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Topological Classification of Defects in Ordered Media

Bachelor Thesis

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Abstract

In this thesis, defects in ordered media will be introduced. Then, they will be classified, first introducing a physical measure of equivalence, and then translating that to mathematics via homotopy theory. It will turn out that point defects in $\mathbb{R}^2$ and line defects in $\mathbb{R}^3$ are classified by conjugacy classes of the fundamental group. Finally, after some examples of defects, a relatively simple way to compute the fundamental group is discussed.

Contents

1 Introduction

Physicists like studying ordered media, because they are great for modelling many things. Examples include metals and liquid crystals. Often, though, these media are not fully ordered - they have certain mistakes in their composition, called defects. It is generally very useful to know what types of defects one might encounter, and whether or not two defects cause the same problems in some medium. This is where the maths comes in, to characterise defects for any given ordered medium. In this thesis, we will look at classifying such defects.

2 Phase Transitions and Landau Theory

Physicists often observe phase transitions in materials they study. This can be from a solid to a liquid, or from a magnetised piece of iron to a non-magnetised piece of iron. Many phase transitions are thermal phase transitions, which means they occur when varying the temperature. There is some critical temperature at which a phase transition occurs. At lower temperatures, there is order, and at higher temperatures, there is disorder. The notion of 'order' can be different in different situations.

The most well-known model to study phase transitions and order is probably the two-dimensional Ising model. We consider a square lattice, with a spin at each lattice point, which can point either up (1) or down (-1). The assumption is that the energy is lower when adjacent spins are equal, and higher when adjacent spins are opposite. This can be modelled by introducing some coupling constant $J > 0$, and then the energy will be equal to the sum of $-JS_iS_j$ for each pair of adjacent spins. The energy function reads as follows:

$$E = -\frac{J}{2} \sum_{\langle i,j \rangle} S_i S_j,$$

where S_i is the spin at location i, so either 1 or -1, and $\langle i, j \rangle$ indicates i and j are adjacent. The factor $\frac{1}{2}$ is there to correct for overcounting - we are counting each pair twice, since both spins can fulfil the role of i.

In the Ising model, the order is described by an order parameter m, which is the average direction of the spins. Since up spins are 1 and down spins are -1, this is simply the average of all the spins. We can show, both by doing the maths[1] and by running a simulation, that this system has a temperature where a phase transition happens. This is called the critical temperature. The phase transition, in this case, is the disappearance of any sort of preferred direction, or in other words, $|m|$ becoming equal to 0. If a system has a temperature below T_c, it is in the ordered phase. If it has a temperature above T_c, it is in the disordered phase.

To make a relatively accurate prediction about the behaviour of phase transitions, Landau theory was developed. For the simple form of Landau theory, we Taylor expand the energy as a function of the order parameter. This is allowed, since the order parameter is small near the critical temperature by definition. Then, we cut this Taylor polynomial off at the fourth power. This is a good compromise between simplicity and accuracy. Notice how the coefficients still depend on temperature. Next, we make sure the polynomial shares the symmetries of the energy. This will be demonstrated with the Ising model in the next paragraph. We also make sure the fourth power has a positive coefficient, to avoid letting the system get arbitrarily low energy by letting the order parameter tend to infinity. Then, we find the minimal energy for each value of the temperature.

As an example, we will use the Ising model again. Its order parameter m is simply the average of all the spins, as we have seen above. First of all, we remark that all we care about is the location of the minima in the energy for a given temperature. Therefore, any constant term is irrelevant and can be discarded. Second of all, we remark that reversing the direction of all spins indeed does not change the energy - which direction is up and which is down is merely a matter of perspective, and does not change the energy. Therefore, it is a Taylor expansion with only even terms, like required, and we find

$$E(m, T) - E(0, T) \approx a(T)m^2 + b(T)m^4.$$

[1] L. Onsager, *Crystal Statistics I*, *Physical Review*, 1944, Vol. 65, Nos. 3 and 4, pag. 117-149.

As mentioned above, there is some critical temperature where the minimal energy changes from being achieved at $m = 0$ to $m \neq 0$. Since $T - T_c$ is small when T is near this critical temperature, we can Taylor expand the coefficients in $T - T_c$. For $b(T)$, we see it has a constant term $b > 0$ (since $b(T) > 0$ for all T), and for $a(T)$, in order for the location of the minimum in energy to change at $T = T_c$, we need $a(T) = a \cdot (T - T_c)$. Thus, we find

$$E(m, T) - E(0, T) \approx a \cdot (T - T_c)m^2 + bm^4.$$

Minimising the energy (for any fixed temperature) is now easy. We could take the derivative, but it is even easier to complete the square:

$$E(m, T) - E(0, T) \approx b\left(m^2 + \frac{a \cdot (T - T_c)}{2b}\right)^2 - \left(\frac{a \cdot (T - T_c)}{2b}\right)^2.$$

Since the final term is a constant, we can discard it again. Since the other term is a square, its minimal value is 0, if and only if the expression we are squaring is 0. However, if $\frac{a \cdot (T - T_c)}{2b}$ is positive, we cannot make $m^2 + \frac{a \cdot (T - T_c)}{2b}$ equal to 0, since m^2 is also positive (m is real, after all). The best we can do is setting $m = 0$, to make $m^2 + \frac{a \cdot (T - T_c)}{2b}$ as small as possible. Therefore, we see

$$m^2 = \begin{cases} -\frac{a \cdot (T - T_c)}{2b} & \text{if } T < T_c \\ 0 & \text{if } T \geq T_c \end{cases}$$

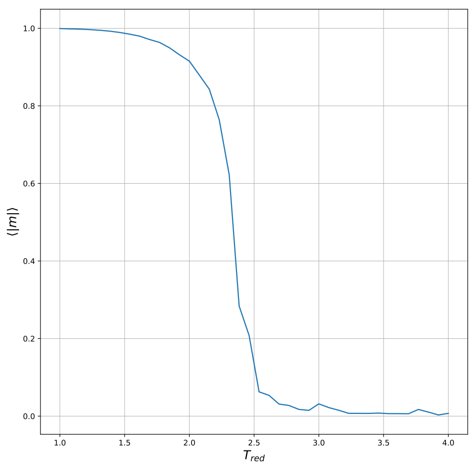

Figure 1: Graph of the Ising model's magnetisation, based on a simulation. The jaggedness arises from the combination of a finite lattice and a Monte Carlo process. The average of the absolute value of m is plotted against T, multiplied by some constant to get T_{red}. The phase transition is visible here, and m does follow a square root curve near the critical point.

Despite the large number of approximations we have made in Landau theory, it works surprisingly well for predicting behaviour near the critical point. However, this very basic approximation is not always enough to describe a system, so we need to expand the theory to deal with more complicated situations. There are two things we need to change.

The first change we need to make is that we need to go from a global order parameter to a local one. The order parameter, therefore, is now a function. In the case of the Ising model, it is the function that assigns the spin value to every point on the grid. To make it more general, however, we want to allow varying magnitude of each spin, so we choose them as elements of $\mathbb{R}$ instead of as elements of $\{-1, 1\}$. Additionally, now that we are working with functions, we do not want to have a discrete domain (like a square grid), but we want to change it to a continuous domain (like $\mathbb{R}^n$). If we 'zoom out' enough, a square grid becomes fine enough that it is approximately $\mathbb{R}^2$, so for the Ising model, we will use $f : \mathbb{R}^2 \to \mathbb{R}$. In a more general case, this will be some $f : \mathbb{R}^n \to U$, where U is some subset of $\mathbb{R}^k$ for some k, and $n = 2$ or $n = 3$.

The second change we need to make is to include (mis)alignment terms in the energy formula. Now that we have a continuous domain and a local order parameter, we can talk about derivatives. The derivative is useful, because it represents alignment - full alignment gives a constant function, so any deviation from 0 indicates misalignment. We will add a term with the 'square' of the derivative, because that is analytic, and a positive coefficient, so that misalignment costs energy. 'Square' here means 'sum of squares of all elements in the Jacobian matrix'. If we denote the Jacobian matrix as df, then this is $\mathrm{tr}(df^T df)$.

Now, since we never demanded that f be differentiable, we cannot compute the derivative everywhere. This means we will use some other theory for the regions where it is not defined.

Since we are working with a function of space now, and since we want to talk about the energy of the system as a whole, we need to find a way to go from the local order parameter to the total energy. On a lattice, this would be simple - it is the sum over all lattice sites of the energies per lattice site. We are using an approximation of a very fine grid, and in the limit of going from a grid to all of $\mathbb{R}^n$, a sum becomes an integral. Hence, the formula becomes

$$E(f,T) - E(0,T) \approx \int_{\mathbb{R}^2} \left(a(T)|f|^2 + b(T)|f|^4 + c(T)\mathrm{tr}(df^T df) \right) \mathrm{d}\vec{x}.$$

It is not very pleasant to work with such a general space, so we will make some assumptions and simplify what we are working with. Since we want to look at the ordered phase, where the function is differentiable almost everywhere,[2] we will assume that $\mathrm{tr}(df^T df) \ll |f|$. This implies we can practically ignore the derivative term when we are trying to minimise this for a fixed T. But that means we can simply try to minimise $a(T)|f|^2 + b(T)|f|^4$ everywhere, and that will minimise the integral. This gives us a very similar result to the one we got for the Ising model earlier - $|f|$ is constant for a fixed temperature.[3] We might have some other relations, such as in the case of nematics (which will be discussed later), where we use a sphere with antipodal points glued together, so in general, we will always have some combination of products and quotients of spheres.[4] This new space, where $|f|$ is constant, will be called the order parameter space X. A more useful definition of X will follow in the next chapter.

In short, we see that in a general case, when observing the ordered phase, we have a local order parameter $f : \mathbb{R}^n \to X$, with X the order parameter space, and $n = 2$ or $n = 3$, that is differentiable almost everywhere.

3 Ordered Media and Defects

Our trip into Landau theory has motivated us to define an 'ordered medium'. We have seen that some models have a phase transition, with the ordered state occurring below the critical temperature, and the disordered state occurring above the critical temperature. We want an ordered medium to be far in the ordered phase ($T \ll T_c$). We have already seen it is described by a local order parameter, and we will call the function that describes this local order parameter the configuration.

Definition 3.1. An **order parameter space** is some connected topological manifold[5] X with metric.

A **configuration** is a map $f : \mathbb{R}^n \to X$, where X is an order parameter space and where $n \in \{2, 3\}$. Since we are looking only at $T \ll T_c$, this map is merely discontinuous in a set with measure 0 and a finite number of connected components.

An **ordered medium** is a physical object that is described by a configuration.

Note that a connected topological manifold is also path-connected.

For an example of the discontinuities in f, we can look at the Ising model. Recall that, since we discovered in the previous chapter that the magnitude had to be fixed for any order parameter space, we have a

[2] The set of points where f is not differentiable has measure 0 and has a finite number of connected components. We are approximating a model with a finite number of particles, after all.

[3] Technically, $|f|$ can vary slightly, due to some fluctuations as a result of temperature, but it's practically constant.

[4] 'Sphere' refers to any dimension here, so a circle is also considered to be a sphere.

[5] In this thesis, all topological manifolds will be second countable. Not everyone seems to agree whether or not this is included in the definition of a topological manifold.

function $f : \mathbb{R}^2 \to \{-1, 1\} = S^0$. There, $f^{-1}(\{1\})$ and $f^{-1}(\{-1\})$ will generally only have a small number of connected components, each with nonempty interior. In the interior of each of the connected components, the function is constant. The boundaries of the connected components are known as 'domain walls', which are an example of the much more general concept of 'defects'.

Definition 3.2. A **defect** is a set on which a configuration is discontinuous or undefined.

For more examples of defects, we will introduce another model - the $O(2)$-model, as introduced in Chaikin-Lubensky.[6] Here, instead of having $f : \mathbb{R}^2 \to \{1, -1\} = S^0$ like in the Ising model, we have $f : \mathbb{R}^2 \to S^1 \subset \mathbb{R}^2$. Notice how the local order parameter is now a vector field, instead of the scalar field it was for the Ising model. This shows how different local order parameters can be in different models.

This $O(2)$-model has much more diverse and interesting defects. One example would be a configuration f where all spins simply point outward from the origin. In other words, for any point $(x, y)! = (0, 0) \in \mathbb{R}^2$, we have

$$f(x, y) = \frac{(x, y)}{|(x, y)|}.$$

This defect is an example of a 'vortex'. What exactly is meant by 'vortex' will be clarified, after we describe how to classify defects. We can see it does not have perfect alignment, so the derivative contribution in the Landau formula is nonzero. It can be seen as the leftmost image in Figure 2.

However, this is obviously not the only type of defect. We could also have all spins rotated a quarter turn counterclockwise compared the defect described above. This is another example of a vortex:

$$f(x, y) = \frac{(-y, x)}{|(x, y)|}.$$

Like the defect discussed before, it also has nonzero derivative contribution in the Landau formula. It is the middle image in Figure 2.

One final example of a defect would be the configuration

$$f(x, y) = (\cos(u), \sin(u)),$$

where

$$u = \frac{2\pi x}{10|(x, y)|^2}.$$

Since this function is perfectly continuous everywhere outside of the origin, but undefined at the origin, we clearly have a defect at the origin on our hands. This defect looks almost uniform outside of a small region around the origin, and does not seem to turn around the origin like the previous two examples of vortices do. It is displayed as the rightmost image in Figure 2.

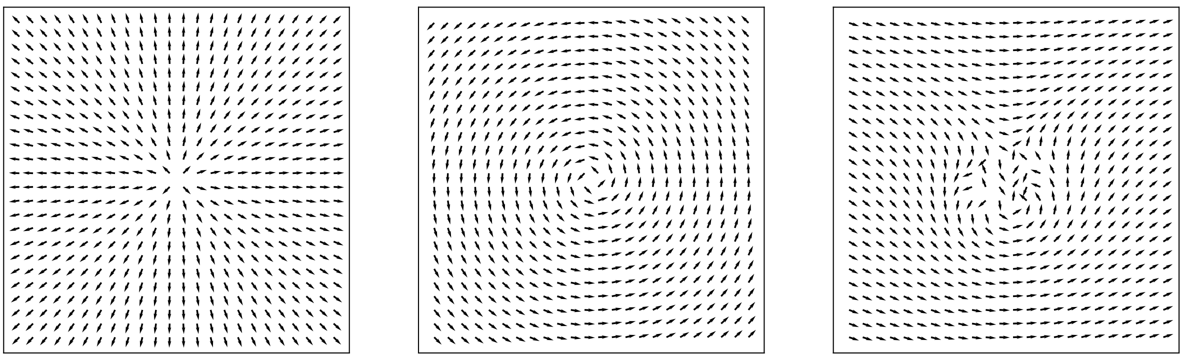

Figure 2: An image showing the three defects discussed above in the region $[-1, 1]^2$.

[6] P.M. Chaikin and T.C. Lubensky, *Principles of Condensed Matter Physics*, 1995.

4 Local Surgery

We have already seen some defect states, and noticed they have nonzero derivative contributions. The reason they do not spontaneously transform into a uniform state does not come from the fact they have minimal energy[7], but from the fact that any transformation from a defect state to a defect-less state must necessarily be discontinuous.

Why do these transformations matter? Because systems generally evolve over time due to fluctuations in the configuration. These fluctuations are called 'thermal fluctuations', since they occur as a result of temperature. The higher the temperature, the more often fluctuations occur, and the larger the areas they can affect at once. At the low temperatures we are working at, only small areas of the configuration can transform at once, and only in a way that keeps the configuration continuous. It explains why defect states do not simply transform into uniform states, and it will lead us to a concept called 'local surgery', as also used in Mermin.[8]

In this thesis, we will only look at point or line defects (the latter of which only in $\mathbb{R}^3$). In these cases, the configuration is continuous everywhere except for a point or a line. In the case of a point, we will place it at the origin, and in the case of a line, we will place it at the z-axis. Physically, it now makes sense to look at a small region of space around the defect (although in the case of a line defect, such a set is an infinitely long cylinder, so 'small' is relative). There are two reasons for this. Firstly, we look at a region around the defect, because changing a region that does not include the defect does not fundamentally change the defect in the configuration. Secondly, we look at a small region, because thermal fluctuations can only affect a small region at once, as we have discussed above.[9]

Based on this observation, we will consider two configurations equivalent if we can make one of them equal to the other in a small set around the origin or z-axis (depending on the type of defect we are considering) in such a way that there is only a small transitional region between the two configurations, and such that this new configuration is continuous everywhere except possibly the origin or z-axis. We have specifically avoided 'defect' here, since we want to include defect-less configurations as well.

This is still a very vague description, so we will make this a bit more rigourous.

Definition 4.1. The open disk with radius r will be denoted $D_r^n = \{x \in \mathbb{R}^n : |x| < r\}$. The open cylinder with radius r will be denoted $K_r = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2) < r\}$.

Definition 4.2. If $f, g : \mathbb{R}^n \to X$ are configurations with potential point defects (at the origin) and with order parameter space X, then the point defects in f and g are equivalent if and only if there exists a configuration $h : \mathbb{R}^n \to X$ and $r_2 > r_1 > 0$ such that

1. $h|_{\overline{D_{r_1}^n}} = g|_{\overline{D_{r_1}^n}}$
2. $h|_{\mathbb{R}^n - D_{r_2}^n} = f|_{\mathbb{R}^n - D_{r_2}^n}$
3. h is continuous on $\mathbb{R}^n - \{0\}$

Such a function h will be called a **local surgery** of point defect states.

If $f, g : \mathbb{R}^3 \to X$ are configurations with potential line defects (on the z-axis) and with order parameter space X, then the line defects in f and g are equivalent if and only if there exists a configuration $h : \mathbb{R}^3 \to X$ and $r_2 > r_1 > 0$ such that

1. $h|_{\overline{K_{r_1}}} = g|_{\overline{K_{r_1}}}$

[7]If you are currently wondering 'Then what did we need Landau theory for, which describes energy?', it is because Landau theory is the reason order parameters exist, which we need for our entire reasoning. Skipping Landau theory would not give us a motivation to define order parameters.

[8]N.D. Mermin, *The topological theory of defects in ordered media*, *Reviews of Modern Physics*, 1979, Vol. 51, No. 3, pag. 591-648.

[9]Since actual, physical versions of ordered media are obviously bounded, the cylinder would be finitely long, but we approximate it as infinitely long. This introduces a new problem, namely defects being able to get out of the medium by moving to the boundary and disappearing, however, if we demand the boundary changes continuously, this cannot happen.

2. $h|_{\mathbb{R}^n - K_{r_2}} = f|_{\mathbb{R}^n - K_{r_2}}$

3. h is continuous on $\mathbb{R}^3 - \{(0,0)\} \times \mathbb{R}$

Such a function h will also be called a **local surgery** of line defects states.

A defect will be called **removable** if there is a local surgery between its configuration and a uniform configuration.

This will be explained with an example.

Example 4.3. Let f be the uniform configuration in $\mathbb{R}^2$ pointing right, and let g be the uniform configuration in $\mathbb{R}^2$ pointing left. In other words,

$$f : \mathbb{R}^2 \approx \mathbb{C} \to S^1, re^{i\theta} \mapsto 1,$$

$$g : \mathbb{R}^2 \approx \mathbb{C} \to S^1, re^{i\theta} \mapsto -1,$$

where we use complex numbers.

Now, we need to construct a function h that is continuous on $\mathbb{R}^2 - \{0, 0\}$, matches g inside $D^2_{r_1}$, and matches f outside of $D^2_{r_2}$. The most obvious thing to try would be to linearly increase the angle of the result as we move from radius r_1 to r_2, keeping the result independent of θ. In other words,

$$h : \mathbb{R}^2 \approx \mathbb{C} \to S^1, re^{i\theta} \mapsto \begin{cases} 1 & \text{if } r > r_2 \\ -1 & \text{if } r < r_1 \\ e^{i\pi \frac{r-r_2}{r_1-r_2}} & \text{if } r_1 \leq r \leq r_2 \end{cases}$$

Since this new function h also does not depend on θ at all, it is well-defined.[10] It is also continuous, since we can verify that $e^{i\pi\frac{r_1-r_2}{r_1-r_2}} = e^{i\pi} = -1$ and $e^{i\pi\frac{r_2-r_2}{r_1-r_2}} = e^0 = 1$. h also, by construction, is equal to f and g on the correct domains, so it appears we have found a local surgery. It is displayed in Figure 3.

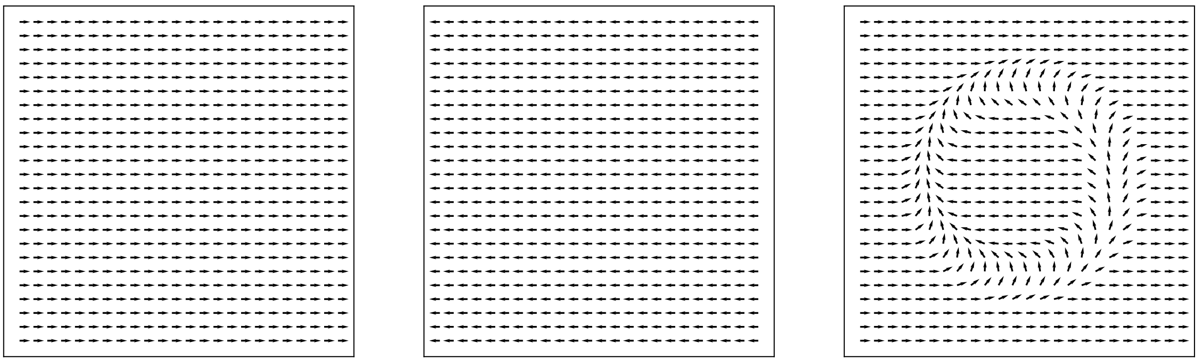

Figure 3: An image showing the above interpolation in the region $[-1, 1]^2$, with $r_1 = 0.4$ and $r_2 = 0.8$.

However, we can see it is not a very nice formula, and it seems like a lot of work for an argument that two uniform states with a different direction of spins are equivalent. We want to find a better way to argue that configurations are equivalent, or if the goal is to find an explicit local surgery, we want to at least know whether or not one exists before putting in a lot of effort to find one. We also do not have the ability to prove the lack of existence of a local surgery yet, which is problematic. This will be the main quest for most of the rest of this thesis - to find mathematical tools that give us a way to quickly determine whether or not two configurations are equivalent.

[10]It does not depend on the representative of the angle, since any multiple of 2π can be added to any angle without changing it.

5 Homotopies

With this thought about making local surgeries simpler in the back of our minds, let us look at a seemingly unrelated topic, that will turn out to be very useful, namely homotopies. A homotopy is basically a continuous interpolation between two functions.

Definition 5.1. Given two topological spaces A and B, and two functions $f, g : A \to B$, a homotopy is a function $h : A \times [0,1] \to B$ such that

1. h is continuous
2. $h(x,0) = f(x)$ for all $x \in A$
3. $h(x,1) = g(x)$ for all $x \in A$

Definition 5.2. The space of continuous functions from A to B is denoted $C(A,B)$.

To see where the interpolation comes in, we can introduce $\tilde{h}$:

$$\tilde{h} : [0,1] \to C(A,B),\ \ \tilde{h}(t)(x) = h(x,t).$$

Now write $\tilde{h}(t) = h_t$, so that we have $h(x,t) = h_t(x)$, $h_0 = f$, and $h_1 = g$. Each value of t gives a different continuous function, that changes from f to g as t changes from 0 to 1. The continuity criterion for this version is much harder to verify, however, so $h : A \times [0,1] \to B$ is used for that anyway.

In this thesis, we will mainly apply homotopies to loops in some space X, which are functions of the form $\gamma : S^1 \to X$. Since a circle is equivalent to an interval with its endpoints identified (in other words, glued together), we can also see this as a function $\gamma' : [0,1] \to X$, with $\gamma'(0) = \gamma'(1)$.

Definition 5.3. A **loop** is a function $\gamma : [0,1] \to B$ for some topological space B such that $\gamma(0) = \gamma(1)$. $\gamma(0)$ is called the basepoint.

We will clarify the notion of homotopy by using an example.

Example 5.4. Let

$$\gamma_1 : [0,1] \to \mathbb{R}^2,\ \ \gamma_1(s) = (1 - \cos(2\pi s), \sin(2\pi s))$$
$$\gamma_2 : [0,1] \to \mathbb{R}^2,\ \ \gamma_2(s) = (-1 + \cos(2\pi s), \sin(2\pi s))$$

be two loops starting at the origin. In other words, γ_1 moves clockwise through a unit circle that starts at the origin and γ_2 moves counterclockwise through a unit circle that starts at the origin, too. Now, the function

$$h : [0,1]^2 \to \mathbb{R}^2, h(s,t) = ((1-2t)(1-\cos(2\pi s)),\ \sin(2\pi s)) = (1-t)\gamma_1(s) + t\gamma_2(s)$$

is a homotopy between the two loops. It slowly squishes γ_1 into a vertical line segment, and then opens up the vertical line segment into γ_2. It is displayed in Figure 4. We can easily see that h is continuous, since it is simply made up of sums, products, and compositions of continuous functions. By construction, $h(s,0) = \gamma_1(s)$ and $h(s,1) = \gamma_2(s)$, so h is indeed a homotopy. Therefore, γ_1 and γ_2 are homotopic.[11]

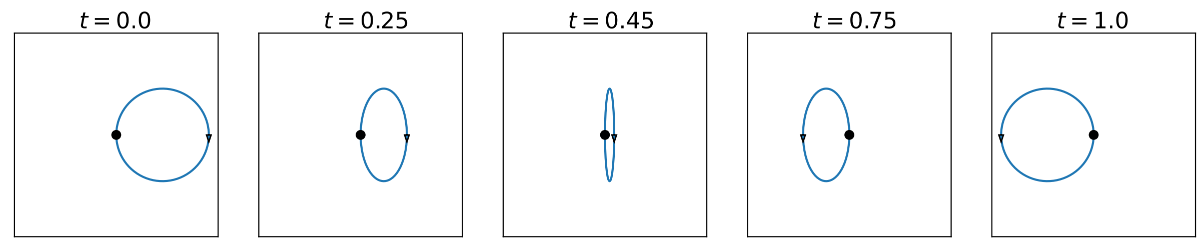

Figure 4: An image showing the above interpolation in the region $[-1,1]^2$, with $r_1 = 0.4$ and $r_2 = 0.8$.

[11] As a matter of fact, all loops in $\mathbb{R}^2$ are homotopic to each other, using the same interpolation as here. In general, any two loops in a convex space are homotopic to each other, using the same principle of interpolation.

We will prove two lemmas that will be very useful for the next theorem.

Definition 5.5. We will denote $C_r^n : [0,1] \to \mathbb{R}^n, C_r^n(s) := r(\cos(2\pi s), \sin(2\pi s), 0, \dots, 0)$.

Lemma 5.6. *Let $n = 2$ or $n = 3$. Given any two circles $C_{r_1}^n$ and $C_{r_2}^n$, and any configuration $f : \mathbb{R}^n \to X$ that is continuous everywhere except for possibly the origin if $n = 2$ and the z-axis if $n = 3$, $f(C_{r_1}^n)$ is homotopic to $f(C_{r_2}^n)$.*

Proof. There is clearly a homotopy h from $C_{r_1}^n$ to $C_{r_2}^n$ for any r_1, r_2 in $\mathbb{R}^2 - \{0,0\}$ or $\mathbb{R}^3 - \{0,0\} \times \mathbb{R}$, depending on whether $n = 2$ or $n = 3$ - simply grow or shrink the circle. Now, $f(h)$ is a homotopy from $f(C_{r_1}^n)$ to $f(C_{r_2}^n)$ - it restricts to the correct functions by definition, and it is continuous, since f is continuous on the entire range of h. This proves the statement. □

Lemma 5.7. *Given two configurations $f, g : \mathbb{R}^3 \to X$ that are continuous everywhere except for possibly on the z-axis, we have that $f|_{\partial K_{r_2}}$ and $g|_{\partial K_{r_1}}$ are homotopic if and only if $f(C_{r_2}^3)$ and $g(C_{r_1}^3)$ are homotopic.*

Proof. If $f|_{\partial K_{r_2}}$ and $g|_{\partial K_{r_1}}$ are homotopic, then obviously $f(C_{r_2}^3)$ and $g(C_{r_1}^3)$ are homotopic, since we can simply restrict the homotopy to a subset.

If, on the other hand, $f(C_{r_2}^3)$ and $g(C_{r_1}^3)$ are homotopic with homotopy $h : [0,1]^2 \to X$, the proof is trickier. Define

$$k : [0,1] \times \mathbb{R} \times [0,1] \to X, k(s,z,t) = \begin{cases} f(r_2\cos(2\pi s), r_2\sin(2\pi s), (1-3t)z) & \text{if } 0 \le t < \frac{1}{3} \\ h(s, 3t-1) & \text{if } \frac{1}{3} \le t < \frac{2}{3} \\ g(r_1\cos(2\pi s), r_1\sin(2\pi s), (3t-2)z) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}$$

We get $k(-,-,0) = f|_{\partial K_{r_2}}$ and $k(-,-,1) = g|_{\partial K_{r_1}}$. Furthermore, we know $h(s,0) = f(C_{r_2}^3) = f(r_2\cos(2\pi s), r_2\sin(2\pi s), 0)$ and $h(s,1) = g(C_{r_1}^3) = g(r_1\cos(2\pi s), r_1\sin(2\pi s), 0)$, so the pieces of k match up, and k is continuous. Therefore, k is a homotopy between $f|_{\partial K_{r_2}}$ and $g|_{\partial K_{r_1}}$.[12]

This proves the equivalence. □

With these two lemmas proven, we can now move on to the main theorem of this section, which will demonstrate a link between homotopies and local surgeries.

Theorem 5.8. *Let $n = 2$ or $n = 3$. Given two configurations $f, g : \mathbb{R}^n \to X$ that are continuous everywhere except for possibly at the origin if $n = 2$ and the z-axis if $n = 3$, there exists a local surgery of point defect states with radii r_1, r_2 if and only if $f(C_r)$ and $g(C_r)$ are homotopic, for any $r > 0$.*

Proof. This proof will feature the $n = 2$ case, but the $n = 3$ case is essentially the same after applying Lemma 5.7.

Assume we have a local surgery $h : \mathbb{R}^2 \approx \mathbb{C} \to X$ with radii r_1 and r_2, in other words,

1. $h|_{\overline{D_{r_1}^n}} = g|_{\overline{D_{r_1}^n}}$
2. $h|_{\mathbb{R}^2 - D_{r_2}^n} = f|_{\mathbb{R}^2 - D_{r_2}^n}$
3. h is continuous on $\mathbb{R}^2 - \{(0,0)\}$.

Then define $k : [0,1]^2 \to X, k(s,t) = h((tr_1 + (1-t)r_2)e^{2\pi si})$. We can see that, by construction, $k(s,0) = h(r_2 e^{2\pi si}) = h(C_{r_2}) = f(C_{r_2})$, and $k(s,1) = h(r_1 e^{2\pi si}) = h(C_{r_1}) = g(C_{r_1})$. Furthermore, since h is continuous everywhere except for possibly at the origin, and since $r_1, r_2 > 0$, k is continuous in both variables. Therefore, if a local surgery with radii r_1, r_2 exists, then $f(C_{r_2})$ and $g(C_{r_1})$ are homotopic. We can now apply Lemma 5.6 to both f and g and find that $f(C_r)$ and $g(C_r)$ are homotopic for any $r > 0$.

[12] The notation is arguably suboptimal, with the configurations being restricted to a set in one case, and composed with a parametrisation of a set in the other case. This is because the parametrisation made the proof for Theorem 5.8 look nicer.

Conversely, assume we have a homotopy $\tilde{k} : [0,1]^2 \to X$ such that $\tilde{k}(s,0) = f(C_r)$ and $\tilde{k}(s,1) = g(C_r)$. Then applying Lemma 5.6 to both f and g gives a new homotopy k from $f(C_{r_2})$ to $g(C_{r_1})$. Define

$$h : \mathbb{R}^2 \approx \mathbb{C} \to X$$

$$h(re^{i\theta}) = \begin{cases} f(re^{i\theta}) & \text{if } r \geq r_2 \\ g(re^{i\theta}) & \text{if } r \leq r_1 \\ k(s_\theta, \frac{r-r_2}{r_1-r_2}) & \text{if } r_1 < r < r_2 \end{cases}$$

where s_θ is defined as the unique solution of $s = \frac{\theta}{2\pi}$ such that $s \in [0,1)$. This is necessary, because any angle can have different representatives that differ by multiples of 2π.

We can see quite easily that h is equal to f and g in the appropriate places, and continuity is guaranteed by k being a homotopy and matching f and g in the appropriate places. Therefore, h is a local surgery, and this proves the equivalence of the two statements. □

Now, we can clearly see the usefulness of homotopies.

6 Fundamental Group

This section assumes the reader is familiar with equivalence relations.

Since we now know how important homotopies are to our problem, we would like to use some of the existing maths involving homotopies to help us simplify the problem. To do that, we must first introduce two new concepts.

Given two loops γ, γ', there is a special situation when $\gamma(0) = \gamma'(0)$. In that case, we can use a concept known as based homotopy. To introduce based homotopy more generally, we first need to introduce another definition

Definition 6.1. A **pointed topological space** is a pair (A, a), where A is a topological space and a is a point in A. Any map $f : (A, a) \to (B, b)$ must have $f(a) = b$.

If we want to have a function from A to B without this property, we will omit the designated point, so we will write this as $g : A \to B$.

For paths and loops, the notation will be slightly different. For a path $\gamma : [0,1] \to (B, b)$, it is implied that the designated point in $[0,1]$ is 0, and for a loop $\eta : [0,1] \to (B, b)$, it is implied that the designated point in $[0,1]$ is the single point represented by both 0 and 1.

Definition 6.2. Given two functions $f, g : (A, a) \to (B, b)$ for some topological spaces A and B, a based homotopy is a function $h : A \times [0,1] \to B$ such that

1. h is continuous
2. $h(x, 0) = f(s)$ for all $x \in A$
3. $h(x, 1) = g(s)$ for all $x \in A$
4. $h(a, t) = b$ for all $t \in [0,1]$

We can see this as an interpolation between two functions with a shared basepoint, that fixes the basepoint (hence the name 'based homotopy').

Another concept we can use when we have two loops (or paths) γ, γ' with a shared basepoint is concatenation.

Definition 6.3. The concatenation of two loops (or paths) γ, γ' with a shared basepoint is the loop (or path) $\gamma'\gamma$, with

$$\gamma'\gamma(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s < \frac{1}{2} \\ \gamma'(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

In other words, it is the loop that first traverses γ and then γ'.

Notice how $\gamma'\gamma$ and $\gamma\gamma'$ are usually not equal.

We see concatenation is a way to combine two loops with a common basepoint into a single new loop, with the same basepoint. It now makes sense to consider the set of loops with a given basepoint, and see if concatenation can be useful.

Definition 6.4. The set of loops with a given basepoint x_0 in some topological space X is defined as $\Omega(X, x_0) = \{\gamma : [0,1] \to X : \gamma(0) = \gamma(1) = x_0\}$.

Concatenation on its own is not very useful - it is not even associative. The loop $\gamma_3(\gamma_2\gamma_1)$ first goes through γ_1 at four times the normal speed, then through γ_2 at four times the normal speed, and finally through γ_3 at twice the normal speed. The loop $(\gamma_3\gamma_2)\gamma_1$, on the other hand, first goes through γ_1 at twice the normal speed, then through γ_2 at four times the normal speed, and finally through γ_3 at four times the normal speed. However, the path traced is identical, the parametrisation is just different.

This is where based homotopies enter the picture. While concatenation is not associative, it is associative up to based homotopy, in the sense that we can find a based homotopy between $\gamma_3(\gamma_2\gamma_1)$ and $(\gamma_3\gamma_2)\gamma_1$. We will introduce a lemma to prove this, which will come in handy again later.

Definition 6.5. A **reparametrisation** of a loop γ is a loop γ', such that $\gamma'(s) = \gamma(f(s))$ for some continuous function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$.

Lemma 6.6. *A reparametrisation γ' of a loop $\gamma : [0,1] \to X$ is homotopic to γ via a based homotopy.*

Proof. Since γ' is a reparametrisation of γ, there exists a continuous $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ such that $\gamma'(s) = \gamma(f(s))$. Because $[0,1]$ is convex, we can define the standard linear homotopy

$$h : [0,1]^2 \to X, h(s,t) = \gamma((1-t)s + tf(s)).$$

h is a composition of continuous functions and therefore continuous. Furthermore, $h(s,0) = \gamma(s)$ and $h(s,1) = \gamma(f(s)) = \gamma'(s)$. Therefore, h is a homotopy. We even have $h(0,t) = \gamma(0)$ and $h(1,t) = \gamma(1) = \gamma(0)$, so this is a based homotopy. □

Lemma 6.7. *Concatenation on $\Omega(X, x_0)$ is associative up to based homotopy.*

Proof. Let $\gamma_1, \gamma_2, \gamma_3 : [0,1] \to X$ be three loops with common basepoint p. Now let

$$f : [0,1] \to [0,1], f(s) = \begin{cases} 2s & \text{if } 0 \leq s < \frac{1}{4} \\ s + \frac{1}{4} & \text{if } \frac{1}{4} \leq s < \frac{1}{2} \\ \frac{1}{2}s + \frac{1}{2} & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

We see that $(\gamma_3(\gamma_2\gamma_1))(f(s)) = ((\gamma_3\gamma_2)\gamma_1)(s)$. In other words, one is a reparametrisation of the other, which means that we can apply Lemma 6.6 and conclude that $\gamma_3(\gamma_2\gamma_1)$ and $(\gamma_3\gamma_2)\gamma_1$ are homotopic. □

We are going to define a group, which we will call the 'first homotopy group' or 'fundamental group', that will very naturally encompass based homotopy, and which will also be very useful. Here, all loops that are equal up to based homotopy will be merged into a single class of loops. In order to do this nicely, we will use an equivalence relation.

Definition 6.8. An **equivalence relation** on a set A is a set $R \subset A \times A$, which fulfils the following three criteria

1. $(a, a) \in R$ for every $a \in A$ (reflexivity).
2. If $(a, b) \in R$, then $(b, a) \in R$ (symmetry).
3. If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ (transitivity).

The usual notation is $a \sim b$ if and only if $(a, b) \in R$. The equivalence relation is also usually referred to simply as $\sim$ (such as '$\sim$ is an equivalence relation'), even though it is technically a set.

The **equivalence class** of an element $a \in X$ under an equivalence relation $\sim$ is defined as the set $[a] := \{b \in X : a \sim b\}$.

Given a set X and an equivalence relation $\sim$ on X, the **quotient set** $X/\sim$ is defined as $X/{\sim} := \{[a] : a \in X\}$.

For readers unfamiliar with the concept, the most common example of an equivalence relation is probably modular arithmetic. If we are working with the integers $\mathbb{Z}$ modulo n, then $a \sim b$ if and only if n divides $a - b$. In modular arithmetic, $a \sim b$ is usually denoted $a \equiv b \pmod{n}$. The reader can verify that this is indeed an equivalence relation. The equivalence class $[k]$ of a number k consists of all the numbers that have the same remainder upon division by n as k does. The quotient set when working modulo n is then simply $\{[0], [1], \ldots, [n-1]\}$.

It turns out that based homotopy is also an equivalence relation.

Lemma 6.9. *Based homotopy is an equivalence relation.*

Proof. We see $\gamma \sim \gamma$ for all loops γ, since we can define

$$h : [0,1]^2 \to X, h(s,t) = \gamma(s).$$

Given γ_1, γ_2, with $\gamma_1 \sim \gamma_2$, we know there exists a based homotopy h from γ_1 to γ_2. Then $k(s,t) = h(s, 1-t)$ is a based homotopy from γ_2 to γ_1, hence $\gamma_2 \sim \gamma_1$.

Given γ_3, and given a based homotopy k from γ_2 to γ_3, consider

$$l : [0,1]^2 \to X, l(s,t) = \begin{cases} h(s, 2t) & \text{if } 0 \leq t < \frac{1}{2} \\ k(s, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

This is a based homotopy between γ_1 and γ_3, hence $\gamma_1 \sim \gamma_3$.

We see that based homotopy is indeed an equivalence relation. □

Now that we know based homotopy is an equivalence relation, we can take the quotient set of $\Omega(X, x_0)$ to lump loops together in various sets of loops, such that there exist based homotopies between any pair of loops within one set. We can use this to define the first homotopy group, also known as fundamental group.

Definition 6.10. The **fundamental group of** X **at basepoint** x_0 is defined as $\pi_1(X, x_0) := \Omega(X, x_0)/{\sim}$, where $\gamma_1 \sim \gamma_2$ if and only if there exists a based homotopy between the two.

Although we call it 'group' already, notice that we have not given this quotient set a group structure yet. We have not even proven that there exists a meaningful binary operation on this set, since we have only seen concatenation on $\Omega(X, x_0)$. We have yet to check that it is well-defined on $\pi_1(X, x_0)$. Define concatenation as the map from $\pi_1(X, x_0) \times \pi_1(X, x_0)$ to $\pi_1(X, x_0)$, where $[\gamma_2][\gamma_1] := [\gamma_2\gamma_1]$.

Lemma 6.11. *Concatenation is well-defined as operation from* $\pi_1(X, x_0) \times \pi_1(X, x_0)$ *to* $\pi_1(X, x_0)$.

Proof. We need to check that concatenation yields the same resulting equivalence class, regardless of which representatives are used. In other words, if $\gamma_1 \sim \gamma_1'$, and $\gamma_2 \sim \gamma_2'$, then $\gamma_2\gamma_1 \sim \gamma_2'\gamma_1'$, where $\gamma_1 \sim \gamma_2$ if and only if there exists a based homotopy between the two.

Let h be the based homotopy between γ_1 and γ_2, and let h' be the based homotopy between γ_1' and γ_2'. Define the function

$$k : [0,1]^2 \to X, k(s,t) = \begin{cases} h(2s,t) & \text{if } 0 \leq s < \frac{1}{2} \\ h'(2s-1,t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

By construction, $l(s,0) = (\gamma_2\gamma_1)(s)$ and $l(s,1) = (\gamma_2'\gamma_1')(s)$. Since $h(1,t) = p = h'(0,t)$ by the fact that they are based homotopies, the two pieces of the function agree at $s = \frac{1}{2}$. We also see each piece individually is continuous, since they are both compositions of continuous functions. Finally, we see $l(0,t) = l(1,t) = p$. Therefore, this is a based homotopy between $\gamma_2\gamma_1$ and $\gamma_2'\gamma_1'$, and that proves that concatenation is well-defined on $\pi_1(X,x_0)$. □

Now that we know concatenation carries over to $\pi_1(X,x_0)$, we can use lemma 6.7 to immediately see that concatenation is associative on $\pi_1(X,x_0)$. There is no more need to include 'up to homotopy', since we have already dealt with that by using equivalence classes.

Since we have called the fundamental group a group, we are now also going to prove the other two properties a group needs to have, namely the existence of an identity element and the existence of an inverse for each element. The most logical option for the identity would be the loop $\gamma_{x_0} : [0,1] \to (X,x_0), s \mapsto x_0$ that just stays at the basepoint x_0 all the time, so we will now verify that γ_{x_0} is indeed the identity.

Definition 6.12. The constant loop at x_0 will be denoted $\gamma_{x_0} : [0,1] \to (X,x_0), s \mapsto x_0$.

Lemma 6.13. *The equivalence class $[\gamma_{x_0}]$ of the constant loop at x_0 acts as the identity for concatenation on $\pi_1(X,x_0)$. In other words, $[\gamma][\gamma_{x_0}] = [\gamma_{x_0}][\gamma] = [\gamma]$ for any $[\gamma] \in \pi_1(X,x_0)$.*

Proof. Let $[\gamma] \in \pi_1(X,x_0)$. Then $\gamma : [0,1] \to X$ is a loop with basepoint p. Since $[\gamma_{x_0}][\gamma] = [\gamma_{x_0}\gamma]$, we need to prove that $[\gamma_{x_0}\gamma] = [\gamma]$. Because concatenation is well-defined, it is enough to verify that there exists a based homotopy between $\gamma_{x_0}\gamma$ and γ.

We know

$$(\gamma_{x_0}\gamma)(s,t) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s < \frac{1}{2} \\ p & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Therefore, we can define

$$f : [0,1] \to [0,1], f(s) = \begin{cases} 2s & \text{if } 0 \leq s < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Since $(\gamma_{x_0}\gamma)(s) = \gamma(f(s))$, one is a reparametrisation of the other, and Lemma 6.6 gives us the desired based homotopy.

A completely analogous proof shows that there is also a based homotopy from $\gamma\gamma_{x_0}$ to γ, so $[\gamma_{x_0}]$ is indeed the identity for concatenation on $\pi_1(X,x_0)$. □

We now only need to prove that concatenation has inverses, and then we will have proven that $\pi_1(X,x_0)$ is actually a group.

Definition 6.14. Given a loop (or path) $\gamma : [0,1] \to (X,x_0)$, we define $\gamma^{-1}(s) = \gamma(1-s)$, or in other words, the loop (or path) γ but in reverse.

Lemma 6.15. *The inverse of the equivalence class $[\gamma]$ is the equivalence class $[\gamma^{-1}]$. In other words, $[\gamma][\gamma^{-1}] = [\gamma^{-1}][\gamma] = [\gamma_{x_0}]$.*

Proof. Since $[\gamma^{-1}][\gamma] = [\gamma^{-1}\gamma]$, we need to find a based homotopy between γ_{x_0} and $\gamma^{-1}\gamma$.

Define

$$h : [0,1]^2 \to X, h(s,t) = \begin{cases} \gamma(2(1-t)s) & \text{if } 0 \leq s < \frac{1}{2} \\ \gamma((1-t)(2-2s)) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

We see $h(s,0) = \gamma^{-1}\gamma$ and $h(s,1) = \gamma(0) = p$. Furthermore, the two pieces line up for all t, and each piece individually is continuous thanks to being a composition of continuous functions, hence h is continuous. Finally, $h(0,t) = \gamma(0) = h(1,t)$, so this is in fact a based homotopy.

An analogous proof works to prove that there is a based homotopy between $\gamma\gamma^{-1}$ and γ_{x_0}, which completes our proof. □

As a matter of fact, this lemma also works for paths. We will state it here without proof, since the proof is the same as for loops.

Lemma 6.16. *Given a path γ with basepoint x_0, there is a based homotopy between the loop $\gamma\gamma^{-1}$ and the constant loop γ_{x_0}.*

With the help of all of these lemmas, we can easily prove that $\pi_1(X, x_0)$ is a group.

Theorem 6.17. *The fundamental group $\pi_1(X, x_0)$ has a group structure.*

Proof. Simply combine Lemmas 6.7, 6.13, and 6.15. □

This group is actually far more useful than might initially be apparent, because in path-connected spaces, it does not depend on the choice of basepoint, as is described in Hatcher.[13]

Theorem 6.18. *For any order parameter space X, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, y_0)$ for any $x_0, y_0 \in X$.*

Proof. Since X is a connected topological manifold, it is path-connected. Let η be a path from q to p. Define

$$h : \pi_1(X, x_0) \to \pi_1(X, y_0), [\gamma] \mapsto [\eta^{-1}\gamma\eta],$$

where $\eta^{-1}\gamma\eta$ is a concatenation of paths, but with a loop as result. h is a homomorphism, since we can see $h([\gamma_2][\gamma_1]) = h([\gamma_2\gamma_1]) = [\eta^{-1}\gamma_2\gamma_1\eta] = [\eta^{-1}\gamma_2\eta\eta^{-1}\gamma_1\eta] = [\eta^{-1}\gamma_2\eta][\eta^{-1}\gamma_1\eta] = h([\gamma_2])h([\gamma_1])$. Here, we use Lemma 6.16. Additionally, h has an inverse. Define

$$k : \pi_1(X, y_0) \to \pi_1(X, x_0), [\gamma'] \mapsto [\eta\gamma'\eta^{-1}].$$

We can verify $h(k[\gamma']) = h([\eta\gamma'\eta^{-1}]) = [\eta^{-1}\eta\gamma'\eta^{-1}\eta] = [\gamma']$, and similarly $k(h([\gamma])) = [\gamma]$. Therefore, since h has is a homomorphism and has an inverse, it must be an isomorphism. □

In other words, the basepoint we choose is irrelevant. Therefore, we sometimes denote the fundamental group by $\pi_1(X)$, with the basepoint dropped.

7 Classifying Defects

So, our question now is: how does the fundamental group connect to the classification of defects? We have seen Theorem 5.8, which tells us how homotopies connect to local surgeries. We already know how local surgeries classify defects. We will go back to this, and see where the fundamental group comes in.

Let us take a closer look at Theorem 5.8. Since f and g are continuous, $f(C_r)$ and $g(C_r)$ are loops in X, and we can use the fundamental group of X to classify these loops. There are still two problems.

Firstly, $f(C_r)$ and $g(C_r)$ need not share the same basepoint. Secondly, an element of the fundamental group is a class of loops that are equivalent up to based homotopy, whereas we want to consider classes of loops that are equivalent up to regular homotopy.[14]

Fortunately, the solutions to both of these issues are fairly simple to understand, although they are much harder to prove.

[13] A. Hatcher, *Algebraic Topology*, 2001.

[14] These notions are different even when the loops share a basepoint.

Let us define the space we would like to work on, namely classes of homotopic loops on X, not necessarily with the same basepoint.

Definition 7.1. Define $[A, X] := C(A, X)/\sim$, where $f \sim g$ if and only if f and g are homotopic for $f, g \in C(A, X)$.

Evidently, $[S^1, X]$ is the space of loops we want to work with, and we have seen there exists a bijection between classes of equivalent point or line defects (depending on the dimension of the domain of the configurations) on a space X and this space $[S^1, X]$ - since a local surgery guarantees equivalence and exists if and only if a homotopy exists, the homotopy classes of loops correspond to the classes of equivalent defects. Now, we need to find how $[S^1, X]$ relates to $\pi_1(X, x_0)$ for any point $x_0 \in X$.

In order to find this relation, we will follow Hatcher[15], Chapter 0. Specifically, the part about the homotopy extension property, most importantly Proposition 0.16. We will not need this in its full generality. All we want is the following lemma, however, it does not seem like there is a simpler proof for this than using the general case and filling in the specific spaces we want to work with. The reader who is not interested in the proof can skip it.

Lemma 7.2. *Given a map $f : (S^n, *) \to (X, x_0)$ for some pointed order parameter space (X, x_0) and given some path or loop $\gamma : [0, 1] \to (X, x_0)$, we can find a homotopy $h : S^n \times [0, 1] \to X$ such that $h(s, 0) = f(s)$ and $h(*, t) = \gamma(t)$.*

Proof. First, we remark that S^n is a CW-complex like required in Hatcher's Proposition 0.16, and $\{*\}$ is a subcomplex. It does not really matter what this means, and explaining the concept would be well outside the scope of this thesis. However, for the reader familiar with CW-complexes, this should be a very simple observation.

Then, we remark we can see $\gamma : [0, 1] \to (X, x_0)$ as a homotopy $\tilde{\gamma} : \{*\} \times [0, 1] \to X$, or in Hatcher's notation, $\tilde{\gamma}_t : \{*\} \to X$, with $t \in [0, 1]$.

Finally, we can apply Hatcher's Proposition 0.16 and the homotopy extension property as discussed in Hatcher. □

From this lemma, we will be able to easily derive that the basepoints of $f(C_r)$ and $g(C_r)$ can be made to match, in the sense that we can find two loops, one of which is homotopic to $f(C_r)$ and the other of which is homotopic to $g(C_r)$, that both have the same, predetermined basepoint.[16]

Lemma 7.3. *Given a map $f : S^n \to X$ for some order parameter space X, given some point $* \in S^n$ and given some point $x_0 \in X$, we can find a map $g : (S^n, *) \to (X, x_0)$ such that f is homotopic to g.*

Proof. Use Lemma 7.2. Since X is path-connected, we can find a path $\gamma : [0, 1] \to X$, $\gamma(0) = f(*), \gamma(1) = x_0$. Then we get a homotopy $h : S^n \times [0, 1] \to X$ with $h(s, 0) = f(s)$ and $h(*, t) = \gamma(t)$. Specifically, $h(*, 1) = \gamma(1) = x_0$. Therefore, we can define $g : (S^n, *) \to (X), x_0, g(s) = h(s, 1)$, and this map is homotopic to f. □

Therefore, since we wish to consider two loops $f(C_r)$ and $g(C_r)$ up to homotopy (regular homotopy, and not based homotopy), we can transform them from two loops that arise from looking at an actual configuration into two loops $\alpha, \beta : [0, 1] \to X$ at some given basepoint x_0, such that $f(C_r)$ is homotopic to α and $g(C_r)$ is homotopic to β. This means that, for classification, we can completely ignore the configurations for a moment, and simply look at the problem of classifying loops in X at a given basepoint x_0 up to (regular) homotopy.

In order to do this, we will first have to define a group action from $\pi_1(X, x_0)$ on homotopy groups.

[15] A. Hatcher, *Algebraic Topology*, 2001.

[16] For a more visual reasoning, but without proof (since the proof is not very nice if made rigourous), let p be the basepoint of $f(C_r)$ and let q be the basepoint of $g(C_r)$. Let γ be a path from q to p. Then $\gamma^{-1}g(C_r)\gamma$ is a loop homotopic to $g(C_r)$ with the same basepoint as $f(C_r)$. The new loop will look like the old loop with a string attached, and the homotopy involves slowly squishing the string into a point.

Definition 7.4. A **group action** of a group G on a set X is a homomorphism φ from G to S_X, where S_X is the group of bijections $f : X \to X$ under composition. The notation is generally that $g(x) := \varphi(g)(x)$.

The **orbit** of a certain $x \in X$ is defined as $G(x) := \{g(x) : g \in G\} \subset X$. If the group action is conjugation, the resulting orbits are called **conjugacy classes**.

The **identity** of a group will be denoted e.

Definition 7.5. The ***n*-th homotopy group of a topological space Y at basepoint** y is given by $\pi_n(Y, y) := \{\alpha : S^n \to Y : \alpha(*) = y\}/\sim$, for some fixed point $* \in S^n$, where $\alpha, \beta : S^n \to Y, \alpha \sim \beta$ if and only if there exists a based homotopy between α and β.

It should be noted that we can identify S^n with $[0,1]^n$, but where the entire boundary acts as a single point. Therefore, any map $k : (S^n, *) \to (Y, y)$ has an associated map $\tilde{k} : [0,1]^n \to (Y, y)$, where this notation means that $\tilde{k}(\partial([0,1]^n)) = \{y\}$. Like the loop notation, the designated point in $[0,1]^n$ with its boundary identified as a single point is implied.

The operation on this group is a form of concatenation. Given $f, g : [0,1]^n \to (Y, y)$, define the concatenation

$$gf : [0,1]^n \to (Y, y), (gf)(x_1, x_2, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n) & \text{if } 0 \leq x_1 < \frac{1}{2} \\ g(2x_1 - 1, x_2, \dots, x_n) & \text{if } \frac{1}{2} \leq x_1 \leq 1 \end{cases}$$

Now, with the n-th fundamental group defined, we can define a group action from $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Given a function $\alpha : (S^n, *) \to (X, x_0)$, and a loop $\gamma : [0,1] \to (X, x_0)$, we can use Lemma 7.2 once again. By definition, $\alpha(*) = \gamma(0) = x_0$. Therefore, we can find a homotopy $h : [0,1]^2 \to X$ with $h(s, 0) = \alpha(s)$ and $h(*, t) = \gamma(t)$. We can then define $[\gamma]([\alpha])(s) := [h(s, 1)]$, and this is the group action we want.

Lemma 7.6. *The group action of the fundamental group $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ described above is indeed a group action.*

Proof. We can see that if we have γ, η loops in (X, x_0), and $\alpha : (S^n, *) \to (X, x_0)$ then $[\eta]([\gamma]([\alpha])) = [\eta\gamma]([\alpha])$. Namely, let h be the homotopy from α to a representative α' of $[\gamma]([\alpha])$ as used in the definition above, and let k be the homotopy from α' to a representative α'' of $[\eta]([\gamma]([\alpha]))$, also as used in the definition above. Then $h(s, 0) = \alpha(s), h(0, t) = \gamma(t), k(s, 0) = \alpha' = h(s, 1)$, and $k(0, t) = \eta$. We can now define

$$m : S^n \times [0,1] \to X, m(s, t) = \begin{cases} h(s, 2t) & \text{if } 0 \leq t < \frac{1}{2} \\ k(s, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

This homotopy from α to α'' clearly shows that $[\eta]([\gamma]([\alpha])) = [\eta\gamma]([\alpha])$.

The action is also well-defined, in the sense that if we have loops η, γ with basepoint x_0 with $\eta \in [\gamma]$, and if we have $\alpha, \beta : (S^n, *) \to (X, x_0)$ with $\beta \in [\alpha]$, then $[\eta]([\beta]) = [\gamma]([\alpha])$.[17] The proof for that is very tricky and well outside the scope of this thesis, but it is documented in Hatcher, in Proposition 4A.1. $\square$

In the case $n = 1$ (which is the case we will be mostly interested in for now), this group action becomes a rather simple one, namely conjugation.

Lemma 7.7. *Given loops α, γ with basepoint x_0 in an order parameter space X, $[\gamma]([\alpha]) = [\gamma\alpha\gamma^{-1}]$.*

Proof. We have to prove there exists a homotopy h between α and $\gamma(\alpha\gamma^{-1})$ such that $h(0, t) = \gamma(t)$. When we pair this with the well-definedness of the group action, we get the desired result.

Define

$$h : [0,1]^2 \to X, h(s, t) = \begin{cases} \gamma^{-1}(4(s - \frac{t}{4}) + 1) & \text{if } 0 \leq s < \frac{t}{4} \\ \alpha((1 - \frac{t}{2})^{-1}(s - \frac{t}{4})) & \text{if } \frac{t}{4} \leq s < 1 - \frac{t}{4} \\ \gamma(4(s - (1 - \frac{t}{4}))) & \text{if } 1 - \frac{t}{4} \leq s \leq 1 \end{cases}$$

[17] In fact, the homotopies used for the definitions of the group actions are homotopic to each other - homotopies are functions, too, after all, and can be homotopic to each other.

For a fixed t, this becomes the loop with basepoint $\gamma(t)$ that first traverses γ in reverse order, from $\gamma(t)$ to $\gamma(0)$, then traverses α, and finally traverses γ from $\gamma(0)$ back to $\gamma(t)$. Notice how $h(s,0) = \alpha(s)$ and $h(s,1)$ is homotopic to $\gamma(\alpha\gamma^{-1})$ (by Lemma 6.6). Furthermore, $h(0,t) = \gamma^{-1}(1-t) = \gamma(t) = h(1,t)$. Finally, since all pieces are compositions of continuous functions, and since they agree where they join, h is continuous. This proves that h is a homotopy with the required criteria.

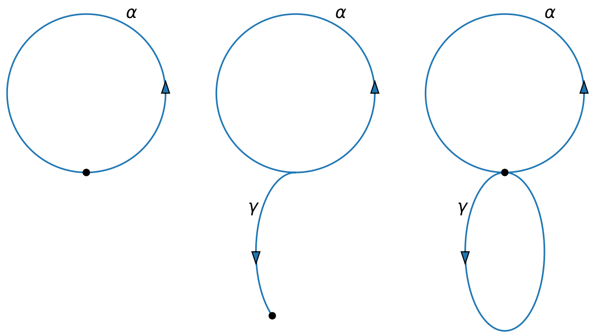

Figure 5: Image showcasing the homotopy. In order to move the basepoint through the loop γ, the resulting loop first moves through a piece of γ, but in opposite direction (of which the size increases as t increases), then through α, and then through the same piece of γ but now in the normal direction, to get back to the basepoint. Over all, the resulting loop is $\gamma\alpha\gamma^{-1}$.

□

Now, we can find how to classify $[S^1, X]$.

Lemma 7.8. *For any order parameter space X, the map*

$$\mathrm{id} : \pi_1(X, x_0) \to [S^1, X], [\alpha] \mapsto [\alpha]$$

induces a bijection $\tilde{\mathrm{id}}$ *between the set of conjugacy classes* $\{[\gamma\alpha\gamma^{-1}] : [\alpha], [\gamma] \in \pi_1(X, x_0)\}$ *and* $[S^1, X]$.

Proof. All equivalence classes in the following proof that are actually notated as $[\alpha]$, with square brackets, will be those in $\pi_1(X, x_0)$. The ones in $[S^1, X]$ are merely described with words.

We can first use 7.3 to show that any element[18] of $[S^1, X]$ has a representative with basepoint x_0. Therefore, id is surjective. Because each element of $\pi_1(X, x_0)$ is in some conjugacy class, $\tilde{\mathrm{id}}$ must also be surjective.

Since conjugation is the action discussed in Lemma 7.6 in the case $n = 1$, we can use that description to more easily prove the well-definedness of $\tilde{\mathrm{id}}$. Two elements $[\alpha], [\beta]$ of the same orbit in $\pi_1(X, x_0)$ will fulfil the equation $[\beta] = [\gamma]([\alpha])$ for some $[\gamma] \in \pi_1(X, x_0)$. In other words, there will be a (regular, not based) homotopy h between any two[19] representatives α and β, where $h(0,t) = \gamma(t)$. But this implies any two elements of the same orbit will map to homotopic loops.[20] Therefore, $\tilde{\mathrm{id}}$ is well-defined - two different elements of the same conjugacy class will map to the same element of $[S^1, X]$.

If we have two loops α, β with basepoint x_0, which are representatives of the same element of $[S^1, X]$, then by definition, these two loops are homotopic via a (regular, not based) homotopy h. Then by definition of the group action in Lemma 7.6 (which is conjugation in the case of $n = 1$), $[\beta] = [h(0,t)]([\alpha])$. Therefore, if $\tilde{id}([\alpha]) = \tilde{id}([\beta])$, then $[\alpha]$ and $[\beta]$ are in the same conjugacy class, and $\tilde{\mathrm{id}}$ is injective.

It follows that $\tilde{\mathrm{id}}$ is a bijection. □

With all of these lemmas taken care of, we can finally prove the main result of this chapter (and also one of the main results of the thesis), which will classify point defects in $\mathbb{R}^2$ and line defects in $\mathbb{R}^3$.

[18]Recall that such an element is an equivalence class of loops that have (regular, not based) homotopies between them.
[19]The group action is well-defined, after all.
[20]These loops are only homotopic by regular homotopy and not by based homotopy.

Theorem 7.9. *Point defects in configurations $f : \mathbb{R}^2 \to X$, and line defects in configurations $f : \mathbb{R}^3 \to X$ are classified by conjugacy classes of $\pi_1(X)$, in the sense that a set of equivalent defects (via the local surgery definition of equivalence) corresponds to a single conjugacy class.*

Proof. We know from Theorem 5.8 that these defects are elements of $[S^1, X]$. Theorem 7.8 tells us that there is a bijection between $[S^1, X]$ and the set of conjugacy classes of $\pi_1(X, x_0)$. This proves the desired statement. □

8 Defect Classification Examples

To show just how useful this theory is for classifying defects, we will discuss some examples here. Let us start with the $O(2)$-model. We have already seen these defects, but for clarity's sake, here they are again, in Figure 6.

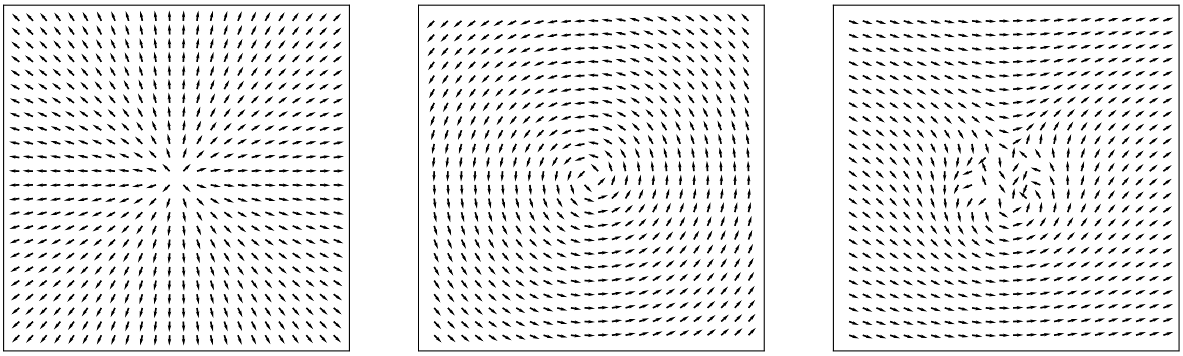

Figure 6: An image showing three defects in the region $[-1, 1]^2$.

It is a well-known fact that $\pi_1(S^1) = \mathbb{Z}$. Since $\mathbb{Z}$ is abelian, its conjugacy classes are simply its elements. Therefore, each loop corresponds to a number, which is called its winding number. This is how many full rotations around the circle the loop makes, namely. If we look at the leftmost configuration in Figure 6 and move around the origin once counterclockwise, we can see that the resulting values when we apply the configuration[21] also rotate around S^1 once counterclockwise. Therefore, it corresponds to $1 \in \mathbb{Z}$. The same applies to the configuration in the middle. It follows that these point defects are equivalent. We can now also explain that a vortex is a defect with winding number 1. Notice that we now immediately now that these defects are not removable, because any uniform state has a winding number of 0.

The configuration on the right is different. While we cannot really tell what happens when we get close to the origin, thanks to the low resolution of the image, we already know that the classification is independent of loop size. If we take a larger loop around the origin and apply the configuration, we see the values wobble around a bit, but do not actually fully rotate around the circle at all. Therefore, this is a removable defect.

Instead of limiting ourselves to S^1, we can also allow the configuration to rotate in another direction, so that it becomes a function from $\mathbb{R}^2$ to S^2. In other words, arrows can now also point into or out of the screen (or paper). It is also well-known that $\pi_1(S^2) = \{e\}$, the trivial group containing only the identity. Therefore, all defects must be removable. We can an example when we look at Figure 6. By also allowing all arrows to rotate out of the screen (or paper), we can actually define a local surgery between both the left and middle configuration and the uniform state. It is equal to the uniform state pointing out of the screen (or paper) in the middle, then slowly rotating down as we move away from the origin, and turning into either of the two vortices displayed.

Another model often studied by physicists is the nematic model. Here, instead of having vectors pointing

[21]The configuration is a function, after all.

on S^1 or S^2, we have line segments[22] that can point in any direction. This means that we cannot distinguish the configuration if all line segments are flipped. This is equivalent to pointing on S^1 or S^2, but with opposite points glued together. This gives $\mathbb{RP}^1$ or $\mathbb{RP}^2$, a projective space, as the order parameter space X.

Let us first look at the case $X = \mathbb{RP}^1$. Now, it turns out that $\mathbb{RP}^1$ is simply homeomorphic to S^1.[23] This implies that it has the same fundamental group as S^1, so $\pi_1(\mathbb{RP}^1) = \mathbb{Z}$. However, instead of the corresponding element being the number of full rotations, it is now the number of half rotations. This makes sense, because line segments are invariant under half rotations, whereas vectors need a full rotation to stay invariant. For an example with a half rotation, see Figure 7. For an example with a full rotation, see Figure 8.

Now, we will discuss what happens when we add the extra direction of rotation again, to get $X = \mathbb{RP}^2$. It is known that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, the two element cyclical group. This implies there is one class of removable defects, and one class of non-removable defects. For the configuration in Figure 8, we can do the same as what we did with $X = S^2$, namely rotate the line segments out of the screen (or paper). It has now become a removable defect. For the configuration in Figure 7, however, this cannot be done in a continuous manner. Therefore, this is an element of the only class of non-removable defects with $X = \mathbb{RP}^2$.

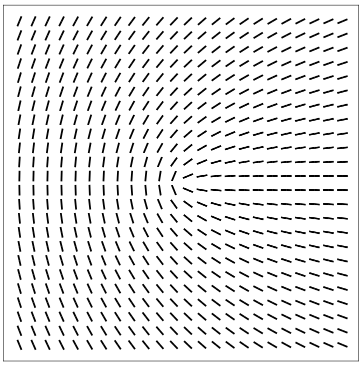

Figure 7: An image showing a defect in the region $[-1, 1]^2$.

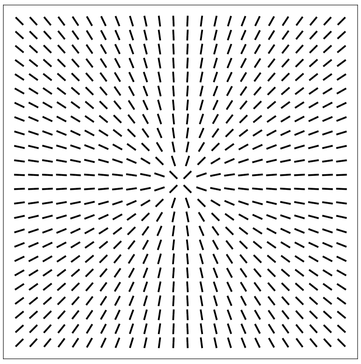

Figure 8: An image showing another defect in the region $[-1, 1]^2$.

We can see how quickly we can classify defects once we know the fundamental group, especially if it is abelian. However, we still need a way to calculate the fundamental groups easily once the spaces we use get more complicated.

[22] Or cylindrical rods, if you are a physicist.

[23] As a matter of fact, we do not even need the spaces to be homeomorphic. We only need them to be homotopy equivalent, but since we did not introduce that concept in the thesis, it was not named here. Intuitively, it should be clear homeomorphisms do not change the fundamental group.

9 Stabilisers and Cosets

This section assumes some basic group theory knowledge. If the reader does not have this yet, it is explained well in, for example, Armstrong.[24]

Since we want to compute the fundamental group, a bit of group theory is going to come in handy. We can use it to represent our order parameter space X in a convenient way, that will make computing the fundamental group much easier.

Firstly, let us recall the definition of a group action, and some associated terms.

Definition 9.1. A **topological group** is a group that is also a topological space, where the multiplication and inversion maps are continuous. We also demand that any group action G induces on a set X be continuous, in the sense that the associated map $\tilde{\varphi} : G \times X \to X, \tilde{\varphi}(g, x) := \varphi(g)(x) = g(x)$ is continuous.

The **stabiliser** of x is defined as $H_x := \{g \in G : g(x) = x\} \subset G$.

A group action of G on X is **transitive** when there is only one orbit, in other words, $G(x) = X$ for any $x \in X$. It is said that G acts transitively on X.

Notice that $G(x) = X$ for any $x \in X$ implies that the action is transitive, since distinct orbits must be disjoint, and therefore this must be the only orbit.

A relatively simple example of a group acting transitively would be $G = \mathrm{SO}(2), X = S^1$. $\mathrm{SO}(2)$ is the group of all planar rotations, and S^1 is a (unit) circle, so we see we can use $\mathrm{SO}(2)$ to rotate any point of the circle to any other point. To be a bit more rigourous, we can embed S^1 in $\mathbb{C}$, and represent $\mathrm{SO}(2)$ as a subset of $\mathbb{C}$, too. Specifically, any element of $\mathrm{SO}(2)$ acting on S^1 can be represented as multiplication by a complex number with norm 1:

$$R(\theta) : \mathbb{C} \to \mathbb{C}, z \mapsto e^{i\theta} z.$$

It follows that we can restrict this to

$$\tilde{R}(\theta) : S^1 \to S^1, z \mapsto e^{i\theta} z,$$

since $|e^{i\theta} z| = |e^{i\theta}||z| = 1$.

We can easily see that the multiplication works as expected - $R(\theta)R(\varphi) = R(\theta + \varphi)$, by the fact that multiplying exponentials means adding their exponents.

We can verify that the point 1 can be rotated to any other point in S^1:

$$e^{i\theta} \cdot 1 = e^{i\theta}.$$

Since S^1 is parametrised by $e^{i\theta}$, the map $G(1) = X$, and G does indeed act transitively on X.

The stabiliser of any point $x \in S^1$ is given by the set of elements such that $\tilde{R}(\theta)(x) = x$. This implies that $e^{i\theta} = 1$, or in other words, $\tilde{R}(\theta)$ has to be the identity map. Therefore, the stabiliser of any x is equal to $H_x = \{\mathrm{id}\}$.

The fact all stabilisers are equal is not surprising at all, since a well-known fact in group theory is the following.

Lemma 9.2. *Let G be a group that acts on a set X. Let x, y be elements of X. If $y = g(x)$ for some $g \in G$, then $H_y = gH_xg^{-1} := \{ghg^{-1} : h \in H_x\}$. The associated map $f : H_x \to H_y, h \mapsto ghg^{-1}$ is an isomorphism.*

Proof. See Armstrong.[25] □

While this is a very common fact, something interesting happens when we demand G be a topological group that acts transitively on a topological manifold with metric X.

[24] M.A. Armstrong, *Groups and Symmetry*, 1988.
[25] M.A. Armstrong, *Groups and Symmetry*, 1988.

Lemma 9.3. *Let G be a Lie group, and let X be a set. Then H_p and H_q are homeomorphic as topological spaces for all $p, q \in X$.*

Proof. The first thing we need to do is remark that a Lie group is a topological group.

We know that $C_g(h) := ghg^{-1}$ is a bijection, from Lemma 9.2. To prove it is a homeomorphism, we only have to prove that both C_g and C_g^{-1} are continuous.

Let us look at $C_g(h) = ghg^{-1}$. By the definition of a topological group, the multiplication map and inversion map are both continuous in all arguments, so it follows that $C_g(h)$ is continuous (in both g and h, although we only need continuity in h). Similarly, since $C_g^{-1}(h) = g^{-1}hg$, $C_g^{-1}(h)$ is continuous in both g and h, too.

It follows that both C_g and C_g^{-1} are continuous bijections, and therefore homeomorphisms. This proves the desired result. □

It follows that H_x is independent of x up to isomorphism and homeomorphism.

Let us introduce some more notation.

Definition 9.4. Given a group G and a subgroup H, the **left cosets** of H are all the sets of the form $gH := \{gh : h \in H\}$.

The collection of all left cosets is called the **left coset space** and is denoted as $(G/H)^l := \{gH : g \in G\}$.[26]

Recall that distinct cosets are disjoint, so that left coset space actually makes sense.

Now, we can use an existing theorem to find something that represents X.

Lemma 9.5 (Orbit-stabiliser theorem)**.** *For any $x \in X$, we have that there is a bijective correspondence*

$$g(x) \in G(x) \mapsto gH_x \in (G/H_x)^l.$$

Proof. See Armstrong.[27] □

Notice how H is not necessarily a normal subgroup, as seen in Lemma 9.2, so $(G/H_x)^l$ is generally not a group, but just a set. However, the topology of G does carry over - $(G/H_x)^l$ comes equipped with the quotient topology. Let

$$\pi : G \to (G/H_x)^l, g \mapsto gH$$

be the projection map that sends every element of a coset to the same coset. Then a set $U \subset (G/H_x)^l$ is open if and only if its preimage $\pi^{-1}(U) \subset G$ is open. Now that we have a topology, we would also like for there to be continuity.

First, remark that we can use Lemma 9.5 and combine it with the fact that for every x, $G(x) = X$. Specifically, $G(x_0) = X$, and that gives a bijection $f : X \to (G/H_{x_0})^l, g(x_0) \mapsto gH_{x_0}$. With this bijection being defined, we can prove a very useful theorem. We will first need a lemma.

Lemma 9.6. *Let $\varphi : G \times X \to X$ be a group action from a Lie group G on an order parameter space X. Let $x_0 \in X$ be a point. Then $h : G \to X, h(g) := \tilde{\varphi}(g, x_0) = g(x_0)$ is an open map.*

While this is true, we will not provide a proof for this, because it was very complicated and no source that could properly be referenced was found.

Theorem 9.7. *Given a Lie group G acting transitively on an order parameter space X and a point $x_0 \in X$, the function f as defined above is a homeomorphism between X and $(G/H_{x_0})^l$.*

[26] This is done to avoid confusion with the quotient group, when H is a normal subgroup, and to also make the difference between left and right coset space clearer.

[27] M.A. Armstrong, *Groups and Symmetry*, 1988.

Proof. First, remark that we can use Lemma 9.5 and combine it with the fact that for every x, $G(x) = X$. Specifically, $G(x_0) = X$, and that gives a bijection $f : X \to (G/H_{x_0})^l, g(x_0) \mapsto gH_{x_0}$.

Now, we need to prove that this map and its inverse are continuous. In other words, we have to prove that the preimage of an open set is open both under f and under f^{-1}. Since this is a bijection, the preimage of f^{-1} is equal to the image of f. This means we do not need to work with f^{-1} at all, and we can replace its continuity with the fact f maps open sets to open sets.

Given an open set $V \subset (G/H_x)^l$, we know that the set $\pi^{-1}(V) := \{g \in G : gH \in V\} \subset G$ is open. Now, $f^{-1}(V) := \{x \in X : x = g(x_0),\ gH \in V\} \subset X$. However, we know that the map $\tilde{\varphi} : G \times X \to X$ describing the homomorphism is continuous. It follows that $h : G \to X, h(g) := \tilde{\varphi}(g, x_0) = g(x_0)$ is a continuous function, and therefore, $h(\pi^{-1}(V)) := \{x \in X : x = g(x_0),\ gH \in V\} = f^{-1}(V)$. Since h is an open map, f is continuous.

Given an open set $U \subset X$, let us consider $f(U) := \{gH : g(x_0) \in U\} \subset (G/H_x)^l$. This set is open if and only if $\pi^{-1}(f(U)) = \{g : g(x_0) \in U\} \subset G$ is open. Now, consider $h^{-1}(U) := \{g : g(x_0) \in U\} = \pi^{-1}(f(U))$. Since h is continuous, $h^{-1}(U)$ is open, and therefore, so is $\pi^{-1}(f(U))$. Finally, we conclude $f(U)$ is open.

This proves f is a homeomorphism. □

Now we see why we wanted G to be a Lie group. We need continuity for homotopies, and therefore, it really is necessary to make sure X is homeomorphic to $(G/H_{x_0})^l$. Another nice thing is that H_x is actually independent of x up to isomorphism and homeomorphism, so we can take any point x as x_0, and the result will not change. We will drop the subscript x_0 for the next section.

10 Computing Fundamental Groups

In this chapter, we will find a very simple and intuitive way to compute the fundamental group.

First, we will look at the stabiliser H from last chapter, where we had a Lie group G acting transitively on an order parameter space X. In this chapter, we also demand that G is simply connected. In other words, $\pi_1(G) = \{e\}$.

We have no idea what H actually looks like, so first, we will assume H is discrete. For that, we will first need to look at H_0, the path component of the identity of H.

Definition 10.1. The **path component of the identity in *H*** is denoted H_0.

If we suppose that H is discrete, in other words, H_0 is the trivial group, then we can define an action from H on G, where $h(g) = gh^{-1}$. Note that $h'(h(g)) = gh^{-1}(h')^{-1} = g(h'h)^{-1} = (h'h)(g)$. Now, each orbit will be a set of the form $\{gh^{-1} : h \in H\} = \{gh : H \in H\} = gH$, so a left coset of H.

It can be proven that any action of a discrete group H on a Lie group G has the desired property that for any $g \in G$, there exists a neighbourhood U such that $U \cap h(U) = 0$ for every $h \in H$ with $h \neq e$. However, like Lemma 9.6, the proof is very complicated, and no good source could be found.

With this knowledge, we can apply Hatcher[28], Proposition 1.40, where we use G as the Y Hatcher uses, and H as the G Hatcher uses. Then it follows that $\pi_1((G/H)^l)$ is isomorphic to H.

However, this is not too useful, since many isotropy groups are not discrete. We will need to find a group that is closely related to H, but is discrete. The following lemma will be of help.

Lemma 10.2. *H_0 is a normal subgroup of H.*

Proof. Firstly, we need to prove that H_0 is a subgroup of H. Let g, h be elements of H_0, and let γ, η be paths connecting g and h to the identity, respectively. Then we can define a new path $\alpha : [0, 1] \to H_0, \alpha(s) = \gamma(s)\eta^{-1}(s)$. Notice how this new path uses the group operation, and not concatenation. This

[28] A. Hatcher, *Algebraic Topology*, 2001.

path η starts at the identity and ends at gh^{-1}. But this implies gh^{-1} is in the path component of the identity for any $g, h \in H$, which proves it is a subgroup.

Secondly, we need to prove that H_0 is a normal subgroup of H. Let g be an element of H_0, let γ be the path connecting g to the identity, and let h be an element of H. Now, define a path $\beta : [0,1] \to H_0, \beta(s) = h\gamma(s)h^{-1}$. This path starts at $hh^{-1} = e$ and ends at hgh^{-1}. Therefore, for any $g \in H_0$ and $h \in H$, we have that $hgh^{-1} \in H_0$, so H_0 is a normal subgroup. □

Now that we have a normal subgroup of H, we can look at the quotient group H/H_0. This will be a discrete group, since every path component is of the form hH_0 for some $h \in H$, and therefore, every path component is collapsed to a single point.

For an application of this quotient group, we will look at Mermin[29] again. Mermin proves the following:

Theorem 10.3. *Given a simply connected Lie group G acting transitively on an order parameter space X, with stabiliser H, the fundamental group $\pi_1(X)$ is isomorphic to the group H/H_0.*

Proof. Mermin proves that $\pi_1((G/H)^l)$ is isomorphic to H/H_0. Combine this with theorem 9.7, and we are done. □

We expect the proof can also be completed by using $(G/H_0)^l$ and H/H_0 in Hatcher's proposition, but we could not find how to do this properly.

11 Conclusion

In conclusion, we have looked at defects in ordered media. Specifically, point defects in $\mathbb{R}^2$ and line defects in $\mathbb{R}^3$. These can be classified by conjugacy classes of the fundamental group. If we have a group action of a simply connected Lie group on an order parameter space, we can use the stabiliser to compute the fundamental group of the order parameter space easily.

Further interesting subjects that could be looked into would be point defects in $\mathbb{R}^3$, and more different shapes.

12 References

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