Utrecht University

Faculty of Science

Classical and stable local limit theorem with application to long-range random walks

Bachelor Thesis

Floris Dorrestijn

Mathematics

Supervisor:

Dr. Wioletta Ruszel
Mathematical Institute

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Abstract

The central limit theorem states that the sum of n independent and identically distributed properly scaled random variables with finite mean and finite variance converges to a normal distribution. One question arises: is there a discrete version of this theorem for discrete random variables? The answer is yes, and this is what we show with the local central limit theorem. Another question that arises is what happens in the case when there is no finite mean or finite variance. In the case where we are dealing with n random variables with a Pareto like tail we can deduce that when properly scaled their sum converges to a stable distribution. We will in this case again consider a discrete version of this theorem, which is called the stable local central limit theorem. We will prove and discuss both theorems in the context of long range random walks.

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1 Introduction

The central limit theorem states that if we take a sequence of independent and identically distributed 1-dimensional random variables $(X_n)_{n\in\mathbb{N}}$ where each X_n has mean μ and variance σ^2 that their average - when properly scaled - converges to a standard normal distribution. In mathematical notation we can write this as:

$$\lim_{n\to\infty} F_{Y_n}(x) = F_Z(x)$$

for each real x, with $Y_n = \sqrt{n}\frac{\bar{X}_n-\mu}{\sigma}$ and where $Z \sim N(0,1)$.

The local central limit theorem is an adaption of this theorem that gives a discrete version of the central limit theorem. If $X_1, ..., X_m$ are independent and identically distributed discrete random variables with mean zero and finite variance σ^2, and if we denote with S_m their sum then we can approximate the probability mass function of S_m by:

$$\mathbb{P}(S_m = k) = \mathbb{P}\left(\frac{k}{\sqrt{m}} \leq \frac{S_m}{\sqrt{m}} < \frac{k+1}{\sqrt{m}}\right) \approx P\left(\frac{k}{\sqrt{m}} \leq Z_m \leq \frac{k+1}{\sqrt{m}}\right)$$

where $Z_m \sim N(0,\sigma^2)$.
The local central limit theorem tells us under what conditions and how fast this approximation converges. We will do so using random walks consisting of steps which we model with random variables X_i on the integer lattice.

The other theorem we will consider is the stable local central limit theorem. For the central limit theorem to work we need finite variance on each of our steps. But what happens if we don't have finite variance? Do we have convergence at all?

It turns out we do under certain conditions. If we consider $X_1, ..., X_l$ independent and identically distributed discrete random variables, where each random variable has a Pareto-like tails with $\alpha \in (0,2)$ then we can find that their sum S_l converges to a stable random variable. Both normal distributed random variables and Cauchy distributed random variables belong to this class.

We will now provide an overview of the structure of this thesis. We will start with a list of basic definitions and theorems in measure theory and probability theory. In chapter 3 we will consider classical central limit theorems, which both the central limit theorem and the local central limit theorem are part of. In chapter 4 we will discuss and prove the stable local central limit theorem. Finally, in chapter 5 we will discuss the rates of convergence we found and give examples of the usage of both theorems.

2 Basic definitions and theorems

In order to understand what will be discussed in this paper, we will first give some definitions. We will when appropriate also accompany these definitions with theorems that we will use later on.

2.1 General

For functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ we will write $f \leq g$ if for all $u \in U$ and all $v \in V$ we have that $f(u) \leq g(v)$.
We will use $\mathbb{R}_{\geq 0}$ to denote the set of reals larger or equal to 0, e.g.

$$\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \ : \ x \geq 0\}$$

When we speak about an infinite sequence $(a_n)_{n\geq 1}$ where each $a_n \in \mathbb{R}^d$, we implicitly assume that the sequence a_n is countable infinite.

Definition 2.1.1 (Big O notation)**.** For functions $f_1 : \mathbb{R}^d \to \mathbb{R}$ and $g_1 : \mathbb{R}^d \to \mathbb{R}$ for some $d \in \mathbb{N}$ we write:

$$f_1(\mathbf{x}) = O(g_1(\mathbf{x}))$$

as $\mathbf{x} \to \mathbf{y}$ if there is some $C \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^d, \delta > 0$ such that $||f(\mathbf{x})|| \leq C||g(\mathbf{x})||$ for all $\mathbf{x} \in B(\mathbf{a}, \delta)$.
Similarly, we write for functions $f_2 : \mathbb{N} \to \mathbb{R}$ and $g_2 : \mathbb{N} \to \mathbb{R}$:

$$f_2(n) = O(g_2(n))$$

if there is some $C' > 0$ such that:

$$|f_2(n)| \leq C'|g_2(n)|$$

for all $n \in \mathbb{N}$.

Definition 2.1.2 (Norm and unit vectors)**.** We denote with $|| \cdot ||$ the standard Euclidean norm in $\mathbb{R}^d$, for some positive integer d. We denote with $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$ the unit vectors in $\mathbb{R}^d$, where each $\boldsymbol{e}_i$ for $1 \leq i \leq d$ has a 1 at position i and a 0 everywhere else.

Definition 2.1.3 (Multidimensional vectors)**.** If $\boldsymbol{x} \in \mathbb{R}^d$, then we implicitly assume that we can write $\boldsymbol{x}$ as:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_d \end{pmatrix}$$

where $x_i \in \mathbb{R}$ for all $1 \leq i \leq d$.

When we integrate a function $f : \mathbb{R}^d \to \mathbb{C}$ in multiple dimensions over some set $A \subset \mathbb{R}^d$, we will often use the notation

$$\int_A f(\boldsymbol{x}) d\boldsymbol{x}$$

to denote the integral over all $\boldsymbol{x} \in A$.

Definition 2.1.4 (Partition)**.** Recall that a sequence of sets $(A_i)_{i\in I}$ for some index set I partitions a set A, if:

1. for all $i \in I$ we have $A_i \neq \emptyset$,
2. for all $i, j \in I$ we have that $A_i \cap A_j = \emptyset$, and
3. $\bigcup_{i\in I} A_i = A$

Finally, we will define what the convolution of two real-valued functions is in the particular case that their domain is equal to $\mathbb{Z}^d$:

Definition 2.1.5. Let $f : \mathbb{Z}^d \to \mathbb{R}$ and $g : \mathbb{Z}^d \to \mathbb{R}$. Then we define the convolution $f * g$ by:

$$f * g(\boldsymbol{y}) = \sum_{\boldsymbol{x} \in \mathbb{Z}^d} f(\boldsymbol{x}) g(\boldsymbol{y} - \boldsymbol{x})$$

Note that it is not always the case that for every pair of functions f, g the convolution $f * g$ is defined for every $\boldsymbol{y} \in \mathbb{Z}^d$. However, in the case that we are dealing with two probability distributions $p_{\boldsymbol{X}}$ and $p_{\boldsymbol{Y}}$ with $\boldsymbol{X}$ and $\boldsymbol{Y}$ two d-dimensional discrete random variables, the convolution is bounded from above as:

$$p_{\boldsymbol{X}} * p_{\boldsymbol{Y}}(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in \mathbb{Z}^d} p_{\boldsymbol{X}}(\boldsymbol{x}) p_{\boldsymbol{Y}}(\boldsymbol{y} - \boldsymbol{x}) \leq \sum_{\boldsymbol{x} \in \mathbb{Z}^d} p_{\boldsymbol{X}}(\boldsymbol{x}) = 1$$

Furthermore the sum is bounded below by 0 and if we define the sequence a_m by:

$$a_m = \sum_{\boldsymbol{x} \in \mathbb{Z}^d \cap B(\boldsymbol{0}, m)} p_{\boldsymbol{X}}(\boldsymbol{x}) p_{\boldsymbol{Y}}(\boldsymbol{y} - \boldsymbol{x})$$

then it is easy to see that a_m is monotonically increasing and as such a_m is a converging sequence. Note that the convolution of $p_{\boldsymbol{X}}$ and $p_{\boldsymbol{Y}}$ coincides with the probability distribution of $p_{\boldsymbol{X}+\boldsymbol{Y}}$:

$$\begin{aligned} p_{\boldsymbol{X}+\boldsymbol{Y}}(\boldsymbol{x}) &= \mathbb{P}(\boldsymbol{X} + \boldsymbol{Y} = \boldsymbol{x}) = \mathbb{P}(\boldsymbol{X} = \boldsymbol{0}, Y = \boldsymbol{x}) + \mathbb{P}(\boldsymbol{X} = -\boldsymbol{e}_1, Y = \boldsymbol{x} + \boldsymbol{e}_1) + \mathbb{P}(\boldsymbol{X} = \boldsymbol{e}_1, \boldsymbol{Y} = \boldsymbol{x} - \boldsymbol{e}_1) + ... \\ &= \sum_{\boldsymbol{x} \in \mathbb{Z}^d} p_{\boldsymbol{X}}(\boldsymbol{x}) p_{\boldsymbol{Y}}(\boldsymbol{y} - \boldsymbol{x}) \end{aligned}$$

2.2 Measure theory

Since in later parts of this thesis we will use the dominated convergence theorem, we will state the necessary definitions and theorems to understand under what conditions we can use this theorem. We make use of the extended real line, which is explained in appendix B of [1].
This next section follows the definitions and theorems given in the book "Measure theory" (see [1], chapters 1 and 2).

2.2.1 Measure

Definition 2.2.1 (σ-algebra)**.** Let Ω be an arbitrary set, and let $\mathscr{A}$ be a collection of subsets of Ω. Then we call $\mathscr{A}$ an *algebra* on Ω if:

1. $\Omega \in \mathscr{A}$,
2. If $A \in \mathscr{A}$ then $A^c \in \mathscr{A}$, and
3. If $(A_n)_{n \geq 1}$ is an infinite sequence of subsets that belong to $\mathscr{A}$, then $\cup_{n=1}^{\infty} A_i \in \mathscr{A}$ and $\cap_{n=1}^{\infty} A_i \in \mathscr{A}$.

Note that in the case of (3), not all A_i need to be distinct: in particular, any finite sequence $A_1, ..., A_n$ can be extended to an infinite sequence by rewriting it as $A_1, ..., A_n, A_n, ...$

Definition 2.2.2 (Countable additive)**.** Let Ω be a set and let $\mathscr{A}$ be a σ-algebra on Ω. A function $\mu : \mathscr{A} \to [0, \infty]$ is said to be *countably additive* if for each infinite sequence $(A_i)_{i \geq 1}$ of disjoint sets belonging to $\mathscr{A}$ we have:

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

A function $\mu : \mathscr{A} \to [0, \infty]$ is said to be a *measure* if it is countable additive and $\mu(\emptyset) = 0$.

A typical example of a measure is the following.

Example 2.2.3 (Point mass measure)**.** Let Ω be a set and $\mathscr{A}$ a σ-algebra on Ω. Then the *point mass* $\delta_x : \mathscr{A} \to [0, \infty]$ concentrated on some $x \in \Omega$ is given by:

$$\delta_x(A) = \mathbb{1}_A(x)$$

It is easy to see that this is indeed a measure: we have for any countable sequence $(A_i)_{i\geq 1}$ of disjoint sets belonging to $\mathscr{A}$ that:

$$\mathbb{1}_{\cup_{i=1}^{\infty} A_i}(x) = \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(x)$$

and we also have that $\delta_x(\emptyset) = 0$, hence the point mass concentrated on x is a measure. △

Definition 2.2.4 ($\mathscr{A}$-measurable)**.** Let Ω be a set and $\mathscr{A}$ be a σ-algebra on Ω. Let $A \in \mathscr{A}$, and let $f : A \to [-\infty, \infty]$. Then we call f $\mathscr{A}$-measurable if the set

$$\{x \in A \mid f(x) \leq t\}$$

belongs to $\mathscr{A}$, for each $t \in \mathbb{R}$.

The constant functions are always $\mathscr{A}$-measurable. The following proposition will state this. The following corollary is not from [1].

Corollary 2.2.5. *For any set Ω and any σ-algebra $\mathscr{A}$ on Ω we have that the constant functions, defined by $f : \Omega \to \mathbb{R}$ with $f(x) = C$ with $C \in \mathbb{R}$ for all $x \in \Omega$, are $\mathscr{A}$-measurable.*

Proof. Either we have that for some $t \in \mathbb{R}$:

$$\{x \in \Omega \mid f(x) \leq t\} = \Omega$$

or

$$\{x \in \Omega \mid f(x) \leq t\} = \emptyset$$

As the former is an element of $\mathscr{A}$ by (1) of definition 2.2.1 and the latter is an element of $\mathscr{A}$ by (1) and (2) of definition 2.2.1, we see that the constant functions are indeed $\mathscr{A}$-measurable. □

Notice that if f is $\mathscr{A}-$measurable, we have by (2) of definition 2.2.1 that also for each $t \in \mathbb{R}$ the set

$$\{x \in A \mid f(x) > t\}$$

belongs to $\mathscr{A}$. In particular, if f is simple, $\mathscr{A}$-measurable and real-valued and its domain consists of the points $a_1, ..., a_n$, then we can partition A into the set $(A_k)_{1\leq k\leq n}$ where

$$A_k = \{x \in A \mid f(x) > a_{k-1}\} \cap \{x \in A \mid f(x) \leq a_k\}$$

for $1 \leq k \leq n$ and $a_0 = -\infty$. Notice that these sets all belong to $\mathscr{A}$ by (2) of definition 2.2.1.

Definition 2.2.6 (Measure space)**.** A triplet $(\Omega, \mathscr{A}, \mu)$ where Ω is a set, $\mathscr{A}$ a σ-algebra on Ω and μ a measure on $\mathscr{A}$ is called a *measure space*.

If Ω is a set and $\mathscr{A}$ is a σ-algebra on Ω, then we call $\mathscr{L}$ the set of real-valued simple $\mathscr{A}$-measurable functions on Ω and $\mathscr{L}^+$ the set of functions in $\mathscr{L}$ which are nonnegative. As we just have seen, we may write any $f \in \mathscr{L}^+$ as $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$ for some $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and $A_i \in \mathscr{A}$ for all $1 \leq i \leq n$. We can additionally assume that for all $1 \leq i \leq n$ we have that $a_i > 0$, and that the A_i partition Ω.

2.2.2 Lebesgue integral

The idea behind the Riemann integral for a function f over a set A contained in the domain of f is that we split this set A of f into n parts and for each part calculate the value of f to get its total area under A. Then as we let n to infinity the area we find will converge to the true area of f over A.

The idea behind the Lebesgue integral on the other hand is that for a function g and a set B contained in the range of f we will split this set B of the range into n parts instead. Then in similar fashion we calculate the total area of g we find this way, and as we let n to infinity this will converge to the true value of g over B.

We will now start with simple functions, for which it is most intuitive to define what their Lebesgue integral is.

Definition 2.2.7 (Simple function)**.** A function f is said to be simple if it only has finite many distinct values.

The Lebesgue integral for simple functions is easy to define: we split the function over its range and then sum their areas. We will give a precise definition below.

Definition 2.2.8 (Lebesgue integral for simple functions)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Let $f \in \mathscr{L}^+$, and write $f(x) = \sum_{i=1}^n a_i \mathbb{1}_{A_i}(x)$ where the A_i partition the domain of f and each A_i belongs to $\mathscr{A}$. Then we define the *Lebesgue integral* of f, which we write as $\int f \, d\mu$, as:

$$\int f \, d\mu = \sum_{i=1}^n a_i \mu(A_i)$$

Note that this definition is unambiguous. Indeed, assume additionally that we may write $f(x) = \sum_{j=1}^m b_j \mathbb{1}_{B_j}(x)$ where $b_j \in \mathbb{R}, b_j > 0$, $B_j \in \mathscr{A}$ for $1 \leq j \leq m$ and the B_j partition the domain of f. Then it follows that:

$$\bigcup_{j=1}^m B_j = \bigcup_{i=1}^n A_i$$

Additionally, if $A_i \cap B_j \neq \emptyset$ for some $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$, then $a_i = b_j$, since otherwise we would get for $x \in A_i \cap B_j$ two different definitions of $f(x)$. Hence we see that:

$$\sum_{i=1}^n a_i \mu(A_i) = \sum_{i=1}^n a_i \mu\left(\cup_{j=1}^m B_j \cap A_i\right) = \sum_{i=1}^n \sum_{j=1}^m a_i \mu(B_j \cap A_i)$$
$$= \sum_{i=1}^n b_j \mu(\cup_{i=1}^n A_i \cap B_j) = \sum_{j=1}^m b_j \mu(B_j)$$

which confirms that the definition is indeed unambiguous.

We will now define the Lebesgue integral in other cases. We will gradually generalize the definition, using previous definitions for Lebesgue integrals. As to avoid repetition, we will in the two definitions below mentioning the Lebesgue integral implicitly assume that $(X, \mathscr{A}, \mu)$ is a measure space.

Definition 2.2.9 (Lebesgue integral for nonnegative functions)**.** Let f be a $[0, \infty]$-valued $\mathscr{A}$-measurable function. Then we define:

$$\int f \, d\mu = \sup\left\{ \int g \, d\mu \mid g \in \mathscr{L}^+, g \leq f \right\}$$

Definition 2.2.10. (Lebesgue integral for real functions.) Let f be a $[-\infty, \infty]$-valued $\mathscr{A}$-measurable function. Define f^+ and f^- by:

$$f^+(x) = \max(f(x), 0)$$

$$f^-(x) = -\min(f(x), 0)$$

then we define $\int f\ d\mu$ by:

$$\int f\ d\mu = \int f^+\ d\mu - \int f^-\ d\mu$$

We say that a function f is Lebesgue integrable (or just integrable) if we have that $\int f\ d\mu$ is finite.

2.2.3 Convergence

For the dominated convergence theorem we need to know what "μ-almost every $x \in \Omega$" (or just almost everywhere if μ and Ω are known) means.

Definition 2.2.11. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. A property holds μ-everywhere if there is a set $A \in \mathscr{A}$ such that $\mu(A) = 0$ and every $x \in \Omega$ where the property fails to hold is an element of A.

We now state Lebesgue's dominated convergence theorem, which is a powerful tool that allows us to prove all sorts of convergence.

Theorem 2.2.12 (Lebesgue's dominated convergence theorem)**.** *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Let g be a $[0, \infty]$-valued integrable function on Ω and let f and $(f_n)_{n\geq 1}$ be $[-\infty, \infty]$ valued $\mathscr{A}$-measurable functions on Ω such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

and

$$|f_n(x)| \leq g(x), n = 1, 2...$$

hold at $\mu-$almost every $x \in \Omega$. Then f and $(f_n)_{n\geq 1}$ are integrable, and

$$\int f\ d\mu = \lim_{n\to\infty} \int f_n\ d\mu$$

Proof. A proof will not be given here. Instead we will refer to [1], section 2.4 for a proof. □

2.3 Probability theory

To allow for more precise definitions of what we will encounter in this thesis, we will first more precisely define several terms in probability theory. We will do so in this section using measure theory.

2.3.1 General

Definition 2.3.1 (Probability space)**.** We say that $(\Omega, \mathscr{A}, \mathbb{P})$ is a probability space if $\mathbb{P}(\Omega) = 1$.

We call $\mathbb{P}$ the *probability measure*. Note that this definition indeed agrees with the definition from classical probability theory (e.g. $\mathbb{P}(A) \geq 0$ for all $A \in \mathscr{A}$, $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}$ is countably additive).

Definition 2.3.2 (Random variable)**.** Let Ω be a set and $\mathscr{A}$ be a σ-algebra on Ω. A *real-valued random variable* X is a $\mathscr{A}-$measurable function $X : \Omega \to \mathbb{R}$.

We will from now on assume that all random variables we encounter are real-valued.

Definition 2.3.3 (d-dimensional random variable)**.** A random variable $\boldsymbol{X}$ in d dimensions is a vector where its components consist of 1-dimensional random variables. We will use for any $k \in \mathbb{N}, k \leq d$ the notation X_k to denote the $k-$th component of $\boldsymbol{X}$. We can thus write:

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ ... \\ X_d \end{pmatrix}$$

When we talk about a discrete random variable $\boldsymbol{Y}$ in m dimensions, we mean a random variable in m dimensions where each of its components is a 1-dimensional discrete random variable.

We will now define the support of a random variable, their expected value and their covariance matrix.

Definition 2.3.4 (Support of a random variable)**.** Let $\mathbf{X}$ be a d-dimensional continuous random variable. Then we define the support of $\mathbf{X}$ to be all the $x \in \mathbb{R}^d$ for which we have that $f_{\mathbf{X}}(x) > 0$. Similarly, the support of a $d-$dimensional discrete random variable $\mathbf{Y}$ is defined to be all the $y \in \mathbb{Z}^d$ such that $\mathbb{P}(\mathbf{Y} = y) > 0$.

Definition 2.3.5 (Expected value)**.** Let $\boldsymbol{X}$ be a $d-$dimensional random variable. Then we define the expected value of $\boldsymbol{X}$ as:

$$\mathbb{E}[\boldsymbol{X}] = \begin{pmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \ldots \\ \mathbb{E}[X_d] \end{pmatrix}$$

On the other hand, we define the expected value of $||\boldsymbol{X}||$ as:

$$\mathbb{E}[||\boldsymbol{X}||] = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} ||\boldsymbol{k}|| \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{k}\right)$$

Given a random variable $\boldsymbol{X}$ in d dimensions, we say that the m-th moment of $\boldsymbol{X}$ exists if $\mathbb{E}[||\boldsymbol{X}||^m]$ exists. It turns out that if a higher moment of some discrete $d-$dimensional random variable $\boldsymbol{X}$ exists that then its lower moments also exist. We will prove that statement now.

Lemma 2.3.6. *Let $\boldsymbol{X}$ be a discrete d-dimensional random variable. If $\mathbb{E}[||\boldsymbol{X}||^m]$ exists for some $m \in \mathbb{N}$, then $\mathbb{E}[||\boldsymbol{X}||^k]$ also exists for k where $1 \leq k \leq m$.*

Proof. For $k = m$ the result is obvious. Assume therefore that $k < m$. Expanding $\mathbb{E}[||\boldsymbol{X}||^m]$ gives us:

$$\mathbb{E}[||\boldsymbol{X}||^m] = \sum_{\boldsymbol{x} \in \mathbb{Z}^d} ||\boldsymbol{x}||^k ||\boldsymbol{x}||^{m-k} \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{x}\right)$$

as $||\boldsymbol{x}||^{m-k} \geq 1$ for all $\boldsymbol{x} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ the result follows. □

It is easy to see that if $\boldsymbol{X} = (X_1, ..., X_d)^T$ is a d-dimensional random variable, and its k-th moment exists that then the $k-$th moment of X_i also exists where $1 \leq i \leq d$.

Definition 2.3.7 (Covariance matrix)**.** Given a random variable $\boldsymbol{X}$ in d dimensions, we will with Γ denote its own covariance matrix, thus Γ is the $d \times d$ matrix given by:

$$\Gamma_{i,j} = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]$$

for all $1 \leq i, j \leq d$.

The covariance matrix only exists if the second moment of $\boldsymbol{X}$ exists. In the case of all our random walks, this matrix reduces to a diagonal matrix as the steps $\boldsymbol{X}_k$ we take are independent of each other.

2.3.2 Characteristic functions

The characteristic function is a function with a lot of important properties. One of its most important properties is that convergence in characteristic function implies convergence in distribution. We will start with a formal definition of the characteristic function.

Definition 2.3.8. Let $\mathbf{X} = (X_1, X_2, ..., X_d)$ be a discrete random variables supported on a domain $D \subset \mathbb{Z}^d$. We call the characteristic function of X the function $\phi_{\boldsymbol{X}} : \mathbb{R}^d \to \mathbb{C}$ given by

$$\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) = \mathbb{E}[e^{i\boldsymbol{\theta} \cdot \boldsymbol{X}}] = \sum_{\boldsymbol{k} \in D} e^{i\boldsymbol{k} \cdot \boldsymbol{\theta}} \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{k}\right)$$

If $\boldsymbol{Y} = (Y_1, ..., Y_d)$ is a d-dimensional continuous random variable supported on a domain $A \subset \mathbb{R}^d$, then its characteristic function $\phi_{\boldsymbol{Y}} : \mathbb{R}^d \to \mathbb{C}$ is defined by:

$$\phi_{\boldsymbol{Y}}(\boldsymbol{\theta}) = \mathbb{E}\left[e^{i\boldsymbol{\theta} \cdot \boldsymbol{Y}}\right] = \int_A e^{i\boldsymbol{\theta} \cdot \boldsymbol{y}} f_{\boldsymbol{Y}}(y) d\boldsymbol{y}$$

We will now give an example of the characteristic function of a random variable.

Example 2.3.9 (Characteristic function of a normal distribution).
Let $X \sim N(0,1)$. Then:

$$\phi_X(\theta) = \mathbb{E}[e^{i\theta X}] = \int_{-\infty}^{\infty} e^{i\theta x} f_X(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\theta x} e^{-\frac{1}{2}x^2} dx$$

The integral stated above on the right hand side is equal to:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\theta x} e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\theta x) e^{-\frac{1}{2}x^2} dx$$

since $\int_{-\infty}^{\infty} \sin(\theta x) e^{-\frac{1}{2}x^2} = 0$. Now notice that:

$$\begin{aligned}
\frac{d}{d\theta}\phi_X(\theta) &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \sin(\theta x) e^{-\frac{1}{2}x^2} dx \\
&= -\frac{1}{\sqrt{2\pi}} \left(-\sin(\theta x) e^{-\frac{1}{2}x^2} \Big|_{x\to-\infty}^{x\to\infty} + \theta \int_{-\infty}^{\infty} \cos(\theta x) e^{-\frac{1}{2}x^2} \right) \\
&= -\frac{\theta}{\sqrt{2\pi}} \phi_X(\theta)
\end{aligned}$$

Notice that we are allowed to interchange the differentiating and the integral here by Leibniz' integral rule (see for example [2]). Hence $\phi_X(\theta)$ is of the form $Ae^{-\frac{\theta^2}{2}}$ with A some constant. Since $\phi_X(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$ we have that:

$$\phi_X(\theta) = e^{-\frac{\theta^2}{2}}$$

If we now want to find the characteristic function of a normal distribution with parameters $\mu, \sigma > 0$ we can notice that $Y = \sigma X + \mu$ is $N(\mu, \sigma^2)$ distributed. Now it follows that:

$$\phi_Y(\theta) = \mathbb{E}(e^{itY}) = e^{it\mu}\mathbb{E}(e^{it\sigma X}) = e^{it\mu}\phi_X(\sigma t) = e^{it\mu} e^{-\frac{(\sigma t)^2}{2}} = e^{it\mu - \frac{(\sigma t)^2}{2}}$$

△

One might notice the similarities between the characteristic function and the Fourier transform of a function. We will give the definition of the Fourier transform and the Fourier inversion theorem in only 1 dimension. We will later give an inversion theorem which works in multiple dimensions.
The definition and the Fourier inversion theorem are from [3].

Definition 2.3.10. (Fourier transform.) Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function, such that $\int_{-\infty}^{\infty} |f(x)|dx$ exists. Then we define its Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ by:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2i\pi xt} dx$$

We can thus in the continuous case represent the characteristic function in terms of the Fourier transform - the characteristic function of a continuous random variable X is the Fourier transform of the probability distribution of X corrected by a factor of 2π.

Theorem 2.3.11. *(Fourier inversion theorem.) Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function such that $\int_{-\infty}^{\infty} |f(x)|dx$ is finite. Then we have that:*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{-2i\pi xt} dt$$

Corollary 2.3.12. *Let X be a continuous random variable with probability distribution f_X. Then we have that:*

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(t)e^{itx}dt$$

We will now show one of the most important properties of the characteristic function, however before we can do that we need to define what it means to "converge in distribution".

Definition 2.3.13. Let $(X_n)_{n\geq 1}$ be a sequence of random variables and X be another random variable. We say that X_n converges to X in distribution (notation: $X_n \xrightarrow{d} X$) if for every point x in the domain of F_X where F_X is continuous and F_X is a proper CDF we have that:

$$\lim_{n\to\infty} F_{X_n}(u) = F_X(u)$$

We will use $X \overset{\mathrm{d}}{=} Y$ when we want to state that X has the same distribution as Y, which is equivalent with saying that the cumulative distribution functions of X and Y are the same.
We will now prove some important properties of the characteristic function in multiple dimensions. This proposition is based on proposition 2.2.1 from [4].

Proposition 2.3.14. *Let $\boldsymbol{X} = (X_1, X_2, ..., X_d)$ be a discrete random variable, and let $\boldsymbol{\theta} \in \mathbb{R}^d$. Denote by $\phi_{\boldsymbol{X}}$ the characteristic function of $\boldsymbol{X}$. Then we have that:*

1. *$|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})| \leq 1$ and $\phi_{\boldsymbol{X}}(\boldsymbol{0}) = 1$.*
2. *$\phi_{\boldsymbol{X}}$ is uniformly continuous.*
3. *$\phi_{a\boldsymbol{X}}(\boldsymbol{\theta}) = \phi_{\boldsymbol{X}}(a\boldsymbol{\theta})$ for all $a \in \mathbb{R}$. Furthermore, if $\boldsymbol{X}_1, \boldsymbol{X}_2, ...\boldsymbol{X}_k$ are independent identically distributed d-dimensional random variables with $\boldsymbol{S}_k = \sum_{j=1}^k X_j$ their sum then $\phi_{\boldsymbol{S}_k}(\boldsymbol{\theta}) = (\phi_{\boldsymbol{X}_1}(\boldsymbol{\theta}))^k$.*

Proof.

1. The fact that $\phi_{\boldsymbol{X}}(0) = 1$ follows from the definition of a probability distribution. We will therefore only show that $|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})| \leq 1$ for all $\boldsymbol{\theta} \in \mathbb{R}^d$. Since

$$\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) = \mathbb{E}[e^{i\boldsymbol{\theta}\cdot\boldsymbol{X}}] = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{i(\boldsymbol{k}\cdot\boldsymbol{\theta})}\mathbb{P}\left(\mathbf{X} = \boldsymbol{k}\right)$$

we find for $|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})|$:

$$|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})| = \left|\sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{i(k\cdot t)}\mathbb{P}\left(\mathbf{X} = \boldsymbol{k}\right)\right|$$

By the repeated triangle inequality (see theorem B.0.1 for a proof) we find that:

$$\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |e^{i(\boldsymbol{k}\cdot\boldsymbol{\theta})}|\mathbb{P}\left(\mathbf{X} = k\right) \leq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathbb{P}\left(\mathbf{X} = k\right)$$

As this sum is precisely 1, we see that $|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})| \leq 1$ for all $\boldsymbol{\theta} \in \mathbb{R}^d$.

2. We will prove this statement by looking at the difference between $\phi_{\boldsymbol{X}}(\boldsymbol{\theta}+\boldsymbol{h})$ and $\phi_{\boldsymbol{X}}(\boldsymbol{\theta})$ for each $\boldsymbol{\theta} \in \mathbb{R}^d$ and $\boldsymbol{h} \in \mathbb{R}^d$ small enough. It then follows that:

$$\begin{aligned} |\phi_{\boldsymbol{X}}(\boldsymbol{\theta}+\boldsymbol{h}) - \phi_{\boldsymbol{X}}(\boldsymbol{\theta})| &= \left|\mathbb{E}\left[e^{i\boldsymbol{X}\cdot\boldsymbol{\theta}}(e^{i\boldsymbol{X}\cdot\boldsymbol{h}} - 1)\right]\right| \\ &\leq \mathbb{E}\left[\left|e^{i\boldsymbol{X}\cdot\boldsymbol{\theta}}(e^{i\boldsymbol{X}\cdot\boldsymbol{h}} - 1)\right|\right] \\ &\leq \mathbb{E}\left[\left|e^{i\boldsymbol{X}\cdot\boldsymbol{h}} - 1\right|\right] \end{aligned} \tag{2.1}$$

Now we will expand the last term. We see that:

$$\mathbb{E}\left[\left|e^{i\mathbf{X}\cdot\mathbf{h}} - 1\right|\right] = \sum_{\mathbf{k}\in\mathbb{Z}^d} \left|e^{i\mathbf{k}\cdot\mathbf{h}} - 1\right| \mathbb{P}\left(\mathbf{X} = \mathbf{k}\right) \tag{2.2}$$

Now let $(\mathbf{h}_n)_{n\geq 1}$ be a sequence such that:

$$\lim_{n\to\infty} \boldsymbol{h}_n = \mathbf{0}$$

and $\boldsymbol{h}_n \neq 0$ for all $n \in \mathbb{N}$. It is clear that we have that:

$$\lim_{n\to\infty} \left|e^{i\boldsymbol{X}\cdot\boldsymbol{h}_n} - 1\right| = 0$$

We will show that for this particular case that we may interchange limits in (2.2) and later on use theorem A.0.1 to show that this is equivalent with taking the limit $\boldsymbol{h} \to 0$ to conclude the proof.

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be the associated probability space. Define $f_n : \mathbb{Z}^d \to [0,\infty)$ by:

$$f_n(x) = |e^{i\boldsymbol{x}\cdot\boldsymbol{h}_n} - 1| \leq 2$$

and let $f : \mathbb{Z}^d \to [0,\infty)$ be defined by:

$$f(\boldsymbol{x}) = \lim_{n\to\infty} f_n(\boldsymbol{x}) = \lim_{n\to\infty} |e^{i\boldsymbol{x}\cdot\boldsymbol{h}_n} - 1| = 0$$

then it is clear that both f and f_n for all $n \in \mathbb{N}$ satisfy theorem 2.2.12 (as the constant function 2 is integrable in the probability space). We thus see by theorem 2.2.12 that:

$$\lim_{n\to\infty} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |e^{i\boldsymbol{k}\cdot\boldsymbol{h}_n} - 1|\mathbb{P}\left(\boldsymbol{X} = \boldsymbol{k}\right) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \lim_{n\to\infty} |e^{i\boldsymbol{k}\cdot\boldsymbol{h}_n} - 1|\mathbb{P}\left(\boldsymbol{X} = \boldsymbol{k}\right) = 0$$

As we can repeat this proof for any sequence $(\boldsymbol{h}_n)_{n\geq 1}$ where $\boldsymbol{h}_n \neq \mathbf{0}$ for all $n \in \mathbb{N}$ we see that this is by A.0.1 equivalent with saying that:

$$\lim_{\boldsymbol{h}\to\mathbf{0}} \mathbb{E}[|e^{i\boldsymbol{X}\cdot\boldsymbol{h}} - 1|] = 0$$

which shows that $\phi_{\boldsymbol{X}}$ is uniformly continuous.

3. The first statement follows directly from the definition of a characteristic function. The second statement we can prove by noticing that:

$$\phi_{\boldsymbol{S}_k}(\boldsymbol{\theta}) = \mathbb{E}[e^{i\boldsymbol{\theta}\cdot\left(\sum_{j=1}^k \boldsymbol{X}_j\right)}] = \mathbb{E}[e^{i\boldsymbol{\theta}\cdot\boldsymbol{X}_1}]^k = \left(\phi_{\boldsymbol{X}_1}(\boldsymbol{\theta})\right)^k$$

by independence.

□

We have seen that the characteristic function is almost exactly like a (discrete) Fourier transform. We will state the inversion formula for characteristic functions below in d dimensions. This proposition is based on proposition 2.2.2 from [4].

Proposition 2.3.15 (Inversion formula)**.** *Let $\boldsymbol{X}$ be a d-dimensional discrete random variable. Then for every $\boldsymbol{x} \in \mathbb{Z}^d$ we have:*

$$\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi_{\boldsymbol{X}}\left(\boldsymbol{\theta}\right) e^{-i\boldsymbol{x}\cdot\boldsymbol{\theta}} d\boldsymbol{\theta}$$

Proof. Notice that we have that for $\boldsymbol{\theta} \in \mathbb{R}^d$:

$$\phi_{\mathbf{X}}\left(\boldsymbol{\theta}\right) = \sum_{\boldsymbol{y}\in\mathbb{Z}^d} e^{i\boldsymbol{y}\cdot\boldsymbol{\theta}}\mathbb{P}\left(\mathbf{X} = \boldsymbol{y}\right)$$

Hence:

$$\int_{[-\pi,\pi]^d} \phi_{\mathbf{X}}\left(\boldsymbol{\theta}\right) e^{-i\boldsymbol{x}\cdot\boldsymbol{\theta}} = \int_{[-\pi,\pi]^d} \sum_{\boldsymbol{y}\in\mathbb{Z}^d} e^{i(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\theta}}\mathbb{P}\left(\mathbf{X} = \boldsymbol{y}\right) d\boldsymbol{\theta}$$

Let

$$a_m = \sum_{\boldsymbol{y} \in \mathbb{Z}^d \cap B(\mathbf{0},m)} \mathbb{P}\left(\mathbf{X} = \boldsymbol{y}\right) \int_{[-\pi,\pi]^d} e^{i(\boldsymbol{y}-\boldsymbol{x})\boldsymbol{\theta}} d\boldsymbol{\theta}$$

be a constant function. Then it is clear that:

$$a = \lim_{m\to\infty} a_m = \int_{[-\pi,\pi]^d} \sum_{\boldsymbol{y}\in\mathbb{Z}^d} e^{i(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\theta}} \mathbb{P}\left(\mathbf{X} = \boldsymbol{y}\right) d\boldsymbol{\theta}$$

We can furthermore notice that:

$$|a_m| = \left| \int_{[-\pi,\pi]^d} \sum_{\boldsymbol{y}\in\mathbb{Z}^d \cap B(0,m)} e^{i(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\theta}} \mathbb{P}\left(\mathbf{X} = \boldsymbol{y}\right) d\boldsymbol{\theta} \right| \leq \int_{[-\pi,\pi]^d} \left| \sum_{\boldsymbol{y}\in\mathbb{Z}^d} e^{i(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\theta}} \mathbb{P}\left(\mathbf{X} = \boldsymbol{y}\right) \right| d\boldsymbol{\theta}$$

$$\leq \int_{[-\pi,\pi]^d} 1 d\boldsymbol{\theta} = (2\pi)^d$$

hence a_m is bounded. It now follows by the dominated convergence theorem that we can interchange the sum and the integral, and the result follows by noticing that the integral given by:

$$\int_{[-\pi,\pi]^d} e^{i(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{\theta}} d\boldsymbol{\theta}$$

is equal to 0 if $\boldsymbol{x} \neq \boldsymbol{y}$ and $(2\pi)^d$ otherwise. □

In the case where we are dealing with 1-dimensional random variables, there are some other important properties that the characteristic function has.

Proposition 2.3.16. *Let X be a 1-dimensional random variable, and let $t \in \mathbb{R}$. Denote by ϕ_X the characteristic function of X. Then:*

1. *If X is discrete and $\mathbb{E}[|X|^m]$ exists for some $m \in \mathbb{N}$ we have that ϕ is m times differentiable and in particular:*

$$\phi^{(m)}(0) = i^{-m}\mathbb{E}[X^m]$$

2. *If ϕ_X is continuous around 0, $(X_n)_{n\geq 1}$ is a sequence of 1-dimensional random variables with characteristic function ϕ_{X_n} for each $n \in \mathbb{N}$ and:*

$$\lim_{n\to\infty} \phi_{X_n}(s) \to \phi_X(s)$$

for all $s \in \mathbb{R}$ then $X_n \xrightarrow{d} X$.

3. *If Y is another 1-dimensional random variable with $\phi_X(t) = \phi_Y(t)$ for all $t \in \mathbb{R}$ then $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$ where F_X is continuous and F_X is a proper cumulative distribution function.*

Proof. 1. We will give a proof in the case that $m = 1$, as for any $m > 1$ the proof follows analogously. We want to show that:

$$\lim_{h\to 0} \frac{\phi_X(h) - 1}{h} = \lim_{h\to 0} \sum_{k\in\mathbb{Z}} \left(\frac{e^{ikh} - 1}{h} \right) \mathbb{P}(X = k) = i\mathbb{E}[X]$$

Let $(h_n)_{n\geq 1}$ be a sequence such that $\lim_{n\to\infty} h_n = 0$ and $h_n \neq 0$ for all $n \in \mathbb{N}$. We will show that for this specific sequence that the statement is true, and then use theorem A.0.1 from the appendix to show that we may first differentiate and then calculate the infinite sum. Define $f_n : \mathbb{R} \to \mathbb{C}$ by:

$$f_n(t) = \frac{e^{ith_n} - 1}{h_n}$$

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be the associated probability space. We can bound f_n for all $t \in \mathbb{R}$ by:

$$|f_n(t)| = \left| \frac{e^{ith_n} - 1}{h_n} \right| \leq 1$$

which is integrable. Furthermore define $f : \mathbb{R} \to \mathbb{C}$ by:

$$f(t) = \lim_{n\to\infty} f_n(t) = it$$

Then by the dominated convergence theorem we may write:

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} f_n(k)\mathbb{P}(X = k) = \sum_{k\in\mathbb{Z}} f(k)\mathbb{P}(X = k) = i \sum_{k\in\mathbb{Z}} k\mathbb{P}(X = k) = i\mathbb{E}[X]$$

As we can repeat this for every sequence $(h_n)_{n\geq 1}$ such that $h_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} h_n = 0$ we see that by theorem A.0.1:

$$\lim_{h\to 0} \frac{\phi_X(h) - 1}{h} = i\mathbb{E}[X]$$

This concludes the proof.

2. We will not prove this statement here. Instead we will refer to [5] in which a proof is given.

3. This is a direct consequence of (2): assume to the contrary that there is some $x \in \mathbb{R}$ where F_X is continuous and a proper cumulative distribution function such that $F_X(x) \neq F_Y(y)$. By (2) we know that for the sequence defined by $X_n = Y$ for all $n \in \mathbb{N}$ we have that

$$\lim_{n\to\infty} F_{X_n}(y) = F_Y(y) = F_X(y)$$

for all $y \in \mathbb{R}$ where F_X is continuous and a proper cumulative distribution function, which is a contradiction.

□

2.3.3 Stable random variables

The following section follows the definitions and theorems as stated [6], chapter 1. We will start with a definition of a stable random variable.

Definition 2.3.17. A stable random variable X is a random variable that satisfies the following condition: for every X_1, X_2 which are identically and independently distributed like X, and for all $a, b > 0$, there exists a $c > 0$ and $d \in \mathbb{R}$ such that:

$$aX_1 + bX_2 \stackrel{\mathrm{d}}{=} cX + d \tag{2.3}$$

Theorem 2.3.18. *For any stable random variable X there is some $\alpha \in (0, 2]$ such that for all $a, b, c > 0$ and $d \in \mathbb{R}$ that satisfy*

$$aX_1 + bX_2 \stackrel{\mathrm{d}}{=} cX + d$$

with X_1 and X_2 independent copies of X we have:

$$c^\alpha = a^\alpha + b^\alpha$$

Proof. We will not prove this statement here, but refer to [6] for a proof. □

We will call a stable random variable X with the number α as above a stable random variable with scale parameter α.
Although most stable random variables do not have well defined probability distributions, it does turn out ([6], page 9) that every stable random variable is continuous.

A typical example of a stable distribution is the normal distribution. Indeed, if X is normally distributed with mean μ and variance σ^2 and X_1 and X_2 are independent copies of X, then for some given $a, b > 0$ we can use the fact that:

$$\phi_{aX_1+bX_2}(t) = \mathbb{E}[e^{i(aX_1+bX_2)t}] = \mathbb{E}[e^{iX_1at}]\mathbb{E}[e^{iX_2at}] = e^{it(a+b)\mu-\frac{(a^2+b^2)(\sigma t)^2}{2}}$$

By (3) of proposition 2.3.16 we see that $aX_1 + bX_2$ is distributed as a normal variable with mean $(a+b)\mu$ and variance $(a^2+b^2)\sigma$. We find that $c^2 = a^2 + b^2$ and $d = (a + b - a^2 - b^2)\mu$ satisfy equation 2.3.

Another example of a stable distribution is the Cauchy distribution (see [6]). The probability density function of X with $X \sim Cau(\mu, \sigma)$ is given by:

$$f_X(x) = \frac{\sigma}{\pi\left((x-\mu)^2 + \sigma^2\right)}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $x \in \mathbb{R}$.

While in most cases for stable distributions we cannot find a probability distribution which we can express in simple terms, we can however characterize stable distributions in terms of their characteristic function. We will only give the definition of symmetric α-stable random variables, as those are the kind of stable random variables that we will later have to use.

Definition 2.3.19. (Equivalent to definition 2.3.17.) We say that a random variable X is symmetric α-stable for some $\alpha \in (0, 2]$ if there is some $\sigma \geq 0$ such that:

$$\phi_X(\theta) = e^{-\sigma^\alpha|\theta|^\alpha}$$

We call σ the scale parameter of X.

The fact that this definition is equivalent to the one given in definition 2.3.17 will not be shown here, but we instead refer to [6]. The Cauchy distribution with $\mu = 0$ is an example of a symmetric stable random variable with $\alpha = 1$.

3 Classical local limit theorems

The central limit theorem and the local limit theorem both tell us that if we have a sequence of 1-dimensional random variables $(X_i)_{i\in\mathbb{N}}$ where each X_i has finite variance that the sum of these random variables when properly scaled will converge to a normal random variable. In multiple dimensions their sum will converge to a multivariate normal distribution. A discrete adaption of the central limit theorem is the local central limit theorem which we will state and prove in this section here. We will start with a proof of the 1-dimensional central limit theorem.

3.1 Central limit theorem

The central limit theorem is one of the most important theorems in probability theory. We will give a formal proof of this theorem in this subsection before we go on and prove the local central limit theorem.
Before we prove the central limit theorem we first introduce some tools to prove this theorem.

Lemma 3.1.1. *For any $n \in \mathbb{N}$ and for any $x > -1$ we have that $(1+x)^n \geq 1 + nx$.*

Proof.
We will use induction to n. For $n = 1$ the result is obvious. Assume now that the statement is proven for $n = k$, thus we know that:

$$1 + kx \leq (1+x)^k$$

then we see for $k+1$ that:

$$(1+x)^{k+1} \leq (1+x)(1+kx) = 1 + x + kx + kx^2 \leq 1 + (k+1)x$$

As the statement is also true for $k+1$ we conclude that the statement is true for all $n \in \mathbb{N}$. □

Corollary 3.1.2. *For all $x > -1$ we have that $x \geq \log(1+x) \geq \frac{x}{x+1}$.*

Proof.
We will first consider the upper bound. Since we can write $e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$ it follows by lemma 3.1.1 that $e^x \geq 1+x$ for all $x > -1$. As $\log(x)$ is a strictly monotone increasing function the upper bound follows. For the lower bound we can notice that implies that for all $t > 0$ we have that $t - 1 \geq \log(t)$. In particular we have that $\frac{1}{t} - 1 \geq -\log(t)$, hence $1 - \frac{1}{s+1} \leq \log(s+1)$ where $s = t-1$. Rewriting the left hand side gives the result. □

Lemma 3.1.3. *Let $(a_n)_{n\geq 1}$ be a sequence. Let $a \in \mathbb{R}$, and assume that $\lim_{n\to\infty} a_n = a$. Then*

$$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = e^a$$

Proof.
Write

$$\left(1+\frac{a_n}{n}\right)^n = e^{n\ln\left(1+\frac{a_n}{n}\right)}$$

By the product rule for limits of sequences we see that $\lim_{n\to\infty}\frac{a_n}{n} = 0$, and it follows that we can find $N \in \mathbb{N}$ such that $|\frac{a_n}{n}| < 1$ for $n \geq N$.
We will now lower and upper bound $n\ln\left(1+\frac{a_n}{n}\right)$. By corollary 3.1.2 we know that for $n \geq N$:

$$a_n \geq n\ln\left(1+\frac{a_n}{n}\right) \geq \frac{na_n}{a_n+n}$$

As

$$\lim_{n\to\infty}\frac{na_n}{a_n+n} = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty}\frac{n}{n+a_n}\right) = a$$

we see that by the squeeze theorem:

$$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = e^a$$

□

We are now equipped to prove the central limit theorem.

Theorem 3.1.4. *(Central limit theorem.) Let $(X_n)_{n\geq 1}$ be a sequence of independently identically distributed 1-dimensional random variables with finite positive variance. Denote by $\bar{X}_n$ the average of all the X_i, thus $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. Let μ be the expected value and σ^2 the variance of each X_i. Define*

$$Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$$

then $Z_n \xrightarrow{d} Z$, where Z is $N(0,1)$ distributed.

Proof.
Let $Y_i = \frac{X_i - \mu}{\sigma}$ for $1 \leq i \leq n$. Then

$$\sqrt{n}\left(\sum_{k=1}^{n} Y_k\right) = Z_k$$

It is easy to see that $\mathbb{E}[Y_i] = 0$ and $\mathrm{Var}(Y_i) = 1$. In particular we have seen that we can then write

$$\phi_{Z_n}(t) = \phi_{\sum_{k=1}^{n} Y_k}\left(\frac{t}{\sqrt{n}}\right) = \left(\phi_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

for $t \in \mathbb{R}$.

By Taylor's theorem, we may write for some $s \in \mathbb{R}$:

$$\phi_{Y_1}\left(\frac{s}{\sqrt{n}}\right) = \phi_{Y_1}(0) + \frac{s}{\sqrt{n}}\phi'_{Y_1}(0) + \frac{s^2}{2n}\phi''_{Y_1}(0) + \frac{s^2}{n}R_2\left(\frac{s}{\sqrt{n}}\right)$$

where

$$\lim_{x\to 0} R_2(x) = 0$$

Since we have that $\lim_{n\to\infty}\frac{s}{\sqrt{n}} = 0$ it is easy to verify that the statement above implies that:

$$\lim_{n\to\infty} R_2\left(\frac{s}{\sqrt{n}}\right) = 0$$

As

$$\phi_{Y_1}(0) = 1$$
$$\phi'_{Y_1}(0) = -i\mathbb{E}[Y_1] = 0$$
$$\phi''_{Y_1}(0) = -\mathbb{E}[Y_1^2] = -1$$

we see that:

$$\phi_{Z_n} = \left(1 - \frac{s^2}{2n} + \frac{a_n}{n}\right)^n$$

where a_n is a sequence that goes to 0 as n goes to infinity. As such we see that by lemma 3.1.3:

$$\lim_{n\to\infty}\left(1 - \frac{s^2}{2n} + \frac{a_n}{n}\right)^n = e^{-\frac{s^2}{2}}$$

As the right hand side is the characteristic function of a standard normal variable, we conclude by (2) of proposition 2.3.16 that $Z_n \xrightarrow{d} Z$. □

3.2 Random walks

A random walk is a sequence of steps in a certain mathematical space. In this paper we will only consider the random walks in $\mathbb{Z}^d$ with $d \geq 1$ an integer. We will begin with a formal definition. The theorems and definitions in this section are similar to the ones given in [4].

Definition 3.2.1. A finite range random walk $\mathbf{S}_n$ in $\mathbb{Z}^d$ is a sum of symmetric, independent, identically distributed random variables $\mathbf{X}_1, ...\mathbf{X}_n$, each of which has probability distribution:

$$\mathbb{P}\left(\mathbf{X}_j = \mathbf{e}_k\right) = \mathbb{P}\left(\mathbf{X}_j = -\mathbf{e}_k\right) = \frac{1}{2}\kappa\left(e_k\right), \quad j, k = 1...n$$

where $\kappa : \mathbb{Z}^n \to (0, 1]$ such that $\sum_{l=1}^{d} \kappa\left(e_l\right) \leq 1$ and $\kappa(\mathbf{0}) = 1 - \sum_{l=1}^{d} \kappa\left(e_l\right)$.

A *long* range random walk $\mathbf{S}_n$ in $\mathbb{Z}^d$ is a sum of symmetric, independent, identically distributed random variables $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ such that for all $\boldsymbol{x} \in \mathbb{Z}^d$ there is some $N \in \mathbb{N}$ such that for $k \geq N$:

$$\mathbb{P}\left(\sum_{i=1}^{k} \boldsymbol{X}_i = \boldsymbol{x}\right) > 0$$

where $\boldsymbol{X}_\mathbf{1}, ..., \boldsymbol{X}_k$ are all independent and identically distributed as $\boldsymbol{X}_1$.

Given a random variable $\boldsymbol{X}$ belonging to a d-dimensional (finite or long range) random walk, we will with $p_{\boldsymbol{X}}^n$ denote the probability distribution of $\mathbf{S}_n$. Thus:

$$p_{\boldsymbol{X}}^n(\mathbf{x}) = \mathbb{P}\left(\mathbf{S}_n = \mathbf{x}\right)$$

$\mathbf{x} \in \mathbb{Z}^d$ where $\boldsymbol{S}_n$ is a (finite or long range respectively) random walk consisting of identically and independently distributed random variables $\boldsymbol{X}_j$ whose distribution is the same as that of $\boldsymbol{X}$. Additionally, if we talk about some random variable $\boldsymbol{X}$ and talk about $\boldsymbol{S}_n$ we will mean a random walk $\boldsymbol{S}_n$ as given above. We will from now on deduce most properties for random walks in terms of their probability mass functions.

Note that the most default case of a finite range random walk (with $\kappa(\mathbf{0}) = 0$) can never return to the origin if an odd amount of steps has been set. However, if $\kappa(\mathbf{0}) \neq 0$ we can return to the origin with an odd amount of steps. We will make a distinction between those two types of random walks.

Definition 3.2.2. We say that the probability distribution $p_{\boldsymbol{X}}$ of some random variable $\boldsymbol{X}$ belong to a random walk is *bipartite* if $p_{\boldsymbol{X}}^k(\mathbf{0}) = 0$ for all odd k, and *aperiodic* otherwise.

3.3 Local central limit theorem

We will now state the local central limit theorem for random walks. We only give it in case of aperiodic walks, but we can give a similar statement for bipartite walks. For that we refer to [4].

Theorem 3.3.1. *(Local central limit theorem.) Let $p_{\boldsymbol{X}}$ be the probability distribution of a random variable $\boldsymbol{X}$ belonging to an aperiodic (long range or finite) random walk in $\mathbb{Z}^d$. Assume that $\mathbb{E}[||\boldsymbol{X}||]^3$ exists. Let Γ be the associated covariance matrix of $\boldsymbol{X}$. Define:*

$$\bar{p}^n_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{\boldsymbol{s}\cdot\boldsymbol{x}}{\sqrt{n}} - \frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} d\boldsymbol{s}$$

Then there is some $c \geq 0$ such that for all $n \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{Z}^d$ we have:

$$|p^n_{\boldsymbol{X}}(\boldsymbol{x}) - \bar{p}^n_{\boldsymbol{X}}(\boldsymbol{x})| \leq \frac{c}{n^{\frac{d+1}{2}}}$$

In particular, if we assume that $\mathbb{E}[||\boldsymbol{X}||^4]$ exists and $\mathbb{E}[||\boldsymbol{X}||^3] = 0$, we can improve this result to:

$$|p^n_{\boldsymbol{X}}(\boldsymbol{x}) - \bar{p}^n_{\boldsymbol{X}}(\boldsymbol{x})| \leq \frac{c'}{n^{\frac{d+2}{2}}}$$

for some $c' \geq 0$.

In a similar style to [4], chapter 2 we will prove this statement in several steps below. The following theorems and their proofs are adaptions of the proofs and theorems given in [4], chapter 2.

Lemma 3.3.2. *Let $\boldsymbol{X}$ be a $d-$dimensional random variable belong to a (finite or long range) random walk in $\mathbb{Z}^d$. For every $\epsilon > 0$ we have that*

$$\sup\{|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})|\ ;\ \boldsymbol{\theta} \in [-\pi,\pi]^d, ||\boldsymbol{\theta}|| \geq \epsilon\} < 1$$

Proof. Notice first of all that we have:

$$\sup\{|\phi_X(\boldsymbol{\theta})|\ ;\ \boldsymbol{\theta} \in [-\pi,\pi]^d, ||\boldsymbol{\theta}|| \geq \epsilon\} \leq \sup\{|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})|\ ;\ \boldsymbol{\theta} \in [-\pi,\pi]^d, \boldsymbol{\theta} \neq \boldsymbol{0}\}$$

In a similar fashion to [4] we will prove this lemma using a contradiction. Assume to the contrary that there is some $\boldsymbol{\theta} \in [-\pi,\pi]^d \setminus \{\boldsymbol{0}\}$ such that

$$|\phi_{\boldsymbol{X}}(\boldsymbol{\theta})| = 1$$

We can now see that:

$$||\mathbb{E}[e^{i\sum_{j=1}^n \mathbf{X}_j\cdot\boldsymbol{\theta}}]|| = |\mathbb{E}[e^{i\mathbf{S}_n\cdot\boldsymbol{\theta}}]| = |\phi_{\mathbf{S}_n}(\boldsymbol{\theta})| = 1$$

Thus we see that:

$$|\phi_{\mathbf{S}_n}(\boldsymbol{\theta})| = |\sum_{\boldsymbol{x}\in\mathbb{Z}^d} p^n_{\boldsymbol{X}}(\boldsymbol{x})e^{i\boldsymbol{x}\cdot\boldsymbol{\theta}}| \leq \sum_{x\in\mathbb{Z}^d} |p^n_{\boldsymbol{X}}(\boldsymbol{x})e^{i\boldsymbol{x}\cdot\boldsymbol{\theta}}| = \sum_{\boldsymbol{x}\in\mathbb{Z}^d} |p^n_{\boldsymbol{X}}(\boldsymbol{x})| = 1$$

Hence by corollary B.0.3 we see that for all $\boldsymbol{x} \in \mathbb{Z}^d$:

$$p^n_{\boldsymbol{X}}(\boldsymbol{x})e^{i\boldsymbol{x}\cdot\boldsymbol{\theta}} = a_{\boldsymbol{x}}e^{i\psi}$$

with $a_{\boldsymbol{x}} \geq 0$. As we can find some $\boldsymbol{y} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ with $p^n_{\boldsymbol{X}}(\boldsymbol{y}) = p^n_{\boldsymbol{X}}(-\boldsymbol{y}) > 0$ we have that:

$$p^n_{\boldsymbol{X}}(\boldsymbol{y})e^{i\boldsymbol{y}\cdot\boldsymbol{\theta}} = p^n_{\boldsymbol{X}}(-\boldsymbol{y})e^{-i\boldsymbol{y}\cdot\boldsymbol{\theta}}$$

which implies that $\boldsymbol{\theta} = \boldsymbol{0}$. This concludes the proof. ◻

A direct implication of this lemma is in fact that we can write for any $\epsilon > 0$:

$$\sup\{|\phi_X(\boldsymbol{\theta})| \; ; \; \boldsymbol{\theta} \in [-\pi, \pi]^d, ||\boldsymbol{\theta}|| \geq \epsilon\} \leq e^{-b}$$

for some $b > 0$.

Lemma 3.3.3. *Let $\boldsymbol{X} = (X_1, ..., X_d)^T$ be a discrete random variable in $\mathbb{Z}^d$ with $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{0}$ and $\mathbb{E}[||\boldsymbol{X}||^2] < \infty$. The Taylor expansion of $\phi_{\boldsymbol{X}}$ is given by:*

$$\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) = 1 - \frac{1}{2}\sum_{k=1}^{d}\sum_{j=1}^{d}\mathbb{E}[X_k X_j]\theta_k\theta_j + O\left(||\boldsymbol{\theta}||^2\right)$$

Proof. The Taylor expansion of $\phi_{\boldsymbol{X}}$ around $\boldsymbol{\theta} = \boldsymbol{0}$ is in this case given by:

$$\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) = \phi_{\boldsymbol{X}}(\boldsymbol{0}) + (\boldsymbol{\theta}\cdot\nabla)\phi_{\boldsymbol{X}}(\boldsymbol{0}) + \frac{(\boldsymbol{\theta}\cdot\nabla)^2\phi_{\boldsymbol{X}}(\boldsymbol{0})}{2} + O\left(||\boldsymbol{\theta}||^2\right)$$

Where ∇ denotes the operator which turns a vector into a gradient vector. That is,

$$\nabla\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) = \left(\frac{\partial\phi_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_1}, ..., \frac{\partial\phi_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_n}\right)^T$$

In a similar way to the proof of case (1) of proposition 2.3.16 we can see that (with p the probability mass function of $\boldsymbol{X}$):

$$(\boldsymbol{\theta}\cdot\nabla)\,\phi_{\boldsymbol{X}}(\boldsymbol{0}) = i\sum_{k=1}^{d}\theta_k\sum_{\boldsymbol{x}\in\mathbb{Z}^d}x_k p(\boldsymbol{x}) = 0$$

$$(\boldsymbol{\theta}\cdot\nabla)^2\,\phi_{\boldsymbol{X}}(\boldsymbol{0}) = -\sum_{k=1}^{d}\sum_{j=1}^{d}\theta_j\theta_k\sum_{x\in\mathbb{Z}^d}x_k x_j p(\boldsymbol{x}) = -\sum_{k=1}^{d}\sum_{j=1}^{d}\theta_j\theta_k\mathbb{E}[X_k X_j]$$

The result now follows.

□

We can express the statement above in terms of the covariance matrix of $\boldsymbol{X}$: if Γ is the covariance matrix of $\boldsymbol{X}$, then we can write $\phi_{\boldsymbol{X}}$ as:

$$\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) = 1 - \frac{\theta\cdot\Gamma\theta}{2} + O\left(||\boldsymbol{\theta}||\right)$$

Note that if we would want to expand this series further we would require higher moment conditions on $\boldsymbol{X}$. In the case of our random walks this expansion reduces to a sum of squares as the random variables X_k and X_j for $k \neq j$ are independent of each other.

Lemma 3.3.4. *Let $\boldsymbol{X}$ be a discrete random variable in $\mathbb{Z}^d$ with finite second moment and $\mathbb{E}[\boldsymbol{X}] = 0$. Then we can find $\epsilon > 0$, $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ with $\boldsymbol{\theta} \in [-\epsilon\sqrt{n}, \epsilon\sqrt{n}]^d$ we can write*

$$\left(\phi_{\boldsymbol{X}}\left(\frac{\boldsymbol{\theta}}{\sqrt{n}}\right)\right)^n = e^{-\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2}}\left(1 + F_n(\boldsymbol{\theta})\right)$$

with

$$|F_n(\boldsymbol{\theta})| \leq e^{\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{4}} + 1$$

Proof. Choose $\delta > 0$ such that for all $\boldsymbol{\theta} \in B(0, \delta)$ we have:

$$|\phi_{\boldsymbol{X}}(\boldsymbol{\theta}) - 1| \leq \frac{1}{2}$$

(this is always possible as $\phi_{\boldsymbol{X}}$ is continuous around $\boldsymbol{\theta} = 0$ and $\phi_{\boldsymbol{X}}(0) = 1$). Using the Taylor expansion of $\log\left(\phi_{\boldsymbol{X}}(\boldsymbol{\theta})\right)$ around $\boldsymbol{\theta} = \boldsymbol{0}$ gives:

$$\log\left(\phi_{\boldsymbol{X}}(\boldsymbol{\theta})\right) = -\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2} + O(||\boldsymbol{\theta}||^2)$$

In particular, it follows that:

$$n\log\left(\phi_{\boldsymbol{X}}\left(\frac{\theta}{\sqrt{n}}\right)\right) = -\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2} + g(\boldsymbol{\theta}, n)$$

with $g(\boldsymbol{\theta}, n) = O(||\boldsymbol{\theta}||^2)$. A direct consequence of this is the fact that we have:

$$\lim_{\boldsymbol{\theta}\to\boldsymbol{0}} -\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2} + 2g(n, \boldsymbol{\theta}) = \boldsymbol{0}$$

This implies that we can find some $\epsilon > 0, \epsilon \leq \delta$ such that for $\boldsymbol{\theta} \in B(\boldsymbol{0}, \epsilon)$ (as then $\frac{|\boldsymbol{\theta}|}{\sqrt{n}} \leq \delta$) we have:

$$|g(\boldsymbol{\theta}, n)| \leq \frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{4}$$

As we can write

$$F_n(\boldsymbol{\theta}) = e^{g(n,\boldsymbol{\theta})} - 1$$

the result follows. □

Note that we can improve the result of lemma 3.3.4 in the case of either $\mathbb{E}[||\boldsymbol{X}||^3] < \infty$ or $\mathbb{E}[||\boldsymbol{X}||^4] < \infty$ and $\mathbb{E}[||\boldsymbol{X}||^3] = 0$. We can see this by expanding the Taylor expansion we found in lemma 3.3.3. We can then again write:

$$n\log\left(\phi_{\boldsymbol{X}}\left(\frac{\boldsymbol{\theta}}{\sqrt{n}}\right)\right) = -\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2} + g(\boldsymbol{\theta}, n)$$

but now with either $g(\boldsymbol{\theta}, n) = O\left(\frac{||\boldsymbol{\theta}||^3}{\sqrt{n}}\right)$ or $g(\boldsymbol{\theta}, n) = O\left(\frac{||\boldsymbol{\theta}||^4}{n}\right)$ respectively.

Lemma 3.3.5. *Let $\boldsymbol{X}$ be a d-dimensional random variable belong to a random walk with finite second moment, $\mathbb{E}[\boldsymbol{X}] = 0$ covariance matrix Γ, and let ϵ, F_n be as in the previous lemma. Then we have that:*

$$p^n_{\boldsymbol{X}}(\boldsymbol{x}) = \bar{p}^n_{\boldsymbol{X}}(\boldsymbol{x}) + O(e^{-bn}) + \frac{1}{(2\pi)^d n^{\frac{d}{2}}}\int_{|\boldsymbol{\theta}|\leq\epsilon\sqrt{n}} e^{-\frac{s\cdot\Gamma s}{2}} F_n(\boldsymbol{\theta}) e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s}$$

Proof. By the inversion formula, we may write:

$$p^n_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \left(\phi_{\boldsymbol{X}}(\boldsymbol{\theta})\right)^n e^{-i\boldsymbol{x}\cdot\boldsymbol{\theta}} d\boldsymbol{\theta}$$

Using a change of variables, this becomes equal to:

$$p^n_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^d n^{\frac{d}{2}}}\int_{[-\sqrt{n}\pi,\sqrt{n}\pi]^d} \left(\phi_{\boldsymbol{X}}\left(\frac{\boldsymbol{s}}{\sqrt{n}}\right)\right)^n e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s}$$

with $z = \frac{\boldsymbol{x}}{\sqrt{n}}$. Let $\epsilon' = \frac{2\epsilon}{\sqrt{d}}$. The hypercube with side lengths of ϵ' is now fully inscribed in the hypersphere with a radius of ϵ. By lemma 3.3.2 we can find some $b > 0$ such that for $||\boldsymbol{\theta}|| > \epsilon'$ we have that $|\phi(\boldsymbol{\theta})| \leq e^{-b}$. Let

$$A = [-\sqrt{n}\pi, \sqrt{n}\pi]^d \cap \{\boldsymbol{\theta} \in \mathbb{R}^d \ : \ ||\boldsymbol{\theta}|| > \epsilon\sqrt{n}\}$$
$$B = \left[-\sqrt{n}\pi, -\frac{2\epsilon\sqrt{n}}{\sqrt{d}}\right]^d \cup \left[\frac{2\epsilon\sqrt{n}}{\sqrt{d}}, \sqrt{n}\pi\right]^d$$

It now follows that:

$$\begin{aligned}
\frac{1}{(2\pi)^d n^{\frac{d}{2}}}\int_A \left(\phi_{\boldsymbol{X}}\left(\frac{s}{\sqrt{n}}\right)\right)^n e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} &\leq \frac{1}{e^{bn}(2\pi)^d n^{\frac{d}{2}}}\int_A e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} \\
&\leq \frac{1}{e^{bn}(2\pi)^d n^{\frac{d}{2}}}\int_B e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} \qquad (3.1)\\
&= \frac{2}{e^{bn}(2\pi)^d n^{\frac{d}{2}}}\int_{[\frac{2\epsilon\sqrt{n}}{\sqrt{d}},\sqrt{n}\pi]^d} \cos(\boldsymbol{z}\cdot\boldsymbol{s}) d\boldsymbol{s} = O(e^{-bn})
\end{aligned}$$

Hence we can write:

$$p_n(x) = O(e^{-bn}) + \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \leq \epsilon\sqrt{n}} \left(\phi\left(\frac{s}{\sqrt{n}}\right)\right)^n e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s}$$

However, we also know that for this particular ϵ we can write for $||\boldsymbol{s}|| \leq \epsilon\sqrt{n}$:

$$\left(\phi_{\boldsymbol{X}}\left(\frac{\boldsymbol{s}}{\sqrt{n}}\right)\right)^n = e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} + e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} F_n(\boldsymbol{\theta}) \geq e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}}$$

As

$$\bar{p}^n_{\boldsymbol{X}}(x) = \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \leq \epsilon\sqrt{n}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} + \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \geq \epsilon\sqrt{n}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s}$$

We get for the term on the right hand side in a similar way to what we saw before for some $C \geq 0$:

$$\left|\frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \geq \epsilon\sqrt{n}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s}\right| \leq \frac{C}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \geq \epsilon\sqrt{n}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} d\boldsymbol{s} = O(e^{-bn})$$

The result now follows. □

We will now prove the local central limit theorem.

Proof. (Of the local central limit theorem). Assume that $\mathbb{E}[||\boldsymbol{X}||^3]$ exists. We can now use the bound from lemma 3.3.5 and we may write:

$$\begin{aligned} &\bar{p}^n_{\boldsymbol{X}}(x) + O(e^{-bn}) + \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \leq \epsilon\sqrt{n}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} F_n(\boldsymbol{\theta}) e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} \\ &= \bar{p}^n_{\boldsymbol{X}}(x) + O(e^{-bn}) + \frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{||\boldsymbol{\theta}|| \leq n^{1/8}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} F_n(\boldsymbol{\theta}) e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} + O(e^{-bn^{1/4}}) \end{aligned} \tag{3.2}$$

The fact that

$$\frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{n^{1/8} \leq ||\boldsymbol{\theta}|| \leq \epsilon\sqrt{n}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} F_n(\boldsymbol{\theta}) e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s} = O(e^{-bn^{1/4}})$$

is confirmed by using the fact that for $\boldsymbol{\theta} \in B(\boldsymbol{0}, \epsilon\sqrt{n})$ we can apply the bound for $F_n(\boldsymbol{\theta})$ from lemma 3.3.4 to see that:

$$\left|\frac{1}{(2\pi)^d n^{\frac{d}{2}}} \int_{|\boldsymbol{\theta}| \geq n^{1/8}} e^{-\frac{\boldsymbol{s}\cdot\Gamma\boldsymbol{s}}{2}} F_n(\boldsymbol{\theta}) e^{-i\boldsymbol{z}\cdot\boldsymbol{s}} d\boldsymbol{s}\right| \leq \frac{C}{(2\pi)^d n^{\frac{d}{2}}} \int_{|\boldsymbol{\theta}| \geq n^{1/8}} e^{-\frac{s\cdot\Gamma s}{4}} ds$$

for some $C > 0$. On the other hand, we have seen in lemma 3.3.4 that for $\boldsymbol{\theta} \in B(\boldsymbol{0}, n^{1/8})$ we have that $g(n, \boldsymbol{\theta}) = O\left(\frac{|\boldsymbol{\theta}|^3}{\sqrt{n}}\right)$. As $F_n(\boldsymbol{\theta}) = e^{g(n,\boldsymbol{\theta})} - 1$, we see that by a taylor expansion of $F_n(\boldsymbol{\theta})$ around $\boldsymbol{\theta} = \boldsymbol{0}$ that:

$$|F_n(\boldsymbol{\theta})| \leq c|\boldsymbol{\theta}|^3 n^{-\frac{3}{2}}$$

for some $c > 0$. In particular, this means that:

$$\left|\int_{||\boldsymbol{\theta}|| \leq n^{1/8}} e^{-i\boldsymbol{s}\cdot\boldsymbol{\theta}} e^{-\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2}} F_n(\boldsymbol{\theta}) d\boldsymbol{\theta}\right| \leq \frac{c}{\sqrt{n}} \int_{\mathbb{R}^d} ||\boldsymbol{\theta}||^3 e^{-\frac{\boldsymbol{\theta}\cdot\Gamma\boldsymbol{\theta}}{2}} d\boldsymbol{\theta} \leq \frac{c'}{\sqrt{n}} \tag{3.3}$$

For some $c' > 0$. We refer for a proof of the last inequality to the appendix (**??**). In the case that we assume that $\mathbb{E}[||\boldsymbol{X}||^4] < \infty$ and $\mathbb{E}[||\boldsymbol{X}||^3] = 0$ we can show the same by bounding the integral from (3.3) by $\frac{b'}{n}$ for some $b' > 0$. This concludes the proof.

□

4 Stable local central limit theorem

In this section we will consider the local central limit theorem in the case of stable random variables. We will follow the definitions and proofs given in [7]. We will when possible use the notation from the previous chapter.

4.1 Admissible distributions

Definition 4.1.1. Let $\alpha \in (0, 2]$. Let furthermore $R_\alpha \subset (\alpha, 2+\alpha)$ be a finite set. Then we call a 1-dimensional discrete symmetric random variable X *admissible* of index α and set R_α if we can write:

$$\phi_X(x) = 1 - \kappa_\alpha|\theta|^\alpha + \sum_{\beta \in R_\alpha} \kappa_\beta |\theta|^\beta + O(|\theta|^{2+\alpha})$$

as $|\theta| \to 0$, where $\kappa_\alpha > 0$ and $\kappa_\beta \neq 0$ for all $\beta \in \mathbb{R}_\alpha$.

We will group these admissible distributions in three different cases, and give for each case a different proof of the stable local central limit theorem.

Definition 4.1.2. Let X be a 1-dimensional discrete symmetric random variable, admissible of index α and set $R_\alpha \in \{\emptyset, \{2\}\}$. Then we say that:

1. X is repaired if $R_\alpha = \emptyset$.
2. X is locally repairable if $R_\alpha = \{2\}$ and $\kappa_2 > 0$.
3. X is asymptotically repairable if $R_\alpha = \{2\}$ and $\kappa_2 < 0$.

For the locally repairable and asymptotically repairable cases we can define independent random variables (which we call "repairers") Z and $\bar{Z}$ which allow us to make X repaired by adding X to said random variables. The probability distribution of Z is given by:

$$p_Z(x) = \begin{cases} \frac{\kappa_2}{M^2} \text{ if } |x| = M \\ 1 - \frac{2\kappa_2}{M^2} \text{ if } |x| = 0 \\ 0 \text{ otherwise} \end{cases} \tag{4.1}$$

With $M = \lceil \sqrt{2\kappa_2} \rceil$. The probability distribution of $\bar{Z}$ is given by the probability distribution of a $N(0, 2|\kappa_2|)$ distributed random variable. We will for both of these cases show that when adding them with X we get a repaired random variable. However, we first show that Z is in fact admissible of index $\alpha = 2$ and $R_\alpha = \emptyset$.

Proposition 4.1.3. *The repairer Z as defined above, is repaired.*

Proof.
Notice that:

$$\phi_Z(\theta) = \sum_{x=-\infty}^{\infty} e^{ix\theta} p_Z(x) = e^{iM\theta}\frac{\kappa_2}{M^2} + e^{-iM\theta}\frac{\kappa_2}{M^2} + 1 - \frac{2\kappa_2}{M^2} = 1 + \frac{2\kappa_2(\cos(M\theta) - 1)}{M^2}$$

Using a Taylor expansion of $\cos(M\theta)$ around $\theta = 0$:

$$= 1 + \frac{2\kappa_2\left(1 - \frac{M^2\theta^2}{2} + O(\theta^4) - 1\right)}{M^2} = 1 - \kappa_2\theta^2 + O(\theta^4)$$

as $\theta \to 0$. Similarly, in the case of the asymptotic repairer, we can notice using the characteristic function we calculated earlier for the normal distribution that:

$$\phi_{\bar{Z}}(\theta) = e^{-\frac{2\kappa_2 t^2}{2}} = 1 - \kappa_2\theta^2 + O(\theta^4)$$

as $\theta \to 0$, by a Taylor expansion of $\exp(\cdot)$. ☐

We will now show that adding the repairer Z to a locally repairable random variable X will make it repaired.

Proposition 4.1.4. *Let X be a 1-dimensional, discrete symmetric random variable admissible of index α and set R_α. Then if X is locally repairable, $X+Z$ is repaired.*

Proof. We can notice that for $\theta \in \mathbb{R}$ we have:

$$\phi_{X+Z}(\theta) = \sum_{n=-\infty}^{\infty} e^{i\theta n} p_X * p_Z(n) = \sum_{n=-\infty}^{\infty} e^{i\theta n} \sum_{m=-\infty}^{\infty} p_Z(m) p_X(n-m)$$

$$= \sum_{n=-\infty}^{\infty} e^{i\theta n} \left(C_1 p_X(n-M) + C_1 p_X(n+M) + C_2 p_X(n)\right)$$

with $C_1 = \frac{\kappa_2}{M^2}$, $C_2 = 1 - 2C_1$. Now we see that:

$$\begin{aligned}
&= C_1 \left(\left(\sum_{n=-\infty}^{\infty} e^{i\theta n} p_X(n-M) \right) + \left(\sum_{n=-\infty}^{\infty} e^{i\theta n} p_X(n+M) \right) \right) + C_2 \phi_X(\theta) \\
&= C_1 \left(\left(\sum_{n=-\infty}^{\infty} e^{i\theta (n+M)} p_X(n) \right) + \left(\sum_{n=-\infty}^{\infty} e^{i\theta (n-M)} p_X(n) \right) \right) + C_2 \phi_X(\theta) \\
&= C_1 \left(e^{i\theta M} \phi_X(\theta) + e^{-i\theta M} \phi_X(\theta) \right) + C_2 \phi_X(\theta) \\
&= \phi_X(\theta) \left(2C_1 \cos(M\theta) + 1 - 2C_1\right) \\
&= \phi_X(\theta) \left(1 - C_1 M^2 \theta^2 + O(\theta^4)\right) \\
&= \phi_X(\theta) \left(1 - \kappa_2 \theta^2 + O(\theta^4)\right)
\end{aligned} \tag{4.2}$$

using a Taylor expansion of $\cos(M\theta)$ around $\theta = 0$. Now using the expansion we got for $\phi_X(\theta)$ around $\theta = 0$ gives us the following:

$$= \left(1 - \kappa_\alpha |\theta|^\alpha + \kappa_2 |\theta|^2 + O(|\theta|^{2+\alpha})\right) \left(1 - \kappa_2 |\theta|^2 + O(\theta^4)\right)$$

Anything multiplied by $O(\theta^4)$ in the left term will be at least of order $O(|\theta|^4)$ and hence of order $O(|\theta|^{2+\alpha})$. As such in the right term to focus on the other two terms. We thus can see the following:

$$\begin{aligned}
&\left(1 - \kappa_\alpha |\theta|^\alpha + \kappa_2 |\theta|^2 + O(|\theta|^{2+\alpha})\right) \left(1 - \kappa_2 |\theta|^2\right) \\
&= \left(1 - \kappa_\alpha |\theta|^\alpha + \kappa_2 |\theta|^2 + O(|\theta|^{2+\alpha})\right) - \left(\kappa_2 |\theta|^2 - \kappa_\alpha \kappa_2 |\theta|^{\alpha+2} + \kappa_2^2 |\theta|^4 + O(|\theta|^{2+\alpha})\right) \\
&= \left(1 - \kappa_\alpha |\theta|^\alpha + O(|\theta|^{2+\alpha})\right)
\end{aligned} \tag{4.3}$$

hence $X+Z$ is repaired. □

In the proof of the stable local central we require a set which we denote by J_α which depends on the set R_α. This set is given by:

$$J_\alpha = \{mx \mid m \in \mathbb{N}, x \in R_\alpha \cup \{\alpha\}\} \cap (\alpha, 2+\alpha)$$

The reason why we need this set is because of sums of the form:

$$\sum_{k=1}^{\infty} a_k \left(|x|^\alpha + O(|x|^{2+\alpha})\right)$$

with $(a_k)_{k\geq 1}$ a sequence of strictly positive numbers and $x \in \mathbb{R}$. When we expand this term for each $k \in \mathbb{N}$ we see that any term that contains a factor $O(|x|^{2+\alpha})$ is again of the form $O(|x|^{2+\alpha})$. The only terms that are not $O(|x|^{2+\alpha})$ are of the form $b_k x^{k\alpha}$ with $b_k > 0$ unless of course $k\alpha \geq 2+\alpha$.
We will prove one final lemma that we will use in the proof of the stable local central limit theorem.

Lemma 4.1.5. *Let $f, g : \mathbb{R} \to \mathbb{R}_{\geq 0}$. Let $a \geq 0$ and assume that g is bounded on the interval $[a, \infty)$. Let $M = \sup_{x \in [a,\infty)} g(x)$ and let $m = \inf_{x \in [a,\infty)} g(x)$. Assume further that $\int_a^\infty f(x)dx$ exists. Then we have that:*

$$m \int_a^\infty f(x)dx \leq \int_a^\infty f(x)g(x)dx \leq M \int_a^\infty f(x)dx$$

Proof. Notice that for all $x \in [a, \infty)$:

$$mf(x) \leq f(x)g(x) \leq Mf(x)$$

As $\int_a^\infty f(x)dx$ exists and as $\int_a^\infty f(x)g(x)dx \geq 0$ it follows that $\int_a^\infty f(x)g(x)dx$ exists and the result follows. □

4.2 Stable local central limit theorem

We will now state the stable local central limit theorem for each of the three cases possible for an admissible distribution.

Theorem 4.2.1 (Stable local central limit theorem)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables admissible of index α and set $R_\alpha \in \{\emptyset, \{2\}\}$, and let $\bar{X}$ be a stable random variable with scale parameter $\kappa_\alpha{}^{\frac{1}{\alpha}}$. Let p_X be the probability distribution of the X_i, let $p_{\bar{X}}$ be the probability distribution of $\bar{X}$, let p_Z be the probability distribution of the local repairer and let $p_{\bar{Z}}$ be the probability distribution of the asymptotic repairer. Then there are $C_1, C_2, C_3 \geq 0$ such that:*

1. *If X is repaired,*
$$\sup_{n\in\mathbb{Z}} |p_X^n(x) - p_{\bar{X}}^n(x)| \leq C_1 n^{-1-\frac{1}{\alpha}}$$
2. *If X is locally repairable,*
$$\sup_{n\in\mathbb{Z}} |p_{X+Z}^n(x) - p_{\bar{X}}^n(x)| \leq C_2 n^{-1-\frac{1}{\alpha}}$$
3. *If X is repaired,*
$$\sup_{n\in\mathbb{Z}} |p_X^n(x) - p_{\bar{X}+\bar{Z}}^n(x)| \leq C_3 n^{-1-\frac{1}{\alpha}}$$

Proof. This proof is similar to the one given for the local central limit theorem. The structure and the proof itself is based on the proof given in [7] of this statement. We write X to indicate a random variable with the same distribution as any X_i and independently distributed from all the X_i's.

1. We start again - as with the proof of the LCLT - with proposition 2.3.15. Thus we may write:
$$p_X^n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\phi_X(\theta))^n e^{-ix\theta} d\theta \tag{4.4}$$
By the inverse Fourier transform we see that:
$$p_{\bar{X}}^n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\phi_{\bar{X}}(x))^n e^{isx} ds$$
As $(\phi_{\bar{X}}(s))^n = e^{-n\kappa_\alpha |s|^\alpha}$ it follows that:
$$p_{\bar{X}}^n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-n\kappa_\alpha |s|^\alpha} e^{isx} ds$$

We can again use lemma 3.3.2 as we are dealing with discrete symmetric random variables. Let $\epsilon \in (0,1)$, then we find that:

$$\begin{aligned} p_X^n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\phi_X(\theta))^n e^{-ix\theta} d\theta \\ &= \frac{1}{2\pi}\int_{\theta\in[-\pi,-\epsilon]\cup[\epsilon,\pi]} (\phi_X(\theta))^n e^{-ix\theta} d\theta + \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} (\phi_X(\theta))^n e^{-ix\theta} d\theta \\ &= \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} (\phi_X(\theta))^n e^{-ix\theta} d\theta + O(e^{-bn}) \end{aligned} \tag{4.5}$$

for some $b > 0$.

We use a similar substitution again by setting $s = \frac{\theta}{n^{1/\alpha}}$ in what we found in 4.5, and we get:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\phi_X(s))^n e^{-isx} ds = \frac{1}{2\pi n^{\frac{1}{\alpha}}}\int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} \left(\phi_X\left(\frac{\theta}{n^{1/\alpha}}\right)\right)^n e^{-ix\frac{\theta}{n^{1/\alpha}}} d\theta + O(e^{-bn})$$

We can do the same for the probability distribution of the stable random variable and we find that:

$$\begin{aligned} p_{\bar{X}_n}^n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-n\kappa_\alpha|\theta|^\alpha} e^{i\theta x} d\theta \\ &= \frac{1}{2\pi}\int_{\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha} e^{i\theta\pi n^{\frac{1}{\alpha}}} d\theta + \frac{1}{2\pi}\int_{|\theta|>\epsilon n^{\frac{1}{\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha} e^{i\theta\pi n^{\frac{1}{\alpha}}} d\theta \end{aligned}$$

In particular, we can bound this term on the right hand side by:

$$\begin{aligned} \left|\frac{1}{2\pi}\int_{|\theta|>\epsilon n^{\frac{1}{\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha} e^{i\theta\pi n^{\frac{1}{\alpha}}} d\theta\right| &\leq \frac{C}{\pi}\int_{\theta>\epsilon n^{\frac{1}{\alpha}}}^{\infty} e^{-\kappa_\alpha|\theta|^\alpha} d\theta \\ &= \frac{C2^{1/\alpha}}{\pi}\int_{\theta>\frac{\epsilon n^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}}}^{\infty} \left(e^{-\kappa_\alpha\theta^\alpha}\right)^2 d\theta \end{aligned}$$

for some $C \geq 0$. We can now apply lemma 4.1.5 and we find that:

$$\frac{C2^{1/\alpha}}{\pi}\int_{\theta>\frac{\epsilon n^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}}}^{\infty} \left(e^{-\kappa_\alpha\theta^\alpha}\right)^2 d\theta \leq \frac{C2^{\frac{1}{\alpha}}e^{-c'n}}{\pi}\int_0^{\infty} e^{-\kappa_\alpha\theta^\alpha} d\theta = O(e^{-c'n})$$

for some $c' > 0$. Note that the fact that the integral

$$\int_0^{\infty} e^{-\theta^\alpha} d\theta$$

converges can be seen by using a substitution $\theta = u^{\frac{1}{\alpha}}$ and noting that the resulting integral is the gamma function of α, which is known to converge for $\alpha > 0$.

We can now pay attention to integrating:

$$\frac{1}{2\pi n^{\frac{1}{\alpha}}}\int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} \left(\phi_X\left(\frac{\theta}{n^{1/\alpha}}\right)\right)^n e^{-ix\frac{\theta}{n^{1/\alpha}}}$$

We can rewrite $\left(\phi_X\left(\frac{\theta}{n^{1/\alpha}}\right)\right)^n$ in terms of a different function which we call $F_n(\theta)$:

$$\left(\phi_X\left(\frac{\theta}{n^{1/\alpha}}\right)\right)^n = (1 + F_n(\theta))\, e^{-\kappa_\alpha|\theta|^\alpha}$$

It follows that since we took ϵ to be small enough that we may write:

$$\begin{aligned}\phi\left(\frac{\theta}{n^{\frac{1}{\alpha}}}\right)^n &= \left(1-\kappa_\alpha\frac{|\theta|^\alpha}{n}+O\left(\frac{|\theta|^{2+\alpha}}{n\cdot n^{\frac{2}{\alpha}}}\right)\right)^n \\ &= \left(1-\frac{\kappa_\alpha|\theta|^\alpha - O\left(\frac{|\theta|^{2+\alpha}}{n^{\frac{2}{\alpha}}}\right)}{n}\right)^n \\ &\leq e^{-\kappa_\alpha|\theta|^\alpha}e^{O\left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}}\right)}\end{aligned}$$

by lemma C.0.2. Hence we find that for $F_n(\theta)$ we have that:

$$F_n(\theta) \leq \sum_{k=1}^{\infty}\frac{1}{k!}O\left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}}\right)^k$$

As $|\theta|^{2\alpha}$ is a monotonically strictly increasing function, it follows that for $|\theta| < \epsilon n^{\frac{1}{\alpha}}$ we have:

$$O\left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}}\right)^k \leq \frac{|\theta|^{2\alpha}}{n}$$

and as such $F_n(\theta) \leq e^{\frac{|\theta|^{2\alpha}}{n}}$ for $|\theta| < \epsilon n^{\frac{1}{\alpha}}$. In particular, we can conclude that:

$$\begin{aligned}|p^n_X(x) - p^n_{\bar{X}}(x)| =& \frac{1}{2\pi n^{1/\alpha}}\left|\int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha}e^{-ix\frac{\theta}{n^{1/\alpha}}}F_n(\theta)d\theta + O(e^{-bn})\right| \\ \leq& \frac{C}{n^{1+1/\alpha}}\int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} e^{-\kappa_\alpha|\theta|^\alpha}|\theta|^{2\alpha}d\theta\end{aligned}$$

for some $b, C > 0$. We have seen that the above stated integral converges when we let $n \to \infty$, and as such this concludes the proof.

2. In this case, the proof is almost equal to the one given in case (1); as $X+Z$ is repaired, the characteristic function of $X+Z$ is equal to the one given in the case of X, and we can simply repeat the proof of case (1).

3. This is also very similar to case (1). In particular, we see that using the characteristic function of the normal distribution we found earlier:

$$\phi_{\bar{X}+\bar{Z}}(\theta) = \phi_{\bar{X}}(\theta)\phi_{\bar{Z}}(\theta) = e^{-\kappa_\alpha|\theta|^\alpha - |\kappa_2||\theta|^2}$$

And thus:

$$\begin{aligned}p^n_{\bar{X}+\bar{Z}}(x) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-n\kappa_\alpha|\theta|^\alpha - n|\kappa_2||\theta|^2}e^{-ix\theta}d\theta \\ =& \frac{1}{2\pi n^{1/\alpha}}\int_{-\infty}^{\infty} e^{-\kappa_\alpha|\theta|^\alpha - n^{1-\frac{2}{\alpha}}|\kappa_2||\theta|^2}e^{-\frac{ix\theta}{n^{1/\alpha}}}d\theta\end{aligned}$$

For any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$ we have:

$$e^{-\kappa_\alpha|\theta|^\alpha - n^{1-\frac{2}{\alpha}}|\kappa_2||\theta|^2} \leq e^{-\kappa_\alpha|\theta|^\alpha}$$

The result now follows similar to case (1).

□

We can find a simple generalization of this theorem which works for every finite set $R_\alpha \in (\alpha, 2+\alpha)$ which we present now:

Theorem 4.2.2. *Let $\alpha \in (0,2)$, let $(X_i)_{i\in\mathbb{N}}$ be a sequence of identical and independent distributed with a probability distribution p_X that is admissible of index α and set R_α. Let $p_{\bar{X}}$ denote the probability distribution of a symmetric α-stable random variable with scale parameter $(\kappa_\alpha)^{\frac{1}{\alpha}}$. Then there exists a collection of real constants $\{C_j, j \in J_\alpha\}$ and some $C \geq 0$ such that:*

$$\sup_{x\in\mathbb{Z}} \left| p_X^n(x) - p_{\bar{X}}^n(x) - \sum_{j\in J_\alpha} \frac{C_j}{2\pi n^{(1+j-\alpha)/\alpha}} \int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha|\theta|^\alpha} \cos\left(\frac{\theta x}{n^{1/\alpha}}\right) d\theta \right| \leq C n^{-\frac{3}{\alpha}}$$

Proof. We can once again start in similar way of case (1) of the proof of the stable local central limit theorem. We start by writing:

$$p_X^n(x) = \frac{1}{2\pi n^{1/\alpha}} \int_{\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} (1 + F_n(\theta))\, e^{-\kappa_\alpha|\theta|^\alpha} e^{-i\frac{x\theta}{n^{1/\alpha}}} d\theta + O(e^{-cn^{1/\alpha}})$$

with again $c > 0$. Notice that

$$1 + F_n(\theta) = \left(\phi_X\left(\frac{\theta}{n^{\frac{1}{\alpha}}}\right)\right)^n e^{-\kappa_\alpha|\theta|^\alpha}$$

We can now see that:

$$\left(\phi_X\left(\frac{\theta}{n^{\frac{1}{\alpha}}}\right)\right)^n = \left(1 + \frac{-\kappa_\alpha|\theta|^\alpha + \sum_{\beta\in R_\alpha} \kappa_\beta|\theta|^\beta n^{1-\frac{\beta}{\alpha}} + O\left(\frac{|\theta|^{2+\alpha}}{n^{\frac{2}{\alpha}}}\right)}{n}\right)^n$$

We can again apply lemma C.0.2 to find:

$$1 + F_n(\theta) \leq e^{\sum_{\beta\in R_\alpha} \kappa_\beta|\theta|^\beta n^{1-\beta/\alpha}} e^{O\left(\frac{|\theta|^{2+\alpha}}{n^{\frac{2}{\alpha}}}\right)}$$

Using a Taylor expansion around $\theta = 0$ gives:

$$\begin{aligned} 1 + F_n(\theta) &\leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{\beta\in R_\alpha} \kappa_\beta|\theta|^\beta n^{1-\beta/\alpha} + O\left(\frac{|\theta|^{2+\alpha}}{n^{\frac{2}{\alpha}}}\right)\right)^k \\ &= 1 + \sum_{j\in J_\alpha} \kappa_j|\theta|^j n^{1-j/\alpha} + \sum_{k=1}^{\infty} \frac{1}{k!}\left(O\left(\frac{|\theta|^{2+\alpha}}{n^{\frac{2}{\alpha}}}\right)\right)^k \end{aligned} \tag{4.6}$$

Similarly to case (1) of the proof of the stable local central limit theorem we find that:

$$\begin{aligned} F_n(\theta) &\leq \sum_{j\in J_\alpha} C_j n^{1-j/\alpha}|\theta|^j + \sum_{k=1}^{\infty} \frac{1}{k!} O\left(\frac{|\theta|^{2+\alpha}}{n^{\frac{2}{\alpha}}}\right)^k \\ &\leq \sum_{j\in J_\alpha} C_j n^{1-j/\alpha}|\theta|^j + \frac{|\theta|^{2\alpha}}{n} \end{aligned} \tag{4.7}$$

It is easy to see that:

$$\int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha|\theta|^\alpha} \cos\left(\frac{\theta x}{n^{1/\alpha}}\right) = \int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha|\theta|^\alpha} e^{\frac{i\theta x}{n^{1/\alpha}}}$$

as the function to be integrated is symmetric. We can again notice in a way that is very similar to how we showed that:

$$\int_{\mathbb{R}} e^{-\kappa_\alpha|\theta|^\alpha} d\theta = O(e^{-c'n})$$

for some $c' > 0$ that now we have that:

$$\int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha |\theta|^\alpha} d\theta = O(e^{-bn})$$

for every $j \in J_\alpha$. As such we can now see that:

$$\left| p_X^n(x) - p_{\bar{X}}^n(x) - \sum_{j \in J_\alpha} \frac{C_j}{2\pi n^{(1+j-\alpha)/\alpha}} \int_{\mathbb{R}} |\theta|^j e^{-\kappa_\alpha |\theta|^\alpha} \cos\left(\frac{\theta x}{n^{1/\alpha}}\right) d\theta \right|$$
$$= \left| \frac{1}{2\pi n^{\frac{1}{\alpha}}} \int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} \left(\left(F_n(\theta) - \sum_{j \in J_\alpha} C_j \frac{|\theta|^j}{n^{\frac{j-\alpha}{\alpha}}} \right) e^{-\kappa_\alpha |\theta|^\alpha} e^{ix\theta n^{-\frac{1}{\alpha}}} d\theta \right) + O(e^{-bn}) + O(e^{-cn^{\frac{1}{\alpha}}}) \right|$$

for some $b > 0$. In particular for n large enough, we can again notice that:

$$\leq C' \left| \frac{1}{2\pi n^{\frac{1}{\alpha}}} \int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} \left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}} e^{-\kappa_\alpha |\theta|^\alpha} d\theta \right) \right| + O(e^{-bn})$$

for some $C' > 0$. We have:

$$\int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} \left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}} e^{-\kappa_\alpha |\theta|^\alpha} d\theta \right) \leq 2 \int_0^\infty \left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}} e^{-\kappa_\alpha |\theta|^\alpha} d\theta \right)$$

As the latter integral converges (see eq. (D.2) from the appendix) we conclude that:

$$C' \left| \frac{1}{2\pi n^{\frac{1}{\alpha}}} \int_{-\epsilon n^{\frac{1}{\alpha}}}^{\epsilon n^{\frac{1}{\alpha}}} \left(\frac{|\theta|^{2+\alpha}}{n^{2/\alpha}} e^{-\kappa_\alpha |\theta|^\alpha} d\theta \right) \right| + O(e^{-bn}) \leq C n^{-\frac{3}{\alpha}}$$

for some $C > 0$.

□

One may now wonder what exactly are the bounds of convergence that we can put on our 1-dimensional general case. It turns out that under the conditions of the previous theorem that (see [7], corollary 3.3) with $\beta_1 = \min(J_\alpha^+)$ that we have:

$$\sup_{x \in \mathbb{Z}} |p_X^n(x) - p_{\bar{X}}^n(x)| \leq C n^{-(\beta_1 + 1 - \alpha)/\alpha}$$

We will discuss these rates of convergence in the next section.
We will now give a generalization of the stable local central limit theorem in d dimensions. A simple generalization can be found as follows.

Theorem 4.2.3 (Stable local central limit theorem in d dimensions)**.** *Let $(\boldsymbol{X}_n)_{n\geq 1}$ be a sequence of d-dimensional independent and identically distributed random variables where each component of every $\boldsymbol{X}_i$ satisfies the 1-dimensional stable local central limit theorem with scale parameter α. Then we have that $\boldsymbol{\theta} \cdot \sum_{k=1}^n \boldsymbol{X}_k$ converges to a univariate stable random variable, for some $\boldsymbol{\theta} \in \mathbb{R}^d$.*

Proof. Let $\boldsymbol{\theta} \in \mathbb{R}^d$ and write $\boldsymbol{S} = \sum_{i=1}^n \boldsymbol{X}_i$. We will only in this theorem use the notation $\boldsymbol{X}_i^d$ to denote the d-th component of $\boldsymbol{X}_i$. We now find that:

$$\begin{aligned}
\phi_{\boldsymbol{S}}\left(\frac{\theta}{n^{\frac{1}{\alpha}}}\right) &= \mathbb{E}\left[e^{i\boldsymbol{\theta}\cdot\left(\sum_{i=1}^n \boldsymbol{X}_i\right)n^{-1/\alpha}}\right] \\
&= \mathbb{E}\left[e^{i\sum_{k=1}^d \theta_k \left(\sum_{i=1}^d \boldsymbol{X}_i^k\right)n^{1/\alpha}}\right] \\
&= \prod_{k=1}^d \mathbb{E}\left[e^{i\theta_k\left(\sum_{i=1}^n \boldsymbol{X}_i^k\right)n^{1/\alpha}}\right]
\end{aligned}$$

By the proof of the one dimensional stable local central limit theorem we know that this converges in distribution to the following as $n \to \infty$:

$$= \prod_{k=1}^{n} e^{-\kappa_\alpha |\theta_k|^\alpha} = e^{-\sum_{k=1}^{d} \kappa_\alpha |\theta_k|^\alpha}$$

It now follows from [6], chapter 2 that this is the characteristic function of an univariate stable random variable.

□

5 Examples of admissible distributions and convergence rate

The general probability distribution of long range random walks we are studying are the ones given by the following formula:

$$p(x) = \begin{cases} \frac{c_\alpha}{|x|^{1+\alpha}} & \text{if } x \in \mathbb{Z},\ x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{5.1}$$

where c_α is a constant such that p is a probability distribution and $\alpha \in (0, 2]$. These random walks have Pareto-like tails and only therefore the classical local central limit theorem does not apply but the stable local central limit theorem does. Note that if $\alpha > 2$, and we are in 1 dimension, we can use the central limit theorem. We will now give an example of such a case.

Example 5.0.1. Let $(X_i)_{i\in\mathbb{N}}$ be an infinite sequence of independent, identically distributed discrete random variables with probability distribution:

$$p_{X_i}(x) = \begin{cases} \frac{45}{\pi^4|x|^4} & \text{if } x \in \mathbb{Z},\ x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{5.2}$$

Let $X = X_1$. Then we find that:

$$\mathbb{E}[X] = \mu = \frac{90\zeta(3)}{\pi^4}$$

$$\text{Var}(X) = \frac{\pi^2}{3} - \mu^2 = \frac{\pi^2}{3} - \left(\frac{90\zeta(3)}{\pi^4}\right)^2$$

Here $\zeta(\cdot)$ denotes the zeta function, which is a function from reals to reals defined by:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

This function is known to converge for $s > 1$. We may now apply the central limit theorem and find that:

$$\lim_{n\to\infty} \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z$$

where $\bar{X}_n = \sum_{i=1}^{n} \frac{S_i}{n}$ and Z is $N(0, \sigma^2)$ distributed. △

We would want that obvious cases (such as when we can visit only a finite number of places) the stable local central limit theorem still applies. The following example reassures us that that is indeed the case.

Theorem 5.0.2. *Let S_n be a 1-dimensional finite range random walk with the probability distribution of its steps given by:*

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$$

Then S_n is repaired.

Proof. Write $S_n = \sum_{i=1}^{n} X_i$. We can rewrite this distribution to

$$S_n = \sum_{i=1}^{n} (2Y_i - 1) = 2Z - n$$

where Z is $B(n, \frac{1}{2})$ distributed. The characteristic function is given by:

$$\phi_{S_n}(\theta) = \mathbb{E}[e^{i\theta S_n}] = \sum_{k=-n}^{n} e^{i\theta k}\mathbb{P}(S_n = k)$$

As $\mathbb{P}(S_n = -k) = \mathbb{P}(S_n = k)$ it follows that:

$$= \sum_{k=-n}^{n} \cos(tk)\mathbb{P}(S_n = k) = \sum_{k=0}^{n} \cos(\theta(2k - n))\mathbb{P}(Z = k)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \cos(\theta(2k-n)) \binom{n}{k}$$

Using the taylor expansion $\cos(\cdot)$ around $\theta = 0$ gives us:

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{\theta^2(2k-n)^2}{2} + O(\theta^4)\right)$$

This concludes the proof.

□

In the case of $\alpha \in (0, 2]$, we can use the stable local central limit theorem. In particular, in [7] a proof is given about the fact that if X has the probability distribution from (5.1) with $\alpha \in (0, 2)$, then X is admissible of index α and locally repairable with regularity set $R_\alpha = \{2\}$. In this paper they state the following proposition:

Proposition 5.0.3. *Let $\alpha \in (0, 2)$ and let X have the probability distribution from (5.1) with parameter α. Let ϕ_α denote the characteristic function of X for the given α. Then we can write:*

$$\phi_\alpha(\theta) = 1 - \kappa_\alpha |\theta|^\alpha + \kappa_2 |\theta|^2 + O(|\theta|^{2+\alpha})$$

as $|\theta| \to 0$, where if $\alpha \neq 1$:

$$\kappa_\alpha = -2c_\alpha \cos\left(\frac{\pi\alpha}{2}\right) \Gamma(-\alpha)$$

and

$$\kappa_2 = 2c_\alpha \left(\frac{1}{2(2-\alpha)} - \frac{1}{4} - K_2\right) > 0$$

with

$$K_2 = \frac{1-\alpha}{2} \left(\left(\frac{2^{1-\alpha}-1}{1-\alpha} - \frac{3\left(2^{2-\alpha}-1\right)}{2\left(2-\alpha\right)} \right) + \frac{1}{2\Gamma(\alpha)} \sum_{m=1}^{\infty} \left(\zeta(m+\alpha) - 1\right) \frac{m\Gamma(m+\alpha)}{\Gamma(m+2)(m+2)} \right)$$

where $c_\alpha > 0$. If $\alpha = 1$, then we have that:

$$\kappa_1 = -\frac{3}{\pi}$$
$$\kappa_2 = \frac{3}{2\pi^2}$$

We will not give a proof of this statement but instead give a sketch of the proof that is given in [7]:

- Use the Euler-Maclaurin formula which states that for some function f which is infinitely many times differentiable on $\mathbb{R}$ that we have:

$$\sum_{x=1}^{M} f(x) - \int_1^M f(x)dx = \frac{f(1) + f(M)}{2} + \int_1^M f'(x) \left(x - \lfloor x \rfloor - \frac{1}{2}\right) dx$$

- Use this formula for the function f given by:

$$f(x) = \frac{1 - \cos(\theta x)}{|x|^{1+\alpha}}$$

  and take the limit of M to infinity such that the infinite sum on the left hand side is writeable in terms of the characteristic function of X and some constants;

- Find expansions for the integral on the left hand side and the right hand side;

- Bound the constants that come from these expansions to show that these are either strictly negative or strictly positive.

Let again $(X_i)_{i\in\mathbb{N}}$ be an infinite sequence of independent identically distributed random variable each with probability distribution from (5.1) where $\alpha = 2$. We can in this case calculate c_α to be equal to $\frac{1}{\zeta 3}$. Let Z have the distribution as defined in (4.1). Let $S_n = \sum_{i=1}^n X_i$ and denote its probability distribution by p_X^n. Then by the stable local central limit theorem we know that:

$$\sup_{x\in\mathbb{Z}} |p_{X+Z}^n(x) - p_{\bar{X}}^n(x)| \leq \frac{C}{n^{\frac{3}{2}}}$$

for some $C \geq 0$.
This rate of convergence is as fast as the local central limit theorem when we assume that $\mathbb{E}[|X_1|]^3$ exists. In this case we do not have that $\mathbb{E}[|X|^2] < \infty$, and as such we can not even apply the local central limit theorem. However, one could argue that the distribution we gave was not much different from the one given by 5.1 with $\alpha = 2 + \epsilon$ where $\epsilon > 0$ is small. In that case, we find that by the improved local central limit theorem p_n converges to $\bar{p}_n$ with a rate of $\frac{c}{n^{3/2}}$. In comparison to the local central limit theorem this rate of convergence is optimal.

In the general case of an admissible distribution, the rate of convergence is not as good as in the default case. Indeed, if β_1 is close to α we only will get a convergence rate of $n^{-\frac{1}{2}}$. The best case we can have is when $\beta_1 = 2 + \alpha$, which is the case of a repaired random variable. This shows that under the stable local central limit theorem the case with $R_\alpha \in \{\emptyset, \{2\}\}$ is indeed optimal, which is what we expect as a finite range random walk belongs to this group but also falls under the conditions of the local central limit theorem.

A Limits

The following theorem is similar to the statement made in C.7 on page 386 of [1].

Theorem A.0.1. *Let $A \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$, and let $f : A \to \mathbb{R}$. Then*

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = b$$

for some $b \in \mathbb{R}$ if and only if for each sequence $(\boldsymbol{x}_n)_{n\geq 1}$ in A where $\boldsymbol{x}_n \neq \boldsymbol{a}$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{a}$ we have that

$$\lim_{n\to\infty} f(\boldsymbol{x}_n) = b$$

Proof. We will first assume that

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = b$$

Let $(\boldsymbol{x}_n)_{n\geq 1}$ be a sequence in A such that $\boldsymbol{x}_n \neq \boldsymbol{a}$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{a}$$

Let $\epsilon > 0$. There is some $\delta > 0$ such that $||\boldsymbol{x} - \boldsymbol{a}|| < \delta$ implies that $||f(\boldsymbol{x}) - \boldsymbol{b}|| < \epsilon$. We can also find some $N \in \mathbb{N}$ such that for $n \geq N$ we have that $||\boldsymbol{x}_n - \boldsymbol{a}|| < \delta$. This implies that for any $n \geq N$ we have that $||f(\boldsymbol{x}_n) - \boldsymbol{b}|| < \epsilon$, and as we can repeat this proof for any suitable sequence the proof in this direction follows. We refer for a proof of the other direction to [1], page 386. □

B Complex numbers and infinite series

This appendix will state a few theorems about complex numbers. Note that any $w \in \mathbb{C}$ can be written as:

$$w = ae^{i\theta}$$

with $a \geq 0$, $\theta \in [0, 2\pi)$.

Theorem B.0.1. *(Triangle inequality for countable infinite series). Let $(z_n)_{n\geq 1}$ be an infinite series in $\mathbb{C}$. Assume that $\sum_{i=1}^{\infty} |z_i|$ converges to some point $z \in \mathbb{C}$. Then:*

$$|\sum_{i=1}^{\infty} z_i| \leq \sum_{i=1}^{\infty} |z_i| \tag{B.1}$$

Proof. We will first show that $\sum_{i=1}^{\infty} z_i$ converges. We can write:

$$\sum_{i=1}^{\infty} z_i = \sum_{i=1}^{\infty} a_i + i \sum_{i=1}^{\infty} b_i$$

where a_i and b_i are the real and imaginary component of each z_i for all $i \in \mathbb{N}$ respectively. Notice now that:

$$\sum_{i=1}^{\infty} |z_i| = \sum_{i=1}^{\infty} \sqrt{a_i^2 + b_i^2} \geq \sum_{i=1}^{\infty} |a_i|$$

Thus $\sum_{i=1}^{\infty} a_i$ converges. In a similar way we see that $\sum_{i=1}^{\infty} b_i$ converges and thus that $\sum_{i=1}^{\infty} z_i$ converges. We will now prove equation B.1. Assume to the contrary that there is some $\epsilon > 0$ such that:

$$\epsilon = |\sum_{i=1}^{\infty} z_i| - \sum_{i=1}^{\infty} |z_i|$$

By the definition of convergence we can find some $N \in \mathbb{N}$ such that:

$$|\sum_{i=1}^{\infty} z_i - \sum_{i=1}^{N} z_i| < \frac{\epsilon}{2}$$

This implies that:

$$|\sum_{i=1}^{\infty} z_i| \leq |\sum_{i=1}^{\infty} z_i - \sum_{i=1}^{N} z_i| + |\sum_{i=1}^{N} z_i| < \frac{\epsilon}{2} + |\sum_{i=1}^{N} z_i|$$

In particular, it now follows that:

$$\sum_{i=1}^{N} |z_i| \leq \sum_{i=1}^{\infty} |z_i| < |\sum_{i=1}^{N} z_i|$$

which is a contradiction by the repeated triangle inequality. This concludes the proof.

□

Lemma B.0.2. *For any $w_1, w_2 \in \mathbb{C}$ where we can write $w_1 = a_1 e^{i\theta_1}$ and $w_2 = a_2 e^{i\theta_2}$ with $a_1, a_2 \geq 0$, $\theta_1, \theta_2 \in [0, 2\pi)$ and $\theta_1 \neq \theta_2$ we have that*

$$|w_1 + w_2| < |w_1| + |w_2| \tag{B.2}$$

Proof. By the triangle inequality we have that:

$$|w_1 + w_2|^2 \leq (|w_1| + |w_2|)^2 = |w_1|^2 + |w_2|^2 + 2|w_1||w_2| = |w_1|^2 + |w_2|^2 + 2|a_1||a_2| \tag{B.3}$$

On the other hand, we also have that:

$$|w_1 + w_2|^2 = (w_1 + w_2)(\overline{w_1} + \overline{w_2}) = |w_1|^2 + |w_2|^2 + \overline{w_1} w_2 + w_1 \overline{w_2}$$

$$= |w_1|^2 + |w_2|^2 + a_1 a_2 \left(e^{i(\theta_2 - \theta_1)} + e^{i(\theta_1 - \theta_2)}\right) = |w_1|^2 + |w_2|^2 + 2a_1 a_2 \cos(\theta_1 - \theta_2)$$

In particular we have that:

$$|w_1|^2 + |w_2|^2 + 2a_1 a_2 \cos(\theta_1 - \theta_2) \leq |w_1|^2 + |w_2|^2 + 2|a_1||a_2|$$

Which is equivalent with saying that:

$$2a_1 a_2 \cos(\theta_1 - \theta_2) \leq 2|a_1||a_2|$$

We now see that equation B.2 is precisely true when $\theta_1 \neq \theta_2$. □

Corollary B.0.3. *If $(w_k)_{k\geq 1}$ is a sequence in $\mathbb{C}$ such that*

$$|\sum_{k=1}^{\infty} w_k| = \sum_{k=1}^{\infty} |w_k| = 1$$

then for all $n \in \mathbb{N}$ we have that

$$w_n = a_n e^{i\theta}$$

with $a_n \geq 0$ and $\theta \in [0, 2\pi)$.

Proof. Assume to the contrary that not all w_n are of the form $a_n e^{i\theta}$, e.g. there exists w_i and w_j such that $w_i = a_i e^{i\theta_1}$ and $w_j = a_j e^{i\theta_2}$ with $\theta_1, \theta \in [0, 2\pi), \theta_1 \neq \theta_2$. Call

$$A = \sum_{k=1, k\neq i, k\neq j}^{\infty} w_k$$

then by the triangle inequality for infinite series and lemma B.0.2 we see that:

$$1 = |A + w_i + w_j| \leq |A| + |w_i + w_j| < |A| + |w_i| + |w_j| \leq \sum_{k=1}^{\infty} |w_k| = 1$$

which is a contradiction. As such we conclude that all w_n are writable as:

$$w_n = a_n e^{i\theta}$$

□

C Exponential inequalities

Lemma C.0.1. *Let $x \in \mathbb{R}$. Then:*

$$e^x \geq 1 + x$$

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = e^x - x - 1$$

Then its derivative is given by:

$$f'(x) = e^x - 1$$

for $x = 0$ we see that f has a minimum because:

$$f''(0) > 0$$

As $f(x) > 0$ for all $x > 0$ and $f(x) < 0$ for all $x < 0$ we see that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and the result follows. □

Lemma C.0.2. *Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then:*

$$e^x \geq \left(1 + \frac{x}{n}\right)^n$$

Proof. Notice that:

$$e^{\frac{nx}{n}} \geq \left(1 + \frac{x}{n}\right)^n$$

□

This concludes the proof.

D Error function

In this section we consider integrals of the form

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{d} a_i x_i^2} dx_1 ... dx_d \tag{D.1}$$

where the values a_i are all nonnegative. We will start with the definition of the error function and give some useful properties of the error function. After that we will start with the 1-dimensional case of the integral given in D.1, and later on extend it to the d-dimensional case.

Definition D.0.1. (Error function and the complementary error function).
The error function $\text{erf}(x)$ is given by:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The complementary error function $\text{erfc}(x)$ is given by:

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \text{erf}(x)$$

We will now give some properties of the error function.

Lemma D.0.2. *For each $x \geq 0$ we have that:*

$$1 - e^{-x^2} \leq \text{erf}^2(x) \leq 1 - e^{-2x^2}$$

In particular,

$$1 - e^{-x^2} \leq \text{erf}(x) \leq 1$$

Proof. We can write:

$$\begin{aligned}\operatorname{erf}(x)^2 &= \frac{4}{\pi}\int_0^x\int_0^x e^{-(t^2+s^2)}dtds = \frac{1}{\pi}\int_{-x}^x\int_{-x}^x e^{-(t^2+s^2)}dtds \\ &\geq \frac{1}{\pi}\iint_{t^2+s^2\leq x^2} e^{-(t^2+s^2)}dtds = 1-e^{-x^2}\end{aligned}$$

The other bound follows similarly:

$$\begin{aligned}\operatorname{erf}(x)^2 &= \frac{1}{\pi}\int_{-x}^x\int_{-x}^x e^{-(t^2+s^2)}dtds \\ &\leq \frac{1}{\pi}\iint_{t^2+s^2\leq 2x^2} e^{-(t^2+s^2)}dtds = 1-e^{-2x^2}\end{aligned}$$

Rewriting the right bound now gives us:

$$\operatorname{erf}^2(x) \leq 1-e^{-2x^2} = \left(1-e^{-x^2}\right)\left(1+e^{-x^2}\right)$$

Since $\operatorname{erf}(x)\geq 0$ and as $\operatorname{erf}(x)^2$ is bounded from above by 1 and as $1-e^{-x^2}\geq 0$ we find that:

$$1-e^{-x^2} \leq \operatorname{erf}(x) \leq 1$$

□

Corollary D.0.3. *There exist $c\geq 0$ such that for any $r,a>0$ we have:*

$$\int_{\mathbb{R}\setminus[-r,r]} e^{-ax^2}dx \leq ce^{-ar^2}$$

Proof. We rewrite the above stated integral in terms of the error function and use the bounds from lemma D.0.2:

$$\int_{\mathbb{R}\setminus[-r,r]} e^{-ax^2}dx = \frac{1}{\sqrt{a}}\int_{\mathbb{R}\setminus[-r\sqrt{a},r\sqrt{a}]} e^{-y^2}dy = \sqrt{\frac{\pi}{a}}\operatorname{erfc}(r\sqrt{a}) \leq \sqrt{\frac{\pi}{a}}e^{-ar^2}$$

□

Below we will state the d-dimensional case of the corollary given above.

Corollary D.0.4. *Let $(a_i)_{1\leq i\leq d}$ be a finite sequence, such that for all a_i where $1\leq i\leq d$ we have that $a_i>0$. Then for all $r\geq 0$ we have some $b\geq 0$ such that:*

$$\int_{\mathbb{R}\setminus[-r,r]}...\int_{\mathbb{R}\setminus[-r,r]} e^{-\sum_{i=1}^d a_ix_i^2}dx_1...dx_n = O(e^{-br^2})$$

Another useful consequence of D.0.2 is given below.

Theorem D.0.5. *Let again $(a_i)_{1\leq i\leq d}$ be a finite sequence, such that for all a_i where $1\leq i\leq d$ we have that $a_i>0$. Let furthermore $n\in\mathbb{N}$ and let $r\geq 0$. Write:*

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_d \end{pmatrix}$$

Then we have that the integral:

$$\int_{\mathbb{R}\setminus[-r,r]}...\int_{\mathbb{R}\setminus[-r,r]} ||\boldsymbol{x}||^n e^{-\sum_{i=1}^d a_ix_i^2}dx_1...dx_d \tag{D.2}$$

exists.

Proof. Notice first of all that:

$$||\boldsymbol{x}||^n \leq \left(\sum_{i=1}^{d} |x_i|\right)^n \leq e^{n\left(\sum_{i=1}^{d} |x_i|\right)}$$

Hence it follows that:

$$\int_{\mathbb{R}\setminus[-r,r]} ... \int_{\mathbb{R}\setminus[-r,r]} ||\boldsymbol{x}||^n e^{-\sum_{i=1}^{d} a_i x_i^2} dx_1...dx_d \leq \int_{\mathbb{R}\setminus[-r,r]} ... \int_{\mathbb{R}\setminus[-r,r]} e^{\sum_{i=1}^{d} n|x_i| - a_i x_i^2} dx_1...dx_d$$

$$= 2^d \int_r^\infty ... \int_r^\infty e^{\sum_{i=1}^{d} x_i(n - a_i x_i)} dx_1...dx_d$$

This integral exists, and as the integral from (D.2) is bounded from below by 0 it follows that the integral from (D.2) exists.

□

References

[1] Donald L Cohn. *Measure theory*. Springer, 2013.

[2] Murray H Protter, B Charles Jr, et al. *Intermediate calculus*. Springer Science & Business Media, 2012, pp. 421–426.

[3] Ronald Newbold Bracewell and Ronald N Bracewell. *The Fourier transform and its applications*. Vol. 31999. McGraw-Hill New York, 1986.

[4] Gregory F Lawler and Vlada Limic. *Random walk: a modern introduction*. Vol. 123. Cambridge University Press, 2010.

[5] David Williams. *Probability with martingales*. Cambridge university press, 1991, pp. 181–187.

[6] Gennady Samoradnitsky. *Stable non-Gaussian random processes: stochastic models with infinite variance*. Chapman and Hall, 1994.

[7] Leandro Chiarini, Milton Jara, and Wioletta M Ruszel. "Local central limit theorem and potential kernel estimates for a class of symmetric heavy-tailted random variables". In: *arXiv preprint arXiv:2101.01609* (2021).