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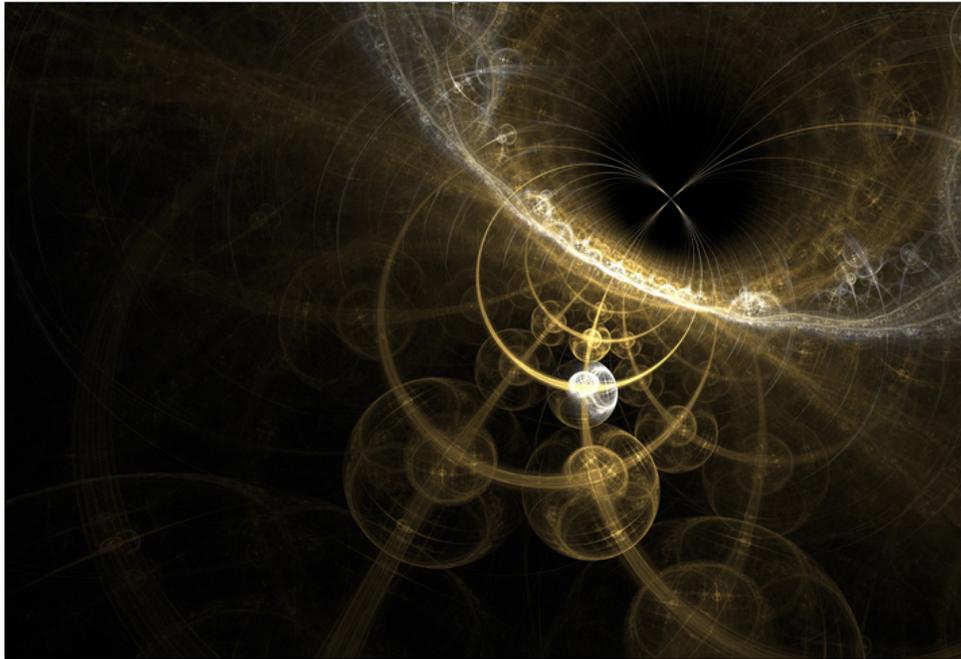
# Quantum theories in de Sitter space

Exploring the connections between representations of  $\mathfrak{sl}(2, \mathbb{R})$  and the quantized Klein-Gordon field on  $dS_2$

BACHELOR THESIS

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## Abstract

A de Sitter space models an infinitely expanding universe with a positive cosmological constant. Motivated by the fact that the symmetry group of two-dimensional de Sitter space  $dS_2$  is the Lie group  $SL(2, \mathbb{R})$ , we study the irreducible representations of  $\mathfrak{sl}(2, \mathbb{R})$ , which are labeled by their *conformal dimension*  $\Delta$ , analogous to the paper '*An invitation to the principal series*' by Anous and Skulte. It turns out that quantum theories allow 4 different types of representations, of which the principal series representations with conformal dimension  $\Delta = \frac{1}{2}(1 + i\nu)$  are of our main interest because they appear in the description of sufficiently massive scalar fields in  $dS_2$ . We show that this gives an equivalence between scalar fields in  $dS_2$  and  $\mathfrak{sl}(2, \mathbb{R})$ -representations acting on wave functions on the circle. We show how this representation can be transformed into the model of the simplest  $\mathfrak{sl}(2, \mathbb{R})$ -invariant scalar field theory, introduced by De Alfaro, Fubini and Furlan (DFF), modified to accommodate principal series representations. We solve the Klein-Gordon equation for the scalar field in  $dS_2$  and proceed to quantize this field and discuss the challenges that arise in this quantum field theory in curved spacetime, exploring the processes of renormalization and Bogolubov transformations and comparing the properties of the different vacuum states that arise in curved spacetime. After summarizing these results, we discuss some potential future directions for research on this topic.

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## 1 Introduction

In the far future, dark energy will dominate the expansion of the universe[1]. Although the nature of dark energy is not fully understood yet and is still a very active field of research, our current understanding is that it is an intrinsic property of spacetime[2]: Einstein's theory of general relativity contains the so-called *cosmological constant*  $\Lambda$  (which is measured to be positive[3]) that gives the energy of empty space, causing the universe to expand and therefore creating more dark energy. This leads to an exponentially expanding universe, of which the mathematical description is a four-dimensional *de Sitter space*[4]. Its isometry group is the group of *conformal* (angle-preserving) transformations[5], and this conformal invariance must be built in quantum theories that describe the universe.

The unification of quantum mechanics and general relativity is an open problem in physics[6][7], because these theories fundamentally disagree on the nature of spacetime. A first step in the right direction in the search for this unifying theory is considering quantum field theories instead of classical quantum mechanics, motivated by two observations: The formulation of field theories is much more in accordance with general relativity than classical mechanics in the way they treat the different spacetime dimensions. Second of all, there is a fundamental flaw in the quantum mechanical description of the universe: It, just like its classical counterparts, can only describe systems with a fixed number of particles, while we know that elementary particles are being created and annihilated constantly according to our knowledge of the subatomic world. This can be explained by the interpretation that there are universal fields in spacetime, of which the excitations can be observed as the corresponding elementary particles. Motivated by this, we're going to investigate scalar fields in de Sitter space, which can describe spin-0 particles like pions and the Higgs boson[8].

We're going to limit ourselves to the two-dimensional de Sitter space  $dS_2$  in this thesis for simplicity, which consists of one temporal and one spatial dimension. Because of its hyperbolic geometry, lower dimensional de Sitter space is also a popular model for studying quantum gravity as it models spacetime around the horizon of a black hole [9][10][11]. We will mostly follow the line of reasoning in the paper '*An invitation to the principal series*'[12] and reproduce, explain and develop the results that are presented here and discuss their consequences. We will mainly be interested in representations of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  of the  $dS_2$ -isometry group  $SL(2, \mathbb{R})$  before we implement our findings when exploring quantum fields in  $dS_2$ . Using the generators, we will explicitly construct the Hilbert spaces that the representations act on, and then we impose some restrictions on the  $\mathfrak{sl}(2, \mathbb{R})$ -representations such as irreducibility and unitarity to be of physical interest. This leads to four types of representations, of which the *principal series* representations will be our main focus because it appears in numerous physical situations:

As  $SL(2, \mathbb{R})$  is the isometry algebra of both  $dS_2$  and  $AdS_2$  [13], its representations appear in several two-dimensional quantum theories. The principal series type of representations appear when studying black holes in two spacetime dimensions [14], massive spin- $\frac{1}{2}$  fermions on the hyperbolic plane [15, Chapter 5][16], the behaviour of charged quantum field near the ergosphere of a Kerr black hole [17], and when studying massive scalar fields in  $dS_2$ [11][17, Appendix B][18], harmonic analysis of the symmetry group  $SL(2, \mathbb{R})$  [19]. Our main interest will be the appearance in the study of massive scalar fields in  $dS_2$ . The so-called DFF model[20] of  $\mathfrak{sl}(2, \mathbb{R})$ -invariant quantum mechanics must be altered slightly to accommodate principal series representations, and this altered version of the DFF model can be transformed into a  $\mathfrak{sl}(2, \mathbb{R})$ -representation the Hilbert space of wave functions on the circle. This representations shows up again when investigating scalar fields on  $dS_2$ , because its spatial component is homeomorphic to the circle.

When you quantize these scalar fields, you immediately encounter several interesting phenomena: Quantum field theories are infamous for the appearance of infinite quantities in calculations, although there are ways to deal with them[21], like renormalization or perturbation theory. Examples of this are self-interacting particles that cause divergent behaviour or even the energy of empty space being infinite, but there are also unsolved problems in physics regarding the properties of the ground states of quantum fields, like the *cosmological constant problem*[22]. Another problem of quantum fields in curved spacetime is that the definition of the vacuum state becomes ambiguous: If an observer in an inertial frame sees empty space, an observer in a different inertial frame might see a large number of particles. This leads to questions on the interpretation

of what it means to observe a particle, for instance in particle physics experiments. It is also interesting to explore behaviour of a quantum field in spacetime with  $n$ -point correlation functions[23], which are important in particle physics experiments[24]. We're going to investigate these three phenomena by quantizing the scalar field on  $dS_2$ , and address the challenges that arise in this quantum field theory.

## 2 De Sitter space

In this thesis, we're going to investigate the properties of quantum physical theories on a so-called *de Sitter space*, motivated by the fact that the geometry of the universe resembles the four-dimensional de Sitter space  $dS_4$  under certain assumptions[4]: Cosmological models are often built on the cosmological principle, which states that the universe is assumed to be homogeneous and isotropic on large enough scales. Our current understanding of the universe is that its expansion will accelerate forever due to the vacuum energy of space (determined by the cosmological constant  $\Lambda$ ) which is measured to be positive[3], and in the far future this will result in the cosmological constant dominating the expansion of the universe. For  $\Lambda > 0$ , we find that the curvature of the universe must be positive, and the corresponding model of the universe would be a de Sitter space, meaning that the geometric model of the universe will asymptotically become a de Sitter space over time. [25, Chapter 8].

Because the curvature of the universe is related to the cosmological constant  $\Lambda$ , we can express relate the cosmological constant to the *de Sitter length*  $\ell$  that gives the curvature of the de Sitter space, which we will explicitly construct with this parameter in the next section. This gives us the following relation for the cosmological constant in  $d$ -dimensional de Sitter space  $dS_d$ [26, p. 2]:

$$\Lambda = \frac{(d-1)(d-2)}{2\ell^2} \quad (2.1)$$

This means that in our universe which we model with  $dS_4$ , we have  $\Lambda = \frac{3}{\ell^2}$

### 2.1 Geometry of de Sitter space

In this thesis, we are specifically interested in quantum theories on  $dS_2$ , reducing the number of spatial dimensions in our model from 3 to 1 for simplicity. We can mathematically realize  $dS_2$  in the following way: We consider the generalized Minkowski space  $\mathbb{R}^{1,2}$  (which is  $\mathbb{R}^3$ , equipped with the Minkowski inner product with  $(-, +, +)$  as the metric signature) and let  $X^0$  denote the temporal coordinate and  $X^1$  and  $X^2$  denote the spatial coordinates of vectors  $X \in \mathbb{R}^{1,2}$ , where we already use superscripts for the indices to indicate that we consider them as coordinates of contravariant vectors. Because the signature of the Minkowski metric tensor  $\eta_{\mu\nu}$  is chosen to be  $(-, +, +)$ , the metric tensor and vectors  $X$  look like

$$X^\mu = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \end{pmatrix} \quad \text{and} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.2)$$

with  $\mu, \nu = 0, 1, 2$ . This gives the following metric, just as usual:

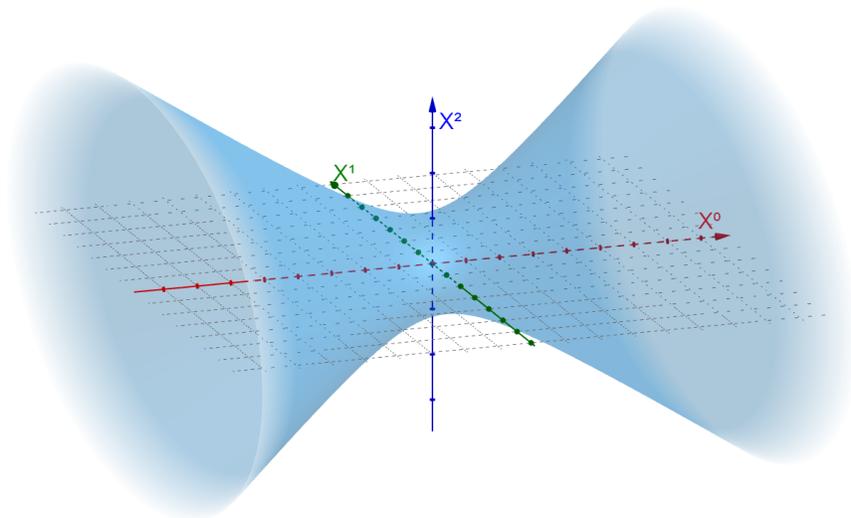
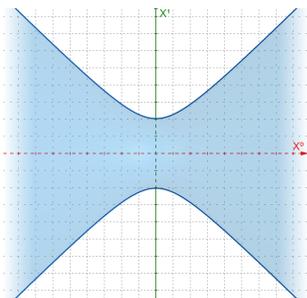
$$ds^2 \equiv dX^\mu dX_\mu = dX^\mu \eta_{\mu\nu} dX^\nu = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 \quad (2.3)$$

Here we make use of the Einstein summation convention, which we will keep doing for most of the remainder of the thesis.

The two-dimensional de Sitter space  $dS_2$  can now be realized as a submanifold of  $\mathbb{R}^{1,2}$  using the following defining equation, where the parameter  $\ell$  is the de Sitter length that is related to the curvature of the hyperboloid.

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = \ell^2 \quad (2.4)$$

This equation describes a hyperboloid, which is the dark blue hyperbola in the  $(X^0, X^1)$ -plane in figure 2, rotated about the  $X^0$ -axis. Figure 1 shows the hyperboloid  $dS_2$  in three dimensions.

Figure 1:  $dS_2$  as a submanifold of  $\mathbb{R}^{1,2}$ Figure 2: The hyperbola that results from the intersection of  $dS_2$  with the  $(X^0, X^1)$ -plane

We can parameterize this hyperboloid with a temporal coordinate  $\tau \in \mathbb{R}$  and an angle  $\theta$  to parameterize the spatial part of the  $dS_2$ : For given  $X^0 \in \mathbb{R}$ , the slice of the hyperboloid in the  $(X^1, X^2)$ -plane is precisely described by an equation defining a circle, which we parameterize with  $\theta \in [0, 2\pi]$  as described before. This means that  $dS_2$  is homeomorphic to the cylinder  $\mathbb{R} \times S^1$ . With these observations in mind, we write down the explicit parameterization of  $dS_2$  as a subspace of  $\mathbb{R}^{1,2}$ , with its points  $x \in dS_2$  given by coordinates  $(\tau, \theta) \in \mathbb{R} \times S^1$ :

$$X^0 = \ell \sinh\left(\frac{\tau}{\ell}\right) \quad X^1 = \ell \cos(\theta) \cosh\left(\frac{\tau}{\ell}\right) \quad X^2 = \ell \sin(\theta) \cosh\left(\frac{\tau}{\ell}\right) \quad (2.5)$$

Expressing  $dX^\mu$  in terms of the differentials  $d\tau$  and  $d\theta$ , we find

$$ds^2 = -d\tau^2 + \ell^2 \cosh^2\left(\frac{\tau}{\ell}\right) d\theta^2 \quad (2.6)$$

This expression for the induced metric on  $dS_2$  can be rewritten in terms of the metric tensor  $g_{\mu\nu}$ , which we can unambiguously determine by expression (2.6) because metric tensors are symmetric:

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \ell^2 \cosh^2\left(\frac{\tau}{\ell}\right) \end{pmatrix} \quad \text{and} \quad x^\mu = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} \tau \\ \theta \end{pmatrix} \quad (2.7)$$

where from now on, we only have  $\mu, \nu = 0, 1$ . Notice that the dimensions of the first and second coordinate don't match in the sense that the dimension of  $\tau$  is length and  $\theta$  is dimensionless as it's an angle, which is something to keep in mind in calculations. For completeness it is good to mention that it will be sufficient to treat points in  $dS_2$  as covariant vectors (although  $dS_2$  is not a vector space, this is just notation!)  $x_\mu \equiv g_{\mu\nu} x^\nu$  that can be explicitly calculated by multiplying the vector  $x^\mu$  by the matrix  $g_{\mu\nu}$  and then transposing the resulting vector (although  $dS_2$  is not a vector space, this is just notation!), which means that expressions like  $x_\mu y^\mu$  become scalars and are therefore invariant under coordinate transformations.

The last detail to mention is that  $g^{\mu\nu}$  is the inverse of  $g_{\mu\nu}$  in the sense that  $g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu$ , and in our calculations it will be sufficient to treat it as the inverse of the matrix  $g_{\mu\nu}$  in expression (2.7), although this neglects the aspects of covariance and contravariance. When performing calculations in  $dS_2$ , we will denote the determinant of the metric with  $g$  (so we will have  $g = -\ell^2 \cosh^2\left(\frac{\tau}{\ell}\right)$  with our parameterization of  $dS_2$ ), and derivatives behave like covariant vectors and will look like

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = (\partial_0, \partial_1) = (\partial_\tau, \partial_\theta) \quad (2.8)$$

## 2.2 Generators of the dS<sub>2</sub>-isometries

The isometries of the hyperboloid (2.4) can easily be found: They include rotations in the  $(X^1, X^2)$ -plane, and Lorentz boosts in both the  $(X^1)$ -direction and the  $(X^2)$ -direction, and these are generated by the following operators:

$$\begin{aligned} J^3 &\equiv -i(X^1\partial_{X^2} - X^2\partial_{X^1}) \\ K^1 &\equiv -i(X^0\partial_{X^1} + X^1\partial_{X^0}) \\ K^2 &\equiv -i(X^0\partial_{X^2} + X^2\partial_{X^0}) \end{aligned} \quad (2.9)$$

These expressions with the prefactor  $-i$  are analogous to [27, Chapter 2.1] and makes these generators hermitian for unitary representations. We can express these generators in terms of the coordinates  $\tau$  and  $\theta$ :

$$\begin{aligned} J^3 &= -i\partial_\theta \\ K^1 &= -i(\ell \cos(\theta)\partial_\tau - \sin(\theta) \tanh(\frac{\tau}{\ell})\partial_\theta) \\ K^2 &= -i(\ell \sin(\theta)\partial_\tau + \cos(\theta) \tanh(\frac{\tau}{\ell})\partial_\theta) \end{aligned} \quad (2.10)$$

These generators satisfy the commutation relations

$$[K^1, K^2] = -iJ^3 \quad [J^3, K^1] = iK^2 \quad [J^3, K^2] = -iK^1 \quad (2.11)$$

Together, these generators form a basis of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ , but before performing calculations we perform a basis transformation given by

$$\begin{aligned} H &\equiv J^3 + K^2 \\ K &\equiv J^3 - K^2 \\ D &\equiv -K^1 \end{aligned} \quad (2.12)$$

This specific transformation is chosen to match the conventions used in [12], of which the calculations will be a common thread in this thesis. This basis transformation results in the following commutation relations:

$$[D, H] = iH \quad [D, K] = -iK \quad [K, H] = 2iD \quad (2.13)$$

Before we start building representations, we present another basis of the  $\mathfrak{sl}(2, \mathbb{R})$ -algebra, namely one where we construct ladder operators from the isometry generators  $J^3$ ,  $K^1$  and  $K^2$ . We can also construct them from the basis  $\{H, K, D\}$ :

$$\begin{aligned} L_0 &\equiv J^3 = \frac{1}{2}(H + K) \\ L_\pm &\equiv K^2 \pm iK^1 = \frac{1}{2}(H - K) \mp iD \end{aligned} \quad (2.14)$$

They take the following form using the expressions for the isometry generators in (2.10):

$$\begin{aligned} L_0 &= -i\partial_\theta \\ L_\pm &= e^{\mp i\theta}(-i \tanh(\frac{\tau}{\ell})\partial_\theta \pm \ell\partial_\tau) \end{aligned} \quad (2.15)$$

We can calculate the commutation relations for these operators, where it is important to realize that these don't follow from the explicit expressions (2.15), but rather from the  $\mathfrak{sl}(2, \mathbb{R})$  commutation relations that we found for the other bases, and therefore, as long as they are consistent with the construction (2.14) we can use these ladder operators in general Hilbert spaces where we define our representations on. The commutation relations for this basis read:

$$[L_\pm, L_0] = \pm L_\pm \quad [L_+, L_-] = 2L_0 \quad (2.16)$$

In quantum mechanics, it is convenient to use this ladder operator basis of  $\mathfrak{sl}(2, \mathbb{R})$  for constructing a countable basis of the Hilbert spaces that we let our  $\mathfrak{sl}(2, \mathbb{R})$ -representations work on. This is because of the fact

that  $L_0 = J^3$  is the infinitesimal generator of the compact subgroup  $U(1)$  of rotations of  $SL(2, \mathbb{R})$ . Motivated by this, we will take the generator  $L_0$  to be compact in our representations, which means that it has a discrete spectrum and its eigenvalues are integers  $n$ . The idea is then to build a countable basis of the Hilbert space with the eigenstates  $|\psi_n\rangle$  corresponding to these eigenvalues. If we then require the ladder operators  $L_{\pm}$  to transform eigenstates into different ones, we will be able to comfortably build and describe a countable basis of the Hilbert space and then investigate the properties of the representation. The relations that we will use to find the eigenbasis will be presented in the next chapters when we perform the explicit calculations following the lines of reasoning in [12], but it is good to already have an idea of the possible applications for these generators of  $\mathfrak{sl}(2, \mathbb{R})$ .

One last interesting aspect of these different bases of the  $\mathfrak{sl}(2, \mathbb{R})$ -algebra is the existence of a central element  $C_2$  in the *universal enveloping algebra* of the Lie algebra. We call  $C_2$  the quadratic Casimir because it will be quadratic in our generators, and it is technically not an element of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  itself, but rather a element in the so-called universal enveloping algebra, which is essentially the expansion of the Lie algebra, which only has the Lie bracket  $[X, Y]$  as a binary operation on it, to an algebra with the binary operation of multiplication  $*$  such that the commutator  $X*Y - Y*X$  corresponds with the Lie bracket  $[X, Y]$ [28].

The quadratic Casimir is called a central element because it commutes with all the generators of the Lie algebra. This can be easily verified using the properties of Lie brackets and the explicit definition

$$C_2 \equiv \frac{1}{2}(HK + KH) - D^2 \quad (2.17)$$

or written in terms of the ladder operators:

$$C_2 = L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) \quad (2.18)$$

We will discuss its relevance in Chapter 3, but the last thing to realize is that we can explicitly calculate  $C_2$  if we use the expressions (2.15) for the ladder operators on  $C_2$ :

$$C_2 = \ell \tanh\left(\frac{\tau}{\ell}\right) \partial_{\tau} + \ell^2 \partial_{\tau}^2 - \frac{1}{\cosh^2\left(\frac{\tau}{\ell}\right)} \partial_{\theta}^2 \quad (2.19)$$

This explicit expression, which looks a little meaningless on its own here, will come up later when describing a simple scalar field on  $dS_2$  and actually play an important role, but for now we leave our discussion on the geometry of de Sitter space for what it is, and focus solely on the Lie algebra of  $\mathfrak{sl}(2, \mathbb{R})$  and its representations that we can build.

### 2.3 Developing the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$

At this point, you should be suspicious: The three systems (2.10), (2.12) and (2.14) are said to be bases of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ , but this deserves explanation: Before we proceed to build quantum theories with these generators, we must show how our different bases arise from the Lie group  $SL(2, \mathbb{R})$ .

We start with the group  $SL(2, \mathbb{R})$ , which consists of the  $2 \times 2$  matrices with real entries that have determinant 1, i.e.  $SL(2, \mathbb{R}) = \{M \in \mathbb{R}^{2 \times 2} | \det(M) = 1\}$ . This is a Lie group under matrix multiplication; it is a group which is also a differentiable manifold (meaning that the group elements depend smoothly on a continuous set of parameters). To construct the corresponding Lie algebra, we use the *Iwasawa decomposition* of  $SL(2, \mathbb{R})$  [29], which tells us that every element  $M \in SL(2, \mathbb{R})$  can be written as the following product, written in terms of the parameters  $\theta$ ,  $r$  and  $x$ :

$$M(\alpha) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \alpha \equiv \begin{pmatrix} \theta \\ r \\ x \end{pmatrix} \quad (2.20)$$

with  $\theta \in [0, 2\pi]$ ,  $r \in \mathbb{R}_{>0}$  and  $x \in \mathbb{R}$ . The identity element of the Lie group is the identity matrix, which is given by  $M(\alpha_e)$  with  $\alpha_e \equiv (0, 1, 0)^T$ . We follow the procedure in [27, Chapter 2] to construct the algebra of

$SL(2, \mathbb{R})$ -generators, which is the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ . The procedure gives a direct correspondence between the representations  $M(\alpha)$  of the elements in the Lie group and the infinitesimal generators  $G_a$  of the group element representations, which form a basis of the corresponding Lie algebra. This correspondence can be formulated in a very elegant way:

$$G_a = -i \frac{\partial}{\partial \alpha_a} M(\alpha) \Big|_{\alpha=\alpha_e} \iff M(\alpha) = e^{i \sum_a \alpha_a G_a} \quad (2.21)$$

With the first formula, we construct the 3 generators  $G_a$  (with  $a = \theta, r, x$ ) and we obtain

$$G_\theta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad (2.22)$$

$$G_r = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (2.23)$$

$$G_x = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \quad (2.24)$$

These satisfy the commutation relations

$$[G_\theta, G_r] = -2iG_\theta - 4iG_x \quad [G_\theta, G_x] = iG_r \quad [G_r, G_x] = -2iG_x \quad (2.25)$$

When describing Lie algebra's, the commutation relations between the elements in the algebra are all you need, and we can immediately forget about the explicit expressions (2.22) that we found for the generators! To show that these commutation relations are equivalent to the ones that we found before, we perform a basis transformation:

$$L_0 = -\frac{i}{2}G_r \quad (2.26)$$

$$L_+ = -G_x \quad (2.27)$$

$$L_- = -G_x - G_\theta \quad (2.28)$$

And this transformation turns the commutation relations (2.25) into the commutation relations (2.16) of the ladder operators as we defined them before:

$$[L_\pm, L_0] = \pm L_\pm \quad [L_+, L_-] = 2L_0 \quad (2.29)$$

This means that these ladder operators indeed form a basis of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ , and the same goes for the basis consisting of  $H$ ,  $K$  and  $D$  and the basis of the  $dS_2$  isometry generators as we described them before. Now that we've established the relation between the two-dimensional de Sitter space and the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ , it is time to explore  $\mathfrak{sl}(2, \mathbb{R})$ -invariant quantum theories, as we will do in the next chapter.

### 3 Building unitary representations

Representation theories of Lie algebras are very natural way to formulate quantum theories: Representations of Lie algebras are constructed by letting the elements of the Lie algebra act as linear transformations on vector spaces[30], while in quantum mechanics physical states are represented by elements in an Hilbert space, which is a complex vector space equipped with an inner product[31, Chapter 3]. Physical observables are then represented by linear operators on this Hilbert space, which is fully in line with the concept of representation theory! Moreover, there is a reason that we are specifically interested in the representation theory of the isometry algebra  $\mathfrak{sl}(2, \mathbb{R})$  of  $dS_2$  to build quantum theories:

Global symmetries of systems are one of the building blocks of physical laws (think for example of Noethers theorem and it's wide applications). To describe quantum mechanical systems, we use labels that represent the state of the particles, but these quantities don't have to be invariant under the transformations corresponding to these symmetries. Examples of this are the momentum in a direction, the spin project on some

axis, and the energy of particles. From representations of the isometry algebra however, you can extract quantum numbers that are invariant under the transformations that are generated by the isometry algebra itself, making them way more useful quantities to describe particles. Examples of such quantities are spin, rest mass, and more quantum numbers like electric charge and isospin which come up a lot when studying the Standard model. In short, labeling particle states is much more meaningful when describing the system using a representation of the isometry algebra of the underlying space, which is  $\mathfrak{sl}(2, \mathbb{R})$  in our case.[23, Chapter 7]

### 3.1 Generators of $\mathfrak{sl}(2, \mathbb{R})$

We build the irreducible unitary representations analogous to [12], starting with generators  $D$ ,  $H$  and  $K$  of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ . The generator  $H$  represents the Hamiltonian in physical systems,  $D$  represents dilations and  $K$  represents special conformal transformations. We will repeat the commutation relations that we found in Chapter 2, because together these form a starting point for all of the calculations in the remainder of the thesis. They read as follows:

$$[D, H] = iH \qquad [D, K] = -iK \qquad [K, H] = 2iD \qquad (3.1)$$

As mentioned before, we can now build an Casimir operator that commutes with these 3 generators:

$$C_2 \equiv \frac{1}{2}(HK + KH) - D^2 \qquad (3.2)$$

When building and researching representations of Lie algebras, we're mainly interested in irreducible representations because these can be seen as the building blocks of general representations in the sense that we can take tensor products of irreducible representations to describe bigger systems. This makes this Casimir element into a powerful tool to describe the representations because of Schur's lemma [32], which tells us that for irreducible representations of the universal enveloping algebra of  $\mathfrak{sl}(2, \mathbb{R})$ , an element that commutes with every element of the representation acts as a scalar, meaning that the operator  $C_2$  will act as a scalar, with a value that depends on the particular representation that we will be working with.

We can start building a basis of the Hilbert space with a *conformal primary*  $|0\rangle$ , which is a state that is annihilated by  $K$  and is an eigenstate of  $D$ , with an eigenvalue given by  $\Delta$ , the *conformal dimension* of the representation, which depends on the particular representation that we're working with:

$$K|0\rangle = 0 \qquad D|0\rangle = i\Delta|0\rangle \qquad (3.3)$$

Now, we calculate the eigenvalue of  $C_2$  by calculating how it acts on the conformal primary:

$$C_2|0\rangle = \left(\frac{1}{2}(HK + KH) - D^2\right)|0\rangle = \left(HK + \frac{1}{2}[K, H] - D^2\right)|0\rangle = 0 - \Delta|0\rangle + \Delta^2|0\rangle = \Delta(\Delta - 1)|0\rangle \qquad (3.4)$$

We can now see why the Casimir element is a powerful tool for describing representations: In a given representation, the eigenvalue of  $C_2$  can be easily calculated by acting on a random state with it, and this leaves us with only 2 possible values for the conformal dimension of the representation: If we find the eigenvalue of  $C_2$  (which must be  $\Delta(\Delta - 1)$  by (3.4)) to be  $\lambda$ , then we find  $\Delta_{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\lambda}$  to be the two possible conformal dimensions. Motivated by the fact that  $1 - \Delta_{\pm} = \Delta_{\mp}$ , we can define the *shadow conformal dimension*  $\Delta_s \equiv 1 - \Delta$ , which will come up a lot in the calculations in the remainder of this thesis.

In physical systems,  $H$  is often the Hamiltonian, so it generates the time evolution of the physical states, which mathematically looks like

$$|t\rangle \equiv e^{-iHt}|0\rangle \qquad (3.5)$$

meaning that  $i\partial_t|t\rangle = H|t\rangle$ . We also want to find how  $D$  and  $K$  act on the states  $|t\rangle$ . We do this by first calculating the commutators  $D$  and  $K$  commute with higher powers of  $H$ , which allows us to calculate  $[D, e^{-iHt}] = tHe^{-iHt}$  and  $[K, e^{-iHt}] = 2te^{-iHt}D + t^2He^{-iHt}$ . These relations, combined with definition (3.5) of states  $|t\rangle$ , can directly be used to calculate how  $D$  and  $K$  act on states  $|t\rangle$ :

$$D|t\rangle = De^{-iHt}|0\rangle = (e^{-iHt}D + tHe^{-iHt})|0\rangle = i(\Delta + t\partial_t)|t\rangle \qquad (3.6a)$$

$$K|t\rangle = (e^{-iHt}K + 2te^{-iHt}D + t^2He^{-iHt})|0\rangle = i(2t\Delta + t^2\partial_t)|t\rangle \quad (3.6b)$$

When performing calculations, it is often advantageous to work in an energy eigenbasis, which we can obtain by Fourier transforming the  $|t\rangle$ -states. We make use of the following convention when it comes to normalization:

$$|E\rangle \equiv \int_{-\infty}^{\infty} dt e^{iEt}|t\rangle \quad (3.7)$$

With this convention for the normalization of the Fourier transform ( $\hat{f}(E) = \int_{-\infty}^{\infty} dt e^{iEt}f(t)$ ), we have  $\int_{-\infty}^{\infty} dt e^{iEt}f^{(n)}(t) = (-iE)^n \hat{f}(E)$ , and with this we can calculate how the operators act on the energy basis:

$$H|E\rangle = E|E\rangle \quad D|E\rangle = -i(E\partial_E + \Delta_s)|E\rangle \quad K|E\rangle = -(E\partial_E^2 + 2\Delta_s\partial_E)|E\rangle \quad (3.8)$$

Here the shadow conformal dimension  $\Delta_s$  of the representation comes up for the first time. Here you can clearly see how the energy basis is an eigenbasis of  $H$ , which results from the fact that  $H$  generates the time evolution of the  $|t\rangle$ -states.

### 3.2 Inner products in the energy eigenbasis

Now we can find some restrictions on the system: For a representation to be unitary, the operators need to be self-adjoint with respect to the inner product on the Hilbert space. If we take two states  $|E\rangle$ , we can evaluate  $\langle E'|H|E\rangle$  in two ways: Letting  $H$  act on the ket-state, and letting its Hermitian adjoint  $H^\dagger = H$  act on the bra-state. This results in  $\langle E'|E|E\rangle$  and  $\langle E'|E'|E\rangle$  respectively, which clearly must be equal, so we find

$$\langle E'|H|E\rangle - \langle E'|H|E\rangle = \langle E'|E|E\rangle - \langle E'|E'|E\rangle = (E - E')\langle E'|E\rangle = 0 \quad (3.9)$$

For this to always hold,  $\langle E'|E\rangle$  must vanish for  $E \neq E'$ , i.e.

$$\langle E'|E\rangle = f(E)\delta(E - E') \quad (3.10)$$

Using the same procedure for the operator  $D$ , we see that the expression  $\langle E'|D|E\rangle$  evaluates to  $-i(E\partial_E + \Delta_s)\langle E'|E\rangle$  and  $i(E'\partial_{E'} + \Delta_s^*)\langle E'|E\rangle$ . Equating these expressions results in

$$(E'\partial_{E'} + E\partial_E)\langle E'|E\rangle = -(\Delta_s + \Delta_s^*)\langle E'|E\rangle \quad (3.11)$$

If we combine this with equation (3.10) and integrate out  $E'$  using the properties of the Dirac delta function (in particular  $\int f(x)\delta'(x - a)dx = -\int f'(x)\delta(x - a)dx = -f'(a)$ ), we end up with the differential equation

$$-f(E) + Ef'(E) = -(\Delta_s + \Delta_s^*)f(E) \quad (3.12)$$

This differential equation has a general solution

$$f(E) = (c_+\Theta(E) + c_-\Theta(-E))|E|^{1-\Delta_s-\Delta_s^*} = \begin{cases} c_+|E|^{1-\Delta_s-\Delta_s^*} & \text{for } E \geq 0 \\ c_-|E|^{1-\Delta_s-\Delta_s^*} & \text{for } E \leq 0 \end{cases} \quad (3.13)$$

for some arbitrary constants  $c_\pm$ , where  $\Theta(E)$  is the Heaviside step function, which makes the inner product distinguish between the negative energy region and the positive energy region of the energy domain (remember that  $f(E)$  is related to the inner products of the states of the energy eigenbasis via (3.10)).

We follow the same procedure for the operator  $K$ , which gives:

$$\langle E'|K|E\rangle - \langle E'|K|E\rangle = [-(E'\partial_{E'}^2 + 2\Delta_s^*\partial_{E'}) + (E\partial_E^2 + 2\Delta_s\partial_E)]f(E)\delta(E - E') = 0 \quad (3.14)$$

Here we already used equation (3.10). Working out these derivatives gives:

$$0 = [-E'f(E)\partial_{E'}^2 + 2\Delta_s^*\partial_{E'} + Ef''(E) + 2f'(E)\partial_E + f(E)\partial_E^2 + 2\Delta_s f'(E) + 2\Delta_s f(E)\partial_E]\delta(E - E') \quad (3.15)$$

Again, we can integrate out  $E'$ . Then the terms with partial derivatives to  $\partial_E$  of  $\delta(E - E')$  will vanish, and we end up with another differential equation:

$$\partial_{E'}^2(-E' f(E)) - \partial_{E'}(2\Delta_s^* f(E)) + E f''(E) + 2\Delta_s f'(E) = E f''(E) + 2\Delta_s f'(E) = 0 \quad (3.16)$$

If we plug our previous solution (3.13) into this equation, we eventually find the restriction

$$(1 - \Delta_s - \Delta_s^*)(\Delta_s - \Delta_s^*) = 0 \quad (3.17)$$

So one of these terms inside the brackets must be zero! There are now two options for the value of  $\Delta$  that satisfy this restriction:

- 1 We can have  $1 - \Delta_s - \Delta_s^* = 0$ . For this to hold, we need  $\text{Re}(\Delta_s) = \frac{1}{2}$ , i.e.  $\Delta = \frac{1}{2}(1 + i\nu)$ , where  $\nu \in \mathbb{R}$ . Representations with this conformal dimension are called *principal series* representations. This gives a function  $f(E)$  which is constant in both the positive and negative energy domain:

$$f(E) = c_+ \Theta(E) + c_- \Theta(-E) \quad (3.18)$$

- 2 The other option is  $\Delta_s - \Delta_s^* = 0$ , which is equivalent to  $\text{Im}(\Delta_s) = 0$  i.e.  $\Delta_s \in \mathbb{R}$  and therefore  $\Delta \in \mathbb{R}$ . This requirement eventually results in a few possible representations, which we will mention in the next section. In this case we have  $1 - \Delta_s - \Delta_s^* = 2\Delta - 1$ , so

$$f(E) = (c_+ \Theta(E) + c_- \Theta(-E)) |E|^{2\Delta-1} \quad (3.19)$$

It is important to notice that these are the only choices for  $\Delta$  that result in a real Casimir eigenvalue  $\Delta(\Delta - 1)$ .

General states  $|\psi\rangle$  in the Hilbert space can be written as a superposition of energy eigenstates:

$$|\psi\rangle \equiv \int_{-\infty}^{\infty} dE \psi(E) |E\rangle \quad (3.20)$$

And for two of these states, the inner product is given by

$$\langle \chi | \psi \rangle = \int_{-\infty}^{\infty} dE f(E) \chi^*(E) \psi(E) \quad (3.21)$$

Here you can see how  $f(E)$  directly determines the overlap between the two states  $|\psi\rangle$  and  $|\chi\rangle$ .

Because states  $|\psi\rangle$  are often represented by their wave functions  $\psi(E)$ , it is useful to formulate the actions of the generators on states  $|\psi\rangle$  in terms of how they act on wave functions. Strictly speaking, this means that we consider a new Hilbert space, namely one that consists of wave functions instead of the kets in our bra-ket notation. However, the transition between those Hilbert spaces is a very natural one to construct: By definition (3.20), we already know how to transform a wave function  $\psi(E)$  of a state into the corresponding ket  $|\psi\rangle$ , and the inverse transformation from kets  $|\psi\rangle$  to wave functions  $\psi(E)$  is given by

$$\psi(E) = \begin{cases} \frac{\langle E | \psi \rangle}{f(E)} & \text{if } f(E) \neq 0 \\ 0 & \text{if } f(E) = 0 \end{cases} \quad (3.22)$$

This is easily verified using expressions (3.10) and (3.20). The factor  $f(E)$  is included because the kets are not necessarily normalized. Because the correspondence between those subtly different Hilbert spaces given by transformations (3.20) and (3.22) is such a natural one, we won't pay any attention to it in the remainder of this thesis, also not when we explore other Hilbert spaces like we do in Chapter 4. If we insert the actions given in (3.8) into definition (3.20) of general states  $|\psi\rangle$  and use integration by parts, we find:

$$H \psi(E) = E \psi(E) \quad D \psi(E) = i(E \partial_E + \Delta) \psi(E) \quad K \psi(E) = -(E \partial_E^2 + 2\Delta \partial_E) \psi(E) \quad (3.23)$$

These equations are easily applicable when working with a given set of wave functions, or when trying to find wave functions when it is given how operators should act on them, as we will see on the next section.

Later on, we will make transformations between bases of the Hilbert spaces we're working with. When doing this, it is important to keep the resolution of the identity in mind, indicating that the new basis is a complete system (it spans the entire Hilbert space). An example could be going from this energy basis to a position basis  $\{|x\rangle | x \in \mathbb{R}\}$  that you can define using a particular position space representation. The completeness relation would look like this:

$$\int_{-\infty}^{\infty} dx |x\rangle\langle x| = \mathbb{1} \quad (3.24)$$

When you've explicitly defined the basis transformation, you can also find expressions for the inner product between general states in terms of the new basis of the Hilbert space, making use of this completeness relation in the process. For the remainder of this chapter however, we will only be working in the energy eigenbasis to find more restrictions on the possible unitary representations.

### 3.3 Ladder operators

We will now make use of the basis transformation to the ladder operator basis of  $\mathfrak{sl}(2, \mathbb{R})$  as described in Chapter 2, so we have

$$L_0 = \frac{1}{2}(H + K) \quad L_{\pm} = \frac{1}{2}(H - K) \mp iD \quad (3.25)$$

which (as we saw before) again satisfy the commutation relations (3.1) up to a factor  $i$ :

$$[L_{\pm}, L_0] = \pm L_{\pm} \quad [L_+, L_-] = 2L_0 \quad (3.26)$$

We repeat that, motivated by the fact that we want one of these operators to be related to a generator of the compact subgroup that  $\text{SL}(2, \mathbb{R})$  has, we proceed to take  $L_0$  to be compact, meaning that its eigenstates span a countable basis of the Hilbert space, so we can index the eigenstates with the integers ( $|\psi_n\rangle$ , represented by wave functions  $\psi_n(E)$ ), and then choose the corresponding eigenvalues of  $L_0$  to be this index  $n$ . Then we define the operators  $L_{\pm}$ , which raise and lower the eigenvalues of the eigenstates  $\psi_n(E)$  by 1, satisfying the following relations:

$$L_0 \psi_n(E) = n \psi_n(E) \quad L_{\pm} \psi_n(E) = -(n \pm \Delta_s) \psi_{n \pm 1}(E) \quad (3.27)$$

These specific relations for  $L_{\pm}$  are chosen to give the Casimir (2.18) acting on  $\psi_n(E)$  the eigenvalue  $\Delta(\Delta - 1)$  for every  $n \in \mathbb{Z}$  to match our findings in (3.4), i.e.

$$C_2 \psi_n(E) = \left[ L_0^2 - \frac{1}{2}(L_+ L_- + L_- L_+) \right] \psi_n(E) = \Delta(\Delta - 1) \psi_n(E) \quad (3.28)$$

We can also replace the shadow conformal dimension  $\Delta_s$  in (3.27) by  $\Delta$  to better match our later calculations, when this ladder operator framework is used again. However, we prefer these actions of  $L_{\pm}$  on the wave functions  $\psi_n(E)$  to match the calculations in the principal series paper [12, Chapter 2].

If we insert the actions in (3.23) of the original operators on the Hilbert space into the definitions of the ladder operators, we find 3 differential equations for the wave functions  $\psi_n$ :

$$E \psi_n''(E) + 2\Delta \psi_n'(E) - (2n + E) \psi_n(E) = 0 \quad (3.29a)$$

$$-(n + \Delta_s) \psi_{n+1}(E) = \frac{E}{2} \psi_n''(E) + (\Delta + E) \psi_n'(E) + \left(\frac{E}{2} + \Delta\right) \psi_n(E) \quad (3.29b)$$

$$-(n - \Delta_s) \psi_{n-1}(E) = \frac{E}{2} \psi_n''(E) + (\Delta - E) \psi_n'(E) + \left(\frac{E}{2} - \Delta\right) \psi_n(E) \quad (3.29c)$$

These differential equations can be solved using Fourier transforms, which allows solutions proportional to

$$\psi_n(E) = \int_{-\infty}^{\infty} (1+it)^{-n-\Delta_s} (1-it)^{n-\Delta_s} e^{-iEt} dt \quad (3.30)$$

In the next section, we will investigate what the consequences of this outcome of the ladder operator framework are for our different representations.

### 3.4 Normalization of the states

If we want the representations to have physical meaning, these states  $|\psi_n\rangle$  need to be normalizable (that is, they need to have a positive finite norm  $\langle\psi_n|\psi_n\rangle$  so they can be rescaled to have norm 1 with the appropriate prefactor. Checking normalizability of states can require long, relatively uninteresting and even problematic calculations, but if we start with investigating only the most simple wave functions, we can quickly address normalization problems that we encounter. This is because when we have a normalizable state  $|\psi_k\rangle$  (with  $k \in \mathbb{Z}$ ) in our framework with ladder operators, then acting on it with  $L_{\pm}$  to obtain the state  $|\psi_{k\pm 1}\rangle$  only magnifies the norm by a finite factor. With the actions (3.27), we find

$$\langle\psi_{k\pm 1}|\psi_{k\pm 1}\rangle = |k \pm \Delta_s|^2 \langle\psi_k|\psi_k\rangle \quad (3.31)$$

So  $|\psi_{k\pm 1}\rangle$  is normalizable as long as  $k \pm \Delta_s \neq 0$ . The rare case that  $\Delta_s$  is an integer can therefore give some special types of representations as we will see later, but in general  $\Delta_s$  is not an integer, meaning that in this case the state  $|\psi_{k\pm 1}\rangle$  is normalizable if  $|\psi_k\rangle$  is normalizable. Because every state  $|\psi_n\rangle$  can be obtained by acting finitely many times with  $L_{\pm}$  on a state, all the states  $|\psi_n\rangle$  in our basis are normalizable as long as they can be obtained from our 'starting point'  $|\psi_k\rangle$ .

To start investigating the normalizability of the solutions (3.30), we take  $n = 0$  as a starting point: We can explicitly calculate the Fourier transform and we find

$$\psi_0(E) = \int_{-\infty}^{\infty} (1+t^2)^{-\Delta_s} e^{-iEt} dt = \frac{\sqrt{8\pi}}{2^{\Delta_s} \Gamma(\Delta_s)} |E|^{\Delta_s - \frac{1}{2}} K_{\Delta_s - \frac{1}{2}}(|E|) \quad (3.32)$$

where  $\Gamma$  is the gamma function and  $K_{\alpha}$  is the modified Bessel function of the second kind[33] with order  $\alpha \in \mathbb{C}$ , which can be expressed as follows for  $x > 0$ :

$$K_{\alpha}(x) = \int_0^{\infty} dt e^{-x \cosh(t)} \cosh(\alpha t) \quad (3.33)$$

Notice that  $K_{\alpha}$  is a real-valued function as long as  $\alpha$  is strictly real or strictly imaginary. If we now want to calculate the norm  $\langle\psi_0|\psi_0\rangle$  using the definition (3.21) of the inner product, we need to now what  $f(E)$  looks like. As discussed before, we have two types of representations to consider:  $\Delta = \frac{1}{2}(1+i\nu)$  and  $\Delta \in \mathbb{R}$ . We start with the first case.

In the case of the principal series representations ( $\Delta = \frac{1}{2}(1+i\nu)$ ), we have  $\Delta_s = \frac{1}{2}(1-i\nu) = \Delta^*$ , which we can combine with the fact that  $\Delta + \Delta_s = 1$  by definition, to switch easily between  $\Delta$ ,  $\Delta_s$  and their complex conjugates in calculations. Using definition (3.21), we find

$$\langle\psi_0|\psi_0\rangle = \frac{8\pi}{2^{\Delta+\Delta_s} \Gamma^*(\Delta_s) \Gamma(\Delta_s)} \int_{-\infty}^{\infty} dE f(E) (K_{-\frac{i\nu}{2}}(|E|))^2 \quad (3.34)$$

Notice that the powers of  $|E|$  cancel out. This expression simplifies nicely due to the properties of the gamma function:

$$\Gamma^*(\Delta_s) \Gamma(\Delta_s) = \Gamma(\Delta_s^*) \Gamma(\Delta_s) = \Gamma(\Delta) \Gamma(1-\Delta) = \frac{\pi}{\sin(\pi\Delta)} = \frac{\pi}{\cosh\left(\frac{\pi\nu}{2}\right)} \quad (3.35)$$

If we combine these results with the fact that the domain of integration is nicely divided into a positive and negative energy region by both  $f(E)$  and the argument  $|E|$  of  $K_\alpha$ , we find the following expression for the norm:

$$\langle \psi_0 | \psi_0 \rangle = 4\pi \cosh\left(\frac{\pi\nu}{2}\right) (c_+ + c_-) \int_0^\infty dE (K_{-\frac{i\nu}{2}}(E))^2 \quad (3.36)$$

We can explicitly calculate this integral, because for  $|\operatorname{Re}(\alpha)| < \frac{1}{2}$ , we have

$$\int_0^\infty dx (K_\alpha(x))^2 = \frac{\pi^2}{4 \cos(\pi\alpha)} \quad (3.37)$$

This gives the following value for the norm of the wave functions  $\psi_0$ :

$$\langle \psi_0 | \psi_0 \rangle = \pi^3 (c_+ + c_-) \quad (3.38)$$

This means that this is a normalizable wave function as long as  $c_+ + c_- \neq 0$ , which loosely speaking means that the inner product on the Hilbert space of energy wave functions can actually be chosen to only consider the positive or negative energy region, as long as they don't cancel each other out over the entire energy domain.

If this is the case, the entire system  $\{|\psi_n\rangle | n \in \mathbb{Z}\}$  is normalizable as we discussed before, and this means that principal series representations as we defined them here can be suitable for describing physical systems with for now only  $c_+ + c_- \neq 0$  as a requirement, which is good news: Later we will take a deeper look into principal series representations, but for now we're going to consider the other options for the value of  $\Delta$ .

In the case of  $\Delta \in \mathbb{R}$ , we will find two possible ways to build representations. The first way is again to take  $\pi_0$  as a starting point, and again plugging expression (3.32) for  $\psi_0$  into the inner product (3.21), we can perform similar calculations (keep in mind that  $f(E)$  contains a term  $|E|^{2\Delta-1}$  as we found before), and we see that the powers of  $|E|$  cancel again, and we find

$$\langle \psi_0 | \psi_0 \rangle = \frac{2^{1+2\Delta}\pi}{\Gamma(\Delta_s)^2} (c_+ + c_-) \int_0^\infty dE (K_{\Delta_s - \frac{1}{2}}(E))^2 \quad (3.39)$$

As mentioned before, this integral only converges for  $\Delta_s - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$ , or equivalently  $\Delta \in (0, 1)$ . This yields the following value for the norm of  $\psi_0$ , where we used the properties of the gamma function again, also making use of the fact that  $\Delta$  is not an integer here:

$$\langle \psi_0 | \psi_0 \rangle = \frac{2^{2\Delta+1}\pi}{\Gamma^2(\Delta_s)} (c_+ + c_-) \frac{\pi^2}{4 \cos(\pi(\Delta_s - \frac{1}{2}))} = 2^{2\Delta-1} \pi \Gamma^2(\Delta) \sin(\Delta\pi) (c_+ + c_-) \quad (3.40)$$

We find that in this case, the same restriction on  $c_\pm$  applies as in the case of principal series representations: As long as  $c_+ + c_- \neq 0$ , we find that the wave function  $\psi_0$  is normalizable for  $\Delta \in (0, 1)$ , and therefore the entire system  $\{|\psi_n\rangle | n \in \mathbb{Z}\}$  is normalizable again. Representations with  $\Delta \in (0, 1)$  are called *complementary series* representations.

In the previous calculations, we used the  $\psi_0$ -wave function to investigate normalizability, but the case of  $\Delta \in \mathbb{R}$  also allows a different starting point of the representation: if  $\Delta$  is an integer (in which case we define  $N \equiv \Delta_s \in \mathbb{Z}^+$ ), there are suddenly two wave functions that get annihilated by the ladder operators and are therefore potentially interesting starting points of our representations, namely the wave functions  $\psi_{\pm N}$ :

$$L_\pm \psi_{\mp N} = -(\mp N \pm \Delta_s) \psi_{\mp N \pm 1} = 0 \quad (3.41)$$

We can also explicitly calculate these wave functions because the Fourier transforms of  $\psi_n(E)$  in (3.30) can be explicitly calculated for  $n = \pm N = \pm \Delta_s$ :

$$\psi_{\pm N}(E) = \frac{\mp 2\pi}{\Gamma(2N)} e^{\pm E} E^{2N-1} \Theta(\mp E) \quad (3.42)$$

And plug this into the inner product just like before, using expression (3.19) for  $f(E)$  that we found earlier. This gives the following values:

$$\langle \psi_{\pm N} | \psi_{\pm N} \rangle = c_{\mp} \frac{4^{1-N} \pi^2}{\Gamma(2N)} \quad (3.43)$$

For now, we focus on  $\psi_N$ : You can see that  $\psi_N(E)$  is normalizable as long as  $c_- \neq 0$ . If we now let the ladder operators  $L_+$  act on this wave function, we can obtain wave functions  $\psi_n(E)$  (which are nonzero on the negative energy domain) for  $n \geq N = \Delta_s$ , and as discussed before, letting  $L_-$  act on  $\psi_{\Delta_s}(E)$  makes the wave functions vanish, so we obtain a representation which has a lowest state  $|\psi_{\Delta_s}\rangle$  and is spanned by states  $|\psi_n\rangle$  with  $n \in \mathbb{Z}_{\geq \Delta_s}$ . Notice that this avoids the problems with normalizability of  $\psi_0(E)$  that we encountered with expression (3.39), because  $|\psi_0\rangle$  is not included in this *discrete lowest weight* representation when  $\Delta_s$  is positive, and this is precisely why we demanded  $\Delta_s \in \mathbb{Z}^+$  before.

The same line of reasoning can be followed when you take  $\psi_{-\Delta_s}(E)$  as the starting point of the representation. Normalizability of  $\psi_{-\Delta_s}(E)$  requires  $c_+ \neq 0$ , and then we find that the representation is spanned by wave functions  $\psi_n(E)$  for  $n \in \mathbb{Z}_{\leq -\Delta_s}$  which are nonzero in the positive energy domain. Such representations are called *discrete highest weight* representations, because the highest state is  $|\psi_{-\Delta_s}\rangle$ .

Similar to the principal series paper[12], we summarize the results about the different types of possible unitary irreducible representations of  $\mathfrak{sl}(2, \mathbb{R})$  and the requirements on the parameters for normalizability of the states  $|\psi_n\rangle$ . Although the parameters  $c_{\pm}$  are not something that we will be working with in the remainder of the thesis, I also include the constraints on  $c_{\pm}$  because they give an insight in how a representation can treat the positive and negative energy regions. The results are summarized in table 1 (remember that  $\Delta_s = 1 - \Delta$ ).

Representation	Possible values of $\Delta$	Range of $n$	Restrictions on $c_{\pm}$
Principal series	$\Delta = \frac{1}{2}(1 + i\nu)$ ( $\nu \in \mathbb{R}$ )	$n \in \mathbb{Z}$	$c_+ + c_- \neq 0$
Complementary series	$\Delta \in (0, 1)$	$n \in \mathbb{Z}$	$c_+ + c_- \neq 0$
Discrete lowest weight	$\Delta \in \mathbb{Z}_{\leq 0}$	$n \in \mathbb{Z}_{\geq \Delta_s}$	$c_- \neq 0$
Discrete highest weight		$n \in \mathbb{Z}_{\leq -\Delta_s}$	$c_+ \neq 0$

Table 1: Summary of the possible unitary irreducible representations of  $\mathfrak{sl}(2, \mathbb{R})$

## 4 Principal series representations

In the previous section, we investigated what properties quantum mechanical representations of  $\mathfrak{sl}(2, \mathbb{R})$  must have. This resulted in a different classes of representations, of which the principal series representations will be the main focus in the remainder of this thesis, because of reasons that will become clear later on, when we will relate the conformal dimension  $\Delta = \frac{1}{2}(1 + i\nu)$  to the mass of a free scalar field in  $dS_2$  that we will investigate because of its simplicity as a quantum field theory.

In this chapter, we're mainly going to investigate two representations: In the paper [20] by De Alfaro, Fubini and Furlan, a representation (which I will refer to as the *DFF model* from now on) is introduced from a Lagrangian of a scalar field that is invariant under *conformal* (angle-preserving) transformations, motivated by the fact that conformal invariance is a crucial building block in many physical theories, like general relativity, any conformal field theories (as the name suggests), going as far as currently developing string theories[34]! In the DFF model, the isometry group is  $SL(2, \mathbb{R})$ , and the conformal transformations are given by translations, dilations and special conformal transformations, of which the generators  $H$ ,  $D$  and  $K$  will therefore form the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ .

We will also investigate the Hilbert space of square-integrable functions on  $S^1$  and see how these representations of  $\mathfrak{sl}(2, \mathbb{R})$  can be related to the DFF model. There are two other reasons to investigate these  $S^1$ -wave functions: First of all, this Hilbert space has a very well-known structure and is a good example of a representation of  $\mathfrak{sl}(2, \mathbb{R})$  because of this. More importantly, these  $S^1$ -wave functions will turn up again when we explore fields in de Sitter space, because the spatial component of  $dS_2$  is homeomorphic to  $S^1$ .

#### 4.1 Introducing the DFF model

To build the DFF model in [20, Chapter 2], a  $\mathfrak{sl}(2, \mathbb{R})$ -invariant Lagrangian for a single field  $Q(t)$  on a one-dimensional space-time is introduced by  $L = \frac{1}{2}(\dot{Q}(t)^2 - \frac{g}{Q(t)^2})$ , where  $g$  is a coupling constant. They introduce the conjugate momentum  $P \equiv \frac{\partial L}{\partial \dot{Q}}$  of  $Q$ , which can be turned into a differential operator by  $P \equiv i\partial_Q$  (notice that this differs slightly from the usual convention  $P \equiv -i\partial_Q$ ). This canonical quantization eventually yields the following expressions for the generators (which are conserved quantities) at  $t = 0$ :

$$\begin{aligned} H &= \frac{1}{2} \left( -\partial_Q^2 + \frac{g}{Q^2} \right) \\ K &= \frac{Q^2}{2} \\ D &= -\frac{i}{2} \left( Q\partial_Q + \frac{1}{2} \right) \end{aligned} \tag{4.1}$$

Up to dimensional constants, we recognize the Hamiltonian as a description of a particle in a one-dimensional inverse square potential (meaning that we treat  $Q$  as a spatial variable, and more specific as the distance to the origin, meaning that  $Q$  can take values in  $(0, \infty)$  and can't be negative). An example of situations where this potential can arise, is when the central force at the origin ( $Q = 0$ ) comes from a dipole moment (for example an electric or magnetic dipole).

For  $g > 0$ , the potential is repulsive and prevents the particle to move through the origin, while negative  $g < 0$  are attractive. Depending on the specific physical system described by the model, this can give problems because the negative  $Q$ -region has no physical meaning, but we will discuss this later on. A bigger problem is the fact that  $g < 0$  makes the DFF model fail to be unitary with respect to the inner product on the positive  $Q$ -region:

$$\langle f, g \rangle \equiv \int_0^\infty dQ f^*(Q)g(Q) \tag{4.2}$$

This is problematic because unfortunately, the principal series representations  $\Delta = \frac{1}{2}(1 + i\nu)$  make this coupling constant  $g$  negative for any  $\nu \in \mathbb{R}$ :

$$g = \frac{(4\Delta - 1)(4\Delta - 3)}{4} = -\nu^2 - \frac{1}{4} \tag{4.3}$$

However, it turns out that the DFF model can be altered slightly to accommodate unitary principal series representations, as we will show in the next sections.

#### 4.2 Wave functions on $S^1$

Before, the generators acted on energy wave functions  $\psi_n(E)$  which had domain  $\mathbb{R}$ . We can also do quantum mechanics on Hilbert spaces of wave functions on  $S^1$ , with variable  $\theta \in [0, 2\pi)$ . In this case, we define the

generators  $H$ ,  $D$  and  $K$  as follows:

$$\begin{aligned} H &= 2i \cos\left(\frac{\theta}{2}\right) \left( \Delta \sin\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right) \partial_\theta \right) \\ K &= -2i \sin\left(\frac{\theta}{2}\right) \left( \Delta \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \partial_\theta \right) \\ D &= -i \left( \Delta \cos(\theta) + \sin(\theta) \partial_\theta \right) \end{aligned} \quad (4.4)$$

Not only do these operators satisfy the  $\mathfrak{sl}(2, \mathbb{R})$  commutation rules (3.1), in this form they make the Casimir element  $C_2$ , which we define like (3.2), have eigenvalue  $\Delta(\Delta - 1)$ , so it can match with the framework that we built in Chapter 3. Again, we need the operators (4.4) to be self-adjoint. We choose the usual inner product on the Hilbert space of square-integrable complex functions on  $S^1$ :

$$\langle f, g \rangle \equiv \int_0^{2\pi} d\theta f^*(\theta)g(\theta) \quad (4.5)$$

We find that out of the possible representations that we found in Chapter 3, the operators are self-adjoint only for the principal series representations. Motivated by this, we will take  $\Delta = \frac{1}{2}(1 + i\nu)$  in the remainder of this thesis.

Just like before, we define the ladder operators  $L_0$  and  $L_\pm$  by (3.25) and we find the explicit expressions

$$L_0 = -i\partial_\theta \quad L_\pm = -e^{\mp i\theta} (i\partial_\theta \pm \Delta) \quad (4.6)$$

We proceed to take  $L_0$  to be compact and we require want its eigenfunctions  $\psi_n(\theta)$  to satisfy relations similar to the relations (3.27), but now we have a term  $\Delta$  in the action of  $L_\pm$  instead of  $\Delta_s$ . Notice that because we're working with  $\Delta = \frac{1}{2}(1 + i\nu)$ , we have  $\Delta_s = \frac{1}{2}(1 - i\nu)$  so taking  $\Delta$  instead of  $\Delta_s$  won't give drastically different calculations. The actions of the operators are given by

$$L_0 \psi_n(\theta) = -n\psi_n(\theta) \quad L_\pm \psi_n(\theta) = -(n \pm \Delta)\psi_{n\pm 1}(\theta) \quad (4.7)$$

If you plug in the expressions for the operators, we obtain 3 differential equations which fortunately are easier to solve than the ones that followed from the ladder operators in Chapter 3. Its solutions are

$$\psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{-in\theta} \quad (4.8)$$

This specific form indeed makes the system of functions  $\{\psi_n(\theta) | n \in \mathbb{Z}\}$  into an orthonormal complete system (it spans the entire Hilbert space), or written in formulas:

$$\langle \psi_n, \psi_m \rangle = \delta_{nm} \quad (4.9a)$$

$$\sum_{n \in \mathbb{Z}} \psi_n^*(\theta) \psi_n(\theta') = \delta(\theta - \theta') \quad (4.9b)$$

This explains why the functions in (4.8) is the most common and natural basis for this Hilbert space. We will now try to transform this Hilbert space into one similar to the DFF model.

### 4.3 Transforming the Hilbert space

We can also work with eigenfunctions of our operators  $H$ ,  $D$  and  $K$ . The eigenfunctions of  $K$  will be particularly interesting, because with the appropriate coordinate transformations for which we will use those eigenfunctions, a model similar to the DFF model arises. Analogous to [12], we first calculate these eigenfunctions  $\rho_\kappa(\theta)$  that satisfy  $K\rho_\kappa(\theta) = \kappa\rho_\kappa(\theta)$ , which results in the differential equation

$$i(\cos(\theta) - 1)\rho'_\kappa(\theta) = (\Delta \sin(\theta) - i\kappa)\rho_\kappa(\theta) \quad (4.10)$$

which results in normalized eigenfunctions of  $K$  of the following form:

$$\rho_\kappa(\theta) = \frac{1}{2\sqrt{\pi}} e^{-i\kappa \cot\left(\frac{\theta}{2}\right)} \left(\sin\left(\frac{\theta}{2}\right)\right)^{-2\Delta} \quad (4.11)$$

We can let the eigenvalues  $\kappa$  of  $K$  range over  $\mathbb{R}$ , and by doing the following calculations (remember that  $\Delta = \frac{1}{2}(1 + i\nu)$ ), we find that the system  $\{\rho_\kappa(\theta) | \kappa \in \mathbb{R}\}$  is an orthonormal basis of the Hilbert space with respect to the inner product (4.5):

$$\langle \rho_\kappa, \rho_{\kappa'} \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\theta e^{i(\kappa' - \kappa) \cot\left(\frac{\theta}{2}\right)} \left(\sin\left(\frac{\theta}{2}\right)\right)^{-2} = \delta(\kappa - \kappa') \quad (4.12a)$$

The completeness relation reads

$$\int_{-\infty}^{\infty} d\kappa \rho_\kappa^*(\theta) \rho_\kappa(\theta') = \delta(\theta - \theta') \quad (4.12b)$$

To make a connection with the DFF model (where the wave functions depend on a spatial variable  $Q > 0$ ), we introduce a coordinate transformation in two steps. In the first place, we Fourier transform wave functions  $\psi(\theta)$  on  $S^1$  to wave functions on  $\mathbb{R} \ni \kappa$  using the eigenfunctions  $\rho_\kappa(\theta)$  of  $K$ . The wave functions in our new Hilbert space look like

$$\hat{\psi}(\kappa) \equiv |2\kappa|^{\frac{3}{4}-\Delta} \langle \rho_\kappa, \psi \rangle = |2\kappa|^{\frac{3}{4}-\Delta} \int_0^{2\pi} d\theta \rho_\kappa^*(\theta) \psi(\theta) \quad (4.13)$$

and we also need an inner product on the new Hilbert space that is defined by this transformation:

$$\langle \hat{\psi}, \hat{\chi} \rangle' \equiv \int_{-\infty}^{\infty} \frac{d\kappa}{|2\kappa|^{1/2}} \hat{\psi}^*(\kappa) \hat{\chi}(\kappa) = \langle \psi, \chi \rangle \quad (4.14)$$

This specific form is chosen to preserve inner products in this transformation of the Hilbert space. We now need to formulate how the operators  $H$ ,  $D$  and  $K$  act on these new transformed wave functions, which comes down to working out

$$A \hat{\psi}(\kappa) = |2\kappa|^{\frac{3}{4}-\Delta} \langle \rho_\kappa, A\psi \rangle = |2\kappa|^{\frac{3}{4}-\Delta} \langle A\rho_\kappa, \psi \rangle \quad (4.15)$$

where  $A$  is one of the generators  $H$ ,  $D$  and  $K$ . In the last step, we used that these are self-adjoint with respect to the inner product (4.5). The next step is to find how the generators act on  $\rho_\kappa(\theta)$ , which turns out to give nice expressions:

$$\begin{aligned} H\rho_\kappa(\theta) &= \left( \kappa \cot^2\left(\frac{\theta}{2}\right) + 2i\Delta \cot\left(\frac{\theta}{2}\right) \right) \rho_\kappa(\theta) \\ K\rho_\kappa(\theta) &= \kappa \rho_\kappa(\theta) \\ D\rho_\kappa(\theta) &= \left( i\Delta - \kappa \cot\left(\frac{\theta}{2}\right) \right) \rho_\kappa(\theta) \end{aligned} \quad (4.16)$$

Now the key insight is that we can use the explicit expression (4.11) for  $\rho_\kappa(\theta)$  to translate these expressions of the generators into expressions that only depend on  $\kappa$  and not on  $\theta$ . You can easily see that  $\cot\left(\frac{\theta}{2}\right)\rho_\kappa(\theta) = i\kappa\partial_\kappa\rho_\kappa(\theta)$ , resulting into:

$$\begin{aligned} H\rho_\kappa &= -\left( \kappa\partial_\kappa^2 + 2\Delta\partial_\kappa \right) \rho_\kappa \\ K\rho_\kappa &= \kappa\rho_\kappa \\ D\rho_\kappa &= i\left( \Delta - \kappa\partial_\kappa \right) \rho_\kappa \end{aligned} \quad (4.17)$$

Notice that we already neglected the  $\theta$ -dependence of the wave functions in our notation with the transformation of the Hilbert space in mind. We can plug these results into expression (4.15), and working out

the final actions of the generators is now a matter of taking the differential operators  $\partial_\kappa$  out of the inner product in expression (4.15), taking into consideration the properties of adjoint operators. The final step is carefully applying the product rule to derivatives in  $\kappa$  of the transformed wave functions  $\hat{\psi}(\kappa)$  defined by (4.13) to relate the derivatives in the action (4.17) to these derivatives of  $\hat{\psi}(\kappa)$ . Eventually, we find that the generators of  $\mathfrak{sl}(2, \mathbb{R})$  act on the transformed wave functions as follows:

$$\begin{aligned} H &= \frac{1}{2} \left( -2\kappa \partial_\kappa^2 - \partial_\kappa + \frac{(4\Delta - 1)(4\Delta - 3)}{8\kappa} \right) \\ K &= \kappa \\ D &= -i \left( \kappa \partial_\kappa + \frac{1}{4} \right) \end{aligned} \tag{4.18}$$

Now, to make the connection to the DFF model, we introduce a new variable  $r \equiv \text{sign}(\kappa) \sqrt{2|\kappa|}$  with the inverse transformation given by  $\kappa = \text{sign}(r) \frac{r^2}{2}$ . The factor  $\text{sign}(\kappa)$  is chosen to make the transformation from  $\mathbb{R}$  to  $\mathbb{R}$  into a bijection, meaning that  $r$  can take negative values. This gives  $\partial_\kappa = \frac{\text{sign}(r)}{r} \partial_r$  and  $\partial_\kappa^2 = (\frac{1}{r^2} \partial_r^2 - \frac{1}{r^3} \partial_r)$ , which gives the following expressions for the actions of the generators on the wave functions  $\hat{\psi}(\kappa)$ :

$$\begin{aligned} H &= \frac{\text{sign}(r)}{2} \left( -\partial_r^2 + \frac{(4\Delta - 1)(4\Delta - 3)}{4r^2} \right) \\ K &= \text{sign}(r) \frac{r^2}{2} \\ D &= -\frac{i}{2} \left( r \partial_r + \frac{1}{2} \right) \end{aligned} \tag{4.19}$$

You can already see that  $H$  looks like a Hamiltonian that arises for particles in a central potential, describing their radial motion. The first problem to notice with this interpretation should be that we allowed  $r$  to take negative values in our model, but before we discuss this result, we can simplify the second term in  $H$  using the fact that  $\Delta = \frac{1}{2}(1 + i\nu)$ . This eventually yields an attractive inverse square potential (because the fraction is always positive, the potential is attractive).

$$H = \frac{\text{sign}(r)}{2} \left( -\partial_r^2 - \frac{4\nu^2 + 1}{4r^2} \right) \tag{4.20}$$

In the framework of unitary representations of  $\mathfrak{sl}(2, \mathbb{R})$ , this is the closest that we can come to the  $\mathfrak{sl}(2, \mathbb{R})$ -invariant quantum mechanics that are described in the DFF paper [20] (identifying the spatial variable  $Q$  in equation (4.1) with our variable  $r$ ). The key difference is that we extended the domain  $\mathbb{R}^+$  of the variable  $r$  to  $\mathbb{R}$  to make the representation unitary: If we only had  $r \in (0, \infty)$ , then  $H$  would fail to be self-adjoint with respect to the inner product (4.2), while our adjusted representation (4.18) actually is unitary only for the principal series representations with respect to the inner product (4.14) on the transformed Hilbert space, which looks like this after the coordinate transformation:

$$\langle \hat{\psi}, \hat{\chi} \rangle = \int_{-\infty}^{\infty} dr \hat{\psi}^* \left( \frac{\text{sign}(r)}{2} r^2 \right) \hat{\chi} \left( \frac{\text{sign}(r)}{2} r^2 \right) \tag{4.21}$$

This is just the original inner product (4.2) on the half-line  $\mathbb{R}^+$  extended to  $\mathbb{R}$ . The problem, as discussed before at the start of this chapter, is that it's not possible to directly assign a sensible physical interpretation to the region  $r < 0$ , especially when considering the fact that the most logical application (a central forcefield from a dipole moment at  $r = 0$ ) of the DFF model and our variant of it is one where  $r$  represents physical distances. Because of the Hamiltonian (4.20) that we found, you could say that  $r = 0$  behaves like a mathematical horizon of the system: For  $r < 0$ , the Hamiltonian flips sign, corresponding to negative energies. An obvious and appropriate physical interpretation of this behaviour is yet unclear, as the appearance of a horizon is often the result of an oversimplified mathematical model failing to describe reality. However, it is nice to see that we can modify the DFF model for a simple  $\mathfrak{sl}(2, \mathbb{R})$ -invariant scalar field theory to accommodate principal series representations, especially when we will make a connection to massive scalar field theories on  $dS_2$  in the next chapter.

## 5 Scalar fields in $\text{dS}_2$

In this section, we are going to investigate a simple classical scalar field in  $\text{dS}_2$ , which will immediately turn out to agree with our previous discussion of the  $\mathfrak{sl}(2, \mathbb{R})$ -isometries of  $\text{dS}_2$  in a very natural way, and in this discussion we will also find some validation for our specific interest in the principal series and the DFF model. Besides this, the purpose of this chapter will be that we can quantize the scalar field afterwards, and we can investigate how far we can extend the conclusions of our previous discussions of the representations of  $\mathfrak{sl}(2, \mathbb{R})$  in this context, exploring some basic objects in quantum field theories.

In general, the interest in (quantum) field theories instead of (quantized) classical systems can briefly be motivated by two observations [35, Chapter 1]: First of all, the formulation of field theories is often by nature much more in accordance with Einstein's theory of special relativity (think for example of how classical electromagnetism can be formulated in a Lorentz covariant way), which makes it a good starting point for unifying quantum mechanics and special relativity. Second of all, classical systems describe systems with a fixed number of particles, which means that the quantum mechanical description of the system doesn't allow creation or annihilation of particles, which is a crucial feature of the subatomic world. Motivated by this, you can for example consider quantized waves in a universal electron field to be manifested by what we observe as electrons. This interpretation does not only fix this problem, it is also in line with the wave-particle duality and it also explains why every elementary particle of a kind always has the exact same fundamental properties like rest mass, charge and quantum spin.

### 5.1 Massive scalar field

To start our exploration of  $\mathfrak{sl}(2, \mathbb{R})$ -invariant quantum fields, we consider a simple classical field theory; a scalar field  $\phi(x)$  on  $\text{dS}_2$  with mass  $m > 0$ , which can be described with the action

$$S \equiv \int d^2x \sqrt{|g|} \mathcal{L} = -\frac{1}{2} \int d^2x \sqrt{|g|} \left( (\partial^\mu \phi)(\partial_\mu \phi) + m^2 \phi^2 \right) \quad (5.1)$$

where the Lagrangian density  $\mathcal{L}$  is written out for the field, and the term  $\sqrt{|g|}$  (where  $g$  is the determinant of the metric tensor  $g_{\mu\nu}$ ) keeps the integral invariant under coordinate transformations, because  $d^2x \sqrt{|g|}$  is a volume element on  $\text{dS}_2$ . Notice that we choose to use the convention that excludes the term  $\sqrt{|g|}$  from the Lagrangian density, which differs from the convention in the discussion in the Carroll book [25, Chapter 9.4], where it is included in  $\mathcal{L}$ . This can and will only affect expressions and results that explicitly depend on  $\mathcal{L}$ , and will be something to keep in mind in this section and especially in Chapter 6.

To write the equations of motion, we introduce the covariant derivative  $\nabla_\mu$  that acts on vector fields like [25, Chapter 3.2]

$$\nabla_\mu V^\lambda \equiv \partial_\mu V^\lambda + \Gamma_{\mu\nu}^\lambda V^\nu \quad (5.2)$$

where  $\Gamma_{\mu\nu}^\lambda = \frac{g^{\lambda\sigma}}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$  is called the Christoffel symbol. This gives the following Lorentz invariant expression for the divergence of a vector field  $V^\mu$ :

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu) \quad (5.3)$$

With this definition, the Euler-Lagrange equations of motion for  $\phi$  in curved spacetime take the elegant form [36]:

$$\nabla_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \quad (5.4)$$

Using the expression for the Lagrangian density  $\mathcal{L}$  in the action (5.1) and plugging  $V^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = -\partial^\mu \phi$  into expression (5.3), we find that this reduces to

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} \partial^\mu \phi) = m^2 \phi \quad (5.5)$$

where we multiplied both sides by  $-1$ . If we define the Lorentz invariant D'Alembertian  $\square \equiv \nabla_\mu \nabla^\mu$ , the equation of motion can be written very compactly, and can be recognized as a curved spacetime-variant of the Klein-Gordon equation:

$$\square\phi = \nabla_\mu \nabla^\mu \phi = m^2 \phi \quad (5.6)$$

Here we used that  $\nabla^\mu f = \partial^\mu f$  for scalar functions  $f$ . With the explicit expressions for the metric  $g_{\mu\nu}$  and the partial derivatives  $\partial_\mu$  (given in equations (2.7) and (2.8)) that follow from our parameterization of dS<sub>2</sub>, we can find more explicit expression for the equations of motion for  $\phi$ :

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} \partial^\mu \phi) = \left( -\frac{\tanh(\frac{\tau}{\ell})}{\ell} \partial_\tau - \partial_\tau^2 + \frac{\partial_\theta^2}{\ell^2 \cosh^2(\frac{\tau}{\ell})} \right) \phi = m^2 \phi \quad (5.7)$$

We recognize the equation (2.19) for the Casimir  $C_2$  that we found in our discussion of the isometries of dS<sub>2</sub>! Plugging this into the equation of motion and requiring  $C_2 = \Delta(\Delta - 1)$  to match our discussion of the representations of  $\mathfrak{sl}(2, \mathbb{R})$  (meaning that we will consider a Hilbert space of wave functions on dS<sub>2</sub>), we find the relation

$$-\frac{1}{\ell^2} C_2 \phi = -\frac{\Delta(\Delta - 1)}{\ell^2} \phi = m^2 \phi \quad (5.8)$$

We see that this equation imposes a restriction on the conformal dimension  $\Delta$  of our representation if we want the wave functions in the Hilbert space to describe a scalar field with mass  $m$ . We find two possible value for  $\Delta$ , namely

$$\Delta_\pm = \frac{1}{2} \pm \frac{\sqrt{1 - 4m^2 \ell^2}}{2} \quad (5.9)$$

You can see how the types of representations that we found in Chapter 3 arise: A mass  $m \leq \frac{1}{2\ell}$  gives  $\Delta \in \mathbb{R}$ , and this leads to a complementary series representation because  $m > 0$  (as we assumed) leads to  $\Delta \in (0, 1)$  in this case. A massive field however (where 'massive' refers to  $m \geq \frac{1}{2\ell}$  from now on because of this distinction), makes the term inside the square root negative, meaning that the second term becomes imaginary and we find a principal series representation. The appearance in the description of a scalar field in dS<sub>2</sub> of these  $\mathfrak{sl}(2, \mathbb{R})$ -representations that we found before justifies our interest in them massively, because quantizing the scalar field will now be a very natural continuation to explore these representations.

From now on, we will consider massive scalar fields, and the conformal dimension  $\Delta = \frac{1}{2}(1 + i\nu)$  is labeled by  $\nu \in \mathbb{R}$ , which we can express in terms of the mass of the field by

$$\nu = \sqrt{4m^2 \ell^2 - 1} \quad (5.10)$$

Here we implicitly chose  $\Delta \equiv \Delta_+$  as in (5.9) to make  $\nu$  positive.

We now return to expression (5.7), because we were not finished with the equation of motion of the scalar field: We can try to look for the solutions to this differential equation, and to make it easy for ourselves we make the Ansatz that in the late time limit, we have

$$\phi(\tau, \theta) = f\left(\frac{\tau}{\ell}\right) \psi(\theta) \quad (5.11)$$

which will later be justified. This separation of variables gives the following differential equation (where  $f'$  denotes the derivative of the function  $f$  with respect to its argument, so it should not be confused with derivatives with respect to  $\tau$  or  $\theta$ ):

$$\tanh\left(\frac{\tau}{\ell}\right) \frac{f'(\frac{\tau}{\ell})}{f(\frac{\tau}{\ell})} + \frac{f''(\frac{\tau}{\ell})}{f(\frac{\tau}{\ell})} - \Delta(\Delta - 1) = \frac{1}{\cosh^2(\frac{\tau}{\ell})} \frac{\psi''(\theta)}{\psi(\theta)} \quad (5.12)$$

where we used  $-m^2 \ell^2 = \Delta(\Delta - 1)$  (which follows from the appearance of the Casimir in equation (5.8)). Notice that we could bring the  $\cosh^2(\frac{\tau}{\ell})$ -term to the other side to obtain a differential equation where the

two variables are fully separated (the left side would not depend on  $\theta$  and the right side would not depend on  $\tau$ ). This would be a difficult equation to solve, but in the late time limit ( $\tau \rightarrow \infty$ ), the equation takes a much simpler after we multiply it by  $f(\frac{\tau}{\ell})$ :

$$f'(\frac{\tau}{\ell}) + f''(\frac{\tau}{\ell}) - \Delta(\Delta - 1)f(\frac{\tau}{\ell}) = 0 \quad (5.13)$$

This differential equation has the two linearly independent solutions

$$f(\frac{\tau}{\ell}) = e^{-\Delta \frac{\tau}{\ell}} \quad \text{and} \quad f(\frac{\tau}{\ell}) = e^{-(1-\Delta) \frac{\tau}{\ell}} \quad (5.14)$$

Realizing that  $1 - \Delta = \Delta_s = \Delta^*$  because we're working with a principal series representation (which also means that these modes are complex conjugates), we find that the Klein-Gordon equation (5.5) is solved in the late time limit in  $dS_2$  by a superposition of these two solutions:

$$\phi(\tau, \theta) \underset{\tau \rightarrow \infty}{\approx} (A e^{-\Delta \frac{\tau}{\ell}} + B e^{-\Delta_s \frac{\tau}{\ell}}) \psi(\theta) \quad (5.15)$$

## 5.2 Klein-Gordon inner product

We want to have an inner product on this Hilbert space that arises from the wave functions that solve the Klein-Gordon equation in  $dS_2$ , and for this we define the Klein-Gordon inner product between wave functions  $\phi_1$  and  $\phi_2$  by

$$\langle \phi_1, \phi_2 \rangle \equiv -i \int_{\Sigma} d\Sigma^{\mu} (\phi_1 \partial_{\mu} \phi_2^* - \phi_2^* \partial_{\mu} \phi_1) \quad (5.16)$$

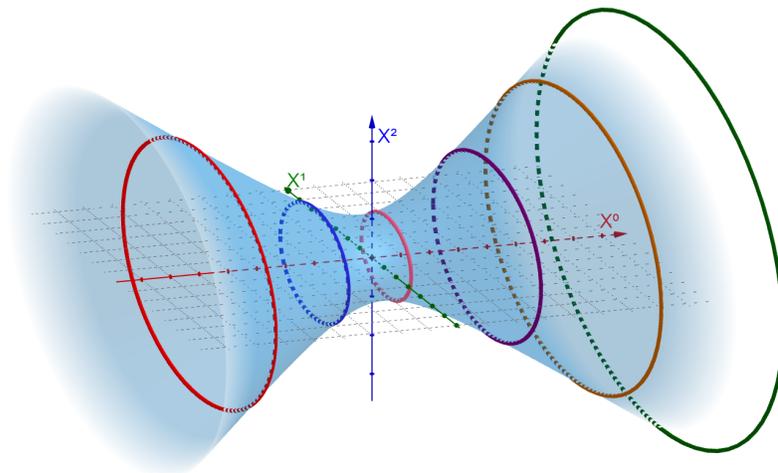


Figure 3: Examples of one-dimensional hypersurfaces  $\Sigma_{\tau}$  in  $dS_2$  that result from different values of  $\tau$

Here  $\Sigma$  denotes a one-dimensional surface in  $dS_2$  defined by  $\tau = \text{constant}$ . The term  $d\Sigma^{\mu}$  can be split into a normalized normal vector  $n^{\mu}$  to the hypersurface and a volume element  $\hat{\epsilon}$  on the hypersurface  $\Sigma$ . Using the discussion on hypersurfaces in manifolds in Appendix D of [25] on our constant time slices  $\Sigma$  of  $dS_2$ , we find a future-directed normal vector

$$n^{\mu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5.17)$$

The volume element is given by the following expression, where  $\gamma$  is the determinant of the induced metric  $\gamma_{ij}$  (where the indices  $i, j$  run over the coordinates  $y^1, \dots, y^{n-1}$  of the  $(n-1)$ -dimensional hypersurface inside a  $n$ -dimensional manifold):

$$\hat{\epsilon} = \sqrt{|\gamma|} d^{n-1}y = \sqrt{|\gamma|} dy^1 \wedge \dots \wedge dy^{n-1} \quad (5.18)$$

and for our one-dimensional hypersurface, the induced metric reduces to the one by one matrix  $\gamma_{ij} = (\ell^2 \cosh^2(\frac{\tau}{\ell}))$  with the only coordinate being  $\theta$ , and we find

$$\hat{\epsilon} = \ell \cosh\left(\frac{\tau}{\ell}\right) d\theta \quad (5.19)$$

We can define the covariant current  $J_\mu$  which using  $\phi_1$  and  $\phi_2$  by

$$J_\mu \equiv -i(\phi_1 \partial_\mu \phi_2^* - \phi_2^* \partial_\mu \phi_1) \quad (5.20)$$

to write the Klein-Gordon inner product in the compact form

$$\langle \phi_1, \phi_2 \rangle = \int_\Sigma \hat{\epsilon} n^\mu J_\mu \quad (5.21)$$

where we used our earlier expressions to write  $d\Sigma^\mu = \hat{\epsilon} n^\mu$ .

Before we investigate the Klein-Gordon inner product for our solutions (5.15), we should question whether or not the Klein-Gordon inner product is even well-defined in the first place! After all, we can choose arbitrary times  $\tau \in \mathbb{R}$  to define our hypersurface  $\Sigma$  which we will now denote by  $\Sigma_\tau$  until we manage to prove that the definition of the Klein-Gordon inner product is disambiguous. In figure 3, some possible hypersurfaces  $\Sigma_\tau$  are shown.

Fortunately, the inner product turns out to be disambiguous, which we will prove using Stokes' theorem, which is presented in the same conventions as we use in the section in [25, p. 455]:

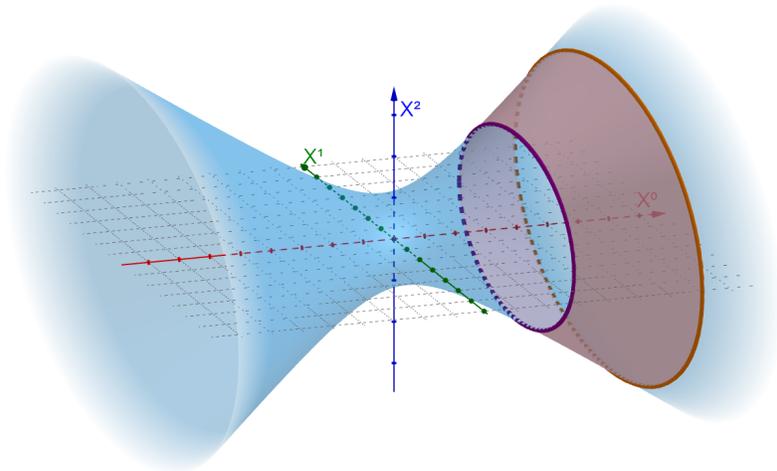


Figure 4: The region  $U$  (in pink) between two hypersurfaces  $\Sigma_\tau$  that is used in Stokes' theorem

$$\int_U d^n x \sqrt{|g|} \nabla_\mu V^\mu = \int_{\partial U} d^{n-1} y \sqrt{|\gamma|} n_\mu V^\mu \quad (5.22)$$

Here  $V^\mu$  is a vector field on  $U$ , a  $n$ -dimensional manifold (in our case, it will be a region in  $dS_2$ ) with coordinates  $x^\mu$  and metric  $g_{\mu\nu}$ , and the boundary  $\partial U$  will therefore be a  $(n-1)$ -dimensional hypersurface with coordinates  $y^i$ , metric  $\gamma_{ij}$  and normal vector  $n^\mu$ .

We define our region  $U$  to be the region in  $dS_2$  enclosed by two hypersurfaces  $\Sigma_{\tau_1}$  and  $\Sigma_{\tau_2}$  with  $\tau_1 < \tau_2$ , i.e.  $U \equiv \{(\tau, \theta) \in dS_2 \mid \tau_1 < \tau < \tau_2\}$ , as is visualized in figure 4. The trick is now to show that the current

$V_\mu \equiv J_\mu$  is a conserved current, making the first integral in (5.22) vanish. To do this, we calculate  $\nabla^\mu J_\mu$  from expression (5.3) using the product rule and the explicit expression for the metric in dS<sub>2</sub>. We find

$$\nabla^\mu J_\mu = \partial^\mu J_\mu - \frac{\tanh(\frac{\tau}{\ell})}{\ell} J_0 \quad (5.23)$$

where the minus sign follows from the derivatives being contravariant vectors in this expression. We calculate  $\partial^\mu J_\mu$  using the product rule, and we end up with the following expression after some terms cancel each other out:

$$\partial^\mu J_\mu = -i \left( \phi_1 \partial^\mu \partial_\mu \phi_2^* - \phi_2^* \partial^\mu \partial_\mu \phi_1 \right) \quad (5.24)$$

Now we can use the equation of motion (5.7) for the fields  $\phi_1$  and  $\phi_2^*$  to simplify this expression even more! Recognizing that this expression can be rewritten into  $\partial^\mu \partial_\mu \phi = m^2 \phi + \frac{\tanh(\frac{\tau}{\ell})}{\ell} \partial_0 \phi$ , we obtain the following after the terms with the mass cancel each other out:

$$\partial^\mu J_\mu = -i \left( \phi_1 \frac{\tanh(\frac{\tau}{\ell})}{\ell} \partial_0 \phi_2^* - \phi_2^* \frac{\tanh(\frac{\tau}{\ell})}{\ell} \partial_0 \phi_1 \right) = \frac{\tanh(\frac{\tau}{\ell})}{\ell} J_0 \quad (5.25)$$

We see that plugging this into equation (5.23) makes the two terms cancel each other out again, and we find that indeed the current  $J_\mu$  is conserved because it is constructed with fields that obey the Klein-Gordon equation:

$$\nabla^\mu J_\mu = 0 \quad (5.26)$$

As we mentioned before, this makes the first integral in Stokes' theorem (5.22) vanish. We now focus on the second integral over the boundary of  $U$ , and the realization that  $\partial U$  only consists of  $\Sigma_{\tau_1}$  and  $\Sigma_{\tau_2}$  (which you can immediately see in figure 4) makes the integral look like the difference between the Klein-Gordon inner product on these two hypersurfaces (after substituting  $\sqrt{|\gamma|} d^{n-1} y = \hat{\epsilon}$ ):

$$\int_{\partial U} d^{n-1} y \sqrt{|\gamma|} n^\mu J_\mu = \left( \int_{\Sigma_{\tau_2}} - \int_{\Sigma_{\tau_1}} \right) \hat{\epsilon} n^\mu J_\mu \quad (5.27)$$

The minus sign follows from our definition (5.17) of normal vectors  $n^\mu$  to the hypersurfaces  $\Sigma_\tau$  which point into the future, while it should point outwards from  $U$ , meaning that the vector normal to  $\Sigma_{\tau_1}$  points to the past instead of the future in the integral over  $\partial U$ , which explains the minus sign. In conclusion, we find that Stokes' theorem (5.22) in this situation reduces to:

$$0 = \int_{\Sigma_{\tau_2}} \hat{\epsilon} n^\mu J_\mu - \int_{\Sigma_{\tau_1}} \hat{\epsilon} n^\mu J_\mu \quad \iff \quad \int_{\Sigma_{\tau_2}} \hat{\epsilon} n^\mu J_\mu = \int_{\Sigma_{\tau_1}} \hat{\epsilon} n^\mu J_\mu \quad (5.28)$$

So indeed the definition (5.16) of the Klein-Gordon inner product is independent of the value of  $\tau$ ! Now that we've proven that the inner product is well-defined, we can investigate inner products between our late time solutions (5.15). This is a huge justification for our interest in them: If we want to calculate the inner product between two solutions to the Klein-Gordon equation, we can perform the calculation in the late time limit as it simplifies the expressions massively, simplifying calculations massively, as we will see soon.

We choose to calculate the inner product  $\langle \phi_1, \phi_2 \rangle$  between two Klein-Gordon fields  $\phi_1, \phi_2$  that look as follows in the late time limit:

$$\phi_i(\tau, \theta) \underset{\tau \rightarrow \infty}{\approx} \sqrt{\frac{2}{\nu}} e^{-\Delta \frac{\tau}{\ell}} \psi_i(\theta) \quad \text{for } i = 1, 2 \quad (5.29)$$

Here  $\psi_1$  and  $\psi_2$  are square-integrable functions on  $S^1$  that determine the spatial behaviour of our Klein-Gordon fields. In other words, we will calculate the inner product between two late time solutions (5.15) with the coefficients chosen as  $A = \sqrt{\frac{2}{\nu}}$  and  $B = 0$ . These choices seem arbitrary, but because the second

mode is just the complex conjugate of the first one, we don't really discard the second mode by setting  $B = 0$ , as we will see it come up again when taking linear combinations of solutions. The choice of  $A$  is just chosen to create a nice correspondence between this Klein-Gordon inner product and other inner products that we investigated, as we will see soon. It doesn't determine the normalization of  $\phi(\tau, \theta)$  as we can always rescale  $\psi(\theta)$  if we need to renormalize  $\phi(\tau, \theta)$ .

Because of our expression (5.17) for  $n^\mu$ , the compact expression (5.21) for the inner product can be simplified even more because we have  $n^\mu J_\mu = J_0$ , which we can explicitly calculate by plugging in expression (5.29) for  $\phi_1$  and  $\phi_2$ .

$$J_0(\tau, \theta) = -i(\phi_1(\tau, \theta)\partial_\tau\phi_2^*(\tau, \theta) - \phi_2^*(\tau, \theta)\partial_\tau\phi_1(\tau, \theta)) = -\frac{2i}{\nu}e^{-\Delta\frac{\tau}{\ell}-\Delta^*\frac{\tau}{\ell}}\left(\frac{\psi_1(\theta)\psi_2^*(\theta)}{\ell}(\Delta - \Delta^*)\right) \quad (5.30)$$

Using  $\Delta = \frac{1}{2}(1 + i\nu)$ , this reduces nicely to

$$J_0(\tau, \theta) = \frac{2}{\ell}e^{-\frac{\tau}{\ell}}\psi_1(\theta)\psi_2^*(\theta) \quad (5.31)$$

Realizing that  $\Sigma$  is parameterized by  $\theta \in [0, 2\pi]$  and that as  $\tau \rightarrow \infty$ , the volume element in expression (5.19) becomes  $\hat{\epsilon} \underset{\tau \rightarrow \infty}{\approx} \frac{\ell}{2}e^{\frac{\tau}{\ell}}d\theta$ , we find:

$$\langle \phi_1, \phi_2 \rangle = \int_{\Sigma} \hat{\epsilon} \frac{2}{\ell}e^{-\frac{\tau}{\ell}}\psi_1(\theta)\psi_2^*(\theta) = \int_0^{2\pi} d\theta \psi_1(\theta)\psi_2^*(\theta) \quad (5.32)$$

This is exactly the inner product (4.5) between square-integrable wave functions  $\psi_1(\theta)$  and  $\psi_2(\theta)$  on  $S^1$  that we used before! If we consider the late time solutions  $\phi(\tau, \theta) = \sqrt{\frac{2}{\nu}}e^{-\Delta\frac{\tau}{\ell}}\psi(\theta)$  as in (5.29) and let the ladder operators (2.15) act on them, we find that the derivative  $\partial_\tau$  of the wave functions gives a factor  $-\frac{\Delta}{\ell}$ , resulting into:

$$L_0\phi(\tau, \theta) = -i\partial_\theta\phi(\tau, \theta) \quad (5.33)$$

$$L_\pm\phi(\tau, \theta) = e^{\mp i\theta}\left(-i\tanh\left(\frac{\tau}{\ell}\right)\partial_\theta \mp \Delta\right)\phi(\tau, \theta) \underset{\tau \rightarrow \infty}{\approx} -e^{\mp i\theta}(i\partial_\theta \pm \Delta)\phi(\tau, \theta) \quad (5.34)$$

We immediately see that they take the same form as the ladder operators (4.6) on the  $S^1$ -wave functions in the late time limit, and we could transform those to make the operators resemble the DFF model, transforming the variable  $\theta$  into  $r$  via the eigenvalues  $\kappa$  of the generator  $K$ . This gives a correspondence between the DFF model of  $\mathfrak{sl}(2, \mathbb{R})$ -invariant quantum mechanics and massive Klein Gordon fields in dS<sub>2</sub>! With this realization, we fill in expression (5.10) for  $\nu$  in terms of the mass  $m$  of the field into the Hamiltonian (4.20) of our version of the DFF model, and we find

$$H = \frac{\text{sign}(r)}{2}\left(-\partial_r^2 - \frac{16m^2\ell^2 - 3}{4r^2}\right) \quad (5.35)$$

which is again an attractive potential for massive fields ( $m \geq \frac{1}{2\ell}$ ) as you should expect. At this point, we have found interesting connections between several principal series representations of  $\mathfrak{sl}(2, \mathbb{R})$  and classical fields in dS<sub>2</sub>. In the next chapter, we're going to quantize the scalar field to explore the consequences of these connections in the context of quantum fields theories. Before we do so, we calculate the full solutions to the equation of motion (5.12) for the Klein-Gordon field instead of the solutions in the late time limit.

### 5.3 Solving the equation of motion

If we insert the solutions (5.11) with separated variables and plug in the  $S^1$ -wave functions  $\psi_n(\theta)$  given by expression (4.8) into the equation of motion, we find  $\psi_n''(\theta) = -n^2\psi_n(\theta)$  which gives us the following differential equation for  $f(x)$ , where  $x \equiv \frac{\tau}{\ell}$  to simplify the expression:

$$f''(x) + \tanh(x)f'(x) - \Delta(\Delta - 1)f(x) = -\frac{n^2}{\cosh^2(x)}f(x) \quad (5.36)$$

We will now solve this differential equation: Analogous to the calculations in [37, Section 3.1], we make the change of variables  $u = u(x) \equiv -e^{-2x}$  (giving us  $\partial_x = -2u\partial_u$  and  $\partial_x^2 = 4u(u\partial_u^2 + \partial_u)$ ) to define a new function  $g(u)$  such that  $f(x) = g(u(x))$ , and we end up with the differential equation

$$\frac{u-1}{u}g''(u) + \frac{1}{2}[3u-1]g'(u) - \left[\frac{\Delta(\Delta-1)}{4} - \frac{n^2}{u-1}\right]g(u) = 0 \quad (5.37)$$

Then we introduce the substitution

$$g(u) \equiv (1-u)^n (-u)^{\frac{\Delta}{2}} h(u) \quad \Longleftrightarrow \quad f(x) = (2 \cosh(x))^n e^{-(n+\Delta)x} h(u(x)) \quad (5.38)$$

which gives the following differential equation for  $h(u)$ :

$$u(1-u)h''(u) + \left[\frac{1}{2} + \Delta - (2n + \Delta + \frac{3}{2})u\right]h'(u) - \left(\frac{1}{2} + n\right)(n + \Delta)h(u) = 0 \quad (5.39)$$

This is solved by Gauss' hypergeometric function [38]  ${}_2F_1(a, b, c; ; u)$ , with  $a = n + \frac{1}{2}$ ,  $b = n + \Delta$  and  $c = \Delta + \frac{1}{2}$ :

$$h(u) = {}_2F_1\left(n + \frac{1}{2}, n + \Delta, \Delta + \frac{1}{2}; u\right) \quad (5.40)$$

The complex conjugate of this function is another linearly independent solution to the differential equation, so just like we did in the late time limit, we discard this conjugate solution  $h^*(u)$  for now (later on, we will use it when constructing a general Klein-Gordon field) and we again normalize the solution  $h(u)$  with a coefficient  $\sqrt{\frac{2}{\nu}}$ , so we end up with:

$$f_n(x) = \sqrt{\frac{2}{\nu}} (2 \cosh(x))^n e^{-(\Delta+n)x} {}_2F_1\left(n + \frac{1}{2}, n + \Delta, \Delta + \frac{1}{2}; -e^{-2x}\right) \quad (5.41)$$

This differs from the solutions in the original principal series paper [12, Chapter 5.3], where the solutions  $f_n$  are identical except for the fact that the absolute value  $|n|$  is used to construct the solutions, instead of  $n$  as we did. This is justified by the fact that the only  $n$ -dependence in the original differential equation (5.36) comes from a  $n^2$ -term, and therefore the equation is invariant under changes in the sign of  $n$ . However, mathematically speaking there is nothing wrong with our solutions (5.41) where we don't take the absolute value of  $n$  and therefore we will proceed using these solutions.

It is worth noticing however that in the Bouso paper [37], which is referenced in the principal series paper, a de Sitter space of general dimension  $d$  is considered, of which the spatial part is part is homeomorphic to  $S^{d-1}$ . In general dimensions, the spatial part of the solutions (5.42) is given by the spherical harmonics on  $S^{d-1}$  and for  $d > 2$  this set of linearly independent functions can only be labeled by nonnegative  $n$  [39, Chapter 4.4], which means that the parameter  $n$  in the solutions to the differential equation only takes nonnegative values, meaning that replacing  $n$  by  $|n|$  doesn't change anything about the solutions in higher dimensions because our solutions would be labeled by  $n \in \mathbb{Z}_{\geq 0}$  anyway.

With these expressions (5.41) for  $f_n(\frac{\tau}{\ell})$ , our modes look like:

$$\phi_n(\tau, \theta) = f_n\left(\frac{\tau}{\ell}\right)\psi_n(\theta) \quad (5.42)$$

This solution matches our previous late time solutions (5.29), because

$$f_n(x) \underset{x \rightarrow \infty}{\approx} \sqrt{\frac{2}{\nu}} e^{-\Delta x} \quad (5.43)$$

We can now calculate the Klein-Gordon inner product between these modes. To save ourselves a lot of work, we recall that the inner product is independent of the value of  $\tau$  that defines the hypersurface  $\Sigma_\tau$ , meaning that we can recycle our calculations (5.32), plug them into our result (4.9a) and we find

$$\langle \phi_m, \phi_n \rangle = \delta_{mn} \quad (5.44)$$

Moreover, the term  $n^\mu J_\mu = J_0$  in the Klein-Gordon inner product (5.21) vanishes when we consider the complex conjugate of the mode  $\phi_n$ :

$$\langle \phi_m, \phi_n^* \rangle = 0 \quad (5.45)$$

And lastly, if we also consider the complex conjugate of  $\phi_m$ , we find

$$\langle \phi_m^*, \phi_n^* \rangle = -\delta_{mn} \quad (5.46)$$

So we see that the modes  $\phi_n$  and  $\phi_n^*$  (with  $n \in \mathbb{Z}$ ) form an orthonormal system in this sense. Acting on these modes, the ladder operators (2.15) on dS<sub>2</sub> take the form (4.7) of the ladder operator actions on  $S^1$ -wave functions:

$$L_0 \phi_n(\tau, \theta) = -n\phi_n(\tau, \theta) \quad L_\pm \phi_n(\tau, \theta) = -(n \pm \Delta)\phi_{n\pm 1}(\tau, \theta) \quad (5.47)$$

This means that the Hilbert space of the Klein-Gordon field wave functions on dS<sub>2</sub> and the Hilbert space of  $S^1$ -wave functions are equivalent, and this model can be transformed into the Hilbert space of the  $\mathfrak{sl}(2, \mathbb{R})$ -invariant quantum mechanics in the modified DFF model! We can now try to extend this framework by quantizing a general Klein-Gordon field, which we will do in the next chapter.

## 6 Quantized Klein-Gordon field in dS<sub>2</sub>

We construct a general Klein-Gordon field by taking a superposition of the modes  $\phi_n(\tau, \theta)$  and  $\phi_n^*(\tau, \theta)$  of the Klein-Gordon field as described in the last chapter:

$$\phi(\tau, \theta) \equiv \sum_{n=-\infty}^{\infty} \left[ a_n \phi_n(\tau, \theta) + a_n^\dagger \phi_n^*(\tau, \theta) \right] \quad (6.1)$$

Here we used arbitrary coefficients  $a_n$  and  $a_n^\dagger$ . The usual way to canonically quantize this theory is to make these coefficients into annihilation and creation operators, following the calculations in [25, Chapter 9.4]. We introduce a vacuum state  $|0_f\rangle$  that gets annihilated by  $a_n$  for every  $n \in \mathbb{Z}$ . The subscript  $f$  is included because the definition of the vacuum state depends on the choice of the basis of Klein-Gordon modes, which in our case consists of the modes  $\phi_n(\tau, \theta)$  given by expression (5.42) and their complex conjugates. We will later see that another basis of wave functions gives a vacuum state that is not equivalent to  $|0_f\rangle$ .

In bra-ket notation, the imposed annihilation of the vacuum state (which we take to be normalized) is given by:

$$a_n |0_f\rangle \equiv 0 \quad \forall n \in \mathbb{Z} \quad (6.2)$$

The canonical commutation relations of the operators are as follows (with  $n, m \in \mathbb{Z}$ ):

$$[a_n, a_m^\dagger] = \delta_{nm} \quad [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad (6.3)$$

In this framework, which describes an infinite-dimensional harmonic oscillator, letting a creation operator  $a_n^\dagger$  act on a state creates a particle of which the state is represented by the index  $n$ , and the same goes for the annihilation operator  $a_n$ . These particle states, which we will refer to as the particle being in the  $n$ -state, should not be confused with general quantum states in a Hilbert space.

We will perform some basic calculations in this chapter where the products  $a_n a_m$ ,  $a_n a_m^\dagger$ ,  $a_n^\dagger a_m$  and  $a_n^\dagger a_m^\dagger$  will appear, and we will let those act on the vacuum state most of the time. The terms with  $a_n |0_f\rangle$  and  $\langle 0_f | a_m^\dagger$  vanish because of definition (6.2) of the vacuum state, realizing that  $a_n^\dagger$  is the hermitian conjugate of  $a_n$ . The only surviving term is the one containing  $a_n a_m^\dagger$ , which we can rewrite into  $a_m^\dagger a_n + \delta_{nm}$  (of which the first term will vanish again) using the canonical commutation relations of the operators. This can be summarized by the following equations, which will be very useful for simplifying expressions with operators:

$$\langle 0_f | a_n a_m | 0_f \rangle = \langle 0_f | a_n^\dagger a_m | 0_f \rangle = \langle 0_f | a_n^\dagger a_m^\dagger | 0_f \rangle = 0 \quad \text{and} \quad \langle 0_f | a_n a_m^\dagger | 0_f \rangle = \delta_{nm} \quad (6.4)$$

You can also build new operators with these creation and annihilation operators, of which a simple yet important example is the *number operator*:

$$N_n \equiv a_n^\dagger a_n \quad (6.5)$$

This operator gives the number of particles in the  $n$ -state, and therefore indeed vanishes when acting on the vacuum state. This way of quantizing the field is in line with the interpretation that particles arise as excitations in quantum fields. To illustrate this, we consider an example of a state that we can construct with these operators:

$$|p_n, q_m\rangle \equiv \frac{1}{\sqrt{p! q!}} (a_n^\dagger)^p (a_m^\dagger)^q |0_f\rangle \quad (6.6)$$

The prefactor is chosen for normalization. You can see that we let the creation operators  $a_n^\dagger$  and  $a_m^\dagger$  act on the vacuum state  $p$  and  $q$  times respectively, which should be interpreted as the system being in a quantum state with  $p$  particles in the  $n$ -state and  $q$  particles in the  $m$ -state, which can be formulated in terms of the number operator:

$$\langle p_n, q_m | N_n | p_n, q_m \rangle = p \quad \text{and} \quad \langle p_n, q_m | N_m | p_n, q_m \rangle = q \quad (6.7)$$

The beautiful thing about this framework is that the states are orthonormal! This can be shown with the commutation relations (6.3): If two states have a different number of particles in an  $n$ -state, then you can manipulate the states using their definition (6.6) until you annihilate the bra or ket and the inner product becomes zero. For our states (6.6) with particles in the  $n$ - and  $m$ -state, inner products look like

$$\langle p'_n, q'_m | p_n, q_m \rangle = \delta_{p'p} \delta_{q'q} \quad (6.8)$$

Of course, a quantum state can describe systems with particles in infinitely many  $n$ -states, but the description of such a state is not any different from the previous examples. With this framework in mind, we should think about appropriate physical interpretations of the index  $n$  of the different particle  $n$ -states:

We know that the modes  $\phi_n$  are eigenfunctions of  $L_0$ , the infinitesimal generator of translations in the spatial part of dS<sub>2</sub> (which are rotations in  $\mathbb{R}^{1,2}$  as discussed in Chapter 2), leading us to the insight that the index  $n$  is related to the spatial momentum of particles that are described by waves that consist of the modes  $\phi_n$ . The fact that the spectrum of possible momenta is discrete, is related to the fact that the spatial component of dS<sub>2</sub> is homeomorphic to the compact space  $S^1$ . This is in contrast with wave functions in flat  $d$ -dimensional Minkowski space, where the possible particle states are determined by the spatial wave vector  $\mathbf{k} \in \mathbb{R}^{d-1}$ , which leads to a continuous system of operators  $a_{\mathbf{k}}$  and  $a_{\mathbf{k}}^\dagger$ , and with those you can build a basis of the Hilbert space called the *Fock Basis*, as is done in [25, Chapter 9.3]. For now, the loose interpretation of the index  $n$  as being related to the (spatial) momentum of waves will suffice, but we will revisit this discussion at the end of the next section.

## 6.1 Energy of the quantized field

With the field made into a superposition of operators, we can calculate the Lagrangian density in terms of  $a_n$  and  $a_n^\dagger$ . In Chapter 5, we introduced the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \left( (\partial^\mu \phi)(\partial_\mu \phi) + m^2 \phi^2 \right) = -\frac{1}{2} \left( -(\partial_0 \phi)(\partial_0 \phi) + g^{11}(\partial_1 \phi)(\partial_1 \phi) + m^2 \phi^2 \right) \quad (6.9)$$

Here we wrote out the summation in anticipation of plugging in the expressions for our fields into the Lagrangian density. We don't plug in  $g^{11} = (\ell \cosh(\frac{\tau}{\ell}))^{-2}$  to keep the expression compact, but in writing out the summation we already used  $g^{00} = -1$  and  $g^{01} = g^{10} = 0$ . To calculate energies of states, we turn into the Hamiltonian description of the field theory by introducing the conjugate momentum of the scalar field:

$$\pi(\tau, \theta) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}(\tau, \theta) = \partial_0 \phi(\tau, \theta) \quad (6.10)$$

The convention to exclude the factor  $\sqrt{|g|}$  gives a new discrepancy with the results in the Carroll book [25, Chapter 9.4], where the conjugate momentum contains an extra term  $\sqrt{|g|}$  because of this. This will affect expressions for quantities like energy densities, but not total energies in a region, because it will appear again when we integrate over the region. We introduce the Hamiltonian density by the following Legendre transform:

$$\mathcal{H} \equiv \pi(\partial_0\phi) - \mathcal{L} \quad (6.11)$$

where we have omitted the  $\tau$ - and  $\theta$ -dependence of the field and its conjugate momentum, and the  $\phi$ - and  $\pi$ -dependence of  $\mathcal{L}$  and  $\mathcal{H}$  in the notation. Filling in our expression (6.9) for  $\mathcal{L}$ , we find:

$$\mathcal{H} = \frac{1}{2} \left[ \pi^2 + g^{11}(\partial_1\phi)^2 + m^2\phi^2 \right] \quad (6.12)$$

Now we want to fill in the solutions (6.1) to the Klein-Gordon equation that we found earlier. We start our calculations in the late time limit  $\tau \rightarrow \infty$ , where our modes  $\phi_n$  look like

$$\phi_n(\tau, \theta) = \sqrt{\frac{2}{\nu}} e^{-\Delta \frac{\tau}{\ell}} \psi_n(\theta) \quad \text{with} \quad \psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{-in\theta} \quad (6.13)$$

This gives the following expression for a general quantized Klein-Gordon field:

$$\phi(\tau, \theta) = \sqrt{\frac{2}{\nu}} \sum_{n=-\infty}^{\infty} \left[ e^{-\Delta \frac{\tau}{\ell}} \psi_n(\theta) a_n + e^{-\Delta_s \frac{\tau}{\ell}} \psi_n^*(\theta) a_n^\dagger \right] \quad (6.14)$$

Now we can calculate the other quantities that appear in the Hamiltonian density (6.12), keeping in mind that  $\partial_0 = \partial_\tau$  and  $\partial_1 = \partial_\theta$  and that  $\pi$  is used both for the number  $\pi$  and the momentum  $\pi(\tau, \theta)$ :

$$\pi(\tau, \theta) = \partial_0\phi(\tau, \theta) = -\frac{1}{\ell} \sqrt{\frac{2}{\nu}} \sum_{n=-\infty}^{\infty} \left[ \Delta e^{-\Delta \frac{\tau}{\ell}} \psi_n(\theta) a_n + \Delta_s e^{-\Delta_s \frac{\tau}{\ell}} \psi_n^*(\theta) a_n^\dagger \right] \quad (6.15a)$$

$$\partial_1\phi(\tau, \theta) = \sqrt{\frac{2}{\nu}} \sum_{n=-\infty}^{\infty} \left[ -ine^{-\Delta \frac{\tau}{\ell}} \psi_n(\theta) a_n + ine^{-\Delta_s \frac{\tau}{\ell}} \psi_n^*(\theta) a_n^\dagger \right] \quad (6.15b)$$

We can calculate the commutation relations for the field and its conjugate momentum in the late time limit using the relations (6.3), and we find the expected relations for fixed  $\tau$ :

$$\begin{aligned} [\phi(\tau, \theta), \phi(\tau, \theta')] &= 0 \\ [\pi(\tau, \theta), \pi(\tau, \theta')] &= 0 \\ [\phi(\tau, \theta), \pi(\tau, \theta')] &= \frac{2i}{\ell} e^{-\frac{\tau}{\ell}} \delta(\theta - \theta') \approx \frac{i}{\sqrt{|g|}} \delta(\theta - \theta') \end{aligned} \quad (6.16)$$

These expressions gives the following Hamiltonian density after some simplification, making use of the fact that  $g^{11} = \frac{4}{\ell^2} e^{-2\frac{\tau}{\ell}}$  in the late time limit:

$$\begin{aligned} \mathcal{H} = \frac{e^{-\frac{\tau}{\ell}}}{\nu \ell^2} \sum_{n, m \in \mathbb{Z}} \left[ (\Delta - 4nm e^{-2\frac{\tau}{\ell}}) e^{-i\nu \frac{\tau}{\ell}} \psi_n(\theta) \psi_m(\theta) a_n a_m \right. \\ (2m^2 \ell^2 + 4nm e^{-2\frac{\tau}{\ell}}) (\psi_n(\theta) \psi_m^*(\theta) a_n a_m^\dagger + \psi_n^*(\theta) \psi_m(\theta) a_n^\dagger a_m) \\ \left. (\Delta_s - 4nm e^{-2\frac{\tau}{\ell}}) e^{i\nu \frac{\tau}{\ell}} \psi_n^*(\theta) \psi_m^*(\theta) a_n^\dagger a_m^\dagger \right] \end{aligned} \quad (6.17)$$

Notice that our we use  $m$  for both the mass of the field and as a summation index, but the only place where it refers to the mass is where it appears in the  $m^2$ -term. With this expression, we can calculate the Hamiltonian  $H = \int_0^{2\pi} d\theta \sqrt{|g|} \mathcal{H}$ , and we can make use of the inner product (4.9a) between wave functions  $\psi_n(\theta)$  (realizing

that  $\psi_n^*(\theta) = \psi_{-n}(\theta)$  to turn this integral into Kronecker deltas, which will then cancel the summation over  $m$  and we end up with:

$$H = \frac{1}{2\nu\ell} \sum_{n \in \mathbb{Z}} \left[ (\Delta + 4n^2 e^{-2\frac{\tau}{\ell}}) e^{-i\nu\frac{\tau}{\ell}} a_n a_{-n} \right. \\ (2m^2\ell^2 + 4n^2 e^{-2\frac{\tau}{\ell}}) (a_n a_n^\dagger + a_n^\dagger a_n) \\ \left. (\Delta_s + 4n^2 e^{-2\frac{\tau}{\ell}}) e^{i\nu\frac{\tau}{\ell}} a_n^\dagger a_{-n}^\dagger \right] \quad (6.18)$$

This is still a bulky expression to handle in calculations, but because of the results (6.4) that we found before, we see that most of these terms vanish when we calculate the vacuum energy i.e. the energy that the non-excited field carries in empty space, and we end up with the following expression:

$$\langle 0_f | H | 0_f \rangle = \frac{1}{\nu\ell} \sum_{n \in \mathbb{Z}} (m^2\ell^2 + 2n^2 e^{-2\frac{\tau}{\ell}}) \quad (6.19)$$

An elegant way to treat the infinite sums is to use the series representation of the Dirac delta:

$$\delta^{(k)}(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (in)^k e^{inx} \implies \sum_{n \in \mathbb{Z}} n^k = 2\pi(-i)^k \delta^{(k)}(0) \quad (6.20)$$

Resulting in a vacuum energy of the entire space at a certain moment  $\tau$

$$H_{0_f} \equiv \langle 0_f | H | 0_f \rangle = \frac{2\pi}{\nu\ell} (m^2\ell^2 \delta(0) - 2\delta''(0) e^{-2\frac{\tau}{\ell}}) \quad (6.21)$$

This is equivalent to a vacuum energy density of

$$\mathcal{H}_{0_f} \equiv \langle 0_f | \mathcal{H} | 0_f \rangle = \frac{2}{\nu\ell^2} (m^2\ell^2 \delta(0) e^{-\frac{\tau}{\ell}} - 2\delta''(0) e^{-3\frac{\tau}{\ell}}) \quad (6.22)$$

The appearance of infinities in the vacuum energy of spacetime seems problematic: It seems wrong that the presence of a scalar field in its ground state contributes an infinite amount of energy, especially considering the fact that we didn't integrate over an infinite space, so where does this infinity come from? In actuality, this result is not surprising when you compare it to other results in quantum field theories: When turning classical theories into quantum theories, infinities arise all the time. In fact, one of the main aspects of a quantum field theory is how it deals with the appearance of such infinities[21]!

When an infinite energy arises, we can change the Hamiltonian up to a constant without changing the resulting physics, as is also the case in classical theories. The fact that this renormalization would require the constant to be infinite may seem scary, but as mentioned before, there is nothing fundamentally wrong with infinities as long as we are careful in our calculations[25, Chapter 9]. This is a simple example of renormalization, of which the basic principles are presented in [40].

With this in mind, we can calculate what the energy levels of excited states of the field are as well. To illustrate this, we introduce a minimally excited state  $|1_n\rangle$ , which describes a state of the field with only one single particle in the  $n$ -state, analogous to (6.6):

$$|1_n\rangle = a_n^\dagger |0_f\rangle \quad (6.23)$$

The energy density of this excited state can be again be calculated by measuring the Hamiltonian density (6.12) in this state. As expected, we recognize this energy density to be the vacuum energy density plus an additional contribution from the excitation:

$$\langle 1_n | \mathcal{H} | 1_n \rangle = \mathcal{H}_{0_f} + \frac{2}{\pi\nu\ell^2} [m^2\ell^2 e^{-\frac{\tau}{\ell}} + 2n^2 e^{-3\frac{\tau}{\ell}}] \quad (6.24)$$

We recognize that this contribution from the excited state  $|1_n\rangle$  consists of a term that only depends on the mass of the field, and a term that only depends on the state  $n$  of this excitation and not on the mass. To further investigate the meaning of this  $n$ -state of particle in the state  $|1_n\rangle$ , we turn to another important object: In field theories, it is more common to work with the energy-momentum tensor for Lagrangian field theories, which for looks as follows[41, p. 192]:

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) + g_{\mu\nu}\mathcal{L} \quad (6.25)$$

We can calculate the energy density component  $T_{00}$  and as we expect, we obtain the Hamiltonian density (6.12):

$$T_{00} = \mathcal{H} \quad (6.26)$$

The benefit of working with the energy-momentum tensor is that it contains more information than the energy density, and it appears in both general relativity and classical field theories, forming a natural 'bridge' between the two as we try to do with quantum field theories. We can also extract the momentum density in the  $\theta$ -direction  $T_{10}$  from the energy-momentum tensor[42]. Using the relations (6.15) and inserting them into  $T_{10} = (\partial_1\phi)(\partial_0\phi)$ , we find that also the momentum density of the scalar field in the vacuum state becomes an infinite quantity:

$$\langle 0_f|T_{10}|0_f\rangle = \frac{2\Delta_s}{\nu\ell} e^{-\frac{\tau}{\ell}} \delta'(0) \quad (6.27)$$

Just like with the energy density, we can calculate the momentum density contribution from a minimal excitation to the vacuum state, a single particle in the  $n$ -state. We do this by again measuring  $T_{10}$ , but now in the state  $|1_n\rangle$ . We find:

$$\langle 1_n|T_{10}|1_n\rangle = \langle 0_f|T_{10}|0_f\rangle + n \frac{e^{-\frac{\tau}{\ell}}}{\pi\ell} \quad (6.28)$$

This is a surprisingly elegant result! The momentum density of the excitation of the field is indeed closely related to the index  $n$  of the  $n$ -state of the particle, as we suggested before. The result simplifies even more if we integrate this contribution over the entire spatial component of  $dS_2$  at a given time  $\tau$ . Because at late times we have  $\sqrt{|g|} \approx \frac{\ell}{2} e^{\frac{\tau}{\ell}}$ , we find

$$\int_0^{2\pi} d\theta \sqrt{|g|} n \frac{e^{-\frac{\tau}{\ell}}}{\pi\ell} = \frac{n}{2\pi} \int_0^{2\pi} d\theta = n \quad (6.29)$$

This means that our interpretation of comparing the index  $n$  with the wave vector  $\mathbf{k}$  in flat space was spot-on! The index  $n$  is precisely the spatial momentum of the particle that is manifested as the  $n$ -state excitation of the scalar field. If we remember that  $n$  is the eigenvalue of the generator  $L_0$  of rotations in  $dS_2$ , we can interpret this as the field oscillating with an increasing frequency in  $dS_2$  as  $n$  increases. We can also calculate the total energy that a particle with momentum  $n$  adds to the system:

$$\int_0^{2\pi} d\theta \sqrt{|g|} \frac{2}{\pi\nu\ell^2} [m^2\ell^2 e^{-\frac{\tau}{\ell}} + 2n^2 e^{-3\frac{\tau}{\ell}}] = \frac{2}{\nu\ell} [m^2\ell^2 + 2n^2 e^{-2\frac{\tau}{\ell}}] \quad (6.30)$$

So again, we recognize the first term that only depends on the mass  $m$  of the scalar field, and the second term is a contribution to the energy because of the oscillation of the field travelling in the spatial direction. It makes sense for this term to decrease when  $\tau$  becomes large, as the spatial component of  $dS_2$  becomes larger as time progresses, meaning that the corresponding wavelength decreases and therefore the energy that the wave oscillation contains decreases too.

We've demonstrated the most basic way to deal with infinities appearing in quantum field theories, to the point where we could actually make sense of infinite vacuum energies! However, there are more problems that follow from vacuum state energies in quantum field theories. One of those is the *Cosmological constant problem*, which is described as 'the largest discrepancy between theory and experiment in all of physics'[22].

This problem arises from the fact that due to Heisenberg's uncertainty principle, even empty space must have a non-vanishing energy density, even after renormalization. Because of the theory of general relativity, this so-called zero-point energy must contribute a gravitational effect and is related to the cosmological constant  $\Lambda$  that we mentioned in Chapter 2. In general quantum field theories, the value of this so-called zero-point energy [43] can be calculated, and can turn out to be as much as 120 orders of magnitude higher than the measured value of the cosmological constant, depending on the specific quantum field theory. The origin of this discrepancy is still an open problem in physics.

## 6.2 Bogolubov transformations

The results of the previous section depended on the basis of wave functions that was used to construct the Klein-Gordon field (6.1). It turns out that quantized fields in curved spacetime, observables don't behave as nicely under basis transformations as in flat spacetime, which is explained in [25, Chapter 9.3]. In flat spacetime, number operators and vacuum states of fields are invariant under Lorentz transformations between different inertial observers, but we will soon see that this doesn't have to be the case for general transformations between bases of modes for the scalar field in curved spacetime. In curved spacetime however, there is no preferred set of basis modes that behaves nicely under the symmetry group of the spacetime like in flat spacetime, and therefore we must investigate what the consequences of a different basis of modes are: Following the procedure in [25, Chapter 9.4], we consider a different set of solutions  $g_n(\frac{\tau}{\ell})$  and  $g_n^*(\frac{\tau}{\ell})$  to (5.36) and construct a general Klein-Gordon field in the same way as in (6.1), but now our modes look like

$$\chi_n(\tau, \theta) = g_n(\frac{\tau}{\ell})\psi_n(\theta) \quad (6.31)$$

instead of expression (5.42). We require the modes  $\chi_n$  to form a complete orthonormal system (together with their conjugates  $\chi_n^*$ ) just like the modes  $\phi_n$  ((5.44),(5.45),(5.46)), i.e. for  $n, m \in \mathbb{Z}$  we must have

$$\langle \chi_m, \chi_n \rangle = \delta_{mn} \quad \langle \chi_m, \chi_n^* \rangle = 0 \quad \langle \chi_m^*, \chi_n^* \rangle = -\delta_{mn} \quad (6.32)$$

This means that we can expand the general Klein-Gordon field  $\phi(\tau, \theta)$  into a superposition of the modes  $\chi_n$ :

$$\phi(\tau, \theta) = \sum_{n=-\infty}^{\infty} \left[ b_n \chi_n(\tau, \theta) + b_n^\dagger \chi_n^*(\tau, \theta) \right] \quad (6.33)$$

Here  $b_n$  and  $b_n^\dagger$  are new coefficients to make this general field match expression (6.1). Just like before, we can quantize the field by promoting these coefficients to creation and annihilation operators, imposing a vacuum state  $|0_g\rangle$  (where the subscript  $g$  refers to the fact that our basis is constructed with functions  $g_n(\frac{\tau}{\ell})$ ) by

$$b_n |0_g\rangle \equiv 0 \quad \forall n \in \mathbb{Z} \quad (6.34)$$

and we impose the same canonical commutation relations as before:

$$[b_n, b_m^\dagger] = \delta_{nm} \quad [b_n, b_m] = [b_n^\dagger, b_m^\dagger] = 0 \quad (6.35)$$

With this new basis of modes, corresponding to an observer in a different inertial frame, we must define a new number operator  $N'_n$  that gives the number of particles in the  $n$ -state that the new observer sees in a system:

$$N'_n \equiv b_n^\dagger b_n \quad (6.36)$$

To make a connection between the two observers, we need to find the explicit basis transformation between the modes, which is called a *Bogolubov transformation*. We can expand one of the systems of modes into a superposition of the other modes, giving us

$$\chi_i(\tau, \theta) = \sum_{j \in \mathbb{Z}} \left[ \alpha_{ij} \phi_j(\tau, \theta) + \beta_{ij} \phi_j^*(\tau, \theta) \right] \quad (6.37)$$

for some coefficients  $\alpha_{ij}$  and  $\beta_{ij}$ . Using the orthonormality relations (5.44),(5.45) and (5.46) for the modes  $\phi_i$ , you can easily find that the coefficients must be:

$$\begin{aligned}\alpha_{ij} &= \langle \chi_i, \phi_j \rangle \\ \beta_{ij} &= -\langle \chi_i, \phi_j^* \rangle\end{aligned}\tag{6.38}$$

With these coefficients, it is easy to find the inverse Bogolubov transformation of (6.37):

$$\phi_i(\tau, \theta) = \sum_{j \in \mathbb{Z}} \left[ \alpha_{ji}^* \chi_j(\tau, \theta) - \beta_{ji} \chi_j^*(\tau, \theta) \right]\tag{6.39}$$

If we plug this expression into the original Bogolubov transformation (6.37), we find (from now on, we will neglect the explicit  $(\tau, \theta)$ -dependence of the modes in our notation)

$$\chi_i = \sum_{k, j \in \mathbb{Z}} \left[ (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) \chi_j + (-\alpha_{ik} \beta_{jk} + \beta_{ik} \alpha_{jk}) \chi_j^* \right]\tag{6.40}$$

which means that these coefficients must satisfy

$$\sum_{k \in \mathbb{Z}} \left( \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^* \right) = \delta_{ij}\tag{6.41}$$

$$\sum_{k \in \mathbb{Z}} \left( \beta_{ik} \alpha_{jk} - \alpha_{ik} \beta_{jk} \right) = 0\tag{6.42}$$

You can think of this as a normalization condition on the coefficients. We can also use the Bogolubov coefficients to transform between the different operators, by evaluating  $\langle \phi, \phi_i \rangle$  and  $\langle \chi_i, \phi \rangle$  using both expansions (6.1) and (6.33) of  $\phi$ , and we find

$$\left. \begin{aligned} a_i &= \sum_{j \in \mathbb{Z}} \left( \alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger \right) \\ b_i &= \sum_{j \in \mathbb{Z}} \left( \alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger \right) \end{aligned} \right\} \implies \left\{ \begin{aligned} a_i^\dagger &= \sum_{j \in \mathbb{Z}} \left( \beta_{ji} b_j + \alpha_{ji}^* b_j^\dagger \right) \\ b_i^\dagger &= \sum_{j \in \mathbb{Z}} \left( -\beta_{ij} a_j + \alpha_{ij} a_j^\dagger \right) \end{aligned} \right.\tag{6.43}$$

So this means that we can express any operator that we can build with the operators  $a_n$  and  $a_n^\dagger$  in terms of the operators  $b_n$  and  $b_n^\dagger$  of the other observer and vice versa, meaning that we can transform observables between different inertial frames! To see how this leads to ambiguities, we calculate the number of particles that the second observer perceives if the first observer sees empty space (meaning that the system is in the vacuum state  $|0_f\rangle$ ):

$$\langle 0_f | N'_n | 0_f \rangle = \langle 0_f | b_n^\dagger b_n | 0_f \rangle = \sum_{i, j \in \mathbb{Z}} \langle 0_f | \left( -\beta_{ni} a_i + \alpha_{ni} a_i^\dagger \right) \left( \alpha_{nj}^* a_j - \beta_{nj}^* a_j^\dagger \right) | 0_f \rangle\tag{6.44}$$

Again, we can make use of the inner products (6.4) to simplify this expression massively and we end up with

$$\langle 0_f | N'_n | 0_f \rangle = \sum_{i, j \in \mathbb{Z}} \beta_{ni} \beta_{nj}^* \langle 0_f | \delta_{ij} | 0_f \rangle = \sum_{i \in \mathbb{Z}} \beta_{ni} \beta_{ni}^* = \sum_{i \in \mathbb{Z}} |\beta_{ni}|^2\tag{6.45}$$

and vice versa, the first observer sees the following amount of  $n$ -state particles in the vacuum state  $|0_g\rangle$  of the second observer:

$$\langle 0_g | N_n | 0_g \rangle = \sum_{i \in \mathbb{Z}} |\beta_{in}|^2\tag{6.46}$$

This means that two observers disagree on whether or not a vacuum state describes empty space if *any* of the Bogolubov coefficients  $\beta_{ij}$  corresponding to the transformation between their inertial frames is nonzero! Intuitively, this makes sense if you look at the Bogolubov transformation (6.43) of the operators: The coefficients  $\beta_{ij}$  are responsible for transforming creation operators into annihilation operators and vice versa, while the coefficients  $\alpha_{ij}$  don't mix these up.

How do we deal with this apparent ambiguity of an observer seeing particles appear in a vacuum state? As we unfortunately can't use a vacuum cleaner to get rid of the particles that the second observer sees in the vacuum state of the first observer, we should think about how we should actually interpret empty space, as apparently emptiness of space and the presence of particles are not concepts that observers in different inertial frames agree on anymore when we work in curved spacetime: What looks like empty space for one observer might look like a brawling sea of particles for another observer.



Figure 5: Unfortunately, a vacuum cleaner is not a solution to the ambiguity of particles appearing in empty space.

This begs the question on the interpretation of particle detections in experimental physics: There might be ambiguity in the results of experiments, as a particle detector might measure a certain outcome of an experiment (think of the research done with particle accelerators at CERN for example [44]) while a different observer can see empty space during the entire experiment. There are ways to overcome these problems as already demonstrated later in [25, Chapter 9] and is further illustrated in [45] and [46, Chapter 4] for example. These ways generally come down to finding ways to describe preferred coordinate systems unambiguously. This is beyond the scope of this thesis however, but in the next section we will exemplify the possible consequences of Bogolubov transformations of our field in  $dS_2$  by introducing a specific change of coordinates.

### 6.3 Euclidean continuation of de Sitter space

We will explore the Euclidean continuation of the global  $dS_2$ -metric, which is obtained by a so-called *Wick rotation* of the time coordinate [47][26, Chapter 2], introducing the real-valued Euclidean time coordinate  $\tau_E$  by

$$\tau = i\tau_E \tag{6.47}$$

The Euclidean time coordinate  $\tau_E \in \mathbb{R}$  must be a periodic coordinate with period  $T = 2\pi\ell$  because the metric (2.7) must be regular everywhere. With this new coordinate, the metric now reads

$$ds^2 = d\tau_E^2 + \ell^2 \cos^2\left(\frac{\tau_E}{\ell}\right) d\theta^2 \tag{6.48}$$

because the factor  $i$  transforms the hyperbolic cosine into a regular cosine. Here you can see why this is called the Euclidean continuation of the  $dS_2$ - metric: Because of the Wick rotation, the metric describing a manifold in Minkowski space has transformed into one which rather resembles a manifold in Euclidean space because of the sign change of the time coordinate of the metric. More specific, without getting into details, the new metric resembles one of a 2-sphere after some coordinate changes. To parameterize this sphere, the range of the Euclidean time coordinate must be  $\tau_E \in [-\frac{\pi}{2}\ell, \frac{\pi}{2}\ell]$ , where the coordinates  $\tau_E = \pm\frac{\pi}{2}\ell$  correspond to the north pole and south pole respectively[48][37].

Allowing the time coordinate  $\tau$  to take on values in  $\mathbb{C}$  might seem like a bizarre thing to do: What would even be an appropriate interpretation of imaginary times? In practice however, it turns out to be a very powerful tool to turn a Minkowski space problem into a related one in Euclidean space which is partly mapped on the complex plane, expanding the set of theorems and convergence properties of integrals that can be used in calculations. This is especially useful in Feynman's path integral formulation of quantum field theories [49].

We now investigate motion of the field on this Euclidean continuation of dS<sub>2</sub>. To do this, we follow the same procedure as we did in Chapter 5: We solve the equation of motion (5.36) for time-dependence  $f(\frac{\tau}{\ell})$  of the scalar field, which we will now label with  $f_E$  to represent the Euclidean region we're considering the function on. The original equation now reads:

$$f_E''(x) + \tanh(x)f_E'(x) - \Delta(\Delta - 1)f_E(x) = -\frac{n^2}{\cosh^2(x)}f_E(x) \quad (6.49)$$

The key difference is that our variable  $x = \frac{\tau}{\ell}$  now takes on imaginary values. Again, we introduce a change of variables by  $u = u(x) = 1 + e^{2x}$  and a function  $g(u)$  such that  $f_E(x) = g(u(x))$ , and we proceed to make the following substitution:

$$g(u) \equiv u^n (u - 1)^{\frac{\Delta}{2}} h(u) \quad \iff \quad f_E(x) = (2 \cosh(x))^n e^{(n+\Delta)x} h(u(x)) \quad (6.50)$$

The differential equation for  $g(u)$  now reads:

$$u(1-u)h''(u) + \left[2n+1 - (2n+\Delta + \frac{3}{2})u\right]h'(u) - (\frac{1}{2}+n)(n+\Delta)h(u) = 0 \quad (6.51)$$

This is almost identical to (5.39), and is also solved by Gauss' hypergeometric function [38]  ${}_2F_1(a, b, c, ; u)$ , with again  $a = n + \frac{1}{2}$ ,  $b = n + \Delta$  but now  $c = 2n + 1$ :

$$h(u) = {}_2F_1\left(n + \frac{1}{2}, n + \Delta, 2n + 1; u\right) \quad (6.52)$$

This gives the following time-dependence of the Euclidean modes  $\phi_n^E(\tau, \theta) \equiv f_n^E(\frac{\tau}{\ell})\psi_n(\theta)$ :

$$f_n^E(x) = C_n \cosh^n(x) e^{(n+\Delta)x} {}_2F_1\left(n + \frac{1}{2}, n + \Delta, 2n + 1; 1 + e^{2\frac{\tau}{\ell}}\right) \quad (6.53)$$

Here  $C_n$  is the normalization constant. We calculate the normalization constant in the late time limit  $\tau \rightarrow \infty$  (remember that the inner product (5.16) is independent of the time it is evaluated at) as it leads to easier calculations when making use of the properties of the Hypergeometric function. We find:

$$C_n = \frac{(-2)^{-(n+1)}}{n!} \sqrt{\frac{2\Gamma(n+\Delta)\Gamma(n+1-\Delta)}{\nu\Gamma(\Delta-\frac{1}{2})\Gamma(\frac{1}{2}-\Delta)}} (1 - e^{\pi\nu}) \quad (6.54)$$

With this expression for  $f_n^E$ , we construct a general Klein-Gordon field using the Euclidean modes  $\phi_n^E(\tau, \theta) \equiv f_n^E(\frac{\tau}{\ell})\psi_n(\theta)$  and its complex conjugates:

$$\phi(\tau, \theta) = \sum_{n \in \mathbb{Z}} b_n \phi_n^E(\tau, \theta) + b_n^\dagger \phi_n^{E*}(\tau, \theta) \quad (6.55)$$

We can do so because these modes form an orthonormal system in the sense that the relations (6.32) are satisfied by taking  $\chi_n = \phi_n^E$ . With this different modes basis, we can explore the Bogolubov transformation between our original modes  $\phi_n$  given by (5.42) and the Euclidean modes! In the late time limit, you can verify that the transformation is given by

$$\phi_n^E = \frac{e^{-i\gamma_n}}{\sqrt{1 - e^{\pi\nu}}} \phi_n - \frac{e^{\frac{\pi\nu}{2} + i\gamma_n}}{\sqrt{1 - e^{\pi\nu}}} \phi_{-n}^* \quad (6.56)$$

where  $\gamma_n$  induces a phase shift of the field, and is given by

$$e^{i\gamma_n} \equiv \sqrt{\frac{\Gamma(\frac{1}{2} - \Delta)\Gamma(n + \Delta)}{\Gamma(\Delta - \frac{1}{2})\Gamma(n + 1 - \Delta)}} \quad (6.57)$$

From this we extract the Bogolubov coefficients

$$\alpha_{ij} = \frac{e^{-i\gamma_i}}{\sqrt{1 - e^{\pi\nu}}} \delta_{ij} \quad \text{and} \quad \beta_{ij} = -\frac{e^{\frac{\pi\nu}{2} + i\gamma_i}}{\sqrt{1 - e^{\pi\nu}}} \delta_{-ij} \quad (6.58)$$

Now we can compare our original vacuum state  $|0_f\rangle$  and the Euclidean vacuum  $|0_E\rangle$ , which by definition gets annihilated by all the coefficients  $b_n$  if we quantize the field (6.55), applying the framework of last section:

$$b_n|0_E\rangle \equiv 0 \quad \forall n \in \mathbb{Z} \quad (6.59)$$

We can now compare observables in the two vacua to get a better understanding of their properties. We can for example calculate how many particles in the  $n$ -state the different observers measure in the other vacuum state, plugging our Bogolubov coefficients into (6.45) and (6.46), which yields

$$\langle 0_E|N_n|0_E\rangle = \langle 0_f|N'_n|0_f\rangle = \frac{e^{\pi\nu}}{|1 - e^{\pi\nu}|} \quad (6.60)$$

So for an observer in the inertial frame that corresponds to our basis of modes (5.42), the Euclidean vacuum looks like it contains infinitely many particles, namely  $\frac{e^{\pi\nu}}{|1 - e^{\pi\nu}|}$  for every index  $n \in \mathbb{Z}$ . For very massive fields, this value approaches 1. The contrary also holds: An observer in the Euclidean inertial frame sees the exact same number of particles in the vacuum state  $|0_f\rangle$ , whatever the precise interpretation of this inertial frame would even be. In the next section, we investigate one last quantity to illustrate the consequences of the Bogolubov transformation (6.56).

## 6.4 Vacuum correlation functions

To further illustrate the properties of the different vacua  $|0_f\rangle$  and  $|0_E\rangle$ , we investigate one last fundamental object in quantum field theories: Two-point correlation functions (also called *correlators*)  $G(x, x')$  in the vacuum state. These indicate how correlated two spacetime points are by the scalar field  $\phi$ , i.e. how much an excitation at spacetime point  $x$  influences the behaviour of the field at spacetime point  $x'$ . The most direct application of these  $n$ -point correlators is in the calculation of scattering amplitudes between  $n$  particles, which are important quantities for experiments in particle physics[23][24]. We define the two-point correlator in the vacuum state  $|0_f\rangle$  for two spacetime points  $x, x' \in dS_2$  by

$$G_f(x, x') \equiv \langle 0_f|\phi(x)\phi(x')|0_f\rangle \quad (6.61)$$

If we insert our expression (6.1) for the scalar field into this correlator, we end up with products of our operators  $a_n$  and  $a_n^\dagger$  that act on the vacuum state again:

$$G_f(x, x') = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle 0_f|\left[\phi_m(x)\phi_n(x')a_m a_n + \phi_m^*(x)\phi_n(x')a_m^\dagger a_n + \phi_m(x)\phi_n^*(x')a_m a_n^\dagger + \phi_m^*(x)\phi_n^*(x')a_m^\dagger a_n^\dagger\right]|0_f\rangle \quad (6.62)$$

We can simplify this expression with the relations (6.4) and the summation over  $m$  contracts by the Kronecker delta. Notice that we haven't used the explicit expressions for the modes  $\phi_n$  to find the following result: This all follows from the expression (6.1) and the properties of the operators  $a_n$ ,  $a_n^\dagger$  and the vacuum state.

$$G_f(x, x') = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle 0_f|\phi_m(x)\phi_n^*(x')\delta_{mn}|0_f\rangle = \sum_{n=-\infty}^{\infty} \phi_n(x)\phi_n^*(x') \quad (6.63)$$

The following calculations will all be done in the late time limit  $\tau \rightarrow \infty$  as this simplifies the expressions massively, and we will only be interested in the generic structure of the correlator. We use the results for our modes (5.42) and (5.43) from the previous chapter, and we find

$$G_f((\tau, \theta), (\tau'\theta')) \approx \frac{2}{\nu} e^{-\Delta\frac{\tau}{\ell} - \Delta_s\frac{\tau'}{\ell}} \sum_{n=-\infty}^{\infty} \psi_n(\theta)\psi_n^*(\theta') \quad (6.64)$$

which, using the completeness relation (4.9b) of the  $S^1$ -wave functions  $\psi_n(\theta)$ , results into:

$$G_f((\tau, \theta), (\tau'\theta')) \approx \frac{2}{\nu} e^{-\frac{\Delta}{\ell}(\tau - \tau') - \frac{\tau'}{\ell}} \delta(\theta - \theta') \quad (6.65)$$

We can also plug in  $\Delta = \frac{1}{2}(1 + i\nu)$ , we find the following structure in the exponential term:

$$G_f((\tau, \theta), (\tau' \theta')) \approx \frac{2}{\nu} e^{-\frac{\tau+\tau'}{2\ell} - i\frac{\nu}{2} \frac{\tau-\tau'}{\ell}} \delta(\theta - \theta') \quad (6.66)$$

The correlator can be split into three components: First of all, the term  $e^{-\frac{\tau+\tau'}{2\ell}}$  (notice that  $\frac{\tau+\tau'}{2}$  gives the average of the two coordinates) indicates that at later times, different points in  $dS_2$  are separated more and more, which follows directly from the metric (2.7). The second part  $e^{-i\frac{\nu}{2} \frac{\tau-\tau'}{\ell}}$  rotates the value correlator in the complex plane, periodic in the time difference  $\tau - \tau'$  with period  $\frac{4\pi\nu}{\ell}$ . Last of all, the term  $\delta(\theta - \theta')$  indicates that at late times, two spacetime points are correlated by the field if and only if their spatial coordinate is the same, as the distance between hypersurfaces of constant  $\theta$  of the hyperboloid  $dS_2$  becomes bigger and bigger at late times.

We can also calculate the two-point correlation in the Euclidean vacuum state:

$$G_E(x, x') \equiv \langle 0_E | \phi(x) \phi(x') | 0_E \rangle \quad (6.67)$$

Following the same procedure (as it did not depend on the choice of basis for the modes  $\phi_n$ ), we find

$$G_E(x, x') = \sum_{n=-\infty}^{\infty} \phi_n^E(x) \phi_n^{E*}(x') \quad (6.68)$$

We're again going to perform the calculation in the late time limit  $\tau, \tau' \rightarrow \infty$ , and as we're already familiar with the late time behaviour of our original modes  $\phi_n$  at late times ((5.42), (5.43)), we plug in the Bogolubov transformation (6.56) to make use of them:

$$G_E(x, x') = \frac{1}{1 - e^{\pi\nu}} \sum_{n=-\infty}^{\infty} \left( e^{-i\gamma_n} \phi_n(x) - e^{\frac{\pi\nu}{2} + i\gamma_n} \phi_{-n}^*(x) \right) \left( e^{i\gamma_n} \phi_n^*(x') - e^{\frac{\pi\nu}{2} - i\gamma_n} \phi_{-n}(x') \right) \quad (6.69)$$

This is particularly desirable because their time dependence  $f_n(\frac{\tau}{\ell}) \approx \sqrt{\frac{2}{\nu}} e^{-\Delta \frac{\tau}{\ell}}$  is actually independent of  $n$  at late times, simplifying our calculations massively. Written in the most compact way, the resulting correlator becomes

$$G_E((\tau, \theta), (\tau' \theta')) \approx \frac{2}{\nu \sinh(\frac{\pi\nu}{2})} \sum_{n=-\infty}^{\infty} \left[ \cos\left(2\gamma_n + \frac{\nu}{2} \frac{\tau+\tau'}{\ell}\right) - \cosh\left(\frac{\nu}{2}(\pi + i\frac{\tau-\tau'}{\ell})\right) \right] \psi_n(\theta) \psi_n^*(\theta') \quad (6.70)$$

This expression is not as easy to evaluate using (4.9b) as previous ones were, because the term  $\gamma_n$  depends on the index  $n$ , making it impossible to take the entire term in the square brackets outside of the sum. However, we can quantitatively describe the behaviour of the resulting correlator: The cosh-term inside the bracket doesn't depend on the index  $n$  and therefore doesn't affect the summation. This results in a term with the Dirac delta  $\delta(\theta - \theta')$  that resembles the correlator (6.66) quite well in the sense that the same three components can be recognized: A factor  $e^{-\frac{\tau+\tau'}{2\ell}}$ , a periodic behaviour in  $\tau - \tau'$ , and the Dirac delta  $\delta(\theta - \theta')$ . If you were still skeptical about the Euclidean coordinate system, this result might convince you that the Wick rotation doesn't result in physical nonsense: The two-point function of the field in the Euclidean vacuum state  $|0_E\rangle$ , although it is a little more complicated, is quite similar to the two-point function of our original vacuum state!

The direct consequences of the Bogolubov transformations are clear: Different observers in a curved spacetime can have a different perception of the vacuum state of a scalar field, and therefore the excited states of the field will be perceived differently as well, as well as any observables that are measured in those states. This means that in quantum field theories in curved spacetime, these transformations give a new challenge: Finding a way to describe these transformations of the different vacuum states, which is hinted at in [12, Chapter 5.5] and worked out in more detail in [50]. Exploring more vacua in de Sitter space and how their properties compare is an appropriate next step in the exploration of difficulties of quantum field theories in curved spacetime, but all in all it is a bridge too far to investigate more deeply in this bachelor thesis.

## 7 Conclusion and Outlooks

In this thesis, we studied how the generators of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  produce the symmetries of two-dimensional de Sitter space  $dS_2$  and we studied what the consequences on irreducible  $\mathfrak{sl}(2, \mathbb{R})$ -representations are of demanding unitarity of the generators and normalizability of the states. As summarized in table 1, there are four types of representations that meet these requirements, of which the principal series representations (with conformal dimension  $\Delta = \frac{1}{2}(1 + i\nu)$ ) became our main interest because of their appearance in the construction of a massive scalar field theory on  $dS_2$ . We saw that the DFF model, which describes the most general  $\mathfrak{sl}(2, \mathbb{R})$ -invariant one-dimensional scalar field theory, actually had to be modified to accommodate principal series representations, but this modification produces a new problem: It expands the model with a mathematical horizon at  $r = 0$  beyond which the Hamiltonian flips sign and the sensible interpretation of the variable  $r$  becomes less obvious.

However, we found another interesting feature of the modified DFF model: It can be constructed from the Hilbert space of wave functions on the circle, which appears again in the description of a scalar field in  $dS_2$  because its spatial component is homeomorphic to the circle. By explicitly solving the Klein-Gordon equation of the scalar field, we even found that the Hilbert space of Klein-Gordon fields in  $dS_2$  is actually equivalent to the Hilbert space of wave functions on the circle, which gives an equivalence with the DFF model! We saw that the mass  $m$  of the scalar field was related to the conformal dimension of the representation, yielding principal series representations for fields with  $m > \frac{1}{2\ell}$ . This justifies the interest in this modified DFF model in the context of scalar field theories in the Sitter space, and raises the question of what the precise physical consequences of this modification to the DFF model are.

We proceeded to quantize the scalar field in de Sitter space, and we immediately encountered some of the general challenges in quantum field theories: First of all, we see that even in the ground state of the system, calculations of the energy density and the total energy in the system yield infinities. This can be fixed by subtracting off these infinite values from the original Hamiltonian, an example of the process in quantum field theories called renormalization.

We also saw how in curved spacetime, Bogolubov transformations result in disagreements between different inertial observers that make us doubt our fundamental ideas about the structure of the universe. In particular, we showed how different observers can disagree on something that seems as trivial as the emptiness of space as different basis modes of the Klein-Gordon field can make the different observers disagree on the amount of observed particles in the vacuum state of the Hilbert space. This means that in curved spacetime, different observers must find ways to disambiguously talk about observables in our universe, for example in experiments. One example of such a Bogolubov transformation arises from a Wick rotation of the time coordinate, describing the Euclidean continuation of the de Sitter space. Although this seems to be a bizarre thing to do, it turns out to be surprisingly useful as it transforms problems in Minkowski space to equivalent ones in Euclidean space which is partly mapped onto the complex plane, extending the collection of mathematical tools to perform physical calculations. Unfortunately, we have not managed to find an example of a situation where the Bogolubov transformation to the Wick rotated frame actually turned out to be helpful, but despite the unclear physical interpretation of this Wick rotation, we saw that it produced perfectly sensible results when we calculated for example the two-point correlation functions of the vacuum state of the scalar field, similar to the original basis of modes that we found in Chapter 5.

### 7.1 Future directions

After the discussions in this thesis on conformal invariance in several quantum theoretical models, we encountered some topics that deserve some further investigation. Here I will briefly address some future directions for research on some of the topics that we touched upon, focusing mostly on generalizations of our theories and unanswered problems that we presented.

First of all, as we considered only two-dimensional de Sitter space  $dS_2$  for simplicity, an obvious future direction for research would be a generalization to a 4-dimensional de Sitter space, or even a de Sitter space

$dS_d$  of general dimension  $d$ . You would again need to describe the symmetries of the space, and investigate the possible representations of the isometry algebra of the de Sitter space, as is done in [51] and [52] for example. One of the biggest differences is that the spherical harmonics in higher-dimensional de Sitter spaces are more complicated than our spherical harmonics (4.8) on  $S^1$ [39]. Principal series representations also arise in higher dimensions when describing massive scalar fields, yielding very similar results to the results that we found for fields in  $dS_2$  in this thesis.

We also ended our discussion in Chapter 4 by questioning the interpretation of the horizon at  $r = 0$  that appeared in our modified DFF model (4.19). Although it is hard to give a clear interpretation of this horizon, it is definitely a topic that deserves further research because of the importance of this general  $\mathfrak{sl}(2, \mathbb{R})$ -invariant one-dimensional scalar field theory and the appearance of a principal series representation. This representation is further investigated in [20][53][54] and the last two articles describe how you can deal with the horizon at  $r = 0$  using a coordinate transformation of the system among other things, depending on the conformal dimension of the representation.

Lastly, we had a brief look at some of the basics in quantum field theories, discussing several challenges that arise in them, particularly when studying quantized fields in curved spacetime. One of the main problems that we encountered was the appearance of infinities, which can be dealt with using techniques like perturbation theory and renormalization. Especially the latter one is a particularly interesting field of research in quantum theories[21][55] and is very important in for example the infamous cosmological constant problem [22]. An elementary demonstration of the consequences of Bogolubov transformations in curved spacetime is given in [25, Chapter 9], and some more examples of calculations with Bogolubov transformations can be found in [45],[46] and [56] for example, but the relevance of Bogolubov transformation goes way beyond these specific topics: When investigating any quantized field, you can switch between different inertial frames by Bogolubov transformations and get to work with their consequences.

Lastly, we introduced  $n$ -point correlation functions (more precisely, we calculated some 2-point correlators) which are important in studying quantum field theories: One of the most important problems in cosmology is the *conformal bootstrap*[57], the procedure that tries to describe the nature of the cosmological constant  $\Lambda$  that drives the inflation of the universe using quantum field theories, and correlators are very important for describing the nature of our universe, as illustrated in [58] and [59] for example. Although these are some specific examples of fields of research that demonstrate the challenges that we found in this thesis, there are way more aspects in quantum field theories and especially conformally invariant field theories that we haven't touched upon here. Any introductory book, article or text on quantum field theories or conformal field theories will be able to present some of these new aspects, but good examples are [23][40][49][60][61, Chapter 4].

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