Tidal Effects for Black Holes with Scalar Hair in EsGB Gravity

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Abstract

After the groundbreaking detection of Gravitational Waves in 2015, a new era looms of unprecedented new data of the strong field gravity regime. The need for well-developed theories beyond General Relativity is therefore high. Einstein-scalar-Gauss-Bonnet gravity is one of the most promising quadratic gravity theories, and has recently had its Lagrangian calculated to 1PN order. In this thesis, we include the tidal effects induced on the scalar field around the black hole in the action and derive its contribution to the 1PN Lagrangian. A small mistake in the Lagrangian is also corrected. Finally, we calculate the binding energy and relative acceleration from the Lagrangian, and we specifically analyse the magnitude of the tidal contribution to the binding energy. The contribution is found to be generally small in most situations, but also to be heavily dependent on yet unknown parameters. The relative magnitude of the tidal contribution is found to be up to 10^{-5} compared to the GR energy in some situations.

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Contents

1 Introduction

After the landmark gravitational wave observation of a black hole merger in 2015 [1] we are at the start of a new era in cosmology. Gravitational wave detection has given us a new tool to look into the universe, and an exciting time of multi-messenger astronomy, revealing more information than ever, lies ahead of us. Moreover, the ability to observe gravitational waves has given us a window into the previously unobservable extreme gravity regime. Many of the problems with General Relativity at high energies are now directly testable, and the demand for well-developed beyond-GR theories is consequently higher than ever [5].

Quadratic gravity theories as first order corrections are natural extensions to GR, and improve its high energy behaviour [23, 4]. They assume GR to be the low-energy expansion of a more complete theory, and therefore add terms that are quadratic in curvature as a first order correction. One of the most appealing quadratic gravity theories, is EsGB gravity, modifying the GR action with a scalar that is quadratic in curvature, coupled to a scalar field [18, 21]. This particular scalar avoids the ghost problems that other quadratic gravity theories tend to have, and black holes can evade the classical no-hair theorem [2]. This last fact will be of particular interest in this thesis, as we will focus on the induced tidal effect on the scalar field of black holes. The modification is supported by the low energy limit of string theory [13].

In recent years, new work has been done to obtain approximate Lagrangians for EsGB [14] and the similar EMD theory [15]. They are built by using the Post-Newtonian approximation on the field equations, which expand them in v/c and $Gm/rc^2 \sim (v/c)^2$. Approximate solutions can then be used to construct a Lagrangian to first order in the Post-Newtonian approximation (1PN). With this Lagrangian, important physical quantities such as the binding energy, relative acceleration, energy flux or the waveform can be calculated and compared to experimental data.

The next step to make this theory more complete would be to include the tidal effects that occur due to the scalar field around the black hole being tidally deformed. An action for these effects have already been derived in GR and analysed for a system of neutron stars [16, 11, 3]. In this thesis, the calculation of the 1PN Lagrangian in EsGB theory is redone with the tidal effects included for the first time. This will give a more complete picture of the behaviour of black holes in EsGB gravity, and give a better framework for future GWs tests for this theory. Furthermore, relative importance of the new tidal effects is analysed through the binding energy of the black hole binary.

The thesis is built up in the following way: in Sections 2, 3, 4 and 5 the theoretical background of the thesis is discussed, starting of from normal GR. First we talk about gravitational waves and how they generate from GR, why we mainly expect them to be generated by binary systems of black holes or neutron stars, and we discuss the stages of such a binary system. In Section 3 we explain what arguments there are for modifying GR with a quadratic gravity theory, specifically EsGB gravity, and we show what EsGB looks like. In Section 5, we argue for the inclusion of tidal effects in the EsGB action, and show how it is derived. In Section 5, we briefly explain the main tool to work with field equations in GR: the Post-Newtonian approximation. In Section 6, we finally calculate the tidal contribution to the 1PN two body Lagrangian. We also correct a mistake in the original two body Lagrangian without tidal effects. The section is concluded with the calculation of the relative acceleration and the binding energy from the Lagrangian. In Section 7, we analyse the tidal contribution to the relative energy and investigate its relative importance for a variety of different situations. Finally, we will conclude the thesis in Section 8 and discuss possible limitations of the results derived in this thesis. All the way in the end there is an Appendix A in which the lengthier full parts of the calculations of the relative acceleration and binding energy without tidal effects are placed.

2 Gravitational Waves

We'll start by discussing gravitational waves (GWs). You can think of GWs like perturbations in spacetime. Just like moving your hands around in water creates regular waves, so does moving masses around in space create GWs. The created waves then propagate through spacetime, and periodically squeeze and stretch space. They were proposed for the first time by Poincaré in 1905 and later predicted by Einstein's theory of General Relativity in 1916. In the next part we will show how GWs follow from linearising Einstein field equations in GR. They were finally experimentally observed in 2015 by the LIGO gravitational wave detectors [1]. The main reason that it took almost a century to finally observe GWs after Einstein's prediction, is that their effect is very small, even in very extreme events. The 4 km long interferometers at LIGO had to observe a distance squeeze/stretch of 1/1000 the width of a proton. The LIGO experiment needed multiple upgrades before it was able to reach this accuracy.

Now that GWs have finally been observed, a new window has opened to look out into the universe, and even gives us the possibility to observe events such as black hole mergers, that were previously impossible to see with more traditional observation techniques. This last feature is especially important for the work in this thesis, as it makes it possible to test beyond-GR theories in the extreme field regime.

2.1 GWs from Linearizing GR

It is fairly straightforward to see how GWs are predicted by GR. See [19] and [12] for a more extensive discussion of this subject. We start by quickly refreshing

GR. The gravitational action looks like $S = S_E + S_M$, with

$$S_E = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R \tag{1}$$

the Einstein-Hilbert action and S_M the matter action. R is the Ricci scalar. Varying the action w.r.t. the metric $g_{\mu\nu}$ gives the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2}$$

To see how GWs follow from this framework, we will expand the Einstein field equations around flat spacetime. Concretely, this means that we expand the metric around the Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3}$$

where we assume $h_{\mu\nu} \ll 1$. It is then possible to expand the Einstein field equations to linear order in $h_{\mu\nu}$, while using the available gauge freedoms that we already have in GR to get the wave equation

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{4}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} h$ and $h = \eta^{\mu\nu} h_{\mu\nu}$. To obtain a fairly easy solution we transform to the transverse-traceless (TT) gauge with

$$h^{0\mu} = 0, \qquad h^i_i = 0, \qquad \partial^j h_{ij} = 0. \tag{5}$$

To get a solution

$$h^{TT}_{ab}(t, z) = \begin{pmatrix} h_+ & h_\times \\ h_\times & h_+ \end{pmatrix}_{ab} \cos[\omega(t - z/c)] \tag{6}$$

So, we can separate any GW into two polarisations, the ’plus’ and ’cross’ polarisation. In the TT-gauge, test masses initially at rest will remain at rest, so they will not be moved by GWs. However, the proper distance between two test masses *does* change. And, since the time it takes light to traverse from one point to another is determined by the proper distance between those two points, GWs affect the travel time of light between points. This is the essential effect of GWs: they stretch and squeeze spacetime itself, and so they manipulate distances between objects without moving those objects themselves. In fig. 1 it is illustrated how the two polarisations would affect a ring of test masses when a GW passes through. The changing of proper distance is also the property that makes GWs observable by very accurate interferometers. The changing proper distance that light need to travel along the arms of the interferometer causes interference, and the change in intensity can be measured.

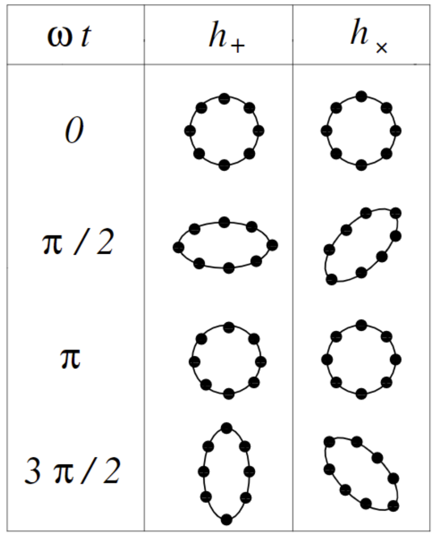

Figure 1: Effect of the 'plus' and 'cross' polarisation parts of a GW on a ring of test masses. Retrieved from [19]

2.2 Quadrupolar Radiation

To actually determine some of the features of GW radiation, we would need to approximate the wave equation (4). This is done in detail in [19], so rather, we will qualitatively analyse the features of this solution, which will show a key difference with electro-magnetic radiation generation

The way to approximate the wave equation is to expand the stress-energy tensor in its multipole moments. This is a low velocity expansion $v << c$, and is a Post-Newtonian approximation, that we will discuss in more detail in Section 5. A similar procedure of a multipole expansion is done in electromagnetism to show that the leading order contribution to EM radiation is from the dipole moment. So, let us analyse what the leading order moments are in GR, based on the analysis done in [24]. First off, let us look at the monopole moment, which is defined as

$$m = \int \rho d^3x, \tag{7}$$

with ρ the charge density of our theory. In EM this is electrical charge, in GR this is the mass-energy distribution, as described by the energy-momentum tensor. In both GR and EM the total charge is conserved over time (to lowest order), since the mass/energy of a system is conserved. This moment therefore does not change in time, and does not contribute to GW generation. The next lowest order is the dipole moment, defined as

$$d_i = \int \rho x_i d^3x. \tag{8}$$

In EM, this moment is not conserved, and is therefore the leading order contributor to EM radiation generation. However, in GR, if we take the derivative of this moment, and use the mass-energy conservation, we see

$$\dot{d}_i = \int \rho v_i d^3x, \tag{9}$$

which is simply the total momentum in the system, also a conserved quantity (at lowest order). So the dipole moment can also not contribute to any GW generation. To find the lowest order contributor, we have to turn to the quadrupole moment, defined as

$$Q_{jk} = \int \rho x_j x_k d^3x. \tag{10}$$

At last, this is not a conserved quantity, and it will contribute to generating GWs. Similar to EM, the amplitude of the radiation is dependent on the second time-derivative of the Q_{jk}, and it falls of as $1/r$. The lowest order solution to h_{jk} will therefore turn out to be

$$h_{jk} = \frac{2G}{rc^4}\ddot{Q}_{jk}. \tag{11}$$

This is the famous quadrupole formula, first derived by Einstein, and it is the lowest order description of how GWs are generated. This also tells us what the most likely source of GWs will be. We need a system with a changing quadrupole moment. So, we do not expect gravitational waves to be generated by a spinning black hole on its own, since its movement is axisymmetric. The most likely source therefore seems to be binary systems, which have large, changing quadrupole moments. Indeed, every GW observation made thus far, has been of binaries of neutron stars and black holes. Other possible sources are rotating stars with bumps (not spherically symmetric), or possibly inflation, if the expansion was not symmetric in all directions. For the purposes of this thesis, we will solely focus on binary black holes systems.

2.3 Black Hole Merger Phases

Even though any movement of mass that breaks axisymmetry generates GWs, practically speaking we are restricted to observing either neutron stars or black holes, as other object are simply not heavy enough to generate GWs that are observable with our current detectors. For this thesis, we will focus on the case of a binary black hole system. A binary black hole system knows three main phases of GW generation (see 2). In the first phase, the black holes are already orbiting each other, but they are still far away. The GWs that are generated

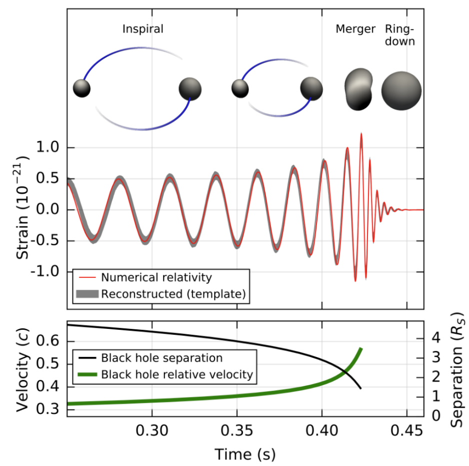

Figure 2: Phases of GW generation for a binary black hole system. Underneath is the associated waveform with each phase, and all the way at the bottom the separation and relative velocity. Retrieved from [1]

are therefore not big and don't carry a lot of energy yet. So, the orbit decays very slowly and the GWs generated are very constant. In the second phase, the black holes get close enough to each other that the forces become very strong, and the GWs that are generated carry a lot of energy. Consequently, the orbit decays rapidly until the black holes finally merge. At the merger, the GWs peak in magnitude, and can radiate away multiple solar masses in energy in seconds. Finally, after the two black holes have merged, the final inhomogeneities are radiated away and leaves behind a new black hole of a combined mass slightly less than the sum of the starting masses.

3 Einstein-scalar-Gauss-Bonnet Gravity

In this section we will briefly discuss why we are looking for theories beyond GR, and why quadratic gravity theories are particularly appealing. See [23] for a more general overview of quadratic gravity theories, and [4] for a discussion about a handful of specific modified gravity theories and their testability with GWs. Afterwards we will quickly argue why, ultimately, we decided to work Einstein-scalar-Gauss-Bonnet gravity, see [21] for a full discussion of how black holes fit in EsGB theory.

Despite the huge success of General Relativity in describing most of our observable universe, it is not without issues. The main problem that concerns us, is its incompatibility with quantum physics, which becomes especially important at higher energies. This is especially interesting to us, as GWs are a unique tool with which high energy situations can be probed. We are therefore at a very exciting moment for modified gravity theories, being at start of an era in which we will gain unprecedented amounts of data in the strong field regime. Modified gravity theories will be rigorously tested against GW data, and they will either be disproven or reinforced. This makes it an excellent time to build up the framework of modified gravity theories, so they are readily usable to interpret the new influx of data.

The Lovelock theorem [17] states that if we're building our theory in the standard way from just the metric, and its first and second order derivatives, the requirements of a theory of gravity will always cause us to end up with the Einstein field equations. Therefore, if we want to modify GR, we have a couple of options. Two of the most natural extensions to GR are the incorporation of an additional scalar degree of freedom in scalar-tensor theories, and the inclusion of quadratic curvature terms in the action in quadratic gravity theories. Scalar-tensor theories are constructed by including a scalar field that is non-minimally coupled to the Ricci scalar. They are fairly simple extensions and are therefore well studied. Quadratic gravity theories on the other hand, include a quadratic curvature term to the action, and often also include a coupling to a scalar field. The idea behind them is that the standard Einstein-Hilbert action

$$S_{EH} = \int \frac{d^4x}{16\pi} \sqrt{-g} R \tag{12}$$

is only the lowest order expansion in curvature of a more complete theory. With this in mind, the logical next step is to include terms that are quadratic in curvature. These theories are motivated from string theory [13], and can solve some of the issues at high energy, and make the theory renormalizable [26]. However, they bring with them their own problems, as most quadratic gravity theories are susceptible to Ostrogradsky instabilities, which appear when the Hamiltonian of the theory is unbounded from below [20].

This is one of the reasons that makes EsGB gravity particularly appealing, since the particular combination of its quadratic term is free of any Ostrogradsky instabilities. The term has to be coupled to a scalar field, otherwise the the quadratic curvature terms would not actually contribute to the equations of motion. The action for EsGB gravity is therefore

$$S = S_{EH} + S_{GB} + S_m, \tag{13}$$

with

$$S_{GB} = \int \frac{d^4x\sqrt{-g}}{16\pi} \left[-2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \alpha f(\phi) R_{GB}^2 \right], \tag{14}$$

where $R_{GB}^2 \equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$ is the Gauss-Bonnet scalar and $2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic term of the scalar field. This particular combination makes sure that the derivatives in the equations of motion are at most of second order, giving one of the most simple high-energy modifications to GR. Most importantly for this thesis, EsGB gravity allows black holes to violate the no-hair theorem, meaning black holes can actually scalarize in this theory, the tidal effects of which will be the main topic of this thesis.

4 Tidal Effects

The tidal force is a gravitational effect that stretches out a body. Since gravity scales as r^{-2}, the result is that the far side of the body experiences less gravitational pull than the the near side, which stretches the body out. The typical example of this is the effect that the Moon has on the Earth. The side of the Earth that is close to the Moon undergoes a greater pull towards the Moon, causing the water to rise (towards the Moon). On the other side, the Earth undergoes a lesser pull, so the water also rises (away from the moon). This is what causes tides.

There are two ways for us to derive the tidal action that we want to add to our theory. The first one, as done by [16, 11] and followed in [3], starts from a Newtonian system, and generalizes to a relativistic action. Here, it is shown that the Newtonian action looks like

$$L_T = \frac{1}{2\lambda^{(s)}\omega_1^2}[\dot{\mathcal{Q}}^i_{(s)}\dot{\mathcal{Q}}^{(s)}_i - \omega_1^2 Q^i_{(s)} Q^{(s)}_i] - \mathcal{E}^{(s)}_i Q^i_{(s)}, \tag{15}$$

with $Q^i_{(s)}$ the internal dynamical dipole, $\mathcal{E}^{(s)}_i$ the external tidal field and $\lambda^{(s)}$ the tidal deformability. Taking the equation of motion of this action

$$\ddot{Q}^i_{(s)} + \omega_1 Q^i_{(s)} = -\lambda_{(s)}\omega_1^2\mathcal{E}^i_{(s)} \tag{16}$$

gives rise to the static solution $Q^i_{(s)} = -\lambda_{(s)}\mathcal{E}^i_{(s)}$. We use this as a solution in the adiabatic limit, and substitute the solution back in (15) to get the Newtonian tidal correction

$$L_T = \frac{1}{2\lambda_{(s)}}\mathcal{E}^{(s)}_i\mathcal{E}^i_{(s)}. \tag{17}$$

To get the relativistic action for the tidal effects generated by the scalar field, we now use that the external field is simply $\mathcal{E}^{(s)}_\mu = \partial_\mu\phi$ the gradient of the scalar field. And so the dipole moment induced by the scalar field is $Q^{(s)}_\mu = -\lambda_{(s)}\mathcal{E}^{(s)}_\mu$. Reintroducing this in (17) finally gives us the tidal action in GR

$$S_{fs} = -\sum_A \frac{1}{2}\lambda^{(s)}_A \int ds_A c(g^{\mu\nu})(\partial_\mu\phi)_A(\partial_\nu\phi)_A. \tag{18}$$

The other way of obtaining this action, as done by [7], is by taking the most general action for the scalar field dependent mass action up to second order derivatives in ϕ or $g_{\mu\nu}$. By then making clever use of the gauge symmetries of the Lagrangian, it can be reduced to (18).

What is interesting about this action is that it is generated by the excitation of a dipole moment, where in GR the dipole moment of an object is zero in its own center of mass frame, and the lowest order moment contributing to GW radiation is quadrupolar. With this action however, we have a nonzero dipole moment, and therefore dipolar GW radiation.

This is the action that we will be using to modify our GB gravity further. It is important to note that these finite size effects appear in explicit contributions to the action, and do not necessarily implicitly modify the mass action of the black holes. We therefore only have to introduce them when building the 1PN test particle Lagrangian in Section 6.3, and not at the starting action. For consistency's sake, we have included the action to the starting action as well, and show that its contribution vanishes when solving the equations of motion.

5 Post-Newtonian Expansion

Finally, we need a tool to work with our action and resulting equations of motion. As is known, the Einstein field equations are highly nonlinear, and therefore difficult to work with. Our modified field equations will be no more easy to solve. We therefore want to calculate an approximate solution to the field equations, and for this we use the Post-Newtonian (PN) expansion. The PN-expansion is a slow-motion, weak-field expansion. So, concretely, we are expanding in terms of $R_S/r = 2Gm/c^2r$ and v/c, with $R_S/r \sim (v/c)^2$ and R_S is the Schwarzschild radius.

In fig. 3 you can get an idea of the viability of the PN expansion depending on the parameters of your system. So, it is not useful for systems in which either v is close to the speed of light or r is close to the Schwarzschild radius. Furthermore, it is also not really useful for systems in which v is very small, as a static system will not generate significant GWs. Conversely, a system with a very small ratio between its Schwarzschild radius and the radius of the system is not governed by gravity in its dynamics, and therefore also not an interesting source for GWs. This leaves a sizeable middle area of systems that are interesting sources of GWs, but can still be effectively approximated with a slow-motion, weak-field expansion. In the next section we will extensively be using this method to obtain approximate solutions to our complicated system. In the end, we formulate a Lagrangian to first order PN (1PN), which we can then use to analyse our system.

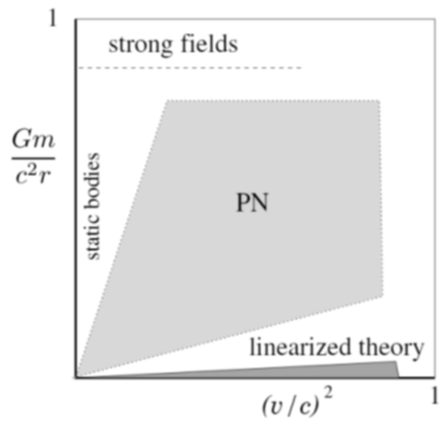

Figure 3: Schematic depiction of viability of the PN expansion in terms of the velocity and the radius of the system compared to the speed of light and the Schwarzschild radius respectively. Retrieved from [19]

6 Calculation of the Two-Body Lagrangian

In the previous sections we argued what the framework of our theory looks like, and we obtained the action for EsGB in (13). We picked the mass action to be that of the point particle, denoted by S_{pp}. Furthermore, we already included the action with the finite size effects to show its effects drop out if already included to the original action.

$$S = S_{EH} + S_{GB} + S_{pp} + S_{fs}, \tag{19}$$

$$\text{with}\quad S_{EH} = \int \frac{d^4x}{16\pi}\sqrt{-g}R \tag{20}$$

$$S_{GB} = \int \frac{d^4x\sqrt{-g}}{16\pi}\left[-2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \alpha f(\phi)R^2_{GB}\right] \tag{21}$$

$$S_{pp} = -\sum_A \int m_A(\phi)ds_A \tag{22}$$

$$S_{fs} = -\frac{1}{2}\sum_A \lambda_A^{(s)} \int (g^{\mu\nu})_A(\partial_\mu\phi)_A(\partial_\nu\phi)_A ds_A, \tag{23}$$

where $g \equiv \det g_{\mu\nu}$, R is the Ricci scalar, α the coupling constant of dimensions length squared, $f(\phi)$ a dimensionless function defining the theory, $ds_A = \sqrt{-g_{\alpha\beta}dx^\alpha_A dx^\beta_A} = \sqrt{-g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A}dt$, $\lambda_A^{(s)}$ is the tidal deformability. Furthermore, the subscript A on $m_A(\phi)$, $(\partial_\mu\phi)_A$ and $(g^{\mu\nu})_A$ indicate that the function needs to be evaluated at the position of object A. In practice this means we will introduce a delta function that localizes the function on the object when deriving the equations of motion. Finally, R^2_{GB} is the Gauss-Bonnet scalar, which is defined as $R_{GB}^2 \equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$.

The goal of this section is to obtain a 1PN, two-body Lagrangian, including tidal effects. We will do this as follows: first, we will derive the equations of motion for this action. Second, we will expand the metric in PN, and solve the equations of motion approximately. Third, we will construct a point particle Lagrangian that includes the tidal effects of the scalar field. After symmetrizing this Lagrangian, we will have a 1PN, two-body Lagrangian that we can work with. A calculation of this Lagrangian has already been done without including tidal effects in [14]. However, I have redone the calculation, and corrected a mistake when constructing the point particle Lagrangian. In this thesis I will therefore follow along with their calculation and show in detail how to include the new tidal effects. When constructing the point particle Lagrangian, I will show the calculation in full to illustrate the mistake. A similar calculation in Einstein-Maxwell-dilaton theory has also been done in [15].

6.1 Varying the Action

Starting with the aforementioned action in (19), we can now derive the equations of motion by varying w.r.t. the metric $g_{\mu\nu}$ and the scalar field ϕ. This gives the following result

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - 4\alpha\left(P_{\mu\alpha\nu\beta} - \frac{1}{2}g_{\mu\nu}P_{\alpha\beta}\right)\nabla^\alpha\nabla^\beta f(\phi) + 8\pi\left(T_{\mu\nu}^{pp} - \frac{1}{2}g_{\mu\nu}T^{pp}\right) + 8\pi\left(T_{\mu\nu}^{fs} - \frac{1}{2}g_{\mu\nu}T^{fs}\right) \tag{24}$$

$$\Box\phi = -\frac{1}{4}\alpha f'(\phi)R_{GB}^2 - 4\pi\Delta S_{pp} - 4\pi\Delta S_{fs}, \tag{25}$$

where

$$T_{\mu\nu}^{pp} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{pp}}{\delta g^{\mu\nu}}, \qquad T^{pp} \equiv g^{\mu\nu}T_{\mu\nu}^{pp} \qquad \Delta S_{pp} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S_{pp}}{\delta\phi} \tag{26}$$

$$T_{\mu\nu}^{fs} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{fs}}{\delta g^{\mu\nu}}, \qquad T^{fs} \equiv g^{\mu\nu}T_{\mu\nu}^{fs} \qquad \Delta S_{fs} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S_{fs}}{\delta\phi} \tag{27}$$

The $T_{\mu\nu}^{pp,fs}$ are the respective distributional stress-energy tensors. Furthermore,

$$P^{\mu\nu}{}_{\rho\sigma} \equiv R^{\mu\nu}{}_{\rho\sigma} - 2\delta^\mu_{[\rho}R^\nu_{\sigma]} + 2\delta^\nu_{[\rho}R^\mu_{\sigma]} + \delta^\mu_{[\rho}\delta^\nu_{\sigma]}R, \tag{28}$$

such that $R_{GB}^2 = R^{\mu\nu\rho\sigma}P_{\mu\nu\rho\sigma}$. The point particle parts are calculated in [14]. We obtain them by introducing the delta function that indicates the localisation at object A.

$$T^{pp}_{\mu\nu} = \sum_A \int ds_A m_A(\phi) \frac{\delta^{(4)}(x - x_A(s_A))}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{dx^\alpha_A}{ds_A} \frac{dx^\beta_A}{ds_A} \tag{29}$$

$$= \sum_A m_A(\phi) \frac{\delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t))}{\sqrt{g g_{\alpha\beta} \dot{x}^\alpha_A \dot{x}^\beta_A}} \dot{x}^A_\mu \dot{x}^A_\nu, \tag{30}$$

$$\text{and}\ \ \Delta S_{pp} = -4\pi \sum_A \int ds_A \frac{dm_A}{d\phi} \frac{\delta^{(4)}(x - x_A(s_A))}{\sqrt{-g}} \tag{31}$$

$$= -4\pi \sum_A \sqrt{\frac{g_{\alpha\beta} \dot{x}^\alpha_A \dot{x}^\beta_A}{g}} \frac{dm_A}{d\phi} \delta^{(3)}(x - x_A(s_A)) \tag{32}$$

Here we denote $\dot{x}^A_\mu = g_{\mu\alpha} \dot{x}^\alpha_A = g_{\mu\alpha} \frac{dx^\alpha_A}{dt}$. We will now calculate the contribution of the finite size part. For this, we will vary the finite size action. First, we vary w.r.t. the metric.

$$\delta S_{fs} = -\frac{1}{2} \sum_A \lambda^{(s)}_A \Big[\int \delta ds_A (g^{\mu\nu})_A (\partial_\mu \phi)_A (\partial_\nu \phi)_A + \int ds_A (\delta g^{\mu\nu})_A (\partial_\mu \phi)_A (\partial_\nu \phi)_A \Big], \tag{33}$$

$$\text{with}\ \ \delta ds_A = \delta \sqrt{-g_{\alpha\beta} \dot{x}^\alpha_A \dot{x}^\beta_A} dt = \frac{-\delta g_{\alpha\beta} \dot{x}^\alpha_A \dot{x}^\beta_A}{2\sqrt{-g_{\gamma\delta} \dot{x}^\gamma_A \dot{x}^\delta_A}} dt \tag{34}$$

Now, using the identity $\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$, we get

$$\delta S_{fs} = -\frac{1}{2} \sum_A \lambda^{(s)}_A \int dt \Big[\frac{\delta g^{\mu\nu} \dot{x}^A_\mu \dot{x}^A_\nu}{2\sqrt{-g_{\gamma\delta} \dot{x}^\gamma_A \dot{x}^\delta_A}} (g^{\alpha\beta})_A (\partial_\alpha \phi)_A (\partial_\beta \phi)_A + \sqrt{-g_{\alpha\beta} \dot{x}^\alpha_A \dot{x}^\beta_A} (\delta g^{\mu\nu})_A (\partial_\mu \phi)_A (\partial_\nu \phi)_A \Big]. \tag{35}$$

We then introduce the delta function to localise the integrand. The resulting stress-energy tensor becomes

$$\text{So,}\ \ T^{fs}_{\mu\nu} = \sum_A \lambda^{(s)}_A \int dt \Big[\frac{\dot{x}^A_\mu \dot{x}^A_\nu}{2\sqrt{g g_{\gamma\delta} \dot{x}^\gamma_A \dot{x}^\delta_A}} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \sqrt{\frac{g_{\alpha\beta} \dot{x}^\alpha_A \dot{x}^\beta_A}{g}} \partial_\mu \phi \partial_\nu \phi \Big] \delta^{(4)}(x - x_A(s_A)). \tag{36}$$

Evaluating the integral then gives

$$T^{fs}_{\mu\nu} = \sum_A \lambda^{(s)}_A \left[\frac{\dot{x}^A_\mu \dot{x}^A_\nu}{2\sqrt{g g_{\gamma\delta}\dot{x}^\gamma_A \dot{x}^\delta_A}} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \sqrt{\frac{g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A}{g}}\partial_\mu\phi\partial_\nu\phi \right]\delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \tag{37}$$

Now the contribution w.r.t. the scalar field

$$\delta S_{fs} = -\sum_A \lambda^{(s)}_A \int dt \sqrt{-g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A}(g^{\mu\nu})_A(\partial_\mu\phi)_A(\partial_\nu\delta\phi)_A. \tag{38}$$

$$= \sum_A \lambda^{(s)}_A \int dt \Big[\sqrt{-g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A}(g^{\mu\nu})_A(\partial_\mu\partial_\nu\phi)_A \tag{39}$$

$$+ \partial_\nu\Big(\sqrt{-g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A}(g^{\mu\nu})_A\Big)(\partial_\mu\phi)_A\Big]\delta\phi \tag{40}$$

So, introducing the delta function

$$\Delta S_{fs} = \sum_A \lambda^{(s)}_A \Bigg[\sqrt{\frac{g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A}{g}} g^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{\partial_\nu\Big(\sqrt{-g_{\alpha\beta}\dot{x}^\alpha_A\dot{x}^\beta_A} g^{\mu\nu}\Big)}{\sqrt{-g}}\partial_\mu\phi \Bigg]\delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \tag{41}$$

6.2 1PN Solution to the Equations of Motion

Now that we calculated all the terms of (24) and (25), we want to obtain a 1PN solution. For this, we expand the metric around Minkowski spacetime, and the scalar field around it's value at spatial infinity. We insert these approximations back in the equations of motion to get a 1PN approximation of them. We will then find a 1PN solution to these using the Green's function.

6.2.1 1PN Equations of Motion

First, we expand the metric following the procedure in [10]

$$g_{00} = -e^{-2U} + \mathcal{O}(v^6) \tag{42}$$

$$g_{0i} = -4g_i + \mathcal{O}(v^5) \tag{43}$$

$$g_{ij} = \delta_{ij}e^{2U} + \mathcal{O}(v^4), \tag{44}$$

where $U = \mathcal{O}(v^2)$ and $g_i = \mathcal{O}(v^3)$.

We will again focus on the contribution from the finite size action, since the rest has already been done in [14]. We are only looking at the lowest order contribution (which will turn out to be second order in v), so we can essentially assume g_i to be zero for our purposes. This means our metric is diagonal, and its inverse will just be the matrix with the inverse diagonal elements. Furthermore, the scalar field is expanded as

$$\phi = \phi_0 + \delta\phi + \mathcal{O}(v^6), \tag{45}$$

with $\delta\phi = \mathcal{O}(v^2)$. A quick look at (37) shows straight away that its lowest order contribution would be at least of order $\mathcal{O}(v^4)$, since $\partial_\mu\phi\partial_\nu\phi \sim (\delta\phi)^2 = \mathcal{O}(v^4)$. We can therefore wholly ignore the energy-momentum tensor for finite size effects. So, (24) will at lowest order be unaffected by finite-size effects. However, (41) does have contribution of order $\mathcal{O}(v^2)$, so there will be a tidal term in (25). We now approximate (41), and we start with the first term.

$$g^{\mu\nu}\partial_\mu\partial\nu\phi = \Big(- e^{2U}\partial_0^2 + e^{-2U}\nabla^2\Big)\phi = \Box_\eta\delta\phi + \mathcal{O}(v^4), \tag{46}$$

where $\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$, with $\eta^{\mu\nu}$ the Minkowski metric. Since $\delta\phi$ is already second order, we only need the zeroth order approximation from its prefactor $\sqrt{(g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta)/g}$. For this we use that

$$g = -e^{4U} = -1 - 4U + \mathcal{O}(v^4) \tag{47}$$

$$g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta = -e^{-2U}(\dot{x}^0)^2 + e^{2U}\mathbf{v}_A^2 = -1 + 2U + \mathbf{v}_A^2 + \mathcal{O}(v^4) \tag{48}$$

So,

$$\sqrt{\frac{g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta}{g}} \approx 1 + \mathcal{O}(v^2) \tag{49}$$

Similarly, for the second term we also see $\partial_\mu\phi \sim \delta\phi = \mathcal{O}(v^2)$, so we also only need the zeroth order of its prefactor.

$$\begin{aligned} \frac{\partial_\nu\Big(\sqrt{-g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta}g^{\mu\nu}\Big)}{\sqrt{-g}} &= g^{\mu\nu}\partial_\nu\sqrt{-g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta} \\ &\quad + \sqrt{-g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta}\partial_\nu g^{\mu\nu} + \mathcal{O}(v^4) \end{aligned} \tag{50}$$

$$\begin{aligned} &= \frac{g^{\mu\nu}}{2\sqrt{-g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta}}\left(-2\partial_\nu U e^{-2U} - \mathbf{v}_A\cdot\partial_\nu\mathbf{v}_A\right) \\ &\quad + \sqrt{-g_{\alpha\beta}\dot{x}_A^\alpha\dot{x}_A^\beta}(\partial_0 g^{\mu 0} + \partial_i g^{\mu i}) + \mathcal{O}(v^4), \end{aligned} \tag{51}$$

which is $\sim \mathcal{O}(v^2)$, so we can disregard this entire term. So, in the end, we can approximate (41) by

$$\Delta S_{fs} = \sum_A \lambda_A^{(s)}\Box_\eta\delta\phi\delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) + \mathcal{O}(v^4) \tag{52}$$

We will add this contribution to the 1PN approximation of (25). This gives a second order differential equation for $\delta\phi$. Furthermore, the 00-term of (24) will give a second order differential equation for U, and the 0i-term for g_i.

$$\begin{aligned}\Box_\eta U = &-4\pi \sum_A m_A^0 \Big[1 + \frac{3}{2}\mathbf{v}_A^2 - U + \alpha_A^0 \delta\phi\Big] \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \\ &+ 4\alpha f'(\phi_0) \left[\Delta\phi\Delta U - (\partial_{ij}\phi)(\partial_{ij}U)\right] + \mathcal{O}(v^6)\end{aligned} \tag{53}$$

$$\Delta g_i = -4\pi \sum_A m_A^0 v_A^i \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) + \mathcal{O}(v^5) \tag{54}$$

$$\begin{aligned}\Box_\eta \delta\phi = 4\pi &\sum_A m_A^0 \alpha_A^0 \Big[1 - \frac{1}{2}\mathbf{v}_A^2 - U + \left(\alpha_A^0 + \frac{\beta_A^0}{\alpha_A^0}\right)\delta\phi\Big] \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \\ &+ 2\alpha f'(\phi_0) \left[(\Delta U)^2 - (\partial_{ij}U)(\partial_{ij}U)\right] \\ &- 4\pi \sum_A \lambda_A^{(s)} \Box_\eta \delta\phi \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) + \mathcal{O}(v^4)\end{aligned} \tag{55}$$

Here, α_A and β_A are defined as

$$\alpha_A(\phi) \equiv \frac{d \ln m_A(\phi)}{d\phi} \tag{56}$$

$$\beta_A(\phi) \equiv \frac{d\alpha_A(\phi)}{d\phi} \tag{57}$$

Such that,

$$m_A(\phi) = m_A(\phi_0) + \dot{m}_A(\phi_0)(\phi - \phi_0) + \frac{1}{2}\ddot{m}_A(\phi_0)(\phi - \phi_0)^2 + ... \tag{58}$$

$$= m_A^0 \left[1 + \alpha_A^0 \delta\phi + \frac{1}{2}({\alpha_A^0}^2 + \beta_A^0)\delta\phi^2\right] + \mathcal{O}(v^6) \tag{59}$$

6.2.2 1PN Solution

We will solve these equation iteratively per order. First for the lowest order in v, then for the lowest order in $\lambda^{(s)}$, and finally for the second-lowest order in v. We will do this explicitly for U for the lowest order in v, and for the lowest order finite size contribution to $\delta\phi$, which will turn out to vanish.

So, for the lowest order in v the differential equations (53), (54) and (55) are

$$\Box_\eta U^{(0)} = -4\pi \sum_A m_A^0 \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \tag{60}$$

$$\Delta g_i^{(0)} = -4\pi \sum_A m_A^0 v_A^i \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \tag{61}$$

$$\Box_\eta \delta\phi^{(0)} = 4\pi \sum_A m_A^0 \alpha_A^0 \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \tag{62}$$

We solve these with standard methods by using the relativistic Green's function (see [8, 9, 15]), which is defined through

$$\Box_\eta G(x,x') \equiv -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{63}$$

This function can then be used to solve differential equation of the form

$$\Box f(x) = g(x) \tag{64}$$

$$\Rightarrow f(x) = (G*g)(x) \equiv -\frac{1}{4\pi}\int d^4x' G(x,x')g(x') \tag{65}$$

The function satisfying this is

$$G(x,x') = \frac{\delta(t-t')}{|\mathbf{x}-\mathbf{x}'|} + \frac{|\mathbf{x}-\mathbf{x}'|}{2}\partial_t^2\delta(t-t') + \dots \tag{66}$$

Applying this to the first order equation of U, we see that

$$U^{(0)} = \sum_A \int d^4x' \left(\frac{\delta(t-t')}{|\mathbf{x}-\mathbf{x}'|} + \frac{|\mathbf{x}-\mathbf{x}'|}{2}\partial_t^2\delta(t-t')\right) m_A^0 \delta^{(3)}(\mathbf{x}'-\mathbf{x}_A(t')) \tag{67}$$

$$= \sum_A m_A^0 \int dt' \left(\frac{\delta(t-t')}{|\mathbf{x}-\mathbf{x}_A(t')|} + \frac{|\mathbf{x}-\mathbf{x}_A(t')|}{2}\partial_t^2\delta(t-t')\right) \tag{68}$$

$$= \sum_A m_A^0 \left(\frac{1}{|\mathbf{x}-\mathbf{x}_A(t)|} + \partial_t^2 \int dt' \frac{|\mathbf{x}-\mathbf{x}_A(t')|}{2}\delta(t-t')\right) \tag{69}$$

$$= \sum_A m_A^0 \left(\frac{1}{|\mathbf{x}-\mathbf{x}_A(t)|} + \partial_t^2 \frac{|\mathbf{x}-\mathbf{x}_A(t)|}{2}\right) \tag{70}$$

We now denote $\mathbf{r}_A = \mathbf{x}_A(t) - \mathbf{x}$ and similarly $r_A = |\mathbf{r}_A|$. Using that

$$\partial_t r_A = \mathbf{n}_A \cdot \mathbf{v}_A, \tag{71}$$

$$\text{with } \mathbf{n}_A \equiv \frac{\mathbf{r}_A}{r_A}, \tag{72}$$

we get that

$$\partial_t^2 r_A = \partial_t(\mathbf{n}_A \cdot \mathbf{v}_A) = \frac{r_A(\mathbf{r}_A\cdot\mathbf{a}_A + \mathbf{v}_A^2) - (\mathbf{r}_A\cdot\mathbf{v}_A)(\mathbf{n}_A\cdot\mathbf{v}_A)}{r_A^2} \tag{73}$$

$$= \frac{1}{r_A}\left[\mathbf{v}_A^2 + \mathbf{r}_A\cdot\mathbf{a}_A - (\mathbf{n}_A\cdot\mathbf{v}_A)^2\right] \tag{74}$$

So,

$$\frac{1}{r_A} + \partial_t^2\frac{r_A}{2} = \frac{1}{r_A}\left[1 + \frac{1}{2}\mathbf{v}_A^2 - \frac{1}{2}(\mathbf{n}_A\cdot\mathbf{v}_A)^2\right] + \frac{1}{2}\mathbf{n}_A\cdot\mathbf{a}_A \equiv \frac{1}{\rho_A} \tag{75}$$

Substituting this back in (70), we finally get that

$$U^{(0)} = \sum_A \frac{m_A^0}{\rho_A} \tag{76}$$

Following the same procedure for $\delta\phi$ and g_i, we get

$$\delta\phi^{(0)} = -\sum_A \frac{m_A^0 \alpha_A^0}{\rho_A} \tag{77}$$

$$g_i^{(0)} = \sum_A \frac{m_A^0 v_A^i}{r_A}, \tag{78}$$

where the differential equation of g^i is already of higher order, so we only have to use $G(x, x') = \frac{\delta(t-t')}{|\mathbf{x}-\mathbf{x}'|}$.

The next step is to solve for the lowest order in $\lambda^{(s)}$. We do this by now adding the lowest order terms of $\lambda^{(s)}$ and substituting the lowest order solutions we just obtained into the equations. We then solve the equations again using the same method. Only (55) has a contribution from $\lambda^{(s)}$, so we only have to solve that equation. We substitute $\Box_\eta \delta\phi^{(0)} = 4\pi \sum_A m_A^0 \alpha_A^0 \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t))$ to get the differential equation

$$\begin{aligned}
\Box_\eta \delta\phi^{(0,\lambda)} &= 4\pi \sum_A m_A^0 \alpha_A^0 \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \\
&\qquad - 4\pi \sum_A \lambda_A^{(s)} \Box_\eta \delta\phi^{(0)} \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \qquad (79) \\
&= 4\pi \sum_A m_A^0 \alpha_A^0 \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \\
&\qquad - 16\pi^2 \sum_{A \neq B} \lambda_A^{(s)} m_B^0 \alpha_B^0 \delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t)) \delta^{(3)}(\mathbf{x} - \mathbf{x}_B(t)). \qquad (80)
\end{aligned}$$

Here, we have excluded the diagonal terms of the sum to exclude the self-energy of the system. It is now clear that the extra term will vanish when we convolute it with the Green's function, as the product of delta functions will always be zero when $x_A(t) \neq x_B(t)$. Since the two black holes can never be in exactly the same place, this will always be the case.

Finally, the equations need to be solved to second-lowest order in v, using the same method as before. As a result we get the solutions

$$U(x) = \sum_A \frac{m_A^0}{\rho_A}\left[1 + \frac{3}{2}\mathbf{v}_A^2 - \sum_{B\neq A}\frac{m_B^0}{r_B}\left(1+\alpha_A^0\alpha_B^0\right)\right] - 4\alpha f'(\phi_0)\sum_{A,B} m_A^0 m_B^0 \alpha_A^0 h_{AB}(\mathbf{x}) \tag{81}$$

$$g_i(x) = \sum_A \frac{m_A^0 v_A^i}{r_A} \tag{82}$$

$$\phi(x) = \phi_0 - \sum_A \frac{m_A^0\alpha_A^0}{\rho_A}\left[1 - \frac{1}{2}\mathbf{v}_A^2 - \sum_{B\neq A}\frac{m_B^0}{r_B}\left(1+\alpha_A^0\alpha_B^0 + \frac{\beta_A^0\alpha_B^0}{\alpha_A^0}\right)\right] + 2\alpha f'(\phi_0)\sum_{A,B} m_A^0 m_B^0 h_{AB}(\mathbf{x}), \tag{83}$$

where h_{12} is the solution to

$$\Delta h_{12} = \left(\frac{\partial^2}{\partial y_1^i \partial y_1^i}\frac{\partial^2}{\partial y_2^j \partial y_2^j} - \frac{\partial^2}{\partial y_1^i \partial y_2^i}\frac{\partial^2}{\partial y_1^j \partial y_2^j}\right)\frac{1}{|\mathbf{x}-\mathbf{y}_1||\mathbf{x}-\mathbf{y}_2|} \tag{84}$$

and $h_{11}(\mathbf{x}) = \frac{1}{2|\mathbf{x}-\mathbf{y}_1|^4}$.

6.3 Two Body Lagrangian with the Tidal Effects

The last thing we now need to do is build a Lagrangian from this. Note that this part is also where a mistake in [14] occurs, so we will show the calculation in full for both the entire Lagrangian, and not just the finite size effects. We first write the Lagrangian of body A as a test particle in the fields of body B. Concretely, this means we assume $m_A^0 = 0$ and $\lambda_A^{(s)} = 0$. Furthermore we write $r_A = r_B = |\mathbf{x}_A - \mathbf{x}_B| \equiv r$ (and consequently $\mathbf{n}_A = -\mathbf{n}_B = (\mathbf{r}_A - \mathbf{r}_B)/r \equiv \mathbf{n}$). The crucial part is that we append the normal test particle Lagrangian with a term that takes the tidal interaction into account. So, we also add the finite size action to our Lagrangian construction.

$$L_A = -m_A(\phi)\frac{ds_A}{dt} + \frac{dS_{fs}}{dt} \tag{85}$$

We first use that

$$\frac{d}{dt}(\mathbf{n}\cdot\mathbf{v}_A) = \frac{1}{r}\left[\mathbf{v}\cdot\mathbf{v}_A - (\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v})\right] + \mathbf{n}\cdot\mathbf{a}_A \tag{86}$$

$$\text{So,}\quad \frac{1}{\rho_A} \equiv \frac{1}{r_A}\left[1 + \frac{1}{2}\mathbf{v}_A^2 - \frac{1}{2}(\mathbf{n}_A\cdot\mathbf{v}_A)^2\right] + \frac{1}{2}\mathbf{n}_A\cdot\mathbf{a}_A \tag{87}$$

$$= \frac{1}{r}\left[1 + \frac{1}{2}\mathbf{v}_A\cdot\mathbf{v}_B - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\right] + \frac{1}{2}\frac{d}{dt}(\mathbf{n}\cdot\mathbf{v}_A), \tag{88}$$

where we can integrate the total derivative away to essentially get $1/\rho_A = 1/\rho_B \equiv 1/\rho$. We now work out the first term of (85).

$$
\begin{aligned}
-m_A(\phi)\frac{ds_A}{dt} &= -m_A(\phi)\sqrt{e^{-2U} + 8g_i v_A^i - e^{2U}\mathbf{v}_A^2} && (89)\\
&= -m_A^0\left[1 + \alpha_A^0\delta\phi + \frac{1}{2}({\alpha_A^0}^2 + \beta_A^0)\delta\phi^2\right]\\
&\quad\times\left[1 - U - \frac{1}{2}\mathbf{v}_A^2 + \frac{1}{2}U^2 + 4g_i v_A^i - \frac{3}{2}U\mathbf{v}_A^2 - \frac{1}{8}\mathbf{v}_A^4\right] + \mathcal{O}(v^6) && (90)\\
&= -m_A^0\Big[1 - U - \frac{1}{2}\mathbf{v}_A^2 + \alpha_A^0\delta\phi + \frac{1}{2}U^2 + 4g_i v_A^i - \frac{1}{8}\mathbf{v}_A^4\\
&\quad+\frac{1}{2}({\alpha_A^0}^2 + \beta_A^0)\delta\phi^2 - \frac{3}{2}U\mathbf{v}_A^2 - \alpha_A^0 U\delta\phi - \frac{1}{2}\alpha_A^0\delta\phi\mathbf{v}_A^2\Big] + \mathcal{O}(v^6) && (91)
\end{aligned}
$$

So, using $m_A^0 = 0$ and $\lambda_A^{(s)} = 0$, we get, to order $\mathcal{O}(v^5)$

$$
\begin{aligned}
U &= \frac{m_B^0}{r}\left[1 + \frac{3}{2}\mathbf{v}_B^2 + \frac{1}{2}\mathbf{v}_A\cdot\mathbf{v}_B - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v_A})(\mathbf{n}\cdot\mathbf{v}_B)\right]\\
&\qquad - 4\alpha f'(\phi_0){m_B^0}^2\alpha_B^0 h_{BB}(\mathbf{x}) && (92)\\
\delta\phi &= -\frac{m_B^0\alpha_B^0}{r}\left[1 - \frac{1}{2}\mathbf{v}_B^2 + \frac{1}{2}\mathbf{v}_A\cdot\mathbf{v}_B - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v_A})(\mathbf{n}\cdot\mathbf{v}_B)\right]\\
&\qquad + 2\alpha f'(\phi_0){m_B^0}^2 h_{BB}(\mathbf{x}) && (93)
\end{aligned}
$$

$$
g_i v_A^i = \frac{m_B^0\mathbf{v}_A\cdot\mathbf{v}_B}{r} \qquad U^2 = \frac{{m_B^0}^2}{r^2} \qquad \delta\phi^2 = \frac{{m_B^0}^2{\alpha_B^0}^2}{r^2} \qquad (94)
$$

$$
U\mathbf{v}_A^2 = \frac{m_B^0\mathbf{v}_A^2}{r} \qquad \delta\phi\mathbf{v}_A^2 = -\frac{m_B^0\alpha_B^0\mathbf{v}_A^2}{r} \qquad U\delta\phi = -\frac{{m_B^0}^2\alpha_B^0}{r^2} \qquad (95)
$$

Substituting this all back into (91) and using that $h_{BB}(\mathbf{x}) = h_{AA}(\mathbf{x}) = (2r^4)^{-1}$, we get

$$
\begin{aligned}
-\,m_A(\phi)\frac{ds_A}{dt} = -m_A^0\Bigg\{&1 - \frac{1}{2}\mathbf{v}_A^2 - \frac{1}{8}\mathbf{v}_A^4 - \frac{m_B^0}{r}\Bigg[1 + \frac{3}{2}\mathbf{v}_B^2 + \frac{1}{2}\mathbf{v}_A\cdot\mathbf{v}_B\\
&-\frac{1}{2}(\mathbf{n}\cdot\mathbf{v_A})(\mathbf{n}\cdot\mathbf{v}_B) + \alpha_A^0\alpha_B^0\left(1 - \frac{1}{2}\mathbf{v}_B^2 + \frac{1}{2}\mathbf{v}_A\cdot\mathbf{v}_B - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v_A})(\mathbf{n}\cdot\mathbf{v}_B)\right)\\
&-4\mathbf{v}_A\cdot\mathbf{v}_B + \frac{3}{2}\mathbf{v}_A^2 - \frac{1}{2}\alpha_A^0\alpha_B^0\mathbf{v}_A^2\Bigg] + \frac{{m_B^0}^2}{r^2}\left[\frac{1}{2} + \frac{1}{2}({\alpha_A^0}^2 + \beta_A^0){\alpha_B^0}^2 + \alpha_A^0\alpha_B^0\right]\\
&+\frac{\alpha f'(\phi_0){m_B^0}^2}{r^4}\left(2\alpha_B^0 + \alpha_A^0\right)\Bigg\} + \mathcal{O}(v^6) \qquad (96)
\end{aligned}
$$

The last thing left for the Lagrangian is to calculate the additional tidal part of (85) with again $\lambda_A^{(s)} = 0$ and $m_A^0 = 0$.

$$\frac{dS_{fs}}{dt} = -\frac{1}{2}\lambda_B^{(s)}\sqrt{-g_{\alpha\beta}\dot{x}_B^\alpha \dot{x}_B^\beta}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{97}$$

We're only interested in the lowest order correction, so

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -e^{2U}(\partial_0\delta\phi)^2 + e^{-2U}(\nabla\delta\phi)^2 = -(\partial_0\delta\phi)^2 + (\nabla\delta\phi)^2, \tag{98}$$

where

$$\partial_0\delta\phi \approx -\partial_0\frac{m_B^0\alpha_B^0}{r} = \frac{m_B^0\alpha_B^0\mathbf{r}\cdot\mathbf{v}}{r^3}, \text{ and } \nabla\delta\phi \approx -\nabla\frac{m_B^0\alpha_B^0}{r} = \frac{m_B^0\alpha_B^0\mathbf{r}}{r^3}. \tag{99}$$

So,

$$(\partial_0\delta\phi)^2 = \frac{(m_B^0\alpha_B^0)^2(\mathbf{n}\cdot\mathbf{v})^2}{r^4} \sim \mathcal{O}(v^6) \text{ and } (\nabla\delta\phi)^2 = \frac{(m_B^0\alpha_B^0)^2}{r^4} \sim \mathcal{O}(v^4) \tag{100}$$

So we will only incorporate the $(\nabla\delta\phi)^2$ term. We furthermore use

$$\sqrt{-g_{\alpha\beta}\dot{x}_B^\alpha\dot{x}_B^\beta} = 1 + \mathcal{O}(v^2). \tag{101}$$

Finally, substituting everything back into (97) we get

$$\frac{dS_{fs}}{dt} = -\frac{1}{2}\lambda_B^{(s)}\frac{(m_B^0\alpha_B^0)^2}{r^4} + \mathcal{O}(v^6) \tag{102}$$

This term is the tidal contribution to the Lagrangian, matching the result of [7]. Adding the two parts together and symmetrising them, we get with a bit of rewriting

$$\begin{aligned} L = &-m_A^0 - m_B^0 + \frac{1}{2}m_A^0\mathbf{v}_A^2 + \frac{1}{2}m_B^0\mathbf{v}_B^2 + \frac{1}{8}m_A^0\mathbf{v}_A^4 + \frac{1}{8}m_B^0\mathbf{v}_B^4 \\ &+ \frac{m_A^0 m_B^0}{r}\Big[1 + \alpha_A^0\alpha_B^0(1-\Lambda) + \frac{\mathbf{v}_A^2 + \mathbf{v}_B^2}{2}(3 - \alpha_A^0\alpha_B^0) + \frac{\mathbf{v}_A\cdot\mathbf{v}_B}{2}(-7 + \alpha_A^0\alpha_B^0) \\ &\qquad - \frac{(\mathbf{n}\cdot\mathbf{v_A})(\mathbf{n}\cdot\mathbf{v}_B)}{2}(1 + \alpha_A^0\alpha_B^0)\Big] \\ &- \frac{m_A^0 m_B^0}{2r^2}\left[m_A^0\left((1+\alpha_A^0\alpha_B^0)^2 + {\alpha_A^0}^2\beta_B^0\right) + m_B^0\left((1+\alpha_A^0\alpha_B^0)^2 + {\alpha_B^0}^2\beta_A^0\right)\right] \\ &- \frac{\alpha f'(\phi_0)m_A^0 m_B^0}{r^4}\left[m_A^0\left(2\alpha_A^0(1-\Lambda) + \alpha_B^0\right) + m_B^0\left(2\alpha_B^0(1-\Lambda) + \alpha_A^0\right)\right] \\ &- \frac{1}{2}\sum_A \lambda_A^{(s)}\frac{(m_A^0\alpha_A^0)^2}{r^4} + \mathcal{O}(v^6) \end{aligned} \tag{103}$$

Note that this is where the mistake in [14] appears. A '+' sign is given for the $f'(\phi_0)$ term, where it should be a '-' sign.

We now introduce the following definitions to make the Lagrangian a bit more compact.

$$\bar{\alpha} \equiv 1 + \alpha_A^0 \alpha_B^0 \qquad \bar{\beta}_A \equiv \frac{\beta_A^0 {\alpha_B^0}^2}{2(1+\alpha_A^0\alpha_B^0)^2} \tag{104}$$

$$\bar{\gamma} \equiv -\frac{2\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0} \qquad \bar{\delta}_A \equiv \frac{2\alpha_A^0 + \alpha_B^0}{(1+\alpha_A^0\alpha_B^0)^2} \tag{105}$$

The Lagrangian then becomes.

$$\begin{aligned}L = -m_A^0 - m_B^0 + \frac{1}{2}m_A^0\mathbf{v}_A^2 + \frac{1}{2}m_B^0\mathbf{v}_B^2 + \frac{\bar{\alpha}m_A^0m_B^0}{r} + \frac{1}{8}m_A^0\mathbf{v}_A^4 + \frac{1}{8}m_B^0\mathbf{v}_B^4 \\ + \frac{\bar{\alpha}m_A^0m_B^0}{2r}\left[3(\mathbf{v}_A^2+\mathbf{v}_B^2) - 7(\mathbf{v}_A\cdot\mathbf{v}_B) - (\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) + 2\bar{\gamma}(\mathbf{v}_A-\mathbf{v}_B)^2\right] \\ - \frac{\bar{\alpha}^2m_A^0m_B^0}{2r^2}\left[m_A^0(1+2\bar{\beta}_B) + m_B^0(1+2\bar{\beta}_A)\right] \\ - \frac{\alpha f'(\phi_0)\bar{\alpha}^2m_A^0m_B^0}{r^4}\left[m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right] - \frac{1}{2}\sum_A \lambda_A^{(s)}\frac{(m_A^0\alpha_A^0)^2}{r^4}.\end{aligned} \tag{106}$$

We recognise the first five terms as the Newtonian Lagrangian for a two body system (except for the rescaling $\bar{\alpha}$ of the gravitational potential). The next couple of terms are the 1PN GR correction to the Newtonian Lagrangian. The terms that are coupled to $be\bar{t}a, gam\bar{m}a$ or $del\bar{t}a$ (and the rescaling $\bar{\alpha}$) are induced by the scalarization of the black hole, coupled to the GB-scalar. Finally, the last term is induced by the tidal effects that work on the scalar field around the black hole. It is also important to state that, despite the fact that it looks like second last (GB) and last (finite size) term look like 3PN terms, they are in fact 1PN terms. This will become more clear once we restore c and G as physical constants in Section 7.

6.4 Physical Effects

6.4.1 Relative Acceleration

Now that we finally have a Lagrangian that we can manipulate, let's calculate the equations of motion using the Euler-Lagrange equations. The equations of motion will be most useful in their relative form

$$\frac{\partial L}{\partial \mathbf{x}_A} = \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_A} \Rightarrow \frac{1}{m_A}\frac{\partial L}{\partial \mathbf{x}_A} - \frac{1}{m_B}\frac{\partial L}{\partial \mathbf{x}_B} = \frac{1}{m_A}\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_A} - \frac{1}{m_B}\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_B}. \tag{107}$$

This is a rather lengthy calculation, so we have relegated the calculation of the non-tidal part ($\lambda^{(s)} = 0$) to the Appendix. Instead we only calculate the contribution from the finite size effects, and add these to the expression we find in the Appendix. The non-tidal part of the relative acceleration is already calculated in [25], and we will follow their notation. The tidal part of the Lagrangian is

$$L_\lambda = -\frac{1}{2}\sum_A \lambda_A^{(s)} \frac{(m_A^0 \alpha_A^0)^2}{r^4} \tag{108}$$

So, Euler-Lagrange gives

$$\frac{1}{m_A^0}\frac{\partial L_\lambda}{\partial \mathbf{x}_A} - \frac{1}{m_B^0}\frac{\partial L_\lambda}{\partial \mathbf{x}_B} = \frac{2m\mathbf{n}}{r^5}\left[\lambda_A^{(s)}\frac{m_A^0 {\alpha_A^0}^2}{m_B^0} + \lambda_B^{(s)}\frac{m_B^0 {\alpha_B^0}^2}{m_A^0}\right] \tag{109}$$

and

$$\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial L_\lambda}{\partial \mathbf{v}_A} - \frac{1}{m_B^0}\frac{d}{dt}\frac{\partial L_\lambda}{\partial \mathbf{v}_B} = 0 \tag{110}$$

A similar result for neutron stars was calculated in [3], but also contains a mistake. In (109), the mass ratios are reversed, i.e. m_B^0/m_A^0 for the first term instead of m_A^0/m_B^0.

The relative acceleration for the rest of the Lagrangian is calculated in the Appendix. If we add the tidal terms to this expression, we get the full relative acceleration to 1PN order

$$\begin{aligned}\mathbf{a} = &-\frac{\bar{\alpha}m\mathbf{n}}{r^2} + \frac{\bar{\alpha}m}{r^2}\Bigg\{\mathbf{n}\left[\frac{3}{2}\eta\dot{r}^2 - (1+3\eta+\bar{\gamma})\mathbf{v}^2\right] + 2\mathbf{v}\dot{r}\left[2-\eta+\bar{\gamma}\right] \\ &+ \frac{2\bar{\alpha}m\mathbf{n}}{r}\left[2+\eta+\bar{\gamma}+\beta_+ - \frac{\Delta m}{m}\beta_- + \frac{2\alpha f'(\phi_0)}{\bar{\alpha}^{3/2}r^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + \frac{\xi}{mr^2}\right]\Bigg\},\end{aligned} \tag{111}$$

with

$$\xi \equiv \lambda_A^{(s)}\frac{m_A^0 {\alpha_A^0}^2}{\bar{\alpha}^2 m_B^0} + \lambda_B^{(s)}\frac{m_B^0 {\alpha_B^0}^2}{\bar{\alpha}^2 m_A^0}, \tag{112}$$

the tidal contribution. Furthermore,

$$\mathcal{S}_\pm \equiv \frac{\alpha_A^0 \pm \alpha_B^0}{2\sqrt{\alpha}} \qquad \beta_\pm \equiv \frac{\bar{\beta}_A \pm \bar{\beta}_B}{2} \tag{113}$$

$$\eta \equiv \frac{m_A^0 m_B^0}{m^2} \qquad \Delta m \equiv m_A^0 - m_B^0, \tag{114}$$

6.4.2 Binding Energy

To get a better idea of the magnitude of the tidal contribution compared to the Newtonian, GR, and other scalar field terms, we will now calculate the energy in circular orbits. This will add the tidal contribution to the calculation done in [25]. In Section 7 we will analyse the magnitude of the tidal contribution to the energy.

Since the orbits are circular we can assume the following

$$\dot{r} = \mathbf{n}\cdot\mathbf{v} = 0 \ \text{ and } \ \ddot{r} = \frac{1}{r}\left[\mathbf{a}\cdot\mathbf{r} + \mathbf{v}^2 - (\mathbf{n}\cdot\mathbf{v})^2\right] = 0 \tag{115}$$

$$\text{So, } \ \mathbf{n}\cdot\mathbf{v} = 0 \ \text{ and } \ \mathbf{v}^2 = -\mathbf{a}\cdot\mathbf{r} \approx \frac{\bar{\alpha}m}{r} \tag{116}$$

So,

$$\begin{aligned}\mathbf{a} = \frac{\bar{\alpha}m}{r^2}\Bigg\{\mathbf{n}\Big[-1-(1+3\eta+\bar{\gamma})\frac{\bar{\alpha}m}{r}\Big] + \frac{2\bar{\alpha}m\mathbf{n}}{r}\Bigg[2+\eta+\bar{\gamma}\\ +\beta_+ - \frac{\Delta m}{m}\beta_- + \frac{2\alpha f'(\phi_0)}{\bar{\alpha}^{3/2}r^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + \frac{\xi}{mr^2}\Bigg]\Bigg\}\end{aligned} \tag{117}$$

It will be convenient to write the energy as a function of the angular velocity ω. Using $\omega = v/r$, so $\omega^2 = \mathbf{v^2}/r^2 = -\mathbf{a}\cdot\mathbf{r}/r^2$ we get

$$\begin{aligned}\omega^2 = \frac{\bar{\alpha}m}{r^3}\Bigg\{1 - \frac{\bar{\alpha}m}{r}\Bigg[3-\eta+\bar{\gamma}+2\beta_+ - 2\frac{\Delta m}{m}\beta_-\\ + \frac{1}{r^2}\left(\frac{4\alpha f'(\phi_0)}{\bar{\alpha}^{3/2}}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + 2\frac{\xi}{m}\right)\Bigg]\Bigg\}\end{aligned} \tag{118}$$

Here, it is useful to introduce the following dimensionless parameters

$$\gamma_{PN} \equiv \frac{\bar{\alpha}m}{r} \qquad x \equiv (\bar{\alpha}m\omega)^{2/3} \tag{119}$$

From (118) and (119), we can obtain the approximate relationship

$$\begin{aligned}x \approx \gamma_{PN}\Bigg\{1 - \frac{\gamma_{PN}}{3}\Bigg[3-\eta+\bar{\gamma}+2\beta_+ - 2\frac{\Delta m}{m}\beta_-\\ + \gamma_{PN}^2\left(\frac{4\alpha f'(\phi_0)}{\bar{\alpha}^{7/2}m^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + 2\frac{\xi}{\bar{\alpha}^2m^3}\right)\Bigg]\Bigg\}\end{aligned} \tag{120}$$

Which is basically a polynomial of the form

$$x = \gamma_{PN} - b\gamma_{PN}^2 - c\gamma_{PN}^4, \tag{121}$$

with

$$a \equiv \frac{1}{3}\left[3-\eta+\bar{\gamma}+2\beta_+ - 2\frac{\Delta m}{m}\beta_-\right] \tag{122}$$

$$b \equiv \frac{1}{3}\left[\frac{4\alpha f'(\phi_0)}{\bar{\alpha}^{7/2}m^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + 2\frac{\xi}{\bar{\alpha}^2m^3}\right] \tag{123}$$

To express γ_{PN} in terms of x, we can now invert the polynomial to get

$$\gamma_{PN} \approx x + ax^2 + bx^4 \tag{124}$$

We would like to write the energy in terms of this single parameter x, so we can easily analyse it. We now calculate the tidal contribution to the energy by performing a Legendre transform to the tidal part of the Lagrangian, which is given by (108).

$$E_\lambda = \mathbf{v}_A \cdot \frac{\partial L_\lambda}{\partial \mathbf{v}_A} + \mathbf{v}_B \cdot \frac{\partial L_\lambda}{\partial \mathbf{v}_B} - L_\lambda = -L_\lambda \tag{125}$$

$$= \mu \left(\frac{\bar{\alpha} m}{r}\right)^2 \frac{1}{2r^2} \frac{\xi}{m} \tag{126}$$

So, taking the energy we found in the Appendix in equation (169), and adding the tidal part to it, we get

$$E = \mu \Bigg\{ \frac{1}{2}\mathbf{v}^2 - \frac{\bar{\alpha} m}{r} + \frac{3}{8}(1-3\eta)\mathbf{v}^4 + \frac{\bar{\alpha} m}{2r}\Big[(3 + 2\bar{\gamma} + \eta)\mathbf{v}^2 + \eta \dot{r}^2\Big] + \left(\frac{\bar{\alpha} m}{r}\right)^2 \left[\frac{1}{2} + \beta_+ - \frac{\Delta m}{m}\beta_- + \frac{1}{r^2}\left(\frac{\alpha f'(\phi_0)}{\bar{\alpha}^{3/2}}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + \frac{\xi}{2} m\right)\right]\Bigg\}. \tag{127}$$

So, now we can substitute every $\bar{\alpha} m / r$ with γ_{PN} and $\mathbf{v}^2 = r^2\omega^2 = \gamma_{PN} - 3a\gamma_{PN}^2 - 3b\gamma_{PN}^4$ to get

$$E = \frac{\mu}{2}\Bigg\{ -\gamma_{PN} + \gamma_{PN}^2 \Bigg[-3a + \frac{19}{4} - \frac{5}{4}\eta + 2\bar{\gamma} + 2\beta_+ - 2\frac{\Delta m}{m}\beta_- \Bigg] - \frac{3}{2} b \gamma_{PN}^4 \Bigg\} \tag{128}$$

Finally, substituting (124), we obtain the following expression for the binding energy as a function of the dimensionless variable $x \propto \omega^{2/3}$

$$E = -\frac{\mu}{2}\Bigg\{ x + \Big[-\frac{3}{4} - \frac{1}{12}\eta - \frac{2}{3}\bar{\gamma} + \frac{2}{3}\beta_+ - \frac{2}{3}\frac{\Delta m}{m}\beta_- \Big] x^2 + \left[\frac{10\alpha f'(\phi_0)}{3\bar{\alpha}^{7/2} m^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) - \frac{5\xi}{3\bar{\alpha}^2 m^3}\right] x^4 \Bigg\} \tag{129}$$

Again, it seems like both the GB term and the tidal term are higher order effects, when they are in fact 1PN effects, as will become clear in the next section.

7 Binding Energy Analysis

We will now analyse the physical effects of the inclusion of the finite size effects using the energy (129). We will use frequency rather than the angular velocity ω or the dimensionless parameter x as a variable for the energy, because it's a more useful physical quantity. However, we will still write the energy in terms of x to keep the cleanest expression. Keep in mind though that they are linked through $\omega = 2\pi f$ and $x = \left(\frac{G\bar{\alpha} m \omega}{c^3}\right)^{2/3}$. Before we can start making plots, we should restore the physical constants that we set to 1. The expression for the energy then becomes

$$E = -\frac{\mu c^2}{2}\Big\{x + \Big[-\frac{3}{4} - \frac{1}{12}\eta - \frac{2}{3}\bar{\gamma} + \frac{2}{3}\beta_+ - \frac{2}{3}\frac{\Delta m}{m}\beta_-\Big]x^2 + \left[\frac{10c^4\alpha f'(\phi_0)}{3G^2\bar{\alpha}^{7/2}m^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) - \frac{5c^4\xi}{3G^2\bar{\alpha}^2 m^3}\right]x^4\Big\} \tag{130}$$

After restoring the constants we can see that the GB and finite size term both have a factor c^4/G^2 associated with them, showing that they are in fact 1PN effects.

We will now first take a look at the relative magnitude of the finite size effects compared to the total energy, see fig. 4. For the chosen values of our

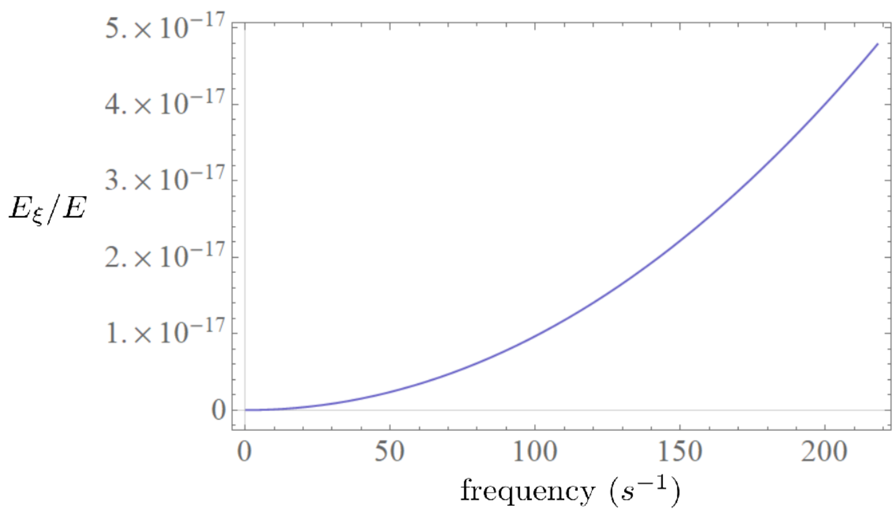

Figure 4: Absolute value of ratio between the tidal contribution to the energy and the total energy $|E_\xi(f)/E(f)|$ for $m_A = m_B = 3M_\odot$, $\lambda_A^{(s)} = \lambda_B^{(s)} = \frac{m_A^3 G^2}{c^4}$, $f(\phi) = e^\phi$, $\alpha = 10\ \mathrm{m}^2$ and $\phi_0 = 0$. The plot is shown from a frequency of 0 to the frequency of the Newtonian innermost stable circular orbit (ISCO), defined with $r_{ISCO} = \frac{6GmM}{c^2}$ and $f_{ISCO} = \sqrt{\frac{Gm}{4\pi^2 r_{ISCO}^3}}$.

constant and our chosen function, the relative magnitude of the tidal energy contribution is very small. However, many values of these constants are still unknown and could have a significant impact on the relative magnitude. The tidal energy is heavily dependent on the tidal deformability $\lambda^{(s)}$, the coupling constant α and the coupling function $f(\phi)$, none of which have been determined yet. There is an upper bound on $\sqrt{\alpha}$ of 1.7 km [22], and for the dimensionless tidal deformability $k^{(s)}$ there is an estimate for boson stars to be around ~ 40

[6]. For black holes we expect the tidal deformability to be less than that of boson stars, so it acts as an upper limit of sorts. The caveat is that this is within Maxwell-Einstein theory, and so does not necessarily have to be an upper limit in our theory. We will use it as a guess for an upper limit regardless.

To show how the tidal energy depends on some of the parameters, we will now make a parameter plot at $f = f_{ISCO}$ over the parameter space of $(\alpha, k^{(s)})$, see fig. 5. Here we took $k^(s) = 40$ en $\lambda^(s) = 10^6$ as upper limits for our graph.

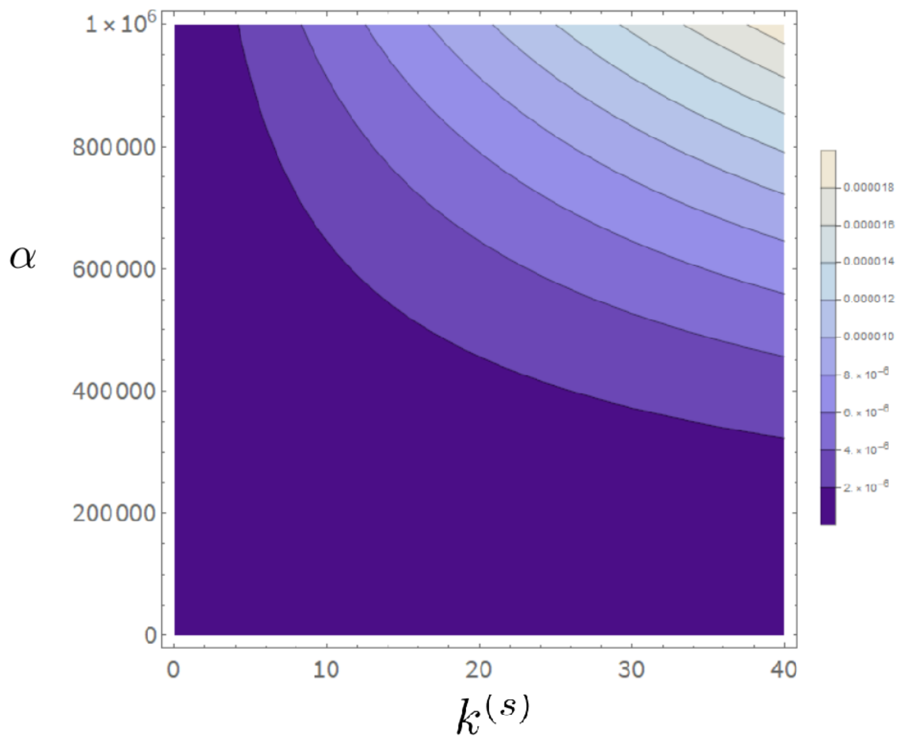

Figure 5: Absolute value of ratio between the tidal contribution to the energy and the total energy $|E_\xi(k^{(s)}, \alpha)/E(k^{(s)}, \alpha)|$ in parameter space $(k^{(s)}, \alpha)$. The other physical constants are the same as in fig. 4, with the chosen frequency $f = f_{ISCO}$. Furthermore, $k^{(s)} = k_A^{(s)} = k_B^{(s)}$, with $k_A^{(s)} = \frac{c^4}{G^2 m_A^3}$ the dimensionless tidal deformability.

We see that on this domain there is already a lot of variation, and at $k^{(s)} = 40$ and $\alpha = 10^6$ the tidal energy has a relative magnitude of $\sim 10^-5$. Over many orbits, this could actually accumulate to a significant influence.

We now take a look at the relative magnitude of all the beyond-GR effects in the energy in the same sort of parameter plot as before, see fig. 6. Here we see

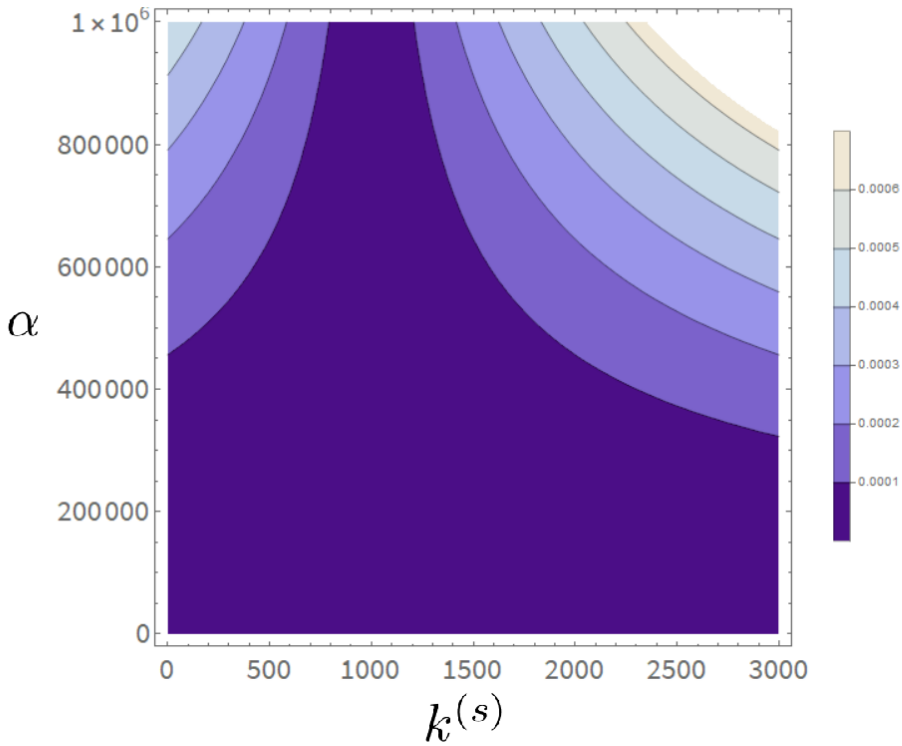

Figure 6: Absolute value of ratio between the beyond GR part of the energy and the GR energy $|(E(k^{(s)}, \alpha) - E_{GR})/E_{GR}|$ in parameter space $(k^{(s)}, \alpha)$. The other physical constants are the same as in 4, with the chosen frequency $f = f_{ISCO}$. Furthermore, $k^{(s)} = k_A^{(s)} = k_B^{(s)}$, with $k_A^{(s)} = \frac{c^4}{G^2 m_A^3}$ the dimensionless tidal deformability.

something interesting happening. As the tidal deformability increases, initially, it reduces the contribution of the beyond-GR energy to the total. We can explain this by looking at (130). Note that the two terms that are proportional to x^4, have opposite signs. This means that initially, as the tidal energy grows along with the increase of $k^{(s)}$, it starts out by cancelling the initially larger GB term. Then, at a certain point around $k^{(s)} = 1000$, it starts to overtake the GB part, and contribute positively again to the total energy. Eventually as $k^{(s)} > 2000$ it starts to dominate the beyond GR terms. However, this switch happens at a very high value of λ if we are to believe the upper limit from boson stars in Einstein-Maxwell theory.

Finally, we will take a look how the magnitude changes for changing mass ratios and total mass of the system, see fig. 7. Here we also see an interesting phenomenon. Initially, the tidal contribution decreases sharply as the total mass

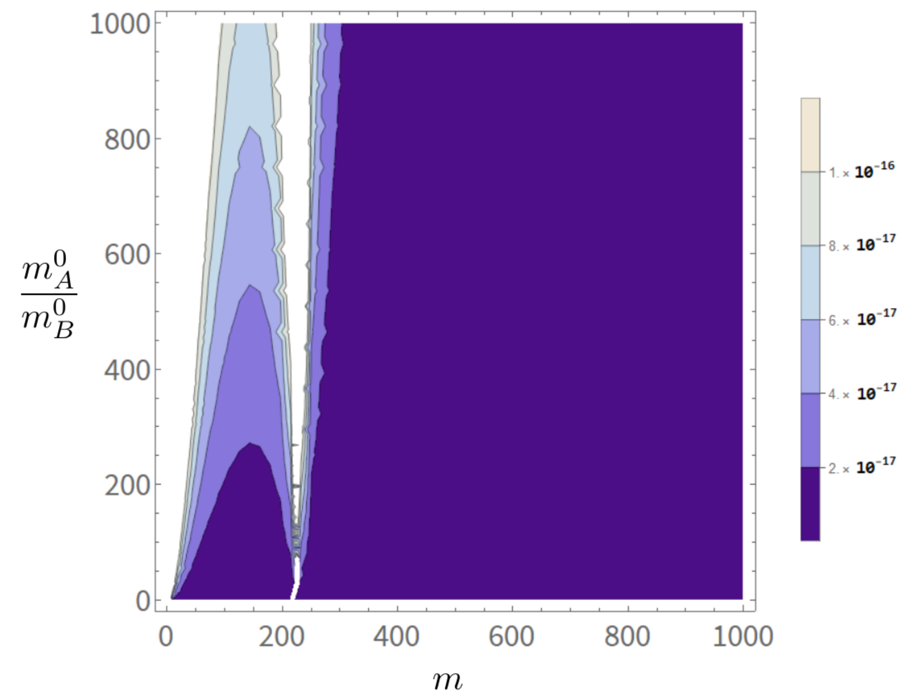

Figure 7: Absolute value of ratio between the tidal part of the energy and the total energy $|E_\xi(m,\rho)/E(m,\rho)|$ in parameter space (m,ρ), with $\rho = m_A^0/m_B^0$ the mass ratio and m is in solar masses. The other physical constants are the same as in 4, with the chosen frequency $f = f_{ISCO}$.

in the system increases, up until a total mass of around $m = 100M_\odot$, where it starts to increase again, peaking at around $200M_\odot$. This increase is surprising, as it is expected that increasing the total mass of the system would make tidal effects less important. The objects become bigger, and the tidal forces less extreme, since the gradient in the potential becomes less extreme. After this peak, the tidal contribution falls again sharply. Furthermore, we see that the tidal contribution goes up in general as the mass ratio increases.

8 Conclusion and Discussion

8.1 Conclusion

In conclusion, we have included an interaction term for the tidal effects of the scalar field around a black hole in EsGB gravity. This term is added when constructing the test particle Lagrangian. From there, the tidal contribution to the 1PN EsGB Lagrangian is calculated, and a mistake in the Lagrangian is corrected. The magnitude of this contribution is then analysed by calculating the binding energy of the binary black hole system, and plotting the relative magnitude of the tidal energy compared to the total energy. For the parameters tested, the tidal contribution was never very large, but is still dependent on many variables, such as the tidal deformability $\lambda^{(s)}$, the coupling constant α, the coupling function $f(\phi)$, the mass ratio $\rho = m_A^0/m_B^0$ and the total mass m. We found a strong positive correlation of the relative contribution with $\lambda^{(s)}$, α and ρ, and a mixed dependence on the total mass of the system. Depending on the actual values of these parameters, the effects could as large as 10^-5, or even larger. This could be a significant accumulative effect over many orbits as the black holes slowly spiral towards each other. On top of that, the tidal effects also become more significant as the orbital frequency increases. Finally, it is interesting to note that the tidal part has opposite sign to the GB contribution, so, as the tidal effect becomes more significant, initially it is cancelled out by the GB contribution. Regardless of these estimates of the significance, the theoretical work done in this thesis with regards to developing a full 1PN Lagrangian with tidal effects will be a useful framework for future work in EsGB theory.

8.2 Discussion and Outlook

The main weakness of any physical conclusions is the large uncertainty when it comes to the choice of parameters, which currently still have loose constraints or are still unknown. So the tidal effects could still range from being significant to completely insignificant. Additionally, the Lagrangian is developed using a weak-field, low-velocity expansion, so we should we careful with the validity of the expression as the system becomes relativistic and close to the merger.

There is still plenty of future work to be done that would expand on this topic. As already mentioned before, greater constraints on physical parameters would give a better idea of the significance of the tidal effects. Furthermore, this calculation could also be taken further to include more physical effects, such as the energy flux. For GW analysis we would need to calculate the effect on the waveform, so the theory can be directly compared to GW data. A calculation that estimates the cumulative effect of the tidal contribution on the system over many orbits could also be interesting to see how significant even a small effect could be.

A Appendix

In this appendix, we will perform the complete calculation of the relative acceleration $\mathbf{a}$. Recall that we found the Lagrangian to be

$$
\begin{aligned}
L = &-m_A^0 - m_B^0 + \frac{1}{2} m_A^0 \mathbf{v}_A^2 + \frac{1}{2} m_B^0 \mathbf{v}_B^2 + \frac{1}{8} m_A^0 \mathbf{v}_A^4 + \frac{1}{8} m_B^0 \mathbf{v}_B^4 \\
&+ \frac{m_A^0 m_B^0}{r} \Big[1 + \alpha_A^0 \alpha_B^0 + \frac{\mathbf{v}_A^2 + \mathbf{v}_B^2}{2} (3 - \alpha_A^0 \alpha_B^0) + \frac{\mathbf{v}_A \cdot \mathbf{v}_B}{2} (-7 + \alpha_A^0 \alpha_B^0) \\
&\qquad - \frac{(\mathbf{n} \cdot \mathbf{v_A})(\mathbf{n} \cdot \mathbf{v}_B)}{2} (1 + \alpha_A^0 \alpha_B^0) \Big] \\
&- \frac{m_A^0 m_B^0}{2r^2} \Big[m_A^0 \Big((1 + \alpha_A^0 \alpha_B^0)^2 + \alpha_A^{0\,2} \beta_B^0 \Big) + m_B^0 \Big((1 + \alpha_A^0 \alpha_B^0)^2 + \alpha_B^{0\,2} \beta_A^0 \Big) \Big] \\
&- \frac{\alpha f'(\phi_0) m_A^0 m_B^0}{r^4} \left[m_A^0 \left(2\alpha_A^0 + \alpha_B^0 \right) + m_B^0 \left(2\alpha_B^0 + \alpha_A^0 \right) \right] + \mathcal{O}(v^6),
\end{aligned}
\tag{131}
$$

which we can rewrite into

$$
\begin{aligned}
L = &-m_A^0 - m_B^0 + \frac{1}{2} m_A^0 \mathbf{v}_A^2 + \frac{1}{2} m_B^0 \mathbf{v}_B^2 + \frac{G\bar{\alpha} m_A^0 m_B^0}{r} + \frac{1}{8} m_A^0 \mathbf{v}_A^4 + \frac{1}{8} m_B^0 \mathbf{v}_B^4 \\
&+ \frac{G\bar{\alpha} m_A^0 m_B^0}{2r} \left[3(\mathbf{v}_A^2 + \mathbf{v}_B^2) - 7(\mathbf{v}_A \cdot \mathbf{v}_B) - (\mathbf{n} \cdot \mathbf{v}_A)(\mathbf{n} \cdot \mathbf{v}_B) + 2\bar{\gamma} (\mathbf{v}_A - \mathbf{v}_B)^2 \right] \\
&- \frac{G\bar{\alpha}^2 m_A^0 m_B^0}{2r^2} \left[m_A^0 (1 + 2\bar{\beta}_B) + m_B^0 (1 + 2\bar{\beta}_A) \right] \\
&- \frac{\alpha f'(\phi_0) G\bar{\alpha}^2 m_A^0 m_B^0}{r^4} \left[m_A^0 \bar{\delta}_A + m_B^0 \bar{\delta}_B \right],
\end{aligned}
\tag{132}
$$

where we reinstated Newton's constant G. Furthermore we used

$$
\bar{\alpha} \equiv 1 + \alpha_A^0 \alpha_B^0 \qquad \bar{\delta}_A \equiv \frac{2\alpha_A^0 + \alpha_B^0}{(1 + \alpha_A^0 \alpha_B^0)^2} \tag{133}
$$

$$
\bar{\beta}_A \equiv \frac{\beta_A^0 \alpha_B^{0\,2}}{2(1 + \alpha_A^0 \alpha_B^0)^2} \qquad \bar{\gamma} \equiv -\frac{2\alpha_A^0 \alpha_B^0}{1 + \alpha_A^0 \alpha_B^0} \tag{134}
$$

The relative acceleration describes the motion between the two black holes, and we can obtain it by using the Euler-Lagrange equations

$$
\frac{1}{m_A^0} \frac{\partial L}{\partial \mathbf{x}_A} - \frac{1}{m_B^0} \frac{\partial L}{\partial \mathbf{x}_B} = \frac{1}{m_A^0} \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_A} - \frac{1}{m_B^0} \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_B}. \tag{135}
$$

To make things more orderly, we will divide the Lagrangian into different parts.

$$L_0 = -m_A^0 + \frac{1}{2}m_A^0\mathbf{v}_A^2 + \frac{G\bar{\alpha}m_A^0 m_B^0}{2r} + \frac{1}{8}m_A^0\mathbf{v}_A^4 + (A \leftrightarrow B) \tag{136}$$

$$\begin{aligned} L_1 = \frac{G\bar{\alpha}m_A^0 m_B^0}{2r}\Big[&3(\mathbf{v}_A^2 + \mathbf{v}_B^2) - 7(\mathbf{v}_A \cdot \mathbf{v}_B) - (\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) \\ &+ 2\bar{\gamma}(\mathbf{v}_A - \mathbf{v}_B)^2 - \frac{G\bar{\alpha}}{r}\left(m_A^0(1+2\bar{\beta}_B) + m_B^0(1+2\bar{\beta}_A)\right)\Big] \end{aligned} \tag{137}$$

$$L_2 = -\frac{\alpha f'(\phi_0)G\bar{\alpha}^2 m_A^0 m_B^0}{r^4}\left[m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right], \tag{138}$$

with $L = L_0 + L_1 + L_2$.

A.1 L_0 Contribution

Let us start with L_0. To reiterate, we use that $\partial r/\partial \mathbf{x}_A = \mathbf{r}/r = -\partial r/\partial \mathbf{x}_B$, giving the following intermediate results that we will use.

$$\frac{\partial}{\partial \mathbf{x}_A}\frac{1}{r^n} = -\frac{n\mathbf{n}}{r^{n+1}} = -\frac{\partial}{\partial \mathbf{x}_B}\frac{1}{r^n} \tag{139}$$

$$\text{So, } \frac{\partial}{\partial \mathbf{x}_A}(\mathbf{n}\cdot\mathbf{v}_A) = \frac{1}{r}\left[\mathbf{v}_A - (\mathbf{n}\cdot\mathbf{v}_A)\mathbf{n}\right] \tag{140}$$

So, applying these on L_0

$$\frac{\partial L_0}{\partial \mathbf{x}_A} = -\frac{G\bar{\alpha}m_A^0 m_B^0\mathbf{n}}{r^2} = -\frac{\partial L_0}{\partial \mathbf{x}_B} \tag{141}$$

$$\frac{d}{dt}\frac{\partial L_0}{\partial \mathbf{v}_A} = \frac{d}{dt}\left[m_A^0\mathbf{v}_A + \frac{1}{2}m_A^0(\mathbf{v}_A^2)\mathbf{v}_A\right] \tag{142}$$

$$= m_A^0\mathbf{a}_A + m_A^0(\mathbf{v}_A\cdot\mathbf{a}_A)\mathbf{v}_A + \frac{1}{2}m_A^0(\mathbf{v}_A^2)\mathbf{a}_A \tag{143}$$

$$\text{where we get } \frac{d}{dt}\frac{\partial L_0}{\partial \mathbf{v}_B} \text{ by simply exchanging } A \leftrightarrow B.. \tag{144}$$

(141) follows from (139) and (144) due to the fact that the Lagrangian is symmetric in $\mathbf{v}_A$ and $\mathbf{v}_B$. We use this to get the contribution to the relative acceleration.

$$\frac{1}{m_A^0}\frac{\partial L_0}{\partial \mathbf{x}_A} - \frac{1}{m_B^0}\frac{\partial L_0}{\partial \mathbf{x}_B} = -\frac{G\bar{\alpha}m\mathbf{n}}{r^2} \tag{145}$$

$$\begin{aligned} \frac{1}{m_A^0}\frac{d}{dt}\frac{\partial L_0}{\partial \mathbf{v}_A} - \frac{1}{m_B^0}\frac{d}{dt}\frac{\partial L_0}{\partial \mathbf{v}_B} &= \mathbf{a} + (\mathbf{v}_A\cdot\mathbf{a}_A)\mathbf{v}_A - (\mathbf{v}_B\cdot\mathbf{a}_B)\mathbf{v}_B \\ &+ \frac{1}{2}(\mathbf{v}_A)^2\mathbf{a}_A - \frac{1}{2}(\mathbf{v}_B)^2\mathbf{a}_B, \end{aligned} \tag{146}$$

with $m = m_A^0 + m_B^0$.

A.2 L_1 Contribution

Now we do the same for L_1

$$\begin{aligned}\frac{\partial L_1}{\partial \mathbf{x}_A} =& -\frac{G\bar{\alpha} m_A^0 m_B^0 \mathbf{n}}{2r^2}\Big[3(\mathbf{v}_A^2+\mathbf{v}_B^2)-7(\mathbf{v}_A\cdot\mathbf{v}_B)-(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\ &+2\bar{\gamma}(\mathbf{v}_A-\mathbf{v}_B)^2-\frac{G\bar{\alpha}}{r}\left(m_A^0(1+2\bar{\beta}_B)+m_B^0(1+2\bar{\beta}_A)\right)\Big]\\ &+\frac{G\bar{\alpha} m_A^0 m_B^0}{2r}\Big[-\frac{\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)+\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)-2\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)}{r}\\ &+\frac{G\bar{\alpha}\mathbf{n}}{r^2}\left(m_A^0(1+2\bar{\beta}_B)+m_B^0(1+2\bar{\beta}_A)\right)\Big]\\ =& \frac{G\bar{\alpha} m_A^0 m_B^0}{2r^2}\Big\{-\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)+\Big[3(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)-3(\mathbf{v}_A^2+\mathbf{v}_B^2)\\ &+7(\mathbf{v}_A\cdot\mathbf{v}_B)-2\bar{\gamma}(\mathbf{v}_A-\mathbf{v}_B)^2+\frac{2G\bar{\alpha}}{r}\left(m_A^0(1+2\bar{\beta}_B)+m_B^0(1+2\bar{\beta}_A)\right)\Big]\mathbf{n}\Big\},\end{aligned} \tag{147}$$

with again

$$\frac{\partial L_1}{\partial \mathbf{x}_A} = -\frac{\partial L_1}{\partial \mathbf{x}_B}. \tag{148}$$

To get the RHS of the Euler-Lagrange equations we first calculate

$$\frac{\partial L_1}{\partial \mathbf{v}_A} = \frac{G\bar{\alpha} m_A^0 m_B^0}{2r}\Big[6\mathbf{v}_A-7\mathbf{v}_B-\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B)+4\bar{\gamma}(\mathbf{v}_A-\mathbf{v}_B)\Big]. \tag{149}$$

Then

$$\begin{aligned}\frac{d}{dt}\frac{\partial L_1}{\partial \mathbf{v}_A} =& -\frac{G\bar{\alpha} m_A^0 m_B^0 \mathbf{n}\cdot\mathbf{v}}{2r^2}\Big[6\mathbf{v}_A-7\mathbf{v}_B-\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B)+4\bar{\gamma}\mathbf{v}\Big]\\ &+\frac{G\bar{\alpha} m_A^0 m_B^0}{2r}\Big[6\mathbf{a}_A-7\mathbf{a}_B-\frac{\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_B)}{r}-\frac{\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_B)}{r}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_B)\\ &+\frac{2\mathbf{n}(\mathbf{n}\cdot\mathbf{v_B})(\mathbf{n}\cdot\mathbf{v})}{r}+4\bar{\gamma}\mathbf{a}\Big]\\ =& \frac{G\bar{\alpha} m_A^0 m_B^0}{2r}\Big\{6\mathbf{a}_A-7\mathbf{a}_B+4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_B)\\ &+\frac{1}{r}\Big[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_B)+(\mathbf{n}\cdot\mathbf{v})\Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B)-6\mathbf{v}_A+7\mathbf{v}_B-4\bar{\gamma}\mathbf{v}\Big)\Big]\Big\},\end{aligned} \tag{150}$$

where we introduced $\mathbf{v} \equiv \mathbf{v}_A - \mathbf{v}_B = \dot{\mathbf{r}}$. We can again get the term with the derivative to $\mathbf{v}_B$ by simply exchanging $A \leftrightarrow B$ (keep in mind that $\mathbf{v} = \mathbf{v}_A - \mathbf{v}_B$, so $\mathbf{v} \to -\mathbf{v}$ under exchange of A and B. Similarly $\mathbf{n} \to -\mathbf{n}$). The contribution to the relative acceleration then becomes

$$\begin{aligned}\frac{1}{m_A^0}\frac{\partial L_1}{\partial \mathbf{x}_A} - \frac{1}{m_B^0}\frac{\partial L_1}{\partial \mathbf{x}_B} = \frac{G\bar{\alpha}m}{2r^2}\Big\{ &-\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B) - \mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A) \\ &+ \Big[3(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) - 3(\mathbf{v}_A^2+\mathbf{v}_B^2) + 7(\mathbf{v}_A\cdot\mathbf{v}_B) - 2\bar{\gamma}\mathbf{v}^2 \\ &+ \frac{2G\bar{\alpha}}{r}\left(m_A^0(1+2\bar{\beta}_B) + m_B^0(1+2\bar{\beta}_A)\right)\Big]\mathbf{n}\Big\} \end{aligned} \tag{151}$$

$$\begin{aligned}\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial L_1}{\partial \mathbf{v}_A} - \frac{1}{m_B^0}\frac{d}{dt}\frac{\partial L_1}{\partial \mathbf{v}_B} = &\frac{G\bar{\alpha}m_B^0}{2r}\Big\{6\mathbf{a} - \mathbf{a}_B + 4\bar{\gamma}\mathbf{a} - \mathbf{n}(\mathbf{n}\cdot\mathbf{a}_B) \\ &+ \frac{1}{r}\Big[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_B) - \mathbf{n}(\mathbf{v}\cdot\mathbf{v}_B) + (\mathbf{n}\cdot\mathbf{v})\Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B) - 6\mathbf{v} + \mathbf{v}_B - 4\bar{\gamma}\mathbf{v}\Big)\Big]\Big\} \\ &- \frac{G\bar{\alpha}m_A^0}{2r}\Big\{-6\mathbf{a} - \mathbf{a}_A - 4\bar{\gamma}\mathbf{a} - \mathbf{n}(\mathbf{n}\cdot\mathbf{a}_A) \\ &+ \frac{1}{r}\Big[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_A) - \mathbf{n}(\mathbf{v}\cdot\mathbf{v}_A) + (\mathbf{n}\cdot\mathbf{v})\Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_A) + 6\mathbf{v} + \mathbf{v}_A + 4\bar{\gamma}\mathbf{v}\Big)\Big]\Big\} \end{aligned} \tag{152}$$

A.3 L_2 Contribution

Finally, doing the same for L_2

$$\frac{\partial L_2}{\partial \mathbf{x}_A} = \frac{4\alpha f'(\phi_0)G\bar{\alpha}^2 m_A^0 m_B^0 \mathbf{n}}{r^5}\left[m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right] = -\frac{\partial L_2}{\partial \mathbf{x}_B} \tag{153}$$

$$\frac{d}{dt}\frac{\partial L_2}{\partial \mathbf{v}_A} = 0 = \frac{d}{dt}\frac{\partial L_2}{\partial \mathbf{v}_B} \tag{154}$$

So,

$$\frac{1}{m_A^0}\frac{\partial L_2}{\partial \mathbf{x}_A} - \frac{1}{m_B^0}\frac{\partial L_2}{\partial \mathbf{x}_B} = \frac{4\alpha f'(\phi_0)G\bar{\alpha}^2 m\mathbf{n}}{r^5}\left[m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right] \tag{155}$$

$$\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial L_2}{\partial \mathbf{v}_A} - \frac{1}{m_B^0}\frac{d}{dt}\frac{\partial L_2}{\partial \mathbf{v}_B} = 0 \tag{156}$$

A.4 Relative Acceleration

We can now substitute these results into (135)

$$
\begin{aligned}
&-\frac{G\bar{\alpha}m\mathbf{n}}{r^2}+\frac{G\bar{\alpha}m}{2r^2}\Bigg\{-\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)+\Bigg[3(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\
&-3(\mathbf{v}_A^2+\mathbf{v}_B^2)+7(\mathbf{v}_A\cdot\mathbf{v}_B)-2\bar{\gamma}\mathbf{v}^2+\frac{2G\bar{\alpha}}{r}\left(m_A^0(1+2\bar{\beta}_B)+m_B^0(1+2\bar{\beta}_A)\right)\Bigg]\mathbf{n}\Bigg\}\\
&\qquad+\frac{4\alpha f'(\phi_0)G\bar{\alpha}^2m\mathbf{n}}{r^5}\left[m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right]\\
&=\mathbf{a}+(\mathbf{v}_A\cdot\mathbf{a}_A)\mathbf{v}_A-(\mathbf{v}_B\cdot\mathbf{a}_B)\mathbf{v}_B+\frac{1}{2}(\mathbf{v}_A)^2\mathbf{a}_A-\frac{1}{2}(\mathbf{v}_B)^2\mathbf{a}_B\\
&+\frac{G\bar{\alpha}m_B^0}{2r}\Bigg\{6\mathbf{a}-\mathbf{a}_B+4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_B)+\frac{1}{r}\Bigg[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_B)\\
&+(\mathbf{n}\cdot\mathbf{v})\Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B)-6\mathbf{v}+\mathbf{v}_B-4\bar{\gamma}\mathbf{v}\Big)\Bigg]\Bigg\}-\frac{G\bar{\alpha}m_A^0}{2r}\Bigg\{-6\mathbf{a}-\mathbf{a}_A-4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_A)\\
&+\frac{1}{r}\Bigg[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_A)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_A)+(\mathbf{n}\cdot\mathbf{v})\Big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_A)+6\mathbf{v}+\mathbf{v}_A+4\bar{\gamma}\mathbf{v}\Big)\Bigg]\Bigg\}
\end{aligned}
\tag{157}
$$

In this expression we can recognise the zeroth order Newtonian part ($\mathbf{a} = -G\bar{\alpha}m\mathbf{n}/r^2$) and the lowest order correction (the rest). To get an explicit expression for $\mathbf{a}$, we will substitute $\mathbf{a} \approx -G\bar{\alpha}m\mathbf{n}/r^2$ in the correctional part. Similarly $\mathbf{a}_A \approx -G\bar{\alpha}m_B\mathbf{n}/r^2$ and $\mathbf{a}_B \approx G\bar{\alpha}m_A\mathbf{n}/r^2$. To simplify, we order the terms into three groups, those proportional to G^2, those proportional to $\mathbf{n}$ and those proportional to $\mathbf{v}_A$, $\mathbf{v}_B$ or $\mathbf{v}$.

$$
\begin{aligned}
\mathbf{a}=&-\frac{G\bar{\alpha}m\mathbf{n}}{r^2}+\frac{G\bar{\alpha}}{2r^2}\Bigg\{\frac{G\bar{\alpha}\mathbf{n}}{r}\Bigg[8m^2+4m_A^0m_B^0+4\bar{\gamma}m^2\\
&\qquad+4m\left(m_A^0\bar{\beta}_B+m_B^0\bar{\beta}_A\right)+\frac{8\alpha f'(\phi_0)m}{r^2}\left(m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right)\Bigg]\\
&+\mathbf{n}\Bigg[3m(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)+3(\mathbf{n}\cdot\mathbf{v})\left(\mathbf{n}\cdot(m_A^0\mathbf{v}_A-m_B^0\mathbf{v}_B)\right)-3m\left(\mathbf{v}_A^2+\mathbf{v}_B^2\right)\\
&\quad+7m(\mathbf{v}_A\cdot\mathbf{v}_B)-2m\bar{\gamma}\mathbf{v}^2-\mathbf{v}\cdot(m_A^0\mathbf{v}_A-m_B^0\mathbf{v}_B)+m_B^0\mathbf{v}_A^2+m_A^0\mathbf{v}_B^2\Bigg]\\
&\quad-m\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)-m\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)+2m_B^0\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_A)+2m_A^0\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_B)\\
&+\left(6m\mathbf{v}-m_B^0\mathbf{v}_B+m_A^0\mathbf{v}_A+4\bar{\gamma}m\mathbf{v}\right)(\mathbf{n}\cdot\mathbf{v})-\mathbf{v}\left(\mathbf{n}\cdot(m_A^0\mathbf{v}_A-m_B^0\mathbf{v}_B)\right)\Bigg\},
\end{aligned}
\tag{158}
$$

where we used $\bar{\alpha} = 1 + \alpha_A^0 \alpha_B^0$. Simplifying this, we get

$$\begin{aligned}
\mathbf{a} = -\frac{G\bar{\alpha}m\mathbf{n}}{r^2} + \frac{G\bar{\alpha}}{r^2}\Bigg\{ \frac{2G\bar{\alpha}\mathbf{n}}{r}\Big[2m^2 + m_A^0 m_B^0 + \bar{\gamma}m^2 \\
+ m\left(m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A\right) + \frac{2\alpha f'(\phi_0)m}{r^2}\left(m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right)\Big] \\
+ \mathbf{n}\Big[\frac{3}{2}\left(m_A^0(\mathbf{n}\cdot\mathbf{v}_A)^2 + m_B^0(\mathbf{n}\cdot\mathbf{v}_B)^2\right) - 2m\mathbf{v}^2 + m_A^0\mathbf{v}_B^2 + m_B^0\mathbf{v}_A^2 - m\bar{\gamma}\mathbf{v}^2\Big] \\
+ \mathbf{v}\Big[4m(\mathbf{n}\cdot\mathbf{v}) - m_A^0(\mathbf{n}\cdot\mathbf{v}_A) + m_B^0(\mathbf{n}\cdot\mathbf{v}_B) + 2\bar{\gamma}m(\mathbf{n}\cdot\mathbf{v})\Big]\Bigg\} \qquad (159)
\end{aligned}$$

To simplify further, we move to the CM-frame, and use the lowest order approximations

$$\mathbf{x}_A \approx \frac{m_B^0}{m}\mathbf{r} \qquad \mathbf{x}_B \approx -\frac{m_A^0}{m}\mathbf{r} \qquad (160)$$

$$\Rightarrow \quad \mathbf{v}_A \approx \frac{m_B^0}{m}\mathbf{v} \qquad \mathbf{v}_B \approx -\frac{m_A^0}{m}\mathbf{v} \qquad (161)$$

Substituting this into the (159), we obtain

$$\begin{aligned}
\mathbf{a} = -\frac{G\bar{\alpha}m\mathbf{n}}{r^2} + \frac{G\bar{\alpha}}{r^2}\Bigg\{ \frac{2G\bar{\alpha}\mathbf{n}}{r}\Big[2m^2 + m_A^0 m_B^0 + \bar{\gamma}m^2 \\
+ m\left(m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A\right) + \frac{2\alpha f'(\phi_0)m}{r^2}\left(m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right)\Big] \\
+ \mathbf{n}\Big[\frac{3}{2}m\frac{m_A^0 m_B^0}{m^2}(\mathbf{n}\cdot\mathbf{v})^2 + \left(\frac{{m_A^0}^3 + {m_B^0}^3}{m^2} - 2m - \bar{\gamma}\right)\mathbf{v}^2\Big] \\
+ 2m\mathbf{v}(\mathbf{n}\cdot\mathbf{v})\Big[2 - \frac{m_A^0 m_B^0}{m^2} + \bar{\gamma}\Big]\Bigg\}. \qquad (162)
\end{aligned}$$

Then, using the definitions

$$\mathcal{S}_\pm \equiv \frac{\alpha_A^0 \pm \alpha_B^0}{2\sqrt{\alpha}} \qquad \beta_\pm \equiv \frac{\bar{\beta}_A \pm \bar{\beta}_B}{2} \qquad (163)$$

$$\eta \equiv \frac{m_A^0 m_B^0}{m^2} \qquad \Delta m \equiv m_A^0 - m_B^0, \qquad (164)$$

and reintroducing $\dot{r} = (\mathbf{n}\cdot\mathbf{v})$, we get

$$\begin{aligned}
\mathbf{a} = -\frac{G\bar{\alpha}m\mathbf{n}}{r^2} + \frac{G\bar{\alpha}m}{r^2}\Bigg\{\mathbf{n}\Big[\frac{3}{2}\eta\dot{r}^2 - (1 + 3\eta + \bar{\gamma})\mathbf{v}^2\Big] + 2\mathbf{v}\dot{r}\Big[2 - \eta + \bar{\gamma}\Big] \\
+ \frac{2G\bar{\alpha}m\mathbf{n}}{r}\Big[2 + \eta + \bar{\gamma} + \beta_+ - \frac{\Delta m}{m}\beta_- + \frac{2\alpha f'(\phi_0)}{\bar{\alpha}^{3/2}r^2}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\Big]\Bigg\} \qquad (165)
\end{aligned}$$

A.5 Binding Energy

With the Lagrangian, we can also calculate the conserved binding energy of the system.

$$E = \mathbf{v}_A \cdot \frac{\partial L}{\partial \mathbf{v}_A} + \mathbf{v}_B \cdot \frac{\partial L}{\partial \mathbf{v}_B} - L \tag{166}$$

We already did these derivatives while calculating $\mathbf{a}$, so substituting these in, we get

$$\begin{aligned} E = {} & m_A^0 + m_B^0 + \frac{1}{2} m_A^0 \mathbf{v}_A^2 + \frac{1}{2} m_B^0 \mathbf{v}_B^2 - \frac{\bar{\alpha} m_A^0 m_B^0}{r} + \frac{3}{8} m_A^0 \mathbf{v}_A^4 + \frac{3}{8} m_B^0 \mathbf{v}_B^4 \\ & + \frac{\bar{\alpha} m_A^0 m_B^0}{2r} \Big[3 \left(\mathbf{v}_A^2 + \mathbf{v}_B^2 \right) - 7 \left(\mathbf{v}_A \cdot \mathbf{v}_B \right) - (\mathbf{n} \cdot \mathbf{v}_A)(\mathbf{n} \cdot \mathbf{v}_B) + 2\bar{\gamma} \mathbf{v}^2 \Big] \\ & + \frac{\bar{\alpha}^2 m m_A^0 m_B^0}{r^2} \left[\frac{1}{2} + \beta_+ - \frac{\Delta m}{m} \beta_- + \frac{\alpha f'(\phi_0)}{\bar{\alpha}^{3/2} r^2} \left(3\mathcal{S}_+ + \frac{\Delta m}{m} \mathcal{S}_- \right) \right] \end{aligned} \tag{167}$$

We can simplify this again in the CM-frame using

$$\mathbf{v}_A \approx \frac{m_B^0}{m} \mathbf{v} \qquad \mathbf{v}_B \approx -\frac{m_A^0}{m} \mathbf{v} \qquad \mu \equiv \frac{m_A^0 m_B^0}{m} \tag{168}$$

Then

$$\begin{aligned} E = m + \mu \bigg\{ & \frac{1}{2} \mathbf{v}^2 - \frac{\bar{\alpha} m}{r} + \frac{3}{8} (1 - 3\eta) \mathbf{v}^4 + \frac{\bar{\alpha} m}{2r} \Big[(3 + \eta + 2\bar{\gamma}) \mathbf{v}^2 + \eta \dot{r}^2 \Big] \\ & + \frac{\bar{\alpha}^2 m^2}{r^2} \left[\frac{1}{2} + \beta_+ - \frac{\Delta m}{m} \beta_- + \frac{\alpha f'(\phi_0)}{\bar{\alpha}^{3/2} r^2} \left(3\mathcal{S}_+ + \frac{\Delta m}{m} \mathcal{S}_- \right) \right] \bigg\} \end{aligned} \tag{169}$$

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