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Irreducible unitary representations of $SU(2,2)$ and the Hyperbolic Higgs model

MASTER THESIS

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Abstract

This thesis consists of two parts: a mathematical and a physical part. In the mathematical part the irreducible unitary representations of a linear connected semisimple group G are studied. We discuss parabolic subgroups, and induce these subgroups to the whole group to get representations on G. We discuss the Langlands classification, which states that any irreducible unitary representation is the unique irreducible quotient of an suitable parabolically induced representation. After that, tools for building discrete series and measuring irreducibility are considered, and an explicit example to find certain irreducible unitary representations of $SU(2,2)$ is discussed.

In the physical part, we build a model which will solve the hierarchy problem of the Standard Model. Before discussing the model itself, we give a reminder of supersymmetry. After the reminder, the model is described. It assumes 5-dimensional supersymmetry, and breaks supersymmetry explicitely by choosing certain Scherk-Schwarz twists. This results in an explicit $SU(2,2)$ symmetry, which is not an irreducible unitary representation.

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Contents

1 Introduction

In the first half of the 20^{th} century both Special Relativity and Quantum Mechanics were constructed. However, both fields of physics were incompatible, so it seemed at first [23, 30]. A lot of work has been done on what we now call Quantum Field Theory (QFT) which unifies the two, and even the Standard Model of elementary particles is a result of this work.

One of the key players in the construction of the theory of QFT was E. Wigner. He showed that all elementary particles can be understood as irreducible unitary representations of the Poincaré group, which is a non-compact group [32]. This was both physically and mathematically innovative. Mathematicians such as G. Mackey built upon this to create a self-contained theory to describe representation theory of the Poincaré group [17].

This is not the only occasion where representation theory is of importance in physics. The Standard Model itself comprises elementary particles and their interactions. These interactions often have symmetries associated with them, and these symmetries allow us to see particles in certain representations. These representations allow us to construct new interactions with a certain coupling strength. Specifically, irreducible unitary representations would correspond to new particles of the model.

The Standard Model (SM) has predicted nature up to high precision. However, there are questions that the SM cannot answer. One of these questions is the hierarchy problem. The hierarchy problem arose a few years ago when the Higgs particle was experimentally discovered. The Higgs particle was found to have a relatively light mass with respect to the theory [18]. As we will discuss in our thesis, it could be merely an aestetic problem, but it could also be a hint of new physics. We will discuss a model that solves the hierarchy problem based on [7]. This model uses supersymmetry, however we do not require soft-breaking terms to break supersymmetry explicitely. In addition, we will find the potential to have an internal $SU(2,2)$ symmetry, which could lead to new particles.

However, the theory of non-compact group representation, such as $SU(2,2)$, is not so well-known among physicists. Therefore we shall discuss the representation theory of linear connected semisimple groups, and show how to construct irreducible unitary representations in the specific case of $SU(2,2)$. We have written this thesis in such a way that the reader can apply their favorite linear connected semisimple group to find some irreducible unitary representations.

This thesis consists of two parts. The first part is the mathematical part, where we describe some structure theory of the groups we will be looking at, and some general representation theory. We will introduce induced representations and the Langlands classification is discussed. The Langlands classification classifies all irreducible unitary representations by considering quotients of induced representations. Thereafter we will give a description to find certain irreducibles, and close the mathematical part with an explicit example. We will then switch gears and focus on the physical part. The hierarchy problem is quickly discussed, after which we review supersymmetry and its possible solution to the hierarchy problem. After that, the Hyperbolic Higgs model is discussed and we show the explicit $SU(2,2)$ symmetry, and argue whether it is irreducible unitary. We end the second part by discussing how this model solves the hierarchy problem.

Part I
Irreducible unitary representations of $SU(2,2)$

2 Structure of non-compact groups

2.1 The Iwasawa decomposition

In the first part of our thesis, we will describe a method to list the irreducible unitary representations of certain non-compact groups. To describe which kind of groups we will be discussing, we start with a definition.

Definition 2.1.1. Let G be a Lie group. We call G a *linear connected reductive group* if G is a closed connected subgroup of $GL(n,\mathbb{C})$ that is stable under the map $g \mapsto g^\dagger$ whereby $g^\dagger$ we mean the Hermitian conjugate of g. We call G *linear connected semisimple* if it is linear connected reductive and has finite center.

For the rest of this thesis, we will assume that G is linear connected reductive, unless it is stated otherwise. Since G is a closed subgroup of a Lie group, G itself is a Lie group, hence has a Lie algebra $\mathfrak{g}$ which we will assume to be a vectorspace over the real numbers.

Lemma 2.1.2. *[14, Prop 1.1] If G is a linear connected semisimple group, then $\mathfrak{g}$ is semisimple. If G is a linear connected reductive group, then $\mathfrak{g}$ is reductive, i.e.*

$$\mathfrak{g} = Z_\mathfrak{g} \oplus [\mathfrak{g},\mathfrak{g}]$$

where $Z_\mathfrak{g}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}]$ is semisimple.

To describe the irreducible unitary representations of these groups, we require the notion of parabolic subgroups. This chapter will be devoted to that.

But first we will find an equivalence of a root decomposition for real Lie algebras. For that define an involution on $\mathfrak{g}$. By an *involution* we mean an automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma^2 = I$.

Definition 2.1.3. Let G be a linear connected reductive group. Define the involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by $X \mapsto -X^\dagger$. Here we used the physical notation, where $\dagger$ means taking the adjoint or the conjugate transpose. This involution is called the *Cartan involution*. We can also define the *global Cartan involution* Θ as the automorphism on G having differential θ. In this case, $\Theta : G \to G$ by $\Theta(g) = (g^\dagger)^{-1}$

Because $\theta^2 = I$, it has as eigenvalues ± 1, hence the linear vector space $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

where $\mathfrak{g}_\pm$ are the eigenspaces of the eigenvalues ± 1 respectively. We define

$$\mathfrak{k} := \mathfrak{g}_+, \qquad \mathfrak{p} := \mathfrak{g}_-. \tag{2.1}$$

It can easily be checked by the features of an automorphism that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad [\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}. \tag{2.2}$$

We call this decomposition the *Cartan decomposition*. Note that this shows that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$.

Theorem 2.1.4. *[14, 15, Prop. 5.5 and 7.19 respectively] Let G be a linear connected reductive group, let θ be the Cartan involution, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let K be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. Then*

1. *K is closed, compact and a maximal compact subgroup of G.*
2. *The mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp(X)$ is a diffeomorphism.*
3. *If G has compact center, then the analytic subgroup G_{ss} with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is closed (hence linear connected semisimple) and G is the commuting product of $G = (Z_G)_0 G_{ss}$.*

The first part of the previous theorem tells us that K is compact, which will be key in analyzing non-compact representation theory later on. With a Cartan involution defined, we consider the trace form

$$B_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \qquad B_0(X, Y) := \mathrm{Tr}(XY).$$

This form is a complex-valued symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$. We note that this form is ad invariant, i.e.

$$B_0(\mathrm{ad}(X)Y, Z) = -B_0(Y, \mathrm{ad}(X)Z).$$

Also note that this form is nondegenerate. To prove this, assume there is a $0 \neq X \in \mathfrak{g}$ such that $B_0(X, Y) = 0$ for all $Y \in \mathfrak{g}$. We then note that

$$B_0(X, \theta X) < 0$$

because

$$\mathrm{Tr}(X\theta X) = -\mathrm{Tr}(XX^\dagger) = -\sum_{i,j} x_{ij} x_{ij}^* = -\sum_{i,j} |x_{i,j}|^2 < 0$$

where $X_{ij} = x_{ij}$, which produces a contradiction.

Therefore we can define a non-degenerate inner product on $\mathfrak{g}$, given by

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad \langle X, Y \rangle := -\mathrm{Re}\, B_0(X, \theta Y). \tag{2.3}$$

In addition, Theorem 2.1.4.3 tells us that if we wish to construct a root decomposition, it is enough to look at G_{ss}, for the other part commutes. Hence we can focus on the semisimple part. So for now, let us consider a linear connected semisimple group.

Lemma 2.1.5. *[14, p. 7 and 117] Let G be linear connected semisimple. The billinear form in Equation (2.3) is a positive definite inner product, that decomposes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ orthogonally. In addition, with respect to this inner product, $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ is anti-symmetric if $X \in \mathfrak{k}$ and symmetric if $X \in \mathfrak{p}$.*

We remember that a nonzero real valued symmetric matrix with real eigenvalues is diagonalizable. This implies that $\mathrm{ad}(X)$ for $X \in \mathfrak{p}$ can be diagonalized with respect to a suitable basis. This allows for a root decomposition of $\mathfrak{g}$ in a similar way as in the complex-valued Lie algebra case.

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. By maximal we mean maximality with respect to the inclusion $\mathfrak{a} \subset \mathfrak{p}$. Since all elements in $\mathfrak{a}$ commute, we can diagonalize them simultaneously.

Definition 2.1.6. Let $\lambda \in \mathfrak{a}^*$. We then define the subspace $\mathfrak{g}_\lambda$ as

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} | [H, X] = \lambda(H)X, \, \forall H \in \mathfrak{a}\}. \tag{2.4}$$

When $\mathfrak{g}_\lambda \neq 0$ and $\lambda \neq 0$, we call λ a *root* in $\mathfrak{g}$, and $\mathfrak{g}_\lambda$ a *root space*. The set of all roots is denoted by $\Sigma(\mathfrak{g}, \mathfrak{a})$.

Theorem 2.1.7. *[14, 27] Let $\mathfrak{g}$ be a real semisimple Lie algebra, and let θ be a Cartan involution with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as before. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra. Then*

1. *$\Sigma(\mathfrak{g}, \mathfrak{a})$ is finite.*
2. *$\mathfrak{g}$ is the orthogonal direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\lambda$ with respect to the inner product given in Equation (2.3).*
3. *Let $\lambda, \mu \in \Sigma(\mathfrak{g}, \mathfrak{a})$. Then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.*
4. *$\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, hence if $\lambda \in \Sigma(\mathfrak{g}, \mathfrak{a})$ then $-\lambda \in \Sigma(\mathfrak{g}, \mathfrak{a})$.*
5. *$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$ and the sum is orthogonal.*

One might think that $\Sigma(\mathfrak{g}, \mathfrak{a})$ is an abstract root system as defined for complex Lie algebras. This need not to be true.

Definition 2.1.8. A possibly non-reduced root system is a pair (E, Σ) consisting of a finite dimensional real vector space E and a finite subset $\Sigma \subset E \setminus 0$ such that

1. The set Σ spans E.
2. For every $\alpha \in \Sigma$ there exists a reflection $s_\alpha : E \to E$ with $s_\alpha(\Sigma) \subset \Sigma$ and $s_\alpha(\alpha) = -\alpha$.
3. For all $\alpha, \beta \in \Sigma$, we have $s_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$

If in addition to the previous three requirements, the following is also true, we call the root system *reduced*:

4. If $\alpha, c\alpha \in \Sigma$ where $c \in \mathbb{R}$, then $c = \pm 1$.

Lemma 2.1.9. *[27, Lemmas 16.6 and 16.21] Let (E, Σ) be a possibly non-reduced root system and let $\alpha \in \Sigma$. Then $-\alpha \in \Sigma$ and there exists $\beta \in \Sigma$ such that $\mathbb{R}\alpha \cap \Sigma \subset \{\pm\frac{1}{2}\beta, \pm\beta\}$. In addition, if both $\beta, \frac{1}{2}\beta \in \mathbb{R}\alpha \cap \Sigma$, then $s_\beta = s_{\frac{1}{2}\beta}$.*

So if α is a root, the only positive multiples of this root is either 2α or $\frac{1}{2}\alpha$. The following theorem tells us that the root system we found is a possibly non-reduced root system.

Theorem 2.1.10. *[27, Lemma 16.9] Let $\mathfrak{g}$ be a real semisimple Lie algebra, and θ be a Cartan involution. Then $(\mathfrak{a}^*, \Sigma(\mathfrak{g}, \mathfrak{a}))$ is a possibly non-reduced root system.*

Definition 2.1.11. A *system of positive roots* for Σ is a subset $\Sigma^+ \subset \Sigma$ such that $\Sigma = \Sigma^+ \cup (-\Sigma^+)$ and such that Σ^+ and $-\Sigma^+$ are separated by a hyperplane in E. In addition, we say a root is *simple* if it cannot be written as a sum of two positive roots. The set of simple roots is given by Π. We say a root $\alpha \in \Sigma^+$ is *reduced* if $\frac{1}{2}\alpha \notin \Sigma^+$.

Definition 2.1.12. The *Weyl group of* Σ is the subgroup of $GL(\mathfrak{a}^*)$ that is generated by s_α for $\alpha \in \Sigma$. We shall denote it as $W(A : G)$.

Now that we have discussed how to construct a root decomposition on a linear connected semisimple group in Theorem 2.1.7, we can extend the notion of a root decomposition to a linear connected reductive group. Let G be a linear connected reductive group, and write $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace. Then certainly $Z_{\mathfrak{g}} \cap \mathfrak{p} \subseteq \mathfrak{a}$, and so

$$\mathfrak{a} = (Z_{\mathfrak{g}} \cap \mathfrak{p}) \oplus ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a}).$$

Note that $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a}$ is a maximal abelian subspace of $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{p}$.

We can define for any $\lambda \in \mathfrak{a}$ the subspaces

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | [H, X] = \lambda(H)X, \, \forall H \in \mathfrak{a}\}$$

and define λ to be a *root* and $\mathfrak{g}_\lambda$ a *root space* if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. It is clear that such a root is obtained by taking a root on $[\mathfrak{g}, \mathfrak{g}]$ (which is semisimple), and extending it from $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a}$ to $\mathfrak{a}$ by letting it be 0 on $Z_{\mathfrak{g}} \cap \mathfrak{a}$. This way the root space decomposition on $[\mathfrak{g}, \mathfrak{g}]$ gives a root space decomposition of $\mathfrak{g}$. In addition, we set $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ so that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. This generalization also shows that previous lemmas are all true in the reductive case as well, and we shall make use of them even in the reductive case.

Definition 2.1.13. Let G be a linear connected reductive group, and choose $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian. We find the root system as described above, and we shall denote the possibly non-reducted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ by $\Sigma(\mathfrak{g}, \mathfrak{a})$. We will denote a system of positive roots for $\Sigma(\mathfrak{g}, \mathfrak{a})$ by Σ^+, and the set of simple roots by Π.

Definition 2.1.14. Let G be a linear connected reductive group. Because the inner product $\langle \cdot, \cdot \rangle$ defined in Equation (2.3) is a nondegenerate inner product, we can define for any $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ the element $H_\alpha \in \mathfrak{a}$ as the unique element such that

$$\alpha(H) = \langle H, H_\alpha \rangle = \operatorname{Re} B_0(H, H_\alpha), \qquad \forall H \in \mathfrak{a}$$

where we used $\theta H_\alpha = -H_\alpha$. In this way, we can define an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*$ by

$$\langle \mu, \nu \rangle = \mu(H_\nu) = \nu(H_\mu) = \langle H_\mu, H_\nu \rangle. \tag{2.5}$$

This also defines a norm on $\mathfrak{a}^*$, given by $\|\mu\|^2 = \langle \mu, \mu \rangle$.

Thus, we can now safely assume G to be linear connected reductive. Then we can find the Iwasawa decomposition.

Theorem 2.1.15 (Infinitesimal Iwasawa decomposition)**.** *[15, Sect. VII.2] Let G be a linear connected reductive group, let θ be a Cartan involution of $\mathfrak{g}$, and choose a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$. Choose a system of positive roots Σ^+. Define the spaces*

$$\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a}), \qquad \mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \qquad \bar{\mathfrak{n}} := \bigoplus_{\alpha \in -\Sigma^+} \mathfrak{g}_\alpha. \tag{2.6}$$

Then each of these sets are subalgebras. Furthermore, the following decomposition holds

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{2.7}$$

Theorem 2.1.16 (The Iwasawa decomposition)**.** *[15, Thm. 7.31] Let G be a linear connected reductive group, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra. Let A and N be the analytic subgroups corresponding with $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Then the map*

$$\phi : K \times A \times N \to G \quad , \quad (k, a, n) \mapsto kan \tag{2.8}$$

is a diffeomorphism, and A, N are simply connected closed subgroups of G.

The Iwasawa decomposition depends on two choices. The first is the choice of the maximal abelian subgroup $\mathfrak{a}$, and the second is the choice of positive roots. However, the choice of $\mathfrak{a}$ is unique up to conjugation by a member $k \in K$.

Lemma 2.1.17. *[15, Prop. 7.29] If $\mathfrak{a}$ and $\mathfrak{a}'$ are two maximal abelian subspaces of $\mathfrak{p}$, then there is an element $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a}' = \mathfrak{a}$. Consequently $\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$.*

Finally, we want to give a definition of a Cartan subalgebra in the case of linear connected reductive groups.

Definition 2.1.18. Let G be a linear connected reductive group. A *θ-stable Cartan subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ that is maximal among abelian θ-stable subalgebras of $\mathfrak{g}$. We will call the dimension of $\mathfrak{h}$ the *rank* of G, written as $\mathrm{rank}(G)$. The term *real rank* is used for the dimension of the subalgebra $\mathfrak{h} \cap \mathfrak{p}$.

Lemma 2.1.19. *[14, Sect. V.4] Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra. Then*

$$\mathfrak{h}_0 = (\mathfrak{h} \cap \mathfrak{k}) \oplus i(\mathfrak{h} \cap \mathfrak{p})$$

is a Cartan subalgebra of the compact Lie algebra $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ and $(\mathfrak{h}_0)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$. Hence we can find roots of $\mathfrak{u}_{\mathbb{C}}$ with respect to $(\mathfrak{h}_0)_{\mathbb{C}}$ and define these as the roots *of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. We will denote this root system by $\Delta(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$.*

Definition 2.1.20. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Write for the moment

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) =: \mathfrak{b} \oplus \mathfrak{a}$$

respectively. Let $\alpha \in \Delta(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$, then α is real valued on $\mathfrak{a} \oplus i\mathfrak{b}$. We say α is *real* or *imaginary* if α vanishes on $\mathfrak{b}$ or $\mathfrak{a}$ respectively.

Definition 2.1.21. Let G be a linear connected reductive group, and let $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ as above. If $\alpha \in \Delta(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$ is a real root, we define H_α as the unique element in $\mathfrak{a}$ such that $\alpha(H) = \mathrm{Re}\, B_0(H, H_\alpha)$ for all $H \in \mathfrak{a}$.

2.2 Parabolic subgroups

Next we turn our attention to the parabolic subgroups. We shall need these subgroups to describe induced representations for linear connected reductive groups G.

Definition 2.2.1. Let G be a linear connected reductive group. A *minimal parabolic subalgebra* of $\mathfrak{g}$ is any subalgebra of $\mathfrak{g}$ that is conjugate via $\mathrm{Ad}(G)$ to

$$\mathfrak{q}_{min} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Remark 2.2.2. In view of the Iwasawa decomposition $G = KAN$, we can equivalently say the conjugacy is done via $\mathrm{Ad}(K)$. In addition, we see that $\mathfrak{q}_{min}$ contains the maximally θ-stable Cartan subalgebra $\mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} \subset \mathfrak{m}$ is maximally abelian.

Definition 2.2.3. Let G be a linear connected reductive group. A *parabolic subalgebra* $\mathfrak{q}$ of $\mathfrak{g}$ is a Lie subalgebra containing some minimal parabolic subalgebra.

Example 2.2.4. One can find examples of parabolic subalgebras, by adding additional root spaces of non-positive roots to the minimal parabolic subalgebra. More concretely, fix a subset $\Pi' \subset \Pi$ and define

$$\Xi := \Sigma^+ \cup \{\beta \in \Sigma | \beta \in \text{span}(\Pi')\}.$$

Then the subalgebra

$$\mathfrak{q} := \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Xi} \mathfrak{g}_\alpha$$

is a parabolic subalgebra, for it contains $\mathfrak{q}_{min}$. $\oslash$

In fact, we will show that these are, up to conjugacy, the only parabolic subalgebras.

Lemma 2.2.5. *[15, Lemma 7.74] Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{q}_{min}$. Then*

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$$

for some subset $\Gamma \subset \Sigma(\mathfrak{g}, \mathfrak{a})$ containing Σ^+.

Idea of the proof. By assumption $\mathfrak{m} \oplus \mathfrak{a}$ is contained in $\mathfrak{q}$ and is invariant under $\text{ad}(\mathfrak{a})$ due to it being a subalgebra of $\mathfrak{g}$. Then we can write $\mathfrak{q}$ as

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})} (\mathfrak{g}_\alpha \cap \mathfrak{q}).$$

The next thing to prove is that $\mathfrak{g}_\alpha \cap \mathfrak{q}$ is either empty or $\mathfrak{g}_\alpha$. Note $\mathfrak{n} \subseteq \mathfrak{q}$ thus we only need to prove $\mathfrak{g}_\alpha \subset \mathfrak{q}$ for $\alpha \in -\Sigma^+$. This can be shown by looking at $(\text{Ad}\,\theta X)^2$ for $X \in \mathfrak{g}_\alpha$. $\square$

Theorem 2.2.6. *[15, Prop. 7.76] All parabolic subalgebras $\mathfrak{q}$ containing the minimal parabolic subalgebra $\mathfrak{q}_{min}$ are parametrized by the set of subsets of the simple restricted roots as in Example 2.2.4.*

This theorem shows that we can generate all parabolic subalgebras (up to a conjugation) by considering all possible subsets of the simple roots Π. Due to Lemma 2.2.5, we can rearrange certain root spaces to get a familiar result back. This description will be put into a lemma, for later convenience.

Lemma 2.2.7. *Let $q_{\Pi'}$ be the parabolic subalgebra corresponding to the subset Π' of the simple roots Π of $\mathfrak{g}$. Then define the following sets*

$$\Gamma := \Sigma^+ \cup \{\beta \in \Sigma | \beta \in \text{span}(\Pi')\} \tag{2.9}$$

$$\mathfrak{a}_{\Pi'} := \bigcap_{\beta \in \Gamma \cap -\Gamma} \ker(\beta) \subset \mathfrak{a} \tag{2.10}$$

$$\mathfrak{a}_{M,\Pi'} := \mathfrak{a}_{\Pi'}^{\perp} \subset \mathfrak{a} \tag{2.11}$$

$$\mathfrak{m}_{\Pi'} := \mathfrak{m} \oplus \mathfrak{a}_{M,\Pi'} \oplus \bigoplus_{\beta \in \Gamma \cap -\Gamma} \mathfrak{g}_\beta \tag{2.12}$$

$$\mathfrak{n}_{\Pi'} := \bigoplus_{\beta \in \Gamma, \beta \notin -\Gamma} \mathfrak{g}_\beta \tag{2.13}$$

$$\overline{\mathfrak{n}_{\Pi'}} = \theta \mathfrak{n}_{\Pi'} \tag{2.14}$$

$$\mathfrak{n}_{M,\Pi'} = \mathfrak{n} \cap \mathfrak{m}_{\Pi'}. \tag{2.15}$$

It then follows that

$$\mathfrak{q}_{\Pi'} = \mathfrak{m}_{\Pi'} \oplus \mathfrak{a}_{\Pi'} \oplus \mathfrak{n}_{\Pi'}. \tag{2.16}$$

This decomposition is called the Langlands decomposition *of* $\mathfrak{q}_{\Pi'}$

Remark 2.2.8. Note that choosing $\Pi' = \emptyset$ is allowed. Going through the definitions, we see that $\mathfrak{a}_\emptyset = \mathfrak{a}$ and $\mathfrak{q}_\emptyset = \mathfrak{q}_{min}$. On the other hand, choosing $\Pi' = \Pi$ results in $\mathfrak{q} = \mathfrak{g}$.

Proposition 2.2.9. *[15, Prop. 7.78 and 7.79] A parabolic subalgebra $\mathfrak{q}_{\Pi'}$ containing the minimal parabolic subalgebra $\mathfrak{q}_{min}$ has the following properties:*

1. *$\mathfrak{m}_{\Pi'}$, $\mathfrak{a}_{\Pi'}$ and $\mathfrak{n}_{\Pi'}$ are Lie subalgebras, and $\mathfrak{n}_{\Pi'}$ is an ideal in $\mathfrak{q}_{\Pi'}$,*
2. *$\mathfrak{a} = \mathfrak{a}_{\Pi'} \oplus \mathfrak{a}_{M,\Pi'}$ orthogonally with respect to the inner product $\langle \cdot, \cdot \rangle$ given in Equation (2.3),*
3. *$\mathfrak{g} = \mathfrak{m}_{\Pi'} \oplus \mathfrak{a}_{\Pi'} \oplus \mathfrak{n}_{\Pi'} \oplus \overline{\mathfrak{n}_{\Pi'}}$ orthogonally with respect to $\langle \cdot, \cdot \rangle$,*
4. *$\mathfrak{m}_{\Pi'} = \mathfrak{m} \oplus \mathfrak{a}_{M,\Pi'} \oplus \mathfrak{n}_{M,\Pi'} \oplus \theta\mathfrak{n}_{M,\Pi'}$,*
5. *Let $0 \neq \eta \in (\mathfrak{a}_{\Pi'})^*$, and define the subspace*

$$\mathfrak{g}_{(\eta)} = \bigoplus_{\beta \in \mathfrak{a}^*,\, \beta|_{\mathfrak{a}_{\Pi'}} = \eta} \mathfrak{g}_\beta.$$

Then $\mathfrak{g}_{(\eta)} \subset \mathfrak{n}_{\Pi'}$ or $\mathfrak{g}_{(\eta)} \subset \overline{\mathfrak{n}}_{\Pi'}$.

Definition 2.2.10. Proposition 2.2.9.5 shows some sort of root decomposition. Motivated by this we define, for $\eta \in (\mathfrak{a}_{\Pi'})^*$, the subspace

$$\mathfrak{g}_{(\eta)} = \{X \in \mathfrak{g} | [H, X] = \eta(H)X \text{ for all } H \in \mathfrak{a}_{\Pi'}\}. \tag{2.17}$$

We say η is a *root* of $\mathfrak{g}$ with respect to $\mathfrak{a}_{\Pi'}$ if $\eta \neq 0$ and $\mathfrak{g}_{(\eta)} \neq 0$. We will call this set of roots *the roots in* $(\mathfrak{g}, \mathfrak{a}_{\Pi'})$. We want to emphasize that these roots do *not* necessarily define a (possibly non-reduced) root system. We define the *positive roots of* $(\mathfrak{g}, \mathfrak{a}_{\Pi'})$ as the roots whose root space lies in $\mathfrak{n}_{\Pi'}$. We should note that if $\Pi' = \emptyset$, the roots in $(\mathfrak{g}, \mathfrak{a}_\emptyset)$ are the same as the possibly non-reduced root system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

Proposition 2.2.9.1 shows that $\mathfrak{m}_{\Pi'}$, $\mathfrak{a}_{\Pi'}$ and $\mathfrak{n}_{\Pi'}$ are Lie subalgebras, hence correspond to Lie subgroups of G. These subgroups together form a parabolic subgroup.

Definition 2.2.11. Consider a (minimal) parabolic subalgebra $\mathfrak{q} = \mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P$. A *(minimal) parabolic subgroup* is a closed subgroup P of G such that $P = M_P A_P N_P$ where $A_P := \exp \mathfrak{a}_P$, $N_P := \exp \mathfrak{n}_P$, and $M_P := Z_K(\mathfrak{a}) \exp \mathfrak{m}_P$. The standard minimal parabolic subgroup is the minimal parabolic subgroup P_{min} defined by setting $\mathfrak{q} = \mathfrak{q}_{min}$. We say a parabolic subgroup P is *standard* if $P_{min} \subset P$.

Lemma 2.2.12. *[15, Prop 7.83] Let P be a parabolic subgroup of a linear connected reductive group G, and define M_P, A_P, N_P as above. Then*

1. *multiplication $M_P \times A_P \times N_P \to P$ is a diffeomorphism,*
2. *P has $\mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P$ as Lie algebra,*
3. *$G = KP = KM_PA_PN_P$,*

4. $(M_P)_0 = \exp \mathfrak{m}_P$ *is a linear connected reductive group with compact center.*

Although the roots in $(\mathfrak{g}, \mathfrak{a}_P)$ are not necessarily an abstract root system, we can still define the Weyl group analytically.

Definition 2.2.13. Let $P = M_P A_P N_P$ be a parabolic subgroup. We define the analytical Weyl group on the roots in $(\mathfrak{g}, \mathfrak{a}_P)$ as

$$W(A_P : G) := N_K(\mathfrak{a}_P)/Z_K(\mathfrak{a}_P) \tag{2.18}$$

Theorem 2.2.14. *[14, Thm. 5.17] If $A_P = A$, then $W(A : G)$ defines the Weyl group for the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$.*

Finally, we can also define the ρ_P function. This will be used later in the thesis.

Definition 2.2.15. Let G be a linear connected reductive group, and let $P = M_P A_P N_P$ be a parabolic subgroup. Consider the roots in $(\mathfrak{g}, \mathfrak{a}_P)$, and define ρ_P by

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Gamma^+} (\dim \mathfrak{g}_{(\alpha)})\alpha \tag{2.19}$$

where Γ^+ is the set of positive roots of $(\mathfrak{g}, \mathfrak{a}_P)$.

To conclude this section, we will mention another decomposition called the KAK-decomposition, as well as integration with respect to Haar measures which will be necessary later on. For a more elaborate discussion on both, we refer kindly to [14, Sect. V.4 and V.6].

Definition 2.2.16. Let G be a Lie group. A nonzero Borel measure on G invariant under left translations is called a *left Haar measure*. A *right Haar measure* is a nonzero Borel measure on G invariant under right translations. We will denote left and right Haar measures by $d_l x$ and $d_r x$ respectively. We say a measure dx is a *Haar measure* if dx is both a left and a right Haar measure.

Proposition 2.2.17. *[15, Prop. 8.44-8.46] Let G be linear connected reductive and $P = M_P A_P N_P$ a parabolic subgroup. Then there is a Haar measure on G, K, M_P, A_P and N_P individually. In addition*

1. *Since $G = KM_P A_P N_P$, we have that the Haar measure can be written as*

$$dx = dk\, d_r(man) = e^{2\rho_P(\log(a))} dk\, dm\, da\, dn.$$

Here dx, dk, dm, da and dn are the Haar measures of G, K, M_P, A_P, N_P respectively.

2. *Since $G = M_P A_P N_P K$, we have that the Haar measure can be written as*

$$dx = d_l(man)\, dk = dm\, da\, dn\, dk$$

3. *Denote $\bar{N}_P = \Theta N_P$, then there exists a Haar measure $d\bar{n}$ on $\bar{N}_P$, and*

$$dx = d\bar{n} d_r(man).$$

In addition, it follows that

$$\int_K f(k)\, dk = \int_{\bar{N}_P} f(\kappa_P(\bar{n})) e^{-2\rho_P[H_P(\bar{n})]} d\bar{n}$$

for any $f \in C(K)$ that is right invariant under $K \cap M_P$. Here $\kappa_P : G \to K$, $\mu_P : G \to \exp(\mathfrak{m}_P \cap \mathfrak{p})$ and $H_P : G \to \mathfrak{a}_P$ are projection maps such that g decomposes according to $G = KM_P A_P N_P$ as

$$g = \kappa_P(g)\mu_P(g)e^{H_P(g)}n.$$

Theorem 2.2.18 (The KAK decomposition)**.** *[15, Thm 7.39] Let G be linear connected reductive. Then every element $g \in G$ can be decomposed as $g = k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A$. In this decomposition, a is uniquely determined up to conjugation by a member of $W(A : G)$. If a is fixed as $\exp(H)$ for $H \in \mathfrak{a}$ and if $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then k_1 is unique up to right multiplication by a member of $Z_K(\mathfrak{a})$.*

3 Representations and K-finite vectors

3.1 Representation theory

With an understanding of parabolic subgroups, we go to representation theory. Before discussing induced representations, which is the focus of the next chapter, we first need some general representation theory. We will start with setting the stage, by reminding ourselves of some definitions and elementary theorems. As before, we will assume G to be linear connected reductive.

Definition 3.1.1. A *continuous representation* of a Lie group G is a pair (π, V) where V is a complex Hilbert space, and $\pi : G \to GL(V)$ is a map such that it is continuous in the sense that the mapping $(g, v) \mapsto \pi(g)v$, $G \times V \to V$ is continuous, as well as

$$\pi(gh) = \pi(g)\pi(h) \qquad\qquad \forall\, g, h \in G.$$

Here $GL(V)$ is the set of bounded operators on V with bounded inverse.
A representation of a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ is a pair (π, V) where V is a complex vector space, and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a linear map such that

$$\pi([x, y]) = [\pi(x), \pi(y)] = \pi(x)\pi(y) - \pi(y)\pi(x)$$

for $x, y \in \mathfrak{g}$. Here $\mathfrak{gl}(V)$ is the set of linear operators on V. Instead of a representation, people often say that a representation of G or $\mathfrak{g}$ is a *module* of G or $\mathfrak{g}$. We will use both terms interchangibly.

Definition 3.1.2. Let (π, V) be a representation of a Lie group G. Denote the inner product on V by $\langle \cdot, \cdot \rangle_V$. If

$$\langle \pi(g)v, \pi(g)w \rangle_V = \langle v, w \rangle_V$$

for all $v, w \in V$ and $g \in G$, we say the representation is *unitary*. Define $\pi(g)^\dagger$ in the usual way. One easily sees that for unitary representations $\pi(g)^\dagger = \pi(g)^{-1} = \pi(g^{-1})$.

Definition 3.1.3. We say two representations (π, V) and (ρ, W) are *equivalent* if there exists a bounded operator $E : V \to W$ with bounded inverse such that $E \circ \pi(g) = \rho(g) \circ E$ for all $g \in G$. We will write $\pi \simeq \rho$. If π and ρ are unitary operators, we say π and ρ are *unitarily equivalent* if $\pi \simeq \rho$ and E is a unitary operator as well.

Definition 3.1.4. Let (π, V) and (ρ, W) be two representations of G. We define the space $\mathrm{Hom}_G(V, W)$ as

$$\mathrm{Hom}_G(V, W) := \{A : V \to W \text{ linear bounded} \mid A \circ \pi(g) = \rho(g) \circ A \text{ for all } g \in G\}.$$

If $A \in \mathrm{Hom}_G(V, W)$ we say A is an *intertwining operator*

Definition 3.1.5. Let (π, V) be a representation of a Lie group G. A vector $v \in V$ is *invariant* if $\pi(G)v = v$. The set of invariant vectors is denoted by V^G. Let (π, V) be a representation of $\mathfrak{g}$. A vector $v \in V$ is *invariant* if $\pi(\mathfrak{g})v = 0$. The set of invariant vectors is denoted by $V^{\mathfrak{g}}$.

Definition 3.1.6. An *invariant subspace* of a representation (π, V) of G or $\mathfrak{g}$, is a subspace $U \subset V$ such that $\pi(G)U \subset U$ or $\pi(\mathfrak{g})U \subset U$ respectively. The representation is said to be *irreducible* if the only *closed* invariant subspaces are $\{0\}$ and V itself. Otherwise the representation is said to be *reducible*. We will denote the set of irreducible representations of G by $\widehat{G}$.

The following lemma characterizes whenever a unitary representation is irreducible or not. In finite dimensions, the proof comes down to looking at orthogonal complements. However, in infinite dimensions one needs to do a bit more work.

Lemma 3.1.7 (Schur's Lemma)**.** *[14, Prop. 1.5] Let (π, V) be a unitary representation of a Lie group G. Then*

$$\pi \textit{ is irreducible} \Leftrightarrow \mathrm{Hom}_G(V, V) = \mathbb{C} I_V \tag{3.1}$$

where I_V is the identity operator on V.

Proof. If π is reducible, then there exists a non-zero closed invariant subspace $U \subsetneq V$. Then there exists a non-scalar projection operator E that commutes with $\pi(G)$. Since U is closed, E is bounded. On the other hand, if L is a non-scalar bounded operator that commutes with $\pi(G)$, then $L^\dagger$ also commutes with $\pi(G)$ due to unitarity of π. Hence the self-adjoint operators $X = \frac{1}{2}\left(L + L^\dagger\right)$ and $Y = \frac{1}{2i}\left(L - L^\dagger\right)$ also commute with $\pi(G)$. At least X or Y is no multiple of the identity operator, so without loss of generality we can assume X to be non-trivial. Then by the Spectral Theorem of Bounded Operators (see for example [19]), there exists a bounded projection E that is non-trivial that also commutes with $\pi(G)$. Then $E(V)$ is a non-trivial closed invariant subspace, hence π is reducible. □

Definition 3.1.8. Let G be a locally compact group. We define the *left regular representation* as $(L, L^2(G, d_l x))$ where $L_g f(x) = f(g^{-1}x)$ for any $g \in G$ and $f \in L^2(G, d_l x)$. Here $d_l x$ is a left-invariant Haar measure. In addition, we define the *right regular representation* as $(R, L^2(G, d_r x))$ where $R_g f(x) = f(xg)$ for any $g \in G$ and $f \in L^2(G, d_r x)$. Here $d_r x$ is a right-invariant Haar measure.

Before moving on, we remind ourselves of differential functions on a manifold. Let $f : \mathbb{R}^n \supset S \to V$ where V is a Hilbert space, and S is an open subset of $\mathbb{R}^n$. We define the partial derivative of f as

$$D_j f : S \to V, \qquad D_j f(x) = \left.\frac{d}{dt}\right|_{t=0} f(x + te_j)$$

with $1 \leq j \leq n$. We say $f \in C^1(S, V)$ if $D_1 f, \ldots, D_n f : S \to V$ exist and are continuous. In the same way we say $f \in C^k(S, V)$ if the mappings $D_1 f, \ldots, D_n f \in C^{k-1}(S, V)$ for any integer $k \geq 2$. We define $f \in C^\infty(S, V)$ if $f \in C^k(S, V)$ for all integers $k \geq 1$. We often say f is a C^∞ function or f is smooth.

If $\Omega, \Omega' \subset \mathbb{R}^n$ are open subsets of $\mathbb{R}^n$, and $\phi : \Omega \to \Omega'$ is a diffeomorphism, we can see that the mapping $\phi_* : f \mapsto f \circ \phi$ defines a continuous map $C^k(\Omega) \to C^k(\Omega')$ by using the chain rule various times. This allows us to define $C^k(M, V)$ functions on a smooth manifold M, hence on a Lie group G.

Definition 3.1.9. Let (π, V) be a continuous representation of G in a Hilbert space V. A vector $v \in V$ is said to be a C^∞*-vector* or a *smooth vector* if $\varphi_v : x \mapsto \pi(x)v$ is a smooth map $G \to V$. The space of C^∞-vectors is denoted by V^∞.

Theorem 3.1.10. *[14, Thm 3.15] If (π, V) is a continuous representation of G, then V^∞ lies dense in V.*

Lemma 3.1.11. *Let (π, V) be a continuous representation of G. Then V^∞ is a G-invariant vector space subspace of V.*

Proof. It is clear that V^∞ is a vectorspace. Let $v \in V^\infty$ and $y \in G$. Then

$$\pi(x)\pi(y)x = \pi(xy)v = \varphi_v(xy)$$

where $\varphi_v : G \to V, x \mapsto \pi(x)v$ is smooth. Since the right multiplication $G \to G, x \mapsto xy$ is smooth, the composition is smooth hence the mapping $x \mapsto \pi(x)[\pi(y)v]$ is smooth. Thus $\pi(x)v \in V^\infty$. □

Note that if $v \in V^\infty$ and $X \in \mathfrak{g}$, we can define

$$\pi_*(X)v := \left.\frac{d}{dt}\right|_{t=0} \pi(\exp tX)v. \tag{3.2}$$

Lemma 3.1.12. *[27, Lemmas 21.4 and 21.5] Let $X \in \mathfrak{g}$ and $v \in V^\infty$. Then $\pi_*(X)v \in V^\infty$, and the bilinear mapping $\mathfrak{g} \times V^\infty \to V^\infty$ defined by $(X, v) \mapsto \pi_*(X)v$ defines a representation of $\mathfrak{g}$. Finally if $x \in G$ we have*

$$\pi(x)\pi_*(X)v = \pi_*(\mathrm{Ad}(x)X)\pi(x)v.$$

Definition 3.1.13. Let (π, V) be a representation of a Lie group G, and assume G has a two-sided Haar measure. Define

$$\langle \pi(f)v, w\rangle_V := \int_G f(x)\langle \pi(x)v, w\rangle_V \, dx. \tag{3.3}$$

where $f \in L^1(G)$ if π is unitary, but for general π we take f to have compact support. Equivalently, $\pi(f)$ can be defined as

$$\pi(f)v := \int_G f(x)\pi(x)\, dx. \tag{3.4}$$

With this definition, one can show that

$$\pi(f)^\dagger = \pi(f^*), \qquad \pi(f)\pi(h) = \pi(f * h)$$

where $f^*(x) = \overline{f(x^{-1})}$, and $(f * h)(x) = \int_G f(xy^{-1})h(y)dy$.

3.2 K-finite vectors

One disadvantages when considering non-compact groups, is that many theorems such as the Peter-Weyl theorem, are not applicable. In fact, we will be concerned with groups that have no finite-dimensional representations.

Lemma 3.2.1. *Let G be linear connected semisimple group such that all the simple ideals of $\mathfrak{g}$ are non-compact, i.e. each simple ideal has a Killing form that is not negative definite. Then the non-trivial irreducible unitary representations of G are infinite dimensional.*

However there is still some use of the theory of compact Lie groups, for the subgroup $K \subset G$ is closed and compact for linear connected reductive groups by Theorem 2.1.4. This allows us to use theorems such as the Peter-Weyl theorem on the restriction to K, which will be useful again when considering representations of G. We will give a reminder of the Peter-Weyl theorem.

Theorem 3.2.2 (The Peter-Weyl theorem)**.** *[14] Let G be a compact Lie group. Then*

1. *Every irreducible unitary representation of G is finite-dimensional.*
2. *Let (π, V) be a unitary representation of G on a Hilbert space V. Then V decomposes into an orthogonal sum of finite-dimensional irreducible invariant subspaces.*
3. *Let (π, V) be a unitary representation of G on a Hilbert space V. For each irreducible unitary representation τ of G, define E_τ to be the orthogonal projection on the closure of the sum of all irreducible invariant subspaces of V that are equivalent with τ. Then $E_\tau = \pi(\bar{\alpha}_\tau)$ with $\alpha_\tau = d_\tau \chi_\tau$ where d_τ is the degree of τ and χ_τ is the character of τ, given by $\chi_\tau(x) = \mathrm{Tr}(\tau(x))$.*

Let us denote the set of (equivalence classes of) continuous irreducible representations of K by $\widehat{K}$ and we write $(\delta, V_\delta) \in \widehat{K}$ when (δ, V_δ) is a continuous irreducible representation of K.

Definition 3.2.3. Let (π, V) be a continuous representation of G. A vector $v \in V$ is called K-*finite* if the linear span $\mathrm{span}(\pi(K)v)$ is a finite dimensional vectorspace. The space of K-finite vectors is denoted as V_K.

Lemma 3.2.4. *If (π, V) is a continuous representation of G such that $\pi|_K$ is unitary, then V_K lies dense in V*

Proof. Since K is compact, we can use Theorem 3.2.2 on $(\pi|_K, V)$. Each invariant subspace is finite dimensional, hence K-finite. □

Definition 3.2.5. Let (π, V) be a continuous representation of G. If $(\delta, V_\delta) \in \widehat{K}$, we define the space

$$V[\delta] := \{v \in V \,|\, \mathrm{span}(\pi(K)v) \text{ decomposes as a } \textit{finite} \text{ direct sum of modules all equivalent to } (\delta, V_\delta)\}\,.$$

We call this set the *isotypical component of type* δ. If $F \subset \widehat{K}$ is a finite subset, we define

$$V[F] := \oplus_{\delta \in F} V[\delta].$$

Lemma 3.2.6. *[27, Prop. 3.5] Let (π, V) be a continuous representation of G. Then*

1. *For each $\delta \in \widehat{K}$, the natural mapping*

$$\varphi_\delta : V_\delta \otimes \mathrm{Hom}_K(V_\delta, V) \to V, \qquad v \otimes T \mapsto T(v)$$

   is a linear isomorphism with image $V[\delta]$, i.e. $V_\delta \otimes \mathrm{Hom}_K(V_\delta, V) \simeq V[\delta]$. The mapping intertwines the K representations $\delta \otimes 1$ and $\pi|_{V[\delta]}$.

2. *The space of K-finite vectors decomposes as*

$$V_K = \bigoplus_{\delta \in \widehat{K}} V[\delta]$$

Together with Lemma 3.2.4 and the previous lemma, it might happen that some isomorphism class only occurs with finite multiplicity.

Definition 3.2.7. Let (π, V) be a continuous representation of G. We say π is *admissible* if $\dim V[\delta] < \infty$ for all $\delta \in \widehat{K}$.

In other words, we say that a representation of G is admissible if the multiplicity of δ is a finite number for all $\delta \in \widehat{K}$. It is interesting to look at admissible representations, for they include all irreducible unitary representations.

Theorem 3.2.8. *[29, Thm. 3.4.10] Let (π, V) be an irreducible unitary representation of G. Then (π, V) is admissible.*

Moreover, admissible representations are of interest because they naturally give way to defining a $\mathfrak{g}$-module on V as well.

Lemma 3.2.9. *[27, Lemma 22.7] Let (π, V) be an admissible representation of G. Then for all $\delta \in \widehat{K}$, we have $V[\delta] \in V^\infty$, and V_K is a $\mathfrak{g}$-invariant subspace of V^∞ with respect to π_*.*

Definition 3.2.10. Let G be a linear connected reductive group. A $(\mathfrak{g}, K)$-module is a linear space V equipped with a representation (π, V) of K, as well as a $\mathfrak{g}$-module such that

1. For all $k \in K$ and $X \in \mathfrak{g}$ we have
$$\pi(k) \circ X = [\mathrm{Ad}(k)X] \circ \pi(k)$$
on V.

2. For all $v \in V$, the linear span $W_v = \pi(K)v$ is finite dimensional and the restricted representation $\pi|_{W_v}$ of K in W_v is continuous.

3. For all $X \in \mathfrak{k}$ and $v \in V$, we have that the two representations are compatible, i.e.
$$Xv = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp[tX])v$$

Remark 3.2.11. By Lemma 3.2.9 any admissible representation (π, V) of G has V_K as a $(\mathfrak{g}, K)$ module. By Lemma 3.1.12, V^∞ is also a $(\mathfrak{g}, K)$-module for any continuous representation.

We define the isotypical component $V[\delta]$ for a $(\mathfrak{g}, K)$-module in the same way as we did for a G-module, and by Lemma 3.2.6 we see that $V[\delta] \simeq V_\delta \otimes \mathrm{Hom}_K(V_\delta, V)$. In addition, since $V_K = V$ for a $(\mathfrak{g}, K)$-module, we also have
$$V = \oplus_{\delta \in \widehat{K}} V[\delta].$$

Definition 3.2.12. Let (π, V) and (ρ, V') be two representations of G. We say π and ρ are *infinitesimally equivalent* if the associated $(\mathfrak{g}, K)$-modules are equivalent, which is to say that there exists a linear invertible operator $A : V \to V'$ such that $A \circ \pi(k) = \rho(k) \circ A$ for all $k \in K$ as well as $A \circ X_\pi = X_\rho \circ A$ for all $X \in \mathfrak{g}$. Here we denoted X_π as the $\mathfrak{g}$-module with respect to π, and X_ρ as the $\mathfrak{g}$-module with respect to ρ.

Definition 3.2.13. Let V be a $(\mathfrak{g}, K)$-module. Then we say V is irreducible if 0 and V are the only closed invariant subspaces for both the representations of K and $\mathfrak{g}$.

Proposition 3.2.14. *[27, Cor. 22.12] Let (π, V) be an admissible representation of G, and let V_K be its associated $(\mathfrak{g}, K)$-module. Then the following statements are equivalent:*

1. *π is irreducible*

2. *V_K is an irreducible $(\mathfrak{g}, K)$-module*

This proposition shows that it is enough to look at the $(\mathfrak{g}, K)$-module V_K to conclude irreducibility of (π, V).

Corollary 3.2.15. *Let (π, V) be an irreducible admissible representation of G. If $L : V_K \to V_K$ is a linear operator commuting with $\pi_*(X)$ for all $X \in \mathfrak{g}$, then L is a scalar operator.*

3.3 The infinitesimal character and matrix coefficients

If (π, V) is irreducible, we know that the center of G acts as scalars in view of Schur's lemma. These scalars contain some structure of the representation. We will take a closer look at these scalars in this section.

To do that, we consider the complexification $\mathfrak{g}_{\mathbb{C}}$, and let $\mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We denote the universal envelopping algebra of $\mathfrak{g}_{\mathbb{C}}$ as $U(\mathfrak{g}_{\mathbb{C}})$. We write $Z(\mathfrak{g}_{\mathbb{C}})$ for the center of $U(\mathfrak{g}_{\mathbb{C}})$. It is known that any representation of $\mathfrak{g}$ can be uniquely lifted to a representation on $U(\mathfrak{g}_{\mathbb{C}})$ [12].

Since $\mathfrak{g}_{\mathbb{C}}$ is a reductive Lie algebra, we can do a root decomposition with respect to $\mathfrak{h}_{\mathbb{C}}$ to find a set of reduced roots $\Delta(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$. Fix a positive system Δ^+ for $\Delta(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$. Define

$$\mathcal{P} := \sum_{\alpha \in \Delta^+} U(\mathfrak{g}_{\mathbb{C}}) E_\alpha$$

where E_α spans $(\mathfrak{g}_{\mathbb{C}})_\alpha$. Note that $\mathfrak{h}_{\mathbb{C}}$ is abelian by construction, so $U(\mathfrak{h}_{\mathbb{C}})$ is the symmetric algebra over $\mathfrak{h}_{\mathbb{C}}$. The Weyl group $W = W(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$ corresponding with the root system $\Delta(\mathfrak{h}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$ acts on $\mathfrak{h}_{\mathbb{C}}$, hence also acts on $U(\mathfrak{h}_{\mathbb{C}})$. Let $\delta := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

Proposition 3.3.1. *[14, Lemma 8.17] $U(\mathfrak{h}_{\mathbb{C}}) \cap \mathcal{P} = \{0\}$ and $Z(\mathfrak{g}_{\mathbb{C}}) \subseteq U(\mathfrak{h}_{\mathbb{C}}) \oplus \mathcal{P}$.*

This proposition shows that if $Z \in Z(\mathfrak{g}_{\mathbb{C}})$, we can write it uniquely as $Z = H + P$, where $H \in U(\mathfrak{h}_{\mathbb{C}})$ and $P \in \mathcal{P}$. Now, let γ'_{Δ^+} be the projection operator $Z(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{h}_{\mathbb{C}})$. In addition to this, define the mapping $\sigma_{\Delta^+} : \mathfrak{h}_{\mathbb{C}} \to U(\mathfrak{h}_{\mathbb{C}})$ by

$$\sigma_{\Delta^+}(H) = H - \delta(H)1.$$

This mapping is an algebra homomorphism, and thus by the universal property of $U(\mathfrak{g})$ can uniquely be lifted to an endomorphism $\tilde{\sigma}_{\Delta^+} : U(\mathfrak{h}_{\mathbb{C}}) \to U(\mathfrak{h}_{\mathbb{C}})$ with $\tilde{\sigma}_{\Delta^+}(1) = 1$. In fact, this mapping $\tilde{\sigma}_{\Delta^+}$ is an automorphism, for

$$\begin{aligned} \tilde{\sigma}_{\Delta^+} \circ \tilde{\sigma}_{-\Delta^+}(H) &= \tilde{\sigma}_{\Delta^+}(H + \delta(H)1) = \tilde{\sigma}_{\Delta^+}(H) + \tilde{\sigma}_{\Delta^+}(\delta(H)1) \\ &= H - \delta(H)1 + \delta(H)1 = H. \end{aligned}$$

for any $H \in \mathfrak{h}_{\mathbb{C}}$, hence also for any $H \in U(\mathfrak{h}_{\mathbb{C}})$. In the same way, we can see $\tilde{\sigma}_{-\Delta^+} \circ \tilde{\sigma}_{\Delta^+} = I$, so $\tilde{\sigma}_{\Delta^+}$ is invertible, hence an automorphism.

Theorem 3.3.2. *[14, Thm. 8.18] Define the* Harish-Chandra homomorphism *as the mapping $\gamma : Z(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{h}_{\mathbb{C}})$ by*

$$\gamma := \tilde{\sigma}_{\Delta^+} \circ \gamma'_{\Delta^+}. \tag{3.5}$$

Then the Harish-Chandra homomorphism is an algebra homomorphism, and is an algebra isomorphism onto the algebra

$$U(\mathfrak{h}_{\mathbb{C}})^W := \{X \in U(\mathfrak{h}_{\mathbb{C}}) \,|\, wX = X\}$$

where W is the Weyl group. The Harish-Chandra homomorphism does not depend upon the choice of the positive system Δ^+.

Definition 3.3.3. Let $\Lambda \in (\mathfrak{h}_{\mathbb{C}})^*$. Then define $\chi_\Lambda : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ by

$$\chi_\Lambda(Z) := \Lambda(\gamma(Z)) \tag{3.6}$$

where we have extended Λ to an algebra homomorphism $U(\mathfrak{h}_{\mathbb{C}}) \to \mathbb{C}$.

Lemma 3.3.4. *[14, Prop. 8.20] Let $\Lambda \in (\mathfrak{h}_{\mathbb{C}})^*$.*

1. *If $w \in W(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ then $\chi_{w\Lambda} = \chi_\Lambda$.*
2. *If $\Lambda' \in (\mathfrak{h})^*$ is such that $\chi_{\Lambda'} = \chi_\Lambda$, then $\Lambda' = w\Lambda$ for some $w \in W(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$.*

Proof. The first assessment is quickly found, if we remember $\mathrm{im}(\gamma) \subset U(\mathfrak{h}_{\mathbb{C}})^W$, so

$$\chi_{w\Lambda}(Z) = w\Lambda(\gamma(Z)) = \Lambda(\mathrm{Ad}(w^{-1})\gamma(Z)) = \Lambda(\gamma(Z)) = \chi_\Lambda(Z).$$

The second assessment can be found by proving the contrapositive. Let $\Lambda' \neq w\Lambda$ for all $w \in W$. Since $\mathfrak{h}_{\mathbb{C}}$ is abelian, $U(\mathfrak{h}_{\mathbb{C}}) \simeq S(\mathfrak{h}_{\mathbb{C}}) \simeq P(\mathfrak{h}_{\mathbb{C}}^*)$ where $S(\mathfrak{h}_{\mathbb{C}})$ is the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$ and $P(\mathfrak{h}_{\mathbb{C}}^*)$ is the set of polynomials on $\mathfrak{h}_{\mathbb{C}}^*$. Since $\Lambda' \neq w\Lambda$ we can define a polynomial p on $(\mathfrak{h}_{\mathbb{C}})^*$ such that it is 1 on the set $W\Lambda$ and is 0 on $W\Lambda'$. By the previous isomorphisms p can be seen as an element of $U(\mathfrak{h})$. Define the polynomial $\tilde{p} := \frac{1}{|W|}\sum_{w\in W} wp$. It follows $\tilde{p}$ has the same properties as p, and is W-invariant. By Theorem 3.3.2 there exists $Z \in Z(\mathfrak{g}_{\mathbb{C}})$ such that $\gamma(Z) = \tilde{p}$. It then follows that $\chi_{\Lambda'}(Z) = \Lambda'(\gamma(Z)) = \gamma(Z)(\Lambda') = \tilde{p}(\Lambda') = 0$ while $\chi_\Lambda(Z) = \tilde{p}(\Lambda) \neq 0$. So $\chi_\Lambda \neq \chi_{\Lambda'}$. ☐

Proposition 3.3.5. *[14, Prop. 8.21] Any homomorphism $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ is of the form χ_Λ for some $\Lambda \in (\mathfrak{h}_{\mathbb{C}})^*$.*

Corollary 3.3.6. *Let (π, V) be an admissible representation of G such that $\pi(Z)$ acts as a scalar on the K-finite vectors for all $Z \in Z(\mathfrak{g}_{\mathbb{C}})$. Then the map $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ given by*

$$\pi(Z) = \chi(Z)I$$

is an algebra homomorphism $Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$, and is of the form χ_Λ for some $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$.

Definition 3.3.7. We say π has *infinitesimal character* Λ if (π, V) is an admissible representation such that $\pi(Z)$ acts as a scalar for all $Z \in Z(\mathfrak{g}_{\mathbb{C}})$ and such that this scalar is given by $\chi_\Lambda(Z)$. By Lemma 3.3.4 Λ is determined up to the action of the Weyl group.

Remark 3.3.8. This means if (π, V) is an irreducible admissible representation, it has an infinitesimal character.

Next, we will discuss the matrix coefficients. They are essential in representation theory in the compact-case, and will play a similar role in the non-compact case. We will introduce them now, but use them in more detail in Chapter 5.

Definition 3.3.9. Let (π, V) be a representation of G on a complex Hilbert space V. We define a *matrix coefficient* of π as the function $m_{u,v} : G \to \mathbb{C}$ where $u, v \in V$, defined as

$$m_{u,v}(x) = \langle \pi(x)u, v\rangle_V.$$

If $u, v \in V_K$, we say $m_{u,v}$ is a *K-finite matrix coefficient.*

Lemma 3.3.10. *[14, Cor. 8.8] If (π, V) and (ρ, W) are infinitesimally equivalent admissible repesentations of G, then π and ρ have the same sets of K-finite matrix coefficients.*

Corollary 3.3.11. *[14, Cor. 8.12] Let (π, V) and (ρ, W) be irreducible admissible representations of G, such that there exist $v, v' \in V_K$ and $w, w' \in W_K$ such that*

$$\langle \pi(x)v, v'\rangle = \langle \rho(x)w, w'\rangle$$

for all $x \in G$. Then π and ρ are infinitesimally equivalent.

Finally, we mention the asymptotics of a representation. To do that, we give a definition.

Definition 3.3.12. Let (τ_1, U_1) and (τ_2, U_2) be finite-dimensional continuous representations of K and set $\tau = (\tau_1, \tau_2)$. We say a function $\psi \in C^\infty(G, \mathrm{Hom}_{\mathbb{C}}(U_2, U_1))$ is *τ-spherical* if

$$\psi(k_1 x k_2) = \tau_1(k_1)\psi(x)\tau_2(k_2).$$

Denote the space of all such functions by $C^\infty(\tau_1{:}G{:}\tau_2)$. The space $C_c^\infty(\tau_1{:}G{:}\tau_2)$ is defined similarly.

It is natural to consider this kind of functions. For if we have an admissible representation (π, V) of G, then $\pi|_K = \sum_{\delta\in\widehat{K}} m_\delta \delta$ where $m_\delta \in \mathbb{N}_0$ is finite. Consider two finite subsets $F_1, F_2 \subset \widehat{K}$, then $\pi|_{V[F_i]}$, seen as a repesentation of K, is given by

$$\pi|_{V[F_i]} = \sum_{\delta\in F_i} m_\delta \delta.$$

We denote $\tau_i := \pi|_{V[F_i]}$, and let $E_i : V \to V[F_i]$ be the orthogonal projection. By the Peter-Weyl theorem this projection exists. Then the function

$$F_\tau : G \to \mathrm{Hom}_{\mathbb{C}}(V[F_2], V[F_1]), \qquad F_\tau(x) := E_1\pi(x)E_2 \tag{3.7}$$

is a τ-spherical function. If in addition π is irreducible, then by Corollary 3.3.6 we see that we get a system of differential equations given by

$$R_Z F_\tau = \chi_\Lambda(Z)F_\tau, \qquad \forall Z \in Z(\mathfrak{g}_{\mathbb{C}}). \tag{3.8}$$

One can solve this system of differential equations by using the τ-spherical properties. These results will not be needed in this thesis, however later on we will find a specific τ-spherical function where solving this system of differential equations is needed to find certain mappings we need. For details on solving this system of differential equations, we refer to [14, Sect. VIII.8].

4 Induced Representations

4.1 Induced Representations

After having encountered some general representation theory, it is time to focus on a specific type of representation: induced representations. We choose to look at this specific type because of the Langlands classification, which classifies all possible irreducible unitary representations as quotients of induced representations. This classification will be described at the end of this chapter.

As stated before, we assume G to be linear connected reductive with Lie algebra $\mathfrak{g}$. In addition, remember that any continuous representation (π, V) has as vector space V a (not necessarily finite dimensional) Hilbert space.

Definition 4.1.1. Let H be a closed subgroup of G, and let (ξ, V_ξ) be a continuous representation of H. Define a subspace of $C(G, V_\xi)$ by

$$C(G/H : \xi) := \{f : G \to V_\xi \text{ continuous} | f(gh) = \xi(h)^{-1} f(g) \, \forall g \in G \text{ and } h \in H\}. \tag{4.1}$$

The *induced representation* $\mathrm{ind}_H^G(\xi, \cdot)$ of G is the representation in $C(G/H : \xi)$ given by

$$[\mathrm{ind}_H^G(\xi, g) f](x) = f(g^{-1}x), \qquad f \in C(G/H : \xi). \tag{4.2}$$

Remark 4.1.2. If (ξ, V_ξ) is a finite dimensional representation, there is a natural geometric way of describing the induced representation as a representation on sections of the associated bundle $(G \times V_\xi)/H$. We will not pursue this interpretation, but the interested reader might consider [25, 27].

Theorem 4.1.3 (Frobenius Reciprocity Theorem)**.** *[25, Thm. 1.2] Let H be a closed subgroup of G, (ξ, V_ξ) be a continuous representation of H (the Hilbert space V_ξ is not necessarily finite dimensional), and (δ, V_δ) be a finite dimensional continuous representation of G. Then the mapping $\varphi : T \mapsto ev_e \circ T$ defines an isomorphism*

$$\mathrm{Hom}_G(V_\delta, C(G/H : \xi)) \simeq \mathrm{Hom}_H(V_\delta, V_\xi). \tag{4.3}$$

Induced representations are widely used in representation theory. For example, they are also used in the representation theory of the Poincaré group, as discussed in Appendix A. In our situation, when G is a linear connected reductive group, we will be interested in setting $H = P = M_P A_P N_P$ a parabolic subgroup of G. We can build a general representation of P by considering a continuous representation (σ, V_σ) of M_P and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. We note that $\mathfrak{a}_P^*$ can be identified as a linear subspace of $\mathfrak{a}^*$ in the following sense: we remember that $\mathfrak{a} = \mathfrak{a}_P \oplus \mathfrak{a}_{M,P}$ orthogonal with respect to $\langle \cdot, \cdot \rangle$. Then the projection $\mathfrak{a} \to \mathfrak{a}_P$ induces an inclusion $\mathfrak{a}_P^* \to \mathfrak{a}^*$ by

$$\mathfrak{a}_P^* = \{\alpha \in \mathfrak{a}^* | \alpha(x) = 0 \text{ for all } x \in \mathfrak{a}_{M,P}\}.$$

This way we can see $\mathfrak{a}_P^*$ as a linear subspace of $\mathfrak{a}^*$.

Then the representations on P are given by

$$\sigma \otimes e^\nu \otimes 1 : M_P A_P N_P \to \mathrm{End}(V_\sigma), \qquad (\sigma \otimes e^\nu \otimes 1)(man) = e^{\nu(\log a)} \sigma(m). \tag{4.4}$$

We see that this is a representation on V_σ. This representation can be induced, giving a representation of the group G. This motivates the following definition.

Definition 4.1.4. Let G be a linear connected reductive group, let $P = M_P A_P N_P$ be a parabolic subgroup of G, let (σ, V_σ) be a continuous representation of M_P and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. Define the *induced representation* for the parabolic subgroup as $\mathrm{Ind}_P^G(\sigma \otimes e^\nu \otimes 1) := \mathrm{ind}_P^G(\sigma \otimes e^{\nu+\rho_P} \otimes 1)$. Remember the definition of ρ_P is given in Equation (2.19). Then the induced representation acts on the following space

$$C(G/P : \sigma : \nu) := \{f : G \to V_\sigma \text{ continuous} | f(xman) = e^{-(\nu+\rho_P)\log(a)}\sigma(m)^{-1}f(x)\}. \tag{4.5}$$

where $x \in G$ and $man \in M_P A_P N_P$. We will take the Hilbert completion of $C(G/P : \sigma : \nu)$ with respect to the following norm

$$\|F\|^2 = \int_K \|F(k)\|^2_{V_\sigma}\, dk.$$

Then the induced representation acts as

$$[\mathrm{Ind}_P^G(\sigma \otimes e^\nu \otimes 1, g)f](x) := f(g^{-1}x). \tag{4.6}$$

We will also use the notation $U(P, \sigma, \nu, g) := \mathrm{Ind}_P^G(\sigma \otimes e^\nu \otimes 1, g)$.

Remark 4.1.5. Notice we introduced ρ_P in the definition of Ind_P^G. This addition is originating from adding a half density to the representation $\sigma \otimes e^\nu \otimes 1$. Multiplying two half-densities gives a density over which we can integrate, which allows us to ensure that the following lemma is true. For more information how to construct such a half density, we refer to [27, Chapt. 20]. Having said that, the Frobenius Recipocity Theorem still holds for the representation Ind_P^G.

Lemma 4.1.6. *[25, Lemma 2.1] Let $P = M_P A_P N_P$ be a parabolic subgroup, σ is a unitary representation of M_P, and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. Then it is possible to define an inner product on $C(G/P : \sigma : \nu)$, given by*

$$\langle \psi, \phi \rangle := \int_K \langle \psi(k), \phi(k) \rangle_{V_\sigma}\, dk \tag{4.7}$$

for $\phi, \psi \in C(G/K : \sigma : \nu)$. In addition, we have that the adjoint of the induced representation, with respect to this inner product, is given by

$$U(P, \sigma, \nu, x)^\dagger = U(P, \sigma, -\bar{\nu}, x^{-1}). \tag{4.8}$$

This also means that if $\nu \in i\mathfrak{a}_P^*$, that $U(P, \sigma, \nu)$ is a unitary representation.

Since G has various decompositions, it is possible to describe the induced representation in various ways. We will describe two equivalent ways of describing the induced representation, both of which will be used in this thesis.

1. *The induced picture*. The induced picture is the picture we have been discussing so far, where we consider $P = M_P A_P N_P$, (σ, V_σ) a unitary representation of M_P and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. Then let the Hilbert space $H_{P,\sigma,\nu}$ be the completion of $C(G/P : \sigma : \nu)$ with respect to $\langle \cdot, \cdot \rangle$ as in Lemma 4.1.6. Then one can describe the elements of $H_{P,\sigma,\nu}$ as measurable functions $\varphi : G \to V_\sigma$ satisfying

$$\varphi(xman) = e^{(-\nu-\rho_P)\log(a)}\sigma(m)^{-1}\varphi(x) \tag{4.9}$$

for $x \in G, m \in M_P, a \in A_P, n \in N_P$. The representation on $H_{P,\sigma,\nu}$ is then the unique extension of $U(P, \sigma, \nu, \cdot)$, which we will denote with the same symbol. Then we see

$$[U(P, \sigma, \nu, g)\varphi](x) = \varphi(g^{-1}x). \tag{4.10}$$

for any $\phi \in H_{P,\sigma,\nu}$.

2. *The compact picture*. By the Iwasawa decomposition, we can write $G = KM_PA_PN_P$ and so any $g \in G$ can be written as $g = kman$. Using the translational rule in Equation (4.9), we thus can define the functions purely on K, and any element in $K_P := K \cap M_P$ can be pulled out by Equation (4.9). In other words, we find a surjective map of $C(G/P : \sigma : \nu)$ to the following space:

$$C(K/K_P : \sigma) := \{f : K \to V_\sigma | f(km) = \sigma(m)^{-1} f(k) \, \forall k \in K, m \in K_P\}. \tag{4.11}$$

Taking the completion with respect to the inner product in Equation (4.7) of the space $C(K/K_P : \sigma)$ will give us the space $[L^2(K, V)]_\sigma$. Here $[L^2(K, V_\sigma)]_\sigma$ is the space of V_σ-valued L^2-functions relative to the Haar measure dk such that the functions transform in the same way as the functions in Equation (4.11). This construction also gives a surjective isometry between $H_{P,\sigma,\nu}$ and $[L^2(K, V_\sigma)]_\sigma$.

Then one can describe the induced representation on $[L^2(K, V_\sigma)]_\sigma$ as well by using said isometry. By the decomposition $G = KM_PA_PN_P$, we define $\kappa : G \to K$, $\mu_P : G \to \exp(\mathfrak{m}_P \cap \mathfrak{p})$ and $H_P : G \to \mathfrak{a}_P$ as projection maps such that $g \in G$ decomposes as

$$g = \kappa_P(g)\mu_P(g)e^{H_P(g)}n. \tag{4.12}$$

Then by using the definition of the induced representation, we see for $g \in G, k \in K$

$$\begin{aligned}[U(P, \sigma, \nu, g)\phi](k) = \phi(g^{-1}k) &= \phi\left(\kappa_P(g^{-1}k)\mu_P(g^{-1}k)e^{H_P(g^{-1}k)}n\right) \\ &= e^{(-\nu-\rho_P)H_P(g^{-1}k)}\sigma(\mu_P(g^{-1}k))^{-1}\phi(\kappa_P(g^{-1}k))\end{aligned} \tag{4.13}$$

for $\phi \in [L^2(K, V_\sigma)]_\sigma$. This description is an equivalent way of describing the induced representation, whereas the former one has a more advantageous definition of the representation, and the latter has a space that is independent of ν.

Lemma 4.1.7. *[14, Sect. VII.2] Let G be a linear connected reductive group, and let $P = M_PA_PN_P$ be a parabolic subgroup of G. Then the following statements are true:*

1. *Let (σ, V_σ) be a unitary representation of M_P. Then for any $\nu \in (\mathfrak{a}_P)^*_\mathbb{C}$, $U(P, \sigma, \nu, \cdot)$ is a continuous representation,*

2. *Let $P_* = M_*A_*N_*$ be a parabolic subgroup of M_P, such that $M_*(A_*A_P)(N_*N_P)$ is a parabolic subgroup of G. If (σ, V_σ) is a unitary representation of M_* and $\nu_* \in (\mathfrak{a}_*)^*_\mathbb{C}$ and $\nu \in (\mathfrak{a}_P)^*_\mathbb{C}$, then there is a canonical equivalence*

$$\mathrm{Ind}_P^G\left(\mathrm{Ind}_{P_*}^{M_P}(\sigma \otimes e^{\nu_*} \otimes 1) \otimes e^{\nu} \otimes 1\right) \simeq \mathrm{Ind}^G_{M_*(A_*A_P)(N_*N_P)}(\sigma \otimes e^{\nu+\nu_*} \otimes 1).$$

The equivalence is given by $f(\cdot) \mapsto f(\cdot)(e)$.

In previous chapter, we stated some theorems regarding admissible representations. It should come to no surprise that we wish to apply these to the induced representation. To do that, we first mention that the induced representation is admissble.

Lemma 4.1.8. *[14, Prop. 8.4] Let $P = M_PA_PN_P$ be a parabolic subgroup of G, and (σ, V_σ) is an irreducible unitary representation on M_P. Then $U(P, \sigma, \nu, \cdot)$ is admissible for any $\nu \in (\mathfrak{a}_P)^*_\mathbb{C}$.*

Thus we can immediately conclude $U(P, \sigma, \nu, \cdot)$ has an associated $(\mathfrak{g}, K)$-module whenever the above lemma is true. There is also a proposition that tells us that the induced representation as a $(\mathfrak{g}, K)$-module is finitely generated.

Proposition 4.1.9. *[29, Prop. 5.5.5] Let $P = M_P A_P N_P$ be a parabolic subgroup of G, and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. Then $U(P, \sigma, \nu)$ as a $(\mathfrak{g}, K)$-module is finitely generated, i.e. there exists a finite dimensional subspace V_F such that $U(P, \sigma, \nu)_*(U(\mathfrak{g}))V_F$ generates $C(G/P : \sigma : \nu)_K$. Here $U(P, \sigma, \nu)_*$ is defined as in Equation (3.2).*

Finally, the induced representation also has an infinitesimal character, given that the representation σ has one. Since we will choose σ to be irreducible unitary, σ does have one, hence giving the following lemma:

Lemma 4.1.10. *[14, Prop. 8.22] Let $P = M_P A_P N_P$ be a parabolic subgroup of G, let $\mathfrak{t}$ be a $\theta-$stable Cartan subalgebra of $\mathfrak{m}$, and let (σ, V_σ) be an irreducible unitary representation of M. If σ has infinitesimal character Λ_σ relative to $\mathfrak{t}_{\mathbb{C}}$, and if $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$, then $U(P, \sigma, \nu, \cdot)$ has infinitesimal character $\Lambda_\sigma + \nu$ relative to $(\mathfrak{a}_P \oplus \mathfrak{t})_{\mathbb{C}}$.*

Now that we have introduced the induced representation, we will give a specific operator. We will call these operators the standard intertwining operators, and these operators will be of use in the next chapter when we look at irreducibility of induced representations.

Definition 4.1.11. Let $P = M_P A_P N_P$ and $P' = M_P A_P N'_P$ be two parabolic subgroups. Let σ be a unitary representation of M_P, and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. We define the *standard intertwining operator* $A(P'{:}P{:}\sigma{:}\nu) : H_{P,\sigma,\nu} \to H_{P',\sigma,\nu}$ formally as

$$A(P'{:}P{:}\sigma{:}\nu)f(x) = \int_{\bar{N} \cap N'} f(x\bar{n})\, d\bar{n} \tag{4.14}$$

where $\bar{N}_P = \Theta N_P$, $x \in G$ and $f \in H_{P,\sigma,\nu}$. Note that the integral in this definition might be divergent. But when it is well-defined, it follows that

$$A(P'{:}P{:}\sigma{:}\nu)U(P, \sigma, \nu, \cdot) = U(P', \sigma, \nu, \cdot)A(P'{:}P{:}\sigma : \nu). \tag{4.15}$$

A proof of the standard intertwining mapping into $H_{P',\sigma,\nu}$ and the validity of (4.15) can be found in [14, Sect. VII.5].

As a result, we can also consider the parabolic subgroups P and $w^{-1}Pw$, where $w \in N_K(\mathfrak{a}_P)$, which gives the following definition.

Definition 4.1.12. For $w \in N_K(\mathfrak{a}_P)$ and $F \in H_{P,\sigma,\nu}$, define the mapping $R(w)$ as

$$R(w)F(x) = F(xw)$$

and define

$$A_P(w, \sigma, \nu) := R(w)A(w^{-1}Pw{:}P{:}\sigma{:}\nu). \tag{4.16}$$

It formally acts on $F \in H_{P,\sigma,\nu}$ as

$$A_P(w, \sigma, \nu)F(x) = \int_{\bar{N} \cap w^{-1}Nw} F(xw\bar{n})\, d\bar{n} \tag{4.17}$$

and thus satisfies, whenever previous integral is well-defined, the equation

$$A_P(w, \sigma, \nu)U(P, \sigma, \nu) = U(P, w\sigma, w\nu)A_P(w, \sigma, \nu).$$

As stated, convergence of the standard intertwining operator is not guaranteed. For example, it could be possible that the function itself is not locally integrable. However, once the integral converges, some interesting results apply. We give a summary in the following theorem.

Theorem 4.1.13. *[14, Thm. 8.38] Let G be a linear connected reductive group with compact center, let $P = M_P A_P N_P$ and $P' = M_P A_P N_P'$ be parabolic subgroups of G, let (σ, V_σ) be an irreducible unitary representation of M_P and let $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. If F is a K-finite function in $H_{P,\sigma,\nu}$, then $A(P'{:}P{:}\sigma,\nu)F$ and $A_P(w,\sigma,\nu)F$ are defined by convergent integrals if ν satisfies*

$$\langle \mathrm{Re}\,\nu - \rho_M, \beta\rangle > 0 \qquad \text{for every restricted root } \beta \in \Sigma(\mathfrak{g},\mathfrak{a}) \text{ such that } \mathfrak{g}_\beta \subset \mathfrak{n} \text{ but not in } \mathfrak{n}'.$$

Here ρ_M is the sum of half the roots (with counted multiplicity) of the linear connected reductive group M_P. In addition, $A(P'{:}P{:}\sigma{:}\nu)F$ and $A_P(w,\sigma,\nu)F$ extend to meromorphic functions of ν on $(\mathfrak{a}_P)^*_{\mathbb{C}}$ and the following formulas are valid on such functions F as identities of meromorphic functions in ν, provided Haar measures are normalized suitably.

1. *$A(P'{:}P{:}\sigma{:}\nu)U(P,\sigma,\nu,X)F = U(P',\sigma,\nu,X)A(P'{:}P{:}\sigma{:}\nu)F$ for all $X \in \mathfrak{g}$.*
2. *$A_P(w\sigma,\nu)U(P,\sigma,\nu,X)F = U(P,w\sigma,w\nu,X)A_P(w,\sigma,\nu)F$ for all $X \in \mathfrak{g}$ and all $w \in N_K(\mathfrak{a}_P)$.*
3. *$A(P''{:}P{:}\sigma{:}\nu)F = A(P''{:}P'{:}\sigma{:}\nu)A(P'{:}P{:}\sigma{:}\nu)F$ for all $P'' = M_P A_P N_P''$ and $P = M_P A_P N_P'$ such that $\mathfrak{n}_P'' \cap \mathfrak{n}_P \subseteq \mathfrak{n}' \cap \mathfrak{n}$.*
4. *$A_P(w_1 w_2, \sigma, \nu)F = A_P(w_1, w_2\sigma, w_2\nu)A_P(w_2,\sigma,\nu)F$ if $w_1, w_2 \in N_K(\mathfrak{a}_P)$ such that $w_1 w_2 \beta < 0$ for each $\mathfrak{n}_P$-positive root β in $(\mathfrak{g},\mathfrak{a}_P)$ with $w_2\beta < 0$*

4.2 The Langlands classification

As a closure of this chapter, we will be discussing the Langlands classification. For this, we need to discuss another type of representations: the tempered and discrete series representations. We assume that G is linear connected reductive with compact center from now on.

Proposition 4.2.1. *[14, 25, Prop. 9.6 and Lemma 4.1 respectively] Let (π, V) be an irreducible unitary representation of G. Then the following statements are equivalent:*

1. *π has a non-zero matrix coefficient that belongs to $L^2(G)$.*
2. *Every matrix coefficient of π belong to $L^2(G)$.*
3. *π is unitarily equivalent to an irreducible subrepresentation of the right regular representation $(R, L^2(G))$*

When these conditions are satisfied, there exists a positive number d_π such that

$$\int_G \langle \pi(x)u_1, v_1\rangle_V \overline{\langle \pi(x)u_2, v_2\rangle_V}\, dx = d_\pi^{-1}\langle u_1, u_2\rangle_V \overline{\langle v_1, v_2\rangle_V}$$

for all $u_1, u_2, v_1, v_2 \in V$.

Definition 4.2.2. If (π, V) satisfies one of the statements in Proposition 4.2.1, we say π *belongs to the discrete series*, or is a *discrete series representation*. We call d_π the *formal degree* of π.

Remark 4.2.3. Note that if G is compact, we immediately have that all irreducible unitary representations are in the descrete series.

Definition 4.2.4. [25, Lemma 4.4] Let (π, V) be an irreducible admissible representation π of G. If all K-finite matrix coefficients of π are in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$, we say π is a *tempered representation*. Note that discrete series representations are tempered representations.

With that, we are finally able to give the Langlands classification. The classification shows that any irreducible unitary representation of a linear connected semisimple group can be described in terms of induced representations.

Definition 4.2.5. Let G be a linear connected semisimple group, and fix a minimal parabolic subgroup $P_{min} = MAN$. We define *Langlands parameters* as a triple (P, σ, ν) where $P = M_P A_P N_P$ is a standard parabolic subgroup of G, σ is an irreducible tempered unitary representation of M_P, and $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$ with $\operatorname{Re}\nu$ in the *open* positive Weyl chamber with respect to P, i.e.

$$\langle \operatorname{Re}\nu, \alpha\rangle > 0 \qquad \text{for all positive roots of } (\mathfrak{g}, \mathfrak{a}_P).$$

Theorem 4.2.6 (The Langlands classification)**.** *[14, 25, Thm. 8.54 and Thm. 5.1 respectively] Let G be a linear connected semisimple group, and fix a minimal parabolic subgroup $P_{min} = MAN$.*

1. *If (P, σ, ν) is a set of Langlands parameters, then $U(P, \sigma, \nu)$, seen as a $(\mathfrak{g}, K)$-module, has a unique irreducible quotient $J(P, \sigma, \nu)$. $J(P, \sigma, \nu)$ is infinitesimally equivalent with the image of $A(\bar{P}{:}P{:}\sigma{:}\nu)$ acting on the K-finite vectors of $U(P, \sigma, \nu)$. Here $\bar{P} = \Theta P$.*

2. *Assume (P_i, σ_i, ν_i) are Langlands parameters for $i = 1, 2$. If $J(P_1, \sigma_1, \nu_1) \simeq J(P_2, \sigma_2, \nu_2)$, then $P_1 = P_2$, $\nu_1 = \nu_2$ and $\sigma_1 \simeq \sigma_2$.*

3. *Every irreducible admissible representation (π, V) is equivalent to $J(P, \sigma, \nu)$ for some Langlands parameters (P, σ, ν).*

For a proof, we refer to [14, 25]. We remind ourselves that by Theorem 3.2.8, any irreducible unitary representation is automatically irreducible admissible. This means that for any irreducible unitary representation of G, there exists Langlands parameters (P, σ, ν) such that $\pi \simeq J(P, \sigma, \nu)$.

This way, we only need to consider quotients of induced representations to find all irreducible unitary representations. However, this characterization has some applicational difficulties. Not every admissible representation is necessarily unitary, so not all induced representations are unitary. Hence the question arises: is there a general method of finding the correct Langlands parameters to render $J(P, \sigma, \nu)$ unitary? In addition, tempered representations are not easily characterized.

The theorem tells us how to construct the irreducible representations, by looking at the image of $A(\bar{P}{:}P{:}\sigma{:}\nu)$, however these standard intertwining operators are tricky to explicitely calculate. Calculating each by hand might be hard work. Especially if it turns out that the induced representation was already irreducible to start with.

These problems discussed are highly non-trivial. We will discuss irreducibility in more detail in the next chapter, and only highlight the ideas to characterize the tempered representations. This will lead to a description of the R-group and a revision of the Langlands classification. The question to characterize the unitarity however is still unanswered at the moment of writing this thesis.

5 The R group

5.1 Discrete series

After the Langlands classification, the following step would be to classify all possible tempered representations. The following theorem characterizes the irreducible tempered representations. The original statement inquires global characters, which we have not discussed, so we will give the spirit of the theorem.

Theorem 5.1.1. *[14, Thm. 14.91] Let G be a linear connected reductive group with compact center. Then every irreducible tempered representation is itself an induced representation from a discrete series representation or a limit of discrete series reprensentation.*

Limits of discrete series are obtained by arguments involving tensor products of discrete series and finite dimensional representations. We will not go into detail, and if the reader is interested, we recommend [13]. We will be focussing on the discrete series representations. First we wish to give a deep theorem which is useful when one wishes to construct discrete series.

Definition 5.1.2. Let G be a connected semisimple Lie group. We call G *cuspidal* if

$$\text{rank}(G) = \text{rank}(K \cap G).$$

Theorem 5.1.3. *[11] A connected semisimple Lie group G has discrete series if and only if rank$(G) =$ rank$(K \cap G)$, i.e. G is cuspidal.*

In order to find tempered representations of M_P, where P is a parabolic subgroup of G, Theorem 5.1.1 tells us we need to consider discrete series and limits of discrete series. Since discrete series and limits of discrete series only appear in cuspidal groups, we can reduce the analysis of tempered representations on M_P to discrete series representations or limits of discrete series representations of cuspidal M_P. In fact, if M_P is linear connected semisimple, we have a characterisation of the discrete series.

Definition 5.1.4. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\lambda \in (\mathfrak{h}_\mathbb{C})^*$. We say λ is *analytically integral* if the following two requirements are fulfilled:

1. When $H \in \mathfrak{h}_\mathbb{C}$ satisfies $\exp(H) = 1$, then $\lambda(H) \in 2\pi i\mathbb{Z}$,
2. There exists a character ξ_λ on $T := \exp \mathfrak{h}$ such that $\xi_\lambda(\exp(H)) = e^{\lambda(H)}$ for all $H \in \mathfrak{h}$.

Theorem 5.1.5. *[14, Thm. 9.20] Let G be a linear connected semisimple cuspidal group. Let $\mathfrak{b} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$, and denote the roots in $(\mathfrak{g}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ by Δ. Let δ_G be half the sum of positive roots of Δ. We can also define the roots in $(\mathfrak{k}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$. Denote this set of roots by Δ_K*

Suppose $\lambda \in (i\mathfrak{b})^*$ is nonsingular, i.e. $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$. Also assume that the positive roots are given by

$$\Delta^+ = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\}.$$

If $\lambda + \delta_G$ is analytically integral, then there exists a discrete series representation π_λ of G with the following properties:

1. *π_λ has infinitesimal character χ_λ.*

2. *$\pi_\lambda|_K$ contains, with multiplicity one, the K-type with highest weight*

$$\Lambda = \lambda + \delta_G - 2\delta_K$$

where δ_K is the half sum of positive roots in $(\mathfrak{k}, \mathfrak{b})$.

3. *If Λ' is the highest weight of a K-type in $\pi_\lambda|_K$, then Λ' is of the form*

$$\Lambda' = \Lambda + \sum_{\alpha \in \Delta^+} n_\alpha \alpha \qquad n_\alpha \in \mathbb{N}_0.$$

Any two such constructed representations π_λ are equivalent if and only if their parameters λ are conjugate in $W(B : G)$ where $B = \exp(\mathfrak{b})$. We call λ the Harish-Chandra parameter *of the discrete series representation.*

However, M_P is not necessarily a linear connected semisimple group. To still be able to characterize the discrete series representations of M_P, we will be considering a broader class of groups.

Write for the moment $M_P = M$. Let M be such that $M = M_0 Z_M$ where $M_0 = \exp \mathfrak{m}_P$ (thus M_0 is a linear connected reductive group with compact center by Lemma 2.2.12.4) and Z_M is the center in M, meaning all elements $m \in M$ commute with M. In the general case $M_0 Z_M$ will be a subgroup of M and we need to induce the found representation on $M_0 Z_M$ to M. For more details, we refer to [14, page 470]. We will however assume $M = M_0 Z_M$ because the parabolic subgroups we will be looking at in Chapter 6 have this property.

So assume $M = M_0 Z_M$. We will describe the discrete series by looking at each factor individually. Let $\mathfrak{b}_P \subset \mathfrak{k} \cap \mathfrak{m}_P$ be a maximal abelian subspace, and define $B_P = \exp \mathfrak{b}_P$. Then $\mathfrak{a}_P \oplus \mathfrak{b}_P$ is a Cartan subalgebra of $\mathfrak{g}$, see Definition 2.1.18. If α is a real root of $\Delta((\mathfrak{a}_P \oplus \mathfrak{b}_P)_\mathbb{C} : \mathfrak{g}_\mathbb{C})$, we can form H_α and consider

$$\gamma_\alpha := \exp(2\pi i \|\alpha\|^{-2} H_\alpha).$$

Restricting α on $\mathfrak{a}_P$ and extending it by 0 on $\mathfrak{a}_{M,P}$, we get a root that is part of the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. The H_α we get in this case (by going through Definition 2.1.14) agrees with the already found value of H_α, and thus induces the same γ_α.

Lemma 5.1.6. *[14, Lemma 12.30] Let $F(B_P)$ be the group generated by all γ_α with α a real root of $\Delta((\mathfrak{a}_P \oplus \mathfrak{b}_P)_\mathbb{C} : \mathfrak{g}_\mathbb{C})$. Then $F(B_P)$ lies in the center of M and $M = M_0 F(B_P)$.*

We can now construct the discrete series of M. First we construct the discrete series of M_0, which is reductive at least. Then we can decompose $M_0 = M_{ss}(Z_M)_0$ where $M_{ss} = \exp[\mathfrak{m}_P, \mathfrak{m}_P]$ by Theorem 2.1.4. Then we can use Theorem 5.1.5 on M_{ss}. To make it explicit, take any $\lambda \in (i\mathfrak{b}_P)^*$ such that $\lambda - \delta_M$ is analytically integral and λ is nonsingular. The roots on M, denoted by Δ_M, are exactly the roots of $\Delta((\mathfrak{a}_P \oplus \mathfrak{b}_P)_\mathbb{C} : \mathfrak{g}_\mathbb{C})$ that vanish on $\mathfrak{a}_P$. Then

$$\delta_M = \frac{1}{2} \sum_{\alpha \in \Delta_M \text{ positive}} \alpha. \tag{5.1}$$

Also note that $(Z_M)_0$ is compact and central in M_0, hence by irreducibility of discrete series will act as a scalar. Therefore we see that we can build a discrete series $\pi_\lambda^{M_0}$ of M_0 whose restriction to M_{ss} is the discrete series π_λ of M_{ss} with Harish-Chandra parameter $\lambda|_{\mathfrak{b}_P \cap \mathfrak{m}_{ss}}$, and whose restriction to $(Z_M)_0$ is the scalar whose differential is given by $\lambda|_{\mathfrak{b}_P \cap Z_{\mathfrak{m}_P}}$. By Theorem 5.1.5, all discrete series of M_0 are found this way and two are equivalent if and only if the Harish-Chandra parameters are conjugate in $W(B_P : M_0)$.

Now we can construct the discrete series of M, by modifying previous argument with Lemma 5.1.6. Since $F(B_P)$ lies in the center of M, it again acts as a scalar for any discrete series we wish to construct. Let $\lambda \in (i\mathfrak{b}_P)^*$ be such that $\lambda - \delta_M$ is analytically integral and λ is nonsingular. Let χ be a character on $F(B_P)$, satisfying $\chi = \xi_{\lambda-\delta_M}$ on $F(B_P) \cap B_P$. Here $\xi_{\lambda-\delta_M}$ is defined in the requirement of analytically integral. Then there exists a discrete series $\pi(\lambda, \chi)$ of M such that

$$\pi(\lambda, \chi)|_{M_0} = \pi_\lambda^{M_0}, \qquad\qquad \pi(\lambda, \chi)|_{F(B_P)} = \chi$$

where $\pi_\lambda^{M_0}$ is a discrete series of M_0 with Harish-Chandra parameter λ, and χ is the scalar on $F(B_P)$. By construction, all discrete series of M are found this way, and two discrete series $\pi(\lambda_1, \chi_1)$ and $\pi(\lambda_2, \chi_2)$ are equivalent if λ_1 and λ_2 are conjugate in $W(B_P : M_0)$ and $\chi_1 = \chi_2$.

This description is constructive, and can be used to explicitely construct the discrete series of M. To refer to this construction, we put the result in a theorem:

Theorem 5.1.7. *[14] Let $P = M_P A_P N_P$ be a standard cuspidal parabolic subgroup of a linear connected semisimple group G such that $M_P = (M_P)_0 Z_{M_P}$. Let $\mathfrak{b}_P \subset \mathfrak{k} \cap \mathfrak{m}_P$ be a maximal abelian subalgebra and $B_P := \exp \mathfrak{b}_P$. Then all discrete series are characterized by the combination (λ, χ) where $\lambda \in (i\mathfrak{b}_P)^*$ such that $\lambda - \delta_{M_P}$ is analytically integral and λ is nonsingular, and χ is a character on $F(B_P)$ such that $\chi|_{F(B_P)\cap B_P} = \xi_{\lambda-\delta_{M_P}}$. Any two discrete series characterized by (λ_1, χ_1) and (λ_2, χ_2) are equivalent if and only if λ_1 and λ_2 are conjugate in $W(B_P : (M_P)_0)$ and $\chi_1 = \chi_2$.*

5.2 Normalized intertwining operators

Now that we know how to construct discrete series, we wish to investigate the irreducibility of the induced representation. We will devote the rest of this chapter to the analysis of irreducibility. Assume for a moment that $U(P, \sigma, \nu, \cdot)$ is a irreducible representation. Then the operator $A(P{:}\bar{P}{:}\sigma{:}\nu)A(\bar{P}{:}P{:}\sigma{:}\nu)$, for $\bar{P} = \Theta P$, must act as a scalar by Schur's lemma in Corollary 3.2.15. Denote this scalar by $\eta(\bar{P}{:}P{:}\sigma{:}\nu)$. This scalar can be factorized, which can be used to modify the standard intertwining operator to become a unitary operator.

But before we can give the lemma that states this, we need to discuss how to construct real rank 1 subgroups of G. These subgroups will be needed in the consideration of the η function.

Definition 5.2.1. Let G be linear connected reductive, and let $P = M_P A_P N_P$ be a parabolic subgroup. If α is a reduced root in $(\mathfrak{g}, \mathfrak{a}_P)$, denote $\mathfrak{g}^{(\alpha)}$ as the algebra generated by $\mathfrak{n}^{(\alpha)} = \sum_{c>0} \mathfrak{g}_{c\alpha}$, and $\overline{\mathfrak{n}} = \sum_{c>0} \mathfrak{g}_{-c\alpha}$. Note $\mathfrak{g}^{(\alpha)}$ is θ-stable, and let $G^{(\alpha)}$ be the corresponding analytic subgroup.

Lemma 5.2.2. *[14, p. 179 and 538] Let G be linear connected reductive, and $P = M_P A_P N_P$ a parabolic subgroup. Then $G^{(\alpha)}$ and $G^{(\alpha)} M_P$ are linear connected reductive and have real rank 1.*

Definition 5.2.3. Let G be linear connected reductive and let $P = M_P A_P N_P$ be a parabolic subgroup. Consider $G^{(\alpha)} M_P$ as in previous lemma. Let $N_P^{(\alpha)} = \exp \mathfrak{n}_P^{(\alpha)}$ and $\overline{N}_P^{(\alpha)} = \exp \overline{\mathfrak{n}}_P^{(\alpha)}$. There are two maximal parabolic subgroups of $G^{(\alpha)} M_P$ which we shall denote by

$$P^{(\alpha)} M_P := M_P[\exp(\mathbb{R} H_\alpha)] N_P^{(\alpha)}, \qquad\qquad \overline{P}^{(\alpha)} M_P := M_P[\exp(\mathbb{R} H_\alpha)] \overline{N}_P^{(\alpha)}.$$

Here H_α is defined as in Definition 2.1.14.

Lemma 5.2.4. *[14, Prop. 14.13 and Lemma 14.1] Let G be a linear connected reductive group with compact center, and let $P_1 = M_P A_P N_1$ and $P_2 = M_P A_P N_2$ be parabolic subgroups. If σ is an irreducible* *unitary representation of M_P, then there exists a complex-valued meromorphic function $\eta(P_2{:}P_1{:}\sigma{:}\nu)$ on $(\mathfrak{a}_P)^*_{\mathbb{C}}$ such that*

$$A(P_1{:}P_2{:}\sigma{:}\nu)A(P_2{:}P_1{:}\sigma{:}\nu) = \eta(P_2{:}P_1{:}\sigma{:}\nu)I \tag{5.2}$$

as an identity of meromorphic functions. The function $\eta(P_2{:}P_1{:}\sigma{:}\nu)$ satisfies the following statements:

1. *Switching the roles of P_1 and P_2 does not matter, i.e. $\eta(P_2{:}P_1{:}\sigma{:}\nu) = \eta(P_1{:}P_2{:}\sigma{:}\nu)$.*
2. *$\eta(P_2{:}P_1{:}\sigma{:}\nu) = \prod_\alpha \eta^{(\alpha)}(\bar{P}^\alpha M_P{:}P^{(\alpha)}M_P{:}\sigma{:}\nu|_{\mathbb{R}H_\alpha})$ where the product on the right side is taken over all reduced roots α that are positive for N_1 and negative for N_2. Here $\eta^{(\alpha)}$ is the η function in Equation (5.2) when we take G to be $G^{(\alpha)}M_P$ and $P_1 = P^{(\alpha)}M_P$ and $P_2 = \bar{P}^\alpha M_P$.*
3. *If $\dim A_P = 1$, we can consider $\eta(P{:}\bar{P}{:}\sigma{:}z\rho_{A_P})$ with ρ_{A_P} relative to P. Then the mapping $\eta(z) := \eta(\bar{P}{:}P{:}\sigma{:}z\rho_{A_P})$ decomposes as $\eta(z) = \gamma(z)\overline{\gamma(-\overline{z})}$ where $\gamma(z)$ is a meromorphic function on $\mathbb{C}$ and is real for $z \in \mathbb{R}$.*
4. *For any parabolic subgroup P we have $\eta(\bar{P}{:}P{:}\sigma{:}\nu) = \gamma(P{:}\bar{P}{:}\sigma{:}\nu)\gamma(\bar{P}{:}P{:}\sigma{:}\nu)$ where*

   $$\gamma(\bar{P}{:}P{:}\sigma{:}\nu) = \prod_\alpha \gamma^{(\alpha)}(\bar{P}^\alpha M_P{:}P^{(\alpha)}M_P{:}\sigma{:}\nu|_{\mathbb{R}H_\alpha})$$

   where the product on the right side is a product of $\gamma^{(\alpha)}$ functions, found by applying this lemma's third item to the second item, and is taken over all reduced roots α that are positive for N_1 and negative for N_2.

Definition 5.2.5. Assume the prerequisites of Lemma 5.2.4, and let σ be a discrete series on M_P. We define the *normalized intertwining operators* for any two parabolic subgroups $P_1 = M_PA_PN_1$, $P_2 = M_PA_PN_2$, and $w \in N_K(\mathfrak{a}_P)$ as

$$\mathscr{A}(P_2 : P_1{:}\sigma{:}\nu) := \gamma(P_2{:}P_1{:}\sigma{:}\nu)^{-1}A(P_2{:}P_1{:}\sigma{:}\nu) \tag{5.3}$$
$$\mathscr{A}_P(w,\sigma,\nu) := \gamma(w^{-1}Pw{:}P{:}\sigma{:}\nu)^{-1}A_P(w,\sigma,\nu). \tag{5.4}$$

Theorem 5.2.6. *[14, Lemma 14.18, 14.19 and 14.21]*

1. *$\mathscr{A}(P_2{:}P_1{:}\sigma{:}\nu)\mathscr{A}(P_1{:}P_2{:}\sigma{:}\nu) = I$, and Theorem 4.1.13 holds with the letter A replaced by $\mathscr{A}$.*
2. *If $P_3 = M_PA_PN_3$ then*

   $$\mathscr{A}(P_3{:}P_1{:}\sigma{:}\nu) = \mathscr{A}(P_3{:}P_2{:}\sigma{:}\nu)\mathscr{A}(P_2{:}P_1{:}\sigma{:}\nu).$$

3. *$\mathscr{A}(P_2{:}P_1{:}\sigma{:}\nu)$ and $\mathscr{A}_P(w,\sigma,\nu)$ extend to holomorphic functions of ν for ν imaginary, and are in addition unitary for ν imaginary. Hence $\mathscr{A}(P_2{:}P_1{:}\sigma{:}\nu)$ renders for ν imaginary $U(P_1,\sigma,\nu)$ unitary equivalent to $U(P_2,\sigma,\nu)$ and $\mathscr{A}_P(w,\sigma,\nu)$ renders $U(P,\sigma,\nu)$ unitary equivalent to $U(P,w\sigma,w\nu)$.*

If $w\sigma \simeq \sigma$ and $w\nu = \nu$, then by previous theorem we have that $\mathscr{A}_P(w,\sigma,\nu)$ is an intertwining operator, hence by Schur's lemma it tests reducibility of $U(P,\sigma,\nu)$. So it feels natural to consider the following group:

$$W_{\sigma,\nu} := \{s \in W(A_P : G) \,|\, s\sigma \simeq \sigma \text{ and } s\nu = \nu\}. \tag{5.5}$$

This group will be of importance later on, and will give an upper bound on the dimension of the space of intertwiners for imaginary ν.

Lemma 5.2.7. *[14, Lemma 14.22] Let $H \subseteq H'$ be locally compact groups with H closed and normal and H'/H cyclic of order n. Let w be an element of H' whose power meet all cosets of H'/H, and let (L, V) be an irreducible unitary representation of H such that L and wL are unitarily equivalent. Then it is possible to define $L(w)$ as an operator on V in exactly n ways, differing only by a n^{th} root of unity as a factor, such that L extends to a unitary representation of H' on V.*

Proposition 5.2.8. *[14, Prop. 14.23 and 14.38] Let G be linear connected reductive with compact center, let $P = M_P A_P N_P$ be a parabolic subgroup of G and σ a discrete series representation of M_P. If $w \in N_K(\mathfrak{a}_P)$ and if $w\sigma \simeq \sigma$, then we can define the normalized operators $\sigma(w)\mathscr{A}_P(w, \sigma, \nu)$. When applied to K-finite vectors of V_σ have the following properties:*

1. *$\sigma(w)\mathscr{A}_P(w, \sigma, \nu)U(P, \sigma, \nu, X) = U(P, \sigma, w\nu, X)\sigma(w)\mathscr{A}_P(w, \sigma, \nu)$ for all $X \in \mathfrak{g}$.*
2. *$[\sigma(w)\mathscr{A}_P(w, \sigma, \nu)]^\dagger = \sigma(w)^{-1}\mathscr{A}_P(w^{-1}, \sigma, -\bar{\nu})$.*

Furthermore, let $\nu \in i\mathfrak{a}_P^*$ and let $w_1, w_2 \in N_K(\mathfrak{a}_P)$ be two representatives of $W_{\sigma,\nu}$.

3. *If w_1 and w_2 are in a cyclic extension of $K_P = K \cap M_P$, and if $\sigma(w_1)$ and $\sigma(w_2)$ are compatibly defined, then*

$$\sigma(w_1)\mathscr{A}_P(w_1, \sigma, \nu)\sigma(w_2)\mathscr{A}_P(w_2, \sigma, \nu) = \sigma(w_1w_2)\mathscr{A}_P(w_1w_2, \sigma, \nu). \tag{5.6}$$

4. *Whether or not w_1 and w_2 are in a cyclic extension of K_P, we get*

$$\sigma(w_1)\mathscr{A}_P(w_1, \sigma, \nu)\sigma(w_2)\mathscr{A}_P(w_2, \sigma, \nu) = c\sigma(w_1w_2)\mathscr{A}_P(w_1w_2, \sigma, \nu). \tag{5.7}$$

   with $c \in \mathbb{C}$ satisfying $|c| = 1$, determined by $\sigma(w_1w_2)\sigma(w_1)^{-1}\sigma(w_2)^{-1} = cI$.

5.3 Eisenstein integrals

The next tool we need is the Eisenstein integral. To get an intuition of the definition, we show that certain spaces are isomorphic. To do that, we remark that we have not fully used the K-finite theory. So let us do that now.

By Proposition 4.1.9 we know that $U(P, \sigma, \nu)$ is finitely generated as a $(\mathfrak{g}, K)$-module. Hence there exists a finite collection of K-finite vectors $\{v_1, \dots, v_n\}$ such that $C(K/K_P : \sigma)_K$ is generated by this collection of vectors. Since each vector is K-finite, we see that the collection $\{v_1, \dots, v_n\}$ lies in the isotypical component of some finite $F \subset \widehat{K}$, i.e.

$$\text{span}\left(\sum_{j=1}^{n} U(P, \sigma, \nu, K)v_j\right) \subset C(K/K_P : \sigma)[F].$$

We know by Lemma 3.2.4 that the K-finite vectors lie dense in the total space. In other words, to describe the representation completely, it is enough to know how the representation behaves on $C(K/K_P : \sigma)[F]$. So fix a set $F \subset \widehat{K}$ that will generate the representation space. We will abbreviate $U(P, \sigma, \nu, \cdot)$ to U_ν for the rest of this chapter.

We will make use of both the left-regular and the right-regular representation. To distinguish the isotypical components of the left-regular from the isotypical components of the right-regular representation, we will denote ${}_FC(K/K_P : \sigma)$ as the isotypical component $C(K/K_P : \sigma)[F]$ with respect to the *left-regular representation*, while we will write $C(K/K_P : \sigma)_F$ as the isotypical component $C(K/K_P : \sigma)[F]$ with respect to the *right regular representation*.

Now that we only need to consider a finite set F of irreducible representations of K, we can look at ${}_FC(K/K_P : \sigma)$. We claim that ${}_FC(K/K_P : \sigma) \simeq ({}_FC(K) \otimes V_\sigma)^{K_P}$. To prove this, we start with $F = \delta$. We can apply Lemma 3.2.6 to get ${}_\delta C(K/K_P : \sigma) \simeq W_\delta \otimes \mathrm{Hom}_K(W_\delta, C(K/K_P : \sigma))$ where the action on the left is given by U_ν and on the right by $L \otimes 1$. Here we denoted W_δ as the finite dimensional subspace of $C(K/K_P : \sigma)$ isomorphic to V_δ such that L is equivalent to δ. By the Frobenius Reciprocity Theorem, given in Theorem 4.1.3, we then get

$$\mathrm{Hom}_K(W_\delta, C(K/K_P : \sigma)) \simeq \mathrm{Hom}_{K_P}(W_\delta, V_\sigma),$$

which in turn is isomorphic to $(W_\delta^* \otimes V_\sigma)^{K_P}$ where the K_P-invariants are with respect to the action $L^\vee \otimes \sigma$. This results in

$${}_\delta C(K/K_P : \sigma) \simeq W_\delta \otimes (W_\delta^* \otimes V_\sigma)^{K_P}.$$

In addition, due to the finite dimensionality we see that $W_\delta \otimes W_\delta^* \simeq \mathrm{End}(W_\delta) \simeq {}_\delta C(K)$ by sending any $T \mapsto \mathrm{Tr}(\delta(\cdot)^{-1} \circ T)$. Therefore, we get that

$${}_\delta C(K/K_P : \sigma) \simeq ({}_\delta C(K) \otimes V_\sigma)^{K_P}$$

where the action on the right handed side is given by $(L \times R) \otimes \sigma$. Doing this procedure for all $\delta \in F$, we get

$${}_FC(K/K_P; \sigma) \simeq ({}_FC(K) \otimes V_\sigma)^{K_P}.$$

We can apply this isomorphism to $\mathrm{End}({}_FC(K/K_P : \sigma))$, giving

$$\begin{aligned}\mathrm{End}({}_FC(K/K_P : \sigma)) &\simeq \mathrm{End}([{}_FC(K) \otimes V_\sigma]^{K_P}) \\ &= \mathrm{End}([{}_FC(K) \otimes (V_\sigma)_{F_P}]^{K_P}) \\ &\simeq (\mathrm{End}({}_FC(K)) \otimes \mathrm{End}((V_\sigma)_{F_P}))^{K_P \times K_P}\end{aligned}$$

where

$$F_P := \{\xi \in \widehat{K}_P | \exists\, \delta \in F \text{ such that } \xi \text{ is a subrepresentation of } \delta|_{K_P}\}.$$

Denote ${}_{F_P^\vee}^{\;0}\mathscr{C}_\sigma(M_P)_{F_P}$ as the span of all K_P-finite matrix coefficients of σ. Then we see that $\mathrm{End}((V_\sigma)_{F_P})$ can be associated with ${}_{F_P^\vee}^{\;0}\mathscr{C}_\sigma(M_P)_{F_P}$ by finite dimensionality. Thus the final product now is:

$$\mathrm{End}({}_FC(K/K_P : \sigma)) \simeq (\mathrm{End}({}_FC(K)) \otimes {}_{F_P^\vee}^{\;0}\mathscr{C}_\sigma(M_P)_{F_P})^{K_P \times K_P}. \tag{5.8}$$

We note that

$$\mathrm{End}({}_FC(K)) \simeq {}_FC(K) \otimes ({}_FC(K))^* \simeq {}_{F \times F^\vee}C(K \times K).$$

The isomorphism sends $f \in {}_{F \times F^\vee}C(K \times K)$ to $\tilde{f} \in \mathrm{End}({}_FC(K))$ defined by

$$\tilde{f}(g)(k) = \int_K f(l, k) g(k) dk.$$

In the end, the final isomorphism becomes

$$\mathrm{End}({}_FC(K/K_P : \sigma)) \simeq ({}_{F \times F^\vee}C(K \times K) \otimes {}_{F_P^\vee}^{\;0}\mathscr{C}_\sigma(M_P)_{F_P})^{K_P \times K_P}. \tag{5.9}$$

Denote $(\tau_1, U_1) = (L|_K, {}_FC(K))$ and $(\tau_2, U_2) = (L^\vee|_K, ({}_FC(K))^*)$. Then $\tau_1 \times \tau_2$ is equivalent to $F \times F^\vee$. This together with previous isomorphism suggests the following definition:

Definition 5.3.1. Define the space ${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ as the space

$${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2) := C^\infty(\tau_1{:}M_P{:}\tau_2) \cap ({}^0\mathscr{C}_\sigma(M_P) \otimes {}_{F\times F^\vee}C(K\times K))$$

where $C^\infty(\tau_1{:}M_P{:}\tau_2)$ is defined in Definition 3.3.12, and ${}^0\mathscr{C}_\sigma(M_P)$ is the finite linear span of all matrix coefficients of σ. In other words, if $\psi \in {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$, then ψ is a function $\psi : M_P \to {}_{F\times F^\vee}C(K\times K)$ with components that are finite linear combinations matrix coefficients of σ such that

$$\psi(k_1mk_2)(l_1,l_2) = \tau_1(k_1)\psi(m)\tau_2(k_2)(l_1,l_2) = \psi(m)(k_1^{-1}l_1, k_2l_2) \tag{5.10}$$

for $k_1,k_2 \in K_P$ and $l_1,l_2\in K$. Here we used the notation $\tau_1(k_1)\psi(m)\tau_2(k_2)$ which is defined by

$$\tau_1(k_1)\psi(m)\tau_2(k_2) := [L\times L^\vee(k_1,k_2)]\psi(m).$$

By previous found isomorphism, we can identify ${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ with $\mathrm{End}({}_FC(K/K_P:\sigma))$. And since U_ν is admissible, $\mathrm{End}({}_FC(K/K_P:\sigma))$ is finite-dimensional. So ${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ is finite dimensional as well. The isomorphism can be explicitely found by Lemma 5.3.3. In fact, this isomorphism is an isometric isomorphism. To show that, we first need to define an inner product on ${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ and $\mathrm{End}({}_FC(K/K_P:\sigma))$.

Definition 5.3.2. We can define two inner products on the two spaces of interest. First note that ${}_FC(K/K_P:\sigma)$ is finite dimensional, so let $\{h_i\}_{1\le j\le n}$ be an orthogonal basis on ${}_FC(K/K_P:\sigma)$. We can define the Hilbert-Schmidt norm on $\mathrm{End}({}_FC(K/K_P:\sigma))$ by

$$d_\sigma^{-1}\langle T,S\rangle_{\text{HS}} := d_\sigma^{-1}\sum_{i=1}^n \langle Th_i, Sh_i\rangle$$

where $T,S\in \mathrm{End}({}_FC(K/K_P:\sigma))$, and d_σ is the formal degree of the discrete series representation σ, see Proposition 4.2.1.
In addition, we see that ${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ is a subspace of square-integrable functions on M_P with values in $L^2(K\times K)$. Hence ${}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ inherits a Hilbert space norm, given by

$$\langle\phi,\psi\rangle = \int_{M_P}\langle\phi(m),\psi(m)\rangle_{L^2(K\times K)}\,dm = \int_{M_P}\int_{K\times K}\phi(m)(k_1,k_2)\overline{\psi(m)(k_1,k_2)}\,dm\,dk_1\,dk_2$$

where $\phi,\psi\in{}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$.

Lemma 5.3.3. *[14, Lemma 14.2] For $T\in\mathrm{End}({}_FC(K/K_P:\sigma))$, define*

$$\psi_T^\sigma(m)(k_1,k_2) = \mathrm{Tr}(e^\dagger\sigma(m)eU_\nu(k_2)TU_\nu(k_1)^{-1})$$

where $e : {}_FC(K/K_P:\sigma)\to(V_\sigma)_{F_P}$ is the evaluation at the identity, and $e^\dagger$ is its adjoint with respect to the inner products. Then $\psi_T^\sigma\in{}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ and $T\mapsto\psi_T^\sigma$ is a linear isometric isomorphism.

Proof. Since the trace is linear, it is enough to prove this for $T = f\otimes\langle\cdot,g\rangle$ where $f,g\in{}_FC(K/K_P:\sigma)$. To show that $\psi_T^\sigma\in{}^0\mathscr{C}_\sigma(\tau_1:M_P:\tau_2)$, let $k_1,k_2\in K_P$ and $m\in M_P$. Then

$$\begin{aligned}\psi_T^\sigma(m)(k_1,k_2) &= \mathrm{Tr}(e^\dagger\sigma(m)eU_\nu(k_2)[f\otimes\langle\cdot,g\rangle]U_\nu(k_1)^{-1})\\ &= \mathrm{Tr}(\sigma(m)U_\nu(k_2)f(e)\otimes\langle U_\nu(k_1)^{-1}e^\dagger(\cdot),g\rangle)\\ &= \mathrm{Tr}(\sigma(m)U_\nu(k_2)f(e)\otimes\langle\cdot,eU_{-\bar\nu}(k_1)g\rangle_{V_\sigma})\\ &= \mathrm{Tr}(\sigma(m)f(k_2^{-1})\otimes\langle\cdot,g(k_1^{-1})\rangle_{V_\sigma}) = \langle\sigma(m)f(k_2^{-1}),g(k_1^{-1})\rangle_{V_\sigma}.\end{aligned}$$

which is just a matrix coefficient. So the coefficients are indeed matrix coefficients. If we take $l_1, l_2 \in K$, we in addition see

$$\begin{aligned}
\psi_T^\sigma(k_1 m k_2)(l_1, l_2) &= \mathrm{Tr}(e^\dagger \sigma(k_1 m k_2) e U_\nu(l_2)[f \otimes \langle \cdot, g\rangle] U_\nu(l_1)^{-1}) \\
&= \mathrm{Tr}(\sigma(k_1)\sigma(m)\sigma(k_2) e U_\nu(l_2) f \otimes \langle \cdot, g\rangle U_\nu(l_1)^{-1} e^\dagger) \\
&= \mathrm{Tr}(\sigma(m)\sigma(k_2) f(l_2^{-1}) \otimes \langle U_\nu(l_1)^{-1} e^\dagger(\cdot), g\rangle)\sigma(k_1)) \\
&= \mathrm{Tr}(\sigma(m) f((k_2 l_2)^{-1}) \otimes \langle \cdot, \sigma(k_1)^{-1} g(l_1^{-1}))\rangle_{V_\sigma}) \\
&= \mathrm{Tr}(\sigma(m) f((k_2 l_2)^{-1}) \otimes \langle \cdot, g((k_1^{-1} l_1)^{-1}))\rangle_\sigma) = \psi_T^\sigma(m)(k_1^{-1} l_1, k_2 l_2).
\end{aligned}$$

so indeed $\psi_T^\sigma(m) \in {}_{F \times F^\vee}C(K \times K)$. Therefore $\psi_T^\sigma \in {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$. Going through the isomorphism above, we see that this mapping describes the isomorphism.

Finally, we need to prove the isometry. Since we know that ${}_FC(K/K_P : \sigma)$ is finite dimensional, there is an orthogonal basis $\{h_i\}_{1 \le i \le n}$ of ${}_FC(K/K_P : \sigma)$. Then the trace of any operator is found by $\mathrm{Tr}(T) = \sum_{i=1}^n \langle T h_i, h_i\rangle$ for any $T \in \mathrm{End}({}_FC(K/K_P : \sigma))$. This way, we see that for any $T, S \in \mathrm{End}({}_FC(K/K_P : \sigma))$

$$\begin{aligned}
\langle \psi_T^\sigma, \psi_S^\sigma\rangle &= \int_{M_P}\int_K\int_K \psi_T^\sigma(m)(k_1,k_2)\overline{\psi_S^\sigma(m)(k_1,k_2)}\,dm\,dk_1\,dk_2 \\
&= \int_{M_P\times K\times K}\sum_{i,j=1}^n \langle e^\dagger\sigma(m)eU_\nu(k_2)TU_\nu(k_1)^{-1}h_i, h_i\rangle\overline{\langle e^\dagger\sigma(m)eU_\nu(k_2)SU_\nu(k_1)^{-1}h_i, h_i\rangle}\,dm\,dk_1\,dk_2 \\
&= \sum_{i,j=1}^n\int_{M_P\times K\times K}\langle\sigma(m)Th_i(k_2^{-1}), h_i(k_1^{-1})\rangle_{V_\sigma}\overline{\langle\sigma(m)Sh_j(k_2^{-1}), h_j(k_1^{-1})\rangle}_{V_\sigma}\,dm\,dk_1\,dk_2 \\
&= d_\sigma^{-1}\sum_{i,j=1}^n\int_{K\times K}\langle Th_i(k_2^{-1}), Sh_j(k_2^{-1})\rangle_{V_\sigma}\overline{\langle h_j(k_1^{-1}), h_i(k_1^{-1})\rangle}_{V_\sigma}\,dk_1\,dk_2 \\
&= d_\sigma^{-1}\sum_{i,j=1}^n\langle Th_i, Sh_j\rangle\overline{\langle h_j, h_i\rangle} \\
&= d_\sigma^{-1}\sum_{i=1}^n\langle Th_i, Sh_i\rangle = d_\sigma^{-1}\langle T, S\rangle_{\mathrm{HS}}.
\end{aligned}$$

where we used Lemma 4.2.1, and the orthogonality of h_i. □

Definition 5.3.4. Define the Eisenstein integral for a given $\psi \in {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ as $E(P,\psi,\nu) : G \to {}_{F\times F^\vee}C(K\times K)$ by

$$E(P,\psi,\nu)(x) = \int_K \psi_\nu(xk)\tau_2(k)^{-1}dk$$

where $\psi_\nu : G \to C(K\times K)$ is defined by $\psi_\nu(kman) = \tau_1(k)e^{(\nu-\rho_P)\log(a)}\psi(m)$.

Lemma 5.3.5. *The Eisenstein integrals generalize the K-finite matrix coefficients of $U(P,\sigma,\nu)$, in such a way that*

$$E(P,\psi_T^\sigma,\nu)(x)(k_1,k_2) = \mathrm{Tr}(E_F U_\nu(k_1^{-1}xk_2)TE_F).$$

Here $E_F : C(K/K_P : \sigma) \to {}_FC(K/K_P : \sigma)$ is the orthogonal projection.

Proof. Again, by linearity we can assume $T = f \otimes \langle \cdot, g\rangle$. Then we get

$$\begin{aligned}
E(P,\psi_T^\sigma,\nu)(x)(k_1,k_2) &= \int \tau_1(\kappa_P(xk))e^{(\nu-\rho_P)H(xk)}\psi_T^\sigma(\mu_P(xk))\tau_2(k)^{-1}(k_1,k_2)dk\\
&= \int_K e^{(\nu-\rho_P)H_P(xk)}\psi_T^\sigma(\mu_P(xk))(\kappa_P(xk)^{-1}k_1, k^{-1}k_2)dk\\
&= \int_K e^{(\nu-\rho_P)H_P(xk)}\operatorname{Tr}(e^\dagger\sigma(\mu_P(xk))eU_\nu(k^{-1}k_2)f\otimes\langle\cdot,g\rangle U_\nu(\kappa_P(xk)^{-1}k_1)^{-1})dk\\
&= \int_K e^{(\nu-\rho_P)H_P(xk)}\operatorname{Tr}(\sigma(\mu_P(xk))f(k_2^{-1}k)\otimes\langle\cdot,g(k_1^{-1}\kappa_P(xk))\rangle)dk\\
&= \int_K \operatorname{Tr}(f(k_2^{-1}k)\otimes\langle\cdot,\sigma^{-1}(\mu_P(xk))e^{(\bar\nu-\rho_P)H_P(xk)}g(k_1^{-1}\kappa_P(xk))\rangle)\\
&= \int_K \operatorname{Tr}(U_\nu(k_2)f(k)\otimes\langle\cdot,U_{-\bar\nu}(x^{-1}k_1)g(k)\rangle)dk\\
&= \int_K \langle U_\nu(k_1xk_2^{-1})f(k),g(k)\rangle dk\\
&= \langle U_\nu(k_1^{-1}xk_2)f,g\rangle = \operatorname{Tr}(E_FU_\nu(k_1^{-1}xk_2)TE_F)
\end{aligned}$$

where μ_P, κ_P and H_P are defined as in the definition of the compact picture, see Equation (4.12), and E_F is the orthogonal projection $C(K/K_P:\sigma)\to {}_FC(K/K_P:\sigma)$. This means $f = E_Ff$ and $g = E_Fg$. This projection operator comes from the Peter-Weyl theorem given in 3.2.2 applied to $U_\nu|_K$. We also used the fact that $U_\nu(x)^\dagger = U_{-\bar\nu}(x^{-1})$ by Lemma 4.1.6. As a remark, if $k_1 = k_2 = 1$, we get

$$E(P,\psi_T^\sigma,\nu)(x)(1,1) = \langle U_\nu(x)f,g\rangle.$$

which is just a matrix-coefficient of U_ν. □

Note that the Eisenstein integrals are τ-spherical functions as defined in Definition 5.3.1, where $\tau_1 = L$ and $\tau_2 = L^\vee$. In addition, the Eisenstein integrals are eigenfunctions of $Z(\mathfrak{g}_\mathbb{C})$, hence they are solutions to the system of differential equations given in Equation (3.8). Using this, one can prove the following theorem. For details, we refer to [14, Sect. XIV.3].

Theorem 5.3.6. *[14, Thm 14.7] Let $P_1 = M_PA_PN_1$ and $P_2 = M_PA_PN_2$ be parabolic subgroups. In addition, let $\nu \in i\mathfrak{a}_P^*$ be regular (i.e. $\langle\nu,\alpha\rangle \neq 0$ for all roots α in $(\mathfrak{g},\mathfrak{a}_P)$). Then there exists a unique element*

$$c_{P_2|P_1}(s{:}\nu) \in \operatorname{Hom}({}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2), {}^0\mathscr{C}_{w\sigma}(\tau_1{:}M_P{:}\tau_2))$$

for $s \in W(A_P:G)$ and $w \in N_K(\mathfrak{a}_P)$ a representative such that

$$e^{\rho_{P_2}(\log a)}E(P_1,\psi,\nu,ma) = \sum_{s\in W(A_P:G)}[c_{P_2|P_1}(s{:}\nu)\psi](m)e^{(s\nu)\log(a)} + \mathcal{O}(1) \qquad (a \xrightarrow{P_2} \infty)$$

where $a \xrightarrow{P} \infty$ means $\log(a) \in \mathfrak{a}_P^+$ and $\alpha(\log(a)) \to \infty$ for all postive roots α of $\mathfrak{a}_P$in P.

Corollary 5.3.7. *[14, Cor. 14.9] Each $c_{P_2|P_1}$ extends to a meromorphic function for $\nu \in (\mathfrak{a}_P)^*_\mathbb{C}$ and is holomorphic for ν imaginary and regular. Furthermore, if $T \in \operatorname{End}({}_FC(K/K_P:\sigma))$ and $s \in W(A:G)$ and $w \in N_K(\mathfrak{a}_P)$ as representative of s, then the following are true*

1. $c_{P|P}(s:\nu)\psi_T^\sigma = c_{P|wPw^{-1}}(1:s\nu)\psi^{w\sigma}_{R(w)TR(w)^{-1}}$.

2. *$c_{P|P}(1:\nu)\psi^{\sigma}_{T} = \psi^{\sigma}_{A(\bar{P}:P:\sigma:\nu)T}$ if the Haar measures are suitably normalized such that Corollary 2.2.17 holds.*

3. *$c_{P|P}(1:\nu)^{-1}c_{P|P}(s:s^{-1}\nu)\psi^{\sigma}_{T} = \psi^{w\sigma}_{T'}$ where $T' = \mathscr{A}_P(w,\sigma,w^{-1}\nu)T\mathscr{A}_P(w,\sigma,w^{-1}\nu)^{-1}$. In addition, if $w\sigma \simeq \sigma$ then $\psi^{w\sigma}_{T'} = \psi^{\sigma}_{T''}$ where $T'' = \sigma(w)T'\sigma(w)^{-1}$.*

5.4 Harish-Chandra's Completeness theorem

With that, we have all the ingredients to prove Harish-Chandra's Completeness Theorem. This theorem tells us that the algebra commuting with $U(P,\sigma,\nu,\cdot)$ is generated by the standard intertwining operators for imaginary valued ν. In this section we assume that ν is imaginary valued.

Theorem 5.4.1 (Harish-Chandra's Completeness Theorem)**.** *[14, Thm 14.31] Let G be a linear connected reductive group with compact center, and let $P = M_PA_PN_P$ be a parabolic subgroup of G. Let σ be a discrete series representation of M_P, and let ν be an imaginary-valued member of $(\mathfrak{a}_P)^*_{\mathbb{C}}$. Then if we denote $\mathcal{U}$ as the algebra of bounded operators commuting with $U(P,\sigma,\nu,\cdot)$, we get*

$$\mathcal{U} = \mathit{span}\{\sigma(w)\mathscr{A}_P(w,\sigma,\nu)\,|[w]\in W_{\sigma,\nu}\}. \tag{5.11}$$

Remark 5.4.2. If span$\{\sigma(w)\mathscr{A}_P(w,\sigma,\nu)\,|[w]\in W_{\sigma,\nu}\}$ is one-dimensional, Schur's lemma shows $U(P,\sigma,\nu)$ is an irreducible representation. Especially note if $W_{\sigma,\nu} = \{1\}$ then $U(P\sigma,\nu)$ is irreducible. Furthermore, the dimension of the commuting algebra is $\leq |W_{\sigma,\nu}|$.

Parts of the proof require Fourier and distribution theory on the group level. It is beyond the scope of this thesis to describe them in detail. However, we will mention them when needed.

Proof of Theorem 5.4.1. The proof consists of several steps. Equation (5.12) is the most involved step, and shall be postponed until later in this section. For simplicity, we define $U_\nu := U(P,\sigma,\nu,\cdot)$, $H := C(K/K_P:\sigma)$ and $H_F := {}_FC(K/K_P:\sigma)$.

We know that U_ν is a unitary representation of K. Hence by the Peter-Weyl theorem, given in Theorem 3.2.2, there exists a projection operator $E_F : H \to H_F$ where $E_F = U_\nu(\alpha_F)$. Here $\alpha_F = \sum_{\delta\in F} d_\delta\overline{\chi}_\delta$ where d_δ is the degree and χ_δ the character of the representation δ. It then holds for any $f\in H$ that

$$E_Ff(k) = U_\nu(\alpha_F)f(k) = \int_K \alpha_F(l)U_\nu(l)f(k)dl = \int_K \alpha_F(l)f(l^{-1}k)dl = \alpha_F *_K f(k),$$

and so if $f\in H_F$ we get $\alpha_F *_K f = f$. Equivalently, we see that $(f *_K \alpha_F)(k) = R(\overline{\alpha_F})f$.

Now consider $T\in \mathrm{End}(H_F)$ such that it commutes with $\sigma(w)\mathscr{A}_P(w,\sigma,\nu)$ for any $[w]\in W_{\sigma,\nu}$. We claim that

$$T \in E_FU_\nu(C_c^\infty(G))E_F. \tag{5.12}$$

Then by previous argument, we see

$$\begin{aligned} E_F\circ U_\nu(C_c^\infty(G))\circ E_F &= U_\nu(\alpha_F)\circ U_\nu(C_c^\infty(G))\circ U_\nu(\alpha_F)\\ &= U_\nu(\alpha_F *_K C_c^\infty(G) *_K \alpha_F) = U_\nu({}_FC_c^\infty(G)_{F^\vee}). \end{aligned}$$

Let $\mathcal{A}$ be the self-adjoint algebra generated by $\sigma(w)\mathscr{A}_P$ for $[w]\in W_{\sigma,\nu}$, and let $\mathcal{A}_F := E_F\mathcal{A}E_F$. Let us denote $'$ for the commutator of an algebra in $\mathrm{End}(H_F)$. By definition of $\mathcal{A}$ and previous calculations, one gets

$$U_\nu({}_FC_c^\infty(G)_{F^\vee}) = (\mathcal{A}_F)'.$$

Remember U_ν is admissible, so H_F is finite dimensional. Therefore, $\mathcal{A}_F$ is finite dimensional and thus automatically closed with respect to the strong operator topology. Hence by the Double Commutant Theorem (see for example [19, Th. 4.1.5]) we get

$$\mathcal{A}_F = (\mathcal{A}_F)'' = (U_\nu({}_F C_c^\infty(G)_{F^\vee}))' \tag{5.13}$$

We wish to prove that $\mathcal{A}$ is the commutator of $U_\nu(G)$ in $\mathrm{End}(H)$. To do that, we define the function

$$\varphi : \mathrm{End}_G(H) \to \mathrm{End}_{{}_F C_c^\infty(G)_{F^\vee}}(H_F), \qquad \varphi(T) = E_F \circ T \circ E_F.$$

We claim that this mapping is a bijection.

To prove injectivity, let $\varphi(T) = 0$. Because E_F is a projection we have $T|_{H_F} = 0$. Since U_ν is finitely generated as a $(\mathfrak{g}, K)$-module by Proposition 4.1.9 and the K-finite vectors are dense by Lemma 3.2.4, we get U_ν to be finitely generated by H_F. So H_F generates H, we thus it must be that $T \equiv 0$. So φ is injective. To show surjectivity, we show that the dimensions are equal. Since U_ν is finitely generated, we can write U_ν as

$$U_\nu = m_1\pi_{\omega_1} \oplus \ldots \oplus m_n\pi_{\omega_n}$$

where (π_{ω_j}, H_j) is an irreducible representation of G that is isomorphic to $\omega_j \in \widehat{G}$, and m_j is the multiplicity. Since U_ν is admissible and finitely generated, both $m_j, n < \infty$ for all j. Then by Schur's Lemma, we get

$$\dim\left(\mathrm{End}_G(H)\right) = \dim\left(\bigoplus_{j=1}^n \mathrm{End}_G(H[\omega_j])\right) = \sum_{j=1}^n m_j^2.$$

On the other hand, we claim that $\dim\left(\mathrm{End}_{{}_F C_c^\infty(G)_{F^\vee}}(H_F)\right) = \sum_{j=1}^n m_j^2$ as well. To show this, we claim $(H_j)_F$ is an irreducible ${}_F C_c^\infty(G)_{F^\vee}$-module. Indeed, we know H is finitely generated, hence there exists a $v \in H$ such that $\overline{U_\nu(C_c^\infty(G))v_j} = H_j$ by irreducibility of (π_{ω_j}, H_j). Then we see that $E_F[U(C_c^\infty(G))v] \subseteq (H_j)_F$ lies dense in $(H_j)_F$, and by finite dimensionality equals H_F. If $v_j \in (H_j)_F$, then obviously $E_F v_j = v_j$ and so

$$\begin{aligned}(H_j)_F &= E_F[U(C_c^\infty(G))v_j] = (E_F \circ U(C_c^\infty(G)) \circ E_F)v_j \\ &= U({}_F C_c^\infty(G)_{F^\vee})v_j.\end{aligned}$$

by previous calculations. So $(H_j)_F$ is an irreducible ${}_F C_c^\infty(G)_{F^\vee}$-module. Then again using Schur's lemma, we get

$$\dim\left(\mathrm{End}_{{}_F C_c^\infty(G)_{F^\vee}}(H_F)\right) = \dim\left(\bigoplus_{j=1}^n \mathrm{End}_{{}_F C_c^\infty(G)_{F^\vee}}((H_j)_F)\right) = \sum_{j=1}^n m_j^2 = \dim\left(\mathrm{End}_G(H)\right)$$

which proves the surjectivity of φ. Since $\varphi(\mathcal{A}) = \mathcal{A}_F = (E_F U(C_c^\infty(G))E_F)'$, we now easily see that

$$\mathcal{A} = \varphi^{-1}[(U({}_F C_c^\infty(G)_{F^\vee}))'] = \mathcal{U}$$

where $\mathcal{U}$ is the algebra of bounded operators commuting with U_ν in $\mathrm{End}(H)$. Finally, we note that $\mathcal{A}$ is given by

$$\mathcal{A} = \mathrm{span}\{\sigma(w)\mathscr{A}_P(w, \sigma, \nu) | [w] \in W_{\sigma,\nu}\}$$

for the latter algebra is already a closed self-adjoint algebra by Lemma 5.2.8. Therefore

$$\mathrm{span}\{\sigma(w)\mathscr{A}_P(w, \sigma, \nu) | [w] \in W_{\sigma,\nu}\} = \mathcal{U}$$

which proves the theorem. $\square$

The only part missing from the proof, is the proof of the validity of (5.12), which we will get into now. For a more detailed description, we refer to papers such as [28].

Definition 5.4.3. We define the space of *Schwartz functions* on $\mathbb{R}^n$ as a subspace of $C^\infty(\mathbb{R}^n)$ by

$$\mathscr{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid (1+\|x\|)^N \partial^\alpha f(x) \in L^2(\mathbb{R}^n) \text{ for all multi-indices } \alpha \in \mathbb{N}^n \text{ and } N \in \mathbb{N}\}. \tag{5.14}$$

Here we used the multi-index notation, defined as $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. In the same way, we define the *Schwartz space* for a linear connected reductive group G as

$$\mathscr{C}(G) := \{f \in C^\infty(G) | (1+\mu)^N L_U R_V f \in L^2(G) \text{ for all } U, V \in U(\mathfrak{g}) \text{ and } N \in \mathbb{N}\} \tag{5.15}$$

where $\mu : KAK \to [0, \infty)$ is defined by $\mu(k_1 a k_2) = \log(a)$ taking the KAK-decomposition into account.

Lemma 5.4.4. *[14, Prop. 12.16] The space $C_c^\infty(G)$ lies dense in $\mathscr{C}(G)$.*

Definition 5.4.5. Let $P = M_P A_P N_P$ be a parabolic subgroup. We define a *τ-spherical Schwartz function* on M_P as a function $f \in \mathscr{C}(M_P) \otimes {}_{F\times F^\vee} C^\infty(K \times K)$ such that

$$f(k_1 m k_2) = \tau_1(k_1) f(m) \tau_2(k_2)$$

where $m \in M_P$ and $k_1, k_2 \in K_P$ Here τ_1 and τ_2 and the definition of $\tau_1(k_1)f(m)\tau_2(k_2)$ are defined in Definition 5.3.1. In other words, f is a Schwartz function on M_P mapping into ${}_{F\times F^\vee} C^\infty(K \times K)$ such that it transforms in the same way as in Definition 5.3.1. We write $\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$ for the τ-spherical Schwartz functions on M_P.

Remember that we defined a τ-spherical function in Definition 3.3.12. For these specific τ_1, τ_2, we see that if $f \in C^\infty(\tau_1{:}G{:}\tau_2)$ we see that $f \in C^\infty(G) \otimes {}_{F\times F^\vee} C^\infty(K \times K)$ such that

$$f(k_1 x k_2) = \tau(k_1) f(x) \tau(k_2)$$

for $x \in G$ and $k_1, k_2 \in K$.

Lemma 5.4.6. *[28] Define the set*

$$^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2) := \oplus_{\pi \in (\widehat{M}_P)_{ds}} {}^0\mathscr{C}_\pi(\tau_1{:}M_P{:}\tau_2)$$

where $^0\mathscr{C}_\pi(\tau_1{:}M_P{:}\tau_2)$ is defined as in Definition 5.3.1. Here $(\widehat{M}_P)_{ds}$ denotes the collection of all discrete series representations of M_P. Then $^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$ is finite dimensional, and $E_\pi : {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2) \to {}^0\mathscr{C}_\pi(\tau_1{:}M_P{:}\tau_2)$ is a continuous orthogonal projection onto a finite dimensional space.

All these definitions and lemmas are needed to be able to define the Fourier transform with respect to the Eisenstein integrals. Just as in the classical Fourier theory, the Fourier transform does not necessarily send C^∞ functions into C^∞ functions, hence Schwartz-functions are needed.

Definition 5.4.7. Define the operator $\mathscr{F}_P : C_c^\infty(\tau_1{:}G{:}\tau_2) \to C^\infty(i\mathfrak{a}_P^*) \otimes {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$ by

$$\langle \mathscr{F}_P f(\nu), \psi \rangle := \int_G \langle f(x), E(P, \psi, \nu)(x) \rangle dx \tag{5.16}$$

where $\psi \in {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ and $E(P, \psi, \nu)(x)$ is the Eisenstein integral. Here $\sigma \in (\widehat{M}_P)_{ds}$. We call the function $\mathscr{F}_P f$ the *Fourier transform* of a τ-spherical function.

Theorem 5.4.8. *[28, Prop. 19.6] The integral given in Equation (5.16) converges absolutely for any $f \in \mathscr{C}(\tau_1{:}G{:}\tau_2)$ and defines a continuous linear map $\mathscr{F}_P : \mathscr{C}(\tau_1{:}G{:}\tau_2) \to \mathscr{S}(i\mathfrak{a}_P^*) \otimes {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$.*

Definition 5.4.9. Let $\psi \in C_c^\infty(i\mathfrak{a}_P^*) \otimes {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$. Then define the wave-packet function $\mathscr{W}_P\psi : G \to {}_{F\times F^\vee}C(K \times K)$ by

$$(\mathscr{W}_P\psi)(x) = \int_{i\mathfrak{a}_P^*} \frac{1}{\eta(\bar{P}{:}P{:}\sigma{:}\nu)} E(P, \psi, \nu)(x)\, d\nu. \tag{5.17}$$

Theorem 5.4.10. *[28, Prop 20.3] The integral given in Equation (5.17) converges absolutely for any $\psi \in \mathscr{S}(i\mathfrak{a}_P^*) \otimes {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2)$ and defines a continuous linear map $\mathscr{W}_P : \mathscr{S}(i\mathfrak{a}_P^*) \otimes {}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2) \to \mathscr{C}(\tau_1{:}G{:}\tau_2)$.*

The previous two definitions can be seen as the Fourier transform and its inverse. It might not be surprising that the composition of two might be of interest. For example, Fourier analysis on $\mathbb{R}$ shows that the Fourier transform and the wave packet are inverses up to a constant. The following theorem shows a similar result.

Theorem 5.4.11. *[14, Modified Lemma 14.36] There exists a $c \in \mathbb{C}\backslash\{0\}$ such that the following equation holds:*

$$E_\sigma(\mathscr{F}_P \circ \mathscr{W}_P)(\phi \otimes \psi)(\nu) = cE_\sigma\left(\sum_{s\in W(A_P:G)} \phi(s^{-1}\nu)c_{P|P}(1{:}\nu)^{-1}c_{P|P}(s{:}s^{-1}\nu)\psi\right). \tag{5.18}$$

for any $\phi \otimes \psi \in C_c^\infty(i\mathfrak{a}_P^*) \otimes {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ and $\nu \in i\mathfrak{a}_P^*$. Here E_σ is the orthogonal projection ${}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2) \to {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ where $\sigma \in (\widehat{M_P})_{ds}$.

The next thing we need, is a property of the Fourier analysis. This will prove to be the vital point of the proof of the validity of (5.12).

Lemma 5.4.12. *Let $f \in \mathscr{C}(\tau_1{:}G{:}\tau_2)$, and $T \in \mathrm{End}(H_F)$. Recall $H_F = {}_FC(K/K_P : \sigma)$. Then*

$$E_\sigma[(\mathscr{F}_P f)(\nu)] = d_\sigma \psi^\sigma_{E_F U(P,\sigma,\nu,\tilde{f})E_F} \tag{5.19}$$

where $\tilde{f} : G \to \mathbb{C}$ is the smooth function defined by $\tilde{f}(x) = f(x^{-1})(1,1)$, E_σ is the orthogonal projection ${}^0\mathscr{C}(\tau_1{:}M_P{:}\tau_2) \to {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$, E_F is the orthogonal projection $C(K/K_P : \sigma) \to {}_FC(K/K_P : \sigma)$ as defined in Lemma 5.3.5 and d_σ is the formal degree of σ.

Proof. Let $T \in \mathrm{End}(H_F)$ be arbitrary. Since $E_\sigma[\mathscr{F}_P f(\nu)] \in {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$ we get

$$\begin{aligned}
\langle E_\sigma[\mathscr{F}_P f(\nu)], \psi^\sigma_T\rangle &= \int_G \langle f(x), E(P, \psi^\sigma_T, \nu)(x)\rangle\, dx \\
&= \int_G \int_{K\times K} f(x)(k_1,k_2)\overline{E(P, \psi^\sigma_T, \nu)(x)(k_1,k_2)}\, dx\, dk_1\, dk_2 \\
&= \int_G \int_{K\times K} f(k_1^{-1}xk_2)(1,1)\overline{\mathrm{Tr}(E_F U_\nu(k_1^{-1}xk_2)TE_F)}\, dx\, dk_1\, dk_2 \\
&= \int_G \int_{K\times K} f(x)(1,1)\overline{\mathrm{Tr}(E_F U_\nu(x)TE_F)}\, dx\, dk_1\, dk_2
\end{aligned}$$

$$
\begin{aligned}
&= \int_G f(x)(1,1)\overline{\mathrm{Tr}(E_F U_\nu(x) T E_F)}\, dx \\
&= \int_G f(x)(1,1)\overline{\mathrm{Tr}(E_F U_\nu(x) E_F T E_F)}\, dx \\
&= \int_G f(x)(1,1)\, \mathrm{Tr}(E_F T^\dagger E_F U_\nu^\dagger(x) E_F)\, dx \\
&= \int_G f(x)(1,1)\, \mathrm{Tr}(E_F T^\dagger E_F U_\nu(x^{-1}) E_F)\, dx \\
&= \mathrm{Tr}(E_F T^\dagger E_F U(P,\sigma,\nu,\tilde{f}) E_F)
\end{aligned}
$$

where we used Lemma 5.3.5 and defined

$$
U(P,\sigma,\nu,\tilde{f}) := \int_G \tilde{f}(x) U(P,\sigma,\nu,x)\, dx. \tag{5.20}
$$

and $\tilde{f}(x) = f(x^{-1})(1,1)$. Note that for any $T, S \in \mathrm{End}(H_F)$ we find

$$
\mathrm{Tr}(S^\dagger T) = \sum_{i=1}^n \langle S^\dagger T h_i, h_i\rangle = \sum_{i=1}^n \langle T h_i, S h_i\rangle = \langle T, S\rangle_{\mathrm{HS}},
$$

hence showing

$$
\mathrm{Tr}(E_F T^\dagger E_F U(P,\sigma,\nu,\tilde{f}) E_F) = \langle E_F U(P,\sigma,\nu,\tilde{f}) E_F, T E_F\rangle_{\mathrm{HS}} = \langle E_F U(P,\sigma,\nu,\tilde{f}) E_F, T\rangle_{\mathrm{HS}}.
$$

Here $\{h_i\}_{1\le i\le n}$ is an orthogonal basis of H_F. Since $E_F U(P,\sigma,\nu,\tilde{f}) E_F$ lies in $\mathrm{End}(H_F)$, the isometric isomorphism in Lemma 5.3.3 tells us that

$$
\langle E_\sigma[\mathscr{F} f(\nu)], \psi_T^\sigma\rangle = \langle E_F U(P,\sigma,\nu,\tilde{f}) E_F, T\rangle_{\mathrm{HS}} = d_\sigma \langle \psi^\sigma_{E_F U(P,\sigma,\nu,\tilde{f}) E_F}, \psi_T^\sigma\rangle \tag{5.21}
$$

for all $T \in \mathrm{End}(H_F)$. Since $E_\sigma[\mathscr{F} f(\nu)] \in {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$, this means that

$$
E_\sigma[\mathscr{F} f(\nu)] = d_\sigma \psi^\sigma_{E_F U(P,\sigma,\nu,\tilde{f}) E_F}.
$$

□

Proof of the validity of the claim in (5.12). Let $T \in \mathrm{End}(H_F)$ be such that it commutes with $\sigma(w)\mathscr{A}_P(w,\sigma,\nu)$ for any $[w] \in W_{\sigma,\nu}$. By Corollary 5.3.7.3 we have

$$
c_{S|S}(1:\nu)^{-1} c_{S|S}(s:s^{-1}\nu)\psi_T^\sigma = \psi_T^\sigma \tag{5.22}
$$

for all $s \in W_{\sigma,\nu}$. We choose $\phi \in C_c^\infty(i\mathfrak{a}^*)$ such that $\phi(\nu) = \frac{1}{c|W_{\sigma,\nu}|}$ and $\phi(s^{-1}\nu) = 0$ for all $s \in W(A_P : G) \setminus W_{\sigma,\nu}$. This is achievable due to the finiteness of $W(A_P : G)$. Applying Theorem 5.4.11 to $\phi \otimes \psi_T^\sigma$ shows

$$
E_\sigma(\mathscr{F}_P \circ \mathscr{W}_P)(\phi \otimes \psi_T^\sigma)(\nu) = c E_\sigma \left(\sum_{s \in W_{\sigma,\nu}} \phi(\nu) \psi_T^\sigma \right) = E_\sigma(\psi_T^\sigma) = \psi_T^\sigma \tag{5.23}
$$

because $\psi_T^\sigma \in {}^0\mathscr{C}_\sigma(\tau_1{:}M_P{:}\tau_2)$. In addition, by Lemma 5.4.12 we see that

$$
E_\sigma[\mathscr{F}_P f(\nu)] = d_\sigma \psi^\sigma_{E_F U(P,\sigma,\nu,\tilde{f}) E_F}.
$$

We remember $\mathscr{W}_P(\phi \otimes \psi_T^\sigma)$ lies in $\mathscr{C}(\tau_1 : G : \tau_2)$. Note that $C_c^\infty(\tau_1 : G : \tau_2)$ lies dense in $\mathscr{C}(\tau_1 : G : \tau_2)$ by Lemma 5.4.4. Hence we can find a sequence $f_n \in C_c^\infty(\tau_1 : G : \tau_2)$ such that $f_n \to \mathscr{W}_P(\phi \otimes \psi)$. It then follows by continuity of the Fourier transform that

$$\begin{aligned}\psi_T^\sigma &= E_\sigma[\mathscr{F}_P\left[\mathscr{W}_P(\phi \otimes \psi_T^\sigma)\right](\nu)] = E_\sigma[\mathscr{F}_P[\lim_{n\to\infty} f_n](\nu)] \\ &= \lim_{n\to\infty} E_\sigma[\mathscr{F}_P(f_n)(\nu)] = \lim_{n\to\infty} d_\sigma \psi^\sigma_{E_F U(P,\sigma,\nu,\tilde{f}_n)E_F} \\ &= \lim_{n\to\infty} \psi^\sigma_{d_\sigma E_F U(P,\sigma,\nu,\tilde{f})E_F} = \lim_{n\to\infty} \psi^\sigma_{E_F U(P,\sigma,\nu,d_\sigma \tilde{f})E_F}.\end{aligned}$$

Hence by the linear isomorphism in Lemma 5.3.3, we get $T = \lim_{n\to\infty} E_F U(P,\sigma,\nu,d_\sigma \tilde{f}_n)E_F$. Since $f_n \in C_c^\infty(\tau_1{:}G{:}\tau_2)$, we see that

$$T \in \overline{E_F U(P,\sigma,\nu,C_c^\infty(G))E_F}.$$

But since $\mathrm{End}(H_F)$ is finite-dimensional, it must be true that $T \in E_F U(P,\sigma,\nu,C_c^\infty(G))E_F$. This proves the validity of (5.12), and completes the proof of Harish-Chandra's Completeness Theorem. □

5.5 R-group and a revised Langlands classification

Now that we know that any bounded operator commuting with $U(P,\sigma,\nu)$ is a linear combination of operators $\sigma(w)\mathscr{A}_P(w,\sigma,\nu)$ with $w \in W_{\sigma,\nu}$ for ν imaginary, we see that this gives an upper bound of $|W_{\sigma,\nu}|$ for the dimension of the commuting algebra. The group $W_{\sigma,\nu}$ will tell us something more about the irreducibility of the induced representation, which we will focus on next. We start with $P = M_P A_P N_P$ with $\dim(A_P) = 1$.

Theorem 5.5.1. *[14, Thm. 14.16] Let G be a linear connected reductive group, let $P = M_P A_P N_P$ be a parabolic subgroup with $\dim(A_P) = 1$, and let σ be a discrete series representation of M_P. Then $U(P,\sigma,0)$ is irreducible unless:*

1. *$|W(A_P : G)| = 2$,*
2. *$W_{\sigma,0} = W(A_P : G)$,*
3. *$\eta(\bar{P}{:}P{:}\sigma{:}\nu)$ has no pole at $\nu = 0$.*

If these conditions are fulfilled, $U(P,\sigma,0)$ is reducible.

Theorem 5.5.2. *[14, Thm. 14.15] Let $P = M_P A_P N_P$ be a parabolic subgroup, and let σ be a discrete series representation of M_P. Then $U(P,\sigma,\nu)$ is irreducible for all regular imaginary $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. Regular here means $\langle \nu, \alpha \rangle \neq 0$ for all roots α of $(\mathfrak{g}, \mathfrak{a}_P)$.*

Theorem 5.5.3. *[14, Thm. 14.27] Let $P = M_P A_P N_P$ be a parabolic subgroup with $\dim(A_P) = 1$, and let σ be a discrete series representation of M_P. Then $\eta(\bar{P} : P : \sigma : \nu)^{-1} = K p_\sigma(\nu)$ for $\nu \in i\mathfrak{a}_P^*$. Here $K \in \mathbb{C} \setminus \{0\}$ and $p_\sigma(\nu)$ is the* Plancherel density *of G.*

We will get to a description of the Plancherel density in a moment. But if we combine these three theorems it follows that if we take $P = M_P A_P N_P$ with $\dim(A_P) = 1$, then the only non-regular value of ν is given by $\nu = 0$. Since the poles of the η-function are described by the Plancherel density, we get the following result.

Corollary 5.5.4. *Let $P = M_P A_P N_P$ be a parabolic subgroup with $\dim(A_P) = 1$, and let σ be a discrete series representation of M_P, and let $\nu \in i\mathfrak{a}_P^*$. Then $U(P,\sigma,\nu)$ is irreducible for all $\nu \neq 0$. If $\nu = 0$, it is irreducible unless*

1. $|W(A_P : G)| = 2$,
2. $W_{\sigma,0} = W(A_P : G)$,
3. $p_\sigma(0) \neq 0$.

If these conditions are fulfilled, $U(P, \sigma, 0)$ is reducible.

The construction of the Plancherel density is somewhat tedious, but doable. We will use the following theorem to describe the Plancherel density.

Theorem 5.5.5. *[14, Thm. 14.26] Let G be linear connected reductive with compact center, and let $P = M_P A_P N_P$ be a parabolic subgroup of G such that $\dim(A_P) = 1$. Let $\nu \in i\mathfrak{a}_P^*$. Take $\mathfrak{b} \subset \mathfrak{k} \cap \mathfrak{m}_P$ a maximal abelian subspace, and let $\pi(\lambda, \chi)$ be a discrete series representation of M_P as constructed in Theorem 5.1.7. Let β be a positive root of $(\mathfrak{g}, \mathfrak{a}_P)$; the unique one if there is just one, or 2α if α is a reduced root and 2α is also a root in $(\mathfrak{g}, \mathfrak{a}_P)$. Then the Plancherel density $p_\sigma(\nu)$ is given by two cases, each involving a product over roots ϵ in $\Delta((\mathfrak{a}_P \oplus \mathfrak{b})_\mathbb{C} : \mathfrak{g}_\mathbb{C})$ and a nonzero constant c_σ depending on σ, but not on ν:*

1. *If rank$(G) >$ rank(K) then*

$$p_\sigma(\nu) = \prod_{\epsilon|_{\mathfrak{a}_P} = c\beta\,,\, c>0} \langle \lambda + \nu, \epsilon \rangle. \tag{5.24}$$

2. *If rank$(G) =$ rank(K), then β extends to a real root of $\Delta((\mathfrak{a}_P \oplus \mathfrak{b})_\mathbb{C} : \mathfrak{g}_\mathbb{C})$ and*

$$p_\sigma(\nu) = \left(\prod_{\epsilon|_{\mathfrak{a}_P} = c\beta\,,\, c>0} \langle \lambda + \nu, \epsilon \rangle \right) f_{\sigma,\beta}(\nu) \tag{5.25}$$

where

$$f_{\sigma,\beta}(\nu) = \begin{cases} \tan\left(\frac{\pi\langle \nu, \beta\rangle}{\|\beta\|^2}\right) & \text{if } \chi(\gamma_\beta) = -(-1)^{2\frac{\langle \rho_\beta, \beta\rangle}{\|\beta\|^2}} \\ \frac{1}{\tan\left(\frac{\pi\langle \nu, \beta\rangle}{\|\beta\|^2}\right)} & \text{if } \chi(\gamma_\beta) = +(-1)^{2\frac{\langle \rho_\beta, \beta\rangle}{\|\beta\|^2}} \end{cases} \tag{5.26}$$

Here ρ_β is half the sum of all members of $\Delta((\mathfrak{a}_P \oplus \mathfrak{b})_\mathbb{C} : \mathfrak{g}_\mathbb{C})$ whose restrictions to $\mathfrak{a}_P$ is a positive multiple of β.

With that, it is possible to calculate if $U(P, \sigma, \nu)$ is irreducible for parabolic subgroups with $\dim(A_P) = 1$. Of course, we wish to calculate the irreducibility of any parabolic subgroup. The following theorem tells us that these η-functions are involved in the irreducibility of the representation again.

Theorem 5.5.6. *Let $P = M_P A_P N_P$ be a parabolic subgroup, let σ be a discrete series representation of M_P, and $\nu \in i\mathfrak{a}_P^*$. Define*

$$R'_{\sigma,\nu} = \{r \in W_{\sigma,\nu} | \eta(r^{-1}Pr{:}P{:}\sigma{:}\nu) \text{ is regular at } \nu\}.$$

Then the operators $\sigma(r)\mathscr{A}_P(r, \sigma, \nu)$ are linearly independent for any $r \in R_{\sigma,\nu}$.

Together with Harish-Chandra's Completeness Theorem, we thus see that the dimensions of the commuting algebra is at maximum given by $|W_{\sigma,\nu}|$, and at minimal given by $|R'_{\sigma,\nu}|$. In fact, it can be shown that the dimension is exactly $|R'_{\sigma,\nu}|$. Moreover, we can decompose the η function by using Lemma 5.2.4. This will allow us to consider η-functions over rank-1 groups, hence allowing us to make use of Lemma 5.5.4. This all gives rise to the following theorem

Theorem 5.5.7. *[14, Thm. 14.43] Let G be a linear connected reductive group with compact center, and $P = M_P A_P N_P$ where M_P is cuspidal, and $\dim(A_P) \geq 2$. Suppose σ is in the discrete series of M_P and $\nu \in i\mathfrak{a}_P^*$. Then $W_{\sigma,\nu}$ (as defined in Equation (5.5)) is a semidirect product of the form*

$$W_{\sigma,\nu} = W'_{\sigma,\nu} \rtimes R_{\sigma,\nu}$$

with $W'_{\sigma,\nu}$ normal. This decomposition has the following properties:

1. *$W'_{\sigma,\nu}$ is the set of $s \in W_{\sigma,\nu}$ for which $\sigma(s)\mathscr{A}_P(s,\sigma,\nu)$ acts as a scalar.*

2. *The set $R_{\sigma,\nu}$ is a subgroup of $W_{\sigma,\nu}$ and is described by*

$$R_{\sigma,\nu} = \{r \in W_{\sigma,\nu} \mid \mu_{\sigma,\alpha}(\nu|_{\mathbb{R}H_\alpha}) \neq 0 \,\forall\alpha \text{ positive roots of } (\mathfrak{g},\mathfrak{a}_P) \text{ such that } r\alpha \text{ is a negative root}\} \tag{5.27}$$

   where $\mu_{\sigma,\alpha}$ is the Plancherel density for the group $G^{(\alpha)}M_P$. The group $G^{(\alpha)}M_P$ is defined in Definition 5.2.3. The Plancherel formula can be calculated by setting $G = G^{(\alpha)}M_P$ and $P = P^{(\alpha)}M_P$ in Theorem 5.5.5. Moreover, the unitary operators $\sigma(r)\mathscr{A}_P(r,\sigma,\nu)$ for $r \in R_{\sigma,\nu}$ are linearly independent and span the commuting algebra of $U(P,\sigma,\nu)$.

This $R_{\sigma,\nu}$ is often called the R group, and we see that if $R_{\sigma,\nu} = \{1\}$, then by Schur's lemma we get $U(P,\sigma,\nu)$ is irreducible. We remind ourselves that this theorem *only* holds for ν *imaginary* valued. To see this theorem in action, we will discuss an example in Section 6.2 where we use $SU(2,2)$ as our group.

Finally, we will adjust the Langlands classification. By Theorem 5.1.1, in order to get any irreducible tempered representation on M_P, we can induce from a discrete series representation or a limit of discrete series. We should however make sure that the induced representation gives an irreducible tempered representation. This is done in detail in [14, Sect. XIV.16]. Note that inducing this obtained tempered representation to G gives a double induced representation. Using Lemma 4.1.7, this is again an induced representation, only now with a different ν. This argument shows that we need to take $\operatorname{Re}\nu$ to lie in the *closed* positive Weyl chamber. This leads to a revised version of the Langlands classification.

Theorem 5.5.8 (Revised Langlands classification)**.** *[14, Thm 14.92] Let G be a linear connected semisimple group and fix a minimal parabolic subgroup $P_{min} = MAN$. Let $P = M_P A_P N_P$ be a cuspidal standard parabolic subgroup of G. Let σ be a discrete series or nondegenerate limit of discrete series of M_P, and let $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$ such that $\operatorname{Re}\nu$ lies in the* closed *positive Weyl chamber relative to P, i.e.*

$$\langle \operatorname{Re}\nu, \alpha\rangle \geq 0 \qquad \forall\alpha \text{ positive roots of } (\mathfrak{g},\mathfrak{a}_P).$$

Also suppose that the induced representation of the discrete series as in Theorem 5.1.1 is irreducible. Then the induced representation $U(P,\sigma,\nu)$ has a unique irreducible quotient $J(P,\sigma,\nu)$ and every irreducible admissible representation of G is of the form $J(P,\sigma,\nu)$ for some such triple (P,σ,ν). If in addition $J(P,\sigma,\nu) \simeq J(P',\sigma',\nu')$, then $P = P'$, $\nu = \nu'$ and $\sigma \simeq \sigma'$.

6 Summary and an example

One of the goals of this thesis, is to write down a path to find the irreducible unitary representations of a linear connected semisimple group G. These kind of representations will be useful in other fields of research, most notably physics. Therefore, it might be of added value if there is a small summary of what we have done, if the reader wants to build their own irreducible unitary representation. In the next section, we have applied such a method ourselves, where we focussed on $SU(2,2)$.

Of course, we wish to make full use of the revised Langlands classification, given in Theorem 5.5.8. To describe the irreducibles, we first of all need to find all standard parabolic subgroups. One describes this by first finding a Cartan involution θ on $\mathfrak{g}$, which defines the sets $\mathfrak{k}$, $K := \exp \mathfrak{k}$ and $\mathfrak{p}$. Choosing a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ gives you the decomposition as seen in Theorem 2.1.7. Note that K is compact. Taking any subset $F \subset \Pi$ of the simple roots of $\mathfrak{g}$ gives you by construction of Proposition 2.2.7 and Definition 2.2.11 the parabolic subgroups. One only need to consider cuspidal parabolic subgroups, as defined in Theorem 5.1.3. The other parabolic subgroups can safely be ignored.

Next is the construction of discrete series and limits of discrete series. We have only discussed discrete series, hence we will focus on them. For a description of the limits of discrete series, we refer to [14, Sect. XII.7]. All discrete series are constructed as in Theorem 5.1.7. Note that for the minimal parabolic subgroup we have $M_{min} \subset K$, so the discrete series are the irreducible unitary representations of M_{min}.

Finally, one simply needs to take any linear functional $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$, as long as $\operatorname{Re}\nu$ lies in the closed Weyl chamber. One can choose $\operatorname{Re}\nu = 0$ i.e. $\nu \in i\mathfrak{a}_P^*$ to immediately generate a unitary representation (see Lemma 4.1.6). Taking any other linear functional does not necessarily give a unitary representation, and should be further researched case-by-case. In our example, we will be concerned only with $\nu \in i\mathfrak{a}_P^*$ for the same reason.

To see whether the set of choices of (P, σ, ν) give an irreducible representation, we can calculate the R group, as described in Theorem 5.5.7, because we chose ν to be imaginary. If one can show that $R = \{1\}$, then the representation $U(P, \sigma, \nu)$ is irreducible, hence giving an irreducible unitary representation.

6.1 SU(2,2): an application of the theory

As stated, we will be looking at the explicit example of $SU(2,2)$. This section is inspired by [16], and we will prove parts of it. The rest of the article also goes into the case of $\operatorname{Re}\nu > 0$, which we shall not discuss. We start by observing that $SU(2,2)$ is a linear connected reductive group that has a simple Lie algebra. It can be described by

$$SU(2,2) := \{A \in GL(4,\mathbb{C}) \mid \det(A) = 1 \text{ and } \langle Ax, Ay\rangle_{2,2} = \langle x, y\rangle_{2,2},\ \forall x, y \in \mathbb{C}^4\} \tag{6.1}$$

where $\langle x, y\rangle_{2,2} = -(\bar{x}_1 y_1 + \bar{x}_2 y_2) + \bar{x}_3 y_3 + \bar{x}_4 y_4$ for $x_i, y_i \in \mathbb{C}$. One easily sees that this sesquilinear product can be written as $\langle x, y\rangle_{2,2} = \langle Jx, y\rangle_{st}$ where $\langle \cdot, \cdot\rangle_{st}$ is the standard inner product on $\mathbb{C}^4$, and

$$J = \begin{pmatrix} -\mathbf{1}_{2\times 2} & \\ & \mathbf{1}_{2\times 2} \end{pmatrix}$$

where $\mathbf{1}_{n\times n}$ is the $n\times n$ identity matrix. Rewriting the defining equations of $SU(2,2)$, gives $\langle JAx, Ay\rangle_{st} = \langle Jx, y\rangle_{st}$. Therefore we find that

$$A \in SU(2,2) \Leftrightarrow JA^\dagger JA = \mathbf{1}_{4\times 4}. \tag{6.2}$$

We remark $J^{-1} = J$. Then if we write $A \in GL(4,\mathbb{C})$ as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \text{Mat}(2, \mathbb{C})$ are 2×2 matrices, we see that

$$JAJ = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{6.3}$$

This way, we find that $SU(2,2)$ is described by

$$SU(2,2) = \left\{ A \in GL(4,\mathbb{C}) \,\middle|\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ where } a,b,c,d \in \text{Mat}(2,\mathbb{C}) \text{ such that } \det(A) = 1, \right. \\ \left. a^\dagger a - c^\dagger c = \mathbf{1}_{2\times 2}, a^\dagger b - c^\dagger d = 0, d^\dagger d - b^\dagger b = \mathbf{1}_{2\times 2} \right\} \tag{6.4}$$

Next, we turn to the Lie algebra. Note $J^2 = \mathbf{1}_{2\times 2}$. This way, if $x \in \text{Mat}(4, \mathbb{C})$, we see $J\exp(x)J = \exp(JxJ)$. Differentiating Equation (6.2) gives

$$\mathfrak{su}(2,2) := \text{Lie}(SU(2,2)) = \{x \in \mathfrak{gl}(4,\mathbb{C}) | \text{Tr}(x) = 0 \text{ and } Jx^\dagger J + x = 0, \}. \tag{6.5}$$

Writing out these equations in a similar way as we did in Equation (6.3), we find

$$\mathfrak{su}(2,2) = \left\{ x \in \mathfrak{gl}(4,\mathbb{C}) \,\middle|\, x = \begin{pmatrix} a & b \\ b^\dagger & d \end{pmatrix} \text{ with } a^\dagger = -a, d^\dagger = -d, \text{Tr}(a+d) = 0 \right\}. \tag{6.6}$$

We should also note that

$$\mathfrak{su}(2,2)_\mathbb{C} = \mathfrak{su}(2,2) \oplus i\mathfrak{su}(2,2) = \mathfrak{sl}(4,\mathbb{C}).$$

We will write $\mathfrak{g} := \mathfrak{su}(2,2)$ and $G := SU(2,2)$ for the rest of this chapter. Remember the Cartan involution is given by $\theta(x) = -x^\dagger$. Then the eigenspaces are

$$\mathfrak{k} := \{x \in \mathfrak{su}(2,2) | \theta x = x\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,\middle|\, \text{Tr}(a+d) = 0, a^\dagger = -a, d^\dagger = -d \right\} = \mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(2)), \tag{6.7}$$

$$\mathfrak{p} := \{x \in \mathfrak{su}(2,2) | \theta x = -x\} = \left\{ \begin{pmatrix} 0 & b \\ b^\dagger & 0 \end{pmatrix} \,\middle|\, b \in \mathfrak{gl}(2,\mathbb{C}) \right\}. \tag{6.8}$$

It is easy to see that $K = S(U(2) \times U(2))$. To find the Iwasawa decomposition, we need to find a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$. There are multiple choices, but to agree with [16], we choose the maximal abelian subalgebra to be

$$\mathfrak{a} = \left\{ \begin{pmatrix} & & c & 0 \\ & & 0 & d \\ c & 0 & & \\ 0 & d & & \end{pmatrix} \,\middle|\, c, d \in \mathbb{R} \right\}. \tag{6.9}$$

To describe the corresponding dual space $\mathfrak{a}^*$, we construct two linear operators that span $\mathfrak{a}^*$, namely $f_1 : \mathfrak{a} \to \mathbb{R}$ and $f_2 : \mathfrak{a} \to \mathbb{R}$ defined by

$$f_1\left(\begin{pmatrix} & & c & 0 \\ & & 0 & d \\ c & 0 & & \\ 0 & d & & \end{pmatrix}\right) = c, \qquad f_2\left(\begin{pmatrix} & & c & 0 \\ & & 0 & d \\ c & 0 & & \\ 0 & d & & \end{pmatrix}\right) = d. \tag{6.10}$$

Next we need to find the root spaces $\mathfrak{g}_\alpha$. We will calculate only one, and give the rest. If for example we try the following matrix, with $a \in \mathbb{R}$

$$\left[\begin{pmatrix} & & c & 0 \\ & & 0 & d \\ c & 0 & & \\ 0 & d & & \end{pmatrix}, \begin{pmatrix} ai & & -ai & \\ & 0 & & \\ ai & & -ai & \\ & & & 0 \end{pmatrix}\right] = \begin{pmatrix} 2aci & & -2aci & \\ & 0 & & \\ 2aci & & -2aci & \\ & & & 0 \end{pmatrix} = 2c \begin{pmatrix} ai & & -ai & \\ & 0 & & \\ ai & & -ai & \\ & & & 0 \end{pmatrix}.$$

we see that $2f_1$ is a root, with the root space given by

$$\mathfrak{g}_{2f_1} = \left\{ \begin{pmatrix} ai & & -ai & \\ & 0 & & \\ ai & & -ai & \\ & & & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}.$$

The rest can be found analogously, and we get the root decomposition as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha \tag{6.11}$$

where the root system is given by

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm 2f_1, \pm 2f_2, \pm(f_1 + f_2), \pm(f_1 - f_2)\}. \tag{6.12}$$

Each root space is given by

$$\mathfrak{g}_{2f_1} = \left\{ \begin{pmatrix} ai & & -ai & \\ & 0 & & \\ ai & & -ai & \\ & & & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}, \qquad \mathfrak{g}_{-2f_1} = \left\{ \begin{pmatrix} ai & & ai & \\ & 0 & & \\ -ai & & -ai & \\ & & & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_{2f_2} = \left\{ \begin{pmatrix} 0 & & & \\ & ai & & -ai \\ & & 0 & \\ & ai & & -ai \end{pmatrix} \middle| a \in \mathbb{R} \right\}, \qquad \mathfrak{g}_{-2f_2} = \left\{ \begin{pmatrix} 0 & & & \\ & ai & & ai \\ & & 0 & \\ & -ai & & -ai \end{pmatrix} \middle| a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_{f_1+f_2} = \left\{ \begin{pmatrix} 0 & \bar{z} & & -\bar{z} \\ -z & 0 & z & \\ & \bar{z} & 0 & -\bar{z} \\ -z & & z & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\}, \qquad \mathfrak{g}_{f_1-f_2} = \left\{ \begin{pmatrix} 0 & \bar{z} & & \bar{z} \\ -z & 0 & z & \\ & \bar{z} & 0 & \bar{z} \\ z & & -z & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

$$\mathfrak{g}_{-f_1+f_2} = \left\{ \begin{pmatrix} 0 & \bar{z} & & -\bar{z} \\ -z & 0 & -z & \\ & -\bar{z} & 0 & \bar{z} \\ -z & & -z & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\}, \qquad \mathfrak{g}_{-f_1-f_2} = \left\{ \begin{pmatrix} 0 & \bar{z} & & \bar{z} \\ -z & 0 & -z & \\ & -\bar{z} & 0 & -\bar{z} \\ z & & z & 0 \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

and finally

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} | [h, x] = 0 \; \forall h \in \mathfrak{a}\} = \mathfrak{a} \oplus \left\{ \begin{pmatrix} ai & & & \\ & -ai & & \\ & & ai & \\ & & & -ai \end{pmatrix} \middle| z \in \mathbb{C} \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} ai & & & \\ & -ai & & \\ & & ai & \\ & & & -ai \end{pmatrix} \middle| z \in \mathbb{C} \right\}.$$

Furthermore, one can see that $\Sigma(\mathfrak{g},\mathfrak{a})$ is a reduced root system of the form C_2. We can choose a system of positive roots, by setting

$$\Sigma^+ = \{f_1 - f_2, 2f_1, f_1 + f_2, 2f_2\}. \tag{6.13}$$

The simple roots are then given by $\Pi = \{f_1 - f_2, 2f_2\}$.

Next, we wish to find the non-conjugate parabolic subgroups $P_{\Pi'}$. For that, we go through Section 2.2, and thus require subsets Π' of the simple roots. We have the choice of $\Pi' = \Pi, \{2f_2\}, \{f_1 - f_2\}$ or $\emptyset$:

Consider $\Pi' = \Pi$: Then $\mathfrak{a}_\Pi = \{0\}$, and therefore $\mathfrak{m}_{\Pi'} = \mathfrak{g}$ and $\mathfrak{n}_{\Pi'} = \{0\}$. Then the parabolic subgroup is simply $P_\Pi = G$.

Consider $\Pi' = \emptyset$: For the empty set, we see that $\mathfrak{a}_\emptyset = \{x \in \mathfrak{a}\} = \mathfrak{a}$, and $\mathfrak{a}^\perp_{M,\Pi'} = \{0\}$, thus the parabolic subalgebra is $\mathfrak{q}_{min}$. Therefore we see that $\mathfrak{m}_\emptyset = \mathfrak{m}$ and $\mathfrak{n}_\emptyset = \mathfrak{n}$. Finding the parabolic subgroup needs some more work, for it contains a centrum part as well. Obviously, $A_\emptyset = A = \exp(\mathfrak{a})$ and $N_\emptyset = N = \exp \mathfrak{n}$. The M-component can be calculated as

$$\begin{aligned}
M_\emptyset = Z_K(\mathfrak{a}) \cdot \exp(\mathfrak{m}) &= \left\{ \left. \begin{pmatrix} e^{i\theta} & & & \\ & \pm e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & \pm e^{-i\theta} \end{pmatrix} \right| \theta \in \mathbb{R} \right\} \cdot \left\{ \left. \begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix} \right| \theta \in \mathbb{R} \right\} \\
&\simeq \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \right\} \times \left\{ \left. \begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix} \right| \theta \in \mathbb{R} \right\} \\
&\simeq \{\mathbf{1}, \gamma\} \times S^1
\end{aligned} \tag{6.14}$$

where $\gamma := \text{diag}(1,-1,1,-1)$, which has the property that $\gamma^2 = \mathbf{1}$, and S^1 is the circle group.

Consider $\Pi' = \{2f_2\}$: This parabolic subgroup is more involved than the previous ones. Using Equation (2.10) we see

$$\mathfrak{a}_{\{2f_2\}} = \ker(2f_2) = \left\{ \left. \begin{pmatrix} & & c & 0 \\ & & 0 & 0 \\ c & 0 & & \\ 0 & 0 & & \end{pmatrix} \right| c \in \mathbb{R} \right\}. \tag{6.15}$$

It is then immediate that

$$\mathfrak{a}_{M\{2f_2\}} = \left\{ \left. \begin{pmatrix} & & 0 & 0 \\ & & 0 & d \\ 0 & 0 & & \\ 0 & d & & \end{pmatrix} \right| d \in \mathbb{R} \right\}.$$

We can then read off of Equations (2.12-2.13) that

$$\mathfrak{m}_{\{2f_2\}} = \mathfrak{m} \oplus \mathfrak{a}_{M\{2f_2\}} \oplus \mathfrak{g}_{2f_2} \oplus \mathfrak{g}_{-2f_2} = \mathfrak{m} \oplus \left\{ \begin{pmatrix} 0 & & & \\ & ai & & \bar{z} \\ & & 0 & \\ & z & & -ai \end{pmatrix} \middle| a \in \mathbb{R}, z \in \mathbb{C} \right\} \tag{6.16}$$

$$\simeq \mathfrak{m} \oplus \mathfrak{su}(1,1), \tag{6.17}$$

$$\mathfrak{n}_{\{2f_2\}} = \mathfrak{g}_{2f_1} \oplus \mathfrak{g}_{f_1-f_2} \oplus \mathfrak{g}_{f_1+f_2}. \tag{6.18}$$

Going to the parabolic subgroup, naturally $A_{\{2f_2\}} = \exp(\mathfrak{a}_{\{2f_2\}})$ and $N_{\{2f_2\}} = \exp(\mathfrak{n}_{\{2f_2\}})$. To find $M_{\{2f_2\}}$ we see

$$Z_K(\mathfrak{a}_{\{2f_2\}}) = \left\{ \begin{pmatrix} e^{i\theta} & & & \\ & e^{i\phi} & & \\ & & e^{i\theta} & \\ & & & e^{i\omega} \end{pmatrix} \middle| \theta, \phi, \omega \in \mathbb{R} \text{ s.t. } \phi + \omega = -2(\theta + k\pi), k \in \mathbb{Z} \right\}$$
$$= \left\{ \begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\} \cdot \left\{ \begin{pmatrix} 1 & & & \\ & e^{i\phi} & & \\ & & 1 & \\ & & & e^{-i\phi} \end{pmatrix} \middle| \phi \in \mathbb{R} \right\}.$$

We claim that $M_{\{2f_2\}} = \exp(\mathfrak{m}) \times SU(1,1)$, i.e. $Z_K(\mathfrak{a}_{\{2f_2\}}) \cdot \exp(\mathfrak{m}_{\{2f_2\}}) = \exp(\mathfrak{m}_{\{2f_2\}})$. To see this, we note, by setting $\omega = -2\phi - \varphi + 2k\pi$, that

$$\begin{pmatrix} e^{i\phi} & & & \\ & e^{i\varphi} & & \\ & & e^{i\phi} & \\ & & & e^{i\omega} \end{pmatrix} \begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \alpha & & \beta \\ & & 1 & \\ & \bar{\beta} & & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{i(\theta+\phi)} & & & \\ & e^{i(\varphi-\theta)}\alpha & & e^{i(\varphi-\theta)}\beta \\ & & e^{i(\theta+\phi)} & \\ & e^{i(\omega-\theta)}\bar{\beta} & & e^{i(\omega-\theta)}\bar{\alpha} \end{pmatrix}$$
$$= \begin{pmatrix} e^{i(\theta+\phi)} & & & \\ & e^{-i(\theta+\phi)} & & \\ & & e^{i(\theta+\phi)} & \\ & & & e^{-i(\theta+\phi)} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & e^{i(\varphi+\phi)}\alpha & & e^{i(\varphi+\phi)}\beta \\ & & 1 & \\ & e^{-i(\varphi+\phi)}\bar{\beta} & & e^{-i(\varphi+\phi)}\bar{\alpha}. \end{pmatrix}$$

Rotating $\alpha \mapsto e^{i(\varphi+\phi)}\alpha$ and $\beta \mapsto e^{i(\varphi+\phi)}\beta$ shows that $Z_K(\mathfrak{a}_{\{2f_2\}})(M_{\{2f_2\}})_0 \subseteq (M_{\{2f_2\}})_0$. Obviously, $\mathbf{1}_{4\times4} \in Z_K(\mathfrak{a}_{\{2f_2\}})$, so we have an equality. Therefore, we conclude that

$$M_{\{2f_2\}} = \exp(\mathfrak{m}) \cdot \left\{ \begin{pmatrix} 1 & & & \\ & \alpha & & \beta \\ & & 1 & \\ & \bar{\beta} & & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\} \simeq \exp(\mathfrak{m}) \times SU(1,1) \tag{6.19}$$

Consider $\Pi' = \{f_1 - f_2\}$: Finally, we determine the last non-conjugate parabolic subgroup. Doing the same as above, we immediately find

$$\mathfrak{a}_{\{f_1-f_2\}} = \left\{ \left. \begin{pmatrix} & & c & 0 \\ & & 0 & c \\ c & 0 & & \\ 0 & c & & \end{pmatrix} \right| c \in \mathbb{R} \right\}, \tag{6.20}$$

$$\mathfrak{a}_{M\{f_1-f_2\}} = \left\{ \left. \begin{pmatrix} & & c & 0 \\ & & 0 & -c \\ c & 0 & & \\ 0 & -c & & \end{pmatrix} \right| c \in \mathbb{R} \right\}, \tag{6.21}$$

$$\mathfrak{n}_{\{f_1-f_2\}} = \mathfrak{g}_{2f_1} \oplus \mathfrak{g}_{2f_2} \oplus \mathfrak{g}_{f_1+f_2}. \tag{6.22}$$

For $\mathfrak{m}_{\{f_1-f_2\}}$ we need to do some more work. By definition we get

$$\begin{aligned} \mathfrak{m}_{\{f_1-f_2\}} &= \mathfrak{m} \oplus \mathfrak{a}_{M\{f_1-f_2\}} \oplus \bigoplus_{\beta=\pm(f_1-f_2)} \mathfrak{g}_\beta \\ &= \left\{ \left. \begin{pmatrix} ia & \alpha & b & \beta \\ -\bar{\alpha} & -ia & \bar{\beta} & -b \\ b & \beta & ia & \alpha \\ \bar{\beta} & -b & -\bar{\alpha} & -ia \end{pmatrix} \right| a, b \in \mathbb{R}, \alpha, \beta \in \mathbb{C} \right\} \\ &= \left\{ \left. \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right| A \in \mathfrak{su}(2), B \in i\mathfrak{su}(2) \right\}. \end{aligned}$$

So we see that $\mathfrak{m}_{\{f_1-f_2\}} \simeq \mathfrak{sl}(2,\mathbb{C})$ by sending $\begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto A+B$. To calculate the parabolic subgroup, we need to find $Z_K(\mathfrak{a}_{\{f_1-f_2\}})$. Going through the definition, we find that the group $Z_K(\mathfrak{a}_{\{f_1-f_2\}})$ is a disjoint set given by

$$\begin{aligned} Z_K(\mathfrak{a}_{\{f_1-f_2\}}) &= \left\{ \left. \begin{pmatrix} \alpha & \beta & & \\ -\bar{\beta} & \bar{\alpha} & & \\ & & \alpha & -\beta \\ & & \bar{\beta} & \bar{\alpha} \end{pmatrix} \right| |\alpha|^2 + |\beta|^2 = 1 \right\} \bigcup \left\{ \left. \begin{pmatrix} \alpha & \beta & & \\ \bar{\beta} & -\bar{\alpha} & & \\ & & \alpha & -\beta \\ & & -\bar{\beta} & -\bar{\alpha} \end{pmatrix} \right| |\alpha|^2 + |\beta|^2 = 1 \right\} \\ &= \left\{ \left. \begin{pmatrix} A & \\ & \gamma A \gamma \end{pmatrix} \right| A \in SU(2) \right\} \cup \left\{ \left. \begin{pmatrix} \gamma A & \\ & A\gamma \end{pmatrix} \right| A \in SU(2) \right\} \end{aligned}$$

Thus applying this set to $\exp(\mathfrak{m}_{\{f_1-f_2\}})$, one can show that

$$M_{\{f_1-f_2\}} \simeq \{1, \gamma\} \ltimes SL(2,\mathbb{C}), \tag{6.23}$$

where the multiplication is defined as $(\alpha, x) \cdot (\beta, y) = (\alpha\beta, \mathrm{Ad}_{\beta^{-1}}(x)y)$.

6.2 R-group

Now that we have found the parabolic subgroups, we can continue to finding some irreducible unitary representations of $SU(2,2)$. To do so, we shall consider discrete series representations of the parabolic subgroup P and a linear functional $\nu \in (\mathfrak{a}_P)^*_{\mathbb{C}}$. Looking back, we have three *cuspidal* subgroup classes, namely $P_\emptyset$, $P_{\{2f_2\}}$ and G itself. By Theorem 5.1.3, we can ignore $P_{\{f_1-f_2\}}$ because $SL(2,\mathbb{C})$ is not cuspidal, hence does not have discrete series representations. We will look at each parabolic subgroup separately.

Consider $P = G$: The induced representation $\mathrm{Ind}_G^G(\sigma \otimes e^\nu \otimes 1)$ acts as the discrete series representation σ of G. The discrete series are characterized in the same way as in Theorem 5.1.5. This has been done in more detail in [16]. We will focus on the induced representation of the other two parabolic subgroups.

Consider $P = P_\emptyset$: Remember $M := M_\emptyset \simeq \{1, \gamma\} \times S^1$ and $\mathfrak{a}_\emptyset = \mathfrak{a}$. Denote

$$X_\theta := \mathrm{diag}(e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta}).$$

Since both sets in the Cartesian product of M are compact groups, the irreducible unitary representations are finite dimensional. The irreducible unitary representations of M are found by tensoring the irreducible unitary representations on $\{1, \gamma\}$ with the irreducible unitary representations on S^1. The irreducible unitary representations of S^1 are given by $\sigma_n : S^1 \to \mathbb{C}, z \mapsto z^n$. On the other hand, we have the group $\{1, \gamma\}$ consisting of just two elements. Since it is an abelian group, the irreducible unitary representations are characters $\xi : \{1, \gamma\} \to \mathbb{C}$ satisfying $\xi(\gamma) = \pm 1$. Therefore, we have that the irreducible unitary representations $\sigma_{n,\epsilon}$ of M are characterized by two parameters $n \in \mathbb{Z}$ and $\epsilon = \pm 1$ such that $\sigma_{n,\epsilon} : M \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$ is defined by

$$\begin{aligned} \sigma_{n,\epsilon}(\gamma, 1) &= \epsilon \\ \sigma_{n,\epsilon}(1, X_\theta) &= e^{in\theta}. \end{aligned} \tag{6.24}$$

Before calculating the R group, we need to place this representation in the setting of discrete series. The discrete series parameters λ and χ are needed in the calculations of the R group. We work in the same way as in Theorem 5.1.7. For that, we first need a maximal abelian subalgebra $\mathfrak{b} \subset \mathfrak{k} \cap \mathfrak{m} = \mathfrak{m}$. We choose $\mathfrak{b} = \mathfrak{m}$. Then $\mathfrak{m} \oplus \mathfrak{a}$ is Cartan subalgebra of $\mathfrak{g}$. Doing the usual calculations for finding roots of Cartan subalgebras, we find that the set of roots are given by

$$\begin{aligned} \Delta((\mathfrak{m} \oplus \mathfrak{a})_\mathbb{C} : \mathfrak{g}_\mathbb{C}) = \{ &\pm 2f_1, \pm 2f_2, \pm(f_1 + f_2 + 2e), \pm(f_1 - f_2 + 2e), \\ &\pm (f_1 + f_2 - 2e), \pm(f_1 - f_2 - 2e)\} \end{aligned} \tag{6.25}$$

where we expanded f_1, f_2 to be defined on $(\mathfrak{m} \oplus \mathfrak{a})_\mathbb{C}$ by $f_i : (\mathfrak{m} \oplus \mathfrak{a})_\mathbb{C} \to \mathbb{C}$:

$$f_1\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & c \\ b & 0 & a & 0 \\ 0 & c & 0 & -a \end{pmatrix}\right) = b \quad , \quad f_2\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & c \\ b & 0 & a & 0 \\ 0 & c & 0 & -a \end{pmatrix}\right) = c$$

and we defined $e : (\mathfrak{m} \oplus \mathfrak{a})_\mathbb{C} \to \mathbb{C}$ by

$$e\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & c \\ b & 0 & a & 0 \\ 0 & c & 0 & -a \end{pmatrix}\right) = a$$

with $a, b, c \in \mathbb{C}$. Note that $\pm 2f_1$ and $\pm 2f_2$ are the only real roots. Thus we form γ_{2f_2} and γ_{2f_1}. It then follows that $\gamma_{2f_2} = \gamma$ and $\gamma_{2f_1} = -\gamma$. Thus we see that

$$F(B) = \{1, -1, \gamma, -\gamma\} = \{(1,1), (1,-1), (\gamma, 1), (\gamma, -1)\}$$

where the last equality is pinpointing the elements of $\{1, \gamma\} \times S^1$. Next we need to find the Harish-Chandra parameter λ. By definition, $\lambda \in (i\mathfrak{m})^*_\mathbb{C}$ such that $\lambda - \delta_M$ is analytically integral. The only option would be $\lambda = \alpha e$ with $\alpha \in \mathbb{C}$. It is given that δ_M is half the sum of all positive roots in $\Delta((\mathfrak{a} \oplus \mathfrak{m})_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ that vanish on $\mathfrak{a}$. In this case $\delta_M = 0$ for no root vanishes on $\mathfrak{a}$. So λ should be analytically integral, i.e.

$$\lambda(\mathrm{diag}(2i\pi k, -2i\pi k, 2i\pi k, -2i\pi k)) \in 2\pi i\mathbb{Z} \qquad \forall k \in \mathbb{Z}.$$

This can only be the case if $\alpha = n \in \mathbb{Z}$. Thus all discrete series of M_0 are characterized by the Harish-Chandra parameter $\lambda = ne$.

To continue the construction given in Theorem 5.1.7, the Harish-Chandra parameter we found should correspond to a Harish-Chandra parameter on $\mathfrak{m} \cap \mathfrak{m}_{ss}$, and to a differential on $\mathfrak{b} \cap Z_{\mathfrak{m}}$. Since $\mathfrak{m}$ is abelian, $\mathfrak{m}_{ss} = \{0\}$ and the Harish-Chandra parameter just corresponds to a differential. To know which discrete series parameters we are considering in our representation, we calculate the differential which gives

$$\left.\frac{d}{dt}\right|_{t=0} \pi_{n,\epsilon}(1, X_{t\theta}) = in\theta = ne.$$

So we see that the Harish-Chandra parameter for our representation is $\lambda = ne$.

Next, we need to find χ. Again by Theorem 5.1.7, χ is defined on $F(B)$. We remember that $\chi = \pi(\lambda, \chi)|_{F(B)}$. This means that

$$\chi(1, -1) = \sigma_{n,\epsilon}(1, -1) = e^{in\pi} = (-1)^n, \qquad \chi(\gamma, 1) = \sigma_{n,\epsilon}(\gamma, 1) = \epsilon. \tag{6.26}$$

We need to check the compatibility on $F(B) \cap B$. Remember $\mathfrak{b} = \mathfrak{m}$, so $F(B) \cap B = \{(1, 1), (1, -1)\}$. Note that $(1, -1) = (1, X_{k\pi})$ with k an odd integer. Hence the compatibility requires us to set

$$\xi_\lambda(1, X_{k\pi}) = e^{\lambda(\mathrm{diag}(ik\pi, -ik\pi, ik\pi, -ik\pi))} = e^{ikn\pi} = (-1)^n$$

which agrees with $\chi(1, -1)$. So they are compatible, and we thus have found the parameters for our discrete series.

Our final ingredient for the induced representation $\mathrm{Ind}_{P_\emptyset}^G(\sigma_{n,\epsilon} \otimes e^\nu \otimes 1)$ is ν itself. Because $\nu \in i\mathfrak{a}^*$ immediately yields a unitary representation, we will only consider those. If the reader is interested in induction for $\mathrm{Re}\,\nu > 0$, we recommend [16]. We shall consider two cases: $\nu = 0$ and $\nu \neq 0$.

But before we go into each case, we note that we will calculate the R group of $\mathrm{Ind}_{P_\emptyset}^G(\sigma_{n,\epsilon} \otimes e^\nu \otimes 1)$ to characterize irreducibility. To determine the R group, we need $W(A : G) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ explicitely. Since $P_\emptyset$ is the minimal parabole, we know $W(A : G)$ is the whole Weyl group, but we do not know how an element $w \in W(A : G)$ acts as $w\sigma_{n,\epsilon}$. Explicit calculations yield

$$\begin{aligned} W(A : G) &= \left\langle I, \begin{pmatrix} 1 &&& \\ & i && \\ && -1 & \\ &&& i \end{pmatrix}, \begin{pmatrix} 1 &&& \\ & i && \\ && 1 & \\ &&& -i \end{pmatrix}, \begin{pmatrix} 0 & 1 && \\ 1 & 0 && \\ && 0 & 1 \\ && 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 && \\ -1 & 0 && \\ && 0 & -1 \\ && 1 & 0 \end{pmatrix} \right\rangle && (6.27) \\ &= \langle 1, s_{2f_1}, s_{2f_2}, s_{f_1 - f_2}, s_{f_1 + f_2} \rangle && (6.28) \end{aligned}$$

where each Weyl group element is identified in the notation term by term.

Consider $\nu = 0$: Let $w \in W(A : G)$. If $s_\alpha = [X]$ where $X \in N_K(\mathfrak{a})$, we see that s_α acts on $\sigma_{n,\epsilon}$ as $s_\alpha \sigma_{n,\epsilon} = \sigma \circ \mathrm{Ad}(X)^{-1}$. It then follows that

$$W_{\sigma_{n,\epsilon}, 0} = \langle 1, s_{2f_1}, s_{2f_2} \rangle.$$

Next, to find $R_{\sigma_{n,\epsilon},0}$, we need to use Equation (5.27). We note that $1 \in R_{\sigma_{n,\epsilon},0}$ by emptiness of the argument. Next we ask ourselves if $s_{2f_2} \in R_{\sigma_{n,\epsilon},0}$. The only root that becomes negative under s_{2f_2} is $2f_2$ itself. So set $\alpha = 2f_2$.

We wish to describe $\mu_{\sigma,\alpha}(0)$. This can be calculated by going to Theorem 5.5.7, which states we need to describe the Plancherel density for the group $G^{(2f_2)}M$, which is to say, we consider the group G in Theorem 5.5.5 to be $G^{(2f_2)}M$ and consider its maximal parabolic subgroups. We find

$$\mathfrak{g}^{(2f_2)} = \mathfrak{a}^{(2f_2)} \oplus \mathfrak{g}_{2f_2} \oplus \mathfrak{g}_{-2f_2}, \qquad G^{(2f_2)} = \left\{ \left.\begin{pmatrix} 1 & & & \\ & a & 0 & b \\ & 0 & 1 & 0 \\ & c & 0 & d \end{pmatrix} \right| a,b,c,d \in \mathbb{R}, ad-bc=1 \right\} \simeq SL(2,\mathbb{R}),$$

where

$$\mathfrak{a}^{(2f_2)} = \left\{ \left.\begin{pmatrix} 0 & & & \\ & 0 & & x \\ & & 0 & \\ & x & & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}.$$

We note that M commutes with $G^{(2f_2)}$. This means that all roots in $\Delta((\mathfrak{a}^{(2f_2)} \oplus \mathfrak{m})_{\mathbb{C}}, (\mathfrak{m} \oplus \mathfrak{g}^{(2f_2)})_{\mathbb{C}})$ are just given by $\pm 2f_2$, for $\mathfrak{m}$ commutes with $\mathfrak{g}^{(2f_2)}$. In addition, we see that $G^{(2f_2)} \simeq SL(2,\mathbb{R})$, meaning rank$(G^{(2f_2)})$ =rank$(K \cap G^{(2f_2)})$. Therefore, the Plancherel density is given by

$$p^{(2f_2)}_{\sigma_{n,\epsilon}}(\nu) = c\langle ne+\nu, 2f_2\rangle \cdot \begin{cases} \tan\left(\frac{\pi\langle \nu, 2f_2\rangle}{\|2f_2\|^2}\right) & \text{if } \chi(\gamma_{2f_2}) = -(-1)^1 = 1 \\ \frac{1}{\tan\left(\frac{\pi\langle \nu, 2f_2\rangle}{\|2f_2\|^2}\right)} & \text{if } \chi(\gamma_{2f_2}) = +(-1)^1 = -1 \end{cases} \tag{6.29}$$

Here $\nu \in (\mathfrak{a}^{(2f_2)})^*_{\mathbb{C}}$, hence $\nu = z \cdot 2f_2$ with $z \in \mathbb{C}$. Since $\langle e, 2f_2\rangle = 0$, we immediately find

$$p^{(2f_2)}_{\sigma_{n,\epsilon}}(\nu) = c'z \cdot \begin{cases} \tan(z\pi) & \text{if } \chi(\gamma_{2f_2}) = 1 \\ \frac{1}{\tan(z\pi)} & \text{if } \chi(\gamma_{2f_2}) = -1 \end{cases}$$

where $c' = c\|2f_2\|^2$. Letting $z \to 0$, we find that $p^{(2f_2)}_{\sigma_{n,\epsilon}}(0) \neq 0$ iff $\chi(\gamma_{2f_2}) = -1$. We remember this is equivalent to $\chi(\gamma, 1) = -1$, so $\epsilon = -1$. No further requirements are put on n. We can therefore conclude that

$$s_{2f_2} \in R_{\sigma_{n,\epsilon},0} \Leftrightarrow (n,\epsilon) = (n,-1).$$

Next, we go to the question whether $s_{2f_1} \in R_{\sigma_{n,\epsilon},0}$. There are three roots that become negative under s_{2f_1}; namely $\alpha = 2f_1$, f_1+f_2, and f_1-f_2. We start with $\alpha = 2f_1$. As before, we need to build $G^{(2f_1)}M$, where $G^{(2f_1)}$ is given by

$$\mathfrak{g}^{(2f_1)} = \mathfrak{a}^{(2f_1)} \oplus \mathfrak{g}_{2f_1} \oplus \mathfrak{g}_{-2f_1}, \qquad G^{(2f_1)} = \left\{ \left.\begin{pmatrix} a & 0 & b & \\ 0 & 1 & 0 & \\ c & 0 & d & \\ & & & 1 \end{pmatrix} \right| a,b,c,d \in \mathbb{R}, ad-bc=1 \right\}$$

where

$$\mathfrak{a}^{(2f_1)} = \left\{ \left.\begin{pmatrix} 0 & 0 & x & \\ 0 & 0 & 0 & \\ x & 0 & 0 & \\ & & & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}.$$

One can go through the same steps as in the $\alpha = 2f_2$ case. This yields

$$p^{(2f_1)}_{\sigma_{n,\epsilon}}(z \cdot 2f_1) = c''z \cdot \begin{cases} \tan(z\pi) & \text{if } \chi(\gamma_{2f_1}) = 1 \\ \frac{1}{\tan(z\pi)} & \text{if } \chi(\gamma_{2f_1}) = -1 \end{cases} \tag{6.30}$$

where $c'' = c\|2f_1\|^2$. Again, letting $z \to 0$, we see $p^{(2f_1)}_{\sigma_{n,\epsilon}}(0) \neq 0$ iff $-1 = \chi(\gamma_{2f_1}) = \chi(-1,\gamma)$. By construction of χ, we get $\chi(\gamma,-1) = (-1)^n\epsilon = -1$. This requirement can only be true in two cases: either $(n,\epsilon) = (2k,-1)$ or $(n,\epsilon) = (2k+1,-1)$ with $k \in \mathbb{Z}$.

Next we go to $\alpha = f_1 - f_2$. Again, we consider $G^{(f_1-f_2)}M$. Doing explicit calculations we find $\mathfrak{a}^{(f_1-f_2)} = \mathfrak{a}_{M\{f_1-f_2\}}$ which we defined in Equation (6.21), and $\mathfrak{g}^{(f_1-f_2)} = \mathfrak{m}_{f_1-f_2} \simeq \mathfrak{sl}(2,\mathbb{C})$. This shows that we are in the situation of Theorem 5.5.5.1, for $SL(2,\mathbb{C})$ is not cuspidal. In other words, the Plancherel density is just a polynomial. In addition, we find

$$\Delta((\mathfrak{m} \oplus \mathfrak{a}^{(f_1-f_2)})_{\mathbb{C}}, \mathfrak{g}^{(f_1-f_2)}) = \{\pm 2(e + f_1 - f_2)\}.$$

Again $\mathfrak{a}^{(f_1-f_2)}$ is only one-dimensional, so take $\nu = z(f_1 - f_2)$. This gives

$$p^{(f_1-f_2)}_{\sigma_{n,\epsilon}}(\nu) = c\langle ne + \nu, 2e + 2(f_1 - f_2)\rangle = 2c(n\|e\|^2 + z\|f_1 - f_2\|^2).$$

Taking $z \to 0$, we get $p^{(f_1-f_2)}_{\sigma_{n,\epsilon}}(0) = 2cn\|e\|$. We wished to investigate when $p^{(f_1-f_2)}_{\sigma_{n,\epsilon}}(0) \neq 0$, which is only the case when $n \neq 0$.

Finally consider $\alpha = f_1 + f_2$. The arguments are the same as with $f_1 - f_2$, thus we immediately conclude $p^{(f_1+f_2)}_{\sigma_{n,\epsilon}}(0) \neq 0$ iff $n \neq 0$. Taking all options together, we find

$$s_{2f_1} \in R_{\sigma_{n,\epsilon},0} \Leftrightarrow n \neq 0 \text{ and either } (n,\epsilon) = (2k,-1) \text{ or } (n,\epsilon) = (2k+1,1) \quad , \quad k \in \mathbb{Z}.$$

Hence we conclude that the R groups are given by

$$\begin{aligned} R_{\sigma_{n,\epsilon},0} &= \{1\} && \text{if } (n,\epsilon) = (2k,1) \\ &= \{1, s_{2f_2}\} && \text{if } (n,\epsilon) = (0,-1) \\ &= \{1, s_{2f_1}, s_{2f_2}, s_{2f_1}s_{2f_2}\} && \text{if } (n,\epsilon) = (2k,-1) \quad k \neq 0 \\ &= \{1, s_{2f_2}\} && \text{if } (n,\epsilon) = (2k+1,-1) \\ &= \{1, s_{2f_1}\} && \text{if } (n,\epsilon) = (2k+1,1) \end{aligned}$$

which agrees with the R groups found in [16]. This also shows that $U(P_{min}, \sigma_{2k,1}, 0)$ is an irreducible unitary representation of $SU(2,2)$, for any $k \in \mathbb{Z}$.
Consider $\nu \neq 0$: Now let us consider $\nu \in i\mathfrak{a}^* \setminus \{0\}$. Note that if $\nu = i(af_1 + bf_2)$ with $a, b \in \mathbb{R}$ such that $a, b \neq 0$, there exists no $s_\alpha \in W(A : G)$ such that $s_\alpha\nu = \nu$. Therefore, we immediately see that $W_{\sigma_{n,\epsilon},i(af_1+f_2)} = \{1\}$ and as such

$$R_{\sigma_{n,\epsilon},i(af_1+bf_2)} = \{1\}.$$

The only option we have not looked at yet is $\nu = iaf_1$ or $\nu = ibf_2$ with $a, b \neq 0$. We first look at $\nu = iaf_1$ and see that only $s_{2f_2}\nu = \nu$. Hence, together with previous arguments, we get

$$W_{\sigma_{n,\epsilon},iaf_1} = \{1, s_{2f_2}\}.$$

To find the R group corresponding to this case, consider the group $G^{(2f_2)}$ again. The Plancherel density is the one we already calculated only now evaluated at $\nu|_{\mathbb{R}H_{2f_2}}$. But $\nu|_{\mathbb{R}H_{2f_2}} = 0$, so we are back at the case we had before. This immediately gives $s_{2f_2} \in R_{\sigma_{n,\epsilon},\nu}$ iff $(n,\epsilon) = (n,-1)$.

Finally we go to $\nu = ibf_2$ with $b \neq 0$. We see that $W_{\sigma_{n,\epsilon},ibf_2} = \{1, s_{2f_1}\}$. By the same arguments as before, we need to consider $\alpha = 2f_1$, $f_1 - f_2$ and $f_1 + f_2$. Considering $\alpha = 2f_1$, we need to construct $G^{(2f_1)}$ again. And as before, we find $\nu|_{\mathbb{R}H_{2f_1}} = 0$, so we immediately find $p^{(2f_1)}_{\pi_{n,\epsilon}}(\nu|_{\mathbb{R}H_{2f_1}}) \neq 0$ iff either $(n,\epsilon) = (2k,-1)$ or $(n,\epsilon) = (2k+1,-1)$ with $k \in \mathbb{Z}$.

Next consider $\alpha = f_1 - f_2$. Unfortunately $\nu|_{\mathbb{R}H_{f_1-f_2}} \neq 0$, so we need to do some work. Note $\nu|_{\mathbb{R}H_{f_1+f_2}} = \frac{-ib}{2}(f_1 - f_2)$. Filling this value in the Plancherel density we found for $G^{(f_1-f_2)}M$, we see

$$p^{(f_1-f_2)}_{\pi_{n,\epsilon}}(\nu|_{\mathbb{R}H_{f_1-f_2}}) = c\langle ne + \nu, 2e + 2(f_1 - f_2)\rangle = c(n\|e\|^2 - ib\|f_1 - f_2\|^2).$$

Since $b \neq 0$ and real, we get that the Plancherel density is never 0. Going over $\alpha = f_1 - f_2$ gives the same result. Hence we can conclude

$$s_{2f_1} \in R_{\sigma_{n,\epsilon},0} \Leftrightarrow \text{ either } (n,\epsilon) = (2k,-1) \text{ or } (n,\epsilon) = (2k+1,1) \quad , \quad k \in \mathbb{Z}.$$

In conclusion, if $\nu \neq 0$, we get the following R groups, where $a, b \neq 0$:

$$\begin{aligned}
R_{\sigma_{n,\epsilon},i(af_1+bf_2)} &= \{1\} && \text{for all } (n,\epsilon)\\
R_{\sigma_{n,\epsilon},iaf_1} &= \{1\} && \text{if } (n,\epsilon) = (n,1)\\
R_{\sigma_{n,\epsilon},iaf_1} &= \{1, s_{2f_2}\} && \text{if } (n,\epsilon) = (n,-1)\\
R_{\sigma_{n,\epsilon},ibf_2} &= \{1\} && \text{if } (n,\epsilon) = (2k,1) \text{ or } = (2k+1,-1)\\
R_{\sigma_{n,\epsilon},ibf_2} &= \{1, s_{2f_1}\} && \text{if } (n,\epsilon) = (2k,-1) \text{ or } = (2k+1,1).
\end{aligned}$$

Consider $P = P_{\{2f_2\}}$: Now that we have dealt with the minimal parabolic subgroup, we go to $P_{\{2f_2\}}$. Remember that $M_{\{2f_2\}} \simeq \exp(\mathfrak{m}) \times SU(1,1)$. As before, any discrete series can be constructed by looking at each component individually. Again, the irreducible unitary representations of $\exp(\mathfrak{m})$ are given by $\sigma_n : S^1 \to \mathbb{C}$ sending $z \mapsto z^n$. However, $SU(1,1)$ is non-compact. We are thus required to look at the discrete series representations and limits of discrete series of $SU(1,1)$, which are known [14, Chapter II]. Let $k \geq 2$ be an integer, and define the Hilbert space H_k as

$$H_k := \left\{ f : \mathbb{C} \to \mathbb{C} \text{ analytic for } |z| < 1 \;\middle|\; \|f\|^2 = \int_{|z|<1} |f(z)|^2(1 - |z|^2)^{k-2} dz < \infty \right\}.$$

Note that this space is non-zero, for $f(z) = 1$ lies in H_k. Then the representation $(\mathscr{D}^+_k, H_k)$, defined by

$$\mathscr{D}^+_k \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = (-\beta z + \bar{\alpha})^{-k} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}}\right), \tag{6.31}$$

is a discrete series representation of $SU(1,1)$. There is another discrete series representation which can be obtained by taking the complex conjugate of $(\mathscr{D}^+_k, H_k)$. That is to say we define on H_k a new scalar multiplication $\bar{\cdot} : \mathbb{C} \times H_k \to H_k$ by $(z, f) \mapsto \bar{z}f$. This mapping is complex linear and we denote the space $\overline{H_k}$ as the space H_k with the scalar multiplication $\bar{\cdot}$. Then the representation $(\mathscr{D}^-_k, \overline{H_k})$, defined by

$$\mathscr{D}^-_k \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = (-\beta z + \bar{\alpha})^{-k} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}}\right), \tag{6.32}$$

is a discrete series representation of $SU(1,1)$. The two representations $(\mathscr{D}^+_k, H_k)$ and $(\mathscr{D}^-_k, \overline{H_k})$ are the only discrete series representations of $SU(1,1)$.

Therefore, setting $M_{\{2f_2\}} \simeq S^1 \times SU(1,1)$, all discrete series are described either by $\sigma^+_{n,k} : M_{\{2f_2\}} \to \mathrm{End}(H_k)$ or $\sigma^-_{n,k} : M_{\{2f_2\}} \to \mathrm{End}(\overline{H_k})$, where $\sigma^\pm_{n,k}$ are defined by

$$\begin{aligned} \sigma^\pm_{n,k}(X_\theta, 1) &= e^{in\theta} I \\ \sigma^\pm_{n,k}(1, A) &= \mathscr{D}^\pm_k(A). \end{aligned}$$

As with the minimal case, we first want to describe the parameters of the discrete series we have found. To do that, we first need a maximal abelian subalgebra $\mathfrak{b}_{\{2f_2\}} \subset \mathfrak{k} \cap \mathfrak{m}_{\{2f_2\}}$. We choose

$$\mathfrak{b}_{\{2f_2\}} = \left\{ \begin{pmatrix} i\theta & & & \\ & i(\alpha-\theta) & & \\ & & i\theta & \\ & & & -i(\alpha+\theta) \end{pmatrix} \middle| \, \theta, \alpha \in \mathbb{R} \right\}.$$

Then $\mathfrak{b}_{\{2f_2\}} \oplus \mathfrak{a}_{\{2f_2\}}$ is a Cartan subalgebra of $\mathfrak{g}$. The roots with respect to this Cartan subalgebra are given by

$$\begin{aligned} \Delta((\mathfrak{b}_{\{2f_2\}} \oplus \mathfrak{a}_{\{2f_2\}})_\mathbb{C} : \mathfrak{g}_\mathbb{C}) = \{ &\pm 2f_1, \pm(f_1 + (e_1 - e_2)), \pm(f_1 - (e_1 - e_2)), \pm(f_1 + (e_1 - e_4)), \\ &\pm (f_1 - (e_1 - e_4)), \pm(e_2 - e_4)\}. \end{aligned} \tag{6.33}$$

where f_1, f_2 are defined as above, and e_j is defined as $e_j : (\mathfrak{b}_{\{2f_2\}} \oplus \mathfrak{a}_{\{2f_2\}}) \to \mathbb{C}$ by

$$e_j \left(\begin{pmatrix} a_1 & 0 & c & \\ 0 & a_2 & 0 & \\ c & 0 & a_3 & \\ & & & a_4 \end{pmatrix} \right) = a_j$$

for all $j = 1, \dots, 4$ and $a_j, c \in \mathbb{C}$ for all j. Also note that $2e_1 + e_2 + e_4 = 0$ on $\mathfrak{b}_{\{2f_2\}}$.

To find the Harish-Chandra parameter of $\sigma^\pm_{n,k}$, we need $\lambda \in (i\mathfrak{b})^*$. So $\lambda = a_1 e_1 + b e_2 + c e_4$ for some $a, b, c \in \mathbb{R}$. Note that $\delta_M = \frac{1}{2}(e_2 - e_4)$. If we look closely, we see that the set of polynomials $\{z^n | n \geq 0\}$ is an orthogonal basis for both H_k and $\overline{H_k}$. In addition, we see that

$$\sigma^\pm_{n,k} \left(\begin{pmatrix} e^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & e^{i\alpha} & & \\ & & 1 & \\ & & & e^{-i\alpha} \end{pmatrix} \right) 1 = e^{in\theta} e^{ik\alpha} 1.$$

This is the lowest K-type, so we get by Theorem 5.1.5 that the Harish-Chandra parameter is given by

$$\begin{aligned} ne_1 + \frac{k}{2}(e_2 - e_4) &= \lambda + \delta_M - 2\delta_{K_M} \\ &= ae_1 + \left(b + \frac{1}{2}\right) e_2 + \left(c - \frac{1}{2}\right) e_4 - 0. \end{aligned} \tag{6.34}$$

So choosing $a = n, b = \frac{k-1}{2}$ and $c = -\frac{k-1}{2}$ gives that the Harish-Chandra parameter of $\sigma^\pm_{n,k}$ is given by

$$\lambda = ne_1 + \frac{k-1}{2}(e_2 - e_4).$$

As a check, one can see that this linear functional is analytically integral. The only real root of $\Delta((\mathfrak{b}_{\{2f_2\}}\oplus \mathfrak{a}_{\{2f_2\}})_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}})$ is given by $\pm 2f_1$. Then constructing H_{2f_1} shows

$$H_{2f_1} = \begin{pmatrix} 0 & & 1 & \\ & 0 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix}$$

and thus $\gamma_{2f_1} = \exp(\pi i H_{2f_1}) = \mathrm{diag}(-1,1,-1,1) = -\gamma$. Therefore $F(B_{\{2f_2\}}) = \{1, \gamma_{2f_1}\} = \{(1,1),(-1,\gamma)\}$ where the last equality is pinpointing the element in $S^1 \times SL(2,\mathbb{R})$. It is worth noting that $(-1,\gamma) \in B_{\{2f_2\}}$, so $F(B_{\{2f_2\}}) \cap B_{\{2f_2\}} = F(B_{\{2f_2\}})$ and thus χ is completely determined by λ, namely

$$\chi(-1,\gamma) = \sigma^{\pm}_{n,k}(-1,\gamma) = e^{(\lambda-\delta_M)(\mathrm{diag}(\pi i, 0, \pi i, -2\pi i))} = e^{i[n+k-2]\pi} = (-1)^{n+k}.$$

Now that we know the discrete series parameters, we can build the R groups. For that we need $W(A_{\{2f_2\}} : G)$. Going through the calculations, we see that

$$W(A_{\{2f_2\}} : G) = \left\langle I, \begin{pmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & i \end{pmatrix} \right\rangle = \{1, s_{2f_1}\} \tag{6.35}$$

where the last equality is the correspondence we have discussed before. This means that we only have to check whether $2f_1 \in R_{\sigma^{\pm}_{n,k},\nu}$. Since $\mathfrak{a}_{\{2f_2\}}$ is only one-dimensional, it is enough to consider $\nu = ixf_1$ with $x \in \mathbb{R}$. If $x \neq 0$, then note that $s_{2f_1}\nu \neq \nu$, hence

$$W_{\sigma^{\pm}_{n,k},ixf_1} = \{1\},$$

and therefore we immediately get that $U(P_{\{2f_2\}}, \sigma^{\pm}_{n,k}, ixf_1)$ is irreducible.

So the only scenario is $x = 0$, i.e. $\nu = 0$. Then obviously $s_{2f_1} \in W_{\sigma^{\pm}_{n,k},0}$. Note that the roots in $(\mathfrak{g}, \mathfrak{a}_{\{2f_2\}})$ are $\{\pm f_1, \pm 2f_1\}$, and $P_{\{2f_2\}}$ has $\dim A_{\{2f_2\}} = 1$, so we can apply Theorem 5.5.5 directly. This means that to prove $s_{2f_1} \in R_{\sigma^{\pm}_{n,k},0}$, we only need to consider $\alpha = 2f_1$. This way, we get

$$\begin{aligned}
p_{\sigma^{\pm}_{n,k}}(ixf_1) =& c\left[\prod_{\varepsilon|_{\mathfrak{a}}=c\cdot 2f_1,\, c>0} \langle \lambda + ixf_1, \epsilon\rangle\right] f_{\sigma^{\pm}_{n,k},2f_1}(ixf_1) \\
=& c\langle\lambda + ixf_1, 2f_1\rangle\langle\lambda + ixf_1, f_1 + (e_1 - e_2)\rangle\langle\lambda + ixf_1, f_1 - (e_1 - e_2)\rangle\cdot \\
& \langle\lambda + ixf_1, f_1 + (e_1 - e_4)\rangle\langle\lambda + ixf_1, f_1 - (e_1 - e_4)\rangle f_{\sigma^{\pm}_{n,k},2f_1}(ixf_1) \\
=& cix\left(\frac{n}{2} + \frac{k-1}{2} + \frac{ix}{2}\right)\left(-\frac{n}{2} - \frac{k-1}{2} + \frac{ix}{2}\right)\cdot \\
& \left(\frac{n}{2} - \frac{k-1}{2} + \frac{ix}{2}\right)\left(-\frac{n}{2} + \frac{k-1}{2} + \frac{ix}{2}\right) f_{\sigma^{\pm}_{n,k},2f_1}(ixf_1) \\
=& \frac{cix}{2}\left((ix)^2 - (n+k-1)^2\right)\left((ix)^2 - (n-k+1)^2\right) f_{\sigma^{\pm}_{n,k},2f_1}(ixf_1)
\end{aligned}$$

where

$$f_{\sigma^{\pm}_{n,k},2f_1}(ixf_1) = \begin{cases} \tanh\left(\frac{\pi x\langle f_1, 2f_1\rangle}{\|2f_1\|^2}\right) = \tanh\left(\frac{\pi x}{2}\right) & \text{if } \chi(\gamma_{2f_1}) = -(-1)^3 = 1 \\ \frac{1}{\tanh\left(\frac{\pi x\langle f_1, 2f_1\rangle}{\|2f_1\|^2}\right)} = \frac{1}{\tanh\left(\frac{\pi x}{2}\right)} & \text{if } \chi(\gamma_{2f_1}) = (-1)^3 = -1. \end{cases}$$

Letting $x \to 0$, we find $p_{\sigma^{\pm}_{n,k}}(ixf_1)(0) \neq 0$ iff $\chi(\gamma_{2f_1}) = -1$ which is only true if $(-1)^{n+k} = -1$. This is equivalent to $n \equiv k+1 \bmod 2$. However if $\chi(\gamma_{2f_1}) = -1$, there is a possibility to get $p_{\sigma^{\pm}_{n,k}}(0) = 0$, because we could choose n such that $p_{\sigma^{\pm}_{n,k}}(0) = \left[\frac{0}{0}\right]$. Using L'Hôpital in this case gives

$$p_{\sigma^{\pm}_{n,k}}(0) = C(n+k-1)^2(n-k+1)^2\frac{1}{\tanh(0)}$$

where $C \in \mathbb{C}$ not equal to 0. Setting $|n| = k-1$ we also get $p_{\sigma^{\pm}_{n,k}}(0) = 0$. Hence we conclude $p_{\sigma^{\pm}_{n,k}}(0) \neq 0$ iff $|n| \neq k-1$ and $n \equiv k+1 \bmod 2$.

Wrapping it up, we get the following R groups for the parabole P_{2f_2}:

$$\begin{aligned} R_{\sigma^{\pm}_{n,k}, ixf_1} &= \{1\} && \text{for all } n \in \mathbb{Z}, k \geq 2 \\ R_{\sigma^{\pm}_{n,k}, 0} &= \{1\} && \text{if } n \equiv k \bmod 2 \text{ or } |n| = k-1 \\ &= \{1, s_{2f_1}\} && \text{all other cases} \end{aligned}$$

which also agrees with [16].

Part II
Hyperbolic Higgs model

7 The hierarchy problem

With that, we switch to the physical side of the thesis. In this part, we will describe a model which solves the hierarchy problem. But before we can discuss the hierarchy problem, we first want to remind ourselves a bit of the Standard Model. The Standard Model is a wonderful model. It describes the behaviour of nature up to high precision, and predicted new particles that have been confirmed experimentally later on such as the Higgs particle. However, there are also phenomena that cannot be described by the Standard Model. One of them is the hierarchy problem.

To discuss the hierarchy problem, one first needs to look at how the Standard Model (SM for short) includes the Higgs particle. The Higgs field is a doublet under $SU(2)$, with each component is a scalar field, i.e.

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

with ϕ^+, ϕ^0 scalar fields. The Lagrangian of the SM that describes purely the Higgs field is given by

$$\mathcal{L}_{\text{SM},\phi} = |D_\mu \phi|^2 + \mu^2|\phi|^2 + \lambda|\phi|^4, \tag{7.1}$$

where the potential is given by $V_{\text{Higgs}} = \mu^2|\phi|^2 + \lambda|\phi|^4$. In both Quantum Mechanics as in Quantum Field Theory, ground states are essential. They describe the vaccuum expecation value for particles when no exitations happen. The ground state are also given by the minimum of the potential. Before calculating the minimum, we first want to make use of the spherical symmetry to redefine our field to

$$\phi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\theta(x)/v}$$

where ρ and θ are now the dynamical fields. Then the potential becomes just $V_{\text{Higgs}} = \frac{1}{2}\mu^2\rho^2 + \frac{1}{2}\lambda\rho^4$. The minimum is now given by

$$0 = \left.\frac{dV}{d\rho}\right|_{\rho=v} = \mu^2 v + 2\lambda v^3 = v(\mu^2 + 2\lambda v^2). \tag{7.2}$$

So either $v = 0$ or $v^2 = \frac{-\mu^2}{2\lambda}$. If λ or $\mu^2 < 0$, we get two extrema of the potential. Assuming $\lambda > 0$ and $\mu^2 < 0$, we get that the global minimum of this potential is found by $v = \frac{|\mu^2|}{\lambda} > 0$. If this is the case, we say that the Higgs particle has an expecation value (VEV for short) at v.

Then we can expand our ϕ around the minimum, giving $\rho = v + h$, and we see that locally the potential looks like a massive free scalar field. This can be seen by

$$\begin{aligned} V(\phi) &= V(v+h) \\ &= V(v) + \left.\frac{dV}{d\rho}\right|_{\rho=v} h + \frac{1}{2}\left.\frac{d^2V}{d\rho^2}\right|_{\rho=v} |h|^2 \\ &= V(v) + \frac{1}{2}\mathcal{M}^2|h|^2 \end{aligned}$$

where $\mathcal{M}^2 = \left.\frac{d^2V}{d\rho^2}\right|_{\rho=v}$ can be seen as the mass of the particle h. Calculating this, we get

$$\mathcal{M}^2 = \mu^2 + 6\lambda v^2 = \mu^2 - 6\lambda\frac{\mu^2}{2\lambda} = -2\mu^2.$$

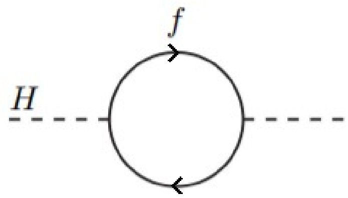
Since we had $\mu^2 < 0$, this gives $\mathcal{M} = \sqrt{2|\mu^2|}$ which is the Higgs mass.

This potential is $SU(2)$ invariant, meaning if $A \in SU(2)$ then $V_{\text{Higgs}}(A\phi) = V_{\text{Higgs}}(\phi)$. However if $v \neq 0$, then ϕ cannot be invariant under all of $SU(2)$ anymore. For example, if we take the expectation value of ϕ to be $\langle\phi\rangle = (0, v)$, then surely $|\langle\phi\rangle| = v$, but the vector $\langle\phi\rangle$ is not invariant under $SU(2)$ anymore. Hence we say that the $SU(2)$ symmetry is spontaneously broken. There is however a slight problem regarding this breaking. From experiments, we know that $v = \frac{-\mu^2}{\lambda} \approx 174$ GeV. With the discovery of the Higgs boson in 2012, the mass was found to be 125 GeV, which implied that the parameter $m_H := \sqrt{-2\mu^2} = 92.9$ GeV when including up to 2-loop corrections [18]. However, the corrections to m_H will be of a bigger order of magnitude than the resulting mass.

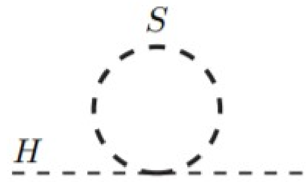
To see this, we remember that in addition to a pure Higgs potential, we also have Yukawa interactions with the fermions in the following way:

$$\mathcal{L}_{\text{SM,Yukawa}} = \sum_{\text{generations}} \left[-\lambda_u \epsilon^{ab}\bar{U}_a\phi_b^\dagger u_R - \lambda_d \bar{D}\cdot\phi d_R - \lambda_e \bar{E}\cdot\phi e_R + \text{hermitian conj.}\right] \tag{7.3}$$

where u_R, d_R and e_R are the right-handed fermion fields of the up quark, down quark and electron respectively, and U, D, L are the left-handed doublet fields of the up quark, down quark and electron respectively. The sum is over all generations, and one should read the corresponding generation quark or lepton for each field in the sum. Also a, b are $SU(2)$ indices, and the $\cdot$ operation is the contraction between $SU(2)$ indices in the natural way. These Yukawa interactions allow for a correction on the mass of the Higgs particle. We will calculate one of these corrections by considering the loop given in Figure 1a,



(a) A fermion loop.



(b) A bosonic loop.

Figure 1: Two one-loop diagrams that correspond to a correction to the Higgs mass. Both figures are from [18]

with a fermion f.

$$\begin{aligned}
\Delta m_H^2 &= -|\lambda_f|^2 \int \frac{(-\not{k}+m_f)_{\alpha\beta}(-\not{k}+m_f)_{\beta\alpha}}{(k^2+m_f^2)^2} \frac{d^4k}{(2\pi)^4} \\
&= -\frac{|\lambda_f|^2}{(2\pi)^4} \int_{S^3} \int_0^{\Lambda_{UV}} \frac{\mathrm{Tr}(\not{k}\not{k} - 2m_f\not{k} + m_f^2)}{(k^2+m_f^2)^2} k^3 dk d\Omega_3 \\
&= \frac{|\lambda_f|^2}{(2\pi)^4} \mathrm{Vol}(S^3) \int_0^{\Lambda_{UV}} \frac{4k^2 - 4m_f^2}{(k^2+m_f^2)^2} k^3 dk \\
&= 4\frac{|\lambda_f|^2}{8\pi^2} \int_0^{\Lambda_{UV}} \left[\frac{1}{k^2+m_f^2} - \frac{2m_f^2}{(k^2+m_f^2)^2} \right] k^3 dk \\
&= \frac{\lambda_f^2}{2\pi^2} \int_0^{\Lambda_{UV}} k + \frac{km_f^2}{k^2+m_f^2} - \frac{2k^3 m_f^2}{(k^2+m_f^2)^2} dk = \frac{\lambda_f^2}{2\pi^2}\Lambda_{UV}^2 + \mathcal{O}(\log(\Lambda_{UV}))
\end{aligned}$$

where Λ_{UV} is the ultra-violet cut-off used to regulate the integral. This cut-off is interpreted as the energy limit in which we can expect physics to behave as expected. Given that previous calculation is true for every fermion, we see that the corrections to the Higgs mass become unnaturally big. In the case of the heaviest quark, the top quark, it is found that $\lambda_t \approx 0.94$ [18]. There are three colours for the top quark, so the previous calculation should be multiplied by 3. Therefore if the order of the cut-off is, say, around $\Lambda_{UV} \approx 10$ TeV, we get the top correction to be around $\Delta m_H^2 \approx 1.2$ TeV, which is about 10 times the order of magnitude of the experimentally observed mass of the Higgs particle. To make the Higgs mass relatively small, one should fine-tune λ_t up to 10 till 20 digits to get to the correct correction. Taking into consideration that the other fermions as well as gauge bosons will also interact with the Higgs, and we also have its self-interaction, it is stunning that the Higgs mass is so small. The problem really becomes apparent if one takes another values of Λ_{UV}. Assuming the cut-off is bigger results in fine-tuning the parameters even more [21]. This necessity of a unnatural fine-tuning is known as *the fine-tuning problem*. It is the question whether nature really chose these parameters to have such a dramatic cancellation this grand a scale, or whether we are missing some essential physics that cancels these divergences in a more natural way.

One might argue that this phenomenon is due to our specific renormalization scheme, and adopting another scheme such as dimensional regularization might solve it. And while the quadratic divergences in the dimensional regularization scheme indeed can be removed, the fine-tuning will still show up. To see this, we remark that we expect the SM to be an effective theory on higher energy scales. If one assumes a heavy particle, with renormalization mass M that has not been found yet, interacting with the Higgs particle, one can show that the corrections to the mass of the Higgs is proportional to M^2. This shows the fine-tuning problem again, for the mass M is bigger than the energy scale we can measure, so getting a small Higgs mass requires again a fine-tuning of parameters [6, 8].

In the literature, many physicists often use the term naturalness or the hierarchy problem for the fine-tuning problem. However closely related, the devil is in the details. The *hierarchy problem* is the question why the electroweak force is so much stronger than gravity [22]. This question can directly be rephrased why the Higgs' mass is that much lighter than expected. *Naturalness* is somewhat more explicit, for a more mathematical definition is given by 't Hooft [24].

Definition 7.0.1. Consider an energy level μ. A set of physical parameters $\{\alpha_i(\mu)\}_i$ is called *natural* if the following situation is present: If setting the parameters $\alpha_i(\mu) = 0$ increases the symmetry of the system, then the parameters are allowed to be only very small relative to the energy scale.

With this definition, one can see that the Higgs mass is unnatural. If we consider the Lagrangian in Equation (7.1) and ignore gauge interactions for the moment, one sees that no extra symmetry is found by just setting $\mu^2 = 0$ because of the $\lambda|\phi|^4$ term. It makes us wonder why the Higgs mass is kept so light, which makes it closely related to the fine-tuning problem.

Now suppose that there exists a bosonic particle S that interacts with the Higgs boson by the interaction

$$\mathcal{L} = -\lambda_S |\phi_H|^2 |S|^2. \tag{7.4}$$

We can then consider the Feynman diagram given in Figure 1b, which gives as a correction to the mass

$$\begin{aligned}\Delta m_H^2 &= -\lambda_S \int \frac{1}{k^2 + m_S^2} \frac{d^4k}{(2\pi)^4} \\ &= -\frac{\lambda_S}{(2\pi)^4} \int_{S^3} \int_0^{\Lambda_{UV}} \frac{k^3}{k^2 + m_S^2} dk d\Omega_3 \\ &= -\frac{\lambda_S}{8\pi^2} \int_0^{\Lambda_{UV}} k + \frac{k m_S^2}{k^2 + m_S^2} dk = -\frac{\lambda_S}{8\pi^2} \Lambda_{UV}^2 + \mathcal{O}(\log(\Lambda_{UV})).\end{aligned} \tag{7.5}$$

Note that there is a minus sign in front of this correction, due to S not being a fermion. In the SM, there is no such particle to have these kind of interactions, except the Higgs boson itself. However, the result seem to diverge at the same order as the Feynman diagram in Figure 1a. Having both contributions would cancel the quadratic divergence. It thus raises the question: are there any bosonic particles that have the interaction term given in Equation (7.4), in such a way that $\lambda_S = |\lambda_f|^2$? If so, that could potentially solve the hierarchy problem. We will first be considering supersymmetry, which assumes new particles and where $\lambda_S = |\lambda_f|^2$ is a byproduct of this assumption. However, supersymmetry assumes certain things that have not been experimentally measured, hence needs a 'patch' in a way that we need to break supersymmetry softly without distorting the parameters too much. This 'patch' however introduces a ton of new parameters, which are difficult to measure. Another way of solving the hierarchy problem is by considering the Hyperbolic Model, which we will discuss later on in the thesis.

8 An introduction to supersymmetry

The hierarchy problem, albeit aesthetic one, is not the only problem that the SM cannot tackle effectively. Big questions such as dark energy, neutrino oscillation and gravity in general, cannot be described by the SM [3]. One way of solving some of these problems was by introducing a new form of symmetry. This symmetry which is called *supersymmetry* will be the focus of this chapter. For a more in-depth discussion on the history of supersymmetry, we refer to [3]. We will closely follow [3, 18] in this chapter.

Before entering the world of supersymmetry, we remind ourselves of the description of a fermion. We have multiple ways of describing a fermionic field, of which the Dirac fermion and the Majorana fermion are the most commonly ones used. A classical Dirac spinor field can be described as $\psi : \mathbb{R}^{1,3} \to \mathbb{C}^4$, where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

that satisfies the Dirac equation

$$-i\partial\!\!\!/\psi + m\psi = 0 \tag{8.1}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. Here we denote

$$\gamma^\mu := \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{8.2}$$

where σ^μ are the Dirac matrices given by

$$\begin{aligned} \sigma^0 = \bar{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad ; \qquad \sigma^1 = -\bar{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 = -\bar{\sigma} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad ; \qquad \sigma^3 = -\bar{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{8.3}$$

Both the Dirac and the Majorana fermion satisfy these equations. However, as we shall see, the Majorana fermion is more restrictive than a Dirac fermion. For the discussion here, we define

$$\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{8.4}$$

This allows us to split the spinor field into two parts: $\psi_L := \frac{1-\gamma_5}{2}\psi$ and $\psi_R := \frac{1+\gamma_5}{2}\psi$, which results in the equality

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} \qquad , \qquad \psi_R = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{8.5}$$

and the useful expression $\psi = \psi_L + \psi_R$. We will call ψ_L the left-, and ψ_R the right-handed fermion field. These two-dimensional spinors are called Weyl-spinors, and so a Dirac spinor is made of two Weyl-spinors. Both ψ_L and ψ_R transform differently under the $SU(2)$ symmetry, hence a distinction between ψ_L and ψ_R will have to be made. The left-handed fermion transforms in the fundamental representation of $SU(2)$, while the right-handed fermion transforms as the trivial representation of $SU(2)$.

This however does not mean that the dynamics of ψ_L and ψ_R are completely independent. In the Dirac representation, the Lagrangian includes a mass term in the form of $m\bar{\psi}\psi = m\psi_R^\dagger\psi_L + m\psi_L^\dagger\psi_R$ where m is the mass of the fermion. This construction leads to an indirect dependence of ψ_R on ψ_L and visa versa by the Euler-Lagrange equation.

Next we go to the Majorana fermion. To do that, we first define the charge conjugation of a spinor, denoted as ψ^c. It is defined as

$$\psi^c = \mathcal{C}\bar{\psi}^T \tag{8.6}$$

where the matrix $\mathcal{C}$ is the charge conjugation matrix satisfying the following equations

$$\mathcal{C}^T = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C} \tag{8.7}$$

$$\mathcal{C}^{-1}\gamma_\mu\mathcal{C} = -\gamma_\mu^T \tag{8.8}$$

$$[\mathcal{C}, \gamma_5] = 0. \tag{8.9}$$

In the chiral basis, it can be shown that the charge conjugation is given by

$$\mathcal{C} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{8.10}$$

We then *define* a Majorana fermion as a spinor that is its own charge conjugation, i.e.

$$\psi = \psi^c = \mathcal{C}\gamma_0^T\psi^* \tag{8.11}$$

where ψ^* is defined as taking the complex conjugate of each component. This has as immediate consequence that a Majorana fermion has less degrees of freedom than a Dirac fermion. For we see that

$$\psi_R = \frac{1+\gamma_5}{2}\psi = \frac{1+\gamma_5}{2}\mathcal{C}\bar{\psi}^T = \frac{1+\gamma_5}{2}\mathcal{C}\gamma_0^T\psi^* = \mathcal{C}\frac{1+\gamma_5}{2}\gamma_0\psi^* = \mathcal{C}\gamma_0\frac{1-\gamma_5}{2}\psi^* = \mathcal{C}\gamma_0\psi_L^* \tag{8.12}$$

This calculation shows that the ψ_R is completely determined by ψ_L, which narrows the degrees of freedom for a spinor field. Thus one can equivalently define a Majorana spinor as two Weyl-spinors with an additional condition. This condition can be seen as an analogue to the condition of a real scalar field having $\phi^\dagger = \phi$. Physically speaking, one has that Majorana fermions are their own anti-particle.

8.1 Wess-Zumino model

The primary toy model one uses for a supersymmetric model, is the Wess-Zumino model [3]. It is therefore useful to discuss this model in somewhat more detail, before going to the general case.

For the Wess-Zumino model, we consider the following Lagrangian:

$$\mathcal{L}_{WZ} = -\frac{1}{2}\left(\partial_\mu A\right)^2 - \frac{1}{2}\left(\partial_\mu B\right)^2 + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{F^2+G^2}{2} - m\left(\frac{1}{2}\bar{\psi}\psi - GA - FB\right) \tag{8.13}$$

where A and B are real scalar fields, ψ is a Majorana fermion and F, G are real scalar fields with mass dimension $[F] = [G] = 2$. We note that F and G have no kinetic terms, hence cannot be dynamical fields at all. Using the Euler-Lagrange formula we find $F = -mB$ and $G = -mA$. Filling this into $\mathcal{L}_{WZ}$, we get the Lagrangian for two free massive scalar fields, and one free massive Majorana field, all having the same mass:

$$\mathcal{L}_{WZ}|_{\{F,G\}} = -\frac{1}{2}\left(\partial_\mu A\right)^2 - \frac{1}{2}\left(\partial_\mu B\right)^2 + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m^2(A^2+B^2) - \frac{1}{2}m\bar{\psi}\psi. \tag{8.14}$$

However, before using the equation of motion, we see that Equation (8.13) contains a symmetry that is absent in the Lagrangian for free fields. Using the notation $\phi \to \phi + \delta\phi$ where ϕ is some field and $\delta\psi$ is some deviation, we can do the following transformations

$$\begin{aligned}
\delta A &= i\bar{\alpha}\gamma_5\psi \\
\delta B &= -\bar{\alpha}\psi \\
\delta\psi &= -F\alpha + iG\gamma_5\alpha + \partial\!\!\!/\gamma_5 A\alpha + i\partial\!\!\!/ B\alpha \\
\delta F &= i\bar{\alpha}\partial\!\!\!/\psi \\
\delta G &= \bar{\alpha}\gamma_5\partial\!\!\!/\psi
\end{aligned}$$

where α is a Majorana spinor that is independent of the space-time coordinates. Under these transformations, the Lagrangian in Equation (8.13) is invariant up to a total derivative. We see that these transformations transform bosonic fields to fermionic fields, and visa versa. These transformations are thus said to be *supersymmetry* transformations for it interchanges bosons with fermions. There is also a constant Majorana spinor, but this can be interpreted as a more general set of parameters that one comes to expect when doing a transformation.

If we look closely, we can even compactify these statements into a more elegant, and more generalizable situation. For if we define

$$\mathcal{S} := \frac{1}{\sqrt{2}}(A + iB) \tag{8.15}$$

$$\mathcal{F} := \frac{1}{\sqrt{2}}(F + iG) \tag{8.16}$$

we see that we can compactify previous transformations to

$$\begin{aligned} \delta\mathcal{S} &= -i\sqrt{2}\bar{\alpha}\psi_L \\ \delta\psi_L &= -\sqrt{2}\mathcal{F}\alpha_L - \sqrt{2}\partial\!\!\!/\mathcal{S}\alpha_R \\ \delta\mathcal{F} &= i\sqrt{2}\bar{\alpha}\partial\!\!\!/\psi_L \end{aligned}$$

where $\psi_L = \frac{1-\gamma_5}{2}\psi$ is the left-handed fermionic field, and equivalently for α_L and α_R. Remember we are talking about Majorana spinors, so ψ_R can be expressed solely in ψ_L as seen in Equation (8.12). Therefore it is enough to discuss only transformations on left-handed fermions. Also note that $\mathcal{S}$ and $\mathcal{F}$ describe complex scalar fields.

If one has a symmetry, one can calculate the current and conserved charges that correspond to them. In this case, we find that the current is given by

$$j^\mu = \partial\!\!\!/(-iA\gamma_5 - B)\gamma^\mu\psi + (G\gamma_5 + iF)\gamma^\mu\psi \tag{8.17}$$

and the conserved charge that corresponds to the current is then found to be

$$Q_a = \int j^0_a(x)d^3x \tag{8.18}$$

where a is the $SU(2)$ index.

As stated before, the Wess-Zumino model in practice describes two free massive scalars with the same mass, and one massive fermion field as seen in Equation (8.14). If we introduce the following interaction term

$$L_{\text{WZ, int}} := -\frac{g}{\sqrt{2}}A\bar{\psi}\psi + \frac{ig}{\sqrt{2}}B\bar{\psi}\gamma_5\psi + \frac{g}{\sqrt{2}}\left(A^2 - B^2\right)G + g\sqrt{2}ABF \tag{8.19}$$

one can show by brute force, plugging in all the transformations, that this interaction term is also invariant under supersymmetry. If we add this to Equation (8.13), the Euler-Lagrange formula gives $F = -mB - g\sqrt{2}AB$ and $G = -mA - \frac{g}{\sqrt{2}}\left(A^2 - B^2\right)$. If we substitute these two non-dynamical fields into the interaction term, we get

$$\begin{aligned} L_{\text{WZ, int}} = &-\frac{g}{\sqrt{2}}A\bar{\psi}\psi + \frac{ig}{\sqrt{2}}B\bar{\psi}\gamma_5\psi - \frac{mg}{\sqrt{2}}A\left(A^2 - B^2\right) \\ &- \frac{g^2}{2}\left(A^2 - B^2\right)^2 - gm\sqrt{2}AB^2 - 2g^2A^2B^2 \end{aligned} \tag{8.20}$$

Although these interactions are complicated by itself, one can note that the interaction terms all have the same coupling constant as well as the same mass parameter. Familiar interactions like the Yukawa interaction and quadratic self-coupling in regards to the scalar field A are present, as well as more nuanced interactions.

8.2 General SUSY

With the Wess-Zumino model, we have constructed a Lagrangian that allows for a symmetry that transforms complex scalar fields into spinor fields and visa versa. Although it seemed accidental, we can generalize this symmetry to a self-consisting theory called supersymmetry (or SUSY in short).

As previous section showed, fermions and bosons are interchanged by a symmetry operator, call it Q. We call Q the *supersymmetry generator*. We assume

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle, \qquad Q|\text{Fermion}\rangle = |\text{Boson}\rangle.$$

One can see that this operator maps the bosonic states into fermionic states, so Q should have fermionic indices as well. In previous section, the transformation translated a boson into a Majorana fermion, so we can assume Q to be a Majorana fermionic operator. There is no reason to assume self-adjointness of the operator, we see that $Q^\dagger$ is also a symmetry operator. Since Q is fermionic, it should anti-commute with itself. Additionally, if we want to have Lorenz covariance, we need the Poincaré algebra to interact with it. In conclusion, we require [18, 23]:

$$\begin{aligned}
\{Q_a, Q_b\} &= 0 \\
\{Q_a, Q_b^\dagger\} &= -2\left(\sigma^\mu\right)_{ab} P_\mu \\
[Q_a, P_\mu] &= 0 \\
[Q_a^\dagger, P_\mu] &= 0 \\
[Q_a, M^{\mu\nu}] &= \left(\sigma^{\mu\nu}\right)_{ab} Q_b
\end{aligned} \tag{8.21}$$

where $a, b \in \{1, 2\}$ are the $SU(2)$ indices and $\sigma^{\mu\nu} := \frac{i}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)$ where σ^μ are the Dirac matrices. Together with the Poincaré group transformations, we get the *super-Poincaré algebra* given by the infinitesimal transformations

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
i[P_\mu, M_{\nu\rho}] &= \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu \\
i[M_{\mu\nu}, M_{\rho\lambda}] &= \eta_{\nu\rho} M_{\mu\lambda} - \eta_{\mu\rho} M_{\nu\lambda} - \eta_{\lambda\mu} M_{\rho\nu} + \eta_{\lambda\nu} M_{\rho\mu} \\
\{Q_a, Q_b^\dagger\} &= -2(\sigma^\mu)_{ab} P_\mu \\
\{Q_a, Q_b\} &= 0 \\
[Q_a, P_\mu] &= 0 \\
[Q_a^\dagger, P_\mu] &= 0 \\
[Q_a, M^{\mu\nu}] &= \left(\sigma^{\mu\nu}\right)_{ab} Q_b.
\end{aligned} \tag{8.22}$$

To connect to Section 8.1, we note that the conserved charge found in Equation (8.18) acts as the SUSY generator, for it satisfies the super-Poincaré algebra requirements in Equation (8.22).

We see that $[Q_a, P_\mu] = 0$, therefore P^2 commutes with Q. It hence shows that to find irreducible representations of this algebra, one can use the same argumentation as for the Poincaré algebra (a little discussion to find the irreducible unitary representations of the Poincaré group can be found in Appendix

A). However, the same does not hold for $[Q, M^{\mu\nu}]$, so multiple spins are allowed. It therefore seems natural to extend our notion of fields we are used to. We will consider a so-called *superfield* that includes both bosonic, as fermionic fields in its description. Referring back to Section 8.1, we thus wish to find a field that includes $\mathcal{S}$, $\mathcal{F}$ and ψ_L in a way such that they are all described on an equal footing.

To construct it consistently, we define a constant Majorana spinor θ that anti-commutes with ψ. Since the components of ψ, denoted by ψ_j, anticommute, we can define

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} \tag{8.23}$$

such that each component θ_i also anticommutes. We also assume that $\{\theta_i, \psi_j\} = 0$ for all $i, j \in \{1, 2, 3, 4\}$. Note that this also means that $\theta_i^2 = 0$. Since θ is Majorana, we have $\bar{\theta} = \theta^T \mathcal{C}$. So the conjugate is uniquely defined if we have θ well-defined. Denote the space of these Grassman numbers $(\theta_1, \theta_2, \theta_3, \theta_4)$ that satisfy these conditions as Θ^4.

The fact that $\theta_i^2 = 0$ induces a contraint on the possible functions of θ that one can have. If we consider any smooth function $\hat{\varphi} : \Theta^4 \to \Theta^4$, we can do a Taylor expansion. However, no θ_i^2 can ever occur, so only have finitely many terms can appear in a Taylor expansion. Explicitely,we get

$$\begin{aligned} \hat{\varphi}(\theta_1, \theta_2, \theta_3, \theta_4) =& \varphi_1 + \varphi_2\theta_1 + \varphi_3\theta_2 + \varphi_4\theta_3 + \varphi_5\theta_4 + \varphi_6\theta_1\theta_2 + \varphi_7\theta_1\theta_3 + \varphi_8\theta_1\theta_4 + \varphi_9\theta_2\theta_3 + \varphi_{10}\theta_2\theta_4 \\ &+ \varphi_{11}\theta_3\theta_4 + \varphi_{12}\theta_1\theta_2\theta_3 + \varphi_{13}\theta_1\theta_2\theta_4 + \varphi_{14}\theta_1\theta_3\theta_4 + \varphi_{15}\theta_2\theta_3\theta_4 + \varphi_{16}\theta_1\theta_2\theta_3\theta_4 \end{aligned} \tag{8.24}$$

where the various prefactors φ_j are constants.

Since we are interested in expanding the field theory, we shall define the *superspace* $\mathbb{R}^{1,3}_{super}$ as $\mathbb{R}^{1,3}_{super} := \mathbb{R}^{1,3} \times \Theta^4$. Any function $\hat{\varphi} : \mathbb{R}^{1,3}_{super} \to \mathbb{C}$ can then be Taylor expanded in the same way as in Equation (8.24), only now each φ_j is a function of x^μ, i.e. $\varphi_j(x)$. Thus each φ_j can now be considered a field on $\mathbb{R}^{1,3}$. As a convention, whenever we have a hat $\wedge$ on top of a function we mean that the function is depending on both x^μ as well as θ, whereas no hat is just a function depending on the space-time coordinates.

Reorganising each term into a more compact and basis independent way, we get for a general superfield $\hat{\varphi}$:

$$\begin{aligned} \hat{\varphi}(x, \theta) =& \mathcal{S} - i\sqrt{2}\bar{\theta}\gamma_5\psi - \frac{i}{2}(\bar{\theta}\gamma_5\theta)\mathcal{M} + \frac{1}{2}(\bar{\theta}\theta)\mathcal{N} + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V^\mu \\ &+ i(\bar{\theta}\gamma_5\theta)\left[\bar{\theta}(\lambda + \frac{i}{\sqrt{2}}\partial\!\!\!/\psi)\right] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\left[\mathcal{D} - \frac{1}{2}\Box\mathcal{S}\right]. \end{aligned} \tag{8.25}$$

where we denoted $\Box := \partial^\mu\partial_\mu$. Many of the coefficients are chosen for future convenience, but can be absorbed by the fields if one prefers. Here $\mathcal{S}, \mathcal{M}, \mathcal{N}$ and $\mathcal{D}$ are complex scalar fields, ψ and λ are Dirac spinor fields, and V^μ is a vector field. It is of no surprise that the form of the superfield in Equation (8.25) is not unique. If we consider the adjoint, we find

$$\begin{aligned} \hat{\varphi}^\dagger(x, \theta) =& \mathcal{S}^\dagger - i\sqrt{2}\bar{\theta}\gamma_5\psi^c - \frac{i}{2}(\bar{\theta}\gamma_5\theta)\mathcal{M}^\dagger + \frac{1}{2}(\bar{\theta}\theta)\mathcal{N}^\dagger + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V^{\mu\dagger} \\ &+ i(\bar{\theta}\gamma_5\theta)\left[\bar{\theta}(\lambda^c + \frac{i}{\sqrt{2}}\partial\!\!\!/\psi^c)\right] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\left[\mathcal{D}^\dagger - \frac{1}{2}\Box\mathcal{S}^\dagger\right]. \end{aligned} \tag{8.26}$$

We see that if $\hat{\varphi}^\dagger = \hat{\varphi}$, this results into restricting all complex scalar fields to be real scalar fields, as well as setting all the Dirac fermions to be Majorana fermions.

When considering the Poincaré algebra, one identifies the momentum operator $\hat{P}_\mu$ to a differential operator acting on the fields, by considering the infinitesimal operation $[P_\mu, \phi]$ on a field ϕ. One can do the same in the super Poincaré algebra by considering the transformations Q induces on superfields. To discuss this, remember that one can take the derivative of Grassmann variables by defining

$$\frac{\partial \theta_a}{\partial \theta_b} = \delta_{ab} \qquad ; \qquad \frac{\partial \bar{\theta}_a}{\partial \bar{\theta}_b} = \delta_{ab}$$

wgere we remember $\frac{\partial \bar{\theta}_b}{\partial \theta_a}$ is not zero due to θ being Majorana. We can also define a slightly different Leibniz' rule, due to having anti-commuting numbers, by

$$\frac{\partial}{\partial \theta_c}(\theta_a \theta_b) = \delta_{ac}\theta_b - \theta_a \delta_{bc}.$$

Using the fact that θ is Majorana, we can find that the operator Q acts on superfields as

$$\delta\varphi := [\bar{\alpha}Q, \hat{\varphi}] = i\left(\bar{\alpha}\frac{\partial}{\partial\bar{\theta}} + i\bar{\alpha}\partial\!\!\!/\theta\right)\varphi. \tag{8.27}$$

For a more in-depth discussion, we refer to [3]. It follows that a general superfield will transform under SUSY as

$$\delta\mathcal{S} = i\sqrt{2}\bar{\alpha}\gamma_5\psi \tag{8.28}$$

$$\delta\psi = -\frac{\alpha\mathcal{M}}{\sqrt{2}} - i\frac{\gamma_5\alpha\mathcal{N}}{\sqrt{2}} - i\frac{\gamma_\mu\alpha V^\mu}{\sqrt{2}} - \frac{\gamma_5\partial\!\!\!/\mathcal{S}\alpha}{\sqrt{2}} \tag{8.29}$$

$$\delta\mathcal{M} = \bar{\alpha}\left(\lambda + i\sqrt{2}\partial\!\!\!/\psi\right) \tag{8.30}$$

$$\delta\mathcal{N} = i\bar{\alpha}\gamma_5\left(\lambda + i\sqrt{2}\partial\!\!\!/\psi\right) \tag{8.31}$$

$$\delta V^\mu = -i\bar{\alpha}\gamma^\mu\lambda + \sqrt{2}\bar{\alpha}\partial^\mu\psi \tag{8.32}$$

$$\delta\lambda = -i\gamma_5\alpha\mathcal{D} - \frac{1}{2}[\gamma_\nu, \gamma_\mu]\partial^\nu V^\mu\alpha \tag{8.33}$$

$$\delta\mathcal{D} = \bar{\alpha}\partial\!\!\!/\gamma_5\lambda. \tag{8.34}$$

where α is a space-time independent anticommuting Majorana spinor parameter.

We can argue that this result shows that we cannot start with a smaller set of parameters. For, say, we state that $\mathcal{D} = 0$, then $\delta\mathcal{D} \neq 0$ as long as $\lambda \neq 0$. Thus we would need to set $\lambda = 0$. But $\delta\lambda$ will be non-zero if $V^\mu \neq 0$, which in turn is generated by ψ. Therefore, we cannot naively set certain parameters to zero and expect SUSY to behave nicely. However, in the Wess-Zumino model we needed less components to describe our system, hence it seems likely that there several components of $\hat{\varphi}$ in this model that are unphysical as well. So if we choose certain conditions on these parameters, one can reduce the amount of degrees of freedom.

One components that was absent in the Wess-Zumino model, is the vector field. This could be because the dynamical part of the vector field, the field strength, was zero i.e.

$$\partial_\mu V^\nu - \partial_\nu V^\mu = 0.$$

If this is the case, we see that $V_\mu = \partial_\mu \xi$ for some scalar field ξ. To have this equality, one must restrict the transformation given in Equation (8.32) to a transformation that does not distort $V_\mu = \partial_\mu \xi$. This can only be realized if the transformation itself is given by a total derivative, which can only be true if $\lambda = 0$. But if $\lambda = 0$, we must also require $\delta\lambda = 0$, which can only be realized if

$$0 = -i\gamma_5 \alpha \mathcal{D} - \frac{1}{2}[\gamma_\mu, \gamma_\nu]\partial^\nu \partial^\mu \xi \alpha.$$

Since ξ is smoothly depending on the space-time coordinates, we have $[\gamma_\mu, \gamma_\nu]\partial^\mu \partial^\nu \xi = 0$, hence $\mathcal{D}$ must be 0 as well. And since we already set $\lambda = 0$, we also see that $\delta\mathcal{D} = 0$, which is consistent.

We conclude that we can choose $\lambda = \mathcal{D} = 0$ and $V^\mu = \partial^\mu \xi$ for some scalar field ξ to reduce the amount of free parameters that seem to be unphysical. Filling these conditions in Equations (8.28)-(8.32) gives the transformations

$$\begin{aligned}
\delta\mathcal{S} &= i\sqrt{2}\bar{\alpha}\gamma_5\psi \\
\delta\psi &= -\frac{\alpha\mathcal{M}}{\sqrt{2}} - i\frac{\gamma_5\alpha\mathcal{N}}{\sqrt{2}} - i\frac{\gamma_\mu\alpha V^\mu}{\sqrt{2}} - \frac{\gamma_5 \partial\!\!\!/ \mathcal{S}\alpha}{\sqrt{2}} \\
\delta\mathcal{M} &= i\sqrt{2}\bar{\alpha}\partial\!\!\!/\psi \\
\delta\mathcal{N} &= -\sqrt{2}\bar{\alpha}\gamma_5\partial\!\!\!/\psi \\
\delta V^\mu &= \sqrt{2}\bar{\alpha}\partial^\mu\psi.
\end{aligned}$$

If we then play around a bit with these transformations, we can write the transformations as

$$\begin{aligned}
\delta\left(\frac{\partial^\mu\mathcal{S} \mp iV^\mu}{\sqrt{2}}\right) &= \mp 2i\bar{\alpha}\partial^\mu\left(\frac{1\pm\gamma_5}{2}\right)\psi \\
\delta\left(\frac{\mathcal{M} \mp i\mathcal{N}}{\sqrt{2}}\right) &= 2i\bar{\alpha}\partial\!\!\!/\left(\frac{1\pm\gamma_5}{2}\right)\psi.
\end{aligned}$$

Rewriting $\frac{1-\gamma_5}{2}\psi = \psi_L$ and $\frac{1+\gamma_5}{2}\psi = \psi_R$ gives

$$\begin{aligned}
\delta\left(\frac{\partial^\mu\mathcal{S} - iV^\mu}{\sqrt{2}}\right) &= -2i\bar{\alpha}\partial^\mu\psi_L \\
\delta\left(\frac{\mathcal{M} - i\mathcal{N}}{\sqrt{2}}\right) &= 2i\bar{\alpha}\partial\!\!\!/\psi_L \\
\delta\psi_L &= -\frac{\mathcal{M} - i\mathcal{N}}{\sqrt{2}}\alpha_L + \frac{\partial^\mu\mathcal{S} - iV^\mu}{\sqrt{2}}\gamma_\mu\alpha_R
\end{aligned} \tag{8.35}$$

which transforms into each other, and

$$\begin{aligned}
\delta\left(\frac{\partial^\mu\mathcal{S} + iV^\mu}{\sqrt{2}}\right) &= 2i\bar{\alpha}\partial^\mu\psi_R \\
\delta\left(\frac{\mathcal{M} + i\mathcal{N}}{\sqrt{2}}\right) &= 2i\bar{\alpha}\partial\!\!\!/\psi_R \\
\delta\psi_R &= -\frac{\mathcal{M} + i\mathcal{N}}{\sqrt{2}}\alpha_R - \frac{\partial^\mu\mathcal{S} + iV^\mu}{\sqrt{2}}\gamma_\mu\alpha_L
\end{aligned} \tag{8.36}$$

transforms into each other. In other words, we have found two subrepresentations of the SUSY algebra by setting only a few components to be a specific value. In fact, we see that the first one is characterized by the left-handed spinor field, while the second one is characterized by the right-handed spinor field. So one can argue that these are the elementary superfields one can consider. If, in addition, the superfield was to be real i.e. $\hat{\varphi} = \hat{\varphi}^\dagger$, then these two multiplets are conjugate to each other. However in the more general case, the one does not need to depend on the other.

To be more explicit, we can define a *left-chiral scalar superfield*. A *left-chiral scalar superfield* or a *left-handed chiral superfield* is a superfield as in Equation (8.25), in such a way that it only contains the left-handed spinor multiplet, i.e. $\psi_R = 0$, $\frac{\partial^\mu \mathcal{S} + iV^\mu}{\sqrt{2}} = 0$ and $\frac{\mathcal{M}+i\mathcal{N}}{\sqrt{2}} = 0$. Equivalently,

$$\psi_R = 0 \qquad , \qquad V^\mu = i\partial^\mu \mathcal{S} \qquad , \qquad \mathcal{N} = i\mathcal{M} =: i\mathcal{F}. \tag{8.37}$$

With this, we can fill in Equation (8.25) to find the description for a left-chiral scalar superfield:

$$\hat{\mathcal{S}}_L = \mathcal{S} + i\sqrt{2}\bar{\theta}\psi_L + i\bar{\theta}\theta_L\mathcal{F} + \frac{i}{2}\left(\bar{\theta}\gamma_5\gamma_\mu\theta\right)\partial^\mu\mathcal{S} - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\cdot\bar{\theta}\partial\!\!\!/\psi_L + \frac{1}{8}\left(\bar{\theta}\gamma_5\theta\right)^2\Box\mathcal{S}. \tag{8.38}$$

Since $\psi_R = 0$, we can equivalently set ψ to be a Majorana fermion, and choose ψ_R to be exactly $\psi_R = \mathcal{C}\gamma_0\psi_L^*$ as in Equation (8.12). As a consequence, the left-chiral scalar superfield then transforms under the SUSY algebra as

$$\delta\mathcal{S} = -i\sqrt{2}\bar{\alpha}\psi_L \tag{8.39}$$
$$\delta\psi_L = -\sqrt{2}\mathcal{F}\alpha_L + \sqrt{2}\partial\!\!\!/\mathcal{S}\alpha_R \tag{8.40}$$
$$\delta\mathcal{F} = i\sqrt{2}\bar{\alpha}\partial\!\!\!/\psi_L \tag{8.41}$$

In the same spirit, we define a *right-chiral scalar superfield* or a *right-handed chiral superfield* by the having no left-handed components, i.e. $\psi_L = \frac{\partial^\mu\mathcal{S}-iV^\mu}{\sqrt{2}} = \frac{\mathcal{M}-i\mathcal{N}}{\sqrt{2}} = 0$. This gives $V^\mu = -i\partial^\mu\mathcal{S}$ and $\mathcal{N} = -i\mathcal{M} =: \mathcal{F}$ which gives the description for the right-handed scalar chiral superfield as

$$\hat{\mathcal{S}}_R = \mathcal{S} - i\sqrt{2}\bar{\theta}\psi_R - i\bar{\theta}\theta_R\mathcal{F} - \frac{i}{2}\left(\bar{\theta}\gamma_5\gamma_\mu\theta\right)\partial^\mu\mathcal{S} - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\cdot\bar{\theta}\partial\!\!\!/\psi_R + \frac{1}{8}\left(\bar{\theta}\gamma_5\theta\right)^2\Box\mathcal{S}. \tag{8.42}$$

As a remark, we note that we can take the adjoint of a left-handed chiral superfield to get a right-chiral scalar superfield. This can be seen by using the fact that $\bar{\chi}\psi = \bar{\psi}^c\chi$ for any two spinor fields, Majorana or Dirac. Since both θ and ψ are Majorana, we get

$$\begin{aligned}\hat{\mathcal{S}}_L^\dagger &= \mathcal{S}^\dagger - i\sqrt{2}\bar{\psi}_L\theta - i\bar{\theta}_L\theta\mathcal{F}^\dagger - \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S}^\dagger - \frac{1}{\sqrt{2}}[-\bar{\theta}\gamma_5\theta]\cdot[-\bar{\theta}\partial\!\!\!/\psi_R] + \frac{1}{8}\left(\bar{\theta}\gamma_5\theta\right)^2\Box\mathcal{S}^\dagger \\ &= \mathcal{S}^\dagger - i\sqrt{2}\bar{\psi}\theta_R - i\bar{\theta}\theta_R\mathcal{F}^\dagger - \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\mathcal{S}^\dagger - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\cdot\bar{\theta}\partial\!\!\!/\psi_R + \frac{1}{8}\left(\bar{\theta}\gamma_5\theta\right)^2\Box\mathcal{S}^\dagger\end{aligned} \tag{8.43}$$

which is indeed a right-handed chiral scalar superfield.

8.3 The Kähler- and superpotential; building the Lagrangian

Next, we will discuss the construction of a SUSY-invariant Lagrangian that includes chiral fermions, like in the SM. If we wish to construct a Lagrangian that is SUSY invariant, we would like the Lagrangian to transform only by a total derivative when considering SUSY transformations. If we look closely at the transformation of any *generic* superfield (as given in Equation (8.28)-(8.34)), the $\mathcal{D}$ part transforms as a total derivative, independent of what λ might be. In addition, any *left-handed* chiral superfield's $\mathcal{F}$ component transforms as a total derivative, as seen in Equation (8.41). These specifically prime our attention, for that means when searching for any SUSY invariant Lagrangians, we only have to take the $\mathcal{D}$-term of a superfield and the $\mathcal{F}$-term of a left-handed chiral superfield into account.

This restricts our possible set of Lagrangians considerably, in such a way that only a few options are possible. To see this we introduce two functions. First we define the function $K(\mathcal{S}_{iL}^\dagger, \mathcal{S}_{iL})$, depending on n different left-handed chiral superfields $\mathcal{S}_{iL}$. Since it depends on both a left-handed chiral superfield $\mathcal{S}_{iL}$, as a right-handed chiral superfield $\mathcal{S}_{iL}^\dagger$, we have that K itself is a generic superfield for which the transformation rules in Equation (8.28)-(8.34) hold. This K function is called the *Kähler potential*. As in the discussion above, we are interested in the $\mathcal{D}$-term of the Kähler potential, for those can only contribute to the Lagrangian.

As for the second function, define the function $\hat{f}(\mathcal{S}_{iL})$. This function only depends on the left-handed chiral superfields, hence is itself a left-handed chiral superfield. We will call this function the *superpotential* and as noted in the above, will only need the the $\mathcal{F}$-component of the superpotential to contribute to the Lagrangian.

We will first focus on the Kähler potential. Because $\{Q_a, Q_b^\dagger\} = -2(\sigma^\mu)_{ab}P_\mu$ by Equation (8.22), and the mass dimension of P is 1, we must have that the mass dimension of Q is $[Q] = [Q^\dagger] = \frac{1}{2}$. By Equation (8.27) we thus find the mass dimension of θ is given by $[\theta] = [\bar{\theta}] = -\frac{1}{2}$.

So, if we only consider renormalizable theories, we must have that each term of the Lagrangian has mass dimension 4 or less. If the $\mathcal{D}$-term is the part that contributes, we see from Equation (8.25) that

$$[K] = [K_{\mathcal{D}-term}] - 2 \leq 4 - 2 = 2.$$

So we see that the Kähler potential has at most mass dimension 2. If we impose that a generic superfield has the same dimensions as in the Wess-Zumino model, we find that the scalar field $\mathcal{S}$ of the superfield has mass dimension $[\mathcal{S}] = 1$, and so $[\hat{\mathcal{S}}] = 1$. This means that the Kähler potential is at most a quadratic polynomial of the superfields we put in. To ensure that $\mathcal{L}$ is a real scalar, we thus enforce

$$K(\hat{\mathcal{S}}_{iL}, \hat{\mathcal{S}}_{iL}^\dagger) = \sum_{i,j=1}^{N} A_{ij} \hat{\mathcal{S}}_{iL}^\dagger \hat{\mathcal{S}}_{jL} \tag{8.44}$$

with A_{ij} some (complex) number for all $1 \leq i, j \leq N$. Without loss of generality, we can transform the fields in such a way that the matrix A is a diagonal matrix. Renormalizing the superfields results in that the most general Kähler potential is given by:

$$K(\hat{\mathcal{S}}_{iL}, \hat{\mathcal{S}}_{iL}^\dagger) = \sum_{i,j=1}^{N} \hat{\mathcal{S}}_{iL}^\dagger \hat{\mathcal{S}}_{iL}. \tag{8.45}$$

If we work out the multiplication of each of these superfields, remembering we only need terms that go like $\left(\bar{\theta}\gamma_5\theta\right)^2$, we find that the $\mathcal{D}$-component of the Kähler potential is given by (where we removed the $\frac{1}{4}(\bar{\theta}\gamma_5\theta)^2$):

$$K(\hat{\mathcal{S}}_{iL}, \hat{\mathcal{S}}_{iL}^\dagger)_{\mathcal{D}-comp} = -2\sum_{i=1}^{N}\left[-\partial_\mu \mathcal{S}_{iL}^\dagger \partial^\mu \mathcal{S}_{iL} + \frac{i}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + \mathcal{F}^\dagger \mathcal{F}\right]. \tag{8.46}$$

This form can be added to the Lagrangian. We can redefine all the fields in such a way that we can get rid of the -2, which gives the canonical expression for a Lagrangian we are used to.

Next, we go to the left-handed chiral superpotential $\hat{f}$. As mentioned before, we only need the $\mathcal{F}$-component of the expansion of left-handed chiral superfields as seen in Equation (8.38). To construct the superpotential, we again count mass-dimensions. We see that

$$[\hat{f}] = [\hat{f}_{\mathcal{F}-term}] - 1$$

by counting the dimensions as in Equation (8.38). So in order to have a renormalizable theory, we use the same argument as above and see that

$$[\hat{f}] \leq 4 - 1 = 3$$

and therefore $\hat{f}_{\mathcal{F}-term}$ is at most a cubic polynomial of left-handed chiral superfields. So in general, the superpotential can have the form of

$$\hat{f}(\mathcal{S}_{iL}) = \sum_{i,j,k=1}^{N} B_{ijk}\hat{\mathcal{S}}_{iL}\hat{\mathcal{S}}_{jL}\hat{\mathcal{S}}_{kL}. \tag{8.47}$$

We already fixed our fields by assuming the Kähler potential to be in the canonical basis, so we cannot absorb the tensor B into the definition of $\hat{S}_{iL}$. To find the terms of order $\mathcal{O}(\bar{\theta}\theta_L)$, we Taylor expand $\hat{f}$ around $\hat{\mathcal{S}} = \mathcal{S}$. This makes the θ dependence explicit such that we can isolate the $\bar{\theta}\theta$ term easily. We get

$$\hat{f}(\hat{\mathcal{S}}_{iL}) = \hat{f}(\mathcal{S}_{iL}) + \sum_{i=1}^{N} \left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_{iL}}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \left(\hat{\mathcal{S}}_{iL} - \mathcal{S}_{iL}\right) + \frac{1}{2}\sum_{i,j=1}^{N} \left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_{iL}\partial \hat{\mathcal{S}}_{jL}}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \left(\hat{\mathcal{S}}_{iL} - \mathcal{S}_{iL}\right)\left(\hat{\mathcal{S}}_{jL} - \mathcal{S}_{jL}\right) + \dots \tag{8.48}$$

whereby $\hat{\mathcal{S}} = \mathcal{S}$ we mean an evaluation of $\hat{\mathcal{S}}_{iL} = \mathcal{S}_{iL}$ for all $i = 1, \dots, N$ and thus setting all other fields equal to 0. Since $\hat{\mathcal{S}}_{iL}$ can be written as in Equation (8.38), each $\left(\hat{\mathcal{S}}_{iL} - \mathcal{S}_{iL}\right)$ term is at least linear in θ. We can conclude that we do not need any more terms than those we have shown in Equation (8.48), for a third order in this expansion would result in at least a cubic order of θ. We also note that the zeroth order of this expansion does not contribute, since it is of zeroth order in θ.

Writing out Equation (8.48) using Equation (8.38), we see at the $\bar{\theta}\theta_L$ order

$$\hat{\mathcal{S}}_{iL} - \mathcal{S}_{iL} = i\mathcal{F}\bar{\theta}\theta_L, \qquad \left(\hat{\mathcal{S}}_{iL} - \mathcal{S}_{iL}\right)\left(\hat{\mathcal{S}}_{jL} - \mathcal{S}_{jL}\right) = (i\sqrt{2}\bar{\theta}\psi_{iL})(i\sqrt{2}\bar{\theta}\psi_{jL}) = \bar{\theta}\theta_L\bar{\psi}_i\psi_{jL} \tag{8.49}$$

where in the last equation we used the fact that ψ is Majorana, the fact that for any spinors $\bar{\theta}\psi_L = \bar{\psi}_L^c\theta^c$ and the equality

$$\theta_i\bar{\theta}_j = -\frac{1}{4}\left[\bar{\theta}\gamma_5\theta(\gamma_5)_{ij} + \bar{\theta}\theta\delta_{ij} - \left(\bar{\theta}\gamma^\mu\gamma_5\right)(\gamma_\mu\gamma_5)_{ij}\right].$$

So, filling in all the components of Equation (8.49) into Equation (8.48) gives the eligible components for a SUSY invariant Lagrangian

$$\hat{f}_{\mathcal{F}-comp}(\hat{\mathcal{S}}_{iL}) = i\sum_{i=1}^{N} \left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_{iL}}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_i + \frac{1}{2}\sum_{i,j=1}^{N} \left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_{iL}\partial \hat{\mathcal{S}}_{jL}}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_i\psi_{jL}. \tag{8.50}$$

Note that previous equation is not necessarily Hermitian. To force this, we also add the adjoint $\hat{f}^\dagger$. Because $\hat{f}^\dagger$ is a right-handed superfield, as seen in Equation (8.43), and the right-handed superfield's $\mathcal{F}$-component is also a total derivative, we can safely add them. By convention, add each term with a minus sign such that $\mathcal{L} = -\hat{f}(\hat{\mathcal{S}}_{iL})_{\mathcal{F}-comp} - \hat{f}(\hat{\mathcal{S}}_{iL})^\dagger_{\mathcal{F}-comp}$. Then, adding Equation (8.46) in the canonical way, the most general Lagrangian density is given by:

$$\begin{aligned}\mathcal{L}_{SUSY,\mathcal{F}} =& K(\hat{\mathcal{S}}_{iL},\hat{\mathcal{S}}^\dagger_{iL})_{\mathcal{D}-comp} - \hat{f}(\hat{\mathcal{S}}_{iL})_{\mathcal{F}-comp} - \hat{f}(\hat{\mathcal{S}}_{iL})^\dagger_{\mathcal{F}-comp} \\ =& \sum_{j=1}^N \left[-\partial_\mu \mathcal{S}^\dagger_{jL}\partial^\mu \mathcal{S}_{jL} + \frac{i}{2}\bar{\psi}_j \partial\!\!\!/ \psi_j + \mathcal{F}^\dagger_j \mathcal{F}_j - i \left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \mathcal{F}_j + i\left(\left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}}\right)^\dagger \mathcal{F}^\dagger_j\right] \\ &- \frac{1}{2}\sum_{j,k=1}^N \left[\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_j \partial \hat{\mathcal{S}}_k}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_j \psi_{kL} + \left(\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_j \partial \hat{\mathcal{S}}_k}\right|_{\hat{\mathcal{S}}=\mathcal{S}}\right)^\dagger \bar{\psi}_j\psi_{kR}\right].\end{aligned} \tag{8.51}$$

Just as in the Wess-Zumino Lagrangian, we see that the $\mathcal{F}$-fields are not dynamical. So if these fields are on-shell, the Euler-Lagrange formula states

$$\mathcal{F} + i\left(\left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}}\right)^\dagger = 0 \quad , \quad \mathcal{F}^\dagger - i\left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}} = 0.$$

These equations can be put back in, to find the general SUSY Lagrangian to be given by

$$\begin{aligned}\mathcal{L}_{SUSY} =& \sum_{j=1}^N \left[-\partial_\mu \mathcal{S}^\dagger_{jL}\partial^\mu \mathcal{S}_{jL} + \frac{i}{2}\bar{\psi}_j \partial\!\!\!/ \psi_j - \left|\left(\left.\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_j}\right|_{\hat{\mathcal{S}}=\mathcal{S}}\right)\right|^2\right] \\ &- \frac{1}{2}\sum_{j,k=1}^N \left[\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_j \partial \hat{\mathcal{S}}_k}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \bar{\psi}_j \psi_{kL} + \left(\left.\frac{\partial^2 \hat{f}}{\partial \hat{\mathcal{S}}_j \partial \hat{\mathcal{S}}_k}\right|_{\hat{\mathcal{S}}=\mathcal{S}}\right)^\dagger \bar{\psi}_j\psi_{kR}\right].\end{aligned} \tag{8.52}$$

We should note that this Lagrangian is found by taking no notion of gauge fields, or gauge symmetry in general. To add gauge fields, we refer to [3, 18]. These involve adding new gauge superfields, but do not change previous Lagrangian. New interactions might crop up due to $\mathcal{F}$ fields being multiplied with these new gauge fields, giving new terms in the Euler-Lagrange equations.

8.4 The Minimal SUSY Standard Model

Now that we have defined and constructed the tools we need, we can discuss an extension of the Standard Model (SM), called the Minimal Supersymmetric Standard Model (MSSM).

In the SM, we have an elegant system of elementary particles, interacting via the gauge group $SU(3)\times SU(2)_L\times U(1)$. We have the quarks, leptons, gauge bosons and the Higgs particle as elementary particles. We wish to extend each of these to a supersymmetric version. This section is mostly based on [3].

We start with the quarks and leptons. Every particle is described by a Dirac fermion, which breaks up into a left-handed and right-handed component due to the $SU(2)_L$ interaction. Each of these left-handed chiral fields can thus be embedded into a left-chiral scalar superfield as given in Equation (8.38). Because the superpotential only depends on left-handed chiral superfields, upgrading the right-handed fermions to right-chiral scalar superfields will not yield desirable results. However, if we consider $(\psi_R)^c$, we see

$$(\psi_R)^c = \left[\frac{1+\gamma_5}{2}\psi\right]^c = \mathcal{C}\overline{\left(\frac{1+\gamma_5}{2}\psi\right)}^T = \mathcal{C}\gamma_0\frac{1+\gamma_5^*}{2}\psi^* = \mathcal{C}\gamma_0\psi_R^* = \mathcal{C}\begin{pmatrix}\psi_3^*\\ \psi_4^*\\ 0\\ 0\end{pmatrix}. \tag{8.53}$$

Given that the charge conjugation matrix, as given in Equation (8.10), does not mix the first two components with the last two, we can conclude that ψ_R^c is a left-handed chiral field, without losing any of the information enveloped in the right handed fermion. This allows us to translate the information in right-handed fermions into left-handed fermions, which can then be injected in the SUSY picture. Hence we can promote each of the fermion fields as

$$\begin{pmatrix} (\nu_i)_L \\ (e_i)_L \end{pmatrix} \rightarrow \hat{L}_i := \begin{pmatrix} (\hat{\nu}_i)_L \\ (\hat{e}_i)_L \end{pmatrix} \tag{8.54}$$

$$(e_{iR})^c \rightarrow \hat{E}_i^c \tag{8.55}$$

$$\begin{pmatrix} (u_i)_L \\ (d_i)_L \end{pmatrix} \rightarrow \hat{Q}_i := \begin{pmatrix} (\hat{u}_i)_L \\ \left(\hat{d}_i\right)_L \end{pmatrix} \tag{8.56}$$

$$(u_{iR})^c \rightarrow \hat{U}_i^c \tag{8.57}$$

$$(d_{iR})^c \rightarrow \hat{D}_i^c \tag{8.58}$$

where $i = 1, 2, 3$ goes over all the generations of quarks and leptons, e.g. $\hat{u}_3$ refers to the superfield that includes the left-handed top field t_L. Remember that we use the convention of having a hat on top of a field to express it is a superfield.

This promotion is not without any repercussions. Since any superfield is described by a scalar field $\mathcal{S}$, a spinor field ψ and an auxiliary field $\mathcal{F}$, we now have more particles than we started with. Especially if the spinor fields correspond to the matter fields of the SM, we have new scalar fields that cannot be identified by just looking at the SM. These newly predicted particles are referred to as *sfermions*. These spin 0 particles are the superpartners of the matter fermions, and for quarks they are often called *squarks* while for leptons they are called *sleptons*. If we are talking about a specific quark or lepton generation, the scalar superpartner is often also called by the same name with an *s* in front of it. So the *stop* particle is the spin 0 superpartner of the top quark, and the *selectron* is the spin 0 superpartner of the electron. We will denote the scalar field superpartner of a fermion with a $\sim$ above the letter. We also note, as was seen before, that the mass of the sfermion is equal to the mass of the fermion.

Since we are considering the left-handed and the right-handed fermion separately, we also get two sfermions for each of them. One needs to note that these newly found scalar fields are *not* the same. To make the distinction clear, we will denote a L or R subscript to denote the superfield to refer to the chirality of the fermion. In other words, $\tilde{t}_L$ is the stop particle that is derived from the left-handed top field t_L.

Next, we turn our attention to the Higgs particle. Since the Higgs doublet is a set of scalar fields itself, promoting these fields to the superfields now introduces new fermionic partners:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \rightarrow \hat{H}_u := \begin{pmatrix} \hat{h}_u^+ \\ \hat{h}_u^0 \end{pmatrix}. \tag{8.59}$$

We will call these new superpartners *higgsinos*. We note that under $SU(2)_L$ the superfield transforms as a doublet, and has a hypercharge of $Y = 1$. However, these higgsinos that have a hypercharge of $Y = 1$ will mess with the triangle anomaly of the SM. To be able to still have a consistent model, we need to introduce a *second* left-chiral scalar superfield, defined by

$$\hat{H}_d := \begin{pmatrix} \hat{h}_d^- \\ \hat{h}_d^0 \end{pmatrix}. \tag{8.60}$$

which has a hypercharge of $Y = -1$ and transforms as $\bar{\mathbf{2}}$ under $SU(2)_L$. Finally, we mention the gauge fields. These fields will live in a so-called *vector superfield*, and is another irreducible representation of the SUSY algebra. They are required in the MSSM, however we will not need these fields in our thesis. For a full discussion, we refer to [3, 18].

To build the model, we need to choose a superpotential. In the MSSM, we choose the superpotential to be of the following form

$$\hat{f}_{\text{MSSM}} = \mu(\hat{H}_u)^a(\hat{H}_d)_a + \sum_{i,j=1,2,3}\left[(\mathbf{f}_u)_{ij}\epsilon_{ab}(\hat{Q}_i)^a(\hat{H}_u)^b\hat{U}^c_j + (\mathbf{f}_d)_{ij}(\hat{Q}_i)^a(\hat{H}_d)_a\hat{D}^c_j + (\mathbf{f}_e)_{ij}(\hat{L}_i)^a(\hat{H}_d)_a\hat{E}^c_j\right] \tag{8.61}$$

where a, b are the $SU(2)_L$ indices that are being summed over. Also μ is a coupling constant, and $\mathbf{f}_u, \mathbf{f}_d, \mathbf{f}_e$ are 3x3 matrices that give the Yukawa interactions between all the superfields. If we purely focus on the top sector and the Higgs fields, the superpotential looks like

$$\hat{f}_{\text{MSSM, top}} = \mu(\hat{H}_u)^a(\hat{H}_d)_a + (\mathbf{f}_u)_{33}\hat{t}_L\hat{h}^0_u\hat{T}^c \tag{8.62}$$

and therefore the Lagrangian interaction between the top sector and the Higgs fields are given by, using Equation (8.52),

$$\begin{aligned}\mathcal{L}_{int} = &-\mu^2(|h_u^+|^2 + |h_u^0|^2 + |h_d^-|^2 + |h_d^0|^2) - |(\mathbf{f}_u)_{33}\tilde{t}_L\tilde{h}^0_u|^2 - |(\mathbf{f}_u)_{33}\tilde{t}_R\tilde{h}^0_u|^2 \\ &- \frac{1}{2}\left[(\mathbf{f}_u)_{33}\tilde{h}^0_u\bar{\psi}_t\psi_{T^cL} + [(\mathbf{f}_u)_{33}]^\dagger\left(\tilde{h}^0_u\right)^\dagger\bar{\psi}_t\psi_{T^cR} + (\mathbf{f}_u)_{33}\tilde{h}^0_u\bar{\psi}_{T^c}\psi_{tL} + [(\mathbf{f}_u)_{33}]^\dagger\left(\tilde{h}^0_u\right)^\dagger\bar{\psi}_{T^c}\psi_{tR}\right].\end{aligned} \tag{8.63}$$

Here we denoted the ψ_t as the Majorana field of t_L and equivalently for ψ_{T^c}. As we can see, the Yukawa interactions are included, as one would expect. With the kinetic terms added as well, we have constructed the SUSY-invariant Lagrangian for the MSSM.

However, as of to today, we have found no evidence of SUSY in nature. If a particle has the exact same mass as the electron but is bosonic, it should have been discovered already. But it has not as of yet. Therefore, it is often assumed that SUSY is a broken symmetry, forcing the masses of fermions and sfermions to not align anymore. Breaking SUSY is a whole topic upon itself, and is beyond the scope of our discussion. We will shortly introduce softly breaking SUSY, but for more details the interested reader might conduct for example [3, 18].

Now that we have introduced the basics of the MSSM, we come back to the hierarchy problem. As stated at the end of Chapter 7, we motivated that the addition of bosonic particles to the SM might solve the hierarchy problem. If we look at Equation (8.63), we see that we the coupling constant for the bosonic particles is given by $\lambda_{\tilde{t}} = |(\mathbf{f}_u)_{33}|^2 = |\lambda_t|^2$ and equivalently for the other generations of particles. Looking back at Equation (7.5), we argued that these specific coupling constants were needed to cancel the quadratic divergences. This thus guarantees the cancellation of quadratic divergences in the corrections of the Higgs mass at first order and therefore the quadratic sensitivity must vanish at all orders in pertubation theory [18].

However, SUSY is a broken symmetry as stated, if it even exists. This concludes that the masses of the fermion and its supersymmetric partner have different masses which probably messes with the cancellation that solved the hierarchy problem. If one wants to let SUSY still be a solution of the hierarchy problem, one must require that the relationship between the coupling constants in the unbroken SUSY picture still holds in the broken SUSY picture. Else quadratic divergences pop up again.

Therefore, we are lead to consider only low-energetic, soft SUSY breaking. We write

$$\mathcal{L} = \mathcal{L}_{\text{SUSY}} + \mathcal{L}_{\text{soft}} \tag{8.64}$$

where $\mathcal{L}_{\text{SUSY}}$ is the SUSY-invariant Lagrangian we have been constructing so far, while $\mathcal{L}_{\text{soft}}$ violates SUSY softly having couplings and masses of positive mass-dimension only [18]. Denote the highest mass-dimensional scale term as m_{soft}. We then note that any corrections to the Higgs potential must vanish if we let $m_{soft} \to 0$, hence it cannot be that the correction goes as $m_{soft}\Lambda_{UV}^2$. Since these corrections originate from loop diagrams which either diverge quadratically or logarithmically, we have that corrections must be of the form

$$\Delta m_{H,soft}^2 \propto m_{soft}^2 \log\left(\frac{\Lambda_{UV}}{m_{soft}}\right) + \dots \tag{8.65}$$

which is logarithmic at best [18]. Hence soft-breaking terms do not change the cancellation of the quadratic terms, hence softly broken SUSY still solves the hierarchy problem.

To be explicit, these soft SUSY breaking terms in Equation (8.64) can be linear, bilinear or trilinear in the scalar field [3]. Extra mass terms are also allowed. Thus the most general soft SUSY breaking Lagrangian is then found to be, ignoring the gauge interactions,

$$\begin{aligned}
\mathcal{L}_{\text{soft}} = &- \Big[\tilde{Q}_i^\dagger \mathbf{m}_{Q_{ij}}^2 \tilde{Q}_j + \tilde{d}_{Ri}^\dagger \mathbf{m}_{D_{ij}}^2 \tilde{d}_{Rj} + \tilde{u}_{Ri}^\dagger \mathbf{m}_{U_{ij}}^2 \tilde{u}_{Rj} + \tilde{L}_i^\dagger \mathbf{m}_{L_{ij}}^2 \tilde{L}_j + \tilde{e}_{Ri}^\dagger \mathbf{m}_{E_{ij}}^2 \tilde{e}_{Rj} \\
&\quad + m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 \,\Big] \\
&+ \Big[(\mathbf{a_u})_{ij}\, \epsilon_{ab} \tilde{Q}_i^a H_u^b \tilde{u}_{Rj}^\dagger + (\mathbf{a_d})_{ij}\, \tilde{Q}_i^a H_{da} \tilde{d}_{Rj}^\dagger + (\mathbf{a_L})_{ij}\, \tilde{L}_i^a H_{da} \tilde{e}_{Rj}^\dagger + \text{hermitian conj.} \\
&\quad + (\mathbf{c_u})_{ij}\, \epsilon_{ab} \tilde{Q}_i^a H_d^{*b} \tilde{u}_{Rj}^\dagger + (\mathbf{c_d})_{ij}\, \tilde{Q}_i^a H_{ua}^* \tilde{d}_{Rj}^\dagger + (\mathbf{c_L})_{ij}\, \tilde{L}_i^a H_{ua}^* \tilde{e}_{Rj}^\dagger + \text{hermitian conj.}\Big] \\
&+ B\mu H_u^a H_{da} + B\mu (H_{da})^\dagger (H_u^a)^\dagger
\end{aligned} \tag{8.66}$$

where i, j are indices that sum over all generations, and a, b sum over the $SU(2)_L$ indices. Here B is a real parameter, and $\mathbf{m}_x, \mathbf{a}_x, \mathbf{c}_x$ are matrices in generation space [3]. These terms break SUSY softly to get the dynamics of the SM back.

We know that in the SM there is only one Higgs field, while in the MSSM there are two Higgs fields. As we know from the SM, the Higgs boson ends up as a neutral spin 0 particle after the spontaneous symmetry breaking of $SU(2)_L \times U(1) \to U(1)_{EM}$. The other physical degrees of freedom get eaten by the gauge bosons $W^\pm$ and Z^0. We would thus expect the same to happen to the two Higgs fields. However there are more degrees of freedom in the Higgs multiplets than the $W^\pm$ and Z^0 bosons can eat. We will examine this shortly. For a more detailed explanation, we refer to [3, Ch. 8].

To connect to the SM, we should break SUSY first. Including terms that would break SUSY results in the Higgs potential for the MSSM to be given by:

$$\begin{aligned}
V_{\text{Higgs}} = &\left(m_{H_u}^2 + \mu^2\right)\left(|h_u^0|^2 + |h_u^+|^2\right) + \left(m_{H_d}^2 + \mu^2\right)\left(|h_d^0|^2 + |h_d^-|^2\right) - B\mu\left(h_u^+ h_d^- + h_u^0 h_d^0 + \text{h.c.}\right) \\
&+ \frac{g^2}{2}\Big[\left(|h_u^+|^2 - |h_u^0|^2 + |h_d^0|^2 - |h_d^-|^2\right)^2 + 4|h_u^+|^2|h_d^0|^2 + 4|h_d^0|^2|h_d^-|^2 - \\
&4\left(h_u^{+*} h_d^{-*} h_u^0 h_d^0 + h_u^{0*} h_d^{0*} h_u^+ h_d^-\right)\Big] + \frac{g'^2}{8}\left[|h_u^+|^2 + |h_u^0|^2 - |h_d^0|^2 - |h_d^-|^2\right].
\end{aligned}$$

Here g and g' are coupling constants coming from the gauge interactions. Since each field is a complex field, we can write $h_u^0 = \frac{1}{\sqrt{2}}(h_{uR}^0 + i h_{uI}^0)$ and equivalently for h_d^0. In this way, we have a potential that is depends on 8 fields, all of which are independent. If we in addition claim that spontaneous symmetry breaking occurs, by having as VEV for H_u given by

$$\left\langle \begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix} \right\rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}$$

(which is possible due to gauge freedom) where $v_u \in \mathbb{R}$, we find that the VEV for H_d is given by setting $\langle h_d^- \rangle = 0$ and $\langle h_d^0 \rangle =: v_d$ where $v_d \in \mathbb{R}$ as well. Filling this all in into V_{Higgs} will generate a set of mass matrices. We start with the charged particles, for which we see that

$$\mathcal{L} \ni (h_u^{+*}, h_d^-)\mathcal{M}^2_{h^\pm} \begin{pmatrix} h_u^+ \\ h_d^{-*} \end{pmatrix} \tag{8.67}$$

where

$$\mathcal{M}_{h^\pm} = \begin{pmatrix} \frac{\partial^2 V_{\text{Higgs}}}{\partial h_u^+ \partial h_u^{+*}}\Big|_{v_u,v_d} & \frac{\partial^2 V_{\text{Higgs}}}{\partial h_u^+ \partial h_u^{+*}}\Big|_{v_u,v_d} \\ \frac{\partial^2 V_{\text{Higgs}}}{\partial h_u^+ \partial h_u^{+*}}\Big|_{v_u,v_d} & \frac{\partial^2 V_{\text{Higgs}}}{\partial h_u^+ \partial h_u^{+*}}\Big|_{v_u,v_d} \end{pmatrix}.$$

Filling this in, we get a mass matrix which has eigenvalues

$$m_1^2 = 0 \text{ and } m_2^2 = B\mu(\tan(\beta) + \frac{1}{\tan(\beta)}) + \frac{g^2}{2}(v_u^2 + v_d^2) \tag{8.68}$$

where $\tan(\beta) := \frac{v_u}{v_d}$. Therefore, we see that we have two charged Higgs bosons that are massless, and thus gets eaten by the $W^\pm$ bosons. We also have two charged particles, which we will denote by $H^\pm$. These two particles of course cannot describe the SM Higgs, for the SM Higgs does not have any charge. If we continue to the neutral states, we get in a similar expression

$$\mathcal{L} \ni (h_{uI}^0, h_{dI}^0)\mathcal{M}^2_{h_I^0} \begin{pmatrix} h_{uI}^0 \\ h_{dI}^0 \end{pmatrix} \tag{8.69}$$

where

$$\mathcal{M}_{h_I^0} = \begin{pmatrix} \frac{\partial^2 V_{\text{Higgs}}}{(\partial h_{uI}^0)^2}\Big|_{v_u,v_d} & \frac{\partial^2 V_{\text{Higgs}}}{\partial h_{uI}^0 \partial h_{dI}^0}\Big|_{v_u,v_d} \\ \frac{\partial^2 V_{\text{Higgs}}}{\partial h_{uI}^0 \partial h_{dI}^0}\Big|_{v_u,v_d} & \frac{\partial^2 V_{\text{Higgs}}}{(\partial h_{dI}^0)^2}\Big|_{v_u,v_d} \end{pmatrix}.$$

Diagonalizing this matrix, gives the mass eigenstates

$$m_1^2 = 0 \text{again and} m_2^2 = B\mu\left(\tan(\beta) + \frac{1}{\tan(\beta)}\right). \tag{8.70}$$

So again, we see one massless Goldstone boson that will be absorbed in the Z^0 boson, and another boson we shall call the A boson. It turns out that this boson is a *pseudoscalar*, hence also not the Higgs particle we are searching for.

Finally, we turn to the real part. This is done in the same manner as the previous two, where

$$\mathcal{L} \ni (h_{uR}^0, h_{dR}^0)\mathcal{M}^2_{h_R^0} \begin{pmatrix} h_{uR}^0 \\ h_{dR}^0 \end{pmatrix} \tag{8.71}$$

with, as before,

$$\mathcal{M}_{h_R^0} = \begin{pmatrix} \frac{\partial^2 V_{\text{Higgs}}}{(\partial h_{uR}^0)^2}\Big|_{v_u,v_d} & \frac{\partial^2 V_{\text{Higgs}}}{\partial h_{uR}^0 \partial h_{dR}^0}\Big|_{v_u,v_d} \\ \frac{\partial^2 V_{\text{Higgs}}}{\partial h_{uR}^0 \partial h_{dR}^0}\Big|_{v_u,v_d} & \frac{\partial^2 V_{\text{Higgs}}}{(\partial h_{dR}^0)^2}\Big|_{v_u,v_d} \end{pmatrix}.$$

Now the eigenvalues give a somewhat more elaborate equation, which is given by

$$m_{\mp}^2 = \frac{1}{2}\left[(m_A^2 + M_Z^2) \mp \sqrt{(m_A^2 + M_Z^2)^2 - 4m_A^2 M_Z^2 \cos(2\beta)}\right] \tag{8.72}$$

where $m_A^2 = B\mu\left(\tan(\beta) + \frac{1}{\tan(\beta)}\right)$ and $M_Z = \frac{g^2+g'^2}{2}(v_u^2 + v_d^2)$. These eigenstates define two real scalar fields of different masses. We denote the lighter particle by h, and the heavier particle by H. Given we are unable to probe physics at arbitrary energy, we can only conclude that the lighter particle with mass m_- is the SM Higgs particle, while the heavier is a new Higgs particle.

9 Hyperbolic Higgs

9.1 5D SUSY and compactifications

After having considered 4D SUSY, we will focus on building another model to solve the hierarchy problem. This system will in turn give way to a $SU(2,2)$ symmetry. Our discussion is based on a paper by Cohen *et al*, given in [7], and we will derive most of the important expressions in that paper.

We will start building the model by assuming an extra dimension that is compact. This extra dimension is chosen to be S_R^1, a circle of radius R. Any function $\varphi : \mathbb{R}^{1,3} \times S_R^1 \to \mathbb{C}$ can then be expanded as a Fourier series

$$\varphi(x,y) = \sum_{n\in\mathbb{Z}} \varphi_n(x) e^{\frac{iny}{R}} \tag{9.1}$$

where $y \in [0, 2\pi R)$ is taken as a parametrization of S_R^1. We see that the fields in 5D can thus be broken up into infinite many particles $\varphi_n(x)$. One could stop there, but as we will see in 5D SUSY, particles will appear that are not present in the 4D MSSM, hence we wish to make use of a system to generate masses for these infinite particles that goes beyond the GUT scale.

To do this, we define an action π of $\mathbb{Z}_2$ on S_R^1, by setting $\pi(-1)y = -y$. This induces an action π^* on φ by $\pi^*(-1)\varphi(x,y) := \varphi(x, \pi(-1)y)$, and has as eigenfunctions $\pi^*(-1)\varphi = \pm\varphi$, i.e. $\varphi(x,-y) = \pm\varphi(x,y)$. Therefore we can identify the eigenfunctions of this $\mathbb{Z}_2$ action as the even and the odd parts of Equation (9.1) i.e.

$$\varphi_1(x,y) = \sum_{n\in\mathbb{N}_0} \phi_n(x)\cos\left(\frac{ny}{R}\right)$$
$$\varphi_2(x,y) = \sum_{n\in\mathbb{N}_0} \phi_n(x)\sin\left(\frac{ny}{R}\right).$$

where $\mathbb{N}_0$ denotes $\{0,1,2,\ldots\}$. Next, we wish to imbue S_R^1 with another condition. In an similar way as we think of fermions, we also wish to get spinor-like behaviour results if we go one way around the circle S_R^1. This means that if we consider $\varphi(x, 2\pi R)$ we could get $\pm\varphi(x,0)$. We are used to $+\varphi(x,0)$ for then the function is well-defined, and taking $-\varphi(x,0)$ results in a multivalued function. However, we will allow both options. Mathematically, we are considering sections of a non-trivial vector bundle. For the more physical interpretation and description, one can think of starting with a circle of radius $2R$ (i.e. $y \in [0, 4\pi R)$) and associate the interval $[0, 2\pi R)$ as our physical space, and assume φ can be analytically continued after that by either setting

$$\varphi(x, 2\pi R + y) = \varphi(x,y) \text{ or } \varphi(x, 2\pi R + y) = -\varphi(x,y)$$

with $y \in [0, 2\pi R)$. Another interpretation is to say that we apply different boundary conditions to the functions at $y = 0$ and $y = 2\pi R$.

In any case, if we assume that the functions are fully described by the Fourier expansion as above in the $(0, 2\pi R)$ range, we require

$$\varphi(x, y + 2\pi R) = \pm\varphi(x, y). \tag{9.2}$$

When a function satisfies the minus sign, we say φ has a *Scherk-Schwarz twist*, or SS-twist in short. Starting with a circle of radius $2R$, we do previous arguments to find the eigenfunctions $\phi_{1,n} = \cos\left(\frac{ny}{2R}\right)$ and $\phi_{2,n} = \sin\left(\frac{ny}{2R}\right)$. Assuming no SS-twist, we see that Equation (9.2) can only be satisfied by having $n = 2k$ with $k \in \mathbb{N}_0$, while having a SS-twist can only be satisfied by having $n = 2k + 1$ for $k \in \mathbb{N}_0$. Concluding, we have two parameters we can characterize the eigenfunctions of our system with, denoted by $(\pm, \pm)$ which correspond with the $\mathbb{Z}_2$ action and having a SS-twist or not respectively. The eigenfunctions are given by:

$$(+,+): \qquad \cos\left(\frac{2ky}{2R}\right) = \cos\left(\frac{ky}{R}\right) \tag{9.3}$$

$$(+,-): \qquad \cos\left(\frac{(2k+1)y}{2R}\right) = \cos\left(\frac{(k+1/2)y}{R}\right) \tag{9.4}$$

$$(-,+): \qquad \sin\left(\frac{2ky}{2R}\right) = \sin\left(\frac{ky}{R}\right) \tag{9.5}$$

$$(-,-): \qquad \sin\left(\frac{(2k+1)y}{2R}\right) = \sin\left(\frac{(k+1/2)y}{R}\right). \tag{9.6}$$

A more common approach to the Scherk-Schwarz twist, is to write down q_ϕ as parameter which defines having a twist or not. Looking at Equations (9.3)-(9.6), having a SS-twist corresponds to having the constant factor $\frac{1}{2}$ in the eigenfunction. We define $q_\phi = \frac{1}{2}$ if ϕ has a SS-twist, and $q_\phi = 0$ if ϕ has no SS-twist.

Now that we have discussed the mathematical properties of the fields of our model, we go to the physics. First off, any field ϕ in our 5 dimensional world should be an eigenfunction of the action of $\mathbb{Z}_2$ and either has a SS-twist or not. Hence it should be of the form

$$\phi(x, y) = \sum_{k \in \mathbb{Z}} \phi_k(x) g_k(y)$$

where $g_k(y)$ is one of the four choices of Equation (9.3)-(9.6). These $\{\phi_k(x)\}_k$ are often called the *Kaluza-Klein modes* or the *Kaluza-Klein tower*. We will often abbreviate the Kaluza-Klein modes to KK-modes.

Next we assume SUSY in 5 dimensions. In five dimensions, the irreducible representations of a SUSY multiplet are again chiral and vector multiplets as discussed. However, chiral does not mean anything anymore, for in 5 dimensions the Clifford algebra spanned by the γ-matrices must also include γ_5, hence automatically rendering chiral multiplet representations irreducible [10, 20]. This so-called hypermultiplet, we shall denote as Ξ, includes the fields $(\psi, \psi^{c\dagger}, \phi, \phi^{c\dagger})$, where $\Psi := (\psi, \psi^{c\dagger})$ is a Dirac spinor. The fields $\psi^{c\dagger}$ and $\phi^{c\dagger}$ transform conjugate under the gauge group with respect to the fields ψ and ϕ [4].

Looking at the degrees of freedom for this hypermultiplet, we see that it has 4 fermionic and 4 bosonic degrees of freedom. Hence if we see the fifth dimension y as a label, we can decompose the 5D hypermultiplet into two 4D chiral supermultiplets $\Phi(\psi, \phi)$ and $\Phi^c(\psi^{c\dagger}, \phi^{c\dagger})$ which have 2 fermionic and 2 bosonic degrees of freedom each. The vector multiplet $\mathcal{A}$ consists in 5D out of a vector A_m where $m = 0, 1, 2, 3, y$, a Dirac fermion $\Lambda = (\lambda, \lambda')$, and a real scalar σ. Decomposing that to 4D, we find a vector multiplet $V(A_\mu, \lambda)$ and a chiral multiplet in the adjoint representation $\Sigma(\lambda', \phi_\Sigma)$ where $\phi_\Sigma = \frac{1}{\sqrt{2}}(\sigma + iA_y)$[4, 5, 10]. We do want to note that each of these superfields are independent of each other, and really describe new particles.

To connect back to the real world, one might disregard all these extra supermultiplets Φ^c and Σ. For surely we have not found proof of the existence of SUSY or a fifth dimension, hence requiring all these new particles would be a stretch. But remember that all particle types should be eigenfunctions of the action of $\mathbb{Z}_2$. Hence we can *choose* Φ to be even under the action of $\mathbb{Z}_2$ and Φ^c to be odd. Then the fields look like this:

$$\begin{aligned} \phi(x,y) &= \sum_{k\in\mathbb{N}_0} \phi_k(x)\cos\left(\frac{(k+q_B)y}{R}\right) \quad ; \quad \phi^{c\dagger}(x,y) = \sum_{k\in\mathbb{N}_0} \phi_k^{c\dagger}(x)\sin\left(\frac{(k+q_B)y}{R}\right) \\ \psi(x,y) &= \sum_{k\in\mathbb{N}_0} \psi_k(x)\cos\left(\frac{(k+q_F)y}{R}\right) \quad ; \quad \psi^{c\dagger}(x,y) = \sum_{k\in\mathbb{N}_0} \psi_k^{c\dagger}(x)\sin\left(\frac{(k+q_F)y}{R}\right). \end{aligned} \tag{9.7}$$

Note that we have not yet chosen the SS-twists, and denoted it just as q_B for the bosonic particles and q_F for the fermionic particles. We choose Φ to be even due to the following argument: we know that the known 4D particles, such as the top particle, are as far as we know independent of the y-coordinate. The only modes in Equation (9.7) that are independent of y, are those with $k = 0$ with no SS-twist. In the case of ϕ or ψ in Equation (9.7) we get $\cos(0) = 1$, and so the 0^{th} mode is independent of y. These should therefore correspond to the particles we have measured, such as the top quark and electron.

In the case of $\phi^{c\dagger}$ and $\psi^{c\dagger}$, we get that the 0^{th} mode has $\sin(0) = 0$. This means that we have no 0^{th} mode of this eigenfunction at all. Putting these unwanted particles from Φ^c in the sine eigenfunctions gives that there is no particle we are used to associated with it. This way Equation (9.7) is justified. We can do the same for the vector multiplet V to be even under $\mathbb{Z}_2$ and Σ to be odd under $\mathbb{Z}_2$.

Of course, we would need to normalize the functions in Equation (9.7). If we assume that $\phi_k^{(c\dagger)}$ and $\psi_l^{(c\dagger)}$ are normalized in 4D space, we only need to normalize the eigenfunctions in Equations (9.3-9.6) over the fifth dimension. Note that

$$\int_0^{2\pi R} \cos^2\left(\frac{k+q}{R}y\right) = \int_0^{2\pi R} \sin^2\left(\frac{k+q}{R}y\right) = \pi R$$

if k or q are nonzero, while if $k = q = 0$, the cosine normalization is $2\pi R$. Normalizing this gives that every field is multiplied by $\frac{1}{\sqrt{\pi R}}$ except the 0^{th} order for the $q = 0$ case, which is multiplied by $\frac{1}{\sqrt{2\pi R}}$. To calculate the masses of each of the KK-modes in Equation (9.7), consider the free Lagrangian of the fields $\phi^{(c\dagger)}$ and $\psi^{(c\dagger)}$. One sees that for the bosonic particles

$$\begin{aligned}
S &= \int\int_0^{2\pi R} -\frac{1}{2}|\partial_m\phi|^2\,dy\,d^4x \\
&= \frac{1}{\pi R}\int\int_0^{2\pi R}\sum_{k,l\in\mathbb{N}_0}\frac{1}{\sqrt{2}^{\delta_{q_B,0}(\delta_{k,0}+\delta_{l,0})}}\left[-\frac{1}{2}\partial^\mu\phi_k^\dagger\partial_\mu\phi_l\cos\left(\frac{(k+q_B)y}{R}\right)\cos\left(\frac{(l+q_B)y}{R}\right)\right. \\
&\qquad\left. -\frac{1}{2}\left(\frac{(k+q_B)y}{R}\right)\left(\frac{(l+q_B)y}{R}\right)\phi_k^\dagger\phi_l\sin\left(\frac{k+q_B}{R}y\right)\sin\left(\frac{l+q_B}{R}y\right)\right]dy\,d^4x \\
&= \frac{1}{2\pi R}\int\int_0^{2\pi R}\sum_{k,l\in\mathbb{N}_0}\frac{1}{\sqrt{2}^{\delta_{q_B,0}(\delta_{k,0}+\delta_{l,0})}}\left[\partial^\mu\phi_k^\dagger\partial_\mu\phi_l\cdot\frac{1}{2}\left[\cos\left(\frac{(k-l)y}{R}\right)+\cos\left(\frac{(k+l+2q_B)y}{R}\right)\right]\right. \\
&\qquad\left. -\left(\frac{k+q_B}{R}\right)\left(\frac{l+q_B}{R}\right)\phi_k^\dagger\phi_l\cdot\frac{1}{2}\left[\cos\left(\frac{(k-l)y}{R}\right)-\cos\left(\frac{(k+l+2q_B)y}{R}\right)\right]\right]dy\,d^4x \\
&= \frac{1}{\pi R}\int\sum_{k,l\in\mathbb{N}_0}-\frac{1}{2}\partial^\mu\phi_k^\dagger\partial_\mu\phi_l\cdot\frac{1}{2}\left[\delta_{k-l,0}\cdot 2\pi R+\delta_{k+l+2q_B,0}\cdot 2\pi R\right] \\
&\qquad -\frac{1}{2}\left(\frac{k+q_B}{R}\right)\left(\frac{l+q_B}{R}\right)\phi_k^\dagger\phi_l\cdot\frac{1}{2}\left[\delta_{k-l,0}\cdot 2\pi R-\delta_{k+l+2q_B,0}\cdot 2\pi R\right]d^4x \\
&= \int\sum_{k\in\mathbb{N}_0}-\frac{1}{2}|\partial_\mu\phi_k|-\frac{1}{2}\left(\frac{k+q_B}{R}\right)^2|\phi_k|^2\,d^4x
\end{aligned}$$

where we used the fact that $\int_0^{2\pi R}\cos\left(\frac{my}{R}\right)=2\pi R\delta_{m,0}$ for any $m\in\mathbb{Z}$. Since k,l are positive or 0, only one of the two Kronecker deltas is non-zero. So we have that each term in the KK tower of ϕ has mass $\frac{k+q_B}{R}$ for $k\geq 0$. The same calculation can be done for $\phi^{c\dagger}$ and gives the same result. On the other hand, we see that for the fermions the mass is given by

$$\begin{aligned}
S &= \int\int_0^{2\pi R} i\bar{\Psi}\partial\!\!\!/\Psi+\overline{\Psi}\gamma_5\partial_y\Psi\,dy\,d^4x = \int\int_0^{2\pi R} i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi-\psi^c\partial_y\psi+i\psi^c\sigma^\mu\partial_\mu\psi^{c\dagger}+\psi^\dagger\partial_y\psi^{c\dagger}\,dy\,d^4x \\
&= \frac{1}{\pi R}\int\int_0^{2\pi R}\sum_{k,l\in\mathbb{N}_0}\left(\frac{1}{2}\right)^{\delta_{q_F,0}\delta_{k,0}} i\psi_k^\dagger\bar{\sigma}^\mu\partial_\mu\psi_l\cos\left(\frac{(k+q_F)y}{R}\right)\cos\left(\frac{(l+q_F)y}{R}\right) \\
&\qquad -\left(\frac{1}{2}\right)^{\delta_{q_F,0}\delta_{k,0}}\frac{l+q_F}{R}\psi_k^c\psi_l\sin\left(\frac{(k+q_F)y}{R}\right)\sin\left(\frac{(l+q_F)y}{R}\right)+\ldots\,dy\,d^4x \\
&= \int\sum_{k,l\in\mathbb{N}_0} i\psi_k^\dagger\bar{\sigma}^\mu\partial_\mu\psi_k+\frac{k+q_F}{R}\psi_k^c\psi_k+i\psi_k^c\sigma^\mu\partial_\mu\psi_k^{c\dagger}+\frac{k+q_F}{R}\psi_k^\dagger\psi_k^c\,d^4x
\end{aligned}$$

where we used the same trick as above. Hence we see that each fermion has a mass of $\frac{k+q_F}{R}$. Taking $q_B\neq q_F$ gives $m_B\neq m_F$ hence the SS-twists explicitely break SUSY. So the choice of the Scherk-Schwarz twist allows us to break SUSY without needing soft breaking terms, as the theory of 4D SUSY required. It also shows us that if we take R to be small enough, we can render the masses of the k^{th} KK-mode to be higher than the GUT-scale for all $k\geq 1$, hence explaining why we have not encountered a KK tower of a specific particle yet. Note that adding a mass term in the bosonic case with mass m leads to the new mass $m_k^2=\left(\frac{k+q_B}{R}\right)^2+m^2$ while for the fermions it leads to a new mass $m_k=m+\frac{k+q_F}{R}$.

9.2 Hyperbolic Higgs

Now that we have defined the space and the functions which we will be working with, we consider the actual model. In our model, we assume the MSSM model in 5D, but disregard the soft-breaking terms. Additionally, we assume that there is a secondary set of particles that is identical to the MSSM. We will call this copy of the MSSM the *Hyperbolic MSSM*, denoted as $\text{MSSM}_{\mathcal{H}}$. We will assume that all particle fields, meaning both the MSSM and the $\text{MSSM}_{\mathcal{H}}$ particles, live in the bulk. This means that we choose every matter field to be depending on the fifth dimension, and we choose the particle fields in such a way that the hypermultiplet correspond to eigenfunctions given in Equation (9.7) as we have discussed before. So to be precise, the actual 4D SUSY particles such as the t_L and $\tilde{t}_L$ particles will be the cos-eigenfunctions, while $t_L^{c\dagger}$ and $\tilde{t}_L^{c\dagger}$ will be in the sin-eigenfunctions, and equivalently for the $\text{MSSM}_{\mathcal{H}}$ particles.

To differentiate these two sets of particles, we choose different Scherk-Schwarz twists. We define the Scherk-Schwarz twists on the MSSM, denoted as $(q_B, q_F)_{\text{MSSM}}$, as $(q_B, q_F)_{\text{MSSM}} = (\frac{1}{2}, 0)$ while for the Hyperbolic MSSM we set $(q_B, q_F)_{\text{MSSM}_{\mathcal{H}}} = (0, \frac{1}{2})$. This breaks SUSY explicitely as discussed, and makes the superpartners of the top quark as well the hyperbolic top particle heavy, while leaving top particl as well as the superpartner of the hyperbolic top particle light. Moreover, having $(q_B, q_F)_{\text{MSSM}} = (\frac{1}{2}, 0)$ renders the 0^{th} order of the KK tower independent of y, hence corresponds to the particle fields in the SM which is expected.

Now that we have introduced the particles, we will add interactions. We assume a Yukawa interaction at the $y = 0$ brane. This can be seen as setting

$$\mathcal{L}_{int} = 2\pi R \int \delta(y)\lambda_t(\hat{Q}\hat{U}\hat{H} + \hat{Q}_{\mathcal{H}}\hat{U}_{\mathcal{H}}\hat{H}_{\mathcal{H}})d^2\theta \tag{9.8}$$

where $\hat{X}$ is the supermultiplet of a given particle, and λ_t is the 4D Yukawa coupling for the top quark. This Yukawa interaction can be added for all particle types, but we restrict ourselves to the third generation because the top quark is the heaviest. The factor of $2\pi R$ is to make the dimensions of $\delta(y)$ cancel, hence making the dimension of the Yukawa coupling the usual one in four dimensions. We can only add these kind of Yukawa interactions for the conjugate X^c is 0 at $y = 0$ by the choice of eigenfunctions in Equation (9.7). In addition note that we have not added any interactions that allows the Hyperbolic MSSM to interact with the visible MSSM. We will do this later.

This interaction at the $y = 0$ brane induces a non-trivial mass change on all particles, which we will derive. The Yukawa interaction is completely symmetric when changing $H \leftrightarrow H_{\mathcal{H}}$, hence we will focus only on the H part. There are multiple ways of finding the masses. We will do so by finding the eigenvalues of the mass matrix, while in Appendix B a derivation will be done by solving the EOM.

To show how this $\delta(y)$ contribution induces a mass, we need to consider the bosonic and the fermionic contributions separately, and find the masses for each seperately. The derivation of having a SS-twist or not is analogous up to a few details. Therefore we will only derive the masses for the $q_B = \frac{1}{2}$ and $q_F = 0$ cases to get an intuition how a change in SS-twist and particle type goes.

Let us first focus on the fermionic part. The Lagrangian is found by filling in the master Lagrangian in Equation (8.51), since the axilliary field does not contribute to the fermionic case. This is given by

$$\begin{aligned}\mathcal{L}_{4D,fermion} = \int & it_Q^\dagger\bar{\sigma}\partial_\mu t_Q + it_Q^c\sigma^\mu\partial_\mu t_Q^{c\dagger} + (Q \leftrightarrow U) \\ & - t_Q^c\partial_y t_Q + t_Q^\dagger\partial_y t_Q^{c\dagger} - t_U^c\partial_y t_U + t_U^\dagger\partial_y t_U^{c\dagger} - \delta(y)\lambda_t\left(t_Q^\dagger t_U + t_U^\dagger t_Q\right)|H|\,dy.\end{aligned} \tag{9.9}$$

where $|H|$ is the VEV of the Higgs particle. Next, we can expand each of these fields in its KK-modes. Due to the nature of the $\delta(y)$, we get a mixing of the KK-modes by evaluation at zero of all the fields.

Therefore, the expansion in the KK-modes gives

$$-\mathcal{L}^{mass}_{4D,fermion} = -\sum_{k=0}^{\infty}\left(\frac{k}{R}t^c_{Q,k}t_{Q,k} + \frac{k}{R}t^{\dagger}_{Q,k}t^{c\dagger}_{Q,k} + (Q \leftrightarrow U)\right) + \alpha\left(t^{\dagger}_Q(0)t_U(0) + t^{\dagger}_U(0)t_Q(0)\right). \tag{9.10}$$

where we defined $\alpha := \lambda_t|H|$. Given the 0^{th} order is normalised differently in the $q_F = 0$ case, the second sum should be separated into three different parts. This gives the following equation

$$\begin{aligned} -\mathcal{L}^{mass}_{4D,fermion} = &-\sum_{k=0}^{\infty}\left(\frac{k}{R}t^c_{Q,k}t_{Q,k} + \frac{k}{R}t^{\dagger}_{Q,k}t^{c\dagger}_{Q,k} + (Q \leftrightarrow U)\right) + \alpha t^{\dagger}_{Q,0}t_{U,0} \\ &+ \sqrt{2}\alpha\sum_{k=1}^{\infty}\left(t^{\dagger}_{Q,k}t_{U,0} + t^{\dagger}_{U,k}t_{Q,0} + t^{\dagger}_{Q,0}t_{U,k} + t^{\dagger}_{U,0}t_{Q,k}\right) \\ &+ 2\alpha\sum_{k,l=1}^{\infty}\left(t^{\dagger}_{Q,k}t_{U,l} + t^{\dagger}_{U,k}t_{Q,l}\right). \end{aligned}$$

This can be put into an infinite dimensional matrix, given as follows

$$-\mathcal{L}^{mass}_{4D,fermion} = \begin{pmatrix} t^{\dagger}_{Q,0} & t^{\dagger}_{Q,k} & t^c_{U,l} \end{pmatrix} \frac{1}{R} \begin{pmatrix} \alpha R & \sqrt{2}\alpha R I^T & 0 \\ \sqrt{2}\alpha R I & 2\alpha R(I \cdot I^T) & -M \\ 0 & -M & 0 \end{pmatrix} \begin{pmatrix} t_{U,0} \\ t_{U,k} \\ t^{c\dagger}_{Q,l} \end{pmatrix} + (Q \leftrightarrow U) \tag{9.11}$$

where the notation is such that we have an infinite long vector summing over all k and then l. Also $M = k\delta_{kl}$ is a diagonal matrix, and $I = (1, 1, 1, 1, \ldots)^T$. We wish to diagonalize this matrix, i.e. find its eigenvalues. To do that, we simply consider the eigenvalue equation, i.e.

$$\frac{1}{R} \begin{pmatrix} \alpha R & \sqrt{2}\alpha R I^T & 0 \\ \sqrt{2}\alpha R I & 2\alpha R(I \cdot I^T) & -M \\ 0 & -M & 0 \end{pmatrix} v = \lambda v.$$

For simplicity, redefine $\lambda \to \lambda R$ to get rid of the constant $\frac{1}{R}$ factor. Defining $v = (v_0, v_k, v^{c\dagger}_l)$, we get the following set of equations

$$\alpha R v_0 + \sqrt{2}\alpha R\left(\sum_{k=1}^{\infty} v_k\right) = \lambda v_0 \tag{9.12}$$

$$\sqrt{2}\alpha R v_0 + 2\alpha R\left(\sum_{k=1}^{\infty} v_k\right) - n v^{c\dagger}_n = \lambda v_n \qquad n \geq 1 \tag{9.13}$$

$$-n v_n = \lambda v^c_n \qquad n \geq 1 \tag{9.14}$$

Note that by Equation (9.14), we get $v^{c\dagger}_n = \frac{-n}{\lambda}v_n$. This is allowed, for λ is an eigenvalue, hence not 0. Plugging this into Equation (9.13), one can easily find that

$$\left(\lambda - \frac{n^2}{\lambda}\right)v_n = \sqrt{2}\lambda v_0.$$

Hence we see that

$$\sum_{k=1}^{\infty} v_n = \sqrt{2}v_0 \sum_{n=1}^{\infty} \frac{\lambda^2}{\lambda^2 - n^2} = \frac{\sqrt{2}v_0}{2}\left(-1 + \lambda\pi\cot(\lambda\pi)\right) \tag{9.15}$$

where we used the identity

$$\sum_{k=0}^{\infty} \frac{1}{\lambda^2 - k^2} = \frac{1}{2\lambda^2}\left(-1 + \lambda\pi\cot(\lambda\pi)\right). \tag{9.16}$$

Filling this all into Equation (9.12), we see that

$$\begin{aligned}\lambda v_0 &= \alpha R v_0 + \sqrt{2}\alpha R \cdot \frac{\sqrt{2}v_0}{2}\left(-1 + \cot(\lambda\pi)\right) \\ &= \alpha R v_0 \left(1 + \left(-1 + \lambda\pi\cot(\lambda\pi)\right)\right) \\ &= v_0 \lambda\pi R\alpha\cot(\lambda\pi)\end{aligned}$$

If $v_0 = 0$, then $v = 0$, which cannot be true. So $v_0 \neq 0$, and $\lambda \neq 0$, giving

$$\lambda = \frac{1}{\pi}\arctan(\pi R\alpha) + k \tag{9.17}$$

where $k \in \mathbb{Z}$. Reinstalling the redefined $\lambda \to \lambda R$, we get that the mass of the KK-modes of the fermions with $q_F = 0$ are given by

$$m^{\pm}_{F,k} = \frac{1}{\pi R}\arctan(\pi R\lambda_t|H|) \pm \frac{k}{R} \qquad k \in \mathbb{Z} \tag{9.18}$$

Concluding, there exists a basis such that the mass matrix becomes diagonal. In fact, the basis index goes over $\mathbb{Z}$, which has two options for a mass, hence predicting $2\mathbb{Z} - 1$ amount of particles, one with mass $m^+_{F,k}$ and one with $m^-_{F,k}$.

Remark 9.2.1. We noted that the 0^{th} order of the KK-modes relates to the measured top particle. Looking at Equation (9.18) the 0^{th} order gives the top mass $m_t(H)$ to be

$$m_t(H) := \frac{1}{\pi R}\arctan(\pi R\lambda_t|H|) \tag{9.19}$$

which means that the mass of the top particle depends non-linearly on the VEV of the Higgs.

After having found the mass of the fermions, we go to the bosonic case. Here the $\mathcal{F}$-component will contribute as well to the Lagrangian, as it did in 4D SUSY. The $\delta(y)$ interaction adds a non-trivial contribution to $\mathcal{F}$-component, which can be found to become

$$\mathcal{F}^\dagger_Q = 2\pi R\delta(y)\lambda_t \tilde{t}^\dagger_U \tilde{H} - \partial_y \tilde{t}^c_Q$$

and equivalently for $\mathcal{F}_U$ [10]. The second term in $\mathcal{F}_Q$ is generated by looking at $\bar{\Psi}\Psi$ and going through the SUSY arguments. The contribution of a $\delta(y)$ in the $\mathcal{F}$-term, means $\mathcal{F}^\dagger\mathcal{F}$ includes a $\delta(y)^2$, which is not well-defined mathematically. Physically speaking, we will have

$$\int \delta(y)^2 dy = \delta(0) = \sum_{k=0}^{\infty} 1 =: \mathcal{D}$$

where $\mathcal{D}$ is obviously infinite. However, one does not need to regularize, for $\mathcal{D}$ will cancel in the final product.

With that, we get that the total Lagrangian becomes[4, 10]

$$\begin{aligned}\mathcal{L}^{mass}_{4D,boson} =& \sum_{k=0}^{\infty}\left(\frac{k+\frac{1}{2}}{R}\right)^2 (\tilde{t}^{\dagger}_{Q}\tilde{t}_{Q,k} + \tilde{t}^{c}_{Q,k}\tilde{t}^{c\dagger}_{Q,k} + (Q \leftrightarrow U)) + \delta(0)\sum_{k,l=0}^{\infty}[4\alpha^2\tilde{t}^{\dagger}_{Q,k}\tilde{t}_{Q,l} + 4\alpha^2\tilde{t}^{\dagger}_{U,k}\tilde{t}_{U,l}] \\ &- \sum_{k,l=0}^{\infty}\left[2\alpha\frac{k+\frac{1}{2}}{R}\tilde{t}^{\dagger}_{U,k}\tilde{t}^{c\dagger}_{Q,l} + 2\alpha\frac{k+\frac{1}{2}}{R}\tilde{t}^{c}_{U,k}\tilde{t}_{Q,l} + (Q \leftrightarrow U)\right] \\ =& \sum_{k=0}^{\infty}\left(\frac{k+\frac{1}{2}}{R}\right)^2 (\tilde{t}^{\dagger}_{Q}\tilde{t}_{Q,k} + \tilde{t}^{c}_{Q,k}\tilde{t}^{c\dagger}_{Q,k} + (Q \leftrightarrow U)) + \mathcal{D}\sum_{k,l=0}^{\infty}[4\alpha^2\tilde{t}^{\dagger}_{Q,k}\tilde{t}_{Q,l} + 4\alpha^2\tilde{t}^{\dagger}_{U,k}\tilde{t}_{U,l}] \\ &- \sum_{k,l=0}^{\infty}\left[2\alpha\frac{k+\frac{1}{2}}{R}\tilde{t}^{\dagger}_{U,k}\tilde{t}^{c\dagger}_{Q,l} + 2\alpha\frac{k+\frac{1}{2}}{R}\tilde{t}^{c}_{U,k}\tilde{t}_{Q,l} + (Q \leftrightarrow U)\right]\end{aligned}$$

This can, as with the fermionic calculations, be rewritten as a infinite dimensional matrix to

$$\mathcal{L}^{mass}_{4D,boson} = \left(\tilde{t}^{\dagger}_{Q,k}, \tilde{t}^{c}_{U,l}\right)\frac{1}{R^2}\begin{pmatrix} M^2 + 4\alpha^2R^2\mathcal{D}(I\cdot I^T) & -2\alpha RI(I^T\cdot M) \\ -2\alpha R(M\cdot I)I^T & M^2\end{pmatrix}\begin{pmatrix}\tilde{t}_{Q,k} \\ \tilde{t}^{c\dagger}_{U,l}\end{pmatrix} \tag{9.20}$$

where $M = \left(k+\frac{1}{2}\right)^2\delta_{k,l}$ is a diagonal matrix, and $I = (1,1,1,\ldots)^T$ similar in the fermionic case. Again, we would like to have the eigenvalues of this mass matrix. For bosonic particles, the eigenvalues are the square of the masses. Therefore, in an equivalent way as Equations $(9.12 - 9.14)$ we find

$$\left(n+\frac{1}{2}\right)^2 v_n + 4R^2\alpha^2\mathcal{D}\sum_{k=0}^{\infty}v_k - 2\alpha R\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)v^c_k = \lambda^2 v_n \quad : \quad n \geq 0 \tag{9.21}$$

$$-2\alpha R\left(n+\frac{1}{2}\right)\sum_{k=0}^{\infty}v_k + \left(k+\frac{1}{2}\right)^2 v^c_n = \lambda^2 v^c_n \quad ; \quad n \geq 0 \tag{9.22}$$

Reshifting the diagonal matrix-contributions to the right gives

$$4R^2\alpha^2\mathcal{D}\sum_{k=0}^{\infty}v_k - 2\alpha R\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)v^c_k = \left(\lambda^2 - \left(n+\frac{1}{2}\right)^2\right)v_n \quad : \quad n \geq 0 \tag{9.23}$$

$$-2\alpha R\left(n+\frac{1}{2}\right)\sum_{k=0}^{\infty}v_k = \left(\lambda^2 - \left(n+\frac{1}{2}\right)^2\right)v^c_n \quad ; \quad n \geq 0. \tag{9.24}$$

Assuming for the moment that $|\sum_{k=0}^{\infty}v_k| < \infty$, one can multiply both sides of Equation (9.24) with $(n+\frac{1}{2})$ to find

$$\begin{aligned}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)v^c_n &= -2\alpha R\left(\sum_{k=0}^{\infty}v_k\right)\left(\sum_{n=0}^{\infty}\frac{\left(n+\frac{1}{2}\right)^2}{\lambda^2 - \left(n+\frac{1}{2}\right)^2}\right) \\ &= -2\alpha R\left(\sum_{k=0}^{\infty}v_k\right)\left(-\sum_{n=0}^{\infty}1 + \sum_{n=0}^{\infty}\frac{\lambda^2}{\lambda^2 - \left(n+\frac{1}{2}\right)^2}\right) \\ &= -2\alpha R\left(\sum_{k=0}^{\infty}v_k\right)\left(-\mathcal{D} + \sum_{n=0}^{\infty}\frac{\lambda^2}{\lambda^2 - \left(n+\frac{1}{2}\right)^2}\right).\end{aligned}$$

Next, we will use another identity involving infinite sums

$$\sum_{k=0}^{\infty} \frac{\lambda^2}{\lambda^2 - \left(k+\frac{1}{2}\right)^2} = (2\lambda)^2 \sum_{k=0}^{\infty} \frac{1}{(2\lambda)^2 - (2k+1)^2} = (2\lambda)^2 \cdot \frac{-\pi\tan(\pi\lambda)}{8\lambda}. \tag{9.25}$$

Using this gives

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)v_n^c = -2\alpha R\left(\sum_{k=0}^{\infty} v_k\right)\left(-\mathcal{D} - \frac{\lambda\pi}{2}\tan(\pi\lambda)\right) = 2\alpha R\left(\sum_{k=0}^{\infty} v_k\right)\left(\mathcal{D} + \frac{\lambda\pi}{2}\tan(\pi\lambda)\right).$$

Injecting this into Equation (9.23) we get

$$\begin{aligned}\left(\lambda^2 - \left(n+\frac{1}{2}\right)^2\right)v_n &= 4\alpha^2R^2\mathcal{D}\sum_{k=0}^{\infty} v_k - 2\alpha R\cdot 2\alpha R\left(\sum_{k=0}^{\infty} v_k\right)\left(\mathcal{D} + \frac{\lambda\pi}{2}\tan(\pi\lambda)\right)\\ &= -2\alpha^2R^2\lambda\pi\left(\sum_{k=0}^{\infty} v_k\right)\tan(\lambda\pi).\end{aligned}$$

Seeing the right-handed side not being dependent on n, we can sum over each n, thus finding

$$\begin{aligned}\sum_{n=0}^{\infty} v_n &= \left(\sum_{k=0}^{\infty}\frac{1}{\lambda^2 - \left(k+\frac{1}{2}\right)^2}\right)\left(-2\alpha^2R^2\lambda\pi\left(\sum_{k=0}^{\infty} v_k\right)\tan(\lambda\pi)\right)\\ &= \frac{-\pi\tan(\pi\lambda)}{2\lambda}\cdot\left(-2\alpha^2R^2\lambda\pi\left(\sum_{k=0}^{\infty} v_k\right)\tan(\lambda\pi)\right)\\ &= \alpha^2R^2\pi^2\tan^2(\lambda\pi)\left(\sum_{k=0}^{\infty} v_k\right)\end{aligned}$$

Since $\sum_{k=0}^{\infty} v_k \neq 0$, we see that $\cot^2(\lambda\pi) = \alpha^2R^2\pi^2$. Shifting λ back to λR, the masses of the KK-modes of the bosonic particles with $q_B = \frac{1}{2}$ are given by

$$m_{B,k}^{\pm} = \frac{1}{\pi R}\arctan\left(\pi R\lambda_t|H|\right) \pm \frac{k+\frac{1}{2}}{R} \qquad k\in\mathbb{Z}. \tag{9.26}$$

Remark 9.2.2. We see that the bosonic masses look similar to the fermionic masses. However, one can clearly see that the SS-twist changes the mass with a factor of $\frac{q}{R}$. Therefore, we see that the SS-twist still breaks SUSY.

As mentioned before, one can do this for all four possibilities of the SS-twists and fermions/ bosons. The mass equations then become

$$m_{B,k} = \frac{1}{\pi R}\arctan(\pi R\lambda_t|H|) + \frac{k+q_B}{R} \tag{9.27}$$

$$m_{B,k} = \frac{1}{\pi R}\arctan(\pi R\lambda_t|H|) + \frac{k+q_F}{R} \tag{9.28}$$

which is a particulary elegant result. A proof of Equation (9.28) using the EOM instead of an infinite matrix can be found in Appendix B.

9.3 The Coleman-Weinberg potential

Now that we know the masses the Yukawa interactions in Equation (9.8) induce, we can address the interactions between particless. The Yukawa coupling also induces interactions between the particles, which gives a one-loop contribution to interactions between the Higgs H and $H_{\mathcal{H}}$. We will be deriving the interaction in the next two sections. This will take some time, but to give the end result, we will find that the potential is of the form

$$V(H, H_{\mathcal{H}}) = -m^2\left(|H|^2 - |H_{\mathcal{H}}|^2\right) + \frac{\lambda}{2}\left(|H|^2 - |H_{\mathcal{H}}|^2\right)^2 \tag{9.29}$$

for some parameters m^2 and λ. This potential has an internal $SU(2,2)$ symmetry, where the representation of $SU(2,2)$ is the fundamental representation. In other words, if we have the vector $(H, H_{\mathcal{H}})^T$, then $V(U \cdot (H, H_{\mathcal{H}})^T) = V((H, H_{\mathcal{H}})^T)$ for any $U \in SU(2,2)$. Note however that this space is finite dimensional, and $SU(2,2)$ is simple, hence by Lemma 3.2.1 this representation cannot be an irreducible unitary representation. This can also be argued by physical arguments, for the kinetic term does not have the $SU(2,2)$ symmetry. To see explicit that this is not an irreducible unitary represntation, we first note that the representation is not unitary with respect to the inner product $\langle x, y\rangle = \sum_{i=1}^4 \bar{x}_i y_i$. One might argue that since we have considered an infinite-dimensional set of particles, the argument of a finite dimensional representation does not hold. However, note that the representation only acts on $H_{(\mathcal{H})}$ per mode. Hence we can choose any mode and see that the $SU(2,2)$ action does not map this mode to any other mode. To be precise, consider the 0$^{\text{th}}$ mode $H_0(x)$ and $H_{\mathcal{H},0}(x)$. Then the action acts as

$$\pi(A)\begin{pmatrix} H_0 \\ H_{\mathcal{H},0}\end{pmatrix} = A\begin{pmatrix} H_0 \\ H_{\mathcal{H},0}\end{pmatrix} \tag{9.30}$$

We see no other modes occuring, and no interaction with the y coordinate has happened. This means that we have found a subrepresentation, hence this representation is neither unitary nor irreducible. Although unfortunate, this is no big problem. This means that $(H, H_{\mathcal{H}})^T$ does not describe an elementary particle, but a composition of particles.

To get the potential in Equation (9.29), we shall consider the Coleman-Weinberg potential. This potential can be interpreted as a correction to the tree potential by including one-loop quantum effects [31] and summing them all up to get an effective 1-loop potential. We will be interested in the Coleman-Weinberg potential regarding to both the Higgs field H, as well as the Hyperbolic Higgs $H_{\mathcal{H}}$. Since no interactions between the MSSM and the MSSM$_{\mathcal{H}}$ are considered, we focus on the MSSM for the moment.

Given previous section, we have the Lagrangian of the following form

$$\mathcal{L} = -\sum_{n=-\infty}^{\infty}\left(\sum_{k=1}^{2}|\partial_\mu\phi_k^n|^2 + m_{k,n}^2|\phi_k^n|^2 + \overline{\psi_k^n}\partial\!\!\!/\psi_k^n - \tilde{m}_{k,n}\overline{\psi_k^n}\psi_k^n\right) \tag{9.31}$$

where ϕ_k^1 is the bosonic top field with mass $m_{k,1}^2 := \left(m_t(H) + \frac{k+q_B}{R}\right)^2$, and ϕ_k^2 has mass $m_{k,2}^2 := \left(m_t(H) - \frac{k+q_F}{R}\right)^2$, and similarly for ψ_k^n. We will use this form to find the Coleman-Weinberg potential for our system. We closely follow [31, Chapter 16].

To describe the Coleman-Weinberg potential, we quickly remind ourselves of some Quantum Field Theory. The vacuum expectation value in the presence of a set of sources J_j is given by

$$Z[J] = \int \exp\left(iI[\phi] + \sum_j \int \phi_j J_j d^4x\right)\left[\prod_{j,x} d\phi_j(x)\right]$$

where we have a finite set of independent fields ϕ_j. If one defines $iW[J]$ as the sum of all connected one-particle irreducible Feynman graphs with a source J, it is known fact that $Z[J] = \exp(iW[J])$. We remind ourselves that a connected one-particle irreducible (1PI) is a graph that is connected and cannot be disconnected if one removes a line. We define the *effective action* $\Gamma(\phi)$ as the action such that

$$iW[J] = \int_{1PI} \exp\left(i\Gamma(\phi) + i\sum_j \int \phi_j(x)J_j(x)d^4x\right)\left[\prod_{j,x} d\phi_j(x)\right].$$

So one can interpret the effective action $\Gamma(\phi)$ as the action such that it takes all the quantum corrections into account [31].

It can be shown that around a fixed field ϕ_0 the effective action can be written as

$$\exp\left(i\Gamma(\phi_0)\right) = \int_{1PI} e^{i\int \mathcal{L}(\phi+\phi_0)d^4x}\left[\prod_{j,x} d\phi_j(x)\right].$$

Calculating the full effective action is equivalent to calculating all the 1PI diagrams by hand, which is hard to do in upon itself. However, we will calculate the first-order loop contribution for our specific Lagrangian in Equation (9.31), in the case $\phi_0(x)$ is a constant field ϕ_0. Since the Lagrangian involves a sum over all KK-modes, we can focus on only one ϕ_j^n, and the contribution becomes

$$\int \mathcal{L}[\phi_j^n + \left(\phi_j^n\right)_0]d^4x = -\,m_{k,n}^2|\left(\phi_j^n\right)_0|^2\left(\int d^4x\right) - m_{k,n}^2\int \left(\phi_j^n\right)_0^\dagger\phi_k^n + (\phi_k^n)^\dagger\left(\phi_j^n\right)_0 d^4x$$
$$-\int |\partial_\mu\phi_k^n|^2 + m_{k,n}^2|\phi_k^n|^2d^4x.$$

The linear term in ϕ does not contribute to the one-particle irreducibles, so the one-loop contributions are then given by

$$\left(\exp\left(i\Gamma(\phi_0)\right)\right)_{1-loop} = \int e^{-\frac{1}{2}i\int|\partial_\mu\phi_k^n|^2+m_{k,n}^2|\phi_k^n|^2d^4x}\prod_x d\phi_k^n(x).$$

This can however be evaluated easily, for it is a Gaussian integral. Therefore, the result is found to be

$$\left(\exp\left(i\Gamma(\phi_{n,0}^+)\right)\right)_{1-loop} = \left[\mathbf{det}\left(\frac{i}{\pi}\left(\partial_\mu\partial^\mu + m_{k,n}^2\right)\right)\right]^{-1/2}.$$

Evaluating this differential operator in the Fourier basis, we can actually diagonalize it to

$$\left[\partial_\mu\partial^\mu + (\xi(H))^2\right]_{p,q} = (p^2 + m_{k,n}^2)\delta^4(p-q). \tag{9.32}$$

Filling this in, we find an expression for the 1-loop approximation of the effective action

$$\begin{aligned}\Gamma_{1-loop}(\phi_{n,0}^+) &= -i\log\left[\det\left(\frac{i}{\pi}\left(\partial_\mu\partial^\mu + m_{k,n}^2\right)\right)^{-1/2}\right]\\
&= \frac{i}{2}\log\left[\det\left(\frac{i}{\pi}\left(\partial_\mu\partial^\mu + m_{k,n}^2\right)\right)\right]\\
&= \frac{i}{2}\mathbf{Tr}\left[\log\left(\frac{i}{\pi}\left(\partial_\mu\partial^\mu + m_{k,n}^2\right)\right)\right]\\
&= \frac{i\cdot 4N_c}{2}\int \log\left(\left[\frac{\partial^2}{\partial x^\mu\partial y_\mu} + m_{k,n}^2\right]_{p,p}\right)\frac{d^4p}{(2\pi)^4}\\
&= \frac{4N_c i\delta(0)}{2}\int \log\left(p^2 + m_{k,n}^2\right)\frac{d^4p}{2\pi}\end{aligned} \tag{9.33}$$

where we used the equality $e^{\mathrm{Tr}(A)} = \det(e^A)$. The $4N_c$ is the contribution of all the physical degrees of freedom of the (s)top particle to the trace, and we removed the $\log(i/\pi)$ due to it contributing only a constant factor which does not influence the physics of the system. Note $\delta(0) = \int e^0 d^4x = \mathrm{Vol}(\mathbb{R}^{1,3})$.

We define the Coleman-Weinberg potential as

$$V_{CW} := -\frac{\Gamma(\phi_0)}{\mathrm{Vol}(\mathbb{R}^{1,3})}.$$

Then the one-loop contribution of ϕ_k^n is given by, using Equation (9.33):

$$V_{CW,\phi_k^n} = -\frac{i \cdot 4N_c}{2} \int \log\left(p^2 + m_{k,n}^2\right) \frac{d^4p}{2\pi} \tag{9.34}$$

The fermionic case can be done in a similar manner. However, due to the anticommutation of the fermionic fields, we see that the Gaussian integral gets another minus sign. Hence we see that

$$V_{CW,\psi_k^n} = \frac{i \cdot 4N_c}{2} \int \log\left(p^2 + \tilde{m}_{n,k}^2\right) \frac{d^4p}{2\pi} \tag{9.35}$$

This can be done for all $n \in \mathbb{Z}$ and $k = 1, 2$, which results in the Coleman-Weinberg potential to be

$$V_{CW} = -\frac{i \cdot 4N_c}{2} \sum_{n=-\infty}^{\infty} \sum_{k=1}^{2} \int \log\left(\frac{p^2 + m_{n,k}^2}{p^2 + \tilde{m}_{n,k}^2}\right) \frac{d^4p}{(2\pi)^4}. \tag{9.36}$$

Filling in that $m_{k,n} = \frac{n+q_B}{R} \pm m_t(H)$ and $\tilde{m}_{k,n} = \frac{n+q_F}{R} \pm m_t(H)$, we get

$$V_{CW} = -\frac{i \cdot 4N_c}{2} \sum_{n=-\infty}^{\infty} \int \log\left(\frac{p^2 + \left(\frac{n+q_B+Rm_t(H)}{R}\right)^2}{p^2 + \left(\frac{n+q_F+Rm_t(H)}{R}\right)^2}\right) + \log\left(\frac{p^2 + \left(\frac{n+q_B-Rm_t(H)}{R}\right)^2}{p^2 + \left(\frac{n+q_F-Rm_t(H)}{R}\right)^2}\right) \frac{d^4p}{(2\pi)^4}. \tag{9.37}$$

9.4 Derivation $SU(2,2)$ potential

Next, we wish to evaluate the integral in Equation (9.37). To do this, we first note that the integral is over $\mathbb{R}^{1,3}$. To go over the Euclidean space $\mathbb{R}^4$, we do a Wick transformation. This translates into sending $-p_0^2 + |\vec{p}|^2 \mapsto p_0^2 + |\vec{p}|^2 =: |p|^2$. This gives

$$\begin{aligned} V_{CW} = \frac{4N_c}{2} \sum_{n=-\infty}^{\infty} \int \log &\left(\frac{R^2|p|^2 + (n + q_B + Rm_t(H))^2}{R^2|p|^2 + (n + q_F + Rm_t(H))^2}\right) \\ &+ \log\left(\frac{R^2|p|^2 + (n + q_B - Rm_t(H))^2}{R^2|p|^2 + (n + q_F - Rm_t(H))^2}\right) \frac{d^4p}{(2\pi)^4}. \end{aligned}$$

Note that $\log(\frac{A}{B}) = \log(A) - \log(B)$, and so we can concentrate on each of the four terms individually which look much alike. Define $\omega_{B,F}^{\pm} := q_{B,F} \pm Rm_t(H)$, and ignore for a moment the sub- and superscript for convenience. Interchanging the sum and the integral gives that we need to evaluate

$$I(\omega) := \frac{1}{2} \int \sum_{n=-\infty}^{\infty} \log\left(R^2|p|^2 + (n + \omega)^2\right) \frac{d^4p}{(2\pi)^4}. \tag{9.38}$$

To evaluate this integral, we closely follow [9]. Note that the integrand only depends on the radial coordinate. Switching to spherical coordinates gives a factor of $2\pi^2$ from the angular dependence and Equation (9.38) becomes

$$I(\omega) = \frac{1}{16\pi^2} \int \sum_{n=-\infty}^{\infty} \log\left(R^2|p|^2 + (n+\omega)^2\right) |p|^3 d|p|. \tag{9.39}$$

Obviously, since $|p|^2 R^2 \geq 1$, the sum itself is infinite. However, remember that we have both a bosonic as an fermionic contribution, hence the resulting integral might be rendered finite. Define $E := |p|$ for simplicity and

$$W(E) := \sum_{n=-\infty}^{\infty} \log\left(R^2 E^2 + (n+\omega)^2\right). \tag{9.40}$$

We assume that this function W is at least continuously differentiable to be able to write

$$W(E) = \int \frac{\partial W(E)}{\partial E} dE. \tag{9.41}$$

Then we can consider $\frac{\partial W(E)}{\partial E}$. If we differentiate $W(E)$ piecewise, we get

$$\frac{\partial W}{\partial E} = \sum_{n=-\infty}^{\infty} \frac{2R^2 E}{R^2E^2 + (n+\omega)^2} = 2R^2 E \sum_{n=-\infty}^{\infty} \frac{1}{R^2E^2 + (n+\omega)^2}.$$

Note that this sum *does* converge, for $|\frac{1}{R^2E^2+(n+\omega)^2}| \leq \frac{1}{n^2}$ for big enough $|n|$. To evaluate the infinite sum that describes $\frac{\partial W}{\partial E}$, we use the Residue Theorem. Define the function

$$f(z) := \frac{1}{(e^{2i\pi(z-\omega)} - 1)(R^2E^2 + z^2)}.$$

Note that there are poles at $z = n + \omega$ and $z = \pm iRE$ for all $n \in \mathbb{Z}$, which are all simple poles. If we take a circular contour integral of radius ϵ around the pole at $z = n + \omega$ and denote these contours by C_n, and sum all these contour integrals, we get one big rectangular contour integral that encloses the real line with an height of ϵ. Call this contour C_ϵ. Hence, by the Residue Theorem, we get

$$\sum_n \oint_{C_i} f(z)dz = \oint_{C_\epsilon} f(z)dz = 2\pi i \sum_n \mathrm{Res}_{n+\omega}(f(z)). \tag{9.42}$$

The residue at $z = n + \omega$ can easily be found, using L'Hôpital:

$$\mathrm{Res}_{n+\omega}(f(z)) = \lim_{z\to n+\omega} \frac{z-n-\omega}{(e^{2\pi i(z-\omega)} - 1)(R^2E^2 + z^2)} = \left[\frac{0}{0}\right] = \frac{1}{2\pi i(R^2E^2 + (n+\omega)^2)}.$$

Filling in Equation (9.42) we can conclude that

$$\sum_n \frac{1}{R^2E^2 + (n+\omega)^2} = \oint_{C_\epsilon} f(z)dz.$$

Explicitely calculating this contour integral is highly non-trivial. Therefore, we use a common trick. We note that a contour line going from $x + i\epsilon$ to $-x + i\epsilon$ for a given $x > 0$ can be closed by adding the contour that is a half-circle going from $-x+i\epsilon$ to $x+i\epsilon$ clockwise while keeping the radius at $|z|=x$. The same can be done with the line starting at $x-i\epsilon$ and ending at $-x-i\epsilon$. When we take $x\to\infty$, we see that these added contours contribute nothing to the given integral. For example, if we parametrize $z=re^{-i\theta}$ with $\theta\in(\pi,2\pi)$ for the upper plane, we see that for very large r we get

$$f(re^{i\theta})=\frac{1}{\left(e^{2\pi i(r\cos(\theta)-ir\sin(\theta)-\omega)}-1\right)\left(R^2E^2+r^2e^{-2i\theta}\right)}\propto\frac{1}{e^{-r}-1}\cdot\frac{1}{R^2E^2+r^2}\propto\frac{1}{r^2}\to 0.$$

Here we used the fact that $r\gg 1$ and $r\gg E$, as well as $\sin(\theta)<0$ for all $\theta\in(\pi,2\pi)$. So the added contour does not contribute when taking $r\to\infty$. The same argument can be done for the other half-circle. Adding these two contours to C_ϵ, it deformed into two disjoint contours that do not encircle the poles anymore that lie on the real line. Instead, they encircle the other two poles, namely $z=\pm iRE$. Therefore, using the Residue Theorem again, we get

$$\sum_n\frac{1}{R^2E^2+(n+\omega)^2}=-2\pi i\mathrm{Res}_{iRE}(f(z))-2\pi i\mathrm{Res}_{-iRE}(f(z))$$

where the minus signs are taken due to taking a clockwise contour instead of counterclockwise. The residue at iRE is found to be

$$\begin{aligned}\lim_{z\to iRE}\frac{z-iRE}{(e^{2\pi i(z-\omega)}-1)(R^2E^2+z^2)}&=\lim_{z\to iRE}\frac{z-iRE}{(e^{2\pi i(z-\omega\pi)}-1)(z-iRE)(z+iRE)}\\&=\lim_{z\to iRE}\frac{1}{(e^{2\pi i(z-\omega)}-1)(z+iRE)}=\frac{1}{2iRE(e^{-2\pi RE-2i\pi\omega}-1)}.\end{aligned}$$

The other one can be seen in the same way and is given by $\mathrm{Res}_{-iRE}(f(z))=\frac{-1}{2iRE(e^{2\pi RE-2i\pi\omega}-1)}$. Therefore, we finally get the formula for $\frac{\partial W}{\partial E}$:

$$\begin{aligned}\frac{\partial W}{\partial E}&=2R^2E\sum_{n=-\infty}^{\infty}\frac{1}{R^2E^2+(n+\omega)^2}=\frac{-4\pi iR^2E}{2iRE}\left(\frac{1}{1-e^{-2\pi RE-2i\pi\omega}}-\frac{1}{1-e^{2\pi RE-2i\pi\omega}}\right)\\&=-2\pi R\left(\frac{1}{1-e^{-2\pi RE-2i\pi\omega}}-\frac{1}{1-e^{2\pi RE-2i\pi\omega}}\right).\end{aligned}$$

Going back to Equation (9.41), we can finally integrate and find $W(E)$. We find thus

$$\begin{aligned}W(E)&=\int-2\pi R\left(\frac{1}{1-e^{-2\pi RE-2i\pi\omega}}-\frac{1}{1-e^{2\pi RE-2i\pi\omega}}\right)dE\\&=\int-2\pi R\left(\frac{1}{1-e^{-2\pi RE-2i\pi\omega}}-\frac{e^{-2\pi RE+2\pi i\omega}}{e^{-2\pi RE+2i\pi\omega}-1}\right)dE\\&=\int\left(\frac{1}{1-\frac{1}{r}e^u}-\frac{re^u}{re^u-1}\right)du\\&=\int\left(\frac{1}{1-\frac{1}{r}e^u}-1-\frac{1}{re^u-1}\right)du\\&=\int\frac{1}{(1-\frac{1}{r}e^u)\frac{1}{r}e^u}d\left(\frac{1}{r}e^u\right)-u-\int\frac{1}{(re^u-1)re^u}d(re^u)\\&=\int\frac{1}{1-\frac{1}{r}e^u}+\frac{1}{\frac{1}{r}e^u}d(\frac{1}{r}e^u)-2\pi RE+\int\frac{1}{1-re^u}-\frac{1}{re^u}d(re^u)\end{aligned}$$

$$= -\log\left|1-\frac{1}{r}e^{-2\pi RE}\right| + \log(\frac{1}{r}e^{-2\pi RE}) - 2\pi RE - \log\left|1-re^{-2\pi RE}\right| + \log(re^{-2\pi RE})$$
$$= -\log\left|1-\frac{1}{r}e^{-2\pi RE}\right| - \log(r) - 2\pi RE - 2\pi RE - \log\left|1-re^{-2\pi RE}\right| + \log(r) - 2\pi RE$$
$$= -\left[6\pi RE + \log|1-re^{-2\pi RE}| + \log\left|1-\frac{1}{r}e^{-2\pi RE}\right| \quad\right]$$

where $u = -2\pi RE$ and $r = e^{2\pi i\omega}$. We can put this into Equation (9.39), resulting into

$$I(\omega) = -\frac{1}{16\pi^2}\int_0^\infty E^3\left[6\pi RE + \log(1-re^{-2\pi RE}) + \log(1-\frac{1}{r}e^{-2\pi RE})\right]dE. \tag{9.43}$$

Before evaluating this integral, we see a E^5 divergence, which was expected since the original sum would diverge. However, as stated before, we are integrating over both the bosonic and fermionic particles which differ by a minus sign. Therefore $\int_0^\infty 6\pi RE^4\,dE$, which is present independently whether we have bosons or fermions, will cancel in the final integral. We can thus safely ignore the $\int_0^\infty 6\pi RE^4\,dE$ contribution, which additionally renders the integral finite.

Performing a partial integration gives

$$\begin{aligned} I(\omega) &= -\frac{1}{16\pi^2}\int_0^\infty \frac{E^4}{4}\left[\frac{2\pi Rre^{-2\pi RE}}{1-re^{-2\pi RE}} + \frac{2\pi R\frac{1}{r}e^{-2\pi RE}}{1-\frac{1}{r}e^{-2\pi RE}}\right]dE \\ &= -\frac{1}{16\pi^2}\int_0^\infty \frac{E^4}{4}\left[\frac{2\pi R}{\frac{1}{r}e^{2\pi RE}-1} + \frac{2\pi R}{re^{2\pi RE}-1}\right]dE \\ &= -\frac{3!}{16\pi^2}\left(\frac{1}{4!}\int_0^\infty \frac{2\pi RE^4}{\frac{1}{r}e^{2\pi RE}-1}dE + \frac{1}{4!}\int_0^\infty \frac{2\pi RE^4}{re^{2\pi RE}-1}dE\right). \end{aligned} \tag{9.44}$$

Next, we use an expression from the Bose-Einstein distribution that states

$$\mathrm{Li}_s(z) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t/z-1}dt \tag{9.45}$$

where $\Gamma(s)$ is the Euler Gamma function, and $\mathrm{Li}_s(z)$ is the polylogarithm given by

$$\mathrm{Li}_s(z) := \sum_{k=1}^\infty \frac{z}{k^s}.$$

Inserting this back into Equation (9.44) we get a closed form given by

$$I(\omega) = -\frac{3!}{16\pi^2}\left(\frac{1}{2^4\pi^4R^4}\mathrm{Li}_5(r) + \frac{1}{2^4\pi^4R^4}\mathrm{Li}_5\left(\frac{1}{r}\right)\right).$$

Remember that $r = e^{2\pi i\omega}$, hence the polylogarithmic function can be simplified to

$$\mathrm{Li}_5(r) + \mathrm{Li}_5\left(\frac{1}{r}\right) = \sum_{k=1}^\infty \frac{e^{2\pi ik\omega}+e^{-2\pi ik\omega}}{k^5} = \sum_{k=1}^\infty \frac{2\cos(2\pi k\omega)}{k^5} = 2\mathrm{Cl}_5(2\pi\omega)$$

where

$$\mathrm{Cl}_s(\theta) := \begin{cases} \sum_{k=1}^\infty \frac{\sin(k\theta)}{k^s} & \text{if } s\in\mathbb{N} \text{ is even} \\ \sum_{k=1}^\infty \frac{\cos(k\theta)}{k^s} & \text{if } s\in\mathbb{N} \text{ is odd.} \end{cases}$$

is the Clausen function. Concluding, we get that $I(\omega)$ takes the form of

$$I(\omega) = -\frac{3}{2^2 \cdot 32\pi^6 R^4}\mathrm{Cl}_5(2\pi\omega). \tag{9.46}$$

Finally, we can go back to the integral we were initially interested in, given in Equation (9.37). Filling the found expression with the correct values of ω in gives

$$V_{CW} = -\frac{3N_c}{32\pi^6 R^4}\left[\mathrm{Cl}_5(2\pi\omega_B^+) - \mathrm{Cl}_5(2\pi\omega_F^+) + \mathrm{Cl}_5(2\pi\omega_B^-) - \mathrm{Cl}_5(2\pi\omega_F^-)\right]. \tag{9.47}$$

We remember the parameters $(q_B, q_F)_{\text{MSSM}} = \left(\frac{1}{2}, 0\right)$ and $(q_B, q_F)_{\text{MSSM}_\mathcal{H}} = \left(0, \frac{1}{2}\right)$. Then Equation (9.47) gives

$$\begin{aligned}
V_{CW}^{\text{MSSM}} &= -\frac{3N_c}{32\pi^4 R^4}\left[\mathrm{Cl}_5(\pi + 2\pi R m_t(H)) - \mathrm{Cl}_5(2\pi R m_t(H)) + \mathrm{Cl}_5(\pi - 2\pi R m_t(H))\right.\\
&\qquad \left. - \mathrm{Cl}_5(-2\pi R m_t(H))\right]\\
V_{CW}^{\text{MSSM}_\mathcal{H}} &= -\frac{3N_c}{32\pi^4 R^4}\left[\mathrm{Cl}_5(2\pi R m_t(H_\mathcal{H})) - \mathrm{Cl}_5(\pi + 2\pi R m_t(H_\mathcal{H})) + \mathrm{Cl}_5(-2\pi R m_t(H_\mathcal{H}))\right.\\
&\qquad \left. - \mathrm{Cl}_5(\pi - 2\pi R m_t(H_\mathcal{H}))\right].
\end{aligned}$$

Given that $\cos(-x) = \cos(x)$ and $\cos(x + k\pi) = \cos(x - k\pi)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, we can reduce the previous expression to find

$$\begin{aligned}
V_{CW}^{\text{MSSM}} &= \frac{3N_c}{16\pi^6 R^4}\left[\mathrm{Cl}_5(2\pi R m_t(H)) - \mathrm{Cl}_5(\pi + 2\pi R m_t(H))\right]\\
V_{CW}^{\text{MSSM}_\mathcal{H}} &= -\frac{3N_c}{16\pi^6 R^4}\left[\mathrm{Cl}_5(2\pi R m_t(H_\mathcal{H})) - \mathrm{Cl}_5(\pi + 2\pi R m_t(H_\mathcal{H}))\right].
\end{aligned}$$

One already sees the minus sign occurrence if we interchange $H \leftrightarrow H_\mathcal{H}$. This is to be expected, for if we look at the Coleman-Weinberg potential in Equation (9.37) we see that switching the MSSM $\leftrightarrow$ MSSM$_\mathcal{H}$ is equivalent to swapping the SS-twists. This is equivalent to swapping the fermionic and bosonic masses, which again is equivalent to taking a minus sign in front of the $\log$ in Equation (9.37). To see this even more explicit we Taylor expand this potential. To do this, we firsts define $\mu := \frac{3N_c}{16\pi^6 R^4}$ for convenience. We define, in analogue to [7], the function

$$F_k(x) = \mathrm{Cl}_k(x) - \mathrm{Cl}_k(x + \pi).$$

We see then that

$$\begin{aligned}
V_{CW}^{\text{MSSM}} &= \mu F_5(2\pi R m_t(H))\\
&= \mu\left[F_5(0) + \frac{dF_5}{dx}(0)x + \frac{1}{2!}\frac{d^2F}{dx^2}(0)x^2 + \frac{1}{3!}\frac{d^3F}{dx^3}(0)x^3\right] + \mathcal{O}(x^4)\\
&= \mu\left[F_5(0) + 0 \cdot x - \frac{1}{2!}F_3(0)x^2 + \frac{1}{3!} \cdot 0 \cdot x^3\right] + \mathcal{O}(x^4)
\end{aligned}$$

where $x = 2\pi R m_t(H)$ and we used in the third equation that $\sin(0) = \sin(k\pi) = 0$ and the fact $\frac{d}{dx}\cos(kx) = k\sin(kx)$ and visa versa. $V_{CW}^{\text{MSSM}_\mathcal{H}}$ can be found analogously.

So the analysis can be reduced to $F_s(0)$ for certain values of s. First, we see

$$\begin{aligned}F_s(0) &= \sum_{k=1}^{\infty}\left(\frac{1}{k^s}-\frac{(-1)^k}{k^s}\right) = 2\sum_{k=1,k\text{ is odd}}^{\infty}\frac{1}{k^s}\\ &= 2\left[\sum_{k=1}^{\infty}\frac{1}{k^s}-\sum_{k=1}^{\infty}\frac{1}{(2k)^s}\right] = 2\left[\sum_{k=1}^{\infty}\frac{1}{k^s}-\frac{1}{2^s}\sum_{k=1}^{\infty}\frac{1}{(k)^s}\right] = 2\left(1-\frac{1}{2^s}\right)\zeta(s)\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function. Therefore, up to second order, we find

$$V_{CW}^{\text{MSSM}} = \mu\left[2\left(1-\frac{1}{2^5}\right)\zeta(5)-\left(1-\frac{1}{2^3}\right)\zeta(3)\cdot(2\pi R m_t(H))^2\right]+\mathcal{O}(m_t^4).$$

We know that $m_t(H) = \frac{1}{\pi R}\arctan(\pi R\lambda_t|H|) = \lambda_t|H| - (\pi R)^2\lambda_t^3|H|^3 + \mathcal{O}(|H|^5)$, so we can fill this in the found approximation. In the end, we find that the effective potential, including both the Higgs and the Hyperbolic Higgs, is given by

$$\begin{aligned}V(H,H_{\mathcal{H}}) =& V_{CW}^{\text{MSSM}} + V_{CW}^{\text{MSSM}_{\mathcal{H}}}\\ =&\mu\left(\frac{2\cdot 31}{32}\zeta(5)-\frac{7}{8}\zeta(3)\cdot 4\pi^2R^2\lambda_t^2|H|^2\right)\\ &-\mu\left(\frac{2\cdot 31}{32}\zeta(5)-\frac{7}{8}\zeta(3)\cdot 4\pi^2R^2\lambda_t^2|H_{\mathcal{H}}|^2\right)+\mathcal{O}(|H|^3)\\ =&-\frac{7\mu\zeta(3)4\pi^2R^2\lambda_t^2}{8}\left(|H|^2-|H_{\mathcal{H}}|^2\right)\\ =&-\frac{21N_c\zeta(3)\lambda_t^2}{32\pi^4R^2}\left(|H|^2-|H_{\mathcal{H}}|^2\right)\end{aligned}$$

which shows the first part in Equation (9.29).

9.5 Breaking the $SU(2,2)$ symmetry

Now that we have found the first part of Equation (9.29), we will continue to prove the second part. But to do that, we need some more modelbuilding, for we have not discussed the gauge fields and any interaction between the Hyperbolic sector and the visible sector. This will give the final contributions to the model, giving the $SU(2,2)$ symmetry. We choose the full hypermultiplet of $SU(2)_W \times U(1)_Y$ gauge sector to be in the $(+,+)$ eigenstate, which means both the gauge bosons as the gluinos. Here we refer back to the eigenstate given in Equation (9.3). For the $SU(3)_c$ we choose only the gauge bosons to be in the $(+,+)$ eigenstate, while the gluinos are in the $(-,+)$ eigenstate. This way we lift the gluinos to a mass that is not observable as of yet. In theory, the Hyperbolic gluinos might interact with the Higgs bosons which would add to the correction of the mass of the Higgs, but in practice these effects are subdominant [7].

Next, we introduce a new $U(1)_X$ gauge symmetry, that charges the Higgs fields H with charge +1, and $H_{\mathcal{H}}$ having charge -1. To fully describe the effects of this, we need the supersymmetry fields of gauge particles, which we have not considered. A full description is done in [3]. To construct the Lagrangian for gauge fields, we can apply similar arguments as the ones we used in constructing the superpotential, to conclude that we can only add certain parts of the gauge superfields to construct SUSY invariant Lagrangians. These parts will give rise to the familiar kinetic part of the gauge field Lagrangian, while it also gives a new term of the form

$$\mathcal{L}_{extra} = \frac{1}{2}\mathcal{D}\mathcal{D} - g_X^2\xi\sum_i \mathcal{S}_i^\dagger\mathcal{D}\mathcal{S}_i$$

where $\mathcal{D}$ is the $\mathcal{D}$-term of the gauge superfield, which is an auxilliary field, and $g_X^2\xi$ is the coupling strength usually associated with gauge interactions. Here ξ is the $\mathcal{D}$-term decoupling factor, measuring the breaking of SUSY in the vector boson hypermultiplet. For a full description of ξ, we refer to [7]. Since $U(1)$ is an abelian group, it turns out that we can add another term to the Lagrangian that is SUSY invariant, namely the Fayet-Iliopoilos term

$$\mathcal{L}_{FI} = -g_X^2 \xi f_X^2 \mathcal{D}.$$

Adding these two together, we get

$$\mathcal{L}_{U(1)_X} = \frac{1}{2}\mathcal{D}\mathcal{D} - g_X^2\xi H^\dagger \mathcal{D} H + g_X^2 \xi H_\mathcal{H}^\dagger \mathcal{D} H_\mathcal{H} - g_X^2 \xi f_X^2 \mathcal{D}$$

Again, $\mathcal{D}$ is not a dynamical field, so the Euler-Lagrange equations give

$$\mathcal{D} = g_X^2\xi |H|^2 - g_X^2 \xi |H_\mathcal{H}|^2 + g_X^2\xi f_X^2$$

Filling this in shows the $U(1)_X$ contribution is given by

$$\mathcal{L}_{U(1)_X} = \frac{g_X^2\xi}{2}\left(|H_\mathcal{H}|^2 - |H|^2 - f_X^2\right)^2 \tag{9.48}$$

Adding this Lagrangian to the one we already had, gives rise to the potential as stated in Equation (9.29).

However, $SU(2,2)$ is not a symmetry we have seen so far in nature. Hence the $SU(2,2)$ symmetry should be a broken one. All Lagrangian contributions we have discussed so far are $SU(2,2)$-invariant. To induce a $SU(2,2)$-breaking, we include the typical Higgs interaction term that is included in the Standard Model. These interactions are of quadratic and quartic order. These quartic interactions are typically from the MSSM $\mathcal{D}$-term, while the quadratic interactions are from brane-localized soft masses and $U(1)$-one loops corrections to these soft masses [7]. Also including our new $U(1)_X$-contributions into these one-loops and soft masses we get

$$V_{\cancel{U(2,2)}} = (\tilde{m}^2 + \tilde{m}_X^2)(|H|^2 + |H_\mathcal{H}|^2) + \frac{g_Z^2}{2}\left(|H|^4 + |H_\mathcal{H}|^4\right) \tag{9.49}$$

where $g_Z^2 = \frac{g^2+(g')^2}{4}$. Including this term, we can explicitely break the $SU(2,2)$-symmetry. This symmetry breaking will affect the Higgs particle's bare mass, in the same way that the Higgs particle in the SM gets a mass. This generation of mass of the Higgs will be the focus for the rest of this section.

All in all, we have the total potential given by

$$\begin{aligned} V =& V_{CW}^{\text{MSSM}} + +V_{CW}^{\text{MSSM}_\mathcal{H}} + V_{U(1)_X} + V_{\cancel{U(2,2)}} \\ =& \frac{3N_c}{16\pi^6 R^4}[F_5(2\pi R m_t(H)) - F_5(2\pi R m_t(H_\mathcal{H}))] + \frac{g_X^2\xi}{2}\left(|H_\mathcal{H}|^2 - |H|^2 - f_X^2\right)^2 \\ &+ (\tilde{m}^2 + \tilde{m}_X^2)(|H|^2 + |H_\mathcal{H}|^2) + \frac{g_Z^2}{2}\left(|H|^4 + |H_\mathcal{H}|^4\right) \end{aligned} \tag{9.50}$$

where we did not approximate $V_{CW}^{\text{MSSM}_{(\mathcal{H})}}$ yet. To say it breaks the $SU(2,2)$-symmetry, we want the potential to be in an extremal point for a non-zero VEV $(v, v_\mathcal{H})$. The extremal points are found when both $\frac{\partial V}{\partial H}$ and $\frac{\partial V}{\partial H_{\mathcal{H}}}$ are zero at these VEVs. These are found by

$$\begin{aligned}0 =& \left.\frac{\partial V}{\partial H}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}}\\ =& -\frac{3N_c\lambda_t\cos^2(\pi Rm_t(v))}{8\pi^5R^3}F_4(2\pi Rm_t(v)) - 2g_X^2\xi\left(vv_{\mathcal{H}}^2 - v^3 - vf_X^2\right) + 2v(\tilde{m}^2+\tilde{m}_X^2) + 2g_Z^2v^3\end{aligned} \tag{9.51}$$

$$\begin{aligned}0 =& \left.\frac{\partial V}{\partial H_{\mathcal{H}}}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}}\\ =& \frac{3N_c\lambda_t\cos^2(2\pi Rm_t(v_{\mathcal{H}}))}{8\pi^5R^3}F_4(2\pi Rm_t(v_{\mathcal{H}})) + 2g_X^2\xi(v_{\mathcal{H}}^3 - v_{\mathcal{H}}v^2 - v_{\mathcal{H}}f_X^2) + 2v_{\mathcal{H}}(\tilde{m}^2+\tilde{m}_X^2) + 2g_Z^2v_{\mathcal{H}}^3\end{aligned} \tag{9.52}$$

Multiplying the first equation with v_H and the second with v and then adding them together, gives the equation

$$\begin{aligned}0 =& \frac{3N_c\lambda_t}{8\pi^5R^3}\left[\cos^2(\pi Rm_t(v_{\mathcal{H}}))F_4(2\pi Rm_t(v_{\mathcal{H}}))v - \cos^2(\pi Rm_t(v))F_4(2\pi Rm_t(v))v_{\mathcal{H}}\right]\\ &+ 4vv_{\mathcal{H}}(\tilde{m}^2+\tilde{m}_X^2) + 2g_Z^2(vv_{\mathcal{H}}^3 + v^3v_{\mathcal{H}})\end{aligned}$$

For convenience, we define $T_{(\mathcal{H})} := \pi R\lambda_t v_{(\mathcal{H})} = \tan(\pi Rm_t(v_{(\mathcal{H})}))$. Using $1+\tan^2(x) = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$ we find

$$\tilde{m}^2+\tilde{m}_X^2 = -\frac{g_Z^2}{2\pi^2R^2\lambda_t^2}(T^2+T_{\mathcal{H}}^2) + \frac{3N_c\lambda_t^2}{32\pi^4R^2}\left(\frac{F_4(2\pi Rm_t(v))}{T(1+T^2)} - \frac{F_4(2\pi Rm_t(v_{\mathcal{H}}))}{T_{\mathcal{H}}(1+T_{\mathcal{H}}^2)}\right) \tag{9.53}$$

Instead of adding, substracting Equation (9.51) times $v_{\mathcal{H}}$ and (9.52) times v gives

$$f_X^2 = \frac{1}{g_X^2\xi}\left(\frac{2g_Z^2+g_X^2\xi}{2\pi^2R^2\lambda_t^2}\left[T_{\mathcal{H}}^2 - T^2\right] - \frac{3N_c\lambda_t^2}{32\pi^4R^2}\left[\frac{F_4(\pi Rm_t(v))}{T(1+T^2)} + \frac{F_4(\pi Rm_t(v_{\mathcal{H}}))}{T_{\mathcal{H}}(1+T_{\mathcal{H}}^2)}\right]\right) \tag{9.54}$$

These two equations can be used to find the mass of the Higgs particle. We remember that the mass matrix is found by expanding the potential around $H_{(\mathcal{H})} = v_{(\mathcal{H})} + h_{(\mathcal{H})}$:

$$\begin{aligned}V(H,H_{\mathcal{H}}) &= V(v+h, v_{\mathcal{H}}+h_{\mathcal{H}})\\ &= V(v,v_{\mathcal{H}}) + \left.\frac{\partial V}{\partial H}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}}h + \left.\frac{\partial V}{\partial H_{\mathcal{H}}}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}}h_{\mathcal{H}} + (h^\dagger, h_{\mathcal{H}}^\dagger)\mathcal{M}^2\begin{pmatrix}h\\ h_{\mathcal{H}}\end{pmatrix}\\ &= V(v,v_{\mathcal{H}}) + (h^\dagger, h_{\mathcal{H}}^\dagger)\mathcal{M}^2\begin{pmatrix}h\\ h_{\mathcal{H}}\end{pmatrix}\end{aligned}$$

where $\mathcal{M}^2$ is the 2x2 mass matrix of the Higgs bosons given by

$$\mathcal{M}^2 = \frac{1}{2}\begin{pmatrix}\left.\frac{\partial^2V}{\partial H^2}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} & \left.\frac{\partial^2V}{\partial H\partial H_{\mathcal{H}}}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}}\\ \left.\frac{\partial^2V}{\partial H_{\mathcal{H}}\partial H}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} & \left.\frac{\partial^2V}{\partial H_{\mathcal{H}}^2}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}}\end{pmatrix}.$$

We can calculate each of these entries explicitely, which gives

$$\begin{aligned}
\left.\frac{\partial^2 V}{\partial H^2}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} =&\frac{3N_c\lambda_t^2\cos^4(\pi R m_t(v))}{4\pi^4 R^2}(TF_4(2\pi R m_t(v)) - F_3(2\pi R m_t(v)))\\
&+2\xi g_X^2(3v^2 - v_{\mathcal{H}}^2 + f_X^2) + 2(\tilde{m}^2 + \tilde{m}_X^2) + 6g_Z^2 v^2\\
\left.\frac{\partial^2 V}{\partial H \partial H_{\mathcal{H}}}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} =&-2\xi g_X^2 v v_{\mathcal{H}}\\
\left.\frac{\partial^2 V}{\partial H_{\mathcal{H}}^2}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} =&-\frac{3N_c\lambda_t^2\cos^4(\pi R m_t(v_{\mathcal{H}}))}{4\pi^4 R^2}(T_{\mathcal{H}}F_4(2\pi R m_t(v_{\mathcal{H}})) - F_3(2\pi R m_t(v_{\mathcal{H}})))\\
&+2\xi g_X^2(3v_{\mathcal{H}}^2 - v^2 - f_X^2) + 2(\tilde{m}^2 + \tilde{m}_X^2) + 6g_Z^2 v_{\mathcal{H}}^2.
\end{aligned}$$

Filling Equation (9.53) and (9.54) in gives

$$\frac{1}{2}\left.\frac{\partial^2 V}{\partial H^2}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} =2(g_X^2\xi + g_Z^2)v - \frac{N_c\lambda_t^4 v^2 G(T)}{4\pi^2} \tag{9.55}$$

$$\frac{1}{2}\left.\frac{\partial^2 V}{\partial H \partial H_{\mathcal{H}}}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} =-2\xi g_X^2 v v_{\mathcal{H}} \tag{9.56}$$

$$\frac{1}{2}\left.\frac{\partial^2 V}{\partial H_{\mathcal{H}}^2}\right|_{H=v,H_{\mathcal{H}}=v_{\mathcal{H}}} =2(g_X^2\xi + g_Z^2)v_{\mathcal{H}} + \frac{N_c\lambda_t^4 v_{\mathcal{H}}^2 G(T_{\mathcal{H}})}{4\pi^2} \tag{9.57}$$

where we defined $G(x) = \frac{3}{2x^2(1+x^2)^2}\left[F_3(2\arctan x) - \frac{1+3x^2}{2x}F_4(2\arctan x)\right]$.

Finding the eigenvalues of this mass matrix give the masses m_h and $m_{h_{\mathcal{H}}}$ of h and $h_{\mathcal{H}}$ respectively. The characteristic equation of $\mathcal{M}^2$ is given by

$$(\lambda - \mathcal{M}_{hh}^2)(\lambda - \mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2) - 2\left(\mathcal{M}_{hh_{\mathcal{H}}}^2\right)^2 = 0$$

where we denoted λ for the eigenvalue of $\mathcal{M}^2$. Physically speaking, one can argue that the lighter mass is the SM-Higgs boson. Therefore concentrating on the Higgs boson's mass, we see

$$\begin{aligned}
m_h^2 &= \frac{1}{2}\left(\mathcal{M}_{hh}^2 + \mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2\right) - \frac{1}{2}\sqrt{\left(\mathcal{M}_{hh}^2 - \mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2\right)^2 + 4\left(\mathcal{M}_{hh_{\mathcal{H}}}^2\right)^2}\\
&= \frac{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}{2}\left(1 + \frac{\mathcal{M}_{hh}^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}\right) - \frac{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}{2}\sqrt{\left(1 - \frac{\mathcal{M}_{hh}^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}\right)^2 + 4\frac{\left(\mathcal{M}_{hh_{\mathcal{H}}}^2\right)^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}}.
\end{aligned}$$

Taking the limit $v \ll v_{\mathcal{H}}$, we get that $\frac{\mathcal{M}_{hh}^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}, \frac{\mathcal{M}_{hh_{\mathcal{H}}}^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2} \ll 1$, hence we can apply a Taylor series with respect to $\frac{\mathcal{M}_{hh_{\mathcal{H}}}^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}$. We define $\kappa = \frac{N_c\lambda_t^4}{8\pi^2 g_Z^2}$ and we get

$$\begin{aligned}
m_h^2 &= \mathcal{M}_{hh}^2 - \frac{1}{1 - \frac{\mathcal{M}_{hh}^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2}}\frac{\left(\mathcal{M}_{hh_{\mathcal{H}}}^2\right)^2}{\mathcal{M}_{h_{\mathcal{H}}h_{\mathcal{H}}}^2} - \mathcal{O}\left(\frac{v^4}{v_{\mathcal{H}}^2}\right)\\
&= 2g_Z^2 v^2\left(1 + \frac{g_X^2\xi}{g_Z^2}\right) - 2\kappa g_Z^2 v^2 G(T) - \frac{1}{1 - \frac{2(g_X^2\xi + g_Z^2)v^2 - 2\kappa g_Z^2 G(T) v^2}{2(g_X^2\xi + g_Z^2)v_{\mathcal{H}}^2 + 2\kappa g_Z^2 G(T_{\mathcal{H}})v_{\mathcal{H}}^2}} \cdot \frac{-4(g_X^2\xi)^2 v^2 v_{\mathcal{H}}^2}{2(g_X^2\xi + g_Z^2)v_{\mathcal{H}}^2 + 2\kappa g_Z^2 G(T_{\mathcal{H}})v_{\mathcal{H}}^2}
\end{aligned}$$

$$
\begin{aligned}
&= 2g_Z^2 v^2\left(1+\frac{g_X^2\xi}{g_Z^2}\right)+\frac{2g_X^2\xi v^2}{\left(1+\frac{g_Z^2}{g_X^2\xi}\right)\left(1-\frac{v^2}{v_{\mathcal{H}}^2}\right)+\kappa\frac{g_Z^2}{g_X^2\xi}\left(G(T_{\mathcal{H}})+\frac{v^2}{v_{\mathcal{H}}^2}G(T)\right)}-2\kappa g_Z^2 v^2 G(T)\\
&\approx 2g_Z^2 v^2\left(1+\frac{g_X^2\xi}{g_Z^2}\right)+\frac{2g_X^2\xi v^2}{\left(1+\frac{g_Z^2}{g_X^2\xi}\right)+\kappa\frac{g_Z^2}{g_X^2\xi}G(T_{\mathcal{H}})}-2\kappa g_Z^2 v^2 G(T)\\
&= 2g_Z^2 v^2\left[\left(1+\frac{g_X^2\xi}{g_Z^2}\right)+\frac{\frac{g_X^2\xi}{g_Z^2}}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}-\kappa G(T)\right]\\
&= 2g_Z^2 v^2\left[\frac{1+\frac{g_Z^2}{g_X^2\xi}(1+\frac{g_X^2\xi}{g_Z^2})(1+\kappa G(T_{\mathcal{H}}))}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}-\kappa G(T)\right]\\
&= 2g_Z^2 v^2\left[\frac{2+\kappa G(T_{\mathcal{H}})+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}-\kappa G(T)\right]\\
&= 2g_Z^2 v^2\left[2+\frac{\kappa G(T_{\mathcal{H}})-\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}-\kappa G(T)\right]
\end{aligned}
$$

where we neglected any order of $\frac{v^4}{v_H^2}$ or higher. Note that in the Standard Model $M_Z^2 = 2g_Z^2v^2$, which can be subsituted into the found equation. In addition, one can show that

$$
G(x)\approx\log(x^2)-c+x^2\left(\frac{23}{10}c+\frac{16}{3}-4\log(x^2)\right)+\mathcal{O}(x^4) \tag{9.58}
$$

where $c=\frac{8}{3}+7\zeta(3)\approx 11.08$. Assuming T is small, we find

$$
\begin{aligned}
m_h^2 &\approx M_Z^2\left[2+\frac{\kappa G(T_{\mathcal{H}})-\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}-\kappa\left(\log\left(\frac{v^2}{v_{\mathcal{H}}^2}\right)+\log(T_{\mathcal{H}}^2)-c\right)\right]\\
&= M_Z^2\left[2-\frac{\left(1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))\right)(\log(T_{\mathcal{H}}^2)-c)-\kappa G(T_{\mathcal{H}})+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}\right]-2M_Z^2\kappa\log\left(\frac{v}{v_{\mathcal{H}}}\right)\\
&= M_Z^2\left[2-\frac{\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))(1+\kappa\log(T_{\mathcal{H}}^2)-c)-\kappa\left(G(T_{\mathcal{H}})-\log(T_{\mathcal{H}}^2)+c\right)}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}\right]+2M_Z^2\kappa\log\left(\frac{v_{\mathcal{H}}}{v}\right)
\end{aligned}
$$

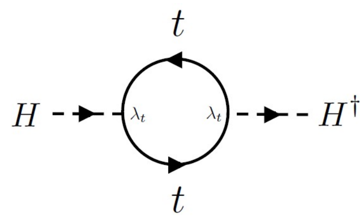
We conclude that

$$
m_h^2\approx M_Z^2(2-\Omega)+\frac{N_c\lambda_t^4}{2\pi^2}v^2\log\left(\frac{v_{\mathcal{H}}}{v}\right)+\mathcal{O}\left(\frac{v^4}{v_{\mathcal{H}}^2}\right) \tag{9.59}
$$

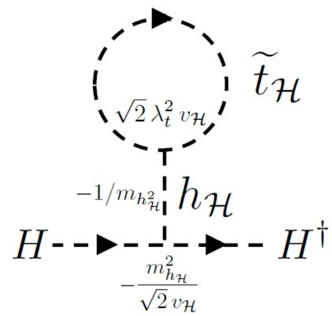
where

$$
\Omega:=\frac{\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))(1+\kappa\log(T_{\mathcal{H}}^2)-c)-\kappa\left(G(T_{\mathcal{H}})-\log(T_{\mathcal{H}}^2)+c\right)}{1+\frac{g_Z^2}{g_X^2\xi}(1+\kappa G(T_{\mathcal{H}}))}.
$$

As a conclusion, Equation (9.59) gives the mass of the lighter Higgs particle, which we can associate with the Higgs particle we can measure. Certain limits are of interest in this equation. When we consider



(a) A fermion loop.



(b) A bosonic loop.

Figure 2: Two one-loop diagrams that correspond to a correction to the Higgs mass. The left is the familiar one, and the right one is newly constructed [7]

the limit where the Hyperbolic interactions dominate, i.e. $g_X \gg g_Z$, we see that $\Omega \to 0$ up to a correction in $T^2_{\mathcal{H}}$, but this will be small if $Rv_{\mathcal{H}}$ is small. This limit means that $m_h^2 \approx 2M_Z^2$, which is around twice as big as the Higgs mass at no loops, i.e. when we do not take soft breaking terms into account in Equation (8.72). On the other hand, the low-energy limit where $g_Z^2 \gg g_X^2$, results in $\Omega \approx 1$. In this limit we get $m_h^2 \approx M_Z^2$ as is expected in the MSSM. Therefore, we can say Ω is the deviation of our model to the predicted value of m_h^2 in the MSSM, which could in theory be measured.

Finally, does this model solve the hierarchy problem? We have not discussed this as of yet, and we will briefly argue that it does solve the hierarchy problem. We remember that the potential we found is of the form

$$V = m^2(|H|^2 - |H_{\mathcal{H}}|^2) + \frac{\lambda}{2}(|H_{\mathcal{H}}|^2 - |H|^2 - f^2).$$

As we have done in the breaking of the $SU(2,2)$, we should not forget that there are supersymmetric additions to the Lagrangian. We will focus on two specific parts [7]

$$V_{\text{SUSY}} = \lambda_t^2(|H_{\mathcal{H}} \cdot \tilde{t}_{L\mathcal{H}}|^2 + |H_{\mathcal{H}}|^2|\tilde{t}_{R\mathcal{H}}|^2).$$

These terms come from the superpotential given in Equation (8.61). We can expand the Higgs around their VEVs to get the real Higgs particles $h_{(\mathcal{H})}$. With these interactions, we can construct the Feynman diagram as given in Figure 2. One can see the interaction terms on each of the crossings in Figure 2b.

Note that there is one propagator without a momentum associated to it. This is due to $H \to H^\dagger$ preserves momentum, hence generating a virtual Higgs particle. The one-loop term however has a running momentum, over which we need to integrate. If we compute this integral, we see that

$$\begin{aligned}
\Delta m_H^2 &= -\frac{m^2_{h_{\mathcal{H}}}}{\sqrt{2}v_{\mathcal{H}}} \cdot \frac{-1}{m^2_{h_{\mathcal{H}}}} \cdot -\sqrt{2}\lambda_t^2 v_{\mathcal{H}} \int \frac{1}{k^2 + m^2_{t_{\mathcal{H}}}} \frac{d^4k}{(2\pi)^4} \\
&= -\frac{\lambda_t}{(2\pi)^4} \int_{S^3} \int_0^{\Lambda_{UV}} \frac{k^3}{k^2 + m^2_{t_{\mathcal{H}}}} dk d\Omega_3 \\
&= -\frac{\lambda_t}{8\pi^2} \int_0^{\Lambda_{UV}} k + \frac{k m^2_{t_{\mathcal{H}}}}{k^2 + m^2_{t_{\mathcal{H}}}} dk = -\frac{\lambda_t}{8\pi^2}\Lambda^2_{UV} + \mathcal{O}(\log(\Lambda_{UV})).
\end{aligned} \tag{9.60}$$

This result is the same as the contribution we predicted in Equation (7.5), hence this diagram also cancels the quadratic divergence, hence the Hyperbolic Higgs also solves the hierarchy problem.

10 Conclusion

Representation theory is a broad subject. In the first part we have discussed a way to find irreducible unitary representations of a linear connected semisimple group G, by inducing from a parabolic subgroup of G and taking discrete series representations on the parabolic subgroup. We discussed a way to find these discrete series, and found a method to show that the induced representation is irreducible for imaginary ν. The R group was a way to characterize the irreducibility of these representations, and we ended with an explicit example.

In the second part we have looked at how the Hyperbolic Higgs model could solve the hierarchy problem. We started with a small review of supersymmetry, and built the model from there. Here we assumed 5 dimensions, where an additional action acted on the 5th dimension to be able to get rid of the extra particles from 5D supersymmetry. We also assumed a Scherk-Schwarz twist difference between the fermionic and bosonic particles to break SUSY explicitely. Adding a new set of MSSM particles with Scherk-Schwarz twist reversed, we were able to get an explicit $SU(2,2)$-symmetric potential, which allowed for a $SU(2,2)$ representation. However, this representation is not irreducible unitary, because the representation acts on a finite dimensional space.

Further research could be done in both directions. On the mathematical side, the problems were already hinted at in the thesis. For example, we worked only with purely imaginary ν. Then the R group characterized the irreducibility. If we choose $\mathrm{Re}\,\nu > 0$, less is known about the irreducibility as well as the unitarity of the induced representations. There is an ambitious project called the "Atlas of Lie Groups and Representations" (see [1]) whose goal was to find all the information of representations of reductive Lie groups, by explicit calculations and expecting repetition to give insight on the irreducible unitary representations. As far as the author is aware, this project has not been completed as of yet.

On the physical side, further investigations might be of interest. Although the model itself has a lot of prerequisites, it does not require the same amount of parameters SUSY requires to break SUSY. And if not for this model, the ideas are interesting enough to build upon further. Some of these ideas could be used to break SUSY in other models as well without the requirement of soft breaking SUSY, which could yield new insights.

A Representations of the Poincaré group

In this thesis, we have been looking at non-compact groups such as $SU(2,2)$. One of the most well-known non-compact groups is the Poincaré group. This group corresponds to the space-time symmetries, and the representation theory is essential in Quantum Field Theory. Hence we will touch upon finding the irreducible unitary representations of the Poincaré group as well. Our analysis is based on [30] and [26].

We start with some Quantum Mechanics. In Quantum Mechanics, the physical state of a system is represented by a one-dimensional subspace of a complex Hilbert space $\mathscr{H}$.

Definition A.0.1. If $v \in \mathscr{H}$, then a *ray* is defined by

$$[v] := \mathbb{C}v.$$

The set of rays $[v]$ with $v \in \mathscr{H} \setminus \{0\}$ is denoted as $\mathbb{P}(\mathscr{H})$.

With this definition any physical state is represented by a ray. We are interested in states that are normalized, so we represent this ray with $\Psi \in [v]$, where $\langle \Psi, \Psi \rangle = 1$. Note that Ψ is not uniquely defined, for $\xi\Psi$ also satisfies this condition for any $\xi \in \mathbb{C}$ such that $|\xi| = 1$.

Let us assume we have a set of mutually orthogonal eigenvectors of $\mathscr{H}$, say $\{\Psi_n\}_{n\in\mathbb{N}}$. Then to see if a physical state lies in any of the rays $[\Psi_n]$, we define the operation (v, w), with $v, w \in \mathscr{H}$, by

$$(v, w) = \frac{|\langle v, w\rangle|^2}{\|v\|^2\|w\|^2}.$$

Note that this operation can also be interpreted as an operation on the rays, for any complex factor gets divided out. So we can equivalently write $([v], [w])$.

The operation $([\Psi], [\Phi])$ can be interpreted as the probability of finding a physical state Ψ in the state Φ after doing an experiment. Note that if Ψ is a physical state that is normalised, then the operation $([\Psi], [\Psi_n]) = |\langle \Psi, \Psi_n\rangle|^2$, and thus $\sum_n([\Psi], [\Psi_n]) = 1$ if Ψ_n is a orthogonal basis of $\mathscr{H}$.

Note that the probability of finding a particular state should not depend on the point of view of the observer that is doing the experiment. This is the principal assumption of the theory of relativity. Let us consider two observers $\mathcal{O}$ and $\mathcal{O}'$. Observer $\mathcal{O}$ has a coordinate system in which he sees the physical system in $[\Psi]$, where $\mathcal{O}$ finds that the basis of $\mathscr{H}$ is $\{\Psi_n\}_n$. The second observer $\mathcal{O}'$ has a different coordinate system in which he measures the physical system in $[\Psi']$ and has the set $\{\Psi'_n\}_n$ as a basis for $\mathscr{H}$.

These two coordinate systems are related by a space-time rotation. To be precise, the coordinate system of $\mathcal{O}'$ is found by applying Lorentz transformations and translations on the coordinate system of $\mathcal{O}$. The group of Lorentz transformations and translations is the Poincaré group, which we will define now.

Definition A.0.2. Let H be a group, and suppose that there is an action of H on $\mathbb{R}^n$ by some matrix multiplication, i.e. there exist a map $\phi : H \to GL(\mathbb{R}^n)$ such that $\phi(gh) = \phi(g)\phi(h)$. Then we define the semidirect product of H by $\mathbb{R}^n$, denoted as $H \ltimes \mathbb{R}^n$, as the group that is $H \times \mathbb{R}^n$ as a set, and has a group multiplication given by

$$(g, x) \cdot (h, y) = (gh, x + \phi(g)y).$$

Definition A.0.3. The Poincaré group is the semidirect product of the Lorentz group and the translation group, i.e. $O(1,3) \ltimes \mathbb{R}^{1,3}$. The action of $O(1,3)$ on $\mathbb{R}^{1,3}$ is $\phi(h)x = hx$.

Since the coordinate system of $\mathcal{O}'$ can be found by applying a space-time rotation to the coordinate system of $\mathcal{O}$, there should be a map $U(\Lambda, a) : \mathbb{P}(\mathscr{H}_1) \to \mathbb{P}(\mathscr{H}_2)$ that depends on the Lorentz transformation Λ and the translation a, such that $[\Psi'] = U(\Lambda, a)[\Psi]$, and $[\Psi'_n] = U(\Lambda, a)[\Psi_n]$. Here $\mathscr{H}_1$ is the Hilbert space $\mathcal{O}$ sees, and $\mathscr{H}_2$ the Hilbert space $\mathcal{O}'$ sees. Note that $\mathscr{H}_1$ and $\mathscr{H}_2$ are the same space, but a different basis is chosen.

Definition A.0.4. Let G be a Lie group. A *projective representation* of G in $\mathscr{H}$ is a map $\rho : G \to \mathrm{Aut}(\mathbb{P}(\mathscr{H}))$ such that

$$\rho(gh) = \rho(g)\rho(h)$$

for any $g, h \in G$ and such that the action map $(g, v) \mapsto \rho(g)v, G \times \mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{H})$ is continuous.

This definition shows that $U(\Lambda, a)$ is a projective representation of the Poincaré group. The principal assumption of relativity states that the laws of physics should be the same in all inertial frames. This means

$$([\Psi], [\Psi_n]) = ([\Psi'], [\Psi'_n]).$$

In other words, the probability of having the system Ψ in the state Ψ_n should be the same as having the system Ψ' in the state Ψ'_n. If we write out the definition, we get

$$([\Psi], [\Psi_n]) = (U(\Lambda, a)[\Psi], U(\Lambda, a)[\Psi_n]). \tag{A.1}$$

The following theorem tells us that if we want to find any such $U(\Lambda, a)$, we need to consider unitary or anti-unitary operators on the Hilbert space itself.

Theorem A.0.5 (Wigner's Theorem, [26])**.** *Let $\mathscr{H}_1, \mathscr{H}_2$ be two complex Hilbert spaces of dimension at least two, with associated canonical projections $p_j : \mathscr{H}_j \to \mathbb{P}(\mathscr{H}_j), v \mapsto [v]$ for $j = 1, 2$. Let Ψ_j be a basisvector of $\mathscr{H}_j$, and $T : \mathbb{P}(\mathscr{H}_1) \to \mathbb{P}(\mathscr{H}_2)$ be continuous such that $(T[v], T[w]) = ([v], [w])$ for all $v, w \in \mathscr{H}_1$ and $T[\Psi_1] = [\Psi_2]$. Then there exists a unique additive $\tilde{T} : \mathscr{H}_1 \to \mathscr{H}_2$ such that*

1. $\tilde{T}\Psi_1 = \Psi_2$
2. $p_2 \circ \tilde{T} = T \circ p_1$

Also, $\tilde{T}$ is invertible and either unitary or anti-unitary.

For convenience, we will also write $U(\Lambda, a)$ for $\tilde{T}$ corresponding to $U(\Lambda, a)$. This shows that there is a (anti)-unitary representation of the Poincaré group on the Hilbert space $\mathscr{H}$. To describe these representations, we can reduce the discussion to considering irreducible unitary representations.

The purpose of this Appendix is to give some insight in how to find the irreducible unitary representations of the Poincaré group, and not the specifics. Hence we will only mention the theorems needed.

Proposition A.0.6. *$SL(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$ is the double cover of the Poincaré group.*

Theorem A.0.7 ([26] Corollary 14.4)**.** *Every irreducible unitary representation of $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$ in the Hilbert space $\mathscr{H}$ induces an irreducible projective representation of $SO(1, 3)_0 \ltimes \mathbb{R}^{1,3}$ on $\mathbb{P}(\mathscr{H})$. Here $SO(1, 3)_0$ denotes the connected component to the identity of the Lorentz group $SO(1, 3)$. This sets up a bijective correspondence between the irreducible projective representations of the connected Poincaré group and irreducible unitary representations of $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$.*

This theorem tells us that all irreducible projective representations of the Poincaré group are found by finding the irreducible unitary representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$. This also means that all irreducible unitary representations of the Poincaré group are found by considering the irreducible unitary representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$. Thus we need to describe the irreducible unitary representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$. Although this group is not linear connected reductive, induced representations are still useful. In this case we use Definition 4.1.1 for specific subgroup of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$.

It turns out that the irreducible unitary representations are found by inducing a tensor product of two representations. This is described in Theorem A.0.8. The first representation is given by

$$\xi_k : \mathbb{R}^{1,3} \to \mathbb{C}, \qquad \xi_k(x) = e^{ix^\mu k_\mu}$$

where $k \in \mathbb{R}^{1,3}$. Here we use the physical notation where $x^\mu k_\mu := \sum_{\mu,\nu=0}^{4} \eta_{\mu\nu} x^\mu k^\nu$ and $\eta = \text{diag}(-1,1,1,1)$ is the metric. One can can consider an action of $SO(1,3)_0$ on the character ξ_k by $h \cdot \xi_k(x) = \xi_k(hx)$. We define H_k to be the subgroup of $SO(1,3)_0$ such that $h \cdot \xi_k = \xi_k$. Since $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ is the double cover of $SO(1,3) \ltimes \mathbb{R}^{1,3}$, we can find a corresponding subgroup $\tilde{H}_k$ of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$. The group H_k is often called the *little group*. The second representation is found by considering any irreducible unitary representation $\rho \in \widehat{\tilde{H}_k}$. Then the irreducible unitary representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ are given in the following theorem.

Theorem A.0.8 ([26] Theorem 16.1)**.** *The irreducible unitary representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ are given by*

$$\pi_{k,\rho} = \text{Ind}_{\tilde{H}_k \ltimes \mathbb{R}^{1,3}}^{SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}} (\rho \otimes \xi_k).$$

Here ρ is an irreducible unitary representation of $\tilde{H}_k$ and $\xi_k : \mathbb{R}^{1,3} \to \mathbb{C}$ by $\xi_k(x) = e^{ix^\mu k_\mu}$.

Remark A.0.9. Note that we switched from ind to Ind with respect to Definition 4.1.1. This involves tensoring the representation $\rho \otimes \xi_k$ with a half density, just as in the main body of the thesis. Since the Poincaré group is not linear connected reductive, this half density will look different than the one we have been discussing in Chapter 4, hence we will only mention the tensoring. For more details we refer to [26].

Some people call $\tilde{H}_k$ the little group. This definition only matters in terms of a double cover, so does not really matter in the end. We will take H_k to be the little group. Theorem A.0.8 shows that we need a representation on $\tilde{H}_k$, and a vector k to construct all irreducible unitary representations of the Poincaré group. To get a more detailed mathematical insight, we recommend the theory of imprimitivity by Mackey [17], or [26].

To get a more physical way of discussing this construction, we first consider the Lie algebra of the Poincaré group. The Lie algebra is given by $\mathfrak{o}(1,3) \ltimes \mathbb{R}^{1,3}$ where

$$\mathfrak{o}(1,3) := \{X \in \text{Mat}(\mathbb{R}^{1,3}) | X + \eta X \eta = 0\}$$

and $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ is the metric. Calculating this gives

$$\mathfrak{o}(1,3) = \left\{ X \in \text{Mat}(\mathbb{R}^{1,3}) \middle| X = \begin{pmatrix} 0 & v^T \\ v & A \end{pmatrix} \text{ where } A \in \mathfrak{o}(3) \text{ and } v \in \mathbb{R}^3 \right\} \tag{A.2}$$

Here $\mathfrak{o}(3)$ is the Lie algebra of $O(3)$.

We see that the generators of $\mathfrak{o}(1,3) \ltimes \mathbb{R}^{1,3}$ are given by the elements $iM^{\mu\nu}$ and iP^μ, with $\mu,\nu = 0,1,2,3$. Here $iM^{\mu\nu}$ are the generators of $\mathfrak{o}(1,3)$ and P^μ are the generators of $\mathbb{R}^{1,3}$. If $\mu,\nu = 1,2,3$, the matrices $iM^{\mu\nu}$ are the generators of the Lie algebra $\mathfrak{o}(3)$, while the $\mu = 0$ and $\nu = 1,2,3$ we have the boost generators. Note that these $M^{\mu\nu}$ matrices are Hermitian by extraction of the i factor, i.e. $(M^{\mu\nu})^\dagger = M^{\mu\nu}$, and are antisymmetric in interchanging $\mu \leftrightarrow \nu$, i.e. $M^{\mu\nu} = -M^{\nu\mu}$.

In addition, $\mathbb{R}^{1,3}$ is an abelian algebra. So the generators of the algebra are the unit vectors, which will be denoted as P^μ for $\mu = 0,1,2,3$. Note these P^μ are Hermitian as well.

To get a physical argument on why we get that the representations are characterized by $\rho \otimes \xi_k$ as in Theorem A.0.8, we go back to the system of rays. Assume that we can let the Lie algebra act on $\mathscr{H}$ by differentiating $U(\Lambda, a)$, for example

$$P_\mu := \frac{d}{dt}\bigg|_{t=0} U(1, te_\mu) \tag{A.3}$$

where e_μ is the unit vector in $(\mathbb{R}^{1,3})^*$ in the μ-direction. Since all the P^μ commute, we can choose an orthogonal basis $\{\Psi_{p,\sigma}\}_{p,\sigma}$ such that each $\Psi_{p,\sigma}$ is an eigenstate of the operator P^μ, i.e.

$$P^\mu \Psi_{p,\sigma} = p^\mu \Psi_{p,\sigma}$$

where $p^\mu \in \mathbb{R}$. The σ label is an abbreviation to account for any other degrees of freedom. We note that

$$\begin{aligned} P^\mu U(\Lambda,0)\Psi_{p,\sigma} &= U(\Lambda,0)U(\Lambda,0)^{-1}P^\mu U(\Lambda,0)\Psi_{p,\sigma} \\ &= U(\Lambda,0)((\Lambda P)^\mu)\Psi_{p,\sigma} = (\Lambda p)^\mu U(\Lambda,0)\Psi_{p,\sigma} \end{aligned}$$

where we used that $U(\Lambda,0)^{-1}P^\mu U(\Lambda,0) = \phi(\Lambda)P^\mu = (\Lambda P)^\mu$ and ϕ is the action as in Definition A.0.3. This can be seen by applying Equation (A.3) and using the group multiplication. So $U(\Lambda,0)\Psi_{p,\sigma}$ is a eigenstate with eigenvector Λp, hence it must be a linear combination of $\Psi_{\Lambda p,\sigma'}$. So

$$U(\Lambda,0)\Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)\Psi_{\Lambda p,\sigma'}$$

for some coefficients $C_{\sigma'\sigma}(\Lambda,p) \in \mathbb{C}$.

Note that p^2 and the sign of p^0 are left invariant under any $\Lambda \in SO(1,3)_0$, hence we get that $\Psi_{p,\sigma}$ can only be mapped into $\Psi_{p',\sigma'}$ if $(p')^2 = p^2$ and $\text{sign}((p')^0) = \text{sign}(p^0)$.

We can choose a fixed four-momentum vector k^μ such that $k^2 = p^2$ and $\text{sign}(k^0) = \text{sign}(p^0)$. Then there exists a Lorentz transformation $L(p)$ such that $p^\mu = (L(p)k)^\mu$. Doing this transformation allows us to define the relation $\Psi_{p,\sigma} := U(L(p),0)\Psi_{k,\sigma}$ up to a normalization factor. This shows that $\{\Psi_{k,\sigma}\}_{k,\sigma}$ is a basis of $\mathscr{H}$. In addition, we get

$$\begin{aligned} U(\Lambda,0)\Psi_{p,\sigma} &= U(\Lambda,0)\,U(L(p),0)\Psi_{k,\sigma} \\ &= U(\Lambda L(p),0)\Psi_{p,\sigma} = U(L(\Lambda p),0)\,U(L(\Lambda p)^{-1}\Lambda L(p),0)\Psi_{k,\sigma}. \end{aligned}$$

Since $[L(\Lambda p)^{-1}\Lambda L(p)]\,k = L(\Lambda p)^{-1}\Lambda p = k$ by definition, we see $U(L(\Lambda p)^{-1}\Lambda L(p),0)\Psi_{k,\sigma}$ must be a linear combination of $\Psi_{k,\sigma'}$, so

$$U(L(\Lambda p)^{-1}\Lambda L(p),0)\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma\sigma'}(L(\Lambda p)^{-1}\Lambda L(p))\Psi_{k,\sigma'}$$

where $D_{\sigma\sigma'}(L(\Lambda p)^{-1}\Lambda L(p)) \in \mathbb{C}$ are some coefficients. This means that

$$U(\Lambda,0)\Psi_{p,\sigma} = \sum_{\sigma'} D_{\sigma\sigma'}(L(\Lambda p)^{-1}\Lambda L(p))U(L(\Lambda p),0)\Psi_{k,\sigma'} = \sum_{\sigma'} D_{\sigma\sigma'}(L(\Lambda p)^{-1}\Lambda L(p))\Psi_{\Lambda p,\sigma'}.$$

Hence we reduced the dependence of $C_{\sigma\sigma'}$ to be depending on matrices $L(\Lambda p)^{-1}\Lambda L(p)$ that leave k^μ invariant. In other words, we can reduce the analysis of $U(\Lambda, 0)$ to just looking at those subgroups $H \subset SO(1,3)_0$ that leave k^μ invariant. This H is again the little group we discussed H_k, and gives the irreducible subrepresentations.

So which way we look at it, physical or mathematical, the conclusion is the same that we need representations of the little group H_k. These could be infinite dimensional. There are three choices of k^μ that are of physical interest. Choose $k^\mu = (m, 0, 0, 0)$ with $m > 0$, then $k^2 = -m^2$ and $k^0 > 0$. This corresponds to a real particle with mass m. The group leaving this vector invariant is $SO(3)$, and its double cover is given by $SU(2)$, which is compact hence has finite-dimensional representations. The second option for k^μ is given by $k^\mu = (\kappa, 0, 0, \kappa)$ with $\kappa \in \mathbb{R}_{>0}$. Then $k^2 = 0$ and $k^0 > 0$, and has as little group $SO(2) \ltimes \mathbb{R}^2$. This corresponds to the massless particles. The third choice is $k^\mu = (0, m, 0, 0)$. This choice corresponds to $k^2 = m^2$ and hence tackles tachyons. The double cover of the little group now is $\tilde{H}_k = SL(2, \mathbb{R})$. Note $SL(2, \mathbb{R})$ is non-compact, and the irreducible unitary representations of $SL(2, \mathbb{R})$ can be described by Langlands classification. Note taking $k^\mu = 0$ just gives $SO(1,3)_0$ as little group and $SL(2, \mathbb{C})$ as double cover, which is also non-compact.

B Deriving mass eigenvalue using the EOM

Another way of proving the mass of the particles, given by Equations (9.27) and (9.28), is to look at the equations of motion. We focus on the fermionic fields. If we denote ψ for t or $t_{\mathcal{H}}$, the Lagrangian of Equation (9.9) is

$$\mathcal{L} = i\psi_Q^\dagger \bar{\sigma}^\mu \partial_\mu \psi_Q + i\psi_Q^c \sigma^\mu \partial_\mu \psi_Q^{c\dagger} - \psi_Q^c \partial_y \psi_Q + \psi_Q^\dagger \partial_y \psi_Q^{c\dagger} + (Q \leftrightarrow U) + 2\pi R\delta(y)(\psi_Q^\dagger \psi_U + \psi_U^\dagger \psi_Q)|H| \tag{B.1}$$

where $|H|$ is the VEV of the Higgs. We use a method inspired by [2] by considering the equations of motion:

$$i\bar{\sigma}^\mu \partial_\mu \psi_Q + \partial_y \psi_Q^{c\dagger} + 2\pi R\delta(y)\lambda_t \psi_U |H| = 0 \tag{B.2}$$

$$i\bar{\sigma}^\mu \partial_\mu \psi_U + \partial_y \psi_U^{c\dagger} + 2\pi R\delta(y)\lambda_t \psi_Q |H| = 0 \tag{B.3}$$

$$i\sigma^\mu \partial_\mu \psi_X^{c\dagger} - \partial_y \psi_X = 0 \tag{B.4}$$

where $X = Q$ or U. First, let $y \in (0, \pi R)$, i.e. let y be around where $\delta(y) = 0$. Then the differential equations become

$$\begin{aligned} i\bar{\sigma}^\mu \partial_\mu \psi_X + \partial_y \psi_X^{c\dagger} &= 0 \\ i\sigma^\mu \partial_\mu \psi_X^{c\dagger} - \partial_y \psi_X &= 0. \end{aligned}$$

Using that $i\bar{\sigma}^\mu \partial_\mu \psi_X = -m\psi_X$ and $i\sigma^\mu \partial_\mu \psi_X^{c\dagger} = -m\psi_X^{c\dagger}$ where m is the mass of the fermion, the differential equations become

$$\begin{aligned} -m\psi_X + \partial_y \psi_X^{c\dagger} &= 0 \\ -m\psi_X^{c\dagger} - \partial_X \psi_Q &= 0. \end{aligned}$$

Filling these equations into each other gives $\partial_y^2 \psi_X^{(c)} = -m^2 \psi_X^{(c)}$, hence the y dependence of each fermion field is given by $C_1 \cos(my) + C_2 \sin(my)$. We can use separation of variables, i.e. $\psi_X^{(c\dagger)}(x, y) =$

$\eta_X^{(c\dagger)}(x)g_X^{(c\dagger)}(y)$ where $g_X^{(c\dagger)} : S^1 \mapsto \mathbb{C}$. Remember by Equation (9.7) each field is either even or odd, hence we conclude

$$\psi_X(x,y) = C_1 \cos(my)\eta_X(x) \tag{B.5}$$

$$\psi_X^{c\dagger}(x,y) = C_2 \sin(my)\eta_X^{c\dagger}(x). \tag{B.6}$$

If $m \neq 0$, normalizing gives $C_1 = C_2 =: C$. If we fill in the solutions into the differential equation, we get

$$0 = -m\psi_X + \partial_y \psi_X^{c\dagger} = [-\eta_X(x) + \eta_X^{c\dagger}(x)]m\cos(my)$$

which is true for all $y \in (0, \pi R)$. This means $\eta_X(x) = \eta_X^{c\dagger}(x)$ for all $x \in \mathbb{R}^{1,3}$. The effect of $\delta(y)$ gives a condition on the values of m. To see this, we can integrate Equation (B.2) and (B.3)from $0-\epsilon$ to $0+\epsilon$ and let $\epsilon \to 0$. This results in

$$\psi_Q^{c\dagger}(0^+) - \psi_Q^{c\dagger}(0^-) + 2\pi R\lambda_t \psi_U^{\dagger}(0)|H| = 0$$
$$\psi_U^{c\dagger}(0^+) - \psi_U^{c\dagger}(0^-) + 2\pi R\lambda_t \psi_Q^{\dagger}(0)|H| = 0.$$

Since $\psi_X^{(c\dagger)}$ is at most discontinuous at a finite amount of points, which are multiples of πR, we get $\int_{0-\epsilon}^{0+\epsilon} \psi = 0$. Using Equation (B.6) we see that $\psi_X^{c\dagger}(0^+) = -\psi_X^{c\dagger}(0^-)$. Therefore, if we define $K := \frac{2\pi R\lambda_t \cos(0^-)}{2\sin(0^-)}$ then

$$\eta_Q^{c\dagger}(x) = K\eta_U(x), \qquad \eta_U^{c\dagger}(x) = K\eta_Q(x).$$

Note $\eta_Q = \eta_Q^{c\dagger} = K\eta_U = K\eta_U^{c\dagger} = K^2\eta_Q$, thus we see that $K^2 = 1$, i.e.

$$\tan(m0^-) = \pm\pi R\lambda|H|.$$

Given that all eigenfunctions are fully described on $(0, \pi R)$, it is enough to state $0^- = \pi R$. Then the mass of the fermions are given by

$$m_{+,k}(H) = \frac{1}{\pi R}\arctan\left(\pi R\lambda|H|\right) + \frac{k}{R} \qquad k \in \mathbb{Z} \tag{B.7}$$

if we choose the $+$ sign, while choosing the - sign gives

$$m_{-,k}(H) = \frac{1}{\pi R}\arctan\left(\pi R\lambda|H|\right) + \frac{k+\frac{1}{2}}{R} \qquad k \in \mathbb{Z}. \tag{B.8}$$

which both agree with Equation (9.28) with having either $q_F = 0$ or $q_F = \frac{1}{2}$ respectively.

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