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An introduction into Lie Group, Lie Algebra, Representations and Spin

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Abstract

In this thesis we will go through what a Lie group and a Lie algebra is. How they are linked to one another and we will have a look at their representations. The main focus will be towards matrix Lie groups and their associated Lie algebras. In particularly we will be looking into the matrix Lie groups $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ with their associated Lie algebras $\mathbf{so}(3)$ and $\mathbf{su}(2)$, resp. This is to describe the spin of a particle. Since the representations of $\mathbf{SO}(3)$ can only describe particles with an integer spin, i.e. bosons, we will look at the matrix Lie group $\mathbf{SU}(2)$ and show that this Lie group can describe particles of half-integer spins through representations, fermions. The main motivations was to understand the spin of an electron. The electron is a fermion and has spin-$\frac{1}{2}$. Therefore we will be focussing on theorems which will assist us to achive our quest to understand the spin.

Acknowledgements

First and foremost, I would like to thank my supervisors Dr. Lennart Meier and Dr. Dirk Schuricht for helping me picking the right research questions, providing me with interesting and usefull literature, always being able to explain my questions clearly. I have not only learned a lot about mathematics and physics, but also about how to write a thesis and how to present your thesis in a structured way. I also want to thank Quirijn Meijer, where I have had some talks with about Lie groups, Lie algebra and other somewhat related topics. He was able to give me other ways to look at certain problems that appeared at the beginning of my thesis.

Introduction

The Stern-Gerlach experiment was done in 1922 by O. Stern and W. Gerlach. In this experiment they send silver atoms flying through an inhomogenous magnetic field and caught them on a detection screen. They observed two distinct spots instead of an oval blur as one would expect from a classical point of view. This raises quenstions. Why a distrete number of possible values? And, why is this discrete value two? This has to do with spins of particles. The silver atoms has the spin of an electron, and we know the electron has spin half.

For us to describe this spin of an electron, or any particle in three dimensions, we have to look at the three dimensional special orthogonal matrix Lie group, $\mathbf{SO}(3)$. This group describes all the rotations in three dimensional real space, $\mathbb{R}^3$. This group is a matrix Lie group and to understand this group, we will dive into properties of Lie groups and in particular matrix Lie groups. We will also consider the two dimensional special unitary group, $\mathbf{SU}(2)$. As it turns out we can only describe integer spins with the matrix Lie group $\mathbf{SO}(3)$. To describe also the half-integer spins, we have to look at $\mathbf{SU}(2)$.

To further describe these (matrix) Lie groups, one has to look at Lie algebras. In the case of matrix Lie groups there is a beautiful connection between matrix Lie groups and their Lie algebra. This connection is done through the matrix-exponential. From there we will study the Lie algebras $\mathbf{so}(3)$ and $\mathbf{su}(2)$ of the matrix Lie groups $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$, respectively. These Lie algebras are on some fronts easier to work with then the Lie groups. What is also confient is that the Lie algebras $\mathbf{so}(3)$ and $\mathbf{su}(2)$ are isomorphic, while the matrix Lie groups $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$ not.

We will look into representations of $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ as well as the representations of the associated Lie algebras. We will start with the irreducible representations of the Lie algebra $\mathbf{so}(3)$. From there we will look at the corresponding representations of the matrix Lie group $\mathbf{SO}(3)$. As it turns out this matrix Lie group doesn't allow us to look at all the dimensions through (irreducible) representations. It only desribes systems of boson particles. To do this we go to the matrix Lie group $\mathbf{SU}(2)$, which can describe every dimension through an irreducible representation. These irreducible representations are linked to projective representations of $\mathbf{SO}(3)$.

Notation: In this thesis we will consider $\mathbb{N} := \{1, 2, 3, \dots\}$. And for the case where we want to include 0, we will define $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

Contents

Chapter 1

The Big Motivation: The Stern-Gerlach Experiment

The Stern-Gerlach experiment is one of the most important experiments concerning quantum mechanics and spin. This experiment is the big motivation from a physical standpoint why we, physicists and mathematicians with an interest in physics, would like to study the Lie group $\mathbf{SU}(2)$ together with the associated Lie algebra $\mathbf{su}(2)$ to create a basic understanding for the description of the electron. This experiment was developed by Otto Stern in 1921 and performed by him in collaboration with Walther Gerlach in 1922 in Frankfurt. In this chapter I mainly follow Chapter 1.1 of [2].

1.1 The Setup

We will go over the setup of this experiment. To do that I will refer to Figure 1.1, where you can see a sketch of the setup, by using the numbers 1 to 5 in this picture. One uses a furnace (1) with a small cavity in it and the furnace is filled with silver atoms and heated. In the furnace the temperature will be so high that the silver atoms will have a great amount of energy. In general the more energy the particle has the more kinetic energy the particle has. The atoms will bounce around against the walls of the furnace and can only leave through the small cavity.

The leaving silver atoms create a beam that will travel through the collimating slits (2) to narrow down the width of the beam and making the atoms travel as parallel to each other as possible. In general there are multiple collimating slits in a row.
After the collimating slits the beam will go through a parallel electromagnet with an

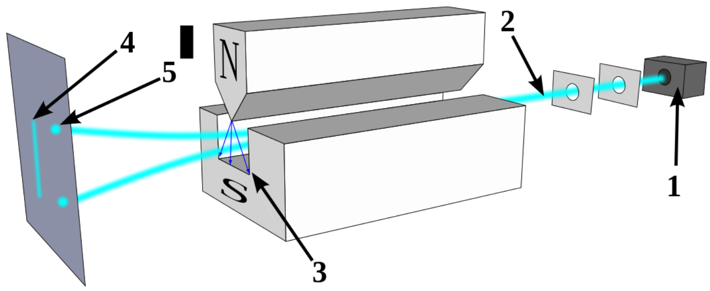

Figure 1.1: A sketch of the setup used in the Stern-Gerlach experiment.
This URL of this picture can be found by the References under [3]

inhomogeneous magnetic field (3). To achieve this inhomogeneous field one might make the north pole pointy, like an axe head, and the south pole flat or concave. One uses an inhomogeneous magnetic field to deviate the particles with different magnetic moments. After leaving the inhomogeneous magnetic field the silver atoms will reach a detection screen (4&5).

One might wonder why they, Stern and Gerlach, used silver atoms. Back in the 1920's there was no simple and cheap way create an controlled amount of free electrons and to use free electrons in experiments. And there might have been a problem with detecting the electrons. Furthermore silver atoms are a lot easier to get, and to make free particles from. One only has to have a furnace with a small cavity in it to create a beam of silver atoms. Silver atoms are a cheap and practical alternative of electrons. This is also because a silver atom behaves as an electron in an magnetic field. Since a silver atoms has 47 electrons around the nucleus, where the first 46 are filling up the first 3 shells and subshell 4p. This will leave us with 1 electron free in the subshell 5s. The 5s subshell is invariant under angular rotation and has only variations in the radial component. Which leaves the electron to move freely in the spherical directions. So the orbital angular momentum of the electron is zero and hence the angular momentum of the electron equals the **spin** angular momentum of the electron.

Furthermore, the magnetic moment of the inner 46 electrons equals zero. Since these electrons fill in complete electron shells and subshells. And these electrons are distributed evenly over the shells and hence symmetrical. Thus the orbital and spin angular momentum of the electrons cancel each other out. So the 46 electrons combined have no angular momentum. The spin of the nucleus can be neglected since it is very small in comparison to the 47th free electron. Thus we can view the inner 46 electrons together with the nuclear as a sphere with no orbital or spin angular momentum. since these electrons fill in complete electron shells and subshells. The magnetic field of the nuclear and the 46 electrons can be set to zero. The mass of the nucleus is around $2 \cdot 10^5$ times the mass an electron. Hence the heavy atom inherits the magnetic moment properties of the 47th free electron, in particular the spin magnetic moment. In conclusion the magnetic field of a silver atom is mostly the same as that of a free electron, i.e. a silver atom behaves like a free electron in a magnetic field.

1.2 The Physics

We have conducted that the magnetic moment of the atom equals the spin magnetic moment of the 47th electron, i.e. the magnetic moment of the silver atom $\vec{\mu}$ is proportional to the spin magnetic moment of the free electron $\vec{S}$. The precise relation, with an accuracy of about $0,2\%$, is

$$\vec{\mu} = \frac{e}{m_e c}\vec{S} \tag{1.1}$$

where $e < 0$ is the electric charge of an electron, m_e is the mass of an electron at rest and c is the speed of light. The interaction energy E from the magnetic moment with the magnetic field is $-\vec{\mu} \cdot \vec{B}$ and the interaction force $\vec{F}$ is the negative of the gradient of the interaction energy E, so $\vec{F} = -\vec{\nabla} E$. We will now have a look at the z-component of the interaction force is:

$$\begin{aligned} F_z &= -\left[\vec{\nabla}\left(-\vec{\mu} \cdot \vec{B}\right)\right]_z = \frac{\partial}{\partial z}\left(\vec{\mu} \cdot \vec{B}\right) = \mu_z \frac{\partial B_z}{\partial z} + B_z \frac{\partial \mu_z}{\partial z} \\ &\simeq \mu_z \frac{\partial B_z}{\partial z}. \end{aligned} \tag{1.2}$$

It is important to note that we have neglected the differential of B_x and B_y to z. We have noticed earlier that the silver atom, compared to the electron, is really heavy, so we would expect the clasical trajectory rules to hold up. This can be shown though Heisenbergs uncertainty relation. Furthermore, the differential of $\vec{\mu}$ to z is zero since the magnetic moment of the silver atom is in proportions to the spin of

the electron and the spin of the electron is a constant vector.
When we look at Figure 1.1 we can deduce that the silver atom travels upwards when the sign of F_z is positive hence $\mu_z < 0$ and $S > 0$, *and on the contrary* the atom experiences a downward motion when the sign of F_z is negative hence $\mu_z > 0$ so $S < 0$.

1.3 The Results and Conclusion

The silver atoms are heated to such a temperature that one would expect the magnetic moments of the silver atoms to be completely randomly orientated. So one would expect a smooth oval blurish spot, like at (4) in Figure 1.1 , but Stern and Gerlach, they detected two distinct circular spots, like at (5) in Figure 1.1. This means there are only two possible values of the magnetic moment in the z-direction of the silver atom, hence only two possible values for the spin (angular moment) in z-direction of the electron, since for every silver atom the differential of B_z to z at the same spot are identical. Since the magnetic moment of the silver atom is identical to the magnetic moment of the electron, the possible values of the spin in the z-direction are also quantized to two distinct values.
The spin angular momentum can be expressed, like the orbital angular momentum, as a 3-dimensional vector in real space, so it is only logical to look at a irreducible representation of the Lie group $\mathbf{SO}(3)$ to describe this group of the spin. But as we have seen in the previous chapter, the Lie group $\mathbf{SO}(3)$ has only irreducible representations of odd dimensions. For us to get even dimensions we have to look at the Lie group $\mathbf{SU}(2)$. With the irreducible representation of dimension 2 of $\mathbf{SU}(2)$, we get a spin of $\frac{1}{2}$, in physics the electron is called a **spin-$\frac{1}{2}$** particle and is a **fermion**.

Chapter 2

Brief Introduction to Group Theory

This chapter is based on my knowledge of group theory which I got by a course I followed at the University of Utrecht. In that course we followed the book [4]. I have used chapter 1 of the book [5] to check all the defintions, this book makes use of an mirrored notation. Keep that in mind if you are reading that book.

Definition 2.1. *A* ***group*** *is a pair* $(G, *)$*, where* G *is a set and* $*$ *is the so called product operator of the group, or is called the group operator. This set with operator have to fulfill the folowing properties, also know as the* ***group axioms****:*

1. $g * h \in G$ *for every pair* $g, h \in G$*,*
2. $(g * h) * k = g * (h * k)$ *for every* $g, h, k \in G$*, i.e. the order of applying the operator should not matter,*
3. *there exists an element* $e \in G$ *such that* $g * e = g = e * g$ *for every* $g \in G$*. This element* e *is called an identity of the group,*
4. *for every* $g \in G$ *there exists an element* $g^{-1} \in G$ *such that* $g * g^{-1} = e = g^{-1} * g$*. This element* g^{-1} *is called an inverse of* g *in* G*.*

An additional property can be that a group is commutative, i.e. for every two elements $a, b \in G$ *the have the property* $a * b = b * a$*. But when one has this property, this group is called an* ***Abelian group*** *or* ***commutative group****. Every finite group, the set* G *is finite, can be denoted by a finite number of elements of the group, called the* ***generators of the group*** *or* ***group generators****, from which one can construct every element of the group* G *by a finite amount of the group generators with the use of the product operator. Since this group* G *is finite, every subset is also finite. Some groups have additional properties where the product* *operator must fulfil upon, also called **additional group axioms**. Most of the time one would take the minimum amount of elements required to form the set of group generators. The group G can then be defined by the following notation: $G = \langle$ 'group generators' $|$ 'additional group axiom(s)' $\rangle$.*

Proposition 2.2. *1. An identity element of a group is unique.*

2. An inverse of an element of a group is unique

Proof. 1. Let e and f be identities of the group G. So we have $g * e = g = e * g$ for every $g \in G$ and $g * f = g = f * g$ for every $g \in G$. Then we can see that $f = f * e = e$, hence the identity element is unique in G.

2. Let g^{-1} and h^{-1} be inverses of the element g the group G. So we have $g * g^{-1} = e = g^{-1} * g$ and $g * h^{-1} = e = h^{-1} * g$ Then we can see that $h^{-1} = h^{-1} * e = h^{-1} * (g * g^{-1})$ and by group axiom 2, we get $h^{-1} = (h^{-1} * g) * g^{-1} = e * g^{-1} = g^{-1}$, hence the inverse of every element of G is unique in G.

∎

We will now co over a nice example for the definitions above.

Example 2.3. *Consider the quaternion group $(Q_8, *)$, where $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ with the properties:*

a. $i^4 = 1$,

b. $i^2 = j^2 = k^2 = -1$,

c. $i * j = k$, $j * k = i$ *and* $k * i = j$,

where $a^n := a * a * \cdots * a$ for $a \in Q_8$, where there are n a's in total. Note that $i^n = i^4 * i^{n-4} = 1 * i^{n-4} = i^{n-4}$ and since $j^2 = i^2$ and $k^2 = i^2$ we have simiar relations for j and k.
From this we can see that group axiom 1 is fulfilled. Before we go on, we will quickly look at the generators of this group. Let us look at i and j. And we have the additional group axioms $i^4 = 1$ and $i^2 = j^2 = -1$ and $j * i * j^{-1} = i^{-1}$. The first couple are copied from above, we only need to check the last one. We have $1 = i^4 = i^2 * i^2 = j^2 * j^2 = j * j^3 = j^3 * j$, hence $j^{-1} = j^3$. We will now construct every element q of Q_8 with these two elements i and j and these additional properties as $q = i^n * j^m$, where $0 \leq n \leq 3$ and $0 \leq m \leq 1$:

$i^4 = 1$ *and* $i^2 = -1$ *and* $k = i * j$.

$i^3 = i^2 * i = -1 * i = -i$ *and* $i^2 * j = -1 * j = -j$ *and* $i^3 * j = i^2 * (i * j) = -1 * k = -k$.

Hence we can write $Q_8 = \langle i, j | i^4 = 1, i^2 = j^2 = -1, j * i * j^{-1} = i^{-1} \rangle$. *From the last property we can see that* $j * i^n * j^{-1} = (j * i * j^{-1})^n = i^{-m}$, *the inverse of* i^m. *We will now check the other group axioms.*

2. *Let* $i^n * j^m, i^a * j^b, i^k * j^l \in Q_8$. *Then*

$$(i^n * j^m * i^a * j^b) * i^k * j^l = \begin{cases} i^{n+a} * j^b * i^k * j^l & \text{if } m = 0, \\ i^{n-a} * j^{b+1} * i^k * j^l & \text{if } m = 1, \end{cases}$$
$$= \begin{cases} i^{n+a+k} * j^l & \text{if } m = 0 \text{ and } b = 0, \\ i^{n+a-k} j^{l+1} & \text{if } m = 0 \text{ and } b = 1, \\ i^{n-a-k} j^{l+1} & \text{if } m = 1 \text{ and } b = 0, \\ i^{n-a+k+2} * j^l & \text{if } m = 1 \text{ and } b = 1, \end{cases} \tag{2.1}$$

and

$$i^n * j^m * (i^a * j^b * i^k * j^l) = \begin{cases} i^n * j^m * i^{a+k} * j^l & \text{if } b = 0, \\ i^n * j^m * i^{a-k} * j^{l+1} & \text{if } b = 1, \end{cases}$$
$$= \begin{cases} i^{n+a+k} * j^l & \text{if } b = 0, \text{ and } m = 0, \\ i^{n-a-k} * j^{l+1} & \text{if } b = 0 \text{ and } m = 1, \\ i^{n+a-k} * j^{l+1} & \text{if } b = 1 \text{ and } m = 0, \\ i^{n+k-a+2} * j^l & \text{if } b = 1 \text{ and } m = 1. \end{cases} \tag{2.2}$$

Hence we can conclude that $(i^n * j^m * i^a * j^b) * i^k * j^l = i^n * j^m * (i^a * j^b * i^k * j^l)$. *We have used above that* $j^2 = i^2$.

3. *We have for* 1 *that* $i * 1 = i = 1 * i$ *and* $1 * j = j = j * 1$, *so* $1 * i^n * j^m = i^n * j^m = i^n * j^m * 1$ *for all* $0 \le n \le 3$ *and* $0 \le m \le 1$. *Thus* 1 *is the identity of* Q_8.

4. *Let* $i^n * j^m$ *be a random element of* Q_8.
If $m = 0$, *then* $(i^n)^{-1} = i^{4-n} = i^{-n}$. *Since we have* $i^{4-n} * i^n = i^4 = 1$ *and* $i^n * i^{4-n} = i^4 = 1$.
If $m = 1$, *then* $j^{-1} = i^{n-2} * j$. *Since we have* $i^{n-2} * j * i^n * j = i^{-2} * j^2 = i^2 * i^2 = i^4 = 1$ *and* $i^n * j * i^{n-2} * j = i^2 * i^2 = i^4 = 1$.

So Q_8 *is a group with the product operator* $*$ *and has generators* i *and* j *such that* $Q_8 = \langle i, j | i^4 = 1, i^2 = j^2 = -1, j * i * j^{-1} = i^{-1} \rangle$.

Definition 2.4. *A* ***subgroup*** H *of a group* G *is a subset of* G *and forms a group with the endowned product operator.*

The elements in a group which are abelian to every other element of the group form a subgroup of the group. So

Definition 2.5. *The set of all commutative elements of the group, defined by:*

$$\mathcal{Z}(G) = \{z \in G | z * g = g * z \text{ for all } g \in G\}, \tag{2.3}$$

is a subgroup of G, this is easily verifiable. This subgroup $\mathcal{Z}(G)$ is called the ***center*** *of the group G.*

Definition 2.6. *Let N be a subgroup of G, then N is a* ***normal subgroup*** *of G if for every $g \in G$ we have $gN = Ng$ or equivalently $N = gNg^{-1}$. Where we define $gN := \{g * n | n \in N\}$, which is named the* ***left coset*** *of N in G, and analogously $Ng := \{n * g | n \in N\}$, which is named the* ***right coset*** *of N in G, and thus $gNg^{-1} := \{g * n * g^{-1} | n \in N\}$.*

Definition 2.7. *A* ***discrete normal subgroup*** *N of G is a normal subgroup of G with the additional property that there is a neighbourhood U of e in G such that $N \cap U = \{e\}$.*

Definition 2.8. *Let us have two groups $(G, *)$ and $(H, \cdot)$, then a* ***group homomorphism*** *of G to H is a function $\varphi : G \to H$ such that*

$$\varphi(g_1 * g_2) = \varphi(g_1) \cdot \varphi(g_2) \tag{2.4}$$

for all $g_1, g_2 \in G$.
When this group homomorphism is inversable, and the inverse is also an group homomorphism, then we say that φ is ***group isomorphism****.*

This requirement of the function gives us an interesting result.

Proposition 2.9. *Let us have two groups $(G, *)$ and $(H, \cdot)$, and a group homomorphism $\varphi : G \to H$. Then the identity of the group G, denoted e_G, will be sent to the identity of group H, denoted e_H.*

Proof. This can be seen by the following trick. We know from group axiom 3 that for $g \in G$,

$$\varphi(g) = \varphi(e_G * g) = \varphi(e_G) \cdot \varphi(g). \tag{2.5}$$

Since $\phi(g) \in H$, by group axiom 4 there exists an inverse in H, namely $\phi(g)^{-1}$. We will multiply from the right with this inverse and we get

$$e_H = \varphi(g) \cdot \varphi(g)^{-1} = \varphi(e_G) \cdot \varphi(g) \cdot \varphi(g)^{-1} = \varphi(e_G) \cdot e_H = \varphi(e_G). \tag{2.6}$$

∎

What if you have three groups and you have a group homomorphism from group 1 to group 2 and a group homomorphism from group 2 to group 3? Can you then construct a group homomorphism from group 1 to group 3 through group 2? The answer is: 'Yes!'

Theorem 2.10. *Let $(G_1, *)$, $(G_2, \cdot)$ and $(G_3, \times)$ be three groups. Let $\varphi : G_1 \to G_2$ and $\phi : G_2 \to G_3$ be two group homomorphisms. Then the **composition** $\psi = \phi \circ \varphi$, defined by $\psi := \phi(\varphi(g))$ for all $g \in G_1$, is again a group homomorphism.*
We call φ the composed function and ϕ the composing function.

Proof. Let $(G_1, *)$, $(G_2, \cdot)$ and $(G_3, \times)$ be three groups. Let $\varphi : G_1 \to G_2$ and $\phi : G_2 \to G_3$ be two group homomorphisms. Let $g, h \in G_1$. The composition of two functions where the range of the composed function is a subset of the domain of the composing fuction is a fucntion. Hence ψ is a function. We will now look at the $\psi(g * h)$:

$$\begin{aligned} \psi(g * h) &= \phi(\varphi(g * h)) = \phi(\varphi(g) \cdot \varphi(h)) \\ &= \phi(\varphi(g)) \times \phi(\varphi(h)) = \psi(g) \times \psi(h). \end{aligned} \tag{2.7}$$

Since g and h are random in G_1, this result holds for every two elements of G_1. Hence $\psi = \phi \circ \varphi$ is a group homomorphism from G_1 to G_3. ∎

Definition 2.11. *Let us have two groups $(G, *)$ and $(H, \cdot)$ with a group homomorphism $\varphi : G \to H$. The **kernel** of φ homomorphism is defined as all the elements in G that are sent to the identity e_H of H. More compactly*

$$\ker(\varphi) = \{g \in G | \varphi(g) = e_H\}. \tag{2.8}$$

Proposition 2.12. *Let us have two groups $(G, *)$ and $(H, \cdot)$ with a group homomorphism $\varphi : G \to H$. The kernel of φ is a normal subgroup of G.*

Proof. It is evident that $\ker(\varphi) \subseteq G$. We will now check all the requirements from Defintion 2.1 to prove that $\ker(\varphi)$ is a group of with the induced operator from $(G, *)$.

1. Let $g, h \in \ker(\varphi)$, then $\varphi(g * h) = \varphi(g) \cdot \varphi(h) = e_H$, thus $g * h \in \ker(\varphi)$.

2. This is evident, since the operator works the same. Thus since G is a group, it holds that $(g * h) * k = g * (h * k)$ for $g, h, k \in \ker(\varphi)$.

3. From Proposition 2.9 we see that $\varphi(e_G) = e_H$, thus $e_G \in \ker(\varphi)$. Since $\ker(\varphi) \subseteq G$, we have that $g * e_G = g = e_G * g$ for all $g \in \ker(\varphi)$.

4. Let $g \in \ker(\varphi) \subseteq G$, then there exists a $g^{-1} \in G$ such that $g * g^{-1} = e_G = g^{-1} * g$. Look at $\varphi(g^{-1}) = \varphi(g^{-1}) \cdot e_H = \varphi(g^{-1}) \cdot \varphi(g) = \varphi(g^{-1} * g) = \varphi(e_G) = e_H$. Thus we have $g^{-1} \in \ker(\varphi)$.

This makes $\ker(\varphi)$ into a subgroup of G.
Now we have to check that it is normal. Let $g \in G$ and let $k \in \ker(\varphi)$. Then look at $\varphi(g * k * g^{-1}) = \varphi(g) \cdot \varphi(k) \cdot \varphi(g^{-1}) = \varphi(g) \cdot e_H \cdot \varphi(g^{-1}) = \varphi(g) \cdot \varphi(g^{-1}) = \varphi(g * g^{-1}) = \varphi(e_G) = e_H$. Thus $g * k * g^{-1} \in \ker(\varphi)$ for every $k \in \ker(\varphi)$ and every $g \in G$. Thus for every $g \in G$ we have $g \ker(\varphi) g^{-1} \subseteq \ker(\varphi)$.
We now want to proof that for every $g \in G$ we have $\ker(\varphi) \subseteq g \ker(\varphi) g^{-1}$. This is equivalent to saying $g^{-1} \ker(\varphi) g \subseteq \ker(\varphi)$ for every $g \in G$. This is equivalent to saying $h \ker(\varphi) h^{-1} \subseteq \ker(\varphi)$ for every $h \in G$, where we have substituted g^{-1} by h. This is the result we have already proven above. So we $\ker(\varphi) = g \ker(\varphi) g^{-1}$ for every $g \in G$. Hence $\ker(\varphi)$ is a normal subgroup of G. ■

Proposition 2.13. *Let G be a group and N be a normal group of G. Let us denote set of all right cosets of N in G by G/N. This set G/N is a group and is called the* ***quotient group*** *of G by N.*

Proof. Let us look at two elements of the set G/N. These elements are right coset of N in G, so there exist $g, h \in G$ such that these element are Ng and Nh. We will define a product between two cosets as following: $(Ng)(Nh) := \{x * y | x \in Ng, y \in Nh\}$. $x \in Ng$ and $y \in Nh$ if and only if there exist $n, m \in N$ such that $x = n * g$ and $y = m * h$. We will look at the product of x and y:

$$\begin{aligned} x * y &= (n * g) * (m * h) = n * g * m * (g^{-1} * g) * h \\ &= n * (g * m * g^{-1}) * g * h \\ &= n * m' * (g * h) \in Ngh. \end{aligned} \tag{2.9}$$

Where $m' := g * m * g^{-1} \in N$ and $Ngh := N(g * h)$. Now we have $(Ng)(Nh) = Ngh = Nf \in G/N$ for some $f = g * h \in G$. And thus we can devine the product operator of G/N by $(Ng)(Nh) := Ngh$. The group axioms will follow from the fact that G is a group. ■

Theorem 2.14. *Let us have two groups $(G, *)$ and $(H, \cdot)$ with a group homomorphism $\varphi : G \to H$. Then $G/\ker(\varphi)$ is isomorphic to $\mathrm{im}(\varphi)$.*

This theorem is known as the First Isomorphism Theorem.

Proof. The proof of this theorem can be found at the bottom of page 86 of [4], in this book it is the proof of Theorem 16.1. ■

For simplicity from now on we will write $g * h$ as gh for $g, h \in G$. We will do the same for other groups, so you have to be carefull about knowing in which group we are working and which group operation we are dealing with. Furthermore, we will therefore only refer to a group by the set alone. So we will talk about a group G instead of $(G, *)$.
Luckily from a certain point we will only look at matrix Lie groups which have, usually, the standard matrix multiplication as group operator.

Chapter 3

Lie Groups and Lie Algebras

In this section we will go over the basic definitions to create a basis of knowledge where we can build upon. We will also introduce some specific cases of Lie groups, namely **SO**(3) and **SU**(2). It is also important to look at some of the properties that these Lie groups posses.

In section 3.1 I follow paragraph 16.2, in section 3.2 I follow paragraph 16.3, in section 3.3 I follow paragraph 16.4 all from [6].

3.1 Lie Groups

Definition 3.1. *The space of all the $n \times n$-matrices with entries from a field $\mathbb{F}$ is denoted by $\boldsymbol{M}_n(F)$.*

Definition 3.2. *The **general linear group**, denoted by $\boldsymbol{GL}(n, \mathbb{F})$, is the group of $n \times n$ invertible matrices with entries from the field $\mathbb{F}$. The **special linear group**, denoted by $\boldsymbol{SL}(n, \mathbb{F})$, is the group of $n \times n$-matrices with entries from the field $\mathbb{F}$ such that the determinant of these matrices are 1.*

Remember a matrix is invertible if and only if the determinant of the matrix is not equal to zero. So when a matrix has determinant 1, the matrix is invertible. This gives us that $\mathbf{SL}(n, \mathbb{F})$ is a subgroup of $\mathbf{GL}(n, \mathbb{F})$. In most cases this field $\mathbb{F}$ will be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Note that both of these fields are continuous manifolds. For $\mathbb{R}$ this statement is trivial and for $\mathbb{C}$ this means we can see $\mathbb{C}$ localy as $\mathbb{R}^m$ for some m. This m is 2, which is no suprise.
In this thesis we will mainly consider matrix Lie groups. We will start with some important definitions.

Definition 3.3. *A subgroup G of $\boldsymbol{GL}(n, \mathbb{C})$ is **closed** if for every sequence A_m in G that converges to a matrix A, either A is in G or A is not in $\boldsymbol{GL}(n, \mathbb{C})$.*

Definition 3.4. *A **matrix Lie group** is a closed subgroup of* $\boldsymbol{GL}(n,\mathbb{C})$ *for some* $n \in \mathbb{N}$.

For completeness it is good to know the general definition of a Lie group.

Definition 3.5. *A **Lie group** G is a smooth manifold G which is also a group and such that the group product*

$$G \times G \to G$$

and the inverse map $G \to G$ are smooth.

This definition is Definition 1.20 of [7].
This smoothness properties makes the Lie group a valid group construction to describe continuous symmetry, such as the rotations in 3 dimensions. We will see later that this Lie group is the $\mathbf{SO}(3)$ group, see Theorem 3.12 for defintion of $\mathbf{SO}(3)$.

Definition 3.6. *If G_1 and G_2 are two Lie groups, a map from G_1 to G_2 is called a **Lie group homomorphism** of G_1 into G_2 if this map is a continuous group homomorphism of G_1 into G_2. If this map is also bijective and has a continuous inverse, then this map is called a **Lie group isomorphism**. If such a Lie group isomorphism exist, we say G_1 and G_2 are isomorphic or isomorphic to each other.*

Since a Lie group is a group as in defintion 2.1 and a Lie group homomorphism is a group homomorphism as in 2.8, we can use all the results from Chapter 2 onto Lie groups and Lie group homomorphisms.
We will now look at some important examples of Lie groups which will be the basis of the Lie groups discussed in this thesis.

Definition 3.7. *A complex $n \times n$-matrix U is called **unitary** if $U^*U = UU^* = \mathbb{I}$.*

Here we have used the transpose-conjugate of U and it is denoted by U^*. When we use the standard complex inner product, namely $\langle v, w\rangle := v^*w$, for $v, w \in \mathbb{C}^n$, we obtain the following result.

Corollary 3.8. *A complex $n\times n$-matrix U is unitary if and only if $\langle Uv, Uw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{C}^n$.*

Proof. Let $U \in \mathbf{M}_n(\mathbb{C})$ and let $\langle Uv, Uw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{C}^n$. We have $\langle Uv, Uw\rangle = (Uv)^*Uw = v^*U^*Uw$ and $\langle v, w\rangle = v^*w$, so $v^*U^*Uw = v^*w$ for all $v, w \in \mathbb{C}^n$. Hence $U^*U = \mathbb{I}$. When we now take the determinant of this, we get $\det(U^*U) = \det(U^*)\det(U) = (\det(U))^2 = \det(\mathbb{I}) = 1$. From $U^*U = \mathbb{I}$ and $|\det(U)| = 1$ and the fact that n is finite we can conclude that U^* is the inverse of U and thus $UU^* = \mathbb{I}$.

Conversely, let a complex $n \times n$-matrix U be unitary, so $U^*U = UU^* = \mathbb{I}$. Take $v, w \in \mathbb{C}^n$ completely abitrary, and have a look at $\langle Uv, Uw\rangle$. We can write this inner product out, since it represents the standard complex inner product. Hence $\langle Uv, Uw\rangle = (Uv)^*Uw = v^*U^*Uw = v^*\mathbb{I}w = v^*w = \langle v, w\rangle$ for all $v, w \in \mathbb{C}^n$. ∎

Theorem 3.9. *The set of complex $n \times n$ unitary matrices with the standard matrix multiplication forms a group, denoted by **U**(n) and it is called the **unitary group**. The subset of **U**(n) of the matrices with determinant equal to 1 forms a group under the standard matrix multiplication and hence it is a subgroup of **SU**(n). This group is denoted by **SU**(n) and it is called the **special unitary group**.*

Proof. We will first proof that $\mathbf{U}(n)$ is a group for $n \in \mathbb{N}$, with the standard matrix multiplication.

1. If $X, Y \in \mathbf{U}(n)$, then we know $X^*X = XX^* = \mathbb{I} = Y^*Y = YY^*$. We have $(XY)^*XY = Y^*X^*XY = Y^*Y = \mathbb{I}$ and similarly $XY(XY)^* = \mathbb{I}$. So XY is an element of $\mathbf{U}(n)$.

2. If $X, Y, Z \in \mathbf{U}(n)$, then $(XY)Z = X(YZ)$, since the standard matrix multiplication is associative.

3. Look at the identity matrix $\mathbb{I}$. It is trivial that $\mathbb{I}^* = \mathbb{I}$. Since we have $\mathbb{I}^*\mathbb{I} = \mathbb{I}$ and $\mathbb{I}\mathbb{I}^* = \mathbb{I}$, the identity matrix is unitary. Hence $\mathbb{I} \in \mathbf{U}(n)$ and is the identity element since for all $X \in \mathbf{U}(n)$ we have $X\mathbb{I} = \mathbb{I}X = X$.

4. If $X \in \mathbf{U}(n)$, then we also have that $X^* \in \mathbf{U}(n)$ since $(X^*)^* = X$. Since $XX^* = X^*X = \mathbb{I}$, the matrix $X^{-1} = X^* \in \mathbf{U}(n)$ is an inverese of X.

So we have proven that $\mathbf{U}(n)$ is a group with the standard matrix multiplication as multiplication. We will now proof that $\mathbf{SU}(n)$ is a subgroup of $\mathbf{U}(n)$.

1. If $X \in \mathbf{SU}(n)$, then X is unitary and the determinant equals one, so therefore $X \in \mathbf{U}(n)$, and thus $\mathbf{SU}(n) \subseteq \mathbf{U}(n)$.

2. If $X, Y \in \mathbf{SU}(n)$, then $XY \in \mathbf{U}(n)$ and the determinant equals one since $\det(XY) = \det(X)\det(Y) = 1 \cdot 1 = 1$. So $XY \in \mathbf{SU}(n)$.

3. If $X \in \mathbf{SU}(n)$, then we know $XX^* = X^*X = \mathbb{I}$ and $\det(X) = 1$, and that an inverse of X is $X^{-1} = X^*$, which is an element of $\mathbf{U}(n)$. Now look at the determinant of XX^* and we get $\det(XX^*) = \det(X)\det(X^*) = \det(X^*)$ and also $\det(XX^*) = \det(\mathbb{I}) = 1$, so $\det(X^{-1}) = \det(X^*) = 1$. Thus $X^{-1} = X^*$ is an element of $\mathbf{SU}(n)$.

Thus $\mathbf{SU}(n)$ is a subgroup of $\mathbf{U}(n)$. ∎

Note, for $U \in \mathbf{U}(n)$ we have the property $U^*U = \mathbb{I}$, which demands that $|\det(U)| = 1$. This is since $1 = \det(\mathbb{I}) = \det(U^*U) = \det(U^*)\det(U) = \overline{\det(U)}\det(U) = |\det(U)|^2$.

Definition 3.10. *A real $n \times n$-matrix O is called **orthogonal** if $O^TO = OO^T = \mathbb{I}$.*

Analogly to the previous lemma when we use the standard real inner product, so $\langle v, w\rangle := v^Tw$, for $v, w \in \mathbb{R}^n$, we can obtain the following lemma.

Corollary 3.11. *A real $n \times n$-matrix O is orthogonal if and only if $\langle Ov, Ow\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$.*

The proof of this Corollary 3.11 is very similar to the proof of Corollary 3.8.

Proof. Let $O \in \mathbf{M}_n(\mathbb{R})$ and let $\langle Ov, Ow\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$. We have $\langle Ov, Ow\rangle = (Ov)^TOw = v^TO^TOw$ and $\langle v, w\rangle = v^Tw$, so $v^TO^TOw = u^Tw$ for all $v, w \in \mathbb{R}^n$. Hence $T^TO\mathbb{I}$. When we now take the determinant of this, we get $\det(O^TO) = \det(O^T)\det(O) = (\det(O))^2 = \det(\mathbb{I}) = 1$. From $O^TO = \mathbb{I}$ and $|\det(O)| = 1$ and the fact that n is finite we can conclude that O^T is the inverse of O and thus $OO^T = \mathbb{I}$.
Conversely, let a real $n \times n$-matrix O be Orthogonal, so $O^*O = OO^T = \mathbb{I}$. Take $v, w \in \mathbb{R}^n$ completely abitrary, and have a look at $\langle Ov, Ow\rangle$. We can write this inner product out, since it represents the standard real inner product. Hence $\langle Ov, Ow\rangle = (Ov)^TUw = v^TO^TOw = v^T\mathbb{I}w = v^Tw = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$. ∎

Theorem 3.12. *The set of real $n \times n$ orthogonal matrices with the standard matrix multiplication forms a group, denoted by $\boldsymbol{O}(n)$ and it is called the **orthogonal group**. The subset of $\boldsymbol{O}(n)$ of the matrices with determinant equal to 1 forms a group under the standard matrix multiplication and hence it is a subgroup of $\boldsymbol{O}(n)$. This group is denoted by $\boldsymbol{SO}(n)$ and it is called the **special orthogonal group**.*

Proof. This proof is very similar to the proof of Theorem 3.9. We will first proof that $\mathbf{O}(n)$ is a group for $n \in \mathbb{N}$, with the standard matrix multiplication.

1. If $X, Y \in \mathbf{O}(n)$, then we know $X^TX = XX^T = \mathbb{I} = Y^TY = YY^T$. We have $(XY)^TXY = Y^TX^TXY = Y^TY = \mathbb{I}$ and similarly $XY(XY)^T = \mathbb{I}$. So XY is an element of $\mathbf{O}(n)$.

2. If $X, Y, Z \in \mathbf{O}(n)$, then $(XY)Z = X(YZ)$, since the standard matrix multiplication is associative.

3. Look at the identity matrix $\mathbb{I}$. it is trivial that $\mathbb{I}^T = \mathbb{I}$. Since we have $\mathbb{I}^T\mathbb{I} = \mathbb{I}$ and $\mathbb{I}\mathbb{I}^T = \mathbb{I}$, the identity matrix is orthogonal. Hence $\mathbb{I} \in \mathbf{O}(n)$ and is the identity element since for all $X \in \mathbf{O}(n)$ we have $X\mathbb{I} = \mathbb{I}X = X$.

4. If $X \in \mathbf{O}(n)$, then we also have that $X^T \in \mathbf{O}(n)$ since $(X^T)^T = X$. Since $XX^T = X^TX = \mathbb{I}$, the matrix $X^{-1} = X^T \in \mathbf{O}(n)$ is an inverse of X.

So we have proven that $\mathbf{O}(n)$ is a group with the standard matrix multiplication as multiplication. We will now proof that $\mathbf{SO}(n)$ is a subgroup of $\mathbf{O}(n)$.

1. If $X \in \mathbf{O}(n)$, then X is orthogonal and the determinant equals one, so therefore $X \in \mathbf{O}(n)$, and thus $\mathbf{SO}(n) \subseteq \mathbf{O}(n)$.

2. If $X, Y \in \mathbf{SO}(n)$, then $XY \in \mathbf{O}(n)$ and the determinant equals one since $\det(XY) = \det(X)\det(Y) = 1 \cdot 1 = 1$. So $XY \in \mathbf{SO}(n)$.

3. If $X \in \mathbf{SO}(n)$, then we know $XX^T = X^TX = \mathbb{I}$ and $\det(X) = 1$, and that an inverse of X is $X^{-1} = X^T$, which is an element of $\mathbf{O}(n)$. Now look at the determinant of XX^T and we get $\det(XX^T) = \det(X)\det(X^T) = \det(X^T)$ and also $\det(XX^T) = \det(\mathbb{I}) = 1$, so $\det(X^{-1}) = \det(X^T) = 1$. Thus $X^{-1} = X^T$ is an element of $\mathbf{SO}(n)$.

Thus $\mathbf{SO}(n)$ is a subgroup of $\mathbf{O}(n)$. ∎

Note, for $O \in \mathbf{O}(n)$ we have the property $O^TO = \mathbb{I}$, which demands that $|\det(O)| = 1$ hence $\det(O) = \pm 1$. This is since $1 = \det(\mathbb{I}) = \det(O^TO) = \det(O^T)\det(O) = \det(O)\det(U) = \det(U)^2$.

Definition 3.13. *A set R is **connected** if for all $A, B \in R$ there is a **continuous path**, which is a function $A(\cdot) : [0,1] \to R$ such that:*

1. $A(0) = A$

2. $A(1) = B$.

Definition 3.14. *A set R is **simply connected** if every **continuous loop** can be continuously shrunken to a single point in R, as demonstrated on S^2 in Figure 3.1. Here a continuous loop is a special case of a coninuous path from Definition 3.13, with the requirement $A = B$.*

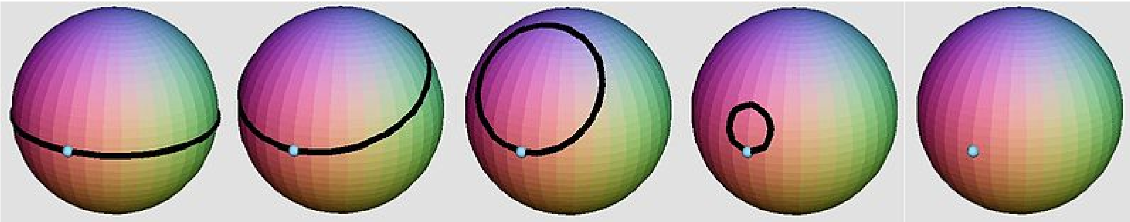

Figure 3.1: A demonstration of a loop being continuously shrunken down to a single point on the 2 dimensional sphere S^2.
The URL of this picture can be found in the Bibliography under [8]

To give you a clear view what the difference is between connected and simply connected have a look at Figure 3.1.

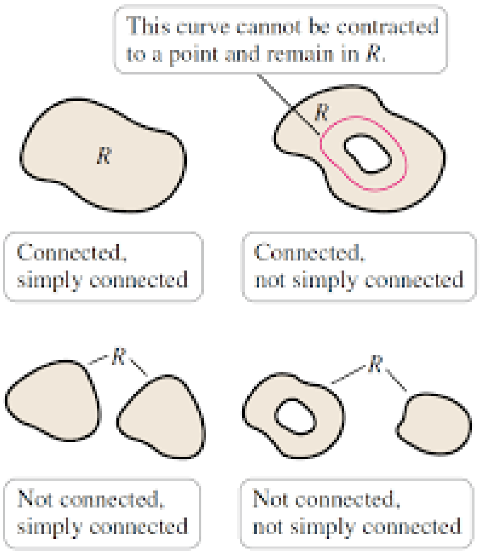

Figure 3.2: A demonstration of the differences between sets that are or are not connected and are or are not simply connected.
The top left set R is connected and simply connected. The top right set R is connected, but not simply connected. The bottom left set R is not connected, but it is simply connected. And lastly, the bottom right set R is neither connected nor simply connected.
The URL of this picture can be found in the Bibliography under [9].

Example 3.15. *Let us have a look at the Lie group $\boldsymbol{SO}(3)$. In the bottom of page 293 of [10] we can see that $\boldsymbol{SO}(3)$ is isomorphic to the projective 3-real space $\mathbb{R}P^3$. This set is the set of all lines going through the origin and through a point on the 3 dimensional sphere. The Lie group $\boldsymbol{SO}(3)$ represents all the rotations in 3 dimensions. Every rotation in 3 dimensions has a rotation axis and a rotation angle α. One can express the rotation axis as a vector in $\vec{\alpha} \in \mathbb{R}^3$ and let the magnitude of the vector be the rotation angle, $\alpha = |\vec{\alpha}|$, and the normalized rotation axis be $\hat{\alpha} = \vec{\alpha}/\alpha$. For clarification we will denote the rotation matrix of a rotation around axis $\hat{\alpha}$ with* *angle α as $R(\vec{\alpha}) = R(\hat{\alpha}, \alpha)$. Now we can express some properties for $\vec{v} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$:*

1. $R(\hat{v}, \alpha) = R(-\hat{v}, -\alpha)$, *note* $\vec{v} = \alpha\hat{v} = (-\alpha)\cdot(-\hat{v})$.
2. $R(\hat{v}, \alpha) = R(\hat{v}, \alpha + 2\pi)$.

This notation is commonly used in physics and sometimes also in mathematics. The Lie group $\boldsymbol{SO}(3)$ is topologically isomorphic to the real projective space $\mathbb{RP}^3$, which is topologically isomorphic to the 3 dimensional sphere with antipolar indentity, $S^3/\{\pm 1\}$. Antipolar identity means that we view two elements x and y the same when $y = -x$. This isomorphism is quite easy to see. One can view the elements of $\mathbb{RP}^3$ as lines that go through the origin and through a point on the three dimensional sphere. These lines have no direction. And when a line goes through the origin and through a point x on S^3 then this line will also go through the point $-x$ on S^3. Thus this line is equal to the line that goes through the origin and through $-x$. With this visualisation, we have connected x and $-x$ on S^3 by a line and we view these points as the same. When a line goes throught the origin then it has only two intersections with S^3 and these two points are antipolars of each other. With this in mind there is a bijection between $\mathbb{RP}^3$ and $S^3/\{\pm 1\}$. This bijection sents a line from $\mathbb{RP}^3$ that goes through a point x of $S^3/\{\pm 1\}$ to x in $S^3/\{\pm 1\}$.

Lemma 3.16. *The matrix Lie group $\boldsymbol{SU}(2)$ can be written as follows:*

$$\boldsymbol{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{3.1}$$

Proof. If $X \in \mathbf{SU}(2)$, then $X^*X = XX^* = \mathbb{I}$ and $\det(X) = 1$. We can write X as a 2×2-matrix, where we use 2 column vectors $\vec{u}, \vec{w} \in \mathbb{C}^2$, such that $X = \begin{pmatrix} \vec{u} & \vec{w} \end{pmatrix}$. Now we will use the relation $X^*X = \mathbb{I}$:

$$X^*X = \begin{pmatrix} \vec{u}^* \\ \vec{w}^* \end{pmatrix} \begin{pmatrix} \vec{u} & \vec{w} \end{pmatrix} = \begin{pmatrix} \vec{u}^*\vec{u} & \vec{u}^*\vec{w} \\ \vec{w}^*\vec{u} & \vec{w}^*\vec{w} \end{pmatrix} = \begin{pmatrix} \langle \vec{u}, \vec{u} \rangle & \langle \vec{u}, \vec{w} \rangle \\ \langle \vec{w}, \vec{u} \rangle & \langle \vec{w}, \vec{w} \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}. \tag{3.2}$$

So we see that $\vec{u}$ as well as $\vec{w}$ are unit vectors in $\mathbb{C}^2$ and that the are perpendicular to each other. Let $\alpha, \beta \in \mathbb{C}$ such that $\vec{u} := \begin{pmatrix} \alpha & \beta \end{pmatrix}^T$. Then we have $\langle \vec{u}, \vec{u} \rangle = |\alpha|^2 + |\beta|^2 = 1$. There are two possibilities for $\vec{w}$, namely $\vec{w} = \pm \begin{pmatrix} -\bar{\beta} & \bar{\alpha} \end{pmatrix}^T$. To determine the sign infront of the vector, we can use the property that $\det(X) = 1$, so

$$\begin{aligned} \det(X) = \det \begin{pmatrix} \vec{u} & \vec{w} \end{pmatrix} = \det \begin{pmatrix} \alpha & \mp\bar{\beta} \\ \beta & \pm\bar{\alpha} \end{pmatrix} &= \pm\alpha\bar{\alpha} \pm \beta\bar{\beta} \\ &= \pm(|\alpha|^2 + |\beta|^2) = \pm\langle \vec{u}, \vec{u} \rangle = \pm 1. \end{aligned} \tag{3.3}$$

To get the determinant of the matrix X to be equal to 1, we have to choose $\vec{w} = \begin{pmatrix}-\bar{\beta} & \bar{\alpha}\end{pmatrix}^T$. And hence $X = \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix}$, for $\alpha, \beta \in \mathbb{C}$, and we now know

$$\mathbf{SU}(2) \subseteq \left\{ \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{3.4}$$

Let $X \in \left\{ \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$, then for some $\alpha, \beta \in \mathbb{C}$, we have $X = \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix}$. We will check the three properties that a 2×-matrix must satisfy:

$$\begin{aligned}
X^*X &= \begin{pmatrix}\bar{\alpha} & \bar{\beta} \\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix} = \begin{pmatrix}\bar{\alpha}\alpha + \bar{\beta}\beta & -\bar{\alpha}\bar{\beta} + \bar{\beta}\bar{\alpha} \\ -\beta\alpha + \alpha\beta & \beta\bar{\beta} + \alpha\bar{\alpha}\end{pmatrix} \\
&= \begin{pmatrix}|\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\beta|^2 + |\alpha|^2\end{pmatrix} = \mathbb{I} \\
XX^* &= \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix}\begin{pmatrix}\bar{\alpha} & \bar{\beta} \\ -\beta & \alpha\end{pmatrix} = \begin{pmatrix}\alpha\bar{\alpha} + \beta\bar{\beta} & -\alpha\bar{\beta} + \bar{\beta}\alpha \\ -\beta\bar{\alpha} + \bar{\alpha}\beta & \beta\bar{\beta} + \alpha\bar{\alpha}\end{pmatrix} \\
&= \begin{pmatrix}|\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\beta|^2 + |\alpha|^2\end{pmatrix} = \mathbb{I} \\
\det(X) &= \det\begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix} = \alpha\bar{\alpha} + \bar{\beta}\beta = |\alpha|^2 + |\beta|^2 = 1.
\end{aligned} \tag{3.5}$$

So X satisfies all the properties related to $\mathbf{SU}(2)$, and we can conclude $X \in \mathbf{SU}(2)$. This result combined with the result which we found earlier, we have proven that:

$$\mathbf{SU}(2) \subseteq \left\{ \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{3.6}$$

■

Theorem 3.17. *The Lie group* $\boldsymbol{SU}(2)$ *is connected and simply connected.*

Proof. Since $\mathbf{SU}(2) \cong S^3$ we can use the little example below Proposition 11 of [11] to conclude that $\mathbf{SU}(2)$ is connected. Note we have used the definition of path-connected in stead of connected, Corollary 14. of [11] tells us that: '*Every path-connected space is connected.*'. So $\mathbf{SU}(2)$ is indeed connected.
We can see from Lemma 3.16 that every element of $\mathbf{SU}(2)$ can be written as

$$\begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix}, \tag{3.7}$$

where $|\alpha|^2 + |\beta|^2 = 1$ with $\alpha, \beta \in \mathbb{C} \cong \mathbb{R}^2$. Let $\alpha = a + ib$ and $\beta = c + id$, with $a, b, c, d \in \mathbb{R}$, then the requirement $|\alpha|^2 + |\beta|^2 = 1$ transforms into $a^2 + b^2 + c^2 + d^2 = 1$. With this one can see that $\mathbf{SU}(2)$ is topologically isomorphic to the 3 dimensional unit sphere, S^3, inside $\mathbb{C}^2 \cong \mathbb{R}^4$. In Proposition 1.14 of [10] they say that S^n has a trivial fundamental group for every $n \geq 2$, hence S^n is simply connected for every $n \geq 2$. Thus in particular S^3 is simply connected.
Furthermore, S^3 is also connected, between every two points runs at least one circle line to connect them. Since simply connectedness is a topological property, $\mathbf{SU}(2)$ is simply connected.

■

Theorem 3.18. *The Lie group* $\boldsymbol{SO}(3)$ *is connected but not simply connected.*

Proof. We have seen that the matrix Lie group $\mathbf{SO}(3)$ is isomorphic to $\mathbb{RP}^3$, which is isomorphic to $S^3/\{\pm\mathbb{I}\}$ (see Example 3.15). We also know from Lemma 3.16 that $\mathbf{SU}(2)$ is isomorphic to S^3. Thus we can conclude that $\mathbf{SO}(3)$ is isomorphic to $\mathbf{SU}(2)/\{\pm\mathbb{I}\}$. Theorem 3.17 tells us that $\mathbf{SU}(2)$ is connected. Further more S^3 is compact, i.e. closed and bounded. So we can use Proposition 2 of [12] to conlcude that $\mathbf{SO}(3)$ is connected, and compact.
In the bottom of page 293 of [10] we can see that $\mathbf{SO}(3)$ is isomorphic to the projective 3-real space $\mathbb{R}P^3$. Furthermore, in Example 1.43 there is proven that the fundamental group of $\mathbb{R}P^n$ equals $\mathbb{Z}_2$, this is the group of two elements, for $n \geq 2$. This means that for $n \geq 2$ the $\mathbb{R}P^n$ is not simply connected. Since simply connectedness is a topological property, it will be caried on by an isomorphism. Thus $\mathbf{SO}(3)$ is also not simply connected. ■

We will later see that this result is of utmost importance to the mathematical argument why we would be interested in studying Lie group and Lie algebra to further understand and describe the spin and thereby the electron.

3.2 Lie Algebras

Since we have developed a basic understanding of Lie groups, we can now dive into the concept of Lie algebras. We will start with a definition where on first glance there is no connection to Lie groups, in the next paragraph we will introduce a very common relation between a Lie group and its Lie algebra.

Definition 3.19. *A* ***Lie algebra*** *over a field* $\mathbb{F}$ *is a vector space* $\mathfrak{g}$ *over* $\mathbb{F}$*, together with a* ***bracket map*** $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, which has the following properties:*

1. *$[\cdot,\cdot]$ is bilinear.*
   This means that $[W+\lambda X, Y+\mu Z] = [W, Y+\mu Z]+\lambda[X, Y+\mu Z] = [W+\lambda X, Y]+\mu[W+\lambda X, Z]$ holds for all $\lambda,\mu \in \mathbb{F}$ and for all $W,X,Y,Z \in \mathfrak{g}$.

2. *$[X,Y] = -[Y,X]$ for all $X,Y \in \mathfrak{g}$.*

3. *$[X,X] = 0$ for all $X \in \mathfrak{g}$.*

4. *For all $X,Y,Z \in \mathfrak{g}$ we have the Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{3.8}$$

Note that from bilinearity of this bracket it follows that $[W + \lambda X, Y + \mu Z] = [W,Y] + \lambda[X,Y] + \mu[W,Z] + \lambda\mu[X,Z]$, for $\lambda,\mu \in \mathbb{F}$ and $W,X,Y,Z \in \mathfrak{g}$. One can check this with first applying linearity in the rightside of the bracket and then in the left side.

Proposition 3.20. *Let $\mathcal{A}$ be an associative algebra and let $\mathfrak{g}$ be a subspace of $\mathcal{A}$ such that for all $X,Y \in \mathfrak{g}$ we have that $XY - YX$ is again in $\mathfrak{g}$. Then by defining the bracket map by:*

$$[X,Y] := XY - YX, \tag{3.9}$$

we have made $\mathfrak{g}$ into a Lie algebra.

Proof. Since $\mathfrak{g}$ is a subspace of $\mathcal{A}$, it is a vector space on its own. To further prove that $\mathfrak{g}$ is a Lie algebra, one has to check the 4 requirements for the bracket map from Definition 3.19.

1. Let $\lambda,\mu \in \mathbb{F}$ and let $W,X,Y,Z \in \mathfrak{g}$, now look at $[W + \lambda X, Y + \mu Z]$.

$$\begin{aligned}
[W + \lambda X, Y + \mu Z] &= (W + \lambda X)(Y + \mu Z) - (Y + \mu Z)(W + \lambda X) \\
&= W(Y + \mu Z) - (Y + \mu Z)W \\
&\quad + \lambda\left(X(Y + \mu Z) - (Y + \mu Z)X\right) \\
&= [W, Y + \mu Z] + \lambda[X, Y + \mu Z] \\
[W + \lambda X, Y + \mu Z] &= (W + \lambda X)(Y + \mu Z) - (Y + \mu Z)(W + \lambda X) \\
&= (W + \lambda X)Y - Y(W + \lambda X) \\
&\quad + \mu\left((W + \lambda X)Z - Z(W + \lambda X)\right) \\
&= [W + \lambda X, Y] + \mu[W + \lambda X, Z]
\end{aligned} \tag{3.10}$$

2. Let $X,Y \in \mathfrak{g}$, then $[X,Y] = XY - YX = -(YX - XY) = -[Y,X]$.

3. Let $X \in \mathfrak{g}$, then $[X,X] = XX - XX = 0$.

4. Let $X, Y, Z \in \mathfrak{g}$, then

$$\begin{aligned}[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] &= [X,YZ]-[X,ZY]+[Y,ZX] \\ &\quad -[Y,XZ]+[Z,XY]-[Z,YX] \\ &= XYZ-XZY+XZY-XYZ \\ &\quad -YZX+YZX-YXZ+YXZ \\ &\quad +ZYX-ZXY+ZXY-ZYX \\ &= 0 \end{aligned} \tag{3.11}$$

Thus the subspace $\mathfrak{g}$ with the bracket map $[X,Y] = ZY - YZ$, where $X, Y \in \mathfrak{g}$, forms a Lie algebra. ■

At a later point we will see that every Lie algebra is isomorphic to such a Lie algebra from Lemma 3.20. This follows from the Poincaré-Birkhoff-Witt theorem

Definition 3.21. *If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras, a map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a* ***Lie algebra homomorphism*** *if ϕ is linear and ϕ satisfies*

$$\phi([x,y]) = [\phi(x), \phi(y)] \tag{3.12}$$

for all $x, y \in \mathfrak{g}_1$. A Lie algebra homomorphism is called a ***Lie algebra isompor-phism*** *if it is bijective.*

3.3 The Matrix Exponential

The well-known exponential function e^x or $\exp(x)$ is defined by taking the xth-power of Eulers number e where x is a scalar of $\mathbb{R}$ or $\mathbb{C}$. However, we are mainly working with matrices, so therefore we are interested in developing a well-defined manner to calculate the exponential of a matrix such that the case of a 1×1-matrix, hence a scalar, gives us the same answer as the traditional scalar exponential function. Matrix multiplication and matrix addition are well-defined and for the case of 1×1-matrix it gives the same results as scalar multiplication and scalar addition. Hence defining the matrix exponential as a power series gives us a natural extension from the scalar domain to a matrix domain.

Definition 3.22. *The* ***matrix exponential*** *of a matrix $X \in \boldsymbol{M}_n(\mathbb{F})$ is given by:*

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}, \tag{3.13}$$

where $X^0 = I$.

Where the field $\mathbb{F}$ is the real or complex numbers, $\mathbb{R}$ or $\mathbb{C}$. From here on we will take $\mathbb{F} = \mathbb{C}$. It is not important to know everything about a field. Especially since we will only look at $\mathbb{R}$ and $\mathbb{C}$ and you can just look at them as usual. But if you are interested you could have a look at [13]. Since we will be taking primarily the exponentials of matrices we will refer to the matrix exponential simply by the exponential.

Proposition 3.23. *For $X \in \boldsymbol{M}_n(\mathbb{C})$ the power series from Definition 3.22 converges for all $X \in \boldsymbol{M}_n(\mathbb{C})$ and e^X is a continuous function.*

Proof. The proof of this proposition can be found in [7] under the proof of Proposition 2.1. ∎

Theorem 3.24. *The matrix exponential has the following properties for all $X, Y \in \boldsymbol{M}_n(\mathbb{C})$:*

1. $e^{\boldsymbol{0}} = \mathbb{I}$.
2. $e^{X^T} = (e^X)^T$ *and* $e^{X^*} = (e^X)^*$.
3. *If A is an invertible $n \times n$-matrix, so $A \in \boldsymbol{GL}(n, \mathbb{C})$, then*
$$e^{AXA^{-1}} = Ae^X A^{-1}. \tag{3.14}$$
4. $\det(e^X) = e^{\mathrm{tr}(X)}$.
5. *If $XY = YX$, then* $e^{X+Y} = e^X e^Y$.
6. e^X *is invertible and* $(e^X)^{-1} = e^{-X}$.
7. *We have*
$$e^{X+Y} = \lim_{k\to\infty} \left(e^{X/k} e^{Y/k}\right)^k. \tag{3.15}$$
This result is known as the ***Lie product formula****.*

Property 7 is in particular interesting when $XY \neq YX$, since in that case we can not use property 5. Property 4, 5 and 7 are of this Theorem are Exercises 5, 6 and 7, resp. of Chapter 16 from [6].

Proof. 1. We assumed $X^0 = \mathbb{I}$ for any $X \in \mathbf{M}_n(\mathbb{C})$

$$\begin{aligned} e^{\mathbf{0}} &= \lim_{X\to\mathbf{0}} e^X = \lim_{X\to\mathbf{0}} \sum_{k=0}^{\infty} \frac{X^k}{k!} = \lim_{X\to\mathbf{0}} \left(\frac{X^0}{0!} + \sum_{k=1}^{\infty} \frac{X^k}{k!} \right) \\ &= \lim_{X\to\mathbf{0}} \left(I + X \sum_{k=1}^{\infty} \frac{X^{k-1}}{k!} \right) = I + \lim_{X\to\mathbf{0}} X \sum_{k=1}^{\infty} \frac{X^{k-1}}{k!} = \mathbb{I}. \end{aligned} \tag{3.16}$$

Since $\left|\sum_{k=1}^{\infty}\frac{X^{k-1}}{k!}\right| \leq \left|\sum_{k=0}^{\infty}\frac{X^{k}}{k!}\right| = |e^X|$, and we know from Proposition 3.23 that e^X converges, than also $|e^X|$ and hence $\sum_{k=1}^{\infty}\frac{X^{k-1}}{k!}$ is finite.

2. For $X_1,\dots,X_n \in \mathbf{M}_n(\mathbb{C})$ the following hold: $(X_1+\dots+X_n)^T = X_n^T+\dots+X_1^T$, $(X_1+\dots+X_n)^* = X_n^*+\dots+X_1^*$, $(X_1\dots X_n)^T = X_n^T\dots X_1^T$ and $(X_1\dots X_n)^* = X_n^*\dots X_1^*$. Since here we are dealing with $X_i = X_j$ for $1 \leq i,j \leq n$ we have $(X^n)^T = (X^T)^n$ and $(X^n)^* = (X^*)^n$. So when plugging this into the power series we find:
$$e^{X^T} = \sum_{k=0}^{\infty}\frac{(X^T)^k}{k!} = \sum_{k=0}^{\infty}\frac{(X^k)^T}{k!} = \left(\sum_{k=0}^{\infty}\frac{X^k}{k!}\right)^T = \left(e^X\right)^T, \tag{3.17}$$
and
$$e^{X^*} = \sum_{k=0}^{\infty}\frac{(X^*)^k}{k!} = \sum_{k=0}^{\infty}\frac{(X^k)^*}{k!} = \left(\sum_{k=0}^{\infty}\frac{X^k}{k!}\right)^* = \left(e^X\right)^*. \tag{3.18}$$

3. Let $A \in \mathbf{GL}(n,\mathbb{C})$, hence A has an inverse denoted by A^{-1}. Let us look at $(AXA^{-1})^k = AXA^{-1}AXA^{-1}\dots AXA^{-1}$ every A^{-1} and A are next to each other expect the outer ones, and $A^{-1}A = \mathbb{I}$. Hence we get $(AXA^{-1})^k = AX\mathbb{I}X\mathbb{I}\dots\mathbb{I}XA^{-1} = AX^kA^{-1}$. When we fill this into the power serie of the exponential of AXA^{-1} we get:
$$e^{AXA^{-1}} = \sum_{k=0}^{\infty}\frac{\left(AXA^{-1}\right)^k}{k!} = \sum_{k=0}^{\infty}\frac{A\left(X\right)^k A^{-1}}{k!} = A\left(\sum_{k=0}^{\infty}\frac{X^k}{k!}\right)A^{-1} = Ae^XA^{-1}. \tag{3.19}$$

4. Let $X \in \mathbf{M}_n(\mathbb{C})$ be diagonalizable, then there exists $Y \in \mathbf{M}_n(\mathbb{C})$ and $A \in \mathbf{GL}(n,\mathbb{C})$ such that $X = AYA^{-1}$. Then we have $\det(e^X) = \det(e^{AYA^{-1}}) = \det(Ae^YA^{-1}) = \det(A)\det(e^Y)\det(A^{-1}) = \det(e^Y)$ where we used the property 3. Since Y is diagonalizable, the indices at on the diagonal are the eigenvalues $\lambda_1,\dots,\lambda_n \in \mathbb{C}$, so
$$\begin{aligned}\det(e^X) = \det(e^Y) &= \det\left(\exp\begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_n\end{pmatrix}\right)\\ &= \det\begin{pmatrix}e^{\lambda_1} & 0 & \dots & 0\\ 0 & e^{\lambda_2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & e^{\lambda_n}\end{pmatrix} = e^{\lambda_1}e^{\lambda_2}\dots e^{\lambda_n} = e^{\lambda_1+\lambda_2+\dots+\lambda_n}\\ &= e^{\mathrm{tr}(Y)} = e^{\mathrm{tr}((AA^{-1})Y)} = e^{\mathrm{tr}(AYA^{-1})} = e^{\mathrm{tr}(X)}.\end{aligned} \tag{3.20}$$

Note that the trace of the product of two matrices $A, B \in \mathbf{M}_n(\mathbb{C})$ is commutative, i.e. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
Now let X not be diagonalizable, then there exists a sequence $(X_k)_{k\in\mathbb{N}} \subseteq \mathbf{M}_n(\mathbb{C})$ such that $\lim k \to \infty X_k = X$ and every X_k is diagonalizable. This sequence exists because the diagonalizable matrices are dense in $\mathbf{M}_n(\mathbb{C})$. So for the sequence $(X_k)_{k\in\mathbb{N}}$ we have two other sequences $(Y_k)_{k\in\mathbb{N}} \subseteq \mathbf{M}_n(\mathbb{C})$ and $(A_k)_{k\in\mathbb{N}} \subseteq \mathbf{GL}(n, \mathbb{C})$, such that for every $k \in \mathbb{N}$ they satisfy $X_k = A_k Y_k A_k^{-1}$. Then we can apply the result from above on X_k, and hence we get $\det(e^{X_k}) = e^{\mathrm{tr}(X_k)}$ for every $k \in \mathbb{N}$. Now we will take the limit and get the following result

$$\det(e^X) = \lim_{k\to\infty} \det(e^{X_k}) = \lim_{k\to\infty} e^{\mathrm{tr}(X_k)} = e^{\mathrm{tr}\, Y}. \tag{3.21}$$

Since the determinant, trace and exponential are continous functions as they can be expressed as polynomials, one can move the limit in and out the functions.

5. Let $XY = YX$, and let us have a look at e^{X+Y}. We will first write it out as a power series

$$e^{X+Y} = \sum_{k=0}^{\infty} \frac{(X+Y)^k}{k!}. \tag{3.22}$$

Here we can use the binomial Theorem, which tells us, when $XY = YX$, then $(X+Y)^k = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} X^n Y^{k-n}$. So when applying this we will get

$$\begin{aligned} e^{X+Y} &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} X^n Y^{k-n} \\ &= \sum_{n=0}^{\infty} \sum_{k-n=0}^{\infty} \frac{X^n}{n!} \frac{Y^{k-n}}{(k-n)!} \\ &= \left(\sum_{n=0}^{\infty} \frac{X^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{Y^m}{m!} \right) \\ &= e^X e^Y, \end{aligned} \tag{3.23}$$

where we have substituted $k - n$ by m since $k - n \geq 0$ and an integer.

6. Since $X \in \mathbf{M}_n(\mathbb{C})$, so is $-X \in \mathbf{M}_n(\mathbb{C})$, and we can apply previous property where we take $X = X$ and $Y = -X$, since $X(-X) = -XX$. And with the use of the first property one gets: $e^X e^{-X} = e^{X-X} = e^{\mathbf{0}} = \mathbb{I}$

7. We will prove this part in three steps. Before that we will first define the logarithm of $A \in \mathbf{M}_n(\mathbb{C})$ by the powerseries:

$$\log(A) := A - \mathbb{I} - \frac{(A-\mathbb{I})^2}{2} + \frac{(A-\mathbb{I})^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{(\mathbb{I}-A)^k}{k}, \tag{3.24}$$

whenever the series converges. Theorem 2.8 from [7] tells us when A is sufficiently close to $\mathbb{I}$, then $\log(A)$ is defined and we have the relation $e^{\log(A)} = A$. The norm $\|\cdot\|$ we will be using is defined a bit above Thoerem 2.8 in Definition 2.2 in [7]. This norm is called the **Hilbert-Schmidt norm**.

i We will prove there exists a constant $C \in \mathbb{R}$ such that for all A with $\|A - \mathbb{I}\| < \frac{1}{2}$ we have $\|\log(A) - (A - \mathbb{I})\| \leq C\|A - \mathbb{I}\|^2$.
Let $A \in \mathbf{M}_n(\mathbb{C})$ such that $\|A - \mathbb{I}\| < \frac{1}{2}$, then

$$\begin{aligned}
\|\log(A) - (A - \mathbb{I})\| &= \left\| \sum_{k=1}^{\infty} \frac{-(\mathbb{I} - A)^k}{k} - (A - \mathbb{I}) \right\| \\
&= \left\| \sum_{k=2}^{\infty} \frac{-(\mathbb{I} - A)^k}{k} \right\| \\
&= \left\| -(\mathbb{I} - A)^2 \sum_{k=0}^{\infty} \frac{(\mathbb{I} - A)^k}{k+2} \right\| \\
&= \left\| \sum_{k=0}^{\infty} \frac{(\mathbb{I} - A)^k}{k+2} \right\| \|A - \mathbb{I}\|^2 \\
&\leq \sum_{k=0}^{\infty} \frac{\|\mathbb{I} - A\|^k}{k+2} \|A - \mathbb{I}\|^2 \\
&\leq \sum_{k=0}^{\infty} \frac{1}{(k+2)2^k} \|A - \mathbb{I}\|^2 \\
&\leq C\|A - \mathbb{I}\|^2.
\end{aligned} \tag{3.25}$$

Where we have chosen $0 < C = \sum_{k=0}^{\infty} \frac{1}{(k+2)2^k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$. The sequence $\left(\frac{1}{(k+2)2^k}\right)_{k \geq 0}$ has only positive elements and the summation has an upper limit, namely 2, hence the sum converges. □

From this we can conclude that

$$\log(A) = A - \mathbb{I} + O\left(\|A - \mathbb{I}\|^2\right). \tag{3.26}$$

ii We will now prove that for all $X, Y \in \mathbf{M}_n(\mathbb{C})$ and sufficiently large $k \in \mathbb{N}$ one has

$$\log\left(e^{X/k} e^{Y/k}\right) = \frac{X}{k} + \frac{Y}{k} + O\left(\frac{1}{k^2}\right). \tag{3.27}$$

Here one choses the X and Y and then pick a $k \in \mathbb{N}$. With this process X and Y are finite, with this we mean the norm is finite, and the k is chosen big enough such that $O\left(\frac{1}{k^2}\right)$ is finite.

Let $X, Y \in \mathbf{M}_n(\mathbb{C})$ and let $k \in \mathbb{N}$ be sufficiently large such that $e^{X/k}e^{Y/k}$ is sufficiently close to $\mathbb{I}$ so that $\log\left(e^{X/k}e^{Y/k}\right)$ is defined.

$$\log\left(e^{X/k}e^{Y/k}\right) = e^{X/k}e^{Y/k} - \mathbb{I} - \frac{\left(e^{X/k}e^{Y/k} - \mathbb{I}\right)^2}{2} + \dots \ . \tag{3.28}$$

Now we will plug in the power series of the exponential, and we get

$$\begin{aligned}\log\left(e^{X/k}e^{Y/k}\right) &= \left(\mathbb{I} + \frac{X}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\mathbb{I} + \frac{Y}{k} + O\left(\frac{1}{k^2}\right)\right) - \mathbb{I} + O\left(\frac{1}{k^2}\right) \\ &= \frac{X}{k} + \frac{Y}{k} + O\left(\frac{1}{k^2}\right).\end{aligned} \tag{3.29}$$

□

iii Lastly, let $X, Y \in \mathbf{M}_n(\mathbb{C})$ and let $k \in \mathbb{N}$ be sufficiently large enough such that we can use the result of part ii. We will now combine the two results from above

$$\begin{aligned}\lim_{k\to\infty}\left(e^{X/k}e^{Y/k}\right)^k &= \lim_{k\to\infty}\left(e^{\log\left(e^{X/k}e^{Y/k}\right)}\right)^k \\ &= \lim_{k\to\infty}\left(e^{X/k+Y/k+O\left(1/k^2\right)}\right)^k \\ &= \lim_{k\to\infty}\left(e^{X+Y+O(1/k)}\right) \\ &= e^{X+Y}.\end{aligned} \tag{3.30}$$

■

Proposition 3.25. *The derivative of the matrix exponential function of tX over $t \in \mathbb{R}$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tX} = Xe^{tX} = e^{tX}X, \tag{3.31}$$

where $X \in \boldsymbol{M}_n(\mathbb{C})$.
And if one takes $t = 0$, one gets

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX} = Xe^{tX}\right)\right]_{t=0} = X, \tag{3.32}$$

where $X \in \boldsymbol{M}_n(\mathbb{C})$.

Proof. The proof of this proposition can be found at the top of page 34 of [7], in this book it is the proof of Proposition 2.4. ■

Chapter 4

The Connection between Matrix Lie Groups and Lie Algebras

In previous chapter we have created the foundation to understand this chapter. In this chapter we will be mainly focussing on matrix Lie groups and their properties. We will also go over the connections between matrix Lie groups and their Lie algebra. Furthermore, we will introduce the Lie algebras $\mathbf{so}(n)$ and $\mathbf{su}(n)$ of the matrix Lie groups $\mathbf{SO}(n)$ and $\mathbf{SU}(n)$, respectively, with a specific look at the case where $n = 3$ and $n = 2$, respectively.

In this chapter I follow paragraphs 16.5 and 16.6 from [6].

4.1 The Lie Algebra of a Matrix Lie Group

Definition 4.1. *If $G \subseteq \boldsymbol{GL}(n,\mathbb{C})$ is a matrix Lie group, then the set $\mathfrak{g}$ defined as follows:*

$$\mathfrak{g} := \left\{ X \in \boldsymbol{M}_n(\mathbb{C}) | e^{tX} \in G \text{ for all } t \in \mathbb{R} \right\} \tag{4.1}$$

*is the **Lie algebra of** G.*

One can wonder if this Lie algebra of G is indeed a Lie algebra. Proposition 4.2 tells us: 'Yes!'. The definition of a Lie algebra of a matrix Lie group might be quite abstract. For that reason have a look at Figure 4.1.

Figure 4.1: A demonstration of the relation between the Lie algebra and Lie group $\mathbf{SO}(3)$. Here one can see that the Lie algebra $\mathbf{so}(3)$ is the tangent space t the identity of the Lie group $\mathbf{SO}(3)$. One can use this picture to visualize the Lie algebra and its connection to the Lie group. Lemma 4.11 tells us there is a diffeomorphism from a neighbourhood of 0 of the Lie algebra to a neighbourhood of $\mathbb{I}$ of the matrix Lie group. Thus this lemma back up this way of visualizing the Lie algebra.
The URL of this picture can be found in the Bibliography under [14]

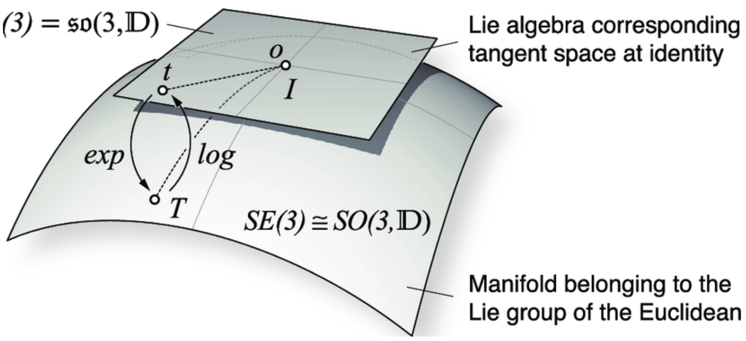

Figure 4.1:

Proposition 4.2. *For any matrix Lie group G, the Lie algebra $\mathfrak{g}$ of G has the following properties:*

1. *The zero matrix $\mathbf{0}$ is an element of $\mathfrak{g}$.*
2. *If $X \in \mathfrak{g}$, then $tX \in \mathfrak{g}$ for all $t \in \mathbb{R}$.*
3. *If $X, Y \in \mathfrak{g}$, then $X + Y \in \mathfrak{g}$.*
4. *If $A \in G$ and $X \in \mathfrak{g}$, then $AXA^{-1} \in \mathfrak{g}$.*
5. *If $X, Y \in \mathfrak{g}$, then the commutator $[X, Y] := XY - YX$ belongs to $\mathfrak{g}$.*

Proof. 1. From Definition 3.3 a Lie group is a subgroup of $\mathbf{GL}(n, \mathbb{C})$, hence it contains the identity element of this group, the identity matrix $\mathbb{I}$. When we use the defintion of a Lie algebra of a group G and choose $X = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix, then with the use of property 1 of Theorem 3.24, we get that the zero matrix is an element of the Lie algebra $\mathfrak{g}$.

2. Let $X \in \mathfrak{g}$, and look at tX with $t \in \mathbb{R}$. Now look at the exponential $e^{t'(tX)} = e^{(t't)X} = e^{t''X}$ with $t, t'' \in \mathbb{R}$, and since $X \in \mathfrak{g}$, we have that $e^{t''X} \in G$. Hence $tX \in \mathfrak{g}$ for all $t \in \mathbb{R}$.

3. Let $X, Y \in \mathfrak{g}$ and let $XY = YX$. Then by property 2 we have $tX, tY \in \mathfrak{g}$ for all $t \in \mathbb{R}$ and with the use of property 5 of Theorem 3.24 we have that $e^{tX}e^{tY} = e^{tX+tY} = e^{t(X+Y)}$ for all $t \in \mathbb{R}$, hence $X + Y \in \mathfrak{g}$.
   On the otherhand, assume $XY \neq YX$. Since $X, Y \in \mathfrak{g}$, we have that $e^{tX/k}, e^{tY/k} \in G$, for $k \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Now we look at the product of these two elements to the power k of the Lie group, $A_k = \left(e^{tX/k}e^{tY/k}\right)^k = \left(e^{X'/k}e^{Y'/k}\right)^k$, where $X' = tX$ and $Y' = tY$. The set of these elements makes a sequence $(A_k)_{k\in\mathbb{N}}$ in G. Since G is a Lie group and hence it is closed, we have that if a sequence converges, then either the limit is an element of G or the limit is not in $\mathbf{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N}$. We can use property 7 of Theorem 3.24:
   $$\lim_{k\to\infty} A_k = \lim_{k\to\infty} \left(e^{X'/k}e^{Y'/k}\right)^k = e^{X'+Y'} = e^{t(X+Y)},$$
   for all $t \in \mathbb{R}$.
   Note that X, Y have a finite norm, hence $X + Y$ has a finite norm. So when we look at the power series of the exponential we find a finite value for $e^{t(X+Y)}$. This results in $e^{t(X+Y)}$ being invertible to $e^{-t(X+Y)}$. Hence $e^{t(X+Y)} \in \mathbf{GL}(n, \mathbb{C})$ and hence also in the Lie group G. Hence $X + Y \in \mathfrak{g}$.

4. Let $A \in G \subseteq \mathbf{GL}(n, \mathbb{C})$, for some $n \in \mathbb{N}$, and $X \in \mathfrak{g}$. Then by property 3 of Theorem 3.24 we have $e^{tAXA^{-1}} = e^{A(tX)A^{-1}} = Ae^{tX}A^{-1}$, where $e^{tX} \in G$, with $t \in \mathbb{R}$, hence $e^{A(tX)A^{-1}} \in G$ and $AXA^{-1} \in \mathfrak{g}$.

5. This point is proven in detail in paragraph 16.5 of [6] in the proof of Proposition 16.20.
   In this proof, it is used that
   $$[X, Y] = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{Xt}Ye^{-Xt}\right)\right]_{t=0}. \tag{4.2}$$
   For completeness, we will prove this claim here with the use of Proposition 3.25:
   $$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{Xt}Ye^{-Xt}\right)\right]_{t=0} &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{Xt}\right)Ye^{-Xt}\right]_{t=0} + \left[e^{Xt}Y\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-Xt}\right)\right]_{t=0} \\ &= \left[Xe^{Xt}Ye^{-Xt}\right]_{t=0} + \left[e^{Xt}Y \cdot (-X)e^{-Xt}\right]_{t=0} \\ &= XY - YX = [X, Y].\end{aligned} \tag{4.3}$$

■

Property 1 upto 3 make the Lie algebra $\mathfrak{g}$ of G a linear space, in fact a subspace of $\mathbf{M}_n(\mathbb{C})$. From property 5 together with Proposition 3.20 we can conlude that the Lie algebra of G is indeed a Lie algebra as we have defined earlier in Defintion 3.19 with as the bracket map $[X, Y] = XY - YX$.

4.2 Important Examples of Lie Algebras

It is a good idea to look at some Lie algebras. For our purpose we will look at the Lie algebra of $\mathbf{SO}(n)$ and $\mathbf{SU}(m)$, since they are of the big importance to latter chapters. We will also look at specific cases of these Lie algebras, namely in the case that $n = 3$ and $m = 2$.

Corollary 4.3. *The Lie algebras $\boldsymbol{u}(n)$,$\boldsymbol{su}(n)$, $\boldsymbol{o}(n)$ and $\boldsymbol{so}(n)$ of $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{O}(n)$ and $\boldsymbol{SO}(n)$, respectively, are given by:*

$$\begin{aligned} &1. \quad \boldsymbol{u}(n) = \{X \in \boldsymbol{M}_n(\mathbb{C}) | X^* = -X\} \\ &2. \quad \boldsymbol{su}(n) = \{X \in \boldsymbol{u}(n) | \operatorname{tr}(X) = 0\} \\ &3. \quad \boldsymbol{so}(n) = \boldsymbol{o}(n) = \{X \in \boldsymbol{M}_n(\mathbb{R}) | X^T = -X\} \end{aligned} \tag{4.4}$$

Proof. 1. Let $X^* = -X$, then by property 2 of Theorem 3.24 we have

$$(e^{tX})^* = e^{tX^*} = e^{-tX} = (e^{tX})^{-1}, \tag{4.5}$$

where $t \in \mathbb{R}$. So $e^{tX} \in \mathbf{U}(n)$ and hence $\{X \in \mathbf{M}_n(\mathbb{C}) | X^* = -X\} \subseteq \mathbf{u}(n)$. In the other direction, let e^{tX} be unitary for all $t \in \mathbb{R}$, then $e^{tX^*} = e^{-tX}$ for all $t \in \mathbb{R}$. With the use of Proposition 3.25, we differentiate around $t = 0$ using $\left[\frac{\mathrm{d}}{\mathrm{d}t} e^{tX}\right]_{t=0} = X$, we get $X^* = -X$ and hence $\mathbf{u}(n) \subseteq \{X \in M | X^* = -X\}$. Thus combining the two inclusions we can conclude that $\mathbf{u}(n) = \{X \in \mathbf{M}_n(\mathbb{C}) | X^* = -X\}$.

2. Let $e^{tX} \in \mathbf{u}(n)$, then $\det(e^{tX}) = e^{t \operatorname{tr}(X)}$ for all $t \in \mathbb{R}$, by property 4 of Theorem 3.24. So if $\det(e^{tX}) = 1$ for all $t \in \mathbb{R}$, then $e^{t \operatorname{tr}(X)} = 1$ for all $t \in \mathbb{R}$, hence $\operatorname{tr}(X) = 0$. So $\mathbf{su}(n) \subseteq \{X \in \mathbf{u}(n) | \operatorname{tr}(X) = 0\}$. Conversely, let $X \in \{X \in \mathbf{u}(n) | \operatorname{tr}(X) = 0\}$, so $\operatorname{tr}(X) = 0$. Then by property 4 of Theorem 3.24 we have $\det(e^{tX}) = e^{t \operatorname{tr}(X)} = e^0 = 1$ for all $t \in \mathbb{R}$, hence $X \in \mathbf{su}(n)$. Since this X was chosen arbitrarily, we can conclude $\{X \in \mathbf{u}(n) | \operatorname{tr}(X) = 0\} \subseteq \mathbf{su}(n)$. This result combined with the previous result implies $\mathbf{su}(n) = \{X \in \mathbf{u}(n) | \operatorname{tr}(X) = 0\}$.

3. Similarly, let $X^T = -X$, then by property 2 of Theorem 3.24 we have

$$(e^{tX})^T = e^{tX^T} = e^{-tX} = (e^{tX})^{-1}, \tag{4.6}$$

where $t \in \mathbb{R}$. So $e^{tX} \in \mathbf{O}(n)$ and hence $\{X \in \mathbf{M}_n(\mathbb{R}) | X^* = -X\} \subseteq \mathbf{o}(n)$. In the other direction, let e^{tX} be orthogonal for all $t \in \mathbb{R}$, then $e^{tX^T} = e^{-tX}$ for all $t \in \mathbb{R}$. With the use of Proposition 3.25, we differentiate around $t = 0$ using $\left[\frac{\mathrm{d}}{\mathrm{d}t} e^{tX}\right]_{t=0} = X$, we get $X^T = -X$. Hence with the previous result we have

$\mathbf{o}(n) = \{X \in \mathbf{M}_n(\mathbb{R}) | X^T = -X\}$.
The proof that $X \in \mathbf{o}(n)$ is in $\mathbf{so}(n)$ if and only if $\mathrm{tr}(X) = 0$ is analogous to the proof for $X \in \mathbf{su}(n)$ since we did not use any specific matrix Lie group or algebra properties. Note when we have $X^T = -X$, then for every diagonal element of X, x_{jj}, satisfies $x_{jj} = -x_{jj} = 0$ so $\mathrm{tr}(X) = 0$. This means $\mathbf{o}(n) \subseteq \mathbf{so}(n)$, and since $\mathbf{so}(n)$ is a subgroup of $\mathbf{o}(n)$ we have $\mathbf{so}(n) = \mathbf{o}(n)$.

■

We will now look at two specific cases of Lie algebra, namely $\mathbf{su}(2)$ and $\mathbf{so}(3)$. We will start with $\mathbf{su}(2)$.

Example 4.4. *We know from Corollary 4.3 that every X in the Lie algebra $\boldsymbol{su}(2)$ must satisfy $X^* = -X$. So we have for the diagonal entries that $X_{jj} = ia$ with $a \in \mathbb{R}$ since $X^*_{jj} = -X_{jj}$. The other two entries, are linked in the following manner: $X^*_{jk} = -X_{kj}$. We can now deduce that $X_{12} = b + ic$ and $X_{21} = -b + ic$ with $b, c \in \mathbb{R}$. So we can describe every element of the Lie algebra $\boldsymbol{su}(2)$ with the use of only 3 variables:*

$$\boldsymbol{su}(2) = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}. \tag{4.7}$$

We can contstruct the following basis of the Lie algebra $\boldsymbol{su}(2)$, by taking one variable equal to 1 and the rest equal to zero and then normalize the matrix:

$$E_1 := \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 := \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.8}$$

When applying the commutator function $[X, Y] = XY - YX$ to matrices in the basis, we obtain the following relations:

$$\begin{aligned} [E_1, E_2] &= E_1E_2 - E_2E_1 \\ &= \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= E_3, \\ [E_2, E_3] &= E_1, \\ [E_3, E_1] &= E_2. \end{aligned} \tag{4.9}$$

The second and third equation can be derived similarly as the first one.
Now we will look at the Lie algebra $\boldsymbol{so}(3)$. From Corollary 4.3 we know that every

element X in the Lie algebra $\boldsymbol{so}(3)$ satisfies $X^T = -X$, so the diagonal entries x_{jj} are zero. And the other entries must satisfy the following $x_{jk} = -x_{kj}$ for $j \neq k$. So we can describe every element of the Lie algebra $\boldsymbol{so}(3)$ with the use of only 3 variables:

$$\boldsymbol{so}(3) = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}. \tag{4.10}$$

We can construct the following basis of the Lie algebra $\boldsymbol{so}(3)$, by taking one variable equal to 1 and the rest equal to zero:

$$F_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad F_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.11}$$

When applying the commutator function $[X, Y] = XY - YX$ to matrices in the basis, we obtain the following relations:

$$\begin{aligned} [F_1, F_2] &= F_1F_2 - F_2F_1 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= F_3, \\ [F_2, F_3] &= F_1, \\ [F_3, F_1] &= F_2. \end{aligned} \tag{4.12}$$

The second and third equation can be derived similarly as the first one. One can also use an index notation with Lêvi-Civita epsilon and gain the above relations in one equation $[F_i, F_j] = \sum_{k=1}^{3} \epsilon_{ijk} F_k$. Here you sum over $k = 1, 2, 3$ on the right handside to let both sides only depend on i and j. And ϵ_{ijk} is 0 if $i = j$, $j = k$ or $i = k$, if this is not the case, then it is 1 if (ijk) is an even permutation, so conjugate to (123), or it is -1 if (ijk) is an odd permutation, so conjugate to (132).

4.3 From a Lie Group Homomorphism to the Lie Algebra Homomorphism

Definition 4.5. *A **one-parameter subgroup** of $\boldsymbol{GL}(n, V)$ is a continuou homomorphism of $(\mathbb{R}, +)$ into $\boldsymbol{GL}(n, \mathbb{C})$. So by Definition 2.8, it is a continuous function $A : \mathbb{R} \to \boldsymbol{GL}(n, \mathbb{C})$ such that $A(s + t) = A(s)A(t)$ for all $s, t \in \mathbb{R}$.*

From Proposition 2.9, we know that the identity of the group $(\mathbb{R}, +)$ has to be sent via $A(\cdot)$ to the identity of the group $\mathbf{GL}(n, \mathbb{C})$, with standard matrix multiplication, i.e. $A(0) = \mathbb{I}$.

Lemma 4.6. *If $A(\cdot)$ is a one-parameter subgroup of $GL(n, \mathbb{C})$, there exists a unique $X \in M_n(\mathbb{C})$ such that*

$$A(t) = e^{tX} \tag{4.13}$$

for all $t \in \mathbb{R}$.

Proof. The proof of this lemma can be found at the bottom of page 41 of [7], in this book it is the proof of Theorem 2.14. ■

Theorem 4.7. *Suppose G_1 and G_2 are matrix Lie groups with Lie algebra $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, and suppose $\Phi : G_1 \to G_2$ is a Lie group homomorphism. Then there exists a unique linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that:*

$$\Phi\left(e^{tX}\right) = e^{t\phi(X)}, \tag{4.14}$$

for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}_1$. This linear map has the following additional properties:

1. *$\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}_1$.*
2. *$\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$ for all $A \in G_1$ and $X \in \mathfrak{g}_1$.*
3. *$\phi(X)$, with $X \in \mathfrak{g}_1$, may be computed by*

$$\phi(X) = \left[\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(e^{tX}\right)\right]_{t=0} \tag{4.15}$$

Note that property 1 makes the map ϕ into a Lie algebra homomorphism.

Proof. The first thing we need to prove is the existence and uniqueness of ϕ. The Lie group homomorphism Φ is continuous and this makes $A : \mathbb{R} \to \mathbf{GL}(n, \mathbb{C})$ defined by $A(t) = \Phi(e^{tX})$ into an one-parameter subgroup og G_2. By Lemma 4.6, there exists a unique $Y \in \mathbf{M}_n(\mathbb{C})$ such that $A(t) = \Phi(e^{tX}) = e^{tY}$. Now let us define for every X that $\phi(X) = Y$.
By putting $t = 1$, we get $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}_1$. By property 2 of Proposition 4.2, we have $sX \in \mathfrak{g}_1$ for every $s \in \mathbb{R}$. So we also have $\Phi(e^{sX}) = e^{\phi(sX)}$. Combining these results we get for every $t, s \in \mathbb{R}$ it must be that $e^{ts\phi(X)} = \Phi(e^{tsX}) = e^{t\phi(sX)}$.

Hence $\phi(sX) = s\phi(X)$ for every $s \in \mathbb{R}$. Since Φ is continuous, we can use the Lie product formula (3.15) (property 7 of Theorem 3.24) in the following way:

$$\begin{aligned} e^{t\phi(X+Y)} &= \Phi(e^{t(X+Y)}) = \Phi\left(\lim_{k\to\infty}\left(e^{tX/k}e^{tY/k}\right)^k\right) \\ &= \lim_{k\to\infty}\left(\Phi\left(e^{tX/k}\right)\Phi\left(e^{tY/k}\right)\right)^k = \lim_{k\to\infty}\left(e^{t\phi(X)/k}e^{t\phi(Y)/k}\right)^k \\ &= e^{t(\phi(X)+\phi(Y))}, \end{aligned} \tag{4.16}$$

for every $t \in \mathbb{R}$ and $X, Y \in \mathfrak{g}_1$. So we get that for every $X, Y \in \mathfrak{g}_1$ we have $\phi(X+Y) = \phi(X) + \phi(Y)$. So the map ϕ is a linear map.
Now we will proof the properties, and we start with proving property 2 and 3 and end with property 1, since we will use property 2 in the proof of property 1.

2. Let $A \in G_1$ and $X \in \mathfrak{g}_1$, then by property 4 of Propositon 4.2 $AXA^{-1} \in \mathfrak{g}_1$. With using property 3 of Theorem 3.24 twice, we get

$$\begin{aligned} e^{t\phi(AXA^{-1})} &= \Phi\left(e^{tAXA^{-1}}\right) = \Phi\left(Ae^{tX}A^{-1}\right) = \Phi(A)\Phi\left(e^{tX}\right)\Phi\left(A^{-1}\right) \\ &= \Phi(A)\Phi\left(e^{tX}\right)\Phi(A)^{-1} = e^{t\Phi(A)\phi(X)\Phi(A)^{-1}}. \end{aligned} \tag{4.17}$$

   Taking the derivative at both sides and setting $t = 0$, we get

$$\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}. \tag{4.18}$$

3. Let $X \in \mathfrak{g}_1$, then with the use of Proposition 3.25, we get

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(e^{tX}\right)\right]_{t=0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}e^{t\phi(X)}\right]_{t=0} = \left[\phi(X)e^{t\phi(X)}\right]_{t=0} = \phi(X). \tag{4.19}$$

1. Let $X, Y \in \mathfrak{g}_1$. By property 5 of Proposition 4.2 we will consider $[X,Y] := XY - YX$ for $X, Y \in \mathfrak{g}_1$ and for $X, Y \in \mathfrak{g}_2$. To proof this result we are going to use the claim formulated in equation (4.2) which we have proven in property 5 of Proposition 4.2. For clearifications we will use the curly brackets for the curved brackets in this equation.

$$\begin{aligned} \phi([X,Y]) &= \phi\left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{Xt}Ye^{-Xt}\right\}\right]_{t=0}\right) = \left[\frac{\mathrm{d}}{\mathrm{d}t}\phi\left(e^{Xt}Ye^{-Xt}\right)\right]_{t=0} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{\Phi\left(e^{tX}\right)\phi(Y)\Phi\left(e^{-tX}\right)\right\}\right]_{t=0} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{t\phi(X)}\phi(Y)e^{-t\phi(X)}\right\}\right]_{t=0} \\ &= [\phi(X), \phi(Y)]. \end{aligned} \tag{4.20}$$

Note we have used property 2 of this Theorem and property 6 of Theorem 3.24. We have also used the commutation property between a linear function and the derivative.

■

4.4 From a Lie Algebra Homomorphism to the Lie Group Homomorphism

Theorem 4.8. *Suppose that G_1 and G_2 are matrix Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, and suppose that $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism. If G_1 is connected and simply connected, then there exists a unique Lie group homomorphism $\Phi : G_1 \to G_2$ such that Φ and ϕ are related as in equation* (4.14).

Proof. The proof of this theorem can be found at the bottom of page 119 of [7], in this book it is the proof of Theorem 5.6. ■

Corollary 4.9. *The Lie algebras* $\boldsymbol{su}(2)$ *and* $\boldsymbol{so}(3)$ *are isomorphic, but the groups* $\boldsymbol{SU}(2)$ *and* $\boldsymbol{SO}(3)$ *are not isomorphic.*

Proof. First we will proof that the Lie algebras $\mathbf{su}(2)$ and $\mathbf{so}(3)$ are isomorphic to each other.
From Example 4.4 we know that the Lie algebra $\mathbf{su}(2)$ has a basis $\{E_1, E_2, E_3\}$ and the commutation relations $[E_1, E_2] = E_3$ with cyclic permutation one finds the other two relations. Similarly we have for the Lie algebra $\mathbf{so}(3)$ a basis $\{F_1, F_2, F_3\}$ and the commutation relations $[F_1, F_2] = F_3$ with cyclic permutation one finds the other two relations.
We can construct a map $\phi : \mathbf{su}(2) \to \mathbf{so}(3)$ such that $\phi(E_j) = F_j$, for $j = 1, 2, 3$, more generally $\phi(aE_1 + bE_2 + cE_3) = aF_1 + bF_2 + cF_3$ for all $a, b, c \in \mathbb{R}$. Since we have the linearity and both Lie algebras satisfy the same commutation relations, this map ϕ is according to Definition 3.21 a Lie algebra homomorphism. We know that $\{E_1, E_2, E_3\}$ form a basis of $\mathbf{su}(2)$ and the same is true for $\{F_1, F_2, F_3\}$ for $\mathbf{so}(3)$. So these matrices are linearly independent and the space they span has a unique composition of $a, b, c \in \mathbb{R}$ to become the zero matrix. In both of these Lie algebras, the only way to get the zero matrix is by taking $a = b = c = 0$. Hence the kernel of ϕ is the trivial kernel, i.e. it only contains the zero matrix. Hence ϕ is injective.
Let $x \in \mathbf{so}(3)$, then there exists a unique way to construct x with the use of three real variables, namely $a, b, c \in \mathbb{R}$, in the following manner $x = aF_1 + bF_2 + cF_3$. For every element y in $\mathbf{su}(2)$ there is a unique way to write it as an sum of the basis matrices multplied by a real number. These real numbers have no restrictions to them, so we

can take $a, b, c \in \mathbb{R}$ as above. Hence $\phi(y) = \phi(aE_1 + bE_2 + cE_3) = aF_1 + bF_2 + cF_3 = x$. Hence ϕ is surjective, we already found it was injective, so ϕ is bijective. And by Definition 3.21 ϕ is a Lie algebra isomorphism. Hence this map is also a Lie algebra homomorphism and hence $\phi : \mathbf{su}(2) \to \mathbf{so}(3)$ is a Lie algebra isomorphism.
If $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$ are isomorphic, then $\mathbf{SU}(2)$ is simply connected if and only if $\mathbf{SO}(3)$ is simply connected. This is since simply connectedness is a topological property, and two isomorphic spaces have the same topological properties. But we have seen in Theorem 3.17 that $\mathbf{SU}(2)$ is simply connected, and we seen in Theorem 3.18 that $\mathbf{SO}(3)$ is not simply connected. Hence they are not isomorphic. ∎

Note that we also obtain a Lie algebra isomorphism when we sent E_j to $F_{(123)j}$, meaning $(aE_1 + bE_2 + cE_3) \mapsto aF_2 + bF_3 + cF_1$ for all $a, b, c \in \mathbb{R}$. We will define this Lie algebra isomorphism by $\phi_{(123)}$. We can do the same for $\phi(aE_1 + bE_2 + cE_3) = aF_3 + bF_1 + cF_2$ for all $a, b, c \in \mathbb{R}$ and we will define this Lie algebra isomorphism by $\phi_{(321)}$. Since the commutation relations are invarient under this transformation. One can easily check this by the Lêvi-Civita epsilon and get the commutation relations: $[F_i, F_j] = \sum_{k=1}^{3} \epsilon_{ijk} F_k$. As you might guess, the corresponding unique Lie group homomorphism of Lie algebra isomorphisms $\phi_{(123)}$ and $\phi_{(321)}$ will be denoted by $\Phi_{(123)}$ and $\Phi_{(321)}$, respectively.

Definition 4.10. *Suppose G is a connected matrix Lie group with Lie algebra $\mathfrak{g}$. A* ***universal cover of*** *G is an ordered pair $(\tilde{G}, \Phi)$ consisting of a connected and simply connected matrix Lie group $\tilde{G}$ and a Lie group homomorphism $\Phi : \tilde{G} \to G$, such that the associated Lie algebra homomorphism $\phi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a Lie algebra isomorphism of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ to $\mathfrak{g}$. The map Φ is called the* ***covering map*** *for $\tilde{G}$.*

If one has a universal cover $(\tilde{G}, \Phi)$ of a connected matrix Lie group G, then by Theorem 4.8 are the Lie group homomorphism Φ and the Lie algebra isomorphism related as in equation (4.14), and is the Lie group homomorphism Φ unique.

Lemma 4.11. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then there exists a neighbourhood U of 0 in $\boldsymbol{M}_n(\mathbb{C})$ and a neighbourhood V of $\mathbb{I}$ in $\boldsymbol{M}_n(\mathbb{C})$ such that the matrix exponential maps U diffeomorphically onto V and such that for all $X \in U$, we have that X belongs to $\mathfrak{g}$ if and only if e^X belongs to G.*

A map maps diffeomorphically if the map is differentiable and bijective and the inverese of the map is also differentiable.

Proof. The proof of this Lemma can be found in [7] as the proof of Corollary 3.44. ∎

This lemma has a corollary that we will use later in Section 5.1 to proof Proposition 5.7.

Corollary 4.12. *If a matrix Lie group G is connected, then for all $A \in G$ there exists a finite sequence $(X_k)_{1 \le k \le n} \subseteq \mathfrak{g}$ such that*

$$A = e^{X_1} e^{X_2} \dots e^{X_n}. \tag{4.21}$$

Proof. The proof of this Corollary can be found in [6] as the proof of Corollary 16.28. ∎

Lemma 4.13. *If G is a connected matrix Lie group and N is a discrete normal subgroup of G, see Definition 2.7. Then N is contained in the center of G, also denoted by $Z(G)$.*

This Lemma is Exercise 1 of Chapter 16 from [6].

Proof. Let G be a connected matrix Lie group and N be a discrete normal subgroup of G. Let $n \in N$ be fixed and $g : [0,1] \to G$ be a path, see Defintion 3.13, from $g(0) = e$ to $g(1) = g$. Then for every $t \in [0,1]$ we have $h(t) := g(t) n g^{-1}(t) \in N$, with for $t = 0$ we have $h(0) = g(0) n g^{-1}(0) = ene^{-1} = n \in N$. This path $g(t)$ is continuous and the path $g^{-1}(t)$ from $g^{-1}(0) = e$ to $g^{-1}(1) = g^{-1}$ is also well-defined and continuous, hence $h(t)$ is well-defined and continious over the entire domain $[0,1]$. Since N is a discrete normal subgroup of G, there exists a neighbourhood U of e in G, that is an open subset of G that contains e, such that $N \cap U = \{e\}$. Look at $h(t)n^{-1}$. We have $h(t)n^{-1} \in N$ for all $t \in [0,1]$ with $h(0)n^{-1} = nn^{-1} = e \in U$ and is conitnuous over $[0,1]$. Since $h(t)n^{-1}$ is well-defined and continuous over $[0,1]$, $h(0)n^{-1} \in U$ and $h(t)n^{-1} \subseteq N$ for all $t \in [0,1]$ and $U \cap N = \{e\}$, it must be that $h(t)n^{-1} = e$ for all $t \in [0,1]$. So $h(t) = g(t) n g^{-1}(t) = n$ for all $t \in [0,1]$ and hence $h(1) = gng^{-1} = n$.
This construction can be done for every $n \in N$ and $g \in G$, so we have that for every $n \in N$ and for every $g \in G$ it is that $gng^{-1} = n$, hence N is a subset of the center $\mathcal{Z}(G)$. ∎

Lemma 4.14. *Let $U, V \in \boldsymbol{SU}(2)$. If U is in the center of $\boldsymbol{SU}(2)$, then $U = \mathbb{I}$ or $U = -\mathbb{I}$. In other words $\mathcal{Z}(\boldsymbol{SU}(2)) = \{\mathbb{I}, -\mathbb{I}\}$.*

This Lemma is Exercise 2 of Chapter 16 from [6]. The eigenspace of an eigenvalue of a matrix corresponds to the space span by all eigenvectors with that exact eigenvalue.

Proof. Let U and V commute, i.e. $UV = VU$. Let V be arbitrarily in $\mathbf{SU}(2)$. We will look at two cases, one where U is diagonalizable and one where U is not diagonalizable.
If U is diagonalizable, then there exists $P \in \mathbf{GL}(2, \mathbb{C})$ and $A \in \mathbf{M}_n(\mathbb{C})$ such that $U = PAP^{-1}$. Let us define $W := P^{-1} V P$, since this will give us the convinient result

of $V = PWP^{-1}$. Before we can calculate with these matrices, we will first define the entries of these matrices:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4.22}$$

where $\alpha, \beta, \gamma, \delta, \lambda, \mu \in \mathbb{C}$. We will now compute UV and VU to then apply the commutation property of U and V.

$$UV = PAP^{-1}PWP^{-1} = PAWP^{-1} = P\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}P^{-1} = \begin{pmatrix} \lambda\alpha & \lambda\beta \\ \mu\gamma & \mu\delta \end{pmatrix}, \tag{4.23}$$

$$VU = PWP^{-1}PAP^{-1} = PWAP^{-1} = P\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}P^{-1} = \begin{pmatrix} \lambda\alpha & \mu\beta \\ \lambda\gamma & \mu\delta \end{pmatrix}. \tag{4.24}$$

Since $UV = VU$ and V is completely random in $\mathbf{SU}(2)$, it must be that $\lambda = \mu$ and we get $A = \lambda\mathbb{I}$. This results in $U = PAP^{-1} = P\lambda\mathbb{I}P^{-1} = \lambda\mathbb{I}$. From Equation (3.1) of Lemma 3.16 we can see that λ must be real and $|\lambda|^2 = \lambda^2 = 1$. Therefore $\lambda = \pm 1$ and thus $U = \pm\mathbb{I}$. Since V is arbitrarily in $\mathbf{SU}(2)$, we have that U commutes with every element of $\mathbf{SU}(2)$ and thus U is in the center of $\mathbf{SU}(2)$.

Let U be not diagonalizable. We will be using Theorem 3.2 of [15], which is the Jordan canonical form theorem. From this theorem we can conclude that we can write every $U \in \mathbf{SU}(2)$, since $\mathbf{SU}(2) \in \mathbf{M}_2(\mathbb{C})$, as PAP^{-1}, where A is the block-diagonal matrix of Jordan-blocks and P the matrix with the corresponding basis of the (generated) eigenvectors of U as columns.

Let $P \in \mathbf{GL}(2, \mathbb{C})$ and $A \in \mathbf{M}_n(\mathbb{C})$ such that $U = PAP^{-1}$. Then A can be a diagonal matrix, which would contradict our assumption of U being not diagonalizable, or A is a Jordan canonical form, this form is as in equation (4.25) or more generally as in equation (3.16) of [15], where you can find more details if you are interested. Since the first option is not possible, we will look at the second option. Furthermore, let us define $W := P^{-1}VP$, since this will give us the convinient result of $V = PWP^{-1}$. Before we can calculate with these matrices, we will first define the entries of these matrices:

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4.25}$$

where $\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{C}$. We will now compute UV and VU to then apply the commutation property of U and V.

$$\begin{aligned} UV &= PAP^{-1}PWP^{-1} = PAWP^{-1} \\ &= P\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}P^{-1} = \begin{pmatrix} \lambda\alpha + \gamma & \lambda\beta + \delta \\ \lambda\gamma & \lambda\delta \end{pmatrix}, \end{aligned} \tag{4.26}$$

$$\begin{aligned} VU &= PWP^{-1}PAP^{-1} = PWAP^{-1} \\ &= P\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}P^{-1} = \begin{pmatrix} \lambda\alpha & \lambda\beta + \alpha \\ \lambda\gamma & \lambda\delta + \gamma \end{pmatrix}. \end{aligned} \tag{4.27}$$

Since $UV = VU$ and V is completely random in $\mathbf{SU}(2)$, it must be that $\gamma = 0$ and $\alpha = \delta$.
If $\beta = 0$, then we have $W = \alpha\mathbb{I}$ and then $V = PWP^{-1} = P\alpha\mathbb{I}P^{-1} = \alpha\mathbb{I}$. From the first case we know $\alpha = \pm 1$ and thus $V = \pm\mathbb{I}$.
If $\beta \neq 0$, then we can write

$$W = Q\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}Q^{-1} = QXQ^{-1}, \tag{4.28}$$

where $\mu \in \mathbb{C}$ and $Q \in \mathbf{GL}(2, \mathbb{C})$. But then we can write $V = (PQ)X(PQ)^{-1}$. X has only one eigenvalue, namely μ, so V also has only one eigenvalue. Since there are also elements in $\mathbf{SU}(2)$ which have 2 eigenvalues, we have that V can not be any element of $\mathbf{SU}(2)$. This is a contradiction, since we demanded that V is completely random in $\mathbf{SU}(2)$.
So it must be that every element of $\mathcal{Z}(\mathbf{SU}(2))$ equals $\mathbb{I}$ or $-\mathbb{I}$. Hence $\mathcal{Z}(\mathbf{SU}(2)) = \{\mathbb{I}, -\mathbb{I}\}$. ■

Theorem 4.15. *Let $\phi : \boldsymbol{su}(2) \to \boldsymbol{so}(3)$ be the unique Lie algebra homomorphism which satifies $\phi(E_j) = F_j$ for $j = 1, 2, 3$. Then there exists a unique Lie group homomorphism $\Phi : \boldsymbol{SU}(2) \to \boldsymbol{SO}(3)$ such that $\Phi(e^{tX}) = e^{t\phi(X)}$, where $X \in \boldsymbol{su}(2)$ and $r \in \mathbb{R}$, such that $\ker(\Phi) = \{\mathbb{I}, -\mathbb{I}\}$ and $(\boldsymbol{SU}(2), \Phi)$ is a universal cover of $\boldsymbol{SO}(3)$.*

Proof. We know from Corollary 4.9 that the Lie algebras $\mathbf{su}(2)$ and $\mathbf{so}(3)$ are isomorphic. We now only have to check that there exists a Lie group homomorphism from the Lie group $\mathbf{SU}(2)$ to the Lie group $\mathbf{SO}(3)$. In Corollary 4.9 we have constructed the Lie algebra isomorphism $\phi : \mathbf{su}(2) \to \mathbf{so}(3)$ such that $\phi(E_j) = F_j$, for $j = 1, 2, 3$. To construct the corresponding Lie group homomorphism $\Phi : \mathbf{SU}(2) \to \mathbf{SO}(3)$ we will use Theorem 4.8, since $\mathbf{SU}(2)$ is connected and simply connected, so we can define a unique Φ as in equation (4.14), so $\Phi(e^{tX}) = e^{t\phi(X)}$ where $X \in \mathbf{su}(2)$. The basis $\{E_1, E_2, E_3\}$ from $\mathbf{su}(2)$ and $\{F_1, F_2, F_3\}$ from $\mathbf{so}(3)$. We have set $E_1 = \frac{1}{2}\operatorname{diag}(i, -i)$, this results in $e^{2\pi E_1} = -\mathbb{I}$. Let us now have a look at the one-parameter subgroup with generator F_1:

$$e^{aF_1} = \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(a) & -\sin(a) \\ 0 & \sin(a) & \cos(a) \end{pmatrix}. \tag{4.29}$$

So when we take $a = 2\pi$, we get $e^{2\pi F_1} = \mathbb{I}$. Combining these two results we get

$$\Phi(-\mathbb{I}) = \Phi(e^{2\pi E_1}) = e^{2\pi\phi(E_1)} = e^{2\pi F_1} = \mathbb{I}. \tag{4.30}$$

This means that $\{\mathbb{I}, -\mathbb{I}\}$ is a subset of the kernel of Φ.
We know from Corollary 4.9 that ϕ is bijective, thus in particular injective. Lemma 4.11 tells us there exists a neighbourhood U of 0 in $\mathbf{M}_n(\mathbb{C})$ and a neighbourhood V of $\mathbb{I}$ in $\mathbf{M}_n(\mathbb{C})$ such that the matrix exponential maps U diffeomorphically onto V and such that for all $X \in U$, we have that X belongs to $\mathfrak{g}$ if and only if e^X belongs to G. Let $A, B \in \mathbf{SU}(2) \cap V$ be distinct, then there exist two distinct elements $X, Y \in \mathbf{su}(2) \cap U$ such that $A = e^X$ and $B = e^Y$. Since X and Y are distinct and ϕ is injective, then also $\phi(X)$ and $\phi(Y)$ are distinct. We can now apply Lemma 4.11 again and get that $\Phi(A) = e^{\phi(X)}$ and $\Phi(B) = e^{\phi(Y)}$ are distinct. Hence Φ is injective in a neighbourhood of $\mathbb{I}$. Using this and Proposition 2.12 we can conclude that $\ker(\Phi)$ is a discrete normal subgroup of $\mathbf{SU}(2)$. We can now apply Lemma 4.13 on the kernel of Φ and we get that $\ker(\Phi) \subseteq \mathcal{Z}(\mathbf{SU}(2))$. From Lemma 4.14, we know that $\mathcal{Z}(\mathbf{SU}(2)) = \{\mathbb{I}, -\mathbb{I}\}$. When we combine all these results we get that $\{\mathbb{I}, -\mathbb{I}\} \subseteq \ker(\Phi) \subseteq \mathcal{Z}(\mathbf{SU}(2)) \subseteq \{\mathbb{I}, -\mathbb{I}\}$, thus $\ker(\Phi) = \{\mathbb{I}, -\mathbb{I}\}$.
We know that we can express every element $A \in \mathbf{SO}(3)$ as $A = e^X$ with $X \in \mathbf{so}(3)$. Since ϕ is injective and $\Phi(A) = \Phi(e^X) = e^{\phi(X)}$, the map Φ is surjective. We know from Theorem 3.17 that $\mathbf{SU}(2)$ is connected and simply connected, hence we $(\mathbf{SU}(2), \Phi)$ is the universal cover of the Lie group $\mathbf{SO}(3)$. ∎

With the same arguments as under the proof of Corollary 4.9, this result also holds for $\phi_{(123)}$ and $\Phi_{(123)}$ instead of ϕ and Φ, respectively, and for $\phi_{(321)}$ and $\Phi_{(321)}$ instead of ϕ and Φ, respectively.
From this Theorem and Theorem 2.14 we can conclude that the Lie group $\mathbf{SO}(3)$ is isomorphic to $\mathbf{SU}(2)/\{\mathbb{I}, -\mathbb{I}\}$. This is since $\mathrm{im}(\Phi) = \mathbf{SO}(3)$ and $\ker(\Phi) = \{\mathbb{I}, -\mathbb{I}\}$. We know from the proof of Theorem 3.17 that $\mathbf{SU}(2)$ is isomorphic to S^3, hence $\mathbf{SO}(3)$ is isomorphic to $S^3/\{\mathbb{I}, -\mathbb{I}\}$. This can help with visualizing the $\mathbf{SO}(3)$.

Chapter 5

A Brief Introduction to Representation Theory

A representation is a way to view a group G in the world of matrices. The map that sends this group into that world of matrices has to be a group homomorphism, since this map has to carry the properties of the group. These matrices, where the elements of the group will be sent to, have to be invertible. This is because we have demanded this map to be a group homomorphism and in the defintion of a group, see Chapter 2, every element of the group is invertible. One might also wonder if there are multiple representations of a group, and wether some are similar i.e. describe the same structures. We will go over the different types of representations in more detail. For the more general definitions, theorems, propositions, etc. I will refer to [5]. And for the more specific definitions, theorems, propositions, etc. I will follow Chapter 16.7 of [6].
We will also be defining representations of Lie algebras. For these definitions I will refer to Chapter 16.7 of [6]. In standard representation theory one only defines a representation for groups. But we will be defining and looking at Lie algebra representations. Be aware of the difference and when we are refering to a (Lie) group representation or a Lie algebra representation.

5.1 Representations of (Lie) Groups and Lie Algebras

Definition 5.1. *Let G be a group, a* ***representation of*** *G over a field $\mathbb{F}$ is a group homomorphism from G into the group of invertible $n \times n$-matrices, $\boldsymbol{GL}(n, \mathbb{F})$. Where $n \in \mathbb{N}$ and $\mathbb{F}$ is the field of the matrices.*

This is Defintion 3.1 from [5]. Sometimes we will denote $\mathbf{GL}(n, \mathbb{F})$ by $\mathbf{GL}(V)$ where $V = \mathbb{F}^n$. Note that a representation is a special case of a group homomorphism, and hence all the results involving group homo- or group isomorphisms we got in Chapter 2 also hold for representations. There is an similar defintion for a representation of a matrix Lie algebra.

Definition 5.2. *A finite-dimensional* ***representation of a Lie algebra*** $\mathfrak{g}$ *is a Lie algebra homomorphism of* $\mathfrak{g}$ *into* $\boldsymbol{gl}(V)$*, the space of all linear transformations of* V*. Here* $\boldsymbol{gl}(V)$ *is considered as a Lie algebra with bracket given by* $[X, Y] = XY - YX$*.*

Example 5.3. *The quaternion group* Q_8 *from Example 2.3 is a group generated by two elements* i *and* j *such that* $i^4 = e$*,* $i^2 = j^2$ *and* $jij^{-1} = i^{-1}$*. Let us define a representation by* $\rho : Q_8 \to \boldsymbol{GL}(2, \mathbb{C})$ *by* $\rho(i) = 2E_1$ *and* $\rho(j) = 2E_2$*, where* E_1 *and* E_2 *are defined as in equation* (4.8)*. We can see this from Example 4 on page 4 of [5]. So we have for* $q = i^n j^m \in Q_8$*, with* $0 \leq n \leq 3$ *and* $0 \leq m \leq 1$*, that* $\rho(i^n j^m) = 2^{n+m} E_1^n E_2^m$*.*

One could wonder if there exists another representation which carries the same properties, i.e. is similar as the respresentation above. And are they related?

Definition 5.4. *Let* $\rho, \sigma : G \to \boldsymbol{GL}(n, \mathbb{F})$ *be two representations of the group* G *over the field* $\mathbb{F}$*. We say* ρ *is* ***equivalant*** *to* σ *if there exists a* $T \in \boldsymbol{GL}(n, \mathbb{F})$ *such that*

$$\rho(g) = T\sigma(g)T^{-1} \tag{5.1}$$

for all $g \in G$*.*

This is Definition 3.3 of [5]. We can construct a similar definition for Lie algebra representations. This definition also holds up for Lie algebra representations. This is since if $X \in \mathfrak{g}$ and $A \in G$ with G a matrix Lie group, then $AXA^{-1} \in \mathfrak{g}$, see property 4 of Proposition 4.2.

Proposition 5.5. *The equivalence of representations is an equivalence relation. In other words, for all representations* ρ*,* σ *and* τ *of a group* G *over a field* $\mathbb{F}$*, we have:*

1. *ρ is equivalent to ρ;*
2. *if ρ is equivalent to σ, then σ is equivalent to ρ;*
3. *if ρ is equivalent to σ and σ is euivalent to τ, then ρ is equivalent to τ.*

This is Excercise 3.4 of [5].

Proof. Let ρ, σ and τ be representations of a group G over a field $\mathbb{F}$. We will now check the properties.

1. $\mathbb{I} \in \mathbf{GL}(n, F)$ and $\rho(g) = \mathbb{I}\rho(g)\mathbb{I}^{-1}$ for all $g \in G$. Hence ρ is equivalent to ρ.

2. Let ρ be equivalent to σ. Then there is a $T \in \mathbf{GL}(n, \mathbb{F})$ such that $\rho(g) = T\sigma(g)T^{-1}$ for all $g \in G$. Since $T \in \mathbf{GL}(n\mathbb{F})$, there exists a $T^{-1} \in \mathbf{GL}(n, \mathbb{F})$ and we get $\sigma(g) = \mathbb{I}\sigma(g)\mathbb{I} = (T^{-1}T)\sigma(g)(T^{-1}T) = T^{-1}(T\sigma(g)T^{-1})T = T^{-1}\rho(g)T$ for all $g \in \mathbf{GL}(n, \mathbb{F})$. Hence σ is equivalent to ρ.

3. Let ρ be equivalent to σ and let σ be euivalent to τ. Then there is a $T \in \mathbf{GL}(n, \mathbb{F})$ such that $\rho(g) = T\sigma(g)T^{-1}$ for all $g \in G$ and there is a $S \in \mathbf{GL}(n, \mathbb{F})$ such that $\sigma(g) = S\tau(g)S^{-1}$ for all $g \in G$. Since $T, S \in \mathbf{GL}(n\mathbb{C})$ and $\mathbf{GL}(n, \mathbb{C})$ is a group, we have that $TS \in \mathbf{GL}(n, \mathbb{F})$ and thus $\rho(g) = T\sigma(g)T^{-1} = TS\tau(g)S^{-1}T^{-1} = TS\tau(g)S^{(}TS)^{-1}$ for all $g \in G$. Hence ρ is equivalent to τ.

Similarly one can proof all these properties also for equivalency of Lie lagebra representations. Thus the equivalence of representations is an equivalence relation. ∎

From now on we will when we are talking about (group) representations, we will consider the entire equivalent class of representation. We will do this by refering to such a class by a representation from within that class. Since, when we have two of representations which are equivalent, then they have the same structure and we can view them as 'the same' representation.

Definition 5.6. *If $\Pi : G \to \boldsymbol{GL}(V)$ is a representation of a (Lie) group G, then a subspace W of V is called an **invariant subspace** if $\Pi(g)w \in W$ for every $g \in G$ and every $w \in W$. Similarly, if $\pi : \mathfrak{g} \to \boldsymbol{gl}(V)$ is a representation of a Lie algebra $\mathfrak{g}$, then a subspace W of V is called **invariant subspace** if $\pi(X)w \in W$ for every $X \in \mathfrak{g}$ and every $w \in W$. A representation of a (Lie) group or a Lie algebra is called **irreducible**, and denoted as an **irreducible representation**, if and only if the invariant subspaces of V are $W = V$ and $W = \{\boldsymbol{0}\}$.*

It looks like we have defined twice what an invariant subspace is: one way through a group representation and the other through a Lie algebra representation. When we are dealing with a Lie group and its associated Lie algebra one would expect there be a connection between invariant subspaces of the Lie group and the Lie algebra. When we are dealing with matrix Lie groups, there is indeed a connection between these invariant subspaces: they are the same!
If the Lie group G is a matrix Lie group with a representation Π, then we can use Theorem 4.7 to construct a unique Lie algebra homomorphism π of the associated Lie algebra $\mathfrak{g}$ such that $\Pi(e^{tX}) = e^{t\pi(X)}$ for $X \in \mathfrak{g}$.

Proposition 5.7. *Suppose G is a connected matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose $\Pi : G \to \boldsymbol{GL}(V)$ is a finite-dimensional representation of G and $\pi : \mathfrak{g} \to$*

gl(V) *is the associated Lie algebra representation. Then a subspace* W *of* V *invariant under the action of* G *if and only if it is invariant under the action of* $\mathfrak{g}$*. In particular,* Π *is irreducible if and only if* π *is irreducible. Furthermore, two representations of* G *are equivalent if and only if the associated Lie algebra representations are equivalent.*

Proof. Let $W \subseteq V$ such that W is invariant under $\pi(X)$ for all $X \in \mathfrak{g}$. Then W is also invariant under $\pi(X)^m$ for all $X \in \mathfrak{g}$ and $m \in \mathbb{N}$. This is since for all $w \in W$ and $X \in \mathfrak{g}$ we have $\pi(X)w \in W$. If $\pi(X)^n w \in W$ for every $m \geq n \in \mathbb{N}$, then $\pi(X)^{m+1}w = \pi(X)\left(\pi(X)^m w\right) \in W$. So by the principle of complete induction we can conlude that $\pi(X)^m w \in W$ for every $m \in \mathbb{N}$ and $X \in \mathfrak{g}$ and $w \in W$. Since V is a finite dimensional space, any subspace of V is finite dimenisonal and thus a closed subspace. Let $w \in W$ and let $X \in \mathfrak{g}$. Then

$$\begin{aligned} \Pi(e^X)w = e^{\pi(X)}w &= \left(\sum_{k=0}^{\infty} \frac{\pi(X)^k}{k!}\right) w \\ &= \sum_{k=0}^{\infty} \frac{\pi(X)^k w}{k!} \in W. \end{aligned} \tag{5.2}$$

Since W is a subspace , we have for every $k \in \mathbb{N}_0$, that $\dfrac{\pi(X)^k w}{k!} \in W$ and also that the finite sums of these fractions are again in W. Since W is closed and the sequence $\left(\sum_{k=0}^{n} \dfrac{\pi(X)^k}{k!}\right)_{n \in \mathbb{N}}$ converges, see Proposition, and $e^{\pi(X)}$ has an inverse, we have that $\Pi(e^X)w \in W$. 3.23 Corollary 4.12 tells us we can write every element A of G as $A = e^{X_1}e^{X_2}\dots e^{X_n}$ where $X_j \in \mathfrak{g}$ for all $1 \leq j \leq n$, since G connected matrix Lie group. Then

$$\begin{aligned} \Pi(A)w &= \Pi\left(e^{X_1}e^{X_2}\dots e^{X_n}\right)w \\ &= \Pi\left(e^{X_1}\right)\Pi\left(e^{X_2}\right)\dots\Pi\left(e^{X_n}\right)w \in W. \end{aligned} \tag{5.3}$$

Hence the subspace W is also invariant under $\Pi(A)$ for all $A \in G$.
Conversely, let $W \subseteq V$ such that W is invariant under $\Pi(A)$ for all $A \in G$. Have a look at:

$$\pi(X) = \pi_0(X) = \lim_{h\to 0} \pi_h(X) = \lim_{h \to 0} \frac{e^{hX} - \mathbb{I}}{h}. \tag{5.4}$$

For every value of $h > 0$ this $\pi_h(X)$ is defined and it is invariant under W, since $e^{hX} \in G$ and $\mathbb{I} \in G$ and W is a subspace. Take X fixed, then the sequence $(\pi_h(X))_{h>0}$ converges and since W is closed. The limit $\pi(X) = \lim_{h\to 0}\pi_h(X)$ is an element of $\mathfrak{g}$. W is invariant under $\pi(X)$.

Thus a subspace W of V is invariant under $\Pi(A)$ for all $A \in G$ if and only if W is invariant under $\pi(X)$ for all $X \in \mathfrak{g}$.
Let Π_1 and Π_2 be two representations of G and let π_1 and π_2 be their associated Lie algebra representation, resp.
Π_1 and Π_2 are equivalent, if and only if there exists a $T \in \mathbf{GL}(n,\mathbb{C})$ such that $\Pi_2(A) = T\Pi_1(A)T^{-1}$ for all $A \in G$. If and only if $e^{\pi_2(X)} = \Pi_2(e^X) = T\Pi(e^X)T^{-1} = Te^{\pi_1(X)}T^{-1} = e^{T\pi_1(X)T^{-1}}$ for all $X \in \mathfrak{g}$. If and only if $\pi_2(X) = T\pi_1(X)T^{-1}$ for all $X \in \mathfrak{g}$ if and only if π_1 and π_2 are equivalent.
Here we have used Corollary 4.12 again in the same manner to go from the equivalency on the Lie algebra representation to the equivalency on the matrix Lie group representation. This is done by describing every $A \in G$ as $A = e^{X_1}e^{X_2}\dots e^{X_n}$ where $X_j \in \mathfrak{g}$ for all $1 \leq j \leq n$. This can be achieved through the following steps:

$$\begin{aligned}
\Pi_2(A) &= \Pi_2(e^{X_1}e^{X_2}\dots e^{X_n}) = \Pi_2\left(e^{X_1}\right)\Pi_2\left(e^{X_2}\right)\dots\Pi_2\left(e^{X_n}\right) \\
&= T\Pi_1\left(e^{X_1}\right)T^{-1}T\Pi_1\left(e^{X_2}\right)T^{-1}\dots T\Pi_1\left(e^{X_n}\right)T^{-1} \\
&= T\Pi_1\left(e^{X_1}\right)\Pi_1\left(e^{X_2}\right)\dots\Pi_1\left(e^{X_n}\right)T^{-1} \\
&= T\Pi_1(e^{X_1}e^{X_2}\dots e^{X_n})T^{-1} \\
&= T\Pi_1(A)T^{-1}.
\end{aligned} \tag{5.5}$$

Conversely, from the equivalency on the matrix Lie group representation to the equivalency on the Lie algebra representation is trivial. Since for every $X \in \mathfrak{g}$ we have $e^X \in G$, see Definition 4.1. ■

5.2 Projective Unitary Representations

An important piece of quantum theory is the Hilbert space. Some would say it is the language in which quantum theory can be described. For this reason we will have a quick look at it.

Definition 5.8. *A* ***Hilbert space*** *is a vector space $\mathcal{H}$ over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, equipped with an* ***inner product*** *$\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{F}$, with the following properties.*

1. *For all $\phi,\psi \in \mathcal{H}$, we have $\langle\phi,\psi\rangle = \overline{\langle\psi,\phi\rangle}$.*
2. *For all $\phi \in \mathcal{H}$, $\langle\phi,\phi\rangle$ is real and non-negative, and $\langle\phi,\phi\rangle = 0$ if and only if $\phi = 0$.*
3. *For all $\phi,\psi \in \mathcal{H}$ and $c \in \mathbb{F}$, we have $\langle c\phi,\psi\rangle = \bar{c}\langle\phi,\psi\rangle$ and $\langle\phi,c\psi\rangle = c\langle\phi,\psi\rangle$.*
4. *For all $\phi,\psi,\chi \in \mathcal{H}$, we have $\langle\phi+\chi,\psi\rangle = \langle\phi,\psi\rangle + \langle\chi,\psi\rangle$ and $\langle\phi,\psi+\chi\rangle = \langle\phi,\psi\rangle + \langle\phi,\chi\rangle$.*

Such that $\mathcal{H}$ *is complete in the norm given by* $\|\phi\| = \sqrt{\langle\phi,\phi\rangle}$.

This is definition is a combination of Definitions A.42 and A.44 and with Equation (A.7) from Proposition A.43 of [7].

Definition 5.9. *Suppose* V *is a finite-dimensional Hilbert space over* $\mathbb{C}$*. Denote by* $\boldsymbol{U}(V)$ *the group of invertible linear transformationsof* V *that preserve the inner product. A (finite-dimensional)* ***unitary representation*** *of a matrix Lie group* G *is a continuous homomorphism* $\Pi : G \to \boldsymbol{U}(V)$*, for some finite-dimensional Hilbert space* V*.*

Proposition 5.10. *Let* $\Pi : G \to \boldsymbol{GL}(V)$ *be a finite-dimensional representation of a connected matrix Lie group* G*. Let* π *be the associated representation of the Lie algebra* $\mathfrak{g}$ *of* G*. Furthermore, let* $\langle\cdot,\cdot\rangle$ *be an inner product on* V*. Then* Π *is unitary with repsect to* $\langle\cdot,\cdot\rangle$ *if and only if*

$$\pi(X)^* = -\pi(X) \tag{5.6}$$

for all $X \in \mathfrak{g}$*.*

Proof. The proof of this Proposition can be found in [6] as the proof of Proposition 16.42. ∎

In quantum mechanics we view two unit-states, ϕ and ψ, as the same when there is some constant on the unit sphere in $\mathbb{C}$, i.e. $e^{i\theta}\mathbb{I}$, that connects the two states, i.e. $\psi = e^{i\theta}\phi$. One could see this action as a rotation of the state ϕ onto the state ψ. For this reason it is only logical to only look at states are really different from each other. To achieve this goal, we will construct an equivalence relation: two states ϕ and ψ of a Hilbert space are equivalent if and only if there exists some $\theta \in \mathbb{R}$ such that $\psi = e^{i\theta}\phi$. With this equivalent relation we will construct a new type of representation.

Definition 5.11. *Suppose* V *is a finite-dimensional Hilbert space over* $\mathbb{C}$*. Then the* ***projective unitary group*** *over* V*, denoted by* $\boldsymbol{PU}(V)$*, is the quotient group*

$$\boldsymbol{PU}(V) = \boldsymbol{U}(V)/\{e^{i\theta}\mathbb{I}\}, \tag{5.7}$$

where $\{e^{i\theta}\mathbb{I}\}$ *denotes the group of matrices of the form* $e^{i\theta}\mathbb{I}$ *with* $\theta \in \mathbb{R}$*.*

This definition is well-defined, since the subgroup $\{e^{i\theta}\mathbb{I}\}$ is a normal subgroup of $\mathbf{U}(V)$. One can construct a group homomorphism from $\mathbf{U}(V)$ to $\mathbf{PU}(V)$ where one sents two states to the same state if and only if they are equivalent to each other. With that we have a group homomorphism with kernel $\{e^{i\theta}\mathbb{I}\}$ and image $\mathbf{PU}(V)$, and applying Theorem 2.14 we get Equation (5.7).

Definition 5.12. *A finite-dimensional* ***projective unitary representation*** *of a matrix Lie group G is a continuous homomorphism Π of G into $\boldsymbol{PU}(V)$, where V is a finite-dimensional Hilbert space over $\mathbb{C}$. A subspace A of V is said to be* ***invariant*** *under Π if for every $A \in G$, W is invariant under U for each $U \in \boldsymbol{U}(V)$ such that $U = e^{i\theta}\Pi(A)$ for some $\theta \in \mathbb{R}$. A projective unitary representation is said to be* ***irreducible*** *if the only invariant subspaces are V or $\{\boldsymbol{0}\}$.*

Chapter 6

Spin

In this chapter we will combine our knowledge of the matrix Lie groups **SO**(3) and **SU**(2) and of the Lie algebras **so**(3) and **su**(2) that we have collected in Chapters 3, 4 and 5.

In this chapter I follow paragraph 17.2 in Section **??**, paragraph 17.4 in Section 6.1, paragraph 17.5 in Section 6.2 and paragraph 17.8 in Section 6.3. All these paragraphs come from the book [6].

6.1 Irreducible Representations of the Lie Algebra so(3)

In Example 4.4 we have looked at the Lie algebra **so**(3) where we defined the basis $\{F_1, F_2, F_3\}$ of this Lie algebra. We will continue using this basis. All of the upcoming representation will be over the complex numbers as their field.

Theorem 6.1. *Let $\pi : \boldsymbol{so}(3) \to \boldsymbol{gl}(V)$ be a finite-dimensional irreducible representation of $\boldsymbol{so}(3)$. Define operators L^+, L^- and L_3 on V by*

$$\begin{aligned} L^+ &= i\pi(F_1) - \pi(F_2) \\ L^- &= i\pi(F_1) + \pi(F_2) \\ L_3 &= i\pi(F_3). \end{aligned} \tag{6.1}$$

Let $l = \frac{1}{2}(\dim(V) - 1)$*, so* $\dim(V) = 2l + 1$*. Then there exists a basis* $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{2l}$ *of* V *such that*

$$L^+\vec{v}_j = \begin{cases} j(2l+1-j)\vec{v}_{j-1} & \text{if } 2l \geq j > 0 \\ 0 & \text{if } j = 0 \end{cases} \tag{6.2}$$

$$L^-\vec{v}_j = \begin{cases} \vec{v}_{j+1} & \text{if } 0 \leq j < 2l \\ 0 & \text{if } j = 2l \end{cases} \tag{6.3}$$

$$L_3\vec{v}_j = (l-j)\vec{v}_j. \tag{6.4}$$

Proof. We have the finite-dimensional irreducible representation π, so any relations on the basis of $\mathbf{so}(3)$ must also hold in the basis $\{\pi(F_1), \pi(F_2), \pi(F_3)\}$ in $\mathbf{gl}(n, V)$. So we have $[\pi(F_i), \pi(F_j)] = \sum_{k=1}^{3} \epsilon_{ijk}\pi(F_k)$. With this knowledge we can look at the commutations of the operators L^+, L^- and L_3:

$$\begin{aligned} [L_3, L^+] &= [i\pi(F_3), i\pi(F_1) - \pi(F_2)] \\ &= -[\pi(F_3), \pi(F_1)] - i[\pi(F_3), \pi(F_2)] \\ &= i\pi(F_1) - \pi(F_2) = L^+ \end{aligned} \tag{6.5}$$

$$\begin{aligned} [L_3, L^-] &= [i\pi(F_3), i\pi(F_1) + \pi(F_2)] \\ &= -[\pi(F_3), \pi(F_1)] + i[\pi(F_3), \pi(F_2)] \\ &= -i\pi(F_1) - \pi(F_2) = -L^- \end{aligned} \tag{6.6}$$

$$\begin{aligned} [L^+, L^-] &= [i\pi(F_1) - \pi(F_2), i\pi(F_1) + \pi(F_2)] \\ &= -[\pi(F_1), \pi(F_1)] + i[\pi(F_1), \pi(F_2)] \\ &\quad - i[\pi(F_2), \pi(F_1)] - [\pi(F_2), \pi(F_2)] \\ &= i[\pi(F_1), \pi(F_2)] - i[\pi(F_2), \pi(F_1)] \\ &= i\pi(F_3) + i\pi(F_3) = 2L_3. \end{aligned} \tag{6.7}$$

We can construct a complex function to find a (complex) eigenvalue with L_3, we can do this with the function $\det(L_3 - \lambda\mathbb{I})$. The main theory of algebra states that every complex function has at least one zero. So when we put this function to zero, we will find at least one eigenvalue λ. To find the corresponding eigenvector $\vec{v}$ one has to solve $L_3\vec{v} - \lambda\vec{v} = \vec{0}$, there are exactly n equations and the vector $\vec{v}$ has n variables, so there exists at least one solution. With the use of equation (6.5), we can derive

$$L_3L^+\vec{v} = (L^+ + L^+L_3)\vec{v} = L^+\vec{v} + L^+L_3\vec{v} = L^+\vec{v} + L^+(\lambda\vec{v}) = (\lambda+1)L^+\vec{v}. \tag{6.8}$$

Then either $L^+\vec{v} = \vec{0}$ or $L^+\vec{v}$ is an eigenvector of L_3 with eigenvalue $\lambda + 1$. From equation (6.8) we can conclude that L^+ raises the obtained eigenvalue of L_3 by the eigenvector by 1. Since $L^+\vec{v}$ is also an eigenvector, we can repeat this proces with the

eigenvector $L^+\vec{v}$ and get $L_3(L^+)^k\vec{v} = (\lambda+k)L^+\vec{v}$. Since L_3 can only have a maximum of n distinct eigenvalues, there has to be a maximum number for this $k \in \mathbb{N}$ such that $L_3(L^+)^k\vec{v} \neq \vec{0}$, let this maximum be l. So we have $L_3(L^+)^{l+1}\vec{v} = \vec{0}$. For convenience define $\vec{v}_0 := (L^+)^k\vec{v} \neq \vec{0}$ and $\mu := \lambda + l$, which satisfies $L^+\vec{v}_0 = \vec{0}$ and $L_3\vec{v} = \mu\vec{v}$.
To find the other eigenvectors we will use a similar trick as we have preformed above, but now with equation (6.6). But first we have to define a series of vectors:

$$\vec{v}_j = (L^-)^j\vec{v}_0, \text{ for } j = 0, 1, 2, \dots. \tag{6.9}$$

With the use of this notation we can derive:

$$\begin{aligned} L_3\vec{v}_j &= L_3(L^-)^j\vec{v}_0 = (L^-L_3 - L^-)(L^-)^{j-1}\vec{v}_0 = L^-L_3(L^-)^{j-1}\vec{v}_0 - (L^-)^j\vec{v}_0 \\ &= L^-\left(L_3(L^-)^{j-1}\vec{v}_0\right) - (L^-)^{j-1}\vec{v}_0 = L^-\left((L^-L_3 - L^-)(L^-)^{j-2}\vec{v}_0\right) - (L^-)^{j-1}\vec{v}_0 \\ &= (L^-)^2L_3(L^-)^{j-2}\vec{v}_0 - (L^-)^j\vec{v}_0 = \dots = (L^-)^j(L_3\vec{v}_0) - (L^-)^j\vec{v}_0 = (L^-)^j(\mu - j)\vec{v}_0 \\ &= (\mu - j)\vec{v}_j. \end{aligned} \tag{6.10}$$

We have found μ distinct eigenvalues so the corresponding eigenvectors are also distinct, and note for $\vec{v}_\mu$ we have $L_3\vec{v}_\mu = \vec{0}$. We now know the effect of the interaction of $\vec{v}_j$ with L_3 and L^-. To get the interaction of $\vec{v}_j$ with L^+, where $j = 1, 2, 3, \dots$, we can use equations (6.7), (6.9) and (6.10):

$$\begin{aligned} L^+\vec{v}_j &= L^+L^-\vec{v}_j = \left(2L_3 + L^-L^+\right)\vec{v}_{j-1} = 2L_3\vec{v}_{j-1} + L^-L^+\vec{v}_{j-1} \\ &= 2(\mu + 1 - j)\vec{v}_{j-1} + L^-L^+(L^-\vec{v}_{j-2}) \\ &= 2(\mu + 1 - j)\vec{v}_{j-1} + L^-\left(2L_3\vec{v}_{j-2} + L^-L^+(L^-\vec{v}_{j-3})\right) \\ &= 2(\mu + 1 - j)\vec{v}_{j-1} + 2(\mu + 2 - j)\vec{v}_{j-1} + (L^-)^2L^+(L^-\vec{v}_{j-3}) = \dots \\ &= \sum_{i=0}^{j-1} 2(\mu - i)\vec{v}_{j-1} + (L^-)^jL^+\vec{v}_0 = j(2\mu + 1 - j)\vec{v}_{j-1} + (L^-)^jL^+\vec{v}_0 \\ &= j(2\mu + 1 - j)\vec{v}_{j-1}. \end{aligned} \tag{6.11}$$

Note we have seen earlier that $L^+\vec{v}_0 = \vec{0}$. Furthermore, L_3 has a finite number of eigenvalues, there must be a $N \in \mathbb{N}$ such that $\vec{v}_{N+1} = \vec{0}$ and $\vec{v}_N \neq \vec{0}$. Using this and equation (6.11), we get

$$0 = L^+\vec{v}_{N+1} = (N+1)(2\mu - N)\vec{v}_N. \tag{6.12}$$

Since $N > 0$ and $\vec{v}_N \neq \vec{0}$, it must be that $2\mu - N = 0$, and thus $\mu = \frac{1}{2}N$. When we put $l = \mu = \frac{1}{2}N$ the found interactions of L^+, L^-, and L_3 with $\vec{v}_j$ are the same as in equations (6.2), (6.3) and (6.4). The found eigenvectors $\vec{v}_j$ all have different eigenvalues with L_3, so they are all distinct, and even linearly independent. Hence the space they span has dimension $2l + 1$. The representation is irreducible, so every invariant subspace of V is either V or $\{\vec{0}\}$. From the equations (6.2), (6.3) and (6.4), we see that the span of the v_j's is invariant under L^+, L^- and L_3. One can write a linear expression to get $\pi(F_k)$ for $k = 1, 2, 3$, so L^+, L^- and L_3 form a basis for the representations of $\mathbf{so}(3)$ in $\mathbf{gl}(n, V)$. Hence the the span of the v_j's are invariant under every element $\mathbf{so}(3)$ and is thus an invariant subspace of V of nonzero dimension, so the subspace spanned by the $\vec{v}_j$'s is the entire space V and thus $\{\vec{v}_0, \vec{v}_1, \dots, \vec{v}_{2l}\}$ is a basis of V and $2l + 1 = \dim(V)$, i.e. $l = \frac{1}{2}(\dim(V) - 1)$. ∎

Definition 6.2. *If (π, V) is an irreducible fininite-dimensional representation of* ***so****(3), then the* ***spin*** *of (π, V) is the largest egeinvalue of the operator L_3, from Theorem 6.1.*

If we have a representation as in Theore 6.1, the largest eigenvalue of L_3 is l. So the spin equals l.

Theorem 6.3. *For any $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ there exists an irreducible representation of* ***so****(3) with dimension $2l + 1$. Any two of these irreducible representations of* ***so****(3) with dimension $2l + 1$ are isomorphic.*

Proof. We will construct a space V and show it is a representation of $\mathbf{so}(3)$ and irreducible. The space V is spanned by the basis $\{\vec{v}_0, \vec{v}_1, \dots, \vec{v}_{2l}\}$ and we will define actions on it by the equations (6.2)), (6.3) and (6.4). In the proof of Theorem 6.1 we have shown that the commutation relations of F_i's are equivalent to the commutation relations of (6.5), (6.6) and (6.7). To show these relations we have to use an arbitrary basis vector $\vec{v}_j$,

$$\begin{aligned}
[L_3, L^+]\vec{v}_j &= L_3 L^+\vec{v}_j - L^+ L_3\vec{v}_j = L_3 \begin{cases} j(2l+1-j)\vec{v}_{j-1} & \text{if } 2l \geq j > 0 \\ 0 & \text{if } j = 0 \end{cases} - (l-j)L^+\vec{v}_j \\
&= \begin{cases} j(2l+1-j)(l+1-j)\vec{v}_{j-1} - j(2l+1-j)(l-j)\vec{v}_{j-1} & \text{if } 2l \geq j > 0 \\ 0 & \text{if } j = 0 \end{cases} \\
&= \begin{cases} j(2l+1-j)\vec{v}_{j-1} & \text{if } 2l \geq j > 0 \\ 0 & \text{if } j = 0 \end{cases} \\
&= L^+\vec{v}_{j-1}
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
[L_3, L^-]\vec{v}_j &= L_3 L^-\vec{v}_j - L^- L_3\vec{v}_j = L_3 \begin{cases} \vec{v}_{j+1} & \text{if } 0 \leq j < 2l \\ 0 & \text{if } j = 2l \end{cases} - (l-j)L^-\vec{v}_j \\
&= \begin{cases} (l-j-1)\vec{v}_{j+1} - (l-j)\vec{v}_{j+1} & \text{if } 0 \leq j < 2l \\ 0 & \text{if } j = 2l \end{cases} \\
&= \begin{cases} -\vec{v}_{j+1}- & \text{if } 0 \leq j < 2l \\ 0 & \text{if } j = 2l \end{cases} \\
&= -L^-\vec{v}_j
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
[L^+, L^-]\vec{v}_j &= L^+ L^-\vec{v}_j - L^- L^+\vec{v}_j \\
&= L^+ \begin{cases} \vec{v}_{j+1} & \text{if } 0 \leq j < 2l \\ 0 & \text{if } j = 2l \end{cases} - L^- \begin{cases} j(2l+1-j)\vec{v}_{j-1} & \text{if } 2l \geq j > 0 \\ 0 & \text{if } j = 0 \end{cases} \\
&= \begin{cases} (j+1)(2l-j)\vec{v}_j & \text{if } 0 \leq j < 2l \\ 0 & \text{if } j = 2l \end{cases} - \begin{cases} j(2l+1-j)\vec{v}_j & \text{if } 2l \geq j > 0 \\ 0 & \text{if } j = 0 \end{cases} \\
&= \begin{cases} (j+1)(2l-j)\vec{v}_j & \text{if } j = 0 \\ (j+1)(2l-j)\vec{v}_j - j(2l+1-j)\vec{v}_j & \text{if } 0 < j < 2l \\ -j(2l+1-j)\vec{v}_j & \text{if } j = 2l \end{cases} \\
&= \begin{cases} 2l\vec{v}_0 & \text{if } j = 0 \\ 2(l-j)\vec{v}_j & \text{if } 0 < j < 2l \\ -2l\vec{v}_{2l} & \text{if } j = 2l \end{cases} \\
&= 2L_3\vec{v}_j.
\end{aligned} \tag{6.15}$$

Since the $\vec{v}_j$'s span the space V, we have that the above relations are true for all $\vec{v} \in V$, and hence we have the commutation relation of $[L_3, L^+] = L^+$, $[L_3, L^-] = -L^-$ and lastly $[L^+, L^-] = 2L_3$. This makes V into a representation of $\mathbf{so}(3)$.
Now we have to proof this representation is irreducible, i.e. that V is irreducible or if $W \neq \{\vec{0}\}$ is an invariant subspace of V, then $W = V$. Let $\vec{w} \in W$ be nonzero, then we can write $\vec{w} = \sum_{k=0}^{2l} a_k \vec{v}_k$. Let N be the largest index such that $a_N \neq 0$. When we apply L^+ to any of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2l}$ we get $2l\vec{v}_0, 2(2l-1)\vec{v}_1, \ldots, 2l\vec{v}_{2l-1}$ back, note that none of these vectors are zero. Let us now apply L^+ N-times on $\vec{w}$, then we get a nonzero multiple of $\vec{v}_0$. Since W is an invariant subspace it must be that $\vec{v}_0 \in W$. Remember how we have generated all the other basis vectors $\vec{v}_j$'s, we did that by $\vec{v}_j = (L^-)^j \vec{v}_0$, and thus with the same argument for all $j = 0, 1, 2, \ldots, 2l$ it must be that $\vec{v}_j \in W$. Since V is the span of these $\vec{v}_j$'s, it follows that $W = V$ and thus V is irreducible.
Theorem 6.1 states that when one has a $(2l+1)$-dimensional irreducible representation of $\mathbf{so}(3)$, then there exists a basis that satisfies the relations (6.2), (6.3) and (6.4). When we have two of these representations of dimension $2l+1$ with bases $\{\vec{v}_j\}$ and $\{\vec{w}\}$ with $j = 0, 1, \ldots, 2l$, we can construct a bijective linear map which sents $\vec{v}_j$ to $\vec{w}$, hence we have a isomorphism and the two representations are isomorphic. ■

6.2 Irreducible Representations of the Lie Group SO(3)

Theorem 6.3 is of great importance. Sicne for every spin, l, there exists an irreducible representation π_l from $\mathbf{so}(3)$ to a space V of dimension $2l+1$. One might wonder if there is a corresponding representation Π_l of the Lie group $\mathbf{SO}(3)$ to this representation of the Lie algebra. If there is such a representation Π_l of $\mathbf{SO}(3)$, how is it related to the representation π_l of the Lie algebra $\mathbf{so}(3)$? Before we can give answers to these questions, we have to proof the following lemma.

Lemma 6.4. *Let G_1, G_2 and G_3 be matrix Lie groups with Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$, respectively. Let $\Phi : G_1 \to G_2$ and $\Psi : G_2 \to G_3$ be Lie group homomorphisms with associated unique Lie algebra homomorphisms ϕ and ψ, repsectively, which satisfy equation* (4.14) *of Theorem 4.7. Then $\Psi \circ \Phi : G_1 \to G_3$ is a Lie group homomorphism and the associated Lie algebra homomorphism is $\psi \circ \phi$ and also satisfies equation* (4.14) *of Theorem 4.7.*

This Lemma is Exercise 10 of Chapter 16 from [6].

Proof. Let G_1, G_2 and G_3 be matrix Lie groups with Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$, respectively. Let $\Phi : G_1 \to G_2$ and $\Psi : G_2 \to G_3$ be Lie group homomorphisms with associated unique Lie algebra homomorphisms ϕ and ψ, repsectively, which satisfy equation (4.14) of Theorem 4.7. Then by Theorem 2.10, $\Psi \circ \Phi : G_1 \to G_3$ is a Lie group homomorphisms, since G_1 and G_3 are matrix Lie groups. We will define $\Pi := \Psi \circ \Phi$. With the use of Theorem 4.7, there is a unique Lie algebra π such that $\Pi(e^{tX}) = e^{t\pi(X)}$ for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}_1$. We will now use property 3 of this theorem.

$$\begin{aligned}\pi(X) &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\Pi\left(e^{tX}\right)\right]_{t=0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi \circ \Phi\right)\left(e^{tX}\right)\right]_{t=0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\Psi\left\{\Phi\left(e^{tX}\right)\right\}\right]_{t=0} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\Psi\left\{e^{t\phi(X)}\right\}\right]_{t=0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}e^{t\psi\{\phi(X)\}}\right]_{t=0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}e^{t(\psi\circ\phi)(X)}\right]_{t=0} \\ &= (\psi \circ \phi)(X),\end{aligned} \tag{6.16}$$

for every $X \in \mathfrak{g}_1$. The last step is done through the use of Proposition 3.25. Hence we can concluse that $\pi = \psi \circ \phi$. ∎

Theorem 6.5. *Let $\pi_l : \boldsymbol{so}(3) \to \boldsymbol{gl}(V)$ be an irreducible representation of $\boldsymbol{so}(3)$, with spin $l = \frac{1}{2}(\dim(V) - 1)$. If l is an integer, then there exists a representation $\Pi_l : \boldsymbol{SO}(3) \to \boldsymbol{GL}(V)$ such that Π_l and π_l are related as in equation (4.14) of Theorem 4.7. If l is a half-integer, then there does not exists such a representation Π_l.*

Proof. Let l be a half-integer and let $\pi_l : \mathbf{so}(3) \to \mathbf{gl}(V)$ be an irreducible representation of $\mathbf{so}(3)$. Then L_3 is diagonal in the basis $\{v_j\}$ with on the diagonal $l - j$, where j is an integer, thus $l - j$ is an half-integer. From Definition 6.2 we can see that $L_3 := i\rho(F_3)$, where ρ is an irreducible finitie-dimensioal representation of $\mathbf{so}(3)$. Thus in our case, we will take ρ to be π_l. Thus we get,

$$e^{2\pi\pi_l(F_3)} = e^{-2\pi i L_3} = -\mathbb{I}. \tag{6.17}$$

On the other hand, we can see that $e^{2\pi F_3} = \mathbb{I}$ by a similar way as in equation (4.29) of the proof of Theorem 4.15. If a corresponding representation of $\mathbf{SO}(3)$ existed, then we would get

$$\Pi_l(\mathbb{I}) = \Pi_l\left(e^{2\pi F_3}\right) = e^{2\pi\pi_l(F_3)} = -\mathbb{I}. \tag{6.18}$$

This is a contradiction, since a representation sends the identity to the identity. Thus such a representation Π_l of $\mathbf{SO}(3)$ does not exist for l being a half-integer.
Let l be an integer and let $\pi_l : \mathbf{so}(3) \to \mathbf{gl}(V)$ be an irreducible representation of $\mathbf{so}(3)$. Let us use the isomorphism $\phi_{(321)} : \mathbf{su}(2) \to \mathbf{so}(3)$ from the nore below Corollary 4.9, where $\phi_{(321)}(aE_1 + bE_2 + cE_3) = aF_3 + bF_1 + cF_2$ for all $a, b, c \in \mathbb{R}$. Remember that $\{E_1, E_2, E_3\}$ is a basis of $\mathbf{su}(2)$ and $\{F_1, F_2, F_3\}$ is a basis of $\mathbf{so}(3)$. Let us now define a representation π_l' of $\mathbf{su}(2)$ by a compostion of π_l on $\phi_{(321)}$ by

$\pi'_l(X) := \pi(\phi_{(321)}(X))$ for $X \in \mathbf{su}(2)$. By Theorem 2.10 we know that this is again a representation. Since $\mathbf{SU}(2)$ is connected and simply connected, see Theorem 3.17, we can apply Theorem 4.8. This theorem tells us there exists a unique representation Π'_l such that it is related to π'_l as in equation (4.14). We can now compute that

$$\begin{aligned}\Pi'_l(-\mathbb{I}) &= \Pi'_l\left(e^{2\pi E_1}\right) = e^{2\pi\pi'_l(E_1)} \\ &= e^{2\pi\pi_l(F_3)} = e^{-2\pi i L_3} = \mathbb{I}.\end{aligned} \tag{6.19}$$

Since π_l is a finite-dimensional irreducible representation of $\mathbf{so}(3)$, we can use Theorem 6.1 and conclude from equation (6.4) that L_3 is diagonal in the basis $\{\vec{v}_0, \vec{v}_1, \dots, \vec{v}_{2l}\}$. The entries on the diagonal are $l - j$ and it is an integer, since l is an integer and $j = 0, 1, \dots, 2l$ is an integer. So this means that $e^{-2\pi i L_3} = \mathbb{I}$. And from Theorem 4.15 we know that there is a surjective homomorphism $\Phi_{(321)}$ from $\mathbf{SU}(2)$ to $\mathbf{SO}(3)$, for which the associated Lie algebra homomorphism is $\phi_{(321)}$, and $\ker(\Phi_{(123)}) = \{\mathbb{I}, -\mathbb{I}\}$. Since the set $\{\mathbb{I}, -\mathbb{I}\}$ is a subset of the kernel of Π'_l, the group homomorphism Π'_l goes through $\mathbf{SO}(3)$. This gives us a representation Π_l of $\mathbf{SO}(3)$ such that $\Pi'_l = \Pi_l \circ \Phi_{(321)}$. By Theorem 4.7 there exists a unique Lie algebra homomorphism σ_l such that Π_l and σ_l are related as in equation (4.14). With the use of Lemma 6.4 we know that the associated Lie algebra representation σ_l of Π_l satisfies $\pi'_l = \sigma_l \circ \phi_{(321)}$. Since $\phi_{(321)}$ is a Lie algebra isomorphism, it has an inverse $\phi^{-1}_{(321)}$, so that $\sigma_l = \pi'_l \circ \phi^{-1}_{(321)} = \pi_l$. And thus Π_l is the desired representation of $\mathbf{SO}(3)$. ■

6.3 Spin

From Theorem 4.15 we know that $\mathbf{SU}(2)$ is the universal cover of $\mathbf{SO}(3)$ and that there is a Lie algebra isomorphism ϕ from $\mathbf{su}(2)$ to $\mathbf{so}(3)$ with a corresponding Lie group homomorphism Φ from $\mathbf{SU}(2)$ to $\mathbf{SO}(3)$. This Lie group homomorphism is unique and it is related to the Lie algebra isomorphism ϕ of $\mathbf{su}(2)$ to $\mathbf{so}(3)$ as in equation (4.14) in Theorem 4.7, i.e. $\Phi(e^{tX}) = e^{t\phi(X)}$ for every $X \in \mathbf{su}(2)$.
We can now construct a representation from $\mathbf{su}(2)$ to $\mathbf{gl}(V)$ through $\mathbf{so}(3)$ by the composition of the irreducible representation π_l of $\mathbf{so}(3)$ on the Lie algebra isoomorphism ϕ. This composition results in an irreducible representation π'_l from $\mathbf{su}(2)$ to $\mathbf{gl}(V)$ and thus is defined by $\pi'_l := \pi_l \circ \phi$. Since ϕ is an isomorphism and π_l is irreducible, the composition π'_l is also irreducible. With the use of Theorem 4.8, we have that for every spin l there exits a unique Lie group homomorphism Π'_l from $\mathbf{SU}(2)$ to $\mathbf{GL}(V)$ which is related to the Lie algebra homomorphism, in this case an irreducible representation, as in equation (4.14). And when we apply Proposition 5.7, we can see that this representation Π'_l of the Lie group $\mathbf{SU}(2)$ is irreducible for every l being an integer or half-integer. These irreducible representations Π'_l of $\mathbf{SU}(2)$ are in one-to-one correspondence to the projective representations of the

matrix Lie group $\mathbf{SO}(3)$. We have seen in the proof of Theorem 6.5 that a representation of $\mathbf{SO}(3)$ to an even dimensional space is not well defined. This is since it sents $\mathbb{I}$ to $-\mathbb{I}$. But this is not a problem when we look at the projective representations. Since in $\mathbf{PU}(V)$ the elements I and $\mathbb{I}$ are equivalent. Thus such a projective representation of $\mathbf{SO}(3)$ to an even dimensional space is a well-defined function. In other words, when we want to describe half-integer spins from $\mathbf{SO}(3)$, we have to look at the projective representations in stead of the 'ordinary' representations.
We have discussed integer spins and half-integer spins, there are also spins that are neither one of these, for instance spin $\frac{1}{3}$.

Definition 6.6. *Let l be a positive number. When a particle has an integer spin we call it a* ***boson****. When a particle has a half-integer spin, we call it a* ***fermion****. Otherwise, we call the particle an* ***anyon****.*

It is important to note that anyons only appear in 2 dimensions. And since Since we where working in three dimensional spaces, these anyons where of no importance.

Chapter 7

Conclusion, Discussion & Future Research

7.1 Conclusion & Discusion

We have proven that the Lie algebras $\mathbf{so}(3)$ and $\mathbf{su}(2)$ are isomorphic to each other. By using that we have proven that matrix Lie group $\mathbf{SU}(2)$ is the universal cover of the matrix Lie group $\mathbf{SO}(3)$. Furthermore we have seen that since $\mathbf{SU}(2)$ is the universal of $\mathbf{SO}(3)$ that for every Lie algebra homomorphism of $\mathbf{so}(3)$ to some Lie algebra $\mathfrak{g}$ associated to a matrix Lie group G, we can compose it with the Lie algebra isomorphism from $\mathbf{su}(2)$ to $\mathbf{so}(3)$ to get a Lie algebra homomorphism of $\mathbf{su}(2)$. And then by applying Theorem 4.8 we can construct a Lie group homomorphism from $\mathbf{SU}(2)$ to the G.
More specifically, we can do this with irreducible representations of $\mathbf{so}(3)$. For every dimension of V there exists an irreducible representation of $\mathbf{so}(3)$ to $\mathbf{gl}(V)$. Theorem 6.5 tells us there are only (irreducible) representations of $\mathbf{SO}(3)$ to $\mathbf{GL}(V)$ when W is of odd dimenion. But with the conclusions from above, we can conclude that there are in fact representations of $\mathbf{SU}(2)$ to $\mathbf{GL}(V)$ with dimension of V being even. These representations are irreducible since the Lie algebra represenation from $\mathbf{su}(2)$ to $\mathbf{gl}(V)$ is irreducible and $\mathbf{SU}(2)$ is a connected matrix Lie group. It is, however, also possible to describe the actions on these even-dimensional spaces through projective representations of $\mathbf{SO}(3)$. So for us to describe systems of boson particles, it is sufficient to only consider the matrix Lie group $\mathbf{SO}(3)$. But when we want to look at systems describing fermion particles, such as an electron, we have to consider unitary groups, in particular the matrix Lie group $\mathbf{SU}(2)$. Or one could also consider projective representations of the matrix Lie group $\mathbf{SO}(3)$ to describe the systems of half-integer particles.

7.2 Future Research

For the enthusiastic reader I have listed some interesting topics that are related to the topic of this thesis and I was not able to tread or tread to the extend they deserve.

1. **Anyons:**
   These are particles which live in two dimensional space. The group that describes the actions on these particles are braid groups. It is also possible to look at the relativistic case.

2. **Projective representations:**
   We have discussed projective representations really shallow. There is a lot more to these representations and one could investigate this much more thoroughly.

3. **Lorentz group:**
   With the Lorentz group one can express the spin actions in a relativistic manner. One has to include the Minkowksi metric and look at the Poincaré group. The Lorentz group is a subgroup of the Poincaré group.

4. **Other dimensions of $\mathbf{SO}(n)$ and $\mathbf{SU}(n)$:** We have restricted ourselfs to the matrix Lie groups $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$. It might be really interesting to look at higher dimensions of these groups.

Bibliography

[1] The University Utrecht logo on the front page.
Link:
https://www.uu.nl/organisatie/huisstijl/downloads/logo, at the date of 20 Januari 2021.

[2] J.J. Sakurai & Jim Napolitano *Modern Quantum Mechanics*, second ed., 5th print, 2019, Cambridge.

[3] The picture used in Section 1.1.
Link:
https://upload.wikimedia.org/wikipedia/commons/thumb/e/ee/Stern-Gerlach_experiment_svg.svg/1200px-Stern-Gerlach_experiment_svg.svg.png, at the date of 20 Januari 2021.

[4] M.A. Armstrong. *Groups and Symmetry*, Undergraduate Texts in Mathematics, 1988, Springer-Verlag, New York.

[5] Gordon James & Martin Liebeck. *Representations and Characters of Groups*, second ed., 2008, Cambridge.

[6] B.C Hall. *Quantum Theory for Mathematicians*, first ed., Graduate Texts in Mathematics, vol. 267, 2013, Springer, New York.

[7] B.C Hall. *Lie Groups, Lie Algebras, and Representations*, second ed., Graduate Texts in Mathematics, vol. 222, 2015, Springer, Switzerland.

[8] The link to the Figure 3.1 used in Section 3.1:
https://www.wikiwand.com/en/Simply_connected_space, this link was lastly visited at the date of 20 Januari 2021.

[9] The Figure 3.1 used in Section 3.1.
Link:
https://www2.math.upenn.edu/~wziller/Math114f19/ch16-3-4.pdf, the

figure can be found on slide 5, this link was lastly visited at the date of 20 Januari 2021.

[10] Allen Hatcher. *Algebraic Topology*, 21st print, 2019, Cambridge.

[11] Charles Staats. *Point-Set Topology II*, 14 September 2010 edition.
Link: `http://math.uchicago.edu/~womp/2010/topology.pdf`.
This link was lastly visited at the date 20 Januari 2021.

[12] Author unkown. *Reminder on Quotient spaces*, 17 Oktober 2013 edition.
Link:
`https://courses.maths.ox.ac.uk/node/view_material/495#:~:text=If%20X%20is%20compact%20(connected,is%20also%20compact%20(connected).&text=The%20projection%20map%20p%20%3A%20X,space%20is%20compact%20(connected).`.
This link was lastly visited at the date 20 Januari 2021.

[13] Frits Beukers. *Ring and Galois Theory*, Department of Mathematics, Utrecht University, 2017. Lecture notes for the course Rings and Galois Theory.
Link:
`https://webspace.science.uu.nl/~beuke106/ringengalois/dic.pdf`.
This link was lastly visited by me at the date 8 Januari 2021.

[14] The Figure 4.1 used in Section 4.1.
Link:
`https://www.researchgate.net/figure/SE-3-and-the-corresponding-Lie-algebra-as-ta`
`fig2_265646878`, this link was lastly visited at the date of 20 Januari 2021.

[15] Gerald Teschl. *Ordinary Differential Equations and Dynamical Systems*, Author's preliminary version made available with permission of the publisher, the American Mathematical Society.
Link:
`https://www.mat.univie.ac.at/~gerald/ftp/book-ode/ode.pdf`.
This link was lastly visited at the date 20 Januari 2021.

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