Utrecht University

Double Bachelor's Thesis

Mathematics and Physics

An Exploration of Gauge Theory and Spinors on a Spacetime Manifold

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Abstract

In this thesis we will explore the mathematical model behind the Standard Model. In the Standard Model, there are two types of particles: force carriers and matter particles. We will see that every force carrier can be modelled by a gauge field, whose structure is given by a Lie group, and spin-half matter particles by a spinor, whose structure depends on the double cover of a rotation group. We will consider these fields first on Minkowski space (special relativity) and later on a general spacetime manifold (general relativity), for which we need to introduce the basics of differential geometry. The focus of this thesis will be that of a specific force carrier and matter particle: the photon and the electron and their interaction.

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Contents

Introduction

The Standard Model

In particle physics, there are two kinds of elementary particles: **force carriers** and **matter particles**. The force carriers have integer spin (**bosons**) and the matter particles half-integer (**fermions**), as can be seen in 1. Matter particles are the building blocks of matter. For example, the quarks can be added to form a proton. Add an electron and we have an atom. In order for the quarks to be so strongly attracted to form a proton, or for the electron to circle around the proton, you need interactions.

These interactions (attraction or repulsion) between different matter particles is mediated by the force carriers. Each one has its own interaction. The photon mediates electromagnetic interactions, the W_+, W_-, Z-bosons the weak interaction and the gluon the strong interaction (excluding the Higgs-particle from our discussion).

How a particle interacts with other particles depends on some internal state. For example, particles can only have an electromagnetic interaction if they have some **charge**. For weak interaction, it is **weak isospin**, and for the strong: **colour**.

To turn this discussion into a more mathematical direction: how can we model these particles?

Force carriers are modeled by **gauge fields**. In Chapter 3, we will see that the structure and symmetry of the considered gauge field are determined by a matrix group. For the electromagnetic interaction, this is $U(1)$. For the weak and the strong, it is respectively $SU(2)$ and $SU(3)$. This is covered in Chapter 1 of *Mathematical Gauge Theory* [6].

For an interaction, a gauge field is coupled to a **matter field**, a complex-valued function $\Psi(t, x, y, z)$, such that, if you square them, you get the probability of the particle at the event (t, x, y, z). Spin-half particles (whose matter fields we will call **spinors**) have an internal symmetry group attached to them, which is called the **spin group** ($SL(2, \mathbb{C})$ in the relativistic scenario). This group describes how the spinor will transform under a rotation of our space, as we shall see in Chapter 1.

This thesis will give the mathematical setup to describe all particles of the standard model, but our focus will be on a very specific interaction: the electron-photon interaction.

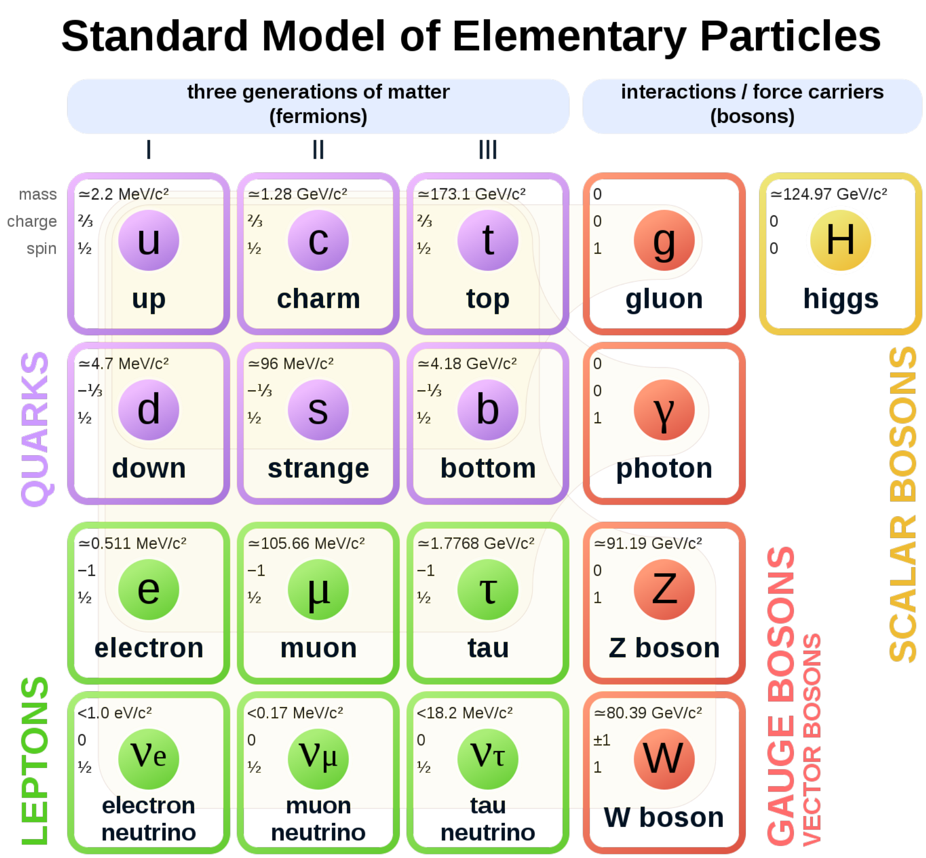


Figure 1: The Standard Model. Source:
`https://en.wikipedia.org/wiki/File:Standard_Model_of_Elementary_Particles.svg`

The world we live in

We may now ask the question: in what space do these particles interact?

In classical physics, this would have been $\mathbb{R}^3$, with time an interval to describe paths in $\mathbb{R}^3$. This neglect of the time-component is corrected by **special relativity**, in which time becomes a proper coordinate, just as x, y, z. Thus, we extend differentials as the curl and the divergence to also differentiate time-components.

Another extension that follows from special relativity is that of the **metric**. A metric is a map that endows all the tangent spaces at points in the considered space with an inner product. For example, in $\mathbb{R}^3$, each tangent space is endowed with the same inner product, the dot product. On $\mathbb{R}^4$, this inner product (and therefore the metric) extends to: $\eta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$. $\mathbb{R}^4$ with this metric is called **Minkowski space** and we will look at the electron-photon interaction for this scenario in Chapter 2.

In $\mathbb{R}^3$ we want the laws of physics to be the same under rotation. In special relativity, this is extended to rotations in $\mathbb{R}^4$, known as Lorentz transformations. These transformations are defined, just like the rotation group, as matrices that keep the metric invariant (and two other properties which we will describe later on).

Spacetime manifold

In **general relativity**, global coordinates (t, x, y, z) do not exist. Spacetime is no longer flat (Cartesian), but curved. This means that only on local regions it looks like $\mathbb{R}^4$ and only observers that are non-accelerating (or accelerating at low speeds compared to the speed of light) will measure with the same metric η. Mathematically, this means that we have to replace $\mathbb{R}^4$ with a general four-dimensional manifold and a flat metric η with a general metric g of signature $(-+++)$, which might no longer give the same inner product for every tangent space on spacetime. This will be properly defined in Chapter 3.

To further illustrate the necessity of a manifold, we introduce the reader to two astronauts: Alice and Bob. They both have their own spaceship with which they explore different parts of the universe. These spaceships can reach velocities up to 50 percent of the speed of light. Alice, from her own frame, counts time and measures space normally, but if she cruises past Bob at high enough speed, then in Bob's frame her clocks run faster and her spaceship is stretched out.

Fortunately, if she's moving at constant speed, we can always do a Lorentz transformation from her frame to Bob's. If, however, she's now accelerating fast enough, her coordinates become curved, and this is no longer possible.

To avoid talking about frames, we need a manifold, which describes events by points $p \in M$. In that way, we can talk about spacetime points without having to consider from which perspective we are viewing the event.

As a condition for this manifold, we require that at every event, there exist **observer frames**, frames in which time and space are just Minkowski space. These are also known as **inertial frames**.

Two other conditions is that spacetime is **orientable** and **time-orientable**. If not time-orientable, Alice and Bob could meet each other, and Bob would find out that Alice now counts time backwards (although she would then have to meet him in the past). If not orientable, Alice and Bob could meet each other and Alice could find out that Bob is now the mirror image of what he was before they started travelling. The details of these requirements will be described in Chapter 1.

Differential geometry

The consequence of throwing coordinates out the window is that now we have to use differential geometry to describe force fields and particle waves. This isn't an easy subject, as intuition can easily be lost, but it also has a certain elegance, as it allows us to write physical laws without boring summations, which we will see in Chapter 2.

Math versus physics

To generalize our take on these subjects we often assume we are dealing with an m-dimensional manifold or an n-dimensional vector space, instead of four, which is the dimension of spacetime. To make this distinction clear, we will always begin with a statement that we are dealing with either spacetime or Minkowski space, when we narrow down to four dimensions.

Rules of notation

- Summations will be written in Einstein notation, which means $\sum_{k=1}^{n} x_n e_n$ is written as $x^\mu e_\mu$.
- If a space is denoted by U, it is always open (either with or without subscripts).
- All manifolds used are smooth (C^∞) manifolds.
- For M a manifold, the space of smooth maps $f : M \longrightarrow \mathbb{R}$ is denoted by: $C^\infty(M)$.
- Our definition of a fibre bundle is actually that of a smooth fibre bundle. The same goes for a vector bundle and a principal bundle.
- Vector spaces are assumed to be over fields $\mathbb{R}$ unless stated otherwise.
- All manifolds denoted by M are m-dimensional, with m a positive integer.
- For the spacetime metric, we will always use the signature $(-+++)$.
- Physical constants as the speed of light, the electron charge, the permittivity of the vacuum and the Planck constant will all be equal to 1.
- Inner products will be inner products without the condition of positive-definiteness.

Chapter 1

Linear algebra on a (co-)tangent space

1.1 The infinitesimal view

Spacetime is a four-dimensional manifold. In special relativity, this manifold is $\mathbb{R}^4$. This will be the subject of Chapter 2, in which we will look at electron-photon interactions in this space. In Chapter 3, $\mathbb{R}^4$ will be generalised to a spacetime manifold M. However, to be able to do any physics in both scenarios, we need to look at the tangent spaces of such a manifold, which is the focus of this chapter.

Given a $p \in M$, we will look at constructions (e.g. tensor products, algebras, inner products, etc.) on a tangent space T_pM and its dual T_p^*M [1]. In the next chapters, these concepts will then be extended to constructions over the whole of spacetime.

To generalise these constructions, we will not define them for the vector spaces T_pM or T_p^*M, but for an unspecified vector space V. As a rule, we will construct exterior algebras only on V^* (even though we define it for any vector space). The reason for this will become clear in Chapter 2.

[1] Definitions of a (co-)tangent space can be found in Chapter 4 of [2]

1.2 Dual space

Definition 1.1. *Given a vector space V, its* ***dual space*** *V^* is the set of all linear maps $f : V \longrightarrow \mathbb{R}$ with addition and scalar multiplication, such that, for all $f \in V^*$, $v, w \in V$ and $a \in \mathbb{R}$:*

- *$f(v + w) = f(v) + f(w)$*
- *$(af)(v) = af(v)$*

Lemma 1.2. *For any finite-dimensional vector space V over a field $\mathbb{R}$, $V = (V^*)^*$*

Proof. For any $v \in V$, $v : V^* \longrightarrow \mathbb{R}, f \mapsto f(v)$ is a linear map. The two conditions from 1.3 are automatically satisfied by definition of V^*. □

For any basis of V, we can also define its dual:

Definition 1.3. *Let $e_1, ...e_n$ be a basis of V. Then, its* ***dual basis*** *are the maps $e^1, ...e^n \in V^*$, such that, for any $c_1, ..., c_n \in \mathbb{R}$: $e^i(c_1 e_1 + .. + c_n e_n) = c_i$*

1.3 Frame space

We can collect all the bases of V in the following space:

Definition 1.4. $Fr(V) = \{e = (e_1, ...e_n) \mid e_1, ..., e_n$ *are a basis of* $V\}$

There is a natural right action associated to this frame space:

$$Fr(V) \times Gl(V) \longrightarrow Fr(V), (e \cdot A)_j = A^i_j e_i$$

This right action is what we call a **basis transformation**. In the third chapter, basis transformations will be extended to **transition functions** on a vector bundle. In vector notation, a basis transformation has the following effect:

Lemma 1.5. *Let $(e_1, ..., e_n) \in Fr(V)$ and $A \in GL(V)$, such that $(d_1, ..., d_n)$ is the basis with: $d = e \cdot A$. Then, for any $v \in V$, $v_d = A^{-1} v_e$ with $\mathbf{v}_d$ and $\mathbf{v}_e$ the vector representations of v in basis d and e respectively.*

Proof. Let $v \in V$. Then $v = v^j_d d_j = v^j_d A^i_j e_i$. But also: $v = v^i_e e_i$. Thus: $A^i_j v^j_d = v^i_e$. In matrix form: $A\mathbf{v}_d = \mathbf{v}_e$. In conclusion: $\mathbf{v}_d = A^{-1}\mathbf{v}_e$ □

1.4 Graded algebras

Vector spaces have as operations addition and multiplication with a scalar. However, we would like to define a multiplication of vectors as well. In physics, for example, we often use the cross product: $\vec{A} \times \vec{B}$. This cross product turns $\mathbb{R}^3$ from a vector space into an **algebra**.

Definition 1.6. *Let V be a vector space over $\mathbb{R}$ with a binary operation $(v, w) \mapsto v \cdot w$ from $V \times V$ to V. Then, V is an **algebra** if the following holds for every $u, v, w \in V$ and any $a, b \in \mathbb{R}$:*

- $(u + v) \cdot w = u \cdot w + v \cdot w$ *(Right distributivity)*
- $u \cdot (v + w) = u \cdot v + u \cdot w$ *(Left distributivity)*
- $(av) \cdot (bw) = abv \cdot w$ *(Compatibility with scalars)*

1.5 Tensor product

The first vector product we consider is the **tensor product** $\otimes$.

Definition 1.7. *Let V be a vector space with basis $e_1, ..., e_n$. The **k-tensor product space $T^k(V)$** is the vector space generated by elements $e_{i_1} \otimes ... \otimes e_{i_k}$, with $i_j = 1, ..., n$ for $j = 1, ...k$.*

All these tensors of different order can be added up to form a graded algebra:

$$T(V) = \bigoplus_{k=1}^{\infty} T^k V = \mathbb{R} \oplus V \oplus T^2(V) \oplus ...$$

For $T(V)$ to be an algebra, we define a multiplication of different tensors:

Definition 1.8. *Let $v = v^{i_1,...,i_k} e_{i_1} \otimes ... \otimes e_{i_k} \in T^k V$ and $w = w^{i_1,...,i_l} e_{i_1} \otimes ... \otimes e_{i_l} \in T^l V$, we define their **tensor product** as:*

$$v \otimes w = v^{i_1,...,i_k} w^{i_1,...,i_l} e_{i_1} \otimes ... \otimes e_{i_k} \otimes e_{i_1} \otimes ... \otimes e_{i_l} \in \otimes^{k+l} V$$

Then:

Theorem 1.9. *$T(V)$ with the tensor product is an algebra.*

Proof. The requirements of 1.6 are readily checked. □

The tensor algebra is a very big algebra, but we can construct smaller algebras by quotienting $T(V)$ with certain subgroups (ideals).

1.6 Exterior algebra

As illustration of the exterior algebra, we look at the set of 2-tensors. Consider two vectors $v, w \in V$. Then, we can separate its symmetric and its antisymmetric part:

$$v \otimes w = \frac{1}{2}(v \otimes w + w \otimes v) + \frac{1}{2}(v \otimes w - w \otimes v)$$

Note that the symmetric part can be written as: $(v + w) \otimes (v + w) - w \otimes w - v \otimes v$. Thus, we can cancel the symmetric part of every 2-tensor with the quotient:

$$V \wedge V = V \otimes V / \{v \otimes v |\ v, w \in V\}$$

This quotient will send, for any $v, w \in V$, the 2-tensor $v \otimes w$ to its antisymmetric part: $v \otimes w - w \otimes v$. Expanding this discussion to tensors of any order, the space of all anti-symmetric tensors form the exterior algebra, which is the following vector space quotient:

$$\Lambda(V) = T(V)/\{v \otimes v |\ v \in V\} \text{ with the canonical surjection } \pi : T(V) \longrightarrow \Lambda(V)$$

More details about this quotient map are given in [2, p. 98]. Just as for $T(V)$, to turn $\Lambda(V)$ into an algebra, we need a multiplication:

Definition 1.10. *Given $v \in \Lambda^k(V), w \in \Lambda^l(V)$ with $v = \pi(\alpha), w = \pi(\beta)$ for some $\alpha \in T^k(V), \beta \in T^l(V)$, then the **wedge product** is defined as:*

$$v \wedge w = \frac{(k+l)!}{k!\ l!}\pi(\alpha \otimes \beta)$$

An important property of the wedge product is:

Lemma 1.11. *For any $v \in \Lambda^k(V), w \in \Lambda^l(V)$, the following holds:*

$$v \wedge w = (-1)^{kl} w \wedge v$$

Proof. This can be proven by writing out v and w in basis components and then taking the wedge product $v \wedge w$ and $w \wedge v$. Reshuffling basis components will then give you the equality. Or see [2, p. 103]. □

Theorem 1.12.

1. *If V is a vector space with basis $e_1, ..., e_n$., then: $e_{i_1} \wedge ... \wedge e_{i_k}$, with $1 \leq i_1 < ... < i_k \leq n$ is a basis for $\Lambda^k(V)$.*
2. *$\Lambda^k(V) = 0$ for $k \geq n$*

Proof. Let $e_1, ..., e_n$ be a basis for V.

1. Because of all the indices, this proof is rather tedious. However, the idea is to check linear independence of elements $e_{i_1} \wedge ... \wedge e_{i_k}$ by using the property of the dual that $E^j(E_i) = \delta_i^j$ for basis elements E_i of $\Lambda^k(V)$. This property can also be used to show that any $\omega \in \Lambda^k(V)$ is equal to: $\omega^{i_1,...,i_k} e_{i_1} \wedge ... e_{i_k}$. Thus, $e_{i_1} \wedge ... e_{i_k}$ span $\Lambda^k(V)$. This proof can be found in either [8, p. 353] or [2, p. 101]
2. Take any $v \in \Lambda^k(V)$ for $k \geq n$. Then: $v = \sum_{i_1,...,i_k} a_{i_1,...,i_k} e_{i_1} \wedge ... \wedge e_{i_k}$ with $a_{i_1,...,i_k} \in \mathbb{R}$. Because $k \geq m$, at least two of these basis elements equal each other in every term of the sum. Using 1.11 we can put these equal elements next to each other. But: $e_i \wedge e_i = 0$ for every basis element, so $v = 0$.

□

From this theorem, we know that: $\dim \Lambda^n(V) = 1$. Thus, $\Lambda^n(V) - \{0\}$ has two connected components. This allows us to define an **orientation** on V.

Definition 1.13. *An **orientation on a vector space V** is a choice of one of the connected components of $\Lambda^n(V) - \{0\}$, which we will denote by $\Lambda^n_+(V)$.*

Now, we can use this property to classify the following set of bases:

Definition 1.14. *A basis $(e_1, ..., e_n) \in Fr(V)$ is **oriented** when:*

$$e_1 \wedge .... \wedge e_n \in \Lambda^n_+(V)$$

These bases will turn up again in section 1.9.

Example 1.15. *On $\mathbb{R}^4$, we will choose as orientation the component of $\Lambda^4(\mathbb{R}^4) - \{0\}$ that contains $dt \wedge dx \wedge dy \wedge dz$. As such, the element $dx \wedge dt \wedge dy \wedge dz = -dt \wedge dx \wedge dy \wedge dz$ is not oriented, but $dx \wedge dy \wedge dt \wedge dz = dt \wedge dx \wedge dy \wedge dz$ is.*

1.7 Inner product

We can also quotient out all the antisymmetric parts of 2-tensors:

$$Sym^2(V) = V \otimes V / \{v \otimes w - w \otimes v | \ v, w \in V\}$$

Elements out of $Sym^2(V^*)$ are bilinear symmetric maps of the form $g : V \times V \longrightarrow \mathbb{R}$. If they are non-degenerate, they are **inner products** on V (without the condition of positive-definiteness):

Definition 1.16. *An element $g \in Sym^2(V^*)$ is an **inner product on V** if, for all $v \in V$, there exists a $w \in V$, such that: $g(v, w) \neq 0$.*

Given a basis $e = (e_1, ..., e_n)$ of V, a general inner product g can be written as a matrix:

$$g(v, w) = \mathbf{v}_e^t \mathbf{g}_e \mathbf{v}_e \text{ for } v, w \in V \text{ and } e \in Fr(V)$$

Here, $\mathbf{g}_{e,ij} = g(e_i, e_j)$. Doing a basis transformation, the inner product transforms as:

Lemma 1.17. *Let g be an inner product on V and $e, f \in Fr(V)$, such that: $e \cdot A^{-1} = f$ for some $A \in GL(n, \mathbb{R})$ and $\mathbf{g}_e$ and $\mathbf{g}_f$ are the matrix representations of g in the respective bases. Then: $\mathbf{g}_e = A^t \mathbf{g}_f A$.*

Proof. From Lemma 1.5: $\mathbf{v}_f = A\mathbf{v}_e$ for any $v \in V$. Thus: $g(v, w) = \mathbf{v}_f^t \mathbf{g}_f \mathbf{v}_e = \mathbf{v}_e^t A^t \mathbf{g}_f A \mathbf{w}_e$. But also: $g(v, w) = \mathbf{v}_e^t \mathbf{g}_e \mathbf{v}_e$. Because $g(v, w)$ should be the same in both frames, we conclude: $\mathbf{g}_e = A^t \mathbf{g}_f A$. □

For Minkowski space, we have the following inner product:

Example 1.18. *Let $\partial_0, \partial_1, \partial_2, \partial_3 \in V$ be a basis in V, a four-dimensional vector space. Then, the **Minkowski inner product on** V is:*

$$\eta = -dx_0 \otimes dx_0 + dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 \in Sym^2(V^*)$$

1.7.1 Inner product on $\Lambda(V)$

We can extend an inner product on V to an inner product on $\Lambda^k(V)$. For this, we first mention that **decomposable k-vectors** are those k-vectors $\omega \in \Lambda^k(V)$ that can be decomposed as $\omega_1 \wedge ... \omega_k$ with $\omega_i \in V$. Then:

Definition 1.19. *Let $g \in Sym^2(V^*)$, then for decomposable elements $\omega, \eta \in \Lambda^k(V)$ the* ***inner product*** *on $\Lambda^k(V)$ is:*

$$\langle \omega, \eta \rangle = \det([\omega_j, \eta_l]) \text{ for all } \omega, \eta \in \Lambda^k(V)$$

Here $[\omega_j, \eta_l]$ is the matrix:

$$\begin{pmatrix} g(\omega_1, \eta_1) & ... & g(\omega_1, \eta_k) \\ ... & ... & ... \\ g(\omega_k, \eta_1) & ... & g(\omega_k, \eta_k) \end{pmatrix}$$

Non-decomposable k-vectors can be written as the sum of decomposable k-vectors, and so the inner product for these k-vectors will just be the sum of inner products of decomposable k-vectors.

Because we declare forms of different degree to be orthogonal, this inner product extends to an inner product on $\Lambda(V)$, which we denote by: $\langle .., .. \rangle_{\Lambda(V)}$.

1.7.2 The Hodge star operator

Definition 1.20. *The* ***Hodge star operator*** *is a map $\star : \Lambda^k(V) \longrightarrow \Lambda^{n-k}(V)$, with the defining property: $\omega \wedge \star\eta = \langle \omega, \eta \rangle_{\Lambda(V)} vol_n$ for any decomposable $\omega, \eta \in \Lambda^k(V)$.*

Note: this definition fully depends on what inner product we have over V.
From this definition, we derive that: $\omega \wedge \star\eta = \eta \wedge \star\omega$

Hodge star for Minkowski space

Given coordinates dt, dx, dy, dz on $\mathbb{R}^4$ with inner product η, we would like to know what the Hodge star of their wedge products are.

Example 1.21. *We take $dt \wedge dy$ and want to know what $\star dt \wedge dy$ is.*

Using definition 1.20, we write:

$$dt \wedge dy \wedge \star(dt \wedge dy) = \det\begin{pmatrix} \eta(dt, dt) & \eta(dt, dy) \\ \eta(dy, dt) & \eta(dy, dy) \end{pmatrix} vol_4 = \det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} vol_4 = -vol_4$$

From this, we conclude: $\star(dt \wedge dy) = dx \wedge dz$.

An easy thumb rule for η with signature (-++++) is:

Let $\omega \in \Lambda(\mathbb{R}^4)$ be a standard basis element of $\Lambda(\mathbb{R}^4)$. Then, $\star\omega$ consists of all the terms that are missing in ω to achieve: $\omega \wedge \star\omega = dt \wedge dx \wedge dy \wedge dz$. If ω has a dt term, an extra minus sign is added.

Example 1.22. *We again take $dt \wedge dy$ and note: $dt \wedge dy \wedge (dz \wedge dx) = dt \wedge dx \wedge dy \wedge dz$. Because $dt \wedge dy$ contains a dt: $\star(dt \wedge dy) = -dz \wedge dx = dx \wedge dy$.*

For other basis elements of $\Lambda(\mathbb{R}^4)$:

0-form/4-form	1-forms/3-forms	2-forms
$\star 1 = dt \wedge dx \wedge dy \wedge dz$	$\star dt = -dx \wedge dy \wedge dz$	$\star(dt \wedge dx) = -dy \wedge dz$
	$\star dx = -dt \wedge dy \wedge dz$	$\star(dt \wedge dy) = dx \wedge dz$
	$\star dy = dt \wedge dx \wedge dz$	$\star(dt \wedge dz) = -dx \wedge dy$
	$\star dz = -dt \wedge dx \wedge dy$	$\star(dx \wedge dy) = dt \wedge dz$
		$\star(dx \wedge dz) = -dt \wedge dy$
		$\star(dy \wedge dz) = dt \wedge dx$

We can also take the double Hodge star, which satisfies the relation: $\star\star\eta = -(-1)^{k(4-k)}\eta$ for $\eta \in \Lambda^k(\mathbb{R}^4)$.

1.8 Clifford algebra

Another famous quotient algebra is the Clifford algebra:

Definition 1.23. *Given a $Q \in Sym^2(V^*)$, a **Clifford algebra** is the quotient space: $Cl(V,Q) = T(V)/\{v \otimes v - Q(v,v)1\}$.*

We can then define the linear map $\gamma : V \longrightarrow Cl(V,Q)$, which satisfies the **Clifford relation**: $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2Q(v,w)1$ for every $v, w \in V$.

A specific form of the Clifford algebra, called the **geometric algebra** is when $V = \mathbb{R}^{s+t}$ and (s,t) is the signature of Q, which means: $Q(e_i, e_i) = 1$ for s standard basis vectors and $Q(e_j, e_j) = -1$ for the other t. This is written as $Cl(s,t)$.

Example 1.24. *An example of a geometric algebra is: $Cl(0,1)$. This vector space has two basis elements, 1 and e_1, for which: $e_1^2 = -1$. Therefore, this algebra is exactly the complex plane $\mathbb{C}$.*

Example 1.25. *If $Q = 0$, then $Cl(V,0) = T(V)/\{v \otimes v\}$ is exactly the exterior algebra.*

The Clifford algebra that will be important for modelling spinors is the one that is constructed on Minkowski space: $Cl(\mathbb{R}^4, \eta)$. It is spanned by:

$$\begin{cases} 1 \\ \gamma(e_t), \gamma(e_x), \gamma(e_y), \gamma(e_z) \\ \gamma(e_t), \gamma(e_x), \gamma(e_t)\gamma(e_y), \gamma(e_t)\gamma(e_z), \gamma(e_x)\gamma(e_y), \gamma(e_x)\gamma(e_z), \gamma(e_y)\gamma(e_z) \\ \gamma(e_t)\gamma(e_x)\gamma(e_y), \gamma(e_t)\gamma(e_x)\gamma(e_z), \gamma(e_t)\gamma(e_y)\gamma(e_z), \gamma(e_x)\gamma(e_y)\gamma(e_z) \\ \gamma(e_t)\gamma(e_x)\gamma(e_y)\gamma(e_z) \end{cases}$$

Thus, $Cl(\mathbb{R}^4, \eta)$ is a 16-dimensional space. For $v, w \in \mathbb{R}^4$ the Clifford elements satisfy: $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2\eta(v, w)1$.

1.9 Lorentz bases

In spacetime, there is a set of special bases for every tangent space, such that their basis transformation is a Lorentz transformation. Those bases have the following three properties (of which we already saw one): orthonormal, oriented and time-oriented. Orthonormal and time-oriented depend on the signature of the inner product, which, for spacetime, is $(- + ++)$ (or $(+ + +-)$ depending on your preference). One could define these properties for other signatures (and vector spaces that are not four-dimensional), which we will not do here. This is discussed in Chapter 6 of [6].

In Minkowski space, with global coordinates t, x, y, z, the Minkowski inner product can be represented as the following diagonal matrix:

$$\eta \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In spacetime, the inner product on T_pM might not be Minkowski in the coordinate basis (as we shall see in chapter 3). Fortunately, using the **Gram-Schmidt process**, we can always choose a different basis in which this *is* the case.

In the coordinate basis, which we denote by e_Φ, the inner product on T_pM might be of the form:

$$g \doteq \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}_\Phi \quad \text{in standard basis } e_\Phi \doteq \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Diagonalising this inner product (such that it looks like Minkowski), we get an **orthonormal basis**:

Definition 1.26. *Let g be an inner product on V. Then, we call an **orthonormal basis of V** a basis $e_0, .., e_3$ of V, such that:*

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3$$

We then say that g has **signature** (-+++) if such a basis exists. Now the last property is defined as:

Definition 1.27. *Let $T \in V$ with $g(T,T) < 0$ and $(e_0, e_1, e_2, e_3) \in Fr(V)$. Then, we call this basis **time-oriented** if $g(T, e_0) < 0$.*

Given an inner product g on a vector space V, we will call orthonormal oriented time-oriented bases **Lorentz bases**. We define: $SO^+(T_pM) = \{e = (e_0, ..., e_3) \in Fr(T_pM) \mid e \text{ is a Lorentz basis}\}$.

In chapter 3, the Lorentz bases will be extended to local Lorentz frames, which are smooth choices of Lorentz bases in every T_pM with $p \in U$, an open in M.

1.10 The Lorentz Group

The special principle of relativity [?] says that the laws of physics must be the same in every inertial frame of reference. This means that if we rotate coordinates (t, x, y, z) in Minkowski space by a Lorentz transformation, we should get the same differential equations as before. On a spacetime manifold, this rotation of coordinates can only be done locally. Mathematically, rotations are **isometries** of a certain inner product:

Definition 1.28. *Let $g \in Sym^2(V)$, then an **isometry** is a map $A \in GL(V)$, such that: $g(Av, Av) = g(v, v)$ for every $v \in V$.*

The isometries of the Minkowski inner product η form the **Poincaré group**:

$$O(3,1) = \{\Lambda \in GL(4, \mathbb{R}) | \ \eta(\Lambda v, \Lambda w) = \eta(v, w) \text{ for all } v, w \in \mathbb{R}^4\}$$

In matrix notation: $\Lambda^t \eta \Lambda = \eta$. Using this notation, $\Lambda \in O(3,1)$ satisfies: $\det^2(\Lambda) = \det(\Lambda^t)\det(\Lambda) = -\det(\Lambda^t \eta \Lambda) = -\det(\eta) = 1$. Lorentz transformations are a subgroup of $O(3,1)$:

Definition 1.29. *The **Lorentz group** is the following collection of matrices:*

$$SO^+(3,1) = \{\Lambda \in SO(3,1) | \Lambda^0_0 > 0\}$$

Here,

$$SO(3,1) = \{\Lambda \in O(3,1) | \det \Lambda = 1\}$$

For the property of $SO^+(3,1)$, we note that any matrix $\Lambda \in GL(4,\mathbb{R})$, can be written in block form:

$$\Lambda = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 \\ \Lambda^1_0 & \Lambda_{11} \end{pmatrix} \text{ with } \Lambda_{11} \text{ a 3x3 matrix.}$$

Then, the condition $\Lambda^0_0 > 0$ (called **orthochronous**) means the matrices preserve time-orientation. We don't want time running backwards after a transformation.

Theorem 1.30. *Let e, f be two Lorentz bases of inner product g on $\mathbb{R}^4$. Then, the basis transformation from e to f is given by a Lorentz transformation. In other words: there exists a $\Lambda \in SO^+(3,1)$, such that: $f = e \cdot \Lambda^{-1}$.*

Proof.

1. Because $\eta = \mathbf{g}_e = \Lambda^t \mathbf{g}_f \Lambda = \Lambda^t \eta \Lambda$, $\Lambda \in O(3,1)$.

2. If e, f are Lorentz bases, they are **oriented** and therefore have the same orientation. Thus, Λ must preserve orientation. This is exactly a property of special orthogonal matrices and therefore: $\Lambda \in SO(3,1)$.

3. If $\Lambda^0_0 < 0$, $g(T, f_0) = g(T, \Lambda^i_0 e_i)$ could be larger than 0. This would mean that a time-oriented frame might be send to one that is not. Therefore: $\Lambda^0_0 > 0$. Thus $\Lambda \in SO^+(3,1)$.

□

From this, we conclude that there is a well-defined right action:

$$SO^+(V) \times SO^+(3,1) \longrightarrow SO^+(V)$$

This right action makes $SO^+(V)$ into a G-structure, whose details can be found in [3, p. 84]

1.11 Lie groups and their algebras

The Lorentz group is an example of a **Lie group**. In the introduction, we also described other symmetry groups of interactions. These are also Lie groups, which allows us to define smooth functions on them.

Definition 1.31. *A **Lie group** is a group that is also a manifold and has a smooth multiplication map:* $m : G \times G \longrightarrow G, (g, h) \mapsto gh$.

We define the following inclusions:

$$i_g(h) = (g, h) \text{ and } \tilde{i}_g(h) = (h, g)$$

Then we define:

$$L_g = m \circ \tilde{i}_g \text{ and } R_g = m \circ i_g$$

Here, L_g is the left action and R_g the right action. They both have their differentials, which will play an important in chapter 3. From these maps we can construct the adjoint action:

$$Ad_g = L_{g^{-1}} \circ R_g$$

1.11.1 Its Lie algebra

Every Lie group has a Lie algebra $\mathfrak{g}$, which is most easily defined as the tangent space at the identity (T_eG). As $\mathfrak{g}$ is an algebra, it has a multiplication. To define it, we first differentiate the adjoint action at the identity: $(dAd_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$ for every $g \in G$. Because we have such a map for every $g \in G$, they induce a representation (which we will use in chapter 3):

$$Ad : G \longrightarrow GL(\mathfrak{g}), g \mapsto (dAd_g)_e$$

This representation also has a differential at the identity:

$$ad = (dAd)_e : \mathfrak{g} \longrightarrow GL(\mathfrak{g}), v \mapsto ad_v$$

Now:

Definition 1.32. *Given a Lie algebra $\mathfrak{g}$, its **Lie bracket** is the map*

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, (v, w) \mapsto ad_v(w)$$

This is a rather abstract definition, but fortunately for matrix algebras (and all the examples in this thesis *are* matrix algebras): $[A, B] = AB - BA$ for $A, B \in \mathfrak{g}$. In general, the Lie bracket has the following properties: it is bilinear, anticommutative and satisfies the Jacobi identity (see [3, p. 51]). The **Jacobi identity** is: $[v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0$ for all $v, w, z \in \mathfrak{g}$.

An equivalent definition for the Lie algebra is the space of all left-invariant vector fields on G. Vector fields on G are smooth choices of tangent vectors for every tangent space of G. This will be properly defined in subsection 3.3.2 of Chapter 3. For continuity, we have chosen to already talk about this concept here.

Definition 1.33. *A vector field $V \in \mathfrak{X}(G)$ is* ***left-invariant*** *if $(dL_g)_h(V_h) = V_{gh}$ for every $h, g \in G$.*

Theorem 1.34.

$$T_eG \cong \mathfrak{X}_{l.i}(G)$$

Proof. ($\Rightarrow$): For any $v \in T_eG$, we can left translate: $(dL_g)_e(v) \in T_gG$ for any $g \in G$. Thus, $v \in T_eG$ induces a vector field. It is left-invariant, because: $(dL_h)_g(dL_g)_e = d(L_h \circ L_g)_e = (dL_{hg})_e$.

($\Leftarrow$): For any $V \in \mathfrak{X}_{l.i}(M)$, $(dL_{g^{-1}})_g V_g = V_e \in T_eG$ for all $g \in G$. This means the whole vector field can be pulled back to a unique element in T_eG. □

The Lie bracket on left-invariant vector fields would take this discussion too far, without introducing manifolds. It is explained in page 186 of [8].

1.11.2 The exponential map

Definition 1.35. *The* ***exponential map*** $\exp : \mathfrak{g} \longrightarrow G$ *is the solution to the following differential equation:*

$$\frac{d\gamma}{dt}(t) = (dL_{\gamma(t)})_e(V_e) \text{ with: } \gamma(0) = e \text{ (the identity of } G)$$

So, given a $v \in \mathfrak{g}$, which induces a vector field V on G, the exponential map describes the path that starts at $e \in G$ and follows the vector field V.It will be easy to work with, because it becomes the matrix exponential when G is a matrix group.

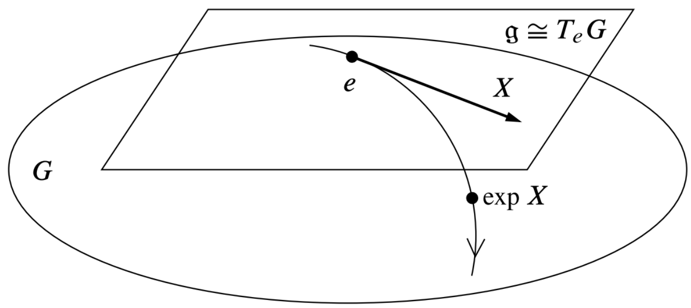


Figure 1.1: The exponential map, [8, p. 519].
Here, you can see that $\exp(X) \in G$ is the path element given by the vector field induced by $X \in \mathfrak{g}$.

The exponential map allows us to write parts of the Lie group in terms of its algebra. Sometimes the whole Lie group can be described as the exponential of the Lie algebra. This occurs when the exponential map is surjective, which is the case for the Lorentz group $SO^+(3,1)$. Now, we will look at some examples of Lie groups:

Example 1.36 ($U(1)$)**.** *As mentioned in the introduction to this chapter, the circle group $U(1)$ will model the electromagnetic interaction. Drawing a circle in the complex plane, it is easy to see that $T_eU(1) = T_{(1,0)}U(1) = i\mathbb{R}$, in other words a vertical line in the $\mathbb{C}$-plane.*

The exponential map of $U(1)$ then is:

$$e : i\mathbb{R} \longrightarrow U(1), i\theta \mapsto e^{i\theta}$$

This is clearly a surjective map.

Example 1.37 (The Lorentz group)**.** *The Lie algebra of the Lorentz group, which we talked about in 1.10, is:*

$$\mathfrak{so}^+(3,1) = \{X \in M_4(\mathbb{R}) \mid \eta X \eta = -X^t\}$$

This algebra has the basis:

$$K_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, any $X \in \mathfrak{so}^+(3,1)$ *can be written as:* $X = \boldsymbol{\zeta} \cdot \boldsymbol{K} + \boldsymbol{\theta} \cdot \boldsymbol{J}$ *with* $\boldsymbol{\zeta}, \boldsymbol{\theta} \in \mathbb{R}^3$*.* $SO^+(3,1)$ *is connected and compact [4, p. 504], which implies that the exponential map is surjective.*

Thus, we get that any $\Lambda \in SO^+(3,1)$ *can be written as:*

$$\Lambda = \exp(\boldsymbol{\zeta} \cdot \boldsymbol{K} + \boldsymbol{\theta} \cdot \boldsymbol{J}) \text{ with } \boldsymbol{\zeta}, \boldsymbol{\theta} \in \mathbb{R}^3$$

This will be particularly useful in quantum field theory, for then we could approximate Lorentz transformations by the first terms of this exponential.

1.12 Gauge and spin-half matter fields

As mentioned in the introduction, there were two types of particle fields: **gauge fields** and **matter fields**. Gauge fields are smooth choices of tangent vectors [2] and matter fields smooth choices of complex vectors for every point $p \in M$. In particular, for spin-half matter fields, the vectors are in $\mathbb{C}^4$, which will be discussed in the next chapter.

For example, the electromagnetic **four-potential** (a gauge field) can be seen at a point $p \in M$ as:

$$A(p) = \phi(p)\partial_t + A_x(p)\partial_x + A_y(p)\partial_y + A_z(p)\partial_z = \begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix}(p) \in T_pM$$

Here $(\partial_t, \partial_x, \partial_y, \partial_z)$ is a basis that we can transform by a Lorentz matrix $\Lambda \in SO^+(3,1)$. For example: $\partial_i' = \Lambda^{-1}\partial_i$. Using Lemma 1.5, we get that $A(p) \in T_pM$ will transform to: $A'(p) = \Lambda A(p)$.

For matter fields of spin-half particles, which are called **spinors**, this is no longer so simple. Doing a Lorentz transformation on T_pM, the spinor does not transform by a Lorentz transformation as well. Instead, it is a lift of $\Lambda \in SO^+(3,1)$ to a different matrix $A \in SL(2,\mathbb{C})$ that transforms the spinor.

[2] And, as we shall see in Chapter 3, choices of the associated Lie algebra

Here: $SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) | \det A = 1\}$, which is called the **spin group**. This is a Lie group and therefore it has a Lie algebra:

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in \mathfrak{gl}(2,\mathbb{C}) | tr(X) = 0\}$$

Now, the Lorentz transformation $\partial_i' = \Lambda^{-1}\partial_i$ in T_pM will transforms the spinor as: $\Psi \mapsto \rho_{\frac{1}{2}}(A)\Psi$ []. Here, $\rho_{\frac{1}{2}}$ is the following complex matrix representation of C into $GL(4,\mathbb{C})$:

$$\rho_{1/2} : SL(2,\mathbb{C}) \longrightarrow Gl(4,\mathbb{C}), A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{\dagger -1} \end{pmatrix}$$

1.13 The double cover of $SO^+(3,1)$

The lift of Λ to a matrix in $SL(2,\mathbb{C})$ is not unique. In fact, for each element of $SO^+(3,1)$, there are two matrices in $SL(2,\mathbb{C})$, an A and a $-A$, that model the transformation of the spinor. Fortunately, in quantum mechanics, a minus sign in the spinor does not change anything in the measurements, and so it does not matter to which matrix we lift the Lorentz transformation [11, p. 76]. In mathematical terms, the above is another way of saying that $SL(2,\mathbb{C})$ is a double cover of $SO^+(3,1)$.

Theorem 1.38. *$SL(2,\mathbb{C})$ is a double cover of $SO^+(3,1)$.*

Proof. This covering is done by first identifying Minkowski space with $H(2,\mathbb{C})$ (hermitian 2x2-matrices), via the isomorphism:

$$f(v) = f((t,x,y,z)) = t\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} t+z & x-yi \\ x+yi & t-z \end{pmatrix} = V \in H(2,\mathbb{C})$$

Then: $\eta(v,v) = -t^2 + x^2 + y^2 + z^2 = -\det(f(v))$. This induces the conjugation map:

$$conj : End(H(2,\mathbb{C})) \longrightarrow End(\mathbb{R}^4), h \mapsto f^{-1} \circ h \circ f$$

Now, we can write, for any $A \in SL(2,\mathbb{C})$, a Lorentz transformation as:

$$\tilde{\Lambda}_A : H(2,\mathbb{C}) \longrightarrow H(2,\mathbb{C}), V \mapsto AV\overline{A}^T$$

We can view this as a map:

$\tilde{\Lambda} : SL(2,\mathbb{C}) \longrightarrow End(H(2,\mathbb{C})), A \mapsto \tilde{\Lambda}_A$ which is 2:1 (but not surjective), because $\tilde{\Lambda}_A = \tilde{\Lambda}_{-A}$.

Using *conj*, we get:

$$\lambda = conj \circ \tilde{\Lambda} : SL(2,\mathbb{C}) \longrightarrow End(\mathbb{R}^4), A \mapsto \Lambda_A$$

Thus, its image are matrices:

$$\Lambda_A = \lambda(A) = f^{-1} \circ \tilde{\Lambda}_A \circ f : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}^{3,1}, v \mapsto f^{-1}(Af(v)A^\dagger)$$

First of all, we note that the transformations are indeed 4×4-matrices, with components:

$$(\Lambda_A)^j_i = \frac{1}{2} tr(\sigma^j A \sigma_i A^\dagger)$$

This can be checked by writing out the map $\Lambda(A)$ in basis components (See pages 86-87 of [11]). We claim that these matrices are Lorentz transformations.

Lemma 1.39. *For every $A \in SL(2,\mathbb{C})$, the map Λ_A has the following properties:*

1. $\eta(\Lambda_A v, \Lambda_A v) = \eta(v, v)$
2. $\det(\Lambda_A) = 1$
3. $\det((\Lambda_A)^0_0) > 0$

Proof. For 1), we just fill in:

$$\eta(\Lambda_A v, \Lambda_A v) = -\det(f(\Lambda_A v)) = -\det(Af(v)\overline{A}^t) =$$

$$-\det(A)\det(f(v))\det(A^\dagger) = -\det(f(v)) = \eta(v,v)$$

■

For 2), we write the inner product in matrix notation: $\eta(v,w) = v^t \eta w$, with:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this notation, the first property is:

$$v^t \Lambda_A^t \eta \Lambda_A v = v \eta v \text{ for every } v \in \mathbb{R}^4$$

Taking the determinant of this equation, we see:

$$\det(\Lambda_A)^2 = \det(\Lambda_A^t)\det(\eta)\det(\Lambda_A) = \det(\Lambda_A^t\eta\Lambda_A) = \det(\eta) = 1$$

Now, for $A = \mathbb{I}$, $\Lambda_{\mathbb{I}} = \mathbb{I}$, so $\det(\Lambda_{\mathbb{I}}) = 1$. If there were a $A \in SL(2,\mathbb{C})$ for which $\det(\Lambda_A) = -1$, then there would have to be a discontinuous jump in the determinant from 1 to -1. This is impossible as the determinant is a continuous function. Thus: $\det(\Lambda_A) = 1$ ■

For the last property, we use that:

$$(\Lambda_A)^0_0 = \frac{1}{2}tr(A\overline{A}^T) = \frac{1}{2}(|a_{11}|^2 + |a_{22}|^2) > 0 \blacksquare$$

□

Therefore, indeed, $\Lambda_A \in SO^+(3,1)$. We have just proven that: $\lambda(SL(2,\mathbb{C})) \subset SO^+(3,1)$. For an equality, we need λ to be surjective. Just as you can write out any $R \in SO(3)$ by Euler rotations, you can also give an explicit form for any Lorentz matrix. Using this, you can directly construct an $A \in SL(2,\mathbb{C})$, such that $\lambda(A)$ is of that form. This is proven in [12, p. 73]. □

We note that its differential at the identity is a Lie algebra isomorphism (see [1, p. 75] for details):

$$\lambda_* : \mathfrak{sl}(2,\mathbb{C}) \xrightarrow{\sim} \mathfrak{so}^+(3,1)$$

Chapter 2

Electrons and photons in Minkowski space

2.1 The local view

In chapter 1, we looked at constructions on general vector spaces V. We then chose $V = T_pM$ (or the cotangent space) at a certain point $p \in M$. Now, we are going to zoom out a little bit and consider a small region $U \subset M$, such that there is a chart:

$$\chi : U \xrightarrow{\sim} \mathbb{R}^m$$

As such, U is essentially $\mathbb{R}^m$ and $\mathbb{R}^m$ has global coordinates $x_1, ..., x_m$. Thus, at this level, you do not really need differential geometry. Certainly not if there is a flat metric:

Definition 2.1. *$\mathbb{R}^m$ has a* ***flat metric*** *if, for every $p \in \mathbb{R}^4$, $T_p\mathbb{R}^4$ has the same inner product g.*

An example of a space with a flat metric is:

Definition 2.2. ***Minkowski space*** *is $\mathbb{R}^4$ with, for every $p \in \mathbb{R}^4$, , the same inner product on $T_p\mathbb{R}^4$ (which has standard basis $\partial_t, \partial_x, \partial_y, \partial_z$): $\eta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$*

On Minkowski space, particles of the Standard Model satisfy certain equations. For example, the photon satisfies **Maxwell's equations** and the electron satisfies the **Dirac equation**. In most physics textbooks, these equations are written in coordinate form. However, to prepare our equations for the complete zooming out that we do in chapter 3, we express them with tools that are defined by coordinates, but can easily be extended to coordinate-independent versions, such that we can still write the laws of physics on a general spacetime. Differential geometry gives us these tools. As we shall see in this chapter, the electric and magnetic field can be written as **k-forms** and the curl and the divergence will be the same operator, called the **exterior derivative**.

2.1.1 Local vector fields and forms

Because we have global coordinates on $\mathbb{R}^m$, we also have global tangent vectors, meaning:

For every $p \in \mathbb{R}^m$, $T_p\mathbb{R}^m$ has the same basis: $\partial_1, ...\partial_m$. Now, a **vector field** is a function that chooses at every $p \in \mathbb{R}^m$ a vector in $T_p\mathbb{R}^m$ in a smooth way. Using our global basis, we can therefore write any vector field as: $X = X^i\partial_i$, where $X^i \in C^\infty(\mathbb{R}^m)$. We denote the space of these local vector fields as: $\mathfrak{X}(\mathbb{R}^m)$.

In much the same way, we can also make smooth choices of k-vectors. These are called k-forms. In the standard basis, we can write any k-form on $\mathbb{R}^m$ as:

Definition 2.3. *On $\mathbb{R}^m$ with coordinates $x_1, ..., x_m$, a **k-form** can be written as: $\omega = \omega^{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k}$ for any $\omega^{i_1,...,i_k} \in C^\infty(\mathbb{R}^m)$.*

We will denote the space of all these forms as: $\Omega^k(\mathbb{R}^m)$.

2.2 Differentiation

In physics, we know how to differentiate elements of $\mathfrak{X}(\mathbb{R}^3)$. We just take the curl. We can generalise this differentiation to k-forms. For this, we introduce an operator d, which will be properly defined in Chapter 3. For now, all we need to know is that:

Definition 2.4. *Let $\omega \in \Omega^k(\mathbb{R}^m)$. Then $\omega = \omega^{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_m}$ with $\omega^{i_1,...,i_k} \in C^\infty(\mathbb{R}^m)$. Then the **exterior derivative on** $\mathbb{R}^m$ is the map $d : \Omega^k(\mathbb{R}^m) \longrightarrow \Omega^{k+1}(\mathbb{R}^m)$, such that:*

$$d\omega = d\omega^{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_m} = \sum_j \frac{\partial \omega^{i_1,...,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_m}$$

Using this definition, it can be readily checked that $d \circ d = 0$.

Example 2.5. *Given the 2-form* $\eta = xdx \wedge dy + y^3 dz \wedge dx$*:*

$$d\eta = d(xdx \wedge dy + y^3 dz \wedge dx) = dx \wedge dx \wedge dy + 3y^2 dy \wedge dz \wedge dx = 3y^2 dy \wedge dz \wedge dx$$

Using the Hodge star operator from Chapter 1, we can define the following tool:

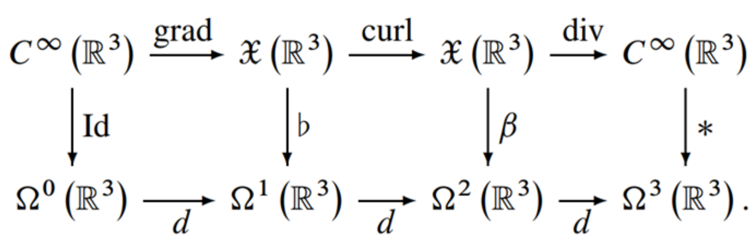
Definition 2.6. *The* ***exterior coderivative*** $\delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ *is defined as* $\delta = \star d \star$.

Together with the exterior derivative, we produce:

Definition 2.7. *The* ***Laplacian*** $\Delta : \Omega^k(\mathbb{R}^4) \longrightarrow \Omega^k(\mathbb{R}^4)$ *is defined as:* $\Delta = d\delta + \delta d$*, which, if written out, equals* $\Delta = -\partial_t \partial^t + \nabla^2$

2.2.1 Linking d to grad, curl and div.

In physics, an electromagnetic field on Euclidean space $\mathbb{R}^3$ is usually seen as a vector field. However, to let d operate on our field, we need to change the vector field into a covector fields. This is what is done in the following diagram from page 368 of Lee [8], and as a result the grad, curl and div can all be described by the De Rham operator:



Here, $\flat$, β and $*$ are the following isomorphisms:

1. $$\flat(X^i \frac{\partial}{\partial x_i}) = X^i dx_i$$

2. $$\beta(X) = i_X(dx \wedge dy \wedge dz) = X^x dy \wedge dz + X^y dz \wedge dx + X^z dx \wedge dy$$

3. $$*(f) = f dx \wedge dy \wedge dz$$

2.3 Integration

Integration on $\mathbb{R}^m$ in differential geometry is just the same as in multivariable calculus, except that the notation is different. For example, $dx_1...dx_n$ turns into a volume form:

$$vol_m = dx_1 \wedge \cdots \wedge dx_m$$

With this form, and assuming that $\Omega(\mathbb{R}^m)$ is a Hilbert space, we can define an inner product on $\Omega(\mathbb{R}^m)$ by:

$$\langle \omega, \eta \rangle_{L_2} = \int_{\mathbb{R}^m} \langle \omega, \eta \rangle_{\Lambda(\mathbb{R}^4)} vol_m \text{ for all } \omega, \eta \in \Omega(\mathbb{R}^m).$$

Here, $\langle \omega, \eta \rangle$ is the inner product on $\Lambda(\mathbb{R}^m)$ as defined in chapter 1. This depends on an inner product g on $\mathbb{R}^m$, which would normally be the Riemannian. For Minkowski space, however, we need the semi-Riemannian inner product η. With the Hodge star, we can also write the inner product as:

$$\langle \omega, \eta \rangle_{L_2} = \int_{\mathbb{R}^m} \omega \wedge \star\eta \text{ for any } \omega, \eta \in \Omega(\mathbb{R}^m)$$

Now we move to the most important theorem of integration, which will be used later on:

Theorem 2.8 (Stokes theorem without boundary)**.** *For any $\omega \in \Omega^k(\mathbb{R}^m)$:*

$$\int_{\mathbb{R}^m} d\omega = 0$$

2.3.1 Lagrangians

We have introduced integration for one purpose only: Lagrangians. For the electromagnetic interaction on Minkowski space, this is a map $\mathcal{L}(A, \Psi) : U \longrightarrow \mathbb{R}$, where U is the considered space and $A \in \Omega^1(\mathbb{R}^4)$ and $\Psi : U \longrightarrow \mathbb{C}^4$ are respectively the electromagnetic and electron field on $\mathbb{R}^4$. The recipe for Lagrangians is to define an inner product for the fields. However, because we like the Lagrangian to be invariant under certain transformations, this inner product has to be chosen carefully. For the electron-photon interaction, the two invariances are **Lorentz** and **gauge invariance**. To move from Lagrangians to equations, we first define the **action**:

$$\mathcal{S}(A, \Psi) = \int_{\mathbb{R}^4} \mathcal{L}(A, \Psi) vol_4$$

Then:

Definition 2.9. *The **action principle** says that, given a Lagrangian $\mathcal{L}(A, \Psi)$ depending on fields A and Ψ, then the equations that these fields have to satisfy are given by:*

$$\frac{d}{dt}\mathcal{S}(A, \Psi + t\sigma)|_{t=0} \ for\ \sigma : \mathbb{R}^4 \longrightarrow \mathbb{C}^4$$

$$\frac{d}{dt}\mathcal{S}(A + t\omega, \Psi)|_{t=0} \ for\ \omega \in \Omega^1(\mathbb{R}^4)$$

This principle also holds for other gauge and matter fields. First, we will look at Lagrangians for a photon and an electron separately. This chapter will then finish with the coupled Lagrangian.

2.4 The photon

The laws of electromagnetic interaction, whose force carrier is the photon, are given by the Maxwell equations. On some open vacuum in $\mathbb{R}^3$ (with an implicit time-coordinate), they are:

1. $\nabla \cdot \mathbf{B} = 0$
2. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
3. $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
4. $\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$

Here, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic vector fields and $\rho \in C^\infty(\mathbb{R}^4)$ and $\mathbf{J}$ (a vector field on $\mathbb{R}^3$) are the charge density and the charge current. They have to satisfy the **continuity equation**: $\nabla \cdot \boldsymbol{J} = -\partial_t \rho$. In classical electrodynamics on $\mathbb{R}^4$, we are taught that we can always choose a scalar potential ϕ and a vector potential $\mathbf{A}$, such that:

$$\mathbf{E} = \nabla \varphi, \ \ \mathbf{B} = \nabla \times \mathbf{A}$$

In that way, the first two equations are immediately satisfied and the last two become:

1. $\nabla^2 \varphi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$
2. $\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\mathbf{A} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\right) = -\mu_0 \mathbf{J}$

In this section, we will show that, in differential geometry terms, these equations can be summarised as:

$$d \star F_A = \star J$$

2.5 The Electromagnetic Lagrangian

For deriving the above equation, we first want to formulate a Lagrangian. In classical field theory, the electromagnetic Lagrangian on $\mathbb{R}^4$ is written in the following form:

$$\mathcal{L} = \frac{1}{4} F_\mu F^\mu + J^\mu A_\mu$$

Here, F is the electro magnetic field tensor, J the conserved current and A the four-potential. The first term can be seen as the density of the EM-field and the second the interaction of the charge with the field [15, p. 98].

Using our tools, we can rewrite A and J as 1-forms:

$$A = \varphi dt + A_x dx + A_y dy + A_z dz \text{ with } \phi, A_i \in C^\infty(\mathbb{R}^4)$$

$$J = -(\rho dt + J_x dx + J_y dy + J_z dz) \text{ with } \rho, J_i \in C^\infty(\mathbb{R}^4) \text{ with } d \star J = 0 \text{ (the continuity equation)}$$

Now, we can take the differential of A and call it:

$$F_A = dA = d\varphi \wedge dt + dA_x \wedge dx + dA_y \wedge dy + dA_z \wedge dz \in \Omega^2(\mathbb{R}^4)$$

To see how this F_A models the electric and magnetic fields we write out the differentials of dA:

$$F_A = \sum_{i=1}^{3} (\frac{\partial \varphi}{\partial x_i} - \frac{\partial A_i}{\partial t}) dx_i \wedge dt + \sum_{i<j} (\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}) dx_i \wedge dx_j = E \wedge dt + B$$

Using the differential form notation, the Lagrangian is:

$$\mathcal{L} vol_4 = \frac{1}{4} F_A \wedge \star F_A + J \wedge \star A$$

We call the associated action the perturbed Yang-Mills action:

$$YM_P(A) = \int_{\mathbb{R}^4} \frac{1}{4} F_A \wedge \star F_A + J \wedge \star A = \langle F_A, F_A \rangle_{L_2} + \langle J, A \rangle_{L_2}$$

For this integration over A, J and F_A to be possible, we assume they are in a Hilbert space. To get to an equation, we determine its critical points.

Proposition 2.10. *The critical points of $YM_P(A)$ are those $A \in \Omega^1(\mathbb{R}^4)$, such that: $d \star dA = \star J$*

Proof. Thus we vary A by $t\omega \in \Omega_1(\mathbb{R}^4)$:

$$YM_P(A + t\omega) = \int_{\mathbb{R}^4} \frac{1}{4} (F_A + td\omega) \wedge \star (F_A + td\omega) + J \wedge \star A + tJ \wedge \star \omega$$

$$= YM_P(A) + \int_{\mathbb{R}^4} \frac{1}{4}(t\omega \wedge \star F_A + F_A \wedge t \star d\omega + t^2 d\omega \wedge \star d\omega) + tJ \wedge \star\omega$$

Differentiating with respect to t, we get:

$$\frac{d}{dt} YM_P(A + t\omega)|_{t=0} = \int_{\mathbb{R}^4} \frac{1}{4}(d\omega \wedge \star F_A + F_A \wedge \star d\omega) + J \wedge \star\omega = \int_{\mathbb{R}^4} \frac{1}{2} d\omega \wedge \star F_A + \omega \wedge \star J$$

We note that: $d(\omega \wedge \star F_A) = d\omega \wedge \star F_A + \omega \wedge d \star F_A$. Thus:

$$\int_{\mathbb{R}^4} \frac{1}{2} d\omega \wedge \star F_A = \int_{\mathbb{R}^4} \frac{1}{2} d(\omega \wedge \star F_A) - \int_{\mathbb{R}^4} \frac{1}{2} \omega \wedge d \star F_A$$

Because $\mathbb{R}^4$ does not have a boundary, Stokes' theorem gives us:

$$\int_{\mathbb{R}^4} \frac{1}{2} d(\omega \wedge \star F_A) = 0$$

Thus:

$$\frac{d}{dt} YM_P(A + t\omega)|_{t=0} = \int_{\mathbb{R}^4} \frac{1}{2} \omega \wedge (\star J - d \star F_A)$$

Critical points of this action are therefore when:

$$d \star F_A = \star J \tag{2.1}$$

□

2.5.1 Maxwell's equations revisited

Now we claim:

Lemma 2.11. *Maxwell's equations are: $d \star F_A = \star J$.*

Proof. If we Hodge star both sides of these equation: $\delta F_A = J$. Filling in $F_A = dA$, this equals: $\delta F_A = \delta dA = \Delta A - d\delta A$. The equation can therefore be written as: $\Delta A - d\delta A = J$. To get to Maxwell's equations we compute δA:

$$\star A = \varphi dx \wedge dy \wedge dz - A_x dt \wedge dy \wedge dz + A_y dt \wedge dx \wedge dz - A_z dt \wedge dx \wedge dy$$

Therefore:

$$d \star A = (\partial_t \varphi + \partial_x A_x + \partial_y A_y + \partial_z A_z) dt \wedge dx \wedge dy \wedge dz$$

In conclusion:

$$\delta A = -(\partial_t \varphi + \partial_x A_x + \partial_y A_y + \partial_z A_z) = -(\partial_t \varphi + \nabla \cdot A)$$

Now, we want to separate the time-component. Therefore, we split: $d\delta A = \partial_t \delta A dt + d|_{\mathbb{R}^3} \delta A$, with $d|_{\mathbb{R}^3}$ the derivative of the space coordinates. We also split the 1-forms A and J into: $A = A_t dt + A|_{\mathbb{R}^3}$. Thus, we can write:

$$\Delta A - d\delta A + J = (\Delta\varphi - \partial_t \delta A + \rho)dt + \Delta A|_{\mathbb{R}^3} - d|_{\mathbb{R}^3}\delta A + J|_{\mathbb{R}^3} = 0$$

Now that we have separated the time-form, we can equate:

1. $\Delta\varphi - \partial_t \delta A = -\rho$
2. $\Delta A|_{\mathbb{R}^3} - d|_{\mathbb{R}^3}\delta A = -J|_{\mathbb{R}^3}$

We note: $\Delta\varphi - \partial_t \delta A = -\partial_t\partial^t\varphi + \nabla^2\varphi + \partial_t\partial^t\varphi + \partial_t(\nabla\cdot A) = \nabla^2\varphi + \frac{\partial}{\partial t}(\nabla \cdot A)$ and that $d|_{\mathbb{R}^3}\delta A$ is the same term as: $-\nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right)$. So indeed, $d\star F_A = \star J$ give us Maxwell's equations. □

2.5.2 The gauge freedom

In our discussion of the electromagnetic Lagrangian, we started out with the **four-potential** A. However, these potentials are not unique. We could also have chosen:

$$\phi - \frac{\partial\Lambda}{dt},\ \mathbf{A} + \nabla\Lambda,\ \text{for any } \Lambda \in C^\infty(\mathbb{R}^4)$$

Then, these shifted potentials would give us exactly the same electromagnetic field. In other words: classical particles do not *feel* gauge transformations. This is why physicists first thought that potentials were only a useful tool to simplify computations and that it were the electric and magnetic field that truly existed.

With quantum mechanics, this view changed. The **Aharonov-Bohm experiment** showed that particles, which now are modeled by probability waves, *do* feel gauge transformations. Summarizing their results, they showed that if two electrons move in areas with different magnetic potentials, $\vec{A}$ and $\vec{A}'$, with the property $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, then their probability wave functions are separated by a phase:

$$\Psi'(p) = e^{-i\Lambda(p)}\Psi(p) \text{ [14, p. 147]}$$

But, given an electromagnetic field F, we may ask: for all four-potentials A with $F = dA$, which is the one that truly exists? This question will be answered in the next chapter, where we shall see that the four-potential and all its gauge transformations are described by one mathematical object, called a **connection**, that is defined over $\mathbb{R}^4$ with circles glued to every point, which is a **principal bundle**.

Even though quantum particles might feel a change of gauge, the electromagnetic Lagrangian is the same for every choice of gauge:

Lemma 2.12. *$YM(A)$ is gauge invariant.*

Proof.

$$YM_P(A + d\Lambda) = \int_{\mathbb{R}^4} \frac{1}{4} F_A \wedge \star F_A + J \wedge \star(A + d\Lambda)$$

The only extra term is therefore:

$$\int_{\mathbb{R}^4} d\Lambda \wedge \star J = \int_{\mathbb{R}^4} d(\Lambda \wedge \star J) - \int_{\mathbb{R}^4} \Lambda \wedge d \star J$$

The first term is zero, because of Stokes'. The second term is zero, because: $d \star J = 0$. In conclusion:

$$YM_P(A + d\Lambda) = YM_P(A)$$

□

2.6 The electron

2.6.1 The Dirac equation

In chapter, we constructed the Clifford algebra on a general vector space. In this section we shall motivate why this algebra is necessary for describing any Lagrangian (ergo: differential equations) for fermions. We take $\Psi \in C^\infty(\mathbb{R}^4)$ (neglecting that it should be complex-valued). Then, the Klein-Gordon equation is:

$$(\Delta + m^2)\Psi = 0$$

Here, $\Delta = -\partial_t \partial^t + \nabla^2$ can be written as: $\eta(dx_i, dx_j)\partial^i \partial^j$ with the η the Minkowski inner product on $\mathbb{R}^{4*}$ and m is a number that describes the mass of the particle.

This equation models spinless particles (e.g. pions), but not spin-1/2 particles. This and the fact that **Schrödinger's equation** has a first order time-derivative motivated Dirac to compute the square root of the Klein Gordon equation [14, p. 494].

In other words, we write $\Delta = D^2$ and want to know: what is D? Since $\Delta = \eta(dx_\mu, dx_\nu)\partial^\mu \partial^\nu$, this means we want to split $\eta(dx_\mu, dx_\nu)$ into factors $\gamma(dx_\mu)\gamma(dx_\nu)$, such that: $D = \gamma(dx_\mu)\partial^\mu$.

So can we just say $\eta(dx_\mu, dx_\nu) = \gamma(dx_\mu)\gamma(dx_\nu)$? Because η is symmetric, this would imply: $\gamma(dx_\mu)\gamma(dx_\nu) = \gamma(dx_\nu)\gamma(dx_\mu)$, which we don't know. To be safe we write:

$$\eta(dx_\mu, dx_\nu) = \frac{1}{2}(\gamma(dx_\mu)\gamma(dx_\nu) + \gamma(dx_\nu)\gamma(dx_\mu)).$$

This property is exactly the defining property for the Clifford algebra $Cl(\mathbb{R}^4, \eta)$ that we saw in Chapter 1. Thus: $\gamma(dx_\mu)$ is in $Cl(\mathbb{R}^4, \eta)$. This algebra being 16-dimensional, the wave function Ψ should be a $\mathbb{C}^4$-valued function at least. For then, we can represent $Cl(\mathbb{R}^4, \eta)$ by complex 4×4 matrices and $\gamma(v)\Psi(v)$ would be matrix multiplication. In conclusion, the square root of Klein-Gordon is:

$$(D^2 + m^2)\Psi = 0 \Rightarrow (iD + m)(-iD + m)\Psi = 0 \Rightarrow (\pm i\gamma(dx_u)\partial^\mu + m)\Psi = 0$$

The equation on the right is called the **Dirac equation** and D is called the **Dirac operator**.

2.6.2 Spin freedom

In the previous chapter, we talked about spinors rotating at half the speed of the tangent vectors: a transformation $v \mapsto \Lambda v$ for every $v \in T_p\mathbb{R}^4$ and $p \in \mathbb{R}^4$ implies a transformation $\Psi(p) \mapsto \rho_{1/2}(A)\Psi(p)$ with $\Psi(p) \in \mathbb{C}^4$. We would like the Dirac equation to be the same under a Lorentz transformation, which means:

$$(i\gamma(dx_u)\partial^\mu + m)\Psi \text{ transforms to } \rho_{1/2}(A)(i\gamma(dx_u)\partial^\mu + m)\Psi$$

In other words: we have to check that $D\Psi$ transforms as a spinor. To prove this, we need to know how $\gamma(dx_u)$ transforms.

2.6.3 Gamma matrices

In our proof of the double covering of $SO^+(3,1)$ by $SL(2,\mathbb{C})$, we talked about the following identification:

$$f_+((t,x,y,z)) = t\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} t+z & x-yi \\ x+yi & t-z \end{pmatrix}$$

For the representation of the Dirac algebra as 4×4-matrices, we also need another one:

$$f_-((t,x,y,z)) = -t\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} -t+z & x-yi \\ x+yi & -t-z \end{pmatrix}$$

Again, $\eta(v,v) = -\det(f_-(v))$. We note that both:

$$f_\pm(v)f_\mp(w) = \eta(v,w)\mathbb{I} \tag{2.2}$$

This f_- can also be used to cover the Lorentz group. Now, we define the Dirac matrices.

Theorem 2.13. *The following is a representation of the Clifford algebra $Cl(3,1)$ as complex 4×4-matrices:*

$$\gamma(v) = \begin{pmatrix} 0 & f_+(v) \\ f_-(v) & 0 \end{pmatrix}$$

Proof. We have to check that the matrices indeed satisfy the relations of the Clifford algebra:

1. It is an associative algebra.

2. $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2\eta(v,w)$.

For 1), we note that complex matrices are an associative algebra. For 2), we write:

$$\gamma(v)\gamma(w) = \begin{pmatrix} 0 & f_+(v) \\ f_-(v) & 0 \end{pmatrix}\begin{pmatrix} 0 & f_+(w) \\ f_-(w) & 0 \end{pmatrix} = \begin{pmatrix} f_+(v)f_-(w) & 0 \\ 0 & f_-(v)f_+(w) \end{pmatrix} = \eta(v,w)\mathbb{I}$$

So:

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = (\eta(v,w) + \eta(w,v))\mathbb{I} = 2\eta(v,w)$$

□

It can be readily checked that the basis elements in this representation are:

$$\gamma_i = \gamma(e_i) \begin{cases} \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} & \text{for } i = 0 \\ \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} & \text{for } i = 1,2,3 \end{cases}$$

Theorem 2.14. *Given the representation $\rho_{\frac{1}{2}}$, the Dirac matrices transform as follows:*

$$\gamma(\Lambda_A v) = \rho_{\frac{1}{2}}(A)\gamma(v)\rho_{\frac{1}{2}}(A^{-1})$$

Proof. We do a Lorentz transformation $v \mapsto \Lambda v$:

$$\gamma(\Lambda v) = \begin{pmatrix} 0 & f_+(\Lambda v) \\ f_-(\Lambda v) & 0 \end{pmatrix}$$

Consequently, we would like to know how $f_\pm$ transform. For this, we need the result from section 1.38 that any $\Lambda \in SO^+(3,1)$ can be written in the form:

$$\Lambda = f_+^{-1} \circ \tilde{\Lambda}_A \circ f_+ \text{ for some } A \in SL(2,\mathbb{C})$$

Using this:

$$f_+(\Lambda v) = f_+(f_+^{-1}(\tilde{\Lambda}_A(f_+(v)) = \tilde{\Lambda}_A(f_+(v)) = Af_+(v)A^\dagger \tag{2.3}$$

And from 2.2 we get:

$$f_-(v) = \eta(v,w)(f_+(w))^{-1}$$

Thus:

$$f_-(\Lambda v) = \eta(\Lambda v, \Lambda v)(f_+(\Lambda v))^{-1} = \eta(v,v)(Af_+(v)A^\dagger)^{-1} = A^{\dagger -1}\eta(v,v)(f_+(v))^{-1}A^{-1} = A^{\dagger -1}f_-(v)A^{-1} \tag{2.4}$$

So now, the Dirac matrices transform as:

$$\gamma(\Lambda v) = \begin{pmatrix} 0 & f_+(\Lambda v) \\ f_-(\Lambda v) & 0 \end{pmatrix} = \begin{pmatrix} 0 & Af_+(v)A^\dagger \\ A^{\dagger -1}f_-(v)A^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A & 0 \\ 0 & A^{\dagger -1} \end{pmatrix} \begin{pmatrix} 0 & f_+(v) \\ f_-(v) & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix} = \rho_{\frac{1}{2}}(A)\gamma(v)\rho_{\frac{1}{2}}(A^{-1})$$

□

Lemma 2.15. *Let $A \in SL(2,\mathbb{C})$, such that $\Lambda = \lambda(A)$. Then:* $\rho_{1/2}(A)\gamma_i\rho_{1/2}(A^{-1}) = \Lambda_i^j\gamma_j$

Proof. By definition of γ: $\gamma(\Lambda e_i) = \gamma(\Lambda_i^j e_j) = \Lambda_i^j\gamma(e_j)$. Using 2.14 we also get: $\gamma(\Lambda e_i) = \rho_{1/2}(A)\gamma_i\rho_{1/2}(A^{-1})$
Thus, indeed: $\rho_{1/2}(A)\gamma_i\rho_{1/2}(A^{-1}) = \Lambda_i^j\gamma_j$

□

Lemma 2.16. *$D\Psi$ transforms as a spinor under a Lorentz transformation.*

Proof. Doing a Lorentz transformation, means: $\partial_j' = \Lambda_j^i\partial_i$. Thus:

$$\gamma^j\partial_j' = \gamma^j\Lambda_j^i\partial_i = \rho_{1/2}(A)\gamma^i\rho_{1/2}(A^{-1})\partial_i$$

Because $\Psi \mapsto \rho_{1/2}(A)\Psi$, we get: $\gamma^j\partial_j'\Psi = \rho_{1/2}(A)\gamma^i\partial_i\Psi$.
Thus, the Dirac operator is frame-independent. □

Thus, the Dirac equation indeed transforms (under Lorentz transformation) as:

$$(D+m)\Psi \mapsto \rho_{1/2}(A)(D+m)\Psi \text{ with } A \in SL(2,\mathbb{C})$$

2.6.4 Dirac Lagrangian

As mentioned in section 2.3.1, we need the Lagrangian that describes relativistic particles to be Lorentz invariant. Thus, the terms in the Lagrangian must be **Lorentz scalars**.

Given two spinors Ψ, Φ, we would normally turn it into a complex scalar by: $\Psi^\dagger\Phi$. Unfortunately, doing a Lorentz transformation gives: $\Psi^\dagger\rho_{1/2}(A^\dagger)\rho_{1/2}(A)\Phi$, which is not equal to $\Psi^\dagger\Phi$. Therefore, we define the following inner product on $\mathbb{C}^4$: $\langle\Psi,\Phi\rangle_{\mathbb{C}^4} = \Psi^\dagger i\gamma_0\Phi$. We claim the following:

Lemma 2.17. *Let Ψ,Φ be spinors. Defining $\tilde{\Psi} = \Psi^\dagger i\gamma_0$, $\tilde{\Psi}\Phi$ is a Lorentz scalar.*

Proof. Lorentz transforming $\tilde{\Psi}\Phi$ we get: $\Psi^\dagger\rho_{1/2}(A^\dagger)i\gamma_0\rho_{1/2}(A)\Phi$. However:

$$i\rho_{1/2}(A^\dagger)\gamma_0\rho_{1/2}(A) = \begin{pmatrix} A^\dagger & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A^{\dagger-1} \end{pmatrix} = i\begin{pmatrix} A^\dagger & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} 0 & A^{\dagger-1} \\ -A & 0 \end{pmatrix} = i\gamma_0$$

So indeed: $\Psi^\dagger\rho_{1/2}(A^\dagger)i\gamma_0\rho_{1/2}(A)\Psi = \tilde{\Psi}\Phi$ □

This being settled, the Dirac Lagrangian on $\mathbb{R}^4$ is:

$$\mathcal{L} = Re(\tilde{\Psi}(D - m)\Psi)$$

Here, the first term is the **kinetic term** and the second the **Dirac mass term**. Minimalizing this Lagrangian with respect to Ψ and $\tilde{\Psi}$, we would get back the Dirac equation.

2.7 Electron-photon interaction

An electromagnetic field influences the spinors of charged particles. This coupling to the electromagnetic field is captured in a shift of the Dirac operator by the electromagnetic potential: $D_A = \gamma^j(\partial_j + A_j)$. For the electron-photon interaction, the Lagrangian would have the following form:

$$\mathcal{L}(A,\Psi)vol_4 = \frac{1}{2}F_A \wedge \star F_A + Re(\tilde{\Psi}(D_A - m)\Psi)vol_4 = \frac{1}{2}F_A \wedge \star F_A + Re(A_j\tilde{\Psi}\gamma^j\Psi + \tilde{\Psi}(\gamma^j\partial_j - m)\Psi)vol_4$$

Note that the second term looks a lot like the interaction term in the electromagnetic lagrangian 2.1. This suggests that $J = -i\tilde{\Psi}\gamma^j\Psi dx_i$ and satisfies the continuity equation [4, p. 533]. Here, the minus sign and the i are consequences of our definition of γ_j, which, mathematically, are defined with an extra factor i. Minimizing this action with respect to A and Ψ would give us the following equations:

1. $d \star F_A = \star J = -i\tilde{\Psi}\gamma^j\Psi \star dx_j$

2. $D_A\Psi = m\Psi$

We mentioned that a gauge transformation will change: $\Psi \mapsto e^{-i\Lambda}\Psi$. As such, $\tilde{\Psi}\Phi \mapsto e^{i\Lambda}\Psi^\dagger i\gamma_0 e^{-i\Lambda}\Psi = \tilde{\Psi}\Phi$. Thus, the inner product is gauge invariant.

Writing $J = -i\tilde{\Psi}\gamma^j\Psi dx_i$, we can use the result from section 3.40:

$$S(A + d\Lambda, e^{-i\Lambda}\Psi) = S(A, \Psi)$$

Chapter 3

Electrons and photons on a spacetime manifold

3.1 Spacetime

Now that we have considered the electron-photon interaction in special relativity, we will move to general relativity. In this scenario, $\mathbb{R}^4$ becomes a general spacetime manifold M, whose metric is no longer flat, but wrinkled by gravitional effects. To couple particles to this gravitational field (the wrinkled metric), we need a specific connection, called the Levi-Civita connection, which we shall see in this chapter. Without the generalised metric, there is also another reason why connections are useful.

3.1.1 Gauge/spin freedom

In the previous chapter, we remarked that if two identical particles move in areas with different magnetic potentials, $\vec{A}$ and $\vec{A}'$, with the property $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, then their probability wave functions are separated by a phase:

$$\Psi'(p) = e^{-i\Lambda(p)}\Psi(p)$$

Thus, given a magnetic field, if we choose a corresponding vector potential, without considering the gauge transformations, we would ignore the phase changes of particles.

This is all fine if the vector potential is globally defined, as then all the particles feel the same potential, but in some cases this is not possible (for example, in the case of a magnetic monopole, which is explained in detail in [10, Chapter 0]). In those scenarios, the most we can do is define potentials over smaller regions with some overlap. But how can we keep track of which potential the particle feels?

For this, we need some mathematical object that, apart from describing how charged particles move (the electromagnetic field), also describes how the phases of particles change as they move through a space with different potentials.

To keep track of a particle's phase, we construct a phase bundle. Noting that the phase of charged particles is given by $e^{i\theta} \in U(1)$, this is done by gluing circles onto a spacetime manifold M. Mathematically, the resulting space is called a principal $U(1)$-bundle over M, which we will define properly in section 3.7.

Given an electromagnetic field, and all the potentials that give us these fields, there exists a unique differential form in the phase bundle, which, depending on how you pull it back to M, gives you one of the potentials.

Although our focus remains on gluing circles, we can also construct different 'phase bundles' by gluing different groups. These different groups would then describe different interactions. For example, the matrix group $SU(2)$ models the weak interaction and $SU(3)$ the strong interaction. As we shall see, even the spin freedom of a spinor, is described by a G-bundle with $G = SL(2, \mathbb{C})$. To generalise this description of all these different fields, we will not immediately fill in the details of the group we consider and instead talk about a general G-bundle.

The goal of this chapter shall be to arrive at a Lagrangian or equivalently a set of equations that describes electron-photon interactions on the whole space-time manifold. This Lagrangian then should be both Lorentz and gauge invariant.

3.2 Fibre bundle

Given a manifold M, we can do the following: glue a space F at every point [1]. At first glance, you would expect the resulting space to be $M \times F$. However, it turns out you can tie these F's together in such a way that the bundle looks like $M \times F$ locally (or rather: an open subspace of M times F), but not necessarily globally. This construction is a **fibre bundle**. Formally, a fibre bundle has the following ingredients:

[1] In our case, either a vector space or a Lie group.

- a manifold M (called the **base space**) covered by $U_i, i \in I$.
- a manifold F, which, in our case, is a group or a vector space.
- a manifold B (the bundle) with: a surjective map $\pi : B \longrightarrow M$ and diffeomorphisms $\Phi_i : B|_{U_i} \longrightarrow U_i \times F$, called **local trivialisations**.

We will designate the bundle with: (B, M, π, F). Because of the surjective map, B is made out of fibres: $B_x = \{b \in B \mid \pi(b) = x\}$.

The most important maps on a fibre bundle are:

Definition 3.1. *A **local section** of E is a map $s : U \longrightarrow B$ with $\pi \circ s = Id|_U$ for $U \subset M$.*

Thus, a section can be viewed as choosing for each $x \in M$ a unique element (x, f) in its fibre B_x (and this in a smooth way). Also, note that s is by definition injective.

Sometimes, bundles allow sections $s : M \longrightarrow B$ defined on the whole manifold M. These are called **global sections**. The space of all global sections is denoted by: $\Gamma(B)$.

3.3 Vector bundles

Definition 3.2. *A **vector bundle** is a fibre bundle (E, M, π, V) with V a vector space and its local trivialisations satisfying the condition that $\Phi_i|_{\{x\}} : E_x \longrightarrow \{x\} \times V$ is a vector isomorphism for every $x \in U_i$.*

An unique quality of a vector bundle is that we can construct frames:

Definition 3.3. *A **local frame** $e = (e_1, ..., e_m)$ consists of smooth sections $e_i : U \longrightarrow E|_U$, such that for every $x \in U$, $e_1(x), ..., e_m(x)$ is a basis of E_x.*

A frame allows us to view E locally as a vector space V. As such, they are equivalent to local trivialisations:

Lemma 3.4. *Every local frame on E induces a local trivialisation of E and vice versa.*

Proof. From a local frame e on U we can construct the local trivialisation: $\Psi : U \times \mathbb{R}^k \xrightarrow{\sim} E|_U, (x, v) \mapsto v^i e_i(x)$. Vice versa: Given a $\Psi : U \times \mathbb{R}^k \xrightarrow{\sim} E|_U$, $e_i(x) = \Psi(x, e_i)$ defines a local frame on U. □

As a consequence, in our definition of the vector bundle, instead of the local trivialisations, we could have required the existence of local frames. A similar argument will be made in Chapter 3, with a principal bundle.

An important property of frames is that it allows us to decompose any section $f \in \Gamma(U)$ as:

$$f = f^i e_i, \ f^i \in C^\infty(M)$$

To illustrate sections and frames, here is a picture of a trivial vector bundle. Here, Φ is trivialisation of $\pi^{-1}(U)$ with U an open subset of M.

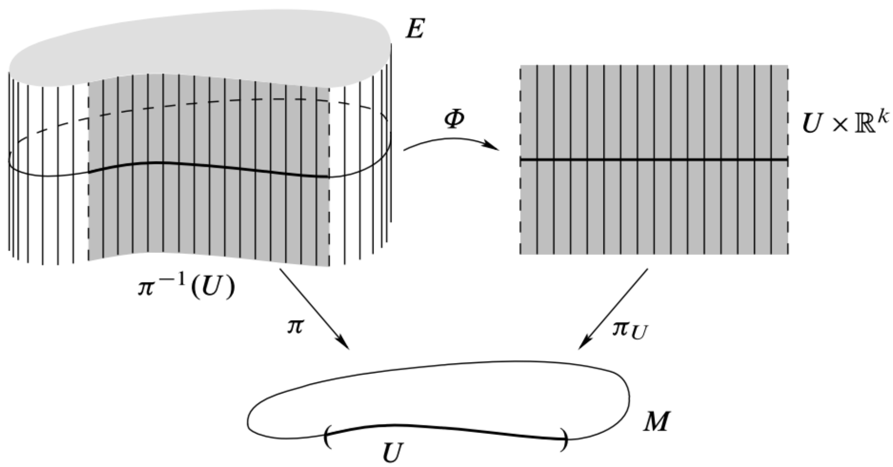


Figure 3.1: Vector bundle from [8, p. 250]

Remark: if there exists a global frame on M, then the bundle is isomorphic to the trivial bundle $M \times V$. Therefore, the frames capture how non-trivial a fibre bundle is.

Now, given two local frames with a non-empty overlap, we need to know how to transition from one frame to another:

Proposition 3.5. *Consider a local frame $e = (e_1, ..., e_m)$ on U and a frame $d = (d_1, ..., d_m)$ on $V \subset M$, such that $U \cap V \neq \varnothing$. Then there is a function: $\tau : U \cap V \longrightarrow GL_m(\mathbb{R}), x \mapsto \tau(x)$ such that:*

$$d_j(x) = \tau_j^i(x) e_i(x) \text{ for } x \in U \cap V$$

Proof. We know from the previous lemma that e and e' both induce local trivialisations, which we call Φ and Φ' with $\Phi^{-1}(x, v) = v^i(e_i)_x$ (same for Φ' and e'). Now we define the projection: π_1 :

$U \cap V \times \mathbb{R}^m \longrightarrow U \cap V$. Then:

$$\pi_1 \circ (\Phi \circ \Phi'^{-1}) = \pi_1$$

This means that: $\Phi(\Phi'^{-1}(x, v)) = (x, \tau(x)v)$, with $\tau : U \cap V \longrightarrow GL_m(\mathbb{R})$ smooth. For more details, see **Lemma 10.5** on page 252 of Lee [8]. □

We call the function described above the **transition function**. In a spacetime manifold, these functions allow us to move, for example, from an accelerating frame to an inertial frame.

In a fibre, a transition function is nothing else than a **basis transformation**. As such: $d_j(x) = \tau_j^i(x)e_i(x)$ implies that any $v \in E_x$ can be written as vector $v = v_d^i d_i$ or $v = v_e^i e_i$, such that, in vector representation: $\mathbf{v_d} = \tau(x)^{-1}\mathbf{v_e}$.

3.3.1 Induced vector bundles

As every fiber E_x of E is a vector space, it has a dual E_x^*. Thus, we can define:

Definition 3.6. *The* ***dual bundle of E*** *is the vector bundle E^* with fibres: $(E^*)_x = E_x^*$*

In much the same manner, we can also define:

Definition 3.7. *Given two vector bundles E and F, the* ***tensor product bundle*** *$E \otimes F$ is the bundle with fibres: $(E \otimes F)_x = E_x \otimes F_x$ for every $x \in M$. The smooth structure is given by smooth local frames: $e \otimes f$ for $e \in \Gamma(E|_U)$ and $f \in \Gamma(E|_U)$.*

3.3.2 The tangent bundle

For each M, there is always a natural vector bundle, called the **tangent bundle**:

$$TM = \bigcup_{p \in M} \{(p, v) | \ v \in T_pM\}$$

Its dual, called the **cotangent bundle** is:

$$T^*M = \bigcup_{p \in M} \{(p, v) | \ v \in T_p^*M\}$$

Vector fields of M are sections of the bundle TM. We denote the space by: $\mathfrak{X}(M) = \Gamma(TM)$. Similarly, **1-forms of** M are sections of T^*M, whose space we denote by $\Omega^1(M)$. Just as we discussed in Chapter 2, but then for the tangent bundle of $\mathbb{R}^4$, vector fields and 1-forms are smooth choices of (co-)tangent vectors for every $p \in M$.

The following operation allows us to send vector fields from one manifold to another:

Definition 3.8. *Given a map $F : M \longrightarrow N$, the* ***push forward*** *is the map:*

$$F_* : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N), X \mapsto F_*(X)$$

Here, $F_*(X)$ is the vector field defined by: $F_*(X)_{F(p)} = (dF)_{F(p)}(X_p)$ for every $p \in M$.

For a vector $v \in T_pM$, this map is: $F_*(v) = dF_{F(p)}(v)$.

Because M is a manifold, there are always natural frames on the tangent bundle (or a coframe on the cotangent bundle) induced by charts. We call this the **coordinate frame**. To define it, we first recall the subtleties of a chart.

Take an open $U \subset M$, such that there exists a chart:

$$\Phi : U \xrightarrow{\sim} \mathbb{R}^m$$

$\mathbb{R}^m$ has the standard basis $e_1, ..., e_m$. This allows us to define local coordinate functions $x_1, ..., x_m \in C^\infty(U)$, with the property: $\Phi(x) = x^i(x)e_i$ for every $x \in U$. By differentiating, Φ induces a local trivialisation of the tangent bundle:

$$(d\Phi) : TM|_U \longrightarrow T\mathbb{R}^m|_U \cong U \times \mathbb{R}^m, (d\Phi)(v_x) = (x, (d\Phi)_x(v_x))$$

Now, the **local coordinate frame** is:

$$e_\Phi = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}), \text{ defined by: } d\Phi((\frac{\partial}{\partial x_i})_p) = (p, e_i) \text{ for every } p \in M$$

From this frame, we can construct the **local coordinate coframe**:

$$e^*_\Phi = (dx_1, ..., dx_m), \text{ defined by: } dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$$

Just as Φ assigned U with coordinates, these frames give coordinates to either $TM|_U$ or its dual.

Example 3.9 (The tangent bundle of $\mathbb{S}$)**.** *The tangent bundle of a circle $\mathbb{S} \subset \mathbb{R}$ has the following global frame:*

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{S})$$

Thus, $T\mathbb{S} \simeq \mathbb{S} \times \mathbb{R}$.

Here is a picture of this bundle:

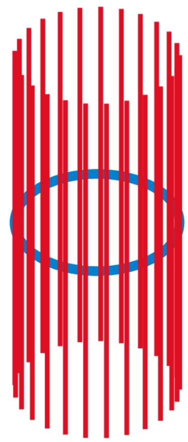


Figure 3.2: The tangent bundle of $\mathbb{S}$. Source: `https://en.wikipedia.org/wiki/Tangent_bundle`

Example 3.10 (The tangent bundle of $\mathbb{S}^2$)**.** *For $T\mathbb{S}^2$, a global frame does not exist:*

Lemma 3.11. *The bundle $T\mathbb{S}^2$ is not trivialisable.*

Proof. Given a global frame of $\mathbb{S}^2$: $e = (e_1, e_2)$. We look at the vector field $e_1 \in \mathfrak{X}(\mathbb{S}^2)$. Because of the **hairy ball theorem** [8, p. 435], there will always be a $p \in M$ for which $(e_1)_p = 0$. This means that in this point $e_p = (0, (e_2)_p)$, and thus is *not* a basis of $T_p\mathbb{S}^2$. Thus, e is not a global frame. ↯ □

3.4 Differential forms

In the first chapter, we constructed a set of k-vectors, $\Lambda^k(V)$ from a vector space V. Filling in $V = T_pM$ as either the tangent or cotangent space, we define the following vector bundle:

$$\Lambda^k T^* M = \bigcup_{p \in M} \{(p, v) | \; v \in \Lambda^k T_p^* M\}$$

This is a vector bundle over M that is isomorphic to $U \times \mathbb{R}^{\binom{m}{k}}$ locally. Its sections are called **k-forms** (more formally: differential forms of degree k) and we denote the space of all k-forms (which is a vector space) by:

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M)$$

Because T_p^*M is m-dimensional, $\Lambda^k(T_pM) = 0$ for $k > m$. Therefore:

$$\Omega(M) = \bigoplus_{k=0}^{m} \Omega^k(M)$$

This is an associative anti-commutative graded algebra, whose multiplication is defined pointwise by: $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p \in \Lambda(T_pM)$ for any $\omega, \eta \in \Omega(M)$ and $p \in M$. Here, the wedge on the right is the wedge product from the exterior algebra.

Now, we are going to define some important operations with differential forms:

Definition 3.12. *For any k-form $\omega \in \Omega^k(N)$, its **pullback by a map** $F : M \longrightarrow N$ is the map:*

$$F^* : \Omega^k(N) \longrightarrow \Omega^k(M) \text{ such that } (F^*\omega)_p = \omega_{F(p)}(dF)_p$$

$F^*\omega$ eats tangent vectors of M, sends them through F to tangent vectors of N and then ω turns them into real numbers. The next operation turns k-forms into $k-1$-forms:

Definition 3.13. *For any $\omega \in \Omega^k(M)$ and $X_1, ..., X_{k-1} \in \mathfrak{X}(M)$ the **interior product** ω **by** V $i_V(\omega)$ is the map:*

$$i_V(\omega)(X_1, ..., X_{k-1}) = \omega(V, X_1, ..., X_{k-1})$$

3.4.1 Differentiation: the De Rham operator

To differentiate our k-forms, we need the De Rham operator d, of which we saw a local version in the previous chapter. It sends k-forms to $k+1$-forms and has three defining properties which we cite from pages 104-105 of Crainic's notes [2]:

Definition 3.14. *The **exterior derivative** on M is the unique operator that satisfies the properties:*

1. *d is $\mathbb{R}$-linear and, on 0-forms (functions), it is the usual differential of functions.*
2. *$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{kl}\omega \wedge d\eta$ for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$.*
3. *$d \circ d = 0$*

If we express k-forms in coordinates (which is always possible locally), the De Rham operator works just the same as described in Chapter 2.

3.4.2 Vector-bundle valued forms

Given a vector bundle E, we can also define k-forms with a twist:

Definition 3.15. *Let E be a vector bundle over M. Then a **vector-bundle valued k-form** is a section of the bundle: $E \otimes \Lambda^k(T^*M)$.*

We denote the space of these forms as: $\Omega^k(M,E) = \Gamma(E \otimes \Lambda^k(T^*M))$. A special type of such a form is the **Lie-algebra valued k-form**, in which case E is the trivial vector bundle $M \times \mathfrak{g}$. The space of these forms is denoted by: $\Omega^k(M,\mathfrak{g})$. A gauge field will be such a form, as will be defined in section 3.8.2.

3.5 Metric

Because any vector space can be endowed with an inner product, we can define a smooth choice of inner products for general vector bundles:

Definition 3.16. *Let E be a vector bundle over M. Then, a **bundle metric of E** is a section of the vector bundle $E^* \otimes E^*$, with the property that, for every $p \in M$, g_p is an inner product on E_p, as defined in chapter 1.*

Now, the metric on a tangent bundle is:

Definition 3.17. *A **metric g on M** is a bundle metric on the tangent bundle with g_p an inner product on T_pM.*

It is a map:

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M), (X,Y) \mapsto g(X,Y)$$

Or on the cotangent bundle:

$$g : \Omega^1(M) \times \Omega^1(M) \longrightarrow C^\infty(M), g(\omega,\eta) \mapsto (\omega,\eta)$$

Example 3.18. *As we saw in chapter 2, on Minkowksi space, this metric was:*

$$\eta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$$

*On a spacetime manifold M, this is generalised to a metric g, such that, at every point $p \in M$, g_p has **signature** $(-+++)$, as mentioned in section 1.9.*

Given a metric g we can also extend the Hodge star operator from Chapter 1:

Definition 3.19. *Let g be a metric on M. Then, the* ***Hodge star operator*** *is a map $\star : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$, with the defining property: for any $\omega \in \Omega^k(M)$ and $p \in M$, $(\star\omega)_p = \star\omega_p$*

Remark: Here, the Hodge star on the right is that for T_p^*M, which we defined in section 1.7.2 of Chapter 1. It fully depends on the inner product g_p on T_p^*M.

3.5.1 Orientation

In chapter 1, we defined an **orientation** on a vector space V as a choice of one of the two connected components of $\Lambda^n(V) - \{0\}$. Setting $V = T_p^*M$, we can extend this to an orientation on M:

Definition 3.20. *M is* ***orientable*** *if there exists an m-form $\omega \in \Omega^m(M)$, such that: $\omega_p \in \Lambda_+^m(T_p^*M)$ for every $p \in M$.*

Orientability therefore means we can make a smooth choice of orientation for each (co-)tangent space. Having endowed each tangent space with an orientation, we can define:

Definition 3.21. *A frame e of $TM|_U$ is* ***oriented*** *if e_p is an oriented basis of T_pM for every $p \in U$.*

With this oriented frames, we can construct a volume form, which we will need for integration.

Definition 3.22. *A* ***volume form*** *μ is a $\mu \in \Omega^m(M)$ with the property that $\mu_p \neq 0$ and $\mu_p \in \Lambda_+^m(T_p^*M)$ for all $p \in M$.*

A volume form induced by the metric g is then (in an oriented coordinate frame $dx_1, ..., dx_m$):

$$vol_g = \sqrt{|g|} dx_1 \wedge \cdots \wedge dx_m$$

For Minkowski space, $g = \eta$, such that $|g| = 1$. Then, the volume form is what we saw in Chapter 2: $dt \wedge dx \wedge dy \wedge dz$.

Just as we chose an orientation for every tangent space, we also want to choose a global time-orientation:

Definition 3.23. *M with metric g of signature $(- + ++)$ is* ***time-orientable****, if there exists a vector field $T \in \mathfrak{X}(M)$, such that: $g(T,T) < 0$.*

3.5.2 Spacetime manifold

Now, we can define a spacetime manifold:

Definition 3.24. *A spacetime manifold M is a four-dimensional manifold with a metric g that has signature $(-+++)$ at every $p \in M$ and is orientable and time-orientable.*

With this definition of spacetime, we can select a Lorentz basis (discussed in section 1.9 of Chapter 1) for every fibre T_pM. Now, for a small bundle of fibres, $TM|_U$ with $U \subset M$, we want to select Lorentz bases of T_pM for every $p \in U$ in a smooth way:

Definition 3.25. *A Lorentz frame is a local frame e of $TM|_U$, such that: $e_p \in SO^+(T_pM)$ for every $p \in M$.*

Locally, you don't need orientability and time-orientability to choose such frames. Therefore, these conditions could be relaxed, which is discussed in [7]. However, we do need this conditions to construct a bundle of Lorentz frames as we shall see in section 3.7.5.

Example 3.26. *An easy example of a spacetime manifold is Minkowski space, which is obviously orientable. In that case, ∂_0 is a vector field with $\eta(\partial_0, \partial_0) < 0$, thus $\mathbb{R}^{3,1}$ is time-orientable. Then, $(\partial_0, \partial_1, \partial_2, \partial_3)$ is a global Lorentz frame. Other examples can be found in Chapter 3 of [11].*

3.5.3 Integration

In chapter 1, we defined an inner product on $\Lambda(T_p^*M)$, which fully depended on the inner product g_p on T_p^*M. Extending g_p to a metric g, we can define an inner product on $\Omega^k(M)$ by:

$$\langle \omega, \eta \rangle_{L_2} = \int_M \langle \omega, \eta \rangle vol_g \text{ for all } \omega, \eta \in \Omega(M).$$

Here, the inner product on the right is the inner product on $\Lambda(T_pM)$ for all $p \in M$. Using the Hodge star operator, we can also write the inner product as:

$$\langle \omega, \eta \rangle_{L_2} = \int_M \omega \wedge \star\eta \text{ for any } \omega, \eta \in \Omega(M)$$

3.6 Connection on a vector bundle

Because of the generalised metric g in general relativity, normal coordinate differentiation is no longer possible, as moving particles can feel a varying gravitational field, that bends their coordinates along their trajectory. In order to keep track of these grooves in spacetime, we need a **vector bundle connection**.

Definition 3.27. *A connection on a vector bundle E is a map:*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

with the properties:

1. $\nabla_X(fs_1 + s_2) = (Xf)s_1 + f\nabla_X s_1 + \nabla_X s_1$
2. $\nabla_{fX+Y} s = f\nabla_X s + \nabla_Y s$

for any $X, Y \in \mathfrak{X}(M), s_1, s_2 \in \Gamma(E)$

If we fill in a vector field $X \in \mathfrak{X}(M)$, a connection sends sections of E to twisted 1-forms:

$$d_\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E), d_\nabla(s) = \nabla_X(s)$$

This can be extended to twisted k-forms:

Definition 3.28. *Let $e_1, ...e_r$ be a local frame of $E|_U$, such that an $\omega \in \Omega^k(M,E)$ locally is equal to: $\omega|_U = \omega^i \otimes e_i$ with $\omega^i \in \Omega(U)$. Then, the **exterior derivative on twisted k-forms** is defined as a map $d_\nabla : \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$, with: $d_\nabla \omega = d\omega^i \otimes e_i + (-1)^k \omega^i \wedge e_i$ for every local frame.*

One would have to check that this map is independent of frame, by doing a frame transformation $e \cdot \tau$ with τ a transition function. d_∇ will then be the same derivative in both frames, as is proven in Lemma 5.12.4 of [6].

Locally, the connection has a more concrete form:

Definition 3.29. *Let (U_α, χ_α) for $\alpha \in I$ be charts over M, with $\cup_{\alpha \in I} U_\alpha$ an open cover of M. Then a **connection of vector bundle E** is a set of operators $d + \omega_\alpha$, with ω_α a $m \times m$-matrix of elements $(\omega_\alpha)^i_j \in \Omega^1(M)$ and d the exterior derivative on V-valued functions, with V the vector space of E.*

Given a frame of E on U_α and another on U_β with transition function $\tau_{\alpha,\beta}$, ω_α transforms as:

$$\omega_\beta = \tau^{-1}_{\alpha,\beta} \omega_\alpha \tau_{\alpha,\beta} + \tau^{-1}_{\alpha,\beta} d\tau_{\alpha,\beta} \text{ on } U_\alpha \cap U_\beta$$

This definition, which can be found on page 12 of [9], is another way of saying that a connection is completely determined by what it looks like in local frames. Linking this definition to the first, over some $U \subset M$, we can choose any local frame of E and the corresponding matrix of 1-forms from will be:

$$\nabla_X(e_j) = \omega_j^i(X)e_i \text{with } \omega_j^i \in \Omega(U)$$

Later, we will see that, globally, this matrix is a connection on a principal bundle.

The Levi-Civita connection

On the tangent bundle with metric g, there is a natural connection, which is called the **Levi-Civita connection** ∇. We will not go into details why it is natural (compatible with g and torsion-free, which is explained in [3, p.29]). However, we will describe what this connection looks like.

In section 3.5.2, we mentioned how in the coordinate frame, the metric g looks like:

$$g \doteq \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}_\Phi \quad \text{with } g_{ij} \in C^\infty(U)$$

Given this matrix, the **Christoffel symbols** are defined as:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

Now, we can construct the Levi-Civita connection as:

Definition 3.30. *The* ***Levi-Civita connection*** ∇ *is a connection on* TM *with the property that for any* (U, χ) *with coordinate frame* $\partial_1, ..., \partial_m$*:*

$$\nabla_X(\partial_i) = \Gamma^i_{p,j} dx^p(X)\partial_i$$

Using definition 3.6, the Levi-Civita connection is completely determined by the matrix of 1-forms that has components: $\omega^i_j = \Gamma^i_{p,j} dx^p$ in a coordinate frame $\partial_1, ..., \partial_m$.

As this connection depends on the derivatives of the metric components, it is intuitively clear that it describes the wrinkles in spacetime and thus the gravitional field.

3.7 Principal bundles

To model gauge fields, one does not need vector bundles, but principal bundles. These are fibre bundles with a few extra conditions.

Definition 3.31. *A **principal bundle** is a fibre bundle* (P, M, π, G) *with:*

1. G *a Lie group*
2. *a smooth right action* $\alpha : P \times G \longrightarrow P, (p, g) \mapsto p \cdot g$ *that is fixed point free.*
3. $\pi : P \longrightarrow M$ *satisfies:* $\pi(p \cdot g) = \pi(p)$ *for every* $p \in P$, $g \in G$ *(**G-invariance**).*
4. *Local trivialisations* $\Psi_i : P|_{U_i} \longrightarrow U_i \times G$ *satisfy the condition:* $\Psi_i(p \cdot g) = \Psi_i(p) \cdot g$ *for every* $p \in P|_{U_i}$, $g \in G$ *(**G-equivariance**). Here, the right action on* $U \times G$*, is defined as:* $(x, g) \cdot h = (x, gh)$ *for every* $(x, g) \in U \times G$ *and every* $h \in G$.

3.7.1 Differential of right action

The right actions induces two smooth maps:

1. $\alpha_p := \alpha \circ i_p : G \longrightarrow P$ with inclusion: $i_p : G \longrightarrow P \times G, i_p(g) = (p, g)$
2. $R_g := \alpha \circ i_g : P \longrightarrow P$ with inclusion: $i_g : P \longrightarrow P \times G, i_g(p) = (p, g)$

Side-note: for the second action, we use the same notation as for the right action on the Lie group. This should not cause confusion as it is always clear what R_g is working on (either an element in P or in G).

Its differentials are:

1. $(d\alpha_p)_g(V_g) = d\alpha_{p,g}(di_p)_g(V_g) = d\alpha_{p,g}(0, V_g)$
2. $(dR_g)_p(V_p) = d\alpha_{p,g}(di_g)_p(V_p) = d\alpha_{p,g}(V_p, 0)$

Thus, for any $V_p \in T_pP$ and any $W_g \in T_gG$:

$$(d\alpha)_{p,g}(V_p, W_g) = (d\alpha)_{p,g}(V_p, 0) + (d\alpha)_{p,g}(0, W_g) = R_g(V_p) + (d\alpha_p)_g(W_g)$$

3.7.2 Local trivialisations

Whereas on a vector bundle local frames were equivalent to local trivialisations, on a principal bundle local sections are already enough:

Theorem 3.32. *Every local trivialisation induces a local section and vice versa*

Proof. ($\Leftarrow$) : On a principal bundle P over manifold M, we take an open cover $(U_i)_{i\in I}$ with local sections: $\sigma_i : U_i \longrightarrow P|_{U_i}$. Now, we define the local diffeomorphism:

$$\Phi_i : U_i \times G \longrightarrow P|_{U_i}, \Phi_i((x,g)) = \sigma_i(x) \cdot g \text{ for } x \in U_i \cap U_j \text{ and } g \in G$$

($\Rightarrow$) : Given a given local trivialisation Ψ, we can define the section: $\sigma_i(x) = \Psi^{-1}((x,e))$ for every $x \in U_i$. This section is smooth because Ψ is smooth. □

3.7.3 Transition function

Just as with the vector bundle, the trivialisations (or sections in this case) induce transition functions on overlaps.

Lemma 3.33. *Given a fibre P_x for an $x \in M$, every $p, q \in P_x$ has a unique element $[p:q] \in G$, such that: $q \cdot [p:q] = p$*

Proof. As a property of the principal bundle, there is an isomorphism: $\phi : \{x\} \times G \longrightarrow P_x$. Thus: $p = \phi(x,a)$ and $q = \phi(x,b)$, for some $a, b \in G$. Because G is a group, there exists a unique element $c \in G$ such that: $a \cdot c = b$, namely $a^{-1}b$. Using that the isomorphism (a trivialisation) is G-equivariant (see 3.31), we get:

$$p \cdot c = \phi(x, a \cdot c) = \phi(x,b) = q$$

□

Thus, the following map is well-defined:

Definition 3.34. *Given two local sections (σ_1, U_1) and (σ_2, U_2), their **transition function** is defined as:*

$$\tau_{12} : U_1 \cap U_2 \longrightarrow G, x \mapsto [\sigma_1(x) : \sigma_2(x)]$$

such that $\sigma_2(x) = \sigma_1(x) \cdot \tau_{12}(x)$.

3.7.4 The frame bundle

Similar to bundling all the bases in $Fr(V)$, as we saw in chapter 1, we define a principal bundle that bundles all the local frames over E:

Definition 3.35. *Given a vector bundle (E, M, π, V), the **frame bundle** is a principal bundle $(Fr(E), M, \tilde{\pi}, GL(V))$ with:*

1. *$Fr(E) = \{(x,u) \mid x \in M,\ u = (u_1, ...u_r)$ a frame of $E_x\}$*

2. *$\tilde{\pi}(x,u) = x$ for every $(x,u) \in Fr(E)$*

3. *a right action: $Fr(E) \times Gl_r(V) \longrightarrow Fr(E), (A,\phi) \mapsto \phi \cdot A$, such that $\phi \cdot A$ is a frame with components: $(\phi \cdot A)_j = A^i_j \phi_i$*

In this thesis, we will only need frame bundles defined over the tangent bundle of a spacetime manifold.

3.7.5 Spacetime frame bundle

Given a spacetime manifold, we can consider its frame bundle over TM. Because $V = \mathbb{R}^4$, this would be a $GL_4(\mathbb{R})$-bundle, with the following right action:

$$Fr(TM) \times GL(4,\mathbb{R}) \longrightarrow Fr(TM), e = (e_0, ..., e_3) \mapsto (e \cdot A) \text{ with components } (e \cdot A)_j = A^i_j e_i$$

Given two frames $e \in Fr(TM)|_U$ and $e' \in Fr(TM)|_{U'}$, there exists a transition function $\tau : U \cap U \longrightarrow GL_4(\mathbb{R})$, such that for every $x \in U \cap U'$, $e'_x = e_x \cdot \tau(x)$.

Because of the three properties in section 3.5.2 that we used to define a spacetime manifold, we can also select only the **Lorentz frames** of the frame bundle $Fr(TM)$. These would then form a $SO^+(3,1)$-bundle.

Proposition 3.36. *Let (M,g) be a spacetime manifold. Then, there exists a $SO^+(3,1)$-bundle $SO^+(M) \subset Fr(TM)$.*

Proof. In section 3.5.2, we said that, given a manifold M with metric g that has signature $(-+++)$ and was (time-)orientable, we could define local Lorentz frames. Thus, we can define a bundle of Lorentz bases $SO^+(M) = \{(p,e) \mid e \in SO^+(T_pM)\} \subset Fr(TM)$ and then the Lorentz frames would be smooth sections of this bundle. This smoothness implies that $SO^+(M)$ indeed is a $SO^+(3,1)$-principal bundle with the right action:

$$SO^+(M) \times SO^+(3,1) \longrightarrow SO^+(M), e = (e_0, ..., e_3) \mapsto (e \cdot \Lambda) \text{ with components } (e \cdot \Lambda)_j = \Lambda^i_j e_i$$

It inherits its quotient map to M from $Fr(TM)$. □

3.8 Connections on a principal bundle

Having discussed frame bundles on spacetime, we already have a useful application for principal bundles in physics. Now, we return to another application (and the primary goal of this chapter): the modelling of a gauge field. The setup is: we have some G-bundle over a base space M with a right action α. This right action allows us to move in the fibres of P (vertically). For $U(1)$, this would correspond to a phase change and thus it is essential for describing gauge transformations.

3.8.1 The infinitesimal action

To construct a connection, we first consider the differential of the right action at e:

$$a_p := (d\alpha_p)_e : \mathfrak{g} \longrightarrow T_pP$$

What does this differential do? We define the curve on G:

$$\gamma : (-\epsilon, \epsilon) \longrightarrow G, t \mapsto e^{tV_e} \text{ such that } \gamma'(0) = V_e$$

$$\frac{d}{dt}\gamma : (-\epsilon, \epsilon) \longrightarrow T_{e^{tV_e}}G, t \mapsto V_{e^{tV_e}}$$

Then:

$$a_p(V_e) = (d\alpha_p)_e(V_e) = (\alpha_p \circ \gamma)'(0) = \frac{d}{dt}(p \cdot e^{tV_e})|_{t=0}$$

Thus, a_p is a map (which is called the **infinitesimal action**):

$$a : \mathfrak{g} \longrightarrow \mathfrak{X}(P), v \mapsto a(v) \text{ with } a(v)_p = \frac{d}{dt}(p \cdot e^{tv})|_{t=0}$$

Essentialy, this map gives you, for each $v \in \mathfrak{g}$, a vector in T_pP that points in the direction of the fibre, which we call **vertical**. Thus, the space of all vertical vectors is: $\mathcal{V}_p = Im(a_p) \subset T_pP$

3.8.2 Connections

Now, we can define the "field" that will model our force carriers:

Definition 3.37. *A connection $\mathcal{A}$ on G-bundle P is a $\mathfrak{g}$-valued 1-form on P with the following properties:*

1. *$\mathcal{A}(a(v)) = v$ for any $v \in \mathfrak{g}$ (a being the infinitisimal action on P)*

2. $R_g^*(\mathcal{A}) = Ad_{g^{-1}}\mathcal{A}$.

Another way of defining a connection is:

Definition 3.38. *A **connection** on P is a choice of a subspace $\mathcal{H}_p \subset T_pP$ for every $p \in P$, such that: $\mathcal{H}_p + \mathcal{V}_p = T_pP, \mathcal{H}_p \cap \mathcal{V}_p = 0$, $\mathcal{H}_{pg} = R_g(\mathcal{H}_p)$ for every $g \in G$.*

We can then define $\mathcal{A}$ as the unique $\mathfrak{g}$-valued 1-form on P with $\mathcal{A}(X_p) = 0$ for every $X_p \in \mathcal{H}_p$ [3, p. 74]. Using this properties, we can write any $X_p \in T_pP$ as: $X_p^h + a_p(v)$ for some $v \in \mathfrak{g}$ and $X_p^h \in \mathcal{H}_p$. Then: $\mathcal{A}(X_p^h + a_p(v)) = \mathcal{A}(X_p^h) + v = v$. Therefore, a connection measures the verticality of tangent vectors on P.

Now, we consider a path $\gamma : [0,1] \longrightarrow M$ with a starting point $\gamma(0) = x \in M$ and $\gamma(1) = x$. Now, a connection makes sure that we can lift the path to a unique $\tilde{\gamma} : [0,1] \longrightarrow P$, with starting point $\tilde{\gamma}(0) \in P_x$ (See Lemma 2.55 of [3]). Then, what is $\tilde{\gamma}(1)$? Is it the starting point of the lifted path? This depends on the curvature. To define it, we need an extension of the Lie bracket: $[.,.] : \Omega^1(P,\mathfrak{g}) \times \Omega^1(P,\mathfrak{g}) \longrightarrow \Omega^2(P,\mathfrak{g})$, which is defined by:

$$[\alpha,\beta](X,Y) = [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)] \text{ for } X,Y \in \mathfrak{X}(P) \text{ and } \alpha,\beta \in \Omega^1(P,\mathfrak{g}).$$

Then:

Definition 3.39. *The **curvature** of the connection $\mathcal{A}$ is defined as:*

$$\mathcal{F} = d\mathcal{A} + [\mathcal{A},\mathcal{A}]$$

It is then a property of the curvature that if $\mathcal{F} = 0$, $\tilde{\gamma}(0) = \tilde{\gamma}(1)$. In other words, the curvature measures the uncommutativity of lifted loops. Here, $d\mathcal{A}$ is the slope of the connection itself and $[\mathcal{A},\mathcal{A}]$ measures the failure of the Lie algebra to be commutative.

3.8.3 Gauge/spin transformation

The following theorem will prove that a connection describes the gauge freedom when pulled back to the base space.

Theorem 3.40. *Given a connection $\mathcal{A} \in \Omega(P,\mathfrak{g})$ defined on a principal G-bundle over M and two local sections (s_1,U_1), (s_2,U_2), with transition function: $\tau : U_1 \cap U_2 \longrightarrow G$ Then,*

$$s_2^*(\mathcal{A}) = Ad_{\tau^{-1}} s_2^*(\mathcal{A}) + (dL_{\tau^{-1}})d\tau$$

Proof. Given two local sections, $s_{1,2} : U_{1,2} \longrightarrow P$, with transition function τ, we can write s_2 in terms of s_1 and τ, by:

$$(s_1, \tau) : U_1 \cap U_2 \longrightarrow P \times G \text{ such that: } s_2(x) = s_1(x) \cdot \tau(x) = \alpha \circ (s_1, \tau)(x).$$

Thus: $s_2 = \alpha \circ (s_1, \tau)$. Given a connection $\mathcal{A} \in \Omega_1(P, \mathfrak{g})$, we can take the pullbacks over both sections:

$$A_1 = s_1^* \mathcal{A}, \;\; A_2 = s_2^* \mathcal{A}$$

Then, for any $X_x \in T_x M$,

$$(s_2^* \mathcal{A})_x(X_x) = \mathcal{A}_{s_2(x)} ds_2(X_x) = \mathcal{A}_{s_1(x)\cdot\tau(x)}(d\alpha)_{(s_1(x),\tau(x))}((ds_1)_x(X_x), (d\tau)_x(X_x))$$

$$(s_2^* \mathcal{A})_x(X_x) = \mathcal{A}_{s_1(x)\cdot\tau(x)}(dR_{\tau(x)})_{s_1(x)}(ds_1)_x(X_x) + \omega_{s_1(x)\cdot\tau(x)}(d\alpha_{s_1(x)})_{\tau(x)}(d\tau)_x(X_x)$$

Using the first property of the connection, we can rewrite the first term as:

$$\mathcal{A}_{s_1(x)\cdot\tau(x)}(dR_{\tau(x)})_{s_1(x)}(ds_1)_x(X_x) = Ad_{\tau^{-1}(x)} \mathcal{A}_{s_1(x)} ds_1(X_x) = Ad_{\tau^{-1}(x)}(s_1^* \mathcal{A})_x(X_x)$$

For the second term, we note:

$$\alpha_{s_1(x)\cdot\tau(x)}(g) = \alpha_{s_1(x)} L_{\tau(x)}(g) \text{ for any } g \in G$$

So:

$$a_{s_1(x)\cdot\tau(x)} = (d\alpha_{s_1(x)\cdot\tau(x)})_e = (d\alpha_{s_1(x)})_{\tau(x)}(dL_{\tau(x)})_e$$

Now, we will rewrite $(d\tau)_x$ as:

$$d\tau_x = (dL_{\tau(x)})_e (dL_{\tau^{-1}(x)})_{\tau(x)} d\tau_x$$

We observe that:

$$(dL_{\tau^{-1}(x)})_{\tau(x)} d\tau_x(X_x) \in T_e G = \mathfrak{g}$$

Thus:

$$(d\alpha_{s_1(x)})_{\tau(x)}(d\tau)_x(X_x) = a_{s_1(x)\cdot\tau(x)}((dL_{\tau^{-1}(x)})_{\tau(x)} d\tau_x(X_x))$$

Using the second property of a connection we get:

$$\mathcal{A}_{s_1(x)\cdot\tau(x)}(a_{s_1(x)\cdot\tau(x)}((dL_{\tau^{-1}(x)})_{\tau(x)} d\tau_x(X_x))) = (dL_{\tau^{-1}(x)})_{\tau(x)} d\tau_x(X_x)$$

In conclusion:

$$(s_2^* \mathcal{A})_x(X_x) = Ad_{\tau^{-1}(x)} s_1^*(\mathcal{A})(X_x) + (dL_{\tau^{-1}(x)})_{\tau(x)} d\tau_x(X_x)$$

Or without filling in points:

$$A_2 = Ad_{\tau^{-1}} A_1 + dL_{\tau^{-1}} d\tau$$

□

What this theorem says, is that we can view a change of gauge or a change of Lorentz frame as a transition from one pullback of the connection to another. For electromagnetism the connection is in $P_{U(1)}$ and for the gravitational field in $SO^+(M)$ (the Levi-Civita connection).

3.9 The photon

As promised, setting $G = U(1)$, we will see that a connection on $P_{U(1)}$ indeed describes an electromagnetic field. In the case of $U(1)$, a connection is described by a $\mathfrak{u}(1)$-valued 1-form, which, as $\mathfrak{u}(1) = i\mathbb{R}$, is just a regular $\mathbb{R}$-valued 1-form times a factor i. Secondly, $U(1)$ is abelian, which means that for any connection: $[\mathcal{A}, \mathcal{A}] = 0$.

3.9.1 Gauge transformation

This implies that for any local section $s : U \longrightarrow P$ with $A = s^*(\mathcal{A})$: $F = s^*(d\mathcal{A}) = dA$.

We can write the pullback of $\mathcal{A}$ over this section as:

$$A = i(\varphi dt + A_x dx + A_y dy + A_z dz) \text{ with: } \varphi, A_i \in C^\infty(U).$$

Note this looks exactly like the potential we considered in the previous chapter except for a factor i. We take two local potentials:

$$A_1 = s_1^*\mathcal{A},\ A_2 = s_2^*\mathcal{A} \text{ with transition function: } \tau : U_1 \cap U_2 \longrightarrow U(1)$$

The gauge transformation then is:

$$A_2 = Ad_{\tau^{-1}} A_1 + (dL_{\tau^{-1}})d\tau$$

For an abelian group: $Ad_{\tau^{-1}} = Id$. So:

$$A_2 = A_1 + (dL_{\tau^{-1}})d\tau$$

Using the exponential map, we can simplify the transition function to: $\tau = e \circ i\Lambda$ with $i\Lambda : U_1 \cap U_2 \longrightarrow i\mathbb{R}$. Thus:

$$d\tau_x = ie^{i\Lambda(x)}(d\Lambda)_x \text{ and: } (L_{\tau^{-1}} d\tau)_x = e^{-i\Lambda(x)} ie^{i\Lambda(x)}(d\Lambda)_x = i(d\Lambda)_x$$

Thus: $A_2 = A_1 + i(d\Lambda)$

In vector notation, $A = i(\varphi, \mathbf{A})$, this becomes:

$$i(\varphi_2, \mathbf{A}_2) = i(\varphi_1 + \frac{\partial \Lambda}{dt}, \mathbf{A}_2 + \nabla \Lambda)$$

which is exactly the gauge transformation of an electromagnetic field (except for the i). The curvature transforms as:

$$F_2 = dA_2 = dA_1 + d(dL_{\tau^{-1}} d\tau) = dA_1 = F_1$$

Note that $(L_{\tau^{-1}})d\tau$ is a closed $\mathfrak{g}$-valued 1-form. In other words: F (which we wrote in chapter 2 as $E \wedge dt + B$) is gauge-invariant, which we know very well as a physics result.

3.10 Associated vector bundle

To be able to couple the electron to an electromagnetic field, we need to associate $\mathbb{C}^4$ to the $U(1)$-bundle. We want to do this in such a way that a phase change in the $U(1)$-bundle, say from p to $p \cdot e^{i\theta}$ equals a change in the spinor from Ψ to $e^{-i\theta}\Psi$.

In general, this is defined as:

Definition 3.41. *Let P be a principal G-bundle over M, V a vector space and $\rho : G \longrightarrow GL(V)$. Then the **associated vector bundle** is:*

$$P \times_\rho V = (P \times V)/\{(p \cdot g, v) \sim (p, \rho(g^{-1})v) |\ g \in G\}$$

Here, $\rho : G \longrightarrow GL(V)$ is called the representation of G.

Example 3.42. *A natural associated vector bundle is the **adjoint bundle** $Ad(P)$. Let $Ad : G \longrightarrow GL(\mathfrak{g})$ be the representation of G. Then: $Ad(P) = P \times_{Ad} G$.*

3.10.1 Sections of the associated vector bundle

Lemma 3.43. *Let $P \times_\rho V$ be an associated vector bundle . Then a section $\Psi \in \Gamma(P \times_\rho V)$ is equivalent to a map $\psi : P \longrightarrow V$ with the property: $\psi(p \cdot g) = \rho(g^{-1})\psi(p)$ for every $p \in P$ and $g \in G$.*

Proof. Let $\Psi \in \Gamma(E)$. Then, we can write any $x \in M$ as $[p]$ for some $p \in P_x$. As such, $\Psi([p]) = [p, \psi(p)]$, with $\psi : P \longrightarrow V$.

By definition, we have the following equality: $\Psi([p \cdot g]) = \Psi([p])$ for every $g \in G$. Thus: $[p \cdot g, \psi(p \cdot g)] = [p, \psi(p)]$. Because of the quotient: $[p \cdot g, \psi(p \cdot g)] = [p, \rho(g)\psi(p \cdot g)]$.

For both equalities to hold ψ must satisfy the following condition: $\psi(p \cdot g) = \rho(g^{-1})\psi(p)$ for every $p \in P$ and $g \in G$, which is called **G-equivariance**. This argumentation can easily be reversed for equivalence. □

In fact, this argument can be extended to $\Omega^k(P,V)$. For this, we need to introduce a special type of form:

Definition 3.44. *A k-form $\omega \in \Omega^k(P,V)$ is called **basic** if:*

1. *$i_{a(v)}\omega = 0$ for all $v \in \mathfrak{g}$ (**horizontal**)*
2. *$R_g^*\omega = \rho(g^{-1})\omega$ for any $g \in G$ (**G-equivariance**)*

Then, the extension of Lemma 3.43:

Lemma 3.45. $\Omega^k(M, P \times_\rho V) \cong \Omega^k(P,V)_{bas}$

Proof. We will not give a full proof of this statement, which can be found in [13, p. 278] or in [3, p. 72]. We will however outline how this isomorphism is constructed.

Let $\omega \in \Omega^k(P,V)_{bas}$ and $v_1,...,v_k \in T_xM$ for an $x \in M$. Then, they can be lifted to $u_1,...,u_k \in T_pP$ for a $p \in P_x$, such that: $d_p\pi(u_i) = v_i$. Now we define the isomorphism (a trivialisation of $P \times_\rho V$): $\Phi : V \longrightarrow (P \times_\rho V)_x, v \mapsto [p,v]$. Then, we construct $\tilde{\omega} \in \Omega^k(M, P \times_\rho V)$ pointwise by: $\tilde{\omega}(v_1,...,v_k) = \Phi(\omega(u_1,...,u_k))$

The converse is very similar: Let $\tilde{\omega} \in \Omega(M, P \times_\rho V)$. Then: $\omega(u_1,...,u_k) = \Phi^{-1}(\tilde{\omega}(v_1,...,v_k))$. We would have to check that ω is basic, which is done in [13]. □

Note that for $k = 0$, we get 3.43.

Lemma 3.46. *Let $\mathcal{A}$ be a connection on P. Then, its curvature $\mathcal{F} \in \Omega^2(P,\mathfrak{g})$ is a basic form with respect to $\rho = Ad : G \longrightarrow GL(\mathfrak{g})$*

Proof. 1. For any $\omega \in \Omega^1(M)$, $d\omega$ can be written as:

$$d\omega(X,Y) = i_X d(\omega(Y)) - i_Y d(\omega(X)) - [\omega(X),\omega(Y)] \text{ [2, p. 107] for } X,Y \in \mathfrak{X}(M)$$

Thus:

$$i_{a(v)} d\mathcal{A}(Y) = i_{a(v)} d(\mathcal{A}(Y)) - i_Y d(\mathcal{A}(a(v))) + [\mathcal{A}(a(v)),\mathcal{A}(Y)] \text{ for } v \in \mathfrak{g}, Y \in \mathfrak{X}(P)$$

Now: $\mathcal{A}(a(v)) = v$, which is a constant for every $p \in P$ and therefore: $d(\mathcal{A}(a(v))) = 0$.

Also, any $Y \in \mathfrak{X}(P)$ can be separated as: $Y^v + Y^h$, where $Y^v = a(w)$ for some $w \in \mathfrak{g}$. Then: $\mathcal{A}(Y) = \mathcal{A}(Y^v) = \mathcal{A}(a(w)) = w$. Because w is a constant: $d\mathcal{A}(Y) = dw = 0$.

Consequently: $i_{a(v)}d\mathcal{A}(Y) = -[\mathcal{A}(a(v)), \mathcal{A}(Y)]$

In conclusion:

$$i_{a(v)}\mathcal{F}(Y) = -[\mathcal{A}(a(v)), \mathcal{A}(Y)] + [\mathcal{A}(a(v)), \mathcal{A}(Y)] = 0$$

2.
$$R_g^*(\mathcal{F}) = d(R_g^*\mathcal{A}) + \frac{1}{2}[R_g^*\mathcal{A}, R_g^*\mathcal{A}] = dAd(g^{-1})\omega + \frac{1}{2}[Ad(g^{-1})\mathcal{A}, Ad(g^{-1})\mathcal{A}] = Ad(g^{-1})\mathcal{F}$$

For the last part, we used that $Ad(g^{-1})$ is a Lie group homomorphism, which means it can be taken out of the Lie bracket, and also the fact that $Ad(g^{-1})$ is unaffected by the exterior derivative. [13, p. 273]

□

In conclusion:

Lemma 3.47. *For a connection $\mathcal{A}$ on P, its curvature $\mathcal{F}$ is equivalent to vector-bundle valued 2-form $\omega_{\mathcal{F}} \in \Omega^k(M, Ad(P))$.*

Proof. Because $\mathcal{F}$ is basic, Lemma 3.45 tells us that it is equivalent to a vector-bundle valued 2-form $\omega_{\mathcal{F}} \in \Omega^k(M, Ad(P))$. □

3.10.2 The Yang-Mills Lagrangian

With this lemma, we can construct a global Lagrangian on M for the electromagnetic interaction. In chapter 2, we saw that the Yang-Mills Lagrangian for a $U(1)$ gauge field on Minkowski space was: $\mathcal{L}vol_4 = F_A \wedge \star F_A$. To get to a global Lagrangian, and thus global laws of physics, we need to find a way to construct a Lagrangian with the curvature $\mathcal{F}$. However, this is a 2-form on P and Lagrangians are defined as maps $\mathcal{L} : M \longrightarrow \mathbb{R}$. But, as proven above, the curvature is equivalent to a 2-form $\omega_F \in \Omega^2(M, Ad(P_{U(1)}))$. The bundle metric of $Ad(P_{U(1)})$ is defined by: $\langle(\omega \otimes f)_x, (\eta \otimes g)_x\rangle = -\langle\omega_x, \eta_x\rangle_{\Lambda(T_x^*M)} f(x)g(x)$ for all $x \in M$ (a minus sign for the i's) and $\omega, \eta \in \Omega^1(M), f, g \in \Gamma(Ad(P))$.

Thus, we can write: $\mathcal{L}_{EM} = \langle\omega_F, \omega_F\rangle_{Ad(P)}$

3.10.3 Associated covariant derivative

Returning to th discussion of the coupling of the spinor to the electromagnetic field, we would like to differentiate our spinor with respect to a connection form $\mathcal{A}$ on the G-bundle. This is possible with the following derivative:

Definition 3.48. *Let $E(P,V)$ an associated vector bundle, $\mathcal{A}$ a connection on P, $\Psi \in \Gamma(P \times_\rho V)$ and $s \in \Gamma(P|_U)$, such that $A = s^*\mathcal{A}$. Then the* ***associated covariant derivative*** *is defined as:*

$$d_A\Psi = d\Psi(-) + \rho_*(A(-))\Psi \in \Omega^1(M, P \times_\rho V)$$

Here: $\rho_* = (d\rho)_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, such that: $\rho_*(A) = A^{ij} \otimes e_{ij} \in \Omega^1(M, \mathfrak{gl}(V))$, where e_{ij} are basis elements of $gl(V)$ and $A^{ij} \in \Omega^1(M)$. Therefore $\rho_*(A)$ is a matrix of 1-forms.

We defined the associated covariant derivative over an open $U \subset M$. However, since the connection is globally defined, we can take $\{U_i \mid i \in U\}$ an open cover of M with sections $s_i \in \Gamma(P|_{U_i})$ and define a covariant derivative on each of them. Taken together, they form a **vector bundle connection**:

Lemma 3.49. *Let $\mathcal{A}$ be a connection of P. Then, an associated covariant derivative $d_\mathcal{A}$ of $P \times_\rho V$ defines a connection $\nabla^\mathcal{A}$on $P \times_\rho V$.*

Proof. As we mentioned above $\rho_*(A(-))$ is an $m \times m$-matrix of 1-forms, just as in definition 3.6. We now check that it transforms as ω in 3.6.

Let $(U_1, s_1), (U_2, s_2)$ be two local sections of P with transition function $\tau : U_1 \cap U_2 \longrightarrow G$. Then:

$$\rho_*(A_2) = \rho_*(Ad_{\tau^{-1}}A_1 + (dL_{\tau^{-1}})d\tau) = \rho_*(\tau^{-1}A_1\tau) + \rho_*(dL_{\tau^{-1}})d\tau)$$

Since s_1, s_2 trivialise P, they also trivialise $P \times_\rho V$, which means they induce local frames on this bundle. Then, $\rho_* \circ \tau$ is the transition function of those frames, which proves. As such, $\rho_*(A(-))$ indeed transforms as in definition 3.6. □

3.11 The electron

3.11.1 Spin bundle

In chapter 1 we doubly covered the Lorentz group by a spin group. Because $SO^+(M)$ is a $SO^+(3,1)$-bundle, we can expect that we can also doubly cover this bundle by a $SL(2,\mathbb{C})$-bundle over M. This is called a **spin bundle**: $Sp^+(M)$.

In constructing the bundle $SO^+(M)$ over a four-manifold M, we required of M to be of signature $(-+++)$, orientable and time-orientable. To construct a spin bundle over a bundle $SO^+(M)$, we need yet another condition, namely that the **second Stiefel-Whitney class $w_2(M)$** is trivial [6, p. 379]. We call these manifolds **spin manifolds**. It would be too much to describe what this classes are. However, here are some examples of spin manifolds:

1. $\mathbb{R}^m$
2. Spheres and tori
3. Products of spin manifolds

We will assume our spacetime manifold is spin and denote the double cover:

$$\beta : Sp^+(M) \longrightarrow SO^+(M)$$

A spin-1/2 particle in a space without any gauge fields can then be described as a section of the following $\mathbb{C}^4$ vector bundle: $\mathcal{S} = Sp^+(M) \times_{\rho_{1/2}} \mathbb{C}^4$.

$SL(2,\mathbb{C})$-equivariance

As $\mathcal{S}$ is an associated vector bundle, a change in a fibre of $Sp^+(M)$ is equalled to a change in the spinor.

To see this, we look at T_xM for $x \in M$. Given $e, f \in SO^+(T_xM)$ with $e \cdot \Lambda = f$, we can lift e, f to elements $\tilde{e}, \tilde{f} \in Sp^+(M)$ by β, such that: $\tilde{e} \cdot C = \tilde{f}$ for some $C \in SL(2,\mathbb{C})$. Then:

$$\Psi_x = [\tilde{e}, \psi_S(\tilde{e})] \mapsto [\tilde{e} \cdot C, \psi_S(\tilde{e})] = [(\tilde{e}, \rho_{1/2}(C^{-1})\psi_S(\tilde{e})] = \rho_{1/2}(C^{-1})\Psi_x$$

This is exactly the then mysterious transformation we saw in chapter 1.

3.11.2 The spin connection

For general relativity, we need to couple the influence of gravity to our particles. The metric g defines how curved spacetime is, and thus how strong particles are accelerated. In that regard, gravity can be seen as a field, just as we saw the electromagnetic potential as a field. In this case, the field is not defined on the $U(1)$-bundle, but on $SO^+(M)$. Using the inclusion $i : SO^+(3,1) \longrightarrow GL(4,\mathbb{R})$, we can see the tangent bundle as: $TM = SO^+(M) \times_i \mathbb{R}^4$.

As such:

Definition 3.50. *Given an* (U,χ) *with coordinate frame* $\partial_0,...,\partial_3$ *and a Lorentz frame* $e \in \Gamma(SO^+(M)|_U)$, *such that* $\tau : U \longrightarrow GL(4,\mathbb{R})$ *is their transition function. Let* ω *be the the matrix of 1-forms on U with components* $\omega^i_j = \Gamma^i_{\mu,j}dx^\mu$. *Then, the* ***Levi-Civita connection*** $\mathcal{A}_{SO}$ *is a connection on* $SO^+(M)$, *such that:*

$$e^*\mathcal{A}_{SO} = \tau^{-1}\omega\tau + \tau^{-1}d\tau$$

Now, the spin connection is a pull-back of this connection:

Definition 3.51. *Given the Levi-Civita connection* $\mathcal{A}_{SO}$ *on* $SO^+(M)$, *the* ***spin connection*** *is the connection* $\mathcal{A}$ *on* $Sp^+(M)$ *that satisfies:*

$$\mathcal{A}_{Sp} = (\lambda_*)^{-1}\beta^*\mathcal{A}_{SO} \in \Omega^1(Sp^+(M),\mathfrak{sl}(2,\mathbb{C}))$$

As a reminder: λ is the double covering map of $SO(3,1)$ that we saw in chapter 1.

3.11.3 The Dirac Operator

Using 3.49, the spin connection $\mathcal{A}_{Sp}$ on the spin bundle $Sp^+(M)$ induces a connection $\nabla^{\mathcal{A}_{Sp}}$ on the spinor bundle $\mathcal{S}$. This allows us to define the Dirac operator.

Definition 3.52. *Let* $e = (e_0,...,e_3)$ *be a Lorentz frame of the tangent bundle* TM. *The Dirac operator is a differential operator* $D : \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S})$

$$D := \gamma_j\nabla^{\mathcal{A}_{Sp}}_{e_j}$$

with γ_j *the matrix representation of the Clifford algebra* $Cl(\mathbb{R}^4,\eta)$ *we saw in chapter 2.*

This can be seen a map: $d_{\mathcal{A}_{Sp}} : \Gamma(\mathcal{S}) \longrightarrow \Omega^1(M,\mathcal{S})$.

Lemma 3.53. *The Dirac operator* $D : \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S})$ *is independent of Lorentz frame.*

Proof. Take two Lorentz frames e, d, such that: $e \cdot \Lambda^{-1} = d$ for some $\Lambda \in SO^+(3,1)$

$$\nabla^{\mathcal{A}_{Sp}}_{d_j} = \nabla^{\mathcal{A}_{Sp}}_{\Lambda^i_j e_i} = \Lambda^i_j\nabla^{\mathcal{A}_{Sp}}_{e_i}$$

The last equality is a property of the connection we mentioned in definition 3.27.

From section 1.13 of chapter 1, we know that there exists a lift of Λ to $C \in SL(2,\mathbb{C})$. Then, we proved in Lemma 2.15: $\rho_{1/2}(C)\gamma_i\rho_{1/2}(C^{-1}) = \Lambda^j_i\gamma_j$. Thus, we get:

$$\gamma^j\nabla^{\mathcal{A}_{Sp}}_{d_j} = \gamma^j\Lambda^i_j\nabla^{\mathcal{A}_{Sp}}_{e_i} = \Lambda^i_j\gamma^j\nabla^{\mathcal{A}_{Sp}}_{e_i} = \rho_{1/2}(C)\gamma^i\rho_{1/2}(C^{-1})\nabla^{A}_{e_i}$$

And thus:

$$\gamma^j \nabla^A_{d_j} \Psi = \rho_{1/2}(C) \gamma^i \nabla^A_{e_i} \Psi$$

□

For some $U \subset M$ with section $s \in \Gamma(Sp^+(M))$, such that $A_{Sp} = s^* \mathcal{A}_{\mathcal{S}}$, the Dirac is:

$$D\Psi = \gamma^i d\Psi(e_i) + \gamma^i \rho_{1/2*}(A_{Sp}(e_i))\Psi$$

Here, $\rho_{1/2*} = (d\rho_{1/2})_e$ is a map from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{gl}(4, \mathbb{C})$, with

$$\rho_{1/2*}(B) = \begin{pmatrix} B & 0 \\ 0 & B^{\dagger -1} \end{pmatrix} \text{ for any } B \in \mathfrak{sl}(2, \mathbb{C})$$

Minkowksi space revisited

On Minkowski space, the metric is flat ($g = \eta$), which means $\mathcal{A}_{\mathcal{S}}$ is zero. For the Dirac, we can take as Lorentz frame the global coordinate frame $\partial_t, \partial_x, \partial_y, \partial_z$. Filling this in we get:

$$D\Psi = \gamma^i d\Psi(\partial_i) = \gamma^i \partial_i \Psi dx_i(\partial_i) = \gamma^i \partial_i \Psi$$

This is exactly the Dirac we saw in chapter 2.

3.12 Electron-photon interaction

3.12.1 Bundle splicing

Now, we have two principal bundles $Spin^+(M)$ and $P_{U(1)}$ with their respective connections. One describes the field of gravity, the other the electromagnetic field. For the interaction of the particle with both fields, we will splice the two bundles together such that we have a $SL(2, \mathbb{C}) \times U(1)$-bundle:

$$Spin^+(M) \times_M P = \{(s, p) \in Spin^+(M) \times P_{U(1)} \mid \pi_S(s) = \pi_P(p)\}$$

Now, to couple our spinor to these connections, we take the associated vector bundle:

$$\mathcal{S} = (Spin^+(M) \times_M P_{U(1)}) \times_\rho \mathbb{C}^4$$

Here, ρ is the representation of $SL(2, \mathbb{C}) \times U(1)$ in the vector space $GL(4, \mathbb{C})$ by:

$$\rho(C, e^{i\theta}) = e^{i\theta} \begin{pmatrix} C & 0 \\ 0 & (C^*)^{-1} \end{pmatrix}$$

Using that the Lie algebra of $SL(2,\mathbb{C}) \times U(1)$ is $\mathfrak{sl}(2,\mathbb{C}) \oplus i\mathbb{R}$, differential of this representation is:

$$\rho_* : \mathfrak{sl}(2,\mathbb{C}) \oplus i\mathbb{R} \longrightarrow \mathfrak{gl}(4,\mathbb{C}), (B, i\theta) \mapsto \begin{pmatrix} C & 0 \\ 0 & (C^*)^{-1} \end{pmatrix} + i\theta \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

3.12.2 $U(1)$-equivariance

As proven in Lemma 3.43, $\Psi \in \mathcal{S}$ is equivalent to a $SL(2,\mathbb{C}) \times U(1)$-equivariant map $\psi_{S,P} : Spin^+(M) \times P_{U(1)} \longrightarrow \mathbb{C}^4$, such that $\Psi(x) = \Psi([(s,p)]) = [(s,p), \psi_S(s)\psi_P(p)]$.

At point $x \in M$, we can do a change of gauge in the fibre $(P_{U(1)})_x$ by sending p to $p \cdot e^{i\theta}$ for a $\theta \in \mathbb{R}$. Then, the spinor transforms as:

$$\Psi_x = [(s,p), \psi_S(s)\psi_P(p)] \mapsto [(s, p \cdot e^{i\theta}), \psi_S(s)\psi_P(p)] = [(s,p), e^{-i\theta}\psi_S(\tilde{e})\psi_P(p)] = e^{-i\theta}\Psi_x$$

This transformation is the one that Aharanov and Bohm measured in their experiment.

3.12.3 Dirac operator

Now, the connection on $Spin^+(M) \times_M P$ will be of the form:

$$\mathcal{A} = \mathcal{A}_{Sp} + \mathcal{A}_{EM} \in \Omega^1(Spin^+(M) \times_M P, \mathfrak{sl}(2,\mathbb{C}) \oplus i\mathbb{R})$$

This principal bundle connection defines a vector bundle connection $\nabla^{\mathcal{A}}$ on $\mathcal{S}$. Given a section $s \in \Gamma(Spin^+(M) \times_M P|_U)$ such that $A = s^*\mathcal{A} \in \Omega(U, \mathfrak{sl}(2,\mathbb{C}) \oplus i\mathbb{R})$, $\nabla^{\mathcal{A}}$ looks like:

$$\nabla^{\mathcal{A}}\Psi = d\Psi + \rho_*(A)\Psi = d\Psi + \rho_{1/2*}(A_{Sp})\Psi + A_{EM}\Psi \text{ with } A_{Sp} \in \Omega^1(U, \mathfrak{sl}(2,\mathbb{C})) \text{ and } A_{EM} \in \Omega^1(U, i\mathbb{R})$$

Now, the associated Dirac operator is:

$$D_{\mathcal{A}}\Psi := \gamma^j \nabla^{\mathcal{A}}_{e_j}\Psi$$

3.12.4 The Yang-Mills-Dirac Lagrangian

Now that we have set up a way to couple spinors to the gauge and spin connection and are able to differentiate them with respect to these fields, we can finally construct a Lagrangians that governs the interaction between an electron and a photon. This Lagrangian will, by virtue, be gauge-invariant, because we define it for connections, such that there is no choice of gauge anymore, and Lorentz invariant. To construct the Lagrangian, we first define a bundle metric on $\mathcal{S}$:

Definition 3.54. *Let $\Psi, \Phi \in \Gamma(\mathcal{S})$, then a **spinor metric** $\tilde{\Psi}\Phi$ is a bundle metric of $\mathcal{S}$, which is defined pointwise as: $(\tilde{\Psi}\Phi)_p = \Psi_p^\dagger i\gamma_0 \Psi_p$ for every $p \in M$*

This metric is an extension of the Dirac adjoint we saw in chapter 2. So now:

Definition 3.55. *Let $\mathcal{A}$ The **Yang-Mills-Dirac Lagrangian** is:*

$$\mathcal{L}(\mathcal{A}, \Psi) = Re(\tilde{\Psi} D_{\mathcal{A}} \Psi - m\tilde{\Psi}\Psi) - \frac{1}{2}\langle \omega_{\mathcal{F}}, \omega_{\mathcal{F}} \rangle_{\Omega(M, Ad(P))}$$

3.12.5 General Lie groups G

The electromagnetic interaction ($U(1)$) is the simplest scenario. However, if we want to describe other interactions, for example the weak interaction, we have a matrix group. In that case, the representation of the spinor bundle would be of the form:

$$\rho = (\rho_1, \rho_2) : SL(2, \mathbb{C}) \times G \longrightarrow GL(4, \mathbb{C}) \otimes GL(r, \mathbb{C})$$

with r depending on the dimension of the matrix group. Also, spinors would be of the form $\Psi = [\psi_S \otimes \psi_P]$ with $\Psi_p \in \mathbb{C}^4 \otimes \mathbb{C}^r$ for any $p \in M$.

Conclusion

In this thesis, we have described the standard procedure for describing interactions of elementary particles in both special and general relativity. In special relativity, because of global coordinates and a flat metric, this could all be done with coordinate-dependent tools, with the setback that the symmetries of our fields were more mysterious. In general relativity, we had the following procedure: we took a gauge group. Then, we defined a principal bundle over spacetime. In this principal bundle, we defined a connection that locally describes a gauge field on M. This connection could then be coupled to the spinor. The laws for this interaction were then captured by a Lagrangian, depending on the connection and the spinor, which was invariant to the symmetries of the spinor and the gauge field. Because of the generalised metric, the symmetry of the spinor depended on the ”spin lift” of the Levi-Civita connection, which described the gravitational grooves in spacetime.

Because of its broad scope, this thesis raises many questions, both physical and mathematical. For example, we could ask: what topology do spacetime manifolds have? [11, p. 170]. This could then lead to a discussion of Donaldson's classification of four-manifolds that allow a smooth structure [10, p. 406].

A more physical question would be if spacetime has to be **time-orientable** to describe the real world. This is discussed in the fascinating paper **The orientability of spacetime** by Mark J. Hadley.

We also could start classifying the solutions to the source-free Maxwell equation $\Delta A = 0$ (which are called harmonic forms) or the Dirac equation. Or we could minimize the **Yang-Mills-Dirac Lagrangian** and ask which connections and which spinors satisfy the corresponding equations. Talking about connections, we could also ask what topology the principal bundles on which they are defined might have.

Furthermore, we could set up a Lagrangian that, apart from the electromagnetic, includes the strong and weak interaction as well. The gauge field would then be a connection on a $U(1) \times SU(2) \times$

$SU(3)$-bundle. This then inspires a discussion about the **Grand Unified Theory** in which all these interactions are described by a connection on a $SU(5)$-bundle.

Although these connections give the setup for quantum field theory, we would have to quantise the gauge fields to be able to calculate anything. This is explained in the standard work *An Introduction to Quantum Field Theory* by Michael E. Peskin and Daniel V. Schroeder.

Finally, we looked at a spin structure on four manifolds with signature $(-+++)$. This can be generalised to spin structures on $s+t$-dimensional manifolds with signature (s,t). In that way, the spin group would not be defined as a matrix group, but as a subset of the Clifford algebra defined on the tangent bundle. This is done in Chapter 6 of [6], in which we also see a double cover of $O(s,t)$ instead of $SO^+(s,t)$, which is called the pin group.

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