Pluralism and Structuralism

On the Compatibility of Sher's Logical Structuralism and Beall and Restall's Logical Pluralism

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ABSTRACT In this thesis, we assess the following question: does Gila Sher's logical structuralism about logic allow for logical pluralism? More specifically, can one consistently endorse both structuralism and the idea that multiple consequence relations are correct? In what follows, we first outline the main principles of Sher's structuralist view. Subsequently, we define logical pluralism, building on Jc Beall and Greg Restall's definition of logical pluralism. In the last chapter, we shall give a positive answer to the stated question: pluralism is attainable within structuralism. More precisely, it is argued that Sher's structuralist view does not commit one to the belief that the world's structure is uniform and that pluralism about consequence can therefore be defended via pluralism about structure or pluralism about interpretations of structure.

Contents

Preface

Throughout the text, logical and mathematical machinery is frequently used: sometimes to illustrate a point, other times as part of the argumentation. It is therefore sensible to explicate a few things beforehand concerning technical terminology.

Set theory In accordance with Sher's mathematical precisification of the formal structural view, we use Zermelo-Fraenkel set theory plus the Axiom of Choice (ZFC). The language of ZFC, $\mathcal{L}_{\text{ZFC}}$, consists of the membership symbol $\in$, the identity symbol $=$, the empty set $\varnothing$, quantifiers $\exists$, $\forall$, connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, variables $x, y, z, ...$ and parentheses (,).[1] For more details on ZFC, the reader can consult the third chapter of Ethan Bloch's *Proofs and Fundamentals* or the first two chapters of Patrick Suppes's *Axiomatic Set Theory*.

Functions A function $f : X \rightarrow Y$ is a relation that associates to each input element $x \in X$ a unique output element $y \in Y$, denoted as $f(x) = y$. For more details, Definition 4.1.1 in Bloch's book may be consulted.

Logical language The employed logical language is, roughly, that of classical predicate logic. It consists of connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, quantifiers $\exists$ and $\forall$, metavariables φ, ψ and χ that range over arbitrary formulas, variables $x, y, z, ...$ and constants $a, b, c, ...$ and parentheses (,) and square brackets [,].

Models and truth values For the semantics of quantifiers, we generally follow Sher's set-theoretical approach as explained later on. Outside that context, we use the semantics of classical predicate logic as described by Sider. First, truth value assignments of formulas are given in a model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ consisting of a non-empty domain $\mathcal{D}$ and interpretation function $\mathcal{I}$ that assigns constants $a, b, c, ...$ to members of the domain $d \in \mathcal{D}$ and predicates P to n-tuples $\langle d_1, ..., d_n \rangle \in \mathcal{D}$. Furthermore, the variable assignment g assigns to each variable $x, y, z, ...$ a $d \in \mathcal{D}$. Lastly, the valuation function $\mathsf{V}_{\mathcal{M},g} : \mathcal{S} \rightarrow \{0, 1\}$ assigns truth values to formulas $s \in \mathcal{S}$ relative to $\mathcal{M}$ and g.[2]

Logical consequence In our discussion about logical consequence, Γ and σ range over arbitrary premise and conclusion sets. "Logical consequence" and validity will figure as close semantic associates: σ is a logical consequence of Γ if and only if the premises $\gamma \in \Gamma$ cannot be true without σ being true. This idea derives from Alfred Tarski, who defined "*X* follows from *K*" as follows: "[f]rom an intuitive standpoint [that] it can never happen that both the class K consists only of true sentences and the sentence X is false".[3]

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[1] Patrick Suppes, *Axiomatic Set Theory* (New York: Dover Publication, Inc., 1972), 14.

[2] Theodore Sider,*Logic for Philosophy* (Oxford: Oxford University Press, 2010), 92-94.

[3] Alfred Tarski, *Logic, Semantics, Metamathematics. Papers from 1923 to 1938,* trans. J.H. Woodger (Oxford: Clarendon Press, 1956), 414.

Consequence relation Given a logic $\mathcal{L}$, a consequence relation is the relation which holds between the premises and conclusion of any valid argument. If $\Gamma \models_{\mathcal{L}} \sigma$ is an arbitrary valid argument in $\mathcal{L}$, then a consequence relation $\mathcal{C}_{\mathcal{L}}$ of $\mathcal{L}$ may be defined as the set of all consequence pairs $\langle \Gamma, \sigma \rangle$ such that $\Gamma \models_{\mathcal{L}} \sigma$. In set-builder notation, this is the set $\mathcal{C}_{\mathcal{L}} := \{\langle \Gamma, \sigma \rangle : \Gamma \models_{\mathcal{L}} \sigma\}$.

Introduction

What is the subject of logic? According to a standard view, logic is the study of validity. In turn, "validity" is typically defined to be "formal" and "necessary": an argument is valid if and only if (1) the conclusion follows from the premises only by the argument's *form*, and (2) it is impossible for the premises to be true *without the conclusion being true*. In the model-theoretic account of validity, these two notions are typically captured as follows:

> σ is a logical consequence of Γ if and only if in all models $\mathcal{M}$, if all $\gamma \in \Gamma$ are true, then σ must be true.[4]

As with the described perspective, model-theoretic consequence requires formality and necessity. First, one only attends to the argument's form — not what Γ and σ stand for. Furthermore, the connection is one of necessity: there cannot exist a model wherein all $\gamma \in \Gamma$ are true without σ being true.[5] Beyond these criteria of formality and necessity, one might however ask *in virtue of what* a given argument is formal and necessary. Why is a given argument formal, and why is it necessary?

In the thesis, we will assess two responses to the stated question: logical structuralism and logical pluralism. In particular, it is evaluated whether Sher's formal-structural view — for short, "structuralism" — allows for Beall and Restall's pluralism about logical consequence — for short, "logical pluralism". As we will see later on, both views provide an independent answer as to why a given argument should be necessary and formal. As we will also see, Sher's structuralism has plausible motivations. However, there are equally legitimate reasons for accepting the pluralist stance. Importantly, structuralism is a view about logical consequence in general. Therefore, a complete exclusion of pluralism would seem, at the very least, suspicious. Moreover, Sher notes that her structuralism offers space for logical pluralism. Still, she has not provided systematic argumentation for this claim. Our objective is to argue whether that claim holds water or not.[6]

But what, exactly, are the central claims of structuralism and pluralism? According to Sher's structuralism, logical consequence ought to take into account the world: it should model the way in which the world allows for truth-preservation.[7] More specifically, any correct logic should hold by virtue of the world's predicational structure: the relations that hold between objects and properties. As we will see, this involves that logical constants ought to depict highly general properties of objects and properties. In turn, logical consequence

[4] Roy T. Cook, *A Dictionary of Philosophical Logic* (Edinburgh: Edinburgh University Press, 2009), 176.

[5] Cook, *A Dictionary of Philosophical Logic,* 62, 305.

[6] Gila Sher, *Epistemic Friction. An Essay on Knowledge, Truth, and Logic* (Oxford: Oxford University Press, 2016), 337-338.

[7] Note: in recent years, Sher has included structuralism into her epistemic theory of foundational holism. Still, it can be viewed in isolation. See for instance pages 183-189 from [38].

models formal laws pertaining to that structure: laws of an especially strong modal force.[8] To precisify this philosophical view, Sher draws on ZFC as a theory of structure — this fact will play a major role in what follows.[9]

On the other hand, logical pluralism holds that there are multiple legitimate codifications of consequence, each correct given some relevant background conditions. Notable candidates of such conditions are cases, contexts and languages.[10] One motivation for pluralism is the prematureness of monism: in the absence of an overarching consequence relation, one should accept the pluralist position.[11] Another motivation is that different topics require different consequence relations. Classical logic, for example, seems accurate enough to model consequence for non-vague topics. By contrast, vague topics are often modelled better by non-classical logic.[12]

To narrow the view down, we will henceforth only consider Jc Beall and Restall's pluralism. This pluralism revolves around the so-called Generalised Tarski Thesis:

> (GTT) An argument is valid$_x$ if and only if in every case$_x$ in which the premises are true, the conclusion must be true.

By specifying the term "case$_x$", one arrives at a consequence relation containing all valid arguments for the class of cases bounded by x.[13] According to Beall and Restall, multiple consequence relations exist because there are multiple correct specifications of the term "case$_x$".[14] If, for example, "case$_x$" is interpreted as an intuitionistic proof stage s, then there are proof stages such that $s \not\models \varphi$ and $s \not\models \neg\varphi$ for some formula φ. On the other hand, if "case$_x$" is interpreted as a Tarskian model $\mathcal{M}$, then we have $\mathcal{M}, g \models \varphi$ or $\mathcal{M}, g \models \neg\varphi$ for any φ. If both specifications are correct, then we end up with two correct consequence relations.[15]

In brief, our problem is as follows. If Sher's logical structuralism is correct, then any consequence relation ought to model the world's formal laws.[16] If that is so, then it seems that logical pluralism is an unwelcome guest within structuralism. For if logical consequence should only model the world's formal laws, then the only relevant interpretation of "case$_x$" seems to be the class of those laws. But this just means that there cannot be multiple correct interpretations

[8] Gila Sher, "The Foundational Problem of Logic," *The Bulletin of Symbolic Logic* 19, 2 (2013): 172.

[9] Sher, "The Foundational Problem of Logic," 163, 171.

[10] Gillian Russell, "Logical Pluralism," *The Stanford Encyclopedia of Philosophy* (Summer 2021 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/sum2021/entries/logical-pluralism/.

[11] Stewart Shapiro, "Structures and Logics: a Case for (a) Relativism," *Erkenntnis (1975-)* 79, 2 (2014): 309, 311, 328.

[12] Graham Priest, "Revising Logic," in *The Metaphysics of Logic,* ed. Penelope Rush (Cambridge: Cambridge University Press, 2014), 217.

[13] Jc Beall and Greg Restall, *Logical Pluralism* (Oxford: Clarendon Press, 2006), 29-30.

[14] Beall and Restall, *Logical Pluralism,* 29.

[15] Ibidem, 66.

[16] Sher, "The Foundational Problem of Logic," 169.

of the term "case$_x$". Consequently, this implies that there cannot be multiple correct consequence relations. In sum, structuralism seems to boil down to a sort of monism about consequence.

Importantly, the stated problem is not just a semantic dispute about the sense of the term "correctness". Taken by itself, Beall and Restall's pluralism *obviously* interprets "correct" differently than Sher's structuralism. For starters, the former's definition of "correct" is not directed at the world's formal laws whereas the latter explicitly is. However, this is besides the point — we are interested in the question whether Sher's structuralism *allows for* pluralism in Beall and Restall's sense. Accordingly, our variety of pluralism *must* adhere to the criteria of correctness given by structuralism. To resolve whether pluralism is attainable within structuralism, then, more is needed than a simple consultation of word meaning.

Seemingly, the pluralist faces a dilemma: either she should grant that the term "correct" is applicable only with respect to those consequences that model the world's formal laws, or she ought to argue that structuralism is, in one way or another, defective. We shall however argue that logical pluralism is possible within the confines of structuralism. In particular, this possibility is given not by Sher's mathematical implementation of structuralism, but by its *philosophical principles*: nowhere does structuralism require that structure is uniform. As we will argue, accepting different theories of structure or interpretations of those structure allows for pluralism about consequence. Hence, logical pluralism can be attained by showing that different background theories or interpretations thereof are acceptable.

At this point, we should say a few words about our methodology. In assessing the views of others, we strive to uphold the principle of charity. Sher's structuralism is mostly based on *The Bounds of Logic*, "The Foundational Problem of Logic" and *Epistemic Friction*; Beall and Restall's pluralism on their book *Logical Pluralism*. Regarding philosophical argumentation, we strive for both clarity and precision. In the first and second chapter, argumentation is restricted to the motivations for structuralism and pluralism. The highest density of philosophical argumentation is in the third chapter, which outlines our main argument.

To close off, a brief overview of what follows. In the first chapter, an overview is given of Sher's structuralism, divided into a philosophical and a mathematical section. In the philosophical section, we discuss in what sense logical consequence is grounded in the world's structure. The mathematical section shows how Sher uses set-theoretic operations and cardinality properties of ZFC to specify the philosophical thesis of structuralism. The second chapter addresses logical pluralism. First, an exposition of Beall and Restall's pluralism is given. Subsequently, we articulate the tension between structuralism and pluralism. In the third and last chapter, we propose a twofold solution to the initially stated problem: logical pluralism is attainable if one accepts sufficiently different theories of structure or different interpretations of the same structure. To illustrate the proposal, we give two examples: one from paraconsistent set theory, and one from Kripke's semantics for intuitionistic logic.

1 Sher's Logical Structuralism

1.1 Philosophical Principles of Structuralism

1.1.1 Structural Realism

A philosophical view is often easiest explained via categorization. Broadly, Sher's formal-structural view can be seen as a species of "structuralism": a family of philosophical views bonded by the idea that a given "world" or "set of entities" is best understood in terms of structure.[17] For the structuralist, specific properties of objects are irrelevant: all that matters is *structure*, that is, the relations that hold between objects. Typically, structuralists define "structure" in terms of set-theoretic constructions and relations or functions on those constructions.[18]

Structuralism in logic has its predecessors in other branches of philosophy. For our purposes, mathematical structuralism is of particular interest: According to mathematical structuralism, mathematics is, essentially, about structure. Roughly, the idea is that mathematical objects are not objects in their own right. Instead, their nature is given by their relations with other mathematical objects. Indeed, mathematical structuralists often dismiss talk about intrinsic, non-relational properties altogether.[19] As Paul Benacerraf, the pioneer of modern mathematical structuralism, puts it: "the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure".[20]

To illustrate the idea, contrast the following two natural number systems: $\mathcal{S}_1 = \langle N, 0, s_1 \rangle$ and $\mathcal{S}_2 = \langle N, 0, s_2 \rangle$, with $0 := \varnothing$, $s_1(x) = x \cup \{x\}$ and $s_2(x) = \{x\}$. In both cases, the natural number set N contains 0, plus all numbers accessed via the successor function s_1 or s_2. For $\mathcal{N}_1$, $1 = s_1(\varnothing) = \varnothing \cup \{\} = \{\varnothing\}$ and so $2 = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$. By contrast, $\mathcal{N}_2$ gives $1 = s_2(\varnothing) = \{\varnothing\}$ and thus $2 = \{\{\varnothing\}\}$.[21] From a non-structuralist point of view, $\{\varnothing, \{\varnothing\}\}$ and $\{\{\varnothing\}\}$ appear to be different objects. From a structuralist point of view, however, all that matters are the *relations* $\{\varnothing, \{\varnothing\}\}$ and $\{\{\varnothing\}\}$ bear to other mathematical objects — for example, that 2 succeeds 1 and precedes 3, that it is the first prime, and so on.[22]

Where mathematical structuralism is a view about the structure of mathematical objects in particular, logical structuralism generalizes the point to the

[17] Theodore Sider, *Writing the Book of the World* (Oxford: Clarendon Press, 2011), 1.

[18] Roman Frigg and James Nguyen, *Modelling Nature: An Opinionated Introduction To Scientific Representation* (Cham, Switzerland: Springer, 2020), 51, 56.

[19] Erich Reck and Georg Schiemer, "Structuralism in the Philosophy of Mathematics," *The Stanford Encyclopedia of Philosophy* (Spring 2020 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/cgi-bin/encyclopedia/archinfo.cgi?entry=structuralism-mathematics.

[20] Paul Benacerraf, "What Numbers Could Not Be," *The Philosophical Review* 74, 1 (1965): 70.

[21] Joan Bagaria, "Set Theory", *The Stanford Encyclopedia of Philosophy* (Spring 2020 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/spr2020/entries/set-theory/.

[22] Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford: Oxford University Press, 1997), 72.

structure of objects in general. A close conceptual sibling of logical structuralism is structural realism about logic: the view that logic is specifically directed at this world's structure. In this view, logical facts are just facts like any other — only very general ones, that is.[23] Although there are multiple ways to specify the relevant notion of structure, we will stick with Sher's specification of structure as predicational structure.[24] By "predicational structure", we understand the relations that hold between objects and properties. If each predicate is interpreted as a set, for instance, then the property "is green" is understood in terms of its extension: that is, the set of green things.[25] Similar to mathematical structuralism, logical structuralism disregards the intrinsic nature of objects and properties and attends to their relations instead.

1.1.2 Logical Structuralism

One main tenet of Sher's structuralism is that logical consequence models "formal laws" or "strong connections" with respect to the world's structure. Amongst Sher's motivations for structuralism, perhaps the strongest and most insightful one is the argument from truth-preservation. As preservation of truth, logical consequence holds by virtue of a strong connection between the truth values of sentences. In turn, truth values of sentences depend, typically, on how things are with respect to the world: a sentence is true only if its designated state of affairs is true.[26] Hence, any truth-preserving connection between sentences relies on a strong connection or law between the states of affairs designated by the sentences. In other words, any correct consequence relation should match the world's formal laws if it is to be truth-preserving.[27]

As Sher puts it, "[i]t would be a mystery [if] a logical theory worked in the world [...] if it were not in tune with the world".[28] Which consequences hold, then, depends on which formal laws exist. As Sher writes:

> [L]ogical consequence is grounded in formal laws governing reality. When a sentence σ stands in the relation of logical consequence to a set of sentences [Γ], this can be explained by a certain formal or structural connection between the situation described by [Γ] and that described by σ [...].[29]

In particular, the notions of a formal element and that of a formal law are of central importance to logical consequence:

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[23] Daniel Cohnitz and Luis Estrada-González, *An Introduction to the Philosophy of Logic* (Cambridge: Cambridge University Press, 2019), 115.

[24] Sher, "The Foundational Problem of Logic," 172.

[25] Cook, *A Dictionary of Philosophical Logic,* 114.

[26] Note: we used the word "typically" to leave open the possibility of a class of sentences whose truth does not depend on the world. Moreover, "sentence" is taken to be restricted to the class of *declarative sentences*: those sentences capable of being true or false.

[27] Sider, *Writing the Book of the World,* vii, 216.

[28] Sher, "The Foundational Problem of Logic," 160.

[29] Ibidem, 169.

> [W]e can say that a given logical consequence is grounded in a universal law connecting formal elements in the truth conditions of its premises and conclusion.[30]

Viewed abstractly, logical consequence depicts the formal laws which hold between formal elements of the world — henceforth called "formal properties". In turn, formal properties determine which laws hold and, thus, which consequences are validated by the world. The formal and necessary aspects of consequence are secured via these notions. Consequence is formal because it depicts formal laws between situations described by Γ and σ by looking at structural connections, not at what Γ and σ stand for. Necessity holds because any formal law ensures that σ is true whenever Γ is — otherwise it would not be a "universal law".[31]

Within Sher's structuralism, logic has a dual nature. As a theory of language, it is concerned with the relation between *sentences*. As a theory of structure, however, logic studies the *world*.[32] In what follows, we will focus on the latter perspective, although we will not completely neglect the linguistic perspective. Before we do so, Sher's technical use of the term "the world" requires clarification. Simply stated, "the world" is associated with a system of knowledge: its extension is the set of objects described by a system of knowledge. More specifically, the world consists of the ontological commitments made by the relevant system of knowledge: the world is all that knowledge says there is. However, that is not to say that the world is nothing but "the totality of facts": instead, "the world" is an open-ended concept, admitting of new objects whenever new objects of knowledge are added to the system of knowledge. We use the term "world" in accordance with this sense, and revisit these remarks in the third chapter.[33]

As indicated, one should distinguish the philosophical principles of Sher's structuralism from its mathematical implementation. Philosophically, Sher aims at a "general and informative characterization". Therefore, the philosophical account is not committed to a specific background theory of formal structure: it does not endorse a particular predicational structure.[34] On the other hand, the mathematical account serves to precisify and *does* build on a specific theory of structure: Zermelo-Fraenkel set theory appended with the Axiom of Choice (ZFC).[35] Despite ZFC's prevalence in Sher's mathematical account, one should nevertheless take it out of the equation when considering Sher's philosophical view. In principle, that view could be precisified using another background theory of structure, as Sher remarks.[36]

[30] Ibidem, 169.
[31] Ibidem, 163.
[32] Sher, *Epistemic Friction,* 272.
[33] Ibidem, 10.
[34] Note: henceforth, we will sometimes shorten "background theory of formal structure" to "background theory of structure" or just "background theory".
[35] Sher, "The Foundational Problem of Logic," 171, 183.
[36] Sher, *Epistemic Friction,* 306-307.

1.1.3 Logical Constants

In formalizing natural language consequence, logicians are particularly attentive to logical concepts in natural language. Broadly put, logical concepts are expressions which seem to play a key role in natural language consequence — for example, "if ..., then ...", "for all" and "not". Logical constants model logical concepts in a rigorous fashion. Contrasting the distinctive inexactness of natural language concepts, logical constants are given a fixed and unambiguous interpretation within the framework of a logical language.[37] Standard logic, for instance, formalizes the logical concept "and" as the constant $\wedge$ such that for any model $\mathcal{M}$ and variable assignment g, $\nu_{\mathcal{M},g}(\varphi \wedge \psi) = 1$ iff $\nu_{\mathcal{M},g}(\varphi) = \nu_{\mathcal{M},g}(\psi) = 1$.

Deciding which natural language concepts are logical and non-logical is, however, not a straightforward task. As Roy T. Cook notes, there is no general consensus amongst logicians about the criteria that divide logical from non-logical concepts — although invariance, topic-neutrality and the possibility of recursive definitions are often believed to be central.[38] Consequently, it is not obvious which natural language concepts should be modeled as logical constants. Although this question is a difficult one, a proper account of logical consequence should answer it: since logical constants determine the form of arguments, they also determine which consequences hold.[39]

1.1.4 Formal Properties and Formal Laws

Sher's structuralist view holds that constants are logical only if they depict formal properties of objects.[40] But what are formal properties, exactly? Broadly speaking, formal properties are properties that remain when one "disregard[s] the particular characteristics of objects".[41] To be more precise, formal properties are properties that only attend to the predicational structure of given objects, properties or relations. If, for example, something with the name "a" is both green and organic, then a stands in the intersection of the set of green things and that of organic things. Importantly, intersection takes into account not the identity, but the relations between objects and properties: whichever objects A and B one takes, if an arbitrary individual is part of both A and B, then a stands in the intersection of A and B.[42] Other notable examples of formal properties are identity, cardinality, intersection, union, reflexitivity, transitivity, and symmetry.[43]

Since formal properties relate to predicational structure, determining these properties relies on some background notion of predicational structure. Interpreting each predicate as a set conforming to the ZFC axioms, for example,

[37] Nick Zangwill, "Logic as Metaphysics," *The Journal of Philosophy* 112, 10 (2015): 531.
[38] Cook, *A Dictionary of Philosophical Logic,* 176.
[39] Mario Gomez-Torrente, "The Problem of Logical Constants," *The Bulletin of Symbolic Logic* 8, 1 (2002): 2.
[40] Sher, "The Foundational Problem of Logic," 170.
[41] Sher, *Epistemic Friction,* 278.
[42] Sher, "The Foundational Problem of Logic," 172.
[43] Ibidem, 184.

assumes a view where for any object and property, either the object has the property or it does not. Arguably, this contrasts with intuitionism: the view that mathematical objects and proofs are mind-dependent constructions. As intuitionists argue, not every mathematical object has a given property or its negation: some things are, so to speak, unsettled. A common example given in favor of intuitionism is the Riemann Hypothesis, an unproven conjecture about infinite series in number theory. Since neither this hypothesis nor its negation is proven, the intuitionist might refuse to say, for instance, that it either belongs to the set of true conjectures or to that of the false ones.[44]

What is so important about formal properties? As Sher writes, "choosing terms that denote *formal* properties (relations, functions) as logical constants is a sound way of constructing a logical system that satisfies the 'job description of logic'". Given the earlier described argument from truth-preservation and our characterization of formal laws, it seems plausible that studying predicational structure is a proper way of securing truth-preservation. That is, if logical constants model formal properties, then it seems plausible that the connections between those logical constants resemble those connections which hold between formal properties.[45]

Importantly, formal properties apply irrespective of whether one considers actual or possible objects: if the same structure is concerned, then the same formal properties apply. Indeed, the same holds for what Sher calls "formal laws".[46] Shortly put, formal laws are especially strong connections between formal properties. Consider, for example, the following connection between intersection and union: if the intersection of two properties A and B is non-empty, then their union is non-empty. Clearly, this is a law of especially strong force: it would be quite surprising if $A \cap B \neq \varnothing$ and $A \cup B = \varnothing$. Similarly, consider self-identity. Plainly, self-identity is not just common to a specific group of objects. Instead, it holds for all objects in general.[47]

[44] Rosalie Iemhoff, "Intuitionism in the Philosophy of Mathematics," *The Stanford Encyclopedia of Philosophy* (Fall 2020 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/fall2020/entries/intuitionism/.
[45] Sher, "The Foundational Problem of Logic," 170.
[46] Ibidem, 183.
[47] Ibidem, 184.

1.2 Logico-Mathematical Outline of Structuralism

1.2.1 The Logicality Criterion

In section 1.1.3, we stumbled upon the question which constants may be properly called logical. In essence, this is the question of logicality: when does something count as *logical*? In Sher's view, constants are logical if they designate formal properties.[48] Formally, this boils down to the following two criteria:

Definition 1. *Let $\mathcal{C}$ be a constant. Then, $\mathcal{C}$ is logical iff*

(i) $\mathcal{C}$ is a formal operator.[49]

(ii) $\mathcal{C}$ satisfies additional constraints that ensure its proper functioning in a logical system.[50]

By "operator", Sher means a function which assigns truth values given a specified condition. Conceptually, this is a mathematical precisification of formal properties: objects or properties have a given formal property iff its assigned formal operator yields 1. For example, conjunction "$\wedge$" holding between two arbitrary properties means that the truth function $f_\wedge : \{0,1\} \to \{0,1\}$ assigns 1 iff the truth value of both conjuncts is 1. For our purposes, we will focus on the set-theoretic aspect of operators. In the case of the quantifiers $\exists, \forall$, formality is spelled out in terms of isomorphic invariance — we will discuss that notion in the next section. By contrast, the connectives $\neg$, $\wedge$, $\vee$, $\to$ and $\leftrightarrow$ are defined as in standard predicate logic. Under the interpretation of two predicates B and C as sets and an arbitrary individual a as a set member,

$$B(a) \wedge C(a) \text{ is true iff } a \in B \cap C \tag{1}$$

$$B(a) \vee C(a) \text{ is true iff } a \in B \cup C \tag{2}$$

$$\neg B(a) \text{ is true iff } a \notin B \tag{3}$$

As usual, the clauses for $\to$ and $\leftrightarrow$ can be defined in terms of the clauses (1)-(3). Notice that these clauses are the clauses of standard predicate logic.[51] In sections 1.2.3 and 3.3.2, we will use these clauses.

For (ii), Sher provides the following list of constraints: $\mathcal{C}$ is a logical constant iff it denotes a formal operator, is extensional, is defined over all models, is given a fixed denotation in all models and is a rigid designator.[52] Unfortunately, it

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[48] Ibidem, 172.

[49] Note: for Sher, Definition 1 constitutes a necessary and sufficient condition for logicality. Still, not everyone agrees. Philosophers as William H. Hanson worry that (a) and (b) wrongly accept non-logical notions as logical (see pages 249-251 of [14]). On the other side of the spectrum, Catarina D. Novaes endorses Sher's criteria as an adequate depiction of the general applicability and topic-neutrality of logic (see page 91 of [23]).

[50] Sher, "The Foundational Problem of Logic," 176.

[51] Ibidem, 171, 174.

[52] Sher, *Epistemic Friction,* 281.

is beyond our purposes here to critically discuss criterion (ii). Briefly put, it requires that any logical constant is well-defined so as to ensure that its workings correspond to the formal operator it depicts.[53] For instance, if the quantifier $\exists$ already satisfies clause (i), one could satisfy (ii) by stipulating syntactic and semantic constraints to include $\exists$ in a logic $\mathcal{L}$: first, what a well-defined formula looks like, and second, what its truth conditions are. Clauses (1)-(3) does this for the connectives. The next section treats the case of the quantifiers $\exists$ and $\forall$.

At this point, one should mention the Feferman-McGee criticism of Sher's proposal. According to this criticism, Sher's extensional approach of logicality fails to show how logical constants remain the same constants across arbitrary domains.[54] As McGee argues, Sher's thesis accepts "a logical connective which acts like disjunction when the size of the domain is an even successor cardinal, like conjunction when the size of the domain is an odd successor cardinal, and like a biconditional at limits".[55] Although it goes beyond our purposes to critically discuss the Feferman-McGee criticism, suffice it to say that (1) it is up for debate whether quantifiers may never change intension between domains, and (2) formality itself only gives a necessary but not a sufficient condition for logicality, as indicated by clause (ii).[56] Moreover, Sher's structuralism defines identity up to numerical structure.[57] In view of this, the Feferman-McGee criticism may be seen as an external criticism of Sher's structuralist view.[58]

1.2.2 The Formality Criterion

In the previous section, we discussed under what conditions connectives may be called formal. In this section, we assess the case of quantifiers: a proper subclass of what Sher calls "operators". In turn, operators will be seen to denote higher-order properties. Contrasting canonical model-theoretic approaches, Sher's structuralism is "objectual": it is concerned primarily with structures and explicitly disregards linguistic elements such as a language or a signature. For the sake of clarity, this stance tallies with the structuralist path that we have taken in section 1.1.2. As also mentioned in that section, Sher's mathematical precisification of logical structuralism draws on ZFC as a background theory of structure. In particular, we will see that set-theoretic cardinality properties plays a significant role in the case of operators.[59] In what follows, we show when operators can be properly called formal, building on ideas of set-theoretic structures and structure-preserving functions.

In the literature, formal properties are made precise in at least two ways: first, via mathematical definability, and second, through the notion of invari-

[53] Gila Sher, *The Bounds of Logic. A Generalized Viewpoint* (Cambridge: The MIT Press, 2001), 64-65.

[54] Solomon Feferman, "Logic, Logics, and Logicism," *Notre Dame Journal of Formal Logic* 40, 1 (1999): 38.

[55] Vann McGee, "Logical Operations," *Journal of Philosophical Logic* 25, 6 (1996): 577.

[56] Sher, *Epistemic Friction,* 312-313.

[57] Sher, "The Foundational Problem of Logic," 188.

[58] Ibidem, 171.

[59] Ibidem, 175, 171.

ance.[60] On the one hand, the definability account classifies properties as formal iff they are definable within a mathematical language.[61] On the other hand, the invariance account holds that properties are formal iff they remain invariant under isomorphic replacements.[62] As the following quote illustrates, Sher adheres to the latter account:

> [a]n operator is formal iff it is invariant under all isomorphisms of its argument-structures.[63]

From now on, we will refer to this requirement as "the formality criterion". Where mathematical structuralism studies mathematical objects via background structures, formality of operators can be studied via structures equally well. Specifically, operator formality is decided via numerical structure: formal operators only attend to numerical structure — the specific properties of objects in that structure play no role. Since formal operators only attend to numerical structure, it will follow that an operator is formal iff any isomorphic replacement of structure has no influence on the operator's truth value ascription in the initial structure.[64] To clarify these ideas, we first need to be familiar with the technical notions "argument-structure" and "isomorphic invariance".

Formally, a structure or "argument-structure" is a set-theoretic n-tuple consisting of a domain A and a set of individuals or constructs of individuals within A — for example, tuples, subsets or relations. For explaining the notion of invariance for $\exists$ and $\forall$, we restrict ourselves to 2-tuple structures $\langle A, B\rangle$, with B a subset of A. The idea is that A models a universe of individuals or objects of the world. In turn, B depicts a monadic first-order property so that each individual in A has the property iff it is in B's extension. In this sense, one can see that argument-structures depict predicational structure.[65]

Conceptually, specifying whether an object $a \in A$ has a property B boils down to giving a yes or no answer. In line with this idea, Sher endorses Andrzej Mostowski's generalization of operators as functions that assign truth values to subsets of domains.[66] Formally, given an argument-structure $\langle A, B\rangle$, each 1-place operator Ω is associated with a function $\Omega_A : \wp(A) \to \{0, 1\}$ that takes a subset B of A and outputs a truth value relative to A, denoted as "Ω_A". Given that operators take B as their input, operators may be seen as properties *of* properties: an operator models a second-order property of a given first-order property depicted by B. In turn, formal operators model second-order properties of a more "general" or "formal" type.[67]

For the semantic clauses of operators, numerical structure plays a significant role: given an operator Ω, a domain A and an associated function $\Omega_A : \wp(A) \to$

[60] Johannes Korbmacher and Georg Schiemer, "What Are Structural Properties?" *Philosophia Mathematica* 26, 3 (2017): 296.

[61] Korbmacher and Schiemer, "What Are Structural Properties?" 310.

[62] Ibidem, 304, 321.

[63] Sher, "The Foundational Problem of Logic," 176.

[64] Ibidem, 188.

[65] Sher, *Epistemic Friction,* 165, 277, 285.

[66] Sher, *Bounds of Logic,* 11.

[67] Sher, "The Foundational Problem of Logic," 174-176.

$\{0, 1\}$, the function definition of Ω_A determines for any subset $B \subseteq A$ under which cardinality of B it holds that $\Omega_A(B) = 1$.[68] Given that a set's cardinality is denoted with $|.|$, the truth clauses for $\exists$ and $\forall$ are as follows:

$$\exists_A(B) = 1 \text{ iff } B \neq \varnothing \text{ iff } 0 < |B| \tag{4}$$

$$\forall_A(B) = 1 \text{ iff } B = A \text{ only if } |B| = |A| \tag{5}$$

As an example, suppose we define an argument-structure $\langle A, B \rangle$ where A models the set of all featherless bipeds and B the set of all humans. Keeping close to reality, suppose we define $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_m\}$ with $1 \leq m < n$. Using clause (1) and noting that $0 < |B|$, we see that, with respect to A, $\exists_A(B) = 1$. On the other hand, since $m < n$ implies that $|A| \neq |B|$, it follows that $\forall_A(B) = 0$.[69]

To recall, formal operators are supposed to model formal properties: those properties that persist when one "disregard[s] the particular characteristics of objects".[70] Mathematically, this intuition is specified via the notion of isomorphic invariance:

> **Definition 2.** *Let $\langle A, B \rangle$ and $\langle A', B' \rangle$ be two argument-structures. Then, $\langle A, B \rangle$ and $\langle A', B' \rangle$ are isomorphic iff there exists a bijective function $f : A \to A'$ such that for all $a \in A$, $a \in B$ iff $f(a) \in B'$.*

The core of isomorphic invariance is that operators only attend to formal, numerical properties of structures instead of particular first-order properties modelled therein. To see this, note that f being bijective involves that $|A| = |A'|$ and that $a \in B$ iff $f(a) \in B'$ for all $a \in A$ implies that $|B| = |B'|$. Conjointly, these two conditions secure preservation of numerical structure across argument-structures.[71]

Using Definition 2, we can redefine the formality criterion as follows:

> **Definition 3.** *Let Ω be an operator and $\langle A, B \rangle$ an argument-structure. Then, Ω is formal iff for any argument-structure $\langle A', B' \rangle$ isomorphic to $\langle A, B \rangle$, $\Omega_A(B) = \Omega_{A'}(B')$ for the associated functions Ω_A and $\Omega_{A'}$.*

Simply put, Definition 3 states that an operator is formal if and only if it assigns the same truth value to an argument-structure as to another argument-structure reached through isomorphic replacement.[72]

We illustrate Definition 3 with $\forall_A$. Let $\langle A, B \rangle$ and $\langle A', B' \rangle$ be argument-structures and $f : A \to A'$ a bijective function such that for all $a \in A$, $a \in B$ iff

[68] Sher, *Bounds of Logic,* 11.
[69] Sher, *Epistemic Friction,* 285.
[70] Ibidem, 278.
[71] Sher, "The Foundational Problem of Logic," 176.
[72] Ibidem, 176.

$f(a) \in B'$. Suppose, furthermore, that $\forall_A(B) = 1$. To prove that $\forall_{A'}(B') = 1$, we use clause (5) and show that $B' = A'$. Since $B' \subseteq A'$ is already given, we only need to show $A' \subseteq B'$. Let $a' \in A'$. By f's surjectivity, there must then exist an $a \in A$ such that $f(a) = a'$. Since $A \subseteq B$, $a \in A$ implies that $a \in B$. By Definition 2, we see that $a \in B$ iff $f(a) \in B'$ and so $a' \in B'$. Conjoining $A' \subseteq B'$ and $B' \subseteq A'$, we see that $A' = B'$ and so $\forall_{A'}(B') = 1$. Alternatively, suppose that $\forall_A(B) = 0$. By clause (5), this means that $|B| \neq |A|$. By Definition 2, we know that $|A| = |A'|$ and that $|B| = |B'|$. By simple substitution, we see that $|B'| \neq |A'|$. Applying clause (5) again, it follows that $\forall_{A'}(B') = 0$.[73]

Although similar arguments can be constructed for the existential quantifier, the main point was to illustrate what the formality criterion requires and how it underpins the intuition that formal quantifiers assign truth values using numerical structure. From the perspective of quantifiers, identity of individuals "identity-up-to-isomorphism": only numerical structure matters.[74] Where connectives depict membership of domain individuals in first-order properties, quantifiers depict second-order properties of those properties. For example, $\neg B(a)$ means that a is not a member of the first-order property B. On the other hand, $\exists_A(B) = 1$ means that the second-order property of non-emptiness holds for the property B.[75]

Obviously, the formality criterion does not hold for non-formal operators. As an illustration, let $\mathcal{R}$ be an operator associated with the following function:

$$\mathcal{R}_A(B) = \begin{cases} 1 & \text{if all } \beta \in B \text{ are red} \\ 0 & \text{else} \end{cases}$$

To see that $\mathcal{R}$ is not formal, let $\langle A, B \rangle$ and $\langle A', B' \rangle$ be two argument-structures defined as follows: $A := \{\alpha, \beta\}$, $B := \{\alpha, \beta\}$, $A' := \{\gamma, \delta\}$, $B' := \{\gamma, \delta\}$, such that all individuals in A are red and all individuals in A' are green. Given Definition 2, there exists an isomorphism between $\langle A, B \rangle$ and $\langle A', B' \rangle$, e.g., $f : A \to A'$ defined by the function graph $\mathcal{F} = \{\langle \alpha, \gamma \rangle, \langle \beta, \delta \rangle\}$. Still, $\mathcal{R}$ does not preserve truth across the given argument-structures: $\mathcal{R}_A(B) = 1$ whereas $\mathcal{R}_{A'}(B') = 0$. In sum, $\mathcal{R}$ is a non-formal operator: it attends not to formal but to specific properties of objects.[76]

1.2.3 The Veridicality Criterion

According to Sher's veridicality criterion, logical consequence must be grounded in the world's structure:

> [A] correct theory of consequence must respect the connection between the conditions that have to hold in the world for the premise-sentences to be true and those that have to hold for the conclusion-sentence to be true.

[73] Ibidem, 176.
[74] Ibidem, 188.
[75] Ibidem, 171.
[76] Ibidem, 176.

On the one hand, logical consequence is *enabled* by the world's structure: permitted consequence relations hold only by virtue of that structure. On the other hand, it is *restricted* by the world: no consequence relation should accept consequences contravening the world's structure.[77] In Sher's terms, logical consequence ought to respect the "friction of the world".[78]

Structure is, as said, specified via ZFC: connectives are defined in terms of set-theoretic operations and quantifiers in terms of functions from subsets of domains to truth values. In turn, logical consequence aims to depict the formal laws which hold between the formal properties modelled by logical constants. In general, we can say that a consequence pair $\langle\Gamma,\sigma\rangle$ is correct iff it depicts a formal law in ZFC or, equivalently, that it is *provable* in ZFC given the view of properties as sets. For example, suppose that $\forall_A(B) = 1$. By definition, this means that $|A| = |B|$ and, since A is non-empty, $\exists_A(B) = 1$. In this particular case, the formal law required for $\langle\forall_A(B), \exists_A(B)\rangle$ is that equality of a subset with its non-empty domain necessitates non-emptiness of the subset.[79] In other words, the truth conditions of the premises facilitate the truth conditions of the conclusion.[80] In what follows, we explain in more detail what Sher means by "condition", and how this relates to logical consequence and formal properties.[81]

A condition may be defined as the situation required for a sentence to hold true. For example, "John exists" is true only if there exists an individual with the name "John" in the relevant domain of discourse.[82] Bearing that in mind, we can say that a sentence follows from another if satisfying the truth conditions of the premises means satisfying the truth conditions of the conclusion. For example, the conditions under which the sentence "John is a liar" holds involve that "John is a liar or John is a mammal", since John being a liar involves that he is a liar or whatnot.

With these notions in mind, we can provide a formal definition of the veridicality criterion:

> **Definition 4.** *Let $\mathcal{S}_\Gamma$ and $\mathcal{S}_\sigma$ stand for situations modeled by formulas Γ and σ, respectively. Then, σ is a logical consequence of Γ only if $\mathcal{S}_\sigma$ is the case whenever each $\mathcal{S}_{\gamma_1}, ..., \mathcal{S}_{\gamma_n} \in \mathcal{S}_\Gamma$ is the case.*[83]

Put differently, logical consequence between two formulas Γ and σ holds only if there exists a formal law between $\mathcal{S}_\Gamma$ and $\mathcal{S}_\sigma$ — that is, satisfying $\mathcal{S}_\Gamma$ means satisfying $\mathcal{S}_\sigma$. If this does not hold, then σ does not follow from Γ.

Notice that Definition 4 only accepts those consequences which model formal laws. Arguably, this tallies with the quote from the beginning of this section: logical consequence ought to match the world's structure.[84] Since the world's

[77] Sher, *Epistemic Friction*, 266.
[78] Ibidem, viii.
[79] Sher, "The Foundational Problem of Logic," 169.
[80] Sher, *Epistemic Friction,* 277.
[81] Ibidem, 264.
[82] Sher, "The Foundational Problem of Logic," 164.
[83] Sher, *Epistemic Friction,* 264.
[84] Sher, "The Foundational Problem of Logic," 163.

structure is specified by ZFC, we can say that any consequence from Γ to σ holds iff it is provable in ZFC using argument-structures, quantifiers and the connective/set-theoretic operator correspondence. In other words, the condition under which ZFC makes Γ true should facilitate σ's truth.[85]

To appreciate this idea, consider the following example. Let $\mathcal{C}_{\text{ZFC}}$ be the consequence relation based on Sher's definitions, A a domain and B, C and D subsets of A. Now, consider the consequence pair $\langle\{\exists_A(B \vee (C \wedge D))\}, \{\exists_A(B \vee C)\}\rangle$. Is this pair included in $\mathcal{C}_{\text{ZFC}}$? By the truth conditions from clause (4) and equivalence between $\vee$ and $\cup$ and $\wedge$ and $\cap$, we see that the antecedent holds iff $B \cup (C \cap D) \neq \varnothing$. From this, it follows that $B \neq \varnothing$ or $C \cap D \neq \varnothing$. Consequently, it holds that $B \cup C \neq \varnothing$. By correspondence of $\vee$ and $\cup$ and by clause (4), we see that $\exists_A(B \vee C)$. Therefore, $\langle\{\exists_A(B \vee (C \wedge D))\}, \{\exists_A(B \vee C)\}\rangle \in \mathcal{C}_{\text{ZFC}}$.

To illustrate the relation with the world, one might think of the domain as a set of worldly objects. Suppose we define $B :=$ "is human", $C :=$ "is an animal" and $D :=$ "has no feathers". Clearly, any situation in which either the class of humans is non-empty or the class of animals and entities without feathers is non-empty implies that the class of humans is non-empty or that the class of animals is non-empty. From a set-theoretical point of view, if the extension of B or $C \cap D$ is non-empty, then it follows that B or C's extension is non-empty as well.[86]

Two things should be noted. Firstly, consequence holds by virtue of formal or structural connections: the specific configuration of argument-structures required for the antecendent's truth implies the specific configuration of argument-structures needed for the truth of the consequent. In this regard, only the configuration matters — not what the antecedent and consequent stand for. Furthermore, formal properties facilitate truth-preservation in a necessary way: each consequence depicts a law which cannot be contradicted. For example, one cannot have that $\exists_A(B \vee (C \wedge D))$ but not $\exists_A(B \vee C)$.[87]

To summarize, the core idea of Sher's structuralism is that logical consequence depicts formal laws. Where formal properties produce formal laws, logical consequence models those laws by defining logical constants in terms of formal properties. Since any correct consequence relation consists of only correct consequence pairs, we can close off by saying that a correct consequence relation consists of only those consequence pairs that model formal laws.[88] Put differently, if $\mathcal{C}_{\text{ZFC}}$ is the correct consequence relation in Sher's structuralist picture, then any consequence pair $\langle\Gamma, \sigma\rangle \in \mathcal{C}_{\text{ZFC}}$ must depict a formal law of ZFC: it must be the case that $\Gamma \vdash_{\text{ZFC}} \sigma$. If it does not model a formal law, then it follows that $\langle\Gamma, \sigma\rangle \notin \mathcal{C}_{\text{ZFC}}$.[89]

[85] Sher, *Epistemic Friction,* 273-274.
[86] Sher, "The Foundational Problem of Logic," 166.
[87] Ibidem, 166.
[88] Ibidem, 165.
[89] Sher, *Epistemic Friction,* 277.

2 Logical Pluralism and The Pluralist Challenge

2.1 What is Logical Pluralism?

2.1.1 Core Principles and Motivations

To be sure, "logical pluralism" is an ambiguous term. Fittingly, this section and the next are devoted to disambiguating and precisifying our employed sense of the term using Beall and Restall's definition. The latter half of this chapter specifies the tension between Sher's structuralism and the defined sense of logical pluralism. Commonly, logical pluralism is contrasted to "logical monism": the view that there exists only one correct logic. Conversely, logical pluralism asserts that there exist multiple logics, each correct with respect to some background condition.[90] Commonly noted candidates of such conditions are language, contexts or, in Beall and Restall's pluralism, cases.[91] Since our discussion is about logical consequence, these observations apply specifically to the consequence relation. Accordingly, we treat logical pluralism as the view that there are multiple correct precisifications of consequence or, more precisely, multiple correct consequence relations with respect to different background conditions.[92]

Why should one endorse logical pluralism? A first motivation is the prematureness of logical monism. In this view, one should not bluntly assume that there exists an overarching consequence relation unless it is shown to exist. Since pluralism fits with the factual existence of competing consequence relations, one ought to remain pluralist unless a plausible overarching theory becomes available.[93] A second motivation is scientific virtue. Pluralism makes possible consistent application and study of theories that would contradict if one adheres to monism.[94] For example, monists cannot consistently endorse both classical infinitesimal analysis and smooth infinitesimal analysis, since the former assumes but the latter rejects the law of the excluded middle.[95] Lastly, there is the motivation that logic is not topic-neutral, but *topic-biased*: different topics require different logics.[96] For example, classical logic is biased towards topics without vagueness. Accordingly, one might suppose that topics with vagueness require non-classical logic, notwithstanding that classical logic is still acceptable for non-vague topics.[97]

[90] Note: by "background condition", we mean the same as Shapiro does when he says that pluralist validity of an argument, the dependent variable Y, is defined against something else, an independent variable X (see page 310 of [34]).

[91] Cook, *A Dictionary of Philosophical Logic,* 178.

[92] Estrada-González and Cohnitz, *An Introduction to the Philosophy of Logic,* 163-164.

[93] Shapiro, "Structures and Logics," 309, 311, 328.

[94] Beall and Restall, *Logical Pluralism,* 30-31.

[95] Shapiro, "Structures and Logics," 323, 325.

[96] Priest, "Revising Logic," 217.

[97] Rosanna Keefe, "Pluralisms: Logic, Truth and Domain-Specificity," in *Pluralisms in Truth and Logic,* ed. Jeremy Wyatt et al. (Cham, Switzerland: Palgrave MacMillan, 2018), 443.

2.1.2 Beall and Restall's Logical Pluralism

A much discussed and insightful definition of pluralism about logical consequence is set forth by Beall and Restall in their book *Logical Pluralism.* Importantly, Beall and Restall's definition is not about a specific consequence relation of a specific logic. Instead, their account is a *metalogical* depiction of consequence across logics.[98] Referred to as the Generalised Tarski Thesis (GTT), Beall and Restall's definition is as follows:

> (GTT) An argument is valid$_x$ if and only if in every case$_x$ in which the premises are true, the conclusion must be true.

What, exactly, is a "case$_x$"? Bluntly put: it depends. Although Beall and Restall do not state necessary and sufficient conditions for something being a case, we can roughly define cases as "things in which claims are true". The idea is that each specification of "case$_x$" gives rise to a notion of validity. To put it differently, specifying the term "case$_x$" produces a consequence relation which holds with respect to the class of cases bounded by the variable x. All arguments contained in that consequence relation are, thus, valid with respect to that class.[99]

For each of these classes of cases, one can assess whether a formula σ is always true if all $\gamma \in \Gamma$ are. If so, then one is permitted to say that the consequence $\Gamma \models_x \sigma$ holds with respect to all cases bounded by x.[100] Observe, however, that an argument being valid$_x$ implies only that its validity holds for those cases bounded by x. Hence, it is not guaranteed that the argument hold with respect to some case$_y$ which falls outside of the scope of x. In other words, an argument's validity is not an absolute property, but one relative to a specified class of cases.[101]

In Beall and Restall's view, the Generalised Tarski Thesis is the settled core of consequence: any specific consequence relation ought to be an admissible instance of the GTT. In particular, a consequence relation is admissible only if it is formal, necessary and normative.[102] As noted in the introduction, formality and necessity require that (1) an argument's conclusion follows from the premises only by the argument's form or structure and (2) for no valid argument, we can have true premises without having a true conclusion. Additionally, normativity requires that one cannot rationally accept the premises and reject the conclusion of an argument validated by an admissible consequence relation. Bluntly put, violating its norms is irrational: one would be incorrect to do so. Within Beall and Restall's framework, formality and necessity must thus hold for any admissible consequence relation instantiated by the GTT.[103]

[98] Sider, *Writing the Book of the World,* 227.
[99] Beall and Restall, *Logical Pluralism,* 89-90.
[100] Ibidem, 36-37, 49, 62.
[101] Steward Shapiro. "Pluralism, Relativism, and Objectivity," in: *The Metaphysics of Logic,* ed. Penelope Rush (Cambridge: Cambridge University Press, 2014), 50-51.
[102] Beall and Restall, *Logical Pluralism,* 35.
[103] Ibidem, 16.

Pluralism enters the game if we ask whether there exist multiple admissible instances of the GTT. Unsurprisingly, Beall and Restall answer in the affirmative: different admissible classes of cases allow for different admissible consequence relations. Given the stated arguments for pluralism, one might contend that interpreting "case$_x$" as a Tarskian model, possible world or proof stage are equally correct — whether one uses the argument from monism, scientific virtue or topic-biasedness. Since their associated consequence relations are moreover formal, necessary and normative, one ought to say, then, that these consequence relations are all equally acceptable.[104]

Consider the consequence pair $\langle \varnothing, \{\varphi \vee \neg\varphi\}\rangle$. Suppose we interpret "case$_x$" as a Tarskian model $\mathcal{M}$. Since each model $\mathcal{M}$ either validates a given formula or its negation, we see that $V_{\mathcal{M},g}(\varphi \vee \neg\varphi) = 1$ for all models $\mathcal{M}$ and variable assignments g. Hence, $\langle \varnothing, \{\varphi \vee \neg\varphi\}\rangle$ is included in the consequence relation $\mathcal{C}_{\mathcal{M}}$. By contrast, interpreting "case$_x$" as a proof stage s yields that $\langle \varnothing, \{\varphi \vee \neg\varphi\}\rangle \notin \mathcal{C}_s$. To see this, note that $\langle \varnothing, \{\varphi \vee \neg\varphi\}\rangle \in \mathcal{C}_s$ requires that all proof stages warrant φ or $\neg\varphi$. Since proof stages allow for incomplete information, one can construct a proof stage s such that neither φ nor $\neg\varphi$ holds in s.[105]

To summarize, a consequence relation is obtained by specifying the term "case$_x$" in the Generalised Tarski Thesis. As we saw, Beall and Restall's main tenet is that there exist at least two admissible specifications of "case$_x$". Therefore, there exist at least two correct consequence relation — "correct" with respect to their specified background condition, that is. Importantly, asking which is the correct consequence relation is a misguided question: each admissible specification of "case$_x$" produces a consequence relation correct with respect to that specification. Since there are multiple correct consequence relations, there exists no overarching correct consequence relation which may be said to capture consequence for all types of cases.[106]

[104] Ibidem, 29-31.
[105] Ibidem, *Logical Pluralism,* 62, 66.
[106] Ibidem, 29.

2.2 Specification of the Problem

2.2.1 Two Senses of Pluralism

In this section, we connect the defined sense of logical pluralism to Sher's structuralism and specify the core issue of this thesis. As we noted, a very important corollary of logical pluralism is that there exists no generally correct consequence relation: any consequence relation is correct with respect to a restricted class of cases. However, this correctness does not involve validation over and above these cases. To recall, Sher's framework seemed to imply that there exists a single correct consequence relation: namely, the one including all consequence pairs $\langle \Gamma, \sigma \rangle$ such that there exists a formal law between the situation $\mathcal{S}_\Gamma$ depicted by Γ and the situation $\mathcal{S}_\sigma$ depicted by σ. Still, there does not yet seem to be an obvious tension between pluralism and Sher's structuralism. As Gillian Russell remarks, existence of multiple *views* on the correct consequence relation need not imply that there *exists* more than one correct consequence relation.[107]

To articulate the tension, we will instead start from the question whether Sher's structuralism allows for pluralism. Given the Generalised Tarski Thesis, this requires that there exist multiple correct specifications of "case$_x$". Our sense of pluralism can be compared with Theodore Sider's distinction between a weak and strong interpretation of Beall and Restall's pluralism. According to the weak variant, there exist multiple correct precisifications of the *concept* of logical consequence. By contrast, the strong variant contends that there exist multiple correct consequence relations *true to the world's structure*: none is "metaphysically privileged" above the other. The strain of pluralism we have in mind is best described by the strong variant.[108] However, this sort of pluralism seems implausible from a structuralist point of view: any consequence relation should depict formal laws in the world's structure. Since that structure is the only legitimate interpretation of "case$_x$", it seems to follows that there can only be one correct consequence relation — namely, the one depicting the world's formal laws.

2.2.2 The Problem: One Correct Consequence Relation?

Let us go through the argument step by step. Given the definition of logical pluralism from section 2.2.1, we see that pluralism within structuralism implies that there exist, minimally, two different consequence relations which both depict the world's formal laws — call them $\mathcal{C}_1$ and $\mathcal{C}_2$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are different, we have $\mathcal{C}_1 \neq \mathcal{C}_2$. Without loss of generality, this implies that there exists, minimally, one correct consequence pair $\langle \Gamma, \sigma \rangle$ included in the union $\mathcal{C}_1 \cup \mathcal{C}_2$ but excluded from the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$. For if $\langle \Gamma, \sigma \rangle \in \mathcal{C}_1 \cap \mathcal{C}_2$, we would contradict the assumption that $\mathcal{C}_1 \neq \mathcal{C}_2$: after all, $\langle \Gamma, \sigma \rangle$ is supposed to constitute the difference between $\mathcal{C}_1$ and $\mathcal{C}_2$.

In Sher's view, any consequence from Γ to σ holds iff the argument-structures required for the truth of all $\gamma \in \Gamma$ facilitate σ's truth. A correct consequence

[107] Gillian Russell, "One True Logic?" *Journal of Philosophical Logic* 37, 6 (2008): 593-594.
[108] Sider, *Writing the Book of the World,* 268-269.

relation, then, captures the formal laws which hold with respect to the world's structure as modelled via argument-structures. However, this seems to imply that there exists only one correct consequence relation: namely, the one consisting of all pairs $\langle \Gamma, \sigma \rangle$ where the argument-structures required for Γ's truth involve σ's truth. For if there would exist a correct consequence pair $\langle \Gamma, \sigma \rangle$ such that $\langle \Gamma, \sigma \rangle \in \mathcal{C}_1 \cup \mathcal{C}_2$ and $\langle \Gamma, \sigma \rangle \notin \mathcal{C}_1 \cap \mathcal{C}_2$, then it seems that either of $\mathcal{C}_1$ or $\mathcal{C}_2$ just has not sufficiently captured the world's formal laws. Supposing that $\mathcal{C}_1$ and $\mathcal{C}_2$ agree on all other consequence pairs, we have $\mathcal{C}_1 \subseteq \mathcal{C}_2$ or $\mathcal{C}_2 \subseteq \mathcal{C}_1$.[109] Arguably, this would just involve that one of $\mathcal{C}_1$ and $\mathcal{C}_2$ better captures the world's formal laws and is thus more correct — not that pluralism is true.

In sum, Sher's structuralism seems to be incompatible with the employed sense of logical pluralism. At first glance, the disagreement might seem to stem from equivocation: the term "correct" means something different for the structuralist than it does for Beall and Restall's pluralism. However, this is mistaken: our question is not whether logical structuralism and pluralism are consistent, but whether pluralism can be included *within* Sher's view. The problem stemming from this question is more than just a semantic dispute. Rather, the disagreement appears to boil down to the question whether the world's structure allows for multiple consequence relations or not.

[109] Note: this assumption is acceptable since (1) consequence relation should be, intuitively, non-empty and (2) the symmetric difference $\mathcal{C}_1 \Delta \mathcal{C}_2$ was defined as the set $\{\langle \Gamma, \sigma \rangle\}$. Given (1), the set without $\langle \Gamma, \sigma \rangle$ should contain at least one different consequence pair — call it $\langle \Gamma^+, \sigma^+ \rangle$. By (2), we have $\langle \Gamma^+, \sigma^+ \rangle \in \mathcal{C}_1 \cap \mathcal{C}_2$. Reiterating this reasoning, we see that $\mathcal{C}_1$ and $\mathcal{C}_2$ should agree on all consequence pairs different from $\langle \Gamma, \sigma \rangle$, given that $\mathcal{C}_1 \Delta \mathcal{C}_2 = \{\langle \Gamma, \sigma \rangle\}$.

3 Resolving the Problem

3.1 Further Specifications

If the dispute between the structuralism and pluralism boils down to a dispute about whether the world's structure allows for multiple consequence relations, then we should be clear about the relevant sense of "the world". To recall, the term "world" covers the objects of knowledge advanced within a system of knowledge. Everything within knowledge's scope is part of the world. Still, the perimeters of knowledge do not coincide with those of the world : "world" is an open-ended concept, allowing for change in extension whenever the perimeters of knowledge shift. In short, "the world" consists of the ontological commitments made by a system of knowledge.[110] Bearing the above in mind, our problem may be phrased as follows: does Sher's structuralism allow for multiple correct consequence relations with respect to the world's formal laws as depicted in a system of knowledge?

In what follows, we argue for the affirmative. First, two candidate solutions are evaluated: the intersection proposal and the logic-as-modeling proposal. Although these solutions turn out to be inadequate, aspects of them will recur in our twofold proposal: logical pluralism can be attained (1) via pluralism about structure and (2) by changing one's interpretation of a given structure. More specifically, we argue that different structures or interpretations are acceptable if these accurately depict the structure as given in the relevant branches of knowledge. In turn, pluralism follows if these different structures or interpretations produce different consequence relations. To illustrate the point, we apply the argument to a case where both the structure and the interpretation are changed — Thierry Libert's paraconsistent set theory — and one where only the interpretation is changed — Saul Kripke's intuitionistic modal logic.

[110] Sher, *Epistemic Friction,* 10.

3.2 Two Proposals

3.2.1 The Intersection Proposal

Suppose that $\mathcal{C}_1, ..., \mathcal{C}_n$ denote all n correct consequence relations, where $2 \leq n$. According to the intersection proposal, we can retrieve the one and only correct consequence relation $\mathcal{O}$ by defining $\mathcal{O}$ as the intersection $\mathcal{C}_1 \cap ... \cap \mathcal{C}_n$. Conjoining all consequence relations, this proposal holds, resolves the presumed tension: the one and only correct consequence relation just consists of those pairs contained in all consequence relations endorsed by the pluralist.[111] However, there are a number of problems with this proposal. First, notice this trivial one: the proposal goes awry if $\mathcal{C}_1 \cap ... \cap \mathcal{C}_n = \varnothing$. Plainly, a set without consequence pairs is no consequence relation at all: this is just the unattractive position of logical nihilism, the view that there exists no correct consequence relation.[112]

Furthermore, the proposal appears to contradict the pluralist intuition that at least some consequences are valid in different classes of cases. For example, excluded middle holds primarily in non-vague cases but fails when vaguities enter. Nonetheless, this does not seem to imply that excluded middle is therefore *completely* invalid. In other words, the intersection proposal just seems to do away with pluralism rather than including it within Sher's framework.[113]

More specifically, $\mathcal{O}$ is always less specific than its conjuncts $\mathcal{C}_1, ..., \mathcal{C}_n$ individually: it fails to capture at least part of the types of cases of which $\mathcal{C}_1, ..., \mathcal{C}_n$ are consequence relations. In the best case, $\mathcal{O}$ is only slightly different from its conjuncts. Suppose, for the sake of argument, that the difference $\mathcal{C}_1 \setminus \mathcal{C}_2$ is one pair. Let $\langle \Gamma, \sigma \rangle$ be that pair. Since $\mathcal{C}_1 \setminus \mathcal{C}_2 = \{\langle \Gamma, \sigma \rangle\}$, we know that $\langle \Gamma, \sigma \rangle \notin \mathcal{C}_1 \cap \mathcal{C}_2$ and so it follows that $\langle \Gamma, \sigma \rangle \notin \mathcal{O}$. Clearly, this means that $\mathcal{O}$ fails to capture at least part of the consequence relation endorsed by either $\mathcal{C}_1$ or $\mathcal{C}_2$. Without loss of generality, suppose that $\langle \Gamma, \sigma \rangle \in \mathcal{C}_1$. Then, $\mathcal{O}$ is less equipped to grasp truth-preservation as it holds with respect to the type of cases $\mathcal{C}_1$ is supposed to model: indeed, $\mathcal{C}_1$ seems better equipped to do that.

As Beall and Restall note, the intersection proposal worsens if more difference consequence relations enter the game. As a rule of thumb, the more different consequence relations there are, the more minimal their intersection is.[114] For any $2 \leq n$ and two different consequence relations $\mathcal{C}_i, \mathcal{C}_j$, it follows that $\mathcal{C}_i, \mathcal{C}_j \in \mathcal{C}_1, ..., \mathcal{C}_n$ must be such that $\mathcal{C}_i \neq \mathcal{C}_j$. Since each $\mathcal{C}_i$ is supposed to differ from each $\mathcal{C}_j$ by at least one consequence pair, $\mathcal{O}$ will be less and less equipped to capture truth-preservation in the divergent types of cases modelled by $\mathcal{C}_1, ..., \mathcal{C}_n$ as n gets bigger.[115] Furthermore, as Rosanna Keefe notes, it is not

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[111] Beall and Restall, *Logical Pluralism,* 92.

[112] Gillian Russell, "Varieties of Logical Consequence by Their Resistance to Logical Nihilism," in *Pluralism in Truth and Logic*, ed. N.J.L.L. Pedersen et al. (Cham, Switzerland: Palgrave MacMillan, 2019), 340-341.

[113] Beall and Restall, *Logical Pluralism,* 92.

[114] Ibidem, 92.

[115] Note: to see this, one could use proof by induction. If $n = 2$, we see that, minimally, $\mathcal{C}_1 \Delta \mathcal{C}_2 = \{\langle \Gamma, \sigma \rangle\}$. Thus, the intersection $\mathcal{O} := \mathcal{C}_1 \cap \mathcal{C}_2$ must exclude at least one consequence pair. For each $2 \leq n+1$, one can see that the same reasoning holds and that $\mathcal{C}_{n+1}$ thus minimally differs by one pair $\langle \Gamma, \sigma \rangle$ from each $\mathcal{C}_i \in \mathcal{C}_1, ..., \mathcal{C}_n$. Therefore, $\mathcal{O} := \mathcal{C}_1, ..., \mathcal{C}_n$ will

obvious that the consequence relation of the one logic is contained within the other. Therefore, the difference between consequence relations might be significantly worse than just one consequence pair, leading to an even more minimal consequence relation than we imagined.[116]

3.2.2 The Logic-As-Modeling Proposal

In Roy T. Cook's logic-as-modeling view, logic is not a perfect representation of natural language. Instead, it is but a *model*, and as such a simplification and idealization of natural language. No perfect model exists; each logic only models a *segment* of language. Multiple correct logics can exist if all these logics are correct with respect to their modelling goals. In Cook's words, different logics can be "correct relative to different theoretical goals, or relative to different ways of simplifying, idealizing, or precisifying the phenomena in question".[117] Turning this into a view about logical consequence, one can say that multiple correct consequence relations can exist, given that each such relation captures its relevant segment of natural language. Importantly, correctness applies only insofar as the consequence relation is an accurate model of the relevant segment.

Arguably, the logic-as-modeling view has something going for it. For one thing, it is backed by the intuition that consequence relations are not inerrant but, as models, prone to limitations.[118] Moreover, the view restrains overconfidence concerning the scope of logical consequence: for a consequence relation to count as correct, we first need to see whether it sufficiently accomplishes the relevant modelling task. Still, the prognosis that logical pluralism arises merely from ambiguity in natural language has its problems. If ambiguity is the sole ground for pluralism, then monism is not at all exempted: resolving ambiguity might still facilitate the one true consequence relation and exclude pluralism, a conclusion that Cook happily endorses.[119]

In particular, it is not sufficiently clear how ambiguity in natural language resolves the veridical conflict between structuralism and logical pluralism. The fact that natural language ambiguity leaves it *conceptually* indeterminate which consequence relation is correct does not show that there are multiple correct consequence relations *true to the world's structure*. That is, it is left unspecified whether the different consequence relations in the defined sense are correct with respect to the world's formal laws. For all we know, ambiguity in natural language might be the sole basis for pluralism, meaning that full-blown pluralism cannot be included into structuralism.

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exclude at least $n-1$ pairs included in $\mathcal{C}_1, ..., \mathcal{C}_n$ individually.

[116] Keefe, "Pluralisms: Logic, Truth and Domain-Specificity," 433.

[117] Roy T. Cook, "Let a Thousand Flowers Bloom: a Tour of Logical Pluralism," *Philosophy Compass* 5/6, 1 (2010): 500-501.

[118] Cook, "Let a Thousand Flowers Bloom," 500.

[119] Ibidem, 502.

3.3 Background Theories and Pluralism about Structure

3.3.1 Logical Consequence and ZFC

In this section, we argue that Sher's structuralism allows for multiple consequence relations. As stated in the beginning of the chapter, our proposal is twofold: logical pluralism is attainable by endorsing sufficiently different background theories of structure or by accepting sufficiently different interpretations of a given background theory. By "sufficiently different", we just mean that these theories or interpretations, indeed, produce different consequence relations: after all, one could have different theories or interpretations *without* having different consequence relations.

But, as Graham Priest argues, one can consistently endorse different background theories without endorsing logical pluralism. More specifically, pluralism about consequence in structures need not entail pluralism about consequence with respect to the world. After all, structures do not necessarily mimic the world and its laws.[120] Arguably, the holds in our case, too: one can consistently accept pluralism about truth-preservation regarding background theories or interpretations thereof without accepting pluralism about truth-preservation simpliciter.

Still, one *should* accept logical pluralism if the relevant background theories or interpretations thereof accurately depict truth-preservation with respect to the world. As we will see, nothing in the structuralist view commits one to suppose that the world's structure is uniform. Therefore, the structuralist should be happy to endorse pluralism if (1) background theories or interpretations involve different consequence relations and (2) truth-preservation in these theories or interpretations accurately depicts truth-preservation with regard to the world. In what follows, we will develop this argument in more detail.

At the outset, the idea of corresponding truth-preservation with the world's laws is in line with the basic principles of structuralism. In particular, it matches the idea that logical consequence models formal laws of the world, i.e., connections of especially strong modal force between given situations. Still, it is not the case that logical consequence models these laws *directly.* Instead, the world's formal laws are laid down *indirectly* into a background theory of structure — a theory that specifies which properties are formal and which formal laws obtain.[121]

As noted, Sher uses ZFC as a background theory. Firstly, connectives were defined in terms of set-theoretic operations and quantifiers in terms of functions from subsets of domains to truth values. Second, consequences were grounded in formal laws of ZFC: for all consequence pairs $\langle \Gamma, \sigma \rangle$ contained in the consequence relation $\mathcal{C}_{\mathrm{ZFC}}$, the condition under which ZFC makes Γ true ought to facilitate σ's truth.[122] Another way of saying the same thing: all consequences are contained in $\mathcal{C}_{\mathrm{ZFC}}$ iff they are provable in ZFC. Roughly, this comes down

[120] Graham Priest, "A Note on Mathematical Pluralism and Logical Pluralism," *Synthese* 1,1 (2019): 6.
[121] Sher, "The Foundational Problem of Logic," 163, 183.
[122] Sher, *Epistemic Friction,* 273-274.

to $\langle \Gamma, \sigma \rangle \in \mathcal{C}_{\text{ZFC}}$ iff $\Gamma \vdash_{\text{ZFC}} \sigma$, bearing in mind that proofs are constructed against the background of argument-structures. As Sher notes, $\mathcal{C}_{\text{ZFC}}$ contains the theorems of standard first-order logic with a number of extensions.[123]

To illustrate that point, consider the following three examples of consequence pairs contained in $\mathcal{C}_{\text{ZFC}}$: (a) the law of the excluded middle, (b) the principle of existential generalization and (c) the principle of explosion. Let $\langle A, B \rangle$ be an argument-structure. Then,

(a) By $B \subseteq A$, every $a \in A$ is either in B or in B's complement. Given clause (2), $\langle \varnothing, \{B(a) \vee \neg B(a)\} \rangle \in \mathcal{C}_{\text{ZFC}}$.

(b) Suppose that for some $a \in A$, $a \in B$. Then, $0 < |B|$. Using clause (4), we see that $\exists_A(B) = 1$ and so $\langle \{B(a)\}, \{\exists_A(B)\} \rangle \in \mathcal{C}_{\text{ZFC}}$.

(c) For any $a \in A$, $a \in B$ iff not $a \notin B$ and $a \notin B$ iff not $a \in B$. Since $B \cap \overline{B} = \varnothing$, $a \in B$ and $a \notin B$ is impossible. Using clause (1), $\langle \{B(a) \wedge \neg B(a)\}, \{\varphi\} \rangle \in \mathcal{C}_{\text{ZFC}}$, where φ stands for an arbitrary formula.

Unsurprisingly, (a)-(c) hold in $\mathcal{C}_{\text{ZFC}}$ because they are provable in ZFC using the semantics given in clauses (1)-(5). On the one hand, these semantics depend on ZFC as a background theory. On the other hand, an aspect of interpretation is involved. For example, these clauses depend on the idea that each predicate may be interpreted as a set. To put it differently, one needs to define what the elements of a given logic denote in the relevant background theory.[124]

Before deciding whether one could accept multiple background theories or interpretations thereof, we should first take a few steps back and delve deeper into the concept of a "background theory". To recall, Sher's structuralism views logical consequence as a model of the world's formal laws. Note, however, that "modelling the world's formal laws" means nothing but "modelling the world's formal laws *in relation to* a system of knowledge".[125] Therefore, logical consequence is always measured relative to the truths given by a system of knowledge. At its core, this relation therefore contains three important relata: (1) the world's general structure, (2) the background theory and (3) the consequence relation. In general, (2) is supposed to capture (1) and (3) follows, in one way or another, from (2).[126]

At this point, it is worth repeating that a background theory of structure aims to depict the world's predicational structure. By its very definition, then, a background theory models that structure: it aims to capture its structural features. As with any model, however, one should judge any background theory by its "intended target", the thing being modelled. In other words, one has to ask the following question: does the background theory accurately track the world's structure?[127] If it does, then one may say that the theory accurately

[123] Sher, *Epistemic Friction,* 287.
[124] Note: clearly, this is just what clause (ii) of Definition 1 requires. In the semantics outlined in the Preface, the interpretation function $\mathcal{I}$ does just that.
[125] Sher, *Epistemic Friction,* 10.
[126] Ibidem, 284.
[127] Frigg and Nguyen, *Modelling Nature,* 53, 56.

divided structural features from their non-structural counterparts. If not, then one ought to consider choosing another background theory.

But how, then, does a background theory track the world's predicational structure?[128] Importantly, modelling structure requires a distinction between structure and non-structure in that world: it should be clear which properties are structural and which are not. If this were not clear, then it would be unclear what the world's predicational structure is.[129] Arguably, the distinction between structure and non-structure may not be dictated by the background theory itself: its task is to *model* structure, not to decide over substantial questions about structure. Rather, the distinction must be given by the system of knowledge wherein the theory models the world's structure: knowledge ought to stipulate which features are structural and which are not.[130]

If a background theory ought to model the world's structure as given in a system of knowledge, then its correctness stands or falls with its ability of modelling that structure.[131] For example, a background theory which does not model self-identity in a system of knowledge where all objects are self-identical is, simply, a bad theory. But why should one evaluate a background theory's value relative to an *entire* system of knowledge? Plainly, this stance *is* justified if predicational structure is sufficiently homogeneous for all branches of knowledge in that system. However, the more heterogeneous the structure/non-structure distinction across branches of knowledge, the less likely it is that an all-purpose background theory will successfully model the structure of each branch. More specifically, neglecting differences in structure produces a background theory with poor modelling capacities for exactly those differences it neglects.

Suppose, for instance, that n branches of knowledge $\mathcal{B}_1, ..., \mathcal{B}_n$ endorse non-emptiness for the union of any property B and its complement $\overline{B}$ whereas $\mathcal{B}_{n+1}$ accepts occassional emptiness of $B \cup \overline{B}$. Suppose, further, that Sher's sense of ZFC is taken as an all-purpose background theory for $\mathcal{B}_1, ..., \mathcal{B}_{n+1}$. Clearly, this would make ZFC a bad model in the case of $\mathcal{B}_{n+1}$: it would neglect that for some property B, $B \cup \overline{B} \neq \varnothing$.

More generally, then, considerable heterogeneity of structure across branches of knowledge ought to be met with a case by case assessment of structure rather than assuming an all-purpose background theory in advance. As Ole Hjortland notes, making restricted generalizations is a common characteristic of science.[132] Given that background theories model structure, it seems reasonable to suggest that their applicability should be restricted to those branches of knowledge where they are known to function well. Claiming applicability for all branches of knowledge, however, appears to be an unwarranted generalization,

[128] Note: one should not confuse the two relevant senses of the word "structure". First, we have structure as it pertains to the world: its predicational structure. Secondly, there is the notion of mathematical structure, which is explicitly a mathematical theory and, in this context, a model of the world's predicational structure.
[129] Marc Gasser, "Structuralism and its Ontology," *Ergo* 2, 1 (2015): 6-7, 11.
[130] Frigg and Nguyen, *Modelling Nature,* 79.
[131] Gasser, "Structuralism and its Ontology," 6-7, 11.
[132] Ole T. Hjortland, "Anti-Exceptionalism About Logic," *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition* 174, 3 (2017): 642.

and should be replaced by a kind of pluralist stance: one should accept that different branches of knowledge may require different background theories, if one wants a background theory to be an accurate model of structure.[133]

Still, background theories are not all there is to logical consequence. In particular, an aspect of interpretation is involved. For example, the semantics from clauses (1)-(5) given in sections 1.2.1 and 1.2.2 depends not only on ZFC, but also on the interpretation of each predicate as a set conforming to the ZFC axioms. Alternative interpretations of predicates may however produce other semantic clauses, as we will see in Kripke's intuitionistic modal logic. Notice that acceptance of different interpretations of a given background theory is equally non-trivial as accepting different background theories. Interpreting each predicate as a set, for instance, presupposes that predicational structure works the same as the behavior of sets within ZFC. Choosing another interpretation, however, might produce a different predicational structure.

3.3.2 Case Study: Paraconsistent Set Theory and Intuitionistic Modal Logic

Pluralism about background theories or interpretations alone is insufficient for pluralism about consequence. To recall, the relation between logical consequence and the world consists of three relata: (1) the world's predicational structure, (2) the background theory and (3) the consequence relation. Granting that there are different theories or interpretations, it still needs to be shown that these, indeed, involve different consequence relations. To secure logical pluralism, one thus needs to find examples of two branches of knowledge that require (1) different background theories such that (2) different consequence relations follow from these theories. In what follows, we apply this line of reasoning to paraconsistent set theory and intuitionistic modal logic and contrast their consequence relations with Sher's $\mathcal{C}_{\mathrm{ZFC}}$. Although our main argument does not depend on these examples, it does serve to show how one can attain logical pluralism within Sher's structuralism.

In branches of knowledge free from contradiction, one can safely say that for any object a and property F, it cannot be that both F holds of a and F does not hold of a.[134] Under Sher's interpretation of ZFC, this is shown to hold: as (c) from the previous section shows, for no argument-structure $\langle A, B\rangle$ and individual $a \in A$ can it occur that $a \in B \cap \overline{B}$. Given that B stands for an arbitrary property and a for an arbitrary individual, this matches the idea that no object a can be both F and not F. Still, the principle of non-contradiction is controversial for at least some branches of knowledge. For

[133] At first glance, this view is similar to Stewart Shapiro's thesis of structure-relativity as defended in [34]. According to structure-relativity, logical validity differs with respect to different mathematical structures. Since each consistent structure is legitimate by itself, Shapiro argues, one ought to accept logical pluralism. However, there are two main differences: (1) our view is not restricted to mathematics but applies to each branch of knowledge, and (2) Shapiro restricts legitimate structures to consistent structures of mathematics.

[134] N.J.L.L. Pedersen, "Pluralism x 3: Truth, Logic, Metaphysics," *Erkenntnis (1975-)* 79, 2 (2014): 271, 273.

example, Carnielli and Coniglio note that contradictions often occur in physical theories — sometimes between different well-established theories, sometimes within the same theory. Consequently, physicians occasionally prefer to suspend the principle of explosion.[135]

Consider the following example from topology, used by Priest in favor of paraconsistency. Take a one-dimensional real line and divide it into two disjoint parts, L and R. Next, consider the point of division, p. Now, ask the following question: where is p located? Since p divides L and R, it seems that p must be contained in both L and R. However, L and R being disjoint means that $L \cup R = \varnothing$ and so $p \in L$ iff $p \notin R$ and $p \in R$ iff $p \notin L$. Consequently, we have that $p \in L$ and $p \notin L$ as well as $p \in R$ and $p \notin R$.[136] Secondly, take R.M. Sainsbury's example of genetic hybridization. Conceivably, hybridizing n plants $p_1, ..., p_n$ of different types $T_1, ..., T_n$ sufficiently long will conjure up "borderline cases": plants $p_i \in p_1, ..., p_n$ that cannot be clearly categorized as either T_j or $\neg T_j$ for a given plant type $T_j \in T_1, ..., T_n$. In particular, one could have both $p_i \in T_j$ and $p_i \notin T_j$ or neither $p_i \in T_j$ nor $p_i \notin T_j$ for a plant p_i.[137] If one wants to endorse that something can have both a property and its negation, then one should change one's background theory or one's interpretation thereof.[138]

In Sher's argument-structures, each domain A can be said to be split into two disjoint sets: all individuals with the property B and all those without it.[139] In Thierry Libert's paraconsistent set theory, this is different: a paraconsistent set is defined as an ordered pair of two collections: the extension set and the anti-extension set. Where the former contains all objects supposedly belonging to the set, the latter consists of all objects supposedly excluded from the set. In turn, a paraconsistent structure $\mathcal{S} = \langle A, [\cdot]_{\mathcal{S}} \rangle$ consists of

(i) a non-empty domain set A, and

(ii) an extension function $[\cdot]_{\mathcal{S}} : A \to \mathcal{P}_p(A)$ given by $[\cdot]_{\mathcal{S}} = \langle [\cdot]^+_{\mathcal{S}}, [\cdot]^-_{\mathcal{S}} \rangle$, where $\mathcal{P}_p(A) := \{\langle X, Y \rangle : X \cup Y = A\}$.

Notice that $\mathcal{P}_p(A)$ is the set of all pairs of extension and anti-extension set. In turn, membership is defined as membership in the extension set and non-membership as membership in the anti-extension set. For any $a, b \in A$,

[135] Walter Carnielli and Marcelo Esteban Coniglio, "Paraconsistency and Philosophy of Science: Foundations and Perspectives." In *Paraconsistent Logic: Consistency, Contradiction and Negation,* ed. Andreas Holger and Peter Verdée (Cham, Switzerland: Springer, 2016), 371.

[136] Priest, "A Note on Mathematical Pluralism and Logical Pluralism," 3.

[137] R.M. Sainsbury, "Concepts Without Boundaries," in *Vagueness: A Reader,* ed. Rosanna Keefe and Peter Smith (Cambridge, Massachusetts: The MIT Press, 1996), 264-265.

[138] Note: regarding the second example, nominalists might argue that vagueness of plant types and other predicates is only a feature of linguistic categorization (see pages 279-280 of [41]). Still, this is besides the point if genetics takes it to be a *fact* that a plant is of a certain type: background theories should neutrally model the branch of knowledge's ontological commitments, not contest it.

[139] Gila Sher and Chen Bo, "Foundational Holism, Substantive Theory of Truth, and a New Philosophy of Logic: Interview with Gila Sher by Chen Bo," *The Philosophical Forum* 50, 1 (2019): 54-55.

(iii) $a \in_{\mathcal{M}} b$ iff $a \in [b]^{+}_{\mathcal{M}}$, and

(iv) $a \notin_{\mathcal{M}} b$ iff $a \in [b]^{-}_{\mathcal{M}}$.

Importantly, it is allowed that $a \in [b]^{+}_{\mathcal{M}}$ and $a \in [b]^{-}_{\mathcal{M}}$. Contrasting ZFC, we can therefore have $a \in_{\mathcal{M}} b$ and $a \notin_{\mathcal{M}} b$. In predicative terms, this means that for some a, b, it can occur that b holds of a and b does not hold of a.[140]

One can do the same in Sher's framework by adding to each argument-structure $\langle A, B\rangle$ an extension function $f_{\pm} : A \to \mathcal{P}_p(A)$ mapping subsets $B \subseteq A$ to ordered pairs $\langle B^+, B^-\rangle$. Analogous to Libert's framework, one can define $\mathcal{P}_p(A) := \{\langle X, Y\rangle : X \cup Y = A\}$ and define membership thus: $a \in B$ iff $a \in B^+$ and $a \notin B$ iff $a \in B^-$ for any $a \in A$. Adding to this that the intersection $B^+ \cap B^-$ can be non-empty, we can have that for some a, $a \in B^+ \cap B^-$ and so $a \in B$ and $a \notin B$: the Libertian interpretation's semantics thus differs from that found in Sher's ZFC-based account. Via these semantics and the correspondence between $\neg$ and $\notin$, we have $\langle\{B(a) \wedge \neg B(a)\}, \{\varphi\}\rangle \notin \mathcal{C}_{\mathrm{LIB}}$, where $\mathcal{C}_{\mathrm{LIB}}$ is the Libertian consequence relation. Since $\langle\{B(a) \wedge \neg B(a)\}, \{\varphi\}\rangle \in \mathcal{C}_{\mathrm{ZFC}}$, it is easily seen that $\mathcal{C}_{\mathrm{ZFC}} \neq \mathcal{C}_{\mathrm{LIB}}$: Libert's paraconsistent interpretation of sets delivers a consequence relation different from C_{ZFC}, although predicates are still interpreted as sets.

Another contentious principle is the law of the excluded middle: is it always the case that for any object a and property F, F holds of a or it does not? One reason for doubt might be that excluded middle is inconsistent with a given branch of knowledge. Shapiro gives an example from mathematical analysis, the study of continuous functions on real and complex numbers. Classical infinitesimal analysis (CIA) is consistent with excluded middle and ZFC in general.[141] However, smooth infinitesimal analysis (SIA) is not. Consider "nilsquares": numbers $n \in \mathbb{R}$ such that $n^2 = 0$. In CIA, only 0 satisfies this criterion: for all n in the set of nilsquares N, $n = 0$. However, SIA implies that 0 is not the only nilsquare and that no nilsquare is not 0. As Shapiro notes, introducing excluded middle would render SIA inconsistent.[142]

One could also provide more theoretical reasons for refusing excluded middle, such as the one given by intuitionists. According to intuitionism, mathematical objects and proofs are mind-dependent constructions. Since some properties are left unsettled, it is not the case that any object as a given property or its negation.[143] One might even go a step further and defend the idea that any truth should be warranted. Supposing that there are objects of knowledge for which we lack warrant for believing it has a given property or its negation, excluded middle not be the best option.[144]

[140] Thierry Libert, "Models for a Paraconsistent Set Theory," *Journal of Applied Logic* 3, 1 (2005): 18.
[141] Geoffrey Hellman, "Mathematical Pluralism: the Case of Smooth Infinitesimal Analysis," *Journal of Philosophical Logic* 35, 1 (2006): 621, 623.
[142] Shapiro, "Structures and Logics," 316.
[143] Iemhoff, "Intuitionism in the Philosophy of Mathematics," https://plato.stanford.edu/archives/fall2020/entries/intuitionism/.
[144] Pedersen, "Pluralism x 3," 264-265.

In Kripkean quantificational intuitionistic modal logic (QIML), a different consequence relation can be attained by changing the interpretation of ZFC whilst keeping ZFC as a background theory. The model structure for QIML is an ordered triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, f_{\mathcal{D}} \rangle$ consisting of

(i) a non-empty set of worlds $\mathcal{W}$,

(ii) a reflexive and transitive relation $\mathcal{R}$ on $\mathcal{W}$, and

(iii) a domain function $f_{\mathcal{D}}$ that assigns to each $w \in \mathcal{W}$ a non-empty domain set $\mathcal{D}$ such that for each $w, w' \in \mathcal{W}$ with $w\mathcal{R}w'$, $f_{\mathcal{D}}(w) \subseteq f_{\mathcal{D}}(w')$.[145]

In turn, a model in $\mathcal{M}$ is given by adding an interpretation function $\mathcal{I}$ defined as follows. For any n-ary predicate P^n, interpretations are relativized to worlds: the extension of P^n at $w \in \mathcal{W}$ is the set of n-tuples $\langle d_1, ..., d_n \rangle$ such that $\langle d_1, ..., d_n, w \rangle \in \mathcal{I}(P^n)$. By contrast, constants are not relativized: if a is a constant, then a is assigned to $\mathcal{I}(a) \in \mathcal{D}$.[146]

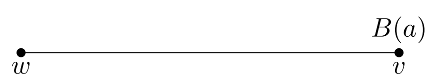
Although the background theory of this logic is still ZFC, its interpretation is drastically different. First of all, Sher's argument-structures were defined to have one domain. In this case, we have a set of worlds, each with an assigned domain. In turn, extensions of predicates are measured relative to those worlds. Conceptually, something being true at a world w means that it is proven at that world. By contrast, a negated formula is true iff the formula is not true at any accessible world.[147] These observations coincide with the intuitionist notions that something is true only if it is proven and false only if it cannot be proven.[148] Of importance to our purposes are the following truth clauses:

(a) $\mathsf{v}_{\mathcal{M},g}(P(a_1, ..., a_n), w) = 1$ iff $\langle [a_1]_{\mathcal{M},g}, ..., [a_n]_{\mathcal{M},g}, w \rangle \in \mathcal{I}(P)$,

(b) $\mathsf{v}_{\mathcal{M},g}(\varphi \vee \psi, w)$ iff $\mathsf{v}_{\mathcal{M},g}(\varphi, w) = 1$ or $\mathsf{v}_{\mathcal{M},g}(\psi, w) = 1$, and

(c) $\mathsf{v}_{\mathcal{M},g}(\neg\varphi, w) = 1$ iff for all $w' \in \mathcal{W}$ such that $w\mathcal{R}w'$, $\mathsf{v}_{\mathcal{M},g}(\varphi, w') = 0$.[149]

From these clauses, it can be shown that the law of the excluded middle fails. Let B be an arbitrary predicate and a an object in the domain and consider the following countermodel:



[145] Saul A. Kripke, "Semantical Analysis of Intuitionistic Logic I," in *Formal Systems and Recursive Functions,* ed. J.N. Crossley and M.A.E. Dummett (Amsterdam: North-Holland, 1965), 95.
[146] Sider, *Logic for Philosophy,* 230.
[147] Kripke, "Semantical Analysis of Intuitionistic Logic I," 98-99.
[148] Iemhoff, "Intuitionism in the Philosophy of Mathematics," https://plato.stanford.edu/archives/fall2020/entries/intuitionism/.
[149] Kripke, "Semantical Analysis of Intuitionistic Logic I," 95-96.

Since $B(a)$ is not true at w, we have $\nu_{\mathcal{M},g}(B(a),w) = 0$. Moreover, because $w\mathcal{R}v$ and $\nu_{\mathcal{M},g}(B(a),v) = 1$, it follows that $\nu_{\mathcal{M},g}(\neg B(a),w) = 0$. Using clause (b), it follows that $\nu_{\mathcal{M},g}(\neg B(a) \vee \neg B(a),w) = 0$.[150]

Again, a similar result may be attained in Sher's framework: one could add $\mathcal{R}$ and $\mathcal{W}$ and assign to each $w \in \mathcal{W}$ an argument-structure $\langle A, B\rangle$. With w a world, one can rewrite the clauses as follows:

(a′) $B(a)$ is true at w iff $a \in B$ at w,

(b′) $B(a) \vee C(a)$ is true for w iff $a \in B$ or $a \in C$, and

(c′) $\neg B(a)$ is true at w iff for all $w' \in \mathcal{W}$ such that $w\mathcal{R}w'$, $B(a)$ is false at w'.

Using (a′)-(c′), one can construct a countermodel similar to the one given earlier by making sure that in a given world $w \in \mathcal{W}$, $B(a)$ is not true at w without $B(a)$ being false for every $w' \in \mathcal{W}$ such that $w\mathcal{R}w'$. Stressing the point once more, (a′)-(c′) are still formulated in ZFC but differs from Sher's clauses (1)-(5) in their interpretations — those of predicates and negation in particular.

In general, one can characterize the line of reasoning as follows. Let $\mathcal{S}$ be a system of knowledge consisting of branches of knowledge $\mathcal{B}_1, ..., \mathcal{B}_n$. To show that logical pluralism is acceptable from the structuralist point of view, one needs to show (1) that there are two branches of knowledge $\mathcal{B}_i, \mathcal{B}_j$ that each require a different background theory of structure or interpretation and (2) that these theories or interpretations each produce different consequence relations. Given that any consequence pair $\langle\Gamma, \sigma\rangle$ is contained in a consequence relation iff it is provable in the relevant background theory, it is sufficient to show that there exists a consequence pair $\langle\Gamma, \sigma\rangle$ contained in the one consequence relation but not in the other. As a philosophical view, then, pluralism within structuralism may be attained by pluralism about structure or interpretations thereof.

[150] Ibidem, 99.

Conclusion

In the introduction, we put forward the question whether Sher's structuralism allows for logical pluralism. It seemed that these views were incompatible. However, we have argued that one can secure logical pluralism within structuralism — namely, by endorsing pluralism about structure or interpretations thereof. To provide a diagnosis, one can say that structuralism and pluralism's supposed incompatibility rested on the assumption that the world's predicational structure should be uniform. This, we argued, need not be the case.

More specifically, logical pluralism can be secured by showing that there are, at the very least, two branches of knowledge which validate (1) different background theories of structure or interpretations thereof such that (2) these theories or interpretations validate different consequence relations. If the pluralist succeeds in doing that, then she will be left with a kind of pluralism that respect the system of knowledge and its commitments — or, as Sher puts it, one that respects "the friction of the world".

In our case, the viability of pluralism within structuralism depends on profound substantial questions that surpass logic. In particular, it depends upon the following question: is predicational structure the same for all branches of knowledge? If not, is it then sufficiently heterogeneous to validate different background theories or interpretations thereof with different consequence relations? Prior to the last point, answers should be given by the branches of knowledge, their ontological commitments and which features they count as structural and non-structural. Only thereafter may logic decide whether the appropriate theories or interpretations, indeed, result in different consequence relations.

From a practical perspective, our proposal is, then, a conditional one: it depends upon the question whether knowledge validates sufficiently different background theories of structure or interpretations thereof. To endorse correctness of different consequence relations, the pluralist ought to provide evidence: she needs to show that there are, in fact, branches of knowledge that require background theories or interpretations that produce different consequence relations. Given the contentious status of classical laws in selected realms of knowledge, however, such a task seems practically attainable.

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