Universiteit Utrecht

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Constructing Distributions in Low Dimensions

Bachelor Thesis

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Mathematics

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Contents

Introduction

The main focus of this thesis is *distributions*, which are objects in differential geometry. Distributions are subbundles of the tangent bundle of a manifold, and we come across them in several different areas of research, such as control theory, thermodynamics, the theory of hypoelliptic operaters, smooth topology and more. In figure 1 we see an illustration of a (smooth) distribution on $\mathbb{R}^3$.

In this thesis, we will start by defining distributions, and looking at several properties and statements about these objects (Chapter 1). For instance, the *Frobenius Theorem* (section 1.1) is an important result about some equivalent properties regarding distributions. Often, we study distributions by looking at vector fields which are tangent to it. For example, the distribution in figure 1 can be described by the span of the vector fields ∂_y and $\partial_x + y\partial_z$. The *Lie bracket* is an important operator on vector fields, and it defines several properties of distributions. For example *bracket-generating distributions* are defined using the Lie bracket in section 1.3.

In Chapter 2 we focus on a special type of bracket-generating distributions called *contact structures.* These are the type of distributions which have been studied the most. Contact structures can be defined on any odd-dimensional manifold and in this thesis we mainly focus on contact structures defined on 3-dimensional manifolds. The distribution in figure 1 is in fact called the *standard contact structure.* One of the questions you can ask, is whether we can transform a given distribution into a contact structure. It turns out we need some extra data about this given distribution, which is why we introduce *formal contact structures.* In this thesis we show that indeed, on open manifolds, every formal contact structure is homotopic to a genuine contact structure.

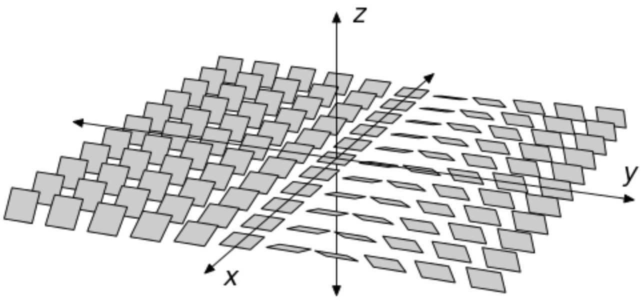


Figure 1: The standard contact structure on $\mathbb{R}^3$ given by $\langle \partial_y, \partial_x + y\partial_z \rangle$ ([6]).

Mikhael Gromov was the first to prove this result in 1969 in his dissertation *Stable mappings of foliations into manifolds* ([14]). If two spaces of geometric structures and formal geometric structures are homotopy equivalent, we say the problem “satisfies the *h-principle*”. Thus, in other words, in Chapter 2 we prove the h-principle for contact structures on open manifolds. The term “h-principle” was introduced and popularized by Gromov in his book *Partial Differential Relations* (1986, [13]).

In Chapter 3 we discuss *Engel structures.* These are distributions defined on 4-dimensional manifolds. In many ways Engel structures and contact structures are very similar. For example, they are both bracket-generating distributions. However, with contact structures we work on 3-dimensional manifolds, while with Engel structures we work on 4-dimensional manifolds, which does lead to some evident differences. Just like in the contact case we define *formal Engel structures* and we show that, on open manifolds, every formal Engel structure is homotopic to a genuine Engel structure. In other words, we show that the h-principle holds for Engel structures on open manifolds. Although Gromov did not state it explicitly, this result also follows from his work *Stable mappings of foliations into manifolds* ([14]).

This thesis has five appendices, which provide the reader with more background information on material discussed in the chapters. In Appendix A we discuss *bundles*, *bundle morphisms* and *subbundles*, as distributions are subbundles of the tangent bundle. The space of distributions, as well as for example the space of contact structures, is endowed with a *Whitney C^k-topology*. These topologies are discussed in Appendix B. In order to transform distributions into contact or Engel structures, we need to work with manifolds. Manifolds are easier to work with when we use a *triangulation* of the manifold, which is explained in Appendix C. Lastly, Appendix D and Appendix E provide two important techniques we use in the proofs of the h-principle for contact and Engel structures. In order to preserve the flow of the story, these are included in the appendices.

To write this thesis I mostly used the book *Introduction to Smooth Manifolds* by J.M. Lee ([17]), the lecture notes *Topological Aspects in the study of Tangent Distributions* ([10]) by Á. del Pino, and the weekly discussions with my supervisor Álvaro del Pino, who I want to thank for his time, knowledge and comments on my work!

Chapter 1

Distributions

The following object will be the main focus of this thesis.

Definition 1.1. *Let M be a smooth manifold. A **distribution on M of rank** k is a rank-k subbundle of TM.*

Let ξ be a rank-k distribution on a smooth manifold M. Then we can write

$$\xi = \cup_{p \in M} \xi_p,$$

where ξ_p is a linear subspace of T_pM of dimension k. ([17, p. 491])

A distribution is considered to be **smooth** if it is a smooth subbundle of TM. That is, a distribution is smooth if and only if each point in your manifold M has a neighbourhood U such that there are smooth vector fields $X_1, ..., X_k : U \to TM$ such that $X_1|_p, ..., X_k|_p$ form a basis for ξ_p. In this case we say that ξ is the distribution (locally) spanned by the vector fields $X_1, ..., X_k$. ([17, p. 491]) In this thesis we will only be working with smooth distributions.

By $\mathrm{Dist}(M, k)$ we denote the space of all distributions on M of rank k. We endow $\mathrm{Dist}(M, k)$ with the C^0-topology. See Appendix B for more on Whitney topologies.

In this chapter we will discuss some important properties of distributions, many of which have to do with the *Lie bracket*. Certain definitions and results, like *bracket-generating distributions* and the *curvature* will be very important in Chapter 2 and 3 as well. The largest part of the material that we discuss in this chapter is either retrieved from [17] or [10].

1.1 Frobenius Theorem

In this section we will discuss a theorem named after the German mathematician Ferdinand Georg Frobenius (1849-1917). The theorem states necessary and sufficient conditions for finding solutions of certain systems of first-order partial differential equations. Although Frobenius was not the first to prove this result, his work led to the application of this theorem in differential topology. This version of the theorem is the one we will be discussing in this section, but we will first need to look at the following definitions.

Definition 1.2. *Let M be a smooth manifold of dimension n and let ξ be a smooth distribution of rank k. A **foliation chart** for ξ at a point $p \in M$ is an embedding*

$$\varphi : [0,1]^n \to M,$$

with $p \in \varphi([0,1]^n)$ and $\varphi^*\xi = \ker(dx_{k+1}, ..., dx_n)$. Here $(x_1, ..., x_n)$ are the coordinates in $[0,1]^n$. ([10, p.7])

Definition 1.3. *We say a distribution is **integrable** if it admits a foliation chart at every point in your manifold. ([10, p.7])*

This means that if ξ is an integrable distribution of rank k, that we can locally describe ξ by the span of the vector fields $\partial_{x_1}, ..., \partial_{x_k}$.

Definition 1.4. *A distribution ξ is said to be **involutive** if for every two vector fields X and Y tangent to ξ, their lie bracket $[X, Y]$ is also tangent to ξ. ([17, p. 492])*

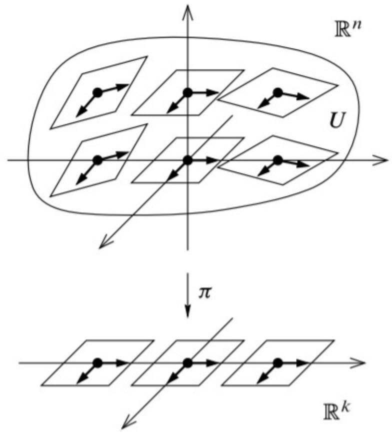


Figure 1.1: Proof of Frobenius ([17, p. 498]).

The Frobenius Theorem tells us that these two properties are actually equivalent.

Theorem 1.5 (Frobenius)**.** *A distribution is integrable if and only if it is involutive.*

Sketch of Proof. Let us briefly think about the proof of this theorem. When a distribution is integrable, we can find a foliation chart at every point in the manifold. Any two vector fields tangent to ξ, can locally (in a neighbourhood around $p \in M$) be described as a linear expression of the vector fields $\partial_{x_1}, ..., \partial_{x_k}$. Taking their Lie bracket yields a vector field which is also a linear expression of $\partial_{x_1}, ..., \partial_{x_k}$. This implies that ξ is involutive.

Now let ξ be an involutive, smooth distribution of rank k on an n-dimensional smooth manifold M. Since M is smooth we can find a neighbourhood U which is isomorphic to $\mathbb{R}^n$. Since ξ is a smooth distribution we can (by maybe restricting the neighbourhood further) find a local framing $X_1, ..., X_k$. Since ξ is of rank k, by reordering the coordinates of $\mathbb{R}^n$ if necessary, we can say that ξ is complementary to the span of the vector fields $\partial_{x_{k+1}}, ..., \partial_{x_n}$.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection on this neighbourhood, and we define the following vector fields:

$$V_i|_q := (d\pi|_{\xi_q})^{-1}(\partial_{x_i}|_{\pi(q)})$$

for $i = 1, ..., k$ and $q \in \mathbb{R}^n$. These vector fields form a new local frame for ξ. Due to the naturality of the Lie bracket and the fact that ξ is involutive, one can see that the vector fields V_i commute on the neighbourhood. By Theorem 9.46 in [17, p. 234], we then know that ξ is integrable, as the theorem provides a flat coordinate chart for all distributions locally spanned by commuting vector fields. □

For the full proof see [17, p. 497-498].

Remark. Suppose X is a vector field on a manifold M. Then let $\xi = \langle X \rangle$ be the distribution of rank 1 spanned by X. It automatically follows that ξ is involutive, and therefore by the Frobenius Theorem, integrable. This means we can locally describe ξ, and thus X, by a coordinate direction. This particular application of the theorem is known as the **flowbox theorem**. ([10, p. 7]) ♦

Example 1.6. Let ξ be the smooth distribution on the manifold $\mathbb{R}^3$ spanned by the two vector fields

$$X = x\partial_x + \partial_y + x(y+1)\partial_z,$$
$$Y = \partial_x + y\partial_z.$$

We see that

$$[X, Y] = [x\partial_x + \partial_y + x(y+1)\partial_z, \partial_x + y\partial_z] = \partial_z - \partial_x - (y+1)\partial_z = -\partial_x - y\partial_z = -Y.$$

Therefore $[X, Y] = -Y$ is tangent to ξ. Let A and B be two vector fields tangent to ξ. Then A and B are linear combinations of the vector fields X and Y, and thus their Lie bracket will be a linear combination of the vector fields X, Y and $[X, Y]$. Therefore $[A, B]$ will also be tangent to ξ. We conclude that ξ is involutive.

By the Frobenius Theorem we now know that ξ is integrable, and thus admits a foliation chart at every point in $\mathbb{R}^3$. ([17, p. 498-499]) △

1.2 Integral manifolds and Foliations

In this section we look at integral manifolds and foliations. By the Frobenius Theorem we know that integrability and involutivity of a distribution are equivalent. In this section we will see that there is another property which is equivalent to the former. To see this we look at the following definition.

Definition 1.7. *Let ξ be a smooth distribution on a smooth manifold M. An* ***integral manifold of*** *ξ is a non-empty, immersed submanifold N of M such that $T_pN = \xi_p$ for each $p \in N$. ([17, p. 491])*

In fact, ξ is an integrable distribution on a smooth manifold M, if and only if each $p \in M$ is contained in an integral manifold of ξ. Let us prove this result.

Lemma 1.8. *Let ξ be a distribution on a smooth manifold M. Then ξ is integrable if and only if each $p \in M$ is contained in an integral manifold of ξ.*

Proof. Suppose each $p \in M$ is contained in an integral manifold of ξ. Let X and Y be two vector fields tangent to ξ. Let $p \in M$ and N an integral manifold such that $p \in N$. Then $X_p, Y_p \in \xi_p = T_pN$. Thus X and Y are tangent to N. By Corollary 8.32 in [17, p. 189], $[X, Y]$ is then also tangent to N, and thus $[X, Y]_p \in T_pN = \xi_p$. We conclude that ξ is involutive, and by Frobenius Theorem ξ is integrable.

Now suppose ξ is integrable. For every $p \in M$ there exists a foliation chart $\varphi : [0, 1]^n \to M$ with coordinates $(x_i)_{1 \le i \le n}$ as in Definition 1.2. Then every slice of the form $x_{k+1} = c_{k+1}, ..., x_n = c_n$ for constants $c_{k+1}, ..., c_n$ is an integral manifold of ξ. Therefore p must be contained in some integral manifold ξ. ([17, p. 496]) □

In [17] and [10] different definitions of *integrable* are used, and the lemma above shows these are equivalent.

We find that, under the right conditions, the integral manifolds of a distribution can partition a manifold into submanifolds. We call this partition a *foliation* and we define it as follows.

Definition 1.9. *Let M be a smooth manifold of dimension n. We say $\mathcal{F}$ is a* ***foliation of dimension*** *k* ***on*** *M if $\mathcal{F}$ is a collection of disjoint, connected, non-empty, immersed k-dimensional submanifolds of M such that:*

- *the union of submanifolds is M,*
- *for each $p \in M$ there is a chart (U, φ) such that $\varphi(U)$ is a cube in $\mathbb{R}^n$ and for each submanifold $N \in \mathcal{F}$ we have $N \cap U = \varnothing$ or $N \cap U$ is equal to a countable union of $k-$dimensional slices of the form $x^{k+1} = c^{k+1}, ..., x^n = c^n$.*

The submanifolds $N \in \mathcal{F}$ are called the ***leaves of the foliation****. ([17, p. 501])*

Example 1.10. The following are examples of foliations (retrieved from [17, p. 501]).

(i) Let $M = \mathbb{R}^n$. Then the collection of all submanifolds parallel to $\mathbb{R}^k \times \{0\}$ is a foliation of $\mathbb{R}^n$ of dimension k.

(ii) Let $M = \mathbb{R}^n \backslash \{0\}$. The collection of all spheres centered at 0 with varying radius is a foliation of M of dimension $n - 1$.

(iii) Figure 1.2 shows three different foliations of the torus.

△

Theorem 1.11. *Let ξ be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of ξ forms a foliation of M.*

For a proof of this theorem see [17, p. 502].

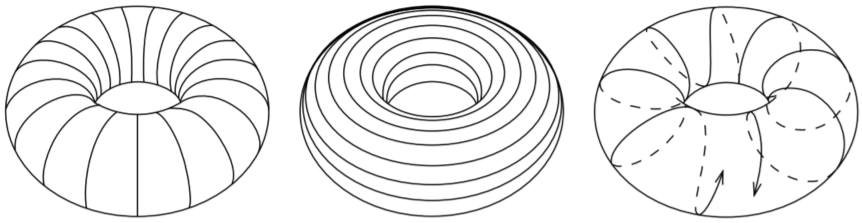


Figure 1.2: Foliations of the torus ([17, p. 502]).

1.3 The associated Lie flag

The Lie bracket is a very useful operations when studying distributions. It produces a new vector field out of two given vector fields. This means we can apply it repeatedly, also on the newly formed vector fields. This will play an important role in this section. The concepts that we will discuss now, will help us define an important map called the *curvature* in section 1.4.

Definition 1.12. *We call a string "a", where a is a formal variable, a* ***bracket expression*** *of length 0. A bracket expression of length* $i + j + 1$ *is a string of the form*

$$[A(a_0, ..., a_i), B(a_{i+1}, ..., a_{i+j+1})],$$

where $A(-)$ *and* $B(-)$ *are bracket expressions of length* i *and* j *respectively. ([10, p. 5])*

These bracket expressions can for example be applied to vector fields tangent to a distribution ξ. The space of all vector fields tangent to a distribution ξ is denoted by $\Gamma(\xi)$. This space is a C^∞-module, which is an algebraic structure defined as follows (the following definition was retrieved from [17, p. 617]).

Definition 1.13. *Let* $\mathcal{R}$ *be a commutative ring. A* ***module over*** $\mathcal{R}$ *is a set* V *endowed with two operations* $+ : V \times V \to V, (v, w) \mapsto v + w$ *and* $\cdot : \mathcal{R} \times V \to V, (a, v) \mapsto a \cdot v$ *satisfying*

(i) $(V, +)$ *is an abelian group,*

(ii) $a \cdot (b \cdot v) = (ab) \cdot v$ *for all* $v \in V$ *and* $a, b \in \mathcal{R}$*,*

(iii) $1 \cdot v = v$ *for all* $v \in V$*,*

(iv) $(a + b) \cdot v = a \cdot v + b \cdot v$ *for all* $v \in V$ *and* $a, b \in \mathcal{R}$*,*

(v) $a \cdot (v + w) = a \cdot v + a \cdot w$ *for all* $v, w \in V$ *and* $a \in \mathcal{R}$*.*

The space of all smooth functions from a manifold M to $\mathbb{R}$ is a commutative ring, and by a $C^\infty-$module we mean a module over the space of these smooth functions.

Lemma 1.14. *Let* ξ *be a distribution on a smooth manifold* M*, then* $\Gamma(\xi)$ *is a* C^∞*-module.*

Proof. Let $X, Y \in \Gamma(\xi)$ be two vector fields tangent to ξ and let $f \in C^\infty(M)$ be a smooth function on M. We then use the standard vector field addition and multiplication by a smooth function for the module operations, i.e.

$$\begin{aligned} (X + Y)_p &= X_p + Y_p, \\ (fX)_p &= f(p)X_p. \end{aligned}$$

We know that $\mathfrak{X}(M)$ is a C^∞-module with these operations ([17, p. 177]), and thus we know that conditions *(i)-(v)* are fulfilled. We only need to show that $X + Y$ and fX are indeed elements of $\Gamma(\xi)$. Let $p \in M$, then $\xi_p \subseteq T_pM$ is a (smooth) linear subspace. Since $X_p, Y_p \in \xi_p$, it follows that $(X + Y)_p = X_p + Y_p \in \xi_p$. And since $f(p)$ is a scalar, also $(fX)_p = f(p)X_p \in \xi_p$. This concludes the proof. □

Definition 1.15. *Let* ξ *be a distribution. We define the* ***associated Lie flag*** *to be the following increasing sequence of* $C^\infty-$*modules*

$$\Gamma^{(0)}(\xi) \subset \Gamma^{(1)}(\xi) \subset \cdots \subset \Gamma^{(i)}(\xi) \subset \Gamma^{(i+1)}(\xi) \subset \cdots,$$

where $\Gamma^{(i)}(\xi) := \langle A(a_0, ..., a_i) : a_0, ..., a_i \in \Gamma(\xi)\rangle_{C^\infty}$ *is the* C^∞*-span of bracket expressions of length* $\leq i$*. ([10, p. 5])*

Remark. Observe that when ξ is involutive that $\Gamma^{(0)}(\xi) = \Gamma^{(i)}(\xi)$ for all $i \geq 0$. ♦

Example 1.16. Let $M = \mathbb{R}^3$ be a smooth manifold and $\xi = \langle \partial_x + z^2\partial_y, \partial_z \rangle$ a distribution on M. We can compute the associated Lie flag by computing Lie brackets. We have that

$$[\partial_z, \partial_x + z^2\partial_y] = \partial_z(\partial_x + z^2\partial_y) = 2z\partial_y,$$

$$[\partial_z, 2z\partial_y] = 2\partial_y.$$

All the other combinations of vector fields which we can apply the Lie bracket on will yield 0. This means that

$$\Gamma^{(0)}(\xi) = \langle \partial_x + z^2\partial_y, \partial_z \rangle_{C^\infty},$$

$$\Gamma^{(1)}(\xi) = \langle \partial_x + z^2\partial_y, \partial_z, 2z\partial_y \rangle_{C^\infty},$$

$$\Gamma^{(2)}(\xi) = \langle \partial_x + z^2\partial_y, \partial_z, 2\partial_y \rangle_{C^\infty} = \langle \partial_x, \partial_y, \partial_z \rangle_{C^\infty} = \mathfrak{X}(M).$$

We note that not every $\Gamma^{(i)}(\xi)$ is necessarily a distribution. In this setting we have that $\Gamma^{(1)}(\xi)$ has rank 2 when $z = 0$ and rank 3 when $z \neq 0$, which means it cannot correspond to a distribution. Furthermore, observe that $\Gamma^{(2)}(\xi)$ yields the module of all vector fields on M. ([10, p. 6]) △

In the previous example we have seen that not every module in the associated Lie flag is necessarily a distribution. This leads to the following definition.

Definition 1.17. *A distribution is* ***weakly regular*** *if all the modules in the associated Lie flag are distributions. In this case we use the notation $\xi^{(i)} = \Gamma^{(i)}(\xi)$. ([10, p. 6])*

When a distribution is weakly regular, the associated Lie flag will stabilise, as shown in the next lemma.

Lemma 1.18. *Let ξ be a weakly regular distribution on a manifold M. Then there exists an $i \geq 0$ such that $\xi^{(i)} = \xi^{(j)}$ for all $j \geq i$. ([10, p. 6])*

Proof. Because ξ is weakly regular, all the modules in the associated Lie flag are distributions. We have that $\xi^{(k+1)}$ is not equal to $\xi^{(k)}$ for a $k \geq 0$ if and only if the rank of $\xi^{(k+1)}$ is strictly larger than the rank of $\xi^{(k)}$. However, the ranks of all the modules in the Lie flag are bound by the rank of TM, and therefore the increasing sequence must stabilise. □

From now on, unless stated otherwise, we will assume that distributions are weakly regular.

Definition 1.19. *We call a distribution ξ* ***bracket generating*** *if there exists an $m \in \mathbb{N}$ such that $\xi^{(m)} = \mathfrak{X}(M)$. ([10, p. 8])*

Remark. Since $\xi^{(m)}$ is also a distribution, this means that we can see $\xi^{(m)}$ as a subbundle of the tangent bundle TM. For every $v \in T_pM$ there is a smooth global vector field $X \in \mathfrak{X}(M)$ such that $X_p = v$ ([17, p. 177]). Since $\xi^{(i)} = \mathfrak{X}(M)$, $\xi^{(i)}$ must contain all the possible tangent vectors at every point in your manifold, and thus $\xi^{(i)} = TM$ when we view $\xi^{(i)}$ as a distribution. ♦

In Chapter 2 and 3 we discuss special types of distributions called *contact structures* and *Engel structures*, which are bracket-generating distributions. This means that if we (repeatedly) apply the Lie bracket on vector fields tangent to these distributions, new vector fields will be formed, and we will be able to span the whole tangent bundle using these vector fields. In the next section we discuss how we can keep track of what happens to a distribution when we apply the Lie bracket to it.

1.4 The curvature

We will now be looking at a map called the curvature, which we can define for every given distribution. The Frobenius Theorem tells us that a distribution is integrable if and only if it is involutive. The curvature tells us precisely how a distribution acts under the application of the Lie bracket, which is why it is useful to study this map when studying the non-integrability of a distribution. The definition of the curvature map was retrieved from [10, p. 12].

Lemma 1.20. *Let ξ be a distribution on a smooth manifold M and let $\xi^{(i)}$ denote the distributions of the Lie flag. The Lie bracket yields a well-defined bundle morphism:*

$$\Omega_{i,j}(\xi) : \xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)} \to \xi^{(i+j+1)}/\xi^{(i+j)}.$$

*We call this map the (i,j)-**curvature**.*

Proof. First we show that the map is well-defined, which means that the image is indeed contained in $\xi^{(i+j+1)}/\xi^{(i+j)}$. Let $(u,v) \in \xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)}$. Since $u \in \xi^{(i)}/\xi^{(i-1)}$ and $v \in \xi^{(j)}/\xi^{(j-1)}$, $[u,v]$ is a bracket expression of length $i+j+1$, and thus an element of $\xi^{(i+j+1)}/\xi^{(i+j)}$.

Next, consider the following diagram:

$$\begin{tikzcd}
\xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)} \arrow[r, "\Omega_{i,j}"] \arrow[d, "\pi_i \times \pi_j"] & \xi^{(i+j+1)}/\xi^{(i+j)} \arrow[d, "\pi_{i+j+1}"] \\
C^\infty(M) \times C^\infty(M) \arrow[r, "\varphi"] & C^\infty(M)
\end{tikzcd}$$

Here $C^\infty(M)$ is the space of all smooth functions from M to $\mathbb{R}$, and $\pi_i : \xi^{(i)}/\xi^{(i-1)} \to C^\infty(M)$ is the map which projects an element in the $C^\infty-$module to the corresponding smooth function, i.e. $\pi_i(fX) = f$, where $fX \in \xi^{(i)}/\xi^{(i-1)}$ and $f \in C^\infty(M)$. With these projections, the spaces $\xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)}$ and $\xi^{(i+j+1)}/\xi^{(i+j)}$ are bundles. The map φ is the multiplication of two smooth functions, which yields another smooth function.

Let $(v,w) \in \xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)}$ and $f,g \in C^\infty$, we see the following:

$$\begin{aligned}
\Omega_{i,j}(fv,gw) = [fv,gw] = (fv)(gw) - (gw)(fv) &= f(dg(v)w) + fg(vw) - g(df(w)v) - gf(wv) \\
&\equiv fg(vw) - gf(wv) = fg[v,w] \in \xi^{(i+j+1)}/\xi^{(i+j)},
\end{aligned}$$

since $f(dg(v)w) \in \xi^j \subset \xi^{i+j}$ and $g(df(w)v) \in \xi^i \subset \xi^{i+j}$. This means that $\pi_{i+j+1} \circ \Omega_{i,j} = \varphi \circ (\pi_i \times \pi_j)$, and thus the diagram commutes. We conclude that $\Omega_{i,j}$ is a well-defined bundle morphism. □

The proof above is based on the proof in [10], but the bundle morphism in question is explained in more detail. More information on bundles and bundle morphisms can be found in Appendix A.

Example 1.21. Let ξ be the distribution on $\mathbb{R}^5$ of rank 4 defined as follows:

$$\xi = \langle \partial_v, \partial_w, \partial_x + v\partial_z, \partial_y + f(w)\partial_z \rangle,$$

with $(v,w,x,y,z) \in \mathbb{R}^5$ and $f : \mathbb{R} \to \mathbb{R}$ a function with $f \equiv 0$ if $w \geq 0$ and strictly increasing if $w < 0$. You can easily check that the only non-trivial Lie brackets of the vector fields that span ξ are the following:

$$[\partial_v, \partial_x + v\partial_z] = \partial_z,$$
$$[\partial_w, \partial_y + f(w)\partial_z] = f'(w)\partial_z.$$

When $w \geq 0$ the function f is constantly 0 and thus $f'(w)$ is also equal to 0, and when $w < 0$ f is increasing and thus $f'(w)$ is strictly positive. In both cases we see that $\xi^{(1)}$ is equal to the whole tangent bundle $T\mathbb{R}^5$.

The first curvature is the map

$$\Omega_{0,0} : \xi \times \xi \to \xi^{(1)}/\xi,$$

and we can view it as the 4 by 4 matrix

$$\begin{bmatrix} 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & f'(w)\partial_z \\ -\partial_z & 0 & 0 & 0 \\ 0 & -f'(w)\partial_z & 0 & 0 \end{bmatrix}.$$

We see that when $w \geq 0$ the curvature has rank 2 and when $w < 0$ that it has maximal rank, i.e. rank 4. So even though the $\xi^{(1)} = T\mathbb{R}^5$ in both cases, we see that the curvature differs in rank, and thus provides us with more information. ([10, p. 13]) △

Chapter 2

Contact structures

In this chapter we will study a type of bracket-generating distributions called *contact structures*. We will mainly focus on contact structures of rank-2 on 3-dimensional manifolds. First, we will give the formal definition (section 2.1) and we will investigate when distributions are contact on $\mathbb{R}^3$ (section 2.2). A question we will treat in this chapter, is when and whether we can deform a given distribution into a contact structure. In fact, we need some extra data about a distribution if we want to homotope it into a contact structure, which is why we introduce *formal contact structures* (section 2.3). In section 2.5 we will prove that every formal contact structure is homotopic to a genuine one, which we call the *h-principle* for contact structures. As mentioned before, this was first proven by Mikhael Gromov in 1969, and it will be the main result we discuss in this chapter.

2.1 Definition contact structure

Let us formally define contact structures.

Definition 2.1. *Let ξ be a distribution of rank $2n$ on a manifold M of dimension $2n+1$. Let $\Omega_{0,0} : \xi \times \xi \to TM/\xi$ be its $(0,0)-$curvature. We say that ξ is a* ***contact structure*** *if the rank of $\Omega_{0,0}$ is maximal, i.e. equal to $2n$. This means that for any $0 \neq v \in \xi$ there is an $w \in \xi$ such that $\Omega_{0,0}(v,w) \neq 0$. ([10, p. 16])*

This last property can be defined on a more general set of maps, called 2-forms. These are defined as follows.

Definition 2.2. *A* ***2-form*** *on a vector space V is a map $f : V \times V \to \mathbb{R}$ such that f is bilinear and anti-symmetric. ([17, p. 312])*

Definition 2.3. *A 2-form $f : V \times V \to \mathbb{R}$ is said to be* ***nondegenerate****, if for any $0 \neq v \in V$ there is an $w \in V$ for which $f(v,w) \neq 0$. ([17, p. 343])*

Since TM/ξ is 1-dimensional we can identify it with $\mathbb{R}$. This means that when ξ is a contact structure, its (0,0)-curvature is a non-degenerate 2-form.

We write $Cont(M)$ for the space of all contact structures on a manifold M. We endow $Cont(M)$ with the C^1-topology (see Appendix B).

Example 2.4. In Example 1.21 we looked at the distribution $\xi = \langle \partial_v, \partial_w, \partial_x + v\partial_z, \partial_y + f(w)\partial_z \rangle$ of rank 4 on the manifold $\mathbb{R}^5$ of dimension 5. We concluded that when $w \geq 0$ the first curvature has rank 2 and that when $w < 0$ it has rank 4, i.e. maximal rank. Therefore ξ is contact when $w < 0$ and not contact when $w \geq 0$. △

From now on, we will be focusing on contact structures of rank 2 on a smooth manifold of dimension 3. Suppose ξ is such a contact structure on a manifold M. Since it is a smooth distribution, it is locally spanned by linearly independent vector fields, i.e. locally $\xi = \langle X, Y \rangle$ for vector fields X and Y on M. The first curvature of ξ has maximal rank if and only if $[X,Y]$ is linearly independent of X and Y. Therefore, in this particular case, we can rewrite the definition as follows.

Definition 2.5. *Let ξ be a smooth distribution of rank 2 on a 3-dimensional manifold M with locally $\xi = \langle X, Y \rangle$ on an open U in M. Then ξ is a* ***contact structure*** *on U if and only if $TM = \langle X, Y, [X,Y] \rangle$ on U.*

Example 2.6. Let $\xi = \langle \partial_x + \partial_y, \partial_y + 2x\partial_z \rangle$ be a distribution on $\mathbb{R}^3$. We then have that

$$[\partial_x + \partial_y, \partial_y + 2x\partial_z] = 2\partial_z.$$

Therefore $\langle \partial_x + \partial_y, \partial_y + 2x\partial_z, 2\partial_z \rangle = \langle \partial_x, \partial_y, \partial_z \rangle = T\mathbb{R}^3$, and we may conclude that ξ is a contact structure. △

2.2 Contact structures on $\mathbb{R}^3$

In this section we will be focusing on distributions on $\mathbb{R}^3$. Since we are working with smooth manifolds, which locally have Euclidean coordinates, this will be quite useful. The following lemma tells us more about when distributions of rank 2 on $\mathbb{R}^3$ are contact.

Lemma 2.7. *Consider the following setting:* $(\mathbb{R}^3, \xi = \langle \partial_x, \cos(f)\partial_y + \sin(f)\partial_z \rangle)$ *where* (x, y, z) *are the coordinates in* $\mathbb{R}^3$ *and* $f : \mathbb{R}^3 \to \mathbb{R}$ *is a smooth function. Then:*

- ξ *is contact if and only if* $\frac{\partial f}{\partial x} \neq 0$,
- ξ *is involutive if and only if* $\frac{\partial f}{\partial x} = 0$.

Proof. To prove this result we compute the Lie bracket of the two vector fields:

$$\begin{aligned}
[\partial_x, \cos(f)\partial_y + \sin(f)\partial_z] &= \partial_x(\cos(f)\partial_y + \sin(f)\partial_z) - (\cos(f)\partial_y + \sin(f)\partial_z)\partial_x \\
&= -\sin(f)f_x\partial_y + \cos(f)f_x\partial_z - 0 \\
&= -\sin(f)f_x\partial_y + \cos(f)f_x\partial_z.
\end{aligned}$$

The resulting vector field is linearly independent of ∂_x and $\cos(f)\partial_y + \sin(f)\partial_z$ if and only if $f_x = \frac{\partial f}{\partial x} \neq 0$, and thus ξ is contact if and only if $\frac{\partial f}{\partial x} \neq 0$ as the three linearly independent vector fields will span the whole tangent bundle. From this also immediately follows that ξ is involutive if and only if $\frac{\partial f}{\partial x} = 0$. □

Intuitively, when we consider ξ as a plane in $\mathbb{R}^3$, then ξ is contact if and only if the plane is turning when moving in the x-direction. And ξ is involutive if and only if the plane is still.

Remark. The set $\{(x, y, z) \in \mathbb{R}^3 : \frac{\partial f}{\partial x} \neq 0\}$ is an open set (it is the complement of a pre-image of a single point). So when $\frac{\partial f}{\partial x} \neq 0$ at some point in your manifold, the same is true for a neighbourhood of this point. And thus if a distribution is contact in some area, it is also contact in a neighbourhood of that area. However, the set $\{(x, y, z) \in \mathbb{R}^3 : \frac{\partial f}{\partial x} = 0\}$ is closed (it is the pre-image of a single point). Therefore a distribution being involutive at a point does not necessarily imply that it is also involutive in a neighbourhood of that point. ♦

We now prove the so called Darboux-lemma.

Lemma 2.8 (Darboux)**.** *Let ξ be a contact structure of rank 2 on a 3-dimensional smooth manifold M. Let $p \in M$. Then there exist local coordinates (x, y, z) around p for which $\xi = \langle \partial_y, \partial_x + y\partial_z \rangle$. ([10, p. 23])*

Proof. Since ξ is a smooth distribution there is a neighbourhood U around p such that $\xi = \langle X, Y \rangle$ on U. Therefore Y is a vector field tangent to ξ on U. We pick a 2-dimensional submanifold S on M such that Y is transverse to S on U (for the definition of transversality see Appendix D).

As S is transverse to ξ, their intersection is a line field, and we can pick a vector field $Z \in \mathfrak{X}(S)$ such that Z spans the line field. By the flow box theorem we can find local coordinates (x, z) around p such that $Z = \partial_z$. Since Y is tranverse to S, we can use the flow of Y as a third coordinate direction. We then obtain local coordinates (x, y, z) in a neighbourhood around p where $Y = \partial_y$ and $Z = \partial_z$.

By construction we now know that locally around p we have $\xi = \langle \partial_y, W \rangle$ for some $W \in \mathfrak{X}(M)$ which we can express as $W = \partial_x + f\partial_z$ for a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$. For ξ to be contact we must then have that $\frac{\partial f}{\partial y} \neq 0$. The implicit function theorem implies that we can reparametrise in the $y-$coordinate such that $f(x, y, z) = y$.

We have therefore found local coordinates (x, y, z) around p such that $\xi = \langle \partial_y, \partial_x + y\partial_z \rangle$, as desired. □

Remark. In the proof of this lemma we used the flow of the vector field Y to build one of the coordinate directions. This is a technique which we will use quite often later on, and will be explained in more detail in section 2.4. ♦

Definition 2.9. *The contact structure on* $\mathbb{R}^3$ *given by* $\langle \partial_y, \partial_x + y\partial_z \rangle$ *is called the* ***standard contact structure*** *on* $\mathbb{R}^3$.

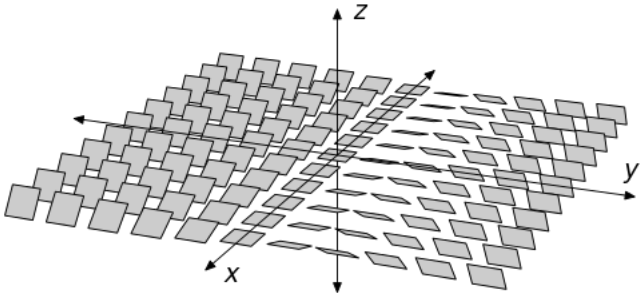


Figure 2.1: The standard contact structure on $\mathbb{R}^3$ given by $\langle \partial_y, \partial_x + y\partial_z \rangle$ ([6]).

2.3 Formal contact structures

In this section we will be looking at formal contact structures. These are distributions together with a separate piece of data. For any geometric structure, we can decouple the derivatives from the object and obtain a "formal geometric structure". The definition for formal contact structures is as follows.

Definition 2.10. *A* ***formal contact structure*** ξ *is a rank 2-distribution on a 3-dimensional manifold* M *together with a map* $\Omega : \xi \times \xi \to TM/\xi$*, which is a non-degenerate 2-form. ([10, p. 22])*

We write $Cont^f(M)$ for the space of all formal contact structures on a manifold M, and we endow it with the C^0-topology (see Appendix B).

The map Ω plays the role of the curvature in the contact case. We know that if ξ is a contact structure, then the (0,0)-curvature $\Omega_{0,0} : \xi \times \xi \to TM/\xi$ is a non-degenerate 2-form. Therefore each genuine contact structure is also a formal contact structure.

In the remaining part of this chapter, we are going to investigate whether we can locally deform a formal contact structure to be genuinely contact. It turns out that this is indeed the case on open manifolds, and we will eventually be able to prove the following result:

Theorem 2.11. *Let* M *be an open* 3*-dimensional manifold. Then every formal contact structure is homotopic to a contact structure. I.e.* $\pi_0(Cont(M)) \to \pi_0(Cont^f(M))$ *is a surjection. ([10, p. 25])*

The theorem actually states that the *h-principle* holds for contact structures on open manifolds. Let us explain this some more.

When taking a contact structure, we can look at its curvature as a separate piece of data and thereby construct a formal contact structure. The next thing someone might ask themselves is if we can construct a genuine contact structure out of a formal one. As said before, we can do the same for any other geometric structures. We can decouple the derivatives from the object and obtain a "formal geometric structure". Again, we could investigate if we can also turn a formal structure into a genuine one. This leads to the following definition.

Definition 2.12. *If two spaces of geometric structures and formal geometric structures are homotopy equivalent, we say "the problem satisfies the* h***-principle****". ([10, p. 25])*

In the remaining part of this chapter we will work towards proving the h-principle for contact structures on open manifolds. Later on in the thesis we will also prove the h-principle for another type of distributions, called Engel structures.

2.4 Building boxes

Let M be a 3-dimensional manifold and ξ a smooth distribution of rank 2 on M. By Lemma D.2 we know that there exists a triangulation of M such that ξ is transverse to every simplex. In this section we explain how we can build boxes around each simplex in the 2-skeleton such that ξ is given by $\langle \partial_x, \cos(f)\partial_y + \sin(f)\partial_z \rangle$ in these boxes. This allows us to use Lemma 2.7 in the proof of the h-principle in section 2.5. The transversality is an important ingredient in this process, and in Appendix D it is explained in detail how one can find such a triangulation. The proofs of the lemmas in this section are all based on the construction in the proof of Theorem 9.46 in [17, p. 235].

Because ξ is a smooth distribution we can locally write $\xi = \langle X, Y \rangle$ for vector fields X and Y on M. We want to find local coordinates such that X 'corresponds' to one of the coordinate direction. By this last statement we mean the following.

Definition 2.13. *Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let X be a vector field on M and Y a vector field on N. We say X and Y are $F-$**related** if for every $p \in M$ we have $dF_p(X_p) = Y_{F(p)}$. ([17, p. 182])*

First, let $p \in M$ be a point contained in the image of a 2-simplex, i.e. there exists a map σ_i^2 such that $p \in \sigma_i^2(\mathring{\Delta}^2)$.

Lemma 2.14. *Let $p \in M$ such that $p \in \sigma_i^2(\Delta^2)$ for some $i \in I_2$. Let $\theta_t : M \to M$ be the flow of X with $t \in \mathbb{R}$. Let Φ be defined as follows:*

$$\Phi : \Delta^2 \times [-\epsilon, \epsilon] \to U, (x, y, t) \mapsto \theta_t(\sigma_i^2(x, y)).$$

Here U is a neighbourhood of $\sigma_i^2(\Delta^2)$ and $\epsilon > 0$ is small such that indeed $\theta_t(\sigma_i^2(x, y)) \in U$. Note that $\Phi(\Delta^2 \times \{0\}) = \sigma_i^2(\Delta^2)$. Then Φ is a diffeomorphism in a neighbourhood of $\Delta^2 \times \{0\}$ and ∂_t is $\Phi-$related to X.

Proof. First we show that ∂_t is Φ-related to X.

Let $p_0 = (x_0, y_0, t_0) \in \Delta^2 \times [-\epsilon, \epsilon]$. Let $f \in C^\infty(U)$, we see the following

$$d\Phi_{p_0}(\partial_t|_{p_0})f = \partial_t|_{p_0}(f \circ \Phi)(x, y, t) = \partial_t|_{p_0}(f(\Phi(x, y, t))) = \partial_t|_{p_0}(f(\theta_t(\sigma_i^2(x, y)))).$$

Since θ_t is the flow of X, we know that $t \mapsto \theta_t(q)$ for $q \in M$ is an integral curve of X which means that $\partial_t|_q \theta_t(q) = X_{\theta_t(q)}$. Therefore

$$d\Phi_{p_0}(\partial_t|_{p_0})f = \partial_t|_{p_0}(f(\theta_t(\sigma_i^2(x, y)))) = X_{\theta_{t_0}(\sigma_i^2(x_0, y_0))}f = X_{\Phi(p_0)}f.$$

This tells us that ∂_t is $\Phi-$related to X. In particular we have that $d\Phi_{(x,y,0)}(\partial_t|_{(x,y,0)}) = X_{\Phi(x,y,0)} = X_{\sigma_i^2(x,y)}$.

We note that $\Phi(x, y, 0) = \theta_0(\sigma_i^2(x, y)) = \sigma_i^2(x, y)$, and thus Φ is here equal to the map σ_i^2. We know that σ_i^2 is a homeomorphism as it comes from the triangulation. Let $q \in \Delta^2$ and $p = \sigma_i^2(q)$. We set $u_p := d\sigma_i^2|_q(\partial_x|_q)$ and $v_p := d\sigma_i^2|_q(\partial_y|_q)$. Since the vector field X is transverse to the simplex, we have that (u_p, v_p, X_p) form a basis of T_pM. We have shown before that $d\Phi|_{(q,0)}(\partial_t|_{(q,0)}) = X_{\sigma_i^2(q)} = X_p$ and thus Φ takes the basis $(\partial_x|_{(q,0)}, \partial_y|_{(q,0)}, \partial_t|_{(q,0)})$ of $\Delta^2 \times [-\epsilon, \epsilon]$ to the basis (u_p, v_p, X_p) of T_pM.

This means that $d\Phi_{(x,y,0)}$ is invertible for all $(x, y, 0) \in \Delta^2 \times \{0\}$. Since Φ is a smooth function, we can use the inverse function theorem ([17, p. 657]) to conclude that Φ is a diffeomorphism in a neighbourhood of $\Delta^2 \times \{0\}$ to an open in U containing the 2-simplex of p. □

By Lemma 2.14 there exists a $\delta > 0$ such that Φ is a diffeomorphism on $\Delta^2 \times [-\delta, \delta]$, where ∂_t is Φ-related to X with $(x, y, t) \in \Delta^2 \times [-\delta, \delta]$. Intuitively, we have therefore build a box around the 2-simplex in M in which p is contained, and where the flow of the vector field X corresponds to the third coordinate direction. In figure 2.2 this idea is illustrated.

Secondly, we assume that p is contained in a 1-simplex Δ^1 of the triangulation. In order to build the box around Δ^1, we will use a similar method as in the 2-dimensional simplex case. However, now we are using two vector fields tangent to ξ to build the box.

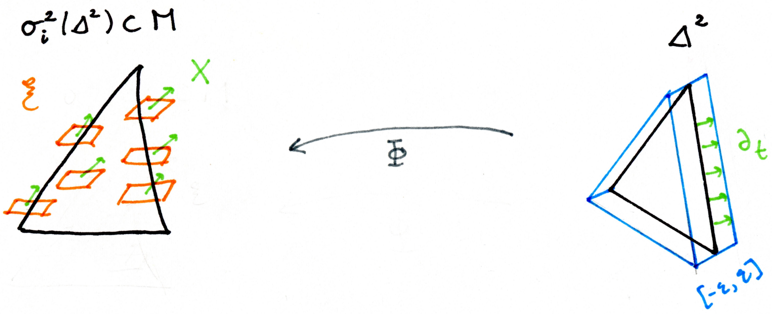


Figure 2.2: Illustration of proof of Lemma 2.14.

Lemma 2.15. *Let $p \in M$ such that $p \in \sigma_i^1(\Delta^1)$ for some $i \in I_1$. Let $\theta_s^X : M \to M$ and $\theta_t^Y : M \to M$ be the flows of X and Y respectively, with $s, t \in \mathbb{R}$. Let Φ be defined as follows:*

$$\Phi : \Delta^1 \times [-\epsilon, \epsilon]^2 \to U, (x, s, t) \mapsto \theta_s^X \circ \theta_t^Y(\sigma_i^1(x)).$$

Here U is a neighbourhood of $\sigma_i^1(\Delta^1)$ and $\epsilon > 0$ is small such that indeed $\theta_s^X \circ \theta_t^Y(\sigma_i^1(x)) \in U$. Note that $\Phi(\Delta^1 \times \{0\}^2) = \sigma_i^1(\Delta^1)$. Then Φ is a diffeomorphism in a neighbourhood of $\Delta^1 \times \{0\}^2$ and ∂_s is Φ−related to X.

Proof. The proof is very similar to the proof of Lemma 2.14. □

See figure 2.3 for an illustration of the idea of this proof.

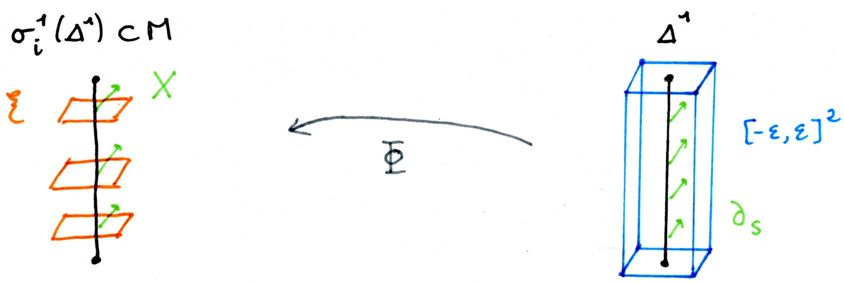


Figure 2.3: Illustration of proof of Lemma 2.15.

Lastly, assume p is contained in the image of a 0-simplex Δ^0. Building a box around a vertex where X corresponds to one of the coordinate directions is somewhat simpler than the other two previous cases. Namely, we can use the flow box theorem. From this theorem it follows that locally, there are coordinates (x, y, z) around p where $X = \partial_x$. This local neighbourhood can be restricted to a smaller box inside it, and thus we have reached our goal.

Remark. The idea of building boxes using flows of vector fields is something that we will see more than once in this thesis. That is why we payed quite a lot attention to it in this section, but after this we won't go into detail that much anymore. The proofs of comparable statements can be easily computed by carrying out a similar process as we have seen here. ♦

2.5 Proof of h-principle

In this section we will give a proof of the h-principle for contact structures. Indeed, we now have gathered all the tools in order to prove Theorem 2.11. The proof below is based on several discussions with my supervisor Álvaro del Pino, but the original proof can be found in the paper *Stable mappings of foliations into manifolds* by Mikhael Gromov (1969), see [14]. Let us state the theorem again for completeness:

Theorem 2.11. *Let M be an open 3-dimensional manifold. Then every formal contact structure is homotopic to a contact structure. I.e. $\pi_0(Cont(M)) \to \pi_0(Cont^f(M))$ is a surjection.*

Proof. Let $\xi \in Cont^f(M)$. Then ξ is a rank-2 distribution, and locally we can describe ξ as $\langle X, Y \rangle$.

As explained in Appendix D, by Thurston's jiggling, we can find a triangulation T of M such that ξ is transverse to every simplex. In section 2.4 we have seen that for every simplex in the 2-skeleton, we can build a box with local coordinates (x, y, z) around it such that the vector field X corresponds to ∂_x.

We can then write $\xi = \langle \partial_x, \cos(f)\partial_y + \sin(f)\partial_z \rangle$, for a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, in such a box. By Lemma 2.7 we know that ξ is contact if and only if $\frac{\partial f}{\partial x} \neq 0$. The idea is to inductively alter the function f on each simplex in order to make ξ contact in all the boxes. Whether we want to alter the function f such that $\frac{\partial f}{\partial x} > 0$ or $\frac{\partial f}{\partial x} < 0$ is determined by the non-degenerate 2-form which comes with the formal contact structure.

The formal contact structure comes with a non-degenerate 2-form $\Omega : \xi \times \xi \to TM/\xi$. Since ξ is a distribution of rank 2, $TM \backslash \xi$ is a line bundle. Locally, let $\xi = \langle X, Y \rangle$. Since Ω is a 2-form, it is anti-symmetric, and thus $\Omega(X, X) = \Omega(Y, Y) = 0$ and $\Omega(X, Y) = -\Omega(Y, X) \neq 0$ (as it is non-degenerate). The map Ω yields a global orientation of M by setting an oriented basis of TM, namely the basis $\{X, Y, \Omega(X, Y)\}$. We choose the sign of $\frac{\partial f}{\partial x}$ such that the basis $\{X, Y, [X, Y]\}$ provides the same orientation. In other words, $[X, Y]$ should point in the same direction as $\Omega(X, Y)$. We want to make the choice between $\frac{\partial f}{\partial x} > 0$ or $\frac{\partial f}{\partial x} < 0$ globally, as otherwise the altered distribution will have some critical points when switching from the turning direction. For now let's assume we want $\frac{\partial f}{\partial x} > 0$.

Let Δ^0 be a 0-simplex. Let (x, y, z) be the coordinates of the box around it. Then $\xi = \langle \partial_x, \cos(f)\partial_y + \sin(f)\partial_z \rangle$. We look at the following family of functions:

$$f_{(y,z)} := f(-, y, z) : \mathbb{R} \to \mathbb{R}, x \mapsto f(x, y, z).$$

We note that by the way we constructed the boxes, we can assume that the 0-simplex is contained in the level set $\{x = 0\}$. For each (y, z) we want to perturb the function $f_{(y,z)}$ very slightly to yield $\tilde{f}$ such that $\frac{\partial \tilde{f}}{\partial x} > 0$ around $x = 0$.

We can look at a compact neighbourhood around $x = 0$ in the box, and thus we can look at the minimum

$$\min_{Op(x=0)} \partial_x f =: h.$$

This $h \in \mathbb{R}$ may be positive, negative or zero, but we assume it to be negative or zero because otherwise the distribution would already be contact around the 0-simplex and we would be done. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function. In figure 2.4 the values of χ in the different neighbourhoods of the box around the simplex are indicated. We note that $\chi = 1$ in a small neighbourhood around Δ^0, and we use $Op(\Delta^0)$ to denote this neighbourhood.

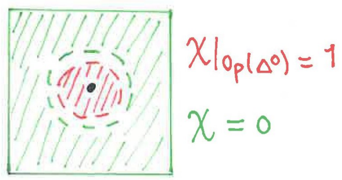


Figure 2.4: The values of the function χ in the box around Δ^0.

We now choose a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and $\partial_x(\chi g)|_{Op(\Delta^0)} > |h|$. We then define the function

$$\tilde{f} := f + \chi g.$$

We note that

$$\partial_x \tilde{f}|_{Op(\Delta^0)} = \partial_x f|_{Op(\Delta^0)} + \partial_x(\chi g)|_{Op(\Delta^0)} > 0.$$

We have now constructed a function $\tilde{f}$ which has positive derivative in the neighbourhood $Op(\Delta^0)$, and it equals f on the boundary and outside the box. We call the function χ the *cut-off* function, as it makes sure the function $\tilde{f}$ equals f outside of a small neighbourhood around the simplex.

The functions f and $\tilde{f}$ are homotopic to each other, by the simple homotopy $f_s := (1-s)f + s\tilde{f} = f + \chi g \cdot s$ for $s \in [0, 1]$.

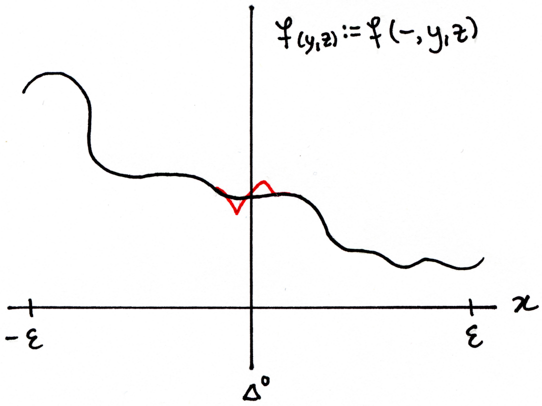


Figure 2.5: Perturbing the function $f_{(y,z)}$ by adding the function χg (in red).

Now we look at 1-simplices. Again, by section 2.4 we can find coordinates (x, y, z) to build a box around Δ^1 such that $\Delta^1 \subset \{x = 0\}$. We choose these boxes small enough such that boxes around different 1-simplices do not overlap, see figure 2.6. This box around Δ^1 we call B. Since every 1-simplex is attached to 0-simplices, we should also be careful about those boxes overlapping. We have already altered the functions f around the 0-simplices to functions $\tilde{f}$ such that $\partial_x \tilde{f} > 0$, so we can choose small (closed) boxes around the 0-simplices, B_1 and B_2, in which the distribution is contact. Because these boxes are compact the following minimum exists:

$$\min_{B_1 \cup B_2} \partial_x f =: c > 0.$$

This minimum is larger than 0 because the distribution is contact on $B_1 \cup B_2$.

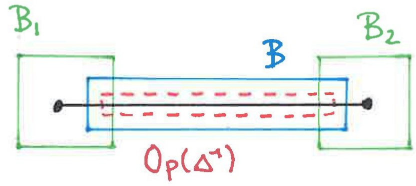


Figure 2.6: Boxes around the 1-simplex Δ^1 and its vertices.

We now want to alter the function f on the box B around the 1-simplex such that the distribution becomes contact, and does not destroy what we have already done in the boxes B_1 and B_2. Again, we use a cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi|_{Op(\Delta^1)} = 1$ and $\chi = 0$ outside a very small neighbourhood around Δ^1. The box around the 1-simplex B is compact and thus the following minimum exists:

$$\min_B \partial_x f =: h \leq 0.$$

We assume this minimum is less or equal to zero, because otherwise the distribution would already be contact around this 1-simplex and thus we would be done.

We then choose a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and $\partial_x(\chi g)|_{Op(\Delta^1)} > |h|$. Then we define the function $\tilde{f}$ on the box B around Δ^1 by

$$\tilde{f} := f + \chi g.$$

Then, just as before, $\partial_x \tilde{f} > 0$ around Δ^1. However, towards the boundary of the box, the derivative of χg can become very negative, as $\chi = 0$ on the boundary of B. We must therefore also choose χ and g such that $\partial_x(\chi g) > -c$. In this way, also for the overlap in the boxes around the vertices, we have that $\partial_x \tilde{f} > 0$. Therefore the distribution also remains contact in these areas. In figure 2.7 we see an illustration of how to make the derivative $\partial_x(\chi g)$ less negative away from zero, but keeping the derivative at $x = 0$ the same.

We conclude that we have altered the functions f in B to yield functions $\tilde{f}$ for which $\partial_x \tilde{f} > 0$. This can be done for all 1-simplices. These functions f and $\tilde{f}$ are homotopic by the homotopy $f_s = f + \chi g \cdot s$ for $s \in [0, 1]$.

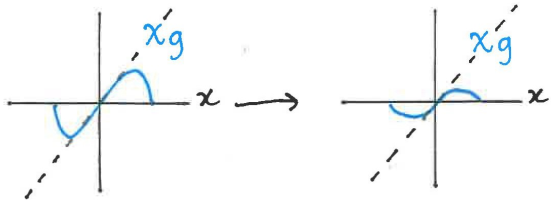


Figure 2.7: Illustration of the function χg, such that the derivative $\partial_x(\chi g)$ away from $x = 0$ is less negative.

Lastly, we want to make ξ contact around the 2-simplices. We carry out the same process, adding a function with a very positive derivative along the simplex, and using again the fact that ξ is already contact at the boundary.

We have been inductively working on the simplices, perturbing the functions f to homotopic functions $\tilde{f}$. Therefore, we have also been homotoping the distribution ξ, step by step, to a distribution $\tilde{\xi}$. This distribution $\tilde{\xi}$ is contact in a neighbourhood of the 2-skeleton, and is homotopic to ξ.

In Appendix E we have found a diffeomorphism $\Phi : M \to U$ where U lies in the complement of the barycenters of the simplices of our triangulation. Furthermore, this diffeomorphism is part of an isotopy of embeddings $(\phi_t)_{t\in[0,1]} : M \to M$ where $\phi_0 = id$ and $\phi_1 = \Phi$.

Let $\{a_j\}$ denote the top simplices of our triangulation. Since U lies in $M\backslash\{a_j\}$ there is a deformation retraction of U to a small neighbourhood of the 2-skeleton. See figure 2.8 for an illustration of this. Let $\psi_{s\in[0,1]} : U \to U$ denote this deformation retraction.

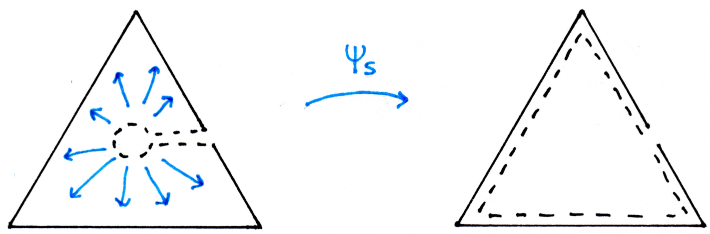


Figure 2.8: Deformation retract of (part of) a 2−simplex to a neighbourhood of the boundary.

We now look at the distribution $\phi_1^*\psi_1^*\tilde{\xi}$. Then $\phi_1^*\psi_1^*\tilde{\xi}$ is homotopic to $\tilde{\xi} = \phi_0^*\psi_0^*\tilde{\xi}$ by the follwing homotopy:

$$\varphi_t\tilde{\xi} := \begin{cases} \phi_0^*\psi_{2t}^*\tilde{\xi} & \text{if } t \in [0, \frac{1}{2}] \\ \phi_{2t-1}^*\psi_1^*\tilde{\xi} & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

By the way we constructed $\tilde{\xi}$, we know that ξ is homotopic to $\tilde{\xi}$, and thus ξ is homotopic to $\phi_1^*\psi_1^*\tilde{\xi}$. We know that $\tilde{\xi}$ is contact around the 2-skeleton, and by pulling back by ψ_s, we obtain a distribution which is contact on U. Next, by pulling back by ϕ_s, we obtain a distribution which is contact on the whole of M. We have therefore shown that for every formal contact structure ξ, there exists a distribution $\phi_1^*\psi_1^*\tilde{\xi}$ which is genuinely contact on the whole manifold. This completes the proof. □

Chapter 3

Engel Structures

In the previous chapters we have focused on distribution of rank 2 on a 3-dimensional manifold. We will now be looking at distributions of rank 2 on a 4-dimensional manifold. Let us study the curvatures in this setting. Let ξ be a distribution of rank 2 on a 4-dimensional manifold M, then

$$\Omega_{0,0} : \xi \times \xi \to \xi^{(1)}/\xi,$$

denotes the first curvature. Since ξ has rank 2, the image of this map has at most dimension 1. If ξ is involutive, and thus a foliation, the image of $\Omega_{0,0}$ has dimension 0. Assume now the image has dimension 1. Then $\xi^{(1)}$ is a distribution of rank 3. We then look at the $(0, 1)$-curvature:

$$\Omega_{0,1} : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}.$$

Since $\xi^{(1)}$ is a distribution of rank 3 the space $TM/\xi^{(1)}$ is a line bundle. Again two things can happen. The map $\Omega_{0,1}$ can be the zero map, which means that $\xi^{(1)}$ is an involutive distribution and thus a foliation. The other situation which can occur is that $\Omega_{0,1}$ is surjective. Since $TM/\xi^{(1)}$ is a line bundle, the kernel is also a line bundle contained in ξ. This last situation describes a special type of distributions called *Engel structures*. ([10, p. 17])

In this chapter we will define these Engel structures, which are bracket-generating distributions of rank 2 on 4-dimensional manifolds (section 3.1). In section 3.2 we will investigate when distributions on $\mathbb{R}^4$ are Engel. Just like in the contact case, we are interested in when and how we can deform a given distribution into an Engel structure. This is why we define *formal Engel structures* in section 3.3. In section 3.4 and 3.5 we will discuss some results which we will need in the last section, where we prove that every formal Engel structure is homotopic to a genuine Engel structure. In other words, we will prove the *h-principle* for Engel structures on open manifolds, which is the main result of this chapter. As mentioned in the introduction, this result was first proven by Mikhael Gromov in 1969 (see [14]).

3.1 Definition Engel structure and even-contact

Let us now formally define Engel structures.

Definition 3.1. *Let ξ be a distribution of rank 2 on a manifold M of dimension 4. We call the distribution ξ an **Engel structure** if the following hold:*

- *The $(0, 0)-$curvature $\Omega_{0,0} : \xi \times \xi \to \xi^{(1)}/\xi$ has maximal rank (i.e. rank 2),*
- *The $(0, 1)$-curvature $\Omega_{0,1} : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}$ is surjective.*

*The line bundle $W \in \xi$ which is the kernel of $\Omega_{0,1}$ is called the **characteristic line field** of ξ. ([10, p. 17])*

We write $Engel(M)$ for the space of all Engel structures on a manifold M. We endow $Engel(M)$ with the C^2-topology (see Appendix B).

Let ξ be an Engel structure on M. Then $\xi^{(1)}$ is a distribution of rank 3, and when we apply the Lie bracket on vector fields tangent to $\xi^{(1)}$ we can produce the whole tangent space TM. However, since $\xi^{(1)}$ is of rank 3 and M is 4-dimensional it is clear that the first curvature of $\Omega_{0,0}(\xi^{(1)})$ cannot be fully non-degenerate. This is why we previously only defined contact structures of even rank on manifolds of odd dimension.

Definition 3.2. *Let ξ be a distribution of rank $2n+1$ on a manifold M of dimension $2n+2$. We call ξ* ***even-contact*** *if the $(0,0)-$curvature $\Omega_{0,0} : \xi \times \xi \to TM/\xi$ is surjective. ([23, p. 6])*

Lemma 3.3. *A distribution ξ is Engel if and only if $\xi^{(1)}$ is even-contact. ([10, p. 18])*

Proof. First assume that ξ is an Engel structure. Then $\xi^{(1)}$ is a distribution of rank 3 for which the map $\Omega_{0,1} : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}$ is surjective. Since ξ and $\xi^{(1)}/\xi$ are subsets of $\xi^{(1)}$, it immediately follows that

$$\Omega_{0,0}(\xi^{(1)}) : \xi^{(1)} \times \xi^{(1)} \to TM/\xi^{(1)}$$

is also surjective, and thus $\xi^{(1)}$ is an even-contact structure.

Now assume that $\xi^{(1)}$ is an even-contact structure. This means that $\xi^{(1)}$ has rank 3 and thus the map $\Omega_{0,0} : \xi \times \xi \to \xi^{(1)}/\xi$ must have maximal rank. Since $\xi^{(1)}$ is even-contact, the map $\Omega_{0,0}(\xi^{(1)}) : \xi^{(1)} \times \xi^{(1)} \to TM/\xi^{(1)}$ is surjective. Since $\Omega_{0,0} : \xi \times \xi \to \xi^{(1)}/\xi$ already has maximal rank, and $\xi^{(1)}/\xi$ is a line field, the line field $TM/\xi^{(1)}$ must be obtained by applying the Lie bracket to vector fields in ξ with vector fields in $\xi^{(1)}/\xi$. Therefore the map $\Omega_{0,1} : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}$ is surjective. We conclude that ξ is an Engel structure. □

We also use $W \subset \xi \subset \mathcal{E}$ to denote an Engel structure ξ. Here W is the characteristic line field and $\mathcal{E}$ is the even contact structure. We then also write $[\xi, \xi] = \mathcal{E}$ and $[\mathcal{E}, \mathcal{E}] = TM$.

Example 3.4. Let ξ be the distribution of rank 2 on $\mathbb{R}^4$ given by $\xi = \langle \partial_w, \partial_x + z\partial_y + w\partial_z \rangle$. Then

$$[\partial_w, \partial_x + z\partial_y + w\partial_z] = -\partial_z,$$

and thus $\xi^{(1)} = \langle \partial_w, \partial_x + z\partial_y + w\partial_z, \partial_z \rangle = \langle \partial_w, \partial_x + z\partial_y, \partial_z \rangle$. This means that $\Omega_{0,0} : \xi \times \xi \to \xi^{(1)}/\xi$ has maximal rank. Furthermore

$$[\partial_z, \partial_x + z\partial_y] = -\partial_y,$$

and thus $\xi^{(2)} = \langle \partial_w, \partial_x + z\partial_y, \partial_z, \partial_y \rangle = \langle \partial_w, \partial_x, \partial_z, \partial_y \rangle = TM$. This means that the curvature $\Omega_{0,1} : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}$ is surjective. Therefore ξ is an Engel structure. We notice that indeed $\mathcal{E} := \xi^{(1)}$ is an even-contact struture, and that $W := \partial_w \in \xi$ is the characteristic line field. ([10, p. 17]) △

3.2 Engel structures on $\mathbb{R}^4$

Just like for contact structures, we are going to set some conditions for when distributions on $\mathbb{R}^4$ are Engel structures. To do this, we need the following definitions which appear in the statement and proof of Lemma 3.7.

Definition 3.5. *We say a curve $\gamma : \mathbb{R} \to \mathbb{S}^2$ is* ***convex*** *if the velocity and acceleration vectors form a positive basis for the tangent space. We say γ is* ***concave*** *if the velocity and acceleration form a negative basis for the tangent space.*

Definition 3.6. *Let M and N be smooth manifolds. A smooth map $F : M \to N$ is called a* ***smooth immersion*** *if the differential is injective in every point in M. ([17, p. 78])*

We use $Imm(M, N)$ to denote the space of all smooth immersions from M to N, and we endow it with the C^1-topology (see Appendix B).

The following lemma will be very useful later on.

Lemma 3.7. *Let ξ be a smooth distribution on $\mathbb{R}^4$ which we can describe by $\xi = \langle \partial_t, X \rangle$. We can write $X = f\partial_x + g\partial_y + h\partial_z$ for smooth functions $f, g, h : \mathbb{R}^4 \to \mathbb{R}$. We may assume X has unit length. Therefore for each (x, y, z), we can look at the map*

$$X_{(x,y,z)} := X(x, y, z, -) : \mathbb{R} \to \mathbb{S}^2.$$

Then ξ is Engel at (x, y, z, t) if and only if at least one of the following conditions holds:

- $X_{(x,y,z)}$ *is convex at* t,
- $\langle X_t, \dot{X}_t\rangle$ *is contact at* (x,y,z). *([4, p. 425])*

Proof. For ξ to be an Engel structure at (x,y,z) we must certainly have that $[\partial_t, X] = \dot{X}$ produces a new linearly independent vector field. Clearly $\dot{X}$ is linearly independent of ∂_t, so we require that X and $\dot{X}$ are linearly independent. This means that we want the curve $X_{(x,y,z)}$ to be immersed. We then have that $\mathcal{E} = \langle \partial_t, X, \dot{X}\rangle$.

Now there are two things that can happen. We have that ξ is Engel if and only if either $TM = \langle \partial_t, X, \dot{X}, [\partial_t, \dot{X}]\rangle$ or $TM = \langle \partial_t, X, \dot{X}, [X, \dot{X}]\rangle$. Here $\ddot{X} = [\partial_t, \dot{X}]$ corresponds to the acceleration of $X_{(x,y,z)}$. We have $TM = \langle \partial_t, X, \dot{X}, \ddot{X}\rangle$ if and only if $X, \dot{X}, \ddot{X}$ are linearly independent. This is equivalent to the curve $X_{(x,y,z)}$ being convex.

Let $X_t := f(-,-,-,t)\partial_x + g(-,-,-,t)\partial_y + h(-,-,-,t)\partial_z$ and $\dot{X}_t$ defined similarly. Since X_t and $\dot{X}_t$ both don't have a component in the t-direction, and the component functions do not depend on t, we can view $\langle X, \dot{X}\rangle$ as a distribution of rank 2 on $\mathbb{R}^3$ with $(x,y,z) \in \mathbb{R}^3$. Then $[X, \dot{X}]$ being linearly independent of ∂_t, X and $\dot{X}$ at the point (x,y,z,t) is the same as the distribution $\langle X_t, \dot{X}_t\rangle$ being contact at (x,y,z). This proves the claim. $\square$

3.3 Formal Engel structures

In chapter 2 we showed how to locally deform formal contact structure to be contact. In this chapter we are going to do something similar but then for Engel structures. Again we can decouple the derivatives, i.e. the curvatures, from the distribution and see them as a separate piece of data, and obtain a formal structure. We define formal Engels structures as follows.

Definition 3.8. *A **formal Engel structure** ξ is a rank 2-distribution on a 4-dimensional manifold M such that we have $W \subset \xi \subset \mathcal{E} \subset TM$ with W a line field (i.e. a rank-1 distribution) and $\mathcal{E}$ a distribution of rank 3 together with the following isomorphisms:*

- $\wedge^2\xi \simeq \mathcal{E}/\xi$
- $\xi/W \wedge \mathcal{E}/\xi \simeq TM/\mathcal{E}$. *([5, p. 2])*

We write $Engel^f(M)$ for the space of all formal Engel structures on a manifold M. We endow this space with the C^0-topology (see Appendix B).

We invite the reader to compare the definition above to the definition of an Engel structure (Definition 3.1). We observe that $Engel(M) \subset Engel^f(M)$.

The following result will be the main focus of the remaining part of this chapter, and it is very similar to Theorem 2.11.

Theorem 3.9. *Let M be an open 4-dimensional manifold. Then every formal Engel structure is homotopic to a genuine Engel structure, i.e. the map $\pi_0(Engel(M)) \to \pi_0(Engel^f(M))$ is a surjection.*

This theorem states the h-principle holds for Engel structures on open manifolds. Just like in the proof of 2.11, we going to build boxes and locally deform the formal Engel structures. In this process Lemma 3.7 will be very useful.

3.4 Building boxes

Let M be a 4-dimensional manifold and ξ a smooth distribution of rank 2 on M. By Lemma D.2 we know that there exists a triangulation of M such that ξ is transverse to every simplex. Similarly like in section 2.4, in this section we discuss how we can build boxes around each simplex in the 3-skeleton such that ξ is given by $\langle \partial_t, X\rangle$ in these boxes. This allows us to use Lemma 3.7. Again, the transversality is an important ingredient in this process, and in Appendix D it is explained in detail how one can find such a triangulation.

Because ξ is a smooth distribution we can locally write $\xi = \langle T, X\rangle$ for vector fields T and X on M. We want to find local coordinates such that T 'corresponds' to one of the coordinate direction, i.e. to ∂_t. Although we are working with a 4-dimensional manifold now, we can still build the boxes in a very similar way. We first look at a 3-simplex.

Lemma 3.10. *Let $p \in M$ such that $p \in \sigma_i^3(\Delta^3)$ for some $i \in I_3$. Let $\theta_t : M \to M$ be the flow of T with $t \in \mathbb{R}$. Let Φ be defined as follows*

$$\Phi : \Delta^3 \times [-\epsilon, \epsilon] \to U, (x, y, z, t) \mapsto \theta_t(\sigma_i^3(x, y, z)).$$

Here U is a neighbourhood of $\sigma_i^3(\Delta^3)$ and $\epsilon > 0$ small such that indeed $\theta_t(\sigma_i^3(x, y, z)) \in U$. Note that $\Phi(\Delta^3 \times \{0\}) = \sigma_i^3(\Delta^3)$. Then Φ is a diffeomorphism in a neighbourhood of $\Delta^3 \times \{0\}$ and T is Φ-related to ∂_t.

Proof. The proof is almost analogous to the proof of Lemma 2.14. □

For the 2-simplex case we also use the vector field $X \in \xi$.

Lemma 3.11. *Let $p \in M$ such that $p \in \sigma_i^2(\Delta^2)$ for some $i \in I_2$. Let $\theta_s^X : M \to M$ and $\theta_t^T : M \to M$ be the flows of X and T respectively, with $s, t \in \mathbb{R}$. Let Φ be defined as follows:*

$$\Phi : \Delta^2 \times [-\epsilon, \epsilon]^2 \to U, (x, y, s, t) \mapsto \theta_t^T \circ \theta_s^X(\sigma_i^2(x, y)).$$

Here U is a neighbourhood of $\sigma_i^2(\Delta^2)$ and $\epsilon > 0$ is small such that indeed $\theta_t^T \circ \theta_s^X(\sigma_i^2(x, y)) \in U$. Note that $\Phi(\Delta^2 \times \{0\}^2) = \sigma_i^2(\Delta^2)$. Then Φ is a diffeomorphism in a neighbourhood of $\Delta^2 \times \{0\}^2$ and ∂_t is $\Phi-$related to T.

Proof. The proof is very similar to the proof of Lemma 2.14. □

For the 1-simplex case we need an extra tool. Let Δ^1 be a 1-simplex in M. Let S be a 2-dimensional submanifold of M which contains Δ^1 and is transverse to ξ. Let $Z \in \mathfrak{X}(S)$ be a vector field in S such that S is spanned by Z and the 1-simplex. In this way we can create the four directions of our box.

Lemma 3.12. *Let $p \in M$ such that $p \in \sigma_i^1(\Delta^1)$ for some $i \in I_1$. Let $\theta_r^Z : M \to M$, $\theta_s^X : M \to M$ and $\theta_t^T : M \to M$ be the flows of Z, X and T respectively, with $r, s, t \in \mathbb{R}$. Let Φ be defined as follows:*

$$\Phi : \Delta^1 \times [-\epsilon, \epsilon]^3 \to U, (x, r, s, t) \mapsto \theta_t^T \circ \theta_s^X \circ \theta_r^Z(\sigma_i^1(x)).$$

Here U is a neighbourhood of $\sigma_i^1(\Delta^1)$ and $\epsilon > 0$ is small such that indeed $\theta_t^T \circ \theta_s^X \circ \theta_r^Z(\sigma_i^1(x)) \in U$. Note that $\Phi(\Delta^1) = \sigma_i^1(\Delta^1)$. Then Φ is a diffeomorphism in a neighbourhood of $\Delta^1 \times \{0\}^3$ and ∂_t is $\Phi-$related to T.

Proof. The proof is very similar to the proof of Lemma 2.14. □

Lastly, for a 0-simplex Δ^0 we can apply the flowbox theorem. Recall that this states that around a point in your manifold we can find local coordinates such that one of the coordinate directions corresponds to a given vector field. So in our case we can find local coordinates (x, y, z, t) such that ∂_t corresponds to T.

3.5 Formal immersions

Let ξ be a formal Engel structure with $W \subset \xi \subset \mathcal{E} \subset TM$ and the corresponding isomorphisms as in the definition. Since ξ is a smooth distribution, it is locally spanned by vector fields. Let T be a vector field tangent ξ and locally transverse to W. Then locally $\xi = \langle T, X\rangle$ for some vector field $X \in \mathfrak{X}(M)$.

Remark. There is a reason we choose a vector field $T \in \xi$ transverse to W. This is because W will play the role of the characteristic line field when we deform the formal Engel structure ξ to a genuine Engel structure. Choosing T this way, means that we can apply the Lie bracket twice such that it actually yields a new vector field. So, if we want to respect the formal data, i.e. the isomorphisms $\wedge^2\xi \simeq \mathcal{E}/\xi$ and $\xi/W \wedge \mathcal{E}/\xi \simeq TM/\mathcal{E}$, we choose T transverse to W. ♦

In the previous section we have shown how to build boxes around the simplices of the 3-skeleton such that ξ is given by $\langle \partial_t, X\rangle$. So the vector field T corresponds to ∂_t. In these boxes we can look at the following maps:

$$X_{(x,y,z)} : \mathbb{R} \to \mathbb{S}^2$$

as in Lemma 3.7. As expressed in the proof of Lemma 3.7, we want these maps to be immersions if we want ξ to become Engel. In this section we will show that these maps are actually homotopic to immersions. In order to achieve this we look at the following definition.

Definition 3.13. *Let M and N be smooth manifolds. A **formal immersion** of M into N consists of a smooth function $f \in C^\infty(M,N)$ and a family of linear monomorphisms $F_p : T_pM \to T_{f(p)}M$ which vary smoothly with respect to $p \in M$. So we have the following commutative diagram:*

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ TM \arrow[u, "\pi_M"] \arrow[r, "F"] & TN \arrow[u, "\pi_N"'] \end{tikzcd}$$

We often write (f,F) for a formal immersion. ([11, p. 1])

We denote $Imm^f(M,N)$ for the space of all formal immersion of M into N and we endow $Imm^f(M,N)$ with the C^0-topology (see Appendix B). We then have the following useful result.

Theorem 3.14 (Smale-Hirsch). *Let M and N be smooth manifolds with dimensions m and n respectively. If $m < n$ then the inclusion*

$$Imm(M,N) \to Imm^f(M,N)$$

is a weak homotopy equivalence.

For the proof see [16].

Switching back to our formal Engel structure, we have the following result. The proof of the lemma below is based on discussions with my supervisor Álvaro del Pino.

Proposition 3.15. *Each of the maps $X_{(x,y,z)}$ for each $(x,y,z) \in \mathbb{R}^3$ as defined above, form a formal immersion from $\mathbb{R}$ into $\mathbb{S}^2$.*

Proof. Since ξ is a formal Engel structure we have the following isomorphism: $\wedge^2\xi \simeq \mathcal{E}/\xi$. Recall $\xi = \langle T, X\rangle$. We can now find a vector field $Y \in \mathfrak{X}(M)$ such that $\mathcal{E} = \langle T, X, Y\rangle$ and Y is orthogonal to X.

We define the following maps for $s \in \mathbb{R}$:

$$F_s : T_s\mathbb{R} \to T_{X_{(x,y,z)}(s)}\mathbb{S}^2, \partial_t|_s = Y_{X_{(x,y,z)}(s)}.$$

The fact that X and Y are orthogonal ensures that indeed $Y_{X_{(x,y,z)}(s)} \in T_{X_{(x,y,z)}(s)}\mathbb{S}^2$. This can be easily seen in figure 3.1. As $X_{(x,y,z)}$ is here a position function on the sphere, and Y is orthogonal to X, Y must be tangent to the sphere.

We then obtain the following commutative diagram:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "X_{(x,y,z)}"] & \mathbb{S}^2 \\ T\mathbb{R} \arrow[u, "\pi_\mathbb{R}"] \arrow[r, "F"] & T\mathbb{S}^2 \arrow[u, "\pi_{\mathbb{S}^2}"'] \end{tikzcd}$$

Where $\pi_\mathbb{R}(\partial_t|_s) = s$ and $\pi_{\mathbb{S}^2}(Y_{X_{(x,y,z)}(s)}) = X_{(x,y,z)}(s)$. In this way we obtain a formal immersion $(X_{(x,y,z)}, F)$. □

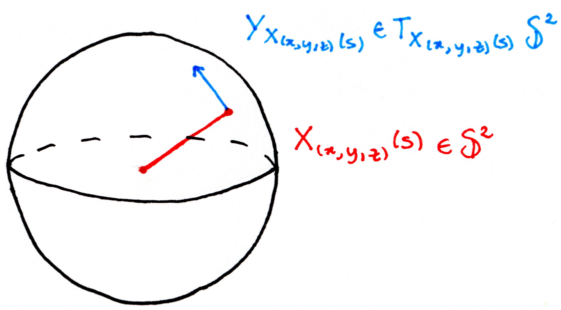


Figure 3.1: Illustration of the map $X_{(x,y,z)} : \mathbb{R} \to \mathbb{S}^2$ and the vector field Y.

Since $(X_{(x,y,z)}, Y_{X_{(x,y,z)}})$ is a formal immersion and the dimension of $\mathbb{R}$ is less than the dimension of $\mathbb{S}^2$, by Smale-Hirsch we know that $(X_{(x,y,z)}, Y_{X_{(x,y,z)}})$ is homotopic to a genuine immersion $(\tilde{X}_{(x,y,z)}, \tilde{X}'_{(x,y,z)})$. These functions $\tilde{X}_{(x,y,z)}$ provide us with a vector field $\tilde{X}$ on $\mathbb{R}^4$ such that $[\partial_t, \tilde{X}]$ corresponds to the vector field which comes from the derivatives $\tilde{X}'_{(x,y,z)}$.

3.6 Proof of h-principle

In this section we will give a proof of the h-principle for Engel structures. Again, we have now obtained all the tools in order to prove Theorem 3.9. The proof below is based on several discussions with my supervisor Álvaro del Pino, but the statement follows immediately from the paper *Stable mappings of foliations into manifolds* by Mikhael Gromov (1969), see [14]. Let us state the theorem again for completeness:

Theorem 3.9. *Let M be an open 4-dimensional manifold. Then every formal Engel structure is homotopic to a genuine Engel structure, i.e. the map $\pi_0(Engel(M)) \to \pi_0(Engel^f(M))$ is a surjection.*

Proof. Let ξ be a formal Engel structure. Let W be the characteristic line field, and let $T \in \xi$ be a vector field locally transverse to W. Let $\xi = \langle T, X \rangle$ for a vector field $X \in \mathfrak{X}(M)$.

By Lemma D.2 we know there exists a triangulation of M such that ξ is transverse to all the simplices of this triangulation. In section 3.4 we saw how to build boxes around each simplex in the 3-skeleton such that ξ is given by $\langle \partial_t, X \rangle$. This allows us to use Lemma 3.7.

Because ξ is a formal Engel structure, we have the isomorphism $\wedge^2 \xi \simeq \mathcal{E}/\xi$. Let $Y \in \mathfrak{X}(M)$ such that $\langle Y \rangle = \mathcal{E}/\xi$. Using the coordinates (x, y, z, t) from the boxes, we can look at the maps $X_{(x,y,z)} : \mathbb{R} \to \mathbb{S}^2$ just like in Lemma 3.7. We have shown in section 3.5 that the functions $X_{(x,y,z)}$ are formal immersions with formal derivatives induced by the vector field Y (Proposition 3.15). By Smale-Hirsch we then know that they are homotopic to genuine immersions $\tilde{X}_{(x,y,z)}$. These functions give us a vector field $\tilde{X}$ on $\mathbb{R}^4$ which is homotopic to X. Therefore $\mathcal{E} = \langle T, X, Y \rangle$ is also homotopic to $\tilde{\mathcal{E}} = \langle T, \tilde{X}, \tilde{X}' \rangle$, where $\tilde{X}' = [\partial_t, \tilde{X}]$, such that $\tilde{\mathcal{E}}$ is a rank-3 distribution.

We note that we have locally, using the coordinates of the boxes, transformed families of formal immersions into genuine immersions. Just like in the contact case (see the proof of Theorem 2.11) we actually need to be a bit careful when making these adjustments. We want to work inductively on the dimensions of the simplices, and we need to pay attention to overlapping boxes. In this way we do not destroy any immersions we have already created, when applying Smale-Hirsch in a different box.

Let us now focus on a box around a 0-simplex Δ^0 with coordinates (x, y, z, t) such that $\Delta^0 \subset \{t = 0\}$. We want to show that the functions $\tilde{X}_{(x,y,z)}$ are homotopic to functions $\overline{X}_{(x,y,z)}$ which are convex functions around the 0-simplex. By Lemma 3.7, the distribution $\overline{\xi} = \langle T, \overline{X} \rangle$ is then Engel around the vertex. Like in the contact case, we are going to deform the functions $\tilde{X}_{(x,y,z)}$ for them to become convex, and we shall see that the method is very similar. See figure 3.2 for an illustration of this.

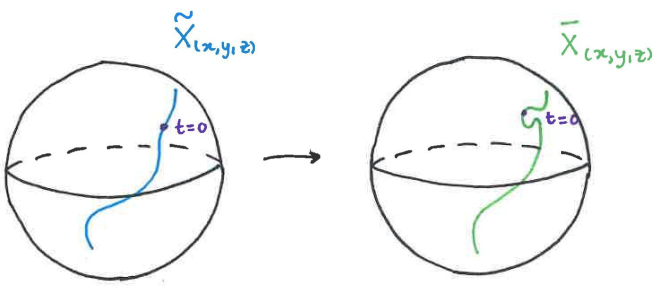


Figure 3.2: Introducting convexity of $X_{(x,y,z)}$ around $t = 0$.

For the function $\overline{X}_{(x,y,z)}$ to be convex, means that the position function $\overline{X}_{(x,y,z)}$, the velocity $\overline{X}'_{(x,y,z)}$ and the acceleration $\overline{X}''_{(x,y,z)}$ are linearly independent. The isomorphisms $\wedge^2 \xi \simeq \mathcal{E}/\xi$ and $\xi/W \wedge \mathcal{E}/\xi \simeq TM/\mathcal{E}$ provide us with an oriented basis of the tangent bundle TM, namely $\{W, \xi/W, \mathcal{E}/\xi, TM/\mathcal{E}\}$, and we want $\{T, \overline{X}, \overline{X}', \overline{X}''\}$ to have the same orientation on TM. This formal data therefore tells us how we should adjust the functions to obtain convexity.

We start with defining a cut-off function on the box around Δ^0. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a function such that $\chi|_{Op(\Delta^0)} = 1$ and $\chi = 0$ away from a small neighbourhood around Δ^0. See figure 2.4 for an illustration of this. We then choose a function $g : \mathbb{R} \to \mathbb{S}^2$ such that $g(0)$ and its velocity at 0 agree with $\tilde{X}_{(x,y,z)}$, and its acceleration is large. We then define the function

$$\overline{X}_{(x,y,z)} = \tilde{X}_{(x,y,z)} + \chi g : \mathbb{R} \to \mathbb{S}^2.$$

The functions $\tilde{X}_{(x,y,z)}$ and $\overline{X}_{(x,y,z)}$ are homotopic to each other, by the simple homotopy $Y_s = (1-s)\tilde{X}_{(x,y,z)} + s\overline{X}_{(x,y,z)} = \tilde{X}_{(x,y,z)} + s\chi g$ for $s \in [0,1]$.

Since the neighbourhood $Op(\Delta^0)$ is compact, we can look at the following minimum:

$$\min_{Op(\Delta^0)} \det(\tilde{X}_{(x,y,z)} | \tilde{X}'_{(x,y,z)} | \tilde{X}''_{(x,y,z)}).$$

The determinant tells us how convex the function $\tilde{X}_{(x,y,z)}$ is, and thus this minimum tells us how we should choose the function g such that $\overline{X}_{(x,y,z)} = \tilde{X}_{(x,y,z)} + \chi g$ is indeed convex on the whole neighbourhood $Op(\Delta^0)$. We carry out this process for every 0-simplex in the triangulation.

By Lemma 3.7, we have now altered the formal distribution ξ to a distribution which is Engel on the 0-skeleton. We now look at a 1-simplex Δ^1, and a box B around it, as in figure 2.6. We choose these boxes such that the different boxes around the 1-simplices do not overlap. We choose two boxes around the vertices of Δ^1 in which the distribution is already Engel, and we name these B_1 and B_2. As you can see in figure 2.6 these boxes do overlap.

We then proceed as before. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut-off function with $\chi|_{Op(\Delta^1) \subset B} = 1$ and $\chi = 0$ away from a small neighbourhood of the simplex. We then choose a function $g : \mathbb{R} \to \mathbb{S}^2$ such that $g(0)$ and the velocity at 0 agree with $\tilde{X}_{(x,y,z)}$, and the acceleration is very large. Again the minimum

$$\min_{Op(\Delta^1)} \det(\tilde{X}_{(x,y,z)} | \tilde{X}'_{(x,y,z)} | \tilde{X}''_{(x,y,z)}),$$

determines how large the acceleration of g must be. We then define the functions

$$\overline{X}_{(x,y,z)} = \tilde{X}_{(x,y,z)} + \chi g.$$

Again these functions $\tilde{X}_{(x,y,z)}$ and $\overline{X}_{(x,y,z)}$ are homotopic to each other, by the homotopy $Y_s = \tilde{X}_{(x,y,z)} + s\chi g$ for $s \in [0,1]$.

In this way we have created convex functions in the neighbourhood $Op(\Delta^1)$. However, since χ is 0 away from this neighbourhood, we must be careful what happens in the areas where χ goes from 1 to 0. As you can also see in figure 3.2, by introducing convexity around the simplex, we also get some very concave parts away from $Op(\Delta^1)$. We want to minimize these concave parts such that we do not destroy the convexity we have already introduced in B_1 and B_2. Since B_1 and B_2 are compact spaces, we can look at the minimum

$$\min_{B_1 \cup B_2} \det(\tilde{X}_{(x,y,z)} | \tilde{X}'_{(x,y,z)} | \tilde{X}''_{(x,y,z)}).$$

This minimum tell us how we should choose the function g such that the functions in B_1 and B_2 remain convex. We can do this for all the 1-simplices, and in this way we obtain a distribution which is Engel around the 1-skeleton.

We can then carry out the same process for the 2- and 3-simplices, by adding very convex functions around the simplices and paying attention to the overlapping parts with boxes around simplices of lower dimension. Working in this inductive way on the simplices, homotoping the functions $\tilde{X}_{(x,y,z)}$ to homotopic (convex) functions $\overline{X}_{(x,y,z)}$, we have been homotoping the distribution $\tilde{\xi} = \langle T, \tilde{X} \rangle$ to a distribution $\overline{\xi} = \langle T, \overline{X} \rangle$. This distribution $\overline{\xi}$ is Engel around the 3-skeleton of M by Lemma 3.7, and is homotopic to ξ. We now proceed again very similar as in the contact case.

Just as in the proof of Theorem 2.11 we can use the diffeomorphism found in Appendix E and the deformation retraction illustrated in figure 2.8 to find a distribution which is homotopic to $\overline{\xi}$ and is Engel on the whole of our manifold M. Namely the distribution $\phi_1^* \psi_1^* \overline{\xi}$ (see the proof of Theorem 2.11 for a more detailed explanation). We already know that $\overline{\xi}$ is homotopic to ξ, and thus ξ is homotopic to $\phi_1^* \psi_1^* \overline{\xi}$. This completes the proof.

□

Appendix A

Bundles and Bundle morphisms

Distributions are the main focus of this thesis. As explained in Chapter 1, these are subbundles of the tangent bundle of a smooth manifold. In this appendix we discuss several concepts like bundles, bundle morphisms and subbundles, which can shed more light on structures like distributions. The material in this appendix is retrieved from chapter 10 of [17].

A.1 Bundles

Definition A.1. *A **bundle** is a space E together with a topological space B and a surjective continuous map $\pi : E \to B$ such that:*

(1) For each $p \in B$ the fiber $E_p := \pi^{-1}(p)$ has the structure of a k-dimensional real vector space,

(2) For every point $p \in B$, there is an open neighbourhood $U \subset B$ of p and a homeomorphism

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

- *$\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \to U$ is the natural projection,*
- *For each $q \in U$, the restriction $\Phi|_{E_q}$ is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k$.*

*The space E is called the **total space**, B is called the **base space** and the map π the **projection**. The open neighbourhood U with the map Φ is called a **local trivialization** of the vector bundle.*

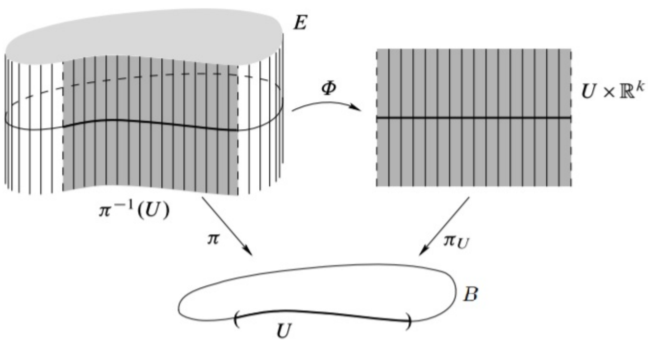


Figure A.1: Local trivialization of a vector bundle ([17, p. 250]).

Definition A.2. *A bundle E which is 1-dimensional, is also called a **line bundle**.*

Definition A.3. *A bundle E with base space B and projection $\pi : E \to B$ is called a **smooth bundle** if E and B are smooth manifolds, $\pi : E \to B$ is smooth and the local trivializations are diffeomorphisms.*

Example A.4. Let M be a smooth manifold. As the name probably already gives away, the tangent bundle TM is a bundle. The tangent bundle is defined as the disjoint union of the tangent spaces of each point in the manifold,

$$TM = \coprod_{p \in M} T_p M.$$

We write v_p for a tangent vector in T_pM, i.e. p is the point at which v_p is tangent to the manifold. We then have the following natural projection

$$\pi : TM \to M, v_p \mapsto p.$$

Now let $p \in M$ and let (U, φ) be any coordinate chart such that $p \in U$. Let (x^i) be the corresponding coordinate functions. We then define the following homeomorphism

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n, v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto (p, (v^1, ..., v^n)).$$

Since $(\frac{\partial}{\partial x^i}\big|_p)$ form a basis of T_pM, the fiber T_pM is endowed with the structure of a n-dimensional real vector space. We note that for $v \in T_pM$ we have $\pi_U \circ \Phi(v) = \pi_U(p, (v^1, ..., v^n)) = p$ and $\pi(v) = p$ so indeed $\pi_U \circ \Phi = \pi$. We also note that $\Phi\big|_{T_qM} : T_qM \to \mathbb{R}^n$ which sends every vector $v^i \frac{\partial}{\partial x^i}\big|_q$ to $(q, (v^1, ..., v^n))$, is a vector space isomorphism between T_qM and $\{q\} \times \mathbb{R}^n$.

In this example TM is the total space, M is the base space, π is the projection and the functions Φ are the local trivializations. In fact, since M and TM are smooth manifolds, the projection π is smooth and the local trivializations are diffeomorphisms, TM is a smooth bundle. △

A.2 Bundle morphisms

Definition A.5. *Let E and F be bundles with base spaces M and N, respectively, and projections $\pi_E : E \to M$ and $\pi_F : F \to N$. A **bundle morphism** is a pair of continuous functions $\varphi : E \to F$ and $f : M \to N$ such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi_E"] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

*i.e. $\pi_F \circ \varphi = f \circ \pi_E$, and the restriction $F|_{E_q} : E_q \to F_{f(q)}$ is a linear map. We also say that φ **covers** f in this case.*

Example A.6. Let M and N be smooth manifolds and $F : M \to N$ a smooth function. The differential of F at a point $p \in M$ is defined as follows:

$$dF_p : T_pM \to T_{F(p)}N, v \mapsto dF_p(v),$$

such that $dF_p(v)(f) = v(f \circ F)$ for each $f \in C^\infty(N)$. Here $dF(v) := dF_p(v)$ for $v \in T_pM$. We are going to show that dF and F form a bundle morphism. Indeed, the following diagram commutes.

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d, "\pi_M"] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

Let $v \in T_pM$ for some $p \in M$. Then $dF_p(v) \in T_{F(p)}N$ by definition, and thus $\pi_N \circ dF(v) = F(p)$. Also, $\pi_M(v) = p$, and thus $F \circ \pi_M(v) = F(p)$, which means that $\pi_N \circ dF = F \circ \pi_M$.

Now let $p \in M$, and look at the restriction $dF|_{T_pM} = dF_p : T_pM \to T_{F(p)}N$. This restriction is linear as the differential is a linear map. We conclude that indeed dF covers F. △

A.3 Subbundles

Definition A.7. *Let E be a bundle with base space B and projection $\pi_E : E \to B$. A topological subspace D of E is a* ***subbundle*** *of E if D is a bundle with base space B and a projection $\pi_D : D \to B$ such that the following hold*

- *π_D is the restriction of π_E to D such that for each $p \in B$ the space $D_p = D \cap E_p$ is a linear subspace of E_p,*
- *the vector space structure of D_p is the one inherited from E_p.*

Definition A.8. *Let E be a bundle with base space B and projection $\pi_E : E \to B$. A subbundle D of E is a* ***smooth subbundle*** *if D is a smooth bundle and an embedded submanifold of E.*

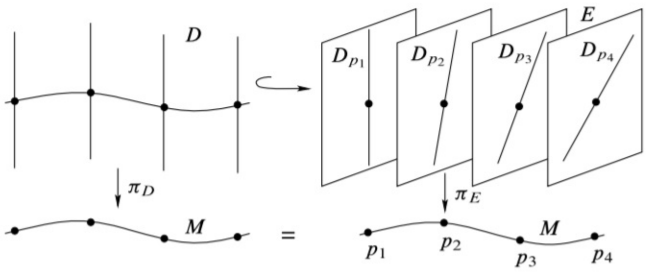


Figure A.2: Subbundle D of a bundle E ([17, p. 265]).

Example A.9. Let M be a smooth manifold and let X be a smooth vector field on M. We have seen that TM is a smooth bundle with base space M and the projection $\pi : TM \to M, v_p \mapsto p$. For every $p \in M$ we have that $X_p \in T_pM$. We then look at the subspace D of TM with fibers defined by $D_p := \langle X_p \rangle$. Then

$$D := \coprod_{p \in M} \langle X_p \rangle.$$

We claim that D is a subbundle of TM.

First of all, together with the projection

$$\pi_D : D \to M, \lambda X_p \mapsto p,$$

the fibers $\pi_D^{-1}(q) = \langle X_q \rangle$ are endowed with a 1-dimensional vector space structure. Also, with the same local trivializations as in Example A.4, D is indeed a bundle.

Now to show that D is a subbundle of TM, we look at the fibers D_p. We see that π_D is the restriction of $\pi : TM \to M$ and for all $p \in M$ we have that $D_p = \langle X_p \rangle = D \cap T_pM$ is a linear subspace of T_pM. Also, the vector space structure of D_p is clearly the one inherited from T_pM. We conclude that D is a subbundle of TM. △

Example A.10. Let $D \subset E$ be a subbundle of a bundle E with base space B. Then the quotient E/D also forms a bundle. We define it as follows:

$$E/D := \coprod_{p \in B} E_p/D_p,$$

together with the projection

$$\pi : E/D \to B, E_p/D_p \ni y \mapsto p.$$

We call the bundle E/D the **quotient bundle** of E and D. △

Appendix B

Jet Bundles & Whitney C^k-Topologies

One of the first things we come across in this thesis, is the space of all distribution on a smooth manifold M, which we denote by $Dist(M)$. We endowed this space with the C^0-topology. Similarly we also endowed the spaces $Cont(M)$, $Cont^f(M)$, $Engel(M)$, $Engel^f(M)$, $Imm(M,N)$ and $Imm^f(M,N)$ with a C^k-topology. In this appendix we explain what this topology precisely entails. To do this, we first need to look at objects called *jet bundles*. This appendix is based on pages 37-42 from the book *Stable Mapping and Their Singularities* by M. Golubitsky and V. Guillemin ([12]).

B.1 Jet Bundles

In this section we are going to look at jet bundles. In order to define these structures we first look at an equivalence relation defined on certain smooth maps between manifolds.

Definition B.1. *Let M and N be smooth manifolds, $p \in M$ and $f, g : M \to N$ smooth maps with $f(p) = g(p) = q \in N$. Then*

- *f has **first order contact** with g if $(df)_p = (dg)_p$ where $(df)_p, (dg)_p : T_pM \to T_qN$,*
- *f has k^{th} **order contact** with g is $(df) : TM \to TN$ has $(k-1)^{th}$ order contact with (dg) at every point in T_pM. We denote this by $f \sim_k g$ at p for $k \in \mathbb{Z}_{>0}$.*

We also use the notation $f \sim_0 g$ at p if and only if $f(p) = g(p)$. ([12, p. 37])

Example B.2. Let $p \in \mathbb{R}$ a constant and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2 + (p-x)^2,$$

and let $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = x^2 + (p-x)^3.$$

Then $f(p) = g(p) = p^2$, and $f'(p) = g'(p) = 2p$. However, $f''(p) = 4$ and $g''(p) = 2$. This means that $f \sim_1 g$ at p but $f \not\sim_2 g$ at p. △

This example suggest a result which is stated in the following lemma.

Lemma B.3. *Let U in $\mathbb{R}^n$ be open, $p \in U$ and $f, g : U \to \mathbb{R}^m$ be smooth functions. Let $k \in \mathbb{Z}_{\geq 0}$. Then $f \sim_k g$ at p if and only if*

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

for every multi-index with $|\alpha| \leq k$ and $1 \leq i \leq m$. ([12, p. 38])

The proof of this result can be found in [12, p. 38]. A corollary of this lemma is that f and g as in the lemma above have k^{th} order contact at p, if and only if their Taylor expansions coincide at p up to and including order k. We will now show that the relation $\sim_k$ defines an equivalence relation.

Lemma B.4. *This $\sim_k$ at p defines an equivalence relation on smooth functions $f, g : M \to N$ with $f(p) = g(p) = q \in N$.*

Proof. Let $f, g, h : M \to N$ be smooth functions with $f(p) = g(p) = h(p) = q$. One can clearly see that $f \sim_k f$ because when functions are equal, all their derivatives coincide. Furthermore, if $f \sim_k g$ then automatically it also holds that $g \sim_k f$, as this relation has to do with the equality of derivatives, which is symmetric. Lastly, if $f \sim_k g$ and $g \sim_k h$ then also $f \sim_k h$. Namely, since $f \sim_k g$ and $g \sim_k h$ this implies that $f \sim_i g$ and $g \sim_i h$ for all $1 \leq i \leq k$, which ensures that $f \sim_k h$. □

Having discussed this equivalence relation $\sim_k$, we can now define the main objects of this section.

Definition B.5. *Let M and N be smooth manifolds, $p \in M$, $f : M \to N$ a smooth map with $f(p) = q \in N$ and Let $k \in \mathbb{Z}_{\geq 0}$.*

- *Let $J^k(M,N)_{p,q}$ denote the set of equivalence classes under "$\sim_k$ at p" of smooth maps $f : M \to N$ with $f(p) = q$,*
- *Let $J^k(M,N) := \sqcup_{(p,q) \in M \times N} J^k(M,N)_{p,q}$ be the disjoint union. We call $J^k(M,N)$ the* ***jet bundle from** M **to** N. An element σ in $J^k(M,N)$ is called a k-**jet from** M **to** N.*

([12, p. 37])

Let M and N be smooth manifolds of dimension m and n respectively. The jet bundle $J^k(M,N)$ is a finite-dimensional manifold, and its fibres can be identified with Taylor polynomials from $\mathbb{R}^m$ to $\mathbb{R}^n$ of degree k (as in Lemma B.3). Every smooth manifold admits a metric, and thus a topology. This is why we can talk about opens in $J^k(M,N)$.

Lastly, we define the notion of a *k-jet* of a smooth map.

Definition B.6. *Let M and N be smooth manifolds and $f : M \to N$ a smooth map. Let $k \in \mathbb{Z}_{\geq 0}$. Then we can define the map $j^k f : M \to J^k(M,N)$ for $p \in M$ by*

$$j^k f(p) := \text{ the equivalence class of } f \text{ in } J^k(M,N)_{p,f(p)}.$$

*We call $j^k f$ the k-**jet of** f. ([12, p. 37])*

B.2 Whitney C^k-Topologies

Having discussed jet bundles in the previous section, we can now define Whitney C^k-topologies. As said before, several spaces of distributions which we come across in this thesis are endowed with a Whitney C^k-topology, such that we can talk about open and closed neighbourhoods in these spaces. In this section we define these topologies, and see how they are related to a couple of these spaces.

Let M and N smooth manifolds and let $C^\infty(M,N)$ denote the space of smooth functions from M to N.

Definition B.7. *Let $k \in \mathbb{Z}_{\geq 0}$. Let U be an open subset of the jet bundle $J^k(M,N)$. Then we define*

$$M(U) := \{f \in C^\infty(M,N) : j^k f(M) \subset U\}.$$

*The family of set $\{M(U)\}$ where U is an open in $J^k(M,N)$ form a basis for a topology on the space of smooth functions $C^\infty(M,N)$, called the **Whitney C^k-topology**. ([12, p. 42])*

We note that we can also use C^k functions, i.e. functions which are k times differentiable, instead of smooth functions. All the definitions and lemmas up until now will still hold and make sense.

Example B.8. Let us now look at the space of immersions from a smooth manifold M to a smooth manifold N, i.e. the space $Imm(M,N)$. An immersion is a smooth function, and thus $Imm(M,N) \subset C^\infty(M,N)$. Therefore we can easily endow $Imm(M,N)$ with the subspace topology of the Whitney C^1-topology.

We choose the C^1-topology here, because when working with immersions we are interested in the first derivative. Whereas when working with formal immersions, we do not look at derivatives, but at the formal data which plays the role of the derivatives. We therefore endow the set $Imm^f(M,N)$ with the C^0-topology. △

We note that the other spaces we endowed with the Whitney topologies, such as $Dist(M)$ and $Cont(M)$, are not spaces of smooth functions. Therefore, we need to 'redefine' certain concepts that we have introduced. First, let us look at the following definition.

Definition B.9. *Let M be a smooth manifold. Let E be a finite-dimensional smooth bundle over M with the projection $\pi : E \to M$. A* ***section of*** *E is a smooth function $s : M \to E$ such that $s \circ \pi = id$. We denote the space of smooth sections of E by $C^\infty(E)$. ([17, p. 255])*

For example, vector fields are sections of the tangent bundle.

Definition B.10. *Let M be a smooth manifold. Let E be a finite-dimensional, smooth bundle over M with the projection $\pi : E \to M$. Let $k \in \mathbb{Z}_{\geq 0}$ Let $J^k(E)$ be the* ***k-th order jet bundle over*** *E, by which we mean the regular jet bundle as in Definition B.5 but now defined on the space of smooth sections $C^\infty(E)$.*

We note that indeed Definition B.1, Lemma B.4, Definition B.5, Definition B.6 and still hold or make sense, which justifies the previous definition. We then also redefine the Whitney topologies.

Definition B.11. *Let $k \in \mathbb{Z}_{\geq 0}$. Let U be an open subset of the jet bundle $J^k(E)$. Then we define*

$$M(U) := \{s \in C^\infty(E) : j^k s(M) \subset U\}.$$

The family of sets $\{M(U)\}$ where U is an open in $J^k(E)$ form a basis for a topology on the space of smooth sections $C^\infty(E)$, called the ***Whitney C^k-topology****.*

Also for this definition we can replace $C^\infty(E)$ by $C^k(E)$, so sections which are k times differentiable, and the definition still makes sense.

In this way we can also define the Whitney topologies on spaces of sections. We note that the notations and terms of the two versions are very similar, but often we can retrieve from the context which of the two we are considering.

Example B.12. Let us look at the space of all smooth distributions of rank k on a smooth manifold M, i.e. $Dist(M, k)$. In fact, this is a space of sections of a bundle called the *Grassmannian bundle.* These objects are defined in Appendix D, but we take a quick look at them now to show that we can endow $Dist(M, k)$ with a Whitney topology.

The space $Gr(T_pM, k)$ is the space of all k-dimensional subspaces of T_pM. Let $\xi \in Dist(M, k)$. Since ξ is a rank-k distribution on M, we know that $\xi = \coprod_{p \in M} \xi_p$ where ξ_p is a k-dimensional linear subspace of T_pM. This means we can view ξ as the map

$$\xi : M \to Gr(TM, k), p \mapsto \xi_p \in Gr(T_pM, k),$$

where $Gr(TM, k) = \coprod_{p \in M} Gr(T_pM, k)$ is the Grassmannian bundle with the projection $\pi : Gr(TM, k) \to M$ such that $\pi^{-1}(p) = Gr(T_pM, k)$ for each $p \in M$ (see Example D.5). Since $\xi \circ \pi = id$, we conclude that ξ is a section of $Gr(TM, k)$. This means that $Dist(M, k)$ is a space of sections of $Gr(TM, k)$, and thus we can endow it with the Whitney C^0-topology. △

Example B.13. We now look at the space of all Engel structures on a smooth manifold M, i.e. $Engel(M)$. Let $\xi \in Engel(M)$ then ξ is a rank-2 distribution, and thus each ξ_p is an element of $Gr(T_pM, 2)$. We can define the map

$$\xi : M \to Gr(TM, 2), p \mapsto \xi_p \in Gr(T_pM, 2),$$

which is a section of the bundle $Gr(TM, 2)$. So $Engel(M)$ is also a space of sections, and we can endow it with the C^2-topology. Here we use the C^2-topology because the first and second 'derivative' (or curvature) of an Engel structure are important as they define the object. △

In a similar way, we can show that we can endow the Whitney topologies on the spaces $Cont(M)$, $Cont^f(M)$ and $Engel^f(M)$. We endow these spaces with the C^1-, C^0- and C^0-topologies, respectively. These choices of topologies make the inclusions $Imm(M, N) \hookrightarrow Imm^f(M, N)$, $Cont(M) \hookrightarrow Cont^f(M)$ and $Engel(M) \hookrightarrow Engel^f(M)$ continuous.

Appendix C

CW- and Simplicial complexes

In this appendix we will be discussing CW- and simplicial complexes. These are structures in algebraic topology which are defined inductively. CW-complexes were first introduced by J.H.C. Whitehead ([24]), a British mathematician who was one of the founders of homotopy theory. Simplicial complexes are a special type of CW-complexes and in certain cases these objects have some very useful properties. We will define these structures, cover their relations to manifolds and discuss some important properties.

C.1 CW-complexes

Definition C.1. *A* ***CW-complex of dimension 0*** *is a set X endowed with the discrete topology. Each $x \in X$ is a 0-cell, or vertex. A* ***CW-complex of dimension*** *k is a topological space X given by*

$$X := Y \cup_{\partial \mathbb{D}_i^k \sim \varphi_i(\partial \mathbb{D}_i^k)} (\sqcup_{i \in I} \mathbb{D}_i^k) = (Y \cup (\sqcup_{i \in I} \mathbb{D}_i^k))/(\partial \mathbb{D}_i^k \sim \varphi_i(\partial \mathbb{D}_i^k))$$

where

- *Y is a CW-complex of dimension $k-1$,*
- *The $\mathbb{D}_i^k$ are copies of the k-dimensional closed disc indexed by a set I (the k-**cells**),*
- *The map $\varphi_i : \partial \mathbb{D}_i^k \to Y$ attaches k-cells to Y by identifying $\partial \mathbb{D}_i^k$ with $\varphi_i(\partial \mathbb{D}_i^k)$. This means that X is endowed with the quotient topology.*
- *The map $\varphi_i : \mathbb{D}_i^k \to X$ also uniquely defines a continuous map $\mathring{\varphi}_i : \mathbb{D}_i^k \to Y$ which is a homeomorphism between the interior $\mathring{\mathbb{D}}_i^k$ and its image $\mathring{\varphi}_i(\mathring{\mathbb{D}}_i^k)$. We call the map $\varphi_i : \mathbb{D}_i^k \to X$ the* ***characteristic map*** *of the k-cell $\mathbb{D}_i^k$.*

*We call the subset $X^n \subseteq X$ consisting of all the cells of dimension at most n the n-**skeleton** for $n \leq k$. In particular $X^k = X$ if X is a CW-complex of dimension k. A set $A \subset X$ is open if and only if $A \cap X^n$ is open for each n.*

This definition is based on the definition in [15, p. 519].

Example C.2. One can construct CW-complexes in very different ways. For example, the square model of the torus T^2 is a CW-complex. Namely, we start with one vertex p. Then we attach two 1-cells a and b to p, and lastly we attach a 2-cell C in the way indicated in figure C.1. In this way we have $X^0 = \{p\}$, $X^1 = a \cup b \cup X^0$ and $X^2 = C \cup X^1 = T^2$. △

Example C.3. The projective plane of dimension 2, $\mathbb{RP}^2$, is the space of all straight lines through the origin in $\mathbb{R}^3$. One can also view $\mathbb{RP}^2$ as the 2-sphere $\mathbb{S}^2$ where antipodal points are identified. Therefore we can also view $\mathbb{RP}^2$ as the one of the hemispheres where the antipodal points on the boundary are identified ([7, p.23]). Thus $\mathbb{RP}^2$ is homeomorphic to $\mathbb{D}^2/\sim$ where $\sim$ indicates the equivalence relation of identifying the antipodal points on the boundary of $\mathbb{D}^2$. We can construct this representation of $\mathbb{RP}^2$ as a CW-complex. We start with a vertex p and attach a 1-cell a. We then attach a 2-cell, i.e. a copy of $\mathbb{D}^2$ to a in the way indicated in figure C.2. In this way we have obtained a closed disc, and because of the way the 2-cell is attached, the antipodal points on the boundary are identified. ([15, p. 6]) △

Example C.4. The $n-$sphere $\mathbb{S}^n$ can also be easily constructed as a CW-complex with just two cells, a vertex p and a n-cell, i.e. copy of $\mathbb{D}^n$. We obtain $\mathbb{S}^n$ by identifying the boundary of $\mathbb{D}^n$ with the point p. ([15, p. 6]) $\triangle$

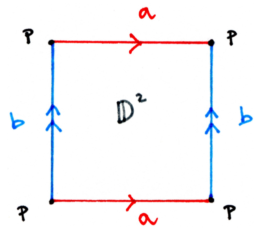


Figure C.1: The 2-torus as a CW-complex.

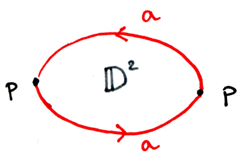


Figure C.2: The projective plane as a CW-complex.

C.2 Simplicial complexes & Triangulations

CW-complexes can sometimes be hard to work with, because the maps φ_i^k can be quite complicated. We will therefore be looking at simplicial complexes, which are build up out of (n-dimensional) triangles instead of discs.

Definition C.5. *The standard n-**simplex** Δ^n is defined by*

$$\Delta^n := \{(x_1, ..., x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } 1 \leq i \leq n\}.$$

*We call the subsets $\partial_i \Delta^n := \{(x_1, ..., x_n) \in \Delta^n : x_i = 0\}$ the **faces** of Δ^n. A simplex Δ^n has $n+1$ **vertices**, namely $v_0 = (0, ..., 0)$ and $v_i = (0, ..., 1, ..., 0)$ for $1 \leq i \leq n$. [15, p. 103]*

Intuitively, an n-simplex is the analog of an n-dimensional triangle. In figure C.3 simplices of dimension 0 up to dimension 3 are drawn.

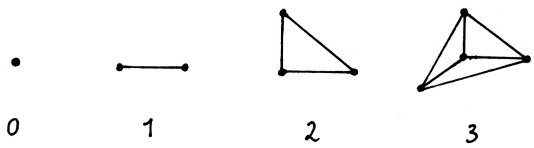


Figure C.3: Simplices of dimension 0 up to 3.

In a similar way we defined CW-complexes we can also define simplicial complexes. This definition is retrieved from lecture notes on algebraic topology, see [9].

Definition C.6. *A **simplicial complex of dimension 0** is a set X endowed with the discrete topology. Each $x \in X$ is a 0-simplex, or vertex. A **simplicial complex of dimension** k is a topological space X together with a collection of maps $\sigma_i^n : \Delta^n \to X$ with $n \leq k$ and $i \in I_n$ an index set, for which the following hold:*

(i) The map σ_i^n restricted to the open simplex, i.e. $\sigma_i^n : \mathring{\Delta}^n \to X$, is a homeomorphism, and for every $x \in X$ there exists exactly one $i \in I_n$ for an $n \leq k$ such that $x \in \sigma_i^n(\mathring{\Delta}^n)$,

(ii) The restriction of σ_i^n to a face of Δ^n, i.e. $\sigma_i^n \circ \partial_j$ is equal to the map σ_l^{n-1} for some $l \in I_{n-1}$.

(iii) The intersection $\sigma_i^n(\Delta^n) \cap \sigma_j^m(\Delta^m)$ for $i \in I_n, j \in I_m$ is either empty or the image of a unique common face of the simplex.

(iv) A set $A \subseteq X$ is open if and only if $(\sigma_i^n)^{-1}(A)$ is open in Δ^n for each $i \in I_n$ and $n \leq k$.

The space X with this structure can also be seen as a quotient space of disjoint simplices. This quotient can be obtained by identifying each face of an n-simplex to a $n-1-$simplex, which is a consequence of (ii). You can think of this process as starting with a discrete set of vertices, and then attaching 1-simplices, then 2-simplices, etcetera. We can endow X with the quotient topology as a consequence of (iv). Point (iii) ensures that when we are gluing simplices, two simplices cannot share more than 1 common face.

Definition C.7. *A **triangulation** of a manifold M is a simplicial complex X homeomorphic to M such that the following diagram commutes for every σ_i^n:*

$$\begin{tikzcd} M \arrow[r, "\simeq"] & X \\ & \Delta^n \arrow[ul, "\varphi_i^n"] \arrow[u, "\sigma_i^n"'] \end{tikzcd}$$

*and φ_i^n is smooth. We call $\varphi(\Delta^n) \subset M$ an n-**simplex of** M and the collection of all simplices of dimension at most n is called the n-**skeleton of M**.*

Remark. For simplicity, we will often just see the maps σ_i^n as maps from Δ^n to M instead of X. This is justified by the fact that M and X are homeomorphic. ♦

Example C.8. In figure C.4, on the left, a triangulation of the torus T^2 is portrayed. We see that T^2 can be build as a quotient space by starting with 18 copies of Δ^2 and attaching them in the way indicated in the figure. You can see that indeed the intersection of two simplices in the triangulation is either empty or a unique common face. For example the second construction in figure C.4 is not a triangulation of the torus as there are simplices which share more than one common face. △

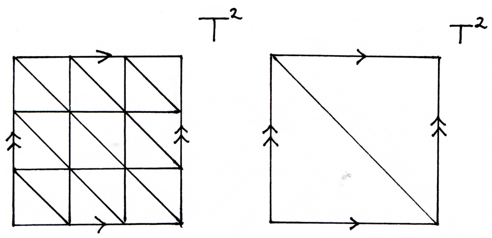


Figure C.4: An example of a triangulation of the torus (left) and a non-example of a triangulation of the torus (right).

Example C.9. We can triangulate $\mathbb{S}^2$ by noticing that $\mathbb{S}^2$ is homeomorphic to the tetrahedron. This means we can start with four 2-simplices and glue them accordingly to obtain the triangulation. See figure C.5. We note that triangulations are certainly not unique. One can find several ways to triangulate $\mathbb{S}^2$. △

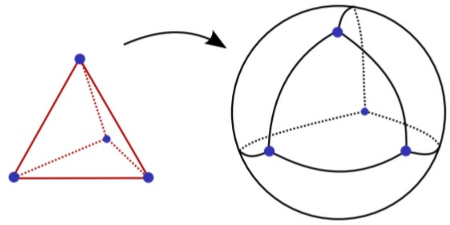


Figure C.5: Triangulation of the sphere. ([8, p. 33])

Given a smooth manifold M one can investigate whether we can find a triangulation of M. This can be useful because a triangulation provide us with local (flat) coordinates coming from the simplices.

The following result was first proven by Whitehead.

Theorem C.10 (Whitehead)**.** *Every smooth manifold is triangulable.*

A proof of this result can be found in [3].

C.3 Subdivisions of triangulations

We have seen that for every smooth manifold we can find a triangulation. However, sometimes we want to triangulate a manifold in a more finely manner. In this section we discuss two methods to subdivide a triangulation and their properties.

Let us begin with a quite straightforward way of subdividing a triangulation. Let T be a triangulation of a smooth manifold M. First we look at the 1-skeleton of M. We locate all the barycenters, i.e. the midpoints of the simplices, and subdivide the 1-simplices into two new 1-simplices. We then continue with the 2-simplices. We locate the barycenters, and subdivide every 2-simplex into six new 2-simplices in the way indicated in figure C.6. We can do this inductively for simplices of any dimension. This subdivision of the triangulation yields a new triangulation. ([15, p. 119-120])

Definition C.11. *The subdivision of a triangulation described as above is called* ***barycentric subdivision****. If K is a triangulation of a manifold M, then we denote by* **sd**K *the barycentric subdivision of K.*

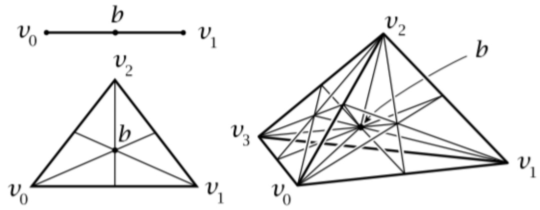


Figure C.6: Barycentric subdivision of a 1-, 2- and 3-simplex. ([15, p. 120])

Another way to refine a triangulation is the following. For each simplex, we have coordinates $(x_1, ..., x_l) \in \mathbb{R}^l$ such that $\sum_{i=1}^{l} x_i \leq 1$ and $x_i \geq 0$. We can build a box around each simplex in these coordinates, and then divide this box into smaller boxes. This also cuts the simplex into smaller objects, which we subdivide where necessary to obtain a genuine triangulation. This method is illustrated for a 2-simplex in figure C.7. ([1, p. 4])

Definition C.12. *The subdivision of a triangulation described as above is called* ***crystalline subdivision****. By the i^{th} crystalline subdivision of a simplicial complex, we mean that each edge of the simplices is subdivided i times.*

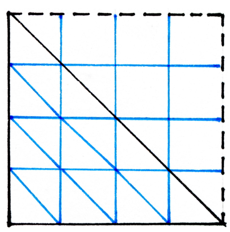


Figure C.7: The third crystalline subdivision of a 2-simplex.

Remark. Different subdivision have different properties. Let us discuss two properties in relation to barycentric and crystalline subdivision.

Firstly, we note that in both subdivisions, the measures of the simplices will become smaller and smaller as we subdivide further. Besides this, in both subdivisions (barycentric and crystalline) the simplices will not only decrease in measure, but will eventually be contained in small balls as we subdivide. This is not the case for all subdivisions, see for example figure C.8. Here the measures of the simplices will get smaller, but they won't be contained in small balls.

Secondly, when we focus on one vertex p in a triangulation, what happens to the number of edges, or simplices, touching this vertex? We note that with the subdivision illustrated in figure C.8 and with barycentric subdivision, the number of simplices touching a vertex can be become very large as we keep refining the triangulation. This is not the case for crystalline subdivisions, which can be a useful property. ♦

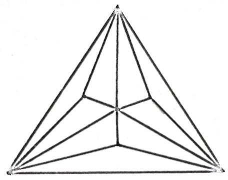


Figure C.8: Subdivision of a 2-simplex.

Crystalline subdivision will be used later on to refine triangulations of manifolds. The following lemma will be used to refer to its useful properties.

Lemma C.13. *Let K be a simplicial complex and let K^i denote the i^{th} crystalline subdivision. We then have the following two properties:*

- *For every $r > 0$ and simplex $\Delta^l \in K$ we can choose i large enough such that Δ^l is contained in the ball B_r of radius r,*
- *For every $i \leq 1$, the number of simplices touching a vertex inside the triangulation K^i will be bounded.*

C.4 Triangulations of manifolds in $\mathbb{R}^N$

Let M be an n-dimensional smooth manifold. We know that by Whitehead's Theorem there exists a triangulation of M. However, sometimes we want a triangulation of M with certain properties which allow us to nicely refine this triangulation as many times as we need, by for example crystalline subdivision. In this section we will discuss a method of triangulating a manifold M by embedding it in a Euclidean space.

Lemma C.14. *Let M be an n-dimensional smooth manifold. We can embed M into $\mathbb{R}^N$ for N large enough. Then there exists a triangulation of M such that each top-simplex comes with a parametrization of the coordinates coming from $\mathbb{R}^N$, and their restrictions to subsimplices differ only by a translation and/or rotation.*

Sketch of proof. Let N be large enough such that we can embed our manifold into $\mathbb{R}^N$, i.e. $M \hookrightarrow \mathbb{R}^N$ (see Theorem 6.19 in [17, p. 135]). Let $\nu(M)$ be the bundle of all vectors which are perpendicular to M, where $\nu(M)(p)$ are all vectors perpendicular to M at a point $p \in M$.

We now cover M by very small boxes such that the 'vertical' axes of these squares corresponds to the vectors which are perpendicular to the manifold. For example a box around a point $p \in M$ should have axes corresponding to the vectors inside $\nu(M)(p)$. Since we are taking these boxes very small, the intersection with the manifold and the box looks like a little square of dimension n. Another consequence of these boxes being very small, is that the intersection of overlapping boxes will (by maybe performing a slight tilt) lie in a level set of the axes which are perpendicular to the manifold. This is because when we look at a very small neighbourhood in our manifold, the set of vectors which are perpendicular to it, will only change very slightly.

This means that when we look at the intersections of all these boxes with the manifold, these are squares (of dimension n) which overlap, and we can further subdivide these squares to obtain a genuine triangulation. See figure C.9 for an illustration of this method of triangulating a manifold.

Because we have embedded M into $\mathbb{R}^N$ and used these Euclidean coordinates to divide M into simplices, we can switch between coordinates of one simplex to another by simply a translation and/or a rotation. □

The idea of this proof comes from the article *A simple triangulation method for smooth manifolds* by Stewart S. Cairns ([3]), but the proof above uses boxes instead of balls.

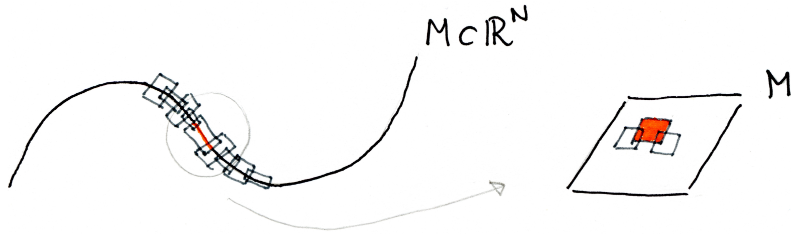


Figure C.9: Illustration for the triangulation method described in Lemma C.14.

Remark. When we take a triangulation of M and then use crystalline subdivision, it could happen that two touching simplices are subdivided differently because each simplex has its own coordinates. See figure C.10 for an illustration of this. When we use this triangulation obtained from $\mathbb{R}^N$ as described in the lemma above, the coordinates of every two simplices only differ by an affine transformation. This means that when we chop one simplex into smaller simplices by chopping an edge i times, this will lead to compatible results in other (touching) simplices. ♦

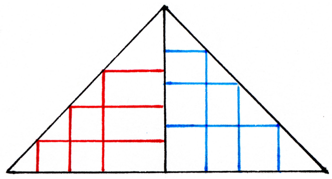


Figure C.10: Crystalline subdivision of two touching simplices.

Appendix D

The Grassmannian & Thurston's Jiggling

In section 2.4 we build boxes around the simplices of a triangulation. When building these boxes, we use a vector field tangent to a distribution for one of the coordinate directions. For us to be able to do this we need one important ingredient. Namely, we require the distribution to be *transverse* to every simplex. This is defined as follows:

Definition D.1. *Let M be a smooth manifold, L a submanifold of M and ξ a distribution on M. We say ξ is* ***transverse*** *to L if $TL \oplus \xi$ is as large as possible. If L and ξ are transverse, we write $L \pitchfork \xi$. ([10, p. 15])*

Given a manifold M and a distribution ξ, we will prove that there is a triangulation of M such that ξ is transverse to every simplex. We will do this using *Thurston's jiggling.* William Thurston (1946-2012) was an American mathematician. In his paper 'The Theory of Foliations of Codimension Greater than One' ([22]) he explains how we can find such a triangulation. The term 'jiggling' refers to the slightly tilting of simplices such that they become transverse. The following lemma is the main result of this appendix.

Lemma D.2 (Thurston's jiggling)**.** *Let M be an n-dimensional manifold and ξ a distribution on M of rank k. Then there exists a triangulation of M such that ξ is transverse to every simplex.*

In order to prove this lemma we first look at spaces called Grassmannians.

D.1 The Grassmannian

In this section we will be studying spaces called Grassmannians. Julies Plücker was a German mathematician who studied the 2-dimensional subspaces of a 4-dimensional space, which is the earliest work in this field. The objects are named after Herman Grassmann, also a German mathematician, who later generalized the concept. ([18, p. 6])

Definition D.3. *The* ***Grassmannian*** *$Gr(V,k)$ for a given vector space V, is the space of all k-dimensional subspaces of V. By $Gr(n,k)$ we denote all the k-planes in $\mathbb{R}^n$. ([17, p. 22])*

Example D.4. Let $V = \mathbb{R}^3$ and let $k = 1$. Then the Grassmanian $Gr(3,1)$ is the space of all lines through the origin, and so it is the same as the real projective space $\mathbb{RP}^2$. △

Example D.5. We can define the following bundle

$$Gr(TM,l) := \sqcup_{p\in M} Gr(T_pM,l)$$

with a projection $\pi : Gr(TM,l) \to M$ such that $\pi^{-1}(p) = Gr(T_pM,l)$ for each $p \in M$. △

When V is a real or complex vector space, Grassmannians are compact smooth manifolds (see Lemma 5.1 in [19, p. 57]). In fact when V is real, we can see $Gr(V,k)$ as the graphs of linear functions.

Lemma D.6. *Let V be an n-dimensional real vector space, and let $1 \le k \le n$. Then $Gr(V,k)$ is a smooth manifold of dimension $k(n-k)$. ([2])*

Proof. Let P be an k-dimensional subspace of V. Then there exists a subspace Q of V (of dimension $n-k$) such that $V = P \oplus Q$. Because V is finitely dimensional, it is clear that we can find such a subspace Q which complements P.

We now look at the space of all linear functions from P to Q, which we denote by $L(P,Q)$. Let $A \in L(P,Q)$. We can then look at the graph of A, which is defined as

$$\Gamma(A) := \{x + Ax : x \in P\}.$$

Then $\Gamma(A) \subset V$ is a k-dimensional linear subspace of V. We then denote by U_Q the set of elements in $Gr(V,k)$ such that the intersection with Q is trivial, i.e. $\{0\}$. We then define the following map

$$\psi : L(P,Q) \to U_Q, A \mapsto \Gamma(A).$$

First we show that this map is well-defined, so we show that indeed $\Gamma(A) \in U_Q$. Suppose that $y \in \Gamma(A) \cap Q$, then $y = x + Ax$ for some $x \in P$. We know that $Ax \in Q$, and since Q is a linear subspace, we must have that $y - Ax = x \in Q$. However this can only happen when $x = 0$, and thus $\Gamma(A) \in U_Q$.

We now show that ψ is bijective. Let $T \in U_Q$. Then $T \cap Q = \{0\}$. For all $t \in T$ we of course have that $t \in V$ and thus there is a unique decomposition $t = p + q$ where $p \in P$ and $q \in Q$. We define a map $B : P \to Q$ by setting $Bp = q$ if $t = p + q$ for some $t \in T$. We first show that B is well-defined. Let $\tilde{t} \in T$ be such that $\tilde{t} = p + \tilde{q}$ with $\tilde{q} \in Q$. Then $Bp = q$ and $Bp = \tilde{q}$, but since $Q \ni \tilde{q} - q = \tilde{t} - t \in T$ and $T \cap Q = \{0\}$ we have that $q = \tilde{q}$. Thus B is well-defined.

Let now $p_1, p_2 \in P$ such that $t_1 = p_1 + q_1$ and $t_2 = p_2 + q_2$ for some $t_1, t_2 \in T$ and $q_1, q_2 \in Q$. Then also $p_1 + p_2 \in P$, $t_1 + t_2 \in T$ and $q_1 + q_2 \in Q$ with $t_1 + t_2 = (p_1 + p_2) + (q_1 + q_2)$. Then $B(p_1 + p_2) = q_1 + q_2 = Bp_1 + Bp_2$, and thus B is linear.

Lastly, we see that $\Gamma(B) = T$, and thus for $T \in U_Q$ we have found a linear map $B \in L(P,Q)$ such that $\Gamma(B) = T$. We may conclude that ψ is surjective.

We now show that ψ is injective. Let $A, B \in L(P,Q)$ such that $\Gamma(A) = \Gamma(B)$. We want to show that $A = B$, so $Ax = Bx$ for every $x \in P$. We note that $x \in P$ and $Ax \in Q$ and that $x + Ax$ is the only element in $\Gamma(A)$ for which x is the component of P, when we use the decomposition $V = P \oplus Q$. This is because A is a well-defined function, and an element x can only be sent to one other element. The same holds for $x + Bx \in \Gamma(B)$. Since $\Gamma(A) = \Gamma(B)$, and in both sets $x + Ax$ and $x + Bx$ are the only elements for which the p-component is x, we must have that $x + Ax = x + Bx$, and thus $Ax = Bx$. This holds for all $x \in P$ and thus $A = B$. We conclude that ψ is injective, and hence bijective.

Since ψ is a bijection, we can define $\varphi := \psi^{-1} : U_Q \to L(P,Q)$. We can identify $L(P,Q)$ with linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$, which we in turn can identify with $\mathbb{R}^{k(n-k)}$. This means we can think of (U_Q, φ) as a coordinate chart. In [2, p. 153-154] it is also shown that the transition maps between the different coordinate charts are smooth, which shows that $Gr(V,k)$ is indeed a smooth manifold of dimension $k(n-k)$. □

From this proof we can derive the following result.

Corollary D.7. *Let V be an n-dimensional real vector space, and let $1 \leq k \leq n$. Then the elements of $Gr(V,k)$ can (locally) be represented as graphs of linear maps $A : \mathbb{R}^k \to \mathbb{R}^{n-k}$.*

The proof of the next lemma uses the previous result, and is based on discussions with my supervisor Álvaro del Pino.

Lemma D.8. *Let V be an n-dimensional real vector space and let $1 \leq k, l \leq n$. Let K be a k-dimensional subspace of V. Then the set of all l-dimensional subspaces of V not transverse to K, is a subvariety of $Gr(V,l)$ of positive codimension.*

Proof. First of all, we consider the case where $l < k$ and $k + l \leq n$. This means that for an l-plane to be transverse to K is to intersect trivially. Let L be an element of $Gr(V,l)$. By Corollary D.7 we know that there exists a linear map $A : \mathbb{R}^l \to \mathbb{R}^{n-l}$ such that $\Gamma(A) = L$. We have split $\mathbb{R}^n$ into $\mathbb{R}^l$ and $\mathbb{R}^{n-l}$, and we now investigate the graphs of linear maps from $\mathbb{R}^l$ to $\mathbb{R}^{n-l}$. We note that not every element of $Gr(V,l)$ can be represented as a graph with this choice of $\mathbb{R}^l$ and $\mathbb{R}^{n-l}$, but most l-planes are. We are interested in the subset of $Gr(V,l)$ for which the planes are non-transverse to

K. However, it is enough to study the graphical planes to determine the dimension of the space we are investigating.

We are free to choose the way we split $\mathbb{R}^n$ into $\mathbb{R}^l$ and $\mathbb{R}^{n-l}$, and thus we can assume that the reference $\mathbb{R}^l$ is contained in K. Then we can split $\mathbb{R}^{n-l}$ into $K\backslash\mathbb{R}^l$ and $\mathbb{R}^{n-k}$.

Let J be an l-plane which is graphical over the choice of splitting, so there exists a linear map $B : \mathbb{R}^l \to \mathbb{R}^{n-l} \simeq K\backslash\mathbb{R}^l \times \mathbb{R}^{n-k}$ such that $\Gamma(B) = J$. This map yields the map $\tilde{B} : \mathbb{R}^l \to \mathbb{R}^{n-k}$, and its graph, and thus J, is not transverse to K if it has some kernel.

Since $k + l \leq n$, it follows that $n - k \geq l$, which means that the dimension of $\mathbb{R}^{n-k}$ is larger than the dimension of $\mathbb{R}^l$. Maps from $\mathbb{R}^l$ to $\mathbb{R}^a$, where $a \geq l$, have kernel if and only if all full minors vanish. This condition is a non-trivial algebraic equation in the space of all matrices. This means that the space of linear maps with kernel, as above, has positive codimension. This means that in this case we can also conclude that the set of all l-dimensional subspaces of V not transverse to K has positive codimension.

The other cases can be shown similarly, also making use of the identification of $Gr(V, l)$ with graphs of linear maps. $\square$

Since all metrics are equivalent on compact spaces it does not really matter what metric we consider on the Grassmannian ([21, p. 215]). In the next section we will be talking about measure and open/closed balls inside the Grassmannian. For this to make sense, let us discuss one quite intuitive metric on this space.

We define $d : Gr(n, k) \times Gr(n, k) \to \mathbb{R}$ such that

$$d(L, L') := \sup_{x \in L \cap \mathbb{S}^{n-1}} d(x, L')$$

where $d(x, L') = \inf_{y \in L'} d_E(x, y)$ where d_E is the Euclidean metric in $\mathbb{R}^n$. So when we look at two k-planes in $\mathbb{R}^n$, we first look at the intersection of one of these planes with $\mathbb{S}^{n-1}$. Then we take the supremum of the (Eucledian) distance between these points in the intersection and the other plane. See [20] for a proof that this indeed defines a metric on $Gr(n, k)$.

D.2 Distributions and the Grassmannian

In this section we will prove two preparatory lemmas and a corollary concerning Grassmanians, simplices and distributions. We will need these results in the next section. The proofs in this section are based on discussions with my supervisor Álvaro del Pino.

Let M be an n-dimensional smooth manifold. Suppose N^l is a submanifold of dimension l of M. We then have the following map

$$N^l \to Gr(TM, l), x \mapsto T_xN \in Gr(T_xM, l).$$

Because M is a smooth n−dimensional manifold, we have that $T_pM \simeq \mathbb{R}^n$ for every $p \in M$. This is why we can also view $Gr(T_xM, l)$ as $Gr(n, l)$. Similarly we have $TM \simeq \mathbb{R}^n \times \mathbb{R}^n$, and thus we obtain a map

$$N^l \to Gr(\mathbb{R}^n \times \mathbb{R}^n, l), x \mapsto T_xN \in Gr(n, l).$$

In the next lemma we see that we can compute a similar map for simplices in a triangulation of M, using the coordinates of the simplex in question.

Lemma D.9. *Let M be a smooth manifold of dimension n and T the triangulation of M obtained by embedding M in $\mathbb{R}^N$ as explained in Lemma C.14. Let $\triangle^l$ be an l-dimensional simplex of T. Then, if we use the coordinates of the simplex to trivialise the Grassmanian $Gr(TM, l)$, the map*

$$\Delta^l \to Gr(TM, l), x \mapsto T_x\Delta^l \in Gr(n, l),$$

is constant.

Proof. Let Δ^l be a simplex of the triangulation of M (as described in Lemma C.14). Then Δ^l comes with coordinates $x_1, ..., x_l$. We can complete these coordinates, such that $x_1, ..., x_n$ are local Eucledian coordinates for M. Using these coordinates, we can again identify $Gr(TM, l)$ with $Gr(\mathbb{R}^n \times \mathbb{R}^n, l)$ and $Gr(T_xM, l)$ with $Gr(n, l)$.

We therefore have a map

$$\Delta^l \to Gr(TM, l), x \mapsto T_x\Delta^l \in Gr(n, l).$$

Because in these coordinates Δ^l is a convex combination of points in $\mathbb{R}^n$, the tangent spaces in each point in the simplex lie in the same subspace of $\mathbb{R}^n$. We may therefore conclude that this map is constant.

□

Lemma D.10. *Let M be an n-dimensional smooth manifold and ξ a rank k distribution on M. Let T be a triangulation of M. We define a new triangulation T^i by the i^{th} crystalline subdivision of T. Given a simplex $\Delta_i \in T^i$, the space*

$$\{\xi_p : p \in \Delta_i\}$$

is a subset of $Gr(n, k)$ (given a certain chart), and has measure less or equal to $\mathcal{O}((\frac{1}{i})^{dim(Gr(n,k))})$.

Proof. Let $p \in M$. Since ξ is a smooth distribution of rank k on M, then we can describe ξ as the span of k vector fields $X_1, ..., X_k$ around p. Locally, we can view M as $\mathbb{R}^n$, and we can find a basis such that $X_1|_p, ..., X_k|_p$ form the first k basis vectors. Using this as a basis, we can identify $Gr(M, k)$ with $Gr(n, k)$. In these coordinates ξ_p is a k-dimensional subspace of M and thus an element $Gr(n, k)$. This is why we can view $\{\xi_p : p \in \Delta_i\}$ as a subset of $Gr(n, k)$, which proves the first part of the statement.

Again, let $p \in M$. We can then look at $\xi_p \subset T_pM$. Suppose we move very slightly away from p to the point $p + h$ (here we use that M locally looks like $\mathbb{R}^n$ and thus the addition is defined). Then we can write $\xi_{p+h} = \xi_p + \mathcal{O}(h)$. We can do this because we know ξ is a smooth distribution. This means that ξ can locally be described as the span of the smooth vector fields $X_1, ..., X_k$. Since M locally looks like $\mathbb{R}^n$ we have for every $1 \leq j \leq k$ that

$$X_j|_p = f_1(p)\partial_{x_1} + ... + f_n(p)\partial_{x_n},$$

for smooth functions $f_1, ..., f_n : \mathbb{R}^n \to \mathbb{R}$. We can then approximate $X_j|_{p+h}$ using the derivatives of these functions. We have

$$\begin{aligned}
X_j|_{p+h} &\approx (f_1'(p)h + f_1(p))\partial_{x_1} + ... + (f_n'(p)h + f_n(p))\partial_{x_n} \\
&= (f_1(p)\partial_{x_1} + ... + f_n(p)\partial_{x_n}) + h(f_1'(p)\partial_{x_1} + ... + f_n'(p)\partial_{x_n}) \\
&= X_j|_p + \mathcal{O}(h).
\end{aligned}$$

So for every $1 \leq j \leq k$ we can approximate $X_j|_{p+h}$ by $X_j|_p + \mathcal{O}(h)$. This justifies why we can write $\xi_{p+h} = \xi_p + \mathcal{O}(h)$ for $h > 0$ small.

When given a triangulation, suppose we subdivide it by crystalline subdivision i times such that for each subsimplex $\Delta_i \in T^i$ we have that Δ_i is contained in a ball of radius $\frac{r}{i}$ for some fixed $r > 0$.

For all $p, q \in \Delta_i$ we then have that $d(p, q) < \frac{r}{i}$, and thus $p = q + h$ for some $h < \frac{r}{i}$. Then $\xi_p = \xi_q + \mathcal{O}(h) < \xi_q + \mathcal{O}(\frac{r}{i}) = \xi_q + \mathcal{O}(\frac{1}{i})$ and thus if $||.||$ denotes the norm on $Gr(n, k)$ we have

$$||\xi_p - \xi_q|| < |\mathcal{O}(\frac{1}{i})|.$$

This means that the radius of the set

$$\{\xi_p : p \in \Delta_i\} \subset Gr(n, k),$$

goes to zero as i becomes very large, and

$$\text{measure}(\{\xi_p : p \in \Delta_i\}) \leq \mathcal{O}((\frac{1}{i})^{dim(Gr(n,k))}).$$

□

Let Δ be a simplex. We define the following space:

$$Gr^{\xi|_\Delta}(n,l) := \{\text{all } l\text{-planes not transverse to some } \xi_p \text{ for a } p \in \Delta\}.$$

Then $Gr^{\xi|_\Delta}(n,l) \subset Gr(n,l)$.

We then have the following corollary of Lemma D.10.

Corollary D.11. *The measure of the space*

$$Gr^{\xi|_{\Delta_i}}(n,l) := \{\text{all } l\text{-planes not transverse to some } \xi_p \text{ for a } p \in \Delta_i\} \subset Gr(n,l)$$

goes to zero as i becomes large.

Proof. We note that not only the measure of the set $\{\xi_p : p \in \Delta_i\}$ goes to zero as i becomes large, but the radius of the set also tends to zero as we have seen in the proof of Lemma D.10. Therefore the set $\{\xi_p : p \in \Delta_i\}$ converges to a single element $\{\xi_p\}$ for some $p \in \Delta_i$ as i becomes large. This means that the set $Gr^{\xi|_{\Delta_i}}(n,l)$ converges to the set

$$Gr^{\xi_p}(n,l) = \{\text{all } l\text{-planes not transverse to } \xi_p\} \subset Gr^{\xi|_{\Delta_i}}(n,l).$$

By Lemma D.8 we know this set has positive codimension, and thus measure 0. Therefore the measure of the space $Gr^{\xi|_{\Delta_i}}(n,l)$ converges to zero as i becomes large.

□

D.3 Thurston's Jiggling

We have now obtained all the ingredients to look at the proof of the main result of this appendix. Since the proof of this statement is quite complicated, we provide a sketch of the proof. You can find the original proof in the paper *The Theory of Foliations of Codimension Greater than One* (1973) by William Thurston (see [22, p. 227-229]).

Lemma D.12 (Thurston's jiggling)**.** *Let M be an n-dimensional manifold and ξ a distribution on M of rank k. Then there exists a triangulation of M such that ξ is transverse to every simplex.*

Sketch of proof. First of all, we triangulate M as in Lemma C.14. Let T denote this triangulation. We then use crystalline subdivision i times to refine T, and denote this triangulation by T^i. We fix the number i at this point in the proof, but how we choose it will be explained later on in the proof. We know by Lemma D.9 that for any simplex Δ^l in the triangulation of T the map $\Delta^l \to Gr(TM,l), x \mapsto T_x\Delta^l \in Gr(n,l)$ is constant (in the chart provided by the coordinates of Δ^l). We note that if $\tilde{\Delta}$ appears in the subdivision of Δ^l, the map associated to $\tilde{\Delta}$ is also constant in the chart provided by the coordinates of Δ^l.

We now proceed by induction on the vertices of the triangulation T^i to show that we can 'tilt' every simplex such that eventually ξ is transverse to all the simplices. We note that by tilting a simplex by moving one vertex, still yields a constant map $\Delta^l \to Gr(TM,l), x \mapsto T_x\Delta^l \in Gr(n,l)$.

We define the following set for (distinct) vertices $v_0,...,v_j$ in the triangulation T^i of M:

$$\Delta(v_0,...,v_j) := \{\text{simplices with all vertices lying in the set } \{v_0,...,v_j\}\}.$$

Let v_0 and v_1 be two vertices in the triangulation T^i. There are two things that can happen. Either v_0 and v_1 don't span a 1-simplex, or they do. So either $\Delta(v_0,v_1)$ is empty or contains a 1-simplex. We want to show that all the simplices in $\Delta(v_0,v_1)$ are transverse to ξ. If $\Delta(v_0,v_1)$ is empty this is certainly the case, so we assume $\Delta(v_0,v_1)$ is not empty.

Suppose Δ^1 is the 1-simplex in T^i spanned by $\{v_0,v_1\}$. If ξ is transverse to Δ^1 then we are done, so we assume that $\Delta^1 \in Gr^{\xi|_{\Delta^1}}(n,1)$. Let $0 < \delta < \epsilon$ be fixed. By Corollary D.11; for every $\epsilon > 0$ we can choose our i large enough (i.e. subdivide many times) such that

$$\text{measure}(Gr^{\xi|_{\Delta^1}}(n,1)) < \text{measure}(B_\epsilon(\Delta^1)),$$

where $B_\epsilon(\Delta^1)$ is an open ball in $Gr(n,1)$ around Δ^1 of radius ϵ. This means that

$$B_\epsilon(\Delta^1) \backslash Gr^{\xi|_{\Delta^1}}(n,1) \neq \varnothing.$$

We can therefore by slightly moving the vertex v_1 and thus tilting the simplex Δ^1, obtain a simplex which does not lie in $Gr^{\xi|_{\Delta^1}}(n, 1)$ and is $\epsilon-$close to the original simplex. In fact, we can choose this ϵ-tilt in such a way that the angle the tilted simplex makes with ξ is at least δ. Now we have that all simplices in $\Delta(v_0, v_1)$ are transverse to ξ, and the angles of the simplices with respect ξ are bounded from below by δ. This is our base case. See figure D.1 for an illustration of this idea.

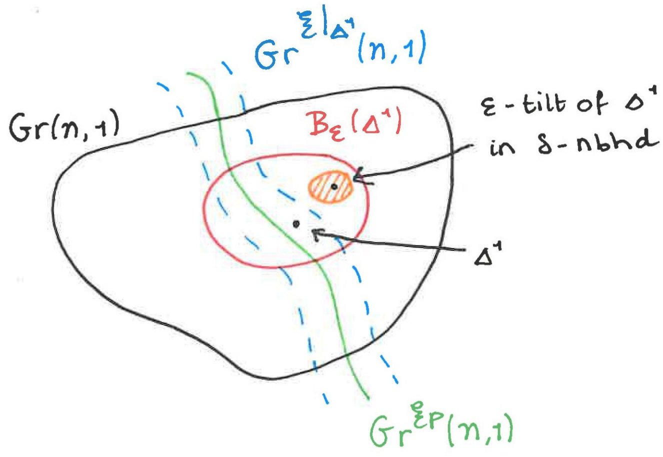


Figure D.1: Illustration for proof of Lemma D.2.

Now suppose that all the simplices in $\Delta(v_0, ..., v_j)$, for j different vertices $v_0, ..., v_j$ in the subdivided triangulation T^i, are transverse to ξ, and the angles the simplices make with respect to ξ are bounded from below by δ (this is our induction hypothesis). For an arbitrary vertex v_{j+1} different from v_i for $1 \leq i \leq j$, we look at the set $\Delta(v_0, ..., v_{j+1})$. Because of our hypothesis we only need to look at simplices in $\Delta(v_0, ..., v_{j+1}) \backslash \Delta(v_0, ..., v_j)$, so simplices which actually arise because we added the vertex v_{j+1}.

Let $\Delta^l \in \Delta(v_0, ..., v_{j+1}) \backslash \Delta(v_0, ..., v_j)$ be a simplex of dimension l in T^i. Then the face opposite of the vertex v_{j+1} is a simplex of dimension $l - 1$ and an element of $\Delta(v_0, ..., v_j)$. This means that ξ is already transverse to this face. We quotient out this face and obtain a 1-simplex Δ^1 in $Gr(n - l + 1, 1)$. For now assume that $n - l + 1 > k$. Because ξ is transverse to the simplex we quotient out, the distribution after the projection, $\pi(\xi)$, still remains a rank-k distribution locally in a neighbourdhood of this 1-simplex we obtain.

So we now have a 1-simplex Δ^1 in $Gr(n - l + 1, 1)$ and a rank-k distribution $\pi(\xi)$ which may or may not be transverse to it. We can locally do this for all simplices in $\Delta(v_0, ..., v_{j+1}) \backslash \Delta(v_0, ..., v_j)$. Because we use the triangulation described in Lemma C.14 and because of Lemma C.13 we know that the number of simplices touching the vertex v_{j+1} is bounded independently of i, and thus we only need to do this finitely many times.

Having now obtained a 1-simplex Δ^1 in $Gr(n - l + 1, 1)$ and a rank-k distribution $\pi(\xi)$, we still want to use Lemma D.10 and Corollary D.11 to make Δ^1 transverse to $\pi(\xi)$. This means that we want the measure of the space $\{\pi(\xi)_p : p \in \Delta^1\}$ to go to zero (and to decrease in radius) as i becomes large, with a rate independent of π.

To show this we now use the fact that we can guarantee that the angles of the simplices in $\Delta(v_0, ..., v_j)$ with respect to ξ are bounded from below by δ. We know that the space of directions which make an angle larger than δ with ξ, is a compact set. Since this is a compact space we can take the following supremum over the possible projections,

$$\sup_{\pi} \mu(\{\pi(\xi)_p : p \in \Delta^1\}),$$

where μ indicates the measure of the set. So regardless of the π we choose, we can take this supremum, as a bound from above for the relevant measures. We can choose i accordingly, independently of the π we choose.

This now allows us to use Lemma D.10 and Corollary D.11 again, and just like before, we can tilt the vertex v_{j+1} such that the 1-simplices become transverse to $\pi(\xi)$. Then, when we go back to the original space, we have that ξ is transverse to all the simplices in $\Delta(v_0, ..., v_{J+1})$. This completes the induction.

In the beginning of the proof we choose an i which determines the subdivision of our triangulation. We have seen that how we choose this i is based on a couple of factors. First of all we note that the number of simplices touching a vertex is bounded independently of i. For each i, we know the measures of the resulting 'problematic sets'. By these we mean the sets of simplices non-transverse to the distribution. The constants ϵ and δ are given. For each i we therefore know how much the problematic measures add up to, and we can choose i such that the total bad measure is much smaller than $\epsilon - \delta$. We do not only require that the measure is smaller than that, but also that the radius of this problematic set becomes small.

Lastly, we note that we made the assumption that $n-l+1 > k$. What happens when $n-l+1 \leq k$? If $n - l + 1 \leq k$, then $n \leq k + l - 1$. We already had that ξ was transverse to the $(l-1)$-simplex, and thus together they already span the whole tangent bundle if $n \leq k + l - 1$. Therefore ξ is automatically transverse to the l-simplex.

We conclude that we can construct a (jiggled) triangulation of M, such that all the simplices are transverse to ξ.

$\square$

Appendix E

A diffeomorphism on open manifolds

As you might have noticed, in section 2.4 we only focused on building boxes around the simplices of dimension 0, 1 and 2. And in the proof of Theorem 2.11 we only make adjustments to our formal contact structure in these boxes. We do this because for every open n-dimensional manifold M endowed with a CW-structure, there exists a diffeomorphism to an open U which lies in the complement of the barycenters of the cells of the CW-structure. In this section we will prove this result, and besides using it for Theorem 2.11, we also use it when we discuss Engel structures. The proof of the following lemma is based on Remark 3.36 of [10, p. 29].

Lemma E.1. *Let M be a smooth open n-dimensional manifold endowed with a CW-structure. Then there exists a diffeomorphism $\Phi : M \to U$, where U lies in the complement of the barycenters of the n-cells of the CW-structure, and an isotopy of embeddings $(\phi_t)_{t\in[0,1]}$ with $\phi_0 = id$ and $\phi_1 = \Phi$.*

Proof. By Proposition A.60 of [17, p. 612] we know that M admits an exhaustion of compact sets, i.e. there exist compact sets M_k for $k \in \mathbb{N}$ such that

$$M \supset ... \supset M_k \supset ... \supset M_1 \supset M_0.$$

Suppose M is given as an CW-structure, and we locate all the barycenters of the n-cells (i.e. the cells of highest dimension). The idea is that we construct maps which 'connect' these centers to infinity, and then invert the maps to obtain an open subset which lies in the complement of the barycenters.

We will proceed by induction. The induction hypothesis is that there is a diffeomorphism $\varphi_i : M \to M$ which is the identity outside of M_i and which will fulfill the following condition. It will push the *critical points* in M_{i-1}, which we denote by $\{a_k^{i-1}\}$, into M_i. By the set of critical points in M_{i-1} we mean either the barycenters of the n-cells of $M_{i-1}\backslash M_{i-2}$ or points in $M_{i-1}\backslash M_{i-2}$ which are the image of barycenters in M_{i-2} under the maps $\varphi_0, ..., \varphi_{i-1}$ (or compositions of these maps).

For $i = 0$ this is automatically true, as there is no M_{-1}. We note that φ_0 is the identity. Therefore the base case holds. Assuming the induction hypothesis, we now want to construct a diffeomorphism $\varphi_{i+1} : M \to M$ with the same properties as above.

We now look at the set $\{a_j^i\}$ of critical points in M_i. This means that each a_j^i is either a barycenter of an n-cell in $M_i\backslash M_{i-1}$ or a_j^i is the image of a barycenter in M_{i-1} which has pushed into M_i by the maps $\varphi_0, ..., \varphi_i$. In both cases, so for each critical point $a_j^i \in M_i\backslash M_{i-1}$, we choose a path γ_j^i connecting it to a point c_j^i in $M_{i+1}\backslash M_i$. Because M_i is compact, there are only finitely many cells, and thus finitely many barycenters (see Proposition A.1 in [15, p. 520]). This means that the set $\{a_j^i\}$ of critical points in $M_i\backslash M_{i-1}$ is also finite. Therefore we only need to pick finitely many paths γ_j^i, and we assume them to be disjoint and embedded (so they also do not self-intersect).

We then define the map $\varphi_{i+1} : M \to M$. First, we let φ_{i+1} be the identity outside of a little neighbourhood of the critical points and their maps $\{(a_j^i, \gamma_j^i)\}$. Secondly, we require φ_{i+1} to take each path γ_j^i, and shrink it such that the critical point and the map lie fully in $M_{i+1}\backslash M_i$. The shrinking of this path is a diffeomorphism, and will happen 'along' the path γ_j^i itself. This means that $\varphi_{i+1} : M \to M$ is a diffeomorphism which is the identity outside of M_{i+1}, and which pushes all critical points in M_i into M_{i+1}. See figure E.1 for an illustration of this. This completes the induction.

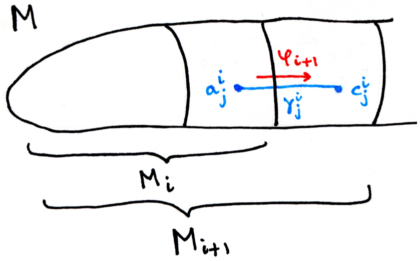


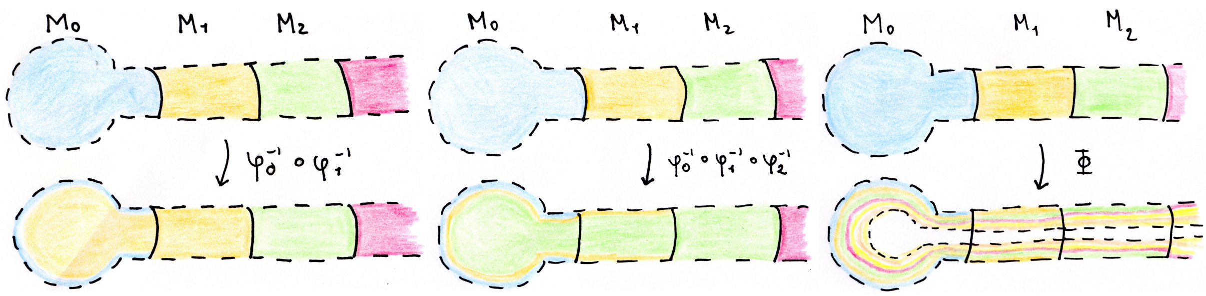
Figure E.1: Illustration of the function φ_{i+1}.

We have now obtained a sequence of diffeomorphisms $(\varphi_i)_{i\in\mathbb{N}}$. We are going to look at the following map:

$$\Phi := \lim_{i\to\infty} \varphi_0^{-1} \circ \varphi_1^{-1} \circ \cdots \circ \varphi_{i-1}^{-1} \circ \varphi_i^{-1},$$

which is the C^∞_{loc}-limit of the sequence $(\varphi_0^{-1} \circ \cdots \circ \varphi_i^{-1})_{i\in\mathbb{N}}$.

First let us think about what the function $\varphi_0^{-1} \circ \cdots \circ \varphi_i^{-1}$ does. Let a^0_j be a barycenter in M_0. Then this point will get mapped to M_i by the map $\varphi_0 \circ \cdots \circ \varphi_i$. Since each φ_j shrinks the paths connected to critical points, such that they lie in $M_{j+1}\backslash M_j$, each inverse φ_j^{-1} will push points out of $M_{j+1}\backslash M_j$ into $M_j\backslash M_{j-1}$. Therefore, the composition $\varphi_0^{-1} \circ \cdots \circ \varphi_i^{-1}$ will push points out of M_{i+1} in the neighbourhood of the path into M_0. This is illustrated in figure E.2. Here this tube indicates the neighbourhood around the barycenter and the path. We see that $\varphi_0^{-1} \circ \varphi_1^{-1}$ pushes points out of $M_1\backslash M_0$ into M_0 to the edge of the neighbourhood. The second picture illustrates the map $\varphi_0^{-1} \circ \varphi_1^{-1} \circ \varphi_2^{-1}$. When we take the limit of these compositions, this means that an empty space will appear in the middle of the neighbourhood, because all points are being pushed to the edge of the neighbourhood. This is illustrated in the third picture in figure E.2.



(a) (b) (c)

Figure E.2: Illustrations for the functions $\varphi_0^{-1} \circ \varphi_1^{-1}$ (a), $\varphi_0^{-1} \circ \varphi_1^{-1} \circ \varphi_2^{-1}$ (b) and Φ (c).

We want to check if this limit actually exists. Let us focus on one compact of the exhaustion M_k for $k \in \mathbb{N}$. We are going to check what happens to M_k under the map $\varphi_0^{-1} \circ \cdots \circ \varphi_i^{-1}$ for every $i \in \mathbb{N}$. If $i < k$ then $\varphi_0^{-1} \circ \cdots \circ \varphi_i^{-1}|_{M_k} = id$. Now let $i \geq k$ and $p \in M_k$. Then $(\varphi_0^{-1} \circ \cdots \circ \varphi_k^{-1} \circ \varphi_{k+1}^{-1} \circ \cdots \circ \varphi_i^{-1})(p) = (\varphi_0^{-1} \circ \cdots \circ \varphi_k^{-1})(p)$, because the maps $\varphi_{k+1}^{-1}, ..., \varphi_i^{-1}$ do not act on M_k. Because this holds for every $i \geq k$, we can conclude that the limit exists on the compact M_k. This is means that the limit exists on every compact, and this is what it means for the C^∞_{loc}-limit to exists on the whole of M. We conclude that the function Φ is well-defined.

We now show that Φ is indeed a diffeomorphism from M to its image $U := \Phi(M)$, where U lies in the complement of the barycenters. We are going to check that Φ is bijective. Of course a map is surjective on its image, so we check that it is injective. Let $p, q \in M$, then $p, q \in M_k$ for some $k \in \mathbb{N}$. Then $\Phi(p) = (\varphi_0 \circ \cdots \circ \varphi_k^{-1})(p)$ and $\Phi(q) = (\varphi_0 \circ \cdots \circ \varphi_k^{-1})(q)$, and since all these maps are diffeomorphisms, we must have that $\Phi(p) = \Phi(q)$.

Now we check that Φ has full rank. This means that we check if $d\Phi_p$ has full rank for every $p \in M$. Let $p \in M_k$, then

$$d\Phi_p = d(\lim_{i\to\infty} \varphi_0^{-1} \circ \cdots \circ \varphi_i^{-1})_p = d(\varphi_0^{-1} \circ \cdots \circ \varphi_k^{-1})_p = d(\varphi_0^{-1})_{(\varphi_1^{-1}\circ\cdots\circ\varphi_i^{-1})(p)} \circ \cdots \circ d(\varphi_k^{-1})_p.$$

Here we use again that for $p \in M_k$ the only functions which may act on p are $\varphi_0, ..., \varphi_k$. Since φ_j is a diffeomorphism for every $j \in \mathbb{N}$, $d\varphi_j^{-1}$ is an isomorphism. Therefore the composition is also an isomorphism, and we conclude that $d\Phi_p$ must have full rank.

Lastly, using the same trick again, we show that Φ is smooth. On every compact Φ is actually the composition of finitely many smooth function, and hence is smooth itself. We can now conclude by Theorem 4.14 in [17, p. 83] that Φ is a diffeomorphism from M to U.

To finish the proof, we are going to construct an isotopy of embeddings $(\phi_t)_{t \in [0,1]}$ such that $\phi_0 = id$ and $\phi_1 = \Phi$. We know that the shrinking of paths is an isotopy of embeddings which means there exists such an isotopy for φ_j for each $j \in \mathbb{N}$. The same holds for its inverse, i.e. there exist embeddings $\varphi_{j,t}^{-1}$ for $t \in [0,1]$ such that $\varphi_{j,0}^{-1} = id$ and $\varphi_{j,1}^{-1} = \varphi_j^{-1}$. We then define the maps

$$\phi_t := \lim_{i \to \infty} \varphi_{0,t}^{-1} \circ \cdots \circ \varphi_{i,t}^{-1},$$

for $t \in [0,1]$. For each compact M_k there are only finitely many of these embeddings $\varphi_{j,t}^{-1}$ which act on it, and thus by a similar argument as before, we can conclude that ϕ_t is also an embedding (by showing it is an injective immersion, and a homeomorphism with its image). Therefore $(\phi_t)_{t \in [0,1]}$ with $\phi_0 = id$ and $\phi_1 = \Phi$ is, an isotopy of embeddings.

□

Bibliography

[1] Athanasiadis, C.A. (2016). *The local h-polynomial of the edgewise subdivision of the simplex.*

[2] Baralic, D. (2011). How to understand Grassmannians? *The Teaching of Mathematics, 14*, 147-157.

[3] Cairns, S.S. (1961). A Simple Triangulation Method for Smooth Manifolds. *Bull. Amer. Math. Soc., 67*, 389-390.

[4] Casals, R., Pérez, J.L, del Pino, Á., Presas, F. (2017). Existence h-principle for Engel structures. *Invent. math., 210*, 417–451.

[5] Casals, R., del Pino, Á. & Presas, F. (2018). *Loose Engel structures.*

[6] *Contact geometry.* `https://en.wikipedia.org/wiki/Contact_geometry`. Consulted on January 2, 2021.

[7] Crainic, M. (2017). *Inleiding Topologie Lecture Notes.*

[8] De Laat, D. (2011). *Approximating Manifolds by Meshes: Asymptotic Bounds in Higher Codimension* (master thesis University of Groningen). Available at `https://fse.studenttheses.ub.rug.nl/9776/1/David_de_Laat_MW_2011.pdf`.

[9] Del Pino, Á. (2020). Lecture notes course *'Topologie en Meetkunde'.*

[10] Del Pino, Á. (2018). *Lecture Notes: Topological Aspects in the study of Tangent Distribution.*

[11] Francis, J. (2011). *The H-principle, Lectures 1 and 2: Overview.* Notes for a lecture on January 5, 2011.

[12] Golubitsky, M. & Guillemin, V. (1973). *Stable Mappings and Their Singularities.* New York: Springer-Verlag New York Inc.

[13] Gromov, M. (1986). *Partial differential relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Berlin: Springer.

[14] Gromov, M.L. (1969). Stable mappings of foliations into manifolds. *Izv. Akad. Nauk SSSR Ser. Mat., 33, issue 4*, 707–734.

[15] Hatcher, A. (2001). *Algebraic Topology.* Cambridge: Cambridge University Press.

[16] Hirsh, M.W. (1959). Immersions of Manifolds. *Transactions of the American Mathematical Society, 93*, 242-276.

[17] Lee, J.M. (2013). *Introduction to Smooth Manifolds.* New York: Springer.

[18] Michalek, M. (2018). *Linear Spaces and Grassmannians.* Notes for the lecture on May 08, 2018, in the IMPRS Ringvorlesung 'Introduction to Nonlinear Algebra'.

[19] Milnor, J.W. & Stasheff, J.D. (1974). *Characteristic Classes.* Princeton: Princeton University Press.

[20] Mishra, S. (2016). *Grassmannian as a metric space.*

[21] Tao, T. *Poincarés legacies: pages from year two of a mathematical blog.* Available at `https://terrytao.files.wordpress.com/2009/01/whatsnew.pdf`. Consulted at December 30 2020.

[22] Thurston, W. (1973). The Theory of Foliations of Codimension Greater than One. *Commentarii Mathematici Helvetici, 49*, 214–231.

[23] Vogel, T. (2009). Existence of Engel structures. *Annals of Mathematics, 169*, 79-137.

[24] Whitehead, J.H.C. (1949). Combinatorial homotopy. I. *Bull. Amer. Math. Soc., 55*, 213–245.