Utrecht University

Faculty of Science

Dynamic Pólya urn models

Bachelor Thesis

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Mathematics and Applications

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June 14, 2021

Contents

Introduction

In this thesis, a dynamic, non-linear generalization of the Pólya urn model will be discussed. Before we treat this model, we will first discuss the classic and non-linear Pólya urn model. Next to this, to motivate the usefulness of this subject, two applications of this model will be discussed.

The classic Pólya urn model, introduced by Eggenberger and Pólya in 1923 [2], is a model consisting of one bin with a balls of colour A and b balls of colour B. Every time a ball with a certain colour is drawn from the bin, the ball, together with another ball of the same colour, is replaced in the bin. There are versions of the Pólya urn model with more than two colours, but we will only consider the two-colour case here.

This classical situation is isomorphic to the following situation considering two bins, where the two bins can be thought of as the two colours. Here, the balls are added one by one, with the probability of a ball landing in a certain bin proportional to the number of balls already in the bin. The last situation will be used to extend in this thesis. Note that we could also say in this case that the probability of a ball landing in a bin with m balls, is proportional to $f(m) = m$. The function $f(m)$ is the so-called feedback function of the process. In the non-linear Pólya urn model, the only property that differs will be the type of feedback function.

In the non-linear Pólya urn model again two bins are considered. Now the probability that a ball lands in a certain bin is proportional to some function $f(m)$. In this thesis the feedback function $f(m) = m^\alpha$ is chosen, where $\alpha > 0$. The model where $\alpha = 1$ corresponds with the classic Pólya urn model. When $\alpha > 1$, we are in the so-called positive feedback scenario. In this situation, the feedback function is convex. A bin with more balls has an higher probability to get new balls, than a bin with less balls. When $0 < \alpha < 1$, we are in the negative feedback case, meaning that the bin containing more balls will probably get less balls.

In this thesis a time-dependent version of the non-linear Pólya urn model will be discussed. That is, at each time $n \in \mathbb{N}$ not one ball, but σ_n balls are added, where σ_n is some function of n. Important is that the balls land independently in the bins, meaning that a part goes in the first bin and the rest in the second bin. The probability for a single ball landing in a certain bin is the same as in the above non-linear model, with $f(m) = m^\alpha$. In this thesis all possibilities for $\alpha > 0$ will considered. It will be shown that for $\alpha < 1$, the proportion of balls in both bins converges to an equilibrium. When $\alpha = 1$, the proportion of balls in a bin converges to a random variable. When $\alpha > 1$, we will show that the proportion converges to 1 for a certain bin and to 0 for the other bin. This event will be called dominance.

In some cases, a much stronger event, monopoly, will occur with positive probability. Then, there exists a point in time $N \in \mathbb{N}$ such that for all timesteps $n \geq N$, one bin will receive all the balls every time and the other bin won't get a ball anymore. The positive feedback case is the most interesting case, since several theorems can be proven. That's why this case is what is focussed on in this thesis.

There are many interesting applications of this simple model, for example in network theory and economic competition. A good example of the first application is the Barábasi-Albert random graph model, which is constructed dynamically. The nodes are interpreted as balls and the different bins are the degrees in the network. At each time a new node is arriving and connecting to an existing node with probability proportional to $f(m) = m^\alpha$, where m is the number of nodes in the particular network. In the positive feedback case we have that highly connected nodes will get higher connected. This is called the "rich-gets richer" phenomenon. For more information about this network, see [1].

Another interesting application is that of economic competition. Most of the time the market dynamics of the current economy have some form of positive feedback. This can be seen as follows. Suppose we have companies, representing the bins, and consumers, which can be seen as the balls. Suppose there are two companies, where one has 60 precent of the market share and the other 40 precent. Then, it will often hold that the company with more market share will grow, and the smaller one will shrink. That is, one company will in the end have a monopoly on the market. An even stronger example, is when both companies have equal market share, until one obtains a non-negligible advantage in the market share. After this happens, the share of the company with the advantage could rapidly grow. The reason both of this could happen is that the market dynamics are strongly driven by the desire of the users to choose the company that has or will have the most users. This is exactly what happens in the case of positive feedback, the stronger gets stronger and the weaker gets weaker. This is a simplification of the real world, but it is a good example. This, and more applications of the model, can be found in [7, chapter 7].

The content of the coming chapters will be as follows. In Chapter 1, the necessary theoretical framework about martingales and convergence is discussed. After that, in Chapter 2 the notations of this thesis will be made clear , together with some general lemmas and examples. In this chapter also the notions of different regimes will be discussed, which will be generally used throughout this thesis. In Chapter 3, we will show that in the supercritical regime monopoly does not occur. Then after that, in Chapter 4 we will show that dominance does occur almost surely when $\alpha > 1$, independent of the regime. For this, we need a lemma that shows that the proportion of balls in a bin does not get stuck at an equilibrium, but deviates significantly infinitely often. After this, in Chapter 5, we will consider the two more difficult cases, first the subcritical regime and after that the critical regime. In Chapter 6 the other possibilities of the feedback function, with $0 < \alpha < 1$ and $\alpha = 1$ respectively, will be discussed. An overview of all the notations and theorems of this thesis can be found in the next section.

This thesis is highly inspired by the work of Nadia Sidorova [8]. The structure of this thesis is based upon it, as are most of the proofs. In some proofs I used a different approach, since the original paper had some minor mistakes or since I was convinced that it was better when it was written down differently. Furthermore, I wanted to complete the story of Sidorova, by also considering $\alpha < 1$, the negative feedback scenario. This way, my thesis has become a complete story.

At last, I would like to thank my supervisor Wioletta Ruszel for introducing me in the fascinating world of balls and bins models and for her guidance during this whole process. Your ideas and thoughts about this subject pushed me to sharpen my thinking and brought my work to an higher level.

Overview

This is an extra chapter to give a clear overview of the notations of this thesis and to make clear which theorems are discussed in every chapter.

Notation and symbols

Throughout this thesis, the following two conventions will hold. In general we will use that zero is a natural number. That means that $\mathbb{N} = \{0, 1, 2, 3, ...\}$. When we are dealing with a sequence $(A_n)_{n\in\mathbb{N}}$ (possibly of random variables), we will often write (A_n).

Below, a list of the notations used in this thesis is written down. Everything is discussed in this thesis as well, this is here solely to give an overview.

Ω	the sample space
ω	an elementary event
$\mathcal{F}$	sigma-algebra of events
$\mathbb{P}$	the probability measure
$(\Omega, \mathcal{F}, \mathbb{P})$	the probability space
(σ_n)	sequence of number of balls added at time $n \in \mathbb{N}$
(τ_n)	sequence of total numbers of balls
(T_n)	sequence of number of balls in the first bin
$(\hat{T}_n)$	sequence of number of balls in the second bin
(Θ_n)	sequence of proportion of balls in the first bin
$(\hat{\Theta}_n)$	sequence of proportion of balls in the second bin
B_n	binomial distributed variable in relation with T_n
ε_n	standardized version of B_n
$\mathcal{D}$	the event dominance
$\mathcal{M}$	the event monopoly
β	the growth parameter
$\beta = \infty$	the supercritical regime
$\beta = 0$	the subcritical regime
$\beta \in (0, \infty)$	the critical regime
$\mathcal{E}$	the set of equilibrium points
$\mathcal{L}$	the set of stationary points

Theorems

In this section an overview of the theorems, that are proven in this thesis, is given.

In chapters 3, 4 and 5 we assume that $\alpha > 1$.

Chapter 3

Theorem 3.1: In the supercritical regime holds that $\mathbb{P}(\mathcal{M}) = 0$.

Chapter 4

Theorem 4.1: Suppose that $\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^{\frac{4}{3}} < \infty$. Then, almost surely, dominance holds.

Chapter 5

Theorem 5.1: Assume that $\beta = 0$. If (ρ_n) is bounded, then $\mathbb{P}(\mathcal{M}) = 1$. If $\rho_n \to \infty$, then $\mathbb{P}(\mathcal{M}) \in \{0, 1\}$.

Theorem 5.6: In the critical regime $\mathbb{P}(\mathcal{M}) \in [0, 1)$, depending on the summation $\sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^{\alpha}}$.

Chapter 6

Theorem 6.1: Suppose $0 < \alpha < 1$ and $\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^{2} < \infty$. Then, almost surely the proportion of balls in both bins converges to $(\frac{1}{2}, \frac{1}{2})$.

Theorem 6.2: Assume $\alpha = 1$. Then Θ_n converges to a random variable and $\mathbb{P}(\mathcal{D}) = 0$.

1 Theoretical framework

In this chapter several definitions and theorems that will be used in this paper, are discussed. Most of the definitions and theorems will have to do with martingales and convergence. Readers who are already familiar with these subjects, can skip this chapter and go to Chapter 2.

In this chapter the random variables are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here Ω is the sample space, $\mathcal{F}$ the sigma-algebra of events and $\mathbb{P}$ the probability measure. The definitions and properties of these are not discussed here. For this, the reader is referred to [3, chapter 1]. The followings sections are based on several chapters of the books [3] and [9], unless stated otherwise.

1.1 Martingales

Definition 1.1 (Stochastic process). *[5, pg. 189] A stochastic process X is a collection of random variables parametrized by a set T i.e. $X = (X_t)_{t \in T}$.*

Typically, we are interested in two different cases for the set T.

(1) $T = \mathbb{N}$. This means that T is discrete. The stochastic process is then called a discrete-time stochastic process.

(2) $T = \mathbb{R}_+$. In this case, the stochastic process is a continuous-time stochastic process.

In this article will only encounter discrete-time processes, where $T = \mathbb{N}$. To be able to define martingales, we need a couple more definitions.

Definition 1.2 (Filtration). *A increasing sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of sub-σ-algebras of $\mathcal{F}$ is called a filtration.*

Most of the time the natural filtration is used. That is, $\mathcal{F}_n = \sigma(X_0, ..., X_n)$ for some stochastic process $(X_n)_{n \in \mathbb{N}}$. This means that the information available at time $n \in \mathbb{N}$ about $\omega \in \Omega$ is given by the values

$$X_0(\omega), X_1(\omega), ..., X_n(\omega).$$

Definition 1.3 (Adapted process). *A stochastic process $(X_n)_{n \in \mathbb{N}}$ is adapted to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ if for every $n \in \mathbb{N}$, X_n is $\mathcal{F}_n$ measurable.*

If a process $(X_n)_{n \in \mathbb{N}}$ is adapted, it intuitively means that for $\omega \in \Omega$ $X_n(\omega)$ is known at time n.

Definition 1.4 ($\mathcal{L}^p$ spaces). *Let X be a random variable and $p \geq 1$. We say that $X \in \mathcal{L}^p$ if*

$$\mathbb{E}\left[|X|^p\right] < \infty.$$

Definition 1.5 (Integrable). *Let X be a random variable . This variable X is integrable, when $X \in \mathcal{L}^1$, so*

$$\mathbb{E}[|X|] < \infty.$$

Definition 1.6 (Martingale). *A stochastic process $(X_n)_{n\in\mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if the following three properties hold for all $n \in \mathbb{N}$:*

(1) $(X_n)_{n\in\mathbb{N}}$ is an adapted process.

(2) X_n is integrable.

(3) $\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = X_n$.

The last property is called the martingale property.

1.2 Convergence

In this article a few different notions of convergence will be used. In this section, these will be discussed. The first definition of this section is one that will be used often in this paper.

Definition 1.7 (Almost surely). *An event $A \in \mathcal{F}$ is said to be true almost surely, if*

$$\mathbb{P}(A) = 1.$$

When an event happens almost surely, the set of points where the property does not hold may be non-empty, but it has probability 0 by definition.
Linked to this definition is the following notation of convergence.

Definition 1.8 (Almost sure convergence). *A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges almost surely to a random variable X, written as $X_n \overset{a.s}{\to} X$, if*

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

We will often write X_n converges to X almost surely.

To be able to state two theorems that will be used in this thesis, two more definitions are needed.

Definition 1.9 (Convergence in $\mathcal{L}^p$). *Let $p \geq 1$. A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in $\mathcal{L}^p$ to a random variable X if*

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^p\right] = 0.$$

We write $X_n \overset{\mathcal{L}^p}{\to} X$ then.

Definition 1.10 (Bounded in $\mathcal{L}^p$). *[5, pg. 222]*
Let $p \geq 1$. We say a stochastic process $(X_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}^p$ if,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|X_n|^p\right] < \infty.$$

Theorem 1.1 ($\mathcal{L}^p$ convergence theorem for martingales.). *Let $p > 1$ and $(X_n)_{n\in\mathbb{N}}$ a martingale bounded in $\mathcal{L}^p$. Then there exists a random variable X_∞ with $\mathbb{E}[|X_\infty|] < \infty$ and $X_n \to X_\infty$ almost surely and in $\mathcal{L}^p$.*

Proof. This theorem and proof can be found in [5, pg.222, 223]. □

Theorem 1.2 (Martingale convergence theorem). *Suppose $(X_n)_{n\in\mathbb{N}}$ is a martingale, with*

$$\sup_{n\in\mathbb{N}} \mathbb{E}(|X_n|) < \infty,$$

then X_n converges to $X_\infty = \lim_{n\to\infty} X_n$ a.s and $\mathbb{E}(|X_\infty|) < \infty$.

Proof. The proof of this theorem can be found in [3, pg. 508]. □

Before we continue with the next lemma, we need to get familiar with some specific events. Denote $(A_n)_{n\in\mathbb{N}}$ a sequence of events, meaning the elements of this sequence are measurable subsets of Ω. Then we call A^* the limsup event. This means that

$$A^* = \limsup_{n\to\infty} A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

Take $\omega \in \Omega$. If $\omega \in A^*$, we say that $\omega \in A_n$ infinitely often (shortly i.o.). It holds that for all $m \geq 1$, there exists an $n \geq m$, such that $\omega \in A_n$.

Let

$$A_* = \liminf_{n\to\infty} A_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n.$$

This is the liminf event. We say that $\omega \in A_*$ if and only if $\omega \in A_n$ eventually for all n (shortly a.a, almost always). This means that there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ holds that $\omega \in A_n$. At last, the following relation will be used often

$$\left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c.$$

Knowing all this, an important lemma can be introduced.

Lemma 1.3 (Borel-Cantelli). *Denote $(A_n)_{n\in\mathbb{N}}$ a sequence of events. Then if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, we have that*

$$\mathbb{P}(A^*) = 0.$$

Proof. Assume $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. We need to proof that $\mathbb{P}(\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n) = 0$. Take $B_m = \cup_{n=m}^{\infty} A_n$. This is a decreasing sequence, since we take smaller and smaller unions of elements. By continuity from below and sub-additivity of the probability measure, we get the following:

$$0 \leq \mathbb{P}(A^*) = \mathbb{P}\left(\bigcap_{m=1}^{\infty} B_m\right) = \lim_{m\to\infty} \mathbb{P}(B_m) = \lim_{m\to\infty} \mathbb{P}\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \lim_{m\to\infty} \sum_{n=m}^{\infty} \mathbb{P}(A_n) = 0.$$

In the last step we used that the tail of a convergent sum goes to 0.
We can conclude that $\mathbb{P}(A^*) = 0$. □

2 Terminology

In this chapter the notations of the rest of this thesis will be explained. This chapter is highly inspired by the work of Sidorova [8, Section 1].

Recall we are working with the feedback function $f(m) = m^{\alpha}$. Recall that the most interesting case is when $\alpha > 1$, the case that we consider in the first chapters. We deduce relations here that will be applied later in this thesis.

2.1 Notations

In this paper we are assuming there are two bins.We are analysing the time-inhomogeneous scenario, where at time n there is a certain number of balls, from now on σ_n, added to both bins together. We assume the balls are independently added to both bins, meaning a part of the balls goes into the first bin and the rest of the balls into the second bin.

Let $(\sigma_n)_{n\in\mathbb{N}}$ be the sequence representing the number of balls added at time n, assume it is positive. Denote $(\tau_n)_{n\in\mathbb{N}}$ the total number of balls at time n. We assume that $\tau_0 > 0$, the initial number of balls is positive. The following relation can be deduced:

$$\tau_n = \tau_0 + \sum_{i=1}^{n} \sigma_i.$$

For simplicity, we denote $\tau_0 = \sigma_0$ such that this relation becomes

$$\tau_n = \sum_{i=0}^{n} \sigma_i. \tag{2.1}$$

This can also be written for all $n \geq 1$ as

$$\tau_n = \sigma_n + \tau_{n-1}.$$

Most of the time we will look at the number of balls in the first bin: T_n. The number of balls in the second bin is then $\hat{T}_n = \tau_n - T_n$. We assume that $T_0 > 0$ and $\hat{T}_0 > 0$, both bins already contain balls at time 0.

Since only two bins are considered, the situation we are in is symmetric. That's why it is sufficient to consider one bin most of the time. The proportion of balls in the first bin is denoted by

$$\Theta_n = \frac{T_n}{\tau_n} \in [0, 1].$$

Equivalently, the proportion of balls in the second bin is equal to

$$\hat{\Theta}_n = 1 - \Theta_n = \frac{\hat{T}_n}{\tau_n}.$$

We define

$$T_{n+1} = T_n + B_{n+1} = T_0 + \sum_{i=1}^{n+1} B_i \tag{2.2}$$

where B_{n+1} is binomial distributed random variable with size σ_{n+1} and parameter

$$P_n = \frac{\Theta_n^\alpha}{\Theta_n^\alpha + (1-\Theta_n)^\alpha}.$$

We often write $P_n = \psi(\Theta_n)$, with

$$\psi(x) = \frac{x^\alpha}{x^\alpha + (1-x)^\alpha} \text{ for } x \in [0,1].$$

In exactly the same way, this can be done for the second bin, where $\hat{B}_{n+1}$ is a binomial distributed variable with size σ_{n+1} and parameter $\hat{P}_n = 1 - P_n$.
Note that this way, the probability that a ball lands in a certain bin depends on the number of balls already in the bin to the power α, with $\alpha > 0$. When $\alpha > 1$, $\psi(x)$ is a convex function. This means that the probability that a ball lands in the bin with already many balls, is relatively high. When $\alpha < 1$, $\psi(x)$ is a concave function. Exactly the opposite of the previous case happens. The bin with already a lot of balls is less likely to get a ball than the bin with less balls. In this case, there can be imagined that this system converges, since fluctuations will cancel themselves out. When $\alpha = 1$, we have that $P_n = \Theta_n$ and the probability that a ball lands in a certain bin is proportional to the number of balls already in that bin. All these cases will be discussed in the next chapters.

To proof several theorems, we need to be able to upper and lower bound the function $\psi(x)$ for $x \in [0,1]$. Since in this thesis we are considering the positive feedback function most of the time, we assume from now on that $\alpha > 1$. Then we have, for x in the interval $[0,1]$, that $x^\alpha + (1-x)^\alpha \leq 1$. That means we get

$$x^\alpha \leq \frac{x^\alpha}{x^\alpha + (1-x)^\alpha}.$$

To find a lower bound we need to do more work. For that we consider the denominator as a function itself, $f(x) = x^\alpha + (1-x)^\alpha$. This function has one extremum, $x = \frac{1}{2}$. Checking with the second derivative, we find that $f''(x) > 0$, thus $f(\frac{1}{2}) = 2(\frac{1}{2})^\alpha$ is a minimum. This gives

$$f(x) \geq 2\left(\frac{1}{2}\right)^\alpha = 2^{-(\alpha-1)},$$

and

$$\frac{1}{f(x)} \leq 2^{\alpha-1}.$$

Since $\psi(x) = x^\alpha \frac{1}{f(x)}$, we get

$$\psi(x) \leq x^\alpha 2^{\alpha-1}.$$

To summarize, $\psi(x)$ can be bounded for $x \in [0,1], \alpha > 1$ by:

$$x^\alpha \leq \psi(x) \leq x^\alpha 2^{\alpha-1}. \tag{2.3}$$

The random variable B_{n+1} only depends on $\mathcal{F}_n = \sigma(B_1, ..., B_n)$ through P_n. Intuitively, this means that B_{n+1} does not dependent on the past through anything other than P_n. Instead of this binomial distributed variable B_{n+1}, we will often use the standardized version of this variable. Since B_{n+1} is binomial distributed and is dependent on the past, we get the following conditional mean and variance

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_n}[B_{n+1}] &= \sigma_{n+1} P_n \\ \operatorname{Var}_{\mathcal{F}_n}[B_{n+1}] &= \sigma_{n+1} P_n (1 - P_n). \end{aligned} \tag{2.4}$$

This gives the following standardized version of B_{n+1}:

$$\varepsilon_{n+1} = \frac{B_{n+1} - \sigma_{n+1} P_n}{\sqrt{\sigma_{n+1} P_n (1 - P_n)}} \tag{2.5}$$

The conditional mean and second moment of this variable are as follows

$$\mathbb{E}_{\mathcal{F}_n}\left[\varepsilon_{n+1}\right] = \mathbb{E}_{\mathcal{F}_n}\left[\frac{B_{n+1} - \sigma_{n+1} P_n}{\sqrt{\sigma_{n+1} P_n (1 - P_n)}}\right] = \frac{\sigma_{n+1} P_n - \sigma_{n+1} P_n}{\sqrt{\sigma_{n+1} P_n (1 - P_n)}} = 0.$$

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_n}\left[\varepsilon_{n+1}^2\right] &= \mathbb{E}_{\mathcal{F}_n}\left[\left(\frac{B_{n+1} - \sigma_{n+1} P_n}{\sqrt{\sigma_{n+1} P_n (1 - P_n)}}\right)^2\right] = \mathbb{E}_{\mathcal{F}_n}\left[\frac{B_{n+1}^2 - 2B_{n+1}\sigma_{n+1} P_n + {\sigma_{n+1}}^2 {P_n}^2}{\sigma_{n+1} P_n (1 - P_n)}\right] \\ &= \frac{\sigma_{n+1} P_n (1 - P_n) + (\sigma_{n+1} P_n)^2 - 2(\sigma_{n+1} P_n)^2 + {\sigma_{n+1}}^2 {P_n}^2}{\sigma_{n+1} P_n (1 - P_n)} = 1. \end{aligned}$$

Here we used that for a random variable X

$$\mathbb{E}[X^2] = \operatorname{Var}[X] + \mathbb{E}^2[X].$$

By the properties of the conditional expectation in [3, pg. 471], we get

$$\begin{aligned} \mathbb{E}[\varepsilon_{n+1}] &= \mathbb{E}[\mathbb{E}_{\mathcal{F}_n}[\varepsilon_{n+1}]] = \mathbb{E}[0] = 0. \\ \mathbb{E}[\varepsilon_{n+1}^2] &= \mathbb{E}[\mathbb{E}_{\mathcal{F}_n}[\varepsilon_{n+1}^2]] = \mathbb{E}[1] = 1. \end{aligned}$$

Using the relations (2.2) and (2.5) , T_{n+1} can be expressed differently, because it holds that

$$B_{n+1} = \sigma_{n+1} P_n + \varepsilon_{n+1}\sqrt{\sigma_{n+1} P_n (1 - P_n)}.$$

This gives, where $P_n = \psi(\Theta_n)$,

$$T_{n+1} = T_n + \sigma_{n+1}\psi(\Theta_n) + \varepsilon_{n+1}\sqrt{\sigma_{n+1} P_n (1 - P_n)}. \tag{2.6}$$

The focus of this paper is on two different events, namely dominance and monopoly, denoted by $\mathcal{D}$ and $\mathcal{M}$ respectively. Monopoly is stronger than dominance, in the sense that dominance occurs when monopoly occurs. This means that

$$\mathcal{M} \subset \mathcal{D}.$$

The two events can be denoted as follows

$$\mathcal{D} = \{\lim_{n\to\infty} \Theta_n \in \{0,1\}\}$$
$$\mathcal{M} = \{B_n = 0 \text{ eventually for all } n\} \cup \{B_n = \sigma_n \text{ eventually for all } n\}.$$

Intuitively, the difference between those two events is the following: When dominance occurs, we can say that eventually the number of balls in one of the bins is negligible. We will later see that this corresponds with two of the three equilibrium points of this two bins model. When monopoly occurs we see that eventually all balls are added to one of the bins, so the other bin does not get a ball anymore. Note that the event $\mathcal{M}$ can be written in terms of a liminf event. For this, let $C_n = \{B_n = 0\}$ and $G_n = \{B_n = \sigma_n\}$. Then

$$\begin{aligned}\mathcal{M} &= \{B_n = 0 \text{ eventually for all } n\} \cup \{B_n = \sigma_n \text{ eventually for all } n\} \\ &= \{\liminf_{n\to\infty} C_n\} \cup \{\liminf_{n\to\infty} G_n\}.\end{aligned}$$

2.2 Important parameters

To be able to distinguish between different cases that we will see in this thesis, we need to work with several parameters. We start with the growth parameter β, to be able to show whether or not monopoly occurs:

$$\beta = \lim_{n\to\infty} \alpha^{-n} \log(\tau_n).$$

It holds that this parameter shows the growth of τ_n, because $\lim_{n\to\infty} \alpha^{-n} = 0$ for $\alpha > 1$. Since both $\alpha, \tau_n \geq 1$, we have that $\beta \in [0, \infty]$. By different values of β, there can be distinguished between 3 regimes. Below the three regimes and values for β are described.

(i) Supercritical regime: $\beta = \infty$.

We will show that $\mathbb{P}(\mathcal{M}) = 0$ in this case. No other parameters are needed for this.

(ii) Subcritical regime: $\beta = 0$.

To be able to show whether monopoly occurs or not in this regime, we need to consider $(\rho_n)_{n\in\mathbb{N}}$. This sequence is mainly useful in the subcritical regime, but will be used more in this thesis. The elements of (ρ_n) are as follows

$$\rho_n = \frac{\sigma_{n+1}}{\tau_n}.$$

We will see that the probability on monopoly depends on the fact whether (ρ_n) is bounded or not. In fact, monopoly will happen almost surely here, unless the sequence (σ_n) is irregular.

(iii) Critical regime: $\beta \in (0, \infty)$.

In this regime the probability that the event monopoly occurs will be smaller than 1, depending on some extra conditions of (τ_n). This will be made more clear in the examples at the end of this chapter.

The above suggests that the transition from no monopoly to monopoly happens when $(\tau_n)_{n\in\mathbb{N}}$ changes from growing fast to growing slowly, since $\beta = \infty$ corresponds with $(\tau_n)_{n\in\mathbb{N}}$ growing much faster than α^{-n} goes to 0 in the limit. To the contrary, $\beta = 0$ implies that a^{-n} goes faster to zero than $\log(\tau_n)$ diverges to infinity. However, in the critical regime, this is no longer the case.

Throughout this thesis, we assume the following

$$\sum_{n=0}^{\infty} \frac{\sigma_n}{\tau_n} = \infty. \tag{2.7}$$

Next to this, in some cases we will assume that

$$\sum_{n=0}^{\infty} \frac{\sigma_n^2}{\tau_n^2} < \infty, \tag{2.8}$$

or we will suppose that

$$\sum_{n=0}^{\infty} \frac{\sigma_n^{\frac{4}{3}}}{\tau_n^{\frac{4}{3}}} < \infty. \tag{2.9}$$

This can be interpreted as the fact that the randomness of the balls is not too large. The second assumption will be needed in the negative feedback scenario in Chapter 6. The third will be necessary to prove almost sure dominance in the positive feedback case in Chapter 5. Later in this chapter there will be shown that these assumptions will hold, when we choose (σ_n) such that it is not growing too fast.

The following lemma will show that ρ_n is bounded, if (2.8) holds.

Lemma 2.1. *Suppose that the series $\sum_{n=0}^{\infty} \frac{\sigma_n^2}{\tau_n^2}$ converges. Then*

$$\rho_n = \frac{\sigma_{n+1}}{\tau_n}$$

is bounded.

Proof. Since we assumed the series converges, it follows that

$$\lim_{n\to\infty} \frac{\sigma_n^2}{{\tau_n}^2} = 0.$$

This implies that

$$\lim_{n\to\infty} \frac{\sigma_n}{\tau_n} = 0.$$

Precisely, this means that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{\sigma_n}{\tau_n} < \epsilon$. Choose $\epsilon = \frac{1}{2}$. Now assume by contradiction that

$$\rho_n \to \infty.$$

This means for all $a \in \mathbb{R}$ and for all $K \in \mathbb{N}$ there exists $n > K$ such that $\frac{\sigma_{n+1}}{\tau_n} > a$. Choose $a = \frac{1}{2}$. Then we get that $\frac{\tau_n}{\sigma_{n+1}} + 1 < \frac{3}{2}$. Then it holds that

$$\begin{aligned} \frac{\sigma_{n+1}}{\tau_{n+1}} &= \frac{\sigma_{n+1}}{\tau_n + \sigma_{n+1}} \\ &= \frac{1}{\frac{\tau_n}{\sigma_{n+1}} + 1} > \frac{2}{3} > \frac{1}{2}. \end{aligned}$$

Choose $K = N$. Then we have, by the converge of the series, that for all $n \geq N$ should hold that $\frac{\sigma_n}{\tau_n} < \frac{1}{2}$. But we also proved that there exists an $n > N$ such that $\frac{\sigma_n}{\tau_n} > \frac{1}{2}$. This is a contradiction. Hence we can conclude that (ρ_n) is bounded, thus that $\rho_n \leq M$ for $M \in \mathbb{R}$. □

From this it follows, when we know that ρ_n diverges to infinity, $\sum_{i\geq 0} \frac{\sigma_i^2}{\tau_i^2}$ cannot be finite. When ρ_n is bounded, we can say more about the parameter β.

Lemma 2.2. *Suppose ρ_n is bounded, then it holds that*

$$\beta = \lim_{n\to\infty} \alpha^{-n} \log(\tau_n) = 0.$$

Proof. Suppose (ρ_n) is bounded by $M \in \mathbb{R}$. The following relation can deduced recursively

$$\begin{aligned} \tau_2 &= \tau_1 + \sigma_2 = \tau_1 \left(1 + \frac{\sigma_2}{\tau_1}\right) = (\tau_0 + \sigma_1)(1 + \rho_1) = \tau_0(1 + \rho_0)(1 + \rho_1) \\ \tau_n &= \tau_0 \prod_{i=0}^{n-1}(1 + \rho_i). \end{aligned}$$

Using this relation, we get that

$$\alpha^{-n} \log(\tau_n) = \alpha^{-n}(\log(\tau_0) + \sum_{i=0}^{n-1} \log(1 + \rho_i) \leq \alpha^{-n}(\log(\tau_0) + n \log{(1 + M)}).$$

Here we used that $x \to \log(x)$ is an increasing function for $x \geq 1$. Taking limits on both sides we obtain

$$\beta = \lim_{n\to\infty} \alpha^{-n} \log(\tau_n) \leq \lim_{n\to\infty} \alpha^{-n}(\log(\tau_0) + n \log(1 + M)) = 0.$$

Since $\beta \geq 0$, we can conclude that $\beta = 0$. □

To be able to prove several theorems, the following distinctions are often made:

$$(\rho): \text{ the sequence } (\rho_n) \text{ is either bounded or tends to infinity.}$$

$$(\sigma): \text{ the sequence } (\sigma_n) \text{ is either bounded or tends to infinity.}$$

This last condition may sound a bit strange, since there are no other options for (σ_n) to be bounded or diverge when $n \to \infty$. In the proofs of the theorems that we will encounter in the next chapter, it is mostly used as a tool to distinguish between several cases. We will assume (σ) and (ρ) hold everywhere. Sometimes it will be mentioned that (ρ) holds, to be more clear.

Now the useful conditions are defined, we will show five examples where different functions for σ_n and τ_n are used and where in the first two examples we show we fulfil the three conditions (2.7), (2.8) and (2.9). The goal of these five examples is to get an intuitive idea of the three different regimes.

Examples

(1) σ_n is constant, i.e $\sigma_n = 3$ for all $n \in \mathbb{N}$.

It is trivial now that σ_n is bounded in this case. For simplicity, we write $\sigma_0 = \tau_0 = 3$, but in essence it could be any number. It does not change the outcome of the bounds. Using relation (2.1), it holds that

$$\tau_n = \sum_{i=0}^{n} \sigma_i = 3(n+1).$$

We notice that

$$\rho_n = \frac{\sigma_{n+1}}{\tau_n} = \frac{3}{3(n+1)} = \frac{1}{(n+1)} < 2.$$

Thus ρ_n is also bounded for all $n \in \mathbb{N}$. This implies by Lemma 2.2 that we are in the subcritical regime. The summations can be expressed as follows

$$\sum_{n=0}^{\infty} \frac{\sigma_n}{\tau_n} = \sum_{n=0}^{\infty} \frac{3}{3(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty.$$

$$\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^2 = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty.$$

$$\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^{\frac{4}{3}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\frac{4}{3}}} < \infty.$$

(2) σ_n is linear, i.e. $\sigma_n = n$.

Clearly, σ_n diverges when $n \to \infty$. Since we assumed that $\tau_0 > 0$, set $\sigma_0 = 1$. We have the following expressions for τ_n and ρ_n

$$\tau_n = 1 + \sum_{i=1}^{n} \sigma_i = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}.$$

$$\rho_n = \frac{\sigma_{n+1}}{\tau_n} = \frac{(n+1)}{\frac{n(n+1)+2}{2}} \leq \frac{(n+1)}{\frac{n(n+1)}{2}} = \frac{2}{n} \leq 3.$$

We can conclude that ρ_n is bounded in this case. We are again in the subcritical regime. The assumptions for the summations are also satisfied

$$\sum_{n=0}^{\infty} \frac{\sigma_n}{\tau_n} = \sum_{n=0}^{\infty} \frac{n}{\frac{n(n+1)+2}{2}} = \sum_{n=0}^{\infty} \frac{2}{n+1+\frac{2}{n}} \geq \sum_{n=0}^{\infty} \frac{2}{n+4} = \infty.$$

$$\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^2 = \sum_{n=0}^{\infty} \frac{4}{(n+1+\frac{2}{n})^2} \leq \sum_{n=0}^{\infty} \frac{4}{(n+1)^2} < \infty.$$

$$\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^{\frac{4}{3}} = \sum_{n=0}^{\infty} \frac{4^{\frac{4}{3}}}{(n+1+\frac{2}{n})^{\frac{4}{3}}} \leq \sum_{n=0}^{\infty} \frac{16}{(n+1)^{\frac{4}{3}}} < \infty.$$

(3) $\sigma_n = \lfloor 3^n e^{\alpha^n} \rfloor$, $b > 0$.

It holds that the value of $\tau_n = \sum_{n=0}^{n} \sigma_n$ will be of the same order, hence we can take $\tau_n = \lfloor 3^n e^{\alpha^n} \rfloor$. The regime won't be different to the regime of the actual value of τ_n. We can bound this as follows

$$3^{n-1} e^{\alpha^{n-1}} \leq \lfloor 3^n e^{\alpha^n} \rfloor \leq 3^n e^{\alpha^n}.$$

The value of β can be bounded by calculating it for both the lower and upper bound, call these bounds β_1 and β_2 respectively. For the lower bound, we get

$$\begin{aligned}\beta_1 &= \lim_{n\to\infty} \alpha^{-n} \log\left(3^{n-1} e^{\alpha^{n-1}}\right) = \lim_{n\to\infty} \alpha^{-n} \log\left(3^{n-1}\right) + \lim_{n\to\infty} \alpha^{-n} \log\left(e^{\alpha^{n-1}}\right) \\ &= \lim_{n\to\infty} \alpha^{-n} (n-1) \log(3) + \frac{1}{\alpha} = \frac{1}{\alpha}.\end{aligned}$$

For β_2, exactly the same can be done:

$$\begin{aligned}\beta_2 &= \lim_{n\to\infty} \alpha^{-n} \log\left(3^n e^{\alpha^n}\right) \\ &= \lim_{n\to\infty} \alpha^{-n} n \log(3) + \lim_{n\to\infty} \alpha^{-n} \log\left(e^{\alpha^n}\right) = 1.\end{aligned}$$

We know that $\beta_1 \leq \beta \leq \beta_2$, and thus $\beta \in (0, \infty)$. This means we are in the critical regime for this choice of (σ_n).We will see later that in this regime, the summation

$$\sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^{\alpha}}$$

plays an important role. In fact, whether it converges of diverges will affect the probability on monopoly to occur. To calculate an infinite summation, the ceiling function can be ignored. We get

$$\sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^{\alpha}} = \sum_{n=0}^{\infty} \frac{3^{n+1} e^{\alpha^{n+1}}}{3^{n\alpha} e^{\alpha^{n+1}}} = \sum_{n=0}^{\infty} \frac{3^{n+1}}{3^{n\alpha}} < \infty,$$

since $\alpha > 1$. Later we will see this implies that $\mathbb{P}(\mathcal{M}) \in (0, 1)$.

(4) $\sigma_n = \lfloor \frac{1}{2}^n e^{\alpha^n} \rfloor$.

We can again say that $\tau_n = \lfloor \frac{1}{2}^n e^{\alpha^n} \rfloor$. Note that the calculation of the value of β can be done exactly in the same way of the last example, so that won't be done here. Recall that this means we are again in the critical regime. The interesting difference is in the summation $\sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^{\alpha}}$. In this case, we get

$$\sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^{\alpha}} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}^{n+1} e^{\alpha^{n+1}}}{\frac{1}{2}^{n\alpha} e^{\alpha^{n+1}}} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}^{n+1}}{\frac{1}{2}^{n\alpha}} = \infty.$$

Later, we will see that this will imply that $\mathbb{P}(\mathcal{M}) = 0$.

The last two examples indeed show that in the critical regime, a relatively fast growing τ_n can imply monopoly with positive probability, but when τ_n grows relatively slowly this is not the case anymore. This is indeed exactly the opposite of what happens in the transition from the supercritical to the subcritical regime.

(5) $\sigma_n = \lceil \exp\{e^{\alpha^n}\} \rceil$.

Note, that as in the last two examples, we can say that τ_n is of the same order and that $\tau_n = \lceil \exp\{e^{\alpha^n}\} \rceil$. This gives the following lower bound for β:

$$\begin{aligned}\beta &= \lim_{n\to\infty} \alpha^{-n} \log(\tau_n) = \lim_{n\to\infty} \alpha^{-n} \log\left(\left\lceil \exp\left\{e^{\alpha^n}\right\}\right\rceil\right) \geq \lim_{n\to\infty} \alpha^{-n} \log\left(\exp\left\{e^{\alpha^n}\right\}\right) \\ &= \lim_{n\to\infty} \alpha^{-n} e^{\alpha^n} = \infty,\end{aligned}$$

by the fact that $\alpha > 1$. This means we are in the supercritical regime.

We have seen enough examples to start proving several theorems. This will be done in the next chapters.

3 No monopoly

In this chapter, there will be proven that in the supercritical regime monopoly never takes place. Note that in this chapter we cannot assume that $\sum_{n=0}^{\infty} \frac{\sigma_n^2}{\tau_n^2} < \infty$, since by Lemma 2.2 it would imply that $\beta = 0$. The approach of this chapter is inspired by the proofs of [8, Section 2].
The aim of this chapter is to prove the following theorem:

Theorem 3.1 (Supercritical regime). *Suppose $\alpha > 1$ and $\beta = \infty$. Then $\mathbb{P}(\mathcal{M}) = 0$.*

We start with a lemma. When we can prove this holds, the proof of Theorem 3.1 will follow.

Lemma 3.2. *If*

$$\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^{\alpha}} = \infty,$$

then $\mathbb{P}(\mathcal{M}) = 0$.

Proof. Let $Y_n = \sum_{i=1}^{n} \sigma_i P_{i-1}$. We start proving that the following claim holds

$$\{Y_\infty = \infty\} \subset \left\{ \lim_{n\to\infty} T_n = \infty \right\}. \tag{3.1}$$

Note that the left event is equivalent to $\left\{\sum_{n=1}^{\infty} \mathbb{E}_{\mathcal{F}_{n-1}}[B_n] = \infty\right\}$.
From (2.2) we can deduce that for every $n \in \mathbb{N}$

$$\begin{aligned} T_n &= T_0 + M_n + Y_n \\ M_n &= \sum_{i=1}^{n} (B_i - \sigma_i P_{i-1}) \end{aligned} \tag{3.2}$$

We will show that $(M_n)_{n\in\mathbb{N}}$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(B_1, B_2, ..., B_n)$. It indeed satisfies the properties of Definition 1.6:

(1) M_n is clearly $\mathcal{F}_n$-measurable, since B_n is $\mathcal{F}_n$ measurable by construction. Thus (M_n) is adapted to the filtration $\mathcal{F}_n$.

(2) Since $B_n \leq \sigma_n$ for all $n \in \mathbb{N}$, we have that $|M_n| \leq n \max(\sigma_1, ...\sigma_n) < \infty$. Hence

$$\mathbb{E}[|M_n|] \leq \mathbb{E}[n \max(\sigma_1, ...\sigma_n)] < \infty.$$

Thus $M_n \in \mathcal{L}^1$.

(3) $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[B_{n+1} - \sigma_{n+1}P_n + M_n|\mathcal{F}_n] = \sigma_{n+1}P_n - \sigma_{n+1}P_n + \mathbb{E}[M_n|\mathcal{F}_n] = \mathbb{E}[M_n|\mathcal{F}_n] = M_n$, since the conditional expectation is linear and both $\sigma_{n+1}P_n$ and M_n are $\mathcal{F}_n$ measurable.

Now we know that the process (M_n) is a martingale, we can use the properties of the quadratic variation of discrete martingales, see [9, §12.13]. For this we need to have that $M_n \in \mathcal{L}^2$ for all $n \in \mathbb{N}$ that is, we need to have that $\mathbb{E}[|M_n|^2] = \mathbb{E}[M_n^2] < \infty$. Using the same bound as above, we notice that

$$\mathbb{E}\left[|M_n|^2\right] \leq \mathbb{E}\left[(n \max(\sigma_1, ...\sigma_n))^2\right] < \infty$$

We can conclude that $M_n \in \mathcal{L}^2$ for all $n \in \mathbb{N}$. That means we can use the quadratic variation of a discrete martingale to be able to conclude almost sure divergence of T_n, using (3.2). We call the quadratic variation process $(\langle M \rangle_n)_{n \in \mathbb{N}}$, see [5, pg. 206]. Then we get

$$\begin{aligned}
\langle M \rangle_n &:= \sum_{i=1}^{n} \mathbb{E}[(M_i - M_{i-1})^2 \,|\mathcal{F}_{i-1}] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[\left(\sum_{k=1}^{i} B_k - \sigma_k P_{k-1} - \sum_{l=1}^{i-1} B_l - \sigma_l P_{l-1}\right)^2 |\mathcal{F}_{i-1}\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[(B_i - \sigma_i P_{i-1})^2 \,|\mathcal{F}_{i-1}\right] \\
&= \sum_{i=1}^{n} \left(\mathbb{E}_{\mathcal{F}_{i-1}}[B_i^2] - 2\mathbb{E}_{\mathcal{F}_{i-1}}[B_i \sigma_i P_{i-1}] + \sigma_i^2 P_{i-1}^2\right) \\
&= \sum_{i=1}^{n} (\sigma_i P_{i-1} - \sigma_i P_{i-1}^2 + \sigma_i^2 P_{i-1}^2 - 2\sigma_i^2 P_{i-1}^2 + \sigma_i^2 P_{i-1}^2) \\
&= \sum_{i=1}^{n} \sigma_i P_{i-1} - \sigma_i P_{i-1}^2 \\
&= \sum_{i=1}^{n} \sigma_i P_{i-1}(1 - P_{i-1}) \leq Y_n
\end{aligned} \tag{3.3}$$

The statement (3.1) can be proven by assuming that $Y_\infty = \infty$ almost surely .Both divergence and convergence for the process $(\langle M \rangle_n)$ will be considered to get the desired result.

Suppose that $(\langle M \rangle_n)$ converges, then by [9, § 12.13] the process (M_n) converges almost surely. According to (3.2), this gives that (T_n) is the sum of a constant, an almost surely random variable and a diverging random variable. This means we get the following:

$$\mathbb{P}\left(\lim_{n\to\infty} T_n = \infty\right) = \mathbb{P}\left(\lim_{n\to\infty} T_0 + M_n + Y_n = \infty\right) = \mathbb{P}\left(\lim_{n\to\infty} Y_n = \infty\right) = 1. \tag{3.4}$$

We can conclude that $T_n \to \infty$ almost surely as well.

Suppose that $(\langle M \rangle_n)$ diverges. Then almost surely holds that

$$\lim_{n\to\infty} \frac{M_n}{\langle M \rangle_n} = 0,$$

see [9, §12.14] Since $\langle M \rangle_n \leq Y_n$, this gives that

$$\lim_{n\to\infty} \frac{M_n}{Y_n} = 0$$

almost surely as well. This means the following holds, almost surely, since $\lim_{n\to\infty} Y_n = \infty$ almost surely and $Y_n > 0$ for all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{T_n}{Y_n} &= \frac{T_0}{Y_n} + \frac{M_n}{Y_n} + 1\\ \lim_{n\to\infty}\frac{T_n}{Y_n} &= 0 + 0 + 1\\ \lim_{n\to\infty} T_n &= \lim_{n\to\infty} Y_n = \infty.\end{aligned}$$

This means $T_n \to \infty$ almost surely. This completes the proof of claim (3.2). When it succeeds to show that $Y_\infty = \infty$ almost surely, the lemma is proven. Observe that by (2.3) and the facts that $P_{n-1} = \psi(\Theta_{n-1})$ and $T_n \geq 1$ for all $n \in \mathbb{N}$, almost surely,

$$Y_\infty = \sum_{n=1}^{\infty} \sigma_n \psi(\Theta_{n-1}) \geq \sum_{n=1}^{\infty} \sigma_n \Theta_{n-1}^{\alpha} = \sum_{n=1}^{\infty} \sigma_n \frac{T_{n-1}^{\alpha}}{\tau_{n-1}^{\alpha}} \geq \sum_{n=1}^{\infty} \frac{\sigma_n}{\tau_{n-1}^{\alpha}} = \infty.$$

This means we have that $\mathbb{P}(Y_\infty) = 1$. From (3.1) we get that

$$\mathbb{P}\Big(Y_\infty) \leq \mathbb{P}(\lim_{n\to\infty} T_n = \infty\Big).$$

This gives that $\mathbb{P}(\lim_{n\to\infty} T_n = \infty) = 1$. Recall that T_n is the number of balls in the first bin. Since this argument holds in general (and does not focus on one specific bin), we now also have that $\hat{T}_n \to \infty$ almost surely.
Recall that the event monopoly is defined as follows:

$$\mathcal{M} = \{B_n = 0 \text{ eventually for all } n\} \cup \{B_n = \sigma_n \text{ eventually for all } n\}.$$

Since the balls in both bins are going to infinity, we notice that

$$\mathbb{P}(\mathcal{M}) = \mathbb{P}(\{B_n = 0 \text{ eventually for all } n\}).$$

When one bin does not get anything in the end, the other bin will get everything, and the other way around. This gives the following

$$\begin{aligned}\mathbb{P}(\mathcal{M}) &= \mathbb{P}(\{B_n = 0 \text{ eventually for all } n\})\\ &= 1 - \mathbb{P}(\{B_n = 0 \text{ eventually for all } n\}^c)\\ &= 1 - \mathbb{P}(\{B_n > 0\} \text{ i.o })\end{aligned}$$

Since $T_n \to \infty$ almost surely, it follows that $\mathbb{P}(\{B_n > 0\} \text{ i.o }) = 1$. We can conclude that indeed $\mathbb{P}(\mathcal{M}) = 0$. □

We have enough information to prove the theorem where this chapter is about, Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, it suffices to show that $\beta = \infty$ implies that $\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^{\alpha}} = \infty$. Assume, to the contrary, that $\beta = \infty$, but $\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^{\alpha}} = c$, with a constant $c \in \mathbb{R}$. Then it follows that $\lim_{n\to\infty} \frac{\sigma_{n+1}}{\tau_n^{\alpha}} = 0$. This means that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $\frac{\sigma_{n+1}}{\tau_n^{\alpha}} < 1$. This implies that

$$\sigma_{n+1} < \tau_n^{\alpha}.$$

Using the relation that $\tau_n = \tau_{n-1} + \sigma_n$, we can make the following estimation for all $n \geq N$

$$\begin{aligned}\tau_n &= \tau_{n-1} + \sigma_n \leq \tau_{n-1} + \tau_{n-1}^{\alpha} \leq 2\tau_{n-1}^{\alpha} < (2\tau_{n-2} + 2\tau_{n-2}^{\alpha})^{\alpha} \\ &\leq 2^{1+\alpha}(\tau_{n-2})^{\alpha^2} \leq \ldots\ldots \leq 2^{\sum_{i=0}^{n-N-1}\alpha^i}\tau_N^{\alpha^{n-N}}. \end{aligned} \tag{3.5}$$

Now we need to distinguish between two different cases. Suppose that $1 < \alpha < 2$, then we can upper bound τ_n as follows:

$$\tau_n \leq 2^{\sum_{i=0}^{n-N-1}\alpha^i}\tau_N^{\alpha^{n-N}} \leq (2\tau_N)^{\frac{\alpha^{n-N}}{\alpha-1}}.$$

This implies, since N is a fixed number,

$$\begin{aligned}\beta &= \lim_{n\to\infty} \alpha^{-n}\log(\tau_n) \leq \lim_{n\to\infty} \alpha^{-n}\log((2\tau_N)^{\frac{\alpha^{n-N}}{\alpha-1}}) \\ &= \lim_{n\to\infty} \alpha^{-n}\frac{\alpha^{n-N}}{\alpha-1}\log(2\tau_N) = \frac{\alpha^{-N}}{\alpha-1}\log(2\tau_N) < \infty.\end{aligned}$$

We assumed that $\beta = \infty$, so this is a contradiction.

Suppose now that $\alpha \geq 2$. Then we get from (3.5) the following upper bound for τ_n

$$\tau_n \leq 2^{\sum_{i=0}^{n-N-1}\alpha^i}\tau_N^{\alpha^{n-N}} \leq (2\tau_N)^{\alpha^{n-N}}.$$

This means that

$$\begin{aligned}\beta &= \lim_{n\to\infty} \alpha^{-n}\log(\tau_n) \leq \lim_{n\to\infty} \alpha^{-n}\log((2\tau_N)^{\alpha^{n-N}}) \\ &= \lim_{n\to\infty} \alpha^{-n}\alpha^{n-N}\log(2\tau_N) = \alpha^{-N}\log(2\tau_N) < \infty.\end{aligned}$$

We assumed that $\beta = \infty$, so this is a also contradiction. From the above, we can conclude that $\sum_{n=0}^{\infty}\frac{\sigma_{n+1}}{\tau_n^{\alpha}} = \infty$ for all $\alpha > 1$, and thus that $\mathbb{P}(\mathcal{M}) = 0$ by Lemma 3.2. □

Notice that from this lemma it follows that $\mathbb{P}(\mathcal{M}) = 0$ when $\alpha = 1$ and thus the feedback function is $f(m) = m$. This means there is no feedback in this case. The simple proof can be found in the lemma below.

Lemma 3.3. *Suppose* $\alpha = 1$. *Then* $\mathbb{P}(\mathcal{M}) = 0$.

Proof. Recall that we assume everywhere that $\sum_{n=0}^{\infty}\frac{\sigma_n}{\tau_n} = \infty$. This means, since $\sigma_{n+1} \geq \sigma_n$, that we get the following result

$$\sum_{n=1}^{\infty}\frac{\sigma_{n+1}}{\tau_n^{\alpha}} = \sum_{n=1}^{\infty}\frac{\sigma_{n+1}}{\tau_n} \geq \sum_{n=1}^{\infty}\frac{\sigma_n}{\tau_n} = \infty.$$

By Lemma 3.2 it follows that $\mathbb{P}(\mathcal{M}) = 0$. □

In the last chapter of this thesis we will show that in the no feedback case there is no dominance as well.

4 Dominance

In this chapter almost sure dominance in the positive feedback case will be shown. The theorems and proofs are based on [8, sections 4 & 5]. The aim of this chapter is to prove the following theorem:

Theorem 4.1. *Suppose $\alpha > 1$. Furthermore, assume that $\sum_{n=0}^{\infty} \left(\frac{\sigma_n}{\tau_n}\right)^{\frac{4}{3}} < \infty$. Then $\mathbb{P}(\mathcal{D}) = 1$.*

We will first need to prove an important, difficult lemma to make sure we can prove the main theorem of this chapter. In this lemma, we show that Θ_n deviates from the equilibrium far enough infinitely often. Before we get started with this, we need to know more about the specific equilibrium points of our model.

4.1 Equilibrium points

Equilibrium points can be thought of as points where the proportion Θ_n won't change anymore, hence $\psi(\Theta_{n+1}) = \Theta_n$ holds for all n large enough, see [6]. This means the equilibrium points for both bins can be found by solving the equation

$$\begin{gathered} h(\Theta_n) = \psi(\Theta_n) - \Theta_n = 0 \\ \frac{\Theta_n^\alpha}{\Theta_n^\alpha + (1-\Theta_n)^\alpha} - \Theta_n = 0. \end{gathered}$$

We immediately see that $\Theta = 0$ and $\Theta = 1$ are solutions to this equation. Now assume that $\Theta_n \notin \{0, 1\}$. Then we find

$$\begin{gathered} \frac{\Theta_n^\alpha}{\Theta_n^\alpha + (1-\Theta_n)^\alpha} = \Theta_n \\ \Theta_n^{\alpha-1} = \Theta_n^\alpha + (1-\Theta_n)^\alpha \\ \Theta_n^{\alpha-1}(1-\Theta_n) = (1-\Theta_n)^\alpha \\ \Theta_n = (1-\Theta_n) \\ \Theta = \frac{1}{2}. \end{gathered}$$

The third equilibrium point for the first bin is equal to $\Theta = \frac{1}{2}$. Since the situation we are in is symmetric and we are working with proportions of balls in respectively the first and second bin, we get the following three equilibrium points:

$$(\Theta, \hat{\Theta}) \in \left\{(0,1), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right)\right\} = \mathcal{E}.$$

The set of all equilibrium points is denoted by $\mathcal{E}$. As mentioned earlier, $\mathcal{D}$ corresponds with the first two equilibrium points. The third equilibrium point is used in the next section, to show that Θ_n deviates significantly infinitely often from this point. Note we do not know anything specific about stability or convergence of Θ_n yet. We only know that if it converges, it should converge to one of these 3 points.

4.2 Deviations

The aim of this section is to prove this important lemma.

Lemma 4.2. *Suppose $\alpha > 1$. Let $(\delta_n)_{n\in\mathbb{N}}$ be a positive sequence converging to zero and such that*

$$\sum_{n=0}^{\infty} \delta_n \frac{\sigma_{n+1}}{\tau_{n+1}} < \infty. \tag{4.1}$$

Then

$$\mathbb{P}\left(|\Theta_n - \frac{1}{2}| > \delta_n \text{ infinitely often } \right) = 1.$$

Proof. Denote

$$\mathcal{H}_m = \left\{ |\Theta_n - \frac{1}{2}| \leq \delta_n \text{ for all } n \geq m \right\}.$$

It holds that

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \ldots.$$

and so on, meaning the events $\mathcal{H}_m$ are increasing. This implies

$$\mathbb{P}(\mathcal{H}_1) \leq \lim_{m\to\infty} \mathbb{P}(\mathcal{H}_m)$$

if the right hand side exists. We are going to prove that $\lim_{m\to\infty} P(\mathcal{H}_m) = 0$, giving that $P(\mathcal{H}_i) = 0$ for all $i \geq 1$. Define the sequence of events $A_n = \{|\Theta_n - \frac{1}{2}| > \delta_n\}$. Then will hold that

$$\mathbb{P}\left(\mathcal{H}_k\right) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} A_n^c \right) = \mathbb{P}\left(\liminf_{n\to\infty} A_n^c \right) = \mathbb{P}\left(\left(\limsup_{n\to\infty} A_n \right)^c \right). \tag{4.2}$$

When $\mathbb{P}(\mathcal{H}_k) = 0$, it implies immediately that $\mathbb{P}(\limsup_{n\to\infty} A_n) = \mathbb{P}(|\Theta_n - \frac{1}{2}| > \delta_n \text{ infinitely often }) = 1$. That is exactly what needs to be shown.

By the mean value theorem we have

$$\psi'(\eta_x) = \frac{\psi(\frac{1}{2}) - \psi(x)}{\frac{1}{2} - x}$$

$$\psi(x) = \psi'(\eta_x)(x - \frac{1}{2}) + \frac{1}{2} \tag{4.3}$$

for $\eta_x \in (\frac{1}{2}, x)$. Multiplying both sides of (4.3) with $\frac{\sigma_{n+1}}{\tau_{n+1}}$ and adding the term $\frac{\tau_n}{\tau_{n+1}}x - \frac{1}{2}$, we obtain

$$\frac{\tau_n}{\tau_{n+1}}x - \frac{1}{2} + \frac{\sigma_{n+1}}{\tau_{n+1}}\psi(x) = \frac{\tau_n}{\tau_{n+1}}x - \frac{1}{2} + \frac{\sigma_{n+1}}{\tau_{n+1}}\left(\psi'(\eta_x)\left(x - \frac{1}{2}\right) + \frac{1}{2}\right) := \kappa_n(x)\left(x - \frac{1}{2}\right),$$

where

$$\kappa_n(x) = \frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}}\psi'(\eta_x).$$

This holds since

$$\frac{\sigma_{n+1}}{2\tau_{n+1}} - \frac{1}{2} = \frac{\sigma_{n+1} - \tau_{n+1}}{2\tau_{n+1}} = -\frac{\tau_n}{2\tau_{n+1}}.$$

We can use the expression for κ_n to rewrite the expression we can already deduce for $\Theta_n - \frac{1}{2}$. Dividing relation (2.6) by τ_n and using the properties we know, we get

$$\begin{aligned}\Theta_n &= \frac{T_{n-1}}{\tau_n} + \frac{\sigma_n\psi(\Theta_{n-1})}{\tau_n} + \varepsilon_n\frac{\sqrt{\sigma_n P_{n-1}(1 - P_{n-1})}}{\tau_n}\\ \Theta_n - \frac{1}{2} &= \frac{\tau_{n-1}}{\tau_n}\Theta_{n-1} + \frac{\sigma_n\psi(\Theta_{n-1})}{\tau_n} + \varepsilon_n\frac{\sqrt{\sigma_n P_{n-1}(1 - P_{n-1})}}{\tau_n} - \frac{1}{2}.\end{aligned}$$

In the last expression exactly $\kappa_{n-1}(\Theta_{n-1})(\Theta_{n-1} - \frac{1}{2})$ can be found, hence

$$\Theta_n - \frac{1}{2} = \kappa_{n-1}(\Theta_{n-1})\left(\Theta_{n-1} - \frac{1}{2}\right) + \varepsilon_n\frac{\sqrt{\sigma_n P_{n-1}(1 - P_{n-1})}}{\tau_n}.$$

Notice that the term $\Theta_{n-1} - \frac{1}{2}$ is in the expression of $\Theta_n - \frac{1}{2}$, meaning the same can be done again. This gives

$$\begin{aligned}\Theta_n - \frac{1}{2} &= \kappa_{n-1}(\Theta_{n-1})\left(\kappa_{n-2}(\Theta_{n-2})\left(\Theta_{n-2} - \frac{1}{2}\right) + \varepsilon_{n-1}\frac{\sqrt{\sigma_{n-1}P_{n-2}(1 - P_{n-2})}}{\tau_{n-1}}\right)\\ &+ \varepsilon_n\frac{\sqrt{\sigma_n P_{n-1}(1 - P_{n-1})}}{\tau_n}.\\ &= \kappa_{n-1}(\Theta_{n-1})(\kappa_{n-2}(\Theta_{n-2})\left(\Theta_{n-2} - \frac{1}{2}\right) + \kappa_{n-1}(\Theta_{n-1})\varepsilon_{n-1}\frac{\sqrt{\sigma_{n-1}P_{n-2}(1 - P_{n-2})}}{\tau_{n-1}}\\ &+ \varepsilon_n\frac{\sqrt{\sigma_n P_{n-1}(1 - P_{n-1})}}{\tau_n}.\end{aligned}$$

Now we have seen what happens in the first two steps, this equation can be re-iterated for all m and $n > m$

$$\begin{aligned}\Theta_n - \frac{1}{2} &= \left[\prod_{j=m}^{n-1}\kappa_j(\Theta_j)\right]\left(\Theta_m - \frac{1}{2}\right) + \sum_{k=m+1}^{n}\left[\prod_{j=k}^{n-1}\kappa_j(\Theta_j)\right]\varepsilon_k\frac{\sqrt{\sigma_k P_{k-1}(1 - P_{k-1})}}{\tau_k}\\ &= \left[\prod_{j=m}^{n-1}\kappa_j(\Theta_j)\right]\left(\Theta_m - \frac{1}{2} + \sum_{k=m+1}^{n}\left[\prod_{j=m}^{k-1}\frac{1}{\kappa_j(\Theta_j)}\right]\varepsilon_k\frac{\sqrt{\sigma_k P_{k-1}(1 - P_{k-1}}}{\tau_k}\right).\end{aligned} \tag{4.4}$$

In the next part we are going to analyse the asymptotic behaviour of the product $\Pi_{j=m}^{k-1}\kappa_j(\Theta_j)$.
Note the event $\mathcal{H}_m$ is equivalent to $\mathcal{H}_m = \{-\frac{1}{2} - \delta_n \leq \Theta_n \leq \frac{1}{2} + \delta_n \text{ for all } n \geq m\}$. From this we have by continuity properties of the derivative $\psi'(x)$ and conditioned on the event $\mathcal{H}_m$ that

$$\psi'(\eta_x) = \psi'\left(\frac{1}{2}\right) + O\left(\delta_j\right).$$

For a sequence $(a_j)_{j\in\mathbb{N}}$ it holds that $a_j \in O(\delta_j)$ if there exists an $M \in \mathbb{R}$ such that $|a_j| \leq M\delta_j$ for all sufficiently large j. Note that $\psi'\left(\frac{1}{2}\right) = \alpha$. This can be plugged in the expression for $\kappa_j(\Theta_j)$:

$$\kappa_j(\Theta_j) = \frac{\tau_j}{\tau_{j+1}} + \frac{\sigma_{j+1}}{\tau_{j+1}}\left(\psi'\left(\frac{1}{2}\right) + O\left(\delta_j\right)\right) = \frac{\tau_j}{\tau_{j+1}} + \alpha\frac{\sigma_{j+1}}{\tau_{j+1}} + \frac{\sigma_{j+1}}{\tau_{j+1}}O(\delta_j).$$

Using this, the following expression for $\Pi_{j=m}^{n-1}\kappa_j(\Theta_j)$ can be obtained

$$\begin{aligned}\prod_{j=m}^{k-1}\kappa_j(\Theta_j) &= \prod_{j=m}^{k-1}\left(\frac{\tau_j}{\tau_{j+1}} + \alpha\frac{\sigma_{j+1}}{\tau_{j+1}} + \frac{\sigma_{j+1}}{\tau_{j+1}}O(\delta_j)\right)\\ &= \prod_{j=m}^{k-1}\left(\frac{\tau_j + \alpha\sigma_{j+1}}{\tau_{j+1}}\right)\left(1 + \frac{\sigma_{j+1}O(\delta_j)}{\tau_j + \alpha\sigma_{j+1}}\right).\end{aligned}$$

Let

$$\pi_{m,k} = \prod_{j=m}^{k-1}\left(\frac{\tau_j + \alpha\sigma_{j+1}}{\tau_{j+1}}\right). \tag{4.5}$$

Furthermore, the other expression of the product can be rewritten to

$$\prod_{j=m}^{k-1}\left(1 + \frac{\sigma_{j+1}O(\delta_j)}{\tau_j + \alpha\sigma_{j+1}}\right) = \exp\left\{\sum_{j=m}^{k-1}\log\left(1 + \frac{\sigma_{j+1}O(\delta_j)}{\tau_j + \alpha\sigma_{j+1}}\right)\right\}.$$

Notice that

$$\sum_{j=m}^{k-1}\frac{\sigma_{j+1}\delta_j}{\tau_j + \alpha\sigma_{j+1}} \leq \sum_{j=m}^{\infty}\delta_j\frac{\sigma_{j+1}}{\tau_{j+1}} \to 0$$

if $m \to \infty$, since the latter is the tail of a convergent sum, by the assumption of this lemma. This means that

$$\sum_{j=m}^{k-1}\frac{\sigma_{j+1}\delta_j}{\tau_j + \alpha\sigma_{j+1}} = o(1).$$

A sequence $(a_n)_{n\in\mathbb{N}} = o(1)$ if $\lim_{n\to\infty} a_n = 0$. Using this, we obtain on the event $\mathcal{H}_m$ for all m large enough and $k > m$ that

$$\prod_{j=m}^{k-1}\kappa_j(\Theta_j) = \pi_{m,k}\exp\left\{\sum_{j=m}^{k-1}\log\left(1 + \frac{\sigma_{j+1}O(\delta_j)}{\tau_j + \alpha\sigma_{j+1}}\right)\right\} = \pi_{m,k}(1 + o(1)). \tag{4.6}$$

This can be used to plug in the expression for $\Theta_n - \frac{1}{2}$. Before that, using (4.3), we notice, since $f(x) = \sqrt{x(1-x)} \le \frac{1}{2}$ for all $x \in \mathbb{R}_{\ge 0}$, on the event $\mathcal{H}_m$, as $m \to \infty$ uniformly in ω and $k > m$ holds that

$$\sqrt{P_{k-1}(1-P_{k-1})} = \sqrt{\psi(\Theta_{k-1})(1-\psi(\Theta_{k-1}))} = \frac{1}{2} + O(\delta_{k-1}) = 1 + o(1). \qquad (4.7)$$

Now we have enough information to rewrite the expression for $\Theta_n - \frac{1}{2}$. Using (4.6) and (4.7), together with (4.4), we get the following as $m \to \infty$ uniformly in ω and $n > m$

$$\begin{aligned}\Theta_n - \frac{1}{2} &= \left[\prod_{j=m}^{n-1} \kappa_j(\Theta_j)\right] \left(\Theta_m - \frac{1}{2} + \sum_{k=m+1}^{n} \left[\Pi_{j=m}^{k-1} \frac{1}{\kappa_j(\Theta_j)}\right] \varepsilon_k \frac{\sqrt{\sigma_k P_{k-1}(1-P_{k-1}}}{\tau_k}\right) \\ &= \pi_{m,n}(1+o(1)) \left(\Theta_m - \frac{1}{2} + \sum_{k=m+1}^{n} \frac{\frac{1}{2}+o(1)}{1+o(1)} \varepsilon_k \frac{\sqrt{\sigma_k}}{\tau_k \pi_{m,k}}\right). \qquad (4.8)\end{aligned}$$

Notice that

$$\frac{\frac{1}{2}+o(1)}{1+o(1)} = \frac{1}{2} + o(1) = \frac{1+2o(1)}{2} = \frac{1+o(1)}{2}.$$

Plugging this in what we already have, we obtain the following expression

$$\Theta_n - \frac{1}{2} = \pi_{m,n}(1+o(1)) \left(\Theta_m - \frac{1}{2} + \frac{1+o(1)}{2} \sum_{k=m+1}^{n} \varepsilon_k \frac{\sqrt{\sigma_k}}{\tau_k \pi_{m,k}}\right).$$

In the last part of this proof we will explore the behaviour of $\sum_{k=m+1}^{n} \varepsilon_k \frac{\sqrt{\sigma_k}}{\tau_k \pi_{m,k}}$. For that we use the characteristic function for the variable ε_n and will show it is close to normal under certain conditions. Recall that for a binomial variable X with parameter p, we have the following characteristic function, [5, pg. 304]

$$\mathbb{E}[\exp\{itX\}] = (1-p+pe^{it})^n.$$

Using this, we obtain

$$\begin{aligned}\mathbb{E}\left[\exp\left\{it\frac{X-np}{\sqrt{np(1-p)}}\right\}\right] &= \left(1-p+p\exp\left\{\frac{it}{\sqrt{np(1-p)}}\right\}\right)^n e^{-\frac{it\sqrt{np}}{\sqrt{1-p}}} \\ &= \left(1 + \frac{it\sqrt{p}}{\sqrt{n(1-p)}} - \frac{t^2}{2n(1-p)} + \right. \\ &\qquad \left. O\left(\frac{t^3}{np(1-p)\sqrt{np(1-p)}}\right)\right)^n e^{-\frac{it\sqrt{np}}{\sqrt{1-p}}}.\end{aligned}$$

In the last step we used the Taylor expansion of the exponent. Notice that $O\left(\frac{t^3}{np(1-p)\sqrt{np(1-p)}}\right)$ can be written as $O(\frac{t^3}{n\sqrt{n}})$, if error terms are not too large. For this, we need to make sure that the denominator cannot become really small. This can be obtained by setting $\frac{1}{4} < p < \frac{3}{4}$. Notice, using the Taylor expansion, that

$\log(1+x) \approx x$. Using this and taking the limit of $n \to \infty$ uniformly in $\frac{1}{4} < p < \frac{3}{4}$ and t, we get

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{it\frac{X-np}{\sqrt{np(1-p)}}\right\}\right] &= \left(1+\frac{it\sqrt{p}}{\sqrt{n(1-p)}}-\frac{t^2}{2n(1-p)}+O\left(\frac{t^3}{n\sqrt{n}}\right)\right)^n e^{-\frac{it\sqrt{np}}{\sqrt{1-p}}} \\
&= \exp\left\{n\log\left(1+\frac{it\sqrt{p}}{\sqrt{n(1-p)}}-\frac{t^2}{2n(1-p)}+O\left(\frac{t^3}{n\sqrt{n}}\right)\right)-\frac{it\sqrt{np}}{\sqrt{1-p}}\right\} \\
&= \exp\left\{n\left(\frac{it\sqrt{p}}{\sqrt{n(1-p)}}-\frac{t^2}{2n(1-p)}+O\left(\frac{t^3}{n\sqrt{n}}\right)\right)-\frac{it\sqrt{np}}{\sqrt{1-p}}\right\} \\
&= \exp\left\{-\frac{t^2}{2}+nO\left(\frac{t^3}{n\sqrt{n}}\right)\right\}.
\end{aligned}$$

Since we know ε_k is exactly like the expression in the expected value, we know that the following holds, when $k \to \infty$ uniformly in ω and t,

$$\mathbb{E}_{\mathcal{F}_{k-1}}\left[e^{it\varepsilon_k}\right] = \exp\left\{\frac{-t^2}{2}+\sigma_k O\left(\frac{t^3}{\sigma_k\sqrt{\sigma_k}}\right)\right\}.$$

Now we need to work towards the expression from what we want to know the distribution. First, consider the term $\exp\{\frac{it\sqrt{\sigma_k}}{\mu_{k-1,k}\pi_{k-1,m}\tau_k}\varepsilon_k\}$. Then the conditional expectation is equal to, when $m \to \infty$ uniformly in ω and t

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_{k-1}}\left[\exp\left\{\frac{\varepsilon_k it\sqrt{\sigma_k}}{\mu_{k-1,k}\pi_{k-1,k}\tau_k}\right\}\right] &= \exp\left\{\frac{-t^2}{2}\frac{\sigma_k}{(\mu_{k-1,k}\pi_{k-1,k}\tau_k)^2}+\sigma_k O\left(\frac{t^3\sigma_k^{\frac{3}{2}}}{(\mu_{k-1,k}\pi_{k-1,k}\tau_k)^3\sigma_k\sqrt{\sigma_k}}\right)\right\} \\
&= \exp\left\{-\frac{t^2}{2}\frac{\sigma_k}{(\mu_{k-1,k}\pi_{k-1,k}\tau_k)^2}+\sigma_k O\left(\frac{t^3}{(\mu_{k-1,k}\pi_{k-1,k}\tau_k)^3}\right)\right\} \\
&= \exp\left\{-\frac{t^2\sigma_k}{2\mu_{k-1,k}^2\pi_{k-1,k}^2\tau_k^2}+O(1)\frac{\sigma_k t^3}{(\mu_{k-1,k}\pi_{k-1,k}\tau_k)^3}\right\}.
\end{aligned}$$

Let

$$\mu_{m,n} = \left[\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^2\tau_k^2}\right]^{\frac{1}{2}}.$$

By applying the tower property [5, pg. 174], by noticing that $\mathcal{F}_m \subseteq \mathcal{F}_n$ for $n \geq m$, we obtain the following result for each fixed t, as $m \to \infty$ and $n \geq m$

$$\mathbb{E}_{\mathcal{F}_m}\left[\frac{it}{\mu_{m,n}}\sum_{k=m+1}^{n}\varepsilon_k\frac{\sqrt{\sigma_k}}{\pi_{m,k}\tau_k}\right] = \exp\left\{-\frac{t^2}{2}\sum_{k=m+1}^{n}\frac{\sigma_k}{\mu_{m,n}^2\pi_{m,k}^2\tau_k^2}+O(1)\frac{1}{\mu_{m,n}^3}\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^3\tau_k^3}\right\}. \tag{4.9}$$

Note, by plugging in $\mu_{m,n}$, notice that

$$\sum_{k=m+1}^{n}\frac{\sigma_k}{\mu_{m,n}^2\pi_{m,k}^2\tau_k^2} = \sum_{k=m+1}^{n}\frac{\sigma_k}{[\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,n}^2\tau_k^2}]\pi_{m,n}^2\tau_k^2} = \frac{\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^2\tau_k^2}}{\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^2\tau_k^2}} = 1.$$

When we use this in (4.9), we obtain the following result

$$\mathbb{E}_{\mathcal{F}_m}\left[\frac{it}{\mu_{m,n}}\sum_{k=m+1}^{n}\varepsilon_k\frac{\sqrt{\sigma_k}}{\pi_{m,k}\tau_k}\right] = \exp\left\{-\frac{t^2}{2}+O(1)\frac{1}{\mu_{m,n}^3}\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^3\tau_k^3}\right\}. \tag{4.10}$$

We will show now that

$$\frac{1}{\mu_{m,n}^3}\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^3\tau_k^3} \tag{4.11}$$

tends to zero as $m \to \infty$. We are going to consider two different cases for this, since we assumed that (σ) holds.

(i) $\sigma_i \to \infty$.

(ii) (σ_i) is bounded, i.e. for all $i \geq 1$ we have $\sigma_i \leq \sigma$, with $\sigma \in \mathbb{N}$.

We start with (i). Observe the following holds for all non-negative x_i with $i \in \mathbb{N}$ and for all m

$$\begin{aligned}(x_1+x_2+....x_m)^{\frac{3}{2}} &= x_1\sqrt{(x_1+x_2+....+x_m)}+x_2\sqrt{(x_1+x_2+....+x_m)}+\\ &\quad x_m\sqrt{(x_1+x_2+....+x_m)} \geq x_1\sqrt{x_1}+x_2\sqrt{x_2}+....+x_m\sqrt{(x_m}\\ &= x_1^{\frac{3}{2}}+x_2^{\frac{3}{2}}....+x_m^{\frac{3}{2}}.\end{aligned}$$

Applying this on $\mu_{m,n}^3$, we obtain

$$\begin{aligned}\mu_{m,n}^3 &\geq \sum_{k=m+1}^{n}\frac{\sigma_k^{\frac{3}{2}}}{\pi_{m,k}^3\tau_k^3}\\ &\geq \min_{m+1\leq k\leq n}\sqrt{\sigma_k}\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^3\tau_k^3}.\end{aligned}$$

Giving

$$\frac{1}{\mu_{m,n}^3}\sum_{k=m+1}^{n}\frac{\sigma_k}{\pi_{m,k}^3\tau_k^3} \leq \frac{1}{\min_{(m+1)\leq k\leq n}\sqrt{\sigma_k}} = \max_{(m+1)\leq k\leq n}\frac{1}{\sqrt{\sigma_k}}.$$

When $m \to \infty$ uniformly in n, we will have, since $\sigma_i \to \infty$, that $\max_{(m+1)\leq k\leq n}\frac{1}{\sqrt{\sigma_k}} \to 0$. Since the terms are positive, that is the desired result.

Now the other case (ii) need to be considered. Suppose that $\sigma_i \leq \sigma$ for all $i \in \mathbb{N}$. The idea is to bound the expression (4.11). Both upper- and lower bounds for σ_i and τ_i are needed. We know that $1 \leq \sigma_i \leq \sigma$ for all $i \in \mathbb{N}$. For the other bounds we use that $i \leq \tau_i$ by definition of τ_i. Furthermore it holds that

$$\tau_i \leq \max_{1\leq k\leq i}\sigma_k(i+1) \leq \sigma(i+1) \leq 2\sigma i \text{ for all } i \in \mathbb{N}. \tag{4.12}$$

Since $\pi_{m,k}$ is part of $\mu_{m,n}$ and the summation in question we need both an upper- and lower bound. First, we need to rewrite (4.5). This gives, using that $\tau_{j+1}-\sigma_{j+1}=\tau_j$, the following

$$
\begin{aligned}
\pi_{m,k} &= \prod_{j=m}^{k-1} \frac{\tau_j + \alpha\sigma_{j+1}}{\tau_{j+1}} \\
&= \exp\left\{ \sum_{j=m}^{k-1} \log(1 + (\alpha - 1)\frac{\sigma_{j+1}}{\tau_{j+1}}) \right\} \\
&= \exp\left\{ \sum_{j=m+1}^{k} \log(1 + (\alpha - 1)\frac{\sigma_j}{\tau_j}) \right\}. \qquad (4.13)
\end{aligned}
$$

Using a Taylor expansion for $\log(1 + x)$, there can be derived that $\log(1 + x) = x + o(1)$, the variable itself plus some convergent error term. This way, we can derive the upper and lower bounds for $\pi_{m,k}$. Using this alternative expression and the upper bound for σ_i and lower bound for τ_i, we obtain as $m \to \infty$ uniformly in $k > m$

$$
\begin{aligned}
\pi_{m,k} &= \exp\left\{ \sum_{j=m+1}^{k} \log(1 + (\alpha - 1)\frac{\sigma_j}{\tau_j}) \right\} \leq \exp\left\{ \sum_{j=m+1}^{k} \log(1 + (\alpha - 1)\frac{\sigma}{j}) \right\} \\
&= \exp\left\{ ((\alpha - 1)\sigma + o(1)) \sum_{j=m+1}^{k} \frac{1}{j} \right\} = \exp\left\{ ((\alpha - 1)\sigma + o(1))(\log(k) - \log(m) + o(1)) \right\} \\
&= \frac{k^{(\alpha-1)\sigma+o(1)}}{m^{(\alpha-1)\sigma+o(1)}}.
\end{aligned}
$$

In the same way the lower bound can be found. For this, use (4.12),

$$
\begin{aligned}
\pi_{m,k} &= \exp\left\{ \sum_{j=m+1}^{k} \log(1 + (\alpha - 1)\frac{\sigma_j}{\tau_j}) \right\} \geq \exp\left\{ \sum_{j=m+1}^{k} \log(1 + (\alpha - 1)\frac{1}{2\sigma j}) \right\} \\
&= \exp\left\{ (\frac{((\alpha - 1)}{2\sigma} + o(1)) \sum_{j=m+1}^{k} \frac{1}{j} \right\} = \exp\left\{ (\frac{(\alpha - 1)}{2\sigma} + o(1))(\log(k) - \log(m) + o(1)) \right\} \\
&= \frac{k^{\frac{(\alpha-1)}{\sigma}+o(1)}}{m^{\frac{(\alpha-1)}{\sigma}+o(1)}}.
\end{aligned}
$$

Observe that for all $\gamma > 1$ the following holds

$$
\sum_{k=m+1}^{n} \frac{1}{k^{\gamma+o(1)}} \leq \int_{m}^{\infty} k^{-\gamma+o(1)} dk = \frac{1}{m^{\gamma-1+o(1)}}.
$$

This holds when $m \to \infty$ uniformly in $n > m$, where n is sufficiently large. Since there is an error term in the equation, it holds that for $n > m^2$ the expression will hold with equality. Using what we did above, we can upper bound $\mu_{m,n}^3$ to be able to lower bound $\frac{1}{\mu^3_{m,n}}$ afterwards. Notice that $\frac{1}{\tau_i} \geq \frac{1}{2\sigma i}$. This gives

$$\begin{aligned}
\mu^3_{m,n} &= \left[\sum_{k=m+1}^{n} \frac{\sigma_k}{\pi^2_{m,k}\tau^2_k}\right]^{\frac{3}{2}} \geq \left[\sum_{k=m+1}^{n} \frac{1}{(\frac{k^{(\alpha-1)\sigma+o(1)}}{m^{(\alpha-1)\sigma+o(1)}})^2(2\sigma k)^2}\right]^{\frac{3}{2}} \\
&= \frac{(m^{(\alpha-1)\sigma+o(1)})^3}{8\sigma^3}\left[\sum_{k=m+1}^{n} \frac{1}{(k^{(\alpha-1)\sigma+o(1)})^2 k^2}\right]^{\frac{3}{2}} \\
&= \frac{(m^{3(\alpha-1)\sigma+o(1)})}{8\sigma^3}\left[\sum_{k=m+1}^{n} \frac{1}{k^{2(\alpha-1)\sigma+2+o(1)}}\right]^{\frac{3}{2}} \\
&= m^{3(\alpha-1)\sigma-\frac{3}{2}(2(\alpha-1)\sigma+1)+o(1)} = m^{-\frac{3}{2}+o(1)}.
\end{aligned}$$

Clearly, this gives the following result

$$\frac{1}{\mu^3_{m,n}} \leq m^{\frac{3}{2}+o(1)}. \tag{4.14}$$

Now we are able to upper bound the summation part of the term we want to approximate. Using again the same relations as in the last part, we obtain the following

$$\begin{aligned}
\sum_{k=m+1}^{n} \frac{\sigma_k}{\pi^3_{m,k}\tau^3_k} &\leq \sum_{k=m+1}^{n} \frac{\sigma}{\left(\frac{k^{\frac{(\alpha-1)}{\sigma}+o(1)}}{m^{\frac{(\alpha-1)}{\sigma}+o(1)}}\right)^3 (2\sigma k)^3} \\
&\leq \sigma m^{3\frac{(\alpha-1)}{\sigma}+o(1)} \sum_{k=m+1}^{n} \frac{1}{k^{\frac{3(\alpha-1)}{\sigma}+o(1)+3}} \\
&= \sigma m^{\frac{3(\alpha-1)}{\sigma}-\frac{3(\alpha-1)}{\sigma}-2+o(1)} = \sigma m^{-2+o(1)}.
\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), the following can be derived

$$\frac{1}{\mu^3_{m,n}} \sum_{k=m+1}^{n} \frac{\sigma_k}{\pi^3_{m,k}\tau^3_k} \leq \sigma m^{\frac{3}{2}+o(1)-2} = \sigma m^{-\frac{1}{2}+o(1)} \to 0,$$

since $m \to \infty$ uniformly in $n > m^2$.

We are almost ready to finish the proof of this important lemma. The only thing that is left to do is concluding that indeed $\mathbb{P}(H_m)$ is zero for all m. For this, take a sequence in $\mathbb{N}$, $(n_m)_{m\in\mathbb{N}}$, satisfying the necessary condition that $n_m > m^2$. This is needed to make sure we can use the bounds we made in the last part of this proof.

For the sequence (n_m) holds that it satisfies (4.10), on the event $\mathcal{H}_m$. The distribution of a random variable is uniquely determined by his characteristic function, see [5, pg. 297, 304]. Since the random variable $\frac{X}{\mu^3_{m,n}}$, where $X = \sum_{k=m+1}^{n} \varepsilon_k \frac{\sqrt{\sigma_k}}{\pi_{m,k}\tau_k}$ has in the limit characteristic function $e^{-\frac{t^2}{2}}$, we can conclude that $\frac{X}{\mu^3_{m,n}} = N_{m,n_m}$ converges in distribution to a normal variable N, conditionally on Θ_m. By (4.8), we now can conclude that on the event $\mathcal{H}_m$

$$\Theta_{n_m} - \frac{1}{2} = (1+o(1))\pi_{m,n_m}\left(\Theta_m - \frac{1}{2} + \frac{1+o(1)}{2}\mu_{m,n_m}N_{m,n_m}\right).$$

Since the event H_m holds for all $n \geq m$, we can conclude that

$$\mathcal{H}_m \subseteq \left\{|\Theta_{n_m} - \frac{1}{2}| \leq \delta_{n_m}\right\} = \left\{(1+o(1)\pi_{m,n_m}(\Theta_m - \frac{1}{2} + \frac{1+o(1)}{2}\mu_{m,n_m}N_{m,n_m}) \in [-\delta_{n_m}, \delta_{n_m}]\right\}.$$

This gives that

$$\mathbb{P}(\mathcal{H}_m) \leq \mathbb{P}\Bigg((1+o(1))\pi_{m,n_m}\left(\Theta_m - \frac{1}{2} + \frac{1+o(1)}{2}\mu_{m,n_m}N_{m,n_m}\right) \in [-\delta_{n_m}, \delta_{n_m}]\Bigg).$$

Showing that the expression on the right tends to 0 in the limit will give the desired result. Notice that on the event $\mathcal{H}_m$, $\Theta_m \approx \frac{1}{2}$ when m grows, and $\pi_{m,n_m} < \infty$. Since $\delta_{n_m} \to 0$, when we show that $\pi_{m,n_m}\mu_{m,n_m} \to \infty$, it will imply that the event on the right hand side is impossible and thus

$$\mathbb{P}\left((1+o(1))\pi_{m,n_m}\left(\Theta_m - \frac{1}{2} + \frac{1+o(1)}{2}\mu_{m,n_m}N_{m,n_m}\right) \in [-\delta_{n_m}, \delta_{n_m}]\right) \to 0.$$

Notice the following

$$\pi_{m,m+1} = \frac{\tau_j + \alpha\sigma_{j+1}}{\tau_{j+1}} \leq \alpha\frac{\tau_{j+1}}{\tau_{j+1}} = \alpha.$$

We can lower bound μ_{m,n_m} by $\mu_{m,m+1}$, because the terms are all positive. Using this, the fact that $\sigma_{m+1} \geq 1$ and (4.13), we obtain

$$\pi_{m,n_m}\mu_{m,n_m} \geq \pi_{m,n_m}\frac{\sqrt{\sigma_{m+1}}}{\pi_{m,m+1}\tau_{m+1}} \geq \pi_{m,n_m}\frac{1}{\alpha\tau_{m+1}} = \frac{1}{\alpha\tau_{m+1}}\exp\left\{\sum_{j=m+1}^{k}\log(1+(\alpha-1)\frac{\sigma_j}{\tau_j})\right\}$$

By (2.7) we can conclude that

$$\log\left(1+(\alpha-1)\frac{\sigma_j}{\tau_j}\right) = \infty,$$

giving that $\pi_{m,n_m}\mu_{m,n_m} \to \infty$ when $m \to \infty$. This means we need to choose (n_m) such that it grows sufficiently fast enough to guarantee this condition. Then we can conclude that

$$\lim_{m\to\infty}\mathbb{P}(\mathcal{H}_m) = 0$$
$$\mathbb{P}(\mathcal{H}_m) = 0 \text{ for all } m.$$

As we have already motivated earlier in (4.2), this implies that

$$\mathbb{P}\left(|\Theta_n - \frac{1}{2}| > \delta_n \text{ infinitely often}\right) = 1.$$

This is the desired result. □

4.3 Moving away from the equilibrium

Before we can start proving Theorem 4.1, we need to make sure we are in the situation of the previous lemma. For this, we need to pick a sequence of deviations (δ_n) and make sure it converges. Let

$$\delta_n = \frac{1}{\tau_n^{\frac{3}{10}}\log(\tau_n)}.$$

It indeed holds that $\lim_{n\to\infty}\delta_n = 0$. The necessary convergence of the summation, assumed in the previous lemma, will be shown in the following lemma.

Lemma 4.3. *If* $\sum_{n=0}^{\infty}\left(\frac{\sigma_n}{\tau_n}\right)^{\frac{4}{3}} < \infty$*, then*

$$\sum_{n=0}^{\infty}\delta_n\frac{\sigma_{n+1}}{\tau_{n+1}} = \sum_{n=0}^{\infty}\frac{1}{\tau_n^{\frac{3}{10}}\log(\tau_n)}\frac{\sigma_{n+1}}{\tau_{n+1}} < \infty.$$

Proof. To prove this lemma, we are using the Hölder's inequality of [5, pg. 152], applicated on summations. Note, to use this inequality, we need to choose $p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. In this case we choose $p = \frac{4}{3}$ and $q = 4$. This means we get the following inequality

$$\sum_{n=0}^{\infty}\delta_n\frac{\sigma_{n+1}}{\tau_{n+1}} = \sum_{n=0}^{\infty}\left|\delta_n\frac{\sigma_{n+1}}{\tau_{n+1}}\right| \leq \left(\sum_{n=0}^{\infty}|\delta_n|^4\right)^{\frac{1}{4}}\left(\sum_{n=0}^{\infty}\left|\frac{\sigma_{n+1}}{\tau_{n+1}}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}.$$

Since we assumed that $\sum_{n=0}^{\infty}\left(\frac{\sigma_n}{\tau_n}\right)^{\frac{4}{3}} < \infty$, we get that the right part of the product will be finite as well, since all terms are positive by construction. We only need to show that $\left(\sum_{n=0}^{\infty}|\delta_n|^4\right)^{\frac{1}{4}}$ is finite. For that, since $T_0, \hat{T}_0 > 0$, we have that $\tau_0, \tau_1 \geq 2$. Since $\sigma_n > 0$ for all $n \in \mathbb{N}$, we have that $\tau_n \geq n$ for all $n \geq 2$. Using this, we obtain the following result

$$\sum_{n=0}^{\infty}|\delta_n|^4 = \sum_{n=0}^{\infty}\delta_n^4 = \sum_{n=0}^{\infty}\frac{1}{\tau_n^{\frac{12}{10}}\log^4(\tau_n)} \leq \frac{2}{2\log^4(2)} + \sum_{n=2}^{\infty}\frac{1}{n^{\frac{12}{10}}\log^4(n)} < \infty.$$

From this we can conclude that both summations that upper bound our term in question are finite, this means that

$$\left(\sum_{n=0}^{\infty}|\delta_n|^4\right)^{\frac{1}{4}}\left(\sum_{n=0}^{\infty}\left|\frac{\sigma_{n+1}}{\tau_{n+1}}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}} < \infty.$$

Hence we can conclude that indeed holds that

$$\sum_{n=0}^{\infty}\delta_n\frac{\sigma_{n+1}}{\tau_{n+1}} < \infty.$$

This is the desired result. □

This means, with this choice of the sequence (δ_n), we are exactly in the situation of Lemma 4.2 of the previous section. To be able to prove dominance in the positive feedback case, two more lemmas are needed.

Lemma 4.4. *Assume that* $\sum_{n=0}^{\infty}(\frac{\sigma_n}{\tau_n})^{\frac{4}{3}} < \infty$*. Then*

$$\sum_{k=\xi+1}^{\infty}\frac{\sigma_k}{\tau_k^2} \leq c\tau_\xi^{-\frac{3}{4}}.$$

Proof. Note that ξ is just a number such that $\xi \in \mathbb{N}$. It will be specified later what it is, but not necessary for this proof. By again applying the inequality of Hölder [5, pg. 152], we get the following result:

$$\sum_{k=\xi+1}^{\infty}\frac{\sigma_k}{\tau_k^2} = \sum_{k=\xi+1}^{\infty}\left|\frac{\sigma_k}{\tau_k^2}\right| \leq \left(\sum_{k=\xi+1}^{\infty}\left|\frac{\sigma_k}{\tau_k}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(\sum_{k=\xi+1}^{\infty}\frac{1}{|\tau_k^4|}\right)^{\frac{1}{4}}.$$

Because all the terms we are working with are positive, we can ignore the absolute value. The left part of the upper bound is finite by assumption. For the other part, we use the rules of bounding a summation by an integral, to obtain the following

$$\left(\sum_{k=\xi}^{\infty} \frac{1}{\tau_k^4}\right)^{\frac{1}{4}} \leq a\left(\int_{\tau_\xi}^{\infty} \frac{1}{x^4}dx\right)^{\frac{1}{4}} \leq \frac{1}{3}a\tau_\xi^{-\frac{3}{4}},$$

where $a \in \mathbb{R}$ is a constant. Combining the above and denoting $c \in \mathbb{R}$ for everything that is constant, we can conclude that

$$\sum_{k=\xi+1}^{\infty} \frac{\sigma_k}{\tau_k^2} \leq c\tau_\xi^{-\frac{3}{4}}.$$

□

Lemma 4.5. *For all $x \in [0, \frac{1}{2}]$, it holds that $x \geq \psi(x)$.*

Proof. We need to verify that

$$x \geq \frac{x^\alpha}{x^\alpha + (1-x)^\alpha}$$

It is equivalent to show that $x^\alpha \leq x(1-x)^\alpha + x^{\alpha+1}$. Since $\alpha > 1$ and $0 \leq x \leq \frac{1}{2}$ it holds that,

$$x^{\alpha-1} \leq (1-x)^{\alpha-1}.$$

Multiplying both sides with $x(1-x) \geq 0$, we get

$$x^\alpha(1-x) \leq x(1-x)^\alpha.$$

Rearranging this expression gives the desired result:

$$\begin{gathered} x^\alpha - x^{\alpha+1} \leq x(1-x)^\alpha \\ x^\alpha \leq x^{\alpha+1} + x(1-x)^\alpha. \end{gathered}$$

□

Now we are ready to prove the main theorem of this chapter, Theorem 4.1. The outline of the proof will be as follows. We will use the lemma of the previous section to pick a time ξ where Θ_n deviates from the equilibrium, using (δ_n). We know (and will motivate) that this time ξ is finite almost surely. The variable Θ_n will be decomposed in a martingale part M_n and a so-called bias part R_n. The fluctuations of M_n will be small with high probability and we can show that (M_n) converges almost surely, so this part won't let Θ_n go back to the equilibrium. The bias part only will move Θ_n further away from the equilibrium. Using martingale convergence arguments from Chapter 1, there can be concluded that dominance holds almost surely. Recall that the event dominance equals

$$\mathcal{D} = \{\lim_{n\to\infty} \Theta_n \in \{0, 1\}\}.$$

Proof of Theorem 4.1. Let

$$\xi = \inf\left\{n \geq r : |\Theta_n - \frac{1}{2}| > \delta_n\right\},$$

where $r \geq \lceil e^{10} \rceil$ fixed and (δ_n) used as in Lemma 4.3. By Lemma 4.2 of the previous section we have that almost surely holds that $|\Theta_n - \frac{1}{2}| > \delta_n$ infinitely often . This means that for all $m \in \mathbb{N}$ there exists an $k \geq m$ such that $|\Theta_k - \frac{1}{2}| > \delta_k$. This means that the value of ξ is finite almost surely.

Since the situation we are in is symmetric, rewriting the expression for ξ, it is enough to consider the event

$$\mathcal{E} = \left\{\Theta_\xi < \frac{1}{2} - \delta_\xi\right\}.$$

Given this event, we will proof that $\Theta_n \to 0$ almost surely. This is consistent with the story above, since the deviation at time ξ is below $\frac{1}{2}$. Automatically it holds, because of the three equilibrium points, that $\hat{\Theta}_n \to 1$.

When we would consider the other case

$$\hat{\mathcal{E}} = \left\{\Theta_\xi > \frac{1}{2} + \delta_\xi\right\},$$

we would find that Θ_n is going to 1 as $n \to \infty$ and $\hat{\Theta}_n \to 0$. The proof will go in the same manner as the proof that $\Theta_n \to 0$, that we are doing now.

First, we we are constructing the decomposition of Θ_n in a martingale part and a bias part. By the properties of T_n and Θ_n defined in chapter 2, we can construct the following for each $n \in \mathbb{N}$:

$$\begin{aligned}
\Theta_{n+1} &= \frac{T_{n+1}}{\tau_{n+1}} = \frac{T_n}{\tau_{n+1}} + \frac{B_{n+1}}{\tau_{n+1}} = \frac{\tau_n}{\tau_{n+1}}\Theta_n + \frac{1}{\tau_{n+1}}B_{n+1} \qquad (4.16)\\
&= \frac{\tau_{n+1} - \sigma_{n+1}}{\tau_{n+1}}\Theta_n + \frac{1}{\tau_{n+1}}B_{n+1}\\
&= \Theta_n - \frac{\sigma_{n+1}}{\tau_{n+1}}\Theta_n + \frac{B_{n+1} - \sigma_{n+1}P_n}{\tau_{n+1}} + \frac{\sigma_{n+1}P_n}{\tau_{n+1}}\\
&= \Theta_n + \frac{B_{n+1} - \sigma_{n+1}P_n}{\tau_{n+1}} - \frac{\sigma_{n+1}}{\tau_{n+1}}(\Theta_n - \psi(\Theta_n)). \qquad (4.17)
\end{aligned}$$

Note that $\psi(\Theta_n) = P_n$. Iterating further in the term Θ_n, for each $k \geq \xi$, and $n \geq 1$, we get

$$\begin{aligned}
\Theta_{n+\xi} &= \Theta_\xi + M_n - R_n \qquad (4.18)\\
M_n &= \sum_{k=\xi+1}^{\xi+n} \frac{B_k - \sigma_k P_{k-1}}{\tau_k}\\
R_n &= \sum_{k=\xi+1}^{\xi+n} \frac{\sigma_k}{\tau_k}(\Theta_{k-1} - \psi(\Theta_{k-1})).
\end{aligned}$$

The process (M_n) is a martingale with respect to the filtration $(\mathcal{F}_{\xi+n}) = (\sigma(B_1), \sigma(B_2), ..., \sigma(B_{\xi+n}))$. This can be shown in the same way we proved that (M_n) is a martingale in Chapter 3. The only difference is that in this case we are not summing from $k = 1$, but $k = \xi + 1$, and we divided by τ_k. This does not matter for the proof.

It is time to show that (M_n) converges and is small with high probability, so that (Θ_n) is not driven back to the equilibrium by (M_n). We show that $(M_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}^2$, using Definition 1.10. Let $\mathcal{F}_{(\xi)} = \mathcal{F}_{\xi+n}$.

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_{(\xi)}}[M_n^2] &= \mathbb{E}_{\mathcal{F}_{(\xi)}}\left[\left(M_{n-1} + \frac{B_{\xi+n} - \sigma_{\xi+n}P_{\xi+n-1}}{\tau_{\xi+n}}\right)^2\right] \\
&= \mathbb{E}_{\mathcal{F}_{(\xi)}}\left[M_{n-1}^2 + 2M_{n-1}\frac{B_{\xi+n} - \sigma_{\xi+n}P_{\xi+n-1}}{\tau_{\xi+n}} + \left(\frac{B_{\xi+n} - \sigma_{\xi+n}P_{\xi+n-1}}{\tau_{\xi+n}}\right)^2\right] \\
&= \mathbb{E}_{\mathcal{F}_{(\xi)}}\left[M_{n-1}^2 + \left(\frac{B_{\xi+n} - \sigma_{\xi+n}P_{\xi+n-1}}{\tau_{\xi+n}}\right)^2\right].
\end{aligned}$$

The last step holds since $M_{n-1}, B_{\xi+n}$ and $P_{\xi+n-1}$ are $\mathcal{F}_{(\xi)}$ measurable and $\mathbb{E}_{\mathcal{F}_{(\xi)}}[B_{\xi+n} - \sigma_{\xi+n}P_{\xi+n-1}] = 0$. Doing the same as done in Chapter 3, equation (3.3), we can conclude that

$$\mathbb{E}_{\mathcal{F}_{(\xi)}}\left[\left(\frac{B_{\xi+n} - \sigma_{\xi+n}P_{\xi+n-1}}{\tau_{\xi+n}}\right)^2\right] = \frac{\sigma_{\xi+n}}{\tau_{\xi+n}^2}P_{\xi+n-1}(1 - P_{\xi+n-1}.)$$

Since for all $n \in \mathbb{N}$ $P_n(1 - P_n) \leq 1$, the following iteration holds

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_{(\xi)}}[M_n^2] &= \mathbb{E}_{\mathcal{F}_{(\xi)}}\left[M_{n-1}^2 + \frac{\sigma_{\xi+n}}{\tau_{\xi+n}^2}P_{\xi+n-1}(1 - P_{\xi+n-1})\right] \\
&\leq \mathbb{E}_{\mathcal{F}_{(\xi)}}[M_{n-1}^2] + \frac{\sigma_{\xi+n}}{\tau_{\xi+n}^2} \leq ... \leq \sum_{k=\xi+1}^{\xi+n} \frac{\sigma_k}{\tau_k^2} \leq c\tau_\xi^{-\frac{3}{4}}.
\end{aligned}$$

In the last step we used the result of Lemma 4.4 .This means that

$$\mathbb{E}[|M_n|^2] = \mathbb{E}[\mathbb{E}_{\mathcal{F}_{(\xi)}}[M_n^2]] \leq c\tau_\xi^{-\frac{3}{4}}.$$

We can conclude that M_n is bounded in $\mathcal{L}^2$, since

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|^2] < \infty.$$

Hence (M_n) converges almost surely by Theorem 1.10. This was the convergence part of (M_n).

Now, let's show that it is small with high probability. For that, denote

$$\mathcal{S} = \{\sup_{n\in\mathbb{N}} M_n \leq \frac{\delta_\xi}{2}\}$$

Using Doobs-$\mathcal{L}^p$-inequality of [5, pg. 218], we obtain the following result

$$\mathbb{P}\left(\sup_{1\le k\le n} M_k > \frac{\delta_\xi}{2}\right) \le \mathbb{P}\left(\sup_{1\le k\le n} M_k^2 > \frac{\delta_\xi^2}{4}\right)$$
$$\le \frac{4}{\delta_\xi^2}\,\mathbb{E}[M_n^2] \le 4c\log^2(\tau_\xi)\tau_\xi^{\frac{6}{10}}\tau_\xi^{-\frac{3}{4}} = \frac{4c\log^2(\tau_\xi)}{\tau_\xi^{\frac{3}{20}}}.$$

The function $f(x) = \frac{\log^2(x)}{x^{\frac{3}{20}}}$ is decreasing for $x \in [\lceil e^{10}\rceil, \infty]$. Since $\xi \ge r \ge \lceil e^{10}\rceil$ and thus $\tau_\xi \ge \tau_r$, we obtain

$$\mathbb{P}\left(\sup_{1\le k\le n} M_k > \frac{\delta_\xi}{2}\right) \le \frac{4c\log^2(\tau_\xi)}{\tau_\xi^{\frac{3}{20}}} \le \frac{4c\log^2(\tau_r)}{\tau_r^{\frac{3}{20}}}.$$

Since the events $\{\sup_{1\le k\le n} M_k > \frac{\delta_\xi}{2}\}$ are increasing, we can take the limit inside the probability measure, to obtain the following result about $\mathcal{S}^c$:

$$\mathbb{P}(\mathcal{S}^c) = \mathbb{P}\left(\lim_{n\to\infty}\sup_{1\le k\le n} M_k > \frac{\delta_\xi}{2}\right) = \lim_{n\to\infty}\mathbb{P}\left(\sup_{1\le k\le n} M_k > \frac{\delta_\xi}{2}\right)$$
$$\le \lim_{n\to\infty}\frac{4c\log^2(\tau_r)}{\tau_r^{\frac{3}{20}}} = \frac{4c\log^2(\tau_r)}{\tau_r^{\frac{3}{20}}}.$$

We now show by induction that when the event $\mathcal{S}\cap\mathcal{E}$ occurs, for all $n\in\mathbb{N}$ holds that

$$\Theta_{\xi+n} < \frac{1}{2} - \frac{\delta_\xi}{2}, \tag{4.19}$$

this means, once the proportion of the first bin is below the equilibrium, it will always stay below the equilibrium.

For $n = 0$, it follows directly from the event $\mathcal{E}$, since $\Theta_\xi < \frac{1}{2} - \delta_\xi < \frac{1}{2} - \frac{\delta_\xi}{2}$. Suppose now it is true for all k with $0 \le k \le n-1$. Since we look at (4.19) on the event $\mathcal{S}\cap\mathcal{E}$, we have $\Theta_{k-1} \le \frac{1}{2}$ for all $\xi+1 \le k \le \xi+n$. That means, using Lemma 4.5, that for these values of k, $\Theta_{k-1} \ge \psi(\Theta_{k-1})$. This implies that

$$R_n = \sum_{k=\xi+1}^{\xi+n} \frac{\sigma_k}{\tau_k}(\Theta_{k-1} - \psi(\Theta_{k-1}) \ge 0.$$

Using this, and the decomposition of (4.18), we get the desired result

$$\Theta_{\xi+n} \le \Theta_\xi + M_n < \frac{1}{2} - \delta_\xi + \frac{\delta_\xi}{2} = \frac{1}{2} - \frac{\delta_\xi}{2}. \tag{4.20}$$

Observe since this holds for all $n\in\mathbb{N}$, that (R_n) is positive and increasing on $\mathcal{S}\cap\mathcal{E}$. This does not immediately show that (R_n) is convergent, but we notice the following. By the decomposition in (4.18), we see the convergence of (Θ_n) depends on (M_n) and (R_n). We showed that the process (M_n) is convergent almost surely. Notice that when (R_n) diverges, we can find $c_1, c_2 \in \mathbb{R}$ such that

$$0 \le c_1\sum_{k=\xi+1}^{\xi+n}\frac{\sigma_k}{\tau_k} \le R_n \le c_2\sum_{k=\xi+1}^{\xi+n}\frac{\sigma_k}{\tau_k}.$$

Since we know that (2.7) holds, this means that then $R_n \to \infty$. Because $\Theta_k \in [0,1]$ for all $k \in \mathbb{N}$ and (M_n) convergent almost surely, by the decomposition in (4.18), this is impossible. Hence we can conclude that R_n converges almost surely. From this, we can immediately conclude that Θ_n converges almost surely on $\mathcal{S} \cap \mathcal{E}$.

We are showing something stronger, namely that on $\mathcal{S} \cap \mathcal{E}$

$$\lim_{n\to\infty} \Theta_n = 0.$$

Assume, to the contrary, that this does not hold. That means, there is an $\omega \in \mathcal{S} \cap \mathcal{E}$ such that

$$\lim_{n\to\infty} \Theta_n(\omega) = \Theta > 0.$$

Then by (4.20), we have $\Theta < \frac{1}{2}$, giving that $\Theta > \psi(\Theta)$ and that for values of k large enough holds that

$$\frac{\sigma_k}{\tau_k}(\Theta_{k-1} - \psi(\Theta_{k-1}) \approx \frac{\sigma_k}{\tau_k}(\Theta - \psi(\Theta).$$

This implies, since we assumed (2.7), that

$$\sum_{k=\xi+1}^{\infty} \frac{\sigma_k}{\tau_k}((\Theta_{k-1} - \psi(\Theta_{k-1}) = \infty.$$

This means that $R_n \to \infty$ when $n \to \infty$, and thus that $\Theta = -\infty$, which is not possible as already argued. This means that

$$\mathbb{P}\left(\lim_{n\to\infty} \Theta_n = 0 | \mathcal{S} \cap \mathcal{E}\right) = 1.$$

The only thing left to show is that $\mathbb{P}\left(\lim_{n\to\infty} \Theta_n = 0 | \mathcal{E}\right) = 1$. Since $S^c \cap \mathcal{E} \subseteq S^c$, we have that $\mathbb{P}(S^c \cap \mathcal{E}) \leq \mathbb{P}(S^c)$. We know that $\mathbb{P}(S^c) \leq \frac{4c\log^2(\tau_r)}{\tau_r^{\frac{3}{20}}}$. for all $r \geq \lceil e^{10} \rceil$. This means that

$$\lim_{r\to\infty} \mathbb{P}(S^c) \leq \lim_{r\to\infty} \frac{4c\log^2(\tau_r)}{\tau_r^{\frac{3}{20}}}.$$

$$\mathbb{P}(S^c) \leq \lim_{r\to\infty} \frac{4c\log^2(\tau_r)}{\tau_r^{\frac{3}{20}}} = 0.$$

From this we can see that $\mathbb{P}(\mathcal{S}^c) = 0$ must hold. This means that $\mathbb{P}(\mathcal{S}^c \cap \mathcal{E}) = 0$. Hence we can conclude that

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{S} \cap \mathcal{E}) + \mathbb{P}(\mathcal{S}^c \cap \mathcal{E}) = \mathbb{P}(\mathcal{S} \cap \mathcal{E}).$$

Since $\mathbb{P}(\lim_{n\to\infty} \Theta_n = 0 | \mathcal{S} \cap \mathcal{E}) = 1$, it holds that

$$\mathbb{P}\left(\lim_{n\to\infty} \Theta_n = 0 | \mathcal{E}\right) \geq \mathbb{P}\left(\lim_{n\to\infty} \Theta_n = 0 | \mathcal{S} \cap \mathcal{E}\right) = 1.$$

This means $\mathbb{P}(\mathcal{D}) = 1$ on $\mathcal{E}$.

$\square$

5 | Subcritical and critical regime

In this chapter the subcritical and critical regime will be discussed. In some cases we will refer to [8] for specific steps or proofs. The aim of this chapter is to get an idea of the proofs of the theorems that correspond with the sections.

5.1 Subcritical regime

The theorems and proofs of this section are highly inspired by [8, section 6, 7]. Here, the proof of the following theorem will be discussed. Before we start, let

$$\lambda = \limsup_{n\to\infty} \frac{\sigma_{n+1}\sigma_{n-1}^{\alpha}}{\sigma_n^{\alpha+1}}$$
$$\lambda_n = \frac{\sigma_{n+1}\sigma_{n-1}^{\alpha}}{\sigma_n^{\alpha+1}}$$

This limit will play an important roll in the subcritical regime.

Theorem 5.1. *Suppose $\alpha > 1$ and $\beta = 0$.*

$$\text{If } (\rho_n) \text{ is bounded, then } \mathbb{P}(\mathcal{M}) = 1.$$
$$\text{If } \rho_n \to \infty, \text{ then } \mathbb{P}(\mathcal{M}) = \begin{cases} 1 & \text{if } \lambda < 1 \\ 0 & \text{if } \lambda > 1. \end{cases}$$

By Theorem 4.1, we now that $\mathbb{P}(\mathcal{D}) = 1$. This means we already know that

$$\lim_{n\to\infty} \Theta_n \in \{0, 1\}.$$

By symmetry, consider the event $\mathcal{E} = \{\Theta_n \to 0\}$. When we can proof that T_n is bounded on $\mathcal{E}$, it will imply that

$$\mathbb{P}(\mathcal{M}) = \mathbb{P}(\{B_n = 0 \text{ eventually for all } n\}) = 1.$$

We start with a lemma, from which directly follows a part of the proof.

Lemma 5.2. *Suppose $\rho_n \to \infty$, $\beta = 0$. Then, if $\lambda < 1$,*

$$\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^{\alpha}} < \infty,$$

and if $\lambda > 1$,

$$\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^{\alpha}} = \infty.$$

Proof of Lemma 5.2. First note, if $\rho_n \to \infty$, we can use the following decomposition

$$\lim_{n\to\infty} \frac{\tau_n}{\sigma_n} = \lim_{n\to\infty} \left(\frac{\tau_{n-1}}{\sigma_n} + 1\right) = \lim_{n\to\infty} \left(\frac{1}{\rho_{n-1}} + 1\right) = 1. \tag{5.1}$$

This implies that $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n$. Since we know that
$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n$, we get that

$$\begin{aligned}\limsup_{n\to\infty} \frac{\rho_n}{\rho_{n-1}^\alpha} &= \limsup_{n\to\infty} \frac{\sigma_{n+1}}{\tau_n} \frac{\tau_{n-1}^\alpha}{\sigma_n^\alpha} = \limsup_{n\to\infty} \frac{\tau_{n+1}}{\sigma_n} \frac{\tau_{n-1}^\alpha}{\sigma_n^\alpha} \\ &= \limsup_{n\to\infty} \frac{\sigma_{n+1}}{\sigma_n} \frac{\sigma_{n-1}^\alpha}{\sigma_n^\alpha} = \limsup_{n\to\infty} \lambda_n = \lambda. \end{aligned} \tag{5.2}$$

The following expression can be checked by writing it all out. Observe that

$$\frac{\sigma_{n+1}}{\tau_n^\alpha} = \frac{\sigma_n}{\tau_{n-1}^\alpha} \frac{\rho_n}{\rho_{n-1}^\alpha} \left(\frac{\rho_{n-1}}{1+\rho_{n-1}}\right)^{\alpha-1}.$$

Using (5.1) and (5.2), we obtain,

$$\limsup_{n\to\infty} \frac{\rho_n}{\rho_{n-1}^\alpha} \left(\frac{\rho_{n-1}}{1+\rho_{n-1}}\right)^{\alpha-1} = \lambda.$$

This holds, since

$$\limsup_{n\to\infty} \left(\frac{\rho_{n-1}}{1+\rho_{n-1}}\right)^{\alpha-1} = 1,$$

if $\rho_n \to \infty$. When we apply the ratio test on the summations in question, we obtain by (5.2) the following limit

$$\limsup_{n\to\infty} \frac{\left(\frac{\sigma_{n+1}}{\tau_n^\alpha}\right)}{\left(\frac{\sigma_n}{\tau_{n-1}^\alpha}\right)} = \limsup_{n\to\infty} \frac{\rho_n}{\rho_{n-1}^\alpha} = \lambda.$$

This indeed implies divergence of the series when $\lambda > 1$ and convergence when $\lambda < 1$. $\square$

We can directly prove a part of the theorem of this section.

Proposition 5.3. *When $\rho_n \to \infty$, $\lambda > 1$, then $\mathbb{P}(\mathcal{M}) = 0$.*

Proof. By Lemma 5.2 it holds that

$$\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^\alpha} = \infty.$$

By Lemma 3.2, it immediately follows that $\mathbb{P}(\mathcal{M}) = 0$. $\square$

Now the cases where ρ_n is bounded and $\rho_n \to \infty$, $\lambda < 1$ are left. The idea is to prove this with the help of 2 propositions. The case where ρ_n is bounded will be easier, since (σ_n) is not growing too fast. This makes us able to make the following approximation with the help of a Riemann-integral,

$$\sum_{i=n}^{\infty} \frac{\sigma_{i+1}}{\tau_i^\alpha} \approx \int_{\tau_n}^{\infty} \frac{dx}{x^\alpha} = \frac{1}{\alpha-1} \frac{1}{\tau_n^{\alpha-1}}. \tag{5.3}$$

Proposition 5.4. *Suppose $\alpha > 1$ and (ρ_n) is bounded. Then $\mathbb{P}(\mathcal{M}) = 1$.*

The idea of the proof is as follows. By contradiction, we assume that $T_n \to \infty$ on $\mathcal{E}$. Then we can derive an upper- and lower bound for the expression

$$\sum_{i=n}^{\infty} \frac{T_{i+1} - T_i}{T_i^\alpha}.$$

By ignoring this term afterwards, but only looking at the specific lower- and upper bound, we can show by Lemma 6.4 of [8], that

$$\liminf_{n\to\infty} \Theta_n^{1-\frac{\alpha}{2}} \sqrt{\tau_n} < \infty$$

holds. We can show this is a contraction with $T_n \to \infty$ on the event $\mathcal{E}$. We won't proof Lemma 6.4 of Sidorova here. In this lemma, two upper bounds for summations are elaborated. This is mainly done with use of Theorem 1.1, by showing the martingale in question is bounded in $\mathcal{L}^2$, and with use of Chebychev's inequality [3, pg. 121].

Proof of Proposition 5.4. Assume $T_n \to \infty$ on $\mathcal{E}$ and (ρ_n) is bounded by $M \in \mathbb{R}$. Then by (2.3) and (2.6), we can derive the following inequality:

$$\begin{aligned} T_{n+1} &\le T_n + 2^{\alpha-1}\Theta_n^\alpha \sigma_{n+1} + \varepsilon_{n+1}\sqrt{\sigma_{n+1}P_n(1-P_n)} \\ &= T_n + 2^{\alpha-1}\frac{T_n^\alpha}{\tau_n^\alpha}\sigma_{n+1} + \varepsilon_{n+1}\sqrt{\sigma_{n+1}P_n(1-P_n)}. \end{aligned} \tag{5.4}$$

Giving that

$$\frac{T_{i+1} - T_i}{T_i^\alpha} \le \frac{2^{\alpha-1}}{\tau_i^\alpha}\sigma_{i+1} + \frac{\varepsilon_{i+1}\sqrt{\sigma_{n+1}P_i(1-P_i)}}{T_i^\alpha} = \frac{2^{\alpha-1}}{\tau_i^\alpha}\sigma_{i+1} + \xi_i\varepsilon_{i+1} \tag{5.5}$$

where

$$\xi_i = \frac{\sqrt{\sigma_{n+1}P_i(1-P_i)}}{T_i^\alpha}.$$

We assume (and it is proven in [8] by using a Riemann integral) that

$$\sum_{i=n}^{\infty} 2^{\alpha-1}\frac{\sigma_{i+1}}{\tau_i^\alpha} < \frac{c}{\tau_n^{\alpha-1}}$$

where $c = \frac{2^{\alpha-1}(1+M)^\alpha}{\alpha-1}$. This gives the following upperbound for the summation, by (5.5)

$$\sum_{i=n}^{\infty} \frac{T_{i+1} - T_i}{T_i^\alpha} \le \frac{c}{\tau_n^{\alpha-1}} + \sum_{i=n}^{\infty} \xi_i\varepsilon_{i+1}.$$

We can also find a lower bound for the summation, doing what we also did in (5.3), since $T_{n+1} - T_n \ge 1$ when n is large enough,

$$\sum_{i=n}^{\infty} \frac{T_{i+1} - T_i}{T_i^\alpha} \ge \int_{T_n}^{\infty} \frac{dx}{x^\alpha} = \frac{1}{\alpha-1}\frac{1}{T_n^{\alpha-1}}$$

Combining both bounds, we obtain the following result

$$\frac{1}{\alpha-1}\frac{1}{T_n^{\alpha-1}} \le \sum_{i=n}^{\infty}\frac{T_{i+1}-T_i}{T_i^{\alpha}} \le \frac{c}{\tau_n^{\alpha-1}} + \sum_{i=n}^{\infty}\xi_i\varepsilon_{i+1}$$

$$\frac{1}{\alpha-1}\frac{1}{T_n^{\alpha-1}} \le \frac{c}{\tau_n^{\alpha-1}} + \sum_{i=n}^{\infty}\xi_i\varepsilon_{i+1}.$$

When we multiply both sides with $(\alpha-1)\tau_n^{\alpha-1}$, we get the following

$$\frac{1}{\Theta_n^{\alpha-1}} - c(\alpha-1) \le \tau_n^{\alpha-1}(\alpha-1)\sum_{i=n}^{\infty}\xi_i\varepsilon_{i+1}.$$

By Lemma 6.4 of Sidorova [8, pg. 16] we can upper bound this in such a way, that we can conclude from this that

$$\liminf_{n\to\infty}[\frac{1}{\Theta_n^{\alpha-1}} - c(\alpha-1)]\Theta_n^{\frac{\alpha}{2}}\sqrt{\tau_n} < \infty.$$

Since c is a constant, it follows that

$$\liminf_{n\to\infty}[\frac{1}{\Theta_n^{\alpha-1}}]\Theta_n^{\frac{\alpha}{2}}\sqrt{\tau_n} = \liminf_{n\to\infty}\Theta_n^{-\frac{\alpha}{2}+1}\sqrt{\tau_n} < \infty.$$

The only thing left to show is that this is a contradiction. Suppose $\alpha > 2$. Then $\Theta^{-\frac{\alpha}{2}+1} \to \infty$, but we are on the event $\mathcal{E}$ and this is clearly a contradiction. When $1 < \alpha < 2$, the above implies that

$$\liminf_{n\to\infty}\Theta_n^{-\frac{\alpha}{2}+1}\sqrt{\tau_n} = \liminf_{n\to\infty}T_n\tau_n^{\frac{\alpha-1}{2-\alpha}} = 0,$$

since $\Theta_n \to 0$ and $\frac{\alpha-1}{2-\alpha} > 0$ here. But we assumed that $T_n \to \infty$, what clearly is not possible here anymore. We can conclude that T_n is bounded on the event $\mathcal{E}$, and thus that $\mathbb{P}(\mathcal{M}) = 1$. $\square$

There is only one case left that needs to be proven. This is the most difficult case, since the proof is really long and contains a lot of detailed steps. That's why only an outline of the proof will be given here, by dividing the proof in 3 steps. The whole proof can be found in [8, pg. 19-23]. After this proposition, we have all the tools to prove our main theorem.
Note that in the case below the approximation (5.3) is not valid anymore.

Proposition 5.5. *Suppose $\alpha > 1$, $\beta = 0$, $\rho_n \to \infty$ and $\lambda < 1$. Then $\mathbb{P}(\mathcal{M}) = 1$.*

Outline of proof. As in the last proposition, we are proving that the following event has probability zero:

$$\mathcal{N} = \{\Theta_n \to 0\} \cap \{T_n \to \infty\}.$$

This will imply that $\mathbb{P}(\mathcal{M}) = 1$. Define the following stopping time, with $\kappa_0 = 0$, for the definition see [5, pg. 192]

$$\kappa_0 = 0,$$
$$\kappa_n = \inf\{i > \kappa_{n-1} : \Theta_i^{\alpha}\sigma_{i+1} \le \delta T_i\},$$

for all $n \geq 1$.

Step 1. There is started with proving that κ_n is finite for all n almost surely. Suppose, to the contrary, this is not the case. That means,

$$\hat{n} = \sup\{i \in \mathbb{N} : \Theta_i^\alpha \sigma_{i+1} \leq \delta T_i\}$$

is finite with positive probability, given that $\mathcal{N}$ holds. This may sound a bit strange, but intuitively, this means the following. When the supremum of $\hat{n}$ is finite, it would imply that there is a number k such that after for all $n \geq k$ the inequality does not hold anymore. That would directly imply that κ_n is not finite for all $n \in \mathbb{N}$ anymore, since there is no value such that the inequality holds. By using the fact that this all would mean that in (5.4) the second term plays the main role eventually, an contradiction with $T_n \to \infty$ can be derived. From this can be concluded that κ_n is finite almost surely.

Step 2. In this step we want to find an upper bound for ε_i, with $i > \kappa_n$. For this, consider the event

$$\mathcal{E}_n = \{\kappa_n < \infty \text{ and } \varepsilon_i \leq c_i(i - \kappa_n) \text{ for all } i > \kappa_n\} \cup \{\kappa_n = \infty\},$$

where (c_n) is a real-valued sequence which diverges to infinity. We want to show that

$$\mathbb{P}(\mathcal{E}_n \text{ i.o }) = \mathbb{P}(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{E}_n) = 1.$$

By the properties of an decreasing sequence, and the fact that $\mathcal{E}_n \subseteq \bigcup_{n=m}^{\infty} \mathcal{E}_n$, it suffices to show that

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1.$$

The proof makes use of the Chebychev's inequality and the monotone convergence theorem([5, pg. 93]). By upper bounding $\mathbb{P}(\mathcal{E}_n)$ by an expected value that converges to 1, we can conclude the desired result.

Step 3. Here, we are going to show there exists a ν, depending on $\omega \in \mathcal{N}$, such that for all $n \geq \kappa_\nu$ holds that

$$\varepsilon_n \leq n \tag{5.6}$$

Next to that, in this step we want to upper bound the term $\frac{T_n^{\alpha-1}}{\tau_n^\alpha}\sigma_{i+1}$ more strict than before. This means there needs to be shown that for all $n \geq \kappa_\nu$ holds that

$$\frac{T_n^{\alpha-1}}{\tau_n^\alpha}\sigma_{i+1} \leq \delta q^{n-\kappa_\nu}. \tag{5.7}$$

By induction, there can be proved that there exists an specific upper bound, from what exactly the above follows.

Knowing all this, it can be shown that $\mathbb{P}(\mathcal{N}) = 0$. Starting with (5.4), everything we have done until now can be plugged in. Using (5.6), $P_n(1 - P_n) \leq P_n = \psi(\Theta_n)$, we obtain for all $n \geq \kappa_v$

$$\begin{aligned} T_n &\leq T_{n-1} + 2^{\alpha-1}\frac{T_{n-1}^{\alpha}}{\tau_{n-1}^{\alpha}}\sigma_n + \varepsilon_n\sqrt{\sigma_n P_{n-1}(1-P_{n-1})} \\ &\leq T_{n-1}\left(1 + 2^{\alpha-1}\frac{T_{n-1}^{\alpha-1}}{\tau_{n-1}^{\alpha}}\sigma_n + nT_{n-1}^{\frac{1}{2}}\sqrt{2^{\alpha-1}\sigma_n\frac{T_{n-1}^{\alpha-1}}{\tau_{n-1}^{\alpha}}}\right). \end{aligned}$$

Since we know (5.7) and $T_n \geq 1$, we get the following when we repeat this procedure

$$\begin{aligned} T_n &\leq T_{n-1}\left(1 + 2^{\alpha-1}\delta q^{n-\kappa_\nu-1} + n\sqrt{2^{\alpha-1}\delta q^{n-\kappa_\nu-1}}\right). \\ &\leq T_{\kappa_\nu}\prod_{i=1}^{n-\kappa_\nu}\left(1 + 2^{\alpha-1}\delta q^{i-1} + (\kappa_\nu + i)\sqrt{2^{\alpha-1}\delta q^{i-1}}\right) \\ &\leq T_{\kappa_\nu}\prod_{i=1}^{\infty}\left(1 + 2^{\alpha-1}\delta q^{i-1} + (\kappa_\nu + i)\sqrt{2^{\alpha-1}\delta q^{i-1}}\right). \end{aligned}$$

We want this product to converge to get an contradiction. Indeed, when we use that

$$\prod_{i=1}^{\infty}(1+a_i) < \infty \Longleftrightarrow \sum_{i=1}^{\infty} a_i < \infty,$$

we can show by the comparison test that $T_n < \infty$ for all $n \geq \kappa_\nu$. This is clearly a contradiction with $T_n \to \infty$, and thus $\mathbb{P}(\mathcal{N}) = 0$. □

We are finally ready to prove the main theorem of this section.

Proof of Theorem 5.1. When (ρ_n) is bounded, the result follows from Proposition 5.4. When $\rho_n \to \infty$ and $\lambda > 1$, it follows from Proposition 5.3. When $\rho_n \to \infty$ and $\lambda < 1$, almost sure monopoly is given by Proposition 5.5. □

5.2 Critical regime

Recall, whether the probability on monopoly is bigger than 0 depends on the summation

$$\sum_{n=0}^{\infty}\frac{\tau_{n+1}}{\tau_n^{\alpha}} \tag{5.8}$$

in this regime. When this sum is finite, the probability is strictly between 0 and 1. An infinite sum will give probability zero. Note that in this case monopoly never happens almost surely. To be able to work with these summations, we consider the term

$$\phi_n = \tau_n e^{-\beta\alpha^n}$$

It holds that $\lim_{n\to\infty}\alpha^{-n}\log(\phi_n) = 0$. This can be clearly seen

$$\begin{aligned} \lim_{n\to\infty}\alpha^{-n}\log(\phi_n) &= \lim_{n\to\infty}\alpha^{-n}\log(\tau_n e^{-\beta\alpha^n}) = \lim_{n\to\infty}\alpha^{-n}\log(\tau_n) - \lim_{n\to\infty}\alpha^{-n}\beta\alpha^n \\ &= \beta - \beta = 0. \end{aligned}$$

It can be easily verified that

$$\frac{\tau_{n+1}}{\tau_n^\alpha} = \frac{\phi_{n+1}}{\phi_n^\alpha} \tag{5.9}$$

This means, when we want to consider the sum of (5.8), it is equivalent to check whether

$$\sum_{n=0}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} \tag{5.10}$$

is either finite or infinite. These are the tools that we will use to prove the following theorem.

Theorem 5.6. *Suppose $\alpha > 1$ and $\theta \in (0, \infty)$.*

$$\text{If } \sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^\alpha} = \infty, \text{ then } \mathbb{P}(\mathcal{M}) = 0.$$

$$\text{If } \sum_{n=0}^{\infty} \frac{\tau_{n+1}}{\tau_n^\alpha} < \infty, \text{ it holds that } \mathbb{P}(\mathcal{M}) \in (0, 1).$$

For the proof, there is assumed that (ϕ_n) is unbounded if $\sum_{n=0}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} < \infty$. This can be proven by contradiction and is shown in [8]. This main theorem will be proven with the help of two propositions. For the first, only an outline will be given. The second proof will be done here.

Proposition 5.7. *Suppose $\alpha > 1$ and $\beta \in (0, \infty)$. If*

$$\sum_{n=0}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} < \infty$$

then $\mathbb{P}(\mathcal{M}) < 1$. If

$$\sum_{n=0}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} = \infty, \tag{5.11}$$

then $\mathbb{P}(\mathcal{M}) = 0$.

Proof. When (5.11) holds, it can be verified that also holds that $\sum_{n=0}^{\infty} \frac{\sigma_{n+1}}{\tau_n^\alpha} = \infty$. By Lemma 3.2, it holds that $\mathbb{P}(\mathcal{M}) = 0$.

The case left to show is that in general in the critical regime holds that the probability on monopoly is smaller than 1 when (5.10) converges. This can be done by constructing an event $\mathcal{E}$ such that $\mathbb{P}(\mathcal{E}) > 0$ and both $T_n \to \infty$ and $\hat{T}_n \to \infty$ on $\mathcal{E}$. Then it follows that the $\mathbb{P}(\mathcal{M}) < 1$. For this, let $\gamma > \frac{2}{\alpha - 1}$ and denote

$$\chi_n = \max_{1 \le k \le n} k^\gamma \phi_k.$$

Since we assumed (ϕ_n) is unbounded in this case, it can be seen that $\chi_n \to \infty$. Since $\beta \in (0, \infty)$, it holds that (τ_n) grows faster than (χ_n). Using this, we can construct several bounds for $m \in \mathbb{N}$ large enough to be able to construct the event

$$\mathcal{E} = \{T_m \in [\chi_m, \tau_m - \chi_m]\} \bigcap \{|\varepsilon_{n+1}| \le n \text{ for all } n \ge m\}.$$

It can be shown that $\mathbb{P}(\mathcal{E}) > 0$, this won't be done here.

Left to show is that $T_n, \hat{T}_n$ both tend to infinity on $\mathcal{E}$. We can argue, that by symmetry of the chosen bounds, also $\hat{T}_m \in [\chi_m, \tau_m - \chi_m]$. This means we only need to proof that $T_n \to \infty$ on $\mathcal{E}$. Since we know that $\chi_n \to \infty$, it is enough to proof that $T_n \geq \chi_n$ on $\mathcal{E}$ for all $n \geq m$. This can be shown by induction, see [8, pg. 25]. After that, everything was needed can be concluded. □

We only know now that $\mathbb{P}(\mathcal{M}) < 1$, when the summation in question is finite. There needs to be shown that $\mathbb{P}(\mathcal{M}) > 0$ in this case. This is done in the next proposition.

Proposition 5.8. *Suppose $\alpha > 1$ and $\beta \in (0, \infty)$. If*

$$\sum_{n=0}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} < \infty, \tag{5.12}$$

then $\mathbb{P}(\mathcal{M}) > 0$.

Proof. Rewriting (5.4), noticing that $\sigma_{n+1} \leq \tau_{n+1}$, we get

$$\begin{aligned}
T_{n+1} &\leq T_n + 2^{\alpha-1} \frac{T_n^\alpha}{\tau_n^\alpha} \sigma_{n+1} + \varepsilon_{n+1} \sqrt{\sigma_{n+1} P_n (1 - P_n)} \\
&\leq T_n + 2^{\alpha-1} \frac{T_n^\alpha}{\tau_n^\alpha} \tau_{n+1} + \varepsilon_{n+1} \sqrt{\sigma_{n+1} P_n (1 - P_n)} \\
&\leq T_n + 2^{\alpha-1} \frac{T_n^\alpha \phi_{n+1}}{\phi_n^\alpha} + \varepsilon_{n+1} \sqrt{\sigma_{n+1} P_n (1 - P_n)}.
\end{aligned}$$

Denote

$$\xi_n = T_n^{-\alpha} \sqrt{\sigma_{n+1} P_n (1 - P_n)}.$$

Then we can rewrite the above to

$$\frac{T_{n+1} - T_n}{T_n^\alpha} \leq 2^{\alpha-1} \frac{\phi_{n+1}}{\phi_n^\alpha} + \xi_n \varepsilon_{n+1}. \tag{5.13}$$

With the above, we are going to construct an event $\mathcal{E}$, with $\mathbb{P}(\mathcal{E}) > 0$. When we can show that $\mathbb{P}(\mathcal{E}) \leq \mathbb{P}(\mathcal{M})$, we have the desired result.
Since (5.12) holds, we can pick $m \in \mathbb{N}$ large enough such that the summation of the first term of (5.13) can be upper bounded. We can take

$$2^{\alpha-1} \sum_{n=m}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} < \frac{1}{\alpha - 1} \frac{1}{T_0^{\alpha-1}}.$$

Here we used the property that the tail of a convergent sum goes to 0. Then we can consider the event

$$\mathcal{E} = \left\{ 2^{\alpha-1} \sum_{n=m}^{\infty} \frac{\phi_{n+1}}{\phi_n^\alpha} < \frac{1}{\alpha - 1} \frac{1}{T_m^{\alpha-1}} \right\} \bigcap \left\{ \sum_{n=m}^{\infty} \xi_n \varepsilon_{n+1} \leq 0 \right\}.$$

The probability of the first event is positive, since $B_1 = ...B_m = 0$ happens with positive probability. Then $T_0 = T_m$. Furthermore, we notice that

$$\sum_{n=m}^{\infty} \xi_n \varepsilon_{n+1} = \sum_{n=m}^{\infty} T_n^\alpha (B_{n+1} - \sigma_{n+1} P_n).$$

The conditional expectation of the above, given $\mathcal{F}_m$ will be 0 almost surely, since (2.4) holds. This means that

$$\mathbb{P}\left(\sum_{n=m}^{\infty} \xi_n \varepsilon_{n+1} \leq 0\right) > 0.$$

Hence we can conclude that $\mathbb{P}(\mathcal{E}) > 0$. Now we show by contradiction that $\mathcal{E} \subset \mathcal{M}$. Suppose there is an $\omega \in \mathcal{E}$ such that $T_n \to \infty$. Using the second part of the event $\mathcal{E}$ and (5.13), the following contradiction can be deduced

$$2^{\alpha-1} \sum_{n=m}^{\infty} \frac{\phi_{n+1}}{\phi_n^{\alpha}} \geq \sum_{n=m}^{\infty} \frac{T_{n+1} - T_n}{T_n^{\alpha}} \geq \int_{T_m}^{\infty} \frac{dx}{x^{\alpha}} = \frac{1}{\alpha - 1} \frac{1}{T_m^{\alpha-1}}.$$

The latter clearly contradicts the fact that $\omega \in \mathcal{E}$. This means that
$0 < \mathbb{P}(\mathcal{E}) \leq \mathbb{P}(\mathcal{M})$. This is what we needed to show. □

Proof of Theorem 5.6 . The proof follows exactly by first using that (5.9) holds. After that we can use both Proposition 5.7 and Proposition 5.8 to get the desired results. □

6 Two other cases of the feedback function

As mentioned earlier, the main focus of this thesis is the two bins model with convex feedback function $f(m) = m^\alpha$, with $\alpha > 1$. This is the most interesting case, since there are several different end-states possible, as we have seen in the past chapters. To make the story about this time-dependent balls and bins model complete, the two other cases $0 < \alpha < 1$ and $\alpha = 1$ will be discussed in this chapter.

6.1 Concave feedback function

In this section the feedback function $f(m) = m^\alpha$ with $\alpha < 1$ will be considered. In this case there won't occur dominance or monopoly, because the proportion of balls in both bins will in the end converge almost surely to the equilibrium point $(\Theta, \hat{\Theta}) = (\frac{1}{2}, \frac{1}{2})$, which will be shown and elaborated in this section.
This section is based on specific pages of [6]. This paper discusses general concave and convex functions and was an inspiration for the discussion of our specific concave feedback function.

In this chapter we will prove the following theorem.

Theorem 6.1 (Concave). *Suppose* $0 < \alpha < 1$ *and* $\sum_{n=0}^{\infty} \frac{\sigma_n^2}{\tau_n^2} < \infty$. *Then, almost surely,*

$$\lim_{n\to\infty} (\Theta_n, \hat{\Theta}_n) = (\frac{1}{2}, \frac{1}{2}),$$

i.e. the proportion of balls of both bins converges to an equilibrium.

Because in this thesis two bins are considered, the situation is symmetric. This means we can both look specific at one bin and extend the situation afterwards to two bins, or we can directly look at the two bins simultaneously. The first is done in the previous chapters, but in this section we will do the second.

To be able to proof the almost sure convergence of $(\Theta_n, \hat{\Theta}_n)$ we need to make use of the stochastic approximation technique. Intuitively, the following happens. We are dealing with a two dimensional discrete process, from what we want investigate whether it converges and to which values. By rewriting our process in a recursive way, the same way it is done in (4.17) for one dimension, we can show that this recursive relation behaves like an ordinary differential equation. This is the main idea of the stochastic approximation technique. By making our time steps really small, we can approximate our process by a continuous process. Instead of

$h(\Theta_n) = h(\Theta(n))$, we then can write $h(\Theta(t))$ for all $t \in \mathbb{R}_+$. The differential equation will have certain stationary points, $\mathcal{L}$. It holds now that

$$\mathcal{E} \subseteq \mathcal{L}.$$

This means the set of stationary points of the corresponding differential equation will contain the original equilibrium points of our process, but potentially contains more points. There can be shown that under certain conditions, Θ_n will converge to a stationary point of the differential equation. That is the main goal of this section.

Let's make this more precise. An intuitive idea of the proof will be given here. For more details, the reader is referred to Appendix A of [6].

Proof. As we have seen in (4.17), we can write Θ_{n+1} recursively into Θ_n, a martingale part and an error term:

$$\begin{aligned}(\Theta_{n+1}, \hat{\Theta}_{n+1}) = (\Theta_n, \hat{\Theta}_n) &- \left(\frac{\sigma_{n+1}}{\tau_{n+1}}(\Theta_n - \psi(\Theta_n)), \frac{\sigma_{n+1}}{\tau_{n+1}}(\hat{\Theta}_n - \psi(\hat{\Theta}_n)\right) \\ &+ \left(\frac{B_{n+1} - \sigma_{n+1}P_n}{\tau_{n+1}}, \frac{\hat{B}_{n+1} - \sigma_{n+1}(1-P_n)}{\tau_{n+1}}\right), \end{aligned} \tag{6.1}$$

since $\hat{B_{n+1}}$ is a random variable with size σ_{n+1} and parameter $\hat{P_n} = 1 - P_n$. We can rewrite this to a vector notation, where $(\Theta_{n+1}, \hat{\Theta}_{n+1}) = \theta_{n+1}$ and $h(\theta_n) = (\Theta_n - \psi(\Theta_n), \hat{\Theta}_n - \psi(\hat{\Theta}_n))$. Furthermore, we can write M_{n+1} for the last expression of (6.1). This gives

$$M_{n+1} = \left(B_{n+1} - \sigma_{n+1}P_n, \hat{B}_{n+1} - \sigma_{n+1}(1-P_n)\right).$$

By exactly the same way as in Chapter 4, we can prove this is a martingale increment. This gives the following equation:

$$\theta_{n+1} = \theta_n - \frac{\sigma_{n+1}}{\tau_{n+1}}\left(h(\theta_n) + \frac{M_{n+1}}{\sigma_{n+1}}\right). \tag{6.2}$$

To be able to obtain information about the convergence, we consider the differential equation of the continuous function $F(x) = h(x(t))$, with $x \in \mathbb{R}^2$, defined as follows:

$$\dot{x} = \begin{cases} \frac{x_1(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} - x_1(t) \\ \frac{x_2(t)^\alpha}{x_2(t)^\alpha + x_1(t)^\alpha} - x_2(t). \end{cases}$$

Notice that x_i still represents the proportion of balls in the i-th bin with $i \in \{1, 2\}$, meaning that the relation $x_2(t) = 1 - x_1(t)$ holds. The stationary points of a differential equation can be found by setting

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0.$$

This means we will find the following stationary points, using the same calculation as in the beginning of Chapter 4. Knowing this, we get

$$\mathcal{L} = \left\{(0,1), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}.$$

In this case holds thus that $\mathcal{E} = \mathcal{L}$. It holds that we can use the Hessian matrix of the corresponding differential equation to find out whether these points are stable or not. Such a differential equation has a stable point when the eigenvalues have all negative real parts, when the equilibrium point is plugged in the Hessian matrix [4, pg. 103]. We will prove that θ_n will converge to a specific stable point by using an important theorem from [6].

Before we do this, we note the following. When $x_i(0) = 0$ for some $i \in \{1, 2\}$, then we see that

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0,$$

meaning when the system starts in either $(0, 1)$ or $(1, 0)$ we will have that $x_i(t) = 0$ for all t. Since in this thesis we assume that $0 < T_0 < \tau_0$, we can indeed conclude that $0 < x_1(0), x_2(0) < 1$. Note this holds since $\alpha < 1$ here, in the other chapters were $\alpha > 1$, this is obviously not happening. Since $\alpha < 1$ and we have $0 < x_1(0), x_2(0) < 1$, we can also see that this means these points will never be reached when the initial values are between 0 and 1.

The only point that needs to be checked is $(\frac{1}{2}, \frac{1}{2})$. This calculation is straightforward and can be found in the Appendix of this thesis.
Now we are able to use Theorem A.2 of [6]. We indeed are able to write our model as in (6.2) and we made the general assumptions (2.7) and (2.8) here. The other necessary assumptions can be easily checked. From this there can be concluded by the second statement of Theorem A.2, because $(\frac{1}{2}, \frac{1}{2})$ is the only stable stationary point, that it holds that θ_n converges to this point, giving that

$$\theta_n \to \left(\frac{1}{2}, \frac{1}{2}\right)$$

almost surely. This is the desired result. □

From the above theorem, we can directly conclude that

$$\mathbb{P}(\mathcal{M}) = \mathbb{P}(\mathcal{D}) = 0.$$

6.2 No feedback

This section is devoted to the no feedback case, meaning that the feedback function $f(m)$ is equal to $f(m) = m$. The structure and proofs of this section are based on [8, Section 3].
Note first, that in this case

$$P_n = \frac{\Theta_n^\alpha}{\Theta_n^\alpha + (1 - \Theta_n)^\alpha} = \Theta_n.$$

This means the probability of a ball landing in a certain bin only depends on the number of balls already in that bin. The feedback function is a lineair function here. This will be useful when we prove that there Θ_n converges to a random variable Θ in this case. The aim of this chapter is to get an idea of the proof of the following theorem,

Theorem 6.2. *Suppose $\alpha = 1$. Then Θ_n converges almost surely to a random variable Θ, and $\mathbb{P}(\mathcal{D}) = 0$.*

The first part of this theorem is not hard to prove and will be shown in detail. For the second part, to proof there is almost surely no dominance, an idea of the proof will be given. For the exact details the reader is referred to [8, section 3].

Proof. First we prove the almost sure convergence of Θ_n. Notice we can rewrite Θ_n as in (4.16). We show, with the help of this relation, that (Θ_n) is a bounded martingale. For that, notice that (Θ_n) is clearly adapted to the filtration $\mathcal{F}_n = \sigma(B_1, B_2, .., B_n)$. Since it holds that $\Theta_n \leq 1$, we have that (Θ_n) is integrable. Note, since Θ_n is a proportion, it holds that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\Theta_n] < \infty.$$

The only thing left to show is that (Θ_n) satisfies the martingale property, which is done below:

$$\begin{aligned}
\mathbb{E}[\Theta_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\frac{\tau_n}{\tau_{n+1}}\Theta_n + \frac{1}{\tau_{n+1}}B_{n+1}|\mathcal{F}_n\right] \\
&= \frac{\tau_n}{\tau_{n+1}}\Theta_n + \mathbb{E}\left[\frac{1}{\tau_{n+1}}B_{n+1}|\mathcal{F}_n\right] = \frac{\tau_n\Theta_n + \sigma_{n+1}P_n}{\tau_{n+1}} \\
&= \Theta_n\left(\frac{\tau_n + \sigma_{n+1}}{\tau_{n+1}}\right) = \Theta_n.
\end{aligned}$$

Thus, (Θ_n) is a bounded martingale with respect to $\mathcal{F}_n$. By Theorem 1.2 holds that Θ_n converges almost surely to a random variable Θ.

To prove no dominance occurs, it holds by symmetry of the two considered bins, that it is enough to show that

$$\mathbb{P}\left(\lim_{n\to\infty}\Theta_n = 0\right) = \mathbb{P}\left(\Theta = 0\right) = 0.$$

For this, we are going to use the Laplace transfrom of Θ_n and Θ, defined as follows

$$\begin{aligned}
f_n(\lambda) &:= \mathbb{E}[e^{-\lambda\Theta_n}] \\
f(\lambda) &:= \mathbb{E}[e^{-\lambda\Theta}],
\end{aligned}$$

where $\lambda \in \mathbb{R}$. Because Θ_n is a discrete random variable, it holds that Θ is discrete as well. This means that

$$f(\lambda) = \sum_{k=0}^{n} e^{-\lambda k}\mathbb{P}(\Theta = k) \geq \mathbb{P}(\Theta = 0), \tag{6.3}$$

implying that it is enough to show that there exists a sequence (λ_m) such that

$$\lim_{m\to\infty} f(\lambda_m) = 0.$$

To reach this goal, choose $\lambda_m = c\tau_m \geq 0$ with $c \in (0, 1)$, such that $e^{-x} \leq 1 - x + \frac{x^2}{2}$ for all $x \in [0, c]$. This is useful when the induction is applied. There can be proven by induction over k, using several upper and lower bounds, that for all m, $n > m$ and $1 \leq k \leq n-m$ holds that

$$f_n(\lambda_m) \leq f_{n-k}\left(\lambda_m - \lambda_m^2 \sum_{i=n-k+1}^{n} \frac{\sigma_i}{\tau_i^2}\right). \tag{6.4}$$

where there is is used that

$$\lambda_m - \lambda_m^2 \sum_{i=m+1}^{n-k+1} \frac{\sigma_i}{\tau_i^2} \geq \lambda_m(1-c) \geq 0. \tag{6.5}$$

The proofs of these two statements can be found in [8, section 3]. Knowing this, we are almost ready to make the conclusion. Notice that the function f_m for fixed m is monotonically declining, since for $x < y$ with $x, y \in \mathbb{R}$ we have that $e^{-x} > e^{-y}$ and thus that

$$f_m(x) = \mathbb{E}[e^{-x\Theta_n}] \geq \mathbb{E}[e^{-y\Theta_n}] = f_m(y).$$

Notice, because $\sigma_i, \tau_i > 0$, that

$$\lambda_m > \lambda_m - \lambda_m^2 \sum_{i=m+1}^{n} \frac{\sigma_i}{\tau_i^2}.$$

Substituting $k = n-m$ in (6.4) and using both the monotonicity of f_m and (6.5), the following can be obtained for all m and $n > m$

$$\begin{aligned} f_n(\lambda_m) &\leq f_m\left(\lambda_m - \lambda_m^2 \sum_{m+1}^{n} \frac{\sigma_i}{\tau_i^2}\right) \leq f_m\left(\lambda_m(1-c)\right) = \mathbb{E}\left[e^{-\lambda_m(1-c)\Theta_m}\right] \\ &= \mathbb{E}\left[e^{-\tau_m c(1-c)\Theta_m}\right] = \mathbb{E}\left[e^{-c(1-c)T_m}\right]. \end{aligned} \tag{6.6}$$

Because we know that $\Theta_n \to \Theta$ almost surely, when $n \to \infty$, we can apply the dominated convergence theorem [3, pg. 57] to check the behaviour of f_n, for $\lambda > 0$. Since

$$e^{-\lambda\Theta_n} \leq 1 = \mathbb{E}[1] < \infty,$$

it holds by the dominated convergence theorem that

$$f_n(\lambda) = \mathbb{E}[e^{-\lambda\Theta_n}] \to \mathbb{E}[e^{-\lambda\Theta}] = f(\lambda).$$

as $n \to \infty$. This means, when we take the limit $n \to \infty$ in (6.6), we get that

$$f(\lambda_m) \leq \mathbb{E}[e^{-c(1-c)T_m}],$$

because the righthand side does not depend on n. By Lemma 3.3, it holds that $\mathbb{P}(\lim_{m\to\infty} T_m = \infty) = 1$ in the no-feedback case. By again applying the dominated convergence theorem with the same upper bound, taking the limit $m \to \infty$, we obtain

$$\lim_{m\to\infty} f(\lambda_m) \leq \lim_{m\to\infty} \mathbb{E}\left[e^{-c(1-c)T_m}\right] = \mathbb{E}[\lim_{m\to\infty} e^{-c(1-c)T_m}] = 0,$$

where it is used that $c \in (0,1)$. Since (6.3) holds, we can conclude that

$$\mathbb{P}(\Theta = 0) = 0,$$

and thus that $\mathbb{P}(\mathcal{D}) = 0$. □

This implies directly that $\mathbb{P}(\mathcal{M}) = 0$. Note that Θ_n converges in this case to a random variable Θ with values strictly between 0 and 1, instead of a deterministic value as we have seen in the previous section.

Appendix

In this chapter the stability of the point $(\frac{1}{2}, \frac{1}{2})$ in the case $\alpha < 1$ will be shown with use of the Hessian matrix. Recall we are working with the following differential equation:

$$\dot{x} = \begin{cases} \frac{\partial F}{\partial x_1} = \frac{x_1(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} - x_1(t) \\ \frac{\partial F}{\partial x_2} = \frac{x_2(t)^\alpha}{x_1(t)^\alpha + x_2(t)^\alpha} - x_2(t). \end{cases}$$

The Hessian matrix is defined as

$$H = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 x_2} \\ \frac{\partial^2 F}{\partial x_2 x_1} & \frac{\partial^2 F}{\partial x_1^2} \end{pmatrix}.$$

The corresponding derivatives are as follows:

$$\begin{aligned} \frac{\partial^2 F}{\partial x_1^2} &= \frac{\alpha x_1^{\alpha-1}(x_1^\alpha + x_2^\alpha) - \alpha x_1^{2\alpha-1}}{(x_1^\alpha + x_2^\alpha)^2} - 1 \\ \frac{\partial^2 F}{\partial x_2^2} &= \frac{\alpha x_2^{\alpha-1}(x_1^\alpha + x_2^\alpha) - \alpha x_2^{2\alpha-1}}{(x_1^\alpha + x_2^\alpha)^2} - 1 \\ \frac{\partial^2 F}{\partial x_1 x_2} &= \frac{-\alpha x_2^\alpha x_1^{\alpha-1}}{(x_1^\alpha + x_2^\alpha)^2} \\ \frac{\partial^2 F}{\partial x_2 x_1} &= \frac{-\alpha x_1^\alpha x_2^{\alpha-1}}{(x_1^\alpha + x_2^\alpha)^2} \end{aligned}$$

When we plug in $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$ in these equations, we obtain the following

$$\begin{aligned} \frac{\partial^2 F}{\partial x_1^2}|_{(\frac{1}{2},\frac{1}{2})} = \frac{\partial^2 F}{\partial x_2^2}|_{(\frac{1}{2},\frac{1}{2})} &= \frac{\alpha \frac{1}{2}^{\alpha-1}(\frac{1}{2}^\alpha + \frac{1}{2}^\alpha) - \alpha \frac{1}{2}^{2\alpha-1}}{(\frac{1}{2}^\alpha + \frac{1}{2}^\alpha)^2} - 1 \\ &= \frac{\alpha \frac{1}{2}^{2\alpha-2} - \alpha \frac{1}{2}^{2\alpha-1}}{\frac{1}{2}^{2\alpha-2}} - 1 = \frac{1}{2}\alpha - 1. \\ \frac{\partial^2 F}{\partial x_2 x_1}|_{(\frac{1}{2},\frac{1}{2})} = \frac{\partial^2 F}{\partial x_1 x_2}|_{(\frac{1}{2},\frac{1}{2})} &= \frac{-\alpha \frac{1}{2}^\alpha \frac{1}{2}^{\alpha-1}}{(\frac{1}{2}^\alpha + \frac{1}{2}^\alpha)^2} = \frac{-\alpha \frac{1}{2}^{2\alpha-1}}{\frac{1}{2}^{2\alpha-2}} = -\frac{1}{2}\alpha. \end{aligned}$$

Plugging this in the matrix H and subtracting λI_2, we obtain the following matrix

$$H - \lambda I_2 = \begin{pmatrix} \frac{1}{2}\alpha - 1 - \lambda & -\frac{1}{2}\alpha \\ \frac{1}{2}\alpha & \frac{1}{2}\alpha - 1 - \lambda \end{pmatrix}.$$

The eigenvalues of H can be found be determining the determinant of this matrix. Solving the equation below, we get the following two eigenvalues

$$\left(\frac{1}{2}\alpha - 1 - \lambda\right)^2 + \frac{1}{4}\alpha^2 = 0.$$
$$\lambda_1 = -1 + \frac{1}{2}\alpha - \frac{i}{2}\alpha.$$
$$\lambda_2 = -1 + \frac{1}{2}\alpha + \frac{i}{2}\alpha.$$

The real parts of both eigenvalues are equal to $-1 + \frac{1}{2}\alpha$, which are clearly negative when $\alpha < 1$. Hence we can conclude that $(\frac{1}{2}, \frac{1}{2})$ is a stable stationary point of our system of differential equations.

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