Universiteit Utrecht

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science

at the

Department of Mathematics

Characterization of Lower Semicontinuity and Relaxation of Fractional Integral and Supremal Functionals

Author:
Hidde Schönberger

Supervisor:
Carolin Kreisbeck
Second reader:
Paul Zegeling

July 16, 2021

Universiteit Utrecht

Abstract

In light of recent developments regarding fractional function spaces, new avenues have opened up for studying fractional variational problems and partial differential equations. We take the viewpoint of the calculus of variations by investigating the minimization of nonlocal integral and supremal functionals depending on the Riesz fractional gradient. With the aim of establishing the existence of minimizers, we give full characterizations of the lower semicontinuity of these fractional functionals. Interestingly, the characterizations are in terms of notions intrinsic to variational problems involving classical gradients, that is, quasiconvexity and level-quasiconvexity. The key ingredient in the proofs is an inherent connection between classical and fractional gradients, which we extend to Sobolev functions, enabling us to transition between the two settings.

In the absence of lower semicontinuity, we determine representation formulas for the relaxations, i.e. lower semicontinuous envelopes, of the fractional integral and supremal functionals. They are obtained by taking the relevant convex hulls of the integrand and supremand, but only inside a prescribed region. As such, we observe that, unlike in the classical case, the integrand and supremand change structure through the relaxation process, going from homogeneous to inhomogeneous. Finally, to draw the connection between the integral and supremal case, we present an L^p-approximation result showing the Γ-convergence of the nonlocal integral functionals to their supremal counterpart.

Acknowledgements

I would like to thank my supervisor Carolin Kreisbeck for providing me with a fascinating topic for my thesis and for supporting me throughout. In particular, the large amount of time she spent helping me write my first research article was very valuable to me. Besides, I would like to thank Paul Zegeling for being the second reader and being especially kind and patient during the process. My appreciation also goes out to Javier Cueto for the interesting discussions we had about the Riesz fractional gradient.

Last but not least, I wish to thank my family and friends for always being there for me.

Contents

Chapter 1

Introduction

The calculus of variations is a branch of mathematics that is mostly concerned with the minimization of functionals of the form

$$\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$$

over some infinite-dimensional function space X. The field possesses a rich history dating back to the 17th century with Fermat's principle of least time, Bernoulli's brachistochrone problem, and the subsequent contributions of Euler and Lagrange [41, 73]. During this time, the focus lay on deriving necessary conditions for minimizers, which changed when Weierstrass provided the first example in 1870 of an integral functional without minimizers [75]. In reaction to this discovery, Hilbert posed his 20th problem in 1902 pertaining the existence of minimizers for integral functionals [42], which shaped the field in the following decades. The contributions led to the development of the direct method, which lies at the heart of the modern calculus of variations.

This powerful method establishes the existence of minimizers for general functionals if they are coercive and lower semicontinuous with respect to a suitable topology, see Section 1.1.4 for details. Coercivity is a condition closely related to compactness properties of bounded sequences, and, in infinite-dimensional spaces, this can only be guaranteed for a topology weaker than the one induced by the norm. As such, we also have to verify the lower semicontinuity of the functional with respect to this weak topology, which is often the hardest and most crucial step. In the situation where the functional is not lower semicontinuous, minimizers may fail to exist, and one often resorts to relaxation methods to deduce information about the asymptotic behavior of minimizing sequences.

The most well-known class of functionals are the integral functionals of the form

$$\mathcal{I}(v) = \int_\Omega f(x, v(x), \nabla v(x))\, dx \quad \text{for } v \in g + W^{1,p}_0(\Omega; \mathbb{R}^m), \tag{1.1}$$

where $p \in (1, \infty)$, $\Omega \subset \mathbb{R}^n$ is open and bounded, $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ is a suitable integrand, and the boundary condition g lies in the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$. Variational problems involving (1.1) enjoy a vast number of applications; for example, the classical brachistochrone and isoperimetric problem, applications in physics such as electrostatics, quantum mechanics and hyperelasticity, and uses in economics in the form of optimal saving, see e.g. [64, Chapter 1].

In order to establish the existence of minimizers of $\mathcal{I}$ via the direct method, it is key to characterize the lower semicontinuity of $\mathcal{I}$ with respect to the weak convergence in $W^{1,p}(\Omega;\mathbb{R}^m)$. This is a classical issue, and has been resolved by the introduction of Morrey's celebrated notion of quasiconvexity in 1952 [56]; it turns out that the weak lower semicontinuity of $\mathcal{I}$ is equivalent to f being quasiconvex in its third variable [2, 56]. Moreover, in the absence of quasiconvexity, the relaxation of $\mathcal{I}$, i.e. its weakly lower semicontinuous envelope, is found via quasiconvexification of f [28].

A different class of functionals that has received attention in the last two decades are the supremal functionals of the form

$$\mathcal{S}(v) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, v(x), \nabla v(x)) \quad \text{for } v \in g + W_0^{1,\infty}(\Omega;\mathbb{R}^m), \tag{1.2}$$

where now $g \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. These functionals enable one to minimize pointwise quantities as opposed to their average and are therefore more natural in situations where the best- or worst-case scenario is of importance. Applications of these supremal functionals include, among others, the study of dielectric breakdown [38], polycrystals [1, 18], optical tomography [46], machine learning [36] and imaging [22]. In this setting, the existence of minimizers is closely related to the lower semicontinuity of $\mathcal{S}$ with respect to the weak* convergence in $W^{1,\infty}(\Omega;\mathbb{R}^m)$. It was shown by Barron, Jensen & Wang in [13] that this can be characterized in terms of level-quasiconvexity [1] in the third variable of the supremand f. We study their proof in the thesis, and also delve into other aspects of supremal functionals like Aronsson equations [6], relaxation [60] and L^p-approximation [61].

Besides a review of established literature, this thesis focuses on expanding the theory around a certain class of nonlocal functionals. Nonlocal aspects include any phenomena where points or objects at a distance can influence each other. They have recently sparked interest due to their ability to incorporate global effects and long-range interactions, and are for example useful in applications of peridynamics [53, 70], new approaches to hyperelasticity [15], imaging [9, 40] and machine learning [5, 43]. From a mathematical perspective, nonlocal effects also provide interesting challenges that require novel techniques to overcome.

The type of nonlocality that we will consider arises through the use of fractional derivatives. While fractional partial differential equations have been studied intensively for many years, developing a theory around the fractional calculus of variations in multiple dimensions is a more recent undertaking [15, 68, 69]. This is in contrast to the one-dimensional case, see e.g. [52] and the references therein, which is well-established. The reason for this is because, until recently, a good notion of a multidimensional fractional gradient has largely been missing in the literature. This has been resolved by the introduction of the Riesz fractional gradient by Shieh & Spector in [68], which for $\alpha \in (0,1)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ is defined as

$$\nabla^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{\varphi(y) - \varphi(x)}{|y-x|^{n+\alpha}} \frac{y-x}{|y-x|}\, dy \quad \text{for } x \in \mathbb{R}^n,$$

with a specific real constant $\mu_{n,\alpha}$. It was shown by Šilhavý in [74] that up to a multiplicative constant, the Riesz fractional gradient is the unique rotation- and translation-invariant α-homogeneous operator. In this sense, it can be viewed as the canonical fractional derivative. For more context on the Riesz fractional gradient, we mention the recent works [32, 51, 54, 67, 71].

[1] In [13] this notion is called strong Morrey quasiconvexity.

By extending the definition of the fractional gradient to functions in $L^p(\mathbb{R}^n;\mathbb{R}^m)$ in a distributional sense, one obtains fractional Sobolev spaces defined for $p \in [1,\infty]$ by

$$S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m) = \{u \in L^p(\mathbb{R}^n;\mathbb{R}^m) \mid \nabla^\alpha u \in L^p(\mathbb{R}^n;\mathbb{R}^{m\times n})\}.$$

Due to the results from Shieh & Spector in [68, 69], and the series of papers by Comi & Stefani and co-authors [20, 24, 25], we know that these spaces possess useful properties like Poincaré-type inequalities, compactness results and density of smooth functions with compact support. As a consequence, the fractional Sobolev spaces form a natural setting for the study of variational problems. The most obvious candidate emerges by replacing the gradient in (1.1) by the Riesz fractional gradient and adjusting the local boundary-value condition to a complementary-value condition. At the same time, we are also interested in the fractional analogue of the supremal functionals in (1.2).

Explicitly, the fractional integral functionals that we consider are of the form

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(x, u(x), \nabla^\alpha u(x))\, dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m), \tag{1.3}$$

where $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ is open and bounded, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a suitable integrand and $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$; the complementary-value space is defined as

$$S^{\alpha,p}_g(\Omega;\mathbb{R}^m) = \{u \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m) \mid u = g \text{ a.e. in } \Omega^c\}.$$

In parallel, we also consider the supremal counterpart

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(x, u, \nabla^\alpha u) \quad \text{for } u \in S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m), \tag{1.4}$$

defined on the complementary-value space with exponent $p = \infty$. Proving the existence of minimizers of the functionals in (1.3) and (1.4) is an essential task, and this relies on the lower semicontinuity of $\mathcal{I}_\alpha$ and $\mathcal{S}_\alpha$ with regard to the weak and weak* convergence in $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ and $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$, respectively. As of yet, only the fractional integral functionals have been studied, and it is known that convexity [68, 69] or polyconvexity [15] in the third argument of f are sufficient conditions for the weak lower semicontinuity of the fractional integral functionals.

Contribution of the thesis. In this thesis, we will extend these results by providing a full characterization of the weak lower semicontinuity of $\mathcal{I}_\alpha$ and the weak* lower semicontinuity of $\mathcal{S}_\alpha$. They are stated in Theorem 3.2.6 and Theorem 3.3.4, respectively. There are two notable aspects about these results that we elaborate on.

Firstly, we identify that the weak lower semicontinuity of $\mathcal{I}_\alpha$ can be characterized in terms of quasiconvexity in the third argument of the integrand; in a similar spirit, the weak* lower semicontinuity of $\mathcal{S}_\alpha$ is reduced to level-quasiconvexity of the supremand in its third variable. As such, we obtain that the characterizing conditions are independent of α and rely on the notions intrinsic to the classical setting. This might be somewhat surprising, and to understand this better we also introduce a natural fractional analogue of quasiconvexity, which we call α-quasiconvexity, see Definition 3.2.8. We show that this notion actually coincides with quasiconvexity for any $\alpha \in (0,1)$, which makes it possible to characterize the lower semicontinuity in terms of α-quasiconvexity as well.

Secondly, we point out that, although the functions are defined on the full space $\mathbb{R}^n$, the assumption of quasiconvexity or level-quasiconvexity is only imposed for $x \in \Omega$. This fact is a consequence of a somewhat unexpected property of the complementary-value spaces; namely, for a sequence converging weakly (weak* if $p = \infty$) in the complementary-value space, we actually have that the fractional gradients converge strongly outside Ω, see Lemma 3.1.24. Since, in the presence of strong convergence, no convexity assumption is required for lower semicontinuity, we understand that the conditions are only inside Ω.

The difficulty in proving these results arises from the inherent nonlocal structure of the fractional gradient, and, to a lesser extent, from working on the whole space $\mathbb{R}^n$ instead of a bounded domain. The overall approach for overcoming this is inspired by the known identities involving the Riesz potential I_α and fractional Laplacian $(-\Delta)^{\alpha/2}$, see Section 3.1.1 for their definition. Explicitly, for $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\nabla^\alpha \varphi = \nabla I_{1-\alpha}\varphi \qquad \text{and} \qquad \nabla\varphi = \nabla^\alpha (-\Delta)^{\frac{1-\alpha}{2}}\varphi. \tag{1.5}$$

These identities tell us that we can express the fractional gradient of a function by the gradient of another function and vice versa. By extending (1.5) to the (fractional) Sobolev spaces, cf. Proposition 3.1.30, we are able to switch between the fractional and classical setting, and thus carry over the well-established theory involving ordinary gradients.

However, as the connection from (1.5) goes through nonlocal operators, it does not preserve complementary values. As a consequence, we need to enforce the complementary values using cut-off techniques, and carefully estimate the errors that it induces. As a matter of fact, simply multiplying with cut-off functions causes errors that are too large, so we will need more sophisticated methods depending on the given situation. One important ingredient is the construction of functions with compact support whose fractional gradient attains a prescribed value at a point, Lemma 3.2.3.

Besides the characterization of weak and weak* lower semicontinuity, we are also interested in obtaining representation formulas for the relaxations, i.e. lower semicontinuous envelopes, of $\mathcal{I}_\alpha$ and $\mathcal{S}_\alpha$. They can be found in Theorem 3.2.15 and Theorem3.3.9. These appear to be the first known relaxation results for a class of fractional functionals. Their formulas arise from taking the relevant convex hulls of the integrand and supremand, but only inside Ω. This is due to the property of complementary-value spaces, where we have the strong convergence of gradients outside Ω. As such, we see that starting from an integrand or supremand that does not depend on x, we obtain an inhomogeneous integrand or supremand through the relaxation process. Note that the supremal relaxation formula is only known in the scalar case $m = 1$, similarly to the classical setting. Lastly, we also prove an L^p-approximation result in Theorem 3.4.2, which connects integral functionals of the form $\mathcal{I}_\alpha$ with their supremal version $\mathcal{S}_\alpha$. This uses the language of Γ-convergence, cf. Section 1.1.5, and is inspired by the proof by Champion, de Pascale & Prinari [23] from the classical setting.

The results regarding the integral functionals $\mathcal{I}_\alpha$ can already be found in the preprint article [49]:

Quasiconvexity in the fractional calculus of variations: Characterization of lower semicontinuity and relaxation.
Carolin Kreisbeck and Hidde Schönberger.

Parts of the introduction and Sections 3.1 and 3.2 of the thesis are based on this article.

Outline of the thesis. The thesis is organized as follows. In the remainder of Chapter 1, we discuss the necessary preliminaries consisting of results and techniques regarding Sobolev spaces and the calculus of variations. In Chapter 2, we conduct a study of classical supremal functionals as in (1.2). This starts with characterizing the weak* lower semicontinuity and subsequently moves towards more specialized topics. In Chapter 3, we initiate the study of the fractional calculus of variations involving the Riesz fractional gradient. We begin by introducing tools on the fractional Sobolev spaces, most notably being the extension of the identities from (1.5) to the Sobolev setting in Section 3.1.5. Then, we are in the position to study the fractional integral and supremal functionals in Sections 3.2 and 3.3, where we prove the lower semicontinuity, existence and relaxation results. We finish with the L^p-approximation result and an outlook on two possible applications related to imaging and hyperelasticity.

1.1 Preliminaries

In this section we introduce the notation and some prerequisite definitions and results.

1.1.1 Notation

We write $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ and for $t \in \mathbb{R}$, we write $\lfloor t \rfloor$ for the largest integer smaller or equal to t. We denote the Euclidean norm of a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by $|x| = \big(\sum_{i=1}^n x_i^2\big)^{1/2}$ and similarly, the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ by $|A|$. The ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ is written as $B_r(x) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$. For $E \subset \mathbb{R}^n$, we indicate its complement as $E^c := \mathbb{R}^n \setminus E$, its closure as $\overline{E}$, its interior as $\operatorname{int}(E)$, its boundary as ∂E and its convex hull as $\operatorname{Conv}(E)$. The notation $E \Subset F$ for sets $E, F \subset \mathbb{R}^n$ means that E is compactly contained in F, i.e., $\overline{E} \subset F$ and $\overline{E}$ is compact. We also write $\mathrm{d}(x, E)$ for the distance of a point $x \in \mathbb{R}^n$ to E. Let

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{for } x \in E, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } x \in \mathbb{R}^n,$$

be the indicator function of a set $E \subset \mathbb{R}^n$. Moreover, Γ stands for Euler's gamma function. For functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^p \to \mathbb{R}^n$ their composition is denoted by $f \circ g$.

Let $U \subset \mathbb{R}^n$, then by $C(U)$ we denote the continuous functions from U to $\mathbb{R}$. Assume in the following that U is also open, then the space $C_c^\infty(U)$ symbolizes the smooth functions $\varphi : U \to \mathbb{R}$ with compact support in $U \subset \mathbb{R}^n$, and we write $\operatorname{supp}(\varphi)$ for their support. We use the convention that functions in $C_c^\infty(U)$ are identified with their trivial extension to $\mathbb{R}^n$ by zero. By $C^k(U)$ we denote the class of k-times continuously differentiable functions on U. Further, let $C^\infty(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$ be the spaces of smooth functions on $\mathbb{R}^n$ and continuous functions on $\mathbb{R}^n$ vanishing at infinity, respectively.

By $\mathrm{Lip}(U)$ and $\mathrm{Lip}_b(\mathbb{R}^n)$, we refer to the Lipschitz functions on U and the bounded Lipschitz functions on $\mathbb{R}^n$. We write $\mathrm{Lip}(\psi)$ for the Lipschitz constant of ψ. The space $C^{0,\beta}(\mathbb{R}^n)$ with $\beta \in (0,1]$ consists of all real-valued β-Hölder continuous functions defined on $\mathbb{R}^n$.

The Lebesgue measure of $U \subset \mathbb{R}^n$ is denoted by $|U|$. We use $\mathcal{M}(\mathbb{R}^m)$ and $\mathcal{P}r(\mathbb{R}^m)$ to denote the space of finite signed Borel measures and the set of probability measures on $\mathbb{R}^m$, and for a $\mu \in \mathcal{M}(\mathbb{R}^m)$ we write $\mathrm{supp}(\mu)$ for its support. The product of two measures μ, ν is denoted by $\mu \otimes \nu$. We use the standard notation for Lebesgue spaces, that is, $L^p(U)$ for $p \in [1,\infty]$ is the space of real-valued p-integrable functions on U with the norm

$$\|u\|_{L^p(U)} = \begin{cases} \left(\int_U |u(x)|^p\,dx\right)^{1/p} & \text{if } p \in [1,\infty), \\ \operatorname{ess\,sup}_{x\in U} |u(x)| & \text{if } p = \infty, \end{cases} \quad \text{for } u \in L^p(U);$$

for brevity, we write $\|u\|_{L^p(\mathbb{R}^n)} = \|u\|_p$ when $U = \mathbb{R}^n$.

The spaces of functions that are locally in $L^p(\mathbb{R}^n)$ are denoted by $L^p_{loc}(\mathbb{R}^n)$. Furthermore, $p' \in [1,\infty]$ stands for the dual exponent of p, i.e. $1/p + 1/p' = 1$, and we recall that a sequence $(v_j)_j \subset L^p(U)$ is called p-equi-integrable if $(|v_j|^p)_j$ is equi-integrable.

In general, the definitions above can be extended componentwise to spaces of vector-valued functions. Our notation then explicitly mentions the target space, like, for example, $L^p(U;\mathbb{R}^m)$ consists of all functions $u : U \to \mathbb{R}^m$ whose individual components lie in $L^p(U)$. We do not specify this target space in the norm. Additionally, we set $Q = (0,1)^n$ and denote spaces of functions that are Q-periodic by using the subscript per, e.g. $C^\infty_{per}(Q)$ denotes the Q-periodic functions in $C^\infty(\mathbb{R}^n)$.

Finally, we use C to denote a generic constant, which may change from one estimate to the next without further mention. Whenever we wish to indicate the dependence of C on certain quantities, we add them in brackets.

1.1.2 Weak Convergence in L^p

We present the definition and properties of weak convergence in L^p and weak* convergence in L^∞.

Definition 1.1.1. Let $1 \le p < \infty$, $\Omega \subset \mathbb{R}^n$ open and $(u_j)_j \subset L^p(\Omega)$. Then we say that $(u_j)_j$ converges weakly to $u \in L^p(\Omega)$, denoted by $u_j \rightharpoonup u$, if

$$\lim_{j\to\infty} \int_\Omega u_j \varphi\,dx = \int_\Omega u\varphi\,dx \quad \text{for all } \varphi \in L^{p'}(\Omega).$$

When $p = \infty$ we say that $(u_j)_j \subset L^\infty(\Omega)$ converges weak* to $u \in L^\infty(\Omega)$, and write $u_j \overset{*}{\rightharpoonup} u$, if

$$\lim_{j\to\infty} \int_\Omega u_j \varphi\,dx = \int_\Omega u\varphi\,dx \quad \text{for all } \varphi \in L^1(\Omega). \qquad \triangle$$

We note that any weak(*) convergent sequence is bounded in $L^p(\Omega)$. Conversely, due to the reflexivity for $p \in (1,\infty)$ and the Banach-Alaoglu theorem for $p = \infty$, we actually have for any bounded sequence $(u_j)_j \subset L^p(\Omega)$ (with $p \in (1,\infty]$) a subsequence that converges weak(*) to some $u \in L^p(\Omega)$.

There is one explicit instance of weak* limits that we use, which concerns highly oscillating sequences. This is a well-known fact, which is proven in e.g. [37, Lemma 2.85]. Recall that $Q = (0,1)^n$ and we denote by $L^\infty_{per}(Q)$ the functions in $L^\infty(\mathbb{R}^n)$ that are Q-periodic.

Proposition 1.1.2. *Let $u \in L^\infty_{per}(Q)$ and define the oscillating sequence $(u_j)_j \subset L^\infty(\mathbb{R}^n)$ via $u_j(x) = u(jx)$ for $j \in \mathbb{N}$. Then, the sequence $(u_j)_j$ converges weak* to the average of u over Q, i.e.*

$$u_j \overset{*}{\rightharpoonup} \int_Q u(y)\,dy \quad \text{in } L^\infty(\mathbb{R}^n) \text{ as } j \to \infty.$$

1.1.3 Sobolev Spaces

Here, we go over some of the properties of the Sobolev spaces that we need in the thesis. For more on this topic, see e.g. [34, 35]. Let us start with the definition of the weak gradient.

Definition 1.1.3. Let $\Omega \subset \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$. Then $w \in L^1_{loc}(\Omega;\mathbb{R}^n)$ is called the weak gradient of u if

$$\int_\Omega w\varphi\,dx = -\int_\Omega u\nabla\varphi\,dx \qquad \text{for all } \varphi \in C^\infty_c(\Omega).$$

We write $w = \nabla u$. △

When u is vector-valued, i.e. $u \in L^1_{loc}(\Omega;\mathbb{R}^m)$, then we view ∇u as an element of $L^1_{loc}(\Omega;\mathbb{R}^{m\times n})$. The weak gradient is unique a.e. and it allows us to define the Sobolev spaces.

Definition 1.1.4. Let $\Omega \subset \mathbb{R}^n$ open and $p \in [1,\infty]$. We define the Sobolev space $W^{1,p}(\Omega)$ as the collection of functions in $L^p(\Omega)$ that have a weak gradient lying in $L^p(\Omega;\mathbb{R}^n)$, i.e.

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \nabla u \in L^p(\Omega;\mathbb{R}^n)\}.$$

This space is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

We denote by $W^{1,p}(\Omega;\mathbb{R}^m)$ the vector-valued analogue. △

These spaces are Banach spaces and are reflexive for $p \in (1,\infty)$. By $W^{1,p}_{loc}(\Omega)$ we denote the functions $u : \Omega \to \mathbb{R}$ such that $u \in W^{1,p}(O)$ for all $O \Subset \Omega$. Furthermore, we can define the Sobolev spaces with boundary condition.

Definition 1.1.5. Let Ω be open and $p \in [1,\infty)$ then we define the space with zero boundary values $W^{1,p}_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. When $p = \infty$ we define

$$W^{1,\infty}_0(\Omega) = W^{1,\infty}(\Omega) \cap W^{1,1}_0(\Omega).$$

Additionally, for $g \in W^{1,p}(\Omega)$ we define the affine boundary-value space

$$W^{1,p}_g(\Omega) = g + W^{1,p}_0(\Omega).$$ △

We also recall the weak and weak* convergence on the Sobolev spaces.

Definition 1.1.6. Let $\Omega \subset \mathbb{R}^n$ open then we say that a sequence $(u_j)_j \subset W^{1,p}(\Omega)$ converges weakly to $u \in W^{1,p}(\Omega)$ (weak* if $p = \infty$) if

$$u_j \rightharpoonup u \text{ in } L^p(\Omega) \text{ and } \nabla u_j \rightharpoonup \nabla u \text{ in } L^p(\Omega; \mathbb{R}^n) \ (\overset{*}{\rightharpoonup} \text{ if } p = \infty). \qquad \triangle$$

We note that any weak(*) convergent sequence is bounded in $W^{1,p}(\Omega)$. Conversely, due to the reflexivity for $p \in (1, \infty)$ and the Banach-Alaoglu theorem for $p = \infty$, we actually have for any bounded sequence $(u_j)_j \subset W^{1,p}(\Omega)$ (with $p \in (1, \infty]$) a subsequence that converges weak(*) to some $u \in W^{1,p}(\Omega)$. This compactness property for bounded sequences is the main reason why we use the Sobolev spaces. Next, we mention the Poincaré inequality, [34, §5.6.1 Theorem 3 & §5.8.1 Theorem 1], which gives a bound on the Sobolev norm using only the norm of the gradient.

Theorem 1.1.7. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Then, there is a constant $C = C(\Omega, n, p) > 0$ such that for every $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

The same result holds true for $u \in W^{1,p}(\Omega)$ with zero average if $\partial\Omega$ is Lipschitz.

Now we give some more insight on $W^{1,\infty}(\Omega)$. This space turns out to correspond with the space of Lipschitz functions, cf. [34, §5.8.2 Theorem 4]; strictly speaking, each $u \in W^{1,\infty}(\Omega)$ has a Lipschitz continuous representative, but we simply identify u with this Lipschitz continuous function without further mention.

Theorem 1.1.8. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. Then, $W^{1,\infty}(\Omega) = \mathrm{Lip}(\Omega)$ and for any $u \in W^{1,\infty}(\Omega)$ it holds that $\|\nabla u\|_{L^\infty(\Omega)} = \mathrm{Lip}(u)$.*

As a consequence of the previous theorem, any bounded sequence in $W^{1,\infty}(\Omega)$ is equi-continuous so that the Arzelà-Ascoli theorem implies the following.

Proposition 1.1.9. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary and $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$. Then,*

$$u_j \to u \text{ in } C(\overline{\Omega}).$$

Lastly, there is a result stating the differentiability of Lipschitz functions from [34, §5.8.3 Theorem 5 & 6], widely known as Rademacher's theorem.

Theorem 1.1.10. *Let $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,\infty}_{loc}(\Omega)$, i.e. u is locally Lipschitz continuous. Then, u is differentiable a.e. in Ω, and its gradient equals its weak gradient almost everywhere.*

1.1.4 Calculus of Variations

We assume that the reader is familiar with the modern methods in the calculus of variations, in particular, the existence theory for minimizers of integral functionals. Here, we mention some key results for the sake of completeness, see e.g. [29, 64] for more details.

We first recall the direct method in the calculus of variations, which can establish the existence of minimizers for abstract functionals, see e.g. [64, Theorem 2.1].

Theorem 1.1.11 (Direct method)**.** *Let X be a topological space and $\mathcal{F} : X \to \mathbb{R}_\infty$. Assume that $\mathcal{F}$ satisfies:*

(i) ***Coercivity:*** *For any sequence $(u_j)_j \subset X$ such that $\lim_{j\to\infty} \mathcal{F}(u_j) < \infty$, there exists a subsequence that converges to some $u \in X$.*

(ii) ***Lower semicontinuity:*** *For any sequence $(u_j)_j \subset X$ converging to $u \in X$ it holds that*

$$\mathcal{F}(u) \le \liminf_{j\to\infty} \mathcal{F}(u_j).$$

Then, there exists a minimizer $u_0 \in X$ of $\mathcal{F}$, i.e. $\mathcal{F}(u_0) = \inf_{u\in X} \mathcal{F}(u)$.

The main way to verify the coercivity condition is to show that any sequence $(u_j)_j \subset X$ with $\lim_{j\to\infty} \mathcal{F}(u_j) < \infty$ is contained in a compact set, and thus has a convergent subsequence. However, for infinite-dimensional normed spaces X, closed and bounded sets are in general not compact, in fact, the closed unit ball is never compact [66, Theorem 2.26]. As such, it becomes infeasible to show coercivity with respect to the norm topology, which is why we need to consider a weaker topology that does have suitable compactness properties, for example, the weak convergence defined in Definition 1.1.6. In turn, this means that the lower semicontinuity also needs to be proven with respect to this topology, which is usually very difficult.

We now move to the specific case of integral functionals of the form

$$\mathcal{I}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \quad \text{for } u \in W^{1,p}(\Omega; \mathbb{R}^m),$$

with $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ a suitable integrand. Two classes of integrands that play a major role are the normal and Carathéodory functions.

Definition 1.1.12.

(i) A function $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n} \to \mathbb{R}$ is a normal function if f is Borel measurable and $(z, A) \mapsto f(x, z, A)$ is lower semicontinuous for a.e. $x \in \Omega$.

(ii) A function $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function if $x \mapsto f(x, z, A)$ is measurable for all $(z, A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}$ and $(z, A) \mapsto f(x, z, A)$ is continuous for a.e. $x \in \Omega$. △

In order to find minimizers of the integral functionals via the direct method, a crucial property is the weak lower semicontinuity of $\mathcal{I}$. It is well-known that this is equivalent to quasiconvexity (introduced by Morrey [56]) in the third argument of f, see Theorem 1.1.14 below.

Definition 1.1.13. A Borel measurable function $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called quasiconvex if

$$h(A) \le \int_Q h(A + \nabla\varphi(y))\, dy \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in W^{1,\infty}_{per}(Q; \mathbb{R}^m),$$

where $Q = (0, 1)^n$. △

The weak lower semicontinuity can be characterized as follows, see [29, Theorem 8.1 and 8.11] and [2, Statement II.5].

Theorem 1.1.14. *Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ open and bounded. Suppose that $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory integrand that satisfies*

$$0 \leq f(x,z,A) \leq a(x) + C(|z|^p + |A|^p) \quad \textit{for a.e. } x \in \Omega \textit{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $a \in L^1(\mathbb{R}^n)$ and $C > 0$. Then, the functional

$$\mathcal{I}(u) = \int_\Omega f(x,u(x),\nabla u(x))\,dx \quad \textit{for } u \in W^{1,p}(\Omega;\mathbb{R}^m),$$

is (sequentially) weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if $A \mapsto f(x,z,A)$ is quasiconvex for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^m$.

There exist other convexity notions that help to understand quasiconvexity better. We denote by $T : \mathbb{R}^{m\times n} \to \mathbb{R}^{\tau(m,n)}$ the map consisting of all the minors, i.e. determinants of submatrices, where $\tau(m,n)$ denotes the number of minors of an $m \times n$-matrix.

Definition 1.1.15.

(i) A function $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called polyconvex if there is a convex function $H : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$ such that $h = H \circ T$.

(ii) A function $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called rank-1-convex if for every $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A-B) \leq 1$ and $\lambda \in [0,1]$ it holds that

$$h(\lambda A + (1-\lambda)B) \leq \lambda h(A) + (1-\lambda)h(B). \qquad \triangle$$

The following illuminates the relations between the notions, see [29, Theorem 5.3].

Theorem 1.1.16. *Let $h : \mathbb{R}^{m\times n} \to \mathbb{R}$. Then, the following implications hold:*

$$h \textit{ convex} \;\Rightarrow\; h \textit{ polyconvex} \;\Rightarrow\; h \textit{ quasiconvex} \;\Rightarrow\; h \textit{ rank-1-convex}.$$

In particular, if $m = 1$ or $n = 1$ all notions coincide.

In the case when the functional $\mathcal{I}$ is not weakly lower semicontinuous we can consider its relaxation, i.e. its weakly lower semicontinuous envelope. This is related to the quasiconvex envelope of a function, which, for $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, is defined by

$$f^{\mathrm{qc}}(A) := \inf\{h(A) \mid h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex, } h \leq f\} \quad \text{for } A \in \mathbb{R}^{m\times n}.$$

We have the following relaxation result from e.g. [64, Theorem 7.6], [29, Theorem 9.1].

Theorem 1.1.17. *Let $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ be open and bounded, $g \in W^{1,p}(\Omega;\mathbb{R}^m)$ and*

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} f(\nabla u(x))\,dx \quad \textit{for } u \in W^{1,p}_g(\Omega;\mathbb{R}^m),$$

where $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is continuous and satisfies for* $c, C > 0$

$$c|A|^p \le f(A) \le C|A|^p \quad \text{for all } A \in \mathbb{R}^{m\times n}.$$

Then the relaxation of $\mathcal{I}$ *with respect to the weak convergence in* $W^{1,p}_g(\Omega;\mathbb{R}^m)$ *is given by*

$$\begin{aligned}\mathcal{I}^{\mathrm{rel}}(u) &:= \inf\{\liminf_{j\to\infty} \mathcal{I}(u_j) \mid u_j \rightharpoonup u \text{ in } W^{1,p}_g(\Omega;\mathbb{R}^m)\} \\ &= \int_\Omega f^{\mathrm{qc}}(\nabla u(x))\, dx \quad \text{for } u \in W^{1,p}_g(\Omega;\mathbb{R}^m).\end{aligned}$$

1.1.5 Γ-Convergence

In this section, we briefly recall the definition of Γ-convergence, which is a type of convergence for functionals on metric spaces. It is of particular interest for minimization problems, as the Γ-convergence encodes information about minimizers, see Theorem 1.1.19 below. For more on this topic, see e.g. [19, 31].

Definition 1.1.18 (Γ-convergence)**.** Let X be a metric space and $\mathcal{F}_j : X \to \mathbb{R}_\infty$ for $j \in \mathbb{N}$ a sequence of functionals. We say that $\mathcal{F}_j$ Γ-converges to $\mathcal{F} : X \to \mathbb{R}_\infty$, and write $\mathcal{F} = \Gamma\text{-}\lim_{j\to\infty} \mathcal{F}_j$, if:

(i) For every sequence $u_j \to u$ in X we have

$$\mathcal{F}(u) \le \liminf_{j\to\infty} \mathcal{F}_j(u_j).$$

(ii) There exists a sequence $u_j \to u$ in X such that

$$\mathcal{F}(u) = \lim_{j\to\infty} \mathcal{F}_j(u_j). \qquad \triangle$$

Condition (i) is usually called the liminf-inequality, while the sequence in (ii) is called a recovery sequence. The main reason why Γ-convergence is especially suited for minimization problems is summarized in the following result, see e.g.[19, Theorem 1.21]. We say that a sequence of functionals $(\mathcal{F}_j)_j$ is equi-coercive if for any sequence $(u_j)_j \subset X$ such that $\lim_{j\to\infty} \mathcal{F}_j(u_j) < \infty$ there exists a subsequence of $(u_j)_j$ converging to $u \in X$.

Theorem 1.1.19. *Let* X *be a metric space and* $\mathcal{F}_j : X \to \mathbb{R}_\infty$ *for* $j \in \mathbb{N}$ *a sequence of equi-coercive functionals* Γ*-converging to* $\mathcal{F} : X \to \mathbb{R}_\infty$*. Then,*

$$\min_{u\in X} \mathcal{F}(u) = \lim_{j\to\infty} \inf_{u\in X} \mathcal{F}_j(u)$$

and if $\min_{u\in X} \mathcal{F}(u) < \infty$ *then any sequence* $(u_j)_j \subset X$ *with* $\lim_{j\to\infty} \mathcal{F}_j(u_j) = \min_{u\in X} \mathcal{F}(u)$ *converges up to subsequence to a minimizer* u_0 *of* $\mathcal{F}$*.*

1.1.6 Young Measures

Young measures are a tool that allow us to study the oscillatory behavior of a sequence of functions. Understanding oscillations and their behavior under application of nonlinear functions is very important when studying weak limits. It turns out that Young measures capture precisely the right information, which makes them extremely convenient to use. Notable references include [37, 58, 64].

Let $\Omega \subset \mathbb{R}^n$ be open then we denote by $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$ the collection of essentially bounded, weak* measurable maps $\mu : \Omega \to \mathcal{M}(\mathbb{R}^m)$. This means that μ_x is a signed measure for each $x \in \Omega$, the map $x \mapsto \int_{\mathbb{R}^m} \varphi(x)(\xi)\, d\mu_x(\xi)$ is measurable for each $\varphi \in L^1(\Omega; C_0(\mathbb{R}^m))$, and the total variation of μ_x is essentially bounded in x with respect to the Lebesgue measure. We sometimes call an element $\mu \in L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$ a parametrized measure. As a side remark we note that $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$ is isometrically isomorphic to the dual of $L^1(\Omega; C_0(\mathbb{R}^m))$ via the Riesz representation theorem. Young measures are defined as follows.

Definition 1.1.20 (Young measure)**.** A Young measure μ is an element of $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$ such that μ_x is a probability measure for almost every $x \in \Omega$. Equivalently, we write $\mu \in L^\infty_w(\Omega; \mathcal{P}r(\mathbb{R}^m))$. △

A sequence of measurable functions $u_j : \Omega \to \mathbb{R}^m$ generates the Young measure $\mu \in L^\infty_w(\Omega; \mathcal{P}r(\mathbb{R}^m))$ if for every $h \in L^1(\Omega)$ and $\varphi \in C_0(\mathbb{R}^m)$ it holds that

$$\lim_{j\to\infty} \int_\Omega h(x)\varphi(u_j(x))\, dx = \int_\Omega h(x) \int_{\mathbb{R}^m} \varphi(\xi)\, d\mu_x(\xi)\, dx.$$

With $\langle \mu_x, \varphi\rangle := \int_{\mathbb{R}^m} \varphi(\xi)\, d\mu_x(\xi)$ this is equivalent to saying that $\varphi(u_j) \overset{*}{\rightharpoonup} \langle \mu_x, \varphi\rangle$ in $L^\infty(\Omega)$ for every $\varphi \in C_0(\mathbb{R}^m)$. We will sometimes write

$$u_j \xrightarrow{YM} \mu$$

to say that $(u_j)_j$ generates μ. The following central result is a version of the fundamental theorem for Young measures and can be found in [37, Theorem 8.2 and 8.6] and [64, Theorem 4.1].

Theorem 1.1.21. *Let $\Omega \subset \mathbb{R}^n$ open with $|\Omega| < \infty$ and $p \in [1, \infty]$. If $(u_j)_j \subset L^p(\Omega; \mathbb{R}^m)$ is a bounded sequence, then there exists a subsequence (not relabeled) and a Young measure $\mu \in L^\infty_w(\Omega; \mathcal{P}r(\mathbb{R}^m))$ such that*

$$u_j \xrightarrow{YM} \mu.$$

Furthermore, it holds that:

(i) For any Carathéodory integrand $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that $(f(\cdot, u_j))_j$ is equi-integrable

$$\lim_{j\to\infty} \int_\Omega f(x, u_j(x))\, dx = \int_\Omega \int_{\mathbb{R}^m} f(x, \xi)\, d\mu_x(\xi)\, dx.$$

(ii) For any normal integrand $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that the negative part of $(f(\cdot, u_j))_j$ is equi-integrable

$$\liminf_{j\to\infty} \int_\Omega f(x, u_j(x))\, dx \geq \int_\Omega \int_{\mathbb{R}^m} f(x, \xi)\, d\mu_x(\xi)\, dx.$$

(iii) *If* $\mathrm{d}(u_j, K) \to 0$ *in measure for some closed* $K \subset \mathbb{R}^m$*, then* $\mathrm{supp}(\mu_x) \subset K$ *for a.e.* $x \in \Omega$*. If* $u_j \to u$ *in measure, then* $\mu_x = \delta_{u(x)}$ *for a.e.* $x \in \Omega$*, where* δ_z *denotes the Dirac measure centered at* $z \in \mathbb{R}^m$*.*

Note that, if u_j converges weakly(*) to u in $L^p(\Omega;\mathbb{R}^m)$, then $(u_j)_j$ is equi-integrable. Hence, part (i) with $f = \mathrm{id}$ shows that $u(x) = \langle \mu_x, \mathrm{id}\rangle =: [\mu]_x$, which is called the barycenter of μ. Another useful result is the following.

Lemma 1.1.22 ([37, Corollary 8.10])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded and consider measurable functions* $u, u_j : \Omega \to \mathbb{R}^m$ *and* $v_j : \Omega \to \mathbb{R}^N$ *for* $j \in \mathbb{N}$*. If*

$$u_j \to u \ \text{ pointwise a.e. } \quad \text{and} \quad v_j \xrightarrow{YM} \mu,$$

then the sequence $(u_j, v_j)_j$ *generates the Young measure* $(\delta_{u(x)} \otimes \mu_x)_{x\in\Omega}$*.*

Since we will mostly be concerned with the weak convergence of sequences of gradients it is useful to know what sets them apart from arbitrary Young measures. We call a Young measure μ a $W^{1,p}(\Omega;\mathbb{R}^m)$-gradient Young measure if there is a bounded sequence $(u_j)_j \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\nabla u_j \xrightarrow{YM} \mu$. The following characterization of gradient Young measures is due to Kinderlehrer and Pedregal [47, 48]. It can also be found in [57, Theorem 4.7].

Theorem 1.1.23. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded and* $p \in (1, \infty]$*. A parametrized measure* $\mu \in L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^{m\times n}))$ *is a* $W^{1,p}(\Omega;\mathbb{R}^m)$*-gradient Young measure if and only if* $\mu_x \geq 0$ *a.e. and there exists a* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ *such that*

(i) $$\begin{cases} \int_\Omega \int_{\mathbb{R}^{m\times n}} |A|^p \, d\mu_x(A)\, dx < \infty & \text{if } p < \infty, \\ \text{There is a compact } K \subset \mathbb{R}^{m\times n} \text{ such that } \mathrm{supp}(\mu_x) \subset K \text{ for a.e. } x \in \Omega & \text{if } p = \infty; \end{cases}$$

(ii) $[\mu]_x := \langle \mu_x, \mathrm{id}\rangle = \nabla u(x)$ *for a.e.* $x \in \Omega$*;*

(iii) $\langle \mu_x, h\rangle \geq h([\mu]_x)$ *for a.e.* $x \in \Omega$ *and all quasiconvex* $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ *with* $|h(A)| \leq C(1 + |A|^p)$ *(no growth condition if* $p = \infty$*).*

A lemma that we use multiple times in the thesis is the following simple consequence of the results in this section.

Lemma 1.1.24. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded and* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ *be a Carathéodory function that satisfies*

$$|f(x, z, \xi)| \leq a(x) + C(|z|^p + |\xi|^p) \quad \text{for a.e. } x \in \Omega \text{ and for all } (z, \xi) \in \mathbb{R}^m \times \mathbb{R}^N,$$

with $a \in L^1(\Omega)$ *and* $C > 0$*. If* $u_j \to u$ *in* $L^p(\Omega;\mathbb{R}^m)$*,* $v_j \to v$ *in* $L^p(\Omega;\mathbb{R}^N)$ *and* $(w_j)_j$ *is bounded in* $L^p(\Omega;\mathbb{R}^N)$ *and* p*-equi-integrable, then*

$$\liminf_{j\to\infty} \int_\Omega f(x, u_j, w_j + v_j)\, dx \leq \liminf_{j\to\infty} \int_\Omega f(x, u, w_j + v)\, dx.$$

Proof. By choosing subsequences (not relabeled) we may assume that

$$\liminf_{j\to\infty} \int_\Omega f(x,u,w_j+v)\,dx = \lim_{j\to\infty} \int_\Omega f(x,u,w_j+v)\,dx.$$

By choosing further subsequence (not relabeled) we may additionally impose that $(w_j)_j$ generates the Young measure $(\mu_x)_{x\in\Omega}$ and $u_j \to u$ pointwise almost everywhere. By [37, Corollary 8.7] it follows that $(w_j+v_j)_j$ generates the translated Young measure $(\hat{\mu}_x)_{x\in\Omega}$ defined via its action on $\varphi \in C_0(\mathbb{R}^N)$ as

$$\langle \hat{\mu}_x, \varphi\rangle = \int_{\mathbb{R}^N} \varphi(\xi + v(x))\,d\mu_x(\xi) \quad \text{for } x \in \Omega.$$

From Lemma 1.1.22 we deduce that $(u_j, w_j+v_j)_j$ generates the Young measure $(\delta_{u(x)}\otimes\hat{\mu}_x)_{x\in\Omega}$. Hence, using the p-equi-integrability and p-growth of f in combination with Theorem 1.1.21 (i) twice

$$\begin{aligned}
\lim_{j\to\infty} \int_\Omega f(x,u_j,w_j+v_j)\,dx &= \int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^N} f(x,z,\xi)\,d(\delta_{u(x)}\otimes\hat{\mu}_x)(z,\xi)\,dx \\
&= \int_\Omega \int_{\mathbb{R}^N} f(x,u(x),\xi+v(x))\,d\mu_x(\xi)\,dx \\
&= \lim_{j\to\infty} \int_\Omega f(x,u,w_j+v)\,dx.
\end{aligned}$$

□

Chapter 2

Supremal Functionals

In this chapter we study variational problems involving supremal functionals, that is, the minimization of functionals of the form

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), \nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\Omega; \mathbb{R}^m).$$

Historically, the calculus of variations was concerned with minimizing integral functionals, but from an applied point of view it is very natural to consider supremal functionals. Indeed, as mentioned in the introduction, if we are interested in minimizing certain quantities in a pointwise sense as opposed to minimizing their average, then supremal functionals are the right tools to use.

To prove existence of minimizers for supremal functionals we utilize the direct method in the calculus of variations, cf. Section 1.1.4. This method is applicable to general functionals that are coercive and lower semicontinuous. In order to obtain coercivity one needs to consider a notion of convergence that has suitable compactness properties, which in this case is the weak* convergence in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. As a consequence of choosing the weak* convergence, we also need to establish the lower semicontinuity with respect to this convergence. This is a quite complicated problem and is the main objective of this chapter. We first consider the scalar case $m = 1$ to build intuition and subsequently move to the vectorial case, inspired by the results from [13]. Other considerations in this chapter regarding supremal functionals are deriving Aronsson equations as necessary conditions for minimizers, finding relaxations, and approximating supremal functionals with integral functionals.

2.1 Weak* Lower Semicontinuity

The present section aims to investigate the lower semicontinuity of supremal functionals with respect to the weak* convergence in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\Omega \subset \mathbb{R}^n$ open and bounded. Most of these results are due to [13].

2.1.1 Scalar Case

In the scalar case we consider for $\Omega \subset \mathbb{R}^n$ open and bounded the supremal functional

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\Omega),$$

with $f : \mathbb{R}^n \to \mathbb{R}$ a suitable supremand. We omit the x- and u-dependence as this section is mostly to gain intuition. We wish to determine criteria on f in order for $\mathcal{S}$ to be lower semicontinuous with respect to the weak* convergence in $W^{1,\infty}(\Omega)$. Recall that weak* convergence in $W^{1,\infty}(\Omega)$ is defined as follows, cf. Section 1.1.3.

Definition 2.1.1 (Weak* convergence). For $\Omega \subset \mathbb{R}^n$ open we say that a sequence $(u_j)_j \subset W^{1,\infty}(\Omega)$ converges weak* to $u \in W^{1,\infty}(\Omega)$, and write $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$, if

$$u_j \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\Omega) \;\text{ and }\; \nabla u_j \overset{*}{\rightharpoonup} \nabla u \text{ in } L^\infty(\Omega;\mathbb{R}^n). \qquad \triangle$$

As mentioned in the preliminaries, any bounded sequence in $W^{1,\infty}(\Omega)$ has a subsequence that converges weak* to some $u \in W^{1,\infty}(\Omega)$. This shows that the weak* convergence possesses the desired compactness properties that we need for coercivity.

In order to understand the weak* lower semicontinuity of the functional $\mathcal{S}$, we introduce the notation for the sub-level sets of f

$$L_c(f) := \{\xi \in \mathbb{R}^n \mid f(\xi) \le c\}.$$

Now, let $(u_j)_j \subset W^{1,\infty}(\Omega)$ be a sequence weak* converging to $u \in W^{1,\infty}(\Omega)$. For weak* lower semicontinuity of $\mathcal{S}$ we need that

$$\operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) \le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u_j(x)). \tag{2.1}$$

Assume now that for some $c \in \mathbb{R}$ we have

$$\liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u_j(x)) < c,$$

then this implies that for all j large enough $\nabla u_j(x) \in L_c(f)$ for a.e. $x \in \Omega$. So, in order to have (2.1) we also need that $\operatorname{ess\,sup}_{x\in\Omega} f(\nabla u(x)) \le c$, or, equivalently, $\nabla u(x) \in L_c(f)$ for a.e. $x \in \Omega$. This illustrates that the sub-level sets of f play an important role in the weak* lower semicontinuity. Indeed, what we need is that for every $c \in \mathbb{R}$ the fact that $\nabla u_j \in L_c(f)$ a.e. for all $j \in \mathbb{N}$ implies $\nabla u \in L_c(f)$ almost everywhere. Stated differently, we need that the subset of $W^{1,\infty}(\Omega)$ given by

$$\mathcal{U}_c = \{u \in W^{1,\infty}(\Omega) \mid \nabla u(x) \in L_c(f) \text{ for a.e. } x \in \Omega\}$$

is closed with respect to weak* convergence in $W^{1,\infty}(\Omega)$. This holds in particular if the subset of $L^\infty(\Omega;\mathbb{R}^n)$ given by

$$\mathcal{V}_c = \{v \in L^\infty(\Omega;\mathbb{R}^n) \mid v(x) \in L_c(f) \text{ for a.e. } x \in \Omega\} \tag{2.2}$$

is weak* closed in $L^\infty(\Omega;\mathbb{R}^n)$. It turns out that this is equivalent to $L_c(f)$ being a convex and closed set, which we prove now. This result uses standard methods but does not seem to appear in this form in the literature.

Proposition 2.1.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and define for some $A \subset \mathbb{R}^n$ the set*

$$\mathcal{V} = \{v \in L^\infty(\Omega;\mathbb{R}^n) \mid v(x) \in A \text{ for a.e. } x \in \Omega\}.$$

Then, $\mathcal{V}$ is (sequentially) closed with respect to the weak convergence in $L^\infty(\Omega;\mathbb{R}^n)$ if and only if A is closed and convex.*

Proof. Suppose A is closed and convex, then it is easy to see that that for any $p \in (1,\infty)$ the set

$$\mathcal{V}_p = \{v \in L^p(\Omega;\mathbb{R}^n) \mid v(x) \in A \text{ for a.e. } x \in \Omega\}$$

is convex and closed with respect to the strong convergence in $L^p(\Omega;\mathbb{R}^n)$. The convexity and closedness imply that $\mathcal{V}_p$ is also weakly closed in $L^p(\Omega;\mathbb{R}^n)$. Since, for any sequence $(v_j)_j \subset \mathcal{V}$ with $v_j \overset{*}{\rightharpoonup} v$ in $L^\infty(\Omega;\mathbb{R}^n)$, we also know that $v_j \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^n)$, we find that $v \in \mathcal{V}_p \cap L^\infty(\Omega;\mathbb{R}^n) = \mathcal{V}$, which shows that $\mathcal{V}$ is weak* closed.

On the other hand, if $\mathcal{V}$ is weak* closed then it is in particular strongly closed. For any $a \in \overline{A}$ we can take a sequence $(a_j)_j \in A$ such that $a_j \to a$. Then, simply defining the sequence of constants $v_j \equiv a_j$ shows that $v_j \to v$ in $L^\infty(\Omega;\mathbb{R}^n)$ with $v \equiv a$. Because $(v_j)_j \subset \mathcal{V}$, we get $v \in \mathcal{V}$ showing that $a \in A$, i.e. A must be closed. For the convexity, take $a_1, a_2 \in A$ and $\lambda \in [0,1]$ then we can define

$$v_0 : \mathbb{R}^n \to \mathbb{R}, \quad v_0(x) = \begin{cases} a_1 & 0 \le x_1 - \lfloor x_1 \rfloor \le \lambda, \\ a_2 & \lambda < x_1 - \lfloor x_1 \rfloor < 1. \end{cases}$$

By considering the oscillating sequence $v_j(x) = v_0(jx) \subset \mathcal{V}$ we find by Proposition 1.1.2 that $v_j \overset{*}{\rightharpoonup} v$ in $L^\infty(\Omega;\mathbb{R}^n)$ with $v(x) \equiv \lambda a_1 + (1-\lambda)a_2$ (the average of v_0 on $(0,1)^n$). Since $v \in \mathcal{V}$, this shows that $\lambda a_1 + (1-\lambda)a_2 \in A$ so that A is convex. □

This result shows that $\mathcal{V}_c$ in (2.2) is weak* closed if and only if $L_c(f)$ is closed and convex. The closedness of $L_c(f)$ for each $c \in \mathbb{R}$ corresponds to f being lower semicontinuous on $\mathbb{R}^n$. The convexity of the sub-level sets is often called level-convexity. Note that it is sometimes also called quasiconvexity in the literature, see e.g. [13, 14], but we want to avoid this name as it is in conflict with the quasiconvexity notion from Morrey, Definition 1.1.13.

Definition 2.1.3 (level-convexity)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is level-convex if for every $c \in \mathbb{R}$ its sub-level set $L_c(f)$ is convex. Equivalently, this holds if for every $\xi_1, \xi_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$

$$f(\lambda\xi_1 + (1-\lambda)\xi_2) \le \max\{f(\xi_1), f(\xi_2)\}. \qquad \triangle$$

Clearly, any convex function is level-convex, but the converse is not true as $f(\xi) = \sqrt{|\xi|}$ shows. We now show that lower semicontinuity and level-convexity is sufficient for the weak* lower semicontinuity of $\mathcal{S}$, by formalizing the argument preceding Proposition 2.1.2.

Theorem 2.1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and level-convex and $\Omega \subset \mathbb{R}^n$ open and bounded. Then, the functional*

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\Omega)$$

is weak lower semicontinuous on $W^{1,\infty}(\Omega)$.*

Proof. Let $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$ and take any $c \in \mathbb{R}$ such that

$$\liminf_{j\to\infty} \mathcal{S}(u_j) < c.$$

Then we know that $\nabla u_j \in L_c(f)$ a.e. for all j large enough. Furthermore, the assumptions on f show that $L_c(f)$ is closed and convex. Hence, Proposition 2.1.2 and the fact that $\nabla u_j \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(\Omega;\mathbb{R}^n)$ imply that $\nabla u \in L_c(f)$ almost everywhere. This shows that $\mathcal{S}(u) \leq c$, and since we can do this for any such c we conclude

$$\mathcal{S}(u) \leq \liminf_{j\to\infty} \mathcal{S}(u_j)$$

as desired. □

One might wonder why we restrict the above result to only the scalar case. Indeed, the same result holds in the vectorial case using an identical argument. The reason is that in the scalar case, level-convexity of f is also a necessary condition for weak* lower semicontinuity, while in the vectorial case it is not. In the vectorial case we will derive a weaker condition which is still sufficient for weak* lower semicontinuity. The proof that level-convexity is necessary in the scalar case will be given later where we show more generally that rank-1-level-convexity is necessary, cf. Proposition 2.3.5.

Next, we want to give two extra proofs that level-convexity is sufficient for weak* lower semicontinuity. This is in order to gain intuition and hopefully be able to extend one of the approaches to the vectorial case. The first additional proof is a reduction to integral functionals similar to [13, Theorem 3.3].

Alternative proof to Theorem 2.1.4. Let $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$ and take any $c \in \mathbb{R}$ such that

$$\liminf_{j\to\infty} \mathcal{S}(u_j) < c.$$

We can define the characteristic function of $L_c(f)$

$$\chi_c(\xi) = \begin{cases} 0 & \text{if } \xi \in L_c(f), \\ \infty & \text{else,} \end{cases} \qquad \text{for } \xi \in \mathbb{R}^n,$$

which is an extended-value function, in the sense that it attains the value ∞. It can be readily seen that the lower semicontinuity and level-convexity of f imply that χ_c is lower semicontinuous and convex. Hence, the integral functional

$$\mathcal{I}_c(u) = \int_\Omega \chi_c(\nabla u)\,dx \quad \text{for } u \in W^{1,\infty}(\Omega)$$

is known to be weak* lower semicontinuous, [29, Corollary 3.22]. Since $\mathcal{I}_c(u_j) = 0$ for j large enough we thus find

$$\mathcal{I}_c(u) \leq \liminf_{j\to\infty} \mathcal{I}_c(u_j) = 0.$$

This shows that $\nabla u \in L_c(f)$ a.e. and doing this for all such c proves the result. □

The final proof is one that is inspired by Jensen's inequality. For convex functions $f : \mathbb{R}^n \to \mathbb{R}$ we know that Jensen's inequality states that for all $v \in L^1(\Omega; \mathbb{R}^n)$

$$f\left(\frac{1}{|\Omega|}\int_\Omega v(x)\,dx\right) \le \frac{1}{|\Omega|}\int_\Omega f(v(x))\,dx.$$

For level-convex functions we have a similar inequality, adapted from [13, Theorem 1.2].

Lemma 2.1.5 (Extended Jensen inequality)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and level-convex. Then we have for any $v \in L^1(\Omega; \mathbb{R}^n)$ that*

$$f\left(\frac{1}{|\Omega|}\int_\Omega v(x)\,dx\right) \le \operatorname*{ess\,sup}_{x\in\Omega} f(v(x)).$$

Proof. Let $c := \operatorname{ess\,sup}_{x\in\Omega} f(v(x))$, which implies $v(x) \in L_c(f)$ almost everywhere. Suppose for the sake of contradiction, that with

$$a := \frac{1}{|\Omega|}\int_\Omega v(x)\,dx$$

it holds that $f(a) > c$. Then $a \notin L_c(f)$ and since $L_c(f)$ is closed and convex we may find by the hyperplane separation theorem, see e.g. [29, Theorem 2.10 (i)], a $\zeta \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that

$$\zeta \cdot a > t \quad \text{and} \quad \zeta \cdot \xi \le t \ \text{ for all } \xi \in L_c(f). \tag{2.3}$$

Using linearity of the integral and that $v(x) \in L_c(f)$ almost everywhere yields

$$\zeta \cdot a = \frac{1}{|\Omega|}\int_\Omega \zeta \cdot v(x)\,dx \le t,$$

which contradicts (2.3). □

We can now use this to prove the weak* lower semicontinuity, at least when the limit function is affine.

Proposition 2.1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and level-convex. Then, for any $\xi \in \mathbb{R}^n$ and any sequence $\varphi_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q)$ it holds that*

$$f(\xi) \le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi_j(x)),$$

with $Q = (0,1)^n$.

Proof. By the extended Jensen inequality with $v(x) = \xi + \nabla\varphi_j(x)$ we find

$$f\left(\xi + \int_Q \nabla\varphi_j(x)\,dx\right) \le \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi_j(x)).$$

Since weak* convergence in $L^\infty(Q;\mathbb{R}^n)$ implies convergence of averages (test with a constant function) it follows that $\fint_Q \nabla\varphi_j(x)\,dx \to 0$. Hence, using the lower semicontinuity of f we deduce

$$f(\xi) \le \liminf_{j\to\infty} f\left(\xi + \fint_Q \nabla\varphi_j(x)\,dx\right) \le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi_j(x)).$$

□

The above result states that weak* lower semicontinuity holds when the limit function is affine and, thus, has a constant derivative. It turns out that this is enough to prove the general case, by using the argument of linearization. This approach is worked out in detail in the next section, see Theorem 2.1.11.

2.1.2 Vectorial Case

In this section we consider for $\Omega \subset \mathbb{R}^n$ open and bounded the weak* lower semicontinuity of general supremal functionals of the form

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), \nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\Omega;\mathbb{R}^m), \tag{2.4}$$

with $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n} \to \mathbb{R}$ a suitable supremand. The vectorial case refers to the fact that we allow $m > 1$. The sufficient and necessary conditions were first discovered by Barron, Jensen & Wang in [13].

Let us first come up with an approach to tackle this problem. For simplicity, we initially assume that $f(x, y, A) = f(A)$ only depends on the last argument. Looking back at what we did in the scalar case on p. 16, we noticed that the sub-level sets of f played an important role. In fact, we derived the weak* lower semicontinuity of $\mathcal{S}$ if

$$\mathcal{U}_c = \{u \in W^{1,\infty}(\Omega;\mathbb{R}^m) \mid \nabla u \in L_c(f) \text{ for a.e. } x \in \Omega\}$$

was (sequentially) weak* closed in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ for each $c \in \mathbb{R}$. Subsequently, we argued that this holds in particular if

$$\mathcal{V}_c = \{v \in L^\infty(\Omega;\mathbb{R}^{m\times n}) \mid v \in L_c(f) \text{ for a.e. } x \in \Omega\}$$

is weak* closed in $L^\infty(\Omega;\mathbb{R}^{m\times n})$, which corresponded to f being level-convex. In the scalar case, level-convexity is also necessary for weak* lower semicontinuity, cf. Proposition 2.3.5, but in the vectorial case it is not. This is because sequences of matrix-valued gradients $(\nabla u_j)_j \subset L^\infty(\Omega;\mathbb{R}^{m\times n})$ possess more structure than arbitrary sequences $(v_j)_j \subset L^\infty(\Omega;\mathbb{R}^{m\times n})$. Therefore, we cannot reduce the problem by studying the set $\mathcal{V}_c$ instead of $\mathcal{U}_c$. This leaves us with the problem of determining the weak* closedness of $\mathcal{U}_c$, which is a much harder problem. Drawing inspiration from the results for integral functionals, we know that the characterizing condition for weak lower semicontinuity goes from convexity to quasiconvexity when moving to the vectorial case, cf. Section 1.1.4. Hence, one might guess that instead of convexity of $L_c(f)$, we want the sub-level sets to be quasiconvex in an appropriate sense. There is a systematic way to define generalized convex sets from the corresponding notion for functions. Indeed, it can be checked that a set $K \subset \mathbb{R}^{m\times n}$ is convex and closed if and only if

$$K = \left\{ A \in \mathbb{R}^{m\times n} \mid h(A) \leq \sup_K h \text{ for all } h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ convex} \right\},$$

see e.g. [29, Proposition 2.36]. The set on the right consists exactly of the points that cannot be strictly separated from K by a convex function. When K is closed and convex, the hyperplane separation theorem tells us that any point outside K can be strictly separated by even an affine function, thus, explaining the above identity. In an analogous way we can define the following, cf. [29, Definition 7.25].

Definition 2.1.7 (Quasiconvex set)**.** We say that $K \subset \mathbb{R}^{m\times n}$ is quasiconvex if

$$K = \left\{ A \in \mathbb{R}^{m\times n} \mid h(A) \leq \sup_K h \text{ for all } h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex} \right\}. \qquad \triangle$$

Note that quasiconvex sets are closed since quasiconvex functions are continuous, see e.g. [29, Theorem 5.3 (iv)]. The following result establishes weak* lower semicontinuity when the sub-level sets of the supremand are quasiconvex. This result is rather simple to prove by using properties of gradient Young measures, but it is hard to find anywhere in the literature. It is mentioned in [76, Theorem 3.3] but only proven for $p < \infty$.

Theorem 2.1.8 (Sufficient condition)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be such that*

$$L_c(f) := \{A \in \mathbb{R}^{m\times n} \mid f(A) \leq c\} \text{ is quasiconvex for all } c \in \mathbb{R}.$$

Then the functional

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$$

is sequentially weak lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.*

Proof. As worked out on p. 16, we only have to show that

$$\mathcal{U}_c = \{u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \mid \nabla u \in L_c(f) \text{ a.e.}\}$$

is sequentially weak* closed in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ for each $c \in \mathbb{R}$. Let $(u_j)_j \subset \mathcal{U}_c$ be a sequence weak* converging to $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. Let $(\mu_x)_{x\in\Omega}$ be the Young measure generated by a non-relabeled subsequence $(\nabla u_j)_j$. If we set $r := \sup_j \|\nabla u_j\|_{L^\infty(\Omega)}$ then we find that ∇u_j is a.e. contained in the compact set $L_c(f) \cap \overline{B_r(0)}$. Hence, we find by Theorem 1.1.21 (iii) that $\operatorname{supp}(\mu_x) \subset L_c(f)$ for a.e. $x \in \Omega$. If we take any quasiconvex function $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ then we find by the characterization of gradient Young measures, Theorem 1.1.23 (iii), that for a.e. $x \in \Omega$

$$h(\nabla u(x)) = h([\mu]_x) \leq \int_{\mathbb{R}^{m\times n}} h(A)\, d\mu_x(A) \leq \sup_{L_c(f)} h,$$

with the last inequality following since $\operatorname{supp}(\mu_x) \subset L_c(f)$ and μ_x is a probability measure for a.e. $x \in \Omega$. Since $L_c(f)$ is quasiconvex this shows per definition that $\nabla u(x) \in L_c(f)$ for a.e. $x \in \Omega$, that is, $u \in \mathcal{U}_c$. □

This is an elegant sufficient condition and is closely related to the concept of differential inclusions. However, the above condition is not known to be necessary. For now, we will abandon this approach and consider the other options.

The second approach we used in the scalar case was the reduction to integral functionals. If for all $c \in \mathbb{R}$ the function

$$\chi_c(A) = \begin{cases} 0 & \text{if } A \in L_c(f), \\ \infty & \text{else,} \end{cases} \quad \text{for } A \in \mathbb{R}^{m\times n}$$

is quasiconvex, then we would find that the integral functional

$$\mathcal{I}_c(u) = \int_\Omega \chi_c(\nabla u)\, dx$$

is weak* lower semicontinuous and we could proceed as in the scalar case. The problem with this reasoning is that the integrand χ_c attains the value ∞, and quasiconvexity only guarantees weak* lower semicontinuity for integrands with certain growth bounds. Hence, we cannot conclude the weak* lower semicontinuity of $\mathcal{I}_c$. The notion of polyconvexity (Definition 1.1.15) was introduced by Ball [10], and does give lower semicontinuity even in the presence of extended-valued integrands. His contributions were inspired by applications in hyperelasticity, in which the energy densities would always attain the value ∞. Polyconvexity is not a necessary condition though, so it does not solve our problem at hand. A universal theory for the weak lower semicontinuity of integral functionals with extended-valued integrands is not clear at this point in time.

Our final approach from the scalar case was using the extended Jensen inequality for level-convex functions from Lemma 2.1.5. This inequality guaranteed that for any $A \in \mathbb{R}^{m\times n}$

$$f(A) \le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi_j),$$

for any sequence $\varphi_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q;\mathbb{R}^{m\times n})$, and we claimed that this was enough to conclude the weak* lower semicontinuity of supremal functionals. Inspired by quasiconvexity, one idea is to come up with a different Jensen inequality, that involves gradients, and still implies the above condition. This can indeed be done and gives us the following convexity notion. Observe that in [13] this notion is called strong Morrey quasiconvexity.

Definition 2.1.9 (level-quasiconvexity [13, Definition 2.1]). A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is level-quasiconvex if for every $A \in \mathbb{R}^{m\times n}$, $K > 0$ and $\epsilon > 0$ there exists a $\delta = \delta(A, K, \epsilon)$ such that for all $\varphi \in W^{1,\infty}(Q;\mathbb{R}^m)$ with

$$\|\nabla\varphi\|_{L^\infty(Q)} \le K \quad \text{and} \quad \max_{x\in\partial Q} |\varphi(x)| \le \delta$$

it holds that

$$f(A) \le \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi(x)) + \epsilon. \qquad \triangle$$

One thing to observe about the last inequality is that it resembles quasiconvexity apart from the integral being replaced by an essential supremum. A notable difference is the technicality involving

K, ϵ and δ. Indeed, for quasiconvexity the test functions simply consist of $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$ and so there is no freedom at the boundary. This yields no complications in the integral setting because we can alter boundary conditions by only changing the functions on a set of small measure. The integral will then not change too much and one can obtain for all $A \in \mathbb{R}^{m\times n}$ that

$$f(A) \leq \liminf_{j\to\infty} \int_Q f(A + \nabla\varphi_j)\,dx,$$

for any sequence $\varphi_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q;\mathbb{R}^m)$. For supremal functionals this method does not work, since the supremum takes any change on a small set into account, regardless of its measure. This intuitively explains why the notion

$$f(A) \leq \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi(x)) \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m) \tag{2.5}$$

will not be strong enough and why we need the K, ϵ and δ in the definition. Still, one might hope that (2.5) implies weak* lower semicontinuity of supremal functionals via some different strategy. This is not true as can be seen readily. Indeed, the condition (2.5) is equivalent to the quasiconvexity of the characteristic functions

$$\chi_c(A) = \begin{cases} 0 & \text{if } A \in L_c(f), \\ \infty & \text{else,} \end{cases} \quad \text{for } A \in \mathbb{R}^{m\times n},$$

for all $c \in \mathbb{R}$, see [13, Lemma 1.4]. Here, quasiconvexity is just the usual definition although we allow the value ∞ to be attained. A classical example by Ball and Murat, [11, Example 3.5], shows that such functions need not be rank-1-convex, even when they are lower semicontinuous. Therefore, functions satisfying (2.5) are not necessarily level-rank-1-convex, which is necessary for weak* lower semicontinuity as we will show in Proposition 2.3.5. A similar observation is made in [63, Remark 5.2 & Example 5.3]. Hence, we immediately see that the notion (2.5) will not work, at least without stronger continuity assumptions.

Let us now show that level-quasiconvexity works for our purposes.

Proposition 2.1.10 ([13, Proposition 2.5])**.** *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be level-quasiconvex. Then, for any $A \in \mathbb{R}^{m\times n}$ and any sequence $\varphi_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q;\mathbb{R}^m)$ it holds that*

$$f(A) \leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi_j(x)).$$

Proof. Let $A \in \mathbb{R}^{m\times n}$ and $\varphi_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q;\mathbb{R}^m)$ then there is a $K > 0$ such that $\|\nabla\varphi_j\|_{L^\infty(Q)} \leq K$ for all $j \in \mathbb{N}$. Now fix $\epsilon > 0$ and take the corresponding $\delta = \delta(A, K, \epsilon) > 0$ as in the definition of level-quasiconvexity. Since $\varphi_j \to 0$ in $C(\overline{Q};\mathbb{R}^m)$ by Proposition 1.1.9 we find that for all j large enough $\max_{x\in\partial Q}|\varphi_j(x)| \leq \delta$. This shows by definition of level-quasiconvexity that

$$f(A) \leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi_j(x)) + \epsilon.$$

By the arbitrariness of $\epsilon > 0$ the result follows. □

Note that this proposition implies in particular that any level-quasiconvex function is lower semicontinuous, by choosing a sequence with constant gradients. This contrasts with level-convexity which does not imply lower semicontinuity; take, for example, the negative indicator function $-\mathbb{1}_E$ of a convex set $E \subset \mathbb{R}^{m\times n}$ that is not closed. As claimed, the inequality in the above proposition is enough to conclude the weak* lower semicontinuity via a linearization argument. This is worked out in the proof of following theorem, which presents the main sufficient condition for weak* lower semicontinuity.

Theorem 2.1.11 ([13, Theorem 2.6]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\mathcal{S}$ be as in (2.4). Suppose that $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfies:*

(i) $f(x, z, \cdot)$ is level-quasiconvex for all $(x, z) \in \Omega \times \mathbb{R}^m$.

(ii) There exists a function $\omega : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is continuous in its first variable with $\omega(0, K) = 0$ and non-decreasing in it second variable, such that

$$|f(x_1, z_1, A) - f(x_2, z_2, A)| \leq \omega(|x_1 - x_2| + |z_1 - z_2|, |A|),$$

for any $x_1, x_2 \in \Omega$, $z_1, z_2 \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$.

Then, $\mathcal{S}$ is sequentially weak lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.*

Proof. Let $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ then we need to show

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), \nabla u(x)) \leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u_j(x), \nabla u_j(x)).$$

By possibly choosing a subsequence we may assume that

$$\liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u_j(x), \nabla u_j(x)) = \lim_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u_j(x), \nabla u_j(x)).$$

We denote the sequence $\varphi_j = u_j - u$, which converges to 0 weak* in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Furthermore, Rademacher's theorem (Theorem 1.1.10) shows that for a.e. $x_0 \in \Omega$ we have

$$\lim_{x\to x_0} \frac{|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|}{|x - x_0|} = 0. \tag{2.6}$$

Therefore, it is sufficient to prove

$$f(x_0, u(x_0), \nabla u(x_0)) \leq \lim_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u_j(x), \nabla u_j(x))$$

for those x_0 that satisfy (2.6). To do this, we try to utilize Proposition 2.1.10 by linearizing u and φ_j at these points. Define for such x_0 and $r > 0$ small enough the functions

$$u^r : Q \to \mathbb{R}^m, \qquad u^r(x) = \frac{1}{r}(u(x_0 + rx) - u(x_0))$$

and

$$\varphi_j^r : Q \to \mathbb{R}^m, \qquad \varphi_j^r(x) = \frac{1}{r}(\varphi_j(x_0 + rx) - \varphi_j(x_0)).$$

By (2.6) we find that u^r converges uniformly to the linear function $x \mapsto \nabla u(x_0)x$ as $r \to 0$. Since furthermore $(u^r)_r$ is bounded in $W^{1,\infty}(Q;\mathbb{R}^m)$, we conclude by Urysohn's subsequence principle that even

$$u^r(x) \overset{*}{\rightharpoonup} \nabla u(x_0)x \quad \text{in } W^{1,\infty}(Q;\mathbb{R}^m) \text{ as } r \to 0. \tag{2.7}$$

Next, because $\varphi_j(x_0) \to 0$ as $j \to \infty$ by Proposition 1.1.9, we find that $\varphi_j^r \overset{*}{\rightharpoonup} 0$ in $L^\infty(Q;\mathbb{R}^m)$ as $j \to \infty$. Similarly, since $\nabla\varphi_j^r(x) = \nabla\varphi_j(x_0 + rx)$ we also see

$$\nabla\varphi_j^r \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(Q;\mathbb{R}^{m\times n}) \text{ as } j \to \infty.$$

All in all, this means that for any $r > 0$ we find $\varphi_j^r \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q;\mathbb{R}^m)$ and thus we may choose indices $j(r) \to \infty$ such that

$$\varphi_{j(r)}^r \overset{*}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}(Q;\mathbb{R}^m) \text{ as } r \to 0. \tag{2.8}$$

In light of (2.7) and (2.8), we can apply Proposition 2.1.10 to the matrix $\nabla u(x_0)$ and the sequence $u^r(x) - \nabla u(x_0)x + \varphi_{j(r)}^r(x)$ to obtain with $Q_r(x_0) := \{x_0 + rx \mid x \in Q\}$

$$\begin{aligned}
f(x_0, u(x_0), \nabla u(x_0)) &\le \liminf_{r\to 0} \operatorname*{ess\,sup}_{x\in Q} f(x_0, u(x_0), \nabla u^r(x) + \nabla\varphi_{j(r)}^r(x)) \\
&= \liminf_{r\to 0} \operatorname*{ess\,sup}_{x\in Q} f(x_0, u(x_0), \nabla u_{j(r)}(x_0 + rx)) \\
&= \liminf_{r\to 0} \operatorname*{ess\,sup}_{x\in Q_r(x_0)} f(x_0, u(x_0), \nabla u_{j(r)}(x)) \\
&\le \liminf_{r\to 0} \operatorname*{ess\,sup}_{x\in Q_r(x_0)} f(x, u_{j(r)}(x), \nabla u_{j(r)}(x)) \\
&\quad + \omega(|x - x_0| + |u_{j(r)}(x) - u(x_0)|, |\nabla u_{j(r)}(x)|).
\end{aligned} \tag{2.9}$$

Let $K > 0$ be such that $\sup_j \|\nabla u_j\|_{L^\infty(\Omega)} \le K$ then we find

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in Q_r(x_0)} \omega(|x - x_0| + |u_{j(r)}(x) - u(x_0)|, |\nabla u_{j(r)}(x)|) \\
&\qquad \le \operatorname*{ess\,sup}_{x\in Q_r(x_0)} \omega(|x - x_0| + |u_{j(r)}(x) - u(x)| + |u(x) - u(x_0)|, K) \xrightarrow{r\to 0} 0,
\end{aligned}$$

using the uniform convergence $u_{j(r)} \to u$ on $Q_r(x_0)$ (Proposition 1.1.9) and the continuity of u (Theorem 1.1.8). Hence, (2.9) yields

$$\begin{aligned}
f(x_0, u(x_0), \nabla u(x_0)) &\le \liminf_{r\to 0} \operatorname*{ess\,sup}_{x\in Q_r(x_0)} f(x, u_{j(r)}(x), \nabla u_{j(r)}(x)) \\
&\le \liminf_{r\to 0} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u_{j(r)}(x), \nabla u_{j(r)}(x)) \\
&\le \lim_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u_j(x), \nabla u_j(x)).
\end{aligned}$$

□

Remark 2.1.12. To deal with the second argument of f, the proof only required the uniform convergence of u_j to u. Therefore, we can actually deduce the slightly more general statement that

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x, w(x), \nabla u(x)) \leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x, w_j(x), \nabla u_j(x))$$

for any two sequences $w_j \to w$ in $C(\overline{\Omega}; \mathbb{R}^m)$ and $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. △

This important result shows that level-quasiconvexity gives a sufficient condition for weak* lower semicontinuity. Now we show that it is also necessary. We first prove an intermediate necessary condition interesting in its own right. For simplicity, we assume $f(x, y, A) = f(A)$. We say that $Y \subset \mathbb{R}^n$ is an n-cube if it can be obtained via a rigid motion acting on the unit cube Q. For such an n-cube, we denote $W^{1,\infty}_{per}(Y; \mathbb{R}^m)$ as the functions in $W^{1,\infty}(\mathbb{R}^n)$ that are Y-periodic.

Lemma 2.1.13 ([61, Lemma 2.8])**.** *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be measurable, $\Omega \subset \mathbb{R}^n$ open and bounded and assume that $\mathcal{S}$, given by*

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)),$$

is (sequentially) weak lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Then, f is lower semicontinuous and for every $A \in \mathbb{R}^{m\times n}$, n-cube $Y \subset \mathbb{R}^n$ and $\varphi \in W^{1,\infty}_{per}(Y; \mathbb{R}^m)$ we have*

$$f(A) \leq \operatorname*{ess\,sup}_{x\in Y} f(A + \nabla\varphi(x)).$$

Proof. The lower semicontinuity of f follows simply by considering a sequence with constant derivatives. For the second property, let $A \in \mathbb{R}^{m\times n}$, Y be an n-cube and $\varphi \in W^{1,\infty}_{per}(Y; \mathbb{R}^m)$. Then, the oscillating sequence $\varphi_j(x) = \frac{1}{j}\varphi(jx)$ is bounded in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ and thus converges weak* in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ to its strong limit in $L^\infty(\Omega; \mathbb{R}^m)$, which is zero. Hence, the sequence $u_j(x) = Ax + \varphi_j(x)$ converges weak* to $u(x) = Ax$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. By the weak* lower semicontinuity of $\mathcal{S}$ we find

$$\begin{aligned} f(A) = \mathcal{S}(u) &\leq \liminf_{j\to\infty} \mathcal{S}(u_j) \\ &= \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(A + \nabla\varphi_j(x)) \\ &= \operatorname*{ess\,sup}_{x\in Y} f(A + \nabla\varphi(x)), \end{aligned}$$

because $\nabla\varphi_j(x) = \nabla\varphi(jx)$ is Y/j-periodic. □

We can now prove the necessity of level-quasiconvexity. One subtlety is that we need to assume that the supremal functional is weak* lower semicontinuous for any subdomain of Ω as well. Such a property automatically follows for weakly lower semicontinuous integral functionals by subtracting the part outside this subdomain, see e.g. [29, Lemma 3.17]. For supremal functionals this is different, because it could happen that for an open subset $O \subset \Omega$

$$\operatorname*{ess\,sup}_{x\in O} f(\nabla u(x)) < \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)).$$

Then, if we slightly change the values of u only in O, the supremum over Ω does not change, so that we can not deduce any information about the supremum over O. This difficulty is nicely exhibited in Section 2.7, where we show that supremal functionals on $\mathbb{R}^n$ require strictly weaker conditions for weak* lower semicontinuity than supremal functionals on bounded domains.

Let us state the necessity result.

Theorem 2.1.14 ([13, Theorem 2.7]). *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be measurable, $\Omega \subset \mathbb{R}^n$ open and bounded and assume that for every open subset $O \subset \Omega$ the functional*

$$\mathcal{S}(u, O) = \operatorname*{ess\,sup}_{x\in O} f(\nabla u(x))$$

is weak lower semicontinuous on $W^{1,\infty}(O; \mathbb{R}^m)$. Then, f is level-quasiconvex.*

Proof. We give a proof by contradiction. Suppose there are $A \in \mathbb{R}^{m\times n}$, $K > 0$ and $\epsilon > 0$ such that for all $\delta > 0$ there exists a $\varphi_\delta \in W^{1,\infty}(Q; \mathbb{R}^m)$ with

$$\|\nabla\varphi_\delta\|_{L^\infty(Q)} \le K, \qquad \max_{x\in\partial Q} |\varphi_\delta(x)| \le \delta$$

and

$$f(A) > \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi_\delta(x)) + \epsilon. \tag{2.10}$$

Then, by choosing $\delta = 1/j$ for all $j \in \mathbb{N}$ we obtain a sequence $(\varphi_j)_j \subset W^{1,\infty}(Q; \mathbb{R}^m)$. This sequence has gradients bounded by K and

$$|\varphi_j(x)| \le |\varphi(y)| + |\varphi_j(x) - \varphi(y)| \le 1/j + K|x - y| < K + 1,$$

where y is any point in ∂Q. We conclude that $(\varphi_j)_j$ is bounded in $W^{1,\infty}(Q; \mathbb{R}^m)$ and hence, a subsequence (not relabeled) converges weak* to $\varphi \in W^{1,\infty}(Q; \mathbb{R}^m)$. Because $\max_{x\in\partial Q} |\varphi_j(x)| \le 1/j$ we even have $\varphi \in W^{1,\infty}_0(Q; \mathbb{R}^m)$. Now, we choose a $x_0 \in \Omega$ and $r > 0$ such that $Q_r(x_0) = \{x_0 + rx \mid x \in Q\} \subset \Omega$. Then, by defining $\psi_j(x) := Ax + \frac{1}{r}\varphi_j(x_0 + rx)$ and $\psi(x) := Ax + \frac{1}{r}\varphi(x_0 + rx)$ it follows that $\psi_j \overset{*}{\rightharpoonup} \psi$ in $W^{1,\infty}(Q_r(x_0); \mathbb{R}^m)$. Since $\mathcal{S}(\cdot, Q_r(x_0))$ is weak* lower semicontinuous we conclude

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi) = \operatorname*{ess\,sup}_{x\in Q_r(x_0)} f(\nabla\psi) &\le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q_r(x_0)} f(\nabla\psi_j) \\ &= \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi_j) \\ &\le f(A) - \epsilon, \end{aligned} \tag{2.11}$$

with the last inequality following from (2.10). On the other hand, since φ has zero boundary values it can be Q-periodically extended to a function $\varphi \in W^{1,\infty}_{per}(Q; \mathbb{R}^m)$. Lemma 2.1.13 now yields

$$f(A) \le \operatorname*{ess\,sup}_{x\in Q} f(A + \nabla\varphi),$$

which contradicts (2.11). □

Remark 2.1.15. The above theorem only needed weak* lower semicontinuity on some scaled n-cube in Ω. However, in the case with x- and u-dependence in [13, Theorem 2.7] the hypothesis of weak* lower semicontinuity on subdomains is used to a greater extent. △

2.2 Existence of Minimizers

The goal of this section is to prove the existence of minimizers, which is a simple application of the direct method in the calculus of variations in combination with the weak* lower semicontinuity from the previous section. This is an adaptation of [13, Theorem 2.9].

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $g \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ and let $\mathcal{S}$ be as in (2.4). Suppose that $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfies the hypotheses in Theorem 2.1.11. If additionally f satisfies the coercivity condition*

$$f(x,z,A) \to \infty \text{ as } |A| \to \infty \quad \text{for all } (x,z) \in \Omega \times \mathbb{R}^m,$$

then there exists a minimizer $u_0 \in W_g^{1,\infty}(\Omega;\mathbb{R}^m)$ of $\mathcal{S}$, i.e. $\mathcal{S}(u_0) = \inf_{u \in W_g^{1,\infty}(\Omega;\mathbb{R}^m)} \mathcal{S}(u)$.

Proof. Suppose $c := \inf_{u \in W_g^{1,\infty}(\Omega;\mathbb{R}^m)} \mathcal{S}(u) < \infty$, otherwise the result is trivial. Let $(u_j)_j \subset W_g^{1,\infty}(\Omega;\mathbb{R}^m)$ be a minimizing sequence, i.e. $\lim_{j\to\infty} \mathcal{S}(u_j) = c$, then it follows that $(\mathcal{S}(u_j))_j$ is a bounded sequence in $\mathbb{R}$. The coercivity condition then implies that $(\nabla u_j)_j$ is bounded in $L^\infty(\Omega;\mathbb{R}^{m\times n})$. By Poincaré's inequality (Theorem 1.1.7) we obtain

$$\|u_j\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)} + C(\|\nabla u_j\|_{L^\infty(\Omega)} + \|\nabla g\|_{L^\infty(\Omega)}),$$

so that also $(u_j)_j$ is bounded in $W_g^{1,\infty}(\Omega;\mathbb{R}^m)$. Hence, there is a subsequence $(u_j)_j$ (not relabeled) and a $u_0 \in W_g^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $u_j \overset{*}{\rightharpoonup} u_0$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$. The weak* lower semicontinuity from Theorem 2.1.11 now yields

$$\mathcal{S}(u_0) \leq \liminf_{j\to\infty} \mathcal{S}(u_j) \leq c,$$

which shows that u_0 is a minimizer. □

2.3 Notions of Level-Convexity

Here, we discuss several different types of level-convexity notions and the relations between them. This is rather similar to the integral case where there are the notions of convexity, polyconvexity, quasiconvexity and rank-1-convexity. In the supremal setting, there are level-convexity, level-polyconvexity, level-quasiconvexity and level-rank-1-convexity. In particular, we are able to show how level-quasiconvexity relates to the more simple to understand notions.

Let us briefly provide the definition of level-convexity for functions $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, while we refer to Definition 2.1.9 for level-quasiconvexity.

Definition 2.3.1 (level-convexity)**.** We say that a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is level-convex if for every $c \in \mathbb{R}$ its sub-level set $L_c(f)$ is convex. Equivalently, this holds if for every $A, B \in \mathbb{R}^{m\times n}$ and $\lambda \in [0,1]$ we have

$$f(\lambda A + (1-\lambda)B) \leq \max\{f(A), f(B)\}. \qquad \triangle$$

The third notion we want to discuss is level-polyconvexity. The idea behind this notion, similar to polyconvexity, is to utilize the fact that when a supremand $F : \mathbb{R}^N \to \mathbb{R}$ is level-convex, Proposition 2.1.2 shows that

$$\widehat{\mathcal{S}}(v) = \operatorname*{ess\,sup}_{x\in\Omega} F(v(x))$$

is even weak* lower semicontinuous on $L^\infty(\Omega; \mathbb{R}^N)$. In other words, we have lower semicontinuity with respect to any weak* converging sequence in $L^\infty(\Omega; \mathbb{R}^N)$, not just sequences of gradients. Therefore, if there is some weak* continuous map $T : \mathbb{R}^{m\times n} \to \mathbb{R}^N$ in the sense that $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ implies $T(\nabla u_j) \overset{*}{\rightharpoonup} T(\nabla u)$ in $L^\infty(\Omega; \mathbb{R}^N)$, then, we see for any level-convex $F : \mathbb{R}^n \to \mathbb{R}$ that

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} F(T(\nabla u))$$

is weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Although this is an interesting observation, if T is linear then $F \circ T$ is also level-convex, which means that we do not obtain a more general sufficient condition. Furthermore, it is known that nonlinear maps T are in general not weak* continuous so there might not be such a T. In the scalar case it is true that there are no nonlinear weak* continuous maps T, but in the vectorial case these functions exist, and they are called quasiaffine functions or null Lagrangians, see [29, Chapter 5.3.1]. It turns out that the quasiaffine functions are exactly the collection of all minors (and linear combinations thereof), i.e. determinants of submatrices. In particular, when $n = m = 2$ the only nonlinear minor is the map $T(A) = \det(A)$. Note that $\det(A)$ is not level-convex since with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ we find } \det\left(\frac{1}{2}A + \frac{1}{2}B\right) = 0 > -1 = \max\{\det(A), \det(B)\}.$$

Hence, the addition of the minors yields more general functions than just level-convex functions and this justifies the definition of level-polyconvexity below. We denote by $\tau(m, n)$ the number of minors of a $m \times n$ matrix and $T : \mathbb{R}^{m\times n} \to \mathbb{R}^{\tau(m,n)}$ the map consisting of all minors, i.e.

$$T(A) = (A, \operatorname{adj}_2(A), \dots, \operatorname{adj}_{\min\{m,n\}}(A)),$$

with $\operatorname{adj}_s(A)$ the matrix consisting of all $s \times s$ minors of A.

Definition 2.3.2 (level-polyconvexity [13, Definition 3.5])**.** We say that $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is level-polyconvex if there exists a level-convex function $F : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$ such that $f = F \circ T$. △

As mentioned, level-polyconvexity is indeed sufficient for weak* lower semicontinuity, cf. [13, Proposition 3.6].

Proposition 2.3.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be level-polyconvex and lower semicontinuous. Then,*

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$$

is sequentially weak lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.*

Proof. Let $F : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$ be a level-convex and lower semicontinuous function such that $f = F \circ T$. Then we know by Proposition 2.1.2 that

$$\widehat{\mathcal{S}}(v) = \operatorname*{ess\,sup}_{x \in \Omega} F(v(x))$$

is weak* lower semicontinuous on $L^\infty(\Omega; \mathbb{R}^{\tau(m,n)})$. Let $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ then we find by [29, Theorem 8.20] that $T(\nabla u_j) \overset{*}{\rightharpoonup} T(\nabla u)$ in $L^\infty(\Omega; \mathbb{R}^{\tau(m,n)})$. Thus, we conclude

$$\mathcal{S}(u) = \widehat{\mathcal{S}}(T(\nabla u)) \le \liminf_{j \to \infty} \widehat{\mathcal{S}}(T(\nabla u_j)) = \liminf_{j \to \infty} \mathcal{S}(u_j).$$

□

Lastly, we discuss level-rank-1-convexity, which gives a necessary condition by testing with laminate functions, similar to rank-1-convexity in the integral case.

Definition 2.3.4 (level-rank-1-convexity [13, Definition 3.7])**.** We say that $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is level-rank-1-convex if for any $A, B \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A - B) \le 1$ and $\lambda \in [0, 1]$

$$f(\lambda A + (1 - \lambda)B) \le \max\{f(A), f(B)\}. \qquad \triangle$$

Note that level-rank-1-convexity is similar to level-convexity although we assume that $A - B$ has rank 1, i.e. they are rank-1 connected. The reason for this is that gradients of vectorial functions cannot jump between any two values, since it could be that the functions themselves would then not align properly. Indeed, if we take $A, B \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A - B) \ge 2$, then any $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\nabla u \in \{A, B\}$ almost everywhere in Ω satisfies $\nabla u = A$ a.e. or $\nabla u = B$ a.e., see e.g. [64, Theorem 5.13 (i)]. It is exactly the rank-1 connection that allows the gradient to jump between the two values via a reduction to the scalar case. We show that level-rank-1-convexity is necessary for weak* lower semicontinuity. In particular, this shows that level-convexity is necessary for weak* lower semicontinuity in the scalar case.

Proposition 2.3.5 ([61, Theorem 2.4])**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ be such that*

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x \in \Omega} f(\nabla u(x))$$

is weak lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Then, f is level-rank-1-convex.*

Proof. Let $A, B \in \mathbb{R}^{m \times n}$ such that $\operatorname{rank}(A - B) = 1$, then we can find $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ with $|b| = 1$ such that $B - A = a \otimes b$, where the tensor product is defined as $(a \otimes b)_{ij} = a_i b_j$ for $1 \le i \le m$ and $1 \le j \le n$. Let $\lambda \in (0, 1)$ and denote $C = \lambda A + (1 - \lambda)B$. Define the laminate function $\varphi(x) = \psi(x \cdot b - \lfloor x \cdot b \rfloor)a$ for $x \in \mathbb{R}^n$ with

$$\psi : [0, 1] \to \mathbb{R}, \qquad \psi(t) = \begin{cases} -(1 - \lambda)t & \text{if } t \in [0, \lambda], \\ \lambda t - \lambda & \text{if } t \in (\lambda, 1]. \end{cases}$$

We observe that $\varphi(x+b) = \varphi(x)$ since $|b| = 1$, and that φ is constant in directions orthogonal to b. Hence, we find that $\varphi \in W^{1,\infty}_{per}(Y;\mathbb{R}^m)$ for any n-cube Y with a face orthogonal to b. Moreover, we can calculate for a.e. $x \in \mathbb{R}^n$

$$\nabla\varphi(x) = \psi'(x\cdot b - \lfloor x \cdot b\rfloor)a \otimes b \in \{-(1-\lambda)(B-A), \lambda(B-A)\},$$

since the derivative of ψ satisfies $\psi'(t) \in \{-(1-\lambda), \lambda\}$ for a.e. $t \in \mathbb{R}$. This shows $C + \nabla\varphi(x) \in \{A, B\}$ for a.e. $x \in \mathbb{R}^n$ and we can apply Lemma 2.1.13 to find

$$f(C) \leq \operatorname*{ess\,sup}_{x\in Y} f(C + \nabla\varphi(x)) = \max\{f(A), f(B)\}.$$

□

We can now state the relations between the different notions. We emphasize the similarity with Theorem 1.1.16 from the integral case.

Theorem 2.3.6 ([13, Adaptation of Corollary 3.9]). *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be lower semicontinuous then we have the following implications:*

$$f \text{ level-convex } \Rightarrow f \text{ level-polyconvex } \Rightarrow f \text{ level-quasiconvex } \Rightarrow f \text{ level-rank-1-convex}.$$

In particular, if $m = 1$ or $n = 1$ all notions coincide.

Proof. If f is level-convex then it is clearly level-polyconvex by letting $F : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$, as in the definition of level-polyconvexity (Definition 2.3.2), only depend on the 1×1-minors. If f is level-polyconvex and lower semicontinuous then we know from Proposition 2.3.3 that the corresponding supremal functional is weak* lower semicontinuous (also on subdomains). By Theorem 2.1.14 this shows that f is level-quasiconvex. If f is level-quasiconvex then Theorem 2.1.11, applied to an x- and u-independent integrand, shows that for any $\Omega \subset \mathbb{R}^n$ open and bounded the functional

$$S(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$$

is weak* lower semicontinuous on $W^{1,\infty}(\Omega;\mathbb{R}^m)$. Proposition 2.3.5 now shows that f is level-rank-1-convex. Lastly, when $m = 1$ or $n = 1$ it is easy to see that level-convexity coincides with level-rank-1-convexity and hence, all notions coincide. □

Remark 2.3.7. As mentioned after Proposition 2.1.10, any level-quasiconvex function is lower semicontinuous, while this is not true for level-convex functions. This is why we assumed lower semicontinuity of f in the above theorem. It is also interesting to mention that the implications

$$f \text{ level-convex } \Rightarrow f \text{ level-polyconvex } \Rightarrow f \text{ level-rank-1-convex}$$

still hold if f is not lower semicontinuous. The first implication is clear, whereas the second can be deduced from the fact that for any $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A - B) \leq 1$ and $\lambda \in [0, 1]$ we have

$$T(\lambda A + (1-\lambda)B) = \lambda T(A) + (1-\lambda)T(B),$$

see [29, Lemma 5.5]. This identity should be understood as T being rank-1-affine. △

2.4 Aronsson Equations

In this section we touch upon the Aronsson equations, which are a system of partial differential equations that minimizers of supremal functionals must satisfy. They can equivalently be seen as a type of Euler-Lagrange equation in the supremal context. There are two main difficulties in the derivation of the Aronsson equations. We require the notion of an absolute minimizer, see Definition 2.4.1, which minimizes the functional also on subdomains, in order to conclude that the Aronsson equation is satisfied. Furthermore, when the minimizer is not smooth, we need a definition of weak solution to the Aronsson equation, which is given by the theory of viscosity solutions.

To derive an optimality condition for a functional

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), \nabla u(x)),$$

we could naively try to calculate its first variation. Under certain regularity assumptions, Danskin's theorem [17] shows that for $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ we have

$$\frac{d}{dt}\mathcal{S}(u + t\varphi)|_{t=0} = \max\{D_z f(x, u, \nabla u)\varphi + D_A f(x, u, \nabla u)\nabla\varphi \mid x \in \Omega_0\}, \tag{2.12}$$

where $\Omega_0 = \{x \in \overline{\Omega} \mid \mathcal{S}(u) = f(x, u(x), \nabla u(x))\}$ and $D_z f$, $D_A f$ are the derivative of f with regard to its second and third entry, respectively. When u is a minimizer of $\mathcal{S}$ over the boundary-value space $W_g^{1,\infty}(\Omega; \mathbb{R}^m)$ for some $g \in W^{1,\infty}(\Omega; \mathbb{R}^m)$, we find that $S(u + t\varphi)$ attains a minimum at $t = 0$, whence (2.12) has to be zero. In the integral case we would now proceed with integration by parts and the fundamental lemma in the calculus of variations to derive the Euler-Lagrange equation. It is not clear that we can do anything similar with (2.12). We could try certain choices of φ and see what we find, but this is hard if we do not know what to look for.

The idea of Aronsson in [6] was to deduce an equation by looking at the Euler-Lagrange equation of the approximating functionals

$$\mathcal{I}_p(u) = \left(\int_\Omega f(x, u(x), \nabla u(x))^p \, dx\right)^{1/p}$$

as $p \to \infty$, see also Section 2.6. This approach led to the following equation

$$D(f(x, u(x), \nabla u(x)) \cdot D_A f(x, u(x), \nabla u(x)) = 0, \tag{2.13}$$

which is known as the Aronsson equation. Here $D(f(x, u(x), \nabla u(x)) \in \mathbb{R}^n$ denotes the derivative of $x \mapsto f(x, u(x), \nabla u(x))$ and we take its inner product with the columns of $D_A f(x, u(x), \nabla u(x)) \in \mathbb{R}^{m\times n}$. The approach via L^p-approximation yielded only a formal derivation of (2.13), and in order to conclude that a minimizer of the supremal functional satisfied (2.13), Aronsson realized that it is crucial to assume that it is an absolute minimizer.

Definition 2.4.1 (Absolute minimizer)**.** We say that $u_0 \in W_g^{1,\infty}(\Omega; \mathbb{R}^m)$ is an absolute minimizer of $\mathcal{S}$ if for each open set $O \subset \Omega$ we have that u_0 minimizes

$$\mathcal{S}(u, O) = \operatorname*{ess\,sup}_{x\in O} f(x, u(x), \nabla u(x))$$

over $W_{u_0}^{1,\infty}(O; \mathbb{R}^m)$. △

Intuitively, this definition states that u_0 minimizes $\mathcal{S}$ over every subdomain. This subtle property is one that is automatically satisfied for minimizers of integral functionals due to their additivity. For supremal functionals this is clearly not the case. If we change a minimizer slightly in a neighborhood where the supremum is not attained, then we still end up with a minimizer. Hence, there is no guarantee that it minimizes the functional in this neighborhood. Let us now give a proof that a smooth enough absolute minimizer satisfies the Aronsson equation. This proof is from [13, Theorem 5.2], although this contains an error that is resolved in [45, p. 128] for $f(x,z,A) = |A|^2$. We straightforwardly generalize this argument to arbitrary f.

Theorem 2.4.2. *Let $f \in C^1(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$ and $u_0 \in C^2(\Omega;\mathbb{R}^m) \cap W^{1,\infty}_g(\Omega;\mathbb{R}^m)$ be an absolute minimizer of*

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), \nabla u(x)).$$

Then u_0 satisfies

$$D(f(x,u_0(x),\nabla u_0(x))\cdot D_A f(x,u_0(x),\nabla u_0(x)) = 0 \qquad x \in \Omega. \tag{2.14}$$

Proof. Take $x_0 \in \Omega$ and $r > 0$ small enough such that $B_r(x_0) \Subset \Omega$. Since u_0 minimizes $\mathcal{S}(\cdot, B_r(x_0))$ it follows from Danskin's theorem [17] that for every smooth $\varphi \in W^{1,\infty}_0(B_r(x_0);\mathbb{R}^m)$ we have

$$0 = \frac{d}{dt}\mathcal{S}(u_0 + t\varphi, B_r(x_0))|_{t=0} = \max\{D_z f(x,u_0,\nabla u_0)\varphi + D_A f(x,u_0,\nabla u_0)\nabla\varphi \mid x \in S\} \tag{2.15}$$

with $S = \{x \in \overline{B_r(x_0)} \mid \mathcal{S}(u_0, B_r(x_0)) = f(x,u_0(x),\nabla u_0(x))\}$. Danskin's theorem is applicable since the function

$$\gamma : \mathbb{R}\times\overline{B_r(x_0)} \to \mathbb{R}, \quad \gamma(t,x) = f(x, u_0(x) + t\varphi(x), \nabla u_0(x) + t\nabla\varphi(x))$$

is jointly continuous and continuously differentiable in its first argument. Choose $\varphi \in W^{1,\infty}_0(B_r(x_0),\mathbb{R}^m)$ with its components given by

$$\varphi_i(x) = \frac{r^2}{2} - \frac{|x-x_0|^2}{2} \quad \text{for } i = 1,\dots,m.$$

Then, by (2.15) there is a $x_r \in S$ such that

$$0 = D_z f(x_r, u_0(x_r), \nabla u_0(x_r))\varphi(x_r) + D_A f(x_r, u_0(x_r), \nabla u_0(x_r))(x_r - x_0). \tag{2.16}$$

Observe that $\varphi(x_r)/r \to 0$ as $r \to 0$. Furthermore, x_r is a point at which $f(x,u_0(x),\nabla u_0(x))$ attains its maximum on $\overline{B_r(x_0)}$. Since the derivative $D(f(x,u_0(x),\nabla u_0(x))$ is locally the direction of steepest increase, we deduce that $x_r - x_0$ points in this direction as $r \to 0$, i.e.

$$\lim_{r\to 0}\frac{x_r - x_0}{r} = \frac{D(f(x,u_0(x),\nabla u_0(x))}{|D(f(x,u_0(x),\nabla u_0(x))|}. \tag{2.17}$$

Note that (2.17) is correct unless $D(f(x,u_0(x),\nabla u_0(x)) = 0$, but then (2.14) holds also. We conclude that the result follows from dividing (2.16) by r and letting $r \to 0$. □

Remark 2.4.3. It is the observation of (2.17) made in [45, page 128] that is omitted in [13, Theorem 5.2]. In [13, Theorem 5.2] the authors conclude that dividing (2.16) by r and letting $r \to 0$ yields

$$D_A f(x_r, u_0(x_r), \nabla u_0(x_r)) = 0,$$

but this argument only works in the case $n = 1$. △

Example 2.4.4. When $m = 1$ and $f(x, z, A) = \frac{1}{2}|A|^2$ the Aronsson equation turns into the infinity Laplace equation

$$\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u = \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0.$$

Indeed, $D_A f(x, u(x), \nabla u(x)) = \nabla u(x)$ and $D(f(x, u(x), \nabla u(x))) = D(\frac{1}{2}|\nabla u(x)|^2) = \nabla^2 u(x) \nabla u(x)$. The infinity Laplace equation plays a central role in the theory of Aronsson equations similarly to the Laplace equation for elliptic equations. △

At this point, Theorem 2.4.2 raises two natural questions. The first is whether absolute minimizers exist for general supremal functionals. The second is whether an absolute minimizer also satisfies the Aronsson equation in a weak sense when it is not C^2. The first question is partially answered in [12] where Barron, Jensen & Wang show that under certain assumptions absolute minimizers exist in the scalar case ($m = 1$ or $n = 1$). Their argument, just like the derivation of the Aronsson equation, uses approximation by L^p-functionals. This will be studied further in Section 2.6, see Theorem 2.6.3. At this point in time, I am not aware of any existence results for absolute minimizers in the vectorial case.

The second question is a bit more involved. In the integral case it follows immediately that even when minimizers are not smooth they still satisfy the Euler-Lagrange equation in a type of distributional sense. This is very natural and easy to derive. For supremal functionals, we could say that (2.15) is the weak version of the Aronsson equation. The problem is that this needs to be well-defined, and intuitively, the relation with (2.14) is not clear.

It turns out that the answer is to consider the notion of viscosity solutions introduced by Crandall & Lions in [27]. This notion is a type of weak solution for degenerate nonlinear PDEs like the Aronsson equation. The remarkable result is that absolute minimizers of supremal functionals will be viscosity solutions of the Aronsson equation even when they are not smooth. This result was originally proven for the infinity Laplacian by Jensen in his groundbreaking paper [44]. He was the first to bring this connection to light, and it spurred a lot of interest from both the calculus of variations experts as well as the PDE community. A notable extension is [12], in which Barron, Jensen & Wang establish that absolute minimizers are viscosity solutions for more general supremal functionals in the scalar case. Their argument was later simplified in [26]. Additionally, a nice overview paper on viscosity solutions and the infinity Laplacian is given in [7]. We will not go in depth into the theory of viscosity solutions as it is not the main aim of the thesis, but the reader is encouraged to view the literature.

2.5 Relaxation

We continue by investigating the relaxation of supremal functionals. Abstractly speaking, the relaxation of a general functional $\mathcal{F}$ is the largest functional below $\mathcal{F}$ that is lower semicontinuous with respect to some desired topology. In our case, we are interested in the relaxation of supremal functionals with respect to the weak* convergence in $W^{1,\infty}(\Omega)$. The relevance of relaxations comes from applications in which a given functional is not weak* lower semicontinuous and thus, might not have a minimizer. By finding the relaxation, we have a functional that is weak* lower semicontinuous, and its minimizers give information on the asymptotic behavior of minimizing sequences of the original functional.

We consider the scalar case with $\Omega \subset \mathbb{R}^n$ open and bounded, $f : \mathbb{R}^n \to \mathbb{R}$ and

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\Omega). \tag{2.18}$$

We are interested in the relaxation $\mathcal{S}^{\text{rel}}$ which can be characterized as

$$\mathcal{S}^{\text{rel}}(u) = \inf\{\liminf_{j\to\infty} \mathcal{S}(u_j) \mid u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega)\} \quad \text{for } u \in W^{1,\infty}(\Omega). \tag{2.19}$$

We know that the weak* lower semicontinuity of $\mathcal{S}$ is equivalent to f being lower semicontinuous and level-convex. Hence, we expect to obtain the relaxation by taking the lower semicontinuous and level-convex hull of f. We denote this hull by f^{lc} and it is defined as

$$f^{\text{lc}}(\xi) := \inf\{c \mid \xi \in \overline{\text{Conv}}(L_c(f))\} \quad \text{for } \xi \in \mathbb{R}^n.$$

This definition ensures that the sub-level sets satisfy $L_c(f^{\text{lc}}) = \overline{\text{Conv}}(L_c(f))$, which implies that f^{lc} is lower semicontinuous and level-convex. Additionally, f^{lc} is the largest lower semicontinuous and level-convex function majorized by f. The first proof of the relaxation result appeared in [14] in the one-dimensional scalar case. The proof we consider is for general dimension n and is adapted from [60, Theorem 2.6] by removing the x- and u-dependence. It relies on the following differential inclusion result.

Theorem 2.5.1 ([30, Theorem 2.10])**.** *Let $\Omega \subset \mathbb{R}^n$ open and $E \subset \mathbb{R}^n$. Suppose that $\varphi \in W^{1,\infty}(\Omega)$ satisfies $\nabla\varphi \in E \cup \text{int}(\text{Conv}(E))$ for a.e. $x \in \Omega$ then, for every $\epsilon > 0$ there exists a $u \in W^{1,\infty}_{\varphi}(\Omega)$ such that*

$$\begin{cases} \|u - \varphi\|_{L^\infty(\Omega)} \leq \epsilon, & \\ \nabla u(x) \in E & \text{for a.e. } x \in \Omega. \end{cases}$$

We now present the relaxation result.

Theorem 2.5.2 ([60, Adaptation from Theorem 2.6])**.** *Let $\Omega \subset \mathbb{R}^n$ open and bounded and $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and coercive in the sense that $f(\xi) \to \infty$ as $|\xi| \to \infty$. Then the relaxation of $\mathcal{S}$ in (2.18) with respect to the weak* convergence in $W^{1,\infty}(\Omega)$ is given by*

$$\mathcal{S}^{\text{rel}}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{\text{lc}}(\nabla u(x)) \quad \textit{for } u \in W^{1,\infty}(\Omega).$$

Proof. We denote $\widetilde{\mathcal{S}}(u) = \text{ess sup}_{x\in\Omega} f^{\text{lc}}(\nabla u(x))$. Since f^{lc} is a lower semicontinuous and level-convex function majorized by f, we find that $\widetilde{\mathcal{S}}$ is weak* lower semicontinuous (Theorem 2.1.4) and $\widetilde{\mathcal{S}} \leq \mathcal{S}$. Hence, we have $\widetilde{\mathcal{S}} \leq \mathcal{S}^{\text{rel}}$.

For the converse, take $u \in W^{1,\infty}(\Omega)$ and set $c := \widetilde{\mathcal{S}}(u)$. Then for every $\delta > 0$ we find that $E = \{\xi \in \mathbb{R}^n \mid f(\xi) < c + \delta\}$ is open due to continuity of f, and consequently, $\mathrm{Conv}(E)$ is open as well. Hence, for a.e. $x \in \Omega$,

$$\nabla u(x) \in L_c(f^{\mathrm{lc}}) = \overline{\mathrm{Conv}}(L_c(f)) \subset \mathrm{Conv}(E) = \mathrm{int}(\mathrm{Conv}(E)),$$

where the inclusion also uses continuity of f. We conclude by Theorem 2.5.1 that we have a sequence $(u_j)_j \subset W^{1,\infty}(\Omega)$ such that $\|u_j - u\|_{L^\infty(\Omega)} \leq 1/j$ and

$$\nabla u_j(x) \in E \quad \text{for a.e. } x \in \Omega. \tag{2.20}$$

Since E is a bounded set by the coercivity assumption on f, we conclude that $(u_j)_j$ is a bounded sequence in $W^{1,\infty}(\Omega)$. Therefore, the sequence $(u_j)_j$ converges weak* in $W^{1,\infty}(\Omega)$ to its L^∞-limit u. In view of (2.19), we find

$$\mathcal{S}^{\mathrm{rel}}(u) \leq \liminf_{j\to\infty} \mathcal{S}(u_j) = \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u_j(x)) \leq c + \delta,$$

with the last inequality following from (2.20) and the definition of E. This proves the result by the arbitrariness of $\delta > 0$. □

In [60, Theorem 2.6] this result is proven with additional (x, u)-dependence under the assumption that f is continuous in all arguments. This is done by locally freezing the (x, u)-terms using the continuity of f and subsequently applying the above argument on each localized part of Ω.

Even though the continuity assumption may seem harmless, dropping it changes the situation significantly. Take for example a Carathéodory function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and the corresponding functional

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, \nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\Omega).$$

In this case f need not be continuous in x, and it turns out that then level-convexity of f in the second argument is not anymore necessary for weak* lower semicontinuity of $\mathcal{S}$. This is exhibited in [39, Remark 3.1]. Hence, we cannot expect the relaxation to be obtained by taking the level-convex hull of the integrand. This is confirmed by the explicit example in [39, Example 3.2]. Moreover, the authors show in [39] that the relaxation of 1-homogeneous supremal functionals can naturally be represented as a difference quotient. Whether or not it can also be represented as a supremal functional is not known in general. The specific instance of [60, Theorem 2.9] shows that the relaxation can be represented by a supremal functional under the assumption of a countable supremality condition. Furthermore, the supremand is the level-convex hull of a certain choice of supremand that represents $\mathcal{S}$. This particular choice was introduced in [21], where the representation of supremal functionals was studied. We will not go in depth into these considerations as they are beyond the scope of this thesis.

When we turn to the vectorial case it appears that there are no relaxation results in the literature. However, in the scalar case we saw that the relaxation follows from the differential inclusion Theorem 2.5.1. These types of differential inclusion results have been extensively studied in the vectorial case, see e.g. [30, 57]. Still, none of these results are as general or elegant as in the scalar case. Hence, we could only try to prove the relaxation in rather specific cases, which is not the aim of this section.

2.6 L^p-Approximation

In this section we study the L^p-approximation of supremal functionals by integral functionals. The approximation is done using the language of Γ-convergence, cf. Section 1.1.5, which is especially suited for minimization problems as it encodes information about minimizers, see Theorem 1.1.19. Intuitively, the idea of L^p-approximation is to approximate

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$$

by

$$\mathcal{I}_p(u) = \left(\int_\Omega f^p(\nabla u(x))\,dx\right)^{1/p}$$

similarly to the way that the L^p-norm approximates the L^∞-norm as $p \to \infty$. One thing to note is that in the Γ-limit there is also a relaxation process happening apart from the L^p-approximation. This is because Γ-limits are always lower semicontinuous. In [23], the L^p-approximation is proven under the assumption of a generalized Jensen inequality, different from Lemma 2.1.5, which guarantees that the relaxation does not affect the supremand. We adapt their approach to the fractional setting in Section 2.6. Here, we want to study the proof by Prinari in [61, Theorem 3.2], which does not need any convexity assumption. Because of this, the supremand might change in the Γ-limit. The theorem states the following.

Theorem 2.6.1 ([61, Theorem 3.2]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f : \mathbb{R}^{m\times n} \to [0,\infty)$ a continuous function satisfying for $c, C > 0$*

$$c|A| \le f(A) \le C(1+|A|) \quad \text{for all } A \in \mathbb{R}^{m\times n}.$$

For every $p \ge 1$ we define $\mathcal{I}_p : C(\overline{\Omega};\mathbb{R}^m) \to [0,\infty]$ by

$$\mathcal{I}_p(u) = \begin{cases} \left(\int_\Omega f^p(\nabla u(x))\,dx\right)^{1/p} & \text{if } u \in C(\overline{\Omega};\mathbb{R}^m) \cap W^{1,p}(\Omega;\mathbb{R}^m),\\ \infty & \text{otherwise.}\end{cases}$$

Then $(\mathcal{I}_p)_p$ Γ-converges with respect to the uniform convergence as $p \to \infty$ to $\widetilde{\mathcal{S}} : C(\overline{\Omega};\mathbb{R}^m) \to [0,\infty]$ given by

$$\widetilde{\mathcal{S}}(u) = \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f^\infty(\nabla u(x)) & \text{if } u \in C(\overline{\Omega};\mathbb{R}^m) \cap W^{1,\infty}(\Omega;\mathbb{R}^m),\\ \infty & \text{otherwise.}\end{cases}$$

Here f^∞ is defined as

$$f^\infty(A) = \lim_{p\to\infty} ((f^p)^{\mathrm{qc}}(A))^{1/p} = \sup_{p\ge 1} ((f^p)^{\mathrm{qc}}(A))^{1/p},$$

where $(f^p)^{\mathrm{qc}}$ is the quasiconvexification of f^p.

Proof sketch. If we scale $\mathcal{I}_p$ by $|\Omega|^{1/p}$ then Hölder's inequality shows that we obtain a sequence of non-decreasing functionals. For such functionals the Γ-limit coincides with the pointwise limit of their relaxations [19, Remark 1.40]. Since $|\Omega|^{1/p} \to 1$ as $p \to \infty$ this also holds for the original sequence $(\mathcal{I}_p)_p$. We know that the relaxation of integral functionals on $W^{1,p}(\Omega;\mathbb{R}^m)$ with respect to the weak convergence is given by taking the quasiconvex hull of the integrand (Theorem 1.1.17). By making some adaptations, see [61, Theorem 3.2] for details, we can similarly show that the relaxation of $\mathcal{I}_p$ with respect to the uniform convergence is given by

$$\mathcal{I}_p^{\mathrm{rel}}(u) = \begin{cases} \left(\int_\Omega (f^p)^{\mathrm{qc}}(\nabla u(x))\,dx\right)^{1/p} & \text{if } u \in C(\overline{\Omega};\mathbb{R}^m) \cap W^{1,p}(\Omega;\mathbb{R}^m), \\ \infty & \text{otherwise.} \end{cases}$$

It remains to verify $\mathcal{I}_p^{\mathrm{rel}} \to \widetilde{\mathcal{S}}$ pointwise as $p \to \infty$. It can be shown that $((f^p)^{\mathrm{qc}})^{1/p} \uparrow f^\infty$ as $p \to \infty$ so that for $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$

$$\mathcal{I}_p^{\mathrm{rel}}(u) = \left(\int_\Omega (f^p)^{\mathrm{qc}}(\nabla u(x))\,dx\right)^{1/p} \le |\Omega|^{1/p} \operatorname*{ess\,sup}_{x\in\Omega} f^\infty(\nabla u(x)).$$

This proves $\lim_{p\to\infty} \mathcal{I}_p^{\mathrm{rel}} \le \widetilde{\mathcal{S}}$. For the converse, we may take $u \in C(\overline{\Omega};\mathbb{R}^m)$ such that

$$\sup_{p\in[1,\infty)} \mathcal{I}_p^{\mathrm{rel}}(u) = M < \infty.$$

Hence the lower bound on f, which implies $(f^p)^{\mathrm{qc}}(A) \ge c^p|A|^p$, shows that

$$\sup_{p\in[1,\infty)} \|u\|_{W^{1,p}(\Omega)} \le M/c.$$

In particular, $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ so that $\widetilde{\mathcal{S}}(u) < \infty$. Therefore, we may find for every $\epsilon > 0$ a subset $U_\epsilon \subset \Omega$ with positive measure such that

$$\operatorname*{ess\,sup}_{x\in\Omega} f^\infty(\nabla u(x)) \le f^\infty(\nabla u(x)) + \epsilon \quad \text{for all } x \in U_\epsilon.$$

As a consequence,

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in\Omega} f^\infty(\nabla u(x))|U_\epsilon| &\le \int_{U_\epsilon} f^\infty(\nabla u(x))\,dx + \epsilon|U_\epsilon| \\ &= \lim_{p\to\infty} \int_{U_\epsilon} ((f^p)^{\mathrm{qc}}(\nabla u(x))^{1/p}\,dx + \epsilon|U_\epsilon| \\ &\le \lim_{p\to\infty} \left(\int_{U_\epsilon} (f^p)^{\mathrm{qc}}(\nabla u(x))\,dx\right)^{1/p} |U_\epsilon|^{1-1/p} + \epsilon|U_\epsilon|, \end{aligned}$$

with the second line being the monotone convergence theorem and the last Hölder's inequality. Dividing both sides by $|U_\epsilon|$ yields $\lim_{p\to\infty} \mathcal{I}_p^{\mathrm{rel}}(u) + \epsilon \ge \widetilde{\mathcal{S}}(u)$, which proves the result by the arbitrariness of ϵ. □

Remark 2.6.2. The reason why we consider the Γ-limit with respect to the uniform convergence as opposed to weak* convergence in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ is because uniform convergence is defined by a metric. △

A function for which $f^\infty = f$ is called curl-∞ quasiconvex [4]. Such functions, under the growth conditions of the previous theorem, are in particular level-quasiconvex. This is because the Γ-limit $\widetilde{\mathcal{S}}$ is lower semicontinuous with respect to uniform convergence and thus also to weak* convergence in $W^{1,\infty}(\Omega;\mathbb{R}^m)$. In the scalar case $m = 1$, it follows from [62, Remark 5.2] that, with f satisfying the hypotheses of Theorem 2.6.1, f^∞ is given by the lower semicontinuous and level-convex hull f^{lc} introduced in the previous section. Thus, if f is level-convex then it is curl-∞ quasiconvex. In particular, this means that we could replace f^∞ by f in Theorem 2.6.1.

As an application of the L^p-approximation result we can show the existence of absolute minimizers of supremal functionals, see Definition 2.4.1. This proof is an adaptation of [23, Proposition 4.3] and [12, Lemma 2.4]. We use the notation

$$\mathcal{S}(u,O) = \operatorname*{ess\,sup}_{x\in O} f(\nabla u(x)) \quad \text{or} \quad \mathcal{I}_p(u,O) = \left(\int_O f^p(\nabla u(x))\,dx\right)^{1/p}$$

to emphasize the dependence of the functionals on the domain O.

Theorem 2.6.3. *Let $m = 1$ and Ω and f be as in Theorem 2.6.1. If f is level-convex, then for any $g \in W^{1,\infty}(\Omega)$ there exists an absolute minimizer u_∞ on $W^{1,\infty}_g(\Omega)$ of*

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)).$$

Proof. With $\mathcal{I}_p$ as in Theorem 2.6.1 consider a sequence $(u_p)_{p\in[1,\infty)}$ with $u_p \in W^{1,p}_g(\Omega)$ such that

$$\mathcal{I}_p^p(u_p) \le \inf_{u\in W^{1,p}_g(\Omega)} \mathcal{I}_p^p(u) + \delta_p^p \qquad \text{with } \delta_p \to 0 \text{ as } p \to \infty.$$

We show that the sequence $(u_p)_p$ converges up to subsequence to an absolute minimizer u_∞ of $\mathcal{S}$. We have $\mathcal{I}_p(u_p) \le \mathcal{I}_p(g) + \delta_p \le |\Omega|^{1/p}\mathcal{S}(g) + \delta_p$ which shows that $\mathcal{I}_p(u_p)$ is uniformly bounded in p. Hence, the lower bound on f in conjuction with Poincaré's inequality (Theorem 1.1.7) implies that $(u_p)_{p>q}$ is a bounded sequence in $W^{1,q}(\Omega)$ with a bound uniform in q, i.e. there is a $M > 0$ such that

$$\sup_{p>q} \|u_p\|_{W^{1,q}(\Omega)} \le M \quad \text{for all } q \in [1,\infty). \tag{2.21}$$

Thus, up to a subsequence (not relabeled) $u_p \rightharpoonup u_\infty$ in $W^{1,q}(\Omega)$ for all $1 \le q < \infty$. In particular, we deduce that $u_p \to u_\infty$ uniformly by Morrey's inequality. Since $\|u\|_{W^{1,q}(\Omega)} \le M$ for all $q \in [1,\infty)$ in light of (2.21), we also conclude that $u_\infty \in W^{1,\infty}_g(\Omega)$.

We now show that u_∞ is an absolute minimizer. Let $O \subset \Omega$ open and take a $\varphi \in W^{1,\infty}_0(O)$. Then, we need to show that

$$\mathcal{S}(u_\infty, O) \le \mathcal{S}(u_\infty + \varphi, O). \tag{2.22}$$

Since the part $\varphi = 0$ is not relevant, and by considering the regions where $\varphi > 0$ and $\varphi < 0$ individually, we may assume that $\varphi > 0$. Define for $p \geq 1$ and $\epsilon > 0$ the set

$$O_{p,\epsilon} = \{x \in \Omega \mid u_p(x) + \epsilon < u_\infty(x) + \varphi(x)\},$$

where we have trivially extended φ to Ω as zero. Since $u_p \to u_\infty$ uniformly, we find for $p \geq p_\epsilon$ that $u_p - \epsilon < u_\infty < u_p + \epsilon$ in Ω. This shows that for such p we have $O_{p,\epsilon} \subset O$. As a consequence, on the boundary of $O_{p,\epsilon}$ the identity $u_p = u_\infty + \varphi - \epsilon$ holds. Hence, we can use the additivity of the integral functional $\mathcal{I}_p^p$ to conclude for $p \geq p_\epsilon$

$$\mathcal{I}_p^p(u_p, O_{p,\epsilon}) \leq \mathcal{I}_p^p(u_\infty + \varphi - \epsilon, O_{p,\epsilon}) + \delta_p^p = \mathcal{I}_p^p(u_\infty + \varphi, O_{p,\epsilon}) + \delta_p^p, \tag{2.23}$$

where the second equality uses that the $-\epsilon$ term does not affect the gradient. Additionally, for $p \geq p_\epsilon$ we have that $O_\epsilon = \{x \in O \mid \varphi(x) > 2\epsilon\} \subset O_{p,\epsilon}$. Thus, taking pth roots in (2.23) shows that for such p

$$\mathcal{I}_p(u_p, O_\epsilon) \leq \mathcal{I}_p(u_\infty + \varphi, O_{p,\epsilon}) + \delta_p \leq \mathcal{S}(u_\infty + \varphi, O)|O|^{1/p} + \delta_p.$$

Using the liminf-inequality from the Γ-convergence result Theorem 2.6.1 on the domain O_ϵ, noting that $f = f^\infty$, yields

$$\mathcal{S}(u_\infty, O_\epsilon) \leq \liminf_{p\to\infty} \mathcal{I}_p(u_p, O_\epsilon) \leq \mathcal{S}(u_\infty + \varphi, O).$$

Letting $\epsilon \downarrow 0$ gives (2.22). □

2.7 Supremal Functionals on $\mathbb{R}^n$

Here, we aim to study supremal functionals on $\mathbb{R}^n$ and characterize their weak* lower semicontinuity. This setting is qualitatively different from the bounded domain case and to my knowledge has not yet been considered in the literature. We will apply these results later to the fractional setting, where the functionals are always over $\mathbb{R}^n$.

Consider the scalar case, where $f : \mathbb{R}^n \to \mathbb{R}$ and

$$\mathcal{S}(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla u(x)) \quad \text{for } u \in W^{1,\infty}(\mathbb{R}^n). \tag{2.24}$$

If the supremum was taken over a bounded domain, then Theorem 2.1.4 and Proposition 2.3.5 show that the necessary and sufficient condition for weak* lower semicontinuity would be level-convexity of f. However, it turns out that a weaker condition is needed for the unbounded case. We dub this notion balanced level-convexity.

Definition 2.7.1 (balanced level-convexity)**.** We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is balanced level-convex if for $\xi, \xi_1, \ldots, \xi_k \in \mathbb{R}^n$

$$f(\xi) \leq \max\{f(\xi_1), \ldots, f(\xi_k)\}, \tag{2.25}$$

whenever ξ and 0 lie in $\operatorname{Conv}(\xi_1, \ldots, \xi_k)$. △

If not for the assumption that 0 lies in $\operatorname{Conv}(\xi_1, \ldots, \xi_k)$, this would be equivalent to level-convexity. Observe that if we only assume that (2.25) holds for combinations where $k = 2$, then it does not imply that the function is balanced level-convex, see Example 2.7.2. This is different from the definition for level-convexity, Definition 2.1.3. However, we may restrict $k \leq n + 1$ in light of Carathéodory's theorem, see e.g. [29, Theorem 2.13].

Example 2.7.2. Take $E = \{\zeta_1, \zeta_2, \zeta_3\} \subset \mathbb{R}^2$ with $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}^2$ satisfying

$$0 \in \operatorname{Conv}(\zeta_1, \zeta_2, \zeta_3) \quad \text{and} \quad 0 \notin \operatorname{Conv}(\zeta_i, \zeta_j) \text{ for all } 1 \leq i, j \leq 3. \tag{2.26}$$

We could, for example, choose $\zeta_1 = (1, 0)$, $\zeta_2 = (0, 1)$ and $\zeta_3 = (-1, -1)$. Then, we define $f = -\mathbb{1}_E$, cf. Section 1.1.1. In view of (2.26), it follows for any $\xi, \xi_1, \xi_2 \in \mathbb{R}^2$ with $0, \xi \in \operatorname{Conv}(\xi_1, \xi_2)$ that $\{\xi_1, \xi_2\} \not\subset E$. Therefore, we deduce that

$$f(\xi) \leq 0 = \max\{f(\xi_1), f(\xi_2)\},$$

which gives (2.25) for $k = 2$. However, since $0 \in \operatorname{Conv}(\zeta_1, \zeta_2, \zeta_3)$ we see from

$$f(0) = 0 > -1 = \max\{f(\zeta_1), f(\zeta_2), f(\zeta_3)\}$$

that f is not balanced level-convex. △

Let us now understand balanced level-convexity a bit more. The first thing we notice is that balanced level-convexity is equivalent to the following two properties:

$$\begin{cases} \max\{f, f(0)\} \text{ is level-convex,} \\ f(0) \leq \max\{f(\xi_1), \ldots, f(\xi_k)\} & \text{when } 0 \in \operatorname{Conv}(\xi_1, \ldots, \xi_k). \end{cases} \tag{C1}$$

Clearly, balanced level-convexity implies the above two. On the other hand, if we have ξ and 0 in $\operatorname{Conv}(\xi_1, \ldots, \xi_k)$ then by the first property we find

$$f(\xi) \leq \max\{f(\xi_1), \ldots, f(\xi_k), f(0)\}.$$

Using the second property, we may remove $f(0)$ from the maximum on the right-hand side, thus yielding balanced level-convexity. Taking (C1) as a starting point it is also possible to see balanced level-convexity as a condition on the sub-level sets $L_c(f) := \{\xi \in \mathbb{R}^n \mid f(\xi) \leq c\}$. Explicitly, we have that f is balanced level-convex if and only if

$$\begin{cases} L_c(f) \text{ is convex for } c \geq f(0), \\ 0 \notin \operatorname{Conv}(L_c(f)) \text{ for } c < f(0). \end{cases} \tag{C2}$$

Indeed, $\max\{f, f(0)\}$ is level-convex if and only if

$$L_c(\max\{f, f(0)\}) = L_{\max\{c, f(0)\}}(f) \text{ is convex for all } c \in \mathbb{R},$$

which is equivalent to the first condition in (C2). The equivalence of the second properties in (C1) and (C2), respectively, is readily seen. We now present some examples of balanced level-convex functions that are not level-convex.

Example 2.7.3. a) Take any non-convex set $N \subset \mathbb{R}^n$ such that $0 \notin \operatorname{Conv}(N)$. Then, define $f = -\mathbb{1}_N$. Clearly, f satisfies the characterization (C2). However, the sub-level set $L_{-1}(f) = N$ is not convex, so f is not level-convex.

b) Take $R > 0$ such that the set N from a) is contained in $B_R(0)$. Then, we define

$$h : \mathbb{R}^n \to \mathbb{R}, \quad h(\xi) = \begin{cases} f(\xi) & \text{if } \xi \in B_R(0), \\ |\xi| - R & \text{else,} \end{cases}$$

with f as in part a). We observe that h is balanced level-convex since $L_c(h)$ is empty for $c < -1$, given by N for $-1 \leq c < 0$ and equal to the convex set $\overline{B_{R+c}(0)}$ for $c \geq 0$. Furthermore, h is also coercive in the sense that $h(\xi) \to \infty$ as $|\xi| \to \infty$, while h is not level-convex due to $L_{-1}(h) = N$. △

Let us now show that balanced level-convexity is sufficient for the weak* lower semicontinuity of supremal functionals on $\mathbb{R}^n$.

Theorem 2.7.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and balanced level-convex. Then the functional $\mathcal{S}$ in (2.24) is weak* lower semicontinuous on $W^{1,\infty}(\mathbb{R}^n)$.*

Proof. We claim that

$$\mathcal{S}(u) \geq f(0) \quad \text{for all } u \in W^{1,\infty}(\mathbb{R}^n). \tag{2.27}$$

If we have this, we may replace the supremand f by $\max\{f, f(0)\}$ without changing the functional. Since f is balanced level-convex we find by (C1) that $\max\{f, f(0)\}$ is level-convex (and lower semicontinuous) so we may apply Theorem 2.1.4 to conclude for $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\mathbb{R}^n)$ and $R > 0$

$$\begin{aligned} \operatorname*{ess\,sup}_{x \in B_R(0)} \max\{f(\nabla u), f(0)\} &\leq \liminf_{j \to \infty} \operatorname*{ess\,sup}_{x \in B_R(0)} \max\{f(\nabla u_j), f(0)\} \\ &\leq \liminf_{j \to \infty} \mathcal{S}(u_j). \end{aligned}$$

Letting $R \to \infty$ yields the weak* lower semicontinuity of $\mathcal{S}$.

To show (2.27) assume to the contrary that there is a $u \in W^{1,\infty}(\mathbb{R}^n)$ such that

$$\mathcal{S}(u) = c < f(0). \tag{2.28}$$

By the lower semicontinuity and balanced level-convexity of f it follows by (C2) that the sub-level set $L_c(f)$ is closed and does not contain zero in its convex hull. Hence, by taking $r := \|\nabla u\|_\infty$ and

$$A = \operatorname{Conv}(L_c(f) \cap \overline{B_r(0)}),$$

we find that A is a compact and convex set that does not contain zero. Furthermore, $\nabla u(x)$ is contained in A for a.e. $x \in \mathbb{R}^n$ in view of (2.28). Because of this, the same holds for the gradients of the functions $u_j(x) = \frac{1}{j} u(jx)$ for $j \in \mathbb{N}$ since $\nabla u_j(x) = \nabla u(jx)$. Additionally, the bounded sequence $(u_j)_j$ converges weak* in $W^{1,\infty}(\mathbb{R}^n)$ to its L^∞-limit, which is zero, so in particular

$$\nabla u_j \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(\mathbb{R}^n; \mathbb{R}^n). \tag{2.29}$$

Due to A being compact and convex, Proposition 2.1.2 tells us that the set

$$\mathcal{V}_R = \{v \in L^\infty(B_R(0);\mathbb{R}^n) \mid v(x) \in A \text{ for a.e. } x \in B_R(0)\}$$

is weak* closed in $L^\infty(B_R(0);\mathbb{R}^n)$ for any $R > 0$, from which we infer that

$$\mathcal{V} = \{v \in L^\infty(\mathbb{R}^n;\mathbb{R}^n) \mid v(x) \in A \text{ for a.e. } x \in \mathbb{R}^n\}$$

is weak* closed in $L^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Hence, the fact that $(\nabla u_j)_j \subset \mathcal{V}$ in combination with (2.29) shows that the zero function lies in $\mathcal{V}$. But this implies that $0 \in A$, which yields a contradiction. Therefore, $\mathcal{S}(u) \geq f(0)$ which proves (2.27). □

Now we prove the necessity.

Proposition 2.7.5. *If $\mathcal{S}$ in (2.24) is weak* lower semicontinuous then f is balanced level-convex.*

Proof. We verify the characterization (C1).

Step 1: We start with showing that $\max\{f, f(0)\}$ is level-convex. Take $\xi_1, \xi_2, \xi \in \mathbb{R}^n$ with $\xi \in \operatorname{Conv}(\xi_1, \xi_2)$ and consider a laminate function $w \in W^{1,\infty}(\mathbb{R}^n)$ such that $v(x) = \xi \cdot x + w(x)$ satisfies $\nabla v \in \{\xi_1, \xi_2\}$ almost everywhere, see the proof of Proposition 2.3.5. The problem is that v is not bounded (for $\xi \neq 0$), and thus we define the truncated laminate functions, cf. Figure 2.1, as

$$u_j(x) = \max\left\{0, \min\left\{1, \xi \cdot x + \frac{1}{j} w(jx)\right\}\right\}.$$

The sequence $(u_j)_j$ is bounded in $W^{1,\infty}(\mathbb{R}^n)$ and satisfies $\nabla u_j \in \{\xi_1, \xi_2, 0\}$ almost everywhere. Because of the boundedness, the sequence $(u_j)_j$ converges weak* in $W^{1,\infty}(\mathbb{R}^n)$ to its uniform limit u given by

$$u(x) = \max\{0, \min\{1, \xi \cdot x\}\}.$$

By the weak* lower semicontinuity we get

$$\max\{f(\xi), f(0)\} = \mathcal{S}(u) \leq \liminf_{j\to\infty} \mathcal{S}(u_j) = \max\{f(\xi_1), f(\xi_2), f(0)\}.$$

This proves that $\max\{f, f(0)\}$ is level-convex.

Step 2: Next, we prove that $f(0) \leq \max\{f(\xi_1), \ldots, f(\xi_k)\}$ for $0 \in \operatorname{Conv}(\xi_1, \ldots, \xi_k)$. We assume that $0 = \lambda_1\xi_1 + \cdots + \lambda_k\xi_k$ with $\lambda_i \in (0, 1)$ and $\sum_{i=1}^k \lambda_i = 1$. If this were not the case then we could just remove some of the ξ_i. We want to construct a $u \in W^{1,\infty}(\mathbb{R}^n)$ with $\nabla u(x) \in \{\xi_1, \ldots, \xi_k\}$ for a.e. $x \in \mathbb{R}^n$. To this aim, we consider the standard basis $e_1, \ldots, e_{k-1}$ of $\mathbb{R}^{k-1}$ and the extra vector $\zeta = (-\lambda_1/\lambda_k, \ldots, -\lambda_{k-1}/\lambda_k)$. Each entry of this last vector is strictly negative and hence we find that $0 \in \operatorname{int}(\operatorname{Conv}(e_1, \ldots, e_{k-1}, \zeta))$. By the theory of differential inclusions, Theorem 2.5.1, we can find a $\varphi \in W_0^{1,\infty}((0,1)^{k-1})$ such that $\nabla\varphi(x) \in \{e_1, \ldots, e_{k-1}, \zeta\}$ for a.e. $x \in (0,1)^{k-1}$. We can periodically extend this function to find a $\varphi \in W^{1,\infty}(\mathbb{R}^{k-1})$ with $\nabla\varphi \in \{e_1, \ldots, e_{k-1}, \zeta\}$ almost everywhere. We now define $u : \mathbb{R}^n \to \mathbb{R}$ as

$$u(x) = \varphi(x \cdot \xi_1, \ldots, x \cdot \xi_{k-1}).$$

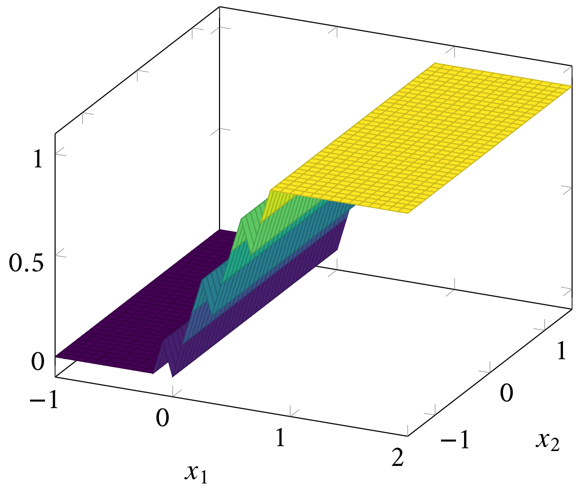

Figure 2.1: An example of the truncated laminate function u_j, where $\xi_1 = (2,0)$, $\xi_2 = (-2,0)$ and $\xi = (1,0)$.

We find that $u \in W^{1,\infty}(\mathbb{R}^n)$ and

$$\nabla u(x) = \nabla\varphi(x\cdot\xi_1,\ldots,x\cdot\xi_{k-1})\begin{bmatrix} -\ \xi_1\ - \\ -\ \xi_2\ - \\ \vdots \\ -\ \xi_{k-1}\ - \end{bmatrix}.$$

It is clear that when $\nabla\varphi \in \{e_1,\cdots,e_{k-1}\}$, then $\nabla u \in \{\xi_1,\ldots,\xi_{k-1}\}$. When $\nabla\varphi = \zeta$ we find

$$\nabla u = \frac{-\lambda_1}{\lambda_k}\xi_1 + \cdots + \frac{-\lambda_{k-1}}{\lambda_k}\xi_{k-1} = \xi_k.$$

Hence, $\nabla u \in \{\xi_1,\ldots,\xi_k\}$ a.e. as desired. Now that we have our u, we simply define the sequence $u_j(x) = \frac{1}{j}u(jx)$ for $j \in \mathbb{N}$, then it follows that $\nabla u_j \in \{\xi_1,\ldots,\xi_k\}$ and $u_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(\mathbb{R}^n)$. By the weak* lower semicontinuity of $\mathcal{S}$ we conclude

$$f(0) = \mathcal{S}(0) \leq \liminf_{j\to\infty}\mathcal{S}(u_j) = \max\{f(\xi_1),\ldots,f(\xi_k)\}.$$

This proves the result. □

Chapter 3

Fractional Calculus of Variations

In this chapter we consider a nonlocal extension of the calculus of variations by replacing the ordinary gradient with the recently introduced Riesz fractional gradient. We study both integral and supremal functionals, and with the goal of proving existence of minimizers, we focus on their weak and weak* lower semicontinuity, respectively. Using a new approach to connect the classical and fractional gradient in the Sobolev spaces, we are able to characterize the weak and weak* lower semicontinuity fully. With similar techniques, we also obtain relaxation formulas and prove an L^p-approximation result in the fractional setting.

In Section 3.1 we lay the groundwork by investigating the fractional Sobolev spaces defined through the Riesz fractional gradient. These results are mostly known in the literature, although notable new results are Lemma 3.1.24 about strong convergence of the fractional gradients outside the domain and especially Section 3.1.5, in which we connect the classical and fractional gradient of Sobolev functions. With these tools we can subsequently study the fractional integral and supremal functionals in Sections 3.2 and 3.3, respectively, with emphasis on their weak and weak* lower semicontinuity. Then, in Section 3.4, we study a relation between the two classes of functionals by means of L^p-approximation. Finally, we present some possible applications of the fractional calculus of variations in Section 3.5

3.1 Fractional Sobolev Spaces

We consider a new type of fractional Sobolev space, which was recently introduced in [68, 69] using the Riesz fractional gradient. These spaces will lay the foundation for studying the fractional variational problems in this thesis. We first introduce the Riesz fractional gradient and related operators for smooth and Lipschitz functions in Section 3.1.1. Secondly, in Section 3.1.2 we use the fractional calculus to define the fractional Sobolev spaces using a distributional approach and go over their main properties. Lastly, we will study the connection between the fractional and classical Sobolev spaces in Section 3.1.5 which is crucial in our endeavor of characterizing the weak and weak* lower semicontinuity of integral and supremal functionals.

3.1.1 Fractional Calculus

The Riesz potential is a singular integral operator, which plays a central part in the fractional calculus. We state a few of its properties.

Definition 3.1.1 (Riesz potential)**.** For $u : \mathbb{R}^n \to \mathbb{R}^m$ measurable and $\alpha \in (0, n)$ we define the Riesz potential $I_\alpha u$ of u as

$$I_\alpha u(x) = \frac{1}{\gamma_{n,\alpha}} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}}\, dy \quad \text{for } x \in \mathbb{R}^n,$$

with $\gamma_{n,\alpha} = \pi^{n/2} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$, provided the integral exists almost everywhere. △

The Riesz potential can be seen as convolution with the locally integrable function $x \mapsto \gamma_{n,\alpha}^{-1}|x|^{\alpha-n}$. In particular, for the Riesz potential to be well-defined we need u to decay to zero sufficiently fast. More precisely, it follows e.g. by [55, Chapter 2 Theorem 1.1] that the Riesz potential $I_\alpha u$ is well-defined if and only if

$$\int_{\mathbb{R}^n} (1 + |y|)^{\alpha-n} |u(y)|\, dy < \infty. \tag{3.1}$$

In this case, $I_\alpha u \in L^1_{loc}(\mathbb{R}^n)$ will be a well-defined locally integrable function. Note that (3.1) holds if $u \in L^p(\mathbb{R}^n)$ for $1 \le p < n/\alpha$ and in this case $I_\alpha u \in L^p_{loc}(\mathbb{R}^n)$, see [55, Chapter 4 Theorem 2.1]. Moreover, when $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have that $I_\alpha \varphi$ is a smooth bounded function ($I_\alpha u \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$), since the Riesz potential is a convolution with a locally integrable, radially symmetric function that decays to zero at infinity. Similarly, when $u \in L^\infty(\mathbb{R}^n)$ with compact support then $I_\alpha u \in L^\infty(\mathbb{R}^n)$. Further properties of the Riesz potential can be found in e.g. [55, 72].

The main object of study is the Riesz fractional gradient. For bounded Lipschitz functions it is defined as follows, see [25, Section 2.3], [74, Definition 2.1].

Definition 3.1.2 (Riesz fractional derivative)**.** Let $\alpha \in (0, 1)$ and $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n)$. Then the (vector-valued) Riesz fractional derivative of φ is defined as

$$\nabla^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{\varphi(y) - \varphi(x)}{|y-x|^{n+\alpha}} \frac{y-x}{|y-x|}\, dy \quad \text{for } x \in \mathbb{R}^n,$$

with $\mu_{n,\alpha} = 2^\alpha \pi^{-n/2} \frac{\Gamma((n+\alpha+1)/2)}{\Gamma((1-\alpha)/2)}$.

△

Note that for $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n)$ we have that the integral exists everywhere and $\nabla^\alpha \varphi \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, see [25, Lemma 2.3]. In fact, by inspecting the proof of [25, Lemma 2.2] and using a substitution $h = y - x$ we have for all $x \in \mathbb{R}^n$ the bound

$$|\nabla^\alpha \varphi(x)| \le |\mu_{n,\alpha}| \int_{\mathbb{R}^n} \frac{\|\varphi(h + \cdot) - \varphi\|_\infty}{|h|^{n+\alpha}}\, dh \le C(n, \alpha) \|\varphi\|_\infty^{1-\alpha} \mathrm{Lip}(\varphi)^\alpha. \tag{3.2}$$

In case $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n; \mathbb{R}^m)$ we view $\nabla^\alpha \varphi(x)$ as an element of $\mathbb{R}^{m \times n}$ by taking the fractional gradient componentwise.

When $\varphi \in C_c^\infty(\mathbb{R}^n)$ we note that $\nabla^\alpha \varphi$ does not necessarily continue to have compact support. Indeed, for any non-negative $\varphi \in C_c^\infty(\mathbb{R})$ with support inside $(0,1)$ it can be seen that $\nabla^\alpha \varphi(x) \neq 0$ for any $x \in (0,1)^c$. Nonetheless, if we define as in [74] the class of functions

$$\mathcal{T}(\mathbb{R}^n) := \{\psi \in C^\infty(\mathbb{R}^n) \mid \partial^a \psi \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n) \text{ for all multi-indices } a \in (\mathbb{N} \cup \{0\})^n\},$$

where $\partial^a \psi$ denotes the ath partial derivative of ψ, then we have $\nabla^\alpha \varphi \in \mathcal{T}(\mathbb{R}^n;\mathbb{R}^n)$ for $\varphi \in C_c^\infty(\mathbb{R}^n)$ ([74, Proposition 5.2]). There is also a useful alternative definition of the fractional gradient which is stated in [24, Proposition 2.2]; for $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\nabla^\alpha \varphi = \nabla I_{1-\alpha} \varphi = I_{1-\alpha} \nabla \varphi. \tag{3.3}$$

This identity will play a major role when we extend this to the (fractional) Sobolev spaces, cf. Section 3.1.5.

A remarkable fact about the Riesz fractional gradient proven in [74] is that (up to a constant) it is the only rotation- and translation-invariant α-homogeneous operator on $C_c^\infty(\mathbb{R}^n)$ that satisfies a weak requirement of continuity. The α-homogeneity means that for any $\lambda > 0$ we have with $\varphi_\lambda = \varphi(\lambda\cdot)$ that

$$\nabla^\alpha \varphi_\lambda = \lambda^\alpha \nabla^\alpha \varphi(\lambda\cdot).$$

Because of the invariance properties, the Riesz fractional gradient is in some sense the canonical fractional derivative. Nonetheless, this derivative has only recently received growing attention. A more well-known object is the fractional Laplacian, which possesses many equivalent definitions [50, 74].

Definition 3.1.3 (Fractional Laplacian)**.** Let $\alpha \in (0,1)$ and $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n)$. Then the (scalar-valued) fractional Laplacian of φ is defined as

$$(-\Delta)^{\alpha/2}\varphi(x) = \nu_{n,\alpha} \int_{\mathbb{R}^n} \frac{\varphi(x+h) - \varphi(x)}{|h|^{n+\alpha}}\, dh \quad \text{for } x \in \mathbb{R}^n, \tag{3.4}$$

with $\nu_{n,\alpha} = 2^\alpha \pi^{-n/2} \frac{\Gamma((n+\alpha)/2)}{\Gamma(-\alpha/2)}$. △

Similarly to [25, Lemma 2.2], we deduce for $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n)$ that $(-\Delta)^{\alpha/2}\varphi$ is well-defined and lies in $L^\infty(\mathbb{R}^n)$, whereas for $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have that $(-\Delta)^{\alpha/2}\varphi \in \mathcal{T}(\mathbb{R}^n)$ by [74, Proposition 5.2]. Again, for vectorial functions we view this operator as acting componentwise.

There are duality relations that the Riesz fractional gradient as well as the fractional Laplacian satisfy; for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathrm{Lip}_b(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \nabla^\alpha \varphi \psi \, dx = -\int_{\mathbb{R}^n} \varphi \nabla^\alpha \psi \, dx \tag{3.5}$$

and

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}\varphi \psi \, dx = \int_{\mathbb{R}^n} \varphi (-\Delta)^{\alpha/2} \psi \, dx. \tag{3.6}$$

The identity for the fractional gradient follows from [25, Proposition 2.8] and the proof for the fractional Laplacian is analogous. Observe that in the first equality, contrary to [25, 74], we do not test with the fractional divergence of vector fields but with the fractional gradient of scalar functions. This is entirely equivalent though, as can be seen by taking a vector field with a single component. Especially the first relation can be seen as a fractional integration by parts formula.

There are a multitude of interesting composition rules for the fractional operators ∇^α, $(-\Delta)^{\alpha/2}$ and the fractional divergence considered in [74, Theorem 5.3], which generalize classical identities. In relation to the results in Section 3.1.5, there is one we wish to point out; for $\varphi \in C_c^\infty(\mathbb{R}^n)$ it holds that

$$\nabla\varphi = \nabla^\alpha(-\Delta)^{\frac{1-\alpha}{2}}\varphi = (-\Delta)^{\frac{1-\alpha}{2}}\nabla^\alpha\varphi. \tag{3.7}$$

Strictly speaking, only the first identity is mentioned in [74, Theorem 5.3], but by using the duality rules (3.5), (3.6) we find that for any $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1-\alpha}{2}}\nabla^\alpha\varphi\psi\, dx = \int_{\mathbb{R}^n} \nabla^\alpha\varphi(-\Delta)^{\frac{1-\alpha}{2}}\psi\, dx = -\int_{\mathbb{R}^n} \varphi\nabla^\alpha(-\Delta)^{\frac{1-\alpha}{2}}\psi\, dx = -\int_{\mathbb{R}^n} \varphi\nabla\psi\, dx.$$

This establishes the last equality in (3.7) via duality.

To prove the fractional Morrey inequality for $p = \infty$, we utilize the following fractional fundamental theorem of calculus, which is interesting in its own right. The proof can be found in [24, Theorem 3.12],[59, Proposition 15.8] or [68, Theorem 2.1].

Theorem 3.1.4 (Fractional fundamental theorem of calculus)**.** *Let $\alpha \in (0,1)$. For any $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$ it holds that*

$$\varphi(y) - \varphi(x) = \mu_{n,-\alpha}\int_{\mathbb{R}^n}\left(\frac{z-x}{|z-x|^{n+1-\alpha}} - \frac{z-y}{|z-y|^{n+1-\alpha}}\right)\cdot\nabla^\alpha\varphi(z)\, dz.$$

We also have a fractional Leibniz rule from [24, Lemma 2.6]. The extra $\nabla^\alpha_{\mathrm{NL}}$ term accounts for the nonlocality.

Proposition 3.1.5. *Let $\alpha \in (0,1)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathrm{Lip}_b(\mathbb{R}^n)$ then*

$$\nabla^\alpha(\psi\varphi) = \psi\nabla^\alpha\varphi + \varphi\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(\varphi,\psi),$$

where $\nabla^\alpha_{\mathrm{NL}}(\varphi,\psi) \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is defined as

$$\nabla^\alpha_{\mathrm{NL}}(\varphi,\psi)(x) = \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(y-x)(\varphi(y)-\varphi(x))(\psi(y)-\psi(x))}{|y-x|^{n+\alpha+1}}\, dy \quad \text{for } x \in \mathbb{R}^n.$$

3.1.2 Definition and Main Properties

In the present section we define the new fractional Sobolev spaces using a distributional approach as in [24] and discuss their main properties. This will lay the foundation for studying fractional variational problems. See also [20, 24, 25, 68, 69] for more details on these spaces.

Motivated by the fractional integration by parts formula (3.5), we define a weak fractional gradient in a similar manner to the classical case. This definition is equivalent to [24, Definition 3.19], where the authors instead use the formulation with the fractional divergence.

Definition 3.1.6 (Weak Riesz fractional gradient)**.** Let $\alpha \in (0,1)$, $p \in [1,\infty]$ and $u \in L^p(\mathbb{R}^n)$. Then $w \in L^1_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ is called the weak α-fractional gradient of u if

$$\int_{\mathbb{R}^n} w\varphi\,dx = -\int_{\mathbb{R}^n} u\nabla^\alpha\varphi\,dx \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

We write $w = \nabla^\alpha u$. △

Observe that the right-hand side integral is well-defined since $\nabla^\alpha\varphi \in \mathcal{T}(\mathbb{R}^n;\mathbb{R}^n)$, cf. Section 3.1.1. Additionally, the weak fractional gradient, if it exists, is unique almost everywhere by the fundamental theorem of the calculus of variations. We can now define the fractional Sobolev spaces.

Definition 3.1.7 (Fractional Sobolev space)**.** Let $\alpha \in (0,1)$ and $p \in [1,\infty]$. We define the fractional Sobolev space $S^{\alpha,p}(\mathbb{R}^n)$ as the collection of functions in $L^p(\mathbb{R}^n)$ that have a weak α-fractional gradient lying in $L^p(\mathbb{R}^n;\mathbb{R}^n)$, i.e.

$$S^{\alpha,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \mid \nabla^\alpha u \in L^p(\mathbb{R}^n;\mathbb{R}^n)\}.$$

This space is endowed with the norm

$$\|u\|_{S^{\alpha,p}(\mathbb{R}^n)} = \|u\|_p + \|\nabla^\alpha u\|_p.$$

We denote by $S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$ the vector-valued analogue. △

We have the following result which can be proven similarly to the classical case, see e.g. [34, §5.2.3 Theorem 2].

Theorem 3.1.8. *Let $\alpha \in (0,1)$ and $p \in [1,\infty]$. Then, $S^{\alpha,p}(\mathbb{R}^n)$ is a Banach space.*

We also define the weak and weak* convergence on these spaces.

Definition 3.1.9 (Weak convergence)**.** Let $\alpha \in (0,1)$ and $p \in [1,\infty)$. We say that a sequence $(u_j)_j \subset S^{\alpha,p}(\mathbb{R}^n)$ converges weakly to u in $S^{\alpha,p}(\mathbb{R}^n)$ if

$$u_j \rightharpoonup u \ \text{ in } L^p(\mathbb{R}^n) \ \text{ and } \ \nabla^\alpha u_j \rightharpoonup \nabla^\alpha u \ \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^n).$$

We write $u_j \rightharpoonup u$ in $S^{\alpha,p}(\mathbb{R}^n)$. △

This notion of weak convergence corresponds to the abstract notion of weak convergence induced by the dual space. This can be seen similarly to the classical case by using the embedding of $S^{\alpha,p}(\mathbb{R}^n)$ into $(L^p(\mathbb{R}^n))^{n+1}$ and the Hahn-Banach theorem to extend functionals on $S^{\alpha,p}(\mathbb{R}^n)$ to $(L^p(\mathbb{R}^n))^{n+1}$.

Definition 3.1.10. (Weak* convergence) Let $\alpha \in (0,1)$, then, we say that a sequence $(u_j)_j \subset S^{\alpha,\infty}(\mathbb{R}^n)$ converges weak* to u in $S^{\alpha,\infty}(\mathbb{R}^n)$ if

$$u_j \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(\mathbb{R}^n) \ \text{ and } \ \nabla^\alpha u_j \overset{*}{\rightharpoonup} \nabla^\alpha u \ \text{ in } L^\infty(\mathbb{R}^n;\mathbb{R}^n).$$

We write $u_j \overset{*}{\rightharpoonup} u$ in $S^{\alpha,\infty}(\mathbb{R}^n)$. △

For this notion it is not clear that it corresponds to the abstract weak* convergence induced by a pre-dual space. However, this will not be a problem as all the relevant properties follow from the weak* convergence on $L^\infty(\mathbb{R}^n)$.

The $S^{\alpha,p}(\mathbb{R}^n)$ spaces were first introduced in [24] by Comi & Stefani and are very similar to the ones earlier introduced by Shieh & Spector in [68] for $p \in (1,\infty)$. However, in the latter they define their spaces as the closure of $C_c^\infty(\mathbb{R}^n)$ in the $S^{\alpha,p}(\mathbb{R}^n)$-norm and prove in [68, Theorem 1.7] that this is equivalent to the well-known Bessel potential spaces, see e.g. [3]. Since it is not clear from the definition that $C_c^\infty(\mathbb{R}^n)$ is dense in $S^{\alpha,p}(\mathbb{R}^n)$ we can not immediately conclude the same for $S^{\alpha,p}(\mathbb{R}^n)$. Nonetheless, it is proven in [20, Appendix A] that $C_c^\infty(\mathbb{R}^n)$ is in fact dense in $S^{\alpha,p}(\mathbb{R}^n)$ for $p \in (1,\infty)$, see also Theorem 3.1.16 below for an alternative proof, so that we indeed have the identification with the Bessel potential spaces.

As a consequence, the spaces denoted by $L^{s,p}(\mathbb{R}^n)$, $X^{s,p}(\mathbb{R}^n)$ in [68] and $H^{s,p}(\mathbb{R}^n)$ in [15,16,69] all coincide with $S^{\alpha,p}(\mathbb{R}^n)$, if s is replaced by α. Hence, we can utilize all the properties proven in those papers for our space $S^{\alpha,p}(\mathbb{R}^n)$. In particular, in relation to other spaces, we mention the continuous embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow S^{\alpha,p}(\mathbb{R}^n)$$

for $\alpha \in (0,1)$ and $p \in (1,\infty)$. Furthermore, the results [68, Theorem 2.2 (g)] and [24, Proposition 3.24] give insight into the relation with the more well-known fractional Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ defined via the Gagliardo semi-norm; see [33] for an elementary introduction to these spaces.

Let us now state the main properties of the fractional Sobolev spaces that we will need. When $p \in (1,\infty)$ the reflexivity of the space $S^{\alpha,p}(\mathbb{R}^n)$ can be deduced from the embedding into $(L^p(\mathbb{R}^n))^{n+1}$, which is reflexive. Alternatively, one can use the identification with the Bessel potential spaces which are reflexive. The embedding into $(L^p(\mathbb{R}^n))^{n+1}$ also inherits the separability for $p \in (1,\infty)$.

Theorem 3.1.11. *Let $\alpha \in (0,1)$ and $p \in (1,\infty)$, then $S^{\alpha,p}(\mathbb{R}^n)$ is reflexive and separable.*

Additionally, there exist strikingly similar inequalities and embeddings to those for the classical Sobolev spaces. We define $p^* = \frac{np}{n-\alpha p}$ when $\alpha p < n$. There is following fractional Sobolev inequality from [68, Theorem 1.8].

Theorem 3.1.12 (Fractional Sobolev inequality)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty)$ and $\alpha p < n$. Then there is a $C = C(n,p,\alpha) > 0$ such that for all $u \in S^{\alpha,p}(\mathbb{R}^n)$*

$$\|u\|_{p^*} \le C\|\nabla^\alpha u\|_p.$$

In the critical exponent case $\alpha p = n$, there is a fractional Trudinger inequality from [68, Theorem 1.10].

Theorem 3.1.13 (Fractional Trudinger inequality)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty)$ and $\alpha p = n$. Then there exist $C_1, C_2 > 0$ (depending on α, p) such that for all $\Omega \subset \mathbb{R}^n$ open and with finite measure and $u \in S^{\alpha,p}(\mathbb{R}^n)$ it holds that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left[\left(\frac{|u(x)|}{C_1\|\nabla^\alpha u\|_p}\right)^{\frac{p}{p-1}}\right] dx \le C_2.$$

The above inequality in particular yields an embedding into $L^q_{loc}(\mathbb{R}^n)$ for any $q \in [1,\infty)$. Lastly, in the regime $\alpha p > n$ we have a fractional Morrey inequality. For $p < \infty$ this follows from [68, Theorem 1.11], while the case $p = \infty$ is new and a proof is given in Section 3.1.4.

Theorem 3.1.14 (Fractional Morrey inequality)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty]$ and $\alpha p > n$. Then there is a $C = C(n,p,\alpha) > 0$ such that for all $u \in S^{\alpha,p}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$ it holds that*

$$|u(x) - u(y)| \leq C\|\nabla^\alpha u\|_p |x-y|^{\alpha - \frac{n}{p}}.$$

The above inequality says that in this exponent regime the fractional Sobolev functions are Hölder continuous with exponent $\alpha - \frac{n}{p}$. In particular, we have the embedding of $S^{\alpha,\infty}(\mathbb{R}^n)$ into $C^{0,\alpha}(\mathbb{R}^n)$.

Additionally, we like to provide an alternative and more elementary proof to [20, Theorem A.1] of density of $C_c^\infty(\mathbb{R}^n)$ in $S^{\alpha,p}(\mathbb{R}^n)$ for $p \in [1,\infty)$. To do this, we begin with the following lemma. A similar result to this is given in [15, Lemma 3.2 & Lemma 3.4] but with different notation.

Lemma 3.1.15. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathrm{Lip}_b(\mathbb{R}^n)$. Then, for any $p \in [1,\infty]$*

$$\|\nabla^\alpha_{\mathrm{NL}}(\varphi,\psi)\|_p \leq C(n,\alpha)\|\psi\|_\infty^{1-\alpha}\mathrm{Lip}(\psi)^\alpha \|\varphi\|_p.$$

Proof. By Minkowski's integral inequality, [72, Section A.1], and Hölder's inequality we find

$$\begin{aligned}
\|\nabla^\alpha_{\mathrm{NL}}(\varphi,\psi)\|_p &\leq |\mu_{n,\alpha}| \left\| \int_{\mathbb{R}^n} \frac{|\varphi(h+\cdot)-\varphi||\psi(h+\cdot)-\psi|}{|h|^{n+\alpha}}\,dh \right\|_p \\
&\leq |\mu_{n,\alpha}| \int_{\mathbb{R}^n} \frac{\big\||\varphi(h+\cdot)-\varphi||\psi(h+\cdot)-\psi|\big\|_p}{|h|^{n+\alpha}}\,dh \\
&\leq 2|\mu_{n,\alpha}| \int_{\mathbb{R}^n} \frac{\|\psi(h+\cdot)-\psi\|_\infty}{|h|^{n+\alpha}}\,dh \|\varphi\|_p \\
&\leq C(n,\alpha)\|\psi\|_\infty^{1-\alpha}\mathrm{Lip}(\psi)^\alpha\|\varphi\|_p.
\end{aligned}$$

The last inequality follows from (3.2). □

We now restate the density result proven in [20, Theorem A.1] with a different proof.

Theorem 3.1.16. *Let $\alpha \in (0,1)$ and $p \in [1,\infty)$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $S^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Take any $u \in S^{\alpha,p}(\mathbb{R}^n)$ then we know by [24, Theorem 3.22] that we can approximate it by its mollified version. Therefore, we may assume without loss of generality that $u \in S^{\alpha,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. We take for each $j \in \mathbb{N}$ a cut-off function $\chi_j \in C_c^\infty(\mathbb{R}^n)$ such that

$$\chi_j|_{B_j(0)} \equiv 1, \quad 0 \leq \chi_j \leq 1 \quad \text{and} \quad \mathrm{Lip}(\chi_j) \leq 1/j. \tag{3.8}$$

We aim to show that $\chi_j u \in C_c^\infty(\mathbb{R}^n)$ converges to u in $S^{\alpha,p}(\mathbb{R}^n)$.

It is readily seen that $\chi_j u \to u$ in $L^p(\mathbb{R}^n)$. Regarding the fractional gradients, we note that $\chi_j u \in C_c^\infty(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$. Their fractional gradients satisfy for any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla^\alpha(\chi_j u)\varphi \, dx &= -\int_{\mathbb{R}^n} \chi_j u \nabla^\alpha \varphi \, dx \\ &= -\int_{\mathbb{R}^n} u\nabla^\alpha(\chi_j\varphi)\, dx + \int_{\mathbb{R}^n} u\varphi\nabla^\alpha \chi_j \, dx + \int_{\mathbb{R}^n} u\nabla^\alpha_{\mathrm{NL}}(\varphi,\chi_j)\, dx \\ &= \int_{\mathbb{R}^n} (\nabla^\alpha u)\chi_j\varphi \, dx \int_{\mathbb{R}^n} u\varphi\nabla^\alpha\chi_j \, dx + \int_{\mathbb{R}^n} u\nabla^\alpha_{\mathrm{NL}}(\varphi,\chi_j)\, dx,\end{aligned}$$

where second line uses the Leibniz rule for $\chi_j\varphi$ (Proposition 3.1.5). Rewriting and taking absolute values yields

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla^\alpha u - \nabla^\alpha(\chi_j u)||\varphi| \, dx \\ &\quad \le \int_{\mathbb{R}^n} |(1-\chi_j)\nabla^\alpha u||\varphi|\, dx + \int_{\mathbb{R}^n} |u\nabla^\alpha\chi_j||\varphi|\, dx + \int_{\mathbb{R}^n} |u||\nabla^\alpha_{\mathrm{NL}}(\varphi,\chi_j)|\, dx \\ &\quad \le \|(1-\chi_j)\nabla^\alpha u\|_p \|\varphi\|_{p'} + C(n,\alpha)\|u\|_p \|\chi_j\|_\infty^{1-\alpha} \mathrm{Lip}(\chi_j)^\alpha \|\varphi\|_{p'},\end{aligned}$$

where the last inequality follows from Hölder's inequality in conjunction with (3.2) and Lemma 3.1.15. Recalling (3.8), we see that in both cases that the term in front of $\|\varphi\|_{p'}$ vanishes as $j \to \infty$ which yields $\|\nabla^\alpha u - \nabla^\alpha(\chi_j u)\|_p \to 0$ by duality. □

Remark 3.1.17. The proof of density for $p = 1$ in [24, Theorem 3.23] is similar to the above, although they do not use a sequence of cut-off functions that becomes less steep. For their proof of the case $p \in (1,\infty)$ in [20, Theorem A.1], they use a completely different approach by using the fractional Laplacian and properties of the Bessel potential spaces. △

3.1.3 Complementary-Value Spaces

Here, we delve into properties related to the complementary-value spaces, which are the setting for the variational problems that we consider. For $\Omega \subset \mathbb{R}^n$ open and bounded we define the spaces with complementary-value zero as

$$S_0^{\alpha,p}(\Omega) = \{u \in S^{\alpha,p}(\mathbb{R}^n) \mid u = 0 \text{ a.e. in } \Omega^c\},$$

which is a closed subspace of $S^{\alpha,p}(\mathbb{R}^n)$. Furthermore, for $g \in S^{\alpha,p}(\mathbb{R}^n)$ fixed we define the affine space

$$S_g^{\alpha,p}(\Omega) = g + S_0^{\alpha,p}(\Omega).$$

When we write $u_j \rightharpoonup u$ in $S_g^{\alpha,p}(\Omega)$ (weak* if $p = \infty$) we simply mean that $(u_j)_j \subset S_g^{\alpha,p}(\Omega)$ and $u \in S_g^{\alpha,p}(\Omega)$ with $u_j \rightharpoonup u$ in $S^{\alpha,p}(\mathbb{R}^n)$ (weak* if $p = \infty$). We first mention a type of fractional Poincaré inequality.

Theorem 3.1.18 (Fractional Poincaré inequality)**.** *Let $\alpha \in (0,1)$ and $p \in (1,\infty]$. Then for any bounded open set $\Omega \subset \mathbb{R}^n$ we have a constant $C = C(\Omega, n, p, \alpha) > 0$ such that for all $u \in S_0^{\alpha,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \le C\|\nabla^\alpha u\|_p.$$

Proof. The case $p \in (1,\infty)$ follows from [68, Theorem 3.3]. For the case $p = \infty$ we use the fractional Morrey inequality (Theorem 3.1.14) and the fact that u is zero outside Ω to obtain

$$|u(x)| \le C(n,p,\alpha)\mathrm{diam}(\Omega)^\alpha \|\nabla^\alpha u\|_\infty.$$

□

Remark 3.1.19. We note that in the case $p \in (1,\infty)$ we do not need the fact that u is zero outside Ω as can be seen from [68, Theorem 3.3]. For $p = \infty$, this assumption is pivotal as the constant function equal to 1 shows. △

There also is a weak compactness result which is crucial for the direct method in the calculus of variations. The case $p \in (1,\infty)$ is from [68, Theorem 2.1] and [15, Theorem 2.3], whereas the case $p = \infty$ is a simple consequence of the fractional Morrey inequality for $p = \infty$.

Theorem 3.1.20. *Let $\alpha \in (0,1)$, $p \in (1,\infty]$, $\Omega \subset \mathbb{R}^n$ open and bounded and $g \in S^{\alpha,p}(\mathbb{R}^n)$. Then for any bounded sequence $(u_j)_j \subset S_g^{\alpha,p}(\Omega)$ there exists a subsequence (not relabeled) and a $u \in S_g^{\alpha,p}(\Omega)$ such that*

$$u_j \to u \text{ in } L^p(\mathbb{R}^n) \qquad \text{and} \qquad \nabla^\alpha u_j \rightharpoonup \nabla^\alpha u \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^n) \text{ (weak* if } p = \infty).$$

Proof. Case $p \in (1,\infty)$: We find by reflexivity that any bounded sequence $(u_j)_j \subset S_g^{\alpha,p}(\Omega)$ has a subsequence which weakly converges to $u \in S^{\alpha,p}(\mathbb{R}^n)$. Since $S_g^{\alpha,p}(\Omega)$ is convex and closed, it is also weakly closed and thus $u \in S_g^{\alpha,p}(\Omega)$. Now we can apply [15, Theorem 2.3].

Case $p = \infty$: The Banach-Alaoglu theorem (applied to L^∞) shows that for any bounded sequence $(u_j)_j \subset S_g^{\alpha,\infty}(\Omega)$ there is a $u \in L^\infty(\mathbb{R}^n)$, $w \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and a subsequence (not relabeled) such that

$$u_j \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}^n) \text{ and } \nabla^\alpha u_j \overset{*}{\rightharpoonup} w \text{ in } L^\infty(\mathbb{R}^n;\mathbb{R}^n).$$

From this one can infer that $\nabla^\alpha u = w$ using the definition of the weak fractional gradient. Furthermore, $(u_j - g)_j \subset S_0^{\alpha,\infty}(\Omega)$ is a sequence weak* converging to $u - g$ and by the fractional Morrey inequality $(u_j - g)_j$ is bounded in $C^{0,\mu}(\overline{\Omega})$. By the compact embedding of $C^{0,\mu}(\overline{\Omega})$ into $C(\overline{\Omega})$ from the Arzelà-Ascoli theorem we conclude that we may upgrade the convergence $u_j - g \to u - g$ to uniform convergence (via the Urysohn subsequence principle). This also shows in particular that $u \in S_g^{\alpha,\infty}(\Omega)$. □

Remark 3.1.21. We note that both the Poincaré inequality as well as the compactness result hold more generally, where we can bound the L^q-norm of u or deduce strong convergence in the L^q-norm for exponents q lying in a certain regime, see [15, 68]. △

In order to construct functions in the complementary-value spaces we will often need a cut-off approach. For this reason, we extend the fractional Leibniz rule to the fractional Sobolev spaces. For a fixed $\psi \in \mathrm{Lip}_b(\mathbb{R}^n)$, we have by Lemma 3.1.15 for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ that

$$\|\nabla^\alpha_{\mathrm{NL}}(\varphi, \psi)\|_p \leq C(n,\alpha)\|\psi\|_\infty^{1-\alpha}\mathrm{Lip}(\psi)^\alpha\|\varphi\|_p \leq C(n,\alpha,\psi)\|\varphi\|_p.$$

Therefore, we can extend $\varphi \mapsto \nabla^\alpha_{\mathrm{NL}}(\varphi,\psi)$ to a bounded linear operator on $L^p(\mathbb{R}^n)$ for which we retain the same notation. For $u \in L^\infty(\mathbb{R}^n)$ and $\psi \in \mathrm{Lip}_b(\mathbb{R}^n)$ we can define $\nabla^\alpha_{\mathrm{NL}}(u,\psi)$ pointwise because

$$\int_{\mathbb{R}^n} \frac{|u(y)-u(x)||\psi(y)-\psi(x)|}{|y-x|^{n+\alpha}}\,dy \leq C\|\psi\|_\infty^{1-\alpha}\mathrm{Lip}(\psi)^\alpha\|u\|_\infty < \infty, \tag{3.9}$$

for a.e. $x \in \mathbb{R}^n$ via the same computation as in Lemma 3.1.15. The proof of the Leibniz rule for $p \in [1,\infty)$ follows by density and has appeared with different notation in [15, Lemma 3.4], while the case $p = \infty$ is new for which we have to use a direct approach.

Lemma 3.1.22. *Let $\alpha \in (0,1)$, $p \in [1,\infty]$ and $\psi \in \mathrm{Lip}_b(\mathbb{R}^n)$. Then, for every $u \in S^{\alpha,p}(\mathbb{R}^n)$ it holds that $\psi u \in S^{\alpha,p}(\mathbb{R}^n)$ with*

$$\nabla^\alpha(\psi u) = \psi\nabla^\alpha u + u\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(u,\psi),$$

and there is a $C = C(n,\alpha) > 0$ such that

$$\|\nabla^\alpha(\psi u) - \psi\nabla^\alpha u\|_p = \|u\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(u,\psi)\|_p \leq C\|\psi\|_\infty^{1-\alpha}\mathrm{Lip}(\psi)^\alpha\|u\|_p. \tag{3.10}$$

Proof. Case $p \in [1,\infty)$: We can take by Theorem 3.1.16 a sequence $(u_j)_j \subset C_c^\infty(\mathbb{R}^n)$ with $u_j \to u$ in $S^{\alpha,p}(\mathbb{R}^n)$. We infer that $\psi u_j \in \mathrm{Lip}_b(\mathbb{R}^n)$ and by the regular Leibniz rule (Proposition 3.1.5)

$$\nabla^\alpha(\psi u_j) = \psi\nabla^\alpha u_j + u_j\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(u_j,\psi).$$

We thus conclude $\psi u_j \to \psi u$ in $L^p(\mathbb{R}^n)$ and

$$\nabla^\alpha(\psi u_j) \to \psi\nabla^\alpha u + u\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(u,\psi) \text{ in } L^p(\mathbb{R}^n).$$

Via the definition of the weak fractional gradient we find $\nabla^\alpha(\psi u) = \psi\nabla^\alpha u + u\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(u,\psi)$ so that $\psi u \in S^{\alpha,p}(\mathbb{R}^n)$. The bound (3.10) follows from Lemma 3.1.15 and (3.2).

Case $p = \infty$: We have that $\nabla^\alpha_{\mathrm{NL}}(\psi,u)$ is well-defined and lies in $L^\infty(\mathbb{R}^n)$ by (3.9). Now we can calculate for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ that

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\psi\nabla^\alpha u + u\nabla^\alpha\psi + \nabla^\alpha_{\mathrm{NL}}(\psi,u))\varphi\,dx \\
&\qquad = \int_{\mathbb{R}^n} -u\nabla^\alpha(\psi\varphi) + u\varphi\nabla^\alpha\psi + \varphi\nabla^\alpha_{\mathrm{NL}}(\psi,u)\,dx \\
&\qquad = \int_{\mathbb{R}^n} -u\psi\nabla^\alpha\varphi - u\nabla^\alpha_{\mathrm{NL}}(\psi,\varphi) + \varphi\nabla^\alpha_{\mathrm{NL}}(\psi,u)\,dx,
\end{aligned}$$

where the third line uses the regular Leibniz rule (Proposition 3.1.5). The first term in the final integral is exactly what we need so it remains to show that the other two cancel out, i.e.

$$\int_{\mathbb{R}^n} -u\nabla^\alpha_{\mathrm{NL}}(\psi,\varphi) + \varphi\nabla^\alpha_{\mathrm{NL}}(\psi,u)\,dx = 0. \tag{3.11}$$

To this end we calculate

$$\begin{aligned}\int_{\mathbb{R}^n} -u\nabla^\alpha_{\mathrm{NL}}(\psi,\varphi)\,dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} -u(x)\frac{(y-x)(\psi(y)-\psi(x))(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\,dy\,dx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)\frac{(y-x)(\psi(y)-\psi(x))(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\,dy\,dx,\end{aligned}$$

where the second line uses Fubini's theorem while also interchanging the x and y variables. As a consequence,

$$\begin{aligned}&\int_{\mathbb{R}^n} -u\nabla^\alpha_{\mathrm{NL}}(\psi,\varphi) + \varphi\nabla^\alpha_{\mathrm{NL}}(\psi,u)\,dx\\ &\qquad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(y-x)(\psi(y)-\psi(x))}{|y-x|^{n+\alpha+1}}(u(y)\varphi(y)-u(x)\varphi(x))\,dy\,dx = 0,\end{aligned}$$

where we use that the last integrand is odd in x and y and thus is equal to zero by Fubini's theorem. This yields (3.11) and thus, the Leibniz identity. The bound (3.10) follows from (3.9) and (3.2). □

As an application of the Leibniz rule, we can derive some useful results. We first extend the compactness result from Theorem 3.1.20 by not fixing any complementary data.

Theorem 3.1.23. *Let $\alpha \in (0,1)$, $p \in (1,\infty]$ and $u_j \rightharpoonup u$ in $S^{\alpha,p}(\mathbb{R}^n)$. Then $u_j \to u$ in $L^p_{loc}(\mathbb{R}^n)$.*

Proof. Take for $R > 0$ a cut-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \chi \le 1$ and $\chi|_{B_R(0)} \equiv 1$. Then, by Lemma 3.1.22 we find that χu_j is a bounded sequence in $S^{\alpha,p}_0(\mathrm{supp}(\chi))$. Hence, Theorem 3.1.20 shows that χu_j converges up to subsequence in $L^p(\mathbb{R}^n)$. But it is clear that this limit must be χu so that we may conclude $\chi u_j \to \chi u$ in $L^p(\mathbb{R}^n)$ for the whole sequence. As $\chi|_{B_R(0)} \equiv 1$ this shows in particular that $u_j \to u$ in $L^p(B_R(0))$ which proves the result. □

There is also a quite interesting property of weak convergence on the complementary-value spaces; it is that outside the domain the convergence of the fractional gradients actually turns into strong convergence. Although simple, this new observation has far-reaching consequences for the rest of the thesis.

Lemma 3.1.24. *Let $\alpha \in (0,1)$, $p \in (1,\infty]$, $\Omega \subset \mathbb{R}^n$ open and bounded and $g \in S^{\alpha,p}(\mathbb{R}^n)$. If $u_j \rightharpoonup u$ in $S^{\alpha,p}_g(\Omega)$ (weak* if $p=\infty$) then for every $\Omega \Subset \Omega'$ we have*

$$\nabla^\alpha u_j \to \nabla^\alpha u \ \text{ in } L^p((\Omega')^c;\mathbb{R}^n).$$

Proof. By linearity it is enough to consider $u_j \rightharpoonup 0$ in $S_0^{\alpha,p}(\Omega)$. Take a cut-off function $\chi \in C_c^\infty(\Omega')$ such that $\chi \equiv 1$ on Ω. Then, $u_j = \chi u_j$ for $j \in \mathbb{N}$ and by Theorem 3.1.20 we deduce $u_j \to 0$ in $L^p(\mathbb{R}^n)$. Hence, we obtain from (3.10) that

$$\begin{aligned}\|\nabla^\alpha u_j\|_{L^p((\Omega')^c)} &= \|\nabla^\alpha(\chi u_j)\|_{L^p((\Omega')^c)} \\ &\leq \|\nabla^\alpha(\chi u_j) - \chi\nabla^\alpha u_j\|_p \leq C(\chi)\|u_j\|_p \to 0 \quad \text{as } j \to \infty,\end{aligned}$$

which yields the lemma. □

We conclude with a result that establishes a continuous inclusion for fractional Sobolev spaces with different orders of integrability. This is needed in the end to prove the L^p-approximation of the supremal functionals.

Proposition 3.1.25. *Let $1 \leq q < p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ open and bounded. Then $S_0^{\alpha,p}(\Omega) \subset S_0^{\alpha,q}(\Omega)$ and there is a constant $C = C(n,\alpha,\Omega,p,q) > 0$ such that for every $u \in S_0^{\alpha,p}(\Omega)$*

$$\|u\|_{S^{\alpha,q}(\mathbb{R}^n)} \leq C\|u\|_{S^{\alpha,p}(\mathbb{R}^n)}.$$

Proof. Let $u \in S_0^{\alpha,p}(\Omega)$, then it is clear that

$$\|u\|_q \leq |\Omega|^{1/q-1/p}\|u\|_p,$$

since u has compact support in Ω. To show that $\nabla^\alpha u \in L^q(\mathbb{R}^n)$, we choose $R > 0$ such that $\Omega \Subset B_R(0)$ and write

$$\|\nabla^\alpha u\|_q \leq \|\nabla^\alpha u\|_{L^q(B_R(0))} + \|\nabla^\alpha u\|_{L^q(B_R(0)^c)}.$$

The first term can be bounded as

$$\|\nabla^\alpha u\|_{L^q(B_R(0))} \leq |B_R|^{1/q-1/p}\|\nabla^\alpha u\|_p.$$

Regarding the second, set $d := \mathrm{d}(\Omega, B_R(0)^c)$ and take a cut-off function $\chi \in \mathrm{Lip}_b(\mathbb{R}^n)$ such that

$$0 \leq \chi \leq 1, \quad \chi_{|B_R(0)^c} \equiv 0, \quad \chi_{|\Omega} \equiv 1 \quad \text{and} \quad \|\nabla\chi\|_\infty \leq 1/d.$$

Then $u = \chi u$ so that by Lemma 3.1.22 we find

$$\begin{aligned}\|\nabla^\alpha u\|_{L^q(B_R(0)^c)} &= \|\nabla^\alpha(\chi u)\|_{L^q(B_R(0)^c)} \\ &\leq \|\nabla^\alpha(\chi u) - \chi\nabla^\alpha u\|_q \\ &\leq C(n,\alpha)d^{-\alpha}\|u\|_q \leq C(n,\alpha)d^{-\alpha}|\Omega|^{1/q-1/p}\|u\|_p,\end{aligned}$$

which finishes the proof. □

By choosing R large enough such that $\mathrm{d}(\Omega, B_R(0)^c) \geq R/2$ we obtain the following more explicit bounds.

Corollary 3.1.26. *Let $1 \leq q < p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ open and bounded. Then, for $R > 0$ such that $\Omega \subset B_R(0)$ and $\mathrm{d}(\Omega, B_R(0)^c) \geq R/2$ there is a constant $C = C(n,\alpha,\Omega) > 0$ such that for every $u \in S_0^{\alpha,p}(\Omega)$*

$$\|\nabla^\alpha u\|_q \leq |B_R|^{1/q-1/p}\|\nabla^\alpha u\|_p + CR^{-\alpha}\|u\|_p.$$

3.1.4 Fractional Morrey Inequality $p = \infty$

This section is devoted to proving the fractional Morrey inequality (Theorem 3.1.14) in the case $p = \infty$. To prove the fractional Morrey inequality we first prove it for functions in $C_c^\infty(\mathbb{R}^n)$.

Lemma 3.1.27. *Let $\alpha \in (0,1)$. There is a constant $C = C(n,\alpha) > 0$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$*

$$|\varphi(x) - \varphi(y)| \leq C|x-y|^\alpha \|\nabla^\alpha \varphi\|_\infty.$$

Proof. By Theorem 3.1.4 and Hölder's inequality we get that

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq |\mu_{n,-\alpha}| \int_{\mathbb{R}^n} |\nabla^\alpha \varphi(y)| \left| \frac{z-x}{|z-x|^{n-\alpha+1}} - \frac{z-y}{|z-y|^{n-\alpha+1}} \right| dz \\ &\leq |\mu_{n,-\alpha}| \, \|\nabla^\alpha \varphi\|_\infty \int_{\mathbb{R}^n} \left| \frac{z-x}{|z-x|^{n-\alpha+1}} - \frac{z-y}{|z-y|^{n-\alpha+1}} \right| dz. \end{aligned}$$

Now, it is proven in [24, Proposition 3.14] that there is a $C = C(n,\alpha) > 0$ such that

$$\int_{\mathbb{R}^n} \left| \frac{z-x}{|z-x|^{n-\alpha+1}} - \frac{z-y}{|z-y|^{n-\alpha+1}} \right| dz \leq C|x-y|^\alpha.$$

This yields the result. □

To extend the fractional Morrey inequality to the fractional Sobolev space we utilize mollification and cut-off arguments. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\operatorname{supp}(\eta) \subset B(0,1), \quad \eta \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \eta \, dx = 1.$$

We define for $\epsilon > 0$ the standard mollifier $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$. Similarly to [24, Lemma 3.5], we can prove the following.

Lemma 3.1.28. *Let $\alpha \in (0,1)$, $p \in [1,\infty]$ and $u \in S^{\alpha,p}(\mathbb{R}^n)$. Then $u_\epsilon = \eta_\epsilon * u \in S^{\alpha,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and $\nabla^\alpha u_\epsilon = \eta_\epsilon * \nabla^\alpha u$.*

We can now proceed with the proof of the fractional Morrey inequality in the case $p = \infty$. The proof of Step 1 takes elements from [34, §5.8.2 Theorem 4].

Theorem 3.1.29 (Fractional Morrey inequality $p = \infty$)**.** *Let $\alpha \in (0,1)$. There exists a constant $C = C(n,\alpha) > 0$ such that for all $u \in S^{\alpha,\infty}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$ it holds that*

$$|u(x) - u(y)| \leq C\|\nabla^\alpha u\|_\infty |x-y|^\alpha.$$

Proof. Step 1: We first consider the case $u \in S^{\alpha,\infty}(\mathbb{R}^n)$ with compact support. By Lemma 3.1.27 we have for any $\epsilon > 0$ and $x, y \in \mathbb{R}^n$ that (note $u_\epsilon \in C_c^\infty(\mathbb{R}^n)$)

$$|u_\epsilon(x) - u_\epsilon(y)| \leq C|x-y|^\alpha \|\nabla^\alpha u_\epsilon\|_\infty.$$

By Lemma 3.1.28 and Young's inequality for convolution we get

$$\|\nabla^\alpha u_\epsilon\|_\infty = \|\eta_\epsilon * \nabla^\alpha u\|_\infty \leq \|\eta_\epsilon\|_1 \|\nabla^\alpha u\|_\infty = \|\nabla^\alpha u\|_\infty.$$

This shows that for all $\epsilon > 0$

$$|u_\epsilon(x) - u_\epsilon(y)| \leq C|x-y|^\alpha \|\nabla^\alpha u\|_\infty. \tag{3.12}$$

Hence, the sequence $(u_\epsilon)_\epsilon$ is bounded and equi-continuous. Since $u_\epsilon \to u$ in $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ by using standard properties of mollifiers, the Arzelà-Ascoli theorem shows that even $u_\epsilon \to u$ uniformly. Taking the limit $\epsilon \downarrow 0$ in (3.12) now yields the result.

Step 2: For general $u \in S^{\alpha,\infty}(\mathbb{R}^n)$ we use a cut-off argument. Take $R > 0$ and $\delta > 0$ and let $\chi \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi|_{B_R(0)} \equiv 1$, $0 \leq \chi \leq 1$ and $\|\nabla\chi\|_{L^\infty(\mathbb{R}^n)} \leq \delta$. Then it follows from Lemma 3.1.22 that $\chi u \in S_0^{\alpha,\infty}(\mathrm{supp}(\chi))$ with

$$\nabla^\alpha(\chi u) = \chi\nabla^\alpha u + u\nabla^\alpha\chi + \nabla^\alpha_{\mathrm{NL}}(\chi, u).$$

We also have by (3.10) that

$$\|\nabla^\alpha(\chi u)\|_\infty \leq \|\nabla^\alpha u\|_\infty + C\delta^\alpha\|u\|_\infty.$$

We thus find by Step 1 that

$$|(\chi u)(x) - (\chi u)(y)| \leq C|x-y|^\alpha(\|\nabla^\alpha u\|_\infty + C\delta^\alpha\|u\|_\infty).$$

If we now choose R large enough so that $x, y \in B_R(0)$, then the left-hand side is equal to $|u(x) - u(y)|$ and by letting $\delta \downarrow 0$ we find the result. □

3.1.5 Connections Between Classical and Fractional Sobolev Spaces

In this section we establish the main tool used in our study of fractional integral and supremal functionals. It consists of a relation between the classical and fractional gradient of Sobolev functions, and it depends on attributes of the Riesz potential and its inverse, the fractional Laplacian; in fact, we use the following two identities

$$\nabla^\alpha\varphi = \nabla I_{1-\alpha}\varphi \qquad \text{and} \qquad \nabla\varphi = \nabla^\alpha(-\Delta)^{\frac{1-\alpha}{2}}\varphi, \tag{3.13}$$

which hold for $\varphi \in C_c^\infty(\mathbb{R}^n)$ by (3.3) and (3.7). The key observation is to interpret (3.13) as a way to express the fractional gradient of a function as the gradient of another function and vice versa. The main aim is to generalize this tool to the context of Sobolev spaces. To accomplish this, we first provide the following reasoning in order to gain intuition. Suppose that $u \in S^{\alpha,p}(\mathbb{R}^n)$ possesses a locally integrable Riesz potential $I_{1-\alpha}u$, cf. (3.1). Then, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} I_{1-\alpha}u\nabla\varphi\,dx = \int_{\mathbb{R}^n} uI_{1-\alpha}\nabla\varphi\,dx = \int_{\mathbb{R}^n} u\nabla^\alpha\varphi\,dx = -\int_{\mathbb{R}^n} \nabla^\alpha u\varphi\,dx, \tag{3.14}$$

where the first equality transfers the convolution to $\nabla\varphi$ via Fubini's theorem, the second relies on (3.3), and the third is simply the definition of the weak fractional gradient. This establishes $\nabla I_{1-\alpha}u = \nabla^\alpha u$, extending the first identity from (3.13). A similar strategy via duality works for the fractional Laplacian to extend the second identity in (3.13). The next proposition provides explicit statements, and is closely related to [24, Lemma 3.28], where instead the authors work on spaces with bounded fractional variation and prove a correspondence between classical and fractional variations.

Proposition 3.1.30. *Let $\alpha \in (0,1)$ and $p \in [1,\infty]$. Then we have the following:*

(i) For every $u \in S^{\alpha,p}(\mathbb{R}^n)$ there exists a $v \in W^{1,p}_{loc}(\mathbb{R}^n)$ such that $\nabla^\alpha u = \nabla v$ on $\mathbb{R}^n$.

(ii) For every $v \in W^{1,p}(\mathbb{R}^n)$ we have $u = (-\Delta)^{\frac{1-\alpha}{2}}v \in S^{\alpha,p}(\mathbb{R}^n)$ with $\nabla v = \nabla^\alpha u$ on $\mathbb{R}^n$ and

$$\|u\|_p \le C(n,\alpha)\|v\|_p^{1-\frac{1-\alpha}{2}}\|\nabla v\|_p^{\frac{1-\alpha}{2}}. \tag{3.15}$$

Proof. Part (i): It suffices to find for every $R > 0$ a $v \in W^{1,p}(B_R(0))$ such that $\nabla^\alpha u = \nabla v$ on $B_R(0)$. We treat the three cases $p \in (1,\infty)$, $p = \infty$ and $p = 1$ individually. In the first two cases, the key idea is to appeal to (3.14) after approximating u by functions with a well-defined Riesz potential.

Case $p \in (1,\infty)$: We can approximate u by a sequence $(u_j)_j \subset C_c^\infty(\mathbb{R}^n)$ in the $S^{\alpha,p}(\mathbb{R}^n)$-norm according to Theorem 3.1.16. Now we define $w_j = I_{1-\alpha}u_j \in L^\infty(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ for $j \in \mathbb{N}$, see Section 3.1.1, then it follows by (3.3) that $w_j \in W^{1,p}(B_R(0))$ and $\nabla w_j = \nabla^\alpha u_j$. By subtracting averages we obtain the sequence $(v_j)_j$ as

$$v_j = w_j - \frac{1}{|B_R|}\int_{B_R(0)} w_j\,dy.$$

We still have that $\nabla v_j = \nabla^\alpha u_j$ in addition to the bound $\|v_j\|_{L^p(B_R(0))} \le C(R,n,p)\|\nabla^\alpha u_j\|_p \le C$ by the Poincaré inequality (Theorem 1.1.7). Hence, $(v_j)_j$ is bounded in $W^{1,p}(B_R(0))$ so that up to subsequence $v_j \rightharpoonup v \in W^{1,p}(B_R(0))$. Since $\nabla v_j = \nabla^\alpha u_j \to \nabla^\alpha u$ we conclude $\nabla v = \nabla^\alpha u$ on $B_R(0)$ as desired.

Case $p = \infty$: Consider a sequence of cut-off functions $(\chi_j)_j \subset C_c^\infty(\mathbb{R}^n)$ such that

$$\chi_j|_{B_R(0)} \equiv 1,\;\; 0 \le \chi_j \le 1 \;\text{ and }\; \mathrm{Lip}(\chi) \le 1/j.$$

Then, by Lemma 3.1.22 we find that $u_j = \chi_j u \in S_0^{\alpha,\infty}(\mathrm{supp}(\chi_j))$ with

$$\|\nabla^\alpha u - \nabla^\alpha u_j\|_{L^\infty(B_R(0))} \le C\|\chi_j\|_\infty^{1-\alpha}\mathrm{Lip}(\chi)^\alpha\|u\|_\infty \le C(1/j)^\alpha\|u\|_\infty. \tag{3.16}$$

Because of the compact support we can define $w_j = I_{1-\alpha}u_j$, which lies in $L^\infty(\mathbb{R}^n)$, cf. Section 3.1.1. We also have, using the calculation (3.14), that $w_j \in W^{1,\infty}(B_R(0))$ and $\nabla w_j = \nabla^\alpha u_j$ on $B_R(0)$. Now we can follow the case $p \in (1,\infty)$ in defining

$$v_j = w_j - \frac{1}{|B_R|}\int_{B_R(0)} w_j\, dy,$$

and finding a limit $v_j \overset{*}{\rightharpoonup} v \in W^{1,\infty}(B_R(0))$ which has $\nabla^\alpha u$ as derivative since $\nabla^\alpha u_j \to \nabla^\alpha u$ uniformly on $B_R(0)$ by (3.16).

Case $p = 1$. This is a straightforward consequence of (3.14), cf. Remark 3.1.31 b).

Part (ii): By [20, Lemma A.4] we have for $1 \le p < \infty$ that $(-\Delta)^{\frac{1-\alpha}{2}}$ can be extended via density to a bounded linear operator from $W^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with the bound given as in (3.15). For $p = \infty$, we have $W^{1,\infty}(\mathbb{R}^n) = \mathrm{Lip}_b(\mathbb{R}^n)$ (Theorem 1.1.8), so then $(-\Delta)^{\frac{1-\alpha}{2}}$ is defined as in (3.4) and (3.15) follows analogously to [25, Lemma 2.2 Step 1]. If we set $u = (-\Delta)^{\frac{1-\alpha}{2}} v$ we can also compute for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} u\nabla^\alpha\varphi\, dx = \int_{\mathbb{R}^n} v(-\Delta)^{\frac{1-\alpha}{2}}\nabla^\alpha\varphi\, dx = \int_{\mathbb{R}^n} v\nabla\varphi\, dx. \tag{3.17}$$

The first equality in the case $p \in (1,\infty)$ is a consequence of the duality of the fractional Laplacian (3.6) after a simple extension to pairs of functions in $W^{1,p}(\mathbb{R}^n)$ and $W^{1,p'}(\mathbb{R}^n)$ using density, see [20, Lemma A.5]; this relies in addition on the fact $\nabla^\alpha\varphi \in \mathcal{T}(\mathbb{R}^n;\mathbb{R}^n) \subset W^{1,p'}(\mathbb{R}^n;\mathbb{R}^n)$. For $p \in \{1,\infty\}$, it suffices to extend the duality (3.6) to pairs in $\mathrm{Lip}_b(\mathbb{R}^n)$ and $W^{1,1}(\mathbb{R}^n)$ in a similar manner, owing to the observation $W^{1,\infty}(\mathbb{R}^n) = \mathrm{Lip}_b(\mathbb{R}^n)$. The second equality in (3.17) follows from the identity $(-\Delta)^{\frac{1-\alpha}{2}}\nabla^\alpha = (-\Delta)^{\frac{1-\alpha}{2}} I_{1-\alpha}\nabla = \nabla$ on $C_c^\infty(\mathbb{R}^n)$ from (3.7). This proves $\nabla^\alpha u = \nabla v$ as required. □

Remark 3.1.31. a) Part (ii) implies in particular that $(-\Delta)^{\frac{1-\alpha}{2}} : W^{1,p}(\mathbb{R}^n) \to S^{\alpha,p}(\mathbb{R}^n)$ is a bounded linear operator, and as such, is weakly continuous for $p \in [1,\infty)$.

b) When $u \in S^{\alpha,p}(\mathbb{R}^n)$ possesses a well-defined Riesz potential $I_{1-\alpha}u \in L^p_{loc}(\mathbb{R}^n)$ we do not need to go through the approximation argument in part (i). Indeed, by taking $v = I_{1-\alpha}u$, we find that it satisfies $\nabla v = \nabla^\alpha u$ on $\mathbb{R}^n$ in view of (3.14). Recalling (3.1), we observe that this argument is applicable in the regime $p < n/(1-\alpha)$, but it fails in general, as e.g. $u(x) = \min\{1, |x|^{-(1-\alpha)}\}$ for $x \in \mathbb{R}^n$ shows. Namely, we have that $u \in W^{1,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ when $p > n/(1-\alpha)$, but u does not possess a well-defined Riesz potential by verifying the criterion (3.1).

c) In part (i) we can not conclude that $v \in W^{1,p}(\mathbb{R}^n)$, i.e. that $v \in L^p(\mathbb{R}^n)$. In the case $p \in (1,\infty)$, this follows if we can find a $u \in S^{\alpha,p}(\mathbb{R}^n)$ with a well-defined Riesz potential $I_{1-\alpha}u \in L^p_{loc}(\mathbb{R}^n)$ such that $I_{1-\alpha}u + C \notin L^p(\mathbb{R}^n)$ for any $C \in \mathbb{R}$, because v is equal to $I_{1-\alpha}u$ up to a constant by b). An example of such a u is $u(x) = \min\{1, |x|^{-n/p+\alpha-1}\}$. We see that $u \in W^{1,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ and $I_{1-\alpha}u$

is well-defined by (3.1). Moreover, we can calculate for $|x| > 1$ that

$$\begin{aligned} I_{1-\alpha}u(x) &= \frac{1}{\gamma_{n,\alpha}} \int_{\mathbb{R}^n} \frac{u(x-y)}{|y|^{n+\alpha-1}}\, dy \\ &\geq \frac{1}{\gamma_{n,\alpha}} \int_{B(0,2|x|)^c} \frac{|x-y|^{-n/p+\alpha-1}}{|y|^{n+\alpha-1}}\, dy \\ &\geq C \int_{B(0,2|x|)^c} |y|^{-n/p-n}\, dy = C|x|^{-n/p}. \end{aligned}$$

The inequality in the last line uses $|x-y| \geq 1/2|y|$ for $y \in B(0,2|x|)^c$. We indeed observe $I_{1-\alpha}u + C \notin L^p(\mathbb{R}^n)$ for any $C \in \mathbb{R}$.

For $p = \infty$, we use a truncation argument to find a function $u \in S^{\alpha,\infty}(\mathbb{R}^n)$ and $v \in W^{1,\infty}_{loc}(\mathbb{R}^n) \setminus L^\infty(\mathbb{R}^n)$ with $\nabla v = \nabla^\alpha u$. The construction is inspired by [15, Lemma 3.1]. Consider for fixed $\beta \in (0, 1-\alpha)$ the function

$$v(x) = \begin{cases} |x| & \text{for } |x| \leq 1, \\ |x|^\beta & \text{for } |x| > 1, \end{cases} \quad \text{for } x \in \mathbb{R}^n,$$

and define its truncation $v_j = \min\{v, j\}$ for $j \in \mathbb{N}$. Per construction, $(v_j)_j \subset W^{1,\infty}(\mathbb{R}^n) \cap C^{0,\beta}(\mathbb{R}^n)$ is a sequence with uniformly bounded Lipschitz and β-Hölder constants. By fixing $u_j := (-\Delta)^{\frac{1-\alpha}{2}} v_j \in S^{\alpha,\infty}(\mathbb{R}^n)$, we can compute

$$\begin{aligned} |u_j| &\leq C(n,\alpha) \int_{\mathbb{R}^n} \frac{|v_j(\cdot + h) - v_j|}{|h|^{n+1-\alpha}}\, dh \\ &\leq C(n,\alpha) \Big(\int_{B(0,1)} \frac{1}{|h|^{n-\alpha}}\, dh + \int_{B(0,1)^c} \frac{1}{|h|^{n+1-\alpha-\beta}}\, dh \Big) < \infty \end{aligned}$$

for all $j \in \mathbb{N}$, where the bounds on the integrals over $B_1(0)$ and $B_1(0)^c$ have been deduced from the Lipschitz and β-Hölder property of v_j, respectively. As a result, $(u_j)_j$ is a bounded sequence in $S^{\alpha,\infty}(\mathbb{R}^n)$, so that up to subsequence $u_j \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^n)$ and $\nabla^\alpha u_j \overset{*}{\rightharpoonup} \nabla^\alpha u$ in $L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ as $j \to \infty$. By virtue of Proposition 3.1.30 (ii), we know that $\nabla^\alpha u_j = \nabla v_j$ for all $j \in \mathbb{N}$, from which we infer that $\nabla^\alpha u = \nabla v$, as required. △

The above proposition almost gives a full identification of fractional and classical gradients. In the case where we consider periodic functions, we actually obtain the full identification. For this, denote $Q = (0,1)^n$ and $W^{1,\infty}_{per}(Q)$ and $S^{\alpha,\infty}_{per}(Q)$ as the Q-periodic functions in $W^{1,\infty}(\mathbb{R}^n)$ and $S^{\alpha,\infty}(\mathbb{R}^n)$, respectively.

Proposition 3.1.32. *Let $\alpha \in (0,1)$ then for every $\varphi \in W^{1,\infty}_{per}(Q)$ there is a $\psi \in S^{\alpha,\infty}_{per}(Q)$ such that $\nabla\varphi = \nabla^\alpha\psi$ on $\mathbb{R}^n$ and vice versa.*

Proof. Take $\varphi \in W^{1,\infty}_{per}(Q)$, then, it follows by Proposition 3.1.30 (ii) that $\psi := (-\Delta)^{\frac{1-\alpha}{2}} \varphi \in S^{\alpha,\infty}(\mathbb{R}^n)$ satisfies $\nabla^\alpha\psi = \nabla\varphi$. It is readily seen from the formula of the fractional Laplacian (3.4) for $\mathrm{Lip}_b(\mathbb{R}^n)$ functions, that ψ is also Q-periodic and hence, $\psi \in S^{\alpha,\infty}_{per}(Q)$.

For the converse, take $\psi \in S^{\alpha,\infty}_{per}(Q)$, then, by Proposition 3.1.30 (i) we can find $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ such that $\nabla\varphi = \nabla^\alpha\psi$. We have that the gradient of φ is Q-periodic on $\mathbb{R}^n$, however, we do not know whether φ itself is Q-periodic. To prove this, we observe that any function in $W^{1,\infty}_{loc}(\mathbb{R}^n)$ with zero derivative is constant, so that there is a vector $\zeta \in \mathbb{R}^n$ such that

$$\varphi(x + e_i) - \varphi(x) = \zeta_i \quad \text{for all } x \in Q \text{ and } i = 1, \ldots, n. \tag{3.18}$$

Here, e_i denotes the ith standard unit vector. It is enough to show that $\zeta = 0$ to conclude that φ is periodic. By the Gauss-Green theorem for Lipschitz functions we can compute

$$\int_Q \nabla^\alpha\psi \, dx = \int_Q \nabla\varphi \, dx = \int_{\partial Q} \varphi\nu \, dx = \zeta, \tag{3.19}$$

with ν an outer normal to the unit cube. The last equality follows from (3.18) and the fact that the normal is in the direction of the unit vectors.

Now we show that the leftmost integral in (3.19) is zero, which yields $\zeta = 0$. By the α-homogeneity of the fractional gradient, cf. Section 3.1.1, the sequence $(\psi_j)_j$ defined by $\psi_j(x) = \frac{1}{j^\alpha}\psi(jx)$ is bounded in $S^{\alpha,\infty}(\mathbb{R}^n)$ and thus, converges weak* to its uniform limit which is zero. In particular, we have

$$\nabla^\alpha\psi_j \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{3.20}$$

At the same time, we find $\nabla^\alpha\psi_j = \nabla^\alpha\psi(j\cdot)$, which is the oscillation of a Q-periodic function and hence, converges by Proposition 1.1.2 weak* to its average

$$\nabla^\alpha\psi_j \overset{*}{\rightharpoonup} \int_Q \nabla^\alpha\psi \, dx \quad \text{in } L^\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{3.21}$$

Combining (3.20) and (3.21) yields $\int_Q \nabla^\alpha\psi \, dx = 0$, which implies $\zeta = 0$ by (3.19). □

3.2 Integral Functionals

We are now in the position to study integral functionals of the form

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(x, u(x), \nabla^\alpha u(x)) \, dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m), \tag{3.22}$$

where $p \in (1,\infty)$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a suitable integrand and $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$. We are interested in minimizing this functional and the weak lower semicontinuity plays an important role here. In Section 3.2.1, we characterize the weak lower semicontinuity fully in terms of quasiconvexity of the integrand in the third argument. As a consequence of the direct method, we provide an existence result under these assumptions. Afterwards, we study the fractional Euler-Lagrange equations, which are a system of fractional PDEs that minimizers of (3.22) must satisfy. Lastly, in Section 3.2.3, we derive a relaxation formula for $\mathcal{I}_\alpha$ providing information about $\mathcal{I}_\alpha$ when it is not weakly lower semicontinuous.

3.2.1 Weak Lower Semicontinuity and Existence

The aim of the present section is to characterize the weak lower semicontinuity of $\mathcal{I}_\alpha$ in (3.22) and obtain existence of minimizers. Using the connection between the classical and fractional gradients from Section 3.1.5 we show that the notion of quasiconvexity is also relevant in the fractional setting. We furthermore answer the question how quasiconvexity is related to a more natural analogue in the fractional setting, which we call α-quasiconvexity.

We denote $Q = (0,1)^n$ and recall that a Borel measurable function $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called quasiconvex (in the sense of Morrey [56]) if

$$h(A) \le \int_Q h(A + \nabla\varphi)\, dy \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m). \tag{3.23}$$

Equivalently, by [29, Proposition 5.13], we can consider Q-periodic test functions as opposed to functions with zero boundary values, that is, f is quasiconvex if and only if

$$h(A) \le \int_Q h(A + \nabla\varphi)\, dy \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in W_{per}^{1,\infty}(Q;\mathbb{R}^m). \tag{3.24}$$

The next theorem shows that the functionals in (3.22) are weakly lower semicontinuous if the integrands f are quasiconvex in the third variable. The proof uses well-known methods in the calculus of variations in combination with the results from Section 3.1.5. Note also that quasiconvexity is not imposed in Ω^c due to the strong convergence of the fractional gradients here (Lemma 3.1.24).

Theorem 3.2.1 (Sufficiency of quasiconvexity)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ open and bounded with $|\partial\Omega| = 0$, $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$ and*

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(x, u(x), \nabla^\alpha u(x))\, dx \quad \text{for } u \in S_g^{\alpha,p}(\Omega;\mathbb{R}^m).$$

Assume $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a normal integrand satisfying

$$0 \le f(x,z,A) \le C(1 + |z|^p + |A|^p) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $a \in L^1(\mathbb{R}^n)$ and $C > 0$. If $A \mapsto f(x,z,A)$ is quasiconvex for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^m$, then $\mathcal{I}_\alpha$ is (sequentially) weakly lower semicontinuous on $S_g^{\alpha,p}(\Omega;\mathbb{R}^m)$.

Proof. Let $u_j \rightharpoonup u$ in $S_g^{\alpha,p}(\Omega;\mathbb{R}^m)$, then we show separately that

$$\int_\Omega f(x,u,\nabla^\alpha u)\, dx \le \liminf_{j\to\infty} \int_\Omega f(x,u_j,\nabla^\alpha u_j)\, dx, \tag{3.25}$$

and

$$\int_{\Omega^c} f(x,u,\nabla^\alpha u)\, dx \le \liminf_{j\to\infty} \int_{\Omega^c} f(x,u_j,\nabla^\alpha u_j)\, dx. \tag{3.26}$$

Regarding (3.25), we find by Theorem 3.1.20 that $u_j \to u$ in $L^p(\mathbb{R}^n;\mathbb{R}^m)$. Additionally, Proposition 3.1.30 (i) allows us to find a sequence $(v_j)_j \subset W^{1,p}(\Omega;\mathbb{R}^m)$ and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\nabla v_j = \nabla^\alpha u_j \text{ on } \Omega \text{ for all } j \in \mathbb{N} \quad \text{and} \quad \nabla v = \nabla^\alpha u \text{ on } \Omega. \tag{3.27}$$

By further assuming that the v_j and v have average zero we conclude via Poincaré's inequality (Theorem 1.1.7) that $(v_j)_j$ is a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. Hence, we find that $v_j \rightharpoonup v$ in $W^{1,p}(\Omega;\mathbb{R}^m)$.

We now resort to well-known techniques involving Young measures, cf. Section 1.1.6. Since (up to a non-relabeled subsequence) $(\nabla v_j)_j$ generates a Young measure $(\mu_x)_{x\in\Omega}$, we find that $(u_j, \nabla v_j)_j$ generates the Young measure $(\delta_{u(x)} \otimes \mu_x)_{x\in\Omega}$ by Lemma 1.1.22. The fundamental theorem of Young measures (Theorem 1.1.21 (ii)) and (3.27) now yields

$$\begin{aligned}\liminf_{j\to\infty} \int_\Omega f(x,u_j,\nabla^\alpha u_j)\,dx &= \liminf_{j\to\infty} \int_\Omega f(x,u_j,\nabla v_j)\,dx \\ &\geq \int_\Omega \int_{\mathbb{R}^{m\times n}} f(x,u,A)\,d\mu_x(A)\,dx \\ &\geq \int_\Omega f(x,u,\nabla v)\,dx \\ &= \int_\Omega f(x,u,\nabla^\alpha u)\,dx.\end{aligned}$$

The third line uses the characterization of gradient Young measures (Theorem 1.1.23 (iii)) relying on the fact that f is quasiconvex in its last argument with p-growth and $[\mu]_x = \nabla v(x)$ for a.e. $x \in \Omega$. This establishes (3.25).

For any $\Omega \Subset \Omega'$ we have by Lemma 3.1.24 that $\nabla^\alpha u_j \to \nabla^\alpha u$ in $L^p((\Omega')^c;\mathbb{R}^{m\times n})$. Therefore, a simple lower semicontinuity result with respect to strong convergence in L^p (e.g. [37, Theorem 6.49]) yields

$$\begin{aligned}\int_{(\Omega')^c} f(x,u,\nabla^\alpha u)\,dx &\leq \liminf_{j\to\infty} \int_{(\Omega')^c} f(x,u_j,\nabla^\alpha u_j)\,dx \\ &\leq \liminf_{j\to\infty} \int_{\Omega^c} f(x,u_j,\nabla^\alpha u_j)\,dx.\end{aligned}$$

By letting Ω' tend to Ω, (3.26) follows by the monotone convergence theorem, the non-negativity of f and the assumption $|\partial\Omega| = 0$. Combining (3.25) and (3.26) finishes the proof. □

Remark 3.2.2. a) We can generalize the bounds on f from the previous theorem. For $x \in \Omega$ we can relax the lower bound to

$$-a(x) - C(|z|^p + |A|^q) \leq f(x,z,A) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

for any $C > 0$, $1 \leq q < p$ and $a \in L^1(\Omega)$. This condition ensures that the negative part of the sequence $(f(\cdot,u_j,\nabla^\alpha u_j))_j$ is equi-integrable, due to the convergence of u_j in L^p and the boundedness of $\nabla^\alpha u_j$ in L^p, so that the fundamental theorem of Young measures can be applied without change. For $x \in \Omega^c$ we may discard the upper bound entirely and relax the lower bound to

$$-b(x) - C(|z|^p + |A|^p) \leq f(x,z,A) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $C > 0$ and $b \in L^1(\Omega^c)$. In this case [37, Theorem 6.49] still applies.

b) In the regime $p > \min\{m, n\}$ we can allow extended-valued integrands $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to [0, \infty]$ that do not satisfy any upper bound, if we replace the quasiconvexity assumption by the stronger condition of polyconvexity. This namely holds in the classical case (e.g. [29, Theorem 8.16]) and we can carry it over, using the same arguments as in the previous theorem. In fact, when $n = m$ it was proven in [15, Theorem 6.1] that an existence result for polyconvex extended-value integrands even holds for $p \geq n - 1$ using a fractional Piola identity. △

In order to prove the necessary condition, we first construct functions with compact support whose fractional gradients have a prescribed value at a point. This is going to be our replacement for the use of affine functions, which have constant gradients, in the classical setting.

Lemma 3.2.3. *Let $\alpha \in (0, 1)$, $\Omega \subset \mathbb{R}^n$ open and bounded and $x_0 \in \Omega$. Then for any $z \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ there exists a $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$ such that $u(x_0) = z$ and $\nabla^\alpha \varphi(x_0) = A$.*

Proof. Using the translation invariance and α-homogeneity of the fractional gradient we may assume that $\Omega = B_1(0)$ and $x_0 = 0$. Since any radially symmetric function has zero fractional derivative at the origin, we can always add such a function to guarantee $u(0) = z$. Combining this with linearity, it is enough to consider $m = 1$ and to find $\nabla^\alpha \varphi(0)$ as a multiple of the first unit vector. For this, take $\chi, \psi \in C_c^\infty((-1, 1))$ to be even and odd functions respectively, i.e. $\chi(t) = \chi(-t)$ and $\psi(-t) = -\psi(t)$ for $t \in (-1, 1)$, which are supported in $(-1/n, 1/n)$. Also assume for $t > 0$ that $\psi(t)\chi(t)$ is non-negative and not identically zero. Then, defining $\varphi \in C_c^\infty(B_1(0))$ by (cf. Figure 3.1)

$$\varphi(x) = \psi(x_1)\chi(x_2)\ldots\chi(x_n)$$

yields (noting that $\varphi(0) = 0$)

$$(\nabla^\alpha \varphi(0))_1 = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{\varphi(y)y_1}{|y|^{n+\alpha+1}}\, dy = 2\mu_{n,\alpha} \int_{\{y\in\mathbb{R}^n \,|\, y_1>0\}} \frac{\varphi(y)y_1}{|y|^{n+\alpha+1}}\, dy > 0.$$

On the contrary, $(\nabla^\alpha \varphi(0))_i = 0$ for $i \neq 1$ as the integrand is then odd with respect to the ith variable. As such, we find that $\nabla^\alpha \varphi(0)$ is a multiple of the first unit vector. □

Remark 3.2.4. If we have $\Omega \subset \mathbb{R}^n$ open and bounded and $g \in S^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^m)$ then we can use Lemma 3.2.3 to find for almost every $x_0 \in \Omega$ a $u \in S_g^{\alpha,p}(\Omega; \mathbb{R}^m)$ such that $u(x_0) = z$, $\nabla^\alpha u(x_0) = A$ and x_0 is a p-Lebesgue point of u and $\nabla^\alpha u$ in the sense that

$$\lim_{r\to 0} r^{-n} \int_{B_r(x_0)} |u(x) - z|^p\, dx = 0 \;\text{ and }\; \lim_{r\to 0} r^{-n} \int_{B_r(x_0)} |\nabla^\alpha u(x) - A|^p\, dx = 0.$$

Indeed, we can just take x_0 to be a p-Lebesgue point of g and $\nabla^\alpha g$ and set $u = g + \varphi$ with a function $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$ such that

$$\varphi(x_0) = z - g(x_0) \;\text{ and }\; \nabla^\alpha \varphi(x_0) = A - \nabla^\alpha g(x_0).$$

△

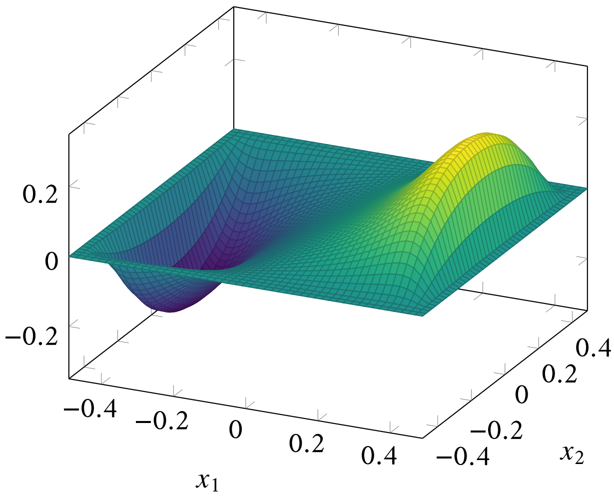

Figure 3.1: An example φ as constructed in Lemma 3.2.3 for $n = 2$, which satisfies $\nabla^\alpha \varphi(0) = (\beta, 0)$ with $\beta > 0$.

Next, we prove that quasiconvexity of the integrands is also necessary for the weak lower semicontinuity. Together with Theorem 3.2.1 this proves the characterization result Theorem 3.2.6. The proof uses the connection between the classical and fractional gradient in combination with cut-off arguments, strong convergence of fractional gradient outside Ω, and the constructed functions from the preceding lemma. It also takes elements from the classical necessity proof [2], as found in [29, Lemma 3.18].

Theorem 3.2.5 (Necessity of quasiconvexity)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ open and bounded and $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$. Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function that satisfies*

$$|f(x,z,A)| \leq a(x) + C(|z|^p + |A|^p) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $a \in L^1(\mathbb{R}^n)$ and $C > 0$. If

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(x, u(x), \nabla^\alpha u(x))\, dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m),$$

is (sequentially) weakly lower semicontinuous on $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ then $A \mapsto f(x,z,A)$ is quasiconvex for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^m$.

Proof. We may assume without loss of generality that $g = 0$. Indeed, the weak lower semicontinuity of $\mathcal{I}_\alpha$ on $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ is equivalent to the weak lower semicontinuity of

$$\mathcal{I}_{\alpha,g}(u) := \int_{\mathbb{R}^n} f_g(x, u, \nabla^\alpha u)\, dx$$

on $S^{\alpha,p}_0(\Omega;\mathbb{R}^m)$ with $f_g(x,z,A) = f(x, z + g(x), A + \nabla^\alpha g(x))$. Moreover, we can check that f_g is also a Carathéodory integrand satisfying the p-growth bound and that $f(x,z,\cdot)$ is quasiconvex if and only if $f_g(x, z - g(x), \cdot)$ is quasiconvex.

Step 1: To show that $f(x,z,\cdot)$ is quasiconvex for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^m$, let $(x_0, z_0, A_0) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$, and consider a $u \in C_c^\infty(\Omega;\mathbb{R}^m)$ such that

$$u(x_0) = z_0 \quad \text{and} \quad \nabla^\alpha u(x_0) = A_0,$$

possible by Lemma 3.2.3. Take $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$ and assume for simplicity that $x_0 + Q \Subset \Omega$. Then we can extend φ periodically to $\mathbb{R}^n$ and consider the sequence $(\varphi_j^\rho)_j \subset W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$ for $j, \rho \in \mathbb{N}$

$$\varphi_j^\rho(x) = \begin{cases} \frac{1}{\rho j}\varphi(\rho j(x-x_0)) & \text{if } x \in Q_\rho := x_0 + (0,1/\rho)^n, \\ 0 & \text{else.} \end{cases}$$

It can be seen that $\varphi_j^\rho \rightharpoonup 0$ as $j \to \infty$ in $W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$, since it is bounded in $W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$ and converges uniformly to zero. Additionally, Proposition 3.1.30 (ii) shows that we have

$$\psi_j^\rho := (-\Delta)^{\frac{1-\alpha}{2}}\varphi_j^\rho \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m) \quad \text{with} \quad \nabla^\alpha \psi_j^\rho = \nabla\varphi_j^\rho \ \text{ on } \mathbb{R}^n. \tag{3.28}$$

By remark 3.1.31 a) and (3.15) we find in the limit $j \to \infty$

$$\psi_j^\rho \rightharpoonup 0 \ \text{ in } S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m) \quad \text{and} \quad \psi_j^\rho \to 0 \ \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^m). \tag{3.29}$$

If we take a cut-off function $\chi \in C_c^\infty(\Omega)$ with $\chi \equiv 1$ on $x_0 + Q$, then we find by Lemma 3.1.22 that the sequence

$$u_j := u + \chi\psi_j^\rho \rightharpoonup u \ \text{ in } S_0^{\alpha,p}(\Omega;\mathbb{R}^m) \text{ as } j \to \infty$$

and in view of (3.29) and (3.10)

$$R_j := \nabla^\alpha u_j - \nabla^\alpha u - \chi\nabla^\alpha\psi_j^\rho \to 0 \ \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^{m\times n}).$$

We also have $\chi\nabla^\alpha\psi_j^\rho = \nabla\varphi_j^\rho$ by (3.28) and the fact that $\nabla\varphi_j^\rho = 0$ outside Q_ρ. Now, using the weak lower semicontinuity of $\mathcal{I}_\alpha$ on $S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$ yields

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x,u,\nabla^\alpha u)\,dx &\le \liminf_{j\to\infty}\int_{\mathbb{R}^n} f(x,u_j,\nabla^\alpha u_j)\,dx \\
&= \liminf_{j\to\infty}\int_{Q_\rho} f(x,u+\chi\psi_j^\rho,\nabla^\alpha u + \nabla\varphi_j^\rho + R_j)\,dx \\
&\qquad + \int_{Q_\rho^c} f(x,u+\chi\psi_j^\rho,\nabla^\alpha u + R_j)\,dx \\
&\le \liminf_{j\to\infty}\int_{\Omega} f(x,u,\nabla^\alpha u + \nabla\varphi_j^\rho)\,dx + \int_{Q_\rho^c} f(x,u+\chi\psi_j^\rho,\nabla^\alpha u + R_j)\,dx \\
&= \liminf_{j\to\infty}\int_{\Omega} f(x,u,\nabla^\alpha u + \nabla\varphi_j^\rho)\,dx + \int_{Q_\rho^c} f(x,u,\nabla^\alpha u)\,dx.
\end{aligned}$$

The second to last line uses the freezing argument from Lemma 1.1.24, which is applicable since R_j and ψ_j^ρ converge strongly to 0 in the L^p-norm and $\nabla\varphi_j^\rho$ is p-equi-integrable because it is an essentially bounded sequence. The last line follows from the dominated convergence theorem. By subtracting the integral over Q_ρ^c from both sides, which is finite by the p-growth of f, we find

$$\int_{Q_\rho} f(x, u, \nabla^\alpha u)\, dx \leq \liminf_{j\to\infty} \int_{Q_\rho} f(x, u, \nabla^\alpha u + \nabla\varphi_j^\rho)\, dx. \tag{3.30}$$

Step 2: Denote

$$\lambda := \|u\|_{L^\infty(\Omega)} + \|\nabla^\alpha u\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^\infty(Q)} < \infty$$

and define the compact set

$$S = \{(z, A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n} \mid |z| + |A| \leq \lambda\}.$$

We can find by the Scorza-Dragoni theorem ([29, Theorem 3.8]) for $l \in \mathbb{N}$ an increasing sequence of compact sets $K_l \subset \Omega$ such that f is continuous on $K_l \times S$ and $|\Omega \setminus K_l| \leq 1/l$. By the Tietze extension theorem ([65, Theorem 20.4]) we find a continuous function $f_l : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ which coincides with f on $K_l \times S$ and such that

$$|f_l| \leq M_l := \max\{|f(x, z, A)| \mid (x, z, A) \in K_l \times S\}.$$

We now fix $\epsilon' > 0$ and we can arrange that for all $l \in \mathbb{N}$

$$\int_{\Omega\setminus K_l} |f_l(x, w(x), W(x))|\, dx \leq \epsilon' \tag{3.31}$$

for any $w \in L^\infty(\Omega; \mathbb{R}^m)$, $W \in L^\infty(\Omega; \mathbb{R}^{m\times n})$. Indeed, this can be achieved by replacing f_l by ηf_l with $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on K_l, $0 \leq \eta \leq 1$ and

$$\int_{\Omega\setminus K_l} \eta(x)\, dx \leq \epsilon'/M_l.$$

We additionally assume that x_0 lies in $\bigcup_{l\in\mathbb{N}} K_l$ and that it is a Lebesgue point of the indicator functions $\mathbb{1}_{\Omega\setminus K_l}$ and $a \cdot \mathbb{1}_{\Omega\setminus K_l}$ for all $l \in \mathbb{N}$. This loses no generality as almost every point $x_0 \in \Omega$ satisfies these conditions. Because of this, we find that for all $l \geq l'$ with $x_0 \in K_{l'}$ that

$$\begin{aligned}\lim_{\rho\to\infty} \rho^n \int_{Q_\rho\setminus K_l} &|f(x, u(x), \nabla^\alpha u(x) + \nabla\varphi_j^\rho(x))|\, dx \\ &\leq \lim_{\rho\to\infty} \rho^n \int_{Q_\rho\setminus K_l} a(x) + 2C\lambda^p\, dx = (a(x_0) + 2C\lambda^p)\mathbb{1}_{\Omega\setminus K_l}(x_0) = 0.\end{aligned}$$

Hence, if we take $\epsilon > 0$ we find that for any $l \geq l'$ fixed, ρ large enough and any $j \in \mathbb{N}$ that

$$\int_{Q_\rho\setminus K_l} |f(x, u(x), \nabla^\alpha u(x) + \nabla\varphi_j^\rho(x))|\, dx \leq \frac{\epsilon}{\rho^n}. \tag{3.32}$$

Step 3: We return to (3.30) and we can write

$$\begin{aligned}\int_{Q_\rho} f(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)\,dx &\le \int_{Q_\rho} f_l(x_0,z_0,A_0+\nabla\varphi_j^\rho)\,dx\\ &\quad+\int_{Q_\rho}|f_l(x_0,z_0,A_0+\nabla\varphi_j^\rho)-f_l(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)|\,dx\\ &\quad+\int_{Q_\rho}|f_l(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)-f(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)|\,dx\\ &=: (I)+(II)+(III).\end{aligned}$$

Fix $l \ge l'$, then we find since $(u(x), \nabla^\alpha u(x)+\nabla\varphi_j^\rho(x)) \in S$ for a.e. $x \in \Omega$ that

$$(III) \le \int_{Q_\rho\setminus K_l}|f_l(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)-f(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)|\,dx \le \frac{\epsilon}{\rho^n}+\epsilon',$$

for all ρ large enough and all $j \in \mathbb{N}$ using (3.31) and (3.32). By the uniform continuity of f_l on compact sets and the continuity of u and $\nabla^\alpha u$ on Ω we find that

$$(II) \le \frac{\epsilon}{\rho^n}$$

for all ρ large enough and all $j \in \mathbb{N}$. Lastly, using the periodicity of φ_j^ρ we see that the first term is equal to

$$(I) = \frac{1}{\rho^n}\int_Q f_l(x_0,z_0,A_0+\nabla\varphi)\,dx = \frac{1}{\rho^n}\int_Q f(x_0,z_0,A_0+\nabla\varphi)\,dy,$$

by also using that $x_0 \in K_l$. All in all we find that for ρ large enough

$$\begin{aligned}\liminf_{j\to\infty}\int_{Q_\rho} f(x,u,\nabla^\alpha u+\nabla\varphi_j^\rho)\,dx &\le \frac{1}{\rho^n}\int_Q f(x_0,z_0,A_0+\nabla\varphi(x))\,dy+\frac{2\epsilon}{\rho^n}+\epsilon'\\ &\le \frac{1}{\rho^n}\int_Q f(x_0,z_0,A_0+\nabla\varphi(x))\,dy+\frac{3\epsilon}{\rho^n},\end{aligned}$$

by choosing $\epsilon' \le \epsilon/\rho^n$. A similar yet simpler computation shows that

$$\int_{Q_\rho} f(x,u,\nabla^\alpha u)\,dx \ge \frac{1}{\rho^n} f(x_0,z_0,A_0) - \frac{3\epsilon}{\rho^n}.$$

Hence, (3.30) becomes after multiplying by ρ^n

$$f(x_0,z_0,A_0) \le \int_Q f(x_0,z_0,A_0+\nabla\varphi)\,dy + 6\epsilon.$$

The proof follows by the arbitrariness of ϵ. □

The combination of Theorem 3.2.1 and Theorem 3.2.5 now yields a characterization of weak lower semicontinuity.

Theorem 3.2.6 (Characterization of weak lower semicontinuity)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ open and bounded with $|\partial\Omega| = 0$ and $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$. Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function that satisfies*

$$0 \le f(x,z,A) \le a(x) + C(|z|^p + |A|^p) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $a \in L^1(\mathbb{R}^n)$ and $C > 0$. Then, the functional

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(x,u(x),\nabla^\alpha u(x))\,dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m),$$

is (sequentially) weakly lower semicontinuous on $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ if and only if $A \mapsto f(x,z,A)$ is quasiconvex for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^m$.

A simple adaptation results in the following characterization result on the full space $S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$.

Theorem 3.2.7. *Let $\alpha \in (0,1)$ and $p \in (1,\infty)$. Suppose that $f : \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function that satisfies*

$$0 \le f(x,z,A) \le a(x) + C(|z|^p + |A|^p) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $a \in L^1(\mathbb{R}^n)$ and $C > 0$. Then, the functional

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(x,u(x),\nabla^\alpha u(x))\,dx \quad \text{for } u \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m),$$

is (sequentially) weakly lower semicontinuous on $S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$ if and only if $A \mapsto f(x,z,A)$ is quasiconvex for a.e. $x \in \mathbb{R}^n$ and all $z \in \mathbb{R}^m$.

Proof. For the sufficiency, let $u_j \rightharpoonup u$ in $S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$ then by Theorem 3.1.23 we find $u_j \to u$ in $L^p_{loc}(\mathbb{R}^n;\mathbb{R}^m)$. Hence, we may argue as in Theorem 3.2.1 to show that for any $R > 0$

$$\begin{aligned}\int_{B_R(0)} f(x,u,\nabla^\alpha u)\,dx &\le \liminf_{j\to\infty} \int_{B_R(0)} f(x,u_j,\nabla^\alpha u_j)\,dx \\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}^n} f(x,u_j,\nabla^\alpha u_j)\,dx.\end{aligned}$$

Then, letting $R \to \infty$ via the monotone convergence theorem yields $\mathcal{I}_\alpha(u) \le \liminf_{j\to\infty} \mathcal{I}_\alpha(u_j)$ as desired.

For the necessity, we note that $\mathcal{I}_\alpha$ is weakly lower semicontinuous on $S^{\alpha,p}_0(B_R(0);\mathbb{R}^m)$ for any $R > 0$ so that we may conclude the result via Theorem 3.2.5. □

It is surprising that the weak lower semicontinuity of fractional integral functionals can be characterized by a notion involving the classical gradient. Because of this, we introduce the following alternative notion, which seems to be the natural analogue of quasiconvexity in the fractional setting.

Definition 3.2.8 (α-quasiconvexity)**.** Let $\alpha \in (0,1)$. A Borel measurable function $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be α-quasiconvex if

$$h(A) \leq \int_Q h(A + \nabla^\alpha \varphi)\, dy \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in S^{\alpha,\infty}_{per}(Q;\mathbb{R}^m). \qquad \triangle$$

Remark 3.2.9. a) In the definition of α-quasiconvexity one could replace Q by any other cube Q' through translation and scaling and test with functions that are Q'-periodic instead. In a similar spirit, if f is continuous we can show that α-quasiconvexity is equivalent to

$$h(A) \leq \int_Q h(A + \nabla^\alpha \varphi)\, dy \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in C^{\infty}_{per}(Q;\mathbb{R}^m). \tag{3.33}$$

Indeed, it is clear that α-quasiconvexity implies (3.33), while for the other direction we can approximate $\varphi \in S^{\alpha,\infty}_{per}(Q;\mathbb{R}^m)$ via mollification by a sequence $(\varphi_j)_j \subset C^\infty_{per}(Q;\mathbb{R}^m)$, as in Lemma 3.1.28, such that $\nabla^\alpha \varphi_j \to \nabla^\alpha \varphi$ in $L^1(Q;\mathbb{R}^{m\times n})$ and $\|\nabla^\alpha \varphi_j\|_\infty \leq \|\nabla^\alpha \varphi\|_\infty$. Then, the dominated convergence theorem shows for any $A \in \mathbb{R}^{m\times n}$

$$h(A) \leq \lim_{j\to\infty} \int_Q h(A + \nabla^\alpha \varphi_j)\, dy = \int_Q h(A + \nabla^\alpha \varphi)\, dy,$$

thus proving that h is α-quasiconvex.

b) Since in classical quasiconvexity we may take test functions in $W^{1,\infty}_0(Q;\mathbb{R}^m)$ or $W^{1,\infty}_{per}(Q;\mathbb{R}^m)$, cf. (3.23) and (3.24), it is natural to expect that

$$h(A) \leq \int_Q h(A + \nabla^\alpha \varphi)\, dy \quad \text{for all } A \in \mathbb{R}^{m\times n} \text{ and } \varphi \in S^{\alpha,\infty}_0(Q;\mathbb{R}^m) \tag{3.34}$$

provides a different meaningful fractional extension of quasiconvexity. However, this is not the case, since in the setting $n = m = 1$ even some convex functions do not satisfy (3.34). Indeed, it is possible to construct $\varphi \in C^\infty_c((0,1))$ with

$$(\nabla^\alpha \varphi)_{(0,1)} := \int_0^1 \nabla^\alpha \varphi\, dy \neq 0, \tag{3.35}$$

which shows that no linear function $h : \mathbb{R} \to \mathbb{R}$ with $h((\nabla^\alpha \varphi)_{(0,1)}) < 0$ satisfies (3.34), because

$$h(0) = 0 > h((\nabla^\alpha \varphi)_{(0,1)}) = \int_0^1 h(\nabla^\alpha \varphi)\, dy.$$

To give an example of (3.35), we may take any $\tilde{\varphi} \in C^\infty_c((0,1))$ that is non-negative and not identically zero. By a straightforward calculation we find $\nabla^\alpha \tilde{\varphi}(0) > 0$ and by exploiting the continuity of $\nabla^\alpha \tilde{\varphi}$, there must be some $\delta < 0$ for which

$$\int_\delta^1 \nabla^\alpha \tilde{\varphi}\, dy \neq 0.$$

A transformation of $\tilde{\varphi}$ on the interval $(-\delta, 1)$ to a function $\varphi \in C^\infty_c((0,1))$, in light of the translation invariance and α-homogeneity of the fractional gradient, yields (3.35). $\triangle$

Actually, it is a simple consequence of Proposition 3.1.32 that the new notion α-quasiconvexity is equivalent to quasiconvexity. From this we conclude that α-quasiconvexity can also be used to characterize the weak lower semicontinuity and that this notion does not depend on α.

Corollary 3.2.10. *Let $\alpha \in (0,1)$, then $h : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if and only if it is α-quasiconvex.*

A simple application of the direct method in the calculus of variations yields the existence of solutions.

Theorem 3.2.11 (Existence)**.** *Assume we are in the setting of Theorem 3.2.1 and furthermore assume the coercivity condition*

$$c|A|^p - b(x) \leq f(x,z,A) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } (z,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n},$$

with $c > 0$ and $b \in L^1(\mathbb{R}^n)$. Then $\mathcal{I}_\alpha$ admits a minimizer in $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$.

Proof. Let $(u_j)_j \subset S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ be a minimizing sequence of $\mathcal{I}_\alpha$, then the coercivity conditions yields that $\|\nabla^\alpha u_j\|_p \leq M < \infty$ for all $j \in \mathbb{N}$. Therefore, the Poincaré inequality (Theorem 3.1.18) yields that

$$\|u_j\|_p \leq \|g\|_p + C(\|\nabla^\alpha u_j\|_p + \|\nabla^\alpha g\|_p)$$

whence $(u_j)_j$ is a bounded sequence in $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$. Using Theorem 3.1.20 we may assume that (up to a non-relabeled subsequence) $u_j \rightharpoonup u$ in $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$. The weak lower semicontinuity from Theorem 3.2.1 now shows that u is a minimizer of $\mathcal{I}_\alpha$ over $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$. □

3.2.2 Fractional Euler-Lagrange Equations

In this section we derive that minimizers of integral functionals $\mathcal{I}_\alpha$ as in (3.22) satisfy a system of fractional partial differential equations in a weak sense. These equations are analogous to the well-known Euler-Lagrange equations, which provide a deep connection between the theory of partial differential equations and the calculus of variations. As an application of the existence of minimizers from the previous section, we conclude the existence of weak solutions to the fractional Euler-Lagrange equations.

We know that if u minimizes $\mathcal{I}_\alpha$ over $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ then for any $\varphi \in C^\infty_c(\Omega;\mathbb{R}^m)$ and $h \in \mathbb{R}$ we have

$$\mathcal{I}_\alpha(u) \leq \mathcal{I}_\alpha(u + h\varphi). \tag{3.36}$$

In Theorem 3.2.12 below, we calculate the derivative of $h \mapsto \mathcal{I}_\alpha(u+h\varphi)$ at $h = 0$, also known as the first variation of I_α at u in the direction φ, which is equal to zero by (3.36). This will provide us with the weak version of the fractional Euler-Lagrange equations. The proof is an adaptation of [15, Theorem 6.2] and [64, Theorem 3.1], where we denote by $D_z f$ and $D_A f$ the derivative of f with respect to the second and third variable, respectively.

Theorem 3.2.12. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a Carathéodory integrand, which is continuously differentiable in its second and third variable and satisfies the bounds*

$$|f(x,z,A)| + |D_z f(x,z,A)| + |D_A f(x,z,A)| \leq a(x) + C(|z|^p + |A|^p)$$

for a.e. $x \in \mathbb{R}^n$ and for all $(z, A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}$ with $a \in L^1(\mathbb{R}^n)$ and $C > 0$. If u minimizes $\mathcal{I}_\alpha$ as in (3.22) over $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n} D_A f(x,u,\nabla^\alpha u)\cdot\nabla^\alpha\varphi + D_z f(x,u,\nabla^\alpha u)\cdot\varphi\,dx = 0 \quad \textit{for all } \varphi \in C_c^\infty(\Omega;\mathbb{R}^m). \tag{3.37}$$

Proof. Take $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$, then, in light of (3.36) it is enough to show that $h \mapsto \mathcal{I}_\alpha(u+h\varphi)$ is differentiable at $h = 0$ with derivative equal to the left-hand side of (3.37). To do this, we observe that the growth bound on f shows that $\mathcal{I}_\alpha(u+h\varphi)$ is well-defined and finite. Next, we calculate using the mean value theorem

$$\begin{aligned}\frac{\mathcal{I}_\alpha(u+h\varphi)-\mathcal{I}_\alpha(u)}{h} &= \int_{\mathbb{R}^n}\frac{f(x,u+h\varphi,\nabla^\alpha u+h\nabla^\alpha\varphi)-f(x,u,\nabla^\alpha u)}{h}\,dx\\ &= \int_{\mathbb{R}^n} D_A f(x,u+t\varphi,\nabla^\alpha u+t\nabla^\alpha\varphi)\cdot\nabla^\alpha\varphi\\ &\qquad + D_z f(x,u+t\varphi,\nabla^\alpha u+t\nabla^\alpha\varphi)\cdot\varphi\,dx,\end{aligned} \tag{3.38}$$

with $t = t(x,h) \in [-h,h]$. By using the boundedness of φ and $\nabla^\alpha\varphi$ and the growth bounds on $D_z f$ and $D_A f$ we find that for $|h| \leq 1$ the integrand in the last integral of (3.38) is bounded by

$$C'(a(x) + C((|u|+|\varphi|)^p + (|\nabla^\alpha u| + |\nabla^\alpha\varphi|)^p),$$

which is an integrable function. Hence, by Lebesgue's dominated convergence theorem we may take the limit of $h \to 0$ and thus $t \to 0$ inside the integral of (3.38) and we end up with (3.37). □

Remark 3.2.13. Theorem 3.2.12 also holds if we generalize the bounds on the derivatives by

$$|D_z f(x,z,A)| + |D_A f(x,z,A)| \leq b(x) + C(|z|^p + |A|^p)$$

for a.e. $x \in \mathbb{R}^n$ and for all $(z,A) \in \mathbb{R}^m\times\mathbb{R}^{m\times n}$ with $b \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and $C > 0$. Indeed, we can simply adapt the bound for (3.38) by estimating

$$\int_{\mathbb{R}^n} b(x)(|\varphi| + |\nabla^\alpha\varphi|)\,dx < \infty,$$

via Hölder's inequality, noting that $\nabla^\alpha\varphi \in \mathcal{T}(\mathbb{R}^n;\mathbb{R}^{m\times n})$, cf. Section 3.1.1. △

Now we take a closer look at (3.37). First, we introduce the fractional divergence for $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, given by

$$\operatorname{div}^\alpha\varphi(x) = \mu_{n,\alpha}\int_{\mathbb{R}^n}\frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy.$$

This object is similar to the fractional gradient, but it turns a vector-valued quantity into a scalar-valued one. With this notion we equivalently write the fractional integration by parts (3.5) as

$$\int_{\mathbb{R}^n}\psi\cdot\nabla^\alpha\varphi\,dx = -\int_{\mathbb{R}^n}\operatorname{div}^\alpha\psi\,\varphi\,dx,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. If we formally apply this to (3.37) we obtain

$$\int_{\mathbb{R}^n} (-\mathrm{div}^\alpha(D_A f(x,u,\nabla^\alpha u)) + D_z f(x,u,\nabla^\alpha u)) \cdot \varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega;\mathbb{R}^m).$$

Via the fundamental theorem of the calculus of variations this shows that u solves the fractional Euler-Lagrange equations

$$\begin{cases} \mathrm{div}^\alpha(D_A f(x,u,\nabla^\alpha u)) = D_z f(x,u,\nabla^\alpha u) & \text{in } \Omega, \\ u = g & \text{in } \Omega^c. \end{cases} \tag{3.39}$$

We remark that (3.39) is a system of m partial differential equations subject to a complementary-value condition. This formal derivation shows that the condition (3.37), which minimizers satisfy, is a type of weak version of the fractional Euler-Lagrange equation (3.39). This understanding provides a deep connection between the fractional integral functionals and the theory of fractional partial differential equations. In fact, under the hypotheses of Theorem 3.2.11 and 3.2.12 we have proven the existence of weak solutions to the system (3.39).

Example 3.2.14. a) Consider the functional

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla^\alpha u|^2 \, dx \quad \text{for } u \in S_g^{\alpha,2}(\Omega).$$

This functional satisfies the hypotheses of Theorem 3.2.11 and 3.2.12 with the more general bound from Remark 3.2.13. For this functional, noting that $\mathrm{div}^\alpha \nabla^\alpha = -(-\Delta)^\alpha$ ([74, Theorem 5.3]), the fractional Euler-Lagrange equation is given by

$$\begin{cases} (-\Delta)^\alpha u = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases}$$

which is a fractional Laplace equation. Therefore, we have existence of weak solutions to this system of equations.

b) More generally, if we consider for $p \in (1,\infty)$ the functional

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} \frac{1}{p}|\nabla^\alpha u|^p \, dx \quad \text{for } u \in S_g^{\alpha,p}(\Omega)$$

then its fractional Euler-Lagrange equation is given by

$$\begin{cases} \mathrm{div}^\alpha(|\nabla^\alpha u|^{p-2}\nabla^\alpha u) = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases}$$

which is a type of fractional p-Laplace equation. Again, we have existence of weak solutions to these systems. △

3.2.3 Relaxation

Here, we give a representation of the relaxed functional of

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(\nabla^\alpha u(x))\,dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$$

with respect to the weak convergence in $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$. Recall that the relaxed functional is the largest weakly lower semicontinuous functional that is majorized by $\mathcal{I}_\alpha$. This functional is of practical interest as it gives information about minimizing sequences of $\mathcal{I}_\alpha$ even when it is not weakly lower semicontinuous. We prove that the relaxed functional is still an integral functional and that it is obtained through quasiconvexification of the integrand, but only inside the domain Ω. This is due to the strong convergence of the fractional gradients outside Ω and, as a result, we observe that through the relaxation process the homogeneous integrand f is turned into an inhomogeneous one.

The quasiconvex envelope of a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, denoted by f^{qc}, is given by

$$f^{\mathrm{qc}}(A) = \sup\{h(A) \mid h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex},\ h \le f\},$$

cf. Section 1.1.4. Furthermore, recall that the relaxed functional can be characterized as

$$\mathcal{I}^{\mathrm{rel}}_\alpha(u) = \inf\{\liminf_{j\to\infty} \mathcal{I}_\alpha(u_j) \mid u_j \rightharpoonup u \text{ in } S^{\alpha,p}_g(\Omega;\mathbb{R}^m)\}.$$

We have the following relaxation result.

Theorem 3.2.15 (Relaxation formula)**.** *Let $\alpha \in (0,1)$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ open and bounded with $|\partial\Omega| = 0$, $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$ and*

$$\mathcal{I}_\alpha(u) = \int_{\mathbb{R}^n} f(\nabla^\alpha u(x))\,dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m).$$

Assume that f is continuous and satisfies for $c, C > 0$

$$c|A|^p \le f(A) \le C|A|^p \quad \text{for all } A \in \mathbb{R}^{m\times n}.$$

Then, the relaxation of $\mathcal{I}_\alpha$ with respect to the weak convergence in $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ is given by

$$\mathcal{I}^{\mathrm{rel}}_\alpha(u) = \int_\Omega f^{\mathrm{qc}}(\nabla^\alpha u(x))\,dx + \int_{\Omega^c} f(\nabla^\alpha u(x))\,dx \quad \text{for } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m).$$

Proof. We denote

$$\widetilde{\mathcal{I}}_\alpha(u) = \int_\Omega f^{\mathrm{qc}}(\nabla^\alpha u)\,dx + \int_{\Omega^c} f(\nabla^\alpha u)\,dx = \int_{\mathbb{R}^n} \tilde f(x, \nabla^\alpha u)\,dx, \quad u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m),$$

with $\tilde f : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$ given by $\tilde f(x,A) = \mathbb{1}_\Omega(x) f^{\mathrm{qc}}(A) + \mathbb{1}_{\Omega^c}(x) f(A)$ for $(x,A) \in \mathbb{R}^n \times \mathbb{R}^{m\times n}$. Since the continuity and growth conditions of f are retained during quasiconvexification, the functional $\widetilde{\mathcal{I}}_\alpha$ is weakly lower semicontinuous on $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ by virtue of Theorem 3.2.1. Hence, we conclude $\widetilde{\mathcal{I}}_\alpha \le \mathcal{I}^{\mathrm{rel}}_\alpha$ because $\widetilde{\mathcal{I}}_\alpha \le \mathcal{I}_\alpha$.

To prove the converse inequality, we fix $u \in S_g^{\alpha,p}(\Omega;\mathbb{R}^m)$. By Proposition 3.1.30 (i) we find a $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\nabla v = \nabla^\alpha u$ on Ω. According to the classical relaxation result (Theorem 1.1.17) we may find a sequence $v_j \rightharpoonup v$ in $W_v^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\liminf_{j\to\infty} \int_\Omega f(\nabla v_j)\,dx \leq \int_\Omega f^{\mathrm{qc}}(\nabla v)\,dx. \tag{3.40}$$

Using a decomposition lemma (see e.g. [29, Lemma 8.15], [8, Lemma 11.4.1]) we may further assume that the sequence $(\nabla v_j)_j$ is p-equi-integrable; for the addition of boundary data we can use a cut-off argument as in [64, Lemma 4.13 Step 3]. Now consider the sequence $(v_j - v)_j$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ which converges weakly to zero. We can extend this sequence as zero outside Ω and define

$$\tilde{u}_j := (-\Delta)^{\frac{1-\alpha}{2}}(v_j - v) \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m).$$

Furthermore, in view of Remark 3.1.31 a), we find

$$\tilde{u}_j \rightharpoonup 0 \ \text{ in } S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m) \tag{3.41}$$

and as $v_j - v \to 0$ in $L^p(\mathbb{R}^n;\mathbb{R}^m)$ by the Rellich-Kondrachov theorem the bound (3.15) yields

$$\tilde{u}_j \to 0 \ \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^m). \tag{3.42}$$

Now, we take a subset $O \Subset \Omega$ and a cut-off function $\chi \in C_c^\infty(\Omega)$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on O. We then define the sequence $u_j := u + \chi\tilde{u}_j$ in $S_g^{\alpha,p}(\Omega;\mathbb{R}^m)$ which converges weakly to u by (3.41) and Lemma 3.1.22. The same lemma in combination with (3.42) gives

$$R_j := \nabla^\alpha u_j - \nabla^\alpha u - \chi\nabla^\alpha \tilde{u}_j \to 0 \ \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^{m\times n}). \tag{3.43}$$

An application of Lemma 1.1.24 in light of the p-equi-integrability of $(\nabla v_j)_j$ and the observation that $R_j = \nabla^\alpha u_j - \nabla v_j$ on O yields

$$\lim_{j\to\infty} \int_O f(\nabla v_j)\,dx \geq \int_O f(\nabla v_j + R_j)\,dx = \lim_{j\to\infty} \int_O f(\nabla^\alpha u_j)\,dx. \tag{3.44}$$

Additionally, for any $\epsilon > 0$ we have that

$$\int_{\Omega\setminus O} f(\nabla^\alpha u_j)\,dx \leq c_2 \int_{\Omega\setminus O} |(1-\chi)\nabla v + \chi\nabla v_j + r_j|^p\,dx \leq \epsilon \tag{3.45}$$

if $|\Omega \setminus O|$ is sufficiently small, which again relies on the p-equi-integrability of $(\nabla v_j)_j$. Lastly, since $\nabla^\alpha u_j = \nabla^\alpha u + R_j \to \nabla^\alpha u$ in $L^p(\Omega^c;\mathbb{R}^m)$ by (3.43) we find using the continuity of f and the dominated convergence theorem that

$$\lim_{j\to\infty} \int_{\Omega^c} f(\nabla^\alpha u_j)\,dx = \int_\Omega f(\nabla^\alpha u)\,dx. \tag{3.46}$$

Summing (3.44), (3.45) and (3.46) together in conjunction with (3.40) yields

$$\begin{aligned}\liminf_{j\to\infty} \mathcal{I}_\alpha(u_j) &\leq \liminf_{j\to\infty} \int_O f(\nabla v_j)\,dx + \epsilon + \int_{\Omega^c} f(\nabla^\alpha u)\,dx \\ &\leq \int_\Omega f^{\mathrm{qc}}(\nabla v)\,dx + \epsilon + \int_{\Omega^c} f(\nabla^\alpha u)\,dx \\ &= \int_\Omega f^{\mathrm{qc}}(\nabla^\alpha u)\,dx + \epsilon + \int_{\Omega^c} f(\nabla^\alpha u)\,dx \\ &= \widetilde{\mathcal{I}}_\alpha(u) + \epsilon.\end{aligned}$$

This proves the result by letting $|\Omega \setminus O| \to 0$, and thus $\epsilon \to 0$. □

Remark 3.2.16. The result about the necessary condition in Theorem 3.2.5 can be alternatively derived from Theorem 3.2.15, in the particular case when f is independent of x and u and the coercivity condition $f(A) \geq c|A|^p$ holds for all $A \in \mathbb{R}^{m\times n}$ with $c > 0$.

Notice that the (sequential) weak lower semicontinuity of $\mathcal{I}_\alpha$ on $S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ implies that it has to coincide with its relaxation, i.e. $\mathcal{I}_\alpha(u) = \mathcal{I}^{\mathrm{rel}}_\alpha(u)$ for all $u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$. By using the p-growth of f, we can subtract the integral over Ω^c, to conclude that

$$\int_\Omega (f - f^{\mathrm{qc}})(\nabla^\alpha u)\,dx = 0 \quad \text{for all } u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m).$$

From the construction in Remark 3.2.4 we find for any $A \in \mathbb{R}^{m\times n}$ a $u \in S^{\alpha,p}_g(\Omega;\mathbb{R}^m)$ and a p-Lebesgue point $x_0 \in \Omega$ of $\nabla^\alpha u$ such that $\nabla^\alpha u(x_0) = A$. Since $f - f^{\mathrm{qc}}$ is non-negative, we infer for any $r > 0$

$$\int_{B_r(x_0)} (f - f^{\mathrm{qc}})(\nabla^\alpha u)\,dx = 0,$$

and following a multiplication by r^{-n}

$$0 = r^{-n} \int_{B_r(x_0)} (f - f^{\mathrm{qc}})(\nabla^\alpha u)\,dx = \int_{B_1(x_0)} (f - f^{\mathrm{qc}})(\nabla^\alpha u(rx))\,dx. \tag{3.47}$$

In the limit $r \to 0$ we have $\nabla^\alpha u(r\,\cdot) \to A$ in $L^p(B_1(x_0);\mathbb{R}^{m\times n})$ since x_0 is a p-Lebesgue point of $\nabla^\alpha u$. As a consequence, the right-hand side of (3.47) tends to $(f - f^{\mathrm{qc}})(A)$ as $r \to 0$, by combining the p-growth and continuity of f and f^{qc} with Lebesgue's dominated convergence theorem. We find that $f = f^{\mathrm{qc}}$, establishing the quasiconvexity of f. △

3.3 Supremal Functionals

This section is devoted to the study of supremal functionals of the form

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(x, u(x), \nabla^\alpha u(x)), \quad \text{for } u \in S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m), \tag{3.48}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a suitable supremand and $g \in S^{\alpha,\infty}(\mathbb{R}^n;\mathbb{R}^m)$. In this setting, the minimization of $\mathcal{S}_\alpha$ is closely related to its lower semicontinuity with respect to the weak* convergence in $S_g^{\alpha,p}(\Omega;\mathbb{R}^m)$, which we characterize fully. As a consequence, we obtain a general existence result for minimizers of (3.48). Subsequently, we study the relaxed version of these supremal functionals in the scalar case $m = 1$, which relies on a new approximate fractional differential inclusion result.

3.3.1 Weak* Lower Semicontinuity and Existence

We now work towards characterizing the weak* lower semicontinuity of fractional supremal functionals. The characterizing condition is in terms of level-quasiconvexity (Definition 2.1.9), which is inherent to the classical supremal functionals studied in Chapter 2. Using a similar approach to the integral case (Theorem 3.2.1), we can carry over the sufficient condition from the classical to the fractional setting, requiring level-quasiconvexity only inside Ω because of the strong convergence of the fractional gradients outside Ω (Lemma 3.1.24).

Theorem 3.3.1 (Sufficiency of level-quasiconvexity)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ open and bounded with $|\partial\Omega| = 0$, $g \in S^{\alpha,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ and*

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(x,u(x),\nabla^\alpha u(x)) \quad \text{for } u \in S_g^{\alpha,\infty}(\Omega;\mathbb{R}^m),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a continuous function. If $A \mapsto f(x,z,A)$ is level-quasiconvex for all $x \in \Omega$ and all $z \in \mathbb{R}^m$, then $\mathcal{S}_\alpha$ is (sequentially) weak lower semicontinuous on $S_g^{\alpha,\infty}(\Omega;\mathbb{R}^m)$.*

Proof. The idea is similar to the proof of Theorem 3.2.1 by splitting the supremum into two parts. Let $u_j \overset{*}{\rightharpoonup} u$ in $S_g^{\alpha,\infty}(\Omega;\mathbb{R}^m)$, then we find by Theorem 3.1.20 and Theorem 3.1.29 that $u_j \to u$ in $C(\mathbb{R}^n;\mathbb{R}^m)$. Additionally, Proposition 3.1.30 (i) enables us to find a sequence $(v_j)_j \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ and $v \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$\nabla v_j = \nabla^\alpha u_j \ \text{ on } \Omega \text{ for all } j \in \mathbb{N} \quad \text{and} \quad \nabla v = \nabla^\alpha u \ \text{ on } \Omega.$$

By assuming that the functions v_j and v have average zero we find by the Poincaré inequality (Theorem 1.1.7) that $(v_j)_j$ is a bounded sequence in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ and hence, $v_j \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$. Now we use the standard lower semicontinuity result, see Theorem 2.1.11 and Remark 2.1.2, which can be applied as f is continuous and level-quasiconvex in its third variable in Ω, to conclude

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in\Omega} f(x,u,\nabla^\alpha u) &= \operatorname*{ess\,sup}_{x\in\Omega} f(x,u,\nabla v)\\ &\le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x,u_j,\nabla v_j)\\ &= \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(x,u_j,\nabla^\alpha u_j).\end{aligned} \tag{3.49}$$

For the part outside Ω, we note that for any $\Omega \Subset \Omega'$ we have that $\nabla^\alpha u_j \to \nabla^\alpha u$ in $L^\infty((\Omega')^c;\mathbb{R}^{m\times n})$ by Lemma 3.1.24. In combination with the uniform convergence $u_j \to u$ and the uniform continuity of f on bounded sets, this yields for any $R > 0$,

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in(\Omega')^c\cap B_R(0)} f(x,u,\nabla^\alpha u) &= \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in(\Omega')^c\cap B_R(0)} f(x,u_j,\nabla^\alpha u_j)\\ &\le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega^c} f(x,u_j,\nabla^\alpha u_j).\end{aligned}$$

By letting $R \to \infty$ and Ω' tend to Ω, considering $|\partial\Omega| = 0$, we obtain

$$\operatorname*{ess\,sup}_{x\in\Omega^c} f(x,u,\nabla^\alpha u) \le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega^c} f(x,u_j,\nabla^\alpha u_j). \tag{3.50}$$

The proof follows by combining (3.49) and (3.50). □

It can also be shown that the above condition is necessary for weak* lower semicontinuity, albeit under an extra assumption. This assumption states that the supremal functional is also weak* lower semicontinuous on all subsets of $\mathbb{R}^n$. While such a condition automatically follows for integral functionals by subtracting the integral outside this set, it is a nontrivial assumption in the supremal case. Without this assumption, localization becomes very hard as the supremum might be attained in a different place, thus providing no information on the region in which we are interested. Observe that such an assumption also appears in the classical case, see Theorem 2.1.14.

We now present the proof of the necessary condition, which utilizes a reduction to the classical necessary condition. The proof is analogous to that of Theorem 3.2.5, apart from the use of a different scaling argument.

Theorem 3.3.2 (Necessity of level-quasiconvexity)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ open and bounded, $g \in S^{\alpha,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ with $\nabla^\alpha g$ continuous and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ a continuous function. If*

$$\mathcal{S}_\alpha(u,O) = \operatorname*{ess\,sup}_{x\in O} f(x,u(x),\nabla^\alpha u(x)) \quad \textit{for } u \in S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$$

is (sequentially) weak lower semicontinuous on $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$ for every open set $O \subset \mathbb{R}^n$ then $A \mapsto f(x,z,A)$ is level-quasiconvex for all $x \in \Omega$ and all $z \in \mathbb{R}^m$.*

Proof. Take $(x_0,z_0) \in \Omega \times \mathbb{R}^m$ and assume for simplicity that $x_0 + Q \Subset \Omega$, where $Q = (0,1)^n$. We show that

$$\operatorname*{ess\,sup}_{x\in x_0+Q} f(x_0,z_0,\nabla v) \le \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in x_0+Q} f(x_0,z_0,\nabla v_j) \tag{3.51}$$

for any sequence $v_j \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(x_0+Q;\mathbb{R}^m)$, which yields that $f(x_0,z_0,\cdot)$ is level-quasiconvex via Theorem 2.1.14, cf. Remark 2.1.15.

Step 1: We construct the required functions in the space $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$. First, take $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$ such that $\varphi(x_0) = z_0 - g(x_0)$ and $\nabla^\alpha\varphi(x_0) = -\nabla^\alpha g(x_0)$ according to Lemma 3.2.3. Then, it follows that $\gamma = g + \varphi \in S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$ satisfies

$$\gamma(x_0) = z_0 \text{ and } \nabla^\alpha\gamma(x_0) = 0, \tag{3.52}$$

with γ and $\nabla^\alpha \gamma$ continuous by Theorem 3.1.29 and the continuity of $\nabla^\alpha g$. Next, we take a sequence $v_j \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(x_0 + Q; \mathbb{R}^m)$ which we can extend to a sequence $v_j \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ such that each v_j and v has support in Ω. Then, Remark 3.1.31 a) shows that the sequence $u_j := (-\Delta)^{\frac{1-\alpha}{2}} v_j$ converges weak* to $u := (-\Delta)^{\frac{1-\alpha}{2}} v$ in $S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ with

$$\nabla v_j = \nabla^\alpha u_j \ \text{ on } \mathbb{R}^n \text{ for all } j \in \mathbb{N} \quad \text{and} \quad \nabla v = \nabla^\alpha u \ \text{ on } \mathbb{R}^n. \tag{3.53}$$

As the sequence $(u_j)_j$ does not yet satisfy the complementary-value condition we are going to scale it and subsequently use cut-off functions. Take $\rho \in \mathbb{N}$ and define the scaled sequence $(u_j^\rho)_j \subset S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ by

$$u_j^\rho(x) = \frac{1}{\rho^\alpha} u_j(\rho(x - x_0) + x_0)$$

and $u^\rho \in S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ idem. By the translation invariance and α-homogeneity of the fractional gradient, cf. Section 3.1.1, we find

$$\nabla^\alpha u_j^\rho(x) = \nabla^\alpha u_j(\rho(x - x_0) + x_0) \tag{3.54}$$

and in particular $u_j^\rho \overset{*}{\rightharpoonup} u^\rho$ in $S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ as $j \to \infty$. Now, we take a cut-off function $\chi \in C_c^\infty(\Omega)$ with $\chi \equiv 1$ on $x_0 + Q$ and define the functions

$$w_j^\rho = \chi u_j^\rho + \gamma \quad \text{and} \quad w^\rho = \chi u^\rho + \gamma,$$

which lie in $S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m)$. Moreover, by Lemma 3.1.22 we find that $w_j^\rho \overset{*}{\rightharpoonup} w^\rho$ in $S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m)$ and

$$\|\nabla^\alpha w_j^\rho - \chi \nabla^\alpha u_j^\rho - \nabla^\alpha \gamma\|_\infty \leq C \|u_j^\rho\|_\infty \leq \frac{C}{\rho^\alpha}. \tag{3.55}$$

We also infer from the definition of w_j^ρ that

$$\|w_j^\rho - \gamma\|_\infty \leq \frac{C}{\rho^\alpha}, \tag{3.56}$$

and we note that both (3.55) and (3.56) hold for w^ρ as well.

Step 2: Setting $Q_\rho = x_0 + Q/\rho$ we find by the weak* lower semicontinuity of $\mathcal{S}_\alpha(\cdot, Q_\rho)$ on $S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m)$ that

$$\operatorname*{ess\,sup}_{x \in Q_\rho} f(x, w^\rho, \nabla^\alpha w^\rho) \leq \liminf_{j \to \infty} \operatorname*{ess\,sup}_{x \in Q_\rho} f(x, w_j^\rho, \nabla^\alpha w_j^\rho). \tag{3.57}$$

Now take $\epsilon > 0$, then we find by the uniform continuity of f on bounded sets that for all $j \in \mathbb{N}$ and ρ large enough

$$\begin{aligned}
\operatorname*{ess\,sup}_{x \in Q_\rho} f(x, w_j^\rho, \nabla^\alpha w_j^\rho) &\leq \operatorname*{ess\,sup}_{x \in Q_\rho} f(x, \gamma, \nabla^\alpha u_j^\rho + \nabla^\alpha \gamma) + \epsilon \\
&\leq \operatorname*{ess\,sup}_{x \in Q_\rho} f(x_0, \gamma(x_0), \nabla^\alpha u_j^\rho + \nabla^\alpha \gamma(x_0)) + 2\epsilon \\
&= \operatorname*{ess\,sup}_{x \in Q_\rho} f(x_0, z_0, \nabla^\alpha u_j^\rho) + 2\epsilon \\
&= \operatorname*{ess\,sup}_{x \in x_0 + Q} f(x_0, z_0, \nabla v_j) + 2\epsilon.
\end{aligned} \tag{3.58}$$

The first inequality combines (3.55) and (3.56) with the fact that $\chi \equiv 1$ on Q_ρ. In the second line we use the continuity of γ and $\nabla^\alpha \gamma$ and the third line uses (3.52). The last line follows from (3.53) and (3.54). A completely analogous calculation shows for ρ large enough

$$\operatorname*{ess\,sup}_{x\in Q_\rho} f(x, w^\rho, \nabla^\alpha w^\rho) \geq \operatorname*{ess\,sup}_{x\in x_0+Q} f(x_0, z_0, \nabla^\alpha v) - 2\epsilon. \tag{3.59}$$

Hence, (3.57) reduces in view of (3.58) and (3.59) to

$$\operatorname*{ess\,sup}_{x\in x_0+Q} f(x_0, z_0, \nabla v) \leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in x_0+Q} f(x_0, z_0, \nabla v_j) + 4\epsilon,$$

which yields (3.51) by the arbitrariness of ϵ. □

Example 3.3.3. We note that the assumption of lower semicontinuity on subdomains cannot be dropped in the presence of an x-dependent supremand. Indeed, by taking any $h : \mathbb{R}^{m\times n} \to [0, 1]$ that is not level-quasiconvex and a $\chi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on Ω we can consider

$$f(x, A) = \chi(x)h(A) + (1 - \chi(x)).$$

Then, the functional

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(x, \nabla^\alpha u(x)), \quad \text{for } u \in S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m),$$

is identically equal to 1 and thus weak* lower semicontinuous on $S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m)$. However, $f(x, \cdot)$ is not level-quasiconvex for $x \in \Omega$. △

Noting that the proof of sufficiency (Theorem 3.3.1) can readily be adapted to supremal functionals on subdomains of $\mathbb{R}^n$, we can obtain the following characterization result.

Theorem 3.3.4 (Characterization of weak* lower semicontinuity)**.** *Let $\alpha \in (0, 1)$, $\Omega \subset \mathbb{R}^n$ open and bounded with $|\partial\Omega| = 0$, $g \in S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ with $\nabla^\alpha g$ continuous and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ a continuous function. Then,*

$$\mathcal{S}_\alpha(u, O) = \operatorname*{ess\,sup}_{x\in O} f(x, u(x), \nabla^\alpha u(x)) \quad \textit{for } u \in S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m)$$

is (sequentially) weak lower semicontinuous on $S_g^{\alpha,\infty}(\Omega; \mathbb{R}^m)$ for every open set $O \subset \mathbb{R}^n$ if and only if $A \mapsto f(x, z, A)$ is level-quasiconvex for all $x \in \Omega$ and all $z \in \mathbb{R}^m$.*

We also have a comparable result on the full space.

Theorem 3.3.5. *Let $\alpha \in (0, 1)$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ a continuous function. Then,*

$$\mathcal{S}_\alpha(u, O) = \operatorname*{ess\,sup}_{x\in O} f(x, u(x), \nabla^\alpha u(x)) \quad \textit{for } u \in S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$$

is (sequentially) weak lower semicontinuous on $S^{\alpha,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ for every open set $O \subset \mathbb{R}^n$ if and only if $A \mapsto f(x, z, A)$ is level-quasiconvex for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{R}^m$.*

Proof. For the sufficiency, let $u_j \overset{*}{\rightharpoonup} u$ in $S^{\alpha,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ then by Theorem 3.1.23 we find $u_j \to u$ locally uniformly. Hence, we may argue as in Theorem 3.3.1 to show that for any $R > 0$

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in O\cap B_R(0)} f(x,u,\nabla^\alpha u) &\leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in O\cap B_R(0)} f(x,u_j,\nabla^\alpha u_j)\\ &\leq \liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in O} f(x,u_j,\nabla^\alpha u_j).\end{aligned}$$

Then letting $R\to\infty$ yields $\mathcal{S}_\alpha(u,O) \leq \liminf_{j\to\infty} \mathcal{S}_\alpha(u_j,O)$ as desired.

For the necessity, we note that $\mathcal{S}_\alpha(\cdot,O)$ is weakly lower semicontinuous on $S_0^{\alpha,\infty}(B_R(0);\mathbb{R}^m)$ for any $R > 0$ so that we may conclude the result via Theorem 3.3.2. □

Trying to understand what happens without the assumption of lower semicontinuity on subsets is rather hard, even in the classical setting. Clearly, Example 3.3.3 illustrates that this is only a fruitful endeavor in the x-independent setting, at least when we do not want to go into the issue of replacing the supremand by an equivalent one. Without x- and u-dependence, a characterization can be obtained in the scalar case when we work on the full space $S^{\alpha,\infty}(\mathbb{R}^n)$. It turns out that the characterizing condition is balanced level-convexity, just like for classical supremal functionals on $\mathbb{R}^n$, cf. Section 2.7.

Theorem 3.3.6. *Let $\alpha\in(0,1)$, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semicontinuous and*

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla^\alpha u(x)) \quad \textit{for } u\in S^{\alpha,\infty}(\mathbb{R}^n).$$

Then $\mathcal{S}_\alpha$ is weak lower semicontinuous on $S^{\alpha,\infty}(\mathbb{R}^n)$ if and only if f is balanced level-convex.*

Proof. The proof of sufficiency is entirely identical to Theorem 2.7.4 except that we consider for $u\in S^{\alpha,\infty}(\mathbb{R}^n)$ the sequence $u_j(x) = \frac{1}{j^\alpha}u(jx)$ for $j\in\mathbb{N}$ which converges weak* to zero.

For necessity, we show that the functional

$$\mathcal{S}(v) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla v(x)) \quad \text{for } v\in W^{1,\infty}(\mathbb{R}^n)$$

is weak* lower semicontinuous on $W^{1,\infty}(\mathbb{R}^n)$. Indeed, if $(v_j)_j \subset W^{1,\infty}(\mathbb{R}^n)$ converges weak* to $v\in W^{1,\infty}(\mathbb{R}^n)$ then by Proposition 3.1.30 (ii) it follows that the sequence $u_j := (-\Delta)^{\frac{1-\alpha}{2}} v_j$ is bounded in $S^{\alpha,\infty}(\mathbb{R}^n)$ and $\nabla^\alpha u_j \overset{*}{\rightharpoonup} \nabla^\alpha u$ with $u := (-\Delta)^{\frac{1-\alpha}{2}} v \in S^{\alpha,\infty}(\mathbb{R}^n)$. Because of the fractional Morrey inequality (Theorem 3.1.29), it follows that $u_j \overset{*}{\rightharpoonup} u$ in $S^{\alpha,\infty}(\mathbb{R}^n)$ up to constants. As these constants are irrelevant to $\mathcal{S}_\alpha$, we may as well assume $u_j \overset{*}{\rightharpoonup} u$. Hence, by the weak* lower semicontinuity of $\mathcal{S}_\alpha$ we get

$$\mathcal{S}(v) = \mathcal{S}_\alpha(u) \leq \liminf_{j\to\infty} \mathcal{S}_\alpha(u_j) = \liminf_{j\to\infty} \mathcal{S}(v_j).$$

We find that $\mathcal{S}$ is weak* lower semicontinuous and by Proposition 2.7.5 we conclude the result. □

As an application of our results on the weak* lower semicontinuity, we prove the existence of minimizers in the complementary-value space.

Theorem 3.3.7 (Existence)**.** *Assume we are in the setting of Theorem 3.3.1 and furthermore assume the coercivity condition*

$$f(x,z,A) \to \infty \text{ as } |A| \to \infty \quad \text{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Then $\mathcal{S}_\alpha$ admits a minimizer on $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$.

Proof. Assume $\mathcal{S}_\alpha \not\equiv \infty$, otherwise the result is trivial. Let $(u_j)_j \subset S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$ be a minimizing sequence of $\mathcal{S}_\alpha$, then the coercivity condition yields that $\|\nabla^\alpha u_j\|_\infty \leq M < \infty$. Hence, we deduce from the Poincaré inequality (Theorem 3.1.18) that

$$\|u_j\|_\infty \leq \|g\|_\infty + C(\|\nabla^\alpha u_j\|_\infty + \|\nabla^\alpha g\|_\infty),$$

whence $(u_j)_j$ is a bounded sequence in $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$. Then, by Theorem 3.1.20 we find that up to a subsequence $u_j \overset{*}{\rightharpoonup} u$ in $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$. The weak* lower semicontinuity from Theorem 3.3.1 now shows that u is a minimizer of $\mathcal{S}_\alpha$ over $S^{\alpha,\infty}_g(\Omega;\mathbb{R}^m)$. □

3.3.2 Relaxation

The topic of this section is the relaxation of the supremal functional

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla^\alpha u(x)) \quad \text{for } u \in S^{\alpha,\infty}_g(\Omega)$$

with respect to the weak* convergence in $S^{\alpha,\infty}_g(\Omega)$. Recall that the relaxation is given by

$$\mathcal{S}^{\text{rel}}_\alpha(u) = \inf\{\liminf_{j\to\infty} \mathcal{S}_\alpha(u_j) \mid u_j \overset{*}{\rightharpoonup} u \text{ in } S^{\alpha,\infty}_g(\Omega)\} \quad \text{for } u \in S^{\alpha,\infty}_g(\Omega).$$

The proof of the relaxation relies on a new extension of a differential inclusion result to the fractional setting. This is the content of the following proposition.

Proposition 3.3.8. *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ open and bounded and $E \subset \mathbb{R}^n$ bounded. Suppose that $\psi \in S^{\alpha,\infty}(\mathbb{R}^n)$ such that $\nabla^\alpha\psi(x) \in E \cup \operatorname{int}(\operatorname{Conv}(E))$ for a.e. $x \in \Omega$. Then, for every $\epsilon > 0$ there exists a $u \in S^{\alpha,\infty}_\psi(\Omega)$ such that*

$$\begin{cases} \|u-\psi\|_\infty \leq \epsilon, \\ \|\nabla^\alpha u - \nabla^\alpha \psi\|_{L^\infty(\Omega^c)} \leq \epsilon, \\ \nabla^\alpha u(x) \in E_\epsilon & \textit{for a.e. } x \in \Omega, \end{cases}$$

where $E_\epsilon = \{x \in \mathbb{R}^n \mid \mathrm{d}(x,E) \leq \epsilon\}$.

Proof. Let $\{Q_i\}_{i\in\mathbb{N}}$ be a collection of disjoint open cubes such that

$$Q_i \Subset \Omega \ \text{ and } \ |\Omega \setminus \cup_{i\in\mathbb{N}} Q_i| = 0.$$

Now, we can consider by Proposition 3.1.30 (i) a $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ such that $\nabla\varphi = \nabla^\alpha\psi$ on $\mathbb{R}^n$. Hence, for every $i \in \mathbb{N}$ and $\delta_i > 0$, we obtain by the classical inclusion result, Theorem 2.5.1, a $v_i \in W^{1,\infty}_\varphi(Q_i)$ such that

$$\begin{cases} \|v_i - \varphi\|_{L^\infty(Q_i)} \leq \delta_i, \\ \nabla v_i(x) \in E & \text{for a.e. } x \in Q_i. \end{cases}$$

Then, we extend $v_i \in W^{1,\infty}_\varphi(Q_i)$ to $\mathbb{R}^n$ as equal to φ and define

$$\tilde{u}_i := (-\Delta)^{\frac{1-\alpha}{2}}(v_i - \varphi) \in S^{\alpha,\infty}(\mathbb{R}^n),$$

which satisfies $\nabla^\alpha \tilde{u}_i = \nabla(v_i - \varphi)$ on $\mathbb{R}^n$ by Proposition 3.1.30 (ii). Furthermore, because of the boundedness of E we find by (3.15) that for any $\epsilon_i > 0$

$$\|\tilde{u}_i\|_\infty \leq \epsilon_i, \tag{3.60}$$

if we choose δ_i accordingly small. We take a cut-off function $\chi_i \in C_c^\infty(\Omega)$ such that $0 \leq \chi_i \leq 1$ and $\chi_i \equiv 1$ on Q_i, from which it follows that $\chi_i\tilde{u}_i \in S_0^{\alpha,\infty}(\Omega)$. Furthermore, $\chi_i\nabla^\alpha\tilde{u}_i = \nabla^\alpha\tilde{u}_i$ since $\nabla^\alpha\tilde{u}_i = \nabla(v_i - \varphi) = 0$ outside Q_i, so that by Lemma 3.1.22 we find

$$\|\nabla^\alpha(\chi_i\tilde{u}_i) - \nabla(v_i - \varphi)\|_\infty \leq C(\chi_i)\epsilon_i. \tag{3.61}$$

Now choose ϵ_i such that

$$\sum_{i\in\mathbb{N}} \epsilon_i \leq \epsilon \text{ and } \sum_{i\in\mathbb{N}} C(\chi_i)\epsilon_i \leq \epsilon, \tag{3.62}$$

then, we claim that

$$u = \psi + \sum_{i\in\mathbb{N}} \chi_i\tilde{u}_i \in S^{\alpha,\infty}(\mathbb{R}^n)$$

fulfills the conditions. Indeed, since each $\chi_i\tilde{u}_i$ lies in $S_0^{\alpha,\infty}(\Omega)$ we find $u = \psi$ a.e. on Ω^c. Next, by (3.60) and (3.62) we find that

$$\|u - \psi\|_\infty \leq \sum_{i\in\mathbb{N}} \|\tilde{u}_i\|_\infty \leq \sum_{i\in\mathbb{N}} \epsilon_i \leq \epsilon.$$

Also, as each $v_i - \varphi$ is zero on Ω^c we find from (3.61) and (3.62)

$$\begin{aligned} \|\nabla^\alpha u - \nabla^\alpha\psi\|_{L^\infty(\Omega^c)} &\leq \sum_{i\in\mathbb{N}} \|\nabla^\alpha(\chi_i\tilde{u}_i)\|_{L^\infty(\Omega^c)} \\ &\leq \sum_{i\in\mathbb{N}} \|\nabla^\alpha(\chi_i\tilde{u}_i) - \nabla(v_i - \varphi)\|_\infty \\ &\leq \sum_{i\in\mathbb{N}} C(\chi_i)\epsilon_i \leq \epsilon. \end{aligned}$$

Lastly, for each $j \in \mathbb{N}$ we find by (3.61) and (3.62) that

$$\begin{aligned} \|\nabla^\alpha u - \nabla v_j\|_{L^\infty(Q_j)} &\leq \|\nabla^\alpha(\chi_j\tilde{u}_j) - \nabla(v_j - \varphi)\|_\infty + \sum_{i\neq j} \|\nabla^\alpha(\chi_i\tilde{u}_i)\|_{L^\infty(Q_j)} \\ &\leq \sum_{i\in\mathbb{N}} \|\nabla^\alpha(\chi_i\tilde{u}_i) - \nabla(v_i - \varphi)\|_\infty \\ &\leq \sum_{i\in\mathbb{N}} C(\chi_i)\epsilon_i \leq \epsilon. \end{aligned} \tag{3.63}$$

The first line uses $\nabla^\alpha \psi = \nabla \varphi$, while the second uses that $\nabla(v_i - \varphi) = 0$ on Q_j for $i \neq j$, recalling that the cubes are disjoint. Since $\nabla v_j \in E$ a.e. on Q_j we find by (3.63) that

$$\nabla^\alpha u(x) \in E_\epsilon \qquad \text{for a.e. } x \in \Omega,$$

as desired. □

The difference with the classical result is that we only have that the fractional gradients are approximately contained in E. However, this is sufficient for our purpose of proving a relaxation formula for supremal functionals on the complementary-value space. We refer to Section 2.5 for the definition of the level-convex hull f^{lc}.

Theorem 3.3.9 (Relaxation formula)**.** *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ open and bounded with $|\partial\Omega| = 0$ and $g \in S^{\alpha,\infty}(\mathbb{R}^n)$. If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and coercive in the sense that $f(\xi) \to \infty$ as $|\xi| \to \infty$, then, the relaxation of*

$$\mathcal{S}_\alpha(u) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla^\alpha u(x)) \quad \text{for } u \in S^{\alpha,\infty}_g(\Omega)$$

with respect to the weak convergence in $S^{\alpha,\infty}_g(\Omega)$ is given by*

$$\mathcal{S}^{\mathrm{rel}}_\alpha(u) = \max\{\operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lc}}(\nabla^\alpha u(x)), \operatorname*{ess\,sup}_{x\in\Omega^c} f(\nabla^\alpha u(x))\} \quad \text{for } u \in S^{\alpha,\infty}_g(\Omega).$$

Proof. Denote

$$\widetilde{\mathcal{S}}_\alpha(u) := \max\{\operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lc}}(\nabla^\alpha u), \operatorname*{ess\,sup}_{x\in\Omega^c} f(\nabla^\alpha u)\} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \tilde f(x, \nabla^\alpha u),$$

where $\tilde f(x,\xi) = \mathbb{1}_\Omega(x) f^{\mathrm{lc}}(\xi) + \mathbb{1}_{\Omega^c}(x) f(\xi)$. Although $\tilde f$ is not continuous, it is when restricted to Ω and Ω^c, so that the proof of Theorem 2.1.11 still applies to show that $\widetilde{\mathcal{S}}_\alpha$ is weak* lower semicontinuous on $S^{\alpha,\infty}_g(\Omega)$. Hence we conclude $\widetilde{\mathcal{S}}_\alpha \leq \mathcal{S}^{\mathrm{rel}}_\alpha$.

For the converse, take $u \in S^{\alpha,\infty}_g(\Omega)$ and set $c := \widetilde{\mathcal{S}}_\alpha(u)$. Define for $\delta > 0$ the set $E = \{\xi \in \mathbb{R}^n \mid f(\xi) < c + \delta\}$, which is open and bounded by the continuity and coercivity of f. In particular, this shows that $\mathrm{Conv}(E)$ is open as well. Thus, we conclude for a.e. $x \in \Omega$ that

$$\nabla^\alpha u(x) \in L_c(f^{\mathrm{lc}}) = \overline{\mathrm{Conv}}(L_c(f)) \subset \mathrm{Conv}(E) = \mathrm{int}(\mathrm{Conv}(E)).$$

Hence, we find by Proposition 3.3.8 that for every $j \in \mathbb{N}$ there is a $u_j \in S^{\alpha,\infty}_g(\Omega)$ such that

$$\begin{cases} \|u_j - u\|_\infty \leq 1/j, \\ \|\nabla^\alpha u_j - \nabla^\alpha u\|_{L^\infty(\Omega^c)} \leq 1/j, \\ \nabla^\alpha u_j(x) \in E_{1/j} & \text{for a.e. } x \in \Omega. \end{cases}$$

Since E is bounded, we see that $(u_j)_j$ is a bounded sequence in $S^{\alpha,\infty}_g(\Omega)$ so that $(u_j)_j$ converges weak* in $S^{\alpha,\infty}_g(\Omega)$ to its uniform limit which is u, cf. Theorem 3.1.20. Furthermore, due to the continuity of f, the fact that $\nabla^\alpha u_j \in E_{1/j}$ a.e. on Ω implies

$$\liminf_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla^\alpha u_j) \leq c + \delta. \tag{3.64}$$

Lastly, since $\nabla^\alpha u_j \to \nabla^\alpha u$ in $L^\infty(\Omega^c;\mathbb{R}^n)$ we find by the uniform continuity of f on bounded sets

$$\lim_{j\to\infty} \operatorname*{ess\,sup}_{x\in\Omega^c} f(\nabla^\alpha u_j) = \operatorname*{ess\,sup}_{x\in\Omega^c} f(\nabla^\alpha u) \leq c. \tag{3.65}$$

Combining (3.64) and (3.65) yields

$$\mathcal{S}_\alpha^{\mathrm{rel}}(u) \leq \liminf_{j\to\infty} \mathcal{S}_\alpha(u_j) \leq c + \delta,$$

which proves the result due to the arbitrariness of δ. □

3.4 L^p-Approximation

In this section we work towards proving a Γ-convergence result. It deals with integral functionals on the fractional Sobolev spaces which converge to supremal functionals as $p \to \infty$. In the classical case, we already considered this in Section 2.6 but we critically used the inclusion $W^{1,p}(\Omega;\mathbb{R}^m) \subset W^{1,q}(\Omega;\mathbb{R}^m)$ for $p > q$ and Ω bounded. Since we do not have a similar inclusion in the fractional case (we always work on $\mathbb{R}^n$), we consider instead $S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$ for which we can use the inclusion proven in Proposition 3.1.25. We need the following well-known lemma from measure theory, see e.g. [65, Exercise 3.4 (e)].

Lemma 3.4.1. *Let $U \subset \mathbb{R}^n$ be open (not necessarily bounded) and $u \in L^q(U)$ for some $1 \leq q < \infty$ then we have*

$$\lim_{p\to\infty} \|u\|_{L^p(U)} = \|u\|_{L^\infty(U)},$$

where the limit could be infinite.

We can now present the Γ-convergence result, which crucially relies on a generalized Jensen inequality, utilizing the theory of Young measures, see Section 1.1.6. The spirit and proof is similar to [23, Theorem 3.1]. We say that $(\mu_x)_{x\in\mathbb{R}^n}$ is a $S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$-gradient Young measure if there is a bounded sequence $(u_j)_j \subset S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$ such that $\nabla^\alpha u_j \xrightarrow{YM} \mu$.

Theorem 3.4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and consider for $p \in [1,\infty)$ the functional $\mathcal{I}_{\alpha,p} : C(\overline{\Omega};\mathbb{R}^m) \to [0,\infty]$ defined as*

$$\mathcal{I}_{\alpha,p}(u) = \begin{cases} \left(\int_{\mathbb{R}^n} f(\nabla^\alpha u)^p \, dx\right)^{1/p} & \text{if } u \in C(\overline{\Omega};\mathbb{R}^m) \cap S_0^{\alpha,p}(\Omega;\mathbb{R}^m), \\ \infty & \text{otherwise}, \end{cases}$$

with $f : \mathbb{R}^{m\times n} \to [0,\infty)$ a lower semicontinuous function. Assume that f satisfies for $C, c > 0$

$$c|A| \leq f(A) \leq C|A| \quad \text{for all } A \in \mathbb{R}^{m\times n},$$

and the generalized Jensen inequality

$$f([\mu]_x) \leq \mu_x\text{-}\operatorname*{ess\,sup}_{A\in\mathbb{R}^{m\times n}} f(A) \quad \text{for all } x \in \mathbb{R}^n,$$

with $(\mu_x)_{x\in\mathbb{R}^n}$ *any* $S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$*-gradient Young measure for all* $p \in [1,\infty)$*. Then the sequence* $(\mathcal{I}_{\alpha,p})_{p\geq 1}$ Γ*-converges as* $p \to \infty$ *to* $\mathcal{S}_\alpha : C(\overline{\Omega};\mathbb{R}^m) \to [0,\infty]$ *defined by*

$$\mathcal{S}_\alpha(u) = \begin{cases} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla^\alpha u(x)) & \text{if } u \in C(\overline{\Omega};\mathbb{R}^m) \cap S_0^{\alpha,\infty}(\Omega;\mathbb{R}^m), \\ \infty & \text{otherwise}, \end{cases}$$

with respect to the uniform convergence.

Proof. For any $u \in C(\overline{\Omega};\mathbb{R}^m)$ we have

$$\limsup_{p\to\infty} \mathcal{I}_{\alpha,p}(u) \leq \mathcal{S}_\alpha(u).$$

Indeed, if $\mathcal{S}_\alpha(u) = \infty$ it is clear, while if $\mathcal{S}_\alpha(u) < \infty$ then we find that $u \in S_0^{\alpha,\infty}(\Omega;\mathbb{R}^m)$ and hence, $u \in S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$ for all $p \in [1,\infty)$ by Proposition 3.1.25. We deduce that $\mathcal{I}_{\alpha,p}(u) \leq C\|\nabla^\alpha u\|_p < \infty$ so that we find by Lemma 3.4.1 that

$$\lim_{p\to\infty} \mathcal{I}_{\alpha,p}(u) = \lim_{p\to\infty} \|f(\nabla^\alpha u)\|_p = \|f(\nabla^\alpha u)\|_\infty = \mathcal{S}_\alpha(u).$$

We conclude that the constant sequence is a recovery sequence.

To prove the liminf-inequality, take any $u \in C(\overline{\Omega};\mathbb{R}^m)$ and a sequence $(u_p)_{p\in[1,\infty)}$ in $C(\overline{\Omega};\mathbb{R}^m)$ such that $(u_p)_p$ converges uniformly to u as $p \to \infty$. We need to establish that

$$\liminf_{p\to\infty} \mathcal{I}_{\alpha,p}(u_p) \geq \mathcal{S}_\alpha(u).$$

Without loss of generality we may assume that we have a $M > 0$ such that $\mathcal{I}_{\alpha,p}(u_p) \leq M$ for all $p \in [1,\infty)$. We can also bound $\|u_p\|_p \leq M$ by virtue of the uniform convergence. Then, for $p > q$ and R large enough we find by Corollary 3.1.26 that

$$\begin{aligned} \|\nabla^\alpha u_p\|_q &\leq |B_R|^{1/q-1/p}\|\nabla^\alpha u_p\|_p + CR^{-\alpha}\|u_p\|_p \\ &\leq \frac{1}{c}|B_R|^{1/q-1/p}\mathcal{I}_{\alpha,p}(u_p) + CR^{-\alpha}\|u_p\|_p \\ &\leq \frac{1}{c}|B_R|^{1/q-1/p}M + CR^{-\alpha}M. \end{aligned} \tag{3.66}$$

We observe that $(u_p)_{p>q}$ is a bounded sequence in $S_0^{\alpha,q}(\Omega;\mathbb{R}^m)$ (even uniformly in q) so that for any $1 \leq q < \infty$ the sequence $(u_p)_p$ converges weakly to u in $S_0^{\alpha,q}(\Omega;\mathbb{R}^m)$ as $p \to \infty$. Therefore, the sequence $(\nabla^\alpha u_p)_{p\in[1,\infty)}$ generates $(\mu_x)_{x\in\mathbb{R}^n}$ which is a $S_0^{\alpha,q}(\Omega;\mathbb{R}^m)$-gradient Young measure for any $q \in [1,\infty)$. This Young measure also satisfies

$$[\mu]_x = \nabla^\alpha u(x) \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{3.67}$$

We can now calculate for any $1 < r < q$

$$
\begin{aligned}
\liminf_{p\to\infty} \mathcal{I}_{\alpha,q}(u_p) &= \liminf_{p\to\infty} \left(\int_{\mathbb{R}^n} f(\nabla^\alpha u_p)^q \, dx \right)^{1/q} \\
&\geq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^{m\times n}} f(A)^q \, d\mu_x(A) \, dx \right)^{1/q} \\
&\geq \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{m\times n}} f(A)^r \, d\mu_x(A) \right)^{q/r} \right)^{1/q},
\end{aligned}
\tag{3.68}
$$

with the first inequality being Theorem 1.1.21 (ii) (extended to $\mathbb{R}^n$) and the second being the standard Jensen inequality for convex functions. Since the left-hand side of (3.68) is finite by (3.66) we conclude that the right-hand sides are as well, whence we may invoke Lemma 3.4.1 to let $q \to \infty$ and obtain

$$
\liminf_{q\to\infty} \liminf_{p\to\infty} \mathcal{I}_{\alpha,q}(u_p) \geq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left(\int_{\mathbb{R}^{m\times n}} f(A)^r \, d\mu_x(A) \right)^{1/r}.
$$

Again, the left-hand side is finite by (3.66) so we may let $r \to \infty$ to get

$$
\begin{aligned}
\liminf_{q\to\infty} \liminf_{p\to\infty} \mathcal{I}_{\alpha,q}(u_p) &\geq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left(\mu_x\text{-} \operatorname*{ess\,sup}_{A\in\mathbb{R}^{m\times n}} f(A) \right) \\
&\geq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} f(\nabla^\alpha u) = \mathcal{S}_\alpha(u),
\end{aligned}
\tag{3.69}
$$

with the first line following from a version of Lemma 3.4.1 for arbitrary measures and the second line following from the generalized Jensen inequality and (3.67). We can also deduce an upper bound for the left-hand side of (3.69) via the calculation

$$
\begin{aligned}
\liminf_{q\to\infty} \liminf_{p\to\infty} \mathcal{I}_{\alpha,q}(u_p) &= \liminf_{q\to\infty} \liminf_{p\to\infty} \|f(\nabla^\alpha u_p)\|_q \\
&\leq \liminf_{q\to\infty} \liminf_{p\to\infty} |B_R|^{1/q-1/p} \|f(\nabla^\alpha u_p)\|_{L^p(B_R(0))} + \|f(\nabla^\alpha u_p)\|_{L^q(B_R(0)^c)} \\
&\leq \liminf_{p\to\infty} \|f(\nabla^\alpha u_p)\|_p + \liminf_{q\to\infty} \liminf_{p\to\infty} C\|\nabla^\alpha u_p\|_{L^q(B_R(0)^c)} \\
&\leq \liminf_{p\to\infty} \|f(\nabla^\alpha u_p)\|_p + \liminf_{q\to\infty} \liminf_{p\to\infty} CR^{-\alpha}\|u_p\|_p \\
&\leq \liminf_{p\to\infty} \|f(\nabla^\alpha u_p)\|_p + CR^{-\alpha}M.
\end{aligned}
$$

The fourth line uses Corollary 3.1.26 again. Letting $R \to \infty$ makes the last term vanish and thus, (3.69) yields

$$
\mathcal{S}_\alpha(u) \leq \liminf_{p\to\infty} \|f(\nabla^\alpha u_p)\|_p = \liminf_{p\to\infty} \mathcal{I}_{\alpha,p}(u_p).
$$

□

Remark 3.4.3. a) We note that if f is level-polyconvex (Definition 2.3.2), then it satisfies the generalized Jensen inequality. Indeed, then we can write $f = F \circ T$ with F level-convex and T the collection of minors (which are quasiaffine). By Proposition 3.1.30 (i), any $S_0^{\alpha,p}(\Omega;\mathbb{R}^m)$-gradient Young measure is locally a $W^{1,p}$-gradient Young measure. Hence, by Theorem 1.1.23 (iii) we find for a.e. $x \in \mathbb{R}^n$

$$\begin{aligned} h\left(T\left(\int_{\mathbb{R}^{m\times n}} A\, d\mu_x(A)\right)\right) &= h\left(\int_{\mathbb{R}^{m\times n}} T(A)\, d\mu_x(A)\right) \\ &\leq \mu_x\text{- }\operatorname*{ess\,sup}_{A\in\mathbb{R}^{m\times n}} h(T(A)), \end{aligned}$$

with the last inequality following from a generalization of Lemma 2.1.5 to arbitrary probability measures. The precise connection between the generalized Jensen inequality and level-quasiconvexity is not known.

b) One could use more general complementary-value conditions in Theorem 3.4.2 with e.g. $g \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$. Then $g \in S^{\alpha,p}(\mathbb{R}^n;\mathbb{R}^m)$ for all $p \in [1,\infty]$ and we have that $\nabla^\alpha g \in C_0(\mathbb{R}^n;\mathbb{R}^{m\times n})$, cf. Section 3.1.1, which allows us to adapt the final computation in the proof. △

3.5 Applications

This final section is meant to highlight some applications of the fractional calculus of variations, which we developed in this chapter. As the models involving the Riesz fractional gradient are quite new, they have not yet been applied in specific areas in the literature. Therefore, we suggest possibly interesting applications inspired by related fractional or classical models. The main effectiveness of the fractional gradient is that it can incorporate long-range interactions and impose less regularity on the solutions depending on the fractional parameter. We exhibit these properties in two applications, one of which concerns hyperelastic materials in the branch of continuum mechanics, while the other is related to imaging in the form of new regularization functionals.

Fractional hyperelasticity. Consider the hypothetical example of $\mathbb{R}^3$ being a solid material, which is deformed by a map $u : \mathbb{R}^3 \to \mathbb{R}^3$. We wish to model the way this material behaves in a domain $\Omega \subset \mathbb{R}^3$ by using a variational model. For this, we consider a bounded open set $\Omega \subset \mathbb{R}^3$ and we assume that all deformations are fixed outside Ω and equal to $g \in S^{\alpha,p}(\mathbb{R}^3;\mathbb{R}^3)$, see Figure 3.2.

Then, we can model the material by minimizing the elastic strain enforced upon the material. As suggested in [15], by using the fractional gradient this can be translated to

$$\text{minimize} \quad \int_{\mathbb{R}^3} W(x,u,\nabla^\alpha u)\, dx \quad \text{over } u \in S_g^{\alpha,p}(\Omega;\mathbb{R}^3), \tag{3.70}$$

where $W : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ is the stored-energy density of the deformable material. We note that this model is not entirely realistic, since we are assuming that the material body is unbounded, which is physically impossible. However, the model can serve as a basis for more realistic models, where the fractional gradient is replaced by an object with a finite horizon. This means that we only consider interactions between points whose distance is less than a fixed range, called the horizon, and this makes the model applicable to bounded domains as well. We could, for instance, use the nonlocal operators from [54], which are finite-horizon versions of the Riesz fractional gradient. The concept of

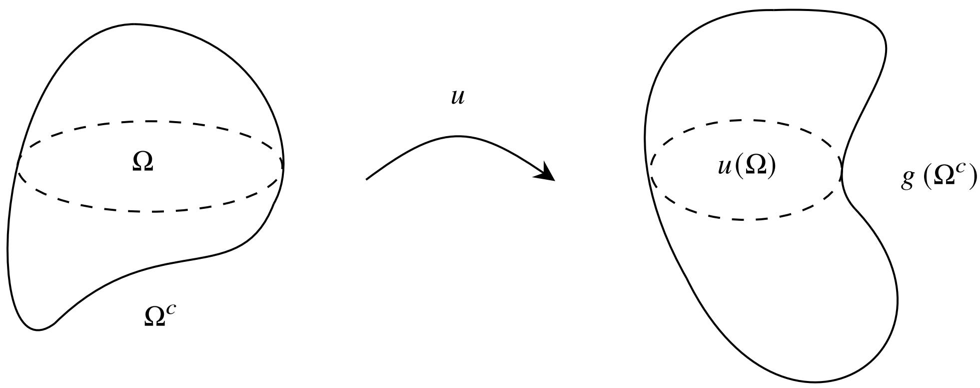

Figure 3.2: A deformation u acting on a solid material with complementary-value g.

finite horizons has already been implemented in the area of peridynamics [53,70], and is readily applied in the study of continuum mechanics.

We return to (3.70); The density W depends on the properties of the material and the size of the expected deformations. In classical hyperelasticity, one can consider for example neo-Hookean, Mooney-Rivlin or Ogden materials, with corresponding densities as in [64, Example 6.2-6.4]. Note that these densities are actually extended-value polyconvex integrands, making them suitable also in the fractional case, cf. Remark 3.2.2 b). Alternatively, one could use densities W different from the classical case to account for the change from a classical to fractional gradient.

The benefits of the fractional model compared to the classical one is that in (3.70) one can incorporate long-range interactions in the material. Furthermore, in classical hyperelasticity all functions are continuous due to the necessary assumption of $p > 3$ for polyconvexity. This can be circumvented in the fractional case by choosing the fractional parameter such that $\alpha p < 3$. Indeed, in this case discontinuous functions can be elements of $S^{\alpha,p}_g(\Omega;\mathbb{R}^3)$ as shown in [15, Lemma 2.5]. The model is therefore compatible with fractures and cavitations in the material, which are interesting if one wants to investigate under which conditions the material can fail.

When instead one is interested in the pointwise strain of the material, while keeping the nonlocal nature, minimization problems of the form

$$\text{minimize} \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} W(x,u,\nabla^\alpha u) \quad \text{over } u \in S^{\alpha,\infty}_g(\Omega;\mathbb{R}^3),$$

can be considered. In this case, one studies the deformation that minimizes the pointwise strain, which is useful in applications where one wants to avoid excessive forces on the material.

Fractional regularization. Nonlocal regularizers are currently an active area of research in both imaging [9,40] and machine learning [5,43]. In this example we focus on models and extensions related to [5]. First of all, in the setting of inverse problems we have a forward operator, say $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and data $d \in L^2(\mathbb{R}^n)$ such that d corresponds to the forward image Ku_0 of the ground truth $u_0 \in L^2(\mathbb{R}^n)$ in addition to noise. One then aims to reconstruct u_0 from the data d, which is very difficult when K is not invertible or ill-posed, as it can magnify the noise in the data. To avoid such problems, one often adds a regularization term $\mathcal{R}$ and tries to reconstruct u_0 by minimizing the functional

$$\|Ku - d\|_2^2 + \lambda\mathcal{R}(u)$$

for some $\lambda > 0$. The first term is the fidelity term, which forces the solution to fit the data, while the regularization term tries to smooth the solution to make it less sensitive to the noise.

We propose using nonlocal regularizers depending on the Riesz fractional gradient. The simplest model would be

$$\text{minimize} \quad \frac{1}{2}\|Ku - d\|_{L^2(\mathbb{R}^n)}^2 + \frac{\lambda}{2}\int_{\mathbb{R}^n} |\nabla^\alpha u|^2\, dx \quad \text{over all } u \in S_0^{\alpha,2}(\Omega). \tag{3.71}$$

Here, we assume that our images have zero complementary-values. If K is a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ then the first term is readily seen to be weakly lower semicontinuous on $S_0^{\alpha,2}(\Omega)$ from which we the obtain existence of minimizers to (3.71) via the direct method (Theorem 3.2.11). One can also derive the Euler-Lagrange equation of (3.71) as in Section 3.2.2, with some adaptations for the forward operator K, to deduce that the solution weakly satisfies the linear equations

$$\begin{cases} (-\Delta)^\alpha u + K^*(Ku - d) = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where K^* is the adjoint of K. The advantage of this model is that the fractional parameter α controls the regularity of functions inside $S_0^{\alpha,2}(\Omega)$. As such, we can tune α in order to get the desired amount of regularization, trading off between insensitivity to noise and the preservation of sharp features of u_0.

Benefiting from the same advantage, we can consider for $p \in (1, \infty)$ the more general reconstruction model A more general model that enjoys the same advantages is

$$\text{minimize} \quad \frac{1}{2}\|Ku - d\|_{L^2(\mathbb{R}^n)}^2 + \frac{\lambda}{p}\int_{\mathbb{R}^n} |\nabla^\alpha u|^p\, dx \quad \text{over all } u \in S_0^{\alpha,p}(\Omega),$$

where $p \in (1, \infty)$. If $p \neq 2$, this would introduce a nonlinear Euler-Lagrange equation of the form

$$\begin{cases} -\operatorname{div}^\alpha(|\nabla^\alpha u|^{p-2}\nabla^\alpha u) + K^*(Ku - d) = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

which might be more versatile in certain applications. The parameter p can also be optimized to enhance the quality of the reconstructions. As a last model we propose

$$\text{minimize} \quad \frac{1}{2}\|Ku - d\|_{L^2(\mathbb{R}^n)}^2 + \lambda \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|\nabla^\alpha u| \quad \text{over all } u \in S_0^{\alpha,\infty}(\Omega).$$

Regularization using the L^∞-norm of the gradient, otherwise known as Lipschitz regularization, has been successfully used in e.g. [22, 36]. The extension to the fractional case is therefore natural and it adds the possibility of varying the smoothness using the fractional parameter.

Fractional regularizers have already proven effective in the literature. In [5], for example, the authors consider the inverse problem of tomography, i.e. reconstructing an image from its Radon transform, by using the L^2-norm of the fractional Laplacian as a regularization, which is closely related to (3.71). They implement a bilevel-optimization scheme involving a neural network in order to optimize the parameters α and λ over training data, and obtain favorable results in comparison to the widely used total variation regularization. Especially the fractional parameter α helps to impose varying levels of smoothing, which is ideal for tuning the regularization term.

Bibliography

[1] F. Abdullayev, M. Bocea, and M. Mihăilescu. A variational characterization of the effective yield set for ionic polycrystals. *Appl. Math. Optim.*, 69(3):487–503, 2014.

[2] E. Acerbi and N. Fusco. Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.*, 86(2):125–145, 1984.

[3] R. A. Adams. *Sobolev spaces*. Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[4] N. Ansini and F. Prinari. Power-law approximation under differential constraints. *SIAM J. Math. Anal.*, 46(2):1085–1115, 2014.

[5] H. Antil, Z. W. Di, and R. Khatri. Bilevel optimization, deep learning and fractional Laplacian regularization with applications in tomography. *Inverse Problems*, 36(6):064001, 22, 2020.

[6] G. Aronsson. Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. *Ark. Mat.*, 6:33–53 (1965), 1965.

[7] G. Aronsson, M. G. Crandall, and P. Juutinen. A tour of the theory of absolutely minimizing functions. *Bull. Amer. Math. Soc. (N.S.)*, 41(4):439–505, 2004.

[8] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 17 of *MOS-SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2014.

[9] G. Aubert and P. Kornprobst. Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems? *SIAM J. Numer. Anal.*, 47(2):844–860, 2009.

[10] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/77.

[11] J. M. Ball and F. Murat. $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.*, 58(3):225–253, 1984.

[12] E. N. Barron, R. R. Jensen, and C. Y. Wang. The Euler equation and absolute minimizers of L^∞ functionals. *Arch. Ration. Mech. Anal.*, 157(4):255–283, 2001.

[13] E. N. Barron, R. R. Jensen, and C. Y. Wang. Lower semicontinuity of L^∞ functionals. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(4):495–517, 2001.

[14] E. N. Barron and W. Liu. Calculus of variations in L^∞. *Appl. Math. Optim.*, 35(3):237–263, 1997.

[15] J. C. Bellido, J. Cueto, and C. Mora-Corral. Fractional Piola identity and polyconvexity in fractional spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(4):955–981, 2020.

[16] J. C. Bellido, J. Cueto, and C. Mora-Corral. Γ-convergence of polyconvex functionals involving s-fractional gradients to their local counterparts. *Calc. Var. Partial Differential Equations*, 60(1):Paper No. 7, 29, 2021.

[17] P. Bernhard and A. Rapaport. On a theorem of Danskin with an application to a theorem of von Neumann-Sion. *Nonlinear Anal.*, 24(8):1163–1181, 1995.

[18] M. Bocea and V. Nesi. Γ-convergence of power-law functionals, variational principles in L^∞, and applications. *SIAM J. Math. Anal.*, 39(5):1550–1576, 2008.

[19] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[20] E. Bruè, M. Calzi, G. E. Comi, and G. Stefani. A distributional approach to fractional sobolev spaces and fractional variation: asymptotics II. *Preprint, arXiv:2011.03928*, 2021.

[21] P. Cardaliaguet and F. Prinari. Supremal representation of L^∞ functionals. *Appl. Math. Optim.*, 52(2):129–141, 2005.

[22] V. Caselles, J.-M. Morel, and C. Sbert. An axiomatic approach to image interpolation. *IEEE Trans. Image Process.*, 7(3):376–386, 1998.

[23] T. Champion, L. De Pascale, and F. Prinari. Γ-convergence and absolute minimizers for supremal functionals. *ESAIM Control Optim. Calc. Var.*, 10(1):14–27, 2004.

[24] G. E. Comi and G. Stefani. A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up. *J. Funct. Anal.*, 277(10):3373–3435, 2019.

[25] G. E. Comi and G. Stefani. A distributional approach to fractional sobolev spaces and fractional variation: asymptotics I. *Preprint, arXiv:1910.13419*, 2021.

[26] M. G. Crandall. An efficient derivation of the Aronsson equation. *Arch. Ration. Mech. Anal.*, 167(4):271–279, 2003.

[27] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

[28] B. Dacorogna. Quasiconvexity and relaxation of nonconvex problems in the calculus of variations. *J. Functional Analysis*, 46(1):102–118, 1982.

[29] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2008.

[30] B. Dacorogna and P. Marcellini. *Implicit partial differential equations*, volume 37 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[31] G. Dal Maso. *An introduction to* Γ*-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[32] M. D'Elia, J. C. D. los Reyes, and A. Miniguano-Trujillo. Bilevel parameter learning for nonlocal image denoising models. *Preprint, arXiv:1912.02347*, 2021.

[33] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[34] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[35] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[36] C. Finlay, J. Calder, B. Abbasi, and A. Oberman. Lipschitz regularized deep neural networks generalize and are adversarially robust. *Preprint, arXiv:1808.09540*, 2019.

[37] I. Fonseca and G. Leoni. *Modern methods in the calculus of variations:* L^p *spaces*. Springer Monographs in Mathematics. Springer, New York, 2007.

[38] A. Garroni, V. Nesi, and M. Ponsiglione. Dielectric breakdown: optimal bounds. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457(2014):2317–2335, 2001.

[39] A. Garroni, M. Ponsiglione, and F. Prinari. From 1-homogeneous supremal functionals to difference quotients: relaxation and Γ-convergence. *Calc. Var. Partial Differential Equations*, 27(4):397–420, 2006.

[40] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[41] H. H. Goldstine. *A history of the calculus of variations from the 17th through the 19th century*, volume 5 of *Studies in the History of Mathematics and Physical Sciences*. Springer-Verlag, New York-Berlin, 1980.

[42] D. Hilbert. Mathematical problems. *Bull. Amer. Math. Soc.*, 8(10):437–479, 1902.

[43] G. Holler and K. Kunish. Learning nonlocal regularization operators. *Mathematical Control & Related Fields*, 2021.

[44] R. Jensen. Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. *Arch. Rational Mech. Anal.*, 123(1):51–74, 1993.

[45] P. Juutinen. Recent advances in the theory of aronsson equations (viscosity solution theory of differential equations and its developments). *RIMS Kokyuroku*, (1545):122–135, 2007.

[46] N. Katzourakis. Inverse optical tomography through PDE constrained optimization L^∞. *SIAM J. Control Optim.*, 57(6):4205–4233, 2019.

[47] D. Kinderlehrer and P. Pedregal. Characterizations of Young measures generated by gradients. *Arch. Rational Mech. Anal.*, 115(4):329–365, 1991.

[48] D. Kinderlehrer and P. Pedregal. Gradient Young measures generated by sequences in Sobolev spaces. *J. Geom. Anal.*, 4(1):59–90, 1994.

[49] C. Kreisbeck and H. Schönberger. Quasiconvexity in the fractional calculus of variations: Characterization of lower semicontinuity and relaxation. *Preprint, arXiv:2104.04833*, 2021.

[50] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017.

[51] C. W. Lo and J. F. Rodrigues. On a class of fractional obstacle type problems related to the distributional riesz derivative. *Preprint, arXiv:2101.06863*, 2021.

[52] A. B. Malinowska, T. Odzijewicz, and D. F. M. Torres. *Advanced methods in the fractional calculus of variations*. SpringerBriefs in Applied Sciences and Technology. Springer, Cham, 2015.

[53] T. Mengesha and Q. Du. On the variational limit of a class of nonlocal functionals related to peridynamics. *Nonlinearity*, 28(11):3999–4035, 2015.

[54] T. Mengesha and D. Spector. Localization of nonlocal gradients in various topologies. *Calc. Var. Partial Differential Equations*, 52(1-2):253–279, 2015.

[55] Y. Mizuta. *Potential theory in Euclidean spaces*, volume 6 of *GAKUTO International Series. Mathematical Sciences and Applications*. Gakkōtosho Co., Ltd., Tokyo, 1996.

[56] C. B. Morrey, Jr. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, 2:25–53, 1952.

[57] S. Müller. Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, volume 1713 of *Lecture Notes in Math.*, pages 85–210. Springer, Berlin, 1999.

[58] P. Pedregal. *Parametrized measures and variational principles*, volume 30 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1997.

[59] A. C. Ponce. *Elliptic PDEs, measures and capacities*, volume 23 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2016. From the Poisson equations to nonlinear Thomas-Fermi problems.

[60] F. Prinari. Semicontinuity and relaxation of L^∞-functionals. *Adv. Calc. Var.*, 2(1):43–71, 2009.

[61] F. Prinari. On the lower semicontinuity and approximation of L^∞-functionals. *NoDEA Nonlinear Differential Equations Appl.*, 22(6):1591–1605, 2015.

[62] F. Prinari and E. Zappale. A relaxation result in the vectorial setting and power law approximation for supremal functionals. *J. Optim. Theory Appl.*, 186(2):412–452, 2020.

[63] A. M. Ribeiro and E. Zappale. Existence of minimizers for nonlevel convex supremal functionals. *SIAM J. Control Optim.*, 52(5):3341–3370, 2014.

[64] F. Rindler. *Calculus of variations*. Universitext. Springer, Cham, 2018.

[65] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[66] B. P. Rynne and M. A. Youngson. *Linear functional analysis*. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, second edition, 2008.

[67] A. Schikorra, T.-T. Shieh, and D. E. Spector. Regularity for a fractional p-Laplace equation. *Commun. Contemp. Math.*, 20(1):1750003, 6, 2018.

[68] T.-T. Shieh and D. E. Spector. On a new class of fractional partial differential equations. *Adv. Calc. Var.*, 8(4):321–336, 2015.

[69] T.-T. Shieh and D. E. Spector. On a new class of fractional partial differential equations II. *Adv. Calc. Var.*, 11(3):289–307, 2018.

[70] S. A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids*, 48(1):175–209, 2000.

[71] D. Spector. An optimal Sobolev embedding for L^1. *J. Funct. Anal.*, 279(3):108559, 26, 2020.

[72] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[73] R. Thiele. Euler and the calculus of variations. In *Leonhard Euler: life, work and legacy*, volume 5 of *Stud. Hist. Philos. Math.*, pages 235–254. Elsevier, Amsterdam, 2007.

[74] M. Šilhavý. Fractional vector analysis based on invariance requirements (critique of coordinate approaches). *Contin. Mech. Thermodyn.*, 32(1):207–228, 2020.

[75] K. Weierstrass. Über das sogenannte Dirichlet'sche Princip. In *Ausgewählte Kapitel aus der Funktionenlehre*, pages 184–189. Vieweg+Teubner Verlag, Wiesbaden, 1988.

[76] B. Yan. On p-quasiconvex hulls of matrix sets. *J. Convex Anal.*, 14(4):879–889, 2007.