Differential relations with delay

Master's thesis
Mathematical Sciences

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Submitted to: Department of Mathematics,
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July 2021

Acknowledgements

I would like to thank my supervisor Álvaro del Pino Gómez for his supervision, his guiding advice and help in academia in general, and for introducing me to the h-principle. For reading, commenting on, and rereading most of my thesis, I would like to thank Aaron Grootjes-Dreesbach. For being part of a weekly support group for people writing their thesis in an otherwise very lonely year, I would like to thank Floor Dogger and Manoy Trip. And I would finally like to thank Anouk Greven, for supporting me every day.

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1 Introduction

The context of this thesis is the **homotopy principle**, usually abbreviated to the h-principle. The main result, theorem 1.4, is an h-principle type result about the d-delay partial differential relation, and is subsequently introduced. This thesis is meant in part as a proof of concept: it shows that an old h-principle technique can be used in a broader setting, and that an h-principle technique for *local* differential relations, namely convex integration, can be used for some *non-local* differential relations. We shall also highlight a few other, more technical results, mainly two generalizations of the Thom transversality theorem and multijet transversality theorem, theorem 4.14 and theorem 4.18. Afterwards, the layout of the thesis is presented.

1.1 Context of this thesis

The homotopy principle, (originally weak homotopy equivalence principle) or h-principle, is a collective term for a number of techniques and results in differential topology and geometry. The results usually provide answers to questions of the form 'what is sufficient topological or smooth data to ensure the existence of a certain smooth map or geometric structure?' A richer question that is often studied is 'how many structures or maps of a certain type exist up to homotopy?' For example, given two smooth manifolds, can one classify up to homotopy the immersions or embeddings between them? Often, a set of 'formal' maps or structures is introduced, which emulate some of the properties of the specific maps or structures under consideration. We say that 'a full h-principle holds for this class of maps/structures' if there is a weak homotopy equivalence between the space of maps or structures under consideration, and their formal counterparts. The techniques usually focus on constructing the desired maps or structures through homotopies, starting at their 'formal' counterparts. The upshot is that the formal data is often better understood in terms of homotopy type, or what the obstructions to the existence of such data are.

Below we give two examples of h-principle results about immersions and smooth embeddings. Heuristically, results about immersions are local results: the behaviour of map in a neighbourhood of a point determines whether or not it is an immersion. In contrast, if a map is an embedding, this is a statement about its global behaviour. This is reflected in the choice of formal data.

The father of h-principle theory is M. Gromov (who coined the term), whose book [6] contains many h-principle results and techniques. The review article by D. McDuff [16] gives an appetizer for the breath and scope of Gromov's book. The book [3] by Y. Eliashberg and N. Mishachev, gives an introduction to the h-principle and Gromov's book, and is accessible for graduate students. Due to its accessibility, it will be used as a reference for the notion of **partial differential relations** (PDRs), which will be used to make concrete the notion of local and non-local partial differential relations, such as the immersion relation and embedding relation, respectively. Their book will also be the main reference for the techique of **convex integration**. This technique is normally used for local PDRs, but we shall argue that under some conditions it can be used for non-local PDRs, too. A famous application of convex integration is the Nash-Kuiper embedding theorem, a proof of which can be found in [3] too.

1.1.1 h-Principles for immersion and embedding

The first example of an h-principle comes from a class of results about immersions, which is referred to as Hirsch-Smale immersion theory. The main references are [12] and [17]. The papers focus on the classification of immersions between a number of fixed spaces, up to homotopy. A broad statement contained in [12] (spread over multiple statements) is the theorem below. Let M and N be smooth manifolds, then we say that $g : TM \to TN$ is a *formal immersion* if g is a continuous bundle monomorphism. I.e. fiberwise a linear, injective map. Any immersion $f : M \to N$ has an associated formal immersion, $Tf : TM \to TN$. Let $\mathrm{Imm}(M, N)$ denote the subset of immersions in $C^\infty(M, N)$, where the latter is endowed with the compact-open Whitney C^∞ topology (see definition 2.5). Let $\mathrm{Mono}(TM, TN)$ denote the subset of $C(TM, TN)$ consisting of continuous bundle monomorphisms, and endow $C(TM, TN)$ with the compact-open topology.

Example 1.1. [Hirsch-Smale Immersion Theorem] Let M and N be manifolds and let N have empty boundary. If $\dim M = \dim N$ and M is open, or if $\dim M < \dim N$, then the map

$$T : \mathrm{Imm}(M, N) \to \mathrm{Mono}(TM, TN), \quad f \mapsto Tf$$

induces a weak homotopy equivalence. △

This result gives powerful information about the homotopy classes of families of immersions, and implies that the existence and classification of immersions is topological in nature. A result of this nature is a 'true' h-principle result, because there is weak homotopy equivalence between the mappings of interest, and the formal mappings. For another example involving immersions, we refer to theorem 3.11.

Another important example appears in the context of smooth embeddings. A well known result, due to H. Whitney, is the embedding theorem named after him [21], which guarantees the existence of a smooth embedding $f : M \to N$ if $\dim N \geq 2 \cdot \dim M$. An h-principle type result about smooth embeddings was obtained by A. Haefliger in [9]. Let M and N be manifolds, then we say that a smooth map $f = (f_1, f_2) : M \times M \to N \times N$ is **equivariant** if for all $(x, y) \in M \times M$ $(f_1, f_2)(y, x) = (f_2, f_1)(x, y)$. A homotopy connecting f to any other map g is called equivariant if all interpolating maps are equivariant.

Example 1.2. [Haefliger Embedding Theorem] Assume that M is compact.

(a) Suppose that $2 \cdot \dim M \leq 3(\dim N + 1)$. An immersion $f : M \to N$ is homotopic through immersions to a smooth embedding if and only if there exists an equivariant homotopy $H : M^2 \times [0, 1] \to N^2$ connecting $f \times f$ to an equivariant map $g : M^2 \to N^2$ that satisfies $g^{-1}(\Delta(N)) = \Delta(M)$, and so that $\Delta(M)$ is open in $H_t^{-1}(\Delta(N))$ for all $t \in [0, 1]$.

(b) Suppose that $2 \cdot \dim M < 3(\dim N + 1)$. A homotopy $H : M \times [0, 1] \to N$ connecting two smooth embeddings is itself homotopic to an isotopy if and only if there exists an equivariant homotopy $G_{\tau,t} : M^2 \to N^2$ connecting $H_\tau^2 = G_{\tau,0}$ to a homotopy $G_{\tau,1}$ that satisfies $G_{\tau,1}^{-1}(\Delta(N)) = \Delta(M)$ for all $\tau \in [0, 1]$, and so that $\Delta(M)$ is open in $G_{\tau,t}^{-1}(\Delta(N))$ for all $\tau, t \in [0, 1]$. △

The formal mappings are now equivariant maps or homotopies, whose preimage of the diagonal in N is the diagonal in M. Since M is assumed to be compact, it follows that $f : M \to N$ is a topological embedding if and only if $(f \times f)^{-1}(\Delta(N)) = \Delta(M)$. The formal embeddings carry more topological data than the formal immersions, which is needed to ensure the existence of embeddings or isotopies. In [18], A. Szücs gave a different proof of the above theorem, communicated to him by M. Gromov and Y. Eliashberg. The technique presented in [18], which is a *removal of singularities*-type argument, is adapted to prove the main result of this thesis, theorem 1.4. It remains an open question how general the class of non-local differential relations is, to which this technique can be adapted.

1.1.2 Transversality

This thesis communicates a slightly more general version of the Thom transversality theorem and multijet transversality theorem, namely theorems 4.14 and 4.18. These two theorems can be used to make statements about the jet transversality and multijet transversality of equivariant maps, such as theorems 5.11 and 5.14.

Thom's theorem is usually used to argue that, for a smooth map $f : M \to N$ and a submanifold $W \subset N$, the map f can be perturbed by an arbitrarily small amount so that $f^{-1}(W)$ is a submanifold. That is, f is made **transverse** to the submanifold W (see chapter 4). In fact, there are more powerful and flexible statements available than this. Whereas Thom's theorem involves the preimage of single maps, and multijet transversality involves the preimage of the product of a map with itself, such as $f \times f : M^2 \to N^2$, the theorems presented in this thesis can be applied to products of many maps, of which some factors may repeat. Moreover, it allows for inputs to be repeated, which can be useful for homotopies of products of maps. These technical results are inspired by similar results claimed in [18].

An example problem for which theorem 4.14 can be used is the following: let $f_1, f_2 : M \to N$ be smooth maps, and suppose that H_1 and H_2 are homotopies connecting the respective maps to new smooth maps g_1, g_2. Suppose we are given a submanifold W of N^2. Can one perturb the combined homotopy $H_1 \times H_2 : M^2 \times [0,1] \to N^2, (x, y, t) \mapsto (H_1(x,t), H_2(y,t))$ which connects $f_1 \times f_2$ to $g_1 \times g_2$, to a new homotopy that is transverse to W, and such that all interpolating maps are products? The theorem gives sufficient conditions for a positive answer to this question.

1.2 Main results

The main result we wish to present, theorem 1.4, is an h-principle type result about solutions of a certain non-local differential relation, the **d-delay differential relations** $\mathcal{R}_d$ (definition 6.2), where $d > 0$ is a real number. Let M denote one of the manifolds $[0,1]$, $\mathbb{R}$, or $\mathbb{S}^1 \cong [0,1]/\{0 \sim 1\}$. A smooth map $f : M \to \mathbb{R}^n$ is a **d-delay solution**, denoted $f \in \mathrm{Sol}(\mathcal{R}_d)$, if and only if for every $x \in M$,

$$f'(x) \neq f'(x \pm d).$$

The inequality in the d-delay relation is non-local, in the sense that it depends on values the map takes at two point that are not nearby. Inspired by Haefliger's embedding theorem (example 1.2), our main results shows there is a class of equivariant maps $g : M^2 \to (\mathbb{R}^n)^2$ to which the d-delay solutions are either weakly homotopy equivalent, or when $M = \mathbb{S}^1$, have isomorphic homotopy groups (up to some degree). We shall first introduce the formal data, before stating the result and giving some idea of the proof.

1.2.1 Formal data

Given a map $g : M \to \mathbb{R}^n$ that satisfies $g(x) \neq g(x \pm d)$ for all $x \in M$, one can wonder if g can be used to find an element $h \in \mathrm{Sol}(\mathcal{R}_d)$ such that h' is equal (up to homotopy) to g. I.e. can we 'integrate' (up to homotopy) g to find a d-delay solution? Maps $g : M \to \mathbb{R}^n$ satisfying $g(x) \neq g(x \pm d)$ for all $x \in M$ are solutions of the **derivative d-delay relation** $\mathcal{R}'_d$ (see also definition 6.2). Notationally, $g \in \mathrm{Sol}(\mathcal{R}'_d)$.

Let $\Delta_{\pm d} = \{(x, y) \in M^2 : y = x \pm d\}$ denote the d-shifted diagonal. Observe that g is an element of $\mathrm{Sol}(\mathcal{R}'_d)$ if and only if $g \times g(\Delta_{\pm d}) \cap \Delta(\mathbb{R}^n) = \varnothing$, where $g \times g : M^2 \times \mathbb{R}^{2n}$ is given by $g \times g(x, y) = (g(x), g(y))$. A generalization of the above question is: given a map $h : M^2 \to \mathbb{R}^{2n}$ satisfying $h(\Delta_{\pm d}) \cap \Delta(\mathbb{R}^n) = \varnothing$, can one find $g \in \mathrm{Sol}(\mathcal{R}'_d)$ such that $g \times g$ is homotopic to h?

We say that $h : M^2 \to \mathbb{R}^{2n}$ is a **formal d-delay solution** if $h(\Delta_{\pm d}) \cap \Delta(\mathbb{R}^n) = \varnothing$, and h is equivariant, i.e. $h_1(x, y) = h_2(y, x)$ for all $(x, y) \in M \times M$, where $h_1, h_2 : M^2 \to \mathbb{R}^n$ are the maps such that $h = (h_1, h_2)$. We denote this by $h \in \mathrm{Sol}_F(\mathcal{R}_d)$ (definition 6.2).

The spaces of solutions and formal solutions inherit the weak Whitney C^∞-topology, as a subspace (see definition 2.5).

1.2.2 The h-principle for the delay relation

Our main result shows that for $M = [0,1]$ and $M = \mathbb{R}$, there is a weak homotopy equivalence between d-delay solutions and their formal counterparts, the formal d-delay solutions. The main point is that the existence and classification question for $\mathbb{S}^k$-families of d-delay solutions can be replaced by the same question for the formal d-delay solutions. The existence and classification of formal solutions is very computable: one can show that formal d-delay solutions, contained in $C^\infty(M^2, \mathbb{R}^{2n})$, can be identified with the space of maps $C^\infty(M, \mathbb{S}^{n-1})$, which is homotopy equivalent to $\mathbb{S}^{n-1}$. Hence the existence and classification of such maps becomes particularly easy.

Remark 1.3. At first glance, this result may look more straightforward than it is: since M is contractible in both cases, perhaps one could prove a weak homotopy equivalence by reducing to the case where M is an arbitrarily small interval. However, the non-locality of the d-delay relation makes deformations of the base space more difficult, and hence the solution more subtle than this. $\triangle$

The main result also shows that, if $M = \mathbb{S}^1$, there are isomorphisms of homotopy groups between true d-delay and formal d-delay solutions, up to a certain dimension of homotopy group, which depends on d and and the dimension of $\mathbb{R}^n$. It is not known to the author if there definitively do or do not exist isomorphisms for the higher dimensional homotopy groups. In this case, the space of formal solutions can be identified with $C^\infty(\mathbb{S}^1, \mathbb{S}^{n-1})$.

As announced, this result is in part a proof of concept: it shows that equivariant maps can be used to prove non-local h-principles, and shows that there is a **removal of singularities**-type technique that can be used more broadly than its original application in [18]. In [18] the embedding relation is studied instead, and the method requires some other results than those we use here. The choice for the d-delay differential relations, as well as M being $[0,1], \mathbb{R}$ or $\mathbb{S}^1$, is to keep this proof of concept simple enough.

Theorem 1.4. *Let $0 < d < 1$, and assume that $M = \mathbb{R}$ or $M = [0,1]$. Then*

$$T \times T : \mathrm{Sol}(\mathcal{R}_d) \to \mathrm{Sol}_F(\mathcal{R}_d), \quad f \mapsto f' \times f'$$

is a weak homotopy equivalence.

Assume that $M = \mathbb{S}^1$, then $T \times T : \mathrm{Sol}(\mathcal{R}_d) \to \mathrm{Sol}_F(\mathcal{R}_d)$ induces maps

$$(T \times T)_k : \pi_k(\mathrm{Sol}(\mathcal{R})_d, f) \to \pi_k(\mathrm{Sol}_F(R_d), f' \times f'),$$

which are

- *isomorphisms for all $k \geq 0$ if d is irrational.*
- *isomorphisms for all $k \geq 0$ if $d = 1/2$.*
- *isomorphisms for all $k < q(n-1) - 2$ if $d = p/q$ is rational, $p, q \in \mathbb{N}_1$ coprime.*

1.2.3 Techniques and further results

The proof of theorem 1.4 is split into two parts: we first show in theorem 6.4 that there is a weak homotopy equivalence between d-delay solutions and formal derivative d-delay solutions, whose proof is based on H. Whitney's classification of regular immersions from $\mathbb{S}^1$ to $\mathbb{R}^2$ [20]. Second, we show in section 6.2 that there are isomorphisms of homotopy groups of derivative d-delay solutions, and formal d-delay solutions (up to a certain dimension of $M = \mathbb{S}^1$).

A minor result of thesis is the application of convex integration to a particular class of non-local partial differential relations, theorem 3.25, such as the d-delay differential relation. That is to say, we can make use of the fact that the d-delay differential relation is not *infinitesimally* non-local, unlike (for example) the embedding problem. Specifically, for every point in M, there is a neighbourhood of that point in which no two points interact through the relation. This can be used to apply convex integration in a small neighbourhood around a point, to deform a formal d-delay solution of the form $F \times F$ to a true solution. In particular, this technique could be used instead of Whitney's technique to prove theorem 6.4.

The proof of the second step in theorem 1.4 requires the most important technical result in this thesis for non-local differential relations, an equivariant jet transversality theorem, and a multijet counterpart: theorem 5.11 and theorem 5.14. When looking at non-local differential relations, such as the embedding relation in theorem 1.2 or at our main result, it appears that a key feature of the formal solutions $g : M^2 \to N^2$ is that they are **equivariant**, i.e. $g = (g_1, g_2) : M^2 \to N^2$ satisfies $g_1(y, x) = g_2(x, y)$ or equivalently $(g_1, g_2)(y, x) = (g_2, g_1)(x, y)$ for every $(x, y) \in M \times M$. It is therefore useful to have a variation of the Thom transversality theorem (theorem 4.2) and multijet transversality theorem (theorem 4.5), which guarantees the existence of an arbitarily small perturbation of an equivariant map g that is transverse to a chosen submanifold W, while remaining equivariant. We shall show that these two theorems can be seen as consequences of theorems 4.14 and 4.18 respectively, which are generalization of the Thom transversality theorem and multijet transversality theorem.

After having written down theorems 4.14 and 4.18, it turned out that a specific case, related to the study of regular homotopies between immersions, had already been published by G.K. Francis and R. Bott in [4]. The language is, accidentally, very similar.

The results presented need some careful analysis of the strong and weak Whitney C^∞ topology with which one can endow the space of smooth functions between manifolds. In particular, the new transversality theorems, and study and deformation of formal d-delay solutions, rely on this. Hence, a substantial part of this thesis and the appendix is devoted to it.

1.3 Layout of the thesis

The structure of the thesis is mostly linear: most chapters start by introducing concepts and auxiliary results, and finish with main or supporting results.

In chapter 2 we recall the definition of a jet bundle of a pair of manifolds, jets of smooth maps between manifolds, and how they can be used to define the strong and weak Whitney topologies on the space of smooth functions. In particular we recall that the space of smooth functions with the (weak or strong) Whitney smooth topology is a Baire space. This lays down the theoretical foundation for many of the new results. The remainder of the chapter is used to prove some preliminary results about the map which sends a tuple $(f_1, \ldots, f_n)$ of smooth maps $f_j : X_j \to Y_j$ to the product map $f_1 \times \ldots \times f_n : \prod_{j=1}^n X_j \to \prod_{j=1}^n Y_j$, and the product of smooth function spaces. In particular, we determine some topological properties of this map, and show that the space of smooth functions, with either the weak or strong Whitney smooth topology, preserves the Baire property under finite Cartesian products. The main reference is [5, sections II.2 and II.3], with an additional reference being [13, section 2.1].

Chapter 3 consists of a collection of preparatory definitions and results, and ends with the proof of theorem 3.25, which concerns non-local convex integration. In the first subsection, we hammer out what we understand by the weak homotopy equivalence mentioned in theorem 1.4, and how it will be studied in this thesis. For the basic definitions, the main reference is [11, chapter 4]. Subsequently, we introduce the h-principle and recall the definition and some examples of a partial differential relation. We introduce a technique which is used in chapter 6, which is the proof of theorem 3.11 (due to H. Whitney). The final subsection of this chapter is spent on recalling a simple version of convex integration, defining delay-type non-local PDRs, and proving a new application of conventional convex integration, theorem 3.25. The main reference for convex integration is [3, chapter 17].

In chapter 4 we take leave of our study of differential relations for a moment, and focus on studying transversality of product maps and families of product maps. In the first subsection we recall what it means for a smooth map $f : X \to Y$ to be transverse to a submanifold $W \subset Y$, and why this is of interest. We recall the Thom transversality and multijet transversality theorems, and in the first two subsections (respectively) generalize these theorems to product maps, as introduced in chapter 2, that possibly depend on the same factor multiple times, theorems 4.14 and 4.18. These theorems are inspired by results claimed in [18], which needed an extra condition to be true, which we discuss briefly. In the final subsection we conclude with a corollary about the transversality of preimage manifolds. The main reference of this chapter is [5, section II.4].

In chapter 5 we return to the studying non-local differential relations, and study the set of equivariant smooth maps as a subset of all smooth maps. We prove some elementary results, and use the results of the previous chapter to find equivariant (multijet) transversality theorems, in particular theorem 5.11 and theorem 5.14. This lays the foundation for the final chapter.

In the final chapter, chapter 6, we introduce the d-delay differential relation, formal d-delay solutions, and the derivative d-delay differential relation. Subsequently, we prove the main result, theorem 1.4. The elementary results about equivariant maps, and the equivariant transversality theorems of the previous chapter are used in a 'removal of singularities'-type argument to prove the final theorem. This argument is based on the argument provided in [18].

The appendix is used to prove the results of chapter 4. Most results are generalizations of [5, sections II.3 and II.4].

2 Jet bundles and Whitney topologies

Here, and throughout this thesis, we shall be making the distinction between product maps and the coordinate maps of a single map. To be exact:

- let M, N_1, N_2 be sets, and let $g : M \to N_1 \times N_2$ be a function. Denote its coordinate functions by $g_1 : M \to N_1$ and $g_2 : M \to N_2$. I.e. $g = (g_1, g_2)$.
- Let M_1, M_2, N_1, N_2 be sets, and let $f_i : M_i \to N_i$ be functions. Denote by $f_1 \times f_2 : M_1 \times M_2 \to N_1 \times N_2$ the map $(x, y) \mapsto (f_1(x), f_2(y))$.

In this section, we first recall the basic definitions and properties of **jet bundles**, which are smooth fiber bundles most often used to study smooth maps f between two manifolds X and Y. The k-th jet bundle of a pair of manifolds X and Y, denoted $J^k(X, Y)$, roughly speaking, records the partial derivatives of a smooth map f up to and including order k.

Jet bundles are used in two ways in this thesis. They are used in chapter 3 to make concrete the notion of partial differential relations, both local and non-local. In this chapter we focus on their second use: jet bundles can be used to endow the space of smooth functions $C^\infty(X, Y)$ with a number of topologies, called the **Whitney topologies**, named after Hassler Whitney. Roughly speaking, two smooth maps $f, g : X \to Y$ are 'nearby' in these topologies if for every point in X, their partial derivatives are close to each other. We recall the *weak* and *strong* Whitney C^∞ topology in particular, which shall be the only two of the Whitney topologies of interest in this thesis. We subsequently recall a number of properties of these two topologies. The most important of these for our purposes is that $C^\infty(X, Y)$ is a **Baire space** with either topology, i.e. a space with the **Baire property**: a countable intersection of open and dense sets in $C^\infty(X, Y)$ is dense.

Definition 2.1. Let A be a subspace of a topological space B. Then A is called a residual set if A is a countable intersection of open and dense sets. △

Remark 2.2. The Baire property of these topologies plays a key role when applying classical *transversality theorems*, theorem 4.2 and 4.5. If a map $f : X \to Y$ is *transverse* to a submanifold $W \subset Y$ (see chapter 4), the preimage $f^{-1}(W)$ is a manifold. Very briefly, the heart of the these theorems is showing that the set of maps f transverse to a fixed submanifold W is residual. The Baire property implies that the set of transverse maps is therefore dense within the set of all smooth maps, which is usually the key property theorem 4.2 and 4.5 are used for. △

In chapters 4 and 5 we introduce variations of the classical transversality theorems. Given families of manifolds $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ and families of maps between them $f_j : X_j \to Y_j$, these theorems focus on product maps

$$f_1 \times \ldots \times f_n : X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$$

and the products of jets

$$j^k f_1 \times \ldots \times j^k f_n : X_1 \times \ldots \times X_n \to \prod_{j=1}^{n} J^k(X_j, Y_j).$$

In the last part of this chapter we preempt these new transversality theorems, and focus on the relation between families of maps in $\prod_{j=1}^{n} C^\infty(X_j, Y_j)$ and the product functions they produce, lying in $C^\infty(X, Y)$. Here X and Y denote the product manifolds, whose factors are the members of the respective families. Specifically, if all factors of the former, as well as the latter, are endowed with the strong or weak Whitney C^∞ topology, we determine if the inclusion map $(f_1, \ldots, f_n) \mapsto f_1 \times \ldots \times f_n$ is continuous or even an embedding. Subsequently we determine for either case if $\prod_{j=1}^{n} C^\infty(X_j, Y_j)$ has the Baire property.

2.1 Jet bundles

We recall the definition of jet bundles, and a number of their basic properties. This summarizes a number of results of [5, section II.2].

Let X and Y be manifolds, then **the k-th jet bundle of X and Y**, $J^k(X,Y)$, is a smooth fiber bundle over $X \times Y$. The fiber $J^k(X,Y)_{(x,y)}$ consists of equivalence classes "$\sim_k$ at x" of smooth mappings $f : X \to Y$ with $f(x) = y$, where the equivalence relation is k-th order contact at x. More precisely, **the k-th order contact at** x condition is defined as follows: define $f, g : X \to Y$ to have 0-th order contact at x if $f(x) = g(x)$. Inductively define f to have k-th order contact at x with g if Tf has $k-1$-th order contact with Tg at x. Here we see Tf and Tg as smooth maps between the manifolds TX and TY.

The k-th order contact at x of two smooth function from $\mathbb{R}^m$ to $\mathbb{R}^n$ can also be formulated in terms of derivatives.

Lemma 2.3 (Lemma II.2.2 of [5])**.** *Let U be an open subset of $\mathbb{R}^m$ and p be a point in U. Let $f, g : U \to \mathbb{R}^n$ be smooth maps. Then $f \sim_k g$ at p if and only if*

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

for every multi-index α with $|\alpha| \leq k$ and $1 \leq i \leq n$ where f_i and g_i are the coordinate functions determined by f and g, respectively, and $x_1, \ldots, x_m$ are coordinates on U.

As a corollary, one finds that smooth maps $f, g : U \to \mathbb{R}^n$ have k-th order contact at x if and only if the Taylor polynomial of f and g agree up to and including order k at x. More generally, for smooth maps $f, g : X \to Y$ to have k-th order contact at x, it is necessary and sufficient that for each choice of charts around x and $f(x) = g(x)$, all mixed partial derivatives of order up to and including k are equal. Equivalently, for each such choice of charts, the Taylor polynomial corresponding to those charts agree up to and including order k at x.

To explain the smooth and bundle structures of $J^k(X,Y)$, we shall first define it over open subsets of Euclidean space: suppose $X = U$ and $Y = V$ are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let A^k_m denote the space of all real valued polynomials in m variables with vanishing constant term, whose order is at most k. By taking the coefficients of the polynomials in A^k_m as coordinates, A^k_m is endowed with a smooth structure, i.e. that of a real vector space. Let $B^k_{m,n} = \bigoplus_{i=1}^n A^k_m$, which then obtains a smooth structure from its factors. For a smooth map $f : U \to \mathbb{R}$, define $T_k f : U \to A^k_m$ by mapping x to the k-th order Taylor polynomial of f at x (omitting the constant term). One can show that the map

$$T_{U,V} : J^k(U,V) \to U \times V \times B^k_{m,n}, \quad [f]_{(x,y)} \mapsto (x, f(x), T_k f_1, \ldots, T_k f_n)$$

is a bijection. The smooth structure of the right hand side defines a smooth structure for the left hand side.

For general manifolds X and Y, let $\phi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ be charts around $x \in X$ and $y \in Y$, respectively. Denote $U' = \phi(U)$ and $V' = \psi(V)$. Then there is a bijection

$$\psi_* \phi^* : J^k(X,Y)_{U,V} \to J^k(U',V'), \quad [f]_{(x,y)} \mapsto [\psi \circ f \circ \phi^{-1}]_{\phi(x),\psi(y)}.$$

See also [5, proposition II.2.5]. The smooth structure of the right hand side again defines a smooth structure for the left hand side. Bundle trivializations of $J^k(X,Y)$ over $U \times V$ are defined by $T_{U,V} \circ \psi_* \circ \phi^*$, and the fiber of $J^k(X,Y)$ is $B^k_{m,n}$, the n-fold space of polynomial of at most order k, in m variables (without constant term).

Theorem II.2.7 of [5] verifies that this choice of bundle charts indeed endows each jet bundle with a smooth structure, such that $\pi : J^k(X,Y) \to X \times Y$ is a submersion (and hence we have a fiber bundle). The transition functions are linear for $J^1(X,Y)$, but generally not even affine for higher order k-jets.

Example 2.4. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the diffeomorphism $x \mapsto x^3 + x$. Using the identity map of $\mathbb{R}$ as a chart twice, we can identify $J^2(\mathbb{R},\mathbb{R})$ with $\mathbb{R}^2 \times B^2_{1,1} = \mathbb{R}^2 \times A^2_1$. Here A^2_1 is the subspace space of polynomials in one variable, of the form $a_2x^2 + a_1x$. The global charts $(\mathrm{id}_{\mathbb{R}}, \mathbb{R})$ and $(\psi, \mathbb{R})$ induce a diffeomorphism $\psi_*\mathrm{id}^*_{\mathbb{R}}$ on $J^k(\mathbb{R},\mathbb{R})$, and the corresponding transition function on $\mathbb{R}^2 \times A^k_1$ is given by $(x, y, a_2z^2 + a_1z) \mapsto (x, y^3 + y, (3ya_2 + 3a_1^2 + a_2)z^2 + (3ya_1 + a_1)z)$. It is clear that these transition functions are not affine. △

Although the transition functions may not be so nice for higher degree jet bundles, there are maps between jet bundles whose degree differs by one,

$$\pi_{k,k-1} : J^k(X,Y) \to J^{k-1}(X,Y), \quad [f]_{(x,y)} \mapsto [f]_{(x,y)}, \quad k \geq 1,$$

which are well-defined, and which give another fiber bundle structure. Its fiber can be identified with all polynomials representing points in $J^k(X,Y)$ that have fixed $(k-1)$-th and lower order terms. The chain rule implies that the transition functions are affine.

Note that for every smooth map $f : X \to Y$ there is now a canonically defined map, the **k-jet of f** or the **k-jet extension of f**, $j^kf : X \to J^k(X,Y)$, defined by $j^kf(p) = [f]_{p,f(p)}$. Theorem II.2.7 also verifies that j^kf is smooth. For an example, observe that $j^0f(p) = (p, f(p))$, and hence we may think of j^0f as the graph of f contained in $J^0(X,Y) = X \times Y$.

It will often be useful to consider not only the bundle projection $\pi : J^k(X,Y) \to X \times Y$, but also the composite projections $\mathfrak{s} : J^k(X,Y) \to X$ and $\mathfrak{t} : J^k(X,Y) \to Y$, which will respectively be referred to as the **source map** and the **target map**. It is important to remark that the k-jet of a function is a section of the fiber bundle $\mathfrak{s} : J^k(X,Y) \to X$.

Another notion we shall make use of is a **multijet bundle**: the **s-fold r-jet of X and Y** or simply **multijet of X and Y** is given by $(J^r(X,Y))^s$. It should be clear that the multijet is a smooth fiber bundle over $(X \times Y)^s$. We call the map induced from $J^r(X,Y)^s$ to X^s the **source map**, and to Y^s the **target map**, in correspondence with the ordinary jet bundles. Similarly, for a smooth map $f : X \to Y$ the **s-fold r-jet of f**, **s-fold r-jet extension of f**, or simply the **multijet of f**, $j^r_sf : X^s \to J^r(X,Y)^s$, is given by

$$(x_1, \ldots, x_n) \mapsto (j^rf(x_1), \ldots, j^rf(x_n)).$$

It should be remarked that sometimes the multijet bundle is instead defined as the restriction of $\mathfrak{s} : J^r(X,Y)^s \to X^s$ to the fibers over the complement of **the large diagonal** $\Delta^{(s)}(X)$ (see below). One of the reasons for this can be found in the formulation of theorem 4.5.

Let S be a set. Here and throughout the thesis, we make the distinction between the **small diagonal** $\Delta^n(S)$ and **large diagonal** $\Delta^{(n)}(S)$ contained in the product of sets S^n for some $n \geq 1$. Here

- $\Delta^n(S)$ is given by $\{(s_1, \ldots, s_n) \in S^n : s_i = s \in S \;\forall i\}$.
- $\Delta^{(n)}(S)$ is given by $\{(s_1, \ldots, s_n) \in S^n : s_i = s_j \text{ for some } 1 \leq i < j \leq n\}$.

2.2 Whitney topologies

Jet bundles allow us to compare two smooth maps $f, g : X \to Y$: we can think of f and g as being close to each other if the images of j^kf and j^kg lie close to another inside $J^k(X,Y)$. I.e. we think of f and g being close to each other if their partial derivatives are close to each other. With this idea, we can endow **the space of smooth functions between X and Y**, $C^\infty(X,Y)$, with a number of topologies.

Definition 2.5. Let X and Y be smooth manifolds.

(i) Fix a non-negative integer k. Let K be a subset of X and U be a subset of $J^k(X,Y)$. Then denote by $M(K,U)$ the set
$$\{f \in C^\infty(X,Y) : j^k f(K) \subset U\}.$$

(ii) In the special case that $K = X$, denote by $M(U)$ the set $M(X,U)$.

(iii) The family of sets $\{M(K,U)\}$ where $K \subset X$ is compact and $U \subset J^k(X,Y)$ is open, form a subbasis for a topology on $C^\infty(X,Y)$. This topology is called *the weak Whitney C^k topology* or *the compact-open C^k topology.*

(iv) The family of sets $\{M(U)\}$ where $U \subset J^k(X,Y)$ is open, form a basis for a topology on $C^\infty(X,Y)$, too. This topology is called *the strong Whitney C^k topology* or simply *the Whitney C^k topology.*

(v) *The weak Whitney C^∞ topology* on $C^\infty(X,Y)$ is the topology whose subbasis is $W = \bigcup_{k=0}^\infty W_k$, where W_k is the subbasis of the weak Whitney C^k topology. If $C^\infty(X,Y)$ is endowed with this topology, we denote this topological space by $C^\infty_W(X,Y)$.

(vi) *The Whitney C^∞ topology* on $C^\infty(X,Y)$ is the topology whose basis is $V = \bigcup_{k=0}^\infty V_k$, where V_k is the basis of the Whitney C^k topology. If $C^\infty(X,Y)$ is endowed with this topology, we denote this topological space by $C^\infty_S(X,Y)$. △

Remark 2.6.
- Let $M(U_1)$ and $M(U_2)$ be open subsets of $C^\infty_S(X,Y)$. Observe that $M(U_1) \cap M(U_2) = M(U_1 \cap U_2)$, but that a similar identity for unions is generally false.
- The weak Whitney C^∞ and strong Whitney C^∞ topology have a well defined (sub)basis, since the $W_k \subset W_l$ and $V_k \subset V_l$ whenever $k \leq l$. To see this, use the canonical mapping $\pi^l_k : J^l(X,Y) \to J^k(X,Y)$, which assigns each $\sigma = [f]_{p,f(p)}$ to the class in $J^k(X,Y)_{(p,f(p))}$ also represented by f. Then $M(K,U) = M(K,(\pi^l_k)^{-1}(U))$ and $M(U) = M((\pi^l_k)^{-1}(U))$.
- The symbol C^∞ is also be referred to as 'smooth'. I.e. $C^\infty_W(X,Y)$ is endowed with the weak Whitney smooth topology. △

In this thesis we will be concerned with the weak and strong Whitney C^∞ topologies. These topologies have a number of important properties, listed in the proposition below. For a more complete background, we refer to [13, chapter 2].

Proposition 2.7. *Let X and Y be manifolds.*

(i) The weak Whitney C^∞ is weaker than the strong Whitney C^∞ topology.

(ii) The weak and strong Whitney C^∞ topologies are equal whenever X is compact.

(iii) $C^\infty_W(X,Y)$ is a complete metric space, and in particular a Baire space.

(iv) $C^\infty_S(X,Y)$ is a Baire space.

Proof. It is easy to verify that for every compact set $K \subset X$ and open set $U \subset J^k(X,Y)$, $M(K,U) = M(U')$, where $U' = (\mathfrak{s}^{-1}(K) \cap U) \cup \mathfrak{s}^{-1}(X \setminus K)$. To see that U' is open, note that U is open and contained in U', and that K is closed. From this it follows that the weak Whitney C^∞ topology is indeed weaker. If X is compact, the strong open sets $M(U) = M(X,U)$ are of course weak open, too. For (iii), [13, theorem 2.4.1] shows that $C^\infty_W(X,Y)$ can be endowed with a metric compatible with the topology, for which the space is complete. The Baire category theorem implies that it is hence Baire. For (iv), we refer to [5, proposition II.3.3]. Below we prove a straightforward generalization of this proposition, and hence omit it here. □

2.3 Product Whitney topologies

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two families of manifolds, and let $f_j : X_j \to Y_j$ be a family of smooth maps. Denote by X the product $X_1 \times \ldots \times X_n$, and denote by Y the corresponding product space. In what follows, we will be interested in product maps $f_1 \times \ldots \times f_n : X \to Y$ given by $(x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n))$. Such product maps form a subspace of $C^\infty(X, Y)$, and it is useful to compare the subspace topology endowed by weak (or strong) Whitney C^∞ topologies with that of the product topology given to $\mathcal{C} := \prod_{j=1}^n C^\infty(X_n, Y_n)$, where each factor has the weak (or strong) topology. We denote by $p : \mathcal{C} \to C^\infty(X, Y)$ the map sending a family of maps $(f_1, \ldots, f_n)$ to its product map $f_1 \times \ldots \times f_n$.

Proposition 2.8. *With the above notation, endow $\mathcal{C}$ with the product topology induced by the weak Whitney C^∞ topology on each factor, and endow $C^\infty(X, Y)$ with the same topology. Then p is an embedding with closed image.*

Proof. Note that for every integer $k \geq 0$, there exists a (smooth) embedding $i_k : \prod_{j=1}^n J^k(X_j, Y_j) \to J^k(X, Y)$ given by $([f_1]_{x_1,y_1}, \ldots, [f_n]_{x_n,y_n}) \mapsto [f_1 \times \ldots \times f_n]_{x,y}$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. To see that this is indeed an embedding, one can use product charts on X and Y to find trivializations of the left and right hand side. For a product map $f = f_1 \times \ldots \times f_n : X \to Y$, it moreover holds that $i_k \circ (j^k f_1 \times \ldots \times j^k f_n) = j^k f$, and hence that $\operatorname{im} j^k f \subset \operatorname{im} i_k$.

Let $K_j \subset X_j$ be compact sets, and $U_j \subset J^{k_j}(X_j, Y_j)$ be open sets. Without loss of generality, we may assume that $k_j = k$ for each $1 \leq j \leq n$. It is easy to verify that

$$p\left(\prod_{j=1}^n M(K_j, U_j)\right) = M\left(\prod_{j=1}^n K_j, i_k\left(\prod_{j=1}^n U_j\right)\right) \cap \operatorname{im} p.$$

From this identity it follows that p is an open map (onto its image): as i_k is an embedding and each U_j is open, it easy to extend $i_k\left(\prod_{j=1}^n U_j\right)$ to an open set.

It is left to show that p is continuous with closed image. Let $K \subset X$ be compact and $U \subset J^k(X, Y)$ be open. Let $(f_1, \ldots, f_n) \in p^{-1}(M(K, U))$, and denote $f = p(f_1, \ldots, f_n)$. Let U' denote $U \cap \operatorname{im} i_k$. We shall indentify the domain of i_k with its image. We construct a cover of compact product neighbourhoods $\{Z_x : x \in K\}$ of $K \subset X$, such that each for each $x \in K$ there exists an open subset $V_x \subset U'$. This subset $V_x = \prod_{j=1}^n V_x^j$ is a product of open sets $V_x^j \subset J^k(X_j, Y_j)$, and for all x' in $Z_x = Z_x^1 \times \ldots \times Z_x^n$, $j^k f(x') \in V_x \subset U'$.

Let $x \in K$. As U' is open in $\prod_{j=1}^n J^k(X_j, Y_j)$ and $j^k f(x) \in U'$, there exists open product sets $V_x^1, \ldots, V_x^n$ with $V_x^j \subset J^k(X_j, Y_j)$, such that $V_x := \prod_{j=1}^n V_x^j \subset U'$ and $j^k f(x) \in V_x$. By the continuity of $j^k f$, there exists a product open neighbourhood $Z' = Z_1' \times \ldots \times Z_n'$ in X of x, such that $j^k f(Z') \subset V_x$. As each X_j is a manifold, we can replace Z' by $Z = Z_1 \times \ldots \times Z_n$, which has the same properties as Z, but each factor of Z is a compact neighbourhood of x. This completes the construction.

By the compactness of K, we can select a finite subcover $V_{x_1}, \ldots, V_{x_m}$ for some $m \geq 0$. It follows that $f_j \in M(Z_j, V_{x_i}^j)$ for each $1 \leq j \leq n$ and $1 \leq i \leq m$, and that

$$\mathcal{V} := \bigcap_{i=1}^m \prod_{j=1}^n M(Z_j, V_{x_i}^j)$$

is an open neighbourhood of $(f_1, \ldots, f_n)$ such that $p(\mathcal{V}) \subset M(K, U)$. This proves the continuity of p.

Finally, to see that the image of p is closed, one can observe that the complement is open. Suppose that $g : X \to Y$ is not a product of functions, then $\operatorname{im} j^1 g$ is not contained in $\operatorname{im} i_1$. In particular, there exists $x \in X$ such that $j^1 g(x)$ does not lie in im i_1. One can show that im i_1 is closed, and hence that $\operatorname{im} i_1 \cap \alpha^{-1}(x)$ is closed. If we let W be the complement of this intersection, it follows that $g \in M(\{x\}, W)$, which is open and disjoint from the set of product functions. $\square$

Remark 2.9. The smooth embeddings $i_k : \prod_{j=1}^n J^k(X_j, Y_j) \to J^k(X, Y)$ will play a role later on, too. As is stated in the above proof, the following diagram commutes:

$$\begin{array}{ccc} \prod_{j=1}^n C^\infty(X_j, Y_j) & \xrightarrow{(j^k)^n} & \prod_{j=1}^n C^\infty(X_j, J^k(X_j, Y_j)) \\ \Big\downarrow p & & \Big\downarrow (i_k)_* \\ C^\infty(X, Y) & \xrightarrow{\quad j^k \quad} & C^\infty(X, J^k(X, Y)) \end{array}$$

Hereafter we shall consider at length products of funtions, and products of jets of functions. Although it is not yet clear, it is crucial for the study of transversality to realize that the jet of a product map, $j^k(f_1 \times \ldots \times f_n)$, takes image in the product of jet bundles $\prod_{j=1}^n J^k(X_j, Y_j)$, seen as a subspace of $J^k(X, Y)$. See also remark 4.7 and the discussion preceding it. $\triangle$

Corollary 2.10. *The space $\mathcal{C}$ with the product topology induced by the weak Whitney C^∞ topology on each factor is a Baire space.*

Proof. As $\mathcal{C}$ is homeomorphic to a closed subset of a complete metric space, it is itself a complete metric space. By the Baire category theorem, it has the Baire property. Alternatively, as a product of complete metric spaces, $\mathcal{C}$ is a complete metric space, and hence has the Baire property by the Baire category theorem. $\square$

Corollary 2.11. *Let $k \geq 0$ be an integer, and let $U \subset J^k(X, Y)$ be an open set. Let V be the complement of U. If $\overline{\mathfrak{s}(V)}$ is compact, then $p^{-1}M(U)$ is open in $\mathcal{C}$, where each factor has the strong Whitney C^∞ topology.*

Proof. Observe that if $\mathfrak{s}(U) \neq X$, then $M(U) = \varnothing$, and hence the result would be trivial. So assume without loss of generality that $\mathfrak{s}(U) \neq X$. The set $\mathfrak{s}(V)$ consists of all those points x in X for which U does not contain the entire fiber $\mathfrak{s}^{-1}(x)$. It follows that $M(U) = M(\mathfrak{s}(V), U) = M(\overline{\mathfrak{s}(V)}, U)$. The proof of proposition 2.8 showed that $p^{-1}M(\overline{\mathfrak{s}(V)}, U)$ was open in $\mathcal{C}$ when each factor had the weak topology. Hence the same preimage is open when each factor has the strong topology. $\square$

Proposition 2.8 proves useful in the study of transversality of product maps. Unfortunately, a similar proposition is false when each factor of $\mathcal{C}$ has the strong Whitney C^∞ topology : the following example will show that, in general, the map p is not continuous in this case. Of course, if all $X_1, \ldots, X_n$ are compact, p will be continuous regardless of the choice of topology.

Example 2.12. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant map $x \mapsto 0$. Let $A \subset \mathbb{R}^4 \cong J^0(\mathbb{R}^2, \mathbb{R}^2)$ be the open set $\{(x, y, z, w) \in \mathbb{R}^4 : |xw| < 1, |yz| < 1\}$. It is clear that $M(A)$ is an open subset of $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, and that $f \times f \in M(A)$. In fact, it is straightforward to verify that $f \times f$ is the only map in the image of $p : C^\infty(\mathbb{R}, \mathbb{R})^2 \to C^\infty(\mathbb{R}^2, \mathbb{R}^2) : (g, h) \mapsto g \times h$ contained in $M(A)$. It follows that p is not continuous: if it were, $p^{-1}(M(A))$ would contain a product neighbourhood of (f, f). One can show that any open neighbourhood of f in $C^\infty(\mathbb{R}, \mathbb{R})$ has multiple maps in it, a contradiction. $\triangle$

Fortunately, $\mathcal{C}$ is still a Baire space, when each factor has the strong Whitney C^∞ topology. The following proposition is a straightforward generalization of [5, proposition II.3.3].

Proposition 2.13. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be manifolds. Endow $\mathcal{C}$ with the product topology induced by the Whitney topology on each factor. Then $\mathcal{C}$ is a Baire space.*

Proof. Endow for every $s \geq 0$ and $1 \leq j \leq n$ the manifold $J^s(X_j, Y_j)$ with a metric $d_{j,s}$, which makes $J^s(X_j, Y_j)$ into a complete metric space. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a countable sequence of open dense subsets of $C^\infty(X, Y)$ and let $\mathcal{V}$ be another non-empty open subset of $\mathcal{C}$. We must show that $\mathcal{V} \cap \bigcap_{i=1}^\infty \mathcal{U}_i \neq \varnothing$. Since $\mathcal{V}$ is open in the the product topology, there exist open subsets $W^1, \ldots, W^n$, $W^j \subset J^{k_j}(X_j, Y_j)$, such that $N := \prod_{j=1}^n M(\overline{W}^j) \subset \mathcal{V}$ and $M := \prod_{j=1}^n M(W^j) \neq \varnothing$. It is enough to show that $N \cap \bigcap_{i=1}^\infty \mathcal{U}_i \neq \varnothing$.

To do this, we inductively choose a sequence of functions $f_{1j}, f_{2j}, \ldots$ for each j; a sequence of integers $k_{1j}, k_{2j}, \ldots$ for each j; and for each $i \in \mathbb{N}$ and j an open subset W_{ij} in $J^{k_{ij}}(A_j, X_j)$ satisfying

(A_i) $(f_{i1}, \ldots, f_{in}) \in M \cap \left(\bigcap_{k=1}^{i-1} \prod_{j=1}^n M(W_{kj})\right) \cap \mathcal{U}_i$.

(B_i) $\prod_{j=1}^n M(\overline{W_{ij}}) \subset \mathcal{U}_i$ and $(f_{i1}, \ldots, f_{in}) \in \prod_{j=1}^n M(W_{ij})$.

(C_i) $(i > 1)$ $d_{j,s}(j^s f_{ij}(x), j^s f_{(i-1)j}(x)) < 1/2^i$ for all $x \in X_j$, $1 \leq s \leq i$, and $1 \leq j \leq n$.

We first show that by choosing the above data we can prove the theorem. Define $g_j^s = \lim_{i\to\infty} j^s f_{ij}(x)$. This is well defined as each $J^s(X_j, Y_j)$ is a complete metric spaces, and condition (C) implies that for each $x \in X_j$ the sequence $j^s f_{1j}(x), j^s f_{2j}(x), \ldots$ is a Cauchy sequence. We can define $g_j : A_j \to X_j$ by $g_j^0(x) = (x, g_j(x))$. We claim that g is smooth. If so, we are done: for each i, the tuple $(f_{i1}, \ldots, f_{in})$ lies in M by (A), and thus $(g_1, \ldots, g_n) \in N$. By (B), the subsets W_{sj} were chosen so that $\prod_{j=1}^n M(\overline{W_{sj}}) \subset \mathcal{U}_s$ and by (A) each f_{ij} for $i > s$ was chosen to be in $M(W_{sj})$, thus $g_j = \lim_{i\to\infty} f_{ij}$ is in $M(\overline{W_{sj}})$ for every s. Hence, $(g_1, \ldots, g_n) \in N \cap \bigcap_{s=1}^\infty \mathcal{U}_s$.

It remains to show that each g_j is smooth and that we can choose the above data. The former is a local question: let $x \in X_j$ and let U be an open neighbourhood of x, and let V be an open neighbourhood of $g(x) \in Y_j$, such that both are chart domains. Select a compact neighbourhoods K of x contained in U and L of $g(x)$ such that $g(K) \subset L \subset V$. It suffices to show that g is smooth at x. Since the metrics $d_{j,s}$ are compatible with the topology on $J^s(X_j, Y_j)$, (C) implies that $(j^s f_{ij})_{i=1}^\infty$ converges uniformly on K for each s. Using the local coordinates of U and V, we see that the coordinate functions of the maps $j^s f_{ij}$ are just $\partial^{|\beta|} f_{ij}/\partial x^\beta$ for $|\beta| \leq s$. Thus locally $\partial^{|\beta|} f_{ij}/\partial x^\beta$ converges uniformly on K. Using a classical theorem [2, 8.6.3], for every $|\beta| \leq s$, the limit $\lim_{i\to\infty}(\partial^{|\beta|} f_{ij}/\partial x^\beta)(x)$ exists, and is equal to $(\partial^{|\beta|} g_j/\partial x^\beta)(x)$. As s is arbitrary, all partial derivatives of g exist at x, and g is smooth at x.

Finally, we show that we can choose the above data inductively. For the base step, choose $(f_{11}, \ldots, f_{1n}) \in M \cap \mathcal{U}_1$. This is possible since M is open and non-empty, while U_1 is dense. Thus (A_1) is satisfied. Since $\mathcal{U}_1$ is open and $(f_{11}, \ldots, f_{1n})$ is in $\mathcal{U}_1$ we may choose integers $k_{1j} \geq 0$ and open sets W_{1j} in $J^{k_{1j}}(X_j, Y_j)$ so that $f_{1j} \in M(W_{1j})$ and $\prod_{j=1}^n M(\overline{W_{ij}}) \subset \mathcal{U}_1$. Thus (B_1) is satisfied. (C_1) is vacuous.

Now assume inductively that the data has been chosen for all $k \leq i - 1$. We will choose f_{ij} satisfying (A_i), (C_i), and $(f_{i1}, \ldots, f_{in}) \in \mathcal{U}_i$. As in the base step, it should be clear that we can then pick k_{ij} and W_{ij} that together with the chosen f_{ij} satisfy (B_i). Consider the set

$$D_i := \{(h_1, \ldots, h_n) \in \mathcal{C} : d_{j,s}(j^s h_j(x), j^s f_{(i-1)j}(x)) < 1/2^i \text{ for } 1 \leq s \leq i \text{ and for all } x \in X_j, 1 \leq j \leq n\}.$$

If D_i is open, then $E_i := M \cap \left(\bigcap_{k=1}^{i-1} \prod_{j=1}^n M(W_{kj})\right) \cap D_i$ open. It is straightforward to check that $(f_{(i-1)1}, \ldots, f_{(i-1)n})$ is in E_i using the inductive hypotheses (A_{i-1}), (B_{i-1}), (C_{i-1}), and the definition of D_i. As $\mathcal{U}_i$ is dense and E_i is open and non-empty, we can choose $(f_{i1}, \ldots, f_{in})$ in $\mathcal{U}_i \cap E_i$. By the definition of E_i, (A_i) is satisfied, and by the definition of D_i, (C_i) is satisfied.

The last step is to show that D_i is open. Let

$$F_{j,s} = \{h \in C^\infty(X_j, Y_j) : d_{j,s}(j^s h(x), j^s f_{(i-1)j}(x)) < 1/2^i \; \forall x \in X_j\}.$$

Since $D_i = \bigcap_{s=1}^{i} \prod_{j=1}^{n} F_{j,s}$, it is enough to show that each $F_{j,s}$ is open in $C_S^\infty(X_j, Y_j)$. Define $B_x = \alpha^{-1}(x) \cap B(1/2^i, j^s f_{(i-1)j}(x))$ where $\alpha : J^s(X_j, Y_j) \to X_j$ is the source map, and

$$B(1/2^i, j^s f_{(i-1)j}(x)) := \{\sigma \in J^s(X_j, Y_j) : d_{j,s}(\sigma, j^s f_{(i-1)j}(x)) < 1/2^i\}.$$

Let $G = \bigcup_{x \in X_j} B_x$. It is easy to see that $F_{j,s} = M(G)$, so we only need to show that G is an open subset of $J^s(X_j, Y_j)$. Let σ be a point in G and $x = \alpha(\sigma)$. Note that the mapping $\Psi : X \to \mathbb{R}$ defined by $q \mapsto d_s(j^s f_{(i-1)j}(q), j^s f_{(i-1)j}(x))$ is continuous by composition. Thus $H = \alpha^{-1}\Psi^{-1}(-\delta/2, \delta/2)$ is an open subset of $J^s(X_j, Y_j)$ where $\delta = 1/2^i - d_{j,s}(\sigma, j^s f_{(i-1)j}(x))$. Note that $\delta > 0$ since $\sigma \in G$. It is clear that $H \cap B(\delta/2, \sigma)$ is open and contains σ, so that if $H \cap B(\delta/2, \sigma) \subset G$, we are done. Let $\tau \in H \cap B(\delta/2, \sigma)$. To show that $\tau \in G$, we need to show that $d_{j,s}(\tau, j^s f_{(i-1)j}(\alpha(\tau))) < 1/2^i$. Indeed,

$$d_{j,s}(\tau, j^s f_{(i-1)j}(\alpha(\sigma))) \leq d_{j,s}(\tau, \sigma) + d_{j,s}(\sigma, j^s f_{(i-1)j}(x)) + d_s(j^s_{(i-1)j} f_{(i-1)j}(x), j^s f_{(i-1)j}(\alpha(\sigma))) <$$

$$\frac{\delta}{2} + \left(\frac{1}{2^i} - \delta\right) + \frac{\delta}{2} = \frac{1}{2^i}.$$

$\square$

3 The h-principle and convex integration

In this chapter, we shall give some context to the notion of a weak homotopy equivalence, and introduce some h-principle concepts which contextualize theorem 1.4. Specifically, we recall the notion of a **partial differential relation**, and introduce the dichotomy between **local** and **non-local** partial differential relations. The main reference for the context is [3, chapters 5 & 6]. Afterwards, we present the technique used in the first step of the proof of the main result, which is the proof of theorem 3.11. We finish by recalling a simple version of convex integration, and adapt this result to a specific class of non-local differential relations.

3.1 Weak homotopy equivalence

We want to recall and discuss the definition of a **weak homotopy equivalence**. To do so, we will first recall the definitions of **homotopy groups**, and of **relative homotopy groups**. We moreover recall the **long exact sequence of (relative) homotopy groups**, associated to a triple (Z, A, z_0), of a topological space Z, a subspace A, and a a point $z_0 \in A$. We round of by giving a concrete definition for a *smooth homotopy* connecting smooth functions, and show that it can be used to replace ordinary homotopies when studying $\pi_*(Z, z_0)$, where $Z = C^\infty_W(X, Y)$ for some manifolds X and Y. The main reference for the basic definitions is [11, section 4.1].

Let $p \in \mathbb{S}^n$ be any point. Recall that for any topological space Z, the **n-th homotopy group of** (Z, z_0), $\pi_n(Z, z_0)$, is the set of equivalence classes

$$\{g \in C(\mathbb{S}^n, Z) : g(p) = z_0\}/\{\text{homotopies relative } p\}, \quad n \geq 1.$$

For $n = 0$, $\pi_0(Z, z_0)$ can be defined as the homotopy classes (not relative a point) of maps $\{\cdot\} \to Z$, which can be identified with the path components of a space. The set $\pi_n(Z, z_0)$ has a group structure for all $n \geq 1$, although we shall not be concerned much with that here. Let X and Y be topological spaces. A **weak homotopy equivalence** is a continuous map $f : X \to Y$, that induces for every $x_0 \in X$ and $n \in \mathbb{N}_0$ isomorphisms

$$f_n : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)).$$

For $n \geq 1$ this means group isomorphisms, and for $n = 0$, a bijection.

Definition 3.1. Let X and Y be smooth manifolds, and let $f, g : X \to Y$ be smooth maps. We say that g and h are smoothly homotopic if there exists a smooth map $H : X \times [0, 1] \to Y$ such that $H_0 = g$ and $H_1 = h$. We say that the smooth homotopy H connects g to/and h. △

In the proof of theorem 1.4, we shall represent homotopies connecting two element g and h in the same class of $\pi_k(\mathrm{Sol}(\mathcal{R}_d), f)$ by smooth homotopies, and the element g and h by smooth maps $g, h : M \times \mathbb{S}^m \to \mathbb{R}^n$, and similarly for other homotopy groups. Here f is any base point. This may seem disingenuous: in principle, g and h are merely continuous maps $\mathbb{S}^n \to C^\infty_W(M, \mathbb{R}^n)$. However, the representations are equivalent: let $g : \mathbb{S}^n \to C^\infty_W(M, \mathbb{R}^n)$ be a continuous map. Let $\tilde{g} : M \times \mathbb{S}^n \to \mathbb{R}^n$ denote the associated map $(x, t) \mapsto g(t)(x)$. The following is a consequence of [1, corollary 4.8(a)]:

Proposition 3.2. *Let $C^{\infty,0}(M \times \mathbb{S}^n, \mathbb{R}^n)$ denote the subspace of $C(M \times \mathbb{S}^n, \mathbb{R}^n)$ consisting of maps g, such that $g_t : M \to \mathbb{R}^n$ is smooth for every $t \in \mathbb{S}^n$ and such that $j^k g_t : M \times \mathbb{S}^n \to J^k(M, \mathbb{R}^n)$ is a continuous map for every $k \geq 0$. Endow $C^{\infty,0}(M \times \mathbb{S}^n, \mathbb{R}^n)$ with the compact-open subspace topology. Then the map*

$$\Phi : C(\mathbb{S}^n, C^\infty_W(M, \mathbb{R}^n)) \to C^{\infty,0}(M \times \mathbb{S}^n, \mathbb{R}^n), \quad g \mapsto \tilde{g}$$

is a homeomorphism, where the left hand side has the compact-open topology too.

Hence, it is permissible to represent g by $\tilde{g}$ instead. Note that $\tilde{g}$ is still not entirely smooth, but at least a continuous map $M \times \mathbb{S}^n \to \mathbb{R}^n$. Fortunately, it is a well known result that $\tilde{g}$ is itself homotopic to a smooth map $G : M \times \mathbb{S}^n \to \mathbb{R}^n$, which fixes g_p (see for example [14, theorem 6.26]). Hence, we can represent an element $[g] \in \pi_k(\mathrm{Sol}(\mathcal{R}_d))$ as a smooth map $g : M \times \mathbb{S}^n \to \mathbb{R}^n$. A similar argument validates the representation of a homotopy H as a smooth map $H : M \times \mathbb{S}^n \times [0,1] \to \mathbb{R}^n$. In conclusion:

Proposition 3.3. *Let X and Y be manifolds, and $p \in \mathbb{S}^n$. Let $[g],[h] \in \pi_n(C^\infty_W(X,Y),\omega)$ for any $n \geq 0$ and $\omega \in C^\infty_W(X,Y)$. Then $[g]$ can be represented by a map $g : (\mathbb{S}^n, p) \to (C^\infty_W(X,Y),\omega)$, such that the associated map*

$$\tilde{g} : X \times S^n \to Y, \quad (x,t) \mapsto g(t)(x)$$

is smooth. Moreover, we can represent $[h]$ by a map h such that the associated map $\tilde{h} : X \times \mathbb{S}^n \to Y$ is smooth, and $[g] = [h]$ if and only if there exists a smooth homotopy $H : X \times \mathbb{S}^n \times [0,1] \to Y$ with $H(x,p,s) = \omega(x)$ for all $(x,s) \in X \times [0,1]$.

We can take this reasoning further: let Z be a topological space, A a subspace, and $z_0 \in A$ a point. Recall that the **n-th relative homotopy group of** (Z,A,z_0), $\pi_n(Z,A,z_0)$ is defined as the set of equivalence classes

$$\{g : (\mathbb{D}^n, \mathbb{S}^{n-1}, p) \to (Z,A,z_0), \text{ continuous}\}/\{\text{homotopies through maps of triples}\}, \quad n \geq 1.$$

Here 'through maps of triples' means that the homotopy $H : \mathbb{D}^n \times [0,1] \to Z$ for each $t \in [0,1]$ is a map of triples $H_t : (\mathbb{D}^n, \mathbb{S}^{n-1}, p) \to (Z,A,z_0)$. We omit the details (which can be found in [11, section 4.1], but our main interest in relative homotopy groups is due to the **long exact sequence of (relative) homotopy groups**,

$$\cdots \to \pi_n(A,z_0) \xrightarrow{i_*} \pi_n(Z,z_0) \xrightarrow{j_*} \pi_n(Z,A,z_0) \xrightarrow{\partial} \cdots \to \pi_0(Z,z_0).$$

Here i_* is the map induced by $i : (A,z_0) \hookrightarrow (Z,z_0)$, j_* is is the map induced map $j : (Z,z_0,z_0) \to (Z,A,z_0)$, and ∂ is the map induced by restricting $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$. The upshot is the following: we can conclude that $i : A \hookrightarrow Z$ is a weak homotopy equivalence if and only if $\pi_n(Z,A,z_0)$ is trivial for every $n \geq 1$ and $z_0 \in A$. In other words, i is a weak homotopy equivalence if and only if for every continuous map $g : (\mathbb{D}^n, \mathbb{S}^{n-1}, p) \to (Z,A,z_0)$, there exists a homotopy $H : \mathbb{D}^n \times [0,1] \to Z$ relative $\mathbb{S}^{n-1}$ connecting g to a map h, with $h(\mathbb{D}^n) \subset A$.

For the final proof of theorem 1.4, we shall be considering two subspaces $A, Z \subset C^\infty_W(X,Y)$ for some manifolds X and Y. Following the steps that led to proposition 3.3, we can conclude the following:

Proposition 3.4. *Let X and Y be manifolds, and let A and Z be subspaces of $C^\infty_W(X,Y)$ with $A \subset Z$. Then $i : A \hookrightarrow Z$ induces for every $f \in A$ an isomorphism*

$$i_n : \pi_n(A,f) \to \pi_n(Z,f)$$

if and only if for $k = n, n+1$ and every smooth map

$$g : X \times \mathbb{D}^k \to Y, \quad g_t \in Z \, \forall t \in \mathbb{D}^k, \quad g_t \in A \, \forall t \in \mathbb{S}^{k-1},$$

there exists a smooth homotopy $H : X \times \mathbb{D}^k \times [0,1] \to Y$ such that

- *$H_{t,s} : X \to Y$ is an element of Z for every $(t,s) \in \mathbb{D}^k \times [0,1]$,*
- *H is relative $X \times \mathbb{S}^{k-1}$,*
- *$H_{t,1} : X \to Y$ is a $\mathbb{D}^k$-family of maps in A.*

3.2 Partial differential relations and the h-principle

In this section we recall the definition of a **partial differential relation**, and introduce some concepts related to it. In particular, we discuss the philosophy of the h-principle for non-local PDRs. This will be the second use in this thesis of jet bundles, as introduced in section 2.1. Partial differential relations are the central object of Gromov's h-principle theory. For manifolds X and Y, we introduce what we mean by *local* and *non-local* PDRs, and rephrase the main result, as well as examples 1.1 and 1.2 in terms of PDRs.

Definition 3.5. Let X and Y be manifolds. Let $f : X \to Y$ be a smooth map.

(i) A local partial differential relation of X and Y is a subset $\mathcal{R}_1$ of $J^k(X, Y)$, for some $k \geq 0$.

(ii) A non-local partial differential relation of X and Y is a subset $\mathcal{R}_2$ of the multijet $J^k(X, Y)^s$ for some $k \geq 0$, $s \geq 2$.

(iii) The map f is a solution of the local PDR $\mathcal{R}_1$, if $\operatorname{im} j^k f \subset \mathcal{R}_1$. This is denoted $f \in \mathrm{Sol}(\mathcal{R}_1)$.

(iv) The map f is a solution of the non-local PDR $\mathcal{R}_2$, if $\operatorname{im} j^k_s f \subset \mathcal{R}_2$. This is denoted $f \in \mathrm{Sol}(\mathcal{R}_2)$.

The above definition of a local PDR agrees with the general definition of a PDR in [3], while the definition of a non-local PDR is new. Formal solutions, as well as the methods to deform them into real solutions, can be quite different for non-local relations than for local relations. Our definition is by no means the most general definition possible, but suffices for our purposes.

Remark 3.6. Partial differential relations $\mathcal{R}$ of either type are often given the topological adjectives like closed, open, or contractible, reflecting their properties as subsets of the jet spaces. They are also often said to have an *order*, depending on the type of jet space they are contained in. For example, if $\mathcal{R} \subset J^2(X, Y)$, then $\mathcal{R}$ is said to be of second order, or a second order partial differential relation. △

Often, to each partial differential relation $\mathcal{R}$ of either type, a class of sections of $\mathcal{R}$ is appointed as the **formal solutions of the relation** $\mathcal{R}$, denoted as a set by $\mathrm{Sol}_F(\mathcal{R})$. For local relations, it is common to let $\mathrm{Sol}_F(\mathcal{R})$ be the space of smooth or continuous sections $F : X \to J^k(X, Y)$ with $\operatorname{im} F \subset \mathcal{R}$. For non-local relations this is usually the space of smooth or continuous section $F : X^s \to J^k(X, Y)^s$ with $\operatorname{im} F \subset \mathcal{R}$, with some additional properties. These additional properties are supposed to mimic the fact that a genuine solution of a non-local relation is a map whose product of jet extensions takes image in $\mathcal{R}$. That is, if one wants to deform a formal solution to a genuine solution, we want to make use of these additional properties to make sure that we can deform to a product of maps. In this thesis we will make the following definition. Let $\mathrm{Sym}(s)$ denote the symmetric group on s integers, i.e. the group of shuffles on the integers $1, \ldots, s$.

Definition 3.7. Let A and B be sets, let X and Y be manifolds, let $\mathcal{R} \subset J^k(X, Y)^s$ be a non-local relation, and let $F : X^s \to J^k(X, Y)^s$ be a section.

(i) For every $\sigma \in \mathrm{Sym}(s)$, let $\sigma : A^s \to A^s$ be the map which shuffles an s-tuple as $(a_j)_{j=1}^s \mapsto (a_{\sigma(j)})_{j=1}^s$, where $a = (a_j) \in A^s$.

(ii) A map $f : A^s \to B^s$ is $\mathrm{Sym}(s)$-equivariant or simply equivariant if, for every $\sigma \in \mathrm{Sym}(s)$, the following diagram commutes:

$$\begin{tikzcd} A^s \arrow[r, "f"] \arrow[d, "\sigma"] & B^s \arrow[d, "\sigma"] \\ A^s \arrow[r, "f"'] & B^s \end{tikzcd}$$

(iii) the section F is a formal solution of $\mathcal{R}$, denoted $F \in \mathrm{Sol}_F(\mathcal{R})$, if F is equivariant and $\operatorname{im} F \subset \mathcal{R}$.

△

Note that this definition of equivariance agrees with the previous definition given for maps $f : X^2 \to Y^2$.

Remark 3.8 (Philosophy of the h-principle)**.** The philosophy of the h-principle is as follows: given a partial differential relation, try to use the properties of formal solutions to find genuine solutions of the relation. Given a relation $\mathcal{R} \subset J^k(X,Y)^s$, we say that 'the h-principle holds for $\mathcal{R}$' if

$$j_s^k : \mathrm{Sol}(\mathcal{R}) \to \mathrm{Sol}_F(\mathcal{R})$$

is a weak homotopy equivalence. This can be interpreted as saying, $\mathbb{S}^n$-families of formal solutions can be deformed in to $\mathbb{S}^n$-families of genuine solutions (surjectivity of the map between n-th homotopy groups), and any two $\mathbb{S}^n$-families of genuine solutions lie in the same homotopy class if and only if they can be connected by a homotopy of formal solutions (injectivity of the map between n-th homotopy groups). If the map j_s^k only induces isomorphisms on homotopy groups up to a certain degree, one can say that a 'weaker version of the h-principle holds for $\mathcal{R}$', or that 'the h-principle holds up to degree k' (where k is the highest degree of homotopy groups for which j_s^k induces isomorphisms). $\triangle$

Example 3.9 (The immersion relation)**.** Let $\mathcal{R}_{\mathrm{imm}} \subset J^1(X,Y)$ denote all those $[f]_{x,y} \in J^1(X,Y)$ with $Tf_x : T_xX \to T_{f(x)}Y$ an injective linear map. One can verify that this is indeed well-defined. The relation $\mathcal{R}_{\mathrm{imm}}$ is called the immersion relation, and $f : X \to Y$ is an immersion if and only if $f \in \mathrm{Sol}(\mathcal{R}_{\mathrm{imm}})$. Using the standard bundle trivializations of $J^1(X,Y)$, one can show that $\mathcal{R}_{\mathrm{imm}}$ is open, and empty if $\dim X > \dim Y$. Note that this implies that the set of immersions in $C^\infty_S(X,Y)$ is open. Before introducing example 1.1, we stated that a *formal immersion* was a continuous bundle monomorphism $F : TX \to TY$. Using the bundle trivializations of $J^1(X,Y)$ once more, one can pointwise show that there is a one-to-one correspondence between linear maps $T_xX \to T_yY$ and elements of $J^1(X,Y)_{(x,y)}$. With this identification, one can identify a bundle monomorphism with a continuous section $\tilde{F} : X \to J^1(X,Y)$, with image contained in $\mathcal{R}_{\mathrm{imm}}$, i.e. an element of $\mathrm{Sol}_F(\mathcal{R}_{\mathrm{imm}})$. Example 1.1 can now be rephrased as 'there exists a weak homotopy equivalence between solutions and formal solutions of the immersion relation, which is given by the 1-jet map $j^1 : f \mapsto j^1f$'. In other words, the h-principle holds for the immersion relation $\mathcal{R}_{\mathrm{imm}}$. $\triangle$

Example 3.10 (The embedding relation)**.** Let $\mathcal{R}_{\mathrm{emb}} \subset J^1(X,Y)^2$ denote all those $([f]_{x_1,y_1}, [g]_{x_2,y_2})$ with Tf_{x_1} and Tg_{x_2} an injective linear map, and $x_1 \neq x_2 \implies y_1 \neq y_2$. The relation $\mathcal{R}_{\mathrm{emb}}$ is called the embedding relation. If $f : X \to Y$ is a smooth proper map, then f is a smooth embedding if and only if $j_2^1 f : X \to J^1(X,Y)^2$ has image in $\mathcal{R}_{\mathrm{emb}}$. Note that $\mathcal{R}_{\mathrm{emb}}$ is an open subset of $\mathcal{R}^2_{\mathrm{imm}}$, and hence open. As the set of proper maps is open in $C^\infty_S(X,Y)$ (even in the strong C^0 Whitney topology), it follows that the set of smooth embeddings is open in $C^\infty_S(X,Y)$. Assume that X is compact (which guarantees properness), then *formal embeddings* in the context of example 1.2 are equivariant sections $g : X^2 \to J^0(X,Y)^2$ whose image lies in $\mathcal{R}'_{\mathrm{emb}} = \pi^2_{1,0}\mathcal{R}_{\mathrm{emb}}$. The formal embeddings of this example are really different from formal solutions of $\mathcal{R}_{\mathrm{emb}}$ in our notation: we forget about the formal immersion property of the formal solutions. This is no accident: Haefliger showed in [8] that the existence of such equivariant map can be used to prove the existence of immersions, too. Haefliger's embedding theorem (example 1.2) can now be rephrased as: the map j^0 induces an isomorphism of 0-th homotopy groups of formal and true smooth embeddings. $\triangle$

3.3 Classification of circle immersions in the plane

In this section, we sketch the technique used by H. Whitney to classify immersions $f : \mathbb{S}^1 \to \mathbb{R}^2$ up to regular homotopy. This technique is the main ingredient of theorem 6.4, which is the first step in the proof of theorem 1.4. Theorem 6.4 shows that we can find solutions of the d-delay relation if and only if we can find true solutions of the derivative d-delay problem $\mathcal{R}'_d$.

Recall that for an immersion $f : \mathbb{S}^1 \to \mathbb{R}^2$, there is an induced map $f'/|f'| : \mathbb{S}^1 \to \mathbb{S}^1$ (the normalized derivative map), and that the turning number of f was defined as the (topological) degree of $f'/|f'|$. The following is the conclusion of [20].

Theorem 3.11. *Two immersions $f, g : \mathbb{S}^1 \to \mathbb{R}^2$ are homotopic through C^1 immersions if and only if f and g have the same turning number.*

Sketch of proof of theorem 3.11. The implication from left to right is simple: if two immersions f and g are homotopic through immersions, the maps $f'/|f'|$ and $g'/|g'|$ are homotopic, and hence must have the same degree. For the converse, one can first homotope any two immersions $f, g : \mathbb{S}^1 \cong [0,1]/\{0 \sim 1\} \to \mathbb{R}^2$ through immersions to maps with $|f'| = |g'| \equiv 1$, $f(0) = g(0)$, and $f'(0) = g'(0)$. For the second step, choose any continuous map $H : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ connecting f' to g'. If we can find $h : S^1 \times [0,1] \to \mathbb{R}^2$ such that $h'_t = H_t$ for every $t \in [0,1]$, and $h_0 = f, h_1 = g$ the proof is complete. The obstacle to finding 'a primitive' of H_t, is that $\int_0^1 H_t(p)dp$ might not be zero, and hence $\tilde{h}_t : [0,1] \to \mathbb{R}^2$ given by $\tilde{h}_t(x) = f(0) + \int_0^x H_t(p)dp$ might not descend to a map $\mathbb{S}^1 \to \mathbb{R}^2$. If the degree of f' and g' are not 0, one can verify that

$$h_t(x) := f(0) + \int_0^x H_t(p)dp - x \int_0^1 H_t(p)dp$$

will always descend, and that h_t is moreover a family of immersions connecting f and g. Here we use that H_t not having degree 0 implies that $\| \int_0^1 H_t(p)dp\| < 1$. If the degree is 0, one has to be a little more careful to make sure that this inequality holds. The inequality will only fail to hold if H_t is constant for some $t \in [0,1]$. It is however possible to perturb H_t so that this does not happen (and we refer to the paper for the details of this). $\square$

3.4 Convex integration

The technique used in the proof of theorem 3.11 makes use of the fact that for any map $f : \mathbb{S}^1 \to \mathbb{S}^1$, the weighted average position $\int_0^1 f(p)dp$ lies in the interior of the convex hull of $\mathbb{S}^1 \subset \mathbb{R}^2$, so long as the degree of a map f is not zero. That is, when we constructed the family of immersions h_t, we made use of the fact that $\| \int_0^1 H_t(p)dp\| < 1$, and hence that $\|(\partial/\partial x)h_t(x)\| > 0$ because $(\partial/\partial x)h_t(x)$ is a difference of an element in $\mathbb{S}^1 \subset \mathbb{D}^2$ and an element of int $\mathbb{D}^2$, and hence never zero.

This technique can be applied in much greater generality, and is referred to as **convex integration**. We introduce it and some useful language briefly. Normally, it is intended for use on local partial differential relations, but we describe how this method can be applied to some non-local differential relations, that are **delay-like** (definition 3.19), such as the d-delay relation.

Convex integration can be applied quite broadly to local relations on maps between manifolds. However, the technique makes use of integrating with respect to coordinates, and hence its results are usually phrased for PDRs contained in $J^1(\mathbb{R}^m, \mathbb{R}^n)$ $(m, n \in \mathbb{N})$. That is to say, to apply the technique one usually needs use charts on the base and target manifold, and some 'globalization argument' to patch together local solutions. Some important local convex integration results can in turn by deduced by 'special cases' of these results for PDRs in $J^1(\mathbb{R}, \mathbb{R}^n)$, and hence we shall focus on non-local first order partial differential relations contained in $J^1(\mathbb{R}, \mathbb{R}^n)^2$. Note that the d-delay differential relation is essentially covered: when $M = \mathbb{S}^1$ or $M = [0,1]$, local charts can be used to reduce to such relations. The key classic result we recall is lemma 3.17, which can be referred to as *one-dimensional convex integration.*

3.4.1 One-dimensional convex integration

Let $\pi : J^1(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R} \times \mathbb{R}^n$ denote the bundle projection. Observe that every fiber of π is an n-dimensional vector space, and that $\pi^{-1}(x, y)$ can be identified with $T_y\mathbb{R}^n$ so that for every smooth map $f : \mathbb{R} \to \mathbb{R}^n$ the 1-jet of f at x, $j^1 f(x)$, can be identified with $T_x f$ applied to $1 \in T_x\mathbb{R} \cong \mathbb{R}$. I.e. $j^1 f(x)$ is identified with the Jacobi matrix of f evaluated at x.

Definition 3.12. Let Ω be a subset of a finite dimensional affine space P, and let $z \in \Omega$. Let $n \geq 1$ be a natural number and let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^n)$ be a partial differential relation.

(i) Denote by $\mathrm{Conn}_z\Omega$ the path component of Ω containing z.

(ii) Denote by $\mathrm{Conv}_z\Omega$ the convex hull of $\mathrm{Conn}_z\Omega$.

(iii) The subset Ω is called ample if $\mathrm{Conv}_z\Omega = P$ for all $z \in \Omega$.

(iv) Let $\mathcal{R}(x, y) := \mathcal{R} \cap \pi^{-1}(x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R}^n$.

(v) A partial differential relation $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^n)$ is called ample if it is fiberwise ample, i.e. if $\mathcal{R}(x, y)$ is ample for every $(x, y) \in \mathbb{R} \times \mathbb{R}^n$. $\triangle$

Notice that the last item makes sense, because $\mathcal{R}(x, y)$ is a subset of the affine space $T_y\mathbb{R}^n$.

Example 3.13. Let $P = \mathbb{R}^2$ and $\Omega_1 = \mathbb{R}^2 \setminus l$ for any line $l \subset \mathbb{R}^2$, and let Ω_2 be $\mathbb{R}^2 \setminus \mathbb{D}^2$. Then Ω_1 is not ample, since $\mathrm{Conv}_z\Omega_1$ is a half-plane for any $z \in \Omega_1$, while Ω_2 is ample, since it only has one path component, whose convex hull is $\mathbb{R}^2$. The immersion relation $\mathcal{R}_{\mathrm{imm}} \subset J^1(\mathbb{R}, \mathbb{R}^n)$ (example 3.9) is an ample PDR if $n \geq 2$: Under the identification of $\pi^{-1}(x, y)$ with $T_y\mathbb{R}^n$, $\mathcal{R}(x, y)$ is identified with $T_x\mathbb{R}^n \setminus \{0\}$, which is ample when $n \geq 2$. $\triangle$

Definition 3.14. Let $n \geq 1$ be an integer, and let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^n)$ be a partial differential relation. Let $F : \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R}^n)$ be a formal solution of $\mathcal{R}$, and denote by $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n$ the composition $\pi \circ F$, where π is the bundle projection of $J^1(\mathbb{R}, \mathbb{R}^n)$ to $\mathbb{R} \times \mathbb{R}^n$.

(i) Denote by $\mathrm{Conn}_{F(x)}\mathcal{R}$ the path component of $F(x)$ in $\mathcal{R}(f(x)) \subset \pi^{-1}(f(x))$.

(ii) Denote by $\mathrm{Conv}_{F(x)}$ the convex hull of $\mathrm{Conn}_{F(x)}\mathcal{R}$ in $\pi^{-1}(f(x))$.

(iii) Denote by Conv_F the partial differential relation

$$\mathrm{Conv}_F := \bigcup_{x \in \mathbb{R}} \mathrm{Conv}_{F(x)}\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^n).$$

(iv) A formal solution F is called short, if $f = \pi \circ F$ is a solution of $\mathrm{Conv}_F\mathcal{R}$. $\triangle$

Example 3.15. Let $n = 2$ and $\mathcal{R}_1$ be fiberwise defined by all those $z \in \pi^{-1}(x, y) \cong T_y\mathbb{R}^n$ that have norm 1. Fiberwise, this means that $\mathcal{R}(x, y)$ can be identified with $\mathbb{S}^1$. Let F be a formal $\mathcal{R}_1$ solution, and let $f = \pi \circ F$. Then $\mathrm{Conn}_{F(x)}\mathcal{R}_1 = \mathcal{R}(f(x))$ for all $x \in \mathbb{R}$, and $\mathrm{Conv}_F\mathcal{R}$ can be fiberwise identified with $\mathbb{D}^2$. The formal solution F is short if and only if $\|(\partial/\partial x)f\| \leq 1$. This differential relation and its short formal solutions are used in the study of isometric immersions (see chapter 21 of [3], for example), and motivates the naming convention 'short'. $\triangle$

Example 3.16. Let $\mathcal{R}_{\mathrm{imm}} \subset J^1(\mathbb{R}, \mathbb{R}^n)$ be the immersion relation. If F is a formal solution, then $\mathrm{Conv}_F\mathcal{R}_{\mathrm{imm}}$ is fiberwise a half-line if $n = 1$, or the entire fiber $\pi^{-1}(f(x))$ if $n \geq 2$, where $f = F \circ \pi$. In the second case, any formal solution is short. In fact, for any ample relation $\tilde{\mathcal{R}}$, any formal solutions is a short formal solution. $\triangle$

The following lemma can be referred to as 'one-dimensional convex integration' and can be used to deduce convex integration statements 'in more variables'. See for example [3, chapter 18].

Lemma 3.17 (Lemma 17.3.1 of [3])**.** *Let $n \geq 1$ be an integer, and let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ be an open differential relation. Let $F : [0, 1] \to J^1(\mathbb{R}, \mathbb{R}^n)$ be a short formal $\mathcal{R}$ solution, and let $f = \pi \circ F$, where $\pi : J^1(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R} \times \mathbb{R}^n$ is the bundle projection. Then there exists a family of short formal solutions*

$$F_t : [0, 1] \to J^1(\mathbb{R}, \mathbb{R}^n), \quad t \in [0, 1]$$

which joins $F_0 = F$ to a genuine solution F_1. That is, there exists a $\mathcal{R}$ solution $f_1 : [0, 1] \to \mathbb{R}^n$ such that $j^1 f_1 = F_1$.

Moreover, the family F_t can be chosen so that

(a) $f_t := \pi \circ F_t$ *is arbitrarily close to* f *for all* $t \in [0,1]$ *in the* C^0 *Whitney topology.*

(b) $F_t(0) = F(0)$ *and* $F_t(1) = F(1)$ *for all* $t \in [0,1]$.

(c) *if* F_0 *is already genuine at* 0 *and* 1 *(i.e.* $j^1 f(0), j^1 f(1) \in \mathcal{R}$*), then* F_t *can be chosen fixed at* 0 *and* 1.

Remark 3.18. The last observation in example 3.15, and lemma 3.17, are the key reasons for introducing ample relations. That is, if $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^n)$ is an ample relation, then any formal solution is automatically a short formal solution. The lemma says that if $\mathcal{R}$ is moreover open, then any formal solution is homotopic through formal solutions to a (true) solution of $\mathcal{R}$. The lemma is in fact more powerful than this, but if one is mainly interested in proving an h-principle, this is the key feature. $\triangle$

3.4.2 Non-local convex integration

We introduce and prove theorem 3.25, which applies lemma 3.17 to a specific class of non-local partial differential relations, the open *delay-type non-local* PDRs. The conclusion is that formal solutions of such PDRs of the form $F \times F : \mathbb{R} \times \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R}^n)^2$ can be used to find true solutions of such relations. We shall first introduce what it means to be infinitesimally non-local.

Definition 3.19. Let X and Y be manifolds, and let $\mathcal{R} \subset J^1(X,Y)^2$ be a first order non-local PDR.

- The PDR $\mathcal{R}$ is called infinitesimally non-local if $\mathfrak{s}(\mathcal{R})$ intersects every open neighbourhood of $\Delta(X) \subset X^2$, where $\mathfrak{s} : J^1(X,Y)^2 \to X^2$ is the source map.
- The PDR $\mathcal{R}$ is called of delay-type if it is not infinitesimally non-local. $\triangle$

Example 3.20. The embedding relation $\mathcal{R}_{\text{emb}} \subset J^1(\mathbb{R}, \mathbb{R}^n)^2$ (example 3.10) is infinitesimally non-local: $\mathfrak{s}(\mathcal{R}_{\text{emb}}) = \mathbb{R}^2 \setminus \Delta(\mathbb{R})$, and hence intersects every neighbourhood of $\Delta(\mathbb{R})$. The d-delay relation $\mathcal{R}_d$ (definition 6.2) is delay-like: $\mathfrak{s}(\mathcal{R}_d) = \Delta_{\pm d}$, which is a closed set that does not intersect $\Delta(\mathbb{R})$. Hence, its complement is a neighbourhood of $\Delta(\mathbb{R})$ disjoint from $\mathfrak{s}(\mathcal{R}_d)$. $\triangle$

Remark 3.21 (Key property of delay-like non-local PDRs)**.** If $\mathcal{R} \subset J^1(X,Y)^2$ is a non-local PDR that is delay-like, then for every $x \in X$, there exists a closed neighbourhood $V \subset X$ of x, such that $\mathfrak{s}(\mathcal{R})$ does not intersect $V \times V$. This is the key property we wish to exploit, in order to define a local relation $\mathcal{R}_V(F)$ to which we can apply one-dimensional convex integration. $\triangle$

Definition 3.22. Let X and Y be manifolds, and let $\mathfrak{s} : J^1(X,Y)^2 \to X^2$ be the source map of the product, and let $\mathfrak{s}' : J^1(X,Y) \to X$ be the source map of the single factor. Let $F : X \to J^1(X,Y)$ be a section, and let $\mathcal{R} \subset J^1(X,Y)^2$ be a delay-like non-local PDR.

(i) if $F \times F$ is a formal $\mathcal{R}$ solution, then we say that F is a spatially holonomic formal solution.

(ii) assume that F is a spatially holonomic formal $\mathcal{R}$ solution. For every $x \in X$ and $x_0 \in X$, and neighbourhood V of x, define

$$\mathcal{R}_{V,x_0}(F) := \{(x,y,v) \in J^1(X,Y) : (x,y,v,F(x_0)) \in \mathcal{R} \text{ and } x \in V\}.$$

(iii) define the local PDR $\mathcal{R}_V(F) \subset J^1(X,Y)$ by

$$\mathcal{R}_V(F) = (\mathfrak{s}')^{-1}(X \setminus V) \cup \bigcap_{x_0 \in X} \mathcal{R}_{V,x_0}(F).$$

We call $\mathcal{R}_V(F)$ the localized $\mathcal{R}$-replacement for F on V. $\triangle$

The localization of $\mathcal{R}$ should be thought of as follows: F traces out some graph in the second factor of $J^1(X,Y)$, and $\mathcal{R}_V(F)$ consists of all those points in the first factor that are 'compatible with the graph of F and $\mathcal{R}$ over V'. That is, it consists of all triples such that the source lies in V, and such that the triple paired with any point in the image of F lies in $\mathcal{R}$. The key property is the following: if we deform F only over V to some smooth section $\tilde{F}$, so that $\tilde{F}$ is a formal $\mathcal{R}_V(F)$-solution, then $\tilde{F} \times \tilde{F}$ will still be a formal $\mathcal{R}$ solution (which is the key step of the next theorem).

Definition 3.23. Let $n \geq 0$ be an integer, and let $\mathcal{R} \subset J^1(\mathbb{R},\mathbb{R}^n)^2$ be a non-local PDR, which is delay-like. We say that $\mathcal{R}$ is ample if for every smooth spatially holonomic formal $\mathcal{R}$ solution $F : \mathbb{R} \to J^1(\mathbb{R},\mathbb{R}^n)$, and for every $x \in \mathbb{R}$, there exists a closed neighbourhood V of x such that $\mathfrak{s}(\mathcal{R}) \cap (V \times V) = \varnothing$ and $\mathcal{R}_V(F)$ is ample. △

Example 3.24. The d-delay relation $\mathcal{R}_d$ is an ample non-local PDR, when $M = \mathbb{R}$ and $n \geq 2$. In example 3.20 we already found that it was delay-like non-local. Let $F : \mathbb{R} \to J^1(\mathbb{R},\mathbb{R}^n)$ be a smooth section such that $F \times F$ is a formal $\mathcal{R}_d$ solution. We shall find a closed neighbourhood V of $0 \in \mathbb{R}$, determine $\mathcal{R}_V(F)$, and conclude that it is ample. It should then be clear that $\mathcal{R}_d$ is indeed ample.

Let $V = [-d/3, d/3]$, then $\mathfrak{s}(\mathcal{R}_d) \cap (V \times V) = \varnothing$. The local relation $\mathcal{R}_V(F) \subset J^1(\mathbb{R},\mathbb{R}^n)$ is given by

$$\left\{(x,y,z) \in J^1(\mathbb{R},\mathbb{R}^n) : x \in V \implies z \neq (F(x \pm d))_3\right\}$$

where $(F(x))_3$ means the third component of $F(x) \in J^1(\mathbb{R},\mathbb{R}^n) \cong \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. To verify this, note that $\mathcal{R}_{V,x_0}(F)$ is equal to $\mathfrak{s}'^{-1}(V)$ without at most two points. One can also check that $\mathcal{R}_V(F) \cap \pi^{-1}(x,y)$, $(x,y) \in \mathbb{R} \times \mathbb{R}^n$, can be identified with $\mathbb{R}^n$ without at most two points, and hence is ample when $n \geq 2$. △

Theorem 3.25. *Let $\mathcal{R} \subset J^1(\mathbb{R},\mathbb{R}^n)$ be an open non-local PDR which is delay-like, and ample. Assume that there exists a smooth section $F : \mathbb{R} \to J^1(\mathbb{R},\mathbb{R}^n)$ such that F is a spatially holonomic formal $\mathcal{R}$ solution. Then there exists a family of sections $F_t : \mathbb{R} \to J^1(\mathbb{R},\mathbb{R}^n)$, $t \in [0,1]$, such that $F_0 = F$, every F_t is spatially holonomic formal $\mathcal{R}$ solution and $F_1 = j^1 f$ for some smooth map $f : \mathbb{R} \to \mathbb{R}^n$.*

Before we prove the theorem, we shall need a lemma, which deals with the openness of the localized $\mathcal{R}$-replacement of F around a closed neighbourhood V. The lemma also verifies that F is, in fact, a formal $\mathcal{R}_V(F)$ solution.

Lemma 3.26. *Let $\mathcal{R} \subset J^1(\mathbb{R},\mathbb{R}^n)$ be an open non-local PDR. Assume that there exists a smooth section $F : \mathbb{R} \to J^1(\mathbb{R},\mathbb{R}^n)$ such that F is a spatially holonomic formal $\mathcal{R}$ solution. Let $x \in \mathbb{R}$, and let V be a closed neighbourhood of x. Then $\mathcal{R}_V(F)$ is open, and F is a formal $\mathcal{R}_V(F)$ solution.*

Proof. Note that it suffices to show that $\mathcal{R}_V(F)$ is open in $(\mathfrak{s}')^{-1}(V)$, where $\mathfrak{s}' : J^1(\mathbb{R},\mathbb{R}^n) \to \mathbb{R}$ is the source map. Note that

$$\mathcal{R}_V(F) \cap (\mathfrak{s}')^{-1}(V) = \left\{(x,y,z) \in J^1(\mathbb{R},\mathbb{R}^n) : x \in V,\ x' \in \mathbb{R},\ \text{and } (x,y,z : F(x')) \in \mathcal{R}\right\}.$$

The openness of $\mathcal{R}$ can be used pointwise to show that this set is indeed open. For the second assertion, one only needs to verify that $F \times F$ being a formal $\mathcal{R}$ solution implies that F is a formal $\mathcal{R}_V$ solution. We will not comment on this more. □

Proof of theorem 3.25. The idea is to apply lemma 3.17 repeatedly. Let $x \in \mathbb{R}$, then without loss of generality there exists $\delta > 0$ such that $\mathfrak{s}(\mathcal{R}) \cap [x-\delta, x+\delta]^2 = \varnothing$ and $\mathcal{R}_V(F)$ is ample, where $V = [x-\delta, x+\delta]$. By lemma 3.26, $\mathcal{R}_V(F)$ is a local open differential relation, and $F|_{[x-\delta,x+\delta]}$ is a formal $\mathcal{R}_V(F)$ solution. By lemma 3.17, there exists a family of formal $\mathcal{R}_V(F)$ solutions $F_t : [x-\delta, x+\delta] \to J^1(\mathbb{R},\mathbb{R}^n)$, $t \in [0,1]$ and $F_0 = F$, such that $F_1 = j^1 f_1$ for some smooth $\mathcal{R}_V(F)$ solution $f_1 : [x-\delta, x+\delta] \to \mathbb{R}^n$. Moreover, this family can be chosen so that $F_1(x-\delta) = F_0(x-\delta)$ and $F_1(x+\delta) = F_0(x+\delta)$.

Replacing F by $\tilde{F}$, defined by

$$\tilde{F}(y) = \begin{cases} F(x) & \text{if } y \notin [x-\delta, x+\delta] \\ F_1(y) & \text{if } y \in [x-\delta, x+\delta] \end{cases}.$$

Because $\mathfrak{s}(\mathcal{R}) \cap [x-\delta, x+\delta]^2 = \varnothing$, it follows that $\tilde{F} \times \tilde{F}$ is a (continuous) formal $\mathcal{R}$ solution, i.e. $\tilde{F} \times \tilde{F}(x,y) \in \mathcal{R}$ for all $(x,y) \in \mathbb{R}^2$:

- if $(x,y) \in V^2$ this is trivially true,
- if $(x,y) \in (\mathbb{R} \setminus V)^2$, this is true because $F \times F$ is a formal solution,
- for all other (x,y), this is true because $\tilde{F}|_V$ is a formal (in fact true) $\mathcal{R}_V(F)$ solution.

For the same reasons, if we denote by $\tilde{F}_t$ the map obtained by replacing $F|_V$ with $F_t|_V$, then $\tilde{F}_t \times \tilde{F}_t$ is a family of formal $\mathcal{R}$ solutions. In principle, $\tilde{F}$ is only continuous, but by the openness of $\mathcal{R}$, one can 'smooth out' $\tilde{F}$ near $x \pm \delta$ so that $\tilde{F}$ is smooth and still a true $\mathcal{R}$ solution on $[x-\delta, x+\delta]$. Moreover, if F was already a true solution on a neighbourhood of $x \pm \delta$, after smoothing out it will still be a true solution over this neighbourhood.

Clearly, we can repeat this process as often as we like around as many points $x \in \mathbb{R}$ as we like. However, to makes sure we obtain one $\mathcal{R}$ solution $f : \mathbb{R} \to \mathbb{R}^n$ we need to be more careful. For every point $x \in \mathbb{R}$ we can find a δ such that $\mathfrak{s}(\mathcal{R}) \cap [x-\delta, x+\delta]^2 = \varnothing$ (as before). Clearly, the collection of all $[x-\delta, x+\delta]^2$ form a cover of $\Delta(R) \subset R^2$. As $\Delta(\mathbb{R})$ is second countable and $\mathcal{R}$ is delay-like non-local, we can find a countable number points $x_1, x_2, \dots$ and real positive numbers $\delta_1, \delta_2, \dots$ such that $\{[x_j - \delta_j, x_j + \delta_j] : j \in \mathbb{N}\}$ is a subcover of $\mathbb{R}$. Working inductively over the points x_j, we can obtain one $\mathcal{R}$ solution f: at the j-th step we are given F_{j-1}, which is a true $\mathcal{R}$ solution over all $[x_i - \delta_i, x_i + \delta_i]$, $1 \leq i \leq j-1$, and we select the closed neighbourhood

$$V := [x_j - \delta_j, x_j + \delta_j] \setminus \bigcup_{i=1}^{j-1} (x_i - \delta_i, x_i + \delta_i).$$

If V is empty, we can skip the j-th step. By the above reasoning (slightly adjusted to V instead of the entire interval), we can find a $[0,1]$-family $(\tilde{F}_j)_t$ of formal $\mathcal{R}$ solutions, so that $\tilde{F}_j = (\tilde{F}_j)_0$, and $(\tilde{F}_j)_1$ is a true solution of $\mathcal{R}$ over $\bigcup_{i=1}^{j} [x_i - \delta_i, x_i + \delta_i]$.

To construct the homotopy $H : \mathbb{R} \times [0,1] \to J^1(\mathbb{R}, \mathbb{R}^n)$ between F and $j^1 f$ so that $H_t \times H_t$ goes through formal $\mathcal{R}$ solutions, we can define H to perform the homotopy between F and $\tilde{F}_1$ on $[0, 1/2]$, and the homotopy between F_j and F_{j+1} on $[1/(j+1), 1/(j+2)]$. As the homotopy stabilizes pointwise in a finite amount of time, this is well defined. By use of bump functions, we can make sure that H is smooth too. This completes the proof. $\square$

3.4.3 Parametric non-local convex integration

As remarked at the start of section 3.4, ordinary one-dimensional convex integration can be seen as the base step for multi-dimensional convex integration for relations in $J^1(\mathbb{R}^m, \mathbb{R}^n)$. In turn, charts on any two manifold X and Y can be used to deduce convex integration results for relations in $J^1(X, Y)$. Here we shall omit a multi-dimensional version of non-local convex-integration, but we remark that such a statement can be proved in the same way as statement 18.2.1 of [3] is proved from lemma 3.17. I.e. one can deduce multi-dimensional statements by first proving a parametric version of the one-dimensional statement. Hence, we shall give a a parametric version of theorem 3.25. This theorem can be proved using statement 17.5.1 of [3] (which is a parametric local convex integration statement) analogously to how the above theorem is proved with lemma 3.17. The details are left to the reader.

Definition 3.27. Let X, Y, and P be manifolds, and let k and $s \geq 2$ be non-negative integers.

(i) A fibered local PDR $\mathcal{R}$ is a subset of $P \times J^k(X, Y)$.

(ii) A formal solution of $\mathcal{R}$ is a smooth map $F : P \times X \to P \times J^k(X, Y)$ such that $\operatorname{im} F \subset \mathcal{R}$ and $(\mathrm{id}_P \times \mathfrak{s}) \circ F = \mathrm{id}_{P \times X}$.

(iii) A genuine solution of $\mathcal{R}$ is a smooth P-family of maps $f_p : X \to Y$ such that the map $P \times X \to P \times J^k(X, Y)$ given by $(p, x) \mapsto (p, j^k f_p(x))$ is a formal solution of $\mathcal{R}$.

(iv) For every $p \in P$, define by $\mathcal{R}_p \subset J^k(X, Y)$ the local relation relation obtained by intersecting $\mathcal{R}$ with $\{p\} \times J^k(X, Y)$.

(v) The fibered local PDR $\mathcal{R}$ is ample, if $\mathcal{R}_p$ is ample for every $p \in P$.

(vi) A fibered non-local PDR is a subset of $P \times J^k(X, Y)^s$.

(vii) A formal solution of $\mathcal{R}$ is a smooth map $F : P \times X^s \to P \times J^k(X, Y)^s$ such that $\operatorname{im} F \subset \mathcal{R}$ and $(\mathrm{id}_P \times \mathfrak{s}) \circ F = \mathrm{id}_{P \times X^s}$, and so that for every $p \in P$, $F_p : X^s \to J^k(X, Y)^s$ is equivariant.

(viii) For every $p \in P$, define by $\mathcal{R}_p \subset J^k(X, Y)^s$ the local relation relation obtained by intersecting $\mathcal{R}$ with $\{p\} \times J^k(X, Y)^s$.

(ix) The fibered non-local PDR $\mathcal{R}$ is ample, if $\mathcal{R}_p$ is ample for every $p \in P$.

(x) A genuine solution of $\mathcal{R}$ is a smooth P-family of maps $f_p : X \to Y$ such that the map $P \times X^s \to P \times J^k(X, Y)^s$ given by $(p, x) \mapsto (p, j^k_s f_p(x))$ is a formal solution of $\mathcal{R}$. △

By abuse of notation, we will say that a non-local spatially holonomic formal solution $F : P \times X \to P \times J^k(X, Y)^s$ is a genuine solution if $F = ((p, x) \mapsto (p, j^k_s f_p(x))$ for some smooth P-family of maps $f_p : X \to Y$.

Theorem 3.28. *Let n, l be non-negative integers. Let $\mathcal{R} \subset [0,1]^l \times J^1(\mathbb{R}, \mathbb{R}^n)^2$ be an open, fibered, ample, non-local PDR. Let $F : [0,1]^l \times \mathbb{R} \to [0,1]^l \times J^1(\mathbb{R}, \mathbb{R}^n)$ be a smooth map so that $F_p : \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R}^n)$ is a spatially holonomic formal solution of $\mathcal{R}_p$ for every $p \in [0,1]^l$, and so that $(\mathrm{id}_{[0,1]^l} \times \mathfrak{s}) \circ F = \mathrm{id}_{[0,1]^l \times \mathbb{R}}$. Suppose that F_p is a genuine solution for all p in a neighbourhood of the boundary of $[0,1]^l$. Then there is a homotopy of fiberwise formal solutions*

$$F_\tau : [0,1]^l \times \mathbb{R} \to [0,1]^l \times J^1(\mathbb{R}, \mathbb{R}^n)$$

which joins $F_0 = F$ to a genuine solution $F_1 = ((x, p) \mapsto (p, j^1 f_p(x)))$, where $f_p : \mathbb{R} \to \mathbb{R}^n$ is a $[0,1]^l$-family of smooth maps, such that

- *F_τ is constant for all p in the neighbourhood of the boundary of $[0,1]^l$ on which F was already a genuine solution.*
- *the first partial derivatives of $f(p, t) : P \times X \to Y$ with respect to the parameter p are (arbitrarily) C^0-close to the respective derivatives of $\mathfrak{t} \circ F_0(p, t)$.*

Remark 3.29. The bullet points in the last theorem have the following interpretation: if one wants to do multi-dimensional convex integration, one wants to work over patches in the base space (modelled by $[0,1]^l$), and 'perform an iterative integral', i.e. work one coordinate direction at a time. The first bullet point says that one can patch solutions together, while the second can be used to make sure that the first order formal derivatives of $\mathfrak{t} \circ F$ with respect to the paramter p, are preserved. △

4 Transversality

In many contexts it is of interest to know when the preimage $f^{-1}(W)$ of a submanifold $W \subset Y$ under a smooth map $f : X \to Y$ is again a smooth manifold. Two usual initial results in this context are the constant rank or submersion level set theorem [14, corollaries 5.13 & 5.14], which state that the preimage of a point is an embedded submanifold if the map has constant rank (or in particular, is a submersion). A sufficient condition for preimages of submanifolds that are not zero dimensional, is usually stated in terms of transversality. A map $f : X \to Y$ is **transverse** to a submanifold $W \subset Y$ if for every $x \in X$ either $f(x) \notin W$ or $f(x) \in W$ and $\operatorname{im} T_x f + T_{f(x)}W = T_{f(x)}Y$. That is to say, for all points where f intersects W, the image of the tangent map of f together with the tangent space of W at that point, should span the tangent space of Y at that point. The common notation for f to be transverse to W is $f \pitchfork W$.

Theorem 4.1 (Theorem 4.4 of [5]). *Let X and Y be smooth manifolds, and W a smooth submanifold of Y. Let $f : X \to Y$ be smooth and assume that $f \pitchfork W$. Then $f^{-1}(W)$ is a submanifold of X. Moreover,* $\operatorname{codim} f^{-1}(W) = \operatorname{codim} W$.

It is straightforward to verify that the above theorem is merely the submersion level-set theorem in the case that W is a point. From the proof of [5] it should be clear that one could prove a similar result when the condition of f is weakened to $\operatorname{im} T_x f + T_{f(x)}W$ being a constant rank subspace of $T_{f(x)}Y$, whenever $f(x) \in W$. When W is again a point, this would be the constant rank level-set theorem. We will not be concerned with this more general statement in this text.

Next to the preimage of a submanifold, it is also often useful to know when two submanifolds V and W of a manifold X have an intersection $V \cap W$ which is again a submanifold of X. Although this may look like a separate question at first, we can use the above theorem and the canonical (smooth) inclusion map $i : V \to X$ to conclude that $V \cap W$ is a submanifold of X of codimension $\operatorname{codim} V + \operatorname{codim} W$ whenever $i \pitchfork W$. Note that $i \pitchfork W$ is equivalent to $T_xV + T_xW = T_xX$ for all $x \in V \cap W$. For this special case, we say that V and W are transverse, and the notation $V \pitchfork W$ is usually used instead of $i \pitchfork W$.

Although we now have have a sufficient condition for the preimage $f^{-1}(W)$ to be a submanifold, one can wonder whether it is reasonable to expect a map f to be transverse to a given submanifold W. Thom's transversaliy theorem [19] gives an affirmative answer. A modern formulation of the result is taken from [5, theorem II.4.9].

Theorem 4.2 (Thom Transversality Theorem). *Let X and Y be smooth manifolds and W a submanifold of $J^k(X, Y)$. Let*

$$T_W := \{f \in C^\infty_S(X, Y) | j^k f \pitchfork W\}.$$

Then T_W is a residual subset of $C^\infty(X, Y)$ in the Whitney C^∞ topology. Moreover, if W is closed, then T_W is open.

Remark 4.3. That T_W is a residual subset of $C^\infty_S(X, Y)$ is by itself not of particular interest. Instead, if A is a residual subset of $C^\infty_S(X, Y)$, then proposition 2.7(iv) implies that A is dense, which is applied in many arguments. It is however more useful to conclude that T_W is residual: a countable intersection of residual sets is still residual, and in a Baire space also dense, in contrast to a countable (or even finite) intersection of merely dense sets. △

Note that the use of jet bundles implies that the theorem has a broader scope than merely smooth maps $f : X \to Y$ and submanifolds $W \subset Y$: we can recover the latter by defining $W' = X \times W \subset J^0(X, Y)$. Thom's transversality theorem has some geometrically interesting consequences. For example, for every two submanifolds V and W of a manifold X, an arbitrarily small movement of V yields a new embedded submanifold V' which is transverse to W [13, theorem 3.2.4].

Example 4.4 (Transversality for smooth embeddings)**.** A context in which transversality is often used, is the study of embeddings in $C^\infty(X,Y)$. Let $f : X \to Y$ be a smooth map, and assume that X is compact. Then f is a smooth embedding if and only if f is an injective immersion. For the injectivity, it is of interest to study the set of double points of f, i.e. the set $\{(x,y) \in X \times X \setminus \Delta(X) : f(x) = f(y)\}$. If the smooth map $f \times f : X \times X \setminus \Delta(X) \to Y \times Y$ is transverse to $\Delta(Y)$, then the set of double points is a submanifold of codimension $\dim Y$. The techniques used to prove theorem 4.2 can be generalized to product maps like $f \times f$, and one can conclude that the set of maps f in $C^\infty_S(X,Y)$ whose product $f \times f|_{X \times X \setminus \Delta(X)}$ is transverse to $\Delta(Y)$ is a residual set. In particular, if $\dim Y > 2 \dim X$, we can immediately deduce the existence of a smooth injection $f : X \to Y$. Using this, and some other techniques found in chapter 2 of [12] which show that the set of immersions is also dense with this dimension condition, one can even prove the existence of embeddings from X to Y. [13, theorem 2.13]. $\triangle$

The generalization of theorem 4.2 alluded to in the example is known as the multijet transversality theorem, and is to the author's best knowledge attributed to [15, proposition 3.3], although a slightly weaker formulation can already be found in [7, theorem 1.10]. Recall that $\mathfrak{s} : J^k(X,Y)^s \to X^s$ was the source map of the multijet of X and Y, and that $\Delta^{(s)}(X)$ is the large diagonal of X in X^s (see section 2.1).

Theorem 4.5 (Multijet transversality theorem)**.** *Let X and Y be smooth manifolds with W a submanifold of $J^k(X,Y)^s$, such that $\mathfrak{s}(W) \cap \Delta^{(s)}(X) = \varnothing$. Let*

$$T_W := \{f \in C^\infty(X,Y) : j^k_s(f) \pitchfork W\}.$$

Then T_W is a residual subset of $C^\infty_S(X,Y)$. Moreover, if W is compact, then T_W is open.

We present a generalization of Thom transversality and multijet transversality, theorem 4.14 and 4.18. It focuses on the transversality of products of maps. Specifically, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two families of smooth manifolds, and denote $J = \prod_{j=1}^n J^{k_j}(X_j, Y_j)$ for some integers $k_j \geq 0$. Let X denote the product manifold whose factors are X_j, and note that the source maps $\mathfrak{s}_j : J^{k_j}(X_j, Y_j) \to X_j$ together form a source map $\mathfrak{s} : J \to X$. For every tuple of maps $(f_1, \ldots, f_n)$ in $\mathcal{C}_S := \prod_{j=1}^n C^\infty_S(X_j, Y_j)$, there exists a smooth map $j(f_1, \ldots, f_n) : X \to J$, given by $(x_1, \ldots, x_n) \mapsto (j^{k_1} f_1(x_1), \ldots, j^{k_n} f_n(x_n))$. Note that, as for the ordinary and multijet jets, the smooth map $j(f_1, \ldots, f_n)$ is a section for $\mathfrak{s}$. Theorem 4.14 implies that for every submanifold W of J, the set of tuples in $\mathcal{C}$ whose associated section $X \to J$ is transverse to W, is a residual subset of $\mathcal{C}_S$. Again, a version of this statement can already be found in [7, theorem 1.9].

However, the scope of theorem 4.14 is larger: it extends to maps that possibly depend on the same variable. An example helps to illustrate the situation: let $X_1 = \mathbb{R} \times Z_1$, $X_2 = \mathbb{R} \times Z_2$ and $X_3 = \mathbb{R} \times Z_3$, where the Z_j are smooth manifolds, and let $f_j : \mathbb{R} \times Z_j \to X_j$ be a collection of smooth maps. Then we can define an associated smooth map

$$f : \mathbb{R} \times Z_1 \times Z_2 \times Z_3 \to Y,\ (t, z_1, z_2, z_3) \mapsto (f_1(t, z_1), f_2(t, z_2), f_3(t, z_3)).$$

Theorem 4.14 implies at least that, for every submanifold W of Y, the set of triples in $\prod_{j=1}^3 C^\infty_S(\mathbb{R} \times Z_j, Y_j)$ whose associated map is transverse to W, is a residual subset. More generally, we shall want to consider the case where each X_j is itself a product of manifolds $Z_{i,j}$, where a number of $Z_{i,j}$ can possibly agree. Moreover, there is an appropriate jet transversality statement for this context. This statement is however not about the jet of a product map, but instead about the product map of jets. We elaborate on this point below.

Theorem 4.14 states that the set of tuples in $\mathcal{C}_S$ whose induced section $X \to J$ is transverse to some submanifold of W, is a residual set. The multijet transversality theorem states that the same is true if $X_1 = \ldots = X_n$, $Y_1 = \ldots = Y_n$, $k_1 = \ldots = k_n$, and we restrict ourselves to the subspace $\Delta^n(C^\infty_S(X_1, Y_1)) \subset \mathcal{C}_S$. That is, we restrict to n-fold tuples of the same smooth map $f : X_1 \to$

Y_n. Unfortunately, we cannot deduce the multijet transversality theorem from the theorem 4.14 immediately. Theorem 4.18 generalizes multijet transversality in the same way that theorem 4.14 generalizes Thom transversality: the scope is extended to products of maps between products of manifolds, where the factors of the maps and spaces can repeat, and the associated map may depend on the same variables multiple times. In the next part we shall give a number of examples.

Remark 4.6 (The difference between products of jets and jets of products)**.** The application of theorem 4.18 in this text is to families of product maps, parameterized by a smooth manifold (usually $\mathbb{D}^k$). The full generality of theorem 4.14 and 4.18 is inspired by [18, theorem 5.3 & 5.4]. Those theorems claimed at least, in the above notation, that for any submanifold of $J^k(X,Y)$, the tuples in $\mathcal{C}_S$ whose induced section $j^k(f_1 \times \ldots \times f_n) : X \to J^k(X,Y)$ are transverse to W, is a residual set. However, these claims are generally false unless W is transverse to the subbundle $\prod_{j=1}^n J^k(X_j,Y_j) \subset J^k(X,Y)$ (see remark 2.9 for the embedding). This was separately observed before in [10, section 4.1]. This observation is the main reason for theorem 4.14 and theorem 4.18 to focus on sections of products bundles $\prod_{j=1}^n J^k(X_j,Y_j)$, instead of sections of $J^k(X,Y)$. The reason why our result does hold, is that the image of the products of jets can 'span' the product of jet bundles, while jet of products cannot attain all values in the jet bundle of the products. The key lemma related to this is lemma A.2, which is used on a local diffeomorphism of the product of jet bundles and perturbations of smooth product maps in the proof of theorems 4.14 and 4.18. The following is a counter example to the theorems in [18]. △

Example 4.7. Let $n = 2$, let $X_1 = X_2 = Y_1 = Y_2 = \mathbb{R}$, and let $W \subset J^1(X,Y) \cong \mathbb{R}^4 \times \mathbb{R}^{2\times 2}$ consist of all tuples of the form

$$\left(x, x, y_1, y_2, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : x, y_i, a_i \in \mathbb{R}\right).$$

Let $f, g \in C^\infty(\mathbb{R},\mathbb{R})$. A dimension count shows that $j^1(f \times g)$ is tranverse to W if and only if $j^1(f \times g)$ does not intersect W. However, regardless of the choice of f and g, $j^1(f \times g)^{-1}(W) = \Delta(X)$. △

4.1 Mixed jet transversality

In this section we introduce mixed jet bundles, which are the appropriate bundles to use, in order to study the transversality of product maps depending on the same variables. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two families of manifolds, and denote by X and Y the product families whose factors are the members of the respective families. Suppose that there exists a third family of manifolds $Z_1, \ldots, Z_m$ such that each X_j is a product of some Z_i. That is to say, suppose there exist n ordered subsets $I_j \subset \{1, \ldots, k\}$ such that $X_j = \prod_{i\in I_j} Z_i$. The ordering on I_j does not have to be the one inherited from $\mathbb{N}$, and we shall suppress the ordering from the notation for ease of reading. Without loss of generality, we may assume that $\bigcup_{j=1}^n I_j = \{1, \ldots, k\}$, and we shall assume this throughout this text.

To every two families $\{X_j\}$ and $\{Y_j\}$, and a choice of n integers $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, we associate the **$k$-product bundle of X and Y** or simply the **product bundle of X and Y**, $J^k(X,Y)$ given by

$$J^k(X,Y) = \prod_{j=1}^n J^{k_j}(X_j, Y_j).$$

Each factor of the product bundle is a fiber bundle over $X_j \times Y_j$, and hence the product bundle itself is a fiber bundle over $X \times Y$. As for the ordinary and multijet bundle, the map $\mathfrak{s} : J^k(X,Y) \to X$ is called the **source map** and the map $\mathfrak{t} : J^k(X,Y) \to Y$ is called the **target map**. We shall define a subbundle of the product bundle, which 'counts' each factor Z_i of the X_j only once. To this end, define $Z = \prod_{i=1}^m Z_i$, and denote by Ω the diffeomorphism

$$X \times Y \to \left(\prod_{i=1}^m \prod_{j:i \in I_j} Z_i \right) \times Y,$$

which assigns each Z_i in X_j to its factor in $\prod_{j:i\in I_j} Z_i$, and the Y_j to themselves. I.e. Ω is merely a reordering. The second product is again ordered by the ordering of I_j. Denote by N_i the integer $|\{j : i \in I_j\}|$. We can embed $Z \times Y$ into the right hand side above, as $\left(\prod_{i=1}^m \Delta^{N_i}(Z_i)\right) \times Y$. We denote the embedding by ι, and it follows that $\Omega^{-1} \circ \iota(Z \times Y)$ is a submanifold of $\prod_{j=1}^n X_j \times Y_j$. Finally, denote by $\mathcal{I}$ the tuple $(I_1, \dots, I_n)$. We are now ready to define the submanifold.

Definition 4.8. Define the $(\mathcal{I}, k)$-mixed jet bundle of X and Y or simply the mixed jet bundle of X and Y, $J^k_{\mathcal{I}}(X, Y)$, to be the restriction of

$$J^k(X, Y) \xrightarrow{(\mathfrak{s}_j \times \mathfrak{t}_j)_{j=1}^n} \prod_{j=1}^n X_j \times Y_j$$

to $\Omega^{-1} \circ \iota(Z \times Y)$. Define $C^\infty_{\mathcal{I},k}(X, Y)$ to be

$$\prod_{j=1}^n C^\infty(X_j, Y_j).$$

Proposition 4.9. *$J^k_{\mathcal{I}}(X, Y)$ is a smooth submanifold of $J^k(X, Y)$. Moreover, it is a fiber bundle over $Z \times Y$, with bundle map given by $\iota^{-1} \circ \Omega \circ (\mathfrak{s}_j \times \mathfrak{t}_j)_{j=1}^n$.*

Proof. It should be clear that the restriction of the product bundle to its fibers over a submanifold of the base, is a smooth submanifold. Moreover, since the product bundle fibers over $\prod_{j=1}^n X_j \times Y_j$ (as each factor does), the restricted bundle has the same fiber. As $\Omega^{-1} \circ \iota$ maps $Z \times Y$ diffeomorphically onto the base of $J^k_{\mathcal{I}}(X, Y)$, it follows that $J^k_{\mathcal{I}}(X, Y)$ is a fiber bundle over $Z \times Y$. $\square$

Remark 4.10. An alternative definition for $J^k_{\mathcal{I}}(X, Y)$ could be the pull-back bundle $(\Omega^{-1} \circ \iota)^* J^k(X, Y)$, and the two are isomorphic as bundles. However, in the remainder of the chapter we make explicit use of $J^k_{\mathcal{I}}(X, Y)$ being embedded, so we choose this definition. $\triangle$

Definition 4.11. Let $\pi : J^k_{\mathcal{I}}(X, Y) \to Z \times Y$ be a $(\mathcal{I}, k)$-mixed jet bundle. A section of $J^k_{\mathcal{I}}(X, Y)$ is a smooth map $F : Z \to J^k(X, Y)$ such that $\mathfrak{s} \circ F = \mathrm{id}_Z$. Let $(f_1, \dots, f_n) \in C^\infty_{\mathcal{I},k}(X, Y)$, and let $\pi_j : Z \to X_j$ denote the projection, which forgets the factors of Z not contained in I_j. The $(\mathcal{I}, k)$-jet extension of $(f_1, \dots, f_n)$, $j^k_{\mathcal{I}}(f_1, \dots, f_n) : Z \to J^k_{\mathcal{I}}(X, Y)$ given by

$$j^k_{\mathcal{I}}(f_1, \dots, f_n) := \left(j^{k_1} f_1 \circ \pi_1, \dots, j^{k_n} f_n \circ \pi_n\right).$$

$\triangle$

The following is a straightforward verification.

Proposition 4.12. *Let $f = (f_1, \dots, f_n) \in C^\infty_{\mathcal{I},k}(X, Y)$, then the $(\mathcal{I}, k)$-mixed jet extension of f, is a smooth section of $\mathfrak{s} : J^k_{\mathcal{I}}(X, Y) \to Z$.*

Example 4.13. We highlight a few examples of $J^k_{\mathcal{I}}(X, Y)$ and $j^k_{\mathcal{I}}(f_1, \dots, f_n)$ for different choices of n, m, k and $\mathcal{I}$.

- If $n = m = 1$, then $J^k_{\mathcal{I}}(X, Y) = J^{k_1}(X_1, Y_1)$, and $j^k_{\mathcal{I}}(f_1) = j^{k_1} f_1$. In other words, in this case we obtain the normal jet bundle and jets of functions.

- If $n = m$ and $I_j = (j)$ for each j, $J^k_{\mathcal{I}}(X,Y) = J^k(X,Y)$, and $j^k_{\mathcal{I}}(f_1,\dots,f_n) = j^{k_1}f_1 \times \ldots \times j^{k_n}f_n$.
- If $n = 3$, $m = 2$, and $\mathcal{I} = ((1),(2),(2,1))$ we have that $X_3 = X_2 \times X_1$, and that $J^k_{\mathcal{I}}(X,Y)$ is the restriction of $\mathfrak{s} : J^k(X,Y) \to X_1 \times X_2 \times X_3$ to the fibers of the form (a,b,b,a). The section $j^k_{\mathcal{I}}(f_1,f_2,f_3)$ is given by $(a,b) \mapsto (j^{k_1}f_1(a), j^{k_2}f_2(b), j^{k_3}f_3(b,a))$.
- If $n = 2, m = 1, k = (r,r)$, and $\mathcal{I} = ((1),(1))$, we have that $J^k_{\mathcal{I}}(X,Y) \cong J^r(X_1, Y_1 \times Y_2)$ and $j^k_{\mathcal{I}}(f_1,f_2) = j^r(f)$, where $f = (f_1,f_2)$. I.e. the following diagram commutes:

$$\begin{tikzcd}
C^\infty(X_1,Y_1)\times C^\infty(X_1,Y_2) \arrow[r, "\sim"] \arrow[d, "j^k_{\mathcal{I}}"] & C^\infty(X_1, Y_1\times Y_2) \arrow[d, "j^r"] \\
C^\infty(X_1, J^k_{\mathcal{I}}(X_1^2, Y_1\times Y_2)) \arrow[r, "\sim"] & C^\infty(X_1, J^r(X_1^2, Y_1\times Y_2))
\end{tikzcd}$$

The horizontal isomorphisms are meant as bijections, but [5, propositions II.3.5 & II.3.6] show that these maps are homeomorphisms, if all sets are endowed with the strong Whitney C^∞ topology. △

The main result about mixed jet transversality we wish to present is the following. As announced, it generalizes Thom's transversality theorem.

Theorem 4.14. *Let $X_1,\dots,X_n$, $Y_1,\dots,Y_n$, and $Z_1,\dots,Z_m$ be three families of manifolds, such that there exist n ordered subsets I_j of $\{1,\dots,n\}$ with $X_j = \prod_{i\in I_j} Z_i$ (ordered). Let X, Y, and Z denote the product manifolds whose factors are the members of the respective families. Let $k \in \mathbb{N}_0^n$, and define $\mathcal{I} = (I_1,\dots,I_n)$. Let W be a submanifold of $J^k_{\mathcal{I}}(X,Y)$. Then*

$$T_W := \{f \in C^\infty_{\mathcal{I},k}(X,Y) : j^k_{\mathcal{I}}f \pitchfork W\}$$

is a residual subset of $C^\infty_{\mathcal{I},k}(X,Y)$ for the topology induced by the strong Whitney C^∞ topology on each factor.

The complete proof of this theorem can be found in appendix A. Here we give a sketch of the proof. The first step (lemma A.1) is to show that the section $j^k_{\mathcal{I}}f$ associated to the tuple $f = (f_1,\dots,f_n) \in C^\infty_{\mathcal{I},k}(X,Y)$ is transverse to W if and only if $j^{k_1}f_1 \times \ldots \times j^{k_n}f_n$ is transverse to $\iota(W)$, where $\iota : J^k_{\mathcal{I}}(X,Y) \to J^k(X,Y)$ is the canonical inclusion map. In other words, the proof is reduced to the special case where $m = n$ and $I_j = (j)$. In the second step, one can choose an appropriate open covering $\{W_x : x \in W\}$ of the submanifold W such that each $\overline{W_x}$ is compact and contained in appropriate charts. By the second countability of W, we can find a precompact countable subcovering $W_1, W_2, \dots$ of W. The central claim is that

$$\{T_{W_j} := \{f \in C^\infty_{\mathcal{I},k}(X,Y) : j^k_{\mathcal{I}}f \pitchfork W \text{ on } \overline{W_j}\}$$

is an open and dense set, from which the residuality then follows. Openness is proved in lemma A.6, while the density is the core of the final proof.

Sometimes, we can even guarantee that T_W is open and dense. Specifically, when W is compact or when W is a product of topologically closed sets, which factors over the product bundle $J^k(X,Y)$. For this, we refer to proposition A.5. For applications of theorem 4.14, it is often useful to have more flexibility than what is stated. The following corollary highlights some of the flexibility. The proof can also be found in the appendix.

Corollary 4.15. *Let $X_1, \dots, X_n$, $Y_1, \dots, Y_n$, and $Z_1, \dots, Z_m$ be three families of manifolds, such that there exist n ordered subsets I_j of $\{1, \dots, m\}$ with*

$$X_j = \prod_{i \in I_j} Z_i \quad \textit{(ordered product).}$$

Define $\mathcal{I} = (I_1, \dots, I_n)$. Let $k \in \mathbb{N}_0^n$, and let X, Y, and Z denote the product manifolds whose factors are the members of the respective manifolds. Let W be a submanifold of $J^k_{\mathcal{I}}(X, Y)$, and let $W' \subset W$ be a compact subset. Then

$$T_W := \{f \in C^\infty_{\mathcal{I},k}(X, Y) : j^k_{\mathcal{I}} f \pitchfork W \ \text{ on } W'\}$$

is an open and dense subset of $C^\infty_{\mathcal{I},k}(X, Y)$ for the topology induced by the strong Whitney C^∞ topology on each factor.

Moreover, let $f \in C^\infty_{\mathcal{I},k}(X, Y)$, let $\mathcal{V}$ be an open neighbourhood of f, and suppose that $\mathfrak{s}(W)$ is contained in an open product set $U = U_1 \times \ldots \times U_m \subset Z_1 \times \ldots \times Z_m$, where $\mathfrak{s} : J^k_{\mathcal{I}}(X, Y) \to Z$ is the source map. The there exists $g \in \mathcal{V}$ such that $j^k_{\mathcal{I}} g = j^k_{\mathcal{I}} f$ off U, and $j^{k,l}_{\mathcal{I}} g \pitchfork W$.

4.2 Symmetric mixed jet transversality

In this section we introduce symmetric mixed jet bundles. Where mixed jet bundles could be used to study transversality of product maps depending on the same variables, the symmetric jet bundles allow us to study the special case where the factors of such products repeat, like in the multijet transversality theorem.

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be two families of manifolds. Fix a tuple l in $\mathbb{N}_1^n$ and another tuple $k \in \mathbb{N}_0^n$, and define the **symmetric product bundle of** $(\{X_j\}, \{Y_j\}, k, l)$ or **symmetric product bundle** as

$$J^{k,l}(X, Y) := \prod_{j=1}^n J^{k_j}(X_j, Y_j)^{l_j}.$$

It should be clear that, similar to all previous jet bundles, the symmetric product bundle is a fiber bundle over $X^l \times Y^l$, where $X^l := \prod_{j=1}^n X_j^{l_j}$ and $Y^l := \prod_{j=1}^n Y_j^{l_j}$. Composing with the projections to X^l and Y^l we obtain the **source and target maps of the symmetric product bundle**. We refer to the l_j as the **multiplicities of the symmetric product bundle**. Let $|l|_s = \sum_{j=1}^s |l_j|$ and define $|l| = |l|_n$. Suppose that there exists a third family of manifolds $Z_1, \dots, Z_m$ such that there exist $|l|$ ordered subsets $I_j \subset \{1, \dots, m\}$ with

$$X^{l_j} = \prod_{j=|l|_{j-1}+1}^{|l|_j} \prod_{i \in I_j}^{n} Z_i \quad \text{(ordered product).}$$

That is to say, there exists a third family of manifolds such that each factor of X is individually a product manifolds Z_i. We stress that the factorization may be chosen differently among equal copies of X_j, which allows for more flexibility. Let $\mathcal{I} = (I_1, \dots, I_{|l|})$, and denote by k_l the $|l|$-fold tuple with $(k_l)_j := k_i$ for all $|l|_{i-1} < j \leq |l|_i$.

Definition 4.16. Define the $(\mathcal{I}, k, l)$-symmetric mixed jet bundle of X and Y or simply the symmetric mixed jet bundle of X and Y, $J^{k,l}_{\mathcal{I}}(X, Y)$, as $J^{k_l}_{\mathcal{I}}(X^l, Y^l)$. Define $C^\infty_{\mathcal{I},k,l}(X, Y)$ to be

$$\prod_{j=1}^n C^\infty(X_j, Y_j).$$

△

It should be clear that the symmetric mixed jet bundle of X and Y is a fiber bundle over $Z \times Y^l$, and projecting further we can again define the **source map** and **target map** of the symmetrical jet bundle. We can again define sections of $\mathfrak{s} : J^{k,l}_{\mathcal{I}}(X,Y) \to Z$ from tuples in $C^\infty_{\mathcal{I},k,l}(X,Y)$. Before we do, we formalize a connection between $C^\infty_{\mathcal{I},k,l}(X,Y)$ and $C^\infty_{\mathcal{I},k_l}(X^l,Y^l)$. The following proposition is a straightforward verification, and does not follow from unique properties of the weak or strong Whitney C^∞ topology. Let f^p denote the p-fold tuple whose entries are f.

Proposition 4.17. *In the above notation, endow both*

$$C^\infty_{\mathcal{I},k,l}(X,Y) = \prod_{j=1}^n C^\infty(X_j,Y_j) \text{ and } C^\infty_{\mathcal{I},k_l}(X^l,Y^l) = \prod_{j=1}^n C^\infty(X_j,Y_j)^{l_j}$$

with the product topology induced by either the weak or strong Whitney C^∞ topology on each of its factors. Then the map $\Delta^l : C^\infty_{\mathcal{I},k,l}(X,Y) \to C^\infty_{\mathcal{I},k_l}(X^l,Y^l)$ given by $(f_1,\ldots,f_n) \mapsto (f_1^{l_1},\ldots,f_n^{l_n})$ is a topological embedding with image $\prod_{j=1}^n \Delta^{l_j}(C^\infty(X_j,Y_j))$.

Let $(f_1,\ldots,f_n) \in C^\infty_{\mathcal{I},k,l}(X,Y))$. It is now straightforward to verify that the **symmetric mixed jet of** $(f_1,\ldots,f_n)$, $j^{k,l}_{\mathcal{I}}(f_1,\ldots,f_n)$, given by $j^{k_l}_{\mathcal{I}}\Delta^l(f_1,\ldots,f_n) : Z \to J^{k,l}_{\mathcal{I}}(X,Y)$, is a smooth section of $\mathfrak{s} : J^{k,l}_{\mathcal{I}}(X,Y) \to Z$.

The main result about symmetric mixed transversality we wish to present is the theorem below. As announced, it generalizes theorem 4.5. Both theorems make reference to a certain generalized diagonal, which is the last object that needs to be introduced before we state our result.

The **l-symmetric diagonal of** X or simply **symmetric diagonal of** X, $\Delta^l(X)$, is given by

$$\prod_{j=1}^n \Delta^{(l_j)}(X_j) \subset X^l.$$

We emphasize that $\Delta^l(X)$ is a product of large diagonals.

Theorem 4.18. *Let $X_1,\ldots,X_n$, $Y_1,\ldots,Y_n$, and $Z_1,\ldots,Z_m$ be three families of manifolds, such that there exist $|l|$ ordered subsets I_j of $\{1,\ldots,m\}$ with*

$$X_j^{l_j} = \prod_{|l|_{j-1}<i\le|l|_j} \prod_{i\in I_j} Z_i \quad \textit{(ordered product).}$$

Define $\mathcal{I} = (I_1,\ldots,I_{|l|})$. Let $k \in \mathbb{N}_0^n$ and $l \in \mathbb{N}_1^n$, let X, Y, and Z denote the product manifolds whose factors are the members of the respective manifolds, and let $X^l := \prod_{j=1}^n X^{l_j}$. Let W be a submanifold of $J^{k,l}_{\mathcal{I}}(X,Y)$ such that $\mathfrak{s}(W) \cap \Delta^l(X) = \varnothing$, where $\mathfrak{s} : J^{k,l}(X,Y) \to X^l$ is the source map of the symmetric mixed bundle. Then

$$T_W := \{f \in C^\infty_{\mathcal{I},k,l}(X,Y) : j^{k,l}_{\mathcal{I}} f \pitchfork W\}$$

is a residual subset of $C^\infty_{\mathcal{I},k,l}(X,Y)$ for the topology induced by the strong Whitney C^∞ topology on each factor.

Example 4.19. We highlight a few examples of $J^{k,l}_{\mathcal{I}}(A,X)$, $j^{k,l}_{\mathcal{I}}(f_1,\ldots,f_n)$ and $\Delta^l(X)$, for different choices of n, m, k, l, and $\mathcal{I}$.

- If $n = m$, $l = (n)$, and $I_j = (j)$ for each j, then $J^{k,l}_{\mathcal{I}}(X,Y) = J^{k_1}(X_1,Y_1)^n$, and $j^{k,l}_{\mathcal{I}} f = j^{k_1}_n f$. In other words, in this special case we obtain the multijet bundle and multijet of f. $\Delta^l(X)$ is the large diagonal $\Delta^{(n)}(X) \subset X^n$.

- Let $n = m = 2$, $l = (2,1)$, and $\mathcal{I} = ((1),(2),(2,1))$. Then $X_2 = X_1^2$, and $J^{k,l}_{\mathcal{I}}(X,Y)$ is the restriction of $J^{k_1}(X_1,Y_1)^2 \times J^{k_2}(X_2,Y_2) \to (X_1)^4 = X^l$ to the fibers of the form (a,b,b,a). The section $j^{k,l}_{\mathcal{I}}(f,g)$ is given by $(a,b) \mapsto (j^{k_1}f(a), j^{k_1}f(b), j^{k_2}g(b,a))$. Moreover, $\Delta^l(X) = \{(a,a,b,c) : a,b,c \in X_1\} \subset X^l$.
- Let $n = 1, m = 3$, $I_j = (1,2,3) \setminus \{j\}$, and $l = (3)$. Then, all Z_i are equal, $Z_i^2 = X_1$, and $J^{k,l}_{\mathcal{I}}(X,Y)$ is the restriction of $J^{k_1}(X_1,Y_1)^3 \to Z_1^6 = X^l$ to the fibers of the form (b,c,a,c,a,b). The section $j^{k,l}_{\mathcal{I}}f$ is given by $(a,b,c) \mapsto (j^{k_1}f(b,c), j^{k_1}f(a,c), j^{k_1}f(a,b))$. Moreover, $\Delta^l(X) = \{(a,a,b,b,c,c) : a,b,c \in Z_1\} \subset X^l$.

△

Again, sometimes T_W is even open and dense, such as when W is compact. For a concrete statement, we refer to proposition A.7. As in corollary 4.15, theorem 4.18 allows for more flexibility then is stated:

Corollary 4.20. *Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, and $Z_1, \ldots, Z_m$ be three families of manifolds, such that there exist $|l|$ ordered subsets I_j of $\{1, \ldots, m\}$ with*

$$X_j^{l_j} = \prod_{|l|_{j-1} < i \leq |l|_j} \prod_{i \in I_j} Z_i \quad \textit{(ordered product).}$$

Define $\mathcal{I} = (I_1, \ldots, I_{|l|})$. Let $k \in \mathbb{N}_0^n$ and $l \in \mathbb{N}_1^n$, let X, Y, and Z denote the product manifolds whose factors are the members of the respective manifolds, and let $X^l := \prod_{j=1}^n X^{l_j}$. Let W be a submanifold of $J^{k,l}_{\mathcal{I}}(X,Y)$ such that $\mathfrak{s}(W) \cap \Delta^l(X) = \varnothing$, where $\mathfrak{s} : J^{k,l}(X,Y) \to X^l$ is the source map of the symmetric mixed bundle. Let $W' \subset W$ be a compact subset. Then

$$T_{W'} := \{f \in C^\infty_{\mathcal{I},k,l}(X,Y) : j^{k,l}_{\mathcal{I}} f \pitchfork W \text{ on } W'\}$$

is an open and dense subset of $C^\infty_{\mathcal{I},k,l}(X,Y)$ for the topology induced by the strong Whitney C^∞ topology on each factor. Moreover, let $f \in C^\infty_{\mathcal{I},k,l}(X,Y)$, let $\mathcal{V}$ be an open neighbourhood of f, and suppose that $\mathfrak{s}(W)$ is contained in an open product set $U = U_1^{l_1} \times \ldots \times U_n^{l_n} \subset X^l$, $U_j \subset X_j$. The there exists $g \in \mathcal{V}$ such that $j^{k,l}_{\mathcal{I}} g = j^{k,l}_{\mathcal{I}} f$ off $U_1 \times \ldots \times U_n$, and $j^{k,l}_{\mathcal{I}} g \pitchfork W$.

4.3 Applications

The aim of this section is to prove corollary 4.24. If $W_1 \subset Y_1$ and $W_2 \subset Y_2$ are submanifolds, then the corollary shows not only that the set of pairs $(f,g) \in C^\infty_S(X_1,Y_1) \times C^\infty_S(X_2,Y_2)$ such that $f \pitchfork W_1$ and $g \pitchfork W_2$ is residual, but that the subset of those pairs satisfying $f^{-1}(W_1) \pitchfork g^{-1}(W_2)$ too, is also residual. The corollary extends also the setting of product maps depending on the same variables. The corollary is not used in the remainder of the thesis, but may be interesting to the reader regardless. This section can by skipped otherwise without a problem. The main idea used is to use a characterization of the tangent space of $f^{-1}(W_1)$ and $g^{-1}(W_2)$ in terms of a map cooked out of f and g.

The following lemma can be used to prove theorem 4.1, although here we are interested in it because it help characterize the tangent space of the preimage of a transverse map.

Lemma 4.21 (II.4.3 of [5])**.** *Let X, Y be manifolds, let $W \subset Y$ be a submanifold, and let $f : X \to Y$ be a smooth map. Let $x \in X$ and $f(x) \in W$. Suppose there is a neighbourhood U of $f(x)$ in Y and a submersion $\phi : U \to \mathbb{R}^k$ ($k = \operatorname{codim} W$) such that $W \cap U = \phi^{-1}(0)$. Then $f \pitchfork W$ iff $\phi \circ f$ is a submersion at x.*

Corollary 4.22. *If $f : X \to Y$ is a smooth map, transverse to a submanifold $W \subset Y$, then for $x \in W_f := f^{-1}(W)$, the tangent space $T_x W_f$ is given by $\{v \in T_x X : T_x f(v) \in T_{f(x)} W\}$.*

Proof. A neighbourhood U around $f(x)$ such as in the above lemma always exists, as W is a submanifold of Y. As $\phi \circ f$ is a submersion at x, it follows that the tangent space $T_x W_f$ is given by $\ker T_x(\phi \circ f)$. As ϕ is also a submersion, and $W \cap U = \phi^{-1}(0)$, it follows that $\ker T_x(\phi \circ f) = \{v \in T_x X : T_x f(v) \in T_{f(x)} W\}$. ◻

We can now give a different characterization for the condition $W_f \pitchfork W_g$, for two maps f and g transverse to submanifolds W_1 and W_2 (respectively).

Lemma 4.23. *Let X, Y, Z be manifolds, let $W \subset Y$ and $V \subset Z$ be submanifolds, and let $f : X \to Y$ and $g : X \to Z$ be smooth maps transverse to W respectively V. Denote $W_f = f^{-1}(W)$ and $V_g = g^{-1}(V)$. Then*

$$W_f \pitchfork V_g \iff F := (f, g) \pitchfork W \times V.$$

Proof. Note that $F^{-1}(W \times V) = W_f \cap V_g$. Hence the result is trivial if $W_f \cap V_g = \varnothing$.

Otherwise, let $x \in W_f \cap V_g$, and let $k = \operatorname{codim} W$ and $l = \operatorname{codim} V$. As W and V are submanifolds, there exist open neighbourhoods U_f of $f(x)$ and U_g of $g(x)$, and submersions $\phi_f : U_f \to \mathbb{R}^k$ and $\phi_g : U_g \to \mathbb{R}^l$ such that $W \cap U_f = \phi_f^{-1}(0)$ and $V \cap U_g = \phi_g^{-1}(0)$. By the above corollary, it follows that $W_f \pitchfork V_g$ at x iff $T_x W_f + T_x V_g = T_x X$ iff $\{v \in T_x X : T_x f(v) \in T_{f(x)} W\} + \{v \in T_x X : T_x g(v) \in T_{g(x)} V\} = T_x X$. Note that $\operatorname{im} T_x F = \{(T_x f(v), T_x g(v) : v \in T_x X\}$. As $\operatorname{im} T_x f + T_{f(x)} W = T_{f(x)} Y$ and $\operatorname{im} T_x g + T_{g(x)} V = T_{g(x)} Z$, it follows that

$$W_f \pitchfork V_g \iff \operatorname{im} T_x F + T_{F(x)} W \times V = T_{F(x)} Y \times Z \iff F \pitchfork W \times V.$$

◻

Corollary 4.24. *Let $X_1, \ldots, X_n$, $Y_{1,1}, \ldots, Y_{1,n}$, $Y_{2,1}, \ldots, Y_{2,n}$ and $Z_1, \ldots, Z_m$ be four families of manifolds, such that there exist n ordered subsets I_j of $\{1, \ldots, n\}$ with $X_j = \prod_{i \in I_j} Z_i$ (ordered). Let X, Y_1, Y_2, and Z denote the product manifolds whose factors are the members of the respective families. Let $k \in \mathbb{N}_0^n$, and define $\mathcal{I} = (I_1, \ldots, I_n)$. Let W be a submanifold of $J^k_{\mathcal{I}}(X, Y_1)$ and V be a submanifold of $J^k_{\mathcal{I}}(X, Y_2)$. Then*

$$T_{W,V} := \{(f, g) \in C^\infty_{\mathcal{I},k}(X, Y_1) \times C^\infty_{\mathcal{I},k}(X, Y_2) : j^k_{\mathcal{I}} f \pitchfork W, j^k_{\mathcal{I}} g \pitchfork V, W_f \pitchfork W_g\}$$

is a residual subset of $C^\infty_{\mathcal{I},k}(X, Y_1) \times C^\infty_{\mathcal{I},k}(X, Y_2)$.

Proof. Note that by theorem 4.14, the set

$$T_W \times T_V = \{(f, g) \in C^\infty_{\mathcal{I},k}(X, Y_1) \times C^\infty_{\mathcal{I},k}(X, Y^2) : j^k_{\mathcal{I}} f \pitchfork W, j^k_{\mathcal{I}} g \pitchfork V\}$$

is a product of residual sets. As a product of open and dense sets is again open and dense, it follows that $T_W \times T_V$ is again residual. Let $Y = Y_1 \times Y_2$ and note that $J^k_{\mathcal{I}}(X, Y_1) \times J^k_{\mathcal{I}}(X, Y_2) \cong J^k_{\mathcal{I}}(X, Y)$, $(j^k_{\mathcal{I}} f, j^k_{\mathcal{I}} g) = j^k_{\mathcal{I}}(f_1, \ldots, f_n, g_1, \ldots, g_n)$, and that $C^\infty_{\mathcal{I},k}(X, Y^1) \times C^\infty_{\mathcal{I},k}(X, Y^2) \cong C^\infty_{\mathcal{I},k}(X, Y)$ (see also example 4.13). By theorem 4.14, the set

$$\tilde{T}_{W,V} = \{(f, g) \in C^\infty_{\mathcal{I},k}(X, Y^1) \times C^\infty_{\mathcal{I},k}(X, Y^2) : F = j^k_{\mathcal{I}}(f, g) \pitchfork W \times V\}$$

is residual. By lemma 4.23, it now follows that $T_{W,V}$ is an intersection of two residual sets, and hence residual. ◻

5 Equivariance and transversality

When one studies a class of maps or structures using h-principle techniques, one can wonder what class of maps or structures should play the role of the formal solutions. As announced in the introduction, in Haefliger's embedding theorem (example 1.2) the formal counterpart to embeddings $f : M \to N$ were at least *equivariant* maps $g : M^2 \to N^2$. Heuristically, the equivariance emulates the property of a product of maps $f \times f$, that swapping the input of such a map would be the same as swapping the output of such map. In this section we shall study some of the topology of equivariant maps, and we shall derive an equivariant jet transversality theorem, and a companion multijet theorem. We expect that equivariant maps can play a role in other h-principles that involve non-local differential relations. Hopefully these results can be of use in the future.

5.1 Smooth equivariant maps

Definition 5.1. Let M, N, and B be manifolds. Let $g : M \times M \times B \to N \times N$ be a smooth map, then g is $\mathbb{Z}/2\mathbb{Z}$-*equivariant* or simply *equivariant* if g the following diagram commutes:

$$\begin{tikzcd} M \times M \times B \arrow[r, "g"] \arrow[d, "\mathbb{Z}_2"] & N \times N \arrow[d, "\mathbb{Z}_2"] \\ M \times M \times B \arrow[r, "g"] & N \times N \end{tikzcd}$$

Here the vertical maps swap the coordinates of M and respectively N. In terms of coordinates, if $g = (g_1, g_2)$, then g is equivariant if and only if

$$(g_1(y, x, t), g_2(y, x, t)) = (g_2(x, y, t), g_1(x, y, t)) \quad \forall (x, y, t) \in M \times M \times B.$$

△

Most often we will consider the case where $B = \mathbb{D}^k$, the closed unit disk in $\mathbb{R}^k$, and we think of an equivariant map $g : M^2 \times \mathbb{D}^k \to N^2$ as a smooth family of equivariant maps, parameterized by $\mathbb{D}^k$.

Definition 5.2. Let M, N, and B be manifolds. Let $g : M^2 \times B \to N^2$ be an equivariant map, and suppose that $N = N_1 \times N_2$ is a product of manifolds N_1 and N_2. We say that $g = (g_1, g_2, g_3, g_4)$ is a holonomic product in the first factor or simply holonomic in the first factor if $(g_1, g_3) = f_t \times f_t : M^2 \times B \to N_1^2$, where $f : M \times B \to N_1$ is a smooth map and $f_t \times f_t$ is the map $(x, y, t) \mapsto (f(x, t), f(y, t))$. We say that g is a holonomic product or simply holonomic if $g = f_t \times f_t$ for some smooth map $f : M \times B \to N$. △

Note that there exists a one-to-one correspondence with equivariant maps and maps $g : M^2 \times B \to N$. I.e. the second coordinate function of an equivariant map is determined by the first. For a map $g : M^2 \times B \to N$, denote its **equivariant product** by $g \owedge g$. Denote by $C^\infty(M^2 \times B, N^2; \mathbb{Z}_2)$ the subset of equivariant maps containted in $C^\infty(M^2 \times B, N^2)$.

Proposition 5.3. *The map $\owedge : C^\infty(M^2 \times B, N) \to C^\infty(M^2 \times B, N^2)$ is a topological embedding for both the strong and weak Whitney C^∞ topology, whose image is precisely $C^\infty(M^2 \times B, N^2; \mathbb{Z}_2)$.*

Proof. As remarked above, one can check that the map $\owedge$ is injective, with image all equivariant maps. For any two manifolds M and N and an automorphism $\phi : M \to M$ it is easy to check that $\phi^* : C^\infty(M, N) \to C^\infty(M, N)$, $f \mapsto f \circ \phi$ is homeomorphism. Of course, the diagonal map $C^\infty(M, N) \to C^\infty(M, N)^2$, $f \mapsto (f, f)$ is an embedding (true for any topological space). Moreover, by proposition II.3.6 of [5] the map $C^\infty(M^2 \times B, N)^2 \to C^\infty(M^2 \times B, N^2)$, $(f, g) \mapsto F = (f, g)$ is also a homeomorphism for the strong topology. For the weak topology, this is much simpler, and is readily verified. It follows that $\owedge = (f \mapsto (f, f \circ \phi))$ is an embedding by composition, where ϕ is the automorphism of $M \times M \times B$ swapping the coordinates on M. □

Corollary 5.4. *The space $C^\infty(M^2 \times B, N^2; \mathbb{Z}_2)$ with the weak or strong Whitney C^∞ topology, is a Baire space.*

If $N = N_1 \times N_2$ is product of manifolds, define $C^\infty(M, N_1, N_2; \mathbb{Z}_2) = C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2)$. Observe that for every pair $(f, g) \in C^\infty(M, N_1, N_2; \mathbb{Z}_2)$, there is a map $E(f, g) : M^2 \times B \to N$ given by $E(f, g)(x, y, t) = (f(x, t), g(x, y, t))$, and that

$$\owedge \circ E : C^\infty(M, N_1, N_2; \mathbb{Z}_2) \to C^\infty(M^2 \times B, N^2; \mathbb{Z}_2)$$

is a well defined map whose image is precisely all smooth equivariant maps that are holonomic in the first factor. Unfortunately, E is generally not continuous if both factors of $C^\infty(M, N_1, N_2; \mathbb{Z}_2)$ have the strong Whitney C^∞ topology, because the product map $(f, g) \mapsto f \times g$ is not continuous for this topology (see also the proposition below). If M is compact on the other hand, E is continuous, and one can verify that $\owedge \circ E$ is an embedding by composition. If both factors are instead endowed with the weak topology, the result is much better:

Proposition 5.5. *Endow both factors of $C^\infty(M, N_1, N_2; \mathbb{Z}_2)$, and $C^\infty(M^2 \times B, N)$ with the weak Whitney C^∞ topology. Then $E : C^\infty(M, N_1, N_2; \mathbb{Z}_2) \to C^\infty(M^2 \times B, N)$ is an embedding.*

Proof. By proposition 2.8, the map

$$C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2) \to C^\infty((M \times B) \times (M^2 \times B), N), \quad (f, g) \mapsto f \times g,$$

is an embedding (where the latter also has the weak topology). One can precompose the map $f \times g$ with the embedding $\iota : M^2 \times B \to (M \times B) \times (M^2 \times B)$, given by $(x, y, t) \mapsto (x, t, x, y, t)$. It is then not hard to show that

$$\iota^* : C^\infty((M \times B) \times (M^2 \times B), N) \to C^\infty(M^2 \times B, N), \quad f \mapsto f \circ \iota$$

is an embedding, too. Hence E is an embedding by composition. □

When each factor of $C^\infty(M, N_1, N_2; \mathbb{Z}_2)$ is endowed with the strong or weak Whitney C^∞ topology, we shall denote the corresponding topological space by $C^\infty_S(M, N_1, N_2; \mathbb{Z}_2)$ respectively $C^\infty_W(M, N_1, N_2; \mathbb{Z}_2)$.

Anticipating the next chapter, we say that a smooth map $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ is **d-isovariant** if h is both equivariant, and $h(x, x \pm d, t) \notin \Delta(\mathbb{R}^n)$ for all $x \in M$ and $t \in \mathbb{D}^k$, where $0 < d < 1$ is a real number. Here M is one of the manifolds $\mathbb{R}, \mathbb{S}^1 \cong [0, 1]/\{0 \sim 1\}$ or $[0, 1]$. We return to this in definition 6.1.

Lemma 5.6. *Let $M = [0, 1]$ or S^1 and let $B = \mathbb{D}^k$. Then the subset of isovariant maps in $C^\infty_W(M, N_1, N_2; \mathbb{Z}_2)$ is open.*

Proof. Note that an equivariant map $h : M^2 \times \mathbb{D}^k \to N^2$ is isovariant if and only if the image of h is contained in the complement of $V = \Delta_\pm \times \Delta(N)$. Let U denote the complement of V, then $M(U)$ is open in $C^\infty_W(M^2 \times B, N^2)$, since the base space is compact. By proposition 5.3 and 5.5, $\owedge \circ E$ is an embedding for the weak topology. Hence the set of isovariant maps, which is given by $(\owedge \circ E)^{-1}(M(U))$, is open. □

Although the above does not hold when $X = \mathbb{R}$, the proof shows the following. For a subset K of $\mathbb{R}$, we say that $h : \mathbb{R}^2 \times \mathbb{D}^k \to N^2$ is **isovariant on** K if $h(x, x \pm d, t) \notin \Delta(N)$ for all $x \in K$ and $t \in B$.

Corollary 5.7. *Let $K \subset \mathbb{R}$ be compact, and let $B = \mathbb{D}^k$. Then*

$$\{(f, g) \in C^\infty(\mathbb{R}, N_1, N_2; \mathbb{Z}_2) : \owedge \circ E(f, g) \text{ is isovariant on } K\}$$

is open.

Remark 5.8. The above propositions also hold when $\mathbb{D}^k$ is replaced by any compact manifold.

Lemma 5.9. *Let $M = [0,1], \mathbb{R}$, or S^1, and let $N_1 = \mathbb{R}^{n_1}$ and $N_2 = \mathbb{R}^{n_2}$. The topological space $im(\oslash \circ E)$ is locally convex. That is, for every $(f,g) \in C^\infty_W(M, N_1, N_2; \mathbb{Z}_2)$ and open neighbourhood $\mathcal{U} \subset im \oslash \circ E$ of $h = \oslash(E(f,g))$, there exists a smaller neighbourhood $\mathcal{V} \subset \mathcal{U}$ of $\oslash(E(f,g))$ such that for all $h' \in \mathcal{V}$, the linear homotopy connecting $\oslash(E(f,g))$ to h' goes through equivariant maps.*

Proof. It is not so hard to see that $C^\infty_W(M^2 \times B, N^2)$ is locally path connected: let $h \in C^\infty_W(M^2 \times B, N^2)$, and let $\mathcal{U}$ be an open set containg h. Select a basic open set $M(K,U) \subset \mathcal{U}$ containing h . Here $K \subset M^2 \times B$ is compact, and $U \subset J^k(M^2 \times B, N^2)$ is open. If necessary, we can shrink U so that $U \cap \mathfrak{s}^{-1}(x,y,t)$ is convex, where $\mathfrak{s} : J^k(M^2 \times B, N^2) \to M^2 \times B$ is the source map. Note that the fibers of $\mathfrak{s}$ are vector spaces, as $N = \mathbb{R}^n$. Let $g \in M(K,U)$, then the linear homotopy $th + (1-t)g : M^2 \times B \times [0,1] \to N^2$ connects h and g. Hence $M(K,U)$ is a path connected neighbourhood of h containd in $\mathcal{U}$. Suppose now that h and g were equivariant maps holonomic in the first factor, then for every $t \in [0,1]$, $th + (1-t)g$ is also equivariant, and holonomic in the first factor. The result now follows. □

5.2 Equivariant jet transversality

The main goal of this section is to formulate and prove transversality statements about equivariant maps. Specifically, for a submanifold W of N^2, one can wonder when

$$\{g \in C^\infty(M^2 \times B, N^2; \mathbb{Z}_2) : g \pitchfork W\} \subset C^\infty(M^2 \times B, N^2; \mathbb{Z}_2)$$

is residual (and hence dense). Moreover, is the same true when $C^\infty(M^2 \times B, N^2; \mathbb{Z}_2)$ is replaced by the set of smooth equivariant maps that are holonomic the first factor? What about a jet transversality statement?

We shall focus on the case where $N = N_1 \times N_2$ and our maps are holonomic in the first factor. The statement about 'merely equivariant' maps is then obtained as the special case of N_1 being a point. We shall also immediately treat a jet transversality statement, from which the non-jet transversality statement follows. Afterwards, we shall also look at a type of multi-jet transversality statement.

The aim is to translate the equivariant jet transversality question into a question of symmetric mixed transversality, to which we can apply theorem 4.14. Below we spell out the details of this identification, culminating in proposition 5.10.

Define three families of manifolds:

$$X_1 = X_2 = M \times B, \quad X_3 = X_4 = M^2 \times B,$$

$$Y_1 = Y_2 = N_1, \quad Y_3 = Y_4 = N_2,$$
$$Z_1 = Z_2 = M, \quad Z_3 = B.$$

Let $\mathcal{I} = ((1,3),(2,3),(1,2,3),(2,1,3))$, and $k = (k_1,k_1,k_2,k_2)$ be a tuple of nonnegative integers. Define the **k-equivariant jet bundle associated with** (M, B, N_1, N_2) as $J^k_{\mathcal{I}}(X,Y)$ (see definition 4.8). Explicitly, $J^k_{\mathcal{I}}(X,Y)$ is the restriction of the product bundle

$$J^k(X,Y) = J^{k_1}(M \times B, N_1)^2 \times J^{k_2}(M^2 \times B, N_2)^2 \to (M \times B)^2 \times (M^2 \times B)^2$$

to the fibers over $\{(x,t,y,t,x,y,t,y,x,t) : x,y \in M, t \in B) \cong M^2 \times B$. If we reorder the second and third factor of the product bundle, we can denote by $J^k(M,B,N_1,N_1)$ the restriction of this reordered bundle over the fibers of $\{(x,t,x,y,t,y,t,x,y,t) : x,y \in M, t \in B\}$. Let Ψ denote the reordering of the product bundle (which is a diffeomorphism), and denote by ψ the restriction of Ψ to $J^k_{\mathcal{I}}(X,Y)$.

For every smooth equivariant map $h : M^2 \times B \to N^2$ that is holonomic in the first factor, we now obtain a section of the bundle $J^k(M, B, N_1, N_2) \to M^2 \times B$: recall from the previous section, that for every such map h exists a unique pair $(f, g) : C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2)$ such that $h = \oslash(E(f, g))$. In particular, recall that $h_4 = g \circ \omega$, where ω is the automorphism of $M^2 \times B = X_4$ swapping the factors of M^2. Define $j^k_E h$ by

$$(x, y, t) \mapsto (j^{k_1} f(x, t), j^{k_2} g(x, y, t), j^{k_1} f(y, t), j^{k_2}(g \circ \omega)(x, y, t)).$$

It is straightforward to verify that $j^k_E h$ is indeed a section.

Let $\omega^* : J^{k_2}(X_4, Y_4) \to J^{k_2}(X_4, Y_4)$ denote the automorphism given by $[f]_{x,y} \mapsto [f \circ \omega^{-1}]_{\omega(x),y}$, and let Ω denote the automorphism of the reordered bundle, which acts as the identity on the first 3 factors, and as ω^* on the last. Observe that $\Omega \circ \psi$ maps $J^k_{\mathcal{I}}(X, Y)$ diffeomorphically onto $J^k(M, B, N_1, N_2)$. Let $m = (k_1, k_2)$, $l = (2, 2)$, and let ϕ denote the restriction of Ω to the image of ψ. Then the following diagram commutes:

$$\begin{tikzcd}
J^{m,l}_{\mathcal{I}}(X, Y) \arrow[rr, "\phi\circ\psi", "\sim"'] & & J^k(M, B, N_1, N_2) \\
& M^2 \times B \arrow[ul, "j^{m,l}_{\mathcal{I}}(f,g)"] \arrow[ur, "j^k_E h"'] &
\end{tikzcd}$$

We have now proved the following proposition:

Proposition 5.10. *Assume the above notation. Let $h : M^2 \times B \to N^2$ be a smooth equivariant map that is holonomic in the first factor, let $(f, g) \in C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2)$ be the unique pair such that $h = \oslash(E(f, g))$, and let W be a submanifold of $J^{m,l}_{\mathcal{I}}(X, Y)$. Denote $\tilde{W} = \phi(\psi(W))$. Then*

$$j^{m,l}_{\mathcal{I}}(f, g) \pitchfork W \iff j^k_E h \pitchfork \tilde{W}.$$

With this proposition, it is now easy to apply theorem 4.18 to determine for a submanifold $\tilde{W}$ of $J^k(M, B, N_1, N_2)$ if there exists an residual set of pairs $C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2)$ such that the associated map $h : M^2 \times B \to N^2$ satisfies $j^k_E h$ is tranverse to $\tilde{W}$. Let $W = (\phi \circ \psi)^{-1}(\tilde{W})$, then observe that

$$\Delta^l(X) \cap \mathfrak{s}(W) = \varnothing \iff \tilde{\mathfrak{s}}(\tilde{W}) \cap \Delta(M) \times B = \varnothing.$$

Here $\mathfrak{s} : J^{m,l}(X, Y) \to X^l$ and $\tilde{\mathfrak{s}} : \tilde{J}^k_{\mathcal{I}}(X, Y) \to M^2 \times B$ are the source maps associated to the respective bundles. We conclude:

Theorem 5.11 (Equivariant jet transversality theorem)**.** *Let M, B, N_1 and N_2 be manifolds, and denote $N = N_1 \times N_2$. Let $k = (k_1, k_1, k_2, k_2)$ be a tuple of non-negative integers, and let W be a submanifold of the bundle $\tilde{\mathfrak{s}} : J^k(M, B, N_1, N_2) \to M^2 \times B$ such that $\tilde{\mathfrak{s}}(W) \cap \Delta(M) \times B = \varnothing$. Then the set*

$$T^E_W := \{(f, g) \in C^\infty(M, B, N_1, N_2; \mathbb{Z}_2) : j^k_E h \pitchfork W,\ h = \oslash(E(f, g))\}$$

is residual in $C^\infty(M, B, N_1, N_2; \mathbb{Z}_2)$.

The following corollary follows from corollary 4.20 in the same as the above theorem follows from theorem 4.18.

Corollary 5.12. *Let M, B, N_1 and N_2 be manifolds, and denote $N = N_1 \times N_2$. Let W be a submanifold of the bundle $\tilde{\mathfrak{s}} : J^k(M, B, N_1, N_2) \to M^2 \times B$ such that $\tilde{\mathfrak{s}}(W) \cap \Delta(M) \times B = \varnothing$. Let W' be a compact subset of W. Then the set*

$$T^E_{W'} := \{(f, g) \in C^\infty(M, B, N_1, N_2; \mathbb{Z}_2) : j^k_E h \pitchfork W \text{ on } W',\ h = \oslash(E(f, g))\}$$

is open and dense in $C^\infty(M, B, N_1, N_2; \mathbb{Z}_2)$. Moreover, let $(f, g) \in C^\infty(M, B, N_1, N_2; \mathbb{Z}_2)$, let $\mathcal{V}$ be an open neighbourhood of (f, g), and suppose that $\tilde{\mathfrak{s}}(W')$ is contained in the product open sets $U = U_1 \times U_2 \times U_3 \subset M \times M \times B$ that is invariant under ω. That is, $\omega(U) = U$ where ω is the automorphisms of $M^2 \times B$ swapping the factors of M. Then there exists $(f', g') \in \mathcal{V}$ such that $\oslash(E(f, g)) = \oslash(E(f', g'))$ off U and $j^k_E h' \pitchfork W$ on W' with $h' = \oslash(E(f', g'))$.

5.3 Equivariant multijet transversality

We shall make a similar identification for multijets of smooth equivariant maps $h : M^2 \times B \to N^2$ that are holonomic in the first factor: let p be an integer and W be a submanifold of $J^k(M, B, N_1, N_2)^p$, then we wish to obtain a theorem which provides sufficient condition on W such that the 'multijet'

$$j^k_E h \times \ldots \times j^k_E h : (M^2 \times B)^p \to J^k(M, B, N_1, N_2)^p$$

is transverse to W for a residual set of pairs in $C^\infty_S(M, B, N_1, N_2; \mathbb{Z}_2)$.

Let $\{X_j\}_{j=1}^4$ and $\{Y_j\}_{j=1}^4$ be the same two families as above. Define a new family $\{Z_{3j-2} = M, Z_{3j-1} = M, Z_{3j} = B : j = 1, \ldots, p\}$, and

$$\mathcal{I}_p = (I_{4j-3} = (3j-2, 3j), I_{4j-2} = (3j-1, 3j), I_{4j-1} = (3j-2, 3j-1, 3j), I_{4j} = (3j-1, 3j-2, 3j))_{j=1}^p,$$

and let $k = (k_1, k_1, k_2, k_2)$. Define the **(k, p)-equivariant jet bundle associated with** (M, B, N_1, N_2) as $J^{k,p}_{\mathcal{I}_p}(X, Y)$ (see definition 4.16). Although heavy on notation, one can verify that $J^{k,p}_{\mathcal{I}_p}(X, Y) \to Z = (M^2 \times B)^p$ is isomorphic as a bundle to $J^k_{\mathcal{I}}(X, Y)^p \to (M^2 \times B)^p$. I.e. $\mathcal{I}_p$ serves the same purpose as $\mathcal{I}$ did for the equivariant jet transversality theorem, but p times. Let Ψ_p denote the reordering of the symmetric product bundle, whose image is

$$\prod_{j=1}^p \left(J^{k_1}(M \times B, N_1) \times J^{k_2}(M^2 \times B, N_2)\right)^2,$$

and let ϕ_p denote restriction of Ψ_p to $J^{k,p}_{\mathcal{I}_p}(X, Y)$. Observe that the image of Ψ is a bundle over $((M \times B)^2 \times (M^2 \times B))^p$, and that $J^k(M, B, N_1, N_2)^p$ is the restriction of this last bundle to the fibers over

$$\{(x_j, t_j, y_t, t_j, x_j, y_j, t_j, x_j, y_j, t_j) : j = 1, \ldots, p, x_j, y_j \in M, t_j \in B\} \cong (M^2 \times B)^p.$$

Let (f, g) again be the unique pair such that $h = \oslash(E(f, g))$. Then we obtain a section $j^{k,p}_E h$ of $J^k(M, B, N_1, N_2)^p \to Z = (M^2 \times B)^p$, given by

$$(x_j, y_j, t_j)_{j=1}^p \mapsto \left(j^{k_1} f(x_j, t_j), j^{k_2} g(x_j, y_j, t_j), j^{k_1} f(y_j, t_j), j^{k_2}(g \circ \omega)(x_j, y_j, t_j)\right)_{j=1}^p.$$

Let Ω_p denote the automorphism of the reordered bundle, that acts as ω^* on every fourth factor, and acts as the identity on all others, and let ϕ_p denote the restriction of Ω to the image of ψ_p. Let $m = (k_1, k_2)$ and let $l_p = (2p, 2p)$, then the following diagram commutes:

$$\begin{tikzcd}
J^{m,l_p}_{\mathcal{I}_p}(X, Y) \arrow[rr, "\phi_p \circ \psi_p", "\sim"'] & & J^k(M, B, N_1, N_2)^p \\
& (M^2 \times B)^p \arrow[ul, "j^{m,l_p}_{\mathcal{I}_p}(f,g)"] \arrow[ur, "j^{k,p}_E h"'] &
\end{tikzcd}$$

We have proved the following proposition:

Proposition 5.13. *Assume the above notation. Let $h : M^2 \times B \to N^2$ be a smooth equivariant map that is holonomic in the first factor, let $(f, g) \in C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2)$ be the unique pair such that $h = \oslash(E(f, g))$, and let W be a submanifold of $J^{m,l_p}_{\mathcal{I}_p}(X, Y)$. Denote $\tilde{W} = \phi_p(\psi_p(W))$. Then*

$$j^{m,l_p}_{\mathcal{I}_p}(f, g) \pitchfork W \iff \tilde{j}^{k,p}_{\mathcal{I}_p} h \pitchfork \tilde{W}.$$

As in the previous remark, let $\tilde{W}$ be a submanifold of $\tilde{J}^{k,p}_{\mathcal{I}_p}(X, Y)$. Theorem 4.18 gives a sufficient condition for the existence of a residual set of tuples in $C^\infty(M \times B, N_1) \times C^\infty(M^2 \times B, N_2)$ such that the associated map $h : M^2 \times B \to N^2$ satisfies $\tilde{j}^{k,p}_{\mathcal{I}_p} h \pitchfork \tilde{W}$. Let $W = (\phi_p \circ \psi_p)^{-1}(\tilde{W})$, then

$$\mathfrak{s}(W) \cap \Delta^{l_p}(X) = \varnothing \iff \tilde{\mathfrak{s}}(\tilde{W}) \cap (\Delta^{(p)}(M^2 \times B) \cup (\Delta(M) \times D^k)^p) = \varnothing.$$

Here $\mathfrak{s} : J^{m,l_p}_{\mathcal{I}_p}(X, Y) \to X^{l_p}$ and $\tilde{\mathfrak{s}} : \tilde{J}^{k,p}_{\mathcal{I}_p}(X, Y) \to (M^2 \times B)^p$ are the source maps of the respective bundles. We conclude:

Theorem 5.14 (Equivariant multijet transversality theorem)**.** *Let M, B, N_1 and N_2 be manifolds, and denote $N = N_1 \times N_2$. Let $k = (k_1, k_1, k_2, k_2)$ be a tuple of non-negative integers, and let W be a submanifold of the bundle $\tilde{\mathfrak{s}} : J^k(M, B, N_1, N_2)^p \to (M^2 \times B)^p$ such that $\tilde{\mathfrak{s}}(W) \cap \big((\Delta(M) \times B)^p \cup \Delta^{(p)}(M^2 \times B)\big) = \varnothing$. Then the set*

$$T^E_W := \{(f, g) \in C^\infty(M, B, N_1, N_2; \mathbb{Z}_2) : j^{k,p}_E h \pitchfork W,\ h = \oslash(E(f, g))\}$$

is residual in $C^\infty(M, B, N_1, N_2; \mathbb{Z}_2)$.

Once more, we comment on some of the flexibility that the above theorem has:

Corollary 5.15. *Let M, B, N_1 and N_2 be manifolds, and denote $N = N_1 \times N_2$. Let W be a submanifold of the bundle $\tilde{\mathfrak{s}} : J^k(M, B, N_1, N_2)^p \to (M^2 \times B)^p$ such that $\tilde{\mathfrak{s}}(W) \cap \big((\Delta(M) \times B)^p \cup \Delta^{(p)}(M^2 \times B)\big) = \varnothing$. Let W' be a compact subset of W. Then the set*

$$T^E_{W'} := \{(f, g) \in C^\infty(M, B, N_1, N_2; \mathbb{Z}_2) : j^{k,p}_E h \pitchfork W \text{ on } W',\ h = \oslash(E(f, g))\}$$

is open and dense in $C^\infty(M, B, N_1, N_2; \mathbb{Z}_2)$. Moreover, let $(f, g) \in C^\infty(M, B, N_1, N_2; \mathbb{Z}_2)$, let $\mathcal{V}$ be an open neighbourhood of (f, g), and suppose that $\tilde{\mathfrak{s}}(W')$ is contained in the product open sets $U = (U_1 \times \ldots \times U_3)^p \subset (M \times M \times B)^p$ that is invariant under ω. That is, $\omega(U) = U$ where ω is the automorphisms of $M^2 \times B$ swapping the factors of M. Then there exists $(f', g') \in \mathcal{V}$ such that $\oslash(E(f, g)) = \oslash(E(f', g'))$ off $U_1 \times U_2 \times U_3$ and $j^{k,p}_E h' \pitchfork W$ on W' with $h' = \oslash(E(f', g'))$.

6 Delayed relations

In this chapter we recall and prove the main result, theorem 1.4. We define the d-delay relation $\mathcal{R}_d$, the derivative d-delay relation $\mathcal{R}'_d$, and what it means to be in $\mathrm{Sol}_F(\mathcal{R}_d)$, i.e. to be a formal d-delay solution. Most of this chapter is devoted to proving the final ingredient of the proof of theorem 1.4: theorem 6.5. Throughout we analyse families of $\mathrm{Sol}(\mathcal{R}'_d)$ and $\mathrm{Sol}_F(\mathcal{R}_d)$ maps. We introduce some more language:

Let M denote one of the distinguished manifolds $\mathbb{R}, [0,1]$, or S^1, and let $N_1 = \mathbb{R}^{n_1}$ and $N_2 = \mathbb{R}^{n_2}$ for some integers $n_1, n_2 \in \mathbb{N}_0$. Let $n = n_1 + n_2$.

Definition 6.1. Let $d > 0$ be a real number, B be a manifold, and let M be one of the distinguished manifolds. Let $g : M^2 \times B \to (\mathbb{R}^n)^2$ be a smooth map. We say that g is d-isovariant or simply isovariant, if g is a B-family of formal $\mathcal{R}_d$-solutions. That is to say, g is equivariant, and

$$g(x, x \pm d, t) \notin \Delta(\mathbb{R}^n) \quad \forall x \in M,\ t \in B.$$

△

We shall denote throughout this chapter by $\Delta_{\pm d} \subset M^2 \times B$ the shifted diagonals

$$\Delta_{\pm d} = \{(x, x \pm d, t) : x \in M, t \in B\}.$$

6.1 The delay relation

We now give the formal definition of the d-delay relation, and define formal d-delay solutions.

Definition 6.2. Let M be the manifold $[0,1], \mathbb{R}$, or $S^1 \cong [0,1]/\{0 \sim 1\}$, and $n \geq 1$ be an integer. The d-delay relation, $\mathcal{R}_d$, is the non-local differential relation in $J^1(M, \mathbb{R}^n)^2$ given by

$$\mathcal{R}_d := \{(x_1, y_1, v_1, x_1, y_2, v_2) \in J^1(M, \mathbb{R}^n)^2 : x_1 = x_2 \pm d \implies v_1 \neq v_2\}.$$

The derivative d-delay relation, $\mathcal{R}'_d$, is the non-local differential relation in $J^0(M, \mathbb{R}^n)^2$ given by

$$\mathcal{R}'_d := \{(x_1, y_1, x_2, y_2) \in J^0(M, \mathbb{R}^n)^2 : x_1 = x_2 \pm d \implies y_1 \neq y_2\}.$$

The formal $\mathcal{R}_d$-solutions are the equivariant sections $M^2 \to J^0(M, \mathbb{R}^n)^2$ with image in $\mathcal{R}'_d$. We denote the set of formal $\mathcal{R}_d$ solutions by $\mathrm{Sol}_F(\mathcal{R}_d)$. △

Theorem 1.4 says that if $M = \mathbb{R}$ or $M = [0,1]$ and $0 < d < 1$ is a real number, then $T \times T, f \mapsto f' \times f'$, is a weak homotopy equivalence between $\mathrm{Sol}(\mathcal{R}_d)$ and $\mathrm{Sol}_F(\mathcal{R}'_d)$. In other words, the h-principle holds for the d-delay relation in this case. Moreover, if $M = \mathbb{S}^1$ and d is either $1/2$ or irrational, the h-principle for $\mathcal{R}_d$ also holds, and if $d = p/q$ with $q > 2$ and p, q coprime integers, then the map $T \times T$ induces isomorphisms between homotopy groups of $\mathrm{Sol}(\mathcal{R}_d)$ and $\mathrm{Sol}_F(\mathcal{R}_d)$ up to a degree depending on n (the dimension of the target space) and q.

The proof theorem 1.4 is split in two parts. First it is shown in theorem 6.4 that there is a weak homotopy equivalence between the set of d-delay solutions, $\mathrm{Sol}(\mathcal{R}_d)$, and derivative d-delay solutions, $\mathrm{Sol}(\mathcal{R}'_d)$. Second, it is shown that there are isomorphisms of homotopy groups of $\mathrm{Sol}(\mathcal{R}'_d)$ and $\mathrm{Sol}_F(\mathcal{R}_d)$, the formal d-delay solutions.

Here we shall briefly give some motivation and a preliminary result used in the second step. Note first that the map $T : \mathrm{Sol}(\mathcal{R}_d) \to \mathrm{Sol}(\mathcal{R}'_d)$ given by $f \mapsto f'$ is a well defined map. I.e. if $f : M \to \mathbb{R}^n$ is a d-delay solutions, its derivative map $f' : M \to \mathbb{R}^n$ is a derivative d-delay solution.

Note that $g \in \mathrm{Sol}(R'_d)$ if and only if $g \times g(\Delta_{\pm d}) \cap \Delta(\mathbb{R}^n) = \varnothing$, while $h \in \mathrm{Sol}_F(\mathcal{R}_d)$ if and only if $h : M^2 \to (\mathbb{R}^n)^2$ is equivariant, and $h(\Delta_{\pm d}) \cap \Delta(\mathbb{R}^n) = \varnothing$. Hence, for every $g \in \mathrm{Sol}(R'_d)$, $g \times g \in$

$\mathrm{Sol}_F(\mathcal{R}_d)$, which means that the map $g \mapsto g \times g$ is a well defined map from $\mathrm{Sol}(\mathcal{R}'_d)$ to $\mathrm{Sol}_F(\mathcal{R}_d)$. Let $h : M^2 \times (\mathbb{R}^n)^2$ be a smooth map, then we say that h **is holonomic** if $h = g \times g$ for some smooth map $g : M \to \mathbb{R}^n$ (as in definition 5.2). We denote by $\mathrm{Sol}_{F,\mathrm{Hol}}(\mathcal{R}_d)$ the subset of all those $h \in \mathrm{Sol}_F(\mathcal{R}_d)$ that are holonomic.

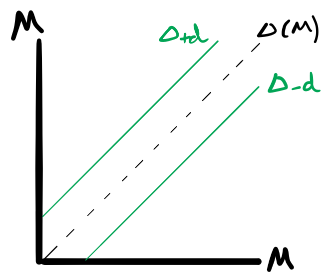

Figure 1: The shifted diagonals in $M.\triangle$

Lemma 6.3. *Let $0 < d < 1$ be a real number. The map*

$$p : \mathrm{Sol}(\mathcal{R}'_d) \to \mathrm{Sol}_F(\mathcal{R}_d), \quad g \mapsto g \times g,$$

is a topological embedding. Its image is $\mathrm{Sol}_{F,Hol}(\mathcal{R}_d)$, which consists of all those $h \in \mathrm{Sol}_F(\mathcal{R}_d)$, $h : M^2 \to (\mathbb{R}^n)^2$, that are holonomic. Moreover, the map p induces for every $g \in \mathrm{Sol}(\mathcal{R}'_d)$ an isomorphism

$$p_n : \pi_n(\mathrm{Sol}(\mathcal{R}'_d), g) \to \pi_n(\mathrm{Sol}_F(\mathcal{R}_d), g \times g)$$

if and only if the inclusion $i : \mathrm{Sol}_{F,Hol}(\mathcal{R}_d) \hookrightarrow \mathrm{Sol}_F(\mathcal{R}_d)$ induces an isomorphism

$$i_n : \pi_n(\mathrm{Sol}_{F,Hol}(\mathcal{R}_d), g \times g) \to \pi_n(\mathrm{Sol}_F(\mathcal{R}_d), g \times g).$$

Proof. By proposition 2.8 and the fact that the diagonal map $g \mapsto (g, g)$ is an embedding, we can identify $\mathrm{Sol}(\mathcal{R}'_d)$ as a topological space with its image in $\mathrm{Sol}_F(\mathcal{R}_d)$. That the image consists precisely of the holonomic maps in $\mathrm{Sol}_F(\mathcal{R}_d)$ is straightforward to verify. As p is an embedding, it is a weak homotopy equivalence onto its image. Hence the moreover part follows from the identity $p = i \circ p$. □

6.2 Proof of main result

In this section we prove the main result, theorem 1.4. The first step is theorem 6.4, which shows that there is a weak homotopy equivalence between solutions of the relation $\mathcal{R}_d$ (d-delay solutions) and solutions of $\mathcal{R}'_d$, which were smooth maps $g : M \to \mathbb{R}^n$ satisfying $g(x \pm d) \neq g(x)$ for all $x \in M$. The equivalence is given by the map $f \mapsto f'$, and the proof is inspired by the proof of theorem 3.11. It then remains to prove that the map $g \mapsto g \times g$ induces isomorphisms of homotopy groups between $\mathrm{Sol}(\mathcal{R}'_d)$ and $\mathrm{Sol}_F(\mathcal{R}_d)$ (for all degrees when $M = [0, 1], \mathbb{R}$ and for some when $M = S^1$). This is done subsequently, using theorem 6.5, which is the topic of the remainder of this chapter.

Theorem 6.4. *The map*

$$T : \mathrm{Sol}(\mathcal{R}_d) \to \mathrm{Sol}(\mathcal{R}'_d), \quad (f : X \to \mathbb{R}^n) \mapsto (f' : X \to \mathbb{R}^n)$$

is a weak homotopy equivalence (for the weak Whitney C^∞-topology).

Proof. Let k be a non-negative integer. Let $f : M \to R^n$ be a smooth map, such $f \in \mathrm{Sol}(\mathcal{R}_d)$. Let

$$T_k : \pi_k(\mathrm{Sol}(\mathcal{R}_d), f) \to \pi_k(\mathrm{Sol}(\mathcal{R}'_d), f')$$

denote the map induced by T on the k-th homotopy group. We need to show that T_k is a bijection for each k. The key claim is the following: let $G : M \times \mathbb{S}^k \to \mathbb{R}^n$ be a smooth map, such that $g_t = G(\cdot, t)$ is a $\mathcal{R}'_d$-solution for every $t \in \mathbb{S}^k$. Then there exists a smooth homotopy $H : M \times \mathbb{S}^k \times [0,1] \to \mathbb{R}^n$ connecting G to $\tilde{G} = H_1$, and a smooth map $F : M \times \mathbb{S}^k \times [0,1] \to \mathbb{R}^n$, such that H_s is a smooth family of $\mathrm{Sol}(\mathcal{R}'_d)$-solutions for every $s \in [0,1]$, and for every $t \in \mathbb{S}^k$, $Tf_t = \tilde{g}_t$. Here $f_t = F(\cdot, t)$ and $\tilde{g}_t = \tilde{G}(\cdot, t)$. Moreover, if $H : M \times \mathbb{S}^k \times [0,1] \to \mathbb{R}^n$ is a smooth homotopy connecting two $\mathbb{S}^k$-families of $\mathrm{Sol}(\mathcal{R}'_d)$ solutions G_0, G_1, lying in the image of T_k, then H is itself homotopic to a smooth homotopy $\tilde{H} : M \times \mathbb{S}^k \times [0,1] \to \mathbb{R}^n$ connecting G_0 and G_1, such that $G_s = \tilde{H}_s$ lies in the image of T_k for every $s \in [0,1]$.

By proposition 3.3 and the claim, we are done: the first part of the proposition shows that T_k is surjective, and the second shows injectivity.

Proof of claim: we make two case distinctions:

- if $M = [0,1]$ or $M = \mathbb{R}$, we can select H to be the constant homotopy, and hence $\tilde{G} = G$. Define $f_t(x) = \int_0^x g_t(p)dp$. By the Fundamental Theorem of Calculus (FTC), $Tf_t = g_t$, and by the smoothness of G, F is also smooth. Moreover, as g_t is a $\mathcal{R}'_d$-solution, f_t is a $\mathcal{R}_d$-solution. The moreover part follows similarly: we can simply select $\tilde{H} = H$ and the homotopy connecting the can be chosen as the constant homotopy. The 'primitives' of G_s can be constructed in a similar fashion, too.

- if $M = \mathbb{S}^1 \cong [0,1]/0 \sim 1$, select $H(x,t,s) = g_t(x) - s\int_0^1 g_t(p)dp$, and hence $\tilde{G}(x,t) = g_t(x) - \int_0^1 g_t(p)dp$. It is straightforward to verify that H_s is a smooth family of $\mathrm{Sol}(\mathcal{R}'_d)$-solutions for every $s \in [0,1]$. Define $f_t(x) = \int_0^x g_t(p)dp - x\int_0^1 g_t(p)dp$. Then $f_t : \mathbb{S}^1 \to \mathbb{R}^n$ is a well defined, smooth function: the FTC gives us that $\tilde{f}_t : [0,1] \to \mathbb{R}^n$ is well defined and smooth, and all order derivatives of $f_{t,s}$ agree at 0 and 1, so $f_{t,s}$ descends to a smooth map on $\mathbb{S}^1$. The FTC also implies that Tf_t and $\tilde{g}_t$ agree. The smoothness of g implies the smoothness of f.

  Finally,
  $$f'_t(x \pm d) - f'_t(x) = \tilde{g}_t(x \pm d) - \tilde{g}_t(x),$$
  and hence f_t is an element of $\mathrm{Sol}(\mathcal{R}_d)$ for every $t \in S^k$.

  The moreover part is analoguous: assume that $G_0 = H_0$ and $G_1 = H_1$ are two $\mathbb{S}^k$-families of $\mathrm{Sol}(\mathcal{R}'_d)$ solutions, and that H is a homotopy connecting them. By linearly subtracting the average position of $H_{t,s} : M \to \mathbb{R}^n$, we can find $\tilde{H}$ and the linear homotopy connecting H to $\tilde{H}$. I.e. the homotopy is given by
  $$M \times \mathbb{S}^k \times [0,1]^2 \ni (x, t, s_1, s_2) \mapsto H_{t,s_1}(x) - s_2 \int_0^x H_{t,s_1}(p)dp.$$
  One can check that this homotopy consists for all $s_1, s_2 \in [0,1]$ of $\mathbb{S}^k$-families of $\mathcal{R}'_d$ solutions, and is relative $s_1 = 0, 1$.

$\square$

The proof of theorem 1.4 is now reduced to showing that the map $\mathrm{Sol}(\mathcal{R}'_d) \to \mathrm{Sol}_F(\mathcal{R}_d)$ given by $g \mapsto g \times g$ induces isomorphisms on homotopy groups of appropriate dimensions. By lemma 6.3, this is equivalent to showing that the inclusion map $i : \mathrm{Sol}_{F,\mathrm{Hol}}(\mathcal{R}_d) \hookrightarrow \mathrm{Sol}_F(\mathcal{R}_d)$ induces isomorphisms on homotopy of appropriate dimensions. The main result of this section is the theorem below, which will be sufficient:

Theorem 6.5. *Let $0 < d < 1$ be a real number. Let M be the manifold $\mathbb{R}$ or $[0,1]$, and let $k \geq 0$ be an integer. Then any smooth $\mathbb{D}^k$ family of $\mathrm{Sol}_F(\mathcal{R}_d)$ maps $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ with $h_t : M^2 \to (\mathbb{R}^n)^2$ in $\mathrm{Sol}_{F,Hol}(\mathcal{R}_d)$ for every $t \in \mathbb{S}^{k-1}$, is homotopic through isovariant maps relative $\mathbb{S}^{k-1}$ to a $\mathbb{D}^k$ family of $\mathrm{Sol}_{F,Hol}(\mathcal{R}_d)$ maps.*

Let $M = S^1$. If d is irrational, $d = 1/2$, or $d = p/q$ with p, q coprime positive integers and $k < q(n-1) - 1$, then the same conclusion holds.

Proof of theorem 1.4 assuming theorem 6.5. By lemma 6.3 and the discussion above it, we only need to show that

$$i_k : \pi_k(\mathrm{Sol}_{F,\mathrm{Hol}}(\mathcal{R}_d), g \times g) \to \pi_k(\mathrm{Sol}_F(\mathcal{R}_d), g \times g)$$

is an isomorphism for all $k \geq 0$ when $M = [0,1]$ or $\mathbb{R}$, or for when $M = S^1$ for all k when $d = 1/2$ or irrational, or all $k < q(n-1) - 1$ when $d = p/q$ rational. By the long exact sequence of (relative) homotopy groups (see section 3.1) and proposition 3.4, i_k is an isomorphism if and only if for $l = k$ and $l = k+1$, every smooth isovariant map $h : M^2 \times \mathbb{D}^l \to (\mathbb{R}^n)^2$ with $h_t : M^2 \to (\mathbb{R}^n)^2$ holonomic for all $t \in \mathbb{S}^{l-1}$, there exists a smooth homotopy $H : M^2 \times \mathbb{D}^l \times [0,1] \to (\mathbb{R}^n)^2$ which connects h to a smooth holonomic isovariant map $\tilde{h}$, which is relative $M^2 \times S^{l-1}$, and which goes through isovariant maps. By theorem 6.5 the latter holds for all $l \geq 0$ when M is $[0,1]$ or $\mathbb{R}$. When $M = S^1$ and d is irrational or $d = 1/2$, the latter holds for all $l \geq 0$ too. If $M = S^1$ and $d = p/q$ is rational with p, q positive comprime integers, the latter holds for all $l < q(n-1) - 1$. Hence i is a weak homotopy equivalence when M is $[0,1]$ or $\mathbb{R}$, and when $M = \mathbb{S}^1$ and d is irrational or $d = 1/2$. If $M = S^1$ and $d = p/q$, then i_k is an isomorphism when $k < q(n-1) - 2$. This completes the proof. □

6.3 Singularities of isovariant maps

It remains to prove theorem 6.5. Let $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ be an isovariant map such that $h_t : M^2 \to (\mathbb{R}^n)^2$ is holonomic for every $t \in \mathbb{S}^{k-1}$. Out of the map h we somehow need to construct a map $\tilde{h}$ such that $\tilde{h}_t$ is holonomic for every $t \in \mathbb{D}^k$, and $\tilde{h}$ is homotopic through isovariant maps to h. Moreover, such a homotopy connecting h to $\tilde{h}$ should be relative $M^2 \times \mathbb{S}^{k-1}$. Our method is a **removal of singularities**, on which we elaborate.

We want to improve $h = (h_1, \ldots, h_{2n})$, $h_j : M^2 \times \mathbb{D}^k \to \mathbb{R}$, by induction over pairs of coordinate functions (h_j, h_{j+n}). Observe that $(h_j, h_{j+n}) : M^2 \times \mathbb{D}^k \to \mathbb{R}^2$ is equivariant. We want to inductively replace (h_j, h_{j+n}) by new coordinate pairs $(\tilde{h}_j, \tilde{h}_{j+n})$ which are holonomic, i.e. $(\tilde{h}_j, \tilde{h}_{j+n}) = f_t \times f_t : M^2 \times \mathbb{D}^k \to \mathbb{R}^2$ for some smooth function $f : M \times \mathbb{D}^k \to \mathbb{R}$. Here the map $f_t \times f_t$ is given by $(x, y, t) \mapsto (f(x,t), f(y,t))$. When replacing the pair (h_j, h_{j+n}) of h by $(\tilde{h}_j, \tilde{h}_{j+n})$, our first worry is that the new map $\tilde{h}^j : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ may not be isovariant anymore: equivariance is guaranteed, since any map $M^2 \to (\mathbb{R}^n)^2$ is equivariant if and only if all its coordinate pairs are equivariant, but the identity

$$\tilde{h}^j(\Delta_{\pm d}) \cap \Delta(\mathbb{R}^n) = \varnothing$$

may no longer hold.

When does this identity fail to hold? Note that the equivariant map $g : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ is isovariant if and only if for every $(x, x \pm d, t) \in M^2 \times \mathbb{D}^k$, there exists a coordinate pair (g_l, g_{l+n}) of g with $(g_l, g_{l+n})(x, x \pm d, t) \notin \Delta(\mathbb{R})$. Hence, the identity may no longer hold if there exists $(x, x \pm d, t) \in M^2 \times \mathbb{D}^k$ such that for all $1 \leq l \leq n, l \neq j$, $(h_l, h_{l+n})(x, x \pm d, t) \in \Delta(\mathbb{R})$.

Definition 6.6. Let B be a manifold.

- Let $h : M^2 \times B \to (\mathbb{R}^n)^2$ be an equivariant map. Define $h_{\hat{l}}$ to be the equivariant map $M^2 \times B \to \mathbb{R}^{2n-2}$ obtained by deleting the l-th and $n+l$-th coordinate functions of h.
- Let $h : M^2 \times B \to (\mathbb{R}^n)^2$ be an isovariant map. Define the l-th singularity of h by

$$\Sigma_l(h) = \left(h_{\hat{l}}\right)^{-1} (\Delta(\mathbb{R}^{n-1})) \cap \Delta_{\pm d}.$$

Remark 6.7. For ease of notation, we will try to suppress the dependence of $\Sigma_l(h)$ on h as much as possible. $\triangle$

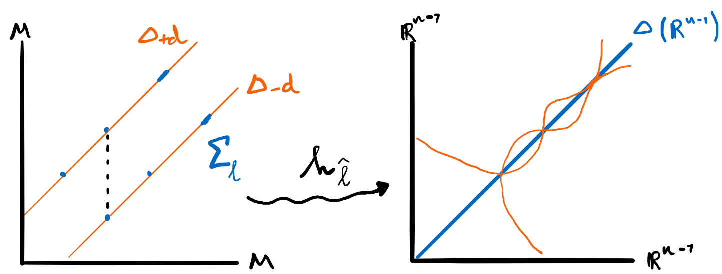

Figure 2: The l-th singularity of h. Points in Σ_l that lie 'above' on another (dashed line) are problematic when defining $f_t \times f_t$. $\triangle$

So, when we replace (h_j, h_{j+n}) by a holonomic pair $(\tilde{h}_j, \tilde{h}_{j+n})$ we can guarantee that new map $\tilde{h}^j : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ is isovariant if and only if

$$(\tilde{h}_j, \tilde{h}_{j+n})(\Sigma_j) \cap \Delta(\mathbb{R}) = \varnothing.$$

Let $H : M^2 \times \mathbb{D}^k \times [0,1] \to (\mathbb{R}^n)^2$ be an equivariant homotopy connecting h to $\tilde{h}^j$ which only changes the j-th coordinate pair. As $\Sigma_j(H_t) = \Sigma_j(h)$ for every $t \in [0,1]$, H goes through isovariant maps if and only if for every $t \in [0,1]$

$$((H_t)_j, (H_t)_{j+n})(\Sigma_j) \cap \Delta(\mathbb{R}) = \varnothing.$$

6.4 Formal data over singularities and the boundary

With the above punchline, the proof of theorem 6.5 now revolves around defining a new holonomic coordinate pair $(\tilde{h}_j, \tilde{h}_{j+n})$ which does not intersect $\Delta(\mathbb{R})$ over the singularity Σ_j. We know that the old coordinate pair already has this property, as h is isovariant, so will use the old coordinate pair to guide the definition of $(\tilde{h}_j, \tilde{h}_{j+n})$. The ideal situation is as follows:

Remark 6.8 (Ideal situation)**.** Assume that Σ_l is a submanifold of $M^2 \times \mathbb{D}^k$, and that the projection $\pi : M^2 \times \mathbb{D}^k \to M \times \mathbb{D}^k, \pi(x,y,t) = (x,t)$ restricts to a smooth embedding $\pi' : \Sigma_l \to M \times \mathbb{D}^k$. Then one can use (h_j, h_{j+n}) with relative ease to construct $f : M \times \mathbb{D}^k \to \mathbb{R}$: define

$$f(x,t) = h_j(x,y,t), \quad (x,t) \in \pi'(\Sigma_l)$$

where (x,y,t) is the unique point in Σ_l in the fiber of π' over (x,t). The map f is well defined, defined over a closed set (as Σ_l is closed) and admits a smooth extension pointwise. To see this last point, one can use local smooth sections of π' and the fact that h_j is smooth. Hence, we can choose some global smooth extension of f to a smooth map $M \times \mathbb{D}^k \to \mathbb{R}$ (see for example [14, lemma 2.26]). Replacing (h_j, h_{j+n}) by $(\tilde{h}_j, \tilde{h}_{j+n}) = f_t \times f_t$ is now ideal: $f_t \times f_t$ is equal to (h_j, h_{j+n}) over Σ_l, and hence $\tilde{h}^j : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ is isovariant and its j-th coordinate pair is holonomic. Moreover, the linear homotopy connecting h to $\tilde{h}^j$ only changes the j-th coordinate pair, and is constant over Σ_l. Therefore, the linear homotopy goes through isovariant maps, too. $\triangle$

There are two primary obstructions to using this idealized solution: first, we shall show that Σ_l can be assumed to be a manifold, but that π' is in general not injective. However, note that $\Sigma_l \subset \Delta_{\pm d}$, and hence fibers of π' contain at most two points. That is, $\pi^{-1}(x,t) \subset \{(x, x+d, t), (x, x-d, t) : x \in M, t \in \mathbb{D}^k\}$. We can however show that the double points of π' can assumed to be a stratified set, and that π' restricted to the strata is a double cover. The aim is then to show that, up to homotopy, we can assume that $h_j(x, x+d, t) = h_j(x, x-d, t)$ for all double points of π'. This will make $f(x,t) = h_j(x, x \pm d, t)$ well defined and we can again argue with local sections that f extends to a global, smooth map. The second obstruction is that we have so far not taken into account that $f_t \times f_t$ should be equal to (h_j, h_{j+n}) on $M^2 \times \mathbb{S}^{k-1}$.

The first obstruction is the topic of the remainder of this chapter. In section 6.5 we show that, up to homotopy, Σ_j can be assumed to be a manifold, and that the double points in Σ_j (or more precisely, a subset thereof) can be stratified. In section 6.6 we argue that (up to homotopy) we can assume that $h_j(x, x+d, t) = h_j(x, x-d, t)$ over the double points. In the final section, section 6.7 we tie everything together and prove theorem 6.5. For the second obstruction we introduce the following lemma. In the remainder of the arguments this lemma shall provide some flexibility in our arguments near the boundary of $\mathbb{D}^k$.

Lemma 6.9. *Let $N = N_1 \times N_2$. Let $h : M^2 \times \mathbb{D}^k \to N^2$ be a smooth isovariant map, holonomic in the first factor N_1, such that $h_t : M^2 \to N^2$ is holonomic for all $t \in \mathbb{S}^{k-1}$. Then there exists a real number $0 < r_1 < 1$ such that h is homotopic through isovariant maps relative $M^2 \times \mathbb{S}^{k-1}$ to a smooth isovariant map $\tilde{h}$ holonomic in the first factor, such that $\tilde{h}_t$ is holonomic for all $|t| \geq r_1$.*

Proof. Let $\phi : \mathbb{D}^k \times [0,1] \to \mathbb{D}^k$ denote a smooth homotopy which blows up $\{t \in \mathbb{D}^k : |t| \leq r_1\}$ radially up to the boundary of $\mathbb{D}^k$. I.e. select any smooth function $\alpha : [0,1] \to [0,1]$ that is non-decreasing, $\alpha(s) = s$ on a neighbourhood of 0, and on $[r_1, 1]$ is identically 1. One can then define ϕ by $(t,s) \mapsto (1-s)t + s \cdot \alpha(\|t\|)t/\|t\|$. This is well defined (and smooth) at 0, since α is equal to the identity near 0. Precomposing h by $(\mathrm{id}_{M^2}, \phi)$, we obtain a smooth homotopy connecting h to $\tilde{h} := h \circ (\mathrm{id}_{M^2}, \phi_1)$. Because the homotopy preserves the slices of M^2 in the product $M^2 \times \mathbb{D}^k$, it follows that this homotopy goes through isovariant maps. Evidently, $\tilde{h}_t = h_{t/|t|}$ for all $|t| \geq r_1$, and for such t it follows that h_t is holonomic. $\square$

6.5 Stratification of the singularity

Let $N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $n = n_1 + n_2$. Let $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ be an isovariant map, holonomic in the first factor $\mathbb{R}^{n_1}$. When we want to replace the $n_1 + 1$ coordinate pair of h by a holonomic pair, we are interested in the singularity Σ_{n_1+1}. Due to lemma 6.9, we can assume without loss of generality that there exists a real number $0 < r_1 < 1$ such that $h_t : M^2 \to (\mathbb{R}^n)^2$ is holonomic for all $|t| \geq r_1$. To guarantee that we do not change h near $M^2 \times S^{k-1}$, we devote our attention to $S := \Sigma_{n_1+1} \cap M^2 \times \mathbb{D}^k_{r_2}$, where $\mathbb{D}^k_{r_2} := \{t \in \mathbb{D}^k : |t| \leq r_2\}$ for some real number $0 < r_1 < r_2 < 1$. Let π_S denote the restriction of $\pi : M^2 \times \mathbb{D}^k \to M \times \mathbb{D}^k$ (as defined before) to S. Let $S_1 \subset S$ denote the double points of π_S. We also analyze $S_1 \subset S$, which leads to **the stratification of the double points in** S, definition 6.12. The main results of this section are lemmas 6.15 and 6.16, which state that S and the stratification of S_1 may be assumed to consist of manifolds.

Remark 6.10. We can guarantee that S is a submanifold of $M^2 \times \mathbb{D}^k$ if $j^0(h_{\widehat{n_1+1}}) : M^2 \times \mathbb{D}^k \to J^0(M^2 \times D^k, \mathbb{R}^{2n-2})$ is transverse to $(\Delta_{\pm d} \cap M^2 \times \mathbb{D}^k_{r_2}) \times \Delta(\mathbb{R}^{n-1})$ (as S is the preimage of a map transverse to a submanifold). Theorem 5.11 will be used to find an isovariant smooth map near h satisfying this condition. △

Let $\widehat{h}$ be shorthand for $h_{\widehat{n_1+1}} : M^2 \times \mathbb{D}^k \to \mathbb{R}^{2n-2}$. Denote $N' = \mathbb{R}^{n-1}$, and note that due to the equivariance of $\widehat{h}$,the following are equivalent: for all $t \in \mathbb{D}^k_{r_2}$,

- $(x, x \pm d, t) \in S_1$.
- $\widehat{h}(x, x-d, t) \in \Delta(N')$ and $\widehat{h}(x, x+d, t) \in \Delta(N')$.
- $\widehat{h}(x, x-d, t) \in \Delta(N')$ and $\widehat{h}(x+d, x, t) \in \Delta(N')$.
- $j^0_2\widehat{h}(x, x-d, t, x+d, x, t) \in \{(x, x-d, t, x+d, x, t)\} \times (\Delta(N'))^2$.

We introduce a stratification of S_1 based on these observations: we partition S_1 into those $(x, x \pm d, t)$ whose repeated shift along $\Delta_{\pm d}$ also lie in S. I.e. we subdivide S_1 into subsets D_L consisting those $(x, x \pm d, t)$ that satisfy $(x + ld, x + (l \pm 1)d, t) \in S$ for at least L subsequent integers (containing 0). This subdivision will be important when we argue that we can assume, up to homotopy, that $h_j(x, x-d, t) = h_j(x, x+d, t)$ for all $(x, x \pm d, t) \in S_1$.

Let $(x + ld, x + (l \pm 1)d, t)_{l=0}^L$ be shorthand for the $3(L+1)$-tuple in $(M^2 \times D^k)^{L+1}$, whose first triple is $(x, x \pm d, t)$ and whose last triple is $(x + Ld, x + Ld \pm d, t)$. Here we use the convention that either all plus-minus signs are positive, or negative.

Definition 6.11. Let $L \geq 1$. The *chains of h of length L* or simply *L-chains of h* are those tuples $(x + ld, x + (l \pm 1)d, t)_{r=0}^L$ such that

$$\widehat{h} \times \ldots \times \widehat{h}(x + ld, x + (l \pm 1)d, t)_{l=0}^L \in \Delta(N')^{L+1}, \quad t \in \mathbb{D}^k_{r_2}.$$

Let $L \geq 0$. Define $W_L \subset (M^2 \times D^k)^{L+1} \times (N')^{2(L+1)}$ to be the submanifold given by

$$\{(x + ld, x + (l \pm 1)d, t)_{l=0}^L : x \in M, t \in \mathbb{D}^k_{r_2}\} \times \Delta(N')^{L+1}.$$

△

Observe that from every L-chain of h, one obtains an $L-1$ chain by omitting the last or first triple. We will say that a triple $(x, x \pm d, t) \in M^2 \times \mathbb{D}^k_{r_2}$ **is a part of a chain** if there exists an L-chain z $(L \geq 1)$ which has $(x, x \pm d, t)$ as one of its triples.

Definition 6.12. The *stratification of S_1* is defined as

$$S_1 = \bigcup_{L \geq 1} D_L$$

where D_L is defined as $D_L = \{(x, x \pm d, t) \in M^2 \times \mathbb{D}^k_{r_2} : (x, x \pm d, t) \text{ is part of a } L-\text{chain}\}$. △

Proposition 6.13. *Let $z = (x + ld, x + (l \pm 1)d, t)_{l=0}^L$. Then z is a L-chain if and only if $j^0_{L+1}\widehat{h}(z) \in W_L$. Assume that $j^0_{L+1}\widehat{h} \pitchfork W_L$, then $j^0_{L+1}\widehat{h}^{-1}(W_L)$ is a manifold of codimension $((L+1)(k+2) - k - 1) + (L+1)(n-1) = L(k+n+1) + n$. In particular, if $L \geq \kappa := \lfloor (k+1)/(n-1) \rfloor$, the set of L-chains is empty.*

Proof. The first assertion is a straightforward verification. The second assertion follows from theorem 4.1 and a dimension count: the codimension of W_L is equal to the codimension of $\{(x + ld, x + (l \pm 1)d, t)_{l=0}^L : x \in M, t \in D^k\} \subset (M^2 \times \mathbb{D}^k)^{L+1}$ plus the codimension of $(\Delta(N'))^{L+1} \subset (N')^{2(L+1)}$. The final assertion follows from the second, as the dimension of $j^0_{L+1}\widehat{h}^{-1}(W_L)$ is $k+1-(L+1)(n-1)$. □

Remark 6.14. In the following lemma, we shall make use of corollaries 5.12 and 5.15. It is important to observe that $J^0(M, \mathbb{D}^k, \mathbb{R}^{n_1}, \mathbb{R}^{n_2-1}) \cong J^0(M^2 \times \mathbb{D}^k, \mathbb{R}^{2n-2})$, and more generally, that the following diagram commutes for all $p \geq 1$ and smooth equivariant maps $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ holonomic in the first factor $\mathbb{R}^{n_1}$:

$$\begin{tikzcd}
J^k(M, \mathbb{D}^k, \mathbb{R}^{n_1}, \mathbb{R}^{n_2-1})^p \arrow[rr, "\sim"] & & (M^2 \times \mathbb{D}^k)^p \times (\mathbb{R}^{n-1})^{2p} \\
& (M^2 \times \mathbb{D}^k)^p \arrow[ul, "j^{k,p}_{\mathcal{I}_p} \widehat{h}"] \arrow[ur, "j^0_p \widehat{h}"'] &
\end{tikzcd}$$

$\triangle$

Lemma 6.15. *Let $h : M^2 \times \mathbb{D}^k \to N^2$ be an isovariant map, such that for all $t \in \mathbb{S}^{k-1} \subset \mathbb{D}^k$, h_t is holonomic, and for all other t, h_t is holonomic in the first factor of $N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Assume that $M = \mathbb{R}$ or $M = [0, 1]$. Then there exists a smooth map $\tilde{h} : M^2 \times \mathbb{D}^k \to N^2$ such that*

- *(a) $\tilde{h}$ is isovariant, there exists $0 < r_1 < 1$ such that $\tilde{h}_t$ is is holonomic for all $|t| \geq r_1$ and holonomic in the first factor for all $t \in \mathbb{D}^k$.*
- *(b) $\tilde{h}$ is homotopic through isovariant maps to h, and the homotopy is relative $M^2 \times \mathbb{S}^{k-1}$.*
- *(c) After deleting $(n_1 + 1)$-th coordinate pair, $\widehat{H} := \tilde{h}_{\widehat{n_1+1}}$ satisfies the following: there exists $r_1 < r_2 < 1$ such that for every $L \in \mathbb{N}_0$,*

$$j^0_{L+1}(\widehat{H}) : (M^2 \times \mathbb{D}^k)^{L+1} \to (M^2 \times \mathbb{D}^k)^{L+1} \times (\mathbb{R}^{n-1})^{2(L+1)}$$

is transverse to the submanifolds W_L.

Assume that $M = \mathbb{S}^1$ and $0 < d < 1$ is irrational or $d = p/q$ is rational, with $p, q \in \mathbb{N}_1$, $q > 2$, p, q coprime and $k < (n-1)q - 1$. Then the same conclusion holds.

Proof. We shall treat each case separately.

Let $M = [0, 1]$. By lemma 6.9 we can assume without loss of generality that there exists $0 < r_1 < 1$ such that h_t is holonomic for all $|t| \geq r_1$. We claim that we can choose an equivariant perturbation of $\widehat{h}$ that is arbitrarily small in the topology of $C^\infty_W(M, \mathbb{R}^{n_1}, \mathbb{R}^{n_2-1}; \mathbb{Z}_2)$, that satisfies (c), and equals $\widehat{h}$ near $M^2 \times \mathbb{S}^{k-1}$. It follows that $\tilde{h}$, given by the (n_1+1)-th coordinate pair and the perturbation of $\widehat{h}$, can be chosen arbitrarily close to h. By lemmas 5.6 and 5.9, it then follows that $\tilde{h}$ is isovariant, and linearly homotopic through isovariant maps to h. Hence, the homotopy connecting h and $\tilde{h}$ is relative $M^2 \times \mathbb{S}^{k-1}$, too. So if the claim holds, then we can find $\tilde{h}$ satisfying (a)-(c).

To prove the claim, we make use of corollary 5.15, and the above commuting diagram which identifies $j^{k,p}_{\mathcal{I}_p} \widehat{h}$ with $j^0_p \widehat{h}$ in our specialized case, for any $p \geq 1$. That is to say, for any $r_2 > 0$ with $r_1 < r_2 < 1$, we may conclude that there exists an equivariant perturbation $H : M^2 \times \mathbb{D}^k \to \mathbb{R}^{2n-2}$ of $\widehat{h}$ that is indeed arbitrarily small, equals $\widehat{h}$ off a small neighbourhood of $M \times M \times \mathbb{D}^k_{r_2}$, is holonomic on $M^2 \times \{t \in \mathbb{D}^k : |t| \in [r_1, 1]\}$ and holonomic in the first factor elsewhere, and so that $j^0_{L+1} H$ is transverse to W_L for each $L \in \mathbb{N}_0$ over $(M^2 \times \mathbb{D}^k_{r_2})^{l+1}$. It should be clear that we can select such an H satisfying the condition for each W_L individually, where we apply the transversality result twice: once to make sure that $j^0_{L+1} H$ is transverse and holonomic at $M^2 \times \{t \in \mathbb{D}^k : r_1 \leq |t| \leq r_2\}$, and once more for $|t| < r_1$, where we only require holonomicity in the first factor. Residuality and the Baire property imply that we can select one such H, for which the multijet of H is transverse to W_L, and so that $H = \widehat{h}$ off a small neighbourhood of $M^2 \times \mathbb{D}^k_{r_2}$. Moreover, we have the freedom to choose H as close to $\widehat{h}$ as we like.

Let $M = \mathbb{S}^1 \cong [0, 1]/\{0 \sim 1\}$. If d is irrational, we can apply the same arguments as above. If $d = p/q$ is rational with p and q coprime, positive integers, we can no longer use 5.15 to determine when

$j^0_{q+r}\widehat{h} \pitchfork W_{q+r-1}$ for every $r \geq 0$. That is, $\mathfrak{s}(W_{q+r-1})$ has non-empty intersection with $\Delta^{q+r}(M^2 \times \mathbb{D}^k)$, where $\mathfrak{s} : J^0(M, \mathbb{D}^k, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \to M^2 \times \mathbb{D}^k$ is the source map. However, using the above arguments we can conclude that we can find a perturbation of h satisfying (a),(b), and (c'), given as

$$\text{(c'):}\quad j^0_{L+1}\widehat{H} \pitchfork W_L, \quad 0 \leq L < q.$$

It follows that $(j^0_q\widehat{H})^{-1}(W_{q-1}) \cap M^2 \times \mathbb{D}^k_{r_2}$ is a manifold of dimension $k + 1 - q(n-1) < 0$ (by assumption). Hence, there are no $q-1$ chains, and there can therefore be no $q+r-1$ chain either. It follows that $j^0_{q+r}\widehat{H} \pitchfork W_{q+r-1}$ for all $r \geq 0$ as W_{q+r-1} does not intersect the image of $j^0_{q+r}(\widehat{H})$.

Let $M = \mathbb{R}$. We shall argue that there exist inductively chosen perturbations $(\tilde{h}_r)_{r=1}^{\infty}$ of h, such that $\tilde{h}_r$ satisfies the conditions (a_r)-(c_r), given by

- (a_r) $\tilde{h}_r$ is isovariant, $(\tilde{h}_r)_t$ is holonomic for all $|t| \geq r_1$, and holonomic in the first factor for all other $t \in \mathbb{D}^k$.
- (b_r) $\tilde{h}_r$ homotopic through isovariant maps to $\tilde{h}_{r-1}$, and the homotopy is relative

$$((-\infty, -r] \cup [-r + 3/2, r - 3/2] \cup [r, \infty))^2 \times \mathbb{D}^k \cup \mathbb{R}^2 \times \mathbb{S}^{k-1}.$$

- (c_r) The map $\widehat{H}_r := (\tilde{h}_r)_{\widehat{n_1+1}} : \mathbb{R}^2 \times \mathbb{D}^k \to \mathbb{R}^{2n-2}$ satisfies for every $L \in \mathbb{N}_0$, $j^0_{L+1}(\widehat{H}_r) \pitchfork W_L$ on $(M^2 \times \mathbb{D}^k_{r_2})^{l+1}$.

Here $\tilde{h}_0 = h$. Indeed, using corollary 5.7 instead of lemma 5.6, we can use the arguments given for $M = [0,1]$ to deduce that there exists a perturbation $\tilde{h}_1$ of h satisfying (a_1)-(c_1), and it is not hard to see that the arguments may be repeated inductively. Define $\tilde{h}$ as $\lim_{r\to\infty} \tilde{h}_r$ (either pointwise or in the C^∞_W-topology). Because for each point this sequence stabilizes after finitely many points, the limit function is smooth, and is straightforward to check that $\tilde{h}$ satisfies (a) and (c). To see (b), denote by G_r the homotopy connecting $\tilde{h}_{r-1}$ to $\tilde{h}_r$, whose existence is guaranteed by condition (b_r). We may concatenate all G_r to form a new homotopy, in which G_1 is completed on the interval $[0, 1/2]$, and G_r is completed on the interval $[1/r, 1/(r+1)]$. Because each G_r changes $\tilde{h}_{r-1}$ only on $((-r, -r+3/2) \cup (r-3/2, r))^2 \times \mathbb{D}^k_{r_2}$, the new homotopy is well defined, and one may smooth out each G_r near the boundaries of $[1/r, 1/(r+1)]$, so that it is also smooth. This completes the proof. $\square$

Lemma 6.16. *Assume that $M = [0,1]$, or $M = \mathbb{R}$, or that $M = \mathbb{S}^1$ and either d is irrational, or $d = p/q$ is rational, $p, q \in \mathbb{N}_1$ are coprime, $q > 2$, and $k < q(n-1) - 1$. Let $h : M^2 \times \mathbb{D}^k \to N^2$ be an isovariant map, and h satisfies condition (a) and (c) of lemma 6.15. Let $\kappa := \lfloor (k+1)/(n-1) \rfloor$ and $L \geq 1$.*

Then S and D_L are topologically closed, S is a submanifold, and $D_L \setminus D_{L+1}$ is a submanifold of dimension $k + 1 - (L+1)(n+1)$ embedded in $M^2 \times \mathbb{D}^k_{r_2}$. Moreover, D_R is topologically embedded in D_L for every $R \geq L$, and D_L is empty for all $L \geq \kappa$.

Proof. By property (c), $V_L := j^0_{L+1}\widehat{h}^{-1}(W_L)$ is a topologically closed submanifold of $(M^2 \times \mathbb{D}^k)^{l+1}$ for every $L \geq 0$, and contained (and hence embedded) in the submanifold of tuples $A_L := \{(x+ld, x+(l\pm 1)d, t)_{l=0}^L : x \in M, t \in \mathbb{D}^k_{r_2}\}$. Its dimension is $k+1-(L+1)(n-1)$, and hence $V_L = \varnothing$ for all $L \geq \kappa$. Note that $S = V_0$, which proves the assertions about S. Observe that $\pi_l : (M^2 \times \mathbb{D}^k_{r_2})^{L+1} \to M^2 \times \mathbb{D}^k_{r_2}$, which projects to the l-th factor, restricts to a smooth embedding $\pi'_l : A_L \to M^2 \times \mathbb{D}^k_{r_2}$. As $D_L = \cup_{l=1}^{L+1} \pi'_l(V_L)$, it follows that D_L is topologically closed. Moreover, as V_L is empty for all $L \geq \kappa$, the same is true for D_L. From the definition of D_L, it follows that $D_R \subset D_L$ whenever $R \geq L$, and as both sets are closed, the inclusion map is a topological embedding.

Let $M = [0,1]$ or $M = \mathbb{R}$. Let $1 \leq r < s \leq l$, then it is straighforward to verify that $(x, x \pm d, t) \in \pi'_r(V_L) \cap \pi_s(V_L) \iff (x, x \pm d, t)$ is part of an $(r+s)$-chain. Moreover, by studying the map $j^0_{L+3}\widehat{h}$, one can verify that the set of L-chains that do not come from $L+1$-chains, form an open subset $\tilde{V}_L$ of V_L. It follows that

$$D_L \setminus D_{L+1} = \bigcup_{l=1}^{L+1} \pi'_l(\tilde{V}_L),$$

and that the right hand side is a union of disjoint manifolds (of dimension $\dim V_l = k+1-(n-1)(l+1)$), and hence a smooth manifold. The manifolds $\pi'_l(\tilde{V}_L)$ will hereafter be referred to as the **rungs of** D_L, with $\pi'_l(\tilde{V}_L)$ being the l-the rung. See also figure 3.

Let $M = S^1$. Assuming that d is irrational, it follows once again that $(x, x\pm d, t) \in \pi'_l(V_L) \cap \pi_s(V_L) \iff (x, x \pm d, t)$ is part of an $(r+s)$-chain. Hence, we can apply the same arguments as before to show that $D_L \setminus D_{L+1}$ is again a submanifold. However, if $d = p/q$ is rational, with p and q coprime positive integers, then any triple part of a $q-1$-chain, is also a $q+r$ chain for every $r \geq 0$, which breaks down the reasoning. However, $\dim V_{q-1} < 0$ by the assumption on k, so $D_{q+r-1} = \varnothing$ for every $r \geq 0$. Hence, we can still conclude that $(x, x \pm d, t) \in \pi'_l(V_L) \cap \pi_s(V_L) \iff (x, x \pm d, t)$ is part of an $(r+s)$-chain, and the remaining arguments can be applied once again. $\square$

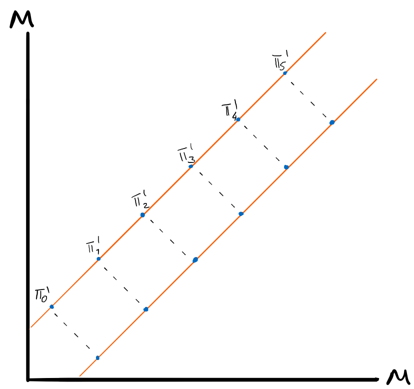

Figure 3: A slice of the manifold $D_5 \setminus D_6$. The equivariance of the map h which produces the singularity implies that $D_L \setminus D_{L+1}$ is symmetric under reflection along $\Delta(M)$. Connecting the symmetric points, we can see the 'rungs in the ladder'. $\triangle$

6.6 Construction of the coordinate pair

Let $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ be an isovariant smooth map, such that h_t is holonomic for every $t \in \mathbb{S}^{k-1}$ and holonomic in the first factor $\mathbb{R}^{n_1}$ for all other $t \in \mathbb{D}^k$. By lemmas 6.15 and 6.16 we can assume without loss of generality that (up to a smooth homotopy through isovariant maps) h **is well positioned**, i.e. h_t is holonomic for every $|t| \geq r_1$, and $j^0_{L+1}\widehat{h} \pitchfork W_L$ for all $L \geq 0$, so that the double points S_1 of S are stratified. Here r_1 is some fixed real number $0 < r_1 < 1$, and

$$S_1 = \bigcup_{L \geq 1} D_L,$$

with each D_L a closed subset of $S_1 \subset M^2 \times \mathbb{D}^k$, $D_R \subset D_L$ for every $R \geq L$, and $D_L \setminus D_{L+1}$ a submanifold. In this section we shall show that the stratification of S_1 can be used to prove the following lemma:

Lemma 6.17. *Let $0 < r_1 < 1$ be a real number, and let $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ be an isovariant smooth map, such that h_t is holonomic for every $t \in \mathbb{S}^{k-1}$, and h_t is holonomic in the first factor $\mathbb{R}^{n_1}$ for all $t \in \mathbb{D}^k$. Assume that h is well positioned. If $M = \mathbb{S}^1$, assume that $d \neq 1/2$. Then there exists a smooth map $\tilde{h} : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ such that*

(i) $\tilde{h}$ is isovariant, h_t is holonomic for every $|t| \geq r_1$, and holonomic in the first factor $\mathbb{R}^{n_1}$ of $\mathbb{R}^n$ for all other $t \in \mathbb{D}^k$.

(ii) $\tilde{h}$ is homotopic through isovariant maps relative $M^2 \times \mathbb{S}^{k-1}$ to h.

(iii) $\tilde{h}$ differs from h only in the $(n_1 + 1)$-coordinate pair, i.e. $(\tilde{h})_{\widehat{n_1+1}} = \widehat{h}$.

(iv) for every $(x, x \pm d, t) \in S_1 \cup M^2 \times \mathbb{D}^k_{[r_1,1]}$,

$$\tilde{h}_{n_1+1}(x, x - d, t) = \tilde{h}_{n_1+1}(x, x + d, t).$$

Here $\mathbb{D}^k_{[r_1,1]} = \{t \in \mathbb{D}^k : r_1 \leq |t| \leq 1\}$.

Remark 6.18. In the following proof, we mean by a **pre-closed neighbourhood** a closed neighbourhood Ω satisfying $\overline{\mathrm{int}(\Omega)} = \Omega$. △

Proof. We shall construct a linear homotopy of the equivariant coordinate pair (h_{n_1+1}, h_{n+n_1+1}) to a new equivariant pair $(\tilde{h}_{n_1+1}, \tilde{h}_{n+n_1+1})$. Recall from section 6.3 that induced homotopy connecting h to $\tilde{h}$ (which then differs from h only in the $(n_1 + 1)$-coordinate pair), goes through isovariant maps if and only if for every interpolating map h_s, $s \in [0, 1]$,

$$((h_s)_{n_1+1}, (h_s)_{n+n_1+1})(\Sigma_{n_1+1}(h)) \cap \Delta(\mathbb{R}) = \varnothing.$$

For later purposes, let S_1^+ be the part of S_1 contained in Δ_{+d}, and S_1^- the part contained in Δ_{-d}. Note that $S_1^+ \cap S_1^-$ is empty (here we use that if $M = \mathbb{S}^1$, then $d \neq 1/2$), and the two are closed subsets covering S_1.

The proof follows by finite induction over the strata of S_1. I.e. recall from lemma 6.16 that there are finitely many non-empty double point strata D_L, with $1 \leq L < \kappa$ (as defined in the lemma). Let $\omega : M^2 \times \mathbb{D}^k \to M^2 \times \mathbb{D}^k$ denote the automorphisms which swaps the coordindates on M. For $j \in \{1, \dots, \kappa - 1\}$, the induction hypothesis is the existence of a smooth map $h^j : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ which satisfies (i)-(iii), and so that the pair $(h^j_{n_1+1}, h^j_{n+n_1+1}) : M^2 \times \mathbb{D}^k \to \mathbb{R}^2$ satisfies (iv) over a pre-closed, ω-invariant neighbourhood Ω_j of $M^2 \times \mathbb{D}^k_{[r_1,1]} \cup D_{\kappa-1} \cup \ldots \cup D_{\kappa-j}$. I.e. for all $(x, x - d, t) \in \Omega_j$,

$$h^j_{n_1+1}(x, x - d, t) = h^j_{n_1+1}(x, x + d, t) = 0.$$

Th proof is then completed by setting $\tilde{h}$ equal to $h^{\kappa-1}$. The purpose of the neighbourhoods Ω_j is to make the inductive step possible: it allows for a clean extension of h^{j-1} to h^j when the strata approach one another.

For the base step, let $h^0 = h$, and recall from lemma 6.16 that $D_{\kappa-1} = D_{\kappa-1} \setminus D_\kappa$, and hence $D_{\kappa-1}$ is a topologically closed manifold. Recall also from that lemma that

$$D_{\kappa-1} = \bigcup_{l=0}^{\kappa-1} \pi'_l(V_{\kappa-1}),$$

where each $\pi'_l : V_{\kappa-1} \to M^2 \times \mathbb{D}^k$ was an embedding and all rungs $\pi'_l(V_{\kappa-1})$ of $D_{\kappa-1}$ are closed and disjoint submanifolds. Let $\phi : M^2 \times \mathbb{D}^k \to M^2 \times \mathbb{D}^k$ denote the diffeomorphism given by $(x, y, t) \mapsto (x + d, y + d, t)$, and observe that

$$\pi'_l(V_{\kappa-1}) = \phi^l \pi'_0(V_{\kappa-1}).$$

Note that $(x, x \pm d, t)$ is in the 0-th rung (which we will also refer to as the bottom rung) if and only if $\phi^l(x, x \pm d, t) \in D_{\kappa-1}$ for all $0 \le l < \kappa$ and $\phi^{-1}(x, x \pm d, t), \phi^{\kappa}(x, x \pm d, t) \notin D_{\kappa-1}$.

Define the smooth map $\tau : M^2 \times \mathbb{D}^k \to \mathbb{R}$ by

$$\tau(x, y, t) = h_{n_1+1}(x, y, t) - h_{n+n_1+1}(x, y, t),$$

Using τ, we define (h_1^1, h_2^1) over the rungs of $D_{\kappa-1}$, using the bottom rung as a reference point. We shall then argue that (h_1^1, h_2^1) admits a smooth extension to all of $M^2 \times \mathbb{D}^k$ so that (iv) is satisfied over some pre-closed ω-invariant neighbourhood of the rungs. Observe that S_1 and all D_L are ω-invariant. Let $(x, x-d, t) \in \pi_0'(V_\kappa)$, then define for $0 \le l \le \kappa - 1$,

$$(h_1^1, h_2^1)(\phi^l(x, x-d, t)) = \left(h_{n_1+1} + \sum_{r=1}^{l} \tau \circ \phi^{-r}, h_{n+n_1+1} + \sum_{r=1}^{l} \tau \circ \phi^{-r} \right) (\phi^l(x, x-d, t)).$$

This means that on the bottom rung we do not change anything, and that for the higher rungs we add the difference between $h_{n_1+1}(x, x-d, t)$ and $h_{n_1+1}(x, x+d, t)$ inductively, which forces (iv) to hold over these rungs. To ensure equivariance, define for all $(y, y+d, t) \in D_{\kappa-1}$

$$(h_1^1, h_2^1)(y, y+d, t) = (h_2^1, h_1^1)(y+d, y, t) \quad \text{where defined.}$$

The new pair is well defined over all of $D_{\kappa-1}$, because $D_{\kappa-1}$ is ω-invariant and all rungs of $D_{\kappa-1}$ are disjoint. It is now straightforward to check that for all $(x, x \pm d, t) \in D_{\kappa-1}$, $h_1^1(x, x-d, t) = h_1^1(x, x+d, t)$, and that $h_1^1 - h_2^1 = h_{n_1+1} - h_{n+n_1+1}$ over $D_{\kappa-1}$. To find the pre-closed neighbourhood Ω_1, note that $h_{n_1+1} - h_{n+n_1+1}$ does not vanish over every component of $D_{\kappa-1}$ because h is isovariant. Because $D_{\kappa-1}$ is closed, $M^2 \times \mathbb{D}^k$ is a manifold, and $h_{n_1+1} - h_{n+n_1+1}$ does not vanish over the components of $D_{\kappa-1}$, we can select for every component $\Sigma \subset D_{\kappa-1}$ an open neighbourhood $U_\Sigma \subset M^2 \times \mathbb{D}^k$ such that $h_{n_1+1} - h_{n+n_1+1}$ has one sign on U_Σ and $\overline{U}_\Sigma$ intersects only the component Σ. Moreover, as $D_{\kappa-1}$ is ω-invariant, we can assume without loss of generality that for every component $\Sigma \subset D_{\kappa-1}$ there exists a unique component Σ' such that $\omega(U_\Sigma) = U_{\Sigma'}$. Define

$$\Omega_1 = \bigcup_{\Sigma \in \pi_0(D_{\kappa-1})} \overline{U}_\Sigma.$$

Over each $\overline{U}_\Sigma$ we can extend the definition of (h_1^1, h_2^1) by the pointwise definition. That is, if $\Sigma \subset \pi_l'(V_{\kappa-1}) \cap S_1^-$, then for ever $(x, y, t) \in \overline{U}_\Sigma$, we define

$$(h_1^1, h_2^1)(x, y, t) = \left(h_{n_1+1} + \sum_{r=0}^{l-1} \tau \circ \phi^r, h_{n+n_1+1} + \sum_{r=0}^{l-1} \tau \circ \phi^r \right) (x, y, t),$$

and for all $(x, y, t) \in \omega(U_\Sigma)$ define

$$(h_1^1, h_2^1)(x, y, t) = (h_2^1, h_1^1)(y, x, t).$$

This defines (h_1^1, h_2^1) over all of Ω_1 well. It is straightforward to check that (iv) holds over Ω_1. The pair $(h_1^1, h_2^1) : \Omega_1 \to \mathbb{R}^2$ admits a smooth extension to all of $M^2 \times \mathbb{D}^k$: it is defined over a closed set, and admits a pointwise smooth extension. To see this last point, note that all $\overline{U}_\Sigma$ can be separated by disjoint opens. Hence the extension of the terms defining the pair can be extend in small neighbourhoods of each point. Moreover, as for all $|t| \ge r_1$, the pointwise definition of (h_1^1, h_2^1) consists of holonomic terms. Hence the smooth extension can be chosen to be holonomic for all $|t| \ge r_1$.

Select any smooth extension, denoted $(\tilde{h}_1^1, \tilde{h}_2^1) : M^2 \times \mathbb{D}^k \to \mathbb{R}^2$, so that $(\tilde{h}_1^1, \tilde{h}_2^1)_t$ is holonomic for all $|t| \ge r_1$. To complete the base step and the construction of (h_1^1, h_2^1), recall that over each $\overline{U}_\Sigma$ the map $h_{n_1+1} - h_{n+n_1+1}$ did not vanish. Select an open ω-invariant neighbourhood of $U \subset M^2 \times \mathrm{int}(\mathbb{D}^k)$ of

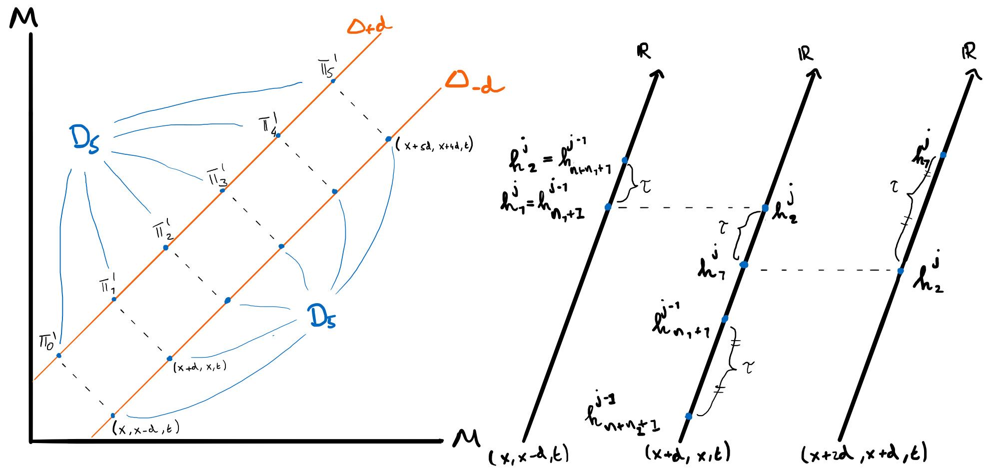

Figure 4: When selecting values over D_5, we make sure that $h_1^j - h_2^j = h_{n_1+1}^{j-1} = h_{n+n_1+1}^{j-1}$.$\triangle$

Ω_1 (which exists by the ω-invariance of $\Omega_1 \subset M^2 \times \mathbb{D}^k_{r_2}$) on which $h_{n_1+1} - h_{n+n_1+1}$ does not vanish, and choose a partition of unity $\{\rho_1, \rho_2\}$ subordinate to $\{U, M^2 \times \mathbb{D}^k \setminus \Omega_1\}$. As both opens in the cover of $M^2 \times \mathbb{D}^k$ are equivariant, we can assume without loss of generality that $\rho_1, \rho_2 : M^2 \times \mathbb{D}^k$ are ω-invariant. Finally, define

$$(h_1^1, h_2^1) = \rho_2 \cdot (h_{n_1+1}, h_{n+n_1+1}) + \rho_1 \cdot (\tilde{h}_1^1, \tilde{h}_2^1).$$

Note that (h_1^1, h_2^1) agrees with $(\tilde{h}_1^1, \tilde{h}_2^1)$ on Ω_1. Hence, it satisfies (iv) on Ω_1. Moreover, $h_{n_1+1} - h_{n+n_1+1}$ has the same sign as $h_1^1 - h_2^1$ on $\Delta_{\pm d} \supset \Sigma_{n_1+1}$. If we denote by $h^1 : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ the map obtained by replacing the pair (h_{n_1+1}, h_{n+n_1+1}) by (h_1^1, h_2^1), it follows that h^1 satisfies (i)-(iii). That is to say, pointwise the linear homotopy between pairs does not cross 0 over Σ_{n_1+1}, which ensures isovariance. This completes the base step.

To induction step is not very different. Recall from lemma 6.16 that $D_{\kappa-j} \setminus D_{\kappa-j+1}$ is a submanifold contained in the topologically closed set $D_{\kappa-j+1}$, and that

$$D_{\kappa-j} \setminus D_{\kappa-j+1} = \bigcup_{l=0}^{\kappa-j} \pi_l'(\tilde{V}_{\kappa-j}),$$

where each $\pi_l' : \tilde{V}_\kappa \to M^2 \times \mathbb{D}^k$ was an embedding of disjoint submanifolds. Here $\tilde{V}_{\kappa-j}$ was the open subset of $V_{\kappa-j}$ consisting of $(\kappa-j)$-chains that did not come from $(\kappa-j+1)$-chains. We shall refer to the 0-th rung as the bottom rung.

Define $D'_{\kappa-j} = D_{\kappa-j} \setminus \mathrm{int}(\Omega_j)$. It is contained in $D_{\kappa-j} \setminus D_{\kappa-j+1}$, and is closed. We refer to $\pi_l'(\tilde{V}_{\kappa-j}) \cap D'_{\kappa-j}$ as the **rungs of** $D'_{\kappa-j}$, and they too are disjoint. Moreover, one can verify that

$$\pi_l'(\tilde{V}_{\kappa-j}) \cap D'_{\kappa-j} = \pi_l'(V_{\kappa-j}) \cap D'_{\kappa-j}$$

from which it follows that the rungs of $D'_{\kappa-j}$ are the intersection of two closed sets, hence closed. Define the smooth map $\tau_j : M^2 \times \mathbb{D}^k \to \mathbb{R}$ by

$$\tau_j(x, y, t) = h_{n_1+1}^j(x, y, t) - h_{n+n_1+1}^j(x, y, t).$$

Using τ_j, we define (h_1^{j+1}, h_2^{j+1}) over the rungs of $D'_{\kappa-j}$, using the bottom rung as a reference point. We use the same formula's as in the base step with τ replaced by τ_j, i.e. first over all $(x, x-d, t) \in \pi'_l(\tilde{V}_{\kappa-j}) \cap D'_{\kappa-j}$, and then over all $(x, x+d, t)$ to ensure equivariance. It is again well defined because all rungs are disjoint and $D'_{\kappa-j}$ is ω-invariant. It follows that for all $(x, x \pm d, t) \in D'_{\kappa-j}$, $h_1^{j+1}(x, x-d, t) = h_1^1(x, x+d, t)$, and that $h_1^{j+1} - h_2^1 = h^j_{n_1+1} - h^j_{n+n_1+1}$ over $D_{\kappa-j} \setminus D_{\kappa-j+1}$.

To find the pre-closed neighbourhood Ω_{j+1}, note that $h^j_{n_1+1} - h^j_{n+n_1+1}$ does not vanish over the rungs of $D'_{\kappa-j} \subset S_1$, because h^j is isovariant. Let Σ be a component of $D_{\kappa-j} \setminus D_{\kappa-j+1}$. Because $D'_{\kappa-j}$ is closed, $M^2 \times \mathbb{D}^k$ is a manifold, and $h_{n_1+1} - h_{n+n_1+1}$ does not vanish over the rungs of $D'_{\kappa-j}$, we can select for every component Σ an open neighbourhood $U_\Sigma \subset M^2 \times \mathbb{D}^k$ of $\Sigma \cap D_{\kappa-j}$, such that $h_{n_1+1} - h_{n+n_1+1}$ has one sign on U_Σ and $\overline{U}_\Sigma$ intersects only the component Σ (i.e. no other component). Moreover, as $D_{\kappa-j} \setminus D_{\kappa-j+1}$ is ω-invariant, we can assume without loss of generality that for every component $\Sigma \subset D_{\kappa-j} \setminus D_{\kappa-j+1}$ there exists a unique component Σ' such that $\omega(U_\Sigma) = U_{\Sigma'}$. Define

$$\Omega_{j+1} = \Omega_j \cup \bigcup_{\Sigma \in \pi_0(D_{\kappa-j} \setminus D_{\kappa-j+1})} \overline{U}_\Sigma.$$

Over each $\overline{U}_\Sigma$ we can extend the definition of (h_1^1, h_2^1) by the pointwise definition, as in the base step. I.e. because all $\overline{U}_\Sigma$ are disjoint, this can again be achieved. It is straightforward to check that (iv) holds each $\overline{U}_\Sigma$. The pair $(h_1^{j+1}, h_2^{j+1}) : \bigcup_\Sigma \overline{U}_\Sigma \to \mathbb{R}^2$ admits a smooth extension to all of $M^2 \times \mathbb{D}^k$: it is defined over a closed set, and admits a pointwise smooth extension. To see this last point, note that all $\overline{U}_\Sigma$ can be separated by disjoint opens. Hence the extension of the terms defining the pair can be extend in small neighbourhoods of each point. Moreover, for all $|t| \geq r_1$ the extension can again be assumed to be holonomic.

Let $(\tilde{h}_1^{j+1}, \tilde{h}_2^{j+1})$ denote any smooth extension, so that for all $|t| \geq r_1$ the map $(\tilde{h}_1^{j+1}, \tilde{h}_2^{j+1})_t$ is holonomic. As $h^j_{n_1+1} - h^j_{n+n_1+1}$ does not vanish over each $\overline{U}_\Sigma$, there exists an open ω-invariant neighbourhood $U \subset M^2 \times \mathrm{int}(\mathbb{D}^k)$ containing all $\overline{U}_\sigma$ on which $h^j_{n_1+1} - h^j_{n+n_1+1}$ does not vanish. Select a partition of unity $\{\rho_1, \rho_2\}$ subordinate to

$$\left\{ U, M^2 \times \mathbb{D}^k \setminus \bigcup_\Sigma \overline{U}_\Sigma \right\}.$$

Finally define

$$(h_1^{j+1}, h_2^{j+1}) = \rho_2 \cdot (h^j_{n_1+1}, h^j_{n+n_1+1}) + \rho_1 \cdot (\tilde{h}_1^{j+1}, \tilde{h}_2^{j+1}).$$

Because the old pair satisfied (iv) over Ω_j and $(\tilde{h}_1^{j+1}, \tilde{h}_2^{j+1})$ satisfies (iv) over $\bigcup_\Sigma \overline{U}_\Sigma$, the new pair satisfies (iv) over Ω_{j+1}. Moreover, by construction the new pair satisfies (i)-(iii). This finishes the induction step. $\square$

6.7 Construction of holonomic isovariant maps

As explained at the start of this chapter, the proof of the main result, theorem 1.4, is reduced to the proof of theorem 6.5. In this section we tie together lemmas 6.15, 6.16, and 1.17 to complete this proof.

Proof of theorem 6.5. Assume that $M = [0,1]$ or $M = \mathbb{R}$. Let $h : M^2 \times \mathbb{D}^k \to (\mathbb{R}^n)^2$ be a smooth map, such that for every $t \in \mathbb{D}^k$ the map $h_t : M^2 \to (\mathbb{R})^2$ lies in $\mathrm{Sol}_F(\mathcal{R}_d)$ and for all $t \in \mathbb{S}^{k-1}$ the map h_t lies in $\mathrm{Sol}_{F,\mathrm{Hol}}(\mathcal{R}_d)$. We need to show that h is smoothly homotopic relative $M^2 \times \mathbb{S}^{k-1}$ to a $\mathbb{D}^k$-family of $\mathrm{Sol}_{F,\mathrm{Hol}}(\mathcal{R}_d)$ maps. This homotopy needs the go through families of $\mathrm{Sol}_F(\mathcal{R}_d)$-maps.

We replace inductively the n coordinate pairs $(h_j, h_{j+1}) : M^2 \times \mathbb{D}^k \to \mathbb{R}^2$, $1 \leq j \leq n$ of h such that in the end, all these coordinate pairs are holonomic. Denote by h^j the map obtained in the j-th step, where the first j coordinate pairs of h have been replaced by a holonomic coordinate pair. The linear homotopy which replaces the old j-th coordinate pair with the new holonomic one, shall induce a linear homotopy between h^{j-1} and h^j that goes through isovariant maps and is relative $M^2 \times \mathbb{S}^{k-1}$. I.e. the linear homotopy is relative $M^2 \times \mathbb{S}^{k-1}$ and goes through families of $\mathrm{Sol}_F(\mathcal{R}_d)$-maps. Let $h = h^0$.

Assume we have completed step $j-1$ and have constructed the map h^{j-1}. This means that h^{j-1} is an isovariant map, holonomic in the first $j-1$ coordinate pairs. We want to select a smooth map $f : M \times \mathbb{D}^k \to \mathbb{R}$ so that $f_t \times f_t$ replaces $(h_j^{j-1}, h_{j+n}^{j-1})$, so that

$$(f(x,t), f(x \pm d, t)) \notin \Delta(\mathbb{R}) \quad \forall (x, x \pm d, t) \in \Sigma_j(h^{j-1}).$$

That is, if we can select such an f and replace $(h_j^{j-1}, h_{j+n}^{j-1})$ by $f_t \times f_t$, then h^j is also isovariant. For all $(x, x \pm d, t) \in \Sigma_j(h^{j-1})$ we ideally define $f(x,t) = h_j^{j-1}(x, x \pm d, t)$. If this can be achieved, then the linear homotopy connecting h^{j-1} to h^j shall goes through isovariant maps. We shall argue that this is indeed possible. I.e. we will argue that f can be defined over a closed set and allows a smooth extension in a neighbourhood of every point in that set.

We partition $\Sigma_j(h^{j-1})$ in two parts: $S = \Sigma_j(h^{j-1}) \cap M^2 \times \mathbb{D}^k_{r_2}$ and $S_{\mathrm{Hol}} = \Sigma_j(h^{j-1}) \cap M^2 \times \mathbb{D}^k_{[r_1,1]}$, where

$$\mathbb{D}^k_{r_2} = \{t \in \mathbb{D}^k : |t| \leq r_2\} \text{ and } \mathbb{D}^k_{[r_1,1]} = \{t \in \mathbb{D}^k : r_1 \leq |t| \leq 1\}$$

with $0 < r_1 < r_2 < 1$ some real numbers. By lemma 6.15 and 6.17 we may assume without loss of generality that $S = j^0 h_{\hat{j}}^{-1}(W_0)$ is a submanifold of $M^2 \times \mathbb{D}^k$, that $(h^{j-1})_t$ is holonomic for all $|t| \geq r_1$ for some real number $0 < r_1 < 1$, and that for all double points $(x, x \pm d, t) \in S_1 \subset S$ (recall: $(x, x-d, t) \in S_1 \iff (x, x+d, t) \in S_1$) it holds that $h_j(x, x-d, t) = h_j(x, x+d, t)$. We define

$$f_j(x,t) = h_j^{j-1}(x,y,t) \quad \forall (x,y,t) \in M^2 \times \mathbb{D}^k_{[r_1,1]}.$$

This is well defined, because $(h_j^{j-1}, h_{j+1}^{j-1})$ is holonomic there. This makes sure that after replacement, $(h^j)_t = (h^{j-1})_t$ for all $t \in \mathbb{S}^{k-1}$ too. We also define

$$f_j(x,t) = h^{j-1}(x, x \pm d, t) \quad \forall (x, x \pm d, t) \in S.$$

This is well defined: it evidently agrees with the definition given for those (x,t) with $|t| \geq r_1$. Moreover, when both $(x, x-d, t)$ and $(x, x+d, t)$ lie S, $h_j(x, x-d, t) = h_j(x, x+d, t)$. We have now defined f over the closed set $M^2 \times \mathbb{D}^k_{[r_1,1]} \cup \pi(S)$, where $\pi : M^2 \times \mathbb{D}^k \to M \times \mathbb{D}^k$ is the map given by $(x,y,t) \mapsto (x,t)$. Recall that π restricted $\Delta_{\pm d} \subset M^2 \times \mathbb{D}^k$ is a smooth two sheeted covering, and that $\Sigma_j(h^{j-1}) \subset \Delta_{\pm d}$. For all (x,t) with $r_1 < |t| \leq 1$, the maps already is defined over a neighbourhood of (x,t), and it is smooth there. For points (x,t) with $|t| < r_2$, one can choose a small enough neighbourhood of (x,t) and a section of $\pi|_{\Delta_{\pm d}} : \Delta_{\pm d} \to M \times \mathbb{D}^k$ to extend the definition of f using h_j^{j-1}. Hence, we can choose a global extension of f to a map $M^2 \times \mathbb{D}^k$. This finishes the proof for $M = [0,1]$ or $M = \mathbb{R}$.

Assume that $M = S^1$. If d is irrational or $d = p/q$ with $q > 2$ and $k < n(q-1) - 1$, we can apply exactly the same arguments. If $d = 1/2$, we can not apply lemma 6.15 or lemma 6.17. However, by lemma 6.9 we can assume without loss of generality that h_t is holonomic for all $|t| \geq r_1$. We can complete the same induction process as above. Partition $\Sigma_j(h^j)$ in two parts, S and S_{Hol}, and define $f(x,t) = h^{j-1}(x,y,t)$ over S_{Hol} and $f(x,t) = h^{j-1}(x, x \pm d, t)$ for all $(x, x \pm d, t) \in S$. Note that in principle S is just a closed set, and that we have no stratification of the double points. The key observation is that $x + d = x - d \in S^1 \cong [0,1]/\{0 \sim 1\}$, from which it follows that $\pi|_{\Delta_{\pm d}} : \Delta_{\pm d} \to M \times \mathbb{D}^k$ is a diffeomorphism, and hence there are no double points. It follows that f is well defined, and it is even easier than before to show that f admits pointwise a smooth extension over a neighbourhood. This completes the proof. $\square$

A Proof of the transversality theorems

In this appendix we prove theorems 4.14 and 4.18, and their subsequent corollaries. We shall first prove three lemmas needed to prove theorem 4.14, and discuss how this proof should be modified for theorem 4.18. The first lemma reduces the proof to the special case that $X_j = Z_j$ and $I_j = (j)$, and the second and third are the main tools for proving density. We separately address the openness of the sets T_W. The second and third lemma, as well as the final proof of both theorems are respectively based on lemma II.4.6, proposition II.4.5, theorem II.4.9, and theorem II.4.13 of [5].

Lemma A.1. *In the context of theorem 4.14, let $f = (f_1, \ldots, f_n) \in C^\infty_{\mathcal{I},k}(X,Y))$. Denote by $\iota_{\mathcal{I}}$ the inclusion of $J^k_{\mathcal{I}}(X,Y)$ into*

$$J^k(X,Y) = \prod_{j=1}^n J^{k_j}(X_j, Y_j).$$

Then

$$j^k_{\mathcal{I}} f \pitchfork W \iff j^{k_1} f_1 \times \ldots \times j^{k_n} f_n \pitchfork \iota_{\mathcal{I}}(W).$$

Proof. Denote by $j^k f$ the map $j^{r_1} f_1 \times \ldots \times j^{r_n} f_n : X \to J^k(X,Y)$. For $(z,y) \in Z \times Y$, denote by $(x_j, y_j)_{j=1}^n$ the element $\Omega^{-1} \circ \iota(z,y) \in \prod_{j=1}^n X_j \times Y_j$, and by x and y the respective n-tuples consisting of the x_j and y_j.

As $W \subset J^k_{\mathcal{I}}(X,Y)$, it is clear that $j^k_{\mathcal{I}} f(x) \in W \iff (j^k(f))(x') \in \iota_{\mathcal{I}} W$. Hence it suffices to show for all $x \in X$ such that $j^k_{\mathcal{I}} f(x) \in W$, that

$$T_{j^k_{\mathcal{I}} f(z)} J^k_{\mathcal{I}}(X,Y) = \operatorname{im} T_z j^k_{\mathcal{I}} f + T_{j^k_{\mathcal{I}} f(z)} W \iff T_{j^k f(x)} J^k(X,Y) = \operatorname{im} T_x (j^k f) + T_{\, j^k f(x)} \iota_{\mathcal{I}}(W).$$

For every factor of $J^k(X,Y)$ one may choose an open product set $U_j \times V_j \subset X_j \times Y_j$ around the corresponding factors of $(x, (f_1 \times \ldots \times f_n)(x))$, that trivializes the bundle. Let $U := \prod_{j=1}^n U_j$, and $S = \prod_{j=1}^n U_j \times V_j$. Let L_j denote the fiber of $J^{k_j}(X_j, Y_j)$, i.e. $J^k(X,Y)|_S \cong \prod_{j=1}^n (U_j \times V_j \times L_j)$. The tangent bundle of U and $J^k(X,Y)|_S$ then factors as

$$TU \cong \prod_{j=1}^n TU_j, \quad T\left(J^k(X,Y)|_S\right) \cong \prod_{j=1}^n T(U_j \times V_j \times L_j).$$

With these identifications, it follows that $T(j^k f)$ can be represented by the matrix

$$\begin{pmatrix} Tj^{k_1} f_1 & & & \\ & Tj^{k_2} f_2 & & \\ & & \ddots & \\ & & & Tj^{k_n} f_n \end{pmatrix}.$$

We can refine our representation of each $Tj^{k_j} f_j$: we may identify $TJ^{k_j}(X_j, Y_j)|_{U_j \times V_j} \cong TU_j \times T(V_j \times L_j)$. It follows that $Tj^{k_j} f_j$ can be represented by

$$\begin{pmatrix} \mathrm{id} \\ p_{V_j \times L_j} \circ Tj^{k_j} f_j \end{pmatrix}.$$

For a further needed refinement, we may assume without loss of generality that each U_j is a product of opens $U_{j,i}$, where the second index runs over the elements of $\{1, \ldots, n\}$ contained in I_j. A short calculation shows that $TJ^k_{\mathcal{I}}(X,Y)|_S$ consists of all those vectors

$$(v_{j,i}, w_j) \in T(\Pi J)|_S \cong \prod_{j=1}^n \left(\prod_{i \in I_j} TU_{j,i} \times T(V_j \times L_j) \right)$$

where $i = i'$ implies $v_{j,i} = v_{j,i'}$. From this calculation, the fact that $T\iota_{\mathcal{I}}(W)|_S \subset TJ^k_{\mathcal{I}}(X,Y)|_S$ and the above block structure, it then follows that

$$\operatorname{im} T_x(j^k f) + T_{j^k f(x)}\iota_{\mathcal{I}}(W) = T_{j^k f(x)} J^k(X,Y) \iff$$

$$\operatorname{im} T_z(j^k f \circ \Omega^{-1} \circ \iota) + T_{j^k f \circ \Omega^{-1} \circ \iota(z)} W = T_{j^k f \circ \Omega^{-1} \circ \iota(z)} J^k_{\mathcal{I}}(X,Y) \iff$$

$$\operatorname{im} T_z j^k_{\mathcal{I}} + T_{j^k_{\mathcal{I}}(a)} W = T_{j^k_{\mathcal{I}} f(a)} J^k_{\mathcal{I}}(X,Y).$$

□

Lemma A.2. *In the setting of theorem 4.14, let $B_1, \ldots, B_n$ be manifolds, and $(g_1, \ldots, g_n) \in \prod_{j=1}^n C^\infty(X_j \times B_j, Y_j)$. Define $B = B_1 \times \ldots \times B_n$. Let $\Phi : Z \times B \to J^k_{\mathcal{I}}(X,Y)$ be given by $\Phi(z,b) = j^k_{\mathcal{I}}(g_{1,b_1}, \ldots, g_{n,b_n})(z)$. Assume that Φ is smooth, and that $\Phi \pitchfork W$. Then the set $\{b \in B : j^k_{\mathcal{I}}(g_{1,b_1}, \ldots, g_{n,b_n}) \pitchfork W\}$ is dense in B.*

Proof. Define $\omega : B \to C^\infty(Z, J^k_{\mathcal{I}}(X,Y))$ by $b \mapsto j^k g_{\mathcal{I}}(g_{1,b_1}, \ldots, g_{n,b_n})$. By lemma 4.6 of [5] and the assumption that $\Phi \pitchfork W$, we are done. □

Remark A.3. Lemma 4.6 of [5] is where the density of sets of transverse maps comes from. Although we shall not go into the details here, the meat of the proof of the lemma is showing that the set of critical values of a smooth map has measure zero (i.e. Sard's lemma).

A.1 Openness of mixed transversality

We want to determine sufficient conditions for openness of T_W for both theorems. We start by analyzing a simple case, from which the general case will follow. Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be manifolds, and let $X = \prod_{j=1}^n X_j$, $Y = \prod_{j=1}^n Y_j$. Let W be a submanifold of Y. In order to show that $T_W := \{(f_1, \ldots, f_n) : f_1 \times \ldots \times f_n \pitchfork W\}$ is open, it suffices to show that for every tuple $(f_1, \ldots, f_n) \in T_W$, there exist integers $k_1, \ldots k_n \geq 0$ and open subsets $U_j \subset J^{k_j}(X_j, Y_j)$, $j = 1, \ldots, n$, such that $f_j \in M(U_j)$ and $M(U_1) \times \ldots \times M(U_n) \subset T_W$. In general it is not obvious what a sufficient condition W must be for T_W to be open, but we will show that T_W is open if W is a product $W_1 \times \ldots \times W_n$ of topologically closed submanifolds $W_j \subset X_j$, or if W is compact, or if W is topologically closed and for every tuple $(f_1, \ldots, f_n) \in T_W$, $(f_1 \times \ldots \times f_n)^{-1}(W)$ is compact in A. The opens U_j will be contained in $J^1(A_j, X_j)$.

We define the set $W^\pitchfork$ by

$$W^\pitchfork = \left\{ (\sigma_1, \ldots, \sigma_n) \in \prod_{j=1}^n J^1(X_j, Y_j) : f_1 \times \ldots \times f_n \pitchfork W \text{ at } x, \text{ where } \sigma_j = [f_j]_{x_j, f_j(x_j)} \right\}.$$

It should be clear that for any tuple $(f_1, \ldots, f_n) \in \prod_{j=1}^n C^\infty(X_j, Y_j)$, $f_1 \times \ldots \times f_n \pitchfork W \iff \operatorname{im} j^1 f_1 \times \ldots j^1 f_n \subset W^\pitchfork$.

Proposition A.4. *With the above notation, let W be a topologically closed submanifold of X. Then $W^\pitchfork$ is an open set.*

Proof. We will show that the complement of $W^\pitchfork$, denoted W^c is closed. Let $\mathfrak{t} : J^{1_n}(X,Y) := \prod_{j=1}^n J^1(X_j, Y_j) \to Y$ denote the target map, and $\mathfrak{t}_j : J^1(X_j, Y_j) \to Y_j$ the target map of each factor. Let moreover $\mathfrak{s} : J^{1_n}(X,Y) \to X$ denote the source map, and $\mathfrak{s}_j$ the source map of each factor. Note that $\sigma = (\sigma_1, \ldots, \sigma_n) \in W^c \iff \mathfrak{t}(\sigma) \in W$ and $\operatorname{im} T_{\mathfrak{s}(\sigma)}(f_1 \times \ldots \times f_n) + T_{\mathfrak{t}(\sigma)} W \neq T_{\mathfrak{t}(\sigma)} X$, where $\sigma_j = [f_j]_{\mathfrak{s}_j(\sigma_j), \mathfrak{t}_j(\sigma_j)}$.

Let $\sigma^k = (\sigma^k_1, \ldots, \sigma^k_n)$, $k \in \mathbb{N}$ be a convergent sequence contained in W^c, whose limit point is σ. We will show that σ is contained in W^c. As W is topologically closed and $\mathfrak{t}$ continuous, it follows that $\mathfrak{t}(\sigma) \in W$. As W is a submanifold, we can select a chart (U, φ) around $\mathfrak{t}(\sigma) \in Y$, such that W is a $\dim W$-dimensional plane in $\varphi(U)$. It follows that the the normal bundle NW of W in TY can locally be identified as $NW|_{W\cap U} \cong (W \cap U) \times \mathbb{R}^{\dim Y - \dim W}$. Similarly identify $TY_U \cong U \times \mathbb{R}^{\dim Y}$. Denote by $p_W : TY|_{W\cap U} \to NW|_{W\cap U}$ the continuous projection corresponding to these identifications, which under the trivializations is given by the projection $p : (W \cap U) \times \mathbb{R}^{\dim Y} \to (W \cap U) \times \mathbb{R}^{\dim Y - \dim W}$. By restricting the neighbourhood U if necessary, we may assume without loss of generality that there exists an open neighbourhood V of $\mathfrak{s}(\sigma)$, such that $V \times U$ is the domain of a bundle trivialization for $J^{1_n}(X,Y)|_{V\times U}$. Let $f^k_j : X_j \to Y_j$ represent σ^k_j, the factors of σ^k. As stated, $\sigma^k \in W^c \iff \mathfrak{t}(\sigma^k) \in W$ and $\operatorname{im} T_{\mathfrak{s}(\sigma^k)}(f_1 \times \ldots \times f_n) + T_{\mathfrak{t}(\sigma^k)}W \neq T_{\mathfrak{t}(\sigma^k)}Y$ (and similar for σ). By selecting product charts (V_j, η_j) and (U_j, φ_j) for the factors X_j and Y_j around $\mathfrak{s}(\sigma)$ and $\mathfrak{t}(\sigma)$ respectively, we can further restrict V and U so that

$$J^{1_n}(X,Y)|_{V\times U} \cong \prod_{j=1}^{n} V_j \times U_j \times \operatorname{Hom}(\mathbb{R}^{n_j}, \mathbb{R}^{m_j}),$$

where $n_j = \dim X_j$ and $m_j = \dim Y_j$. Under this identification, σ is mapped to $(\mathfrak{s}_j(\sigma), \mathfrak{t}_j(\sigma), Df_j)_j$, where Df_j denotes the Jacobi matrix with respect to chosen trivializations. The right hand side of the 'if and only if'-statement is then equivalent to rank deficiency at $(\mathfrak{s}(\sigma^k), \mathfrak{t}(\sigma^k))$ of the matrix

$$p \circ \begin{pmatrix} (Df^k_1) & & \\ & \ddots & \\ & & (Df^k_n) \end{pmatrix} \in V \times U \times \operatorname{Hom}(\mathbb{R}^{\dim X}, \mathbb{R}^{\dim Y - \dim W}),$$

whenever $(\mathfrak{s}(\sigma^k), \mathfrak{t}(\sigma^k)) \in V \times U$. Note that p is not generally the map that forgets the last $\dim W$ columns, as the submanifold charts for W and product charts for the factors X_j and Y_j are possibly different. The rank deficient maps in $\operatorname{Hom}(\mathbb{R}^{\dim Y}, \mathbb{R}^{\dim Y - \dim W})$, form a closed sets, and as the local embedding $J^{1_n}(X,Y)|_{V\times U} \cong \prod_{j=1}^n (U_j \times V_j) \times \operatorname{Hom}(\mathbb{R}^{n_j}, \mathbb{R}^{m_j}) \to U \times V \times \operatorname{Hom}(\mathbb{R}^{\dim X}, \mathbb{R}^{\dim Y - \dim W})$ (block diagonal embedding fiberwise) is continuous, it follows that $W^c \cap J^{1_n}(X,Y)|_{V\times U}$ is closed. Hence $\sigma \in W^c$. □

With this knowledge in hand, we can find sufficient conditions for T_W as in theorem 4.14 to be open.

Proposition A.5. *In the context of theorem 4.14, let $\mathfrak{s}$ denote the source map $J^k_{\mathcal{I}}(X,Y) \to Z$. The set of transverse tuples T_W is open if*

(a) W is a product $W_1 \times \ldots \times W_n$ of topologically closed manifolds $W_j \subset J^{k_j}(X_j, Y_j)$, or

(b) $\overline{\mathfrak{s}(W)}$ is compact.

Proof. By lemma A.1, we may assume without loss of generality that $I_j = (j)$. I.e. if we may replace Z_j by $Z'_j := X$ and I_j by (j), leaving all k_j unchanged and recording everything in the new tuple $\mathcal{H}$. We can then conclude that a tuple in $C^\infty_{\mathcal{I},k}(X,Y)$ produces a section of $J^k_{\mathcal{I}}(X,Y)$ transverse to W if and only if the same tuple produces a section $J^k_{\mathcal{H}}(A,X)$ transverse to $\iota_{\mathcal{I}}(W)$. I.e. we use that $C^\infty_{\mathcal{H},k}(X,Y) = C^\infty_{\mathcal{I},k}(X,Y)$. Observe that this does reduction does not influence conditions (a) or (b).

Let $(f_1, \ldots, f_n) \in T_W$. Let $F_j = j^{k_j} f_j$, and let J^1_j be shorthand for $J^1(X_j, J^{k_j}(X_j, Y_j))$. As observed before proposition A.4, for any tuple $g = (g_1, \ldots, g_n) \in C^\infty_{\mathcal{I},k}(X,Y)$, if we denote $G_j = j^{k_j} g_j$ too, then

$$j^k_{\mathcal{I}} g = G_1 \times \ldots \times G_n \pitchfork W \iff \operatorname{im} j^1 G_1 \times \ldots \times j^1 G_n \subset W^{\pitchfork} \subset \prod_{j=1}^n J^1_j.$$

Assume (a). Because W factors, one can observe that $W^c \subset \prod_{j=1}^n J^1_j$ is the union $\bigcup_{j=1}^n W^c_j$, where $\sigma = (\sigma_1, \ldots, \sigma_n) \in W^c_j \iff \mathfrak{t}_j(\sigma_j) \in W_j$ and $\operatorname{im} T_{\mathfrak{s}_j(\sigma_j)} H_j + T_{\mathfrak{t}_j(\sigma_j)} W_j \neq T_{\mathfrak{t}_j(\sigma_j)} J^1(X_j, Y_j)$, where $\mathfrak{s}_j$ and $\mathfrak{t}_j$ are the source and target map of $J^{k_j}(X_j, Y_j) \to X_j \times Y_j$ and $\sigma_j = [H]_{\mathfrak{s}_j(\sigma_j), \mathfrak{t}_j(\sigma)}$. I.e. we can now check the rank deficiency per factor. Note that $W^c_1 = W'_1 \times \prod_{j=2}^n J^1_j$, where W'_1 is a closed set, and the other W^c_j factor in a similar fashion as whole copies of J^1_j and a closed set $W'_j \subset J^1_j$. Closedness of W'_j follows from proposition A.4. It is readily checked that $W^{\pitchfork} = U_1 \times \ldots \times U_n$, where U_j is the complement of W'_j, and that $M(U_1) \times \ldots \times M(U_n)$ is an open neighbourhood of $(j^1 F_1, \ldots, j^1 F_n)$ contained in $\prod_{j=1}^n C^\infty(X_j, J^1_j)$. By the continuity of the maps $j^1 : C^\infty(X_j, J^{k_j}(X_j, Y_j)) \to C^\infty(X_j, J^1_j)$ and $j^{k_j} : C^\infty(X_j, Y_j) \to C^\infty(X_j, J^{k_j}(X_j, Y_j))$ (prop. II.3.4 of [5]), it follows that there exists an open neighbourhood of $(f_1, \ldots, f_n)$ of tuples whose induced section is transverse to W.

Let $(f_1, \ldots, f_n) \in T_W$ and assume (b). Let $\mathfrak{s}' : \prod_{j=1}^n J^1_j \to X$ be the source map, and observe that $\mathfrak{s}'(W^c) = \mathfrak{s}(W)$. It follows that $\overline{\mathfrak{s}'(W^c)}$ is a compact set. Hence corollary 2.11 tells us that the set

$$\{(g_1, \ldots, g_n) \in \prod_{j=1}^n C^\infty(X_j, J^1_j) : \operatorname{im}\ j^1 g_1 \times \ldots \times j^1 g_n \subset W^{\pitchfork}\}$$

is open. In particular we can select an open neighbourhood of $(j^1 F_1, \ldots, j^1 F_n)$ contained in this open set. By the continuity of the maps $j^1 : C^\infty(X_j, J^{k_j}(X_j, Y_j)) \to C^\infty(X_j, J^1_j)$ and $j^{k_j} : C^\infty(X_j, Y_j) \to C^\infty(X_j, J^{k_j}(X_j, Y_j))$, it follows that there exists an open neighbourhood of $(f_1, \ldots, f_n)$ of tuple whose induced section is transverse to W. □

Let W be a submanifold of $J^k_{\mathcal{I}}(X, Y)$, and let W' be any subset of W. We say that a tuple $f = (f_1, \ldots, f_n) \in C^\infty_{\mathcal{I},k}(X, Y)$ is **transverse to** W **on** W' if for all $z \in Z$, either $j^k_{\mathcal{I}} f(z) \notin W'$ or $\operatorname{im} T_z j^k_{\mathcal{I}} f + T_{j^k_{\mathcal{I}} f(z)} W = T_{j^k_{\mathcal{I}} f(z)} J^k_{\mathcal{I}}(X, Y)$. The following can be proved by retracing the steps in the above two proofs.

Lemma A.6. *Let W be a submanifold of $J^k_{\mathcal{I}}(X, Y)$, and let W' be a compact subset of W. Then*

$$T_{W'} := \{f \in C^\infty_{\mathcal{I},k}(X, Y) : j^k_{\mathcal{I}} f \pitchfork W \text{ on } W'\}$$

is an open set.

Proposition A.7. *In the context of theorem 4.18, let $\mathfrak{s}$ denote the source map $J^{k,l}_{\mathcal{I}}(X, Y) \to Z$. The set of transverse tuples T_W is open if*

(a) W is a product $W_1 \times \ldots \times W_{|l|}$ of topologically closed manifolds $W_i \subset J^{k_j}(X_j, Y_j)$ with $|l|_{j-1} < i \leq |l|_j$, or

(b) $\overline{\mathfrak{s}(W)}$ is compact.

Proof. Most of the hard work has been done in the previous proposition: we know that

$$T'_W := \{(f_1, \ldots, f_{|l|}) \in C^\infty_{\mathcal{I},k^l}(X^l, Y^l) : j^{k^l}_{\mathcal{I}} f \pitchfork W\}$$

is an open set. As $C^\infty_{\mathcal{I},k,l}(X, Y)$ embeds into $C^\infty_{\mathcal{I},k^l}(X, Y)$ and under this embedding T_W is mapped into T'_W, the result follows. □

A.2 Proof of mixed transversality

Proof of theorem 4.14. By lemma A.1, it suffices to prove the theorem in the case that X consists of n factors, and that $I_j = (j)$ for each j. To see this, embed W into the full product bundle, with image $\iota_{\mathcal{I}}(W)$, and $j^k_{\mathcal{I}} f \pitchfork W \iff j^{k_1} f_1 \times \ldots \times j^{k_n} f_n \pitchfork \iota_{\mathcal{I}}(W)$. By setting $Z'_j = X_j$, $k'_j = k_j$, $H_j = (j)$, and recording everything in the new tuple $\mathcal{H}$, we can see that the right hand side is the statement $j^{k'}_{\mathcal{H}} f \pitchfork \iota_{\mathcal{I}}(W)$. Note in particular that $C^\infty_{\mathcal{I},k}(X,Y) = C^\infty_{\mathcal{H},k'}(X,Y)$. So we will assume without loss generality that Z consists of n factors and each $I_j = (j)$, and we will simply write X_j for Z_j. Note that $J^k_{\mathcal{H}}(X,Y) = J^k(X,Y)$, too. We split the remainder of the proof in steps (1) through (4), with an eye on the next proof.

(1) We need to show that T_W is a the countable intersection of open dense subsets. We can choose an open covering $\{W_\alpha\}_{\alpha\in\mathcal{J}}$ for some indexing set $\mathcal{J}$, such that

(a) the closure of W_α in $J^k_{\mathcal{I}}(X,Y)$, $\overline{W_\alpha}$, is contained in W,

(b) $\overline{W_\alpha}$ is compact,

(c) there exist coordinate neighbourhoods $U_{\alpha,j}$ in X_j and $V_{\alpha,j}$ in Y_j such that $\pi(W_\alpha) \subset \prod_{j=1}^n U_{\alpha,j} \times V_{\alpha,j}$, where $\pi : J^k(X,Y) \to X \times Y$ is the bundle projection, and

(d) each $\overline{U_{\alpha,j}}$ is compact.

To see this, let $w \in W$. We can select a bundle trivialization (V,ψ) so that $w \in J^k(X,Y)|_V$, $V \subset X\times Y$, and without loss of generality we may assume that $V = \prod_{j=1}^n U_{w,j} \times \prod_{i=1}^n V_{w,i}$, a product of coordinate neighbourhoods $U_{w,j} \subset X_j$, $V_{w,i} \subset Y_i$. By restricting the $U_{w,j}$ if necessary, we may assume that $\overline{U_{w,j}}$ is compact for each j. As W is a submanifold, we can select a chart (U,ϕ) around w in $J^k(X,Y)$ such that W looks like a $\dim W$-dimensional plane in the chart. By restricting the chart if necessary, we can assume without loss of generality that the chart is contained in $J^k(X,Y)|_V$. By another restriction, we can assume that the image of ϕ is an open ball $B(w)$ in $\mathbb{R}^N$, with $N = \dim J^k(X,Y)$, whose closure is compact, and such that the closure of $\phi^{-1}(B(w)\cap\phi(W))$ is contained in W. It is now easy to see that the constructed chart domains satisfy (a)-(d).

(2) As W is second countable, we may extract a countable subcover $W_1, W_2, \ldots$ from the above open cover $\{W_w\}_{w\in W}$. Let

$$T_{W_r} = \{f \in C^\infty_{\mathcal{I},k}(X,Y) : j^k_{\mathcal{I}} f \pitchfork W \text{ on } \overline{W_r}\}.$$

It is clear that $T_W = \bigcap_{r=1}^\infty T_{W_r}$. Thus the proof reduces to showing that each T_{W_r} is open and dense in $C_{\mathcal{I}}(A,X)$.

(3) To see that the set is open, note that we can simply apply lemma A.6.

(4) To prove denseness, choose charts $\psi_j : U_{r,j} \to \mathbb{R}^{n_j}$ and $\eta_{r,j} : V_r \to \mathbb{R}^{m_j}$, and smooth functions $\rho_j : \mathbb{R}^{n_j} \to [0,1]$ and $\rho'_j : \mathbb{R}^{m_j} \to [0,1]$ such that

$$\rho_j = \begin{cases} 1 & \text{on a neighbourhood of } \psi_j \circ \mathfrak{s}_j(\overline{W}_r) \\ 0 & \text{off } \psi_j(U_{r,j}) \end{cases}, \text{ and } \rho'_j = \begin{cases} 1 & \text{on a neighbourhood of } \eta_j \circ \mathfrak{t}_j(\overline{W_r}) \\ 0 & \text{off } \eta_j(V_{r,j}) \end{cases}.$$

Here $n_j = \dim X_j$, $m_j = \dim Y_j$, $\mathfrak{s}_j : J^k(X,Y) \to X_j$ is the j-th source map, and $\mathfrak{t}_j : J^k(X,Y) \to Y_j$ the j-th target map. The choice of ρ_j and ρ'_j are possible since $\overline{W_r}$ is compact.

Let $f \in C^\infty_{\mathcal{I},k}(X,Y)$, and let $\mathcal{U}$ be an open neighbourhood of f. We will show that we can locally perturb f to $\tilde{f}$ such that $j_{\mathcal{I}}\tilde{f} \pitchfork W$ on $\overline{W_r}$ and $\tilde{f} \in \mathcal{U}$, which will be sufficient for denseness. Denote $f = (f_1, \ldots, f_n)$, and for $x \in X$, denote $x = (x_1, \ldots, x_n)$. Let B'_j denote the space of polynomials from $\mathbb{R}^{n_j}$ to $\mathbb{R}^{m_j}$ of degree at most k_j. Define $B' = \prod_{j=1}^n B'_j$. For $b \in B'$, define $g_b = (g_{b_1,1}, \ldots, g_{b_n,n}) \in$

$C^\infty_{\mathcal{I},k}(X,Y)$ by

$$g_{b_j,j}(x_j) = \begin{cases} f_j(x_j) & \text{if } x_j \notin U_{r,j} \text{ or } f_j(x_j) \notin V_{r,j} \\ \eta_j^{-1}(\rho_j(\psi(x_j))\rho'_j(\eta_j f_j(x_j))b_j(\psi_j(x_j)) + \eta_j f_j(x_j)) & \text{otherwise.} \end{cases}$$

The choice of ρ_j and ρ'_j guarantees that g_b is a smooth function. Define $\Phi : X \times B' \to J^k(X,Y)$ by $\Phi(x,b) = j^k_{\mathcal{I}} g_b(x)$. By inspection of the above formula, Φ is smooth. We aim to apply lemma A.2 to Φ, for which we need to show that $\Phi \pitchfork W$ on $\overline{W_r}$. Although this might not be true for all of B', we will select an open neighbourhood B of 0 in B' for which it does hold. Assuming we have found this neighbourhood B, lemma A.2 tells us that for a dense set in B, $j_{\mathcal{I}} g_b \pitchfork \overline{W_r}$. It is a straightforward verification that for values of b near enough 0 in B, $g_b \in \mathcal{U}$ too. Selecting any such b, it follows that $\tilde{f} = g_b \in \mathcal{U}$, and $j^k_{\mathcal{I}}\tilde{f}$ is transverse to W over $\overline{W_r}$. So it remains to find a neighbourhood B of 0 so that $\Phi|_{X\times B}$ is transverse to W over $\overline{W_r}$.

We will show that there exists a neighbourhood B of 0 in B' on which Φ is a local diffeomorphism whenever $\Phi(x,b) \in \overline{W_r}$, and hence trivially transverse to W over $\overline{W_r}$. Let

$$\epsilon = \frac{1}{2}\min\{d(\mathrm{supp}\rho'_j, \mathbb{R}^{m_j} \setminus \eta_j(V_{r,j})), d(\eta_j \mathfrak{t}_j(\overline{W_r}), (\rho'_j)^{-1}[0,1)) : j = 1,\dots,n\},$$

which is positive by the choice of ρ'_j and the compactness of $\eta_j\mathfrak{t}_j(\overline{W_r})$. Define $B = \{b \in B' : |b_j\psi_j(x_j)| < \epsilon, \forall x_j \in \mathrm{supp}\rho_j, \forall j = 1,\dots,n\}$, which is an open neighbourhood of $0 \in B$. Here we used the compactness of each $\overline{U_{r,j}}$ to find that ρ_j is compactly supported. Suppose that $(x,b) \in X \times B$ such that $\Phi(x,b) \in \overline{W_r}$, then $x_j \in \mathfrak{s}_j(\overline{W_r})$ and $g_{b_j,j}(x_j) \in \mathfrak{t}_j(\overline{W_r})$ for each j. It follows for each j that ρ_j is one on a neighbourhood of x_j, and that ρ'_j is 1 on a neighbourhood of $g_{b_j,j}(x_j)$ containing $f_j(x_j)$: for ρ_j this is immediate, and for ρ'_j, it follows from the definition of ϵ and ρ'_j, and that $d(\eta_j(f_j(x_j)), \eta_j(g_{b_j,j}(x_j))) < \epsilon$. To verify this last point, note that

$$|\eta_j(f_j(x_j)) - \eta_j(g_{b_j,j}(x_j))| = |\rho_j\psi_j(x_j)\rho'_j(\eta_j f_j(x_j))b_j\psi_j(x_j)| \le |b_j\psi_j(x_j)| < \epsilon.$$

Because all bump functions are 1 near their respective source and target of both $\Phi(x,b)$ and $f(x)$, it follows that for all b' in a neighbourhood of $b \in B$ and x' in a neighbourhood of $x \in X$,

$$g_{b'_j,j}(x') = \eta_j^{-1}(\eta_j f_j(x') + \eta_j b_{j'}(x')).$$

It is now clear that Φ is a local diffeomorphism near (x,b): let $\sigma \in J^k(X,Y)$, let $x' = \mathfrak{s}(\sigma)$, and let $(b_1,\dots,b_n)$ denote the unique tuple of polynomials in B such that $\sigma = j^k_{\mathcal{I}}(\tilde{f})(x')$, where $\tilde{f}(x') = (\tilde{f}_1,\dots,\tilde{f}_n)(x')$, $\tilde{f}_j(x') = \eta_j^{-1}(b_j\psi_j + \eta_j f_j)$. Then $\sigma \mapsto (x', b_1,\dots,b_n)$ is a smooth map and the inverse to Φ. □

Proof of corollary 4.15. Let $W' \subset W$ be a compact subset: in step (2) of the proof of theorem 4.14, we replace the set T_{W_r} by the set

$$T'_{W_r} := \{f \in C^\infty_{\mathcal{I}}(A,X) : j_{\mathcal{I}} f \pitchfork W \text{ on } \overline{W_r} \cap W'\}.$$

Steps (3) and (4) show that T'_{W_r} is open and dense. As W' is compact, we need finitely many W_r to cover W', and hence $\{f \in C^\infty_{\mathcal{I},k}(X,Y) : j^k_{\mathcal{I}} f \pitchfork W \mathrm{on} W'\} = \cap_{i=1}^N T'_{W_{r_i}}$ is open and dense. For the last claim: in step (1) we can refine our choice of $U_{w,j}$ so that $\prod_{j=1}^n U_{w,j} \subset U$. In step (4) we then perturb f only within U to make it transverse to W. As the set of values in B' was dense, for which a perturbation of f over $\prod_{j=1}^n U_{r,j}$ was transverse to W on $\overline{W_r}$, it follows that we can choose g to lie in $\mathcal{V}$. □

Proof of theorem 4.18. The proof of the theorem is a modification of the above proof.

Recall that $C^\infty_{\mathcal{I},k,l}(X,Y)$ embeds into $C^\infty_{\mathcal{I},k^l}(X^l,Y^l)$. Denote this embedding by Δ^l. By lemma A.1 and that $j^{k,l}_{\mathcal{I}} f \pitchfork W \iff j^k_{\mathcal{I}} \circ \Delta^l f \pitchfork W$, it suffices to prove the theorem in the case that Z consists of n factors, $Z_j = X_j$, and that $I_j = (j)$. I.e. as before one can construct Z' and $\mathcal{H}$. We continue with the proof by following steps (1)-(4) of the previous proof, highlighting changes/additions to each step.

(1) We choose an open cover $\{W_w\}$ as before, indexed by the set all w in W whose source does not lie in $\Delta^l(X) \subset X^l$, which is possible by assumption. By restricting the $U_{w,i}$ if necessary, we may assume that W_w satisfies properties (a)-(d), and in addition (e): for every $1 \leq q \leq m$ and $|l|_{q-1} < i < j \leq |l|_q$, $U_{w,i} \cap U_{w,j} = \varnothing$.

(2) We can extract a finite subcover $W_1, W_2, \ldots$ that covers W. For each T_{W_r}, we define

$$T_{W_r} := \{f \in C^\infty_{\mathcal{I},k,l}(X,Y) : j^{k,l}_{\mathcal{I}} f \pitchfork W \text{ on } \overline{W_r}\}.$$

As in the previous proof, it follows that $T_W = \bigcap_{r=1}^\infty W_r$. Thus the current proof reduces to showing that each T_{W_r} is open and dense in $C^\infty_{\mathcal{I},k,l}(X,Y)$.

(3) Openness follows from lemma A.6 and the fact that the embedding of $C^\infty_{\mathcal{I},k,l}(X,Y)$ into $C^\infty_{\mathcal{I},k^l}(X^l,Y^l)$ is continuous, and that this embedding maps a tuple f whose induced section $j^{k,l}_{\mathcal{I}}$ is transverse to W, to a new tuple f' whose induced section $j^{k^l}_{\mathcal{I}} f'$ is transverse to W.

(4) To prove density for the sets T_{W_r}, we may simply apply the same argument as before. Note that by our choice of disjoint $U_{w,i}, U_{w,j} \subset X_j$ whenever $|l|_{q-1} \leq i < j \leq |l|_q$, the local perturbation may still be performed per each open. I.e. because the open neighbourhoods are disjoint, we can modify the same function multiple times (so that the resulting functions lies in the image of Δ^l). This finishes the proof. $\square$

Proof of corollary 4.20. This is essentially the same as the proof of corollary 4.15. $\square$

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