Instanton induced vacua from String Theory?

Author:
Shashank Kumar
Master student Theoretical Physics,
Utrecht University

First supervisor:
Dr. Thomas Grimm
Institute for Theoretical Physics
Utrecht University

Second supervisor:
Dr. Nava Gaddam
Institute for Theoretical Physics
Utrecht University

Master Thesis

Institute for Theoretical Physics
Utrecht University
August, 10, 2021

Abstract

We study the one-loop effects on four-dimensional N=1 supergravity in four dimensions with one dimension compactified on a spatial circle. The corresponding non trivial topologies contribute to the gravitational path integral and give rise to an infra red emergent scale. Superpotential is generated from instanton effects which eventually decompactifies the circle. We further extend the study by adding chiral and vector multiplet to the theory. We expect such multiplets to be present in String theory compactifications. This addition generates new vacua in the three dimensional potential. Hence, presence of these multiplets in the theory could produce new vacua.

Contents

1 Introduction

There have been several attempts at understanding quantum gravity. Gravity is particularly hard to quantise. However, a good start would be to considering gravity as any other field theory which is valid at long distances and low curvatures [1]. We start with the path integral,

$$Z = \sum_{topology} \int \mathcal{D}g \; e^{-S_{graviton}} \tag{1.1}$$

where the path integral is wick rotated and there is a sum over all topologies with the same asymptotics. Gravity is represented by the spin 2 graviton field $h_{\mu\nu}$, defined as fluctuations on a background,

$$g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{1.2}$$

The graviton action is the usual Einstein-Hilbert action,

$$S_{graviton} = \int d^4x \; \sqrt{-g}\mathcal{R} \tag{1.3}$$

We wish to explore the contributions of non-trivial topologies to the path integral. The topologies that we are interested in are the four-dimensional spaces with one dimension compactified on a spatial circle. This is generally regarded as the Kaluza Klein reduction. This space is asymptotically flat and is the classical solution to Einstein's equation. However, quantum mechanically there are instabilities associated with it. The most important instability is the Casimir effects [2]. As the graviton goes around in the loops of the spatial circle, it tends to compactify the circle. Whereas, if there were fermions in the theory they push the circle out. Therefore, these two forces compete with one another. Another instability of these classical Kaluza-Klein solutions is the Witten's bubble of nothing [3] instability that is a non-perturbative effect and only becomes significant in absence of Casimir force.
The cure for such instabilities is N=1 supersymmetry. We add the supersymmetric partner of the graviton, the spin 3/2 gravitino field ψ_μ [4]. Spin 3/2 action is given by,

$$S_{gravitino} = \int d^4x \sqrt{-g} \psi_\mu \gamma^{\mu\sigma\nu} \mathcal{D}_\sigma \psi_\nu \tag{1.4}$$

where, $\gamma^{\mu\sigma\nu}$ is the anti-symmetrised gamma product and and $\mathcal{D}_\mu = \nabla_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}$ (a,b being the tangent spcace indices). The supergravity action then becomes,

$$S_{supergravity} = \int d^4x \; \sqrt{-g} \left[\mathcal{R} + \psi_\mu \gamma^{\mu\sigma\nu} \mathcal{D}_\sigma \psi_\nu\right] \tag{1.5}$$

The supersymmetric Kaluza Klein solutions are known to be stable. The presence of equal number of bosons and fermions in the theory balances the Casimir force. However, for N=1 supergravity, a superpotential is generated by gravitational instantons [5] of the form,

$$\mathcal{W} \sim \exp\left(-\frac{\pi R^2}{4G_N} - i\sigma\right) \tag{1.6}$$

where R is the radius of the spatial circle and σ is a periodic scalar associated with graviphoton field arising from Kaluza Klein ansatz. The minima of the potential arising from this superpotential is at $R \sim \infty$, therefore the compact dimension likes to expand. Hence, this superpotential pushes the circle out, and the circle decompactifies [6]. Therefore, the three-dimensional reduction is not stable. However, by adding more N=1 supersymmetric matter we observe the existence of new vacua.

1.1 New N=1 supersymmetric matter

Taking our inspiration from string theory [7] which allows for other N=1 supersymmetric matter to be present in the theory. We add chiral (spin 1/2 and spin 0) and vector (spin 1 and spin 1/2) multiplets to our action. Their actions are,

$$S_{chiral} = \int d^4x \left(\phi\nabla^2\phi + \chi\gamma^\mu\mathcal{D}_\mu\chi\right) \tag{1.7}$$

where ϕ is the scalar field and χ is the spinor field.

$$S_{vector} = \int d^4x \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \chi\gamma^\mu\mathcal{D}_\mu\chi\right) \tag{1.8}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and A_μ is the vector field. Adding these new fields to the theory have interesting consequences. We observe the existence of new vacua apart from the one observed at $R \sim \infty$. This implies that the three-dimensional compactification survives. This has consequences in the swampland project of string theory.

1.2 New emergent scale for gravity

Apart from generating superpotential, another consequence of instanton calculation is the emergence of a new infrared scale. There is an expectation for quantum gravity effects to be present at UV scale [8]. However, in instanton calculation, a new exponentially suppressed scale is found by Tong and Turner. This scale can be thought of as the equivalent of the Λ_{QCD} in Yang Mills theory. This appears as the running of the Gauss-Bonnet term in the path integral [9]. Gauss-Bonnet term is a topological term and has no contribution in loop effects but becomes important when different gravitational instanton effects are considered.

1.3 Overview

We begin by setting up the notation and performing the Kaluza Klein reduction. Then, we calculate the one-loop effects and figure out finite contributions and divergences. Following which we evaluate the gravitational instanton effects and calculate the one-loop determinants on the background of Taub-NUT instanton. Then, piecing everything together we evaluate the superpotential and explore the consequences. Finally, we repeat the process by adding new N=1 supersymmetric matter to the theory and observe how that affects the previous and results. We observe that the presence of more supersymmetric matter results in the existence of new vacua away from $R \to \infty$. While the new gravitational scale Λ_{grav} still exists.

2 Kaluza Klein ansatz

Kaluza Klein ansatz is used for dimensional reduction [10]. One of the dimensions is compactified bringing down the theory from n+1 dimensions to n dimensions. For the reduction, we use the metric in Kaluza Klein ansatz,

$$ds^2_{(4)} = \frac{L^2}{R^2} ds^2_{(3)} + \frac{R^2}{L^2} \left(dz + A_i \, dx^i\right)^2 \tag{2.1}$$

where $z \in [0, 2\pi L)$ is the periodic coordinate whereas x^i are the three-dimensional coordinates. A_i is the graviphoton gauge field and R is the radius of the circle.

3 Three-dimensional reduction

Using the Kaluza Klein ansatz we reduce the Einstein-Hilbert action in three dimensions. We start with classical action and evaluate it on the Kaluza Klein background.

$$\begin{aligned} S &= \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g} \mathcal{R}_{(4)} \\ &= \frac{2\pi L \, M_{pl}^2}{2} \int d^3x \sqrt{-g_{(3)}} \left[\mathcal{R}_{(3)} - 2 \left(\frac{\partial R}{R}\right)^2 - \frac{R^4}{4L^4} F_{ij} F^{ij} \right] \\ &= \frac{2\pi L \, M_{pl}^2}{2} \int d^3x \sqrt{-g_{(3)}} \left[\mathcal{R}_{(3)} - 2 \left(\frac{\partial R}{R}\right)^2 - \frac{R^4}{4L^4} \left(\frac{\partial \sigma}{2\pi}\right) \right] \end{aligned} \tag{3.1}$$

where in the last line the we have picked the three dimensional dual partner of graviphoton, the periodic scalar σ with period 2π. In three dimensions a scalar can be picked up by dualising a 2-tensor ($\partial_\mu \sigma \sim \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho}$). The propagating fields R and Σ form a chiral multiplet,

$$\mathcal{S} = 2\pi^2 M_{pl}^2 R^2 + i\sigma \tag{3.2}$$

In the next section, we look at the corrections to this complex structure.

4 One-loop effects

In this section, we describe the quantum fluctuations in spin 2 graviton and spin 3/2 gravitino field on the background with Kaluza Klein compactification. We look at the contributions at one-loop. We find that there are two kinds of contributions at one loop, the finite corrections, and the divergences. Next, we study them separately. Let's first begin with evaluation of one-loop determinants. For this purpose we use the Faddev Popov method defined next.

4.1 Faddeev Popov method

In this section, we review the Faddeev Popov method of gauge fixing in the partition function. Faddeev and Popov devised a technique for such gauge fixing in the path integrals [11] [12]. This exercise limits the paths in the path integral and the field is defined unambiguously. Gauge fixing the path integral requires us to introduce unphysical fields known as Faddeev Popov ghosts. Gauge fields are connected by gauge transformation $A'_\mu = A_\mu + F^\Lambda$, where F^Λ is the type gauge and Λ is the gauge parameter. This is a source of ambiguity in the theory until the gauge is fixed. As a consequence, the path integral diverges. Consider a gauge field A_μ, the partition function is given by,

$$Z = \int \mathcal{D}A_\mu e^{iS[A]} \tag{4.1}$$

Gauge transforming ($A^\Lambda_\mu = A_\mu + D_\mu\Lambda$) the above integral we have,

$$Z = \int \mathcal{D}A^\Lambda_\mu e^{iS[A]} \int \mathcal{D}\Lambda \tag{4.2}$$

Since, action is gauge invariant, we separate out the contributions of the A'_μ and Λ. The integral over Λ diverges since there are infinite many Λ that satisfy gauge condition. Hence, we need to regularise the path integral by fixing the gauge. We choose the gauge condition $F[A^\Lambda_\mu] = 0$. We need to introduce this condition into our partition function. Now, consider the following,

$$\begin{aligned} \Delta_F^{-1}[A^\Lambda_\mu] &= \int d\Lambda \delta(F[A^\Lambda_\mu]) \\ 1 &= \Delta_F[A^\Lambda_\mu] \int d\Lambda \delta(F[A^\Lambda_\mu]) \end{aligned} \tag{4.3}$$

Adding this identity to the path integral introduces the gauge fixing and regularises the integral. The pre-factor $\Delta_F[A_\mu]$ is called the Faddeev Popov determinant which can be evaluated by performing the integration in the first line. We find,

$$\Delta_F[A_\mu] = det \left|\frac{dF}{d\Lambda}\right|_{\Lambda=0} = det\ M \tag{4.4}$$

Plugging the identity in eq(4.1), we get

$$Z = \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_F[A^\Lambda_\mu] \int d\Lambda \delta(F[A^\Lambda_\mu]) \tag{4.5}$$

We can perform the gauge transformation to remove the gauge parameter dependence from the terms. It can be shown that all the terms are gauge invariant. Hence, we have,

$$Z = \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_F[A_\mu] \delta(F[A_\mu]) \int d\Lambda \tag{4.6}$$

Performing the Λ integral gives an (irrelevant) normalization factor.

$$Z = N \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_F[A_\mu] \delta(F[A_\mu]) \tag{4.7}$$

The Faddeev Popov determinant in the path integral can be thought of as a result of Gaussian integration over some fields. Since the power of the determinant is +1, this new field is represented by anti-commuting complex variables (gaussian integral gives determinant power +1 see appendix F). Since we introduce these fields, these are unphysical and we call them Faddeev Popov Ghosts. Let's call these fields c and its complex conjugate $\bar{c}$. Convert $\Delta_F[A_\mu]$ to $i\ \Delta_F[A_\mu]$ which can be written as the Gaussian integral over these ghost fields.

$$i\ \Delta_F[A_\mu] = det\ iM = \int \mathcal{D}\bar{c}\mathcal{D}c\ e^{i\int d^4x \bar{c}\ M\ c} \tag{4.8}$$

This is the so called ghost action S_{gh}. Hence, replacing the Faddeev Popov determinant with ghost action in 4.5, we get

$$Z = \int \mathcal{D}_\mu \mathcal{D}c\mathcal{D}\bar{c}\ e^{iS[A_\mu]+iS_{gh}} \delta(F[A_\mu]) \tag{4.9}$$

Last, we use another trick to further simplify our partition function first proposed by t'Hooft in [13]. We can modify our gauge condition

$$F[A_\mu] = f(x) \tag{4.10}$$

This modifies the delta function,

$$Z = \int \mathcal{D}_\mu \mathcal{D}c\mathcal{D}\bar{c}\ e^{iS[A_\mu]+iS_{gh}} \delta(F[A_\mu] - f) \tag{4.11}$$

Since f is independent of A_μ, c and $\bar{c}$ we can add $e^{-\frac{i}{\alpha}\int d^4x f^2}$ (α being a parameter)at the cost of change of the (irrelevant) normalization factor,

$$\begin{aligned} Z &= N' \int \mathcal{D}_\mu \mathcal{D}c\mathcal{D}\bar{c}\ e^{iS[A_\mu]+iS_{gh}} \delta(F[A_\mu] - f) e^{-\frac{i}{\alpha}\int d^4x f^2} \\ &= N' \int \mathcal{D}_\mu \mathcal{D}c\mathcal{D}\bar{c}\ e^{iS[A_\mu]+iS_{gh}} e^{-\frac{i}{\alpha}\int d^4x (F[A_\mu])^2} \\ &= N' \int \mathcal{D}_\mu \mathcal{D}c\mathcal{D}\bar{c}\ e^{iS[A_\mu]+iS_{gh}+iS_{gf}} \\ &= N' \int \mathcal{D}_\mu \mathcal{D}c\mathcal{D}\bar{c}\ e^{iS_{eff}} \end{aligned} \tag{4.12}$$

where, we have captured the last exponential as the gauge fixing action. This completes the procedure of gauge fixing the path integrals. The corresponding kinetic operators can be evaluated from the effective action.

4.2 Bosonic sector

Einstein Hilbert action is given by,

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R \tag{4.13}$$

where we have the background metric $g_{\mu\nu}$ and fluctuations $h_{\mu\nu}$,

$$g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{4.14}$$

Using this ansatz we can expand our action in terms of the fluctuation field. For detailed calculation see appendix H. The action takes the form up to second order in $h_{\mu\nu}$,

$$\begin{aligned} S_0 = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\bigg(&\frac{1}{8}h^\mu_\mu h^\nu_\nu R - \frac{1}{2}h^\alpha_\alpha h^{\nu\beta}R_{\nu\beta} - \frac{1}{4}h^\alpha_\beta h^\beta_\alpha R + h^\nu_\alpha h^{\alpha\beta}R_{\nu\beta} + \frac{1}{4}h^\mu_{\mu,\alpha}h^{\nu,\alpha}_\nu \\ &- \frac{1}{4}h^\alpha_{\beta,\mu}h^{\beta,\mu}_\alpha + \frac{1}{2}h^\mu_{\mu,\beta}h^{\beta,\nu}_\nu - \frac{1}{2}h^{\beta,\alpha}_\nu h^\nu_{\alpha,\beta}\bigg) \end{aligned} \tag{4.15}$$

where the curvature tensors are defined on the background metric $g_{\mu\nu}$.

4.2.1 Ghost of Graviton

Next, we look at the gauge freedom available to us in this system. The action is invariant under metric fluctuations $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$, ξ being the gauge parameter. As described in the Fadeev Popov method section there's need for fixing this freedom. This can be done by imposing gauge fixing condition,

$$\nabla^\mu\left(h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h\right) = 0 \tag{4.16}$$

Introduction of the gauge condition is associated with the cost of introducing ghosts fields to the action. This is accomplished by introducing the identity into the action.

$$1 = \int D\alpha\delta(F[\xi_\mu])det\left(\partial F[\xi_\mu]/\partial\xi_\nu\right) \tag{4.17}$$

where F is the gauge condition. Therefore, this identity introduces the gauge condition to the Action. The determinant is known as Faddeev-Popov determinant.

$$\Delta_1(\xi) = det\left(\partial F[\xi_\mu]/\partial\alpha\right) \tag{4.18}$$

The $\Delta_1(\xi)$ can be determined by introducing complex anti-commuting vector fields, called Faddeev Popov ghosts and integrating over them.

$$\begin{aligned} \Delta_1(\xi) &= det\left(\partial F[\xi_\mu]/\partial\alpha\right) \\ &= \int Dc\, D\bar{c}\; exp\left(i\int d^4x\bar{c}\frac{\partial F}{\partial\xi}c\right) \end{aligned} \tag{4.19}$$

Let's calculate it for the case of Graviton, the gauge fixing term is

$$F_\nu = \nabla^\mu \left(h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) \tag{4.20}$$

It's transformation under gauge transformation is given by,

$$\begin{aligned}
F'_\nu &= \nabla^\mu \left(h'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h'_{\alpha\beta} g^{\alpha\beta} \right) \\
&= \nabla^\mu \left(h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \right) - \frac{1}{2} \nabla_\nu \left(h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \right) g^{\alpha\beta} \\
&= F_\nu + \nabla^\mu \nabla_\mu \xi_\nu + [\nabla_\mu, \nabla_\nu] \xi^\mu \\
&= F_\nu + \nabla^\mu \nabla_\mu \xi_\nu + R_\gamma{}^\mu{}_{\nu\mu} \xi^\gamma \\
&= F_\nu + g_{\mu\nu} \nabla^2 \xi^\mu + R_{\mu\nu} \xi^\mu \\
&= F_\nu + (g_{\mu\nu} \nabla^2 + R_{\mu\nu}) \xi^\mu
\end{aligned} \tag{4.21}$$

The determinant is given by,

$$\begin{aligned}
\Delta_1(\xi) &= det \left(\partial F[\xi] / \partial \xi \right) \\
&= det(g_{\mu\nu} \nabla^2 + R_{\mu\nu})
\end{aligned} \tag{4.22}$$

4.2.2 Gauge fixing lagrangian

Using the Faddeev Popov method, we add the gauge fixing action to our path integral. From appendix H we find the gauge fixing action is,

$$S_{gf} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \left(\frac{1}{2} h^{;\mu}_{\mu\alpha} h^{\nu\alpha}_{,\nu} - \frac{1}{2} h^{\beta}_{\beta,\alpha} h^{\nu\alpha}{}_{,\nu} + \frac{1}{8} h^{\mu}_{\mu,\alpha} h^{\nu,\alpha}_{\nu} \right) \tag{4.23}$$

Adding it to our action, we get the final form,

$$\begin{aligned}
S = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \bigg(&\frac{1}{4} \bar{h}^{\mu\nu} g_{\mu\alpha} g_{\nu\beta} \nabla^2 \bar{h}^{\alpha\beta} - \frac{1}{4} \bar{h}^{\mu\nu} g_{\mu\alpha} g_{\nu\beta} R \bar{h}^{\alpha\beta} + \frac{1}{2} \bar{h}^{\mu\nu} g_{\nu\beta} R_{\mu\alpha} \bar{h}^{\alpha\beta} \\
&+ \frac{1}{2} \bar{h}^{\mu\nu} R_{\mu\alpha\nu\beta} \bar{h}^{\alpha\beta} - \frac{1}{16} h \nabla^2 h \bigg)
\end{aligned} \tag{4.24}$$

4.2.3 Loop determinant of Graviton

Now, with all the pieces calculated, we write the final loop determinant as,

$$Z = \int \mathcal{D}c \, \mathcal{D}\bar{c} \, \mathcal{D}\bar{h}_{\mu\nu} \, \mathcal{D}h \exp \left(-\frac{1}{16\pi G} \int d^4x \bar{h}_{\mu\nu} \Delta_2{}^{\mu\nu\alpha\beta} \bar{h}_{\alpha\beta} + h \Delta_0 h + \bar{c} \Delta_1 c \right) \tag{4.25}$$

The contribution from spin 2 field and spin 1 field is that of real bosons with kinetic operators, Δ_2 and Δ_0 and their Gaussian integrals (see appendix F). However, contribution from spin 1 is from integral over c and $\bar{c}$ which are complex anti-commuting variables. Therefore, integrating out Graviton and its ghost we get the one loop determinant as,

$$\Gamma_B = \frac{(det\Delta_1)}{(det\Delta_0)^{1/2} (det\Delta_2)^{1/2}} \tag{4.26}$$

4.3 Fermionic Sector

In this section we describe the Faddeev Popov method of gauge fixing for the case of gravitino field [14] [15] [16].

The action for Majorana spin 3/2 fermion is given by,

$$Z = \int D[\Psi] \exp(i\bar{\Psi}_\mu \gamma^{\mu\theta\nu} D_\theta \Psi_\nu) \tag{4.27}$$

Spin 3/2 field is invariant under transformation,

$$\Psi_\mu = \Psi_\mu + \nabla_\mu \epsilon \tag{4.28}$$

we choose the gauge condition,

$$\begin{aligned} F(\epsilon) &= \gamma^\mu \Psi_\mu \\ &= b \end{aligned} \tag{4.29}$$

where b is a arbitrary field. We find $\frac{\delta F}{\delta \epsilon} = \gamma^\mu D_\mu$. Apart from gauge condition, we need equation of motion to remove all the unphysical degrees of freedom [17].

$$E[\Psi] = \nabla^\mu \Psi_\mu = \gamma^\nu D_\nu \, b = 0 \tag{4.30}$$

Similar to the Graviton case, we introduce the equality,

$$1 = \left(det(\frac{\delta F}{\delta \epsilon}) \right)^{-1} \int d\epsilon \, \delta(F) \tag{4.31}$$

Introducing the gauge condition to the path integral,

$$Z = \int D[\Psi] \delta(\gamma^\mu \Psi_\mu - b) det^{-1} \left(\frac{\delta F}{\delta \epsilon} \right) exp(S_0) \delta(E) \int d\epsilon \tag{4.32}$$

However, introduction of gauge condition doesn't get rid of all the unphysical degrees of freedom.Therefore, we introduce equation of motion with gauge condition,

$$\begin{aligned} E(\chi) &= \gamma^\alpha D_\alpha b \\ &= M \; b \end{aligned} \tag{4.33}$$

Using the t'Hooft device for E,

$$1 = \int db \exp\left(\int d^4 x b M b \right) (det(M))^{-1/2} \tag{4.34}$$

introducing the the path integral,

$$Z = \int D[\Psi] db \delta(\gamma^\mu \Psi_\mu - b) det^{-1}(\gamma^\mu D_\mu) exp(S_0 + \alpha \int d^4 x b M b)(det M)^{-1/2} \tag{4.35}$$

performing the b integral, we get,

$$Z = \int D[\Psi] det^{-1}(\gamma^\mu D_\mu) exp\left(S_0 + \alpha \int d^4 x \gamma^\mu \Psi_\mu \gamma^\alpha D_\alpha \gamma^\nu \Psi_\nu \right) (\gamma^\alpha D_\alpha)^{-1/2} \tag{4.36}$$

where M is identified as $(\gamma^\alpha D_\alpha)$. Finally, we have,

$$Z = \int D[\Psi] det^{-3/2}(\gamma^\mu D_\mu) exp\left(S_0 + \alpha \int d^4x \gamma^\mu \Psi_\mu \gamma^\alpha D_\alpha \gamma^\nu \Psi_\nu\right) \tag{4.37}$$

Therefore, we get three real majorana spin 1/2 fermion ghost fields with compensate the power -3/2 of determinant. The term in the exponential is identified as the gauge fixing Lagrangian.

4.3.1 Spin 3/2 dirac operator

In this section we calculate the dirac operator of gravitino field. The Lagrangian of the field is given by,

$$\mathcal{L} = -i\bar{\Psi}_\mu \gamma^{\mu\theta\nu} D_\theta \Psi_\nu \tag{4.38}$$

where $\gamma^{\mu\theta\nu}$ is the anti symmetrised gamma product given by,

$$\begin{aligned}
\gamma^{\mu\theta\nu} &= \frac{1}{6}\left(\gamma^\mu\gamma^\theta\gamma^\nu + \gamma^\theta\gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\mu\gamma^\theta - \gamma^\mu\gamma^\nu\gamma^\theta - \gamma^\nu\gamma^\theta\gamma^\mu - \gamma^\theta\gamma^\mu\gamma^\nu\right) \\
&= \frac{1}{6}\Big(\gamma^\mu\gamma^\theta\gamma^\nu + (2g^{\theta\nu} - \gamma^\nu\gamma^\theta)\gamma^\mu + \gamma^\nu(2g^{\mu\theta} - \gamma^\theta\gamma^\mu) - \gamma^\mu(2g^{\nu\theta} - \gamma^\theta\gamma^\nu) - \gamma^\nu\gamma^\theta\gamma^\mu \\
&\qquad - (2g^{\theta\mu} - \gamma^\mu\gamma^\theta)\gamma^\nu\Big) \\
&= \frac{1}{6}\left(\gamma^\mu\gamma^\theta\gamma^\nu - \gamma^\nu\gamma^\theta\gamma^\mu - \gamma^\nu\gamma^\theta\gamma^\mu + \gamma^\mu\gamma^\theta\gamma^\nu - \gamma^\nu\gamma^\theta\gamma^\mu + \gamma^\mu\gamma^\theta\gamma^\nu\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\theta\gamma^\nu - \gamma^\nu\gamma^\theta\gamma^\mu\right)
\end{aligned} \tag{4.39}$$

Hence the Lagrangian becomes,

$$\mathcal{L} = -\frac{i}{2}\bar{\Psi}_\mu\left(\gamma^\mu\gamma^\theta\gamma^\nu - \gamma^\nu\gamma^\theta\gamma^\mu\right) D_\theta\Psi_\nu \tag{4.40}$$

The gauge fixing Lagrangian is,

$$\mathcal{L}_{gf} = \frac{i}{2}\bar{\Psi}_\mu\gamma^\mu\gamma^\theta\gamma^\nu D_\theta\Psi_\nu \tag{4.41}$$

Total Lagrangian becomes,

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}_\mu\gamma^\nu\gamma^\theta\gamma^\mu D_\theta\Psi_\nu \tag{4.42}$$

Next, we square the Dirac operator as it is convenient to work with them,

$$\begin{aligned}
\Delta_{3/2} &= (\gamma^{\nu}\gamma^{\theta}\gamma^{\mu}D_{\theta})\gamma^{\lambda}\gamma^{\phi}\gamma_{\nu}D_{\phi} \\
&= \gamma^{\nu}\gamma^{\theta}\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma_{\nu}D_{\theta}D_{\phi} \\
&= (2g^{\nu\theta}\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma_{\nu} - 2g^{\nu\mu}\gamma^{\theta}\gamma^{\lambda}\gamma^{\phi}\gamma_{\nu} + 2g^{\nu\lambda}\gamma^{\theta}\gamma^{\mu}\gamma^{\phi}\gamma_{\nu} \\
&- 2g^{\nu\phi}\gamma^{\theta}\gamma^{\mu}\gamma^{\lambda}\gamma_{\nu} + \gamma^{\theta}\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\nu}\gamma_{\nu})D_{\theta}D_{\phi} \\
&= (2\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta} - 2\gamma^{\theta}\gamma^{\lambda}\gamma^{\phi}\gamma^{\mu} \\
&+ 2\gamma^{\theta}\gamma^{\mu}\gamma^{\phi}\gamma^{\lambda} - 2\gamma^{\theta}\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi} + 4\gamma^{\theta}\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi})D_{\theta}D_{\phi} \\
&= (2\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta} - \cancel{2\gamma^{\theta}(2g^{\lambda\phi})\gamma^{\mu}} + 2\gamma^{\theta}\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu} + \cancel{2\gamma^{\theta}\gamma^{\mu}(\gamma^{\phi}\gamma^{\lambda} + \gamma^{\lambda}\gamma^{\phi})})D_{\theta}D_{\phi} \\
&= (2\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta} + 2\gamma^{\theta}\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu})D_{\theta}D_{\phi} \\
&= (\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta} + \gamma^{\theta}\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu} + \gamma^{\mu}\gamma^{\lambda}(2g^{\theta\phi} - \gamma^{\theta}\gamma^{\phi}) \\
&+ (2g^{\phi\theta} - \gamma^{\phi}\gamma^{\theta})\gamma^{\lambda}\gamma^{\mu})D_{\theta}D_{\phi} \\
&= (2(\gamma^{\mu}\gamma^{\lambda} + \gamma^{\lambda}\gamma^{\mu})D_{\theta}D^{\theta} + \gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta}D_{\theta}D_{\phi} \\
&+ \gamma^{\theta}\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu}D_{\theta}D_{\phi} - \gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta}D_{\phi}D_{\theta} - \gamma^{\theta}\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu})D_{\phi}D_{\theta} \\
&= (4g^{\mu\lambda}D^{2} + (\gamma^{\mu}\gamma^{\lambda}\gamma^{\phi}\gamma^{\theta} + \gamma^{\theta}\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu})[D_{\theta}, D_{\phi}]
\end{aligned} \tag{4.43}$$

The commutator of covariant derivative acting on a fermion can be written as,

$$[D_{\rho}, D_{\lambda}]\Psi_{\nu} = -R^{\sigma}_{\nu\rho\lambda}\psi_{\sigma} + \frac{1}{8}R_{\rho\lambda\theta\phi}[\gamma^{\theta}, \gamma^{\phi}]\Psi_{\nu} \tag{4.44}$$

Plugging this in, we find that the $\Delta_{3/2}$ becomes

$$\Delta_{3/2} = -g^{\mu\lambda}D^{\rho}D_{\rho} + R^{\mu\lambda} - \frac{1}{2}(\gamma^{\rho}\gamma^{\theta})R^{\mu\lambda}{}_{\rho\theta} + \frac{1}{2}\gamma^{\mu}\gamma^{\theta}R^{\lambda}{}_{\theta} - \frac{1}{2}\gamma^{\lambda}\gamma^{\theta}R^{\mu}{}_{\theta} - \frac{R}{4}(\gamma^{\mu}\gamma^{\lambda}) + \frac{R}{4}g^{\mu\lambda} \tag{4.45}$$

4.3.2 Loop determinant of Gravitino

We can rewrite the equation 4.37,

$$Z = \int \mathcal{D}\Psi\mathcal{D}a\,\mathcal{D}b\,\mathcal{D}c\,\exp\left(-\frac{1}{16\pi G_N}\int d^4x(a\gamma^{\mu}D_{\mu}a + b\gamma^{\mu}D_{\mu}b + c\gamma^{\mu}D_{\mu}c + \Psi_{\mu}\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}D_{\alpha}\Psi_{\nu}\right) \tag{4.46}$$

where, a,b,c are commuting fermions. Performing the integrals we find, one loop determinant as,

$$\Gamma_F = \frac{(det(\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}D_{\alpha}))^{1/2}}{(det(\gamma^{\mu}D_{\mu}))^{3/2}} \tag{4.47}$$

It's easier to work with square operators. Therefore,

$$\Gamma_F = \frac{(det\Delta_{3/2})^{1/4}}{(det\Delta_{1/2})^{3/4}} \tag{4.48}$$

Next we look at the finite corrections and divergences arising from these determinants.

4.4 Finite corrections

Finite corrections arise from compactifications on a spatial circle with a loop going around in the circle. We see that these corrections depend on the radius of the circle R, therefore don't have ultra-violet divergences and can be calculated. These corrections were calculated in [2] and [18] and can be manifested as the Casimir forces. These Casimir forces tend to make the compactification unstable. However, the presence of supersymmetry removes the instability and makes the Kaluza Klein compactifications stable.
The one-loop effects renormalizes the kinetic terms and the low energy effective action becomes,

$$\mathcal{L}_{eff} = \frac{1}{2}\left(M_3 + \frac{5L}{16^2}\right)\mathcal{R}_{(3)} - \left(M_3 - \frac{L}{6^2}\right)\left(\frac{\partial R}{R}\right)^2 - \left(M_3 + \frac{11L}{24\pi R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2 \quad (4.49)$$

where, $M_3 = 2\pi L M_{pl}^2$. Comparing this expression with the 3.1, we see that the renormalization of R and σ have different coefficients which give rise to correction to the chiral multiplet of R and σ.

$$\mathcal{S} = 2\pi^2 M_{pl}^2 R^2 + \frac{7}{48}\log(M_{pl}^2 R^2) + i\sigma \quad (4.50)$$

The numerical coefficient of corrections is interesting as the same number arises from determinant calculations.

4.5 Heat kernels and divergences

In this section, we calculate the one-loop divergences arising from the determinants. We use the heat kernel approach to expand the action. We observe that the divergences can be captured by the Gauss-Bonnet term. We begin with the determinants Δ_s (s being the spin 0,1/2,1,3/2,2), which takes the form, (for detailed calulations of determinants see section 7 and 8)

$$\Delta_s = -\nabla^2 - E_s \quad (4.51)$$

and

$$\nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{ab\mu}t^{ab}_{(s)}$$

where $t^{ab}_{(s)}$ is the Lorentz generator. With this we can write the action at one-loop as

$$e^{S_{1-loop}} = (det\Delta_s)^{\zeta_s} \quad (4.52)$$

where, ζ_s is the power of determinants from 6.1.

$$\zeta_s = \left(-\frac{1}{2}, -\frac{3}{4}, +1, +\frac{1}{4}, -\frac{1}{2}\right)$$

for $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ Hence, action at lone loop becomes,

$$S_{1-loop} = -\sum_{s=0}^{2}\zeta \log det\Delta_s \quad (4.53)$$

The number of off-shell degrees of freedom of a field are $d_s = (1,4,4,16,9)$,spin 2 being the symmetric traceless tensor has 9 degrees of freedom. Then supersymmetry dictates that, $\vec{\zeta}\cdot\vec{d}=0$. This property will be used later. Next we use the Pauli Villars regularization (see appendix A) to write the action as,

$$S_{1-loop} = -\sum_{s=0}^{2}\zeta\log det\Delta_{=}\sum_{s=0}^{2}\int\frac{dt}{t}Tr\left[\exp\left(-t(\Delta_s+m^2)\right)\right]_{PV} \tag{4.54}$$

where the second equality is the gamma function of zero argument which diverges. Hence, the equality holds upto a infinite constant which we ignore in this regularization. There are ultra-violet divergences in the t $\rightarrow\ 0^+$ limit. The standard heat kernel expansion gives (for detail follow user manual on heat kernel method [19]),

$$Tr\left[\exp\left(-t\Delta_s\right)\right]\sim t_{-2}B_0+t^{-1}B_2+B_4+O(t) \tag{4.55}$$

where B_k are the heat kernel coefficients defined in terms of different spin operators. The order k represents the number of Laplacian operators associated with the coefficient. Now, let's define the structure of each of the coefficients. The leading divergence is simply a cosmological term,

$$B_0(\Delta_s)=\frac{1}{16\pi^2}\int d^4x\sqrt{g}tr1 \tag{4.56}$$

This term vanishes when summed over all the spins $\vec{\zeta}.\vec{d}=0$. Similarly, quadratic divergences are captured by B_2 given by,

$$B_2(\Delta_s)=\frac{1}{16\pi^2}\int d^4x\sqrt{g}tr\left(E_s+\frac{1}{6}\mathcal{R}\right) \tag{4.57}$$

where the second term vanishes again by virtue of supersymmetry. We further evaluate the first term. It is straightforward to show,

$$tr(E_s)=-b_s\mathcal{R}$$

b_s =(0,1,-1,4,6) we find,

$$\sum_{s\ =0}^{2}\zeta B_2(\Delta_s)=-\frac{\vec{b}.\vec{\zeta}}{16\pi^2}\int d^4x\sqrt{g}\ tr[\mathcal{R}] \tag{4.58}$$

Again, this term can be consumed by renormalization of the Newton's constant. Finally, the logarithmic divergences are contained in B_4 given by,

$$\begin{aligned}B_4(\Delta_s)=\frac{1}{16\pi^2}\int d^4x\sqrt{g}tr\bigg(&\frac{1}{6}\nabla^2E_s+\frac{1}{6}\mathcal{R}E_s+\frac{1}{2}E_s^2+\frac{1}{72}\mathcal{R}^2-\frac{1}{180}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\\&+\frac{1}{180}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}+\frac{1}{48}t^{ab}_{(s)}\mathcal{R}^{ab\mu\nu}t^{cd}_{(s)}\mathcal{R}_{cd}{}^{\rho\sigma}\bigg)\end{aligned} \tag{4.59}$$

Here, again terms without spin dependence vanish. Also,

$$tr_{spin}[t^{ab}_{(s)}t^{cd}_{(s)}] = a_s(-\delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad})$$

where $a_s = (0, 1, 2, 12, 12)$. Plugging in E_s and a_s, we get,

$$\begin{aligned} B_4(\Delta_s) &= \frac{1}{16\pi^2}\int d^4x\sqrt{g}tr\left(\frac{1}{6}\nabla^2\mathcal{R}+\frac{1}{6}\mathcal{R}^2+c_s\frac{1}{2}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}+\frac{1}{48}t^{ab}_{(s)}\mathcal{R}^{ab\mu\nu}t^{cd}_{(s)}\mathcal{R}_{cd}{}^{\rho\sigma}\right) \\ &= \frac{1}{16\pi^2}\int d^4x\sqrt{g}\left(c_s\frac{1}{2}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - a_s\frac{1}{24}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\right) \end{aligned} \tag{4.60}$$

where, in second equality, total derivative term vanishes and $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ are absorbed by the Einstein-Hilbert action by field redefinition. The trace of E_s^2 is given by,

$$tr(E_s^2) = c_s\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} \tag{4.61}$$

with $c_s = (0, 0, 0, 2, 3)$. Finally, we can massage the $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ terms back to get the form of Gauss-Bonnet term. Therefore, we get,

$$\sum_s \zeta_s B_4(\Delta_s) = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left(c_s - \frac{1}{12}a_s\right)\zeta_s\left(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + \mathcal{R}^2\right) \tag{4.62}$$

Evaluating the prefactor and plugging it in the eq (4.54) and applying the Pauli Villars regularization we observe that one-loop effective action has logarithmically divergent term.

$$S_{1-loop} = -\frac{41}{48}\log\left(\frac{\mu^2}{m^2}\right)\chi \tag{4.63}$$

where χ is the Gauss-bonnet term and μ is the Pauli-Villars scheme. This term is interpreted as the running of Gauss-Bonnet term.

5 Gravitational Instantons

The term instanton was first coined about Yang-Mills theory where classical Euclidean solutions which only survived for an instant were found. The field strength was concentrated on one point in four-dimensional space-time. Also, these solutions were (anti) self-dual.

The physical implications of such solutions are profound. Since we have wick rotated the time in the path integral, the potential is inverted. This implies that the maxima of the Minkowski potential is minima in euclidean space and vice versa. Therefore, the instanton solution in the euclidean space gives a first approximation to the tunneling amplitudes between two adjacent vacuum states.

Similarly, Euclidean vacuum solutions were found for gravity as well called the Gravitational instantons [8] [20] [21]. These are the solution to the vacuum Einstein's field equations, therefore are Ricci flat. They are characterised by (anti) self dual Riemannian tensor ($R_{\mu\nu\rho\sigma} = \pm^* R_{\mu\nu\rho\sigma}$) and are asymptotically flat. Since we are interested in gravitational instanton with one compact dimension, the Taub-NUT metric turns out to be the simplest choice.

5.1 Taub-NUT metric

The Taub-NUT metric is given by [8],

$$ds^2 = Ud\mathbf{x}^2 + U^{-1}(dz + A.d\mathbf{x})^2 \tag{5.1}$$

and

$$U = 1 + \frac{R}{2|\mathbf{x} - \mathbf{x}_0|} \tag{5.2}$$

Comparing with the Kaluza Klein ansatz, $U = \frac{L^2}{R^2}$ and goes to 1 asymptotically and the boundary of this space is S^3. The condition for (anti) self-duality of the metric is

$$\nabla \times A = \mp \nabla U \tag{5.3}$$

where, z is the compact dimension, $z \in [0, 2\pi R)$ where R is the radius of the one-dimensional circle.

6 Determinants in anti-self dual background

Adding the bosonic and fermionic parts, we find the total one loop determinant is,

$$\begin{aligned}\Gamma &= \Gamma_B \Gamma_F \\ &= \frac{(det\Delta_{3/2})^{1/4}(det\Delta_1)}{(det\Delta_{1/2})^{3/4}(det\Delta_0)^{1/2}(det\Delta_2)^{1/2}}\end{aligned} \tag{6.1}$$

Now, we examine this loop determinant in the anti-self dual background. The properties of such a background is,

$$*R_{\mu\nu\alpha\beta} = -R_{\mu\nu\alpha\beta} \tag{6.2}$$

The consequence of this property is that that such metrics are Ricci flat, i.e $R_{\mu\nu} = 0$. Further, we work in the chiral basis. The dirac spinor decomposes into left-handed(undotted) and right-handed(dotted) chrial spinors,

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \tag{6.3}$$

and gamma matrices (with tangent space indices) are,

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix} \tag{6.4}$$

The covariant derivative is $D_\mu = \nabla_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}$. The spin connection can also be split in the chiral parts.

$$\begin{aligned}\omega_{\mu ab}\gamma^{ab} &= \frac{1}{2}\omega_{\mu ab}\left(\begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^b \\ \bar{\sigma}^b & 0 \end{pmatrix}\right. \\ &\qquad\qquad \left. - \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^b & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}\right) \\ &= \omega_{\mu ab}\begin{pmatrix} \sigma^{[a}\bar{\sigma}^{b]} & 0 \\ 0 & \bar{\sigma}^{[a}\sigma^{b]} \end{pmatrix}\end{aligned} \tag{6.5}$$

Therefore, the covariant derivative can be split into two different parts depending upon the handedness of the spinor it is acting upon. D^+_μ acts on left-handed spinor whereas D^-_μ acts upon right-handed spinor.

$$\begin{aligned} D^+_\mu &= \nabla_\mu + \omega_{\mu ab}\sigma^{[a}\bar{\sigma}^{b]} \\ &= \nabla_\mu + \omega_{\mu ab}\sigma^{ab} \end{aligned} \tag{6.6}$$

$$\begin{aligned} D^-_\mu &= \nabla_\mu + \omega_{\mu ab}\bar{\sigma}^{[a}\sigma^{b]} \\ &= \nabla_\mu + \omega_{\mu ab}\bar{\sigma}^{ab} \end{aligned} \tag{6.7}$$

where, $\sigma^{ab} = \sigma^{[a}\bar{\sigma}^{b]}$. It is interesting to note that, $\omega_{\mu ab}$ is anti self dual, however, $\bar{\sigma}^{ab}$ is self dual. Hence, their product vanish and left handed covariant derivative doesn't include the spin connection. On the other hand, σ^{ab} is anti-self dual and hence survives. Next, we decompose all the operator in terms of chiral operators.

6.1 Spin 1/2

Now, we attempt to decompose all the other operators in terms of operators acting on left and right handed spinors. First, let's look at spin 1/2 operator,

$$\begin{aligned} \Delta_{1/2} = (i\gamma^\mu D_\mu)^2 &= -\begin{pmatrix} 0 & \sigma^\mu D^-_\mu \\ \bar{\sigma}^\mu D^+_\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu D^-_\nu \\ \bar{\sigma}^\nu D^+_\nu & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma^\mu D^-_\mu \bar{\sigma}^\nu D^+_\nu & 0 \\ 0 & -\bar{\sigma}^\mu D^+_\mu \sigma^\nu D^-_\nu \end{pmatrix} \\ &= \begin{pmatrix} \Delta_{1/2+} & 0 \\ 0 & \Delta_{1/2-} \end{pmatrix} \end{aligned} \tag{6.8}$$

Self duality allows us to further decompose $\Delta_{1/2-}$ acting on right-handed spinors as,

$$\begin{aligned} \Delta_{1/2-} &= -\bar{\sigma}^\mu D^+_\mu \sigma^\nu D^-_\nu \\ &= -\bar{\sigma}^\mu \nabla_\mu \sigma^\nu \nabla_\nu \\ &= -\bar{\sigma}^{(\mu}\sigma^{\nu)}\nabla_{(\mu}\nabla_{\nu)} \\ &= -g_{\mu\nu}\nabla_{(\mu}\nabla_{\nu)} \end{aligned} \tag{6.9}$$

Since, there are two right-handed spinors,

$$= \begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_0 \end{pmatrix} \tag{6.10}$$

Therefore, we can write the determinants as,

$$det\Delta_{1/2-} = (det\Delta_0)^2 \tag{6.11}$$

6.2 Spin 1

We wish to decompose spin 1 operator in terms of chiral operators. For this we use the fact that anti-self dual metric is Ricci flat ($\mathcal{R}_{\mu\nu} = 0$). This implies that the operators acting on different vectors are just Laplacian.
In order to decompose the vector operators, we use the fact that background admits two orthogonal, covariantly constant right-handed spinors, $\xi^{\dot{\alpha}}_{(i)}$ and obey the equation,

$$\nabla^{-}_{\mu}\xi_{(i)} = 0 \tag{6.12}$$

$$\partial_{\mu}\xi_{(i)} = 0 \tag{6.13}$$

Since spin connection on right-handed spinors vanishes. These constant spinors allow us to decompose the vector field A^{α} in the bispinor form.

$$A^{\alpha\dot{\alpha}} = A_a(\sigma^a)^{\alpha\dot{\alpha}} \tag{6.14}$$

We can decompose bispinor in the basis of constant spinors $\xi^{\dot{\alpha}}$,

$$A^{\alpha\dot{\alpha}} = \sum_{i=1}^{2} a^{\alpha}_{(i)}\xi^{\dot{\alpha}}_{(i)} \tag{6.15}$$

The degree of freedom are now the two left-handed spinors $a_{(i)}$. Now, sandwich Δ_1 between two vectors,

$$\tilde{A}(\Delta_1)_{a;b}A^b = \tilde{A}^{\dagger a}\nabla^2\eta_{ab}A^b \tag{6.16}$$

Next we use the idenditity from D,

$${\sigma_{\mu}}^{\alpha\dot{\alpha}}{\sigma_{\nu}}^{\beta\dot{\beta}}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} = 2\eta_{\mu\nu} \tag{6.17}$$

We plug this in,

$$\begin{aligned}\tilde{A}^{\dagger a}\nabla^2\eta_{ab}A^b &= \tilde{A}^{\dagger a}\nabla^2\left({\sigma_a}^{\alpha\dot{\alpha}}{\sigma_b}^{\beta\dot{\beta}}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\right)A^b \\ &= \tilde{A}^{\dagger\alpha\dot{\alpha}}\nabla^2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}A^{\beta\dot{\beta}} \\ &= \sum_{i=1}^{2}\tilde{a}^{\dagger\alpha}_{(i)}\xi^{\dot{\alpha}}_{(i)}\nabla^2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sum_{j=1}^{2}a^{\beta}_{(j)}\xi^{\dot{\beta}}_{(j)}\end{aligned} \tag{6.18}$$

Since, $\xi^{\dot{\alpha}}$ is orthogonal we can ignore the cross term in the product,

$$\begin{aligned}&= \sum_{i=1}^{2}\xi^{\dot{\alpha}}_{(i)}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dot{\beta}}_{(i)}\tilde{a}^{\dagger\alpha}_{(i)}\nabla^2 a^{\beta}_{(i)} \\ &= \sum_{i=1}^{2}\xi_{(i)}\xi_{(i)}\tilde{a}^{\dagger}_{(i)}\nabla^2 a_{(i)}\end{aligned} \tag{6.19}$$

Now, we can write,

$$det(\Delta_1) = (det\Delta_{1/2+})^2 \tag{6.20}$$

the power is squared because we get $det\Delta_{1/2+}$ for each i.

6.3 Spin3/2

In the case of spin 3/2 we can't do the decomposition as above because we have contributions from Riemann tensor. So, we begin with writing spinor in the chiral basis,

$$\psi^{\mu} = \begin{pmatrix} \psi^{\mu\beta} \\ \psi^{\mu\dot{\beta}} \end{pmatrix} \tag{6.21}$$

Further writing the vector index in the bispinor form,

$$\psi^{\mu} = \begin{pmatrix} \psi^{\alpha\dot{\alpha}\beta} \\ \psi^{\alpha\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{6.22}$$

The spin 3/2 operator decomposes into left and right moving parts as seen in the case of spin 1/2,

$$det\Delta_{3/2} = det\Delta^{3/2-} det\Delta_{3/2+} \tag{6.23}$$

Next, we can decompose right-handed spinor$\psi^{\alpha\dot{\alpha}\beta}$ in the basis of ξ

α, we require a combination of two right handed spinors and one left handed spinors. The number of degrees of freedom of this field is 8, hence, we need to introduce 4 spinors each with two degree of freedoms to account for all the degrees of freedom of the right-handed spinor. We introduce, $f_{(i)}$ where i=1,2,3,4. The $f_{(i)}$ are the left moving fields that hold the degree of freedoms. Hence, the decomposition looks like,

$$\psi^{\alpha\dot{\alpha}\dot{\beta}} = f^{\alpha}_{(1)}\xi^{\dot{\alpha}}_{(1)}\xi^{\dot{\beta}}_{(1)} + f^{\alpha}_{(2)}\xi^{\dot{\alpha}}_{(2)}\xi^{\dot{\beta}}_{(2)} + f^{\alpha}_{(3)}\xi^{\dot{\alpha}}_{(3)}\xi^{\dot{\beta}}_{(3)} + f^{\alpha}_{(4)}\xi^{\dot{\alpha}}_{(4)}\xi^{\dot{\beta}}_{(4)} \tag{6.24}$$

Like the case of spin 1/2, the spin connection of covariant derivative on the right handed fermion vanishes. Therefore, for the right handed spinor we get no Riemnannian tensor contribution. So, sandwiching $\Delta_{3/2-}$ between two fermions we get,

$$\begin{aligned} \tilde{\psi}^{\dagger a\dot{\alpha}}\Delta_{a;b}\psi^{b\dot{\beta}} = {} & \xi^{\dagger\dot{\alpha}}_{(1)}\xi^{\dot{\beta}}_{(1)}\xi^{\dagger}_{(1)}\xi_{(1)}f^{\dagger}_{(1)}\nabla^2 f_{(1)} + \xi^{\dagger\dot{\alpha}}_{(1)}\xi^{\dot{\beta}}_{(2)}\xi^{\dagger}_{(1)}\xi_{(2)}f^{\dagger}_{(2)}\nabla^2 f_{(2)} \\ & + \xi^{\dagger\dot{\alpha}}_{(2)}\xi^{\dot{\beta}}_{(1)}\xi^{\dagger}_{(2)}\xi_{(1)}f^{\dagger}_{(3)}\nabla^2 f_{(3)} + \xi^{\dagger\dot{\alpha}}_{(2)}\xi^{\dot{\beta}}_{(2)}\xi^{\dagger}_{(2)}\xi_{(2)}f^{\dagger}_{(4)}\nabla^2 f_{(4)} \end{aligned} \tag{6.25}$$

The orthogonality condition removes any cross terms. Now, the determinant of right-handed spinor can be written in terms of 4 left moving operators on spin 1/2 as $f_{(i)}$ are left moving spin 1/2 fields.

$$det\Delta_{3/2-} = (det\Delta_{1/2+})^4 \tag{6.26}$$

Next we decompose the left-handed spin 3/2 field. The left-handed spinor has 8 degrees of freedom and these can be represented by 2 22 matrices. Further, this matrices can be decomposed into a symmetric and anti-symmetric part. For 22 anti-symmetric matrix, there exist only one degree of freedom, this can be represented by a scalar $\phi(i)$ for i=1,2. The symmetric part has three degrees of freedom and these are represented by $F^{\alpha\beta}_{(i)}$ for i=1,2. The decomposition looks like,

$$\psi^{\alpha\dot{\alpha}\beta} = \sum_{i=1}^{2} F^{\alpha\beta}_{(i)}\xi^{\dot{\alpha}}_{(i)} + \phi_{(i)}\epsilon^{\alpha\beta}\xi^{\dot{\alpha}}_{(i)} \tag{6.27}$$

As before we sandwich $\Delta_{3/2+}$ between two left-handed spinors. But before that, let's decompose the Riemannian tensor in the anti-self dual background. In the bispinor form, anti-self-dual Riemannian tensor can be written as,

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \mathcal{C}_{\alpha\beta\gamma\delta}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} \tag{6.28}$$

where, $\mathcal{C}_{\alpha\beta\gamma\delta}$ is the totally symmetric anti-self-dual Weyl tensor. To evaluate the form of left handed operator, we convert the two indices of $\Delta_{3/2+}$ into bispinor and examine the contraction,

$$\begin{aligned}
&(\bar{\sigma}^a)^{\dot{\alpha}\alpha}(\Delta_{3/2+})_{a\gamma;b}{}^{\delta}(\sigma^b)_{\beta\dot{\beta}} = (\bar{\sigma}^a)^{\dot{\alpha}\alpha}\left(-\eta_{ab}\delta^{\delta}_{\gamma}\nabla^2 - \frac{1}{2}R_{cdab}\sigma^c_{\gamma\dot{\gamma}}\bar{\sigma}^{d\dot{\gamma}\delta}\right)(\sigma^b)_{\beta\dot{\beta}} \\
&= -(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\eta_{ab}\delta^{\delta}_{\gamma}\nabla^2(\sigma^b)_{\beta\dot{\beta}} - (\bar{\sigma}^a)^{\dot{\alpha}\alpha}\frac{1}{2}R_{cdab}\sigma^c_{\gamma\dot{\gamma}}\bar{\sigma}^{d\dot{\gamma}\delta}(\sigma^b)_{\beta\dot{\beta}} \\
&= -2\delta^{\alpha}{}_{\beta}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\delta^{\delta}_{\gamma}\nabla^2 - \frac{1}{2}\mathcal{C}_{\gamma}{}^{\delta\alpha}{}_{\beta}\delta^{\dot{\gamma}}_{\dot{\gamma}}\delta^{\dot{\alpha}}_{\dot{\beta}} \\
&= 2\delta^{\dot{\alpha}}\Delta_C{}^{\alpha}{}_{\gamma:\beta}{}^{\delta}
\end{aligned} \tag{6.29}$$

where,

$$\Delta_C{}^{\alpha}\beta_{\gamma\delta} = -\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\nabla^2 - \frac{1}{2}\mathcal{C}^{\alpha\beta}{}_{\gamma\delta} \tag{6.30}$$

Finally, left moving spin3/2 determinant can be written as,

$$det\Delta_{3/2+} = (det\Delta_C)^2(det\Delta_0)^2 \tag{6.31}$$

6.4 spin 2

The traceless symmetric part of the metric $h^{\bar{\mu}\nu}$ can be decomposed like other fields. We now have two vector indices which can be expressed in bispinor form as,

$$\bar{h}^{\alpha\dot{\alpha}\beta\dot{\beta}} = \bar{h}_{\mu\nu}(\sigma^{\mu})^{\alpha\dot{\alpha}}(\sigma^{\nu})^{\beta\dot{\beta}} \tag{6.32}$$

We now have 9 degrees of freedom that can be represented by 3 symmetric matrices, hence we can write,

$$\bar{h}^{\alpha\dot{\alpha}\beta\dot{\beta}} = H^{\alpha\beta}_{(1)}\xi^{\dot{\alpha}}_{(1)}\xi^{\dot{\alpha}}_{(1)} + H^{\alpha\beta}_{(2)}\xi^{(\dot{\alpha}}_{(1)}\xi^{\dot{\alpha})}_{(1)} + H^{\alpha\beta}_{(3)}\xi^{\dot{\alpha}}_{(2)}\xi^{\dot{\alpha}}_{(2)} \tag{6.33}$$

where in the second term, the symmetry is dictated by symmetric nature of the metric and $H^{\alpha\beta}$. Next, we wish to examine the action of Δ_2 on $H_{(i)}$, we look at the contraction,

$$\begin{aligned}
\bar{\sigma}^{a\dot{\alpha}\alpha}\bar{\sigma}^{b\dot{\beta}\beta}(\Delta_2)_{ab;cd}\sigma^{(c}{}_{\gamma\dot{\gamma}}\sigma^{d)}{}_{\delta\dot{\delta}} &= \bar{\sigma}^{a\dot{\alpha}\alpha}\bar{\sigma}^{b\dot{\beta}\beta}\left(-\frac{1}{4}\eta_{ac}\eta_{bd}\nabla^2 - \frac{1}{2}R_{acbd}\right)\sigma^{(c}{}_{\gamma\dot{\gamma}}\sigma^{d)}{}_{\delta\dot{\delta}} \\
&= -\frac{1}{4}4\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\dot{\beta}}_{\dot{\delta}}\nabla^2 - \frac{1}{4}R^{\alpha\dot{\alpha}}{}_{\gamma\dot{\gamma}}{}^{\beta\dot{\beta}}{}_{\delta\dot{\delta}} - \frac{1}{4}R^{\alpha\dot{\alpha}}{}_{\delta\dot{\delta}}{}^{\beta\dot{\beta}}{}_{\gamma\dot{\gamma}} \\
&= -\frac{1}{2}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta}\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\dot{\beta}}_{\dot{\delta}}\nabla^2 - \frac{1}{2}C^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\dot{\beta}}_{\dot{\delta}} \\
&= \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}}\delta^{\dot{\beta}}_{\dot{\delta}}(\Delta_C)^{\alpha\beta}{}_{\gamma\delta}
\end{aligned} \tag{6.34}$$

Therefore, spin 2 determinant can be written in terms of Δ_C and hence $\Delta_{3/2+}$.

$$\begin{aligned}
det\Delta_2 &= (det\Delta_C)^3 \\
&= \frac{(det\Delta_{3/2+})^{3/2}}{(det\Delta_0)^3}
\end{aligned} \tag{6.35}$$

Finally, the total loop determinant of eq 6.1 can be written in the chiral basis as,

$$\Gamma = \left(\frac{det\Delta_{3/2+}}{det\Delta_{3/2-}}\right)^{-1/2}\left(\frac{det\Delta_{1/2+}}{det\Delta_{1/2-}}\right)^{1/4} \tag{6.36}$$

7 Determinant calculation

Getting fractions of chiral operators, we can compute them explicitly. The ratio of determinants and regularised index are related [22]. We begin by defining the regularised ratio,

$$D(m^2) = \frac{det\Delta_+ + m^2}{det\Delta_- + m^2} \tag{7.1}$$

where m^2 is the infra-red regulator. Next, we define the regularised index as,

$$\begin{aligned}
\mathcal{I}(m^2) &= \frac{\partial \log D}{\partial logm^2} \\
&= Tr\left[\frac{1}{\frac{det\Delta_+ + m^2}{det\Delta_- + m^2}}\frac{\Delta_- - \Delta_+}{(det\Delta_- + m^2)^2}\right] \\
&= Tr\left[\frac{m^2}{\Delta_+ + m^2} - \frac{m^2}{\Delta_- + m^2}\right]
\end{aligned} \tag{7.2}$$

We can treat both spin 1/2 and 3/2 operators this way. Let's denote the operators of these spins as $\hat{\gamma}\cdot\nabla$, where for spin 1/2 $\hat{\gamma}\nabla = \gamma^{\mu}\nabla_{\nu}$ and for spin 3/2 $\hat{\gamma}\nabla = \gamma^{\mu\nu\rho}\nabla_{\nu}$. Also,

$\hat{\gamma}^5 = \hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4 = \gamma^5$.With this we can re-write the index as,

$$\begin{aligned}
\mathcal{I}(m^2) &= Tr\left[m^2 \frac{\Delta_- - \Delta_+}{(\Delta_+ + m^2)(\Delta_- + m^2)}\right] \\
&= Tr\left[\hat{\gamma}^5 \frac{m^2}{-(\hat{\gamma}\cdot\nabla)^2 + m^2}\right] \\
&= Tr\left[\hat{\gamma}^5 \frac{m^2}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^z} + \hat{\gamma}^5 \frac{(\hat{\gamma}\cdot\nabla)^2}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^{1+z}}\right]
\end{aligned} \tag{7.3}$$

where we introduced form of zeta regularization with a new parameter z in the last line. Let's look at both the terms separately, for the first term we use the heat kernel technique and dividing by the zeta function,

$$Tr\left[\hat{\gamma}^5 \frac{m^2}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^z}\right] = Tr\left[[\hat{\gamma}^5 \frac{1}{\Gamma(z)} \int_0^\infty \frac{dt}{t^{1-z}} \exp\left[-(-(\hat{\gamma}\cdot\nabla)^2 + m^2)t\right]\right] \tag{7.4}$$

Here, again we use the heat kernel expansion and get B_k terms for k=0,2,4. However, due to trace over spin indices most of them vanish. For spin independent terms the spinor trace is over γ^5 which vanishes. Only terms with spinor dependence survive,

$$\begin{aligned}
Tr\left[\hat{\gamma}^5 \frac{m^2}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^z}\right] &= Tr\left[\frac{1}{16\cdot 48\pi^2} \int \sqrt{g}\hat{\gamma}^5 t^{ab}_{(s)} \mathcal{R}_{ab\mu\nu} t^{cd}_{(s)} \mathcal{R}_{cd}{}^{\mu\nu}\right] \\
&= \frac{2\ \alpha_s}{16\cdot 48\pi^2} \int \sqrt{g}\epsilon_{abcd}\mathcal{R}_{ab\mu\nu}\mathcal{R}_{cd}{}^{\mu\nu} \\
&= \frac{\alpha_s}{16\cdot 24\pi^2} \int \sqrt{g}^* \mathcal{R}_{ab\mu\nu}\mathcal{R}_{cd}{}^{\mu\nu}
\end{aligned} \tag{7.5}$$

where, in the second line we all other terms of heat kernel expansion vanish due to spinor trace and in the following step we perform the spinor trace and finally use the definition of self-duality of Riemnann tensor to write the final form. The coefficient a_s depends on the spin operator and is given by [6],

$$\alpha_{1/2} = 1 \quad and \quad \alpha_{3/2} = 20$$

We can further evaluate this term by performing the integral. This integral is called Pontryagin class. For Taub-NUT metric, it is found to be,

$$\frac{1}{16\pi^2} \int \sqrt{g}^* \mathcal{R}_{ab\mu\nu}\mathcal{R}_{cd}{}^{\mu\nu} = -2 \tag{7.6}$$

Therefore, we get the evaluated first term as,

$$\frac{a_s}{16\cdot 24\pi^2} \int \sqrt{g}^* \mathcal{R}_{ab\mu\nu}\mathcal{R}_{cd}{}^{\mu\nu} = -\frac{\alpha_s}{12} \tag{7.7}$$

Now, we turn to the second term of the equation (7.3). We begin by setting the z=0 without any consequences as no anomalies are present in this term. We again use the zeta-function regularization as above. Since, this term is a total derivative, we can perform one of the four integrals which results in a surface integral,

$$Tr\left[\hat{\gamma}^5 \frac{(\hat{\gamma}\cdot\nabla)^2}{(-(\hat{\gamma}\cdot\nabla)^2+m^2)^1}\right] = \int_{S^3} dS_\mu \sqrt{g_{bdy}} J^\mu \tag{7.8}$$

where, $\sqrt{g_{bdy}}$ is the induced metric at boundary and J^μ is the current given by,

$$J^\mu = lim_y \frac{1}{2} tr\langle \hat{\gamma}^5 \hat{\gamma}^\mu \frac{\hat{\gamma}\cdot\nabla}{-(\hat{\gamma}\cdot\nabla + m^2} |x\rangle \tag{7.9}$$

we can further simplify it by expanding the denominator in a taylor series expansion, ignoring the higher derivative terms and using the definition of the covariant derivative,$\nabla_\mu = \partial_\mu + \frac{1}{2} t^{ab}\omega_{ab\mu}$,

$$\begin{aligned}
J^\mu &= lim_y \frac{1}{2} tr\langle \hat{\gamma}^5 \hat{\gamma}^\mu \hat{\gamma}^\nu \left(\partial_\nu + \frac{1}{2}\omega_{ab\nu} t^{ab}\right) \left[\frac{1}{-\left(\partial_\nu + \frac{1}{2}\omega_{ab\nu}t^{ab}\right)^2 + m^2}\right] \\
&= lim_y \frac{1}{2} tr\langle \hat{\gamma}^5 \hat{\gamma}^\mu \hat{\gamma}^\nu \left(\partial_\nu + \frac{1}{2}\omega_{ab\nu} t^{ab}\right) \left[\frac{1}{-\left(\partial_\nu + \frac{1}{2}\omega_{ab\nu}t^{ab}\right)^2 + m^2}\right] \\
&= lim_y \frac{1}{2} tr\langle \hat{\gamma}^5 \hat{\gamma}^\mu \hat{\gamma}^\nu \left(\partial_\nu + \frac{1}{2}\omega_{ab\nu} t^{ab}\right) \left[\frac{1}{-\partial^2 + m^2 - \frac{1}{2}\omega_{ab\nu}t^{ab}\partial^\nu + \frac{1}{4}\omega_{ab\nu}t^{ab}\omega_{cd\nu}t^{cd}}\right] \\
&= lim_y \frac{1}{2} tr\langle \hat{\gamma}^5 \hat{\gamma}^\mu \hat{\gamma}^\nu \left(\partial_\nu + \frac{1}{2}\omega_{ab\nu} t^{ab}\right) \left[\frac{1}{(-\partial^2 + m^2)}\left(1 + \frac{1}{2}\omega_{ab\nu}t^{ab}\partial^\nu \frac{1}{(-\partial^2+m^2)} + ...\right)\right]
\end{aligned} \tag{7.10}$$

Using the trace property, $tr[\hat{\gamma}^5\hat{\gamma}^\mu\hat{\gamma}^\nu] = 0$, the first term vanishes. Keeping only the asymptotically non-vanishing terms,

$$J^\mu \to \frac{1}{2} tr[\hat{\gamma}^5\hat{\gamma}^\mu\hat{\gamma}^\nu t^a b]\omega^\rho_{ab}\langle x| \left[\frac{1}{2}\frac{g_{\nu\rho}}{(-\partial^2+m^2)} + \frac{\partial_\nu\partial_r ho}{(-\partial^2+m^2)}\right] x| \tag{7.11}$$

Since, the definition of $\hat{\gamma}^\mu$ is different for spin 1/2 and spin 3/2, we need to evaluate the trace separately,

$$\frac{1}{2} tr[\hat{\gamma}^5\hat{\gamma}^\mu\hat{\gamma}^\nu t^a b] = \beta_s \epsilon^{\mu\nu ab} \tag{7.12}$$

where for spin 1/2, $\beta_{1/2} = 1$ and for spin 3/2, $\beta_{3/2} = 4$. Further, we use the anti-self duality of the spin connection, $\omega^{\mu\nu\rho} = -\epsilon^{\mu\nu ab}\omega_{ab}{}^\rho = -{}^*\omega^{\mu\nu\rho}$, we consume the levi-civita symbol in spin connection and the current take the form,

$$J^\mu \to -\beta_s \omega^{\mu\nu\rho}\langle x| \left[\frac{1}{2}\frac{g_{\nu\rho}}{(-\partial^2+m^2)} + \frac{\partial_\nu\partial_r ho}{(-\partial^2+m^2)}\right] x| \tag{7.13}$$

We can write the current in the Fourier basis over 4 dimensional momenta $k^\mu = (\mathbf{k}, n/L)$.

$$J^\mu \to -\beta_s \omega^{\mu\nu\rho} \frac{1}{\sqrt{g}} \frac{1}{2\pi L} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[\frac{g_{\nu\rho}}{(-k^2+m^2)} - \frac{k_r ho}{(-k^2+m^2)} \right] \tag{7.14}$$

For the present case, our interest is in outward flux, J^i where i is the three-dimensional tangent space index. Also, as the metric is asymptotically flat, we have, $k^2 = \mathbf{k}^2 + n^2/L^2$. We finally used the explicit form of spin connection derived in appendix and find that only $\nu, \rho = 4$ components contribute. Therefore three-dimensional current is given by,

$$\begin{aligned} J^\mu &\to -\beta_s \omega^{i44} \frac{1}{\sqrt{g}} \frac{1}{2\pi L} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[\frac{g_{44}}{(-k^2+m^2)} + \frac{k_4 k_4}{(-k^2+m^2)} \right] \\ &= -\beta_s (\partial_i \log U) \frac{1}{2\pi L} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{(-\mathbf{k}^2 + n^2/L^2 + m^2)} - \frac{2}{(-\mathbf{k}^2 + n^2/L^2 + m^2)} \frac{n^2}{L^2} \right] \\ &= -\beta_s (\partial_i \log U) \frac{1}{2\pi L} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{(-\mathbf{k}^2 + n^2/L^2 + m^2)} - \frac{2}{(-\mathbf{k}^2 + n^2/L^2 + m^2)} \frac{n^2}{L^2} \right] \\ &= -\frac{\beta_s}{32\pi^2 L} \frac{x_i}{|x|^3} \sum_n \left[(n^2 + m^2 L^2)^{1/2} + (n^2 + m^2 L^2)^{-1/2} n^2 \right] \end{aligned} \tag{7.15}$$

where in the last line we performed the k integrals and used the expression $U = 1 + \frac{L}{2|x|}$ to evaluate the derivative. Now, it is straight forward to calculate the second term of eq. 7.3, plugging the expression for current in the equation we get,

$$\begin{aligned} &\sum_n \left[(n^2 + m^2 L^2)^{1/2} + (n^2 + m^2 L^2)^{-1/2} n^2 \right] \int dS_i \sqrt{g_{bdy}} \frac{x^i}{|x|^3} \\ &= \sum_n \left[(n^2 + m^2 L^2)^{1/2} + (n^2 + m^2 L^2)^{-1/2} n^2 \right] \int dx^2 \sqrt{g_{bdy}} \frac{x^i}{|x|^3} \int dz \\ &= \sum_n \left[(n^2 + m^2 L^2)^{1/2} + (n^2 + m^2 L^2)^{-1/2} n^2 \right] 4\pi 2\pi L \end{aligned} \tag{7.16}$$

where, we used the fact that the boundary of Taub-NUT is $S^2 \times S^1$ and then performed the corresponding integrals. Now, we have the expressions for both the terms, we can write the index,

$$\mathcal{I}(m^2) = -\frac{\alpha_s}{12} - \frac{\beta_s}{4} \sum_n \left[(n^2 + m^2 L^2)^{1/2} + (n^2 + m^2 L^2)^{-1/2} n^2 \right] \tag{7.17}$$

Now, with regularized index, next step is the calculation of regularized ratio. The sum in the index are over all integers and hence divergent. Therefore, we again turn back to Pauli-Villars regularization to get the ratio,

$$\begin{aligned} \int_0^{m^2} \partial \log D &= \int_0^{m^2} \int_0^1 \frac{d\lambda}{\lambda} [\mathcal{I}(m^2)]_{PV} \, \partial \log m^2 \\ \log D(m^2) - \log D(0) &= \int_0^1 \frac{d\lambda}{\lambda} [\mathcal{I}(m^2)]_{PV} \end{aligned} \tag{7.18}$$

where in the second equality, we used the fact that any term independent of m^2 will vanish under Pauli-Villars regulator. Further ignoring the m^2 independent terms,

$$\log\frac{D(m^2)}{D(0)} = -\frac{\beta_s}{4}\sum_n\left[2\sqrt{n^2+m^2L^2} - 4|n|\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{m^2L^2}{n^2}}\right)\right]_{PV} \tag{7.19}$$

Now, we use the Pauli-Villars regularization to finally evaluate the determinant. We use the the limit, $m \to 0$ and $M_{UV} \to \infty$ and perform the sum over n,

$$\int_0^1 \frac{d\lambda}{\lambda}[\mathcal{I}(m^2)]_{PV} = \frac{\beta_s}{12}\left[\log(\mu^2R^2) - \log 4 + 1 - 24\zeta'(-1)\right] \tag{7.20}$$

where $\mu^2 = (\gamma-1)M_{UV}^2/\gamma$ is the Pauli Villars scale arising from m^2 expansion and $L \to R$. Note, that this scale was also measured in one-loop divergences. Lastly, $\log D_0 = lim_{\lambda\to 0[\log D(\lambda m^2)]_{PV}}$ is dominated by zero modes. We have $\mathcal{I} = 0$ for spin 1/2 and $\mathcal{I} = 2$ for spin 3/2 which can be easily seen from equation (7.17). Therefore we have,

$$\begin{aligned} D_0 &= \frac{(\lambda m^2)^{\mathcal{I}}(\lambda\gamma M_{UV}^2)^{\mathcal{I}}}{(\lambda M_{UV}^2)^{\mathcal{I}}(\lambda(\gamma-1)M_{UV}^2+\lambda m^2)^{\mathcal{I}}} \\ &\to \left(\frac{m^2}{\mu^2}\right)^{\mathcal{I}} \end{aligned} \tag{7.21}$$

Now, we have all the ingredients to calculate the determinants. We can write the equation (7.18) as,

$$\begin{aligned} \log\frac{D(m^2)}{D(0)} &= \frac{\beta_s}{12}\left[(\log\mu^2R^2) + A\right] \\ \frac{D(m^2)}{D(0)} &= (\mu^2R^2)^{\beta_s/12}\, e^{\frac{\beta_s}{12}A} \\ D(m^2) &= \left(\frac{m^2}{\mu^2}\right)^{\mathcal{I}}(\mu^2R^2)^{\beta_s/12}\, e^{\frac{\beta_s}{12}A} \\ D(m^2) &= \left(\frac{m^2}{\mu^2}\right)^{\mathcal{I}}(\mu^2R^2)^{\beta_s/12}\, e^{\frac{\beta_s}{12}A} \\ D(m^2) &= m^{2\mathcal{I}}(\mu^{\beta_s/6-2\mathcal{I}}R^2)^{\beta_s/12}\, e^{\frac{\beta_s}{12}A} \\ D(m^2) &= m^{2\mathcal{I}}\Gamma' \end{aligned} \tag{7.22}$$

where $A = -\log 4 + 1 - 24\zeta'(-1)$. This is the expression for regularized index for any spin, and Γ' is the truncated determinant. $m^{2\mathcal{I}}$ represents the zero modes (here we have spin3/2 zero modes only). Finally, the truncated determinant for graviton-gravitino supermultiplet becomes using the equation (6.36),

$$\begin{aligned}\Gamma &= \left(\frac{det\Delta_{3/2+}}{det\Delta_{3/2-}}\right)^{-1/2}\left(\frac{det\Delta_{1/2+}}{det\Delta_{1/2-}}\right)^{1/4} \\ &= (D_{3/2}(m^2))^{-1/2}(D_{1/2}(m^2))^{1/4} \\ &= m^{-2}(\mu^2)^{-8/48+1/48+1}(R^2)^{-8/48+1/48}(A')^{-7/48} \\ &= m^{-2}(\mu^2)^{41/48}\left(\frac{R^2}{A'}\right)^{7/48}\end{aligned} \tag{7.23}$$

where A' consumes the constants, given by,

$$A' = 4e^{24\zeta'(-1)-1}$$

Therefore, truncated determinant is given by,

$$\Gamma' = (\mu^2)^{41/48}\left(\frac{R^2}{A'}\right)^{7/48} \tag{7.24}$$

Notice, the interesting powers emerging in the expression. These will be useful as we go ahead.

8 New IR scale

In this section, we show that a new infra-red scale for instanton contributions emerges out of the calculations [6]. This new scale is exponentially suppressed just like the Λ_{qcd} scale. The general expectation for any quantum gravity contribution is thought to be in far Ultra-violet scale. However, the path integral of quantum gravity comes equipped with an infra-red scale. To understand the origins of this scale let's look at the quantum gravity path integral once again.

$$Z = \int \mathcal{D}h\mathcal{D}\Psi \exp\left(-\int d^4x h_{\mu\nu}\mathcal{O}^{\mu\nu\rho\sigma}h_{\rho\sigma} + \Psi_\mu\mathcal{O}^{\mu\nu}\Psi_\nu\right) \tag{8.1}$$

Here, we can add the Gauss-Bonnet term (with coupling coefficient α) and Pontryagin class (with coupling coefficient θ) which is a total derivative term and doesn't contribute to the equation of motion. It is a topological term whose effect can only be seen when summed over gravitational instantons. In supergravity they sit down in a complex structure $\tau_{grav} = \alpha + 2i\theta$. Therefore, add the term to the partition function,

$$\begin{aligned}Z = \int \mathcal{D}h\mathcal{D}\Psi \exp\Big(-\int d^4x\sqrt{g}\,\big[h_{\mu\nu}\mathcal{O}^{\mu\nu\rho\sigma}h_{\rho\sigma} + \Psi_\mu\mathcal{O}^{\mu\nu}\Psi_\nu + \alpha({}^*\mathcal{R}^*_{\mu\nu\sigma\rho}\mathcal{R}^{\mu\nu\sigma\rho}) \\ + 2i\theta{}^*\mathcal{R}_{\mu\nu\sigma\rho}\mathcal{R}^{\mu\nu\sigma\rho}\big]\Big)\end{aligned} \tag{8.2}$$

In the previous sections we have evaluated the these integrals. We integrate out the graviton (and its ghosts) along with gravitino (and its ghosts). We have divergences emerging at one loop which were thought of as the running of Gauss-Bonnet term and in the determinants evaluation. Collecting together all the divergences

$$Z \sim \mu^{41/24} e^{-\tau_{grav}} \tag{8.3}$$

These two combine to give the new RG-invariant scale, defined as,

$$\begin{aligned}\Lambda_{grav} &= \mu e^{\frac{a(\nu)+2i\theta}{41/24}} \\ &= \mu e^{\frac{a(\nu)+2i\theta}{a_o}}\end{aligned} \tag{8.4}$$

where a(μ) and a_0 are the coefficients of Gauss-Bonnet term defined earlier. This new scale is exponentially suppressed as compared to the Pauli Villars scale μ. Also, it sits in a chiral multiplet (τ_{grav}) and hence could appear in the expression of superpotential as we see next.

9 Three-dimensional potential

There is a superpotential generated by instanton effects. Tong and Turner evaluated it by comparing the leading term of of the instanton-generated vertex [23] with the interaction term (eq 4.19 in [6]). It involves the Λ_{grav} scale discussed above and complex structure defined in (section). The superpotential takes the form,

$$\mathcal{W} = CM_3 \left(\frac{\Lambda_{grav}}{M_p l}\right)^{41/24} e^{-S} \tag{9.1}$$

where,

$$C = \frac{(4e^{24\zeta'(-1)-1)^{7/48}})}{2(4\pi)^{3/2}}$$

and the corrected complex structure is given by,

$$S = 2\pi^2 M_{pl}^2 R^2 + \frac{7}{48} log(M_{pl}^2 R^2) + i\sigma$$

The supersymmetric completion of the Yukawa term is a potential. In three-dimensions it is given by,

$$V = M_3 e^K \left((\partial\bar{\partial}K)^{-1} |D\mathbb{W}|^2 - 4|W|^2 \right) \tag{9.2}$$

where the Kahler potential (K) is given by,

$$\begin{aligned}K &= -log(S + S^\dagger) \\ &= -log(4\pi^2 M_{pl}^2 R^2 + \frac{7}{24} log(M_{pl}^2 R^2))\end{aligned} \tag{9.3}$$

and , ∂ on a function $f(R, \sigma) = f_1(R) + i f_2(\sigma)$ is,

$$\partial f = \frac{\partial f_1}{\partial R} - i\frac{\partial f_2}{\partial \sigma} \tag{9.4}$$

$$\bar{\partial} f = \frac{\partial f_1}{\partial R} + i\frac{\partial f_2}{\partial \sigma} \tag{9.5}$$

Next, we evaluate the terms separately,

$$\left(\partial\bar{\partial}K\right)^{-1} = \frac{\left(4\pi^2 M_{pl}^2 R^2 + \frac{7}{24}\log(M_{pl}^2 R^2)\right)^2}{32\pi^4 M_{pl}^4 R^2 - \frac{7}{3}\pi^2 M_{pl}^2 \log(M_{pl}^2 R^2) + \frac{35}{3}\pi^2 M_{pl}^2 + \frac{7}{12R^2}\frac{7}{24}\log(M_{pl}^2 R^2) + \frac{7^2}{12^2 R^2}} \tag{9.6}$$

Now, DW

$$\begin{aligned} DW &= \partial W + (\partial K)W \\ &= W(-\partial S + \partial K) \end{aligned} \tag{9.7}$$

$$|DW|^2 = |W|^2 \left(\left[\left(4\pi^2 M_{pl}^2 R + \frac{7}{24R}\right)\left(1 + \frac{2}{4\pi^2 M_{pl}^2 R^2 + \frac{7}{24}log(M_{pl}^2 R^2)}\right)\right]^2 + 1 \right) \tag{9.8}$$

$$|W|^2 = M_3^2 \left(\frac{\Lambda_{grav}}{M_{pl}}\right)^{41/12} \frac{e^{-4\pi^2 M_{pl}^2 R^2}}{M_{pl}^{7/12} R^{7/12}} \tag{9.9}$$

$$e^K = \frac{1}{4\pi^2 M_{pl}^2 R^2) + \frac{7}{24}\log(M_{pl}^2 R^2)} \tag{9.10}$$

Putting it all together in the potential formula,

$$V = e^K \left((\partial\bar{\partial}K)^{-1}|D\mathcal{W}|^2 - 4|\mathcal{W}|^2\right)$$

$$\begin{aligned} V &= M_3^3 \frac{1}{4\pi^2 M_{pl}^2 R^2}(4\pi^2 M_{pl}^2 R^2)^2 \left(\frac{\Lambda_{grav}}{M_{pl}}\right)^{41/12} e^{-(S+S^\dagger)} \\ &= M_3^3 \frac{1}{4\pi^2 M_{pl}^2 R^2}(4\pi^2 M_{pl}^2 R^2)^2 \left(\frac{\Lambda_{grav}}{M_{pl}}\right)^{41/12} \frac{1}{(M_{pl}^2 R^2)^{\frac{7}{24}}} e^{-4\pi^2 M_{pl}^2 R^2} \\ &= M_3^3 M_{pl}^{-2} \Lambda_{grav}^{41/12} R^{17/12} e^{-4\pi^2 M_{pl}^2 R^2} \end{aligned} \tag{9.11}$$

The striking feature of this expression is that the minima lies at $R \to \infty$. Therefore, Kaluza-Klein compactifications of N=1 supergravity on $R^3 \times S^1$ is not the ground state of the theory. Instead, the instanton-generated potential causes the circle to decompactify, which implies that the compact circle decompactifies. Further, we look at the consequences of adding new N=1 supersymmetric matter to theory have on this potential.

10 New matter

From string theory compactifications we expect chiral and vector multiplets to be accompanied by the graviton multiplet. Chiral multiplet is a spinor (spin 1/2) and scalar pair and vector multiplet is vector and spinor pair (spin 1/2). In this section, we add this new matter to our original path integral. For vector multiplet, we have the following action,

$$S_{vector} = \int d^4x \frac{1}{4} F^{\mu 4} F_{\mu\nu} + \Psi i \gamma^\mu D_\mu \Psi \tag{10.1}$$

where $F_{\mu\nu} = \nabla_{\mu A_\nu}$ where A_μ is the vector field and Ψ is the spinor field. Similarly action for chiral multiplet,

$$S_{chiral} = \int d^4x (-\phi \nabla^2 \phi) + \Psi i \gamma^\mu D_\mu \Psi \tag{10.2}$$

Next, we evaluate the determinants of these multiplets separately using the Faddeev Popov method. There are no associated zero modes, hence the determinants are finite non-zero.

10.1 Spin 1-spin 1/2 multiplet

In this section we introduce a spin 1 field and it's supersymmetric partner spin 1/2 field and calculate it's one-loop determinant.

10.1.1 Bosonic sector

The action of massless spin 1 field (A_μ) is,

$$\begin{aligned} S_1 &= \int d^4x \frac{1}{4} F^{\mu 4} F_{\mu\nu} \\ &= \int d^4x \frac{1}{4} (\nabla_\mu A_\nu - \nabla_\nu A_\mu)(\nabla^\mu A^\nu - \nabla^\nu A^\mu) \\ &= \int d^4x \frac{1}{4} (\nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\nu A_\mu \nabla^\mu A^\nu - \nabla_\mu A_\nu \nabla^\nu A^\mu + \nabla_\nu A_\mu \nabla^\nu A^\mu) \\ &= \int d^4x \frac{1}{2} (\nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\nu A_\mu \nabla^\mu A^\nu) \end{aligned} \tag{10.3}$$

Now, we know that the spin 1 field enjoys the gauge freedom,

$$A_\mu = A_\mu + \nabla_\mu \epsilon \tag{10.4}$$

And subsequently follows the covariant gauge condition,

$$F = \nabla^\mu A_\mu = 0 \tag{10.5}$$

Introduction of this condition requires a complex anti-commuting scalar ghost field, operator of which following appendix B.10 is,

$$\begin{aligned} F' &= \nabla^\mu A'_\mu \\ &= \nabla^\mu (A'_\mu + \nabla_\mu \epsilon) \\ &= F + \nabla^2 \epsilon \end{aligned} \tag{10.6}$$

now, $\det(\frac{\delta F}{\delta \epsilon})$ becomes,

$$det\left(\frac{\delta F}{\delta \epsilon}\right) = g^{\mu\nu}\nabla^2 \tag{10.7}$$

Also, we add the gauge fixing Lagrangian to the action,

$$L_{gf} = \frac{1}{2}\nabla^\mu A_\mu \nabla^\nu A_\nu \tag{10.8}$$

The ghost Lagrangian involving the ghost field a is,

$$L_{gh} = a\nabla^2 a \tag{10.9}$$

Adding everything together, the action becomes,

$$\begin{aligned} S_1 &= \int d^4x L_1 + L_{gf} + L_{gh} \\ &= \int d^4x \frac{1}{2}(\nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\nu A_\mu \nabla^\mu A^\nu) + \frac{1}{2}\nabla^\mu A_\mu \nabla^\nu A_\nu + a\nabla^2 a \\ &= \int d^4x \frac{1}{2}(\nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\nu A_\mu \nabla^\mu A^\nu) + \frac{1}{2}\nabla^\mu A_\mu \nabla^\nu A_\nu + a\nabla^2 a \end{aligned} \tag{10.10}$$

Performing partial integration on spin 1 field terms

$$\begin{aligned} &\int d^4x \frac{1}{2}(-A_\nu \nabla_\mu \nabla^\mu A^\nu + A_\mu \nabla_\nu \nabla^\mu A^\nu - A_\mu \nabla^\mu \nabla^\nu A_\nu) + a\nabla^2 a \\ &= \int d^4x \frac{1}{2}(-A_\mu g^{\mu\nu}\nabla^2 A_\nu - A_\mu[\nabla^\mu, \nabla^\nu]A_\nu) + a\nabla^2 a \\ &= \int d^4x \frac{1}{2}(-A_\mu g^{\mu\nu}\nabla^2 A_\nu - A_\mu R^{\mu\nu} A_\nu) + a\nabla^2 a \end{aligned} \tag{10.11}$$

Hence, we find the following determinants,

$$\Delta_1 = -g^{\mu\nu}\nabla^2 + R^{\mu\nu} \tag{10.12}$$

Spin 0 determinant following wick rotation,

$$\Delta_0 = -\nabla^2 \tag{10.13}$$

Hence, one loop determinant for bosonic sector becomes,

$$\Gamma_B = \frac{det\Delta_0}{(det\Delta_1)^{1/2}} \tag{10.14}$$

10.1.2 Fermionic sector

The action for spin 1/2 field Ψ is,

$$S_{1/2} = \int d^4x \Psi i\gamma^\mu D_\mu \Psi \tag{10.15}$$

where, Ψ is anti-commuting real variable. Also, it is convinient to work with squared operators, hence we square the dirac operator,

$$\begin{aligned} \Delta_1 &= (i\gamma^\mu D_\mu)(i\gamma^\nu D_\nu) \\ &= \nabla^2 + \frac{1}{4}R \end{aligned} \tag{10.16}$$

The determinant for the fermionic sector then becomes,

$$\begin{aligned} \Gamma_F &= (det(i\gamma^\mu D_\mu))^{1/2} \\ &= (det\Delta_{1/2})^{1/4} \end{aligned} \tag{10.17}$$

10.1.3 One loop determinant of multiplet

Finally, we multiply the bosonic and fermionic parts and get the total one loop determinant

$$\Gamma = \frac{det\Delta_0 \ (det\Delta_{1/2})^{1/4}}{(det\Delta_1)^{1/2}} \tag{10.18}$$

using the bispinor decomposition, we reduce the one loop determinant to,

$$\begin{aligned} \Gamma &= \frac{(det\Delta_{1/2-})^{1/2} \ (det\Delta_{1/2-})^{1/4}(det\Delta_{1/2+})^{1/4}}{(det\Delta_{1/2+})} \\ &= \frac{(det\Delta_{1/2-})^{3/4}}{(det\Delta_{1/2+})^{3/4}} \end{aligned} \tag{10.19}$$

10.2 Complex spin 0 spin 1/2 multiplet

The action for a complex scalar field ϕ is,

$$S_0 = \int d^4x(-\phi\nabla^2\phi) \tag{10.20}$$

ϕ is complex commuting scalar, hence one loop determinant (with $\Delta_0 = \nabla^2$)becomes,

$$\Gamma_B = (det\Delta_0)^{-1} \tag{10.21}$$

From the previous sections, we know that,

$$\Gamma_F = (det\Delta_{1/2})^{1/4} \tag{10.22}$$

Hence, one loop determinant of the multiplet becomes,

$$\begin{aligned}\Gamma &= \frac{(det\Delta_{1/2})^{1/4}}{(det\Delta_0)} \\ &= \frac{(det\Delta_{1/2-})^{1/4}(det\Delta_{1/2+})^{1/4}}{(det\Delta_{1/2-})^{1/2}} \\ &= \frac{(det\Delta_{1/2+})^{1/4}}{(det\Delta_{1/2-})^{1/4}}\end{aligned} \tag{10.23}$$

10.3 Gravtion and new multiplets

We now add n_v and n_c number of vector and chiral multiplets to the theory and examine their effects on our results. The determinant with chiral decompostion then takes the form,

$$\Gamma_{new} = \left(\frac{det\Delta_{3/2+}}{det\Delta_{3/2-}}\right)^{-1/2} \left(\frac{det\Delta_{1/2+}}{det\Delta_{1/2-}}\right)^{1/4-3/4n_v+1/4n_c} \tag{10.24}$$

Now, when we calculate the truncated determinant, it becomes,

$$\Gamma'_new = (\mu^2)^{\frac{41-3n_v+n_c}{48}} \left(\frac{R^2}{A'}\right)^{\frac{7-3n_v+n_c}{48}} \tag{10.25}$$

Similarly, the one-loop divergences also change with the same numerical factor,

$$S_{1-loop} = -\frac{41-3n_v+n_c}{48}\log\left(\frac{\mu^2}{m^2}\right)\chi \tag{10.26}$$

Therefore, the structure of path integral is intact and we still get the Λ_{grav} scale as the power of determinant and divergences match. The complex structure get the correction,

$$\mathcal{S} = 2\pi^2 M_{pl}^2 R^2 + \frac{7-3n_v+n_c}{48}\log(M_{pl}^2 R^2) + i\sigma \tag{10.27}$$

Therefore, the superpotential and hence the three-dimensional potential get corrections as well. The new potential becomes,

$$V \sim \Lambda_{grav}^{41/12} R^{(17+n_v-n_c)/12} e^{-4\pi^2 M_{pl}^2 R^2} \tag{10.28}$$

In this case, the minima of the potential don't only lie at $R \to \infty$ but the presence of new factors in the power of R creates new minimas away from infinity. Since we are working in the Planck scale ($Dim[R] = 1/Dim[M_{pl}]$) here, for meaningful results we wish to have these minimas sufficiently away from $R = 1/M_{pl}$. From the analysis from Mathematica, we see that the minimas keep shifting away from the origin with the introduction of more and more vector multiplets (increasing n_v).This requires the presence of a lot of vector multiplets. For minimas to be present at $R = 100/M_{pl}$ we need $n \sim 3,000,000$. This implies that to find new resonable vacua in the theory we need the presence of a large number of new fields.

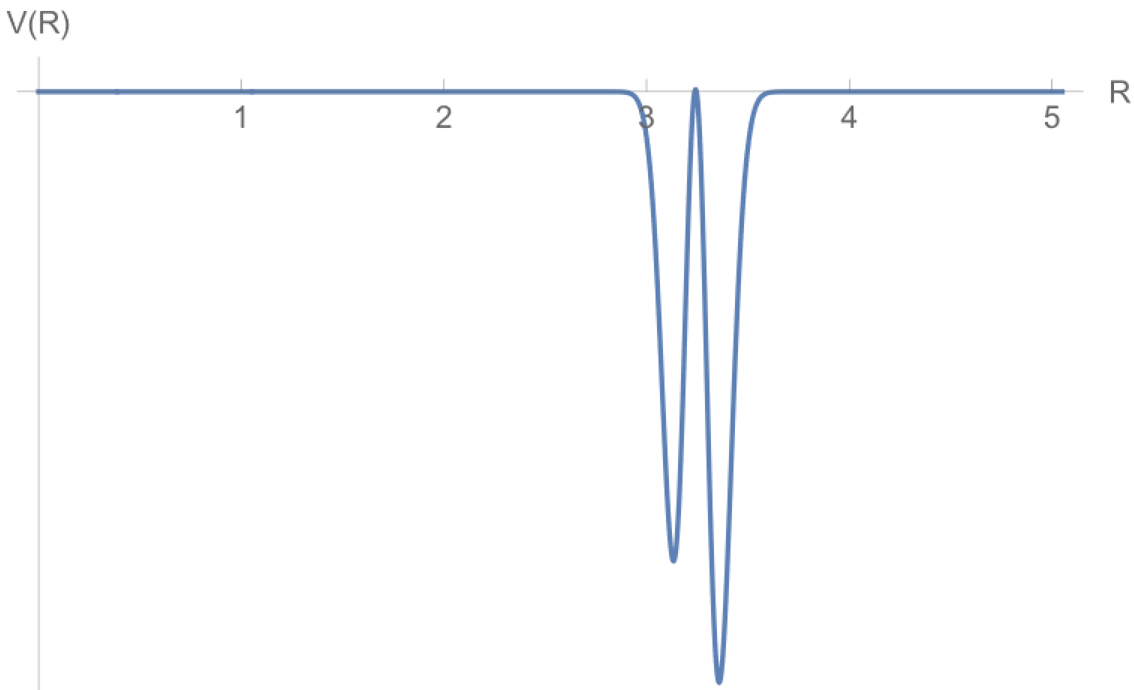

Figure 1: Three-dimnsional potential vs R plot. The presence of minimas at finite R. The minimas are centred around $R = 0.05627\sqrt{n_v}/M_{pl}$

11 Conclusion and Outlook

We started with the Kaluza-Klein compactification of quantum gravity. We calculated the loop effects as well as the instanton effects and found the existence of a new IR scale at which instanton effects contribute. The loop contributions and the instanton contribution fit in perfectly to generate superpotential. However, the minima of this superpotential lied at $R \to \infty$ which meant that the circle would decompactify and the Kaluza-Klein compactification was not stable. However, drawing motivation from String theory, we added more N=1 supersymmetric matter to the theory, namely chiral and vector multiplets. The presence of these new multiplets generated new minima in the three-dimensional potential which points towards the existence of new vacua. However, for the vacua to be in the semi-classical regime, a large number of vector multiplets are required. This has profound interests from the String theory point of view. It'll be interesting to see which String theory compactifications give rise very high number of vector multiplets. On the other hand, if such vacua are forbidden then that puts a cap on the number of vector multiplets allowed in the theory. Therefore, it'll be an exciting endeavor to further explore the consequences of these results.

12 Acknowledgement

I would like to sincerely thank Dr. Nava Gaddam for proposing a very interesting project. His constant support and motivation throughout the my thesis work is highly appreciated. I always enjoyed our long discussions on various topics, they were always very enriching.

A Pauli Villars regularization

Pauli-Villars is a straightforward method of regularization. We start with providing the original fields with small mass m. However, eventually, it acts as the infra-red cut-off as $m \to 0$. On the other hand, UV divergences are taken care of by introducing very heavy ghost particles with mass M_{UV}. The introduction of one ghost field takes care of the logarithmic divergences. However, to deal with higher-order divergences such as quadratic divergences. We have to introduce further fields, a physical field with mass γM_{UV} and a ghost field with mass $(\gamma-1)M_{UV}^2+m^2$, γ being an arbitrary parameter. Therefore, we define the Pauli-Villars notation

$$[fm^2]_{PV} = f(m^2) - f(M_{UV}^2) + f(\gamma M_{UV}^2) - f((\gamma-1)M_{UV}^2 + m^2) \tag{A.1}$$

Finally, we take $M_{UV} \to 0$.

B Manipulating indices of spinor

This excercise is inspired from excercise 3.25 of the Supergravity book. From eq 3.60 we know that the index raising and lowering matrices $\mathcal{C}^{\alpha\beta}$ must follow,

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha$$
$$\mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

It is important to note here that we are using what is called the NW-SE spin convention, which means that the contractions occur along the NW-SE line, for example,

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta$$

Now, given this let's try to figure out what $\mathcal{C}_\alpha{}^\beta$ is. Raising the α index we get,

$$\mathcal{C}_\alpha{}^\beta = \mathcal{C}^{\gamma\beta}\mathcal{C}_{\gamma_\alpha} = \delta_\alpha^\beta$$

C General rules of tensor calculus

- For a general tensor equation $A_{cd}^{ab} = B^{ab}C_{cd}$, we can always swap the order of C and B because there is no matrix product involved in such a product as A_{cd}^{ab} is just the product of a,b component of B with c,d of C. So, in general we can move quantities anywhere in the term, because it's just the scalar product.

- In case of contraction, things are little more interesting,

  $$A_\mu{}^\nu = B_\mu{}^\alpha C^\beta{}_\alpha D_\beta{}^\nu$$

  Still, in tensor form we can move B, C and D around each other but when we need to compute it, we have to turn to matrix representation and the rule for that is,

  $$B_\mu{}^\alpha C_\alpha{}^\beta = B \times C$$

i.e, second index needs to be contracted with first of the other quantity. Also,

$$B_\mu{}^\alpha C^\beta{}_\alpha = B \times C^T$$

as $C^T{}_\alpha{}^\beta = C^\beta{}_\alpha$ Therefore, $A_\mu{}^\nu$ becomes,

$$A = B \times C^T \times D$$

- In case of contracted indices, they can be raised or lowered alternatively anywhere freely,

$$A^{abc}_{efg} = B^{ac}C^{bd}D_{ef}E_{dg} = B^{ac}C^b_i g^{di} D_{ef}E_{dg} = B^{ac}C^b_i D_{ef}E^i_g$$

D Identities with Pauli matrices

$$\sigma_\mu{}^{\alpha\dot\alpha}\sigma_\nu{}^{\beta\dot\beta}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} = -2\eta_{\mu\nu} \tag{D.1}$$

where $\epsilon_{\alpha\beta}$ is given by,

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{D.2}$$

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{D.3}$$

$$\tag{D.4}$$

To prove the above identity, we break down the product into matrix multiplications,we begin with,

$$\begin{aligned}
\sigma_\mu{}^{\alpha\dot\alpha}\sigma_\nu{}^{\beta\dot\beta}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} &= \sigma_\mu{}^\alpha{}_{\dot\alpha}\epsilon_{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma_\nu{}^\beta{}_{\dot\beta} \\
&= \sigma_\mu{}^\alpha{}_{\dot\alpha}\left(\delta_\alpha{}^{\dot\alpha}\delta_\beta{}^{\dot\beta} - \delta_\alpha{}^{\dot\beta}\delta_\beta{}^{\dot\alpha}\right)\sigma_\nu{}^\beta{}_{\dot\beta} \\
&= \sigma_\mu{}^\alpha{}_\alpha\sigma_\nu{}^{\dot\beta}{}_{\dot\beta} - \delta^{\dot\beta}{}_\alpha\sigma_\mu{}^\alpha{}_{\dot\alpha}\delta^{\dot\alpha}{}_\beta\sigma_\nu{}^\beta{}_{\dot\beta} \\
&= \sigma_\mu{}^\alpha{}_\alpha\sigma_\nu{}^{\dot\beta}{}_{\dot\beta} - \sigma_\mu{}^{\dot\beta}{}_{\dot\alpha}\sigma_\nu{}^{\dot\alpha}{}_{\dot\beta} \\
&= \sigma_\mu{}^\alpha{}_\alpha\sigma_\nu{}^{\dot\beta}{}_{\dot\beta} - (\sigma_\mu\cdot\sigma_\nu)^{\dot\beta}{}_{\dot\beta}\,.
\end{aligned} \tag{D.5}$$

Now, this is just the sum of product (i,j) entry of the two matrices ($\sigma_\mu\epsilon$ and $\sigma_\nu\epsilon$) which is equal to right hand side. where $\sigma_\mu = (1, i\sigma_i)$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{D.6}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{D.7}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{D.8}$$

$$\tag{D.9}$$

E Hodge Duality

Hodge star operator maps a p-form in n-dimensional space to n-p form. In four dimensions, Hodge star operators maps a 2-form to a 2-form. It is important for our discussion because curvature can be written as a 2-form.

$$\Omega^{\mu}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\alpha\beta} dx^{\alpha} \wedge dx^{\beta} \tag{E.1}$$

The action of Hodge star on Reimann tensor is given by,

$$*R_{\mu\nu\alpha\beta} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma\alpha\beta} \tag{E.2}$$

where ϵ is the totally antisymmetric levi-civita tensor. A background is said to be self dual if it follows the following condition.

$$*R_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} \tag{E.3}$$

This condition has consequences for the Ricci tensor. Self-duality requires that the metric is Ricci flat. This can be proved as below. Let's take the Hodge star of Ricci tensor

$$\begin{aligned} *R_{\mu\nu} &= g^{\rho\sigma} * R_{\rho\mu\sigma\nu} \\ &= R_{\mu\nu} \end{aligned} \tag{E.4}$$

Let's look at the RHS for first equality,

$$\begin{aligned} R_{\mu\nu} &= g^{\rho\sigma} * R_{\rho\mu\sigma\nu} \\ &= g^{\rho\sigma} \frac{1}{2} \epsilon_{\rho\mu}{}^{\alpha\beta} R_{\alpha\beta\sigma\nu} \\ &= -\frac{1}{2} \epsilon_{\mu}{}^{\sigma\alpha\beta} R_{\alpha\beta\sigma\nu} \\ &= -\frac{1}{2} \epsilon_{\mu}{}^{\alpha\beta\sigma} R_{\alpha\beta\sigma\nu} \end{aligned} \tag{E.5}$$

since Levi Civita tensor is anti-symmetric in all its indices. It is anti-symmetric in its last three indices. Therefore, the first three indices of the Riemann tensor need to be antisymmetric. From Bianchi identity $R_{[\mu\nu\alpha]\beta} = 0$, we find the RHS should vanish.

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{2} \epsilon_{\mu}{}^{\alpha\beta\sigma} R_{[\alpha\beta\sigma]\nu} \\ &= 0 \end{aligned} \tag{E.6}$$

Therefore, self-duality implies that the metric is Ricci flat.

F Gaussian Integrals

Gaussian integrals are evaluated differently based on the integrating variable type. There are two sets of variables that are evaluated differently namely, commuting variables and anti-commuting variables (Grassmann variables). We summarize the calculations below.

F.1 Real commuting variables

Real commuting variables generally represent a single bosonic field. Let's consider a bosonic field it is represented by commuting real variable ϕ. The gaussian integral over ϕ is given by,

$$\begin{aligned} Z[0] &= \int d^n\phi e^{-\frac{1}{2}\phi^T\ M\ \phi} \\ &= (detM)^{-1/2} \end{aligned} \tag{F.1}$$

This integral is the regular Gaussian integral over real variables.

F.2 Complex commuting variables

Complex commuting variables can represent bosons with two degrees of freedom. If xi represents a complex variable, then $\bar{\xi}$ is another independent variable and the gaussian integral is found to be,

$$\begin{aligned} Z[0] &= \int d^n\xi d^n\xi^\dagger\ e^{-\frac{1}{2}\ \xi^\dagger\ M\ \xi} \\ &\propto (detM)^{-1} \end{aligned} \tag{F.2}$$

This integral is evaluated over two independent variables, $\xi\ and\ \bar{\xi}$.

F.3 Real non-commuting variables

Non-commuting or Grassmann variables represent the fermionic degrees of freedom. For example consider a set of real Majorana fermions $\psi_\mu = (\psi_1, \psi_2)$ the gaussian integral over ψ can be evaluated as

$$\begin{aligned} Z[0] &= \int d^2\psi e^{-\frac{1}{2}\psi^T\ M\ \psi} \\ &= \int d^2\psi \left(1 - \frac{1}{2}\psi^T\ M\ \psi + ...\right) \\ &= \int d^2\psi \left(1 - \frac{1}{2}(\psi_1 M_{11}\psi_1 + \psi_1 M_{12}\psi_2 + \psi_2 M_{22}\psi_2 + \psi_2 M_{21}\psi_1)\right) \\ &= \int d^2\psi - \frac{1}{2}\left(\psi_1 M_{12}\psi_2 + \psi_2 M_{21}\psi_1\right) \\ &= -\frac{1}{2}\left(2M_{12}\right) \\ &= (detM)^{\frac{1}{2}} \end{aligned} \tag{F.3}$$

The Matrix M needs to be antisymmetric because of the anti-commuting nature of Grassmann variables. Also, the integration over Grassmann variables is regarded as differentiation. Therefore, writing out the matrix multiplication on the right-hand side, we get a constant time the ψ^n. That constant turns out to be $\sqrt{det(M)}$ and differentiation takes out all the ψ^n. Therefore we have,

F.4 Complex non-commuting variables

Dirac fermion $\psi_\mu = (\psi_1, \psi_2)$ can be represented by the complex Grassmann variables. The Gaussian integral over these variables is straightforward. Also, as in the case of a complex bosonic field, we expect the complex fermionic case to be squared of the real fermionic Gaussian integral.

$$\begin{aligned}
Z[0] &= \int d^n\psi d^n\bar{\psi} e^{-\frac{1}{2}\psi^\dagger\ M\ \psi} \\
&= \int d\bar{\psi}_1 d\bar{\psi}_2 d\psi_1 d\psi_2 e^{-\frac{1}{2}(\bar{\psi}_1 M_{11}\psi_1 + \bar{\psi}_1 M_{12}\psi_2 + \bar{\psi}_2 M_{22}\psi_2 + \bar{\psi}_2 M_{21}\psi_1)} \\
&= \int d\bar{\psi}_1 d\bar{\psi}_2 d\psi_1 d\psi_2 \left(e^{-\frac{1}{2}(\bar{\psi}_1 M_{11}\psi_1} + e^{-\frac{1}{2}\bar{\psi}_1 M_{12}\psi_2} + e^{-\frac{1}{2}\bar{\psi}_2 M_{22}\psi_2} + e^{-\frac{1}{2}\bar{\psi}_2 M_{21}\psi_1)}\right) \\
&= \int d\bar{\psi}_1 d\bar{\psi}_2 d\psi_1 d\psi_2 \left(-\frac{1}{2}(\bar{\psi}_1 M_{11}\psi_1(-\frac{1}{2}\bar{\psi}_2 M_{22}\psi_2) + (-\frac{1}{2}\bar{\psi}_1 M_{12}\psi_2)(-\frac{1}{2}\bar{\psi}_2 M_{21}\psi_1))\right) \\
&= -\frac{1}{4} detM \\
&\propto detM
\end{aligned} \tag{F.4}$$

For complex fermion, we are integrating over two sets of Grassmann variables. So, only the term with all the set of Grassmann variables survives.

G Spin connection of Taub-NUT metric

We use cartan formalism to calculate the Riemann tensor for Multi Taub NUT metric. The metric is given by,

$$ds^2 = U(dx_1^2 + dx_2^2 + dx_3^2) + U^{-1}(dz + A.dx)^2 \tag{G.1}$$

where, U and A are functions of $x^i = x_1, x_2, x_3$. The constraint on A is

$$dA \ = \ ^\star dU \tag{G.2}$$

which implies

$$dA \ = \ \epsilon_{ij}{}^k \frac{1}{U}\frac{\partial U}{\partial x^k}\ \theta^i \wedge \theta^j\ , \tag{G.3}$$

where we choose the following tetrad θ,

$$\theta = \begin{pmatrix} U^{1/2}dx_1 \\ U^{1/2}dx_2 \\ U^{1/2}dx_3 \\ U^{-1/2}(dz + A.dx) \end{pmatrix} \tag{G.4}$$

we use the Caratan's first equation in absence of torsion,

$$d\theta + \omega \wedge \theta = 0 \tag{G.5}$$

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \tag{G.6}$$

let's evaluate $d\theta$ first,

$$\begin{aligned}
d\theta^\mu &= \begin{pmatrix} d\theta^i \\ d\theta^4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{dU^{1/2}}{dx^j} dx^j \wedge dx^i \\ \frac{dU^{-1/2}}{dx^i} dx^i \wedge (dz + A.dx) + U^{-1/2} \frac{dA_k}{dx^j} dx^j \wedge dx^k \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2U^{3/2}} \frac{dU}{dx^j} \theta^j \wedge \theta^i \\ \frac{-1}{2U^{3/2}} \frac{dU}{dx^i} \theta^i \wedge \theta^4 + \frac{1}{2} U^{-3/2} F_{jk} \theta^j \wedge \theta^k \end{pmatrix}
\end{aligned} \tag{G.7}$$

there'll be two components $w^i{}_j$ and $\omega^4{}_j$, let's evaluate them one by one

$$d\theta^4 + \omega^4{}_\nu \wedge \theta^\nu = 0 \tag{G.8}$$

$$\begin{aligned}
d\theta^4 + \omega^4{}_i \wedge \theta^i + \omega^4{}_4 \wedge \theta^4 &= \frac{-1}{2U^{3/2}} \frac{dU}{dx^i} \theta^i \wedge \theta^4 + U^{-3/2} \frac{dA_k}{dx^j} \theta^j \wedge \theta^k + \omega^4{}_i \wedge \theta^i + 0 \\
&= \frac{1}{2U^{3/2}} \frac{dU}{dx^i} \theta^4 \wedge \theta^i + \frac{1}{2U^{3/2}} (\frac{dA_k}{dx^j} \theta^j \wedge \theta^k - \frac{dA_j}{dx^k} \theta^j \wedge \theta^k) + \omega^4{}_i \wedge \theta^i \\
&= \frac{1}{2U^{3/2}} \frac{dU}{dx^i} \theta^4 \wedge \theta^i + \frac{1}{2U^{3/2}} F_{ji} \theta^j \wedge \theta^i + \omega^4{}_i \wedge \theta^i \\
&= \left(\frac{1}{2U^{3/2}} \frac{dU}{dx^i} \theta^4 + \frac{1}{2U^{3/2}} F_{ji} \theta^j + \omega^4{}_i \right) \wedge \theta^i
\end{aligned} \tag{G.9}$$

where $F_{ij} = d_i A_j - d_j A_i$, therefore we have,

$$\frac{1}{2U^{3/2}} \frac{dU}{dx^i} \theta^4 + \frac{1}{2U^{3/2}} F_{ji} \theta^j + \omega^4{}_i = 0 \tag{G.10}$$

$$\begin{aligned}
\omega^4{}_i &= -\omega_i{}^4 \\
&= -\frac{1}{2U^{3/2}} \frac{dU}{dx^i} \theta^4 - \frac{1}{2U^{3/2}} F_{ji} \theta^j
\end{aligned} \tag{G.11}$$

Next, let's evaluate $\omega^i{}_j$

$$d\theta^i + \omega^i{}_\nu \wedge \theta^\nu = 0 \tag{G.12}$$

$$d\theta^i + \omega^i{}_j \wedge \theta^j + \omega^i{}_4 \wedge \theta^4 = 0 \tag{G.13}$$

$$\begin{aligned}
d\theta^i + \omega^i{}_j \wedge \theta^j + \omega^i{}_4 \wedge \theta^4 &= \frac{1}{2U^{3/2}} \frac{dU}{dx^j} \theta^j \wedge \theta^i + \omega^i{}_j \wedge \theta^j + \left(\frac{1}{2U^{3/2}} \frac{dU}{dx_i} \theta^4 + \frac{1}{2U^{3/2}} F_j{}^i \theta^j \right) \wedge \theta^4 \\
&= -\frac{1}{2U^{3/2}} \frac{dU}{dx^j} \theta^i \wedge \theta^j + \omega^i{}_j \wedge \theta^j - \frac{1}{2U^{3/2}} F_j{}^i \theta^4 \wedge \theta^j \\
&= \left(-\frac{1}{2U^{3/2}} \frac{dU}{dx^j} \theta^i + \omega^i{}_j - \frac{1}{2U^{3/2}} F_j{}^i \theta^4 \right) \wedge \theta^j
\end{aligned} \tag{G.14}$$

To preserve the anti-symmetry of ${\omega^i}_j$ we can add a vanishing term to the sum without any loss.

$$0 = \left(-\frac{1}{2U^{3/2}} \frac{dU}{dx^j} \theta^i + \frac{1}{2U^{3/2}} \frac{dU}{dx_i} \theta_j + {\omega^i}_j - \frac{1}{2U^{3/2}} {F_j}^i \theta^4 \right) \wedge \theta^j \tag{G.15}$$

$${\omega^i}_j = \left(\frac{1}{2U^{3/2}} \frac{dU}{dx^j} \theta^i - \frac{1}{2U^{3/2}} \frac{dU}{dx_i} \theta_j + \frac{1}{2U^{3/2}} {F_j}^i \theta^4 \right) \tag{G.16}$$

H Graviton determinant

Let's look at the form of various terms following transformation of (4.14). First, we write down the form of metric tensor with upper indices.

$$\begin{aligned}
g^{\bar{\mu\nu}} g_{\bar{\mu\nu}} &= 4 \\
g^{\bar{\mu\nu}} (g_{\mu\nu} + h_{\mu\nu}) &= 4 \\
g^{\bar{\mu\nu}} g_{\nu\alpha} (\delta^\alpha_\mu + h^\alpha_\mu) &= 4 \\
g^{\bar{\mu\nu}} &= \frac{4}{g_{\nu\alpha}(\delta^\alpha_\mu + h^\alpha_\mu)} \\
g^{\bar{\mu\nu}} &= g^{\nu\alpha} \left(\delta^\mu_\alpha - h^\mu_\alpha + h^\mu_\beta h^\beta_\alpha \right) \\
g^{\bar{\mu\nu}} &= g^{\mu\nu} - h^{\mu\nu} + h^\mu_\beta h^{\beta\nu}
\end{aligned} \tag{H.1}$$

Here, we have expanded upto second order of h. Similarly, the $\delta^{\bar{\mu}}_\nu = \delta^\mu_\nu$. Next, the quantity of interest is the square root of determinant of metric fluctuations. Expanding it to second order in fluctuations as well, we get

$$\begin{aligned}
\sqrt{det\ g} &= \exp\left(\frac{1}{2} tr \ln \bar{g}^\mu_\nu \right) \\
&= \exp\left(\frac{1}{2} tr \ln \left(\delta^\mu_\nu + h^\mu_\nu \right) \right) \\
&= \exp\left(\frac{1}{2} tr (h^\mu_\nu + \frac{1}{2} h^\mu_\alpha h^\alpha_\nu) + \mathcal{O}(h^3) \right) \\
&= \exp\left(\frac{1}{2} (h^\mu_\mu + \frac{1}{2} h^\mu_\alpha h^\alpha_\mu) \right) \\
&= 1 + \frac{1}{2} h^\mu_\mu - \frac{1}{4} h^\mu_\alpha h^\alpha_\mu + \frac{1}{8} h^\mu_\mu h^\nu_\nu + \mathcal{O}(h^3)
\end{aligned} \tag{H.2}$$

The equality in the first line is derived from standard matrix equality, $Tr\ logM = log(detM)$, where M is any matrix. The following equalities use the Taylor series for logarithms and exponentials.

Given these preliminaries, lets expand the Ricci scalar upto second order in metric fluctuations, we start with Christoffel symbols. Christoffel symbol can be expanded as,

$$\bar{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \underline{\Gamma}^\mu_{\alpha\beta} + \underline{\underline{\Gamma}}^\mu_{\alpha\beta} \tag{H.3}$$

Christoffel symbol are given by,

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\sigma}\left(\bar{g}_{\beta\sigma,\alpha} + \bar{g}_{\alpha\sigma,\beta} - \bar{g}_{\alpha\beta,\sigma}\right) \tag{H.4}$$

The first order fluctuations are given by,

$$\begin{aligned}\underline{\Gamma}^{\mu}_{\alpha\beta} &= \frac{1}{2}g^{\mu\sigma}\left(h_{\beta\sigma,\alpha} + h_{\alpha\sigma,\beta} - h_{\alpha\beta,\sigma}\right)\\ &= \frac{1}{2}\left(h^{\mu}_{\beta,\alpha} + h^{\mu}_{\alpha,\beta} - h_{\alpha}{}^{,\mu}_{\beta}\right)\end{aligned} \tag{H.5}$$

Similarly, second order fluctuations are,

$$\underline{\underline{\Gamma}}^{\mu}_{\alpha\beta} = -\frac{1}{2}h^{\mu\sigma}\left(h_{\beta\sigma,\alpha} + h_{\alpha\sigma,\beta} - h_{\alpha\beta,\sigma}\right) \tag{H.6}$$

The negative sign follows due to definition of eq 3.3 of upper indices of metric tensor. Therefore, we expand the action in the power of the h. The next step is to construct Reimannian tensor upto second order,

$$\bar{R}^{\mu}_{\nu\alpha\beta} = R^{\mu}_{\nu\alpha\beta} + \underline{R}^{\mu}_{\nu\alpha\beta} + \underline{\underline{R}}^{\mu}_{\nu\alpha\beta} \tag{H.7}$$

where $R^{\mu}_{\nu\alpha\beta}$ is constructed from the background metric $g_{\mu\nu}$. The Riemannian tensor are given as,

$$R^{\mu}_{\nu\alpha\beta} = D_{\alpha}\bar{\Gamma}^{\mu}_{\beta\nu} - D_{\beta}\bar{\Gamma}^{\mu}_{\alpha\nu} + \bar{\Gamma}^{\mu}_{\lambda\alpha}\bar{\Gamma}^{\lambda}_{\beta\nu} - \bar{\Gamma}^{\mu}_{\lambda\beta}\bar{\Gamma}^{\lambda}_{\alpha\nu} \tag{H.8}$$

The first order fluctuations in Riemannian tensor are,

$$\underline{R}^{\mu}_{\nu\alpha\beta} = D_{\alpha}\underline{\Gamma}^{\mu}_{\beta\nu} - D_{\beta}\underline{\Gamma}^{\mu}_{\alpha\nu} \tag{H.9}$$

Here third and fourth term vanish as none of them supply first order terms. We can contract the first and third index using a delta function and obtain Ricci tensor.

$$\begin{aligned}\underline{R}_{\nu\beta} &= D_{\alpha}\underline{\Gamma}^{\alpha}_{\beta\nu} - D_{\beta}\underline{\Gamma}^{\alpha}_{\alpha\nu}\\ &= \frac{1}{2}\left(h^{\alpha}_{\beta,\nu,\alpha} + h^{\alpha}_{\nu,\beta,\alpha} - h_{\nu\beta}{}^{,\alpha}{}_{,\alpha}\right) - \frac{1}{2}h^{\alpha}_{\alpha,\nu,\beta}\end{aligned} \tag{H.10}$$

Finally, Ricci scalar is computed using contraction by metric tensor,

$$\begin{aligned}\underline{R} &= \bar{g}^{\nu\beta}\bar{R}_{\nu\beta}\\ &= (g^{\nu\beta} - h^{\nu\beta})(R_{\nu\beta} + \underline{R}_{\nu\beta})\\ &= g^{\nu\beta}R_{\nu\beta} - h^{\nu\beta}R_{\nu\beta}\\ &= \frac{1}{2}g^{\nu\beta}\left(h^{\alpha}_{\beta,\nu,\alpha} + h^{\alpha}_{\nu,\beta,\alpha} - h_{\nu\beta}{}^{,\alpha}{}_{,\alpha} - h^{\alpha}_{\alpha,\nu,\beta}\right) - h^{\nu\beta}R_{\nu\beta}\\ &= \frac{1}{2}\left(h^{\alpha}_{\beta,\alpha}{}^{,\beta} + h^{\alpha}_{\beta,\alpha}{}^{,\beta} - h^{\beta}_{\beta}{}^{,\alpha}{}_{,\alpha} - h^{\alpha}_{\alpha,\beta}{}^{,\beta}\right) - h^{\nu\beta}R_{\nu\beta}\\ &= h^{\alpha}_{\beta,\alpha}{}^{,\beta} - h^{\beta,\alpha}_{\beta}{}_{,\alpha} - h^{\nu\beta}R_{\nu\beta}\end{aligned} \tag{H.11}$$

Now, let's compute the second order Ricci scalar using the same procedure. We start with Riemannian tensor of second order,

$$\underline{\underline{R}}^{\mu}_{\nu\alpha\beta} = D_\alpha \underline{\underline{\Gamma}}^{\mu}_{\beta\nu} - D_\beta \underline{\underline{\Gamma}}^{\mu}_{\alpha\nu} + \underline{\Gamma}^{\mu}_{\lambda\alpha}\underline{\Gamma}^{\lambda}_{\beta\nu} - \underline{\Gamma}^{\mu}_{\lambda\beta}\underline{\Gamma}^{\lambda}_{\alpha\nu} \tag{H.12}$$

Ricci tensor of second order is straight forward and is given by,

$$\underline{\underline{R}}_{\nu\beta} = D_\alpha \underline{\underline{\Gamma}}^{\alpha}_{\beta\nu} - D_\beta \underline{\underline{\Gamma}}^{\alpha}_{\alpha\nu} + \underline{\Gamma}^{\alpha}_{\lambda\alpha}\underline{\Gamma}^{\lambda}_{\beta\nu} - \underline{\Gamma}^{\alpha}_{\lambda\beta}\underline{\Gamma}^{\lambda}_{\alpha\nu} \tag{H.13}$$

Again the Ricci scalar is computed by contraction by metric tensor,

$$\begin{aligned}
\underline{\underline{R}} &= \bar{g}^{\nu\beta}\bar{R}_{\nu\beta} \\
&= (g^{\nu\beta} - h^{\nu\beta} + h^{\nu}_{\alpha}h^{\alpha\beta})(R_{\nu\beta} + \underline{R}_{\nu\beta} + \underline{\underline{R}}_{\nu\beta}) \\
&= g^{\nu\beta}\underline{\underline{R}}_{\nu\beta} - h^{\nu\beta}\underline{R}_{\nu\beta} + h^{\nu}_{\alpha}h^{\alpha\beta}R_{\nu\beta} \\
&= g^{\nu\beta}\left(D_\alpha \underline{\underline{\Gamma}}^{\alpha}_{\beta\nu} - D_\beta \underline{\underline{\Gamma}}^{\alpha}_{\alpha\nu} + \underline{\Gamma}^{\alpha}_{\lambda\alpha}\underline{\Gamma}^{\lambda}_{\beta\nu} - \underline{\Gamma}^{\alpha}_{\lambda\beta}\underline{\Gamma}^{\lambda}_{\alpha\nu}\right) \\
&\quad - h^{\nu\beta}\frac{1}{2}\left(h^{\alpha}_{\beta,\nu,\alpha} + h^{\alpha}_{\nu,\beta,\alpha} - h_{\nu\beta}{}^{,\alpha}{}_{,\alpha} - h^{\alpha}_{\alpha,\nu,\beta}\right) + h^{\nu}_{\alpha}h^{\alpha\beta}R_{\nu\beta} \\
&= \left(D_\alpha \underline{\underline{\Gamma}}^{\alpha\nu}_{\nu} - D^{\nu}\underline{\underline{\Gamma}}^{\alpha}_{\alpha\nu} + \underline{\Gamma}^{\alpha}_{\lambda\alpha}\underline{\Gamma}^{\lambda\beta}_{\beta} - \underline{\Gamma}^{\alpha}_{\lambda\beta}\underline{\Gamma}^{\lambda\beta}_{\alpha}\right) \\
&\quad - h^{\nu\beta}\frac{1}{2}\left(h^{\alpha}_{\beta,\nu,\alpha} + h^{\alpha}_{\nu,\beta,\alpha} - h_{\nu\beta}{}^{,\alpha}{}_{,\alpha} - h^{\alpha}_{\alpha,\nu,\beta}\right) + h^{\nu}_{\alpha}h^{\alpha\beta}R_{\nu\beta} \\
&= \left(D_\alpha \underline{\underline{\Gamma}}^{\alpha\nu}_{\nu} - D^{\nu}\underline{\underline{\Gamma}}^{\alpha}_{\alpha\nu}\right) + \frac{1}{2}h^{\gamma,\nu}_{\nu}h^{\alpha}_{\alpha,\gamma} - \frac{1}{4}h^{\alpha,\gamma}_{\alpha}h^{\beta}_{\beta,\gamma} - \frac{1}{2}h_{\nu\beta}{}^{,\gamma}h^{\nu,\beta}_{\gamma} \\
&\quad + \frac{1}{4}h_{\nu\beta}{}^{,\gamma}h^{\beta\nu}{}_{,\gamma} - h^{\nu\beta}\frac{1}{2}\left(h^{\alpha}_{\beta,\nu,\alpha} + h^{\alpha}_{\nu,\beta,\alpha} - h_{\nu\beta}{}^{,\alpha}{}_{,\alpha} - h^{\alpha}_{\alpha,\nu,\beta}\right) + h^{\nu}_{\alpha}h^{\alpha\beta}R_{\nu\beta}
\end{aligned} \tag{H.14}$$

Now, using all these terms we can construct the action with second-order fluctuations.

$$\begin{aligned}
S_0 &= -\frac{1}{16\pi G}\int d^4x\sqrt{g}(1 + \frac{1}{2}h^{\alpha}_{\alpha} - \frac{1}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha} + \frac{1}{8}h^{\mu}_{\mu}h^{\nu}_{\nu})(R + \underline{R} + \underline{\underline{R}}) \\
&= -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\underline{\underline{R}} + \frac{1}{2}h^{\alpha}_{\alpha}\underline{R} - \frac{1}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha}R + \frac{1}{8}h^{\mu}_{\mu}h^{\nu}_{\nu}R\right) \\
&= -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\left(D_\alpha \underline{\underline{\Gamma}}^{\alpha\nu}_{\nu} - D^{\nu}\underline{\underline{\Gamma}}^{\alpha}_{\alpha\nu}\right) + \frac{1}{2}h^{\gamma,\nu}_{\nu}h^{\alpha}_{\alpha,\gamma} - \frac{1}{4}h^{\alpha,\gamma}_{\alpha}h^{\beta}_{\beta,\gamma} - \frac{1}{2}h_{\nu\beta}{}^{,\gamma}h^{\nu,\beta}_{\gamma}\right. \\
&\quad + \frac{1}{4}h_{\nu\beta}{}^{,\gamma}h^{\beta\nu}{}_{,\gamma} - h^{\nu\beta}\frac{1}{2}\left(h^{\alpha}_{\beta,\nu,\alpha} + h^{\alpha}_{\nu,\beta,\alpha} - h_{\nu\beta}{}^{,\alpha}{}_{,\alpha} - h^{\alpha}_{\alpha,\nu,\beta}\right) + h^{\nu}_{\alpha}h^{\alpha\beta}R_{\nu\beta} + \frac{1}{2}h^{\alpha}_{\alpha}h^{\alpha}_{\beta,\alpha}{}^{,\beta} \\
&\quad \left. - \frac{1}{2}h^{\alpha}_{\alpha}h^{\beta,\alpha}_{\beta}{}_{,\alpha} - \frac{1}{2}h^{\alpha}_{\alpha}h^{\nu\beta}R_{\nu\beta} - \frac{1}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha}R + \frac{1}{8}h^{\mu}_{\mu}h^{\nu}_{\nu}R\right)
\end{aligned} \tag{H.15}$$

Performing the partial integration over double derivative terms and ignoring the boundary terms (i.e total derivate terms)

$$\begin{aligned}
S_0 = -\frac{1}{16\pi G}\int d^4x\sqrt{g}&\left(\frac{1}{8}h^{\mu}_{\mu}h^{\nu}_{\nu}R - \frac{1}{2}h^{\alpha}_{\alpha}h^{\nu\beta}R_{\nu\beta} - \frac{1}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha}R + h^{\nu}_{\alpha}h^{\alpha\beta}R_{\nu\beta}\right. \\
&\left. + \frac{1}{4}h^{\mu}_{\mu,\alpha}h^{\nu,\alpha}_{\nu} - \frac{1}{4}h^{\alpha}_{\beta,\mu}h^{\beta,\mu}_{\alpha} + \frac{1}{2}h^{\mu}_{\mu,\beta}h^{\beta,\nu}_{\nu} - \frac{1}{2}h^{\beta,\alpha}_{\nu}h^{\nu}_{\alpha,\beta}\right)
\end{aligned} \tag{H.16}$$

Now, using the procedure of section 2.2 we add the gauge fixing term which is the square of the gauge condition and is given by,

$$\begin{aligned}S_{gf} &= -\frac{1}{32\pi G}\int d^4x\sqrt{g}\nabla^\mu\left(h_{\mu\alpha}-\frac{1}{2}g_{\mu\alpha}h^\beta_\beta\right)\nabla_\nu\left(h^{\nu\alpha}-\frac{1}{2}g^{\nu\alpha}h^\beta_\beta\right)\\ &= -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\frac{1}{2}h^{;\mu}_{\mu\alpha}h^{\nu\alpha}_{,\nu}-\frac{1}{2}h^\beta_{\beta,\alpha}h^{\nu\alpha}{}_{,\nu}+\frac{1}{8}h^\mu_{\mu,\alpha}h^{\nu,\alpha}_\nu\right)\end{aligned} \tag{H.17}$$

Considering only second order terms in $S_0 + S_{gf}$, we get,

$$\begin{aligned}S = &-\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\frac{1}{8}h^\mu_\mu h^\nu_\nu R-\frac{1}{2}h^\alpha_\alpha h^{\nu\beta}R_{\nu\beta}-\frac{1}{4}h^\alpha_\beta h^\beta_\alpha R+h^\nu_\alpha h^{\alpha\beta}R_{\nu\beta}+\frac{1}{4}h^\mu_{\mu,\alpha}h^{\nu,\alpha}_\nu\right.\\ &\left.-\frac{1}{4}h^\alpha_{\beta,\mu}h^{\beta,\mu}_\alpha+\cancel{\frac{1}{2}h^\mu_{\mu,\beta}h^{\beta,\nu}_\nu}-\frac{1}{2}h^{\beta,\alpha}_\nu h^\nu_{\alpha,\beta}+\frac{1}{2}h^{;\mu}_{\mu\alpha}h^{\nu\alpha}_{,\nu}-\cancel{\frac{1}{2}h^\beta_{\beta,\alpha}h^{\nu\alpha}{}_{,\nu}}+\frac{1}{8}h^\mu_{\mu,\alpha}h^{\nu,\alpha}_\nu\right)\\ = &-\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\frac{1}{8}h^\mu_\mu h^\nu_\nu R-\frac{1}{2}h^\alpha_\alpha h^{\nu\beta}R_{\nu\beta}-\frac{1}{4}h^\alpha_\beta h^\beta_\alpha R+h^\nu_\alpha h^{\alpha\beta}R_{\nu\beta}+\frac{1}{8}h^\mu_{\mu,\alpha}h^{\nu,\alpha}_\nu\right.\\ &\left.-\frac{1}{4}h^\alpha_{\beta,\mu}h^{\beta,\mu}_\alpha+-\frac{1}{2}h^{\beta,\alpha}_\nu h^\nu_{\alpha,\beta}+\frac{1}{2}h^{;\mu}_{\mu\alpha}h^{\nu\alpha}_{,\nu}\right)\end{aligned} \tag{H.18}$$

Let's transform this action in terms of transverse traceless part $\bar{h}^{\mu\nu}$ and trace h. $\bar{h}^{\mu\nu}$ is given by,

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{4}g^{\mu\nu}h \tag{H.19}$$

Using this definition in the action, it becomes,

$$\begin{aligned}S = &-\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\frac{1}{4}\bar{h}^{\mu\nu}g_{\mu\alpha}g_{\nu\beta}\nabla^2\bar{h}^{\alpha\beta}-\frac{1}{4}\bar{h}^{\mu\nu}g_{\mu\alpha}g_{\nu\beta}R\bar{h}^{\alpha\beta}+\frac{1}{2}\bar{h}^{\mu\nu}g_{\nu\beta}\bar{h}^{\alpha\beta}R_{\mu\alpha}\right.\\ &\left.+\frac{1}{2}\bar{h}^{\mu\nu}R_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta}-\frac{1}{16}h\nabla^2 h\right)\\ = &-\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(\bar{h}^{\mu\nu}\Delta_2\bar{h}^{\alpha\beta}+h\Delta_0 h\right)\end{aligned} \tag{H.20}$$

where,

$$\Delta_2 = -\frac{1}{4}g_{\mu\alpha}g_{\nu\beta}\nabla^2+\frac{1}{4}g_{\mu\alpha}g_{\nu\beta}R-\frac{1}{2}g_{\nu\beta}R_{\mu\alpha}-\frac{1}{2}R_{\mu\alpha\nu\beta} \tag{H.21}$$

We have negative definite operator for h, therefore we rotate the contour to integrate over imaginary conformal factors. Hence, we have

$$\Delta_0 = -\nabla^2 \tag{H.22}$$

References

[1] Andrew Strominger, Gary T. Horowitz, and Malcolm J. Perry. Instantons in Conformal Gravity. *Nucl. Phys. B*, 238:653–664, 1984.

[2] Thomas Appelquist and Alan Chodos. Quantum Effects in Kaluza-Klein Theories. *Phys. Rev. Lett.*, 50:141, 1983.

[3] Edward Witten. Instability of the Kaluza-Klein Vacuum. *Nucl. Phys. B*, 195:481–492, 1982.

[4] D. G. Cerdeno and C. Munoz. An introduction to supergravity. *PoS*, CORFU98:011, 1998.

[5] Thomas W. Grimm and Raffaele Savelli. Gravitational instantons and fluxes from m/f-theory on calabi-yau fourfolds. *Physical Review D*, 85(2), Jan 2012.

[6] David Tong and Carl Turner. Quantum dynamics of supergravity on r3 $\times$ s1. *Journal of High Energy Physics*, 2014(12), Dec 2014.

[7] K. Becker, M. Becker, and J. H. Schwarz. *String theory and M-theory: A modern introduction*. Cambridge University Press, 12 2006.

[8] S. W. Hawking. *The path integral approach to quantum gravity*. 1980.

[9] Gerard 't Hooft and M. J. G. Veltman. One loop divergencies in the theory of gravitation. *Ann. Inst. H. Poincare Phys. Theor. A*, 20:69–94, 1974.

[10] J.M. Overduin and P.S. Wesson. Kaluza-klein gravity. *Physics Reports*, 283(5-6):303–378, Apr 1997.

[11] L. Faddeev and V. N. Popov. Feynman diagrams for the yang-mills field. *Physics Letters B*, 25:29–30, 1967.

[12] Marco Ornigotti and Andrea Aiello. The faddeev-popov method demystified, 2014.

[13] Gerard 't Hooft. Renormalization of Massless Yang-Mills Fields. *Nucl. Phys. B*, 33:173–199, 1971.

[14] Malcolm J. Perry. Anomalies in Supergravity. *Nucl. Phys. B*, 143:114–124, 1978.

[15] N. K. Nielsen. Ghost Counting in Supergravity. *Nucl. Phys. B*, 140:499–509, 1978.

[16] R. E. Kallosh. Modified Feynman Rules in Supergravity. *Nucl. Phys. B*, 141:141–152, 1978.

[17] Malcolm J. Perry. Anomilies of supergravity. *Nuclear Physics B*, 143(2):114–124, 1978.

[18] Thomas Appelquist and Alan Chodos. The Quantum Dynamics of Kaluza-Klein Theories. *Phys. Rev. D*, 28:772, 1983.

[19] D.V. Vassilevich. Heat kernel expansion: user's manual. *Physics Reports*, 388(5-6):279–360, Dec 2003.

[20] G. W. Gibbons and Malcolm J. Perry. New Gravitational Instantons and Their Interactions. *Phys. Rev. D*, 22:313, 1980.

[21] Andrew Hanson Tohru Eguchi. Gravitational Instantons. *General Relativity and Gravitation*, 11:315–320, 1979.

[22] N. Dorey, V.V. Khoze, M.P. Mattis, D. Tong, and S. Vandoren. Instantons, three-dimensional gauge theory, and the atiyah-hitchin manifold. *Nuclear Physics B*, 502(1-2):59–93, Sep 1997.

[23] B. de Wit, H. Nicolai, and H. Samtleben. Gauged supergravities in three-dimensions: A Panoramic overview. *PoS*, jhw2003:016, 2003.