Universiteit Utrecht

DOUBLE BACHELOR'S THESIS

MATHEMATICS AND PHYSICS

Kaluza-Klein reductions on five dimensional action principles

---

Supervisor mathematics:
DR. G.R. (GIL) CAVALCANTI

Author:
TEUN KLUCK
6152384

Supervisor physics:
DR. T.W. (THOMAS) GRIMM

10 February 2021 - 16 June 2021

Abstract

Since Einstein published his theory of general relativity, scientists wanted to construct a theory which describes both gravity and electromagnetism. Kaluza introduced a five dimensional manifold, which consists of spacetime plus an extra circle dimension. After he reduced the Einstein equations for the five dimensional manifold to four dimensional equations, he could unify gravity and the electromagnetic field in one theory. The derivation done by Kaluza is called the Kaluza-Klein reduction. In this reduction, Kaluza used an ansatz, called the cylindrical condition, which was motivated by Klein. In this thesis we will look at the Kaluza-Klein reduction and at the cylindrical condition. This reduction is a motivation to consider actions in the five dimensional spacetime and look at the results they give in four dimensional spacetime. We are going to look at the action for a $(p-1)$-form gauge field on a five dimensional manifold and we are going to reduce the action to four dimensions. To do this reduction, we cover the mathematics of differential forms. With this mathematics, we will look at Hodge theorem, which relates exact, closed and harmonic forms. With Hodge theorem we will better understand the action for a $(p-1)$-form gauge field.

Acknowledgements

I want to thank my supervisors Thomas Grimm and Gil Cavalcanti for guiding me in the right direction. I really appreciate the personal mini lectures from Gil Cavalcanti informing me about smooth manifolds. Furthermore, I want to thank Damian van de Heisteeg for helping me understand physical concepts and for guiding me in the right direction when I was working on a tough derivation. I want to thank Casper van Hal and Lucía García Alberola for helping me with the style of the thesis. Jelle Draijer helped me with concepts in the theory of manifolds. I want to thank my father, Jeroen Kluck, for helping me make the text better to read and Brechje Walpot for correcting grammatical errors.

Introduction

This thesis is about a unification of gravity and the electromagnetic field. We will discuss the method for this unification done by Theodor Kaluza and Oskar Klein. We have investigated the steps done by Kaluza and Klein to get both the homogeneous Maxwell equations for electromagnetism and the Einstein equations for general relativity. The homogeneous Maxwell equations describe electromagnetic fields without extra charges and currents. Therefore they do not describe charged particles such as electrons.

We will start with a little bit of history. In the beginning of the 20^{th} century, Albert Einstein created the theory of general relativity, in which gravity is described by only using geometrical arguments (2). The idea of general relativity is that spacetime itself is curved due to masses and energies. As the Euclidean space, which was used to describe the spacetime before Einstein introduced general relativity, is not curved, we consider spacetime now to be a manifold, which can be curved. In general relativity we propose that spacetime is a four dimensional space. In the beginning of last century, general relativity could describe gravity more accurately than any other theory, but a major problem at this time was that people did not know how to unify this theory with the physical theories existing at that time, such as electromagnetism. When the standard model was introduced, scientists did not manage to unify this with gravity, even though candidates such as string theory have been constructed. Back in the beginning of the twentieth century, the German mathematician Theodor Kaluza (1885-1954) came up with an idea to combine general relativity and electromagnetism (11). In 1921 he published a paper in which he considered spacetime as a five dimensional manifold, in stead of a four dimensional one. He made the manifold five dimensional by proposing a circle on every point of the four dimensional spacetime.

Kaluza reduced the Einstein equations in five dimensions to equations in four dimensions. These four dimensional equations are the homogeneous Maxwell equations for electromagnetism and the Einstein equations for gravity. This reduction from a theory in five dimensions to a theory in four dimensions, is called the Kaluza-Klein reduction. The result of the reduction has to be four dimensional, as all experiments are done in four dimensions and we need experiments to justify the theory.

To be able to get results in four dimensions, Kaluza made use of an ansatz, called the cylindrical condition. This ansatz is that all fields are constant under transformations on the circle. Kaluza did not show why this ansatz could be made, but the Swedish physicist Oskar Klein (1894-1977) did in 1926 (8).

An equivalent way of the Kaluza-Klein reduction, is to reduce the five dimensional Einstein-Hilbert action to an action in four dimensions. The Einstein-Hilbert action is the action which describes the same system as the Einstein equations.

In this thesis, we will also consider another action than the Einstein-Hilbert action on a five dimensional manifold, as this has been done before and has some interesting results. In the four dimensional spacetime, an important system to consider is the system which describes the Maxwell equations. The corresponding action, the action for a differential 1-form gauge field, can be written in terms of differential forms. We will consider this action, which is dependent on differential forms, in more dimensions. This action is called the action for a $(p-1)$-form gauge field.

To be able to do these dimensional reductions, we have to understand all mathematical terms we are using in the actions. The Einstein-Hilbert action contains the Ricci scalar. To know what the Ricci scalar is, we have to look at the Christoffel symbols, the Riemann tensor and the Ricci tensor first. The action for a $(p-1)$-form gauge field contains differential forms, which are antisymmetric covariant tensors. The notation used for differential forms is different from the component notation which is often used for tensors. We will notice that in some cases computations are more pleasant if we work with differential forms. For these computations we use the exterior product and the exterior derivative.

Wiliam Hodge (1903-1975) stated two important theorems with respect to differential forms. In the first place he proved his decomposition theorem, which states that any differential form can be split in an exact, closed and harmonic form. His other theorem, called Hodge theorem, states that the de Rham cohomology group is isomorphic to the set of harmonic forms. Hodge decomposition theorem will be used in the reduction of the action for a $(p-1)$-form gauge field.

Since the publication of general relativity, new physical theories, such as the standard model, have been proposed. Physicists tried to unify the standard model with general relativity, by adding even more dimensions to the four dimensional spacetime. For this unification a lot more work has to be done than for the unification of the electromagnetic field with gravity. Originally Kaluza only added one extra dimension to the four dimensional spacetime, as he only intended to unify electromagnetism with gravity. String theory, in which a similar reduction as the Kaluza-Klein reduction is used to get expressions in four dimensions, out of a higher dimensional theory, is one of the attempts of this unification. In string theory, particles are considered as strings, which is a fundamental change from how we considered particles before. This gives an impression of the importance of the work done by Kaluza and Klein.

Contents

1 Manifolds

In general relativity, the spacetime we use is a manifold, because this can be curved. A manifold is a more general space than the Euclidean space $\mathbb{R}^n$, as it is locally homeomorphic with $\mathbb{R}^n$, but it can be different globally. In this thesis the spaces we consider are smooth manifolds, so we can differentiate as many times on the manifold as we want. We are going to consider a five dimensional manifold, which is four dimensional spacetime plus a circle.
A manifold which is locally homeomorphic with $\mathbb{R}^n$ is an n dimensional manifold. A manifold can only have one dimensionality. In the end of this chapter, we are going to look at Riemannian manifolds in order to measure distances.

1.1 Smooth manifolds

Before we can give the definition of a smooth manifold, we have to look at some definitions first.

Definition 1.1. *Take U an open subsets of M. A* ***chart*** *on M is a pair (U, ψ), where ψ is a homeomorphism from U to $\psi(U) = U'$ with U' a subset of $\mathbb{R}^n$.*

Definition 1.2. *Two charts (U_1, ψ_1) and (U_2, ψ_2) are called* ***smoothly compatible*** *if the composite map $\psi_2 \circ \psi_1^{-1}$ is a diffeomorphism or if $U_1 \cap U_2 = \varnothing$.*

Definition 1.3. *An* ***atlas*** *is a collection of charts whose union covers the corresponding manifold.*

A smooth atlas is an atlas for which all charts are smoothly compatible. A maximal smooth atlas A, is a smooth atlas, such that there is no larger smooth atlas containing A. Now we can explain what a smooth structure on M is and then what a smooth manifold is.

Definition 1.4. *A* ***smooth structure*** *is a maximal smooth atlas.*

Definition 1.5. *A* ***smooth manifold*** *is a pair (M, A), where M is a manifold and A is a smooth structure on M.*

Normally a smooth manifold is denoted by M only.

1.2 Compactness

Let us define compactness for a topological space in general. We can use this definition for manifolds as a manifold is a topological space. First we must know what a covering is.

Definition 1.6. *Let X be a topological space. A* ***covering*** *of X is a family $\{A_i | i \in I\}$ of subsets of X with*

$$\bigcup_{i \in I} A_i = T \tag{1.1}$$

$\{A_i | i \in I\}$ is called an open covering if for every $i \in I$, $\{A_i\}$ is an open set.

Definition 1.7. *A topological space X is **compact** if, for every open covering $\{A_i | i \in I\}$ of X, there is a finite subset $J \subset I$, with $\{A_j | j \in J\}$ a covering of X.*

As any closed and bounded subset of $\mathbb{R}$ is compact, the circle is compact. Compactness is an important criterion as Klein proved that the Kaluza-Klein reduction can only be done on compact manifolds.

1.3 Tangent vectors

Having introduced smooth manifolds, we will next introduce tangent vectors. Unlike on $\mathbb{R}$, vectors on manifolds are not just the displacement from one point to another. Tangent vectors are not elements of the manifold itself, but elements of a tangent space. These vectors on smooth manifolds are also called tangent vectors. To construct tangent vectors, we must first define a derivation at $p \in M$.

Definition 1.8. *A **derivation** at p is a linear map $V : C^\infty \to \mathbb{R}$, for which also holds that*

$$V(fg) = f(p)Vg + g(p)Vf. \tag{1.2}$$

Here f and g are real-valued smooth functions on M.

Definition 1.9. *The **tangent space** T_pM to M at p, is the set of all derivations at the point p.*

Definition 1.10. *A **tangent vector** at p is an element of T_pM.*

We can now see that a tangent vector at p is a possible direction with length at p which is tangent to the manifold.

Proposition 1.11. *Let M be a smooth manifold. The tangent space T_pM has the same dimensionality as the manifold M.*

We want to find a basis for the tangent space. To find this basis we look at the smooth chart (U, ϕ). Here ϕ is a diffeomorphism from U to $U' \subseteq \mathbb{R}^n$. Therefore there is an isomorphism between T_pM and $T_{\phi(p)}\mathbb{R}^n$. Let us look at a basis of $T_{\phi(p)}\mathbb{R}^n$: $\{\partial_1|_{\phi(p)}, ..., \partial_n|_{\phi(p)}\}$, where we use the notation that $\partial_i|_{\phi(p)} = \frac{\partial}{\partial x^i}|_{\phi(p)}$. Here $\{x^1, ..., x^n\}$ is a basis of $\mathbb{R}^n$. These vectors are exactly the partial derivative operators at $\phi(p)$. As ϕ is a diffeomorphism, the preimages of these basis vectors form a basis for T_pM. This preimages are exactly $\partial_1|_p, ..., \partial_n|_p$.

Proposition 1.12. *For any smooth chart (U, ϕ) with $p \in U$, the set of vectors $\{\partial_1|_p, ..., \partial_n|_p\}$ forms a basis for T_pM.*

This basis is called a **coordinate basis for** T_pM.
Now we can write a tangent vector V_p with respect to its basis: $V_p = V_p^i \partial_i$.

1.4 Tangent bundle

Now we can look at the tangent bundle TM.

Definition 1.13. *The* ***tangent bundle*** *of M is the disjoint union of the tangent spaces at all points of M.*

An element of TM is a pair (p, v), where $p \in M$ and $v \in T_pM$. There is a natural map from the tangent bundle to the manifold. This map is called the projection map π and holds that $\pi(p, v) = p$. The dimension of TM is twice the dimension of M. We can now define vector fields.

Definition 1.14. *A* ***vector field*** *V on M is a smooth map*

$$V : M \to TM, \tag{1.3}$$

such that $V_p \in T_pM$, where we write V_p instead of $V(p)$.

The set of vector fields is denoted by $\Gamma(TM)$.
We want to be able to take derivatives of vector fields. Therefore we must introduce the connection on the tangent bundle. This connection is called an affine connection.

Definition 1.15. *Let V, W be vector fields and let f be a smooth function on the manifold. An* ***affine connection*** *is a bilinear map ∇ from $\Gamma(TM) \times \Gamma(TM)$ to $\Gamma(TM)$, such that*

1. $\nabla_{fV} W = f \nabla_V W$,
2. $\nabla_V (fW) = \partial_V f W + f \nabla_V W$.

In this notation ∂_V is the partial derivative in the direction of the vector V. Further in this thesis we will introduce the Christoffel symbols using an affine connection on a Riemannian manifold.

1.5 Dual space and covectors

In this section we introduce the dual vector space, which is also called the cotangent space and is denoted by T_p^*M. The **cotangent space** is the space of all linear maps from the tangent space to $\mathbb{R}$. Elements of T_p^*M are called **dual vectors, covariant vectors, covectors** or **one-forms**. As T_p^*M is the dual space of T_pM, we can relatively easy construct a basis for T_p^*M.

Proposition 1.16. *Let $\{e_1|_p, ..., e_n|_p\}$ be a basis for T_pM, then $\{E^1|_p, ..., E^n|_p\}$ defined by $E^i e_j = \delta^i_j$ is a basis for T_p^*M.*

Remember the coordinate basis for T_pM: $\{\partial_1|_p, ..., \partial_n|_p\}$. Notice that $\{E^1|_p, ..., E^n|_p\} = \{dx^1|_p, ..., dx^n|_p\}$ fulfills this condition as

$$dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j. \tag{1.4}$$

Here $\{dx^1, ..., dx^n\}$ is called the **coordinate basis for** T_p^*M.
With the same construction as for the tangent bundle, the cotangent bundle is constructed.

Definition 1.17. *The* ***cotangent bundle*** *of M, denoted by T^*M, is the disjoint union of the cotangent spaces at all points of M.*

1.6 Tensors

It is now a simple generalization to go from vectors and covectors to tensors. Let us first define a multilinear map.

Definition 1.18. *A* ***multilinear map*** *is a map which is linear as a function of each separate variable.*

So if S is a multilinear map, we have:

$$S(x_1, ..., x_i + ay_i, ..., x_k) = S(x_1, ..., x_i, ..., x_k) + aS(x_1, ..., y_i, ..., x_k). \tag{1.5}$$

Similar as a covector is a linear map from vectors to $\mathbb{R}$ and a vector is a linear map from covectors to $\mathbb{R}$, a tensor T of rank (k, l) is a multilinear map from a collection of vectors and covectors to $\mathbb{R}$. So T is a map from

$$T : T_p^*M \times ... \times T_p^*M \times T_pM \times ... \times T_pM \to \mathbb{R}. \tag{1.6}$$

Here we have k times T_p^*M and l times T_pM.
To define tensors, we must first introduce the tensor product. The **tensor product** between T a (k, l) tensor and S an (m, n) tensor is the new $(k+m, l+n)$ tensor $T \otimes S$:

$$\begin{aligned} &T \otimes S(v_{a_1}, ..., v_{a_k}, v_{a_{k+1}}, ..., v_{a_{k+m}}, V_{b_1}, ..., V_{b_l}, V_{b_{l+1}}, ..., V_{b_{l+n}}) \\ &= T(v_{a_1}, ..., v_{a_k}, V_{b_1}, ..., V_{b_l}) \times S(v_{a_{k+1}}, ..., v_{a_{k+m}}, V_{b_{l+1}}, ..., V_{b_{l+n}}). \end{aligned} \tag{1.7}$$

Here v_{a_i} are covectors and V_{b_i} are vectors.
In section 1.5, a basis for the tangent space and the cotangent space is introduced. In this basis, the covector v_{a_i} can be written as $v_{a_i} = v_{a_i,\mu_i} dx^{\mu_i}$. Similarly the vector V_{a_i} can be written in its own basis as $V_{a_i} = V_{a_i}^{\nu_i} \partial_{\nu_i}$. A basis for the space of all (k, l) tensors is the tensor products of all these bases. So this basis looks like

$$\partial_{\nu_1} \otimes ... \otimes \partial_{\nu_k} \otimes dx^{\mu_1} \otimes ... \otimes dx^{\mu_l}. \tag{1.8}$$

Finally we can write a (k, l) tensor T as

$$T = T^{\nu_1 ... \nu_k}_{\mu_1 ... \mu_l} \partial_{\nu_1} \otimes ... \otimes \partial_{\nu_k} \otimes dx^{\mu_1} \otimes ... \otimes dx^{\mu_l}. \tag{1.9}$$

1.7 Riemannian manifold

In general relativity we want to measure distances on our space and therefore we use a metric. Let us explain what Riemannian manifolds are.

Definition 1.19. *A* ***Riemannian manifold*** *is a pair* (M, g)*, with* M *a smooth manifold and* g *a Riemannian metric on* M*.*

Here a **Riemannian metric** g is a symmetric metric which is positive at each point. In general, a metric g is often written as follows:

$$g = g_{ij} dx^i dx^j. \tag{1.10}$$

Here we write the metric in components, g_{ij} which is a symmetric $(0, 2)$-tensor. This tensor is called the metric tensor. The inverse metric tensor g^{ij} is defined via:

$$g^{ik} g_{kj} = g_{kj} g^{ik} = \delta^i_j. \tag{1.11}$$

Because g_{ij} is symmetric, g^{ij} is also symmetric.
On a Riemannian manifold, we can define an inner product between two tangent vectors V and W by $g(V, W) \in \mathbb{R}$. Now we can consider $g(V, .)$ as a map from T_pM to $\mathbb{R}$ by $W \mapsto g(V, W)$. The metric tensor acting on a vector $g(V, .)$ is thus a covector. Notice that the metric tensor g is an isomorphism between T_pM and T_p^*M. In the notation where we write the metric as a two tensor g_{ij}, the isomorphism looks as follows: $V_i = g_{ij} V^j$ and $V^i = g^{ij} V_j$. Here $V_i \in T_p^*M$. This mapping between T_pM and T_p^*M is known as raising and lowering indices.

2 Differential forms

In this chapter we will look at differential forms, which are antisymmetrical tensors and are commonly used in physical computations. We will also look at a product and a derivative on differential forms, the exterior product and the exterior derivative. We will have a closer look at exact, closed and harmonic forms. With these differential forms we will introduce two theorems from Hodge.

2.1 Differential forms

Definition 2.1. *A **differential k-form** α, also called a **k-form**, is an alternating $(0,k)$-tensor.*

The integer k is called the **degree** of the differential form. A $(0,k)$-tensor α is alternating if for all vectors $v_1,...,v_k \in V$ and every pair of distinct indices i,j we have

$$\alpha(v_1,...,v_i,...,v_j,...,v_k) = -\alpha(v_1,...,v_j,...,v_i,...,v_k). \tag{2.1}$$

In this thesis we look at differential forms on smooth manifolds. The vector space of k-forms on a smooth manifold M is denoted by $\Omega^k(M)$.
Let us define the exterior product, which is a multiplication such that this product of two differential forms is again a differential form.

2.2 Exterior product

To define the exterior product, we must look first on how it acts on coordinate basis vectors for T_p^*M.

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_n} = dx^{[\mu_1} \otimes dx^{\mu_2} ... \otimes dx^{\mu_n]}. \tag{2.2}$$

Here we used the notation that for a $(0,k)$-tensor $T^{a_1...a_k}$ we have that:

$$T^{[a_1...a_k]} = \frac{1}{k!}(T^{a_1...a_k} + \text{alternating sum over the permutations of indices}). \tag{2.3}$$

We can now write a k-form ω in component notation in the following way:

$$\omega = \frac{1}{k!}\omega_{i_1...i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}. \tag{2.4}$$

With this notation we can look at the **exterior product** between two differential forms $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, which is a new $(k+l)$-form. Their exterior product is as follows:

$$(\omega \wedge \eta)(v_1,...,v_k,...,v_{k+l}) = \frac{1}{k!l!}\omega(v_{[1},...,v_k)\eta(v_{k+1},...,v_{k+l]}). \tag{2.5}$$

Notice that the exterior product of a k-form and an l-form is a $(k+l)$-form, so $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$.
To get a feeling for the exterior product, we look at the following example:

Example 2.2. *Let ω be a 2-form and η a 1-form. Their exterior product is as follows:*

$$\omega \wedge \eta = \omega_{i_1}\omega_{i_2}\eta_{i_3} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3}. \tag{2.6}$$

In general the exterior product of a k-form and an l-form can be written like this:

$$\omega \wedge \eta = \omega_{i_1 \dots i_k}\eta_{j_1 \dots j_l} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}. \tag{2.7}$$

2.3 Exterior derivative

With the exterior product we have seen that the product between two differential forms, is again a differential form. Similarly, we want to take the derivative of a differential form in a way that the result is again a differential form. The exterior derivative fulfils this requirement. Let ω be the $k-$form from equation 2.4.

Definition 2.3. *The* ***exterior derivative*** *of ω, $d\omega$, is the following $(k+1)$-form:*

$$d\omega = \frac{1}{k!}\frac{\partial}{\partial x^i}(\omega_{j_1 \dots j_k}) dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}. \tag{2.8}$$

Here $d : \Omega^k \to \Omega^{k+1}$ is called the **exterior derivative operator**. Notice that the exterior derivative operator only acts on differential forms. The exterior derivative has the important property that $d^2 = 0$. To see this we write $d^2\omega$:

$$d^2\omega = \frac{1}{k!}\frac{\partial^2}{\partial x^l \partial x^i}(\omega_{j_1 \dots j_k}) dx^l \wedge dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}. \tag{2.9}$$

Here $\frac{\partial^2}{\partial x^l \partial x^i}(\omega_{j_1 \dots j_k})$ is symmetric with respect to l and i, but $dx^l \wedge dx^i$ is anti-symmetric. Therefore $\frac{\partial^2}{\partial x^l \partial x^i}(\omega_{j_1 \dots j_k}) dx^l \wedge dx^i = 0$, and also $d^2\omega = 0$.

Lemma 2.4. *Let ω be an k-form and η be a l-form. Then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \tag{2.10}$$

This result follows from writing out $d(\omega \wedge \eta)$.

2.4 Exact and closed forms

Let us first look at the kernel, image and cokernel of a linear map. The **image** of f is $f(V) \subset W$ and is denoted by $im(f)$. The **kernel** of f is $\{v \in V | f(v) = 0\}$ and is denoted by $ker(f)$. The **cokernel**, $coker(f)$ is defined as:

$$coker(f) = \frac{W}{im(f)}. \tag{2.11}$$

We are now going to have a closer look at the differential forms which are exact or closed.

Definition 2.5. *A p-form ω on a smooth manifold M is called* ***exact*** *if there is a $(p-1)$-form η with $\omega = d\eta$.*

We denote the **image** of $d : \Omega^{p-1}(M) \to \Omega^p(M))$ by $\mathcal{B}^p(M)$, which is the set of all exact p-forms on M.

Definition 2.6. *A form ω on a smooth manifold M is called* ***closed*** *if $d\omega = 0$.*

We denote the **kernel** of $d : \Omega^p(M) \to \Omega^{p+1}(M))$ by $\mathcal{Z}^p(M)$, which is the set of all the closed p-forms on M. The image and the kernel are well-defined as the exterior derivative is a linear map.
We notice that exact forms are closed forms, as $d^2\omega = 0$, for any differential form ω. This leads to the fact that the image is a subset of the kernel, so

$$\mathcal{B}^p(M) \subseteq \mathcal{Z}^p(M). \tag{2.12}$$

We can now define the de Rham cohomology group.

Definition 2.7. *The* ***de Rham cohomology group*** *in degree p, denoted by $H^p_{dR}(M)$, is defined as follows:*

$$H^p_{dR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}. \tag{2.13}$$

Two forms are called **cohomologous** if they only differ by an exact form. Later Hodge theorem will show that the de Rham cohomology group is isomorphic to the collection of harmonic forms of M. First we need to introduce some more concepts.

2.5 Adjoint of the exterior derivative

To define the adjoint of the exterior derivative, the Hodge star operator, denoted by $*$, must be introduced first.

Definition 2.8. *Let M be an m-dimensional manifold. Let $\omega \in \Omega^k$ be the following k-form:*

$$\omega = \frac{1}{k!}\omega_{j_1\ldots j_k} dx^{j_1} \wedge \ldots \wedge dx^{j_k}. \tag{2.14}$$

The ***Hodge star operator*** *is a map from $\Omega^k(M)$ to $\Omega^{m-k}(M)$ by:*

$$* \, \omega = \frac{\sqrt{|g|}}{k!(m-k)!}\omega_{l_1\ldots l_k}\epsilon^{l_1\ldots l_k}_{j_{k+1}\ldots j_m} dx^{j_{k+1}} \wedge \ldots \wedge dx^{j_m}. \tag{2.15}$$

The differential form $*\omega$ is called the **Hodge dual** of ω.
In the equation we have that

$$\epsilon^{l_1\ldots l_k}_{j_{k+1}\ldots j_m} = g^{l_1 j_1}\ldots g^{l_k j_k}\epsilon_{j_1\ldots j_m}, \tag{2.16}$$

where $\epsilon_{j_1\ldots j_m}$ is the Levi-Civita symbol, which is 1 for even permutations of $\{1,\ldots,m\}$, -1 for odd permutations and 0 if there are repeated indices. In equation 2.15, the term $|g|$ is the absolute value of the determinant of the metric tensor. Notice that the Hodge star depends on the metric. This dependence is quite tricky as the metric is both found in the prefactor $\frac{\sqrt{|g|}}{k!(m-k)!}$ and in $\epsilon^{l_1 \dots l_k}_{j_{k+1} \dots j_m}$.

Lemma 2.9. *Let η and ω be k-forms. The following exterior product $\eta \wedge *\omega$ is an m-form and is symmetric, so*

$$\eta \wedge *\omega = \omega \wedge *\eta. \tag{2.17}$$

Proof:
The proof of the symmetry property follows by writing the exterior product in component notation:

$$\begin{aligned} \eta \wedge *\omega &= \frac{1}{k!}\eta_{i_1 \dots i_k} \frac{\sqrt{|g|}}{k!(m-k)!}\omega_{j_1 \dots j_k} \epsilon^{l_1 \dots l_k}_{j_{k+1} \dots j_m} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_{k+1}} \wedge \dots \wedge dx^{j_m} \\ &= \frac{1}{k!}\eta_{i_1 \dots i_k} \omega^{j_1 \dots j_k} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^m. \end{aligned} \tag{2.18}$$

This last expression is symmetric as the metric is symmetric.■

Lemma 2.10. *Let ω be as before and let (M, g) be a Riemannian manifold, then*

$$* *\omega = (-1)^{k(m-k)}\omega. \tag{2.19}$$

This result follows from acting the Hodge star operator twice on the differential form ω.

Definition 2.11. *The* ***invariant volume element*** *is $*1$.*

As the simplest 0-tensor is the scalar 1, the most basic volume form is $*1$. Now we look at a Riemannian manifold (M, g). Notice that invariant volume element is as follows:

$$* 1 = \frac{\sqrt{|g|}}{m!}\epsilon_{j_1 \dots j_m} dx^{j_1} \wedge \dots \wedge dx^{j_m} = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^m. \tag{2.20}$$

The following notation of the inner product is useful:

Definition 2.12. *Let ω and η be k-forms on a m-dimensional manifold M. The* ***inner product*** *between ω and η, denoted by (ω, η) is defined as*

$$(\omega, \eta) = \int_M \omega \wedge *\eta = \frac{1}{r!}\int_M \omega_{j_1 \dots j_k} \eta^{j_1 \dots j_{m-k}} \sqrt{|g|} dx^1 \dots dx^m. \tag{2.21}$$

Here $\omega \wedge *\eta$ is an m-form.
The inner product has the following properties:

1. $(\omega, \omega) \geq 0$,

2. If $(\omega, \omega) = 0$, then $\omega = 0$.

The properties follow from the definition of the inner product and the fact that our manifold M is Riemannian.

Let us look at the operator $d^* : \Omega^k \to \Omega^{k-1}$ which is given by:

$$d^* = (-1)^{km+m+1} * d*, \tag{2.22}$$

where d is the exterior derivative operator $d : \Omega^{k-1} \to \Omega^k$. For this operator we have the following lemma.

Lemma 2.13. *Let ω be an r-form and η be an $r-1$-form. Then the following equality holds:*

$$(d\eta, \omega) = (\eta, d^*\omega). \tag{2.23}$$

Proof:
We know from lemma 2.4 and lemma 2.10 that

$$\begin{aligned} d(\omega \wedge *\eta) &= d\omega \wedge *\eta - (-1)^r \omega \wedge d * \eta = d\omega \wedge *\eta - (-1)^{(r-1)m+m+1} \omega \wedge * * d * \eta \\ &= d\omega \wedge *\eta - \omega \wedge *d^*\eta. \end{aligned} \tag{2.24}$$

If we now integrate both $d(\omega \wedge *\eta)$ over the whole manifold, we can set the result equal to zero, by choosing the boundary terms. In total we see that

$$\int_M (d\omega \wedge *\eta - \omega \wedge *d^*\eta) = (d\eta, \omega) + (\eta, d^*\omega) = 0. \tag{2.25}$$

In conclusion we see that $(d\eta, \omega) = (\eta, d^*\omega)$. ■
In this notation, d^* is the adjoint of d and therefore d^* is called the **exterior derivative operator**.
This operator has the property that ${d^*}^2 = 0$ as ${d^*}^2 = *d * *d* = C * d^2 * = 0$, with C a constant.
With the exterior derivative operator we can define coexact forms.

Definition 2.14. *A form ω on a smooth manifold M is called **coexact** if there is a $(k+1)$-form η such that $\omega = d^*\eta$.*

2.6 Harmonic forms

With the exterior derivative and its adjoint, we can define a second derivative operator which sends p-forms to p-forms. This operator is called the Laplacian.

Definition 2.15. *The **Laplacian**, denoted by $\Delta : \Omega^p(M) \to \Omega^p(M)$ is defined as follows:*

$$\Delta = dd^* + d^*d. \tag{2.26}$$

Definition 2.16. *A form ω is called **harmonic** if $\Delta\omega = 0$.*

We will cover some properties of the Laplacian.

Lemma 2.17. *A form ω is harmonic, if and only if $d\omega = d^*\omega = 0$.*

Proof:
Look at the inner product $(\omega, \Delta\omega)$. We claim that this inner product is positive.

$$(\omega, \Delta\omega) = (\omega, (dd^* + d^*d)\omega) = (d\omega, d\omega) + (d^*\omega, d^*\omega) \geq 0, \tag{2.27}$$

because both $(d\omega, d\omega)$ and $(d^*\omega, d^*\omega)$ are non negative. If ω is harmonic, $(\omega, \Delta\omega) = 0$ and thus $d\omega = d^*\omega = 0$. Showing that ω is harmonic if ω is both closed and coclosed is trivial. ∎

Lemma 2.18. *The Laplacian is self-adjoint, so for two p-forms α and β we have that $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$.*

Proof:
This proof is similar to the proof of lemma 2.18.

$$\begin{aligned}(\alpha, \Delta\beta) &= (\alpha, (dd^* + d^*d)\beta) = (d\alpha, d\beta) + (d^*\alpha, d^*\beta) \\ &= ((dd^* + d^*d)\alpha, \beta) = (\Delta\alpha, \beta).\blacksquare\end{aligned} \tag{2.28}$$

The following notation is normally used: The set of harmonic p-forms on M is denoted by $Harm^p(M)$. Let us look at some other properties of the Laplacian.

Lemma 2.19. *The Laplacian commutes with d and d^*.*

Proof:

$$d\Delta = dd^*d + ddd^* = dd^*d. \tag{2.29}$$

$$\Delta d = d^*dd + dd^*d = dd^*d. \tag{2.30}$$

We conclude that $\Delta d = d\Delta$. The proof for d^* is similar. ∎

Lemma 2.20. *The image of the Laplacian, $im(\Delta)$, is closed and has a finite dimensional complement, which is the cokernel of the Laplacian, $coker(\Delta)$.*

Here the image is well-defined as the exterior derivative operator and its adjoint are linear maps and the Laplacian is a combination of them. We use the notation, $im(\Delta)$, for the set of p-forms in the image of the Laplacian, which in our case is an operator on p-forms. Similarly, $coker(\Delta)$, is also a set of p-forms.

Proving lemma 2.20 requires prior knowledge in functional analysis. The results follows from the theory of self-adjoint elliptic operators. We therefore do not give the proof. The proof of lemma 2.20 can be found in chapter 6 of 'Foundations of differential manifolds and Lie groups' by F.W. Warner (6).

Lemma 2.21. *Let the set of harmonic forms and the cokernel be defined as before. Then $Harm^p(M) \cong coker(\Delta)$.*

Proof:
Let β be perpendicular to all exact forms, so $\beta \in im(\Delta)^{\perp}$. Because the Laplacian is self-adjoint we have that for all α:

$$(\alpha, \Delta\beta) = (\Delta\alpha, \beta) = 0. \tag{2.31}$$

As this equation holds for all α, we conclude that β is a harmonic form.
Now let β be a harmonic form. For all α we have that $(\alpha, \Delta\beta) = 0$. Again because the Laplacian is self-adjoint $(\Delta\alpha, \beta) = 0$, so $\beta \in im(\Delta)^{\perp}$. We conclude that $Harm^p(M) = im(\Delta)^{\perp}$.
Now we only have to proof that $coker(\Delta) \cong im(\Delta)^{\perp}$. For the Laplacian we have that $im(\Delta) \subsetneq \Omega(M)$, with $im(\Delta)$ closed. From lemma 2.21 we know that $im(\Delta)^{\perp}$ is finite dimensional. Therefore $coker(\Delta) \cong im(\Delta)^{\perp}$, and thus $Harm^p(M) \cong coker(\Delta)$. ∎

Lemma 2.22. *The Laplacian gives an isomorphism between $Harm^p(M)^{\perp}$ and $im(\Delta)$.*

We have seen that $Harm^p(M) \cong im(\Delta)^{\perp}$. Therefore $Harm^p(M)^{\perp} \cong (im(\Delta)^{\perp})^{\perp}$. We know from linear algebra that for a subspace W in an infinite dimensional Hilbert space, that the complement of the orthogonal complement of W is the closure of W. Therefore we know:

$$Harm^p(M)^{\perp} \cong (im(\Delta)^{\perp})^{\perp} = \overline{im(\Delta)}. \tag{2.32}$$

As $im(\Delta)$ is closed, we conclude that:

$$Harm^p(M)^{\perp} \cong im(\Delta). \blacksquare \tag{2.33}$$

As the Laplacian is an isomorphism between $Harm^p(M)^{\perp}$ and $im(\Delta)$, the inverse of the Laplacian exists. This inverse is called the Green operator G.

2.7 Hodge theorem

Theorem 2.23 (Hodge theorem)**.** *Let (M, g) be a compact orientable Riemannian manifold, then $H^p_{dR}(M)$ is isomorphic to $Harm^p(M)$.*

Corollary 2.24. $Harm^p(M) \cong \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$.

Before we prove the theorem, we will first prove another theorem by Hodge:

Theorem 2.25 (Hodge decomposition theorem)**.** *Let (M, g) be a compact orientable Riemannian manifold, then a p-form $\omega \in \Omega^p(M)$ is written globally as*

$$\omega = d\alpha + d^*\beta + \gamma, \tag{2.34}$$

where $\alpha \in \Omega^{p-1}(M)$, $\beta \in \Omega^{p+1}(M)$ and $\gamma \in Harm^p(M)$.

Proof:

First we show that $d\alpha$, $d^*\beta$ and γ are orthogonal.

$$(d\alpha, d^*\beta) = (d^2\alpha, \beta) = (0, \beta) = 0. \tag{2.35}$$

$$(d\alpha, \gamma) = (\alpha, d^*\gamma) = 0. \tag{2.36}$$

$$(d^*\beta, \gamma) = (\beta, d\gamma) = 0. \tag{2.37}$$

Now we show that ω only consists of $d\alpha$, $d^*\beta$ and γ.
The set of differential p-forms $\Omega^p(M)$ can be decomposed in the set of harmonic p-forms and a set perpendicular to the harmonic p-forms $Harm^p(M)^\perp$:

$$\Omega^p(M) = Harm^p(M) \oplus Harm^p(M)^\perp. \tag{2.38}$$

From before we know that $Harm^p(M)^\perp \cong im(\Delta)$.

Now we can look at a p-form $\omega \in \Omega^p(M)$.

$$\omega = \eta + \gamma, \tag{2.39}$$

where $\eta \in im(\Delta)$ and $\gamma \in Harm^p(M)$. Write $\eta = \Delta\xi$ for a p-form ξ. Now we have:

$$\omega = \eta + \gamma = \Delta\xi + \gamma = dd^*\xi + d^*d\xi + \gamma = d\alpha + d^*\beta + \gamma. \tag{2.40}$$

This proves theorem 2.26. ■

Now we can proof Hodge theorem. First we show that $Harm^p(M) \subset H^p_{dR}(M)$. For any harmonic form γ, $d\gamma = 0$. Thus $\gamma \in \mathcal{Z}^p(M)$. Now look at a $\beta \in \mathcal{B}^p(M)$. We can write $\beta = d\eta$, with η a $(p-1)$-form. Now we have

$$(\beta, \gamma) = (d\eta, \gamma) = (\eta, d^*\gamma) = 0. \tag{2.41}$$

From this follows that $\mathcal{B}^p(M) \bigcap Harm^p(M) = \varnothing$. We conclude that $Harm^p(M) \subset H^p_{dR}(M)$.
Now we show that $H^p_{dR}(M) \subset Harm^p(M)$. Let ω be a p-form in $H^p_{dR}(M)$. We know that $H^p_{dR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$, so ω is a closed p-form. Now look at $(d\omega, \beta)$. According to Hodge decomposition theorem, we can split ω as follows:

$$\omega = d\alpha + d^*\beta + \gamma. \tag{2.42}$$

We now see that:

$$(d\omega, \beta) = 0 = (d^*\beta, d^*\beta). \tag{2.43}$$

Thus $d^*\beta = 0$ and $\omega = d\alpha + \gamma$. We already know that any harmonic form is in the de Rahm cohomology group so $\gamma \in H^p_{dR}(M)$. The de Rham cohomology group is a group under addition, so $\omega - \gamma = d\alpha$ is also in $H^p_{dR}(M)$. As there are per definition no exact forms in $H^p_{dR}(M)$, $d\alpha = 0$ and thus $\omega = \gamma$. Therefore ω any p-form in the de Rham cohomology group is a harmonic form. We conclude that $Harm^p(M)$ is isomorphic to $H^p_{dR}(M)$. With this, Hodge theorem is proven. ■

3 General Relativity

In 1905 Einstein published his theory on special relativity. With this theory he was able to describe particles moving with constant velocity. To explain accelerating particles he created a new theory in 1915, general relativity. With this theory, Einstein gave a new description of gravity. As general relativity could describe phenomena in space, which earlier theories of gravity could not, such as the bending of light and the orbit of Mercury, Einstein's theory was soon seen as the most accurate description of gravity. In general relativity, spacetime is seen as a manifold, which can be curved, instead of a Minkowski-space as in special relativity.

The description of gravity follows by the assumption that non-accelerated test particles, follow geodesics. A test particle is a particle which does not affect the curvature of spacetime. Geodesics can be thought of as the shortest path from one place to another. A curved spacetime has different geodesics than $\mathbb{R}^n$ as the geodesics are bent due to masses. Therefore test particles follow other paths in curved space than in flat space.

In general relativity the Einstein equations are one of the most used results, as they summarise general relativity and as we can calculate with them. We will generate the Einstein equations for a vacuum, because Kaluza used these equations in five dimensions as a starting position for his reduction to four dimensions. To understand the Einstein equations, we will look at the Ricci scalar and the Ricci tensors which are only nonzero on curved manifolds. To understand this scalar and tensors we will look at the Christoffel symbols. In the end of this chapter we will cover the derivation for the Einstein equations in a vacuum.

3.1 Christoffel symbols

In chapter 1, we introduced an affine connection on a smooth manifold. Let us now look at an affine connection on a Riemannian manifold. This connection is called the **covariant derivative**, or the **Levi-Civita connection**. This connection is a derivative which transforms as a vector on an arbitrary manifold. It acts exactly like a partial derivative on a flat space. We want the covariant derivative to transform under a change of coordinates of the tangent space, like a vector. As we have seen, a vector itself does not change under a change of coordinates of the tangent space, only its components change. So does the covariant derivative. A partial derivative changes under a change of coordinates in a curved space, as the partial derivative is a derivative with respect to the coordinates of the tangent space. This makes the covariant derivative better to work with. The third derivative we have discussed is the exterior derivative, which is only defined on differential forms. The covariant derivative on the other hand, can act on any tensor. This connection working on a (p, q)-tensor gives a $(p, q + 1)$-tensor. To let the covariant derivative be independent on the coordinates of the tangent space, we write it as the original derivative plus correction terms. These correction terms are given by the **Christoffel symbols**

$\Gamma^{\nu}_{\mu\lambda}$. So acting on a vector V^{ν} we have:

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}. \tag{3.1}$$

In general we have for tensors that:

$$\begin{aligned}\nabla_{\mu} T^{\nu_1 \ldots \nu_n}_{\nu_{n+1}\ldots\nu_{n+m}} =& \partial_{\mu} T^{\lambda\ldots\nu_n}_{\nu_{n+1}\ldots\nu_{n+m}} + \Gamma^{\nu_1}_{\mu\lambda} T^{\nu_1\ldots\lambda}_{\nu_{n+1}\ldots\nu_{n+m}} + \ldots + \Gamma^{\nu_n}_{\mu\lambda} T^{\nu_1\ldots\nu_n}_{\nu_{n+1}\ldots\nu_{n+m}} \\ & - \Gamma^{\lambda}_{\mu\nu_{n+1}} T^{\nu_1\ldots\nu_n}_{\lambda\ldots\nu_{n+m}} - \ldots - \Gamma^{\lambda}_{\mu\nu_{n+m}} T^{\nu_1\ldots\nu_n}_{\nu_{n+1}\ldots\lambda}.\end{aligned} \tag{3.2}$$

Here we notice that $\nabla_{\mu} V^{\nu}$ is coordinate independent so $\nabla_{\mu} V^{\nu} = \nabla_{\mu'} V^{\nu'}$. We demand two other conditions for the Christoffel symbols.
Firstly they are torsion free, which means that $T^{\lambda}_{\mu\nu} = 0$, where $T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$.
Secondly, they are metric compatible. This means that the covariant derivative of the metric with respect to the connection is zero, so $\nabla_{\lambda} g_{\mu\nu} = 0$.
Now we want to find an expression for the Christoffel symbols. we will do so by expanding $\nabla_{\lambda} g_{\mu\nu}$. By construction we will find that the obtained expression is unique. We know that the following equations hold:

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} = 0, \tag{3.3}$$

$$\nabla_{\mu} g_{\nu\lambda} = \partial_{\mu} g_{\nu\lambda} - \Gamma^{\rho}_{\mu\nu} g_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda} g_{\rho\nu} = 0 \tag{3.4}$$

and

$$\nabla_{\nu} g_{\lambda\mu} = \partial_{\nu} g_{\lambda\mu} - \Gamma^{\rho}_{\nu\lambda} g_{\rho\mu} - \Gamma^{\rho}_{\nu\mu} g_{\rho\lambda} = 0. \tag{3.5}$$

In these expressions we used the property that the Christoffel symbols are torsion free. Combining the three equations, we get the wanted unique expression for the Christoffel symbols:

$$\Gamma^{\rho}_{\nu\mu} = \frac{1}{2} g^{\rho\lambda} (\partial_{\mu} g_{\nu\lambda} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\nu}). \tag{3.6}$$

In the expression we see that the Christoffel symbols are totally dependent on the metric. On a flat manifold, the Christoffel symbols vanish as the metric is constant. In this case the covariant derivative is equal to the partial derivative. In this chapter we are using component notation and not the differential form notation. This is because the tensors we are working with are not antisymmetric. The notation of differential forms will only be used in the next chapter.

3.2 Riemann tensor

We are now going to use the Christoffel symbols for constructing the Riemann tensor and the Ricci tensor. The Riemann tensor, also known as the Riemann curvature tensor, describes the curvature on a manifold. The Riemann tensor gives on every point on the manifold a description of the curvature.
One can transport a vector, which has a direction, over a manifold along a path, while keeping the vector constant. This is called parallel transport. If we

parallel transport a vector over a curved manifold from one place to an other, the resulting vector is path dependent. Parallel transporting a vector to a really small place is equivalent to taking the covariant derivative.
Consider a transport of first taking first the covariant derivative of a vector V^ρ in the x^μ direction and then in the x^ν direction $V^\rho \to V^\rho + \nabla_\mu \nabla_\nu V^\rho dx^\mu dx^\nu$. If the space is curved then this parallel transport is different than taking first the covariant derivative in the x^ν direction and then in the x^μ direction. So then $\nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \neq 0$.
The **Riemann tensor** is defined as follows:

$$R^\rho_{\lambda\mu\nu} V^\lambda = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho - T^\lambda_{\mu\nu} \nabla_\lambda V^\rho. \tag{3.7}$$

Here $T^\lambda_{\mu\nu}$ is the torsion tensor, which is in our cases always equal to zero as we demand the Christoffel symbols to be torsion free.
Now look at $\nabla_\mu \nabla_\nu V^\rho$:

$$\begin{aligned} \nabla_\mu \nabla_\nu V^\rho =& \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho \\ & - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda. \end{aligned} \tag{3.8}$$

$\nabla_\nu \nabla_\mu V^\rho$ gives the same equation but with μ and ν switched positions. In total we find the expression for the Riemann tensor:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \tag{3.9}$$

From the Riemann tensor, we can find the Ricci tensor $R_{\mu\nu}$. The **Ricci tensor** is defined as follows:

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}. \tag{3.10}$$

Notice that the Ricci tensor is by construction symmetric:

$$R_{\mu\nu} = R_{\nu\mu}. \tag{3.11}$$

The trace of the Ricci tensor is the **Ricci scalar** R:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{3.12}$$

It turns out that the Ricci scalar is the simplest scalar dependent on the curvature which invariant on a Riemannian manifold. Therefore it is a really important scalar in general relativity.

3.3 Action principles

An action can be written as the integral over a space of the Lagrange density, which is a function of fields. This could be scalar fields, vector fields or tensor fields. By varying one of these fields with a small value, the action varies too. In physics, a general assumption is that any system is in a state of minimal action. This means that if one varies the action a little, that locally the action does not change as it is in an equilibrium position. This assumption is called the **principle of least action**. This assumption gives a restriction on the fields that leads to the equations of motion. According to Hamilton's principle any system that is described by equations of motion can be reformulated in terms of an action principle and vice versa. In many cases, the Lagrangian density is a more compact formulation and therefore it is often clearer to read what kind of system is described by looking at the Lagrangian density than looking at the equations of motion.

3.4 The Einstein-Hilbert action

In this section we look at the Einstein-Hilbert action. Out of this action, we will later find the Einstein equations, which describe general relativity. Via the principle of least action, Hilbert wanted to construct the Einstein equations. We only consider the vacuum case. In vacuum, the Einstein equations for general relativity are as follows (2):

$$R_{\mu\nu} = 0. \tag{3.13}$$

To get equations which give the Einstein equations and describe the dependence of curvature, the action must be dependent on the metric. An action is always an integral over the whole manifold of a Lagrange density. On a Riemannian manifold, this Lagrange density is a scalar L multiplied with $\sqrt{|g|}$. L is called the Lagrangian scalar. We can write the action as follows:

$$S = \int L * 1. \tag{3.14}$$

A good guess for the Lagrange scalar L, including second order derivatives of the metric, is the Ricci scalar. This is the simplest possible scalar, which is dependent on the second derivative of the metric. One could also add more complex scalars to the Lagrangian density, but this would only give higher order corrections. This is, because any other scalar dependent on the metric, not equivalent to the Ricci scalar, would consist of higher order terms of the metric tensor. We thus look at the action which looks as follows:

$$S = \int R * 1 = \int \sqrt{|g|} R d^n x. \tag{3.15}$$

Here n is the dimensionality of the manifold. This action is called the Einstein-Hilbert action. The Einstein-Hilbert action describes gravity in a vacuum.

3.5 The Einstein vacuum equations

By varying the Einstein-Hilbert action, we can find equations of motion, which will give a description of gravity. To find these equations, we have to minimize the action. Therefore we want to find when $\delta S = 0$.

$$\delta S = \int \delta(\sqrt{|g|} R) d^n x. \tag{3.16}$$

We see that $\delta S = 0$, if $\delta(\sqrt{|g|}R) = 0$.

$$\begin{aligned} \delta(\sqrt{|g|}R) &= \delta(g^{MN} R_{MN}\sqrt{|g|}) \\ &= R_{MN}\delta g^{MN}\sqrt{|g|} + R_5\delta\sqrt{|g|} + g^{MN}\delta R_{MN}\sqrt{|g|}. \end{aligned} \tag{3.17}$$

Here we have $\delta\sqrt{|g|} = -g_{MN}\sqrt{|g|}\delta g^{MN}$.
Now zoom in on δR_{MN}:

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}). \tag{3.18}$$

Luckily, this term vanishes at the boundaries of the integration. In total we find the following equation:

$$\delta(\sqrt{|g|}R) = (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}\sqrt{|g|}. \tag{3.19}$$

We now see that $\delta S = 0$, if the following equation holds:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{3.20}$$

This equation can get simplified by multiplying with the metric tensor.

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}Rg_{\mu\nu} = R - \frac{1}{2}tr(g)R = 0. \tag{3.21}$$

We see now that the Ricci tensor has to be zero. When we use this result in equation 3.20, we conclude that the equations of motion are given by the following equations:

$$R_{\mu\nu} = 0. \tag{3.22}$$

These are the Einstein equations of a vacuum, which we have seen in equation 3.13.

4 Circle reduction

When Einstein published his theory of gravity, one of the big questions was whether general relativity could be combined with other theories. Kaluza made an attempt to unify gravity and electromagnetism. In the first section we will look at what assumption was made for the five dimensional metric which was used for the Kaluza-Klein reduction. With these assumptions we can find the metric and the inverse metric in five dimensions in terms of a four dimensional metric $g_{\mu\nu}$, a vector field A_μ and a scalar field ϕ. With this five dimensional metric, Kaluza could write the five dimensional Einstein equations in terms of $g_{\mu\nu}$, A and ϕ. As the action describes a system equivalently as the equations of motion, we could also take a reduction of the Einstein-Hilbert action in five dimensions. In this process, one has to write the five dimensional Ricci scalar in terms of $g_{\mu\nu}$, A and ϕ. The result is a four dimensional action. In both processes it is necessary to use the cylindrical condition. In 1926 Klein motivated why the cylindrical condition is reasonable. We will reproduce the arguments of Klein. In the end, we will also look at an other action on a five dimensional manifold and reduce it to an action on four dimensional spacetime. The action we consider is:

$$S_{form} = \frac{1}{2}\int_{M_{D+1}} F'_p \wedge *F'_p. \tag{4.1}$$

This action is called the action for a $(p-1)$-form gauge field. We look at this action, because on a four dimensional spacetime it results in the homogeneous Maxwell equations.

4.1 The metric tensor in five dimensions

In this paragraph we argue what the metric tensor, which represents the five dimensional spacetime, looks like. If we know the Ricci tensor, we can find the five dimensional Einstein equations for a vacuum as this is $R_{MN} = 0$. As the Ricci tensor is only dependent on the metric, the metric tensor contains all the information for the five dimensional Einstein equations.

Let us have a look at the five dimensional manifold. The first four dimensions of our space from the manifold just represent spacetime as we are used to. We get a fifth dimension by adding a circle with a radius $R \ll 1$. Do not confuse this R with the Ricci scalar, which is also denoted by R. The radius of the circle has to be really small, as in that case we can not observe the circle.

We use the notation that capital Latin indices run over all five dimensions. The Greek indices run only over the first four dimensions. If we specifically want to appoint the fifth dimension, we use the index 5.

The fifth coordinate is periodic with radius R, so $x^5 = x^5 + 2\pi R$. Now we look at a coordinate transformation $x^M \to x'^M = x^M + \epsilon^M(x^M)$. Here our metric transforms as

$$g_{MN} \to g'_{MN} = g_{MN} - \partial_M \epsilon_N(x^M) - \partial_N \epsilon_M(x^M). \tag{4.2}$$

At this moment we introduce the cylindrical condition for the first time. We say that the transformation ϵ_M is independent on the fifth coordinate. Now we look at the transformation over the fifth coordinate, $x^5 \to x^5 + \epsilon^5(x^\mu)$. Under this transformation $\epsilon_\mu = 0$ and $\partial_5 \epsilon_5 = 0$, so $g_{\mu\nu}$ and g_{55} are invariant under this transformation. The transformation of $g_{\mu 5}$ over the fifth coordinate is as follows:

$$g_{\mu 5} \to g_{\mu 5} - \partial_\mu \epsilon_5. \tag{4.3}$$

This is exactly the same transformation as the gauge transformation of a potential field A_μ. This transformation corresponds with the $U(1)$ symmetry group, which is used to describe the electromagnetic field in the standard model. This is a foresight that in the end the Kaluza-Klein reduction will give results of electromagnetism. To find the metric tensor in five dimensions, write ds:

$$ds^2 = g_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + 2A_\mu dx^\mu dx^5 + g_{55} dx^5 dx^5. \tag{4.4}$$

We take $g_{55} = \phi^2$, where ϕ is a scalar field. We can write ds^2 without losing information as:

$$ds^2 = g_{\mu\nu} + \phi^2 (dx^5 + A_\mu dx^\mu)^2. \tag{4.5}$$

This gives us the five dimensional metric tensor:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}. \tag{4.6}$$

From this we can find the inverse metric. The inverse metric g^{MN} is given by:

$$g^{MN} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & g_{\alpha\beta} A^\alpha A^\beta + \frac{1}{\phi^2} \end{pmatrix}. \tag{4.7}$$

4.2 Kaluza-Klein reduction of the Einstein-Hilbert action

Now we are going to look at the reduction of the five dimensional Einstein equations to four dimensional equations. Also we are going to look at the reduction of the five dimensional Einstein-Hilbert action to a four dimensional action.

As discussed before, the Einstein equations for a vacuum in five dimensions are found by $R_{MN} = 0$. As we know what the metric tensor looks like, we could now calculate the Ricci tensor. This gives 15 independent equations, as R_{MN} is a five by five tensor and $R_{MN} = R_{NM}$. The found equations are as follows (8):

1. $\nabla^2 \phi = \frac{1}{4} \phi^3 F_{\mu\nu} F^{\mu\nu}$,
2. $\nabla^\mu F_{\mu\nu} = -3 \frac{\partial^\mu \phi}{\phi} F_{\mu\nu}$,
3. and $G_{\mu\nu} = \frac{\phi^2}{2} T_{\mu\nu} - \frac{1}{\phi} [\nabla_\mu (\partial_\nu \phi) - g_{\mu\nu} \nabla^2 \phi]$.

Here we used the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}$ and the electromagnetic energy-momentum tensor $T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} - F^{\gamma}_{\mu} F_{\nu\gamma}$.
Originally, Kaluza assumed ϕ to be a constant. In this case both the Einstein equations for matter and the homogeneous Maxwell equations are found:

$$G_{\mu\nu} = \frac{\phi^2}{2} T_{\mu\nu}, \tag{4.8}$$

$$\nabla^{\mu} F_{\mu\nu} = 0. \tag{4.9}$$

We began with the five dimensional Einstein equations of a vacuum and after a reduction, we found the four dimensional Einstein equations which also describes matter. In this way, we can describe matter by using only geometrical arguments. It is important to notice that at least the vacuum Einstein equations can be constructed via the reduction. As we started with the five dimensional vacuum Einstein equations, we expected that at the very least the four dimensional vacuum Einstein equations roll out the reduction. Notice further that ϕ only is a constant if $F_{\mu\nu} F^{\mu\nu} = 0$. The most interesting result is that the homogeneous Maxwell equations are found by the reduction. These equations describe electromagnetism if all source charges and currents are zero. This means that Kaluza succeeded in unifying gravity and the electromagnetic field without charges and currents in one theory. This does not include the whole theory of electromagnetism as the inhomogeneous Maxwell equations, which include non zero charges and currents, are not found via the reduction.

In the appendix, one can find my attempt at finding the equation of motion for $R_{55} = 0$. Unfortunately this result does not match with the result found in the paper by J.M. Overduin and P.S. Wesson on Kaluza-Klein Gravity (8).

A way of describing the system, equivalent to finding the equations of motion, is by finding the action of the system. The action in five dimensions is as follows:

$$S_5 = \int_{M_5} \sqrt{|g_5|} R_5 d^5 x. \tag{4.10}$$

Here g_5 is the five dimensional metric and R_5 is the five dimensional Ricci scalar. Do not confuse R_5 with R_{55}, which is a component of the Ricci tensor. To reduce this action to four dimensions, we have to integrate over the fifth coordinate. The first step is finding the Ricci scalar in terms of objects defined on a four dimensional manifold. As the metric tensor of the five dimensional manifold is given, we can find the Ricci tensor in terms of $g_{\mu\nu}$, A_{μ} and ϕ. Because of the cylindrical condition, all objects defined on a four dimensional manifold are independent on the fifth coordinate. This makes it possible to integrate over the fifth coordinate. In total, the action in four dimensions is given by:

$$S_4 = \int_{M_4} \sqrt{|g_4|} \phi \left(\frac{R_4}{K} + \frac{1}{4} \phi^2 F_{\mu\nu} F^{\mu\nu} + \frac{2}{3} \frac{\partial^{\mu} \phi \partial_{\mu} \phi}{\phi^2} \right) d^4 x. \tag{4.11}$$

Here K is a constant, on which we will not focus more.
In this action we see three different terms. The last term describes a massless

scalar field ϕ, which describes the so-called dilaton particle. The first term is similar to the Hilbert-Einstein action for gravity. The second term is similar to the homogeneous Maxwell actions for electromagnetism. In the result, the equations for gravity and electromagnetism are dependent on the dilaton field. We expected this, as the equations found by $R_{MN} = 0$ are also dependent on the dilaton field. As all terms contain a ϕ, they are connected with each other. In conclusion, Kaluza began with a Hilbert-Einstein action in five dimensions. After the reduction he could describe both gravity and the electromagnetic field without charges and currents in one theory. Further, the equations for a dilaton particle were found.

4.3 Cylindrical condition

The question that we still have to consider from the last chapter is: Why can we make the cylindrical condition? Briefly, Klein came up with an answer for this question by Fourier expanding fields on the manifold over the circle and by showing that all Fourier modes represent massive fields, which are so heavy that they are immeasurable, except the 0^{th} Fourier mode. As we do not have to worry about the immeasurable massive fields and as the 0^{th} Fourier mode is constant under transformation on the circle, we can use the cylindrical condition.
Klein started with the same manifold as Kaluza. As constructed, the circle is periodic, so $x^5 = x^5 + 2\pi R$. Now consider a scalar field ϕ in the five dimensional spacetime. This field is also periodic so $\phi(x^5) = \phi(x^5 + 2\pi R)$. Therefore we can Fourier-expand the scalar field.

$$\phi(x^\mu, x^5) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) e^{inx^5/R}. \tag{4.12}$$

Here ϕ_n refers to the n^{th} Fourier mode.
As ϕ is a scalar field, it fulfils the Klein-Gordon equation:

$$(\Box^2 + \partial_5\partial^5)\phi = -m^2\phi. \tag{4.13}$$

Here $\Box = \partial_\mu\partial^\mu$, with $\mu \in \{0,1,2,3\}$.
Now look at the derivative to the fifth coordinate:

$$\partial_5\partial^5\phi = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)[\frac{in}{R}]^2 e^{inx^5/R} = -\frac{n^2}{R^2}\phi^2. \tag{4.14}$$

We make the assumption that $R \ll 1$. Now the mass becomes really large unless we take $\phi_n(x^\mu) = 0$ for all $n \neq 0$. Already the first excited state of the field would have a mass of $m = \frac{1}{R}$. All other exited states have an even larger mass. Particles with these large masses have energies way above the energies of particles we could possibly measure. As scientists have not been able to measure such particles, we can neglect all modes with $n \neq 0$ (22). The only mode which exists on the circle is therefore the 0^{th} Fourier mode, which describes a massless scalar field.
Now $\phi(x^\mu, x^5)$ becomes:

$$\phi(x^\mu, x^5) = \phi_0(x^\mu). \tag{4.15}$$

In this expression, ϕ is independent on the fifth coordinate.

Similarly we can look at the vector field A_M. First we again Taylor expand the field:

$$A_\mu(x^\mu, x^5) = \sum_{n\in\mathbb{Z}} A_{\mu,n}(x^\mu)e^{inx^5/R}. \tag{4.16}$$

Now instead of using the Klein-Gordon equation, look at the Proca equation in five dimensions:

$$(\Box^2 + \partial_5\partial^5)A_{\mu,n} = -m^2 A_{\mu,n}. \tag{4.17}$$

Again we want to have a mass which is not really large. We thus have to take $A_{M,n} = 0$ for all modes except $A_{\mu,0} = 0$. Now A_μ is invariant under transformations over the fifth coordinate.
To show that the metric is invariant over the fifth coordinate, one has to do similar steps as for the vector and scalar fields. This is a bit more complicated as the metric is a rank two tensor, but Klein stated that also the metric in four dimensions is independent on the fifth coordinate. The cylindrical condition is thus indeed a well considered condition.

4.4 The action for an electromagnetic field

In this section we look at a different action, the action for an electromagnetic field. First we look at the action in four dimensions and focus on the similarities in using the notation of differential forms, where we write the electromagnetic tensor as $F = dA$, and in component notation, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here A is a 1-form. We again begin with writing the action in terms of the Lagrangian scalar:

$$S = \int_M L * 1 \tag{4.18}$$

Now start with a four dimensional manifold, as spacetime. In this action, we take the Lagrangian scalar L as a function of the first derivative of a 1-form A. The simplest possible choice for the Lagrangian scalar is $L = F_{\mu\nu}F^{\mu\nu}$. So we consider the action:

$$S = \int_{M_4} \sqrt{|g|}F_{\mu\nu}F^{\mu\nu}dx^1 \wedge ... \wedge dx^4. \tag{4.19}$$

Now we find the equations of motion by varying the action S with respect to A_μ.

$$\begin{aligned} \delta S &= \int_{M_4} \sqrt{|g|}(\frac{\partial L}{\partial A_\mu}\delta A_\mu + \frac{\partial L}{\partial \partial_\nu A_\mu}\partial_\nu \delta A_\mu)dx^1 \wedge ... \wedge dx^4 \\ &= \int_{M_4} \sqrt{|g|}(F^{\mu\nu}\partial_\nu\delta A_\mu)dx^1 \wedge ... \wedge dx^4. \end{aligned} \tag{4.20}$$

Now using partial integrating we get:

$$\begin{aligned}\delta S &= \sqrt{|g|}F^{\mu\nu}\delta A_\mu|_{\partial M_4} + \int_{M_4}\sqrt{|g|}(-\partial_\nu F^{\mu\nu}\delta A_\mu)dx^1\wedge\ldots\wedge dx^4 \\ &= \int_{M_4}\sqrt{|g|}(-\partial_\nu F^{\mu\nu})\delta A_\mu dx^1\wedge\ldots\wedge dx^4.\end{aligned} \tag{4.21}$$

Fortunately δA is neglectable at the boundary. We now find the equations of motion by putting $\delta S = 0$. Now we can read the equations of motion:

$$\partial_\nu F^{\mu\nu} = 0. \tag{4.22}$$

These are the homogeneous Maxwell equations, which play an important role in electromagnetism. As this four dimensional action gives an important result, we want to consider the same action but then in more dimensions.
First we want to write the action with the notation of differential forms. Our claim is that on a four dimensional manifold, the following way of writing the action is equivalent:

$$\int_{M_4}\sqrt{|g|}F_{\mu\nu}F^{\mu\nu}dx^1\wedge\ldots\wedge dx^4 = \int_{M_4}F_2\wedge *F_2, \tag{4.23}$$

with F_2 the 2-form $F_2 = dA$.
In the form notation we have $A = A_\mu dx^\mu$. Also we see:

$$F_2 = dA = \partial_\nu A_\mu dx^\nu\wedge dx^\mu \tag{4.24}$$

and

$$*F_2 = \frac{\sqrt{|g|}}{4}\partial_\nu A_\mu \epsilon^{\mu\nu}_{\rho\sigma}dx^\rho\wedge dx^\sigma. \tag{4.25}$$

We remember from definition 2.11 the formula for the invariant volume element. In total we see that the following equality holds.

$$\int_{M_4}F_2\wedge *F_2 = \int_{M_4}F_{\mu\nu}F^{\mu\nu} * 1. \tag{4.26}$$

In the next section we are going to look at the electromagnetic tensor in a higher dimensional manifold. Then, as we will see, the notation of differential forms is more useful than using components. The notation of differential forms is more compact and it makes the derivation a lot clearer. Because the gauge field can be written as a differential form we choose to use this notation. In string theory, many physical objects are antisymmetrical and can therefore be written the notation of differential forms.

4.5 Dimensional reduction of the action for a $(p-1)$-form gauge field

We are now in a position to look at a similar action as the action for an electromagnetic field, but now on the five dimensional manifold. As the action in

section 4.4 gave us the homogeneous Maxwell equation, this action, but then for p-forms which are not necessarily 2-forms, is probably also interesting in the five dimensional manifold. In this section we are not looking at the five dimensional manifold, but we generalize the situation by considering a manifold of dimension $D+1$. Here we again take the last dimension to be a small circle. In the end we will focus again on the specific case where $D+1=5$. Also we generalize F_2. Instead of only considering F_2 being a 2-form we look at the exact p-form F'_p. Now $F'_p = dC'_{p-1}$, with C'_{p-1} a $(p-1)$-form.
We use the same action as Federico Bonetti [6], but Bonetti has defined his inner product to be negative, where we defined our inner product to be positive. Therefore, we differ a sign in our action. This action, the action for a $(p-1)$-form gauge field C'_{p-1}, looks as follows:

$$S_{form} = \frac{1}{2}\int_{M_{D+1}} F'_p \wedge *F'_p. \tag{4.27}$$

Here $*$ is the the Hodge star operator which sends p-forms from $\Omega^r(M_{D+1})$ to $\Omega^{D+1-r}(M_{D+1})$.
For this equation in five dimensions, we want to know what result it is equivalent to in four dimensions. To find a result in four dimensions, we have to integrate over the fifth dimension. To do so we start by noticing that we can Fourier expand C'_{p-1} with respect to the circle. Here we split C'_{p-1} in a term C_{p-1} which is a p-form with only values in the first D dimensions and in a term $C_{p-2}\wedge dy$. This last term consists of a $(p-2)$-form in the first D dimensions C_{p-2} and a part on the circle dy. A reasonable way of writing C'_{p-1} would be as follows:

$$C'_{p-1} = \sum_{n\in\mathbb{Z}} [C^{(n)}_{p-1} + C^{(n)}_{p-2}\wedge dy]e^{iny}. \tag{4.28}$$

It turns out that the result will be much more convenient if we shift C'_{p-1}:

$$C'_{p-1} \to C'_{p-1} + \sum_{n\in\mathbb{Z}} C^{(n)}_{p-2}\wedge A e^{iny}. \tag{4.29}$$

Here A is the same 1-form as above. Notice that A lives on the first D dimensions. In our new convention for C'_{p-1} we have the following expression:

$$C'_{p-1} = \sum_{n\in\mathbb{Z}} [C^{(n)}_{p-1} + C^{(n)}_{p-2}\wedge (dy + A)]e^{iny}. \tag{4.30}$$

As with the scalar field and the tensor field, we only consider the modes where $n=0$. All the modes where $n>0$ describe massive fields. Now we have the following equation for C'_{p-1}:

$$C'_{p-1} = [C^{(0)}_{p-1} + C^{(0)}_{p-2}\wedge (dy + A)]. \tag{4.31}$$

Now we look at F'_p.

$$\begin{aligned} F'_p = dC'_{p-1} &= dC^{(0)}_{p-1} + d[C^{(0)}_{p-2}\wedge (dy+A)] \\ &= dC^{(0)}_{p-1} + dC^{(0)}_{p-2}\wedge (dy+A) + (-1)^p C^{(0)}_{p-2}\wedge dA. \end{aligned} \tag{4.32}$$

Now we define the following forms:

$$F_{p-1} = dC^{(0)}_{p-2} \tag{4.33}$$

and

$$F_p = dC^{(0)}_{p-1} + (-1)^p C^{(0)}_{p-2} \wedge dA. \tag{4.34}$$

In total we can write F'_p as follows:

$$F'_p = F_p + F_{p-1} \wedge (dy + A). \tag{4.35}$$

We now see that:

$$\begin{aligned} F'_p \wedge *F'_p =& F_p \wedge *F_p + 2F_p \wedge *[F_{p-1} \wedge (dy + A)] \\ &+ F_{p-1} \wedge (dy + A) \wedge *[F_{p-1} \wedge (dy + A)]. \end{aligned} \tag{4.36}$$

Here we used lemma 2.9, which says that:

$$\omega \wedge *\eta = \eta \wedge *\omega. \tag{4.37}$$

We are looking for an equation in D dimensions. Therefore we want to integrate over the circle, so we want to separate the dy term from the rest of the equation. Write

$$F_p \wedge *F_p = \frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}} F_p \wedge \hat{*} F_p \wedge dy. \tag{4.38}$$

Here the Hodge star $\hat{*}$ is the Hodge star operator for a D-dimensional manifold and not the one for a $(D+1)$-dimensional manifold, as before. This change of dimension corresponding to the Hodge star operator, we get the extra term $\frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}}$. The tensor g_{D+1}, is the metric tensor corresponding to M_{D+1} and the tensor g_D, is the metric tensor corresponding to M_D.
Now we look at the term $*[F_{p-1} \wedge (dy + A)]$.

$$\begin{aligned} &* [F_{p-1} \wedge (dy + A)] = \\ &\frac{\sqrt{|g_{D+1}|}}{p!(D-p)!} F_{p-1,l_1\ldots l_p} (dy + A)_{D+1} g^{l_1 j_1} \ldots g^{l_p j_p} g^{(D+1) j_{D+1}} \epsilon_{j_1 \ldots j_{D+1}} dx^{j_{p+1}} \wedge \ldots \wedge dx^{j_D}. \end{aligned} \tag{4.39}$$

Notice that

$$(dy + A)_{D+1} g^{(D+1) j_{D+1}} \epsilon_{j_1 \ldots j_{D+1}} = \frac{1}{\phi^2} p \epsilon_{j_1 \ldots j_D}, \tag{4.40}$$

as $\begin{pmatrix} A_\mu \\ 1 \end{pmatrix} (-A^\mu, A_\nu A^\nu + \frac{1}{\phi^2}) = \frac{1}{\phi^2}$.
Now we have:

$$\begin{aligned} *[F_{p-1} \wedge (dy + A)] &= \frac{\sqrt{|g_{D+1}|}}{p!(D-p)!} \frac{1}{\phi^2} p F_{p-1,l_1\ldots l_p} \epsilon_{j_1 \ldots j_D} dx^{j_{p+1}} \wedge \ldots \wedge dx^{j_D} \\ &= \frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}} \frac{1}{\phi^2} \hat{*} F_{p-1}. \end{aligned} \tag{4.41}$$

Now we can look at the following exterior product:

$$F_p \wedge *[F_{p-1} \wedge (dy + A)] = \frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}} \frac{1}{\phi^2} F_p \wedge \hat{*} F_{p-1}. \tag{4.42}$$

As F_p is a p-form and $\hat{*}F_{p-1}$ is a $(D-p+1)$-form, $F_p \wedge \hat{*}F_{p-1}$ is a $(D+1)$-form. Both F_p and $\hat{*}F_{p-1}$ live in a D dimensional space, so $F_p \wedge \hat{*}F_{p-1}$ too. Because of antisymmetry, any $(D+1)$-form on a D dimensional space is zero. Therefore we have:

$$F_p \wedge \hat{*}F_{p-1} = 0. \tag{4.43}$$

Finally look at $F_{p-1} \wedge (dy + A) \wedge *[F_{p-1} \wedge (dy + A)]$.

$$F_{p-1} \wedge (dy+A) \wedge *[F_{p-1} \wedge (dy+A)] = \frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}} \frac{1}{\phi^2} F_{p-1} \wedge (dy+A) \wedge \hat{*}F_{p-1}. \tag{4.44}$$

Now we split $F_{p-1} \wedge (dy + A) \wedge \hat{*}F_{p-1}$:

$$F_{p-1} \wedge (dy + A) \wedge \hat{*}F_{p-1} = F_{p-1} \wedge A \wedge \hat{*}F_{p-1} + F_{p-1} \wedge dy \wedge \hat{*}F_{p-1}. \tag{4.45}$$

For $F_{p-1} \wedge A \wedge \hat{*}F_{p-1}$ we use the same argument as before to say that this is zero. In total we get the following expression:

$$F'_p \wedge *F'_p = \frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}} (F_p \wedge \hat{*}F_p + \frac{1}{\phi^2} F_{p-1} \wedge \hat{*}F_{p-1}) \wedge dy. \tag{4.46}$$

Now we look at $\frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}}$. Here we have the following metric for the manifold of dimensionality $D+1$:

$$g_{D+1,MN} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}, \tag{4.47}$$

where μ and ν run over the first D dimensions. The metric for the manifold of dimensionality D is just:

$$g_{D,\mu\nu} = g_{\mu\nu}. \tag{4.48}$$

Now we see that $|g_{D+1}| = \phi^2 |g_D|$, so $\frac{\sqrt{|g_{D+1}|}}{\sqrt{|g_D|}} = \phi$.

$$F'_p \wedge *F'_p = \phi(F_p \wedge \hat{*}F_p + \frac{1}{\phi^2} F_{p-1} \wedge \hat{*}F_{p-1}) \wedge dy. \tag{4.49}$$

Now we can integrate over the extra dimension.

$$\begin{aligned} \frac{1}{2}\int_{M_{D+1}} F'_p \wedge *F'_p &= \frac{1}{2}\int_{M_D}\int_0^{2\pi} \phi(F_p \wedge \hat{*}F_p + \frac{1}{\phi^2} F_{p-1} \wedge \hat{*}F_{p-1}) \wedge dy \\ &= \pi \int_{M_D} \phi(F_p \wedge \hat{*}F_p + \frac{1}{\phi^2} F_{p-1} \wedge \hat{*}F_{p-1}) \end{aligned} \tag{4.50}$$

Now we consider again a four dimensional spacetime with an extra dimension, which is a circle. We thus have:

$$\frac{1}{2}\int_{M_5} F'_p \wedge *F'_p = \pi \int_{M_4} \phi(F_p \wedge \hat{*}F_p + \frac{1}{\phi^2}F_{p-1} \wedge \hat{*}F_{p-1}). \tag{4.51}$$

The left hand side of equation 4.51 is an action which is only dependent on one $(p-1)$-form C'_{p-1} on the five dimensional manifold. The right hand side of the equation, the action contains two such terms, but now on the four dimensional manifold. So out of one single $(p-1)$-form in five dimensions, we can create a theory which describes two different forms at once. Notice that both forms are related as both F_{p-1} and F_p are dependent on the $(p-2)$-form C^0_{p-2}. This means that F_{p-1} and F_p do not change independently. Also F_{p-1} and F_p interact with each other via the scalar field ϕ. This derivation is useful as it is used in string theory (7).

4.6 Hodge theorem and the action for a $(p-1)$-form gauge field

In this section we look at a closed p-form F'_p. In physics we are often interested in closed p-forms which vary under a transformation by an exact form. We are going to use Hodge theorem, that in this case describing a system represented by a closed p-forms, can equivalently be described by an exact p-form. First we want to remember the inner product:

$$(F'_p, F'_p) = \int_{M_{D+1}} F'_p \wedge *F'_p. \tag{4.52}$$

Consider again the action for a $(p-1)$-form gauge field:

$$S_{form} = \frac{1}{2}\int_{M_{D+1}} F'_p \wedge *F'_p. \tag{4.53}$$

To find the equations of motion, we have to minimize the action.
From Hodge decomposition theorem (theorem 2.26), we know that F'_p can be written locally as follows:

$$F'_p = dC + d^*\beta + \gamma, \tag{4.54}$$

with $C \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k+1}(M)$ and $\gamma \in Harm^k(M)$. So here dC is an exact form, $d^*\beta$ is a coexact form and γ is a harmonic form.
By construction we have that $dF'_p = ddC + dd^*\beta + d\gamma = dd^*\beta$. As F'_p is a closed form, $dF'_p = 0$. We thus have that:

$$(dF'_p, \beta) = (dd^*\beta, \beta) = (d^*\beta, d^*\beta) = 0. \tag{4.55}$$

The only solution for this equation is $d^*\beta = 0$.
As proven before, the harmonic form γ is perpendicular to the exact form dC, so $(dC, \gamma) = 0$. We can now write the inner product:

$$(F'_p, F'_p) = (dC, dC) + (\gamma, \gamma). \tag{4.56}$$

Notice that both (dC, dC) and (γ, γ) are positive. The action is minimized when both (dC, dC) and (γ, γ) separately are minimized. In total we get the following action:

$$S_{form} = \frac{1}{2}\int_{M_{D+1}} \gamma \wedge *\gamma + \frac{1}{2}\int_{M_{D+1}} dC \wedge *dC. \tag{4.57}$$

The action has variations in exact directions. As exact forms are perpendicular to the harmonic forms, a variation in the action will only influence (dC, dC). The harmonic form will not influence the equations of motion of the system. Therefore, if we want to find the equations of motion, it is sufficient to consider F'_p to be an exact form. The harmonic part of F'_p does not contribute to the equations of motion of the system. This is the reason that in section 4.5 it is sufficient to consider F'_p to be exact, so $F'_p = dC'_{p-1}$.

4.7 Perception of a five dimensional spacetime

In the reduction we considered a five dimensional spacetime. Even though we cannot perceive any extra circle, it is possible that this five dimensional spacetime exists and that it has a physical meaning. If we would live in a five dimensional spacetime, there has to be a reason why we do not perceive any extra dimension. Therefore, if we want to use a theory containing a higher dimension, we have to reason why we do not perceive extra dimensions. If that is not possible, the whole theory would just be a mathematical description without having the possibility to represent the real world.
It is hard to have an image in mind of a five dimensional manifold. Therefore we consider for now our normally four dimensional space-time to be two dimensional, where we have one spatial direction and the time direction. Now we add the extra circle on this two dimensional manifold. Now our space without the time coordinate looks like $\mathbb{R} \times S^1$, which is a cylinder (see figure 1). As the radius of the circle is very small, any observation of it is impossible. The objects we can measure are ways larger than the scale of the circle. We thus perceive the cylinder as a single line. Similarly we do not perceive the circle in the five dimensional spacetime.
We have also seen that all fields are constant with respect to the extra dimension. We are therefore not able to move on the extra dimension. This means that it is possible that our four dimensional spacetime contains an extra dimension which we cannot perceive. Adding an extra dimension is thus not in contrast with our perception of the world. In fact, any compactified Riemannian manifold, which is so small that we cannot perceive it, can be a part of our space.

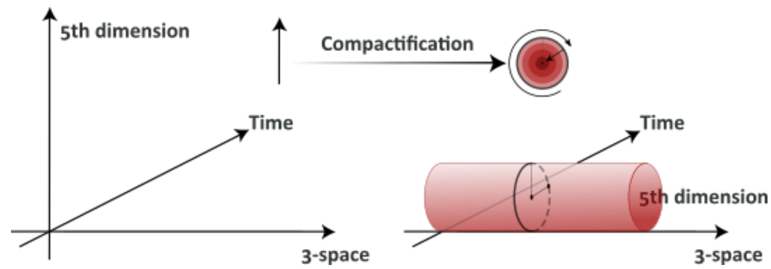

Figure 1: In this figure is visualized how the manifold becomes five dimensional by adding a circle. (25)

5 Beyond a five dimensional spacetime

In this thesis we only considered a reduction from a five dimensional manifold to a four dimensional manifold. As this fifth dimension is combined with the cylindrical condition, this extra dimension is equivalent with the addition of the circle group U(1). In the second half of the twentieth century, when the standard model was formulated, physicists described quantum electrodynamics (QED) as a gauge theory with the symmetry group U(1). In the standard model, the weak force and the strong force also contain symmetries. These symmetries are represented by the groups SU(2) and SU(3) respectively. By adding one extra circle dimension to the four dimensional spacetime, we could describe the electromagnetic field. What is missing in the result of the reduction is the description of charged particles. To add this in one theory, one must consider a higher dimensional spacetime. We also want to conclude the weak force and the strong force in a theory with gravity. Similarly to adding a circle, we must add a manifold in which at the very least concludes all the symmetries of the U(1), SU(2) and SU(3) groups, to be able to describe the standard model with geometrical arguments. In string theory, more assumptions are made to describe the particles in the standard model. This higher order dimensional manifold does not give easily a theory which concludes all wanted theories. In string theory, scientists try to do this unification, but they need more insights than just geometrical arguments. In the compactification in string theory, a similar technique is used as the Kaluza-Klein reduction (24). Studying the Kaluza-Klein reduction is thus of great importance for understanding unification of the standard model and gravity.

Conclusion

In this thesis we investigated the Kaluza-Klein reduction. We started this reduction with the five dimensional Einstein-Hilbert action and the five dimensional Einstein equations. Via the Christoffel symbols and the Riemann tensor we found the equations of motion in four dimensions. These equations are the massive Einstein equations and the homogeneous Maxwell equations under the assumption that the dilaton field is constant. This is a very important result, because with a single theory, both gravity and the electromagnetic field are described.

Further we reduced the dimensions of a manifold containing a circle for the action for a $(p-1)$-form gauge field. After the reduction, the action in the reduced manifold exists of two differential forms, which are dependent on each other. This method of dimensional reduction is used in string theory and is therefore very useful.

After getting to the result of Hodge decomposition theorem, we know that any differential form can be decomposed in an exact form, a coexact form and a harmonic form. Also we have shown that these differential forms are perpendicular to each other. Using this decomposition, we could prove Hodge theorem, which relates the set of harmonic forms with the de Rham cohomology group. We used the result of Hodge decomposition theorem to show that for a closed p-form, one can neglect the harmonic part of the p-form if we describe a p-form which varies under exact forms.

Discussion

We cannot check if a five dimensional spacetime really exists, because we can only do experiments in four dimensional spacetime. Therefore, we can only verify equations in four dimensional spacetime and not the equations in five dimensional spacetime. As we have seen, Kaluza and Klein have found a result in four dimensional equations. The equations they found for electromagnetism and gravity were already found before. The extra terms they found represent a dilaton field. If we could find this dilaton field in our experiments, the existence of extra dimensions would be supported. Also, as the gravity equations, and the homogeneous Maxwell equations are related to the field ϕ, this dilaton field gives a relation between the Maxwell and the Einstein equations. If we would find a dilaton field which corresponds to a relation between the Maxwell and the Einstein equations, which then must be measured, then the existence of an extra dimension would be really supported. Kaluza just assumed the dilaton field to be constant. In this case, the Maxwell and the Einstein equations do not have an extra interaction. In further research we should look for the dilaton field and investigate its impact on gravity and electromagnetism.
A next step in studying the Kaluza-Klein reduction is looking at how a reduction of higher dimension would be. As scientists are trying to unify the standard model and gravity by describing a manifold with multiple extra dimensions, this is a relevant question. We have only considered adding one dimension and in this case, the only compactified manifold which can be added to the four dimensional spacetime is topologically homeomorphic to a circle. If we look at a higher dimensionality, there are many topologically different possible compactified manifolds that we could add to the spacetime. A further study which investigates what possible compactified manifolds exist would be really interesting, as we need manifolds with a higher dimensionality to unify the standard model with gravity.
To understand better what kind of extra manifold we should add to spacetime to do this unification, we must have a closer look at the standard model. The relation between the electromagnetic field and the symmetry group U(1) is well explained in the standard model. Also, we could find an explanation how the weak force is related with the symmetry group SU(2) and how the strong force is related with the symmetry group SU(3). Knowing these relations makes it possible to know what extra manifold we should add to spacetime for the unification.
The Kaluza-Klein reduction gave us the homogeneous Maxwell equations. To describe electromagnetism fully, charges and currents have to be covered. If we want to unify the inhomogeneous Maxwell equations with gravity, we have to focus more on the standard model.
In this thesis we wrote the Fourier expansion for the scalar and tensor fields. Here we neglected fields with high mass. The masses were really high because we chose the radius of the circle to be very small. It might be interesting to look at the massive fields as they give physical results if the radius of the circle would be larger. Also in equation 4.30, we neglected all modes where $n \neq 0$.

These modes form a so-called tower of modes. We should further investigate the massive terms and have a look at the tower of modes.

The four dimensional equations of motion in section 4.2 are copied from the paper by Overduin et al. (8). In my attempt to produce the equations, I did not manage to get the same result. As the result by Overduin et al. is widely used and has survived for many years, I believe this result more than my result. Here are some reasons why I could not find the right result. I could have made a mistake in computing the Ricci tensors, by accidentally adding or missing a term. I missed an insight into how to write my results in terms of covariant derivatives instead of partial derivatives. An other possibility is that the result by Overduin et al. is only found by combining the results for R_{55}, $R_{\mu 5}$ and $R_{\mu\nu}$. I did not check this, because I assumed that I made a mistake. Also in the paper is written that the following equations follow directly from $R_{55} = 0$, $R_{\mu 5} = 0$ and $R_{\mu\nu} = 0$ respectively.

1. $\nabla^2 \phi = \frac{1}{4}\phi^3 F_{\mu\nu}F^{\mu\nu}$,
2. $\nabla^\mu F_{\mu\nu} = -3\frac{\partial^\mu \phi}{\phi}F_{\mu\nu}$,
3. and $G_{\mu\nu} = \frac{\phi^2}{2}T_{\mu\nu} - \frac{1}{\phi}[\nabla_\mu(\partial_\nu\phi) - g_{\mu\nu}\nabla^2\phi]$.

I have not derived the reduction of the five dimensional Einstein-Hilbert action to the four dimensional action myself. This is because I could not find the right equation for R_{55}. The term in the Ricci tensor R_{55} is needed to calculate the five dimensional Ricci scalar, which is part of the five dimensional Einstein-Hilbert action.

As the derivation of the Ricci tensors is not found in the literature, writing them fully would be interesting for other people who are studying the Kaluza-Klein reduction. Adding that derivation in this thesis would be a big addition.

Finally, in section 4.6 we only looked at closed p-forms which vary under transformation by an exact form. In physics we are often interested in these differential forms. We should motivate why especially these differential forms are interesting and why in physics a gauge field only varies by an exact form. Also it is interesting to find out what happens if one looks at an other p-form, which also contains non closed terms. We could also vary these terms by other forms than only the exact forms as done in this thesis.

Appendix

Calculation of the Christoffel symbols

Here we will calculate the Christoffel symbols corresponding to the five dimensional metric used by Kaluza:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}. \tag{5.1}$$

Remember that the Christoffel symbols are found via the following equation:

$$\Gamma^\rho_{\nu\mu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{5.2}$$

Now we will calculate all possible Christoffel symbols corresponding to the five dimensional metric. First we will calculate $\Gamma^\gamma_{\mu\nu}$, where the Greek letters stand for the first four dimensions.

$$\Gamma^\gamma_{\mu\nu} = (\Gamma^\gamma_{\mu\nu})_4 - \frac{1}{2} A^\gamma ((\partial_\mu (\phi^2 A_\nu) + \partial_\nu (\phi^2 A_\mu)). \tag{5.3}$$

The terms $(\Gamma^\gamma_{\mu\nu})_4$ are the four dimensional Christoffel symbols.

$$\Gamma^\gamma_{5\nu} = \frac{1}{2} g^{\gamma\delta} (\partial_\nu (\phi^2 A_\delta) - \partial_\delta (\phi^2 A_\nu)) - \frac{1}{2} A^\gamma \partial_\nu \phi^2. \tag{5.4}$$

$$\Gamma^\gamma_{55} = -\frac{1}{2} g^{\gamma\delta} \partial_\delta \phi^2. \tag{5.5}$$

$$\Gamma^5_{\mu\nu} = -\frac{1}{2} A^\delta (\partial_\mu g_{\nu\delta} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}) + g^{55} (\partial_\mu (\phi^2 A_\nu) + \partial_\nu (\phi^2 A_\mu)). \tag{5.6}$$

$$\Gamma^5_{5\nu} = -\frac{1}{2} A^\delta (\partial_\nu (A_\delta \phi^2) - \partial_\delta (A_\nu \phi^2)) + \frac{1}{2} g^{55} \partial_\nu \phi^2. \tag{5.7}$$

$$\Gamma^5_{55} = -\frac{1}{2} g^{5\delta} \partial_\delta g_{55} = \frac{1}{2} A^\delta \partial_\delta \phi^2 \tag{5.8}$$

Calculation of the Ricci tensor

As we found expressions for the Christoffel symbols, we can now find expressions for the Ricci tensor R_{MN}. Again we are going to find this tensor for the first four coordinates, denoted with Greek indices, and the fifth coordinate. The Ricci tensor is defined as

$$R_{MN} = R^S_{MSN} = \partial_S \Gamma^S_{NM} - \partial_N \Gamma^S_{SM} + \Gamma^S_{SL}\Gamma^L_{NM} - \Gamma^S_{NL}\Gamma^L_{SM}. \tag{5.9}$$

We start looking at R_{55}.

$$R_{55} = \partial_\gamma \Gamma^\gamma_{55} + \Gamma^\gamma_{\gamma\delta}\Gamma^\delta_{55} + \Gamma^\gamma_{\gamma 5}\Gamma^5_{55} - \Gamma^\gamma_{5\delta}\Gamma^\delta_{\gamma 5} - \Gamma^\gamma_{55}\Gamma^5_{\gamma 5}. \tag{5.10}$$

Look at all parts individually.

$$\partial_\gamma \Gamma^\gamma_{55} = -\frac{1}{2}\partial_\gamma \partial^\gamma (\phi^2). \tag{5.11}$$

$$\Gamma^\gamma_{\gamma 5}\Gamma^5_{55} = \frac{1}{4}A^\mu \partial_\mu (\phi^2)(g^{\gamma\delta}[\partial_\gamma (A_\delta \phi^2) - \partial_\delta (A_\gamma \phi^2)] - A^\gamma \partial_\gamma (\phi^2)) \tag{5.12}$$

As $g^{\gamma\delta}$ is symmetric and $\partial_\gamma (A_\delta \phi^2) - \partial_\delta (A_\gamma \phi^2)$ is antisymmetric, their product vanishes. Thus we have

$$\Gamma^\gamma_{\gamma 5}\Gamma^5_{55} = -\frac{1}{4}A^\mu \partial_\mu (\phi^2) A^\gamma \partial_\gamma (\phi^2). \tag{5.13}$$

$$\Gamma^\gamma_{5\delta}\Gamma^\delta_{\gamma 5} = [\frac{1}{2}g^{\gamma\nu}(\partial_\delta (\phi^2 A_\nu) - \partial_\nu (\phi^2 A_\delta)) - \frac{1}{2}A^\gamma \partial_\delta \phi^2][\frac{1}{2}g^{\delta\mu}(\partial_\gamma (\phi^2 A_\mu) - \partial_\mu (\phi^2 A_\gamma)) - \frac{1}{2}A^\delta \partial_\gamma \phi^2] \tag{5.14}$$

$$\Gamma^\gamma_{\gamma\delta}\Gamma^\delta_{55} = \frac{1}{2}\partial^\delta (\phi^2)(-\Gamma^\gamma_{\gamma\delta,4D} + \frac{1}{2}A^\gamma [\partial_\gamma (A_\delta \phi^2) + \partial_\delta (A_\gamma \phi^2)]). \tag{5.15}$$

Finally look at

$$\Gamma^\gamma_{55}\Gamma^5_{\gamma 5} = \frac{1}{4}\partial^\gamma (\phi^2) A^\delta [\partial_\gamma (A_\delta \phi^2) - \partial_\delta (A_\gamma \phi^2)] - \frac{1}{4}\partial^\gamma (\phi^2) g^{55} \partial_\gamma (\phi^2). \tag{5.16}$$

In total we get

$$\begin{aligned} R_{55} = &-\frac{1}{2}\partial_\gamma \partial^\gamma (\phi^2) - \frac{1}{2}\partial^\delta (\phi^2)(\Gamma^\gamma_{\gamma\delta})_4 + \frac{1}{4}\partial^\gamma (\phi^2) g^{55} \partial_\gamma (\phi^2) + \frac{1}{2}\partial^\gamma (A_\delta)\partial_\gamma (\phi^2) A^\delta \phi^2 \\ &- \frac{1}{2}\partial^\lambda (A_\rho)\partial^\rho (A_\lambda)\phi^4 + \frac{1}{2}\partial^\gamma (A_\delta)\partial_\gamma (A_\rho)\phi^4 g^{\delta\rho}. \end{aligned} \tag{5.17}$$

Notice that $\partial^\gamma (\phi^2)$ transforms as a vector and that its covariant derivative is as follows.

$$\nabla_\gamma \partial^\gamma (\phi^2) = \partial_\gamma \partial^\gamma (\phi^2) + (\Gamma^\gamma_{\gamma\delta})_4 \partial^\delta (\phi^2). \tag{5.18}$$

The covariant derivative of a scalar is the same as the partial derivative of a scalar, so

$$\begin{aligned} R_{55} = &-\frac{1}{2}\nabla_\gamma\nabla^\gamma(\phi^2) + \frac{1}{4}\partial^\gamma(\phi^2)g^{55}\partial_\gamma(\phi^2) + \frac{1}{2}\partial^\gamma(A_\delta)\partial_\gamma(\phi^2)A^\delta\phi^2 \\ &-\frac{1}{2}\partial^\lambda(A_\rho)\partial^\rho(A_\lambda)\phi^4 + \frac{1}{2}\partial^\gamma(A_\delta)\partial_\gamma(A_\rho)\phi^4 g^{\delta\rho}. \end{aligned} \tag{5.19}$$

We want to find a term $F_{\mu\nu}F^{\mu\nu}$ in R_{55}. Therefore we focus on $F_{\mu\nu}F^{\mu\nu}$.

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = 2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu. \tag{5.20}$$

I choose to express $F_{\mu\nu}F^{\mu\nu}$ in terms of partial derivatives and not in terms of covariant derivatives, because we have written R_{55} in terms of partial derivatives.
To continue we use the following two equations:

$$\partial_\gamma A_\rho \partial^\gamma A^\rho = \partial_\gamma A_\rho \partial^\gamma(A_\delta g^{\delta\rho}) = g^{\delta\rho}\partial_\gamma A_\rho \partial^\gamma A_\delta + A_\delta \partial_\gamma A_\rho \partial^\gamma(g^{\delta\rho}) \tag{5.21}$$

$$\partial_\gamma A_\rho \partial^\rho A^\gamma = \partial_\gamma A_\rho \partial^\rho(A_\delta g^{\gamma\delta}) = \partial^\delta A_\rho \partial^\rho(A_\delta) + A_\delta \partial_\gamma A_\rho \partial^\rho(g^{\gamma\delta}) \tag{5.22}$$

We now see that

$$\begin{aligned} -\frac{1}{2}\partial^\lambda(A_\rho)\partial^\rho(A_\lambda)\phi^4 + \frac{1}{2}\partial^\gamma(A_\delta)\partial_\gamma(A_\rho)\phi^4 g^{\delta\rho} =&\frac{1}{4}\phi^4 F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\phi^4 A_\delta \partial_\gamma A_\rho \partial^\gamma(g^{\delta\rho}) \\ &+ \frac{1}{2}\phi^4 A_\delta \partial_\gamma A_\rho \partial^\rho(g^{\gamma\delta}). \end{aligned} \tag{5.23}$$

Let us focus on $\nabla_\gamma\nabla^\gamma(\phi^2)$.

$$\frac{1}{2}\nabla_\gamma\nabla^\gamma\phi^2 = \partial_\gamma\phi\partial\gamma\phi + \phi\nabla_\gamma\nabla^\gamma\phi \tag{5.24}$$

and

$$\frac{1}{2}\partial^\gamma\phi^2 = \phi\partial^\gamma\phi. \tag{5.25}$$

We therefore have that

$$-\frac{1}{2}\nabla_\gamma\nabla^\gamma(\phi^2) + \frac{1}{4}\partial^\gamma(\phi^2)g^{55}\partial_\gamma(\phi^2) = -\phi\nabla_\gamma\nabla^\gamma\phi + \frac{1}{4}A^\delta A_\delta \partial^\gamma\phi^2\partial_\gamma\phi^2. \tag{5.26}$$

So

$$\begin{aligned} R_{55} = &-\phi\nabla^2\phi + \frac{1}{4}A^\delta A_\delta \partial^\gamma\phi^2\partial_\gamma\phi^2 + \frac{1}{4}\phi^4 F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\phi^4 A_\delta \partial_\gamma A_\rho \partial^\gamma(g^{\delta\rho}) \\ &+ \frac{1}{2}\phi^4 A_\delta \partial_\gamma A_\rho \partial^\rho(g^{\gamma\delta}) + \frac{1}{2}\partial^\gamma(A_\delta)\partial_\gamma(\phi^2)A^\delta\phi^2. \end{aligned} \tag{5.27}$$

I cannot find why it I do not get the following expression for R_{55}:

$$R_{55} = -\phi\nabla^2\phi + \frac{1}{4}\phi^4 F_{\mu\nu}F^{\mu\nu}. \tag{5.28}$$

As I could not find seemingly the right expression for R_{55}, I put a lot of effort to solve this equation. Therefore, I had no time to construct $R_{\mu 5}$ and $R_{\mu\nu}$.

References

[1] J.M. Lee *Introduction to smooth manifolds*, Second edition

[2] Sean M. Carroll (2019) *Spacetime and geometry*

[3] M. Nakahara (2003) *Geometry, topology and physics*, Second edition

[4] D. Lovelock and H. Rund (1989) *Tensors, differential forms and variational principles*

[5] D.E. Soper *Classical field theory*

[6] F.W. Warner *Foundations of differential manifolds and Lie groups*

[7] F. Bonetti (2014) *Effective actions for F-theory compactifications and tensor theories*
Link: `https://inspirehep.net/files/7e90b5184e5d41e543eef15261b0a2d2`, Page 56 to 60.

[8] J.M. Overduin and P.S. Wesson (2008). *Kaluza-Klein Gravity*
Link: `https://arxiv.org/pdf/gr-qc/9805018.pdf`, Page 12 to 15.

[9] C.V. Johnson *D-Branes*

[10] N. Nowaczyk (2010) *The Hodge Decomposition*
Link: `https://nikno.de/wp-content/uploads/2016/07/hodge.pdf`

[11] Th. Kaluza (translated in 2018)*On the Unification Problem in Physics*
Link: `https://arxiv.org/pdf/1803.08616.pdf`

[12] B. Lowe *The local theory of elliptic operators and the hodge theorem*
Link: `http://math.uchicago.edu/~may/REU2014/REUPapers/Lowe.pdf`

[13] M. Crainic *Mastermath course Differential Geometry 2015/2016*

[14] J. van Dongen (2000) *Einstein and the Kaluza-Klein particle*
Link: `https://arxiv.org/pdf/gr-qc/0009087.pdf`

[15] H. Linander (2011) *Free (2,0) Theory on a Circle Fibration*
Link: `https://publications.lib.chalmers.se/records/fulltext/179151/179151.pdf`

[16] L. Toussaint (2013) *The Hodge Decomposition Theorem*
Link: `https://webspace.science.uu.nl/~caval101/homepage/Students_files/ToussaintBachelor.pdf`

[17] M. Vonk (2019) *Topology and Physics 2019 - lecture 2*
Link: `https://staff.fnwi.uva.nl/h.b.posthuma/TIP2019/Lecture%202-2019.pdf`

[18] V. G. Ivancevic and T. T. Ivancevic (2011) *Undergraduate Lecture Notes in De Rham-Hodge Theory*
Link: https://arxiv.org/abs/0807.4991

[19] G. 't Hooft (2010) *Introduction to general relativity*
Link: https://webspace.science.uu.nl/~hooft101/lectures/genrel_2010.pdf

[20] D. Lowengrub (2014) *Hodge Decomposition*
Link: https://math.berkeley.edu/~lowdanie/hodge_decomposition.pdf

[21] A. M. Patricio *5D Kaluza-Klein theories - a brief review*

[22] C. Pope *Lectures on Kaluza-Klein*
Link: http://people.tamu.edu/~c-pope/ihplec.pdf

[23] https://link.springer.com/content/pdf/bbm%3A978-3-319-49682-5%2F1.pdf

[24] D. Tong *String Theory* http://www.damtp.cam.ac.uk/user/tong/string/string.pdf

[25] Image 1 https://www.quantum-bits.org/?p=1078