848.

[20] Michael G. Dobbins, Linda Kleist, Tillmann Miltzow, and Paweł Rzążewski. $\forall\exists\mathbb{R}$-completeness and area-universality. *ArXiv 1712.05142*, 2017.

[21] Michael Gene Dobbins, Andreas Holmsen, and Tillmann Miltzow. A universality theorem for nested polytopes. *arXiv*, 1908.02213, 2019.

[22] Martin Dyer, Leslie Ann Goldberg, and Mark Jerrum. An approximation trichotomy for boolean# csp. *Journal of Computer and System Sciences*, 76(3-4):267–277, 2010.

[23] Jeff Erickson. Optimal curve straightening is $\exists\mathbb{R}$-complete. *arXiv:1908.09400*, 2019.

[24] Jeff Erickson, Ivor van der Hoog, and Tillmann Miltzow. Smoothing the gap between NP and ER. In *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*, pages 1022–1033. IEEE, 2020.

[25] Jugal Garg, Ruta Mehta, Vijay V. Vazirani, and Sadra Yazdanbod. ETR-completeness for decision versions of multi-player (symmetric) Nash equilibria. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP 2015), part 1*, Lecture Notes in Computer Science (LNCS), pages 554–566, 2015.

[26] Thierry Gensane and Philippe Ryckelynck. Improved dense packings of congruent squares in a square. *Discrete & Computational Geometry*, 34(1):97–109, 2005.

[27] Simon Hengeveld and Tillmann Miltzow. A practical algorithm with performance guarantees for the art˜gallery problem. *Arxivv*, 2007.06920, 2020. accepted to SoCG 2021.

[28] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves.* Cambridge Mathematical Library. Cambridge University Press, 2 edition, 2010.

[29] Joachim Jelisiejew. Pathologies on the Hilbert scheme of points. *arXiv e-prints*, page arXiv:1812.08531, December 2018.

[30] Peter Jonsson and Johan Thapper. Constraint satisfaction and semilinear expansions of addition over the rationals and the reals. *Journal of Computer and System Sciences*, 82(5):912–928, 2016.

[31] Ross Kang and Tobias Müller. Sphere and dot product representations of graphs. In *27th Annual Symposium on Computational Geometry (SoCG)*, pages 308–314. ACM, 2011.

[32] Martijn Kool. Fixed point loci of moduli spaces of sheaves on toric varieties. *arXiv e-prints*, page arXiv:0810.0418, October 2008.

[33] Jan Kratochvíl and Jiří Matoušek. Intersection graphs of segments. *Journal of Combinatorial Theory, Series B*, 62(2):289–315, 1994.

[34] Laurent Lafforgue. *Chirurgie des grassmanniennes*, volume 19 of *CRM Monograph Series*. American Mathematical Society, 2003.

[35] Seok Hyeong Lee and Ravi Vakil. Mnev-Sturmfels universality for schemes. *arXiv e-prints*, page arXiv:1202.3934, February 2012.

[36] Anna Lubiw, Tillmann Miltzow, and Debajyoti Mondal. The complexity of drawing a graph in a polygonal region. In *International Symposium on Graph Drawing and Network Visualization*, 2018.

[37] David Marker, Ya'acov Peterzil, and Anand Pillay. Additive reducts of real closed fields. *The Journal of symbolic logic*, 57(1):109–117, 1992.

[38] Dániel Marx. Parameterized complexity of constraint satisfaction problems. *computational complexity*, 14(2):153–183, 2005.

[39] Jiří Matoušek. Intersection graphs of segments and $\exists\mathbb{R}$. *ArXiv 1406.2636*, 2014.

[40] Colin McDiarmid and Tobias Müller. Integer realizations of disk and segment graphs. *Journal of Combinatorial Theory, Series B*, 103(1):114–143, 2013.

[41] Tillmann Miltzow and Reinier F. Schmiermann. On Classifying Continuous Constraint Satisfaction problems. *arXiv e-prints*, page arXiv:2106.02397, June 2021.

[42] Nicolai Mnëv. The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. In Oleg Y. Viro, editor, *Topology and geometry – Rohlin seminar*, pages 527–543. Springer-Verlag Berlin Heidelberg, 1988.

[43] Gábor Pataki and Aleksandr Touzov. How do exponential size solutions arise in semidefinite programming? *arXiv preprint arXiv:2103.00041*, 2021.

[44] Sam Payne. Moduli of toric vector bundles. *arXiv e-prints*, page arXiv:0705.0410, May 2007.

[45] Ya'acov Peterzil. A structure theorem for semibounded sets in the reals. *The Journal of Symbolic Logic*, 57(3):779–794, 1992.

[46] Ya'acov Peterzil. Reducts of some structures over the reals. *The Journal of Symbolic Logic*, 58(3):955–966, 1993.

[47] Jürgen Richter-Gebert and Günter M. Ziegler. Realization spaces of 4-polytopes are universal. *Bulletin of the American Mathematical Society*, 32(4):403–412, 1995.

[48] Marcus Schaefer. Complexity of some geometric and topological problems. In *Proceedings of the 17th International Symposium on Graph Drawing (GD 2009)*, Lecture Notes in Computer Science (LNCS), pages 334–344. Springer, 2009.

[49] Marcus Schaefer. Realizability of graphs and linkages. In János Pach, editor, *Thirty Essays on Geometric Graph Theory*, chapter 23, pages 461–482. Springer-Verlag New York, 2013.

[50] Marcus Schaefer. Complexity of geometric k-planarity for fixed k. *J. Graph Algorithms Appl*, 25(1):29–41, 2021.

[51] Marcus Schaefer and Daniel Stefankovic. The complexity of tensor rank. *Theory Comput. Syst.*, 62(5):1161–1174, 2018.

[52] Marcus Schaefer and Daniel Štefankovič. Fixed points, Nash equilibria, and the existential theory of the reals. *Theory of Computing Systems*, 60(2):172–193, 2017.

[53] Thomas J Schaefer. The complexity of satisfiability problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 216–226, 1978.

[54] Yaroslav Shitov. A universality theorem for nonnegative matrix factorizations. *ArXiv 1606.09068*, 2016.

[55] Yaroslav Shitov. The complexity of positive semidefinite matrix factorization. *SIAM Journal on Optimization*, 27(3):1898–1909, 2017.

[56] Peter Shor. Stretchability of pseudolines is NP-hard. In Peter Gritzmann and Bernd Sturmfels, editors, *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, DIMACS – Series in Discrete Mathematics and Theoretical Computer Science, pages 531–554. American Mathematical Society and Association for Computing Machinery, 1991.

[57] Ravi Vakil. Murphy's Law in algebraic geometry: Badly-behaved deformation spaces. *arXiv Mathematics e-prints*, page math/0411469, November 2004.

[58] Caterina Viola and Stanislav Zivny. The combined basic lp and affine ip relaxation for promise vcsps on infinite domains. *arXiv preprint arXiv:2007.01779*, 2020.

[59] Dmitriy Zhuk. A proof of the csp dichotomy conjecture. *Journal of the ACM (JACM)*, 67(5):1–78, 2020.

[60] Dmitriy Zhuk and Barnaby Martin. The complete classification for quantified equality constraints. *arXiv preprint arXiv:2104.00406*, 2021.