FDE on Group Knowledge

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Abstract

This thesis is an exploration into how useful it is to use FDE to describe group knowledge. AI agents sometimes have to deal with inconsistent information, which results in inconsistent knowledge. Since it is difficult to use classical logic to describe inconsistent (group) knowledge, this thesis explores to what extent it is possible and useful to define group knowledge based on FDE. We did this by defining group knowledge in FDE and comparing this to the definitions in classical logic and looking the consequences. Results show that defining group knowledge based on FDE solves a few problems, for example, that inconsistent beliefs imply that you believe everything. On the other hand has this way of describing group knowledge some flaws, like that modus ponens is not valid in FDE.

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Introduction

Most of what we do has to do with the things we know or believe. Looking at behaviour of artificial agents, it is important to be interested in the 'knowledge' and 'belief' of an agent. Description and representation of knowledge are subjects that often occur in artificial intelligence. (Meyer and Hoek, 1995). Epistemic logic is used to describe and represent knowledge and beliefs of agents. (Rendsvig and Symons, 2021)

AI agents sometimes have to deal with inconsistent information. For example, this can occur if people feed the agent different information. In this case, it makes less sense to use (classical) epistemic logic to describe and represent knowledge of an agent because in classical is no place for inconsistency. For example, If Christian Huygens tells the agent the following statement: 'light behaves like a wave'. Let us call this proposition p. Suppose that Isaac Newton tells the agent that light does not behave like a wave ($\neg p$). For the agent, it is not rational to believe both p and $\neg p$. In classical logic $p \wedge \neg p$ is always false, which means that $p \wedge \neg p \vdash q$ is always true for any proposition q. So if the agent believes p and $\neg p$, the agent believes everything. This seems untrue. There are logics that can deal with inconsistency, such that $p \wedge \neg p \nvdash q$. These are called paraconsistent logics.

Some computer scientists argue that non-monotonic logic can also be applied to handle inconsistent information. This is a logic that can 'adapt' to new information. This has some benefits that classical logic and many paraconsistent logics do not have. Despite that this is an interesting logic to apply to inconsistent information, the focus in this thesis will be on a paraconsistent logic.

A popular paraconsistent logic is First Degree Entailment logic. In 1977 Belnap and Dunn presented FDE and argued that FDE should be used in some circumstances, for example how a computer should think (Omori and Wansing, 2017). Fifty years later there is still enthusiasm about FDE. Beall even argued that FDE is the one true logic. (Beall, 2019)

FDE is a four-valued logic. In classical logic, there are two values. Namely, True and False. In FDE there is True, False, Both, and None. Where one can interpret True as '(only) told True', False as '(only) told False', Both as 'told both True and False' and None as 'neither told True nor False'.(Belnap and Dunn, 1992) If we apply this in the above example, we get that the proposition 'light behaves like a waves' can receive the value Both.

To reason about inconsistent knowledge and belief, it is maybe a good idea to use FDE logic. It is interesting to look if FDE logic can be used to reason about knowledge of agents and how this affects group knowledge. Which brings us to the following question: To what extent is it possible and useful to define group knowledge based on FDE? In this thesis, I will answer this question.

By looking at the interesting properties of group knowledge based on FDE, we have a new application for FDE and we have a way to reason about knowledge and beliefs of agents who have to deal with inconsistent information. This makes this research relevant for AI.

In this thesis I will first give some background in epistemic logic and group knowledge, then I will talk about FDE and how I will define FDE logic on possible worlds. Lastly, I will apply FDE to Group Knowledge and look at the consequences.

Background Group Knowledge

In AI we are interested in formally describing knowledge of agents. Logic is used to represent knowledge and formalize reasoning methods.(Meyer & Hoek, 1995) We start with describing propositional logic.

Propositional Logic

In propositional logic, there are two truth values: True and False. A proposition p can have the value True (t) or False (f). The function v assigns a value to a proposition, for example $v(p) = t$. Propositional logic has the connectives $\neg$, $\wedge$ and $\vee$. With these connectives, we can make sentences. We denote propositions with the small letters p, q, r and sentences with the Greek letters φ, ψ, ξ. So for example $\varphi = \neg p \vee q$. To determine the value of a sentence we use the following truth tables:

$v_\neg$	
t	f
f	t

$v_\wedge$	t	f
t	t	f
f	f	f

$v_\vee$	t	f
t	t	t
f	t	f

So if $v(p) = t$ and $v(q) = f$, then $v(\varphi) = v(\neg p \vee q) = v_\vee(v_\neg(v(p)), v(q)) = v_\vee(v_\neg(t), f) = v_\vee(f, f) = f$. We can define the implication connectives $\rightarrow$ and $\leftrightarrow$ in term of the connectives $\neg, \vee, \wedge$. Where $\varphi \rightarrow \psi$ is equal to $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ is equal to $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, which is equal to $(\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$.

We say that a sentence φ is true under interpretation v iff the sentence has the value True, so $v(\varphi) = t$. We say that a sentence φ is valid ($\models \varphi$) iff φ is true in every interpretation, so for every valuation of the propositions in φ it holds that $v(\varphi) = t$. We call this a logical truth.

An argument is valid iff it is impossible for the conclusion to be false, while the premises are true. For example $p, p \rightarrow q \models q$ is a valid argument, because if p and $p \rightarrow q$ is true, then q is always true. So it can not be the case that the premises are true and the conclusion is not. (Gensler, 2012)

Modal Logic

We extend propositional logic to modal logic, where we add two operators $\Box$ and $\Diamond$. We use $\Box\varphi$ to say that φ is necessarily true and $\Diamond\varphi$ to say that φ is possible. These operators are often defined on possible worlds. A possible world is a complete description of how the world is or could have been. Now we interpret $\Box\varphi$ as φ being true in every world that we think is possible and $\Diamond\varphi$ as φ being true in one of the worlds that we think is possible.

A Kripke model is a tuple $\mathcal{M} = \langle W, R, V \rangle$. Where W is the set of worlds in the model, R is the accessibility relation and V the valuation function. The valuation function depends on a proposition p and gives the set of worlds where p is true.(Blackburn et al., 2006)

We define the concept of a proposition p being true at a world w in a model $\mathcal{M}$ as follows:

$\mathcal{M}, w \models p$ iff $w \in V(p)$

To extend this to sentences we define a sentence to be true at a world w in the model $\mathcal{M}$ as follows:

$\mathcal{M}, w \models \top$ always,
$\mathcal{M}, w \models \bot$ never,
$\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, x \models \varphi$ for all x such that wRx
$\mathcal{M}, w \models \Diamond\varphi$ iff there exists a world x such that wRx and $M, x \models \varphi$

We say that a sentence φ is valid iff $M, w \models \varphi$ in every world in every model.

Epistemic Logic

The language of epistemic logic is similar to the language of modal logic. Epistemic logic has no $\Box$ and $\Diamond$, but it has the knowledge operator K_a. If $K_a\varphi$, we say that agent a knows φ. A is a set of agent names and $a \in A$. We define $K_a\varphi$ to be true at a world w as follows:

$\mathcal{M}, w \models K_a\varphi$ iff $\mathcal{M}, x \models \varphi$ for all x such that wR_ax, where R_a is the accessibility relation of agent a.

So the operator K_a works very similar as $\Box$. We define the operator $\hat{K}_a$ as $\neg K_a \neg$. We define $\hat{K}_a$ to be true in a world w in a model $\mathcal{M}$ as follows:

$\mathcal{M}, w \models \hat{K}_a\varphi$ iff there exists a world x such that wR_ax and $\mathcal{M}, x \models \varphi$

If $\hat{K}_a\varphi$ $(\neg K_a \neg\varphi)$, the agent a does not know that φ is false. In other words, agents thinks φ could be true.

We say that $K_a\varphi$ is valid iff $\mathcal{M}, w \models K_a\varphi$ for all worlds in all models.

Group Knowledge

We add a group knowledge operator E_B, to describe the knowledge of a group. We say that $E_B\varphi$ iff every agent b in group B knows that φ, in other words:

$$E_B\varphi = \bigwedge_{b \in B} K_b\varphi,$$

where $B \subseteq A$.

Similar to $\hat{K}$, we define $\widehat{E_B}$ as $\neg E_B \neg$, so:

$$\widehat{E_B}\varphi = \neg E_B \neg \varphi = \neg \bigwedge_{b \in B} K_b \neg \varphi = \bigvee_{b \in B} \hat{K}_b \varphi$$

In other words, this means that if $\widehat{E_B}\varphi$, then there is an agent b in group B, that knows φ.

An other interesting kind of knowledge of a group is common knowledge, which is defined as follows:

$$C_B\varphi = E_B\varphi \wedge E_B E_B \varphi \wedge E_B E_B E_B \varphi \wedge \cdots = \bigwedge_{n=0}^{\infty} E_B^n \varphi$$

.

So φ is common knowledge if φ is true and everybody knows φ is true and everybody knows that everybody knows φ is true and so on to infinity. Sometimes common knowledge is described as 'what every fool knows'.

Distributed knowledge is the implicit knowledge of a group, that would be explicit if the group would perfectly work together and share all their knowledge. Distributed knowledge operator is denoted as D_B. Suppose that $B = \{a, b\}$ and $K_a\varphi \wedge K_b(\varphi \rightarrow \psi)$, then $D_B\psi$. (Van Ditmarsch et al., 2007)

To apply group knowledge on possible worlds, we need to define some accessibility relations for E_B, D_B, and C_B. We can do this in terms of the accessibility relation of one agent, above mentioned as R_a:

Note that the accessibility relation of one agent b (R_b) is a set $\{(x, y)|xR_by\}$.
Let $R_{E_B} = \bigcup_{b \in B} R_b$,
Let $R_{D_B} = \bigcap_{b \in B} R_b$ and
let us say that x^+y iff world y is "reachable" by x, more precise:

xR^+y iff either $x = y$ and xRy or for a $n > 1$ holds that there is a sequence $x_1, x_2, \ldots x_n$ such that $x_1 = x, x_n = y$ and for all $i < n$ holds that x_iRx_{i+1}. If xR^+x for all x, we say that $R^+ = R^*$.

Now we can define R_{C_B} as $R^*_{E_B}$.

With these definitions of the relations, we can define the operators E_B, D_B and C_B on worlds:

$\mathcal{M}, w \models D_B\varphi$ iff $\mathcal{M}, x \models \varphi$ for all worlds x such that $wR_{D_B}x$.
$\mathcal{M}, w \models E_B\varphi$ iff $\mathcal{M}, x \models \varphi$ for all worlds x such that $wR_{E_B}x$.
$\mathcal{M}, w \models C_B\varphi$ iff $\mathcal{M}, x \models \varphi$ for all worlds x such that $wR_{C_B}x$.

We had already a formal definition of E_B and C_B. Note that they are equivalent to the above definition.

FDE

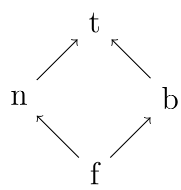
As mentioned in the introduction, in FDE there are four values: True, False, Both and Neither. Let us define a set $\mathcal{V}$ with these values: $\mathcal{V} = \{t, f, b, n\}$. A sentence φ can have one of the values in the set $\mathcal{V}$. The function v assigns a value from $\mathcal{V}$ to φ, for example $v(\varphi) = t$.(Priest, 2008b)

FDE has three connectives: $\neg, \vee$ and $\wedge$. As in classical logic $\neg f = t$ and $\neg t = f$. From this follows that $\neg b = b$ and $\neg n = n$. There is a intuitive explanation for this definition. Suppose that a sentence φ has the value n, then φ is neither told True nor told False. In that case $\neg\varphi$ should also have the value n. If we do not know anything about φ, we also do not know anything about $\neg\varphi$.(Belnap & Dunn, 1992)

In a truth table this looks a follows:

¬	
n	n
t	f
f	t
b	b

To explain how the connectives $\wedge$ and $\vee$ work, it is necessary to order the four values. It is easy to do this with the following lattice:



(Priest, 2008b)

So True is the highest in order, then follow Both and Neither and False is the last in order. The value of a sentence $\varphi \wedge \psi$ is the greatest lower bound of the values of φ and ψ, which is the greatest value in the lattice from where one can get to the value of φ and ψ following the arrows. For example, $b \wedge t = b$ and $b \wedge n = f$.

The value of a sentence $\varphi \vee \psi$ is the least upper bound of the values of φ and ψ, which is the lowest value in the lattice from where one can get to the value of φ and ψ following the arrows in the opposite direction. For example, $b \vee t = t$ and $b \vee n = t$. (Priest, 2008a)

In truth table this looks as follows:

$\wedge$	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

$\vee$	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

The implication connective is defined as in classical logic: $\varphi \rightarrow \psi = \neg\varphi \vee \psi$. (Priest, 2008b)

We say that a sentence φ is true iff $V(\varphi) = t$ or $V(\varphi) = b$. A sentence φ is valid iff for every interpretation φ is true.

There is no sentence that is valid in FDE, for every sentence φ there is a valuation, such that $V(\varphi) = n$.

Where in classical logic the sentence $p \vee \neg p$ is a logical truth, there is in FDE a valuation, such that $V(p \vee \neg p) = n$. Suppose $V(p) = n$, then $V(\neg p) = n$ and $V(p \vee \neg p) = n$, so $p \vee \neg p$ is not valid in FDE.

FDE on Possible Worlds

To define a modal logic based on FDE, the logic **BK** is used in this paper, because it is very close to Kripke semantics. **BK** logic is the Belnapian version of the weakest normal modal logic, named **K**. (Odintsov & Wansing, 2010)

A **BK** model is a tuple $\mathcal{M} = \langle W, R, v^+, v^- \rangle$, where W is a non-empty set of possible worlds, where R is accessibility relation on W and $v^+ : Prop \rightarrow 2^W$ is a function which takes a proposition p and gives the subset of worlds in which p is true and v^- gives the subset of wolds in which p is false.

We define the following relations $\models^+$ and $\models^-$ to express that a proposition p is true or false in a world:

$\mathcal{M}, w \models^+ p \Leftrightarrow w \in v^+(p); \mathcal{M}, w \models^- p \Leftrightarrow w \in v^-(p)$.

To extend this to sentences we define the connectives as follows:

$\mathcal{M}, w \models^+ \neg\varphi$ iff $\mathcal{M}, w \models^- \varphi$
$\mathcal{M}, w \models^- \neg\varphi$ iff $\mathcal{M}, w \models^+ \varphi$

$\mathcal{M}, w \models^+ \varphi \wedge \psi$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \psi$
$\mathcal{M}, w \models^- \varphi \wedge \psi$ iff $\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi$

$\mathcal{M}, w \models^+ \varphi \vee \psi$ iff $\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$
$\mathcal{M}, w \models^- \varphi \vee \psi$ iff $\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi$

To make this a modal logic, we add the $\Box$ and the $\Diamond$ in a very similar way these modal symbols are defined in classical modal logic.

$\mathcal{M}, w \models^+ \Box\varphi$ iff $\mathcal{M}, x \models^+ \varphi$ for all x such that wRx
$\mathcal{M}, w \models^- \Box\varphi$ iff $\mathcal{M}, x \models^- \varphi$ for all x such that wRx

$\mathcal{M}, w \models^+ \Diamond\varphi$ iff there exist a world x such that wRx and $\mathcal{M}, x \models^+ \varphi)$
$\mathcal{M}, w \models^- \Diamond\varphi$ iff there exist a world x such that wRx and $\mathcal{M}, x \models^- \varphi)$

To translate this back to the four values of FDE we define the valuation function v_w as follows:(Priest, 2008b)

$v_w(\varphi) = t$ iff $\mathcal{M}, w \models^+ \varphi$ and it is not the case that $\mathcal{M}, w \models^- \varphi$
$v_w(\varphi) = b$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^- \varphi$
$v_w(\varphi) = n$ iff not $\mathcal{M}, w \models^+ \varphi$ and not $\mathcal{M}, w \models^- \varphi$
$v_w(\varphi) = f$ iff $\mathcal{M}, w \models^- \varphi$ and it is not the case that $\mathcal{M}, w \models^+ \varphi$

We say that φ is true in $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M}, w \models^+ \varphi$ for all $w \in W$. (Odintsov & Wansing, 2010) This means that $\mathcal{M} \models \varphi$ iff ($v_w(\varphi) = t$ or $v_w(\varphi) = b$ for all $w \in W$).We say that φ is valid, if φ is true in each $\mathcal{M}$.

In FDE an argument is valid iff there is no world w in any model $\mathcal{M}$, such that the premises of the argument have the value true or both in world w and the conclusion has not.

Note that $p, p \rightarrow q \models q$ (modus ponens) is not valid in FDE. Take the world x, where $x \in v^+(p), x \in v^-(p)$ and $x \in v^-(q)$, then $v(p) = b$ and $v(q) = f$. Then $v(p \rightarrow q) = v(\neg p \vee q) = b$. So the premises have the value both, while the conclusion has the value false.

FDE on Group Knowledge

In Group Knowledge the knowledge operator is defined as replacement for the box, which looks as follows:

$w \models K_a\varphi$ iff $x \models \varphi$ for all x such that wR_ax.

If we define the knowledge operator in FDE as we defined the box operator in FDE we get the following definition:

$w \models^+ K_a\varphi$ iff $x \models^+ \varphi$ for all x such that wR_ax.
$w \models^- K_a\varphi$ iff $x \models^- \varphi$ for all x such that wR_ax.

Like in epistemic logic we define $\hat{K_a}$ as $\neg K_a \neg$. So,

$w \models^+ \hat{K_a}\varphi$ iff there exists a world x, such that wR_ax and $x \models^+ \varphi$.
$w \models^- \hat{K_a}\varphi$ iff there exists a world x, such that wR_ax and $x \models^- \varphi$.

In FDE $K_a\varphi$ is true iff $v(K_a\varphi) = t$ or $v(K_a\varphi) = b$. From this follows that $K_a\varphi$ is not true iff $v(K_a\varphi) = f$ or $v(K_a\varphi) = n$.

In classical logic $p \wedge \neg p \models q$. Because $p \wedge \neg p$ is a contradiction and always false, it follows that the argument is valid. In epistemic logic $K_a\varphi \wedge K_a\neg\varphi$ is also a contradiction, so the argument $K_a\varphi \wedge K_a\neg\varphi \models K_a\psi$ is valid. This means that if you know two opposite things, you know everything.

In FDE this does not hold. Suppose that $v(\varphi) = b$, then $v(\neg\varphi) = v(\varphi)$, so $v(K_a\neg\varphi) = v(K_a\varphi)$. So $K_a\varphi \wedge K_a\neg\varphi$ is not a contradiction. The question is if this is desirable. If there is a reason to belief $K_a\varphi \wedge K_a\neg\varphi$ is true, then we do not want the argument $K_a\varphi \wedge K_a\neg\varphi \models K_a\psi$ to be valid.

Knowledge is often described as justified true belief. Two opposite things can both be justified to believe. It also seems possible for a person to have opposite beliefs, (maybe unknowingly). To say that two opposite things are both true is a little bit more controversial, but we live in a post-modern period, where people do not think very strictly about truth. I think most people agree if I say that two opposite things can have a part of the truth. Even if two opposite things can not be true and false at the same time, then is a reason to believe you can know two opposite things. (Goldman & McGrath, 2015)

Knowledge as justified true belief is not widely accepted. A lot of philosophers argue that knowledge does not necessarily require truth. With a different definition of knowledge, it can be problematic to consider $K_a\varphi \wedge K_a\neg\varphi$ as a contradiction. A different definition of knowledge is often the case in AI if we talk about agents. The knowledge of an agent is often described as the information the agent is fed. This information is not necessarily true.

If we use FDE to describe knowledge, $K_a\varphi \wedge K_a\neg\varphi$ is not a contradiction and we do not have the 'problem' that $K_a\varphi \wedge K_a\neg\varphi \models K_a\psi$. This can be a reason to use FDE to describe knowledge.

The main difference between knowledge and belief is that we do not need a belief to be true. We write $B_a\varphi$ for "agent a believes that φ". In classical logic, the belief operator in a world w is defined as:

$w \models B_a\varphi$ iff $w \ \models \varphi$ for all x such that wR_a^Bx.

This means that $B_a\varphi \wedge B_a\neg\varphi$ is a contradiction, so $B_a\varphi \wedge B_a\neg\varphi \models B_a\psi$. In other words, if you believe that φ is true and $\neg\varphi$ is true, then you believe everything. This does not seem right, since it seems normal for a person to believe two opposite things. Maybe because the person is not aware of his inconsistent beliefs, or he believes φ and $\neg\varphi$ and realizes it, but he has a justification for both φ and $\neg\varphi$, so he believes both.

Because $B_a\varphi \wedge B_a\neg\varphi$ is a contradiction in classical logic, $B_b(B_a\varphi B_a\neg\varphi)$ is also a contradiction. So if you believe that agent a believes a contradiction, then you believe everything. So even if you are a consistent believer, but you believe somebody else is not, you are not a consistent believer anymore and you believe everything. This seems even more implausible.

In FDE $w \models^+ B_a\varphi$ iff $w \ \models^+ \varphi$ for all x such that wR_a^Bx and
$w \models^- B_a\varphi$ iff $w \ \models^- \varphi$ for all x such that wR_a^Bx

So in FDE, we do not have the problem that an inconsistent belief, causes beliefs in everything, because $B_a\varphi \wedge B_a\neg\varphi$ is not a contradiction.

Everybody Knows

In classical logic the group knowledge operator for ”everybody knows” is defined as:

$$E_B\varphi = \bigwedge_{b\in B} K_b\varphi$$

and on possible worlds E_B is defined as:

$\mathcal{M}, w \models E_B\varphi$ iff $\mathcal{M}, x \models \varphi$ for all worlds x such that $wR_{E_B}x$

In FDE we define this as follows:

$\mathcal{M}, w \models^+ E_B\varphi$ iff $\mathcal{M}, x \models^+ \varphi$ for all worlds x such that $wR_{E_B}x$
$\mathcal{M}, w \models^- E_B\varphi$ iff $\mathcal{M}, x \models^- \varphi$ for all worlds x such that $wR_{E_B}x$

We will discuss some differences and consequences of these definitions.

In classical logic, there are a lot of logical truths. These logical truths are true in every world. Suppose φ is a random logical truth. This means that in every world and for every agent $a \in A, K_a\varphi$ is also true. From which follows that $E_B\varphi$. This means that everybody knows all the logical truths.

Some logical truths are very long and if you see the sentence you can not tell if it is true or not. Is it then justified to say that you know that the sentence is true? People are not perfect reasoners so it is likely that you do not know most of the logical truths. For agents, it is also likely to do not know all the logical truths, since computation costs time and some logical truths are infinite. (Gómez-Torrente, 2019) So it seems unlikely that everybody knows all the logical truths in a group.

In FDE we do not have this problem, because there are no logical truths in FDE.

A related problem is that in classical logic the agent knows everything that follows validly from his knowledge. In other words, if $\varphi, \psi \models \xi$ in classical logic, then $K_a\varphi, K_a\psi \models K_a\xi$. So, for example $K_a\varphi \wedge K_a\psi \models K_a(\varphi \wedge \psi)$, $K_a\varphi \models K_a(\varphi \vee \psi)$ and $K_a\varphi \wedge K_a(\varphi \to \psi) \models K_a\psi$. This does not feel right, because it seems plausible that some things follow from your knowledge, but where you did not think of and it is also human to make mistakes with reasoning.

In FDE we have the same problem. Suppose that $\varphi, \psi \models \xi$ in FDE, then $K_a\varphi, K_a\psi \models K_a\xi$. So, for example $K_a\varphi \wedge K_a\psi \models K_a(\varphi \wedge \psi)$ and $K_a\varphi \models K_a(\varphi \vee \psi)$. A difference with classical logic is that $K_a\varphi \wedge K_a(\varphi \to \psi) \models K_a\psi$ is not valid, because $\varphi \wedge (\varphi \to \psi) \models \psi$ is not valid in FDE. There are enough arguments $\varphi, \psi \models \xi$ that are valid in FDE, so the problem of logical omniscience is not resolved in FDE.

Distributed Knowledge

If an agent a knows φ and agent b knows $\varphi \to \psi$, then $D_{ab}\psi$ in classical logic.

We define the distributed knowledge in FDE on possible worlds as follows:

$\mathcal{M}, w \models^+ D_B\varphi$ iff $\mathcal{M}, x \models^+ \varphi$ for all worlds x such that $wR_{D_B}x$
$\mathcal{M}, w \models^- D_B\varphi$ iff $\mathcal{M}, x \models^- \varphi$ for all worlds x such that $wR_{D_B}x$

Now we show that in FDE $K_a\varphi, K_b\psi \models D_B(\varphi \wedge \psi)$ just like it does in classical logic.

Suppose $K_a\varphi$ and $K_b\psi$ and B = a, b, then $w \models^+ K_a\varphi$ and $w \models^+ K_b\psi$, which means that $w \models^+ \varphi$ for all x such that wR_ax and $w \models^+ \psi$ for all x such that wR_bx. From this follows that $w \models^+ \varphi \wedge \psi$ for all x such that wR_ax and wR_bx. In other worlds $w \models^+ \varphi \wedge \psi$ for all x such that $(w, x) \in \bigcap_{b\in B} R_b$. So $w \models^+ \varphi \wedge \psi$ for all x such that $wR_{D_B}x$, so $w \models^+ D_B(\varphi \wedge \psi)$. From this we can conclude that $K_a\varphi, K_b\psi \models D_B(\varphi \wedge \psi)$ in FDE.

However, there are also arguments concerning distributed knowledge that are valid in classical logic, but not in FDE. We will discuss a few and we begin with modus ponens.

To show that $K_a\varphi, K_b(\varphi \to \psi) \models D_B\psi$ is not valid, we need some counter example were $K_a\varphi$ and $K_b(\varphi \to \psi)$ are valid and $D_B\psi$ is not. Suppose that $w \models^+ K_a\varphi$, $w \models^+ K_b\varphi$, $w \models^- K_b\varphi$ and $w \not\models^+ \widehat{K_b}\psi$, then $K_a\varphi$ and $K_b(\varphi \to \psi)$ are valid, but $w \not\models^+ \widehat{K_b}\psi$, so for all x such that $wR_{D_B}x$ holds that $x \not\models^+ \varphi$, so $D_B\psi$ is invalid. We can conclude that $K_a\varphi, K_b(\varphi \to \psi) \models D_B\psi$.

The question is if it is desirable that modus ponens is not valid in this case. If somebody knows φ and another knows that φ implies ψ, it sounds plausible that they know ψ together. Modus ponens is a very important argument in logic, so if modus ponens does not work, it is at least a flaw. If we use FDE to describe knowledge of agents with inconsistent information, it could be a good thing that modus ponens does not work. It is tricky to make a heavy conclusions based on inconsistent information.

A similar argument that does not work in FDE is $K_a\neg p, K_b(p \vee q) \models D_Bq$, which is the same as modus ponens if we write $p \vee q$ as $\neg p \to q$. We just showed that modus ponens is not valid in FDE, so neither is $K_a\neg p, K_b(p \vee q) \models D_Bq$. This is unfortunate, since in life we often search for the right answer, by eliminating the wrong answers. However, if you work with inconsistent information a wrong answer can also be a right answer, so then it is good that $K_a\neg p, K_b(p \vee q) \models D_Bq$ does not work.

Another argument that does not work in FDE is $K_a(\varphi \to \psi), K_b(\psi \to \xi) \models D_B(\varphi \to \xi)$. Suppose that $v_w(K_a\varphi) = t, v_w(K_a\psi) = b, v_w(K_b\psi) = b$ and $v_w(K_b\xi) = f$, then $v_w(K_a(\varphi \to \psi)) = b$ and $v_w(K_b(\psi \to \xi)) = b$, so $K_a(\varphi \to \psi), K_b(\psi \to \xi)$ is valid. However, for all x such that $wR_{D_B}x$ holds that $v_x(\varphi) = t$ and $v_x(\xi) = f$, so $v_x(\varphi \to \xi) = f$, so $D_B(\varphi \to \xi)$ is not valid. This feels again a little bit strange. (Priest, 2008a)

Common Knowledge

We define common knowledge in FDE:

$\mathcal{M}, w \models^+ C_B\varphi$ iff $\mathcal{M}, x \models^+ \varphi$ for all worlds x such that $wR_{C_B}x$.
$\mathcal{M}, w \models^+ C_B\varphi$ iff $\mathcal{M}, x \models^+ \varphi$ for all worlds x such that $wR_{C_B}x$.

Analogous to the case of "everybody knows", in classical logic all the logical truths are common knowledge ($\models \varphi \implies C_B\varphi$). We already discussed why it is strange to say that everybody knows all the logical truths. Saying that all logical truths are common knowledge is even a stronger claim and even less desirable. It requires more knowledge about knowledge of others, that we often do not have in real life.

Common knowledge is often introduced with the puzzle of the muddy children. The puzzle shows that common knowledge makes a difference in comparison with general knowledge. Some inferences are done, that could not have taken place in general knowledge. We will discuss the puzzle.

Suppose there are three children (Ann, Bill, and Cath), and suppose that Ann and Bill are muddy. Their father says: "At least one of you is muddy" ($m_a \vee m_b \vee m_c$). It is common knowledge that the children are perfect reasoners, that what the father says is true and that they can not see their own face, but they can see each other's faces. Then the father asks: "Please step forward if you are muddy" A child can know if it is muddy when it sees two faces that are not muddy. For Ann, Bill and Cath this is not the case, so nobody steps forward. So then it is common knowledge that nobody knows if their muddy or not. Now Ann knows that Bill is not the only one that is muddy, otherwise, Bill would have stepped forward. So Ann knows that she is muddy and Bill knows he is muddy too, because of similar reasons. When the father asks again: "Please step forward if you are muddy". Ann and Bill step forward and Cath knows that Ann and Bill could only know this if she is self not muddy. So now Cath knows she is not muddy.

Not all these inferences can be done in FDE. For example, Ann knows that $C_B(m_a \vee m_b \vee m_c)$ and $K_b\neg m_a, K_b\neg m_c \models K_b m_b$ and concludes that Bill would have stepped forward if $K_b\neg m_a$. In FDE $K_b(m_a \vee m_b \vee m_c), K_b\neg m_a, K_b\neg m_c \models K_b m_b$ is not valid, because it is possible that $K_b m_a \wedge K_b\neg m_a$, so modus ponens does not work ($K_b\neg m_a, K_b(\neg m_a \to (m_b \vee m_c)) \not\models K_b(m_b \vee m_c)$).

So using FDE instead of classical logic can have some impact on the applications of common knowledge. Since modus ponens is so commonly used, a lot will change if it is not valid anymore.

Conclusion, Discussion

From this research, we can conclude that is possible and partly effective to define group knowledge based on FDE. It is effective, because in FDE $K_a\varphi \wedge K_a\neg\varphi \models K_a\psi$ and $B_a\varphi \wedge B_a\neg\varphi \models B_a\psi$ are not valid, so inconsistent knowledge or belief, does not imply that you know or belief everything. In the case that there is inconsistent information or that something can be true and false at the same time, FDE is a great logic to avoid problems like believing everything as a cause of inconsistent belief.

FDE also solves a part of the logical omniscience problem, namely the part where every agent knows all the logical truths, this is because in FDE there are no logical truths. The other part of the logical omniscience problem is that agents know everything that follows from their knowledge. FDE does not solve this part of the problem, so FDE is here on the same level as classical logic.

A disadvantage of FDE in comparison with classical logic is that modus ponens does not work in FDE. Modus ponens is used very commonly, so not been able to use modus ponens affects what we can describe with distributed knowledge, common knowledge, and just knowledge in general.

That modus ponens does not work in FDE can also be an advantage in (group) knowledge, since we do not expect from people or agents that they know the all implications of their knowledge. So in real life, a not working modus ponens seems appropriate sometimes.

Other things that could be researched are other ways of handling inconsistent information and inconsistent (group) knowledge, for example, other many-valued logics or non-monotonic logic. In addition, FDE could be applied more broadly, for example on different aspects of group knowledge, like public announcements.

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