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Compass and Straightedge Constructions in the Hyperbolic Plane

Bachelor Thesis

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Abstract

At the dawn of mathematics, the straightedge and compass were important tools for classical geometers. A prime example are The Elements of Euclid, which cover a substantial portion of Greek geometry and has been a source of study for generations to come. Up until the 19th century, the geometrical world shaped by Euclid has been the sole focus of the field, when by a slight adjustment in one of the postulates an entirely new theory was born.

This thesis will take the classical approach towards the hyperbolic plane, as we explore straightedge and compass constructions in this space. At last, this inquiry will lead us to necessary and sufficient conditions for line segments and angles to be constructible. As a consequence, it will be demonstrated that some squares and circles of equal area are constructible in the Hyperbolic plane, whereas this is a notorious impossibility in Euclidean space, and precise conditions will be provided.

Furthermore, the straightedge and compass are studied shortly in the Euclidean plane, as well as Hilbert's axioms, which offer a rigorous foundation to synthetic geometry.

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1 Introduction

The main aim of this paper will be to offer a glimpse of straightedge and compass constructions in hyperbolic space. For this, we start with an overview of Euclidean geometry, where we give examples of reasoning and constructions from Euclid's elements. Next, Hilbert's axioms are introduced to make Euclid's theory rigorous and serve as a foundation for all geometry, Euclidean and hyperbolic. In this framework, an additional axiom presented in chapter 4 will show one could develop a synthetic theory of hyperbolic geometry. Further on, Poincaré's model of the upper half-plane will be presented in order to offer more tools from analytic geometry. With these tools it will be shown next what the trigonometry in this space entails, which on its turn is of special use for the elementary constructions we present in chapter 7.

Finally, the two major results of this thesis will be presented in subsequent chapters 8 and 9. In this first chapter a full proof is given that can transform any hyperbolic construction problem to an algebraic one. This theorem has been discovered by mathematician Mordukhai-Boltovskoi and gives a direct correspondence between constructibility in the Euclidean and hyperbolic spaces. Last but not least, we show that some circles and squares with equal area are constructible in the hyperbolic plane, as well as give precise conditions for when they are.

2 Euclidean Geometry

Before diving deep into the hyperbolic theory, this chapter will offer a brief overview of important notions in Euclidean geometry. First off, a description of Euclid's elements, a leading and historic text on this subject, will be given. In particular, we will explore whether Euclid's work holds up to modern standards of rigor. Next, a brief history of Euclid's fifth postulate will be presented and we will see how this has led to the development of a whole new geometry, independent of Euclid's. Furthermore, we consider compass and straightedge constructions in the plane, which were important tools for classical geometers. Here, we also give four construction problems left unsolved by Euclid and his colleagues and explain how later developments have answered these issues. At last, the construction game will be given an algebraic meaning in the final paragraph of this chapter. With this new tool box, some of the constructions of paragraph 2.3 will be further analyzed.

The purpose of this chapter will not be to discuss the aforementioned topics in detail. Instead, we seek to gain a minimal framework on which we can discuss the more interesting questions later on. For example, chapter 3 will show how we can give a rigorous development of Euclid's approach to geometry. Then, in chapter 4 a geometric theory, based on letting the fifth postulate go, will be introduced. Further on, from chapter 7 onward, the subjects of paragraphs 2.2 and 2.3 are combined, as we employ the classical tools of the straightedge and compass onto the more modern theory of hyperbolic geometry. In particular, chapter 9 shows how one of the classical construction problems has a partial solution in the hyperbolic plane.

2.1 Euclid's Elements

Today, studies of geometry often include a coordinate system. Metrics are introduced to induce a notion of distance and vectors might give a sense of direction. For instance, the standard model of the Euclidean plane is the Cartesian plane $\mathbb{R}^2$. With coordinates, geometric problems can be computed in a relatively simple fashion. This approach to geometry is often referred to as analytic. The classical geometers, however, used a different approach called synthetic geometry. Instead of working in a coordinate system, this approach is purely axiomatic; a few properties of the lines in the space were assumed and from this, all propositions had to be logically deduced.

Euclid's elements was the leading text in classical geometry. Although he did not discover many of the results presented himself, Euclid systematized the abundant propositions and molded it into one continuous theory. In his synthetic approach, he did not use a metric and radian measure but spoke of equal line segments and angles. Moreover, the contents of Euclid's books should be familiar to any reader of this text; some examples of propositions found in Euclid concern the existence of equilateral triangles, the bisection of angles, the sum of angles of any triangle and the famous Pythagorean theorem.

Euclid starts out with a list of definitions and follows up with his postulates and common notions. Today, the definitions we pick make use of the undefined notions given by the axioms. As a consequence, the definitions Euclid gives might seem imprecise to the modern reader. For instance, he introduces a point as follows:

> A point is that which has no part. [1, p. 153]

As many have noted before [1, p. 156], there are plenty of ideas that are not divisible in parts but which we do not consider to be points. To address this issue, pre-Euclidean points were defined to be indivisible objects having a location. Nonetheless, this shifts the problem to what exactly it means to have a location or to be indivisible. Nowadays, we elapse this issue by presenting a set of well-defined properties, which we call axioms, and postulating that there exists some objects which comply to these rules. Euclid's postulates and today's axioms are thus similar in the sense that they both represent assumptions, but nowadays we would say that Euclid left too many assumptions implicit.

This distinction accounts to some what would we now consider logical discrepancies in Euclid's work. There are plenty of silent assumptions which Euclid uses but does not name in his postulates. Furthermore, there have been countless critiques on the Elements from its first debut 2500 years ago. Some of these we will explore further when we introduce David Hilbert's axiomatic system, a modern approach to lay the foundations of geometry.

2.2 The Fifth Postulate

One particular issue early readers of the elements had was the formulation of the last postulate. Of the five postulates that Euclid proposed, this one stood out above the rest. For example, Euclid postulated the existence of a line between any pair of points, the existence of a circle for any center and any distance and that all right angles would be of the same size. The fifth postulate, on the other hand, was formulated as follows:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. [1, p.155]

Basically, two lines with any kind of inclination towards each will have to meet eventually. When drawing straight lines on a piece of paper, as long as the piece of paper is big enough and the inclination between the lines is not too negligible, this proposition seems to be reasonable. After all, it is evident that any straight lines on a piece of paper will either move closer together as you extend them in the correct direction, or they would have a constant distance between them. The issue felt by many, however, was not that the fifth postulate was false, but that it was too complex to be considered a postulate. Instead, many reasoned that it ought to be a theorem and supplied their own proofs.

Attempts at proving the fifth postulate began almost as soon as it was raised, but time and again they were shown to be flawed. For instance, Proclus, a Greek philosopher living some generations after Euclid, failed to see that his attempted proof assumed that parallel lines are necessarily equidistant [2, p. 297]. Following Proclus, there were attempts made, among others, by John Wallis, Saccheri, Lambert and Legendre. The interested reader might consult [1, p. 202-217] for their work. Often, the proposed proofs introduced a new axiom and/or as in Proclus' case, relied too much on intuition.

The fifth postulate was embedded so deep into our intuition that it took mathematicians until the 19th century to realize that a world without the parallel postulate would be conceivable. Independently, this realization was made by Carl Friedrich Gauss (1777-1855), Janos Bolyai (1802-1860) and Nicolai Ivanovich Lobachevsky (1793-1856), and it led to the discovery of so-called non-Euclidean geometries. In a letter to his father, Bolyai exclaimed: "Out of nothing I have created a strange new universe" [3, p.98] .

The theory found by Bolyai, Gauss and Lobachevsky became to be known as hyperbolic or Lobachevskian geometry. It tells of a world devoid of the so familiar property that two inclined lines will eventually meet. Consequently, ample results we often take for granted do not hold in this "stange new universe", as Bolyai put it. For example, the triangle's sum of angles is no longer constant nor does the Pythagorean theorem hold.

2.3 Compass and Straightedge Constructions

Euclid's very first proposition is to construct an equilateral triangle on any given finite line segment [1, p. 241]. By 'construct' he means to draw with a straightedge and a compass on a flat surface. Each step, one may only draw a straight line with the straightedge between two points which have been previously obtained, or draw a circle with the compass with an already obtained center and which passes another point already drawn. Points are obtained when they are the intersection of circles and lines that have been drawn. Alternatively, we can choose a random point to lay the compass or straightedge on, as long as the construction does not depend on that choice. For example, a point lying on a certain line could be needed for a specific construction, but it doesn't matter which point on the line we choose. In that case, we count the construction as valid too. Finally, we allow only finite steps using the straightedge and compass.

Euclid also implemented the compass and straightedge in his postulates. The first postulate allows a straight line to be drawn between any pair of points while the third allows drawing a circle with given center and radius.

Let us return to the first proposition, the construction of an equilateral triangle given on a finite line segment. From the proof, we can deduce that to be given a line segment means that we are allowed to use its end points to draw straight lines and circles from. The construction then goes as follows. Let AB be the given line segment such that A and B are the end points of the segment. By postulate 3, we can then draw a circle α with center A passing B, and conversely a circle β with center B passing A . Here, Euclid misses a subtle point. Without proof does he claim that the circles α and β intersect. Though this might seem obvious from a figure, none of the postulates nor other results from Euclid assures the circles in this case intersect. Later on, we will see how Hilbert solves this problem. For now, we assume the circles meet in a point C. By definition, a circle is a figure equidistant from its center and so, the sides AC and BC are both equal to AB. Now, Euclid mentions a different assumption he has listed as a 'common notion', which he distinguished from the postulates. As two things that are equal to the same thing are also equal to one another [1], he writes, the segments AC and BC are therefore also equal. Hence, the triangle with vertices ABC is equilateral, as required.

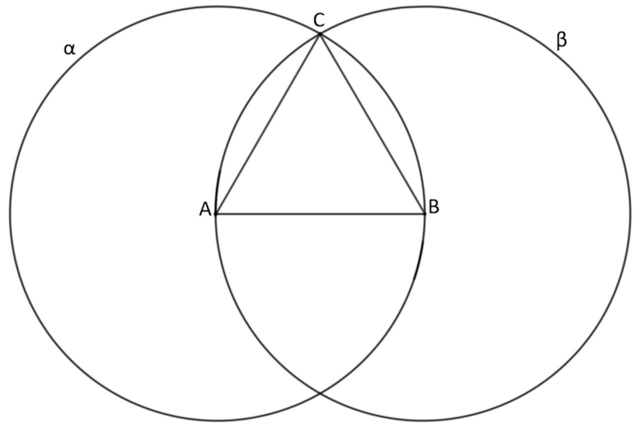


Figure 1: Euclid's construction of an equilateral triangle. Although the figure suggests the circles meet in a point C, this cannot be deduced from the postulates.

As such, the first proposition of the elements give a construction for an equilateral triangle. Naturally, one could ask whether it is possible to construct any regular polygon. It turns out that the construction of a square and a regular hexagon are as trivial as the construction of the triangle. Furthermore, the construction of the regular pentagon is presented as far back as proposition 11 of book 4 and requires all the results developed thus far. Besides these, there were not many constructible regular polygons known to Euclid. Hence, it was known in antiquity how to construct regular polygons of sides 3, 4, 5 and 6 but not of 7. The first major advancement made since antiquity was by Gauss who gave the construction of a regular polygon of 17 sides: a heptadecagon. Moreover, Gauss gave sufficient and necessary conditions for a regular polygon to be constructible, showing for instance that the construction of a heptagon is not possible.

Other famous construction problems seem to arise from similar scaling issues. As regular polygons are constructible with 3,4,5 and 6 sides but not with 7, so too it is possible to bisect an angle, producing two equally sized angles which add up to the original, but not to trisect it. Furthermore, it is possible to construct a square with compass and straightedge twice the area of a given square, but it is impossible to double a cube. The Greek geometers, however, did not possess proofs of the infeasibility of these tasks.

A final problem conceived since antiquity asked whether, given any circle, one can construct a square with equal area. To square a circle is sometimes used to mean to do anything considered impossible. In the Euclidean plane, this indeed is an insurmountable task due to the fact that π is a transcendental number. Nonetheless, Janos Bolyai, by means of his newly found geometry, found a way to do the seemingly impossible, as we will see in chapter 9.

[1] This common notion can be interpreted as the transitivity property of a congruence relation.

2.4 Algebraic Interpretation of Euclidean Constructions

A game changer for straightedge and compass constructions was the introduction of a coordinate system. René Descartes was the first who noticed that by studying constructions in the Cartesian plane, these problems could be expressed algebraically, effectively offering another tool in tackling these problems [2, p. 120]. Since lines and circles over the Cartesian plane are expressed by linear and quadratic equations, he argued that the coordinates of any constructible point must be the solution of a finite set of linear and quadratic equations. In other words, given some points $(1,0),(0,0),(p_1,q_1),...,(p_n,q_n)$ in the Cartesian plane, we can construct a point (x,y) with straightedge and compass if x and y can be expressed in terms of $p_1,q_1,...,p_n,q_n$ by finite operations of $+,-,\cdot,\div,\sqrt{\cdot}$. Moreover, it can be shown that the reverse is also true. So, the coordinates, of any point (x,y) constructible from $(p_1,q_1),...,(p_n,q_n)$, can be expressed in these finite operations. For a proof, see [2, Thm. 13.2]. As a consequence, there is the following theorem.

Theorem 2.4.1. *If we start with two random points and label them as* $(0,0)$ *and* $(1,0)$*, then any point* (x,y) *is constructible if and only if* x,y *lie in the Euclidean field* E*. This field is the smallest field containing* $\mathbb{Q}$ *and that is closed under taking square roots of positive elements.*

In terms of finite field extensions, the elements of E can be characterized as follows. A real $x\in\mathbb{R}$ lies in E if and only if there exist subfields

$$\mathbb{Q}=F_0\subset F_1\subset ...\subset F_k\subset\mathbb{R}$$

such that $x\in F_k$ and the field extension F_i/F_{i-1} is of degree 2 for all $i=1,2,..k$ [2, Prop 28.1].

Usually, we are not interested in constructing particular points. For example, within the construction of a regular triangle, it did not concern us what coordinates the vertices had. Instead, all that mattered was the congruence of the three sides. Therefore, we would like to define what it means for segment lengths and angles to be constructible.

Definition 2.4.2 (Constructible lengths and angles)**.** We call any real $x\in\mathbb{R}$ a constructible length if there exist two constructible points A,B such that the Euclidean distance between these points is $|x|$. In particular, we call 0 constructible and $-x$ is constructible if there exist points with Euclidean distance $|x|$. Furthermore, any real $\alpha\in\mathbb{R}$ is called a constructible angle if $\alpha=0$ or there exist three points A,B,C such that the angle $\angle ABC$ has radian measure α modulo π.

By Euclid's second proposition [1, p. 244], we can construct a line segment starting from a given point and congruent to another given line segment. Therefore, if x is a constructible length, and by the fact that we can use the points $(0,0)$ and $(1,0)$ in our constructions, it is possible to construct the point $(x,0)$. Hence, a real x is said to be constructible if and only if $x\in E$.

Moreover, by Euclid's twenty-third proposition [1, p. 294], it is possible to construct an angle with vertex $(0,0)$ and with one arm as the horizontal axis, which is congruent to a given angle. So, if α is a constructible angle, we can draw a point A such that the angle made by the line through A and the origin $O=(0,0)$ makes an angle of α radians with the horizontal axis. Moreover, line OA will intersect the circle of radius 1 and center O, which we will assume to be in the point A. Dropping the perpendicular from A onto the horizontal axis, we gain a point B as their intersection. The lengths of segments $\overline{OB},\overline{AB}$ are given by $\cos\alpha,\sin\alpha$ respectively. Hence, any angle α is constructible if and only if $\cos\alpha$ and $\sin\alpha$ are constructible lengths.

As such, it could now be shown, using the theory of finite field extensions, that it is not possible to trisect every angle, double a cube and square a circle. These results are fairly well-known and one place to find them is [2, §28]. Furthermore, a regular polygon of n sides happens to be constructible if and only if n can be factorized as

$$n=2^k\cdot F_1\cdots F_m$$

where $F_1,...,F_m$ are Fermat primes. These are prime numbers which can be written as $2^{2^N}-1$ for some $N\in\mathbb{N}$. See also [2, Thm 29.4] for a proof.

To summarize, straightedge and compass constructions solve quadratic equations over the Cartesian plane. Therefore, any point is constructible if and only if its coordinates exist in the Euclidean field. Moreover, the Euclidean field also contains all constructible lengths and the sines and cosines of the constructible angles. With this, one can show that trisecting an angle, doubling a cube and squaring a circle are impossible by straightedge and compass in the Euclidean plane. Moreover, we have seen necessary and satisfactory conditions for a regular polygon to be constructible. In chapter 8 we will see how straightedge and compass constructions can be characterized algebraically in the Hyperbolic plane. Furthermore, in chapters 8 and 9 we will explore the four classical construction problems introduced here in the hyperbolic plane.

3 Hilbert's Axioms

As demonstrated in chapter 2, although Euclid's work has been a staple of classical mathematics, it does not hold up to today's standards of rigor. In order to form a rigorous foundation of geometry, David Hilbert proposed a new set of axioms, comparable to Euclid's postulates [4]. In this chapter, we will introduce an altered version of Hilbert's axioms as can be found in [2]. The main difference between Hilbert's axioms and the ones presented here being our restriction to planar geometry, while Hilbert's theory is based on three-dimensional space.

The major axioms presented will be split into four different groups. The first of which describes the relationship of points lying on a line: the axioms of incidence. The axioms of betweenness describe the relationship of a point lying between any other two. With this we can, for example, define finite line segments, angles and what it means for a point to lie in- or outside a given figure. The third group, the axioms of segment congruence, tells of finite line segments having equal length. Finally, the axioms of angle congruence give us a notion of equal angles.

These four groups form the basis of Hilbert's theory and the study of them is called *neutral geometry*, while the models satisfying this axiomatic system are named *Hilbert planes*. With this foundation, additional axioms can be introduced, such as an axiom concerning circle intersection in paragraph. 3.5. The motivation for this axiom comes from the logical gap in Euclid's first construction, as we have explored in paragraph 2.3. At last, Playfair's axiom will be shortly discussed, which in the context of Hilbert planes is equivalent to the parallel postulate.

3.1 Axioms of Incidence

To our attention comes a collection Π, which we call the plane, of which its elements we call *points* and some of its subsets *lines*. Which subsets we call lines and which we don't, is of no importance as long as it satisfies the axioms which we will shortly introduce. If a point P is contained in a line l, we say that P lies on l or that l passes P. Furthermore, we envision that the plane Π along with the collection of lines satisfies the following properties [2, §6]:

(I1) For any two distinct points $A, B \in \Pi$, there exists a unique line l such that $A, B \in l$.

(I2) Every line contains at least two points.

(I3) There exist three noncollinear points. That is, there exist three distinct point $A, B, C \in \Pi$ such that there is no one line containing all three.

The first axiom can be interpreted as an extension of Euclid's first postulate: "to draw a straight line from any point to any point" [1, p. 154]. The other two axioms, though silently assumed in his proofs, are not listed as postulates by Euclid. The fact that it is necessary to include also these axioms comes from the fact that I1-I3 are independent. That is, there exist models satisfying all but one of the axioms and hence, any particular one does not follow from the other two.

Furthermore, because of the uniqueness part of (I1) and axiom (I2), we can denote any line by two of the points it passes through. Therefore, given two points A, B, we shall henceforth denote the line passing A, B by AB. This notation is well-defined, since by (I1) there is only one line which passes both A as well as B, and for each line l we can find by (I2) two points A, B such that $l = AB$.

Finally, note that by the uniqueness part of axiom (I1) it is impossible for two lines to intersect in two or more points. So, each pair of lines intersects either once or not at all. This immediately rules out the geometry over a sphere centered at the origin of $\mathbb{R}^3$, where the lines are defined as the intersection of planes through the origin and the sphere. For instance, the planes determined by the equations $x = 0$ and $y = 0$ both meet in the north and south pole.

3.2 Axioms of Betweenness

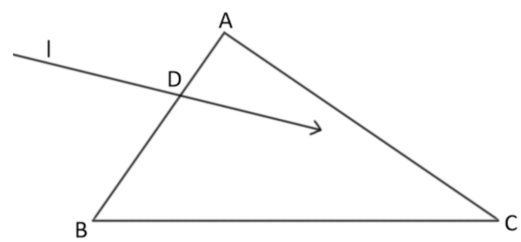
In order to develop the notions of finite line segments and separation we postulate a ternary relation on the set of points Π. For any three points $A, B, C \in \Pi$ we write this relation as $A * B * C$, or say that B lies in between A and C. Moreover, we require the relation of betweenness to satisfy following set of axioms [2, §7]:

(B1) If $A * B * C$, then A, B, C are three distinct points lying on the same line and also, $C * B * A$.

(B2) For any two distinct points A, B, there exists a point C such that $A * B * C$.

(B3) Given three distinct points on a line, one and only one of them is between the other two.

(B4) Let A, B, C be three noncollinear points and let l be a line containing none of them. If l contains a point D such that $A * D * B$, then there also exists a point $E \in l$ such that $B * E * C$ or $A * E * C$. Moreover, if l contains a point in between B and C, then it does not have a point in between A and C, and vice versa.



The fourth axiom is sometimes also referred to as Pasch' axiom and although its formulation is somewhat complicated, it can be easily visualized in terms of a triangle. Namely, Pasch' axiom assures that any line l passing a triangle ABC will meet precisely two of its sides, at least when l does not contain the triangle's vertices A, B, C. In fact, we can make this rigorous by defining finite line segments and subsequently triangles as follows.

Definition 3.2.1 (Segments and Triangles)**.** For a pair of distinct points A, B we define the *line segment* $\overline{AB}$ to be the collection of all points lying between A and B, including A, B themselves. Moreover, we define a *triangle* to be the union of three line segments $\overline{AB}, \overline{BC}, \overline{AC}$, whenever A, B, C are noncollinear. Naturally, the points A, B, C are the *vertices* of a triangle and the segments $\overline{AB}, \overline{BC}, \overline{AC}$ are the *sides*. Finally, a triangle with vertices A, B, C is denoted by ABC.

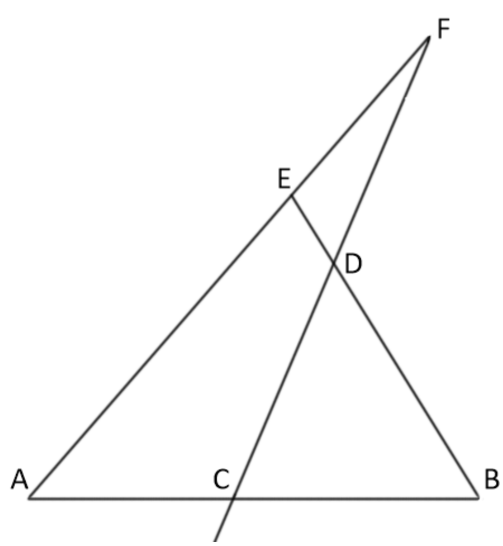
Remark 3.2.2. A brief point on notation. If it does not lead to confusion, we might write AB as a shorthand for $\overline{AB}$.

In order for the notion of a segment to correspond with our intuition, it would be necessary that it contains an infinity of points between the extremities. This follows from the following result:

Lemma 3.2.3. *For any pair of distinct points A, B, there exists a point C in between A, B.*

Proof.
By axiom (I3), there exists a point D not lying on the line AB, which exists by (I1). Moreover, by (B2) we find points E, F such that $B * D * E$ and $A * E * F$. Also, the line DF exists by (I1) and I claim that this line intersects the segment AB.

Note that by $(B1)$, the points B, D, E are distinct and lie on a single line. Moreover, by construction we know that D does not lie on the line AB. So, the line passing B, D, E and the line AB are distinct. By (I1), these two lines meet in at most one point, which happens to be B. In particular, $E \notin AB$ and hence, the points A, B, E are noncollinear. In order to use Pasch' axiom (B4) on the triangle ABE, it remains to prove that the line DF does not contain any of these points.



Because $A * E * F$, we know that these are three distinct and collinear points. If B were to lie on DF, then $BD = DF$ by (I1). Since E also lies on this line, it follows by the same axiom that $BE = EF = AE$. However, we have already shown A, B, E to be noncollinear. Hence, DF does not pass B. Were E to lie on DF, then again by (I1) we have that $DE = DF$ and since B also lies on this line, it follows that $BE = EF$. However, EF also passes A and so, E cannot lie on DF either. Finally, were the line DF to pass A, then again it would be implied that a single line passes A, B, D, E, F. We conclude that DF does not pass the points A, B, E.

By Pasch' axiom (B4), the line DF will meet either $\overline{AB}$ or $\overline{AE}$. Since the lines AE, DF already meet in the point F, they cannot meet in another point by (I1). Therefore, there exists a point C on DF such that $A * C * B$. $\square$

An important notion following from the axioms of betweenness is that of separation. Imagine drawing a straight line on a piece of paper. Assuming the line continues in both directions indefinitely, the paper gets split into two. Moreover, it is impossible to start from a point on any side of the line and continuously move to a point on the other side, without passing the straight line. Using the notion of betweenness, we can make this more rigorous by defining a relation $\sim_l$ for each line l as follows. For any pair of points A, B not lying on l, we say that they lie on the same side of l, and write $A \sim_l B$, if they are the same point, or if the segment $\overline{AB}$ does not intersect the line l. For a proof of the following theorem, see [2, Prop 7.1].

Theorem 3.2.4 (Plane Separation)**.** *The relation $\sim_l$ is an equivalence relation for each line l on the set of points $\Pi \setminus l$ and it has precisely two equivalence classes.*

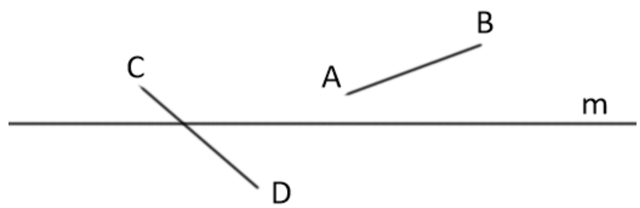


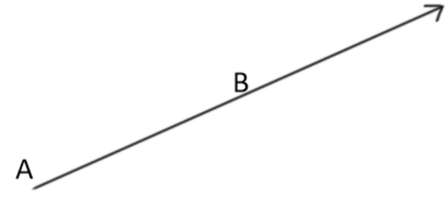
Figure 2: Example of plane separation. Points A, B lie on the same side of line m, while C, D don't.

From now on, we will refer to the equivalence classes of $\sim_l$ as the two sides of l. A direct corollary is the separation of lines. For each point P on a line l, we can consider any other line m through P not equal to l. Note that this line m exists by axiom (I3): there exist three noncollinear points. We now define the relation $\sim_{P,l}$ to be the restriction of $\sim_m$ on the set $l \setminus \{P\}$. This definition of $\sim_{P,l}$ is independent of the choice of a line m through P and from the previous theorem, the following is a direct consequence. Again, for a proof see [2, Prop 7.2].

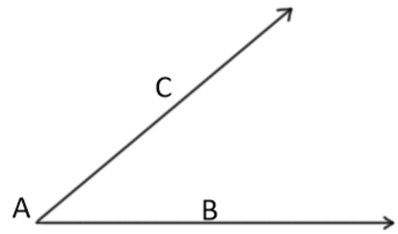
Corollary 3.2.5 (Line Separation)**.** *For each line l and point P on it, the relation $\sim_{P,l}$ is an equivalence relation on $l \setminus \{P\}$ with two equivalence classes.*

To demonstrate the usefulness of plane and line separation, we define a number of important geometric objects.

Definition 3.2.6 (Rays)**.** For any two distinct points A, B, we separate the line AB by the point A and define the *ray* $\overrightarrow{AB}$ to be the set of points on AB lying on the same side as B, including A. Alternatively, we sometimes write Aa for a ray starting at a point A.



Definition 3.2.7 (Angles)**.** An *angle* $\angle BAC$ is the union of two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ which do not lie on the same line. In particular, there do not exist "straight angles" and "zero angles".



Definition 3.2.8 (Interior)**.** The *interior* of an angle $\angle BAC$ is the intersection of the points lying on the same side of AB as C and the points lying on the same side of AC as B. For a triangle ABC, the *interior* is the intersection of the interior of the angles $\angle ABC, \angle BAC, \angle ACB$.

With this terminology, we can extend the result of Pasch' axiom to also include lines which intersect the vertex of a triangle.

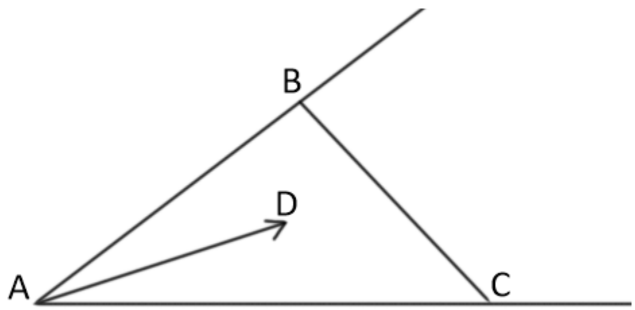
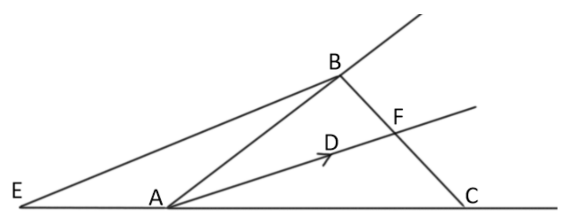


Figure 3: Example of the crossbar theorem.

Theorem 3.2.9 (Crossbar Theorem)**.** *Let $\angle BAC$ be an angle with an interior point D. Then, the ray $\overrightarrow{AD}$ will intersect the segment $\overline{BC}$.*

Proof.

By axiom (B2), there exists a point E such that $C * A * E$. The line AD meets one side of the triangle BCE in the point A. Moreover, since D lies in the interior of $\angle BAC$, we know in particular that D does not lie on the lines AB and AC. Therefore, the lines AB, AC, AD are three distinct lines and since AD already meets the other two lines in the point A, it cannot meet them in another point (I1). Hence, the points B, C, E do not lie on the line AD, which means that we can use Pasch' axiom.



By Pasch' axiom, it follows that line AD will have to meet one other side of the triangle BCE. Since $E*A*C$, the points E, C lie on different sides from the line AD. Furthermore, because D lies in the interior of $\angle BAC$, we also know that B, C lie on different sides of the line AD. So, the points B, E lie on the same side and therefore, the line AD cannot meet the segment $\overline{BE}$. This implies that AD will have to meet the segment BC in a point F. All that remains is to show that $F \in \overrightarrow{AD}$.

This last result follows from the fact that the points B, D lie on the same side of AC, since D is an interior point of $\angle BAC$. Because $B * F * C$, we thus know that the points B, D, F will all have to lie on the same side of AD. In particular, $F \in \overrightarrow{AD}$. $\square$

3.3 Axioms of Segment Congruence

Throughout the elements, Euclid does not use a measure for the length of segments and angles. Instead, he talks about equal segments and angles. To make this approach rigorous, Hilbert introduced a relation $\cong$ on the set of all line segments and gave it the following axioms [2, §8]:

(C1) Given any line segment $\overline{AB}$ and a ray Cr, there exists a unique point D on the ray Cr such that $\overline{AB} \cong \overline{CD}$.

(C2) The congruence relation of line segments $\cong$ is an equivalence relation.

(C3) Let A, B, C, D, E, F be some points satisfying $A * B * C$ and $D * E * F$. If $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

These axioms can be found in Euclid in some shape or form. For instance, axiom C1 happens to be very similar to be proposition 2 of Euclid: "to place at a given point a straight line equal to a given straight line" [1, p. 244]. Furthermore, axiom C2 in particular implies the transitivity and reflexivity property of congruence. These correspond to common notion 1: "things which are equal to the same thing are also equal to one another"; and common notion 4: "things which coincide with one another are equal to one another" [1, p. 155]. Last of all, axiom C3 can be interpreted as common notion 2 of Euclid, which states that if equal things are added, the sums stay equal. In order to further develop this interpretation, we introduce the following operation on line segments.

Definition 3.3.1 (Addition of segments)**.** Given two points A, B, we write $|AB|$ for the congruence class of the segment $\overline{AB}$. Now consider two congruence classes $|AB|, |CD|$. We choose the unique point E, whose existence is ensured by axiom (C1), on the line AB and on the opposite side of B as A, such that $\overline{BE} \cong \overline{CD}$. The congruence class of the segment $\overline{AE}$ is then called the *sum* of $|AB|$ and $|CD|$ and we write $|AB| + |CD| = |AE|$.

Of course, this definition relies on a choice. The following theorem proves that the operation is independent of this choice, the proof of which mainly relies on axiom C3.

Theorem 3.3.2. *The operation of addition + on congruence classes of line segments is well-defined.*

Proof. There are two choices made which could be of influence. First, there is the choice of a segments $\overline{AB}, \overline{CD}$ in given congruence classes $|AB|, |CD|$. Second, because $\overline{AB} = \overline{BA}$, it shouldn't matter if we choose the point E on the other side of B as A, or on the other side of A as B. In order to prove that addition is well-defined, it is required to show that it is independent of these choices.

We first consider the second choice. Let E be the point on the other side of B as A and E' the point on the other side of A as B such that $\overline{BE} \cong \overline{AE'} \cong CD$. In particular, we have that $A * B * E$, $B * A * E'$, $\overline{AB} \cong \overline{BA}$ and $\overline{BE} \cong \overline{AE'}$. Therefore, it follows from axiom (C3) that $\overline{BE'} \cong \overline{AE}$.

Next, we prove that our construction is independent of the segment that represents the equivalence class. For this, consider two pairs of segments $\overline{AB}, \overline{A'B'}$ and $\overline{CD}, \overline{C'D'}$ such that $\overline{AB} \cong \overline{A'B'}$ and $\overline{CD} \cong \overline{C'D'}$. Also let E, E' be the unique points such that $A * B * E$, $A' * B' * E'$, $\overline{BE} \cong \overline{CD}$ and $\overline{B'E'} \cong \overline{C'D'}$. From axiom (C3) it thus follows that $\overline{AE} \cong \overline{A'E'}$. Therefore, we conclude that addition is well-defined on congruence classes of line segments. □

By combining the notions of betweenness and congruence, it is now possible to introduce an ordering on segments.

Definition 3.3.3 (Order of segments)**.** Given two line segments $\overline{AB}, \overline{CD}$, we say that $\overline{AB}$ is *less than* $\overline{CD}$, and write $\overline{AB} < \overline{CD}$, if there exists some point E in between C and D, and $\overline{AB} \cong \overline{CE}$.

Theorem 3.3.4. *The following properties of order hold:*

1. *Given congruent line segments $\overline{AB} \cong \overline{A'B'}$ and $\overline{CD} \cong \overline{C'D'}$, we have that $\overline{AB} < \overline{CD}$ if and only if $\overline{A'B'} < \overline{C'D'}$.*

2. *If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$.*

3. *For any two line segments $\overline{AB}, \overline{CD}$, one and only one of the following conditions may hold:*

$$\overline{AB} < \overline{CD}, \qquad \overline{AB} \cong \overline{CD} \qquad , \qquad \overline{CD} < \overline{AB}.$$

Proof. See [2, Prop. 8.4] □

The last theorem shows in particular that the ordering can also be defined on congruence classes by $|AB| < |CD|$ if $\overline{AB} < \overline{CD}$.

3.4 Axioms of Angle Congruence

As with line segments, we introduce a relation on the set of all angles, which determines whether angles are of equal size. We denote this relation by $\alpha \cong \beta$, for any two angles α, β, and we say in this case that α is congruent to β. Moreover, we require congruence of angles to satisfy the following axioms [2, §9]:

(C4) Given an angle $\angle BAC$ and a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$, on a given side of the line DF, such that $\angle BAC \cong \angle EDF$.

(C5) Angle congruence is an equivalence relation.

(C6) If ABC and DEF are two triangles with congruent angles $\angle BAC \cong \angle EDF$ and congruent sides $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, then the two triangles ABC and DEF are congruent. That is, $\angle ABC \cong \angle DEF$, $\angle ACB \cong \angle DFE$ and $\overline{BC} \cong \overline{EF}$.

First off, axiom (C4) is proposition 24 in Euclid's elements, which he proves using straightedge and compass constructions [1, p. 209]. Since Hilbert does not use these constructions, he had to include it as an axiom. Furthermore, the fact that congruence of angles is an equivalence relation is implied by the common notions Euclid uses. Finally, axiom (C6), also referred to as the side-angle-side property, is proposition 4 from Euclid [1, p. 247]. Here, Euclid argues that one could move triangle DEF onto ABC in such a way that they would coincide. Therefore, Euclid concludes, the triangles ABC, DEF would have to be congruent. However, to incorporate the 'moving' of object rigorously in the theory requires introducing the notion of isometry and its properties. Therefore, the side-angle-side criterion is kept as an axiom here.

As with line segments, the notions of betweenness and congruence combined can give a sense of order on angles. This is defined as follows.

Definition 3.4.1. Let $\angle BAC, \angle DEF$ be two angles. We say that $\angle BAC$ is less than $\angle DEF$, and write $\angle BAC < \angle DEF$, if there exists some point P in the interior of $\angle DEF$ such that $\angle DEP \cong \angle BAC$.

This relation has the usual properties we expect from an ordering and is compatible with congruence. This is summarized in the following theorem, which we give without proof.

Theorem 3.4.2. *For any angles $\alpha, \beta, \gamma, \alpha', \beta'$, the following properties hold:*

1. *If $\alpha \cong \alpha'$ and $\beta \cong \beta'$, then $\alpha < \beta$ if and only if $\alpha' < \beta'$.*

2. *If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

3. *One and only one of the following hold:*

$$\alpha < \beta, \qquad \alpha \cong \beta \qquad , \qquad \beta < \alpha.$$

3.5 Circle Intersection

The axioms of incidence, betweenness and congruence lay the foundation of neutral geometry and a model of this is called a Hilbert plane. Hilbert planes, though they lack the last two axioms we are about to introduce, are already rich in structure and many of the results of Euclid's first book hold. See, also theorem 10.4 from [2]. Moreover, one could explore different geometries from the one presented in Euclid using this foundation and adding different axioms. This approach will be undertaken in chapter 4, where we explore the implications of the hyperbolic axiom. For now, we will mend a small logical gap in Euclid's very first proposition, which we have shown to exist in paragraph 2.3.

Recall from paragraph 2.3 that Euclid's first proposition was to construct an equilateral triangle given any side. The proof relied on producing circles on both edges of the given segment and the fact these circles would have to meet. From all the axioms introduced so far, none guarantee the intersection of the circles. Since intersecting circles are an important tool in the straightedge and compass game, this issue reaches much further than only the first proposition. In order to to solve it, we will need an extra axiom (E), which will ensure us that circles meet when our intuition expects them to. But first, we will properly define what we mean by a circle.

Definition 3.5.1 (Circles)**.** Let A be a point and $\overline{BC}$ a segment. The *circle* with center A and radius $\overline{BC}$ is the set of all points D such that $\overline{AD} \cong \overline{BC}$. Moreover, the *interior* of this circle are all points E such that $\overline{AE} < \overline{BC}$, including the center A itself.

Euclid's postulate 3 assumes the existence of a circle with any center and radius [1, p. 154]. Such an axiom would be unnecessary in Hilbert's axiomatic scheme, since we could always define the set in the above definition. Moreover, from Hilbert's other axioms it follows that this set is nonempty. To see this, consider a circle γ with center A and radius $\overline{BC}$. Then, by axiom (C1) there exists a unique point D on any ray through A such that $\overline{AD} \cong \overline{BC}$. Moreover, axiom (C1) tells us that each line l through A intersects the circle exactly twice, and each ray starting at A once. Nonetheless, it is not clear that any line passing the interior of γ must intersect the circle. This changes once we consider the following axiom, which also solves the intersection issue of Euclid's first proposition.

(E) Two circles Γ, Δ, such that Δ contains a point in the interior and exterior of Γ, intersect.

In order to demonstrate this axiom, we return back to Euclid's construction of an equilateral triangle. There, we had two circles Γ_A, Γ_B with centers A and B, and equal radius $\overline{AB}$. Circle Γ_A contains the center B of Γ_B while circle Γ_B contains the center A of Γ_A. Therefore, by axiom (E) these two circles intersect. Furthermore, a consequence of axiom (E) is the following theorem:

Theorem 3.5.2. *If Π is a Hilbert plane which satisfies axiom (E), then each line l, which contains a point in the interior of a circle Γ, will intersect Γ at least once.*

Proof. See proposition 11.6 from [2]. □

With the last theorem and axiom (E), we have satisfactory conditions for circles to meet other circles and lines. Therefore, the construction game of straightedge and compass makes sense in any Hilbert plane satisfying (E).

3.6 Playfair's Axiom

Playfair's axiom, which is equivalent to the parallel postulate given Hilbert's other axioms, can be introduced using only the notion of line incidence [2, p. 68]:

Definition 3.6.1 (Parallel lines)**.** Two lines are called *parallel* if they do not intersect. Moreover, we say that a line is parallel to itself.

(P) For each point A and each line l, there exists at most one line parallel to l which contains A.

Definition 3.6.2 (Euclidean plane)**.** A *Euclidean plane* is a Hilbert plane satisfying (E) and (P).

4 Introduction to Hyperbolic Geometry

With Hilbert's axioms as a foundation of geometry, in this chapter an additional axiom will be introduced that can describe the hyperbolic space. This axiom is defined in terms of limiting parallel rays, a special kind of rays which are asymptotic to one another and which extend the idea of parallel lines. We will end this chapter with an introduction of the angle of parallelism. This idea is a direct consequence of the hyperbolic axiom and is central in the space's trigonometry, which we will explore in further detail in 6.

4.1 Limiting Parallel Rays

In neutral geometry, there does not necessarily exist for each line l and point A a unique line passing A and parallel to l. In order to remedy this, one could consider in particular the 'closest' parallel rays, which enjoy more properties than the general class. What is meant with 'closest' rays is made precise with the following definitions.

Definition 4.1.1. Two rays $A\alpha, B\beta$ are called *coterminal* if one contains the other. We call a ray $A\alpha$ *limiting parallel* to another ray $B\beta$ when they are coterminal or if they do not lie on the same line, they are parallel and any ray $A\gamma$ inside the interior of $\angle BA\alpha$ meets $B\beta$ in some point. Alternatively, we write $A\alpha \mid\mid\mid B\beta$, if $A\alpha$ is limiting parallel to $B\beta$.

(a) Two coterminal rays $A\alpha$, $B\beta$ (b) $A\alpha$ is limiting parallel to $B\beta$

Figure 4: Examples of coterminal and limiting parallel rays

Example 4.1.2. In the Euclidean plane, if $\overrightarrow{AB}, \overrightarrow{CD}$ are two rays lying on parallel lines, then either $\overrightarrow{AB}, \overrightarrow{CD}$ are limiting parallel or the opposite ray $\overrightarrow{BA}$ and $\overrightarrow{CD}$ are limiting parallel. This is assured by the parallel postulate. For any point X inside $\angle CAB$ we have that $\angle CAX < \angle CAB$ and therefore, the sum of the interior angles $\angle CAX + \angle ACD$ is smaller than two right angles.

Moreover, note that if $\overrightarrow{AB}, \overrightarrow{CD}$ are limiting parallel rays, then their opposites $\overrightarrow{BA}, \overrightarrow{DC}$ are also limiting parallel in the Euclidean plane. In particular, the rays $\overrightarrow{AB}, \overrightarrow{BA}$, which are limiting parallel to both arms of the line CD, lie on the same line. We will see later on, with the introduction of the hyperbolic axiom, that this is quite different in the hyperbolic plane. △

Let us again consider the points A, B and the rays $A\alpha, B\beta$ from definition 4.1.1. It should be noted that, although any ray inside $\angle BA\alpha$ will eventually meet the ray $B\beta$, there is a priori no guarantee that any ray inside $\angle AB\beta$ intersects $A\alpha$. In other words, we have yet to discern that $A\alpha \mid\mid\mid B\beta$ implies $B\beta \mid\mid\mid A\alpha$, i.e. the relation $\mid\mid\mid$ is reflective. The following theorem is therefore of particular use. An extensive proof can be found in [2, §34].

Theorem 4.1.3. *The relation $\mid\mid\mid$ of being limiting parallel is an equivalence relation on the set of rays.*

Definition 4.1.4. An equivalence class of limiting parallel rays is called an **end**. Furthermore, when two rays are contained in the same end, we say that they intersect there. For two distinct points A, B and an end ω the limit triangle $AB\omega$ is the union of the segment $\overline{AB}$ with the rays $A\omega, B\omega$, which start at A, B and are contained in ω.

4.2 The Hyperbolic Axiom

From example 4.1.2 we know that in the Euclidean case, for each line l and point A there exist two distinct rays limiting parallel to the rays of l, that start at A and which both lie on the same line. In the hyperbolic case, the rays propagating from A do not lie on the same line, as is assured by the hyperbolic axiom.

L. For each line l and each point A not on l, there exist two rays $A\alpha, A\beta$ starting from A, not on the same line, such that $A\alpha, B\beta$ are parallel to l and any ray $A\gamma$ in the interior of the angle $\angle\alpha A\beta$ meets l.

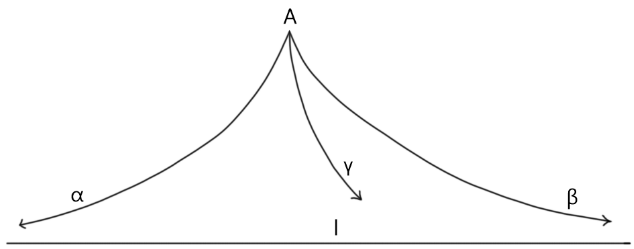


Figure 5: The hyperbolic axiom

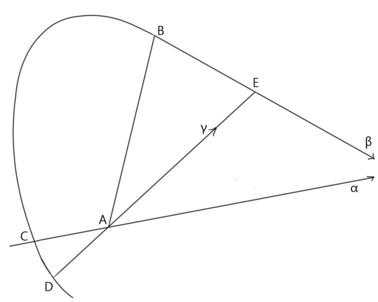
Definition 4.2.1. We define a **hyperbolic plane** to be a Hilbert plane satisfying (L).

Now that we have given all the axioms of hyperbolic geometry, a natural question to ask is whether the theory is consistent. For this, I would refer the reader to section 5, where this question will shortly be discussed. For now, we will take the bold assumption that the theory of hyperbolic geometry does not lead to any contradictions.

Next, we will show that the hyperbolic plane is indeed different from the familiar Euclidean plane.

Proposition 4.2.2. *Hyperbolic geometry is non-Euclidean. In other words, the parallel postulate does not hold in the hyperbolic plane.*

Proof. To show that the hyperbolic plane is non-Euclidean, we demonstrate that parallel lines are not unique. That is, we will prove that there exist a line l, a point A not on l and two lines parallel to l, both containing A. The hyperbolic axiom (L) assures that there always exist two rays through A, not on the same line that are limiting parallel to the rays in l. So, the proof would be finished if we showed that distinct lines carrying two limiting parallel rays do not intersect.



For this, let $A\alpha, B\beta$ be limiting parallel rays which are not coterminal and assume the lines carrying them do meet. Call the point of intersection C and let D be any point such that $B * C * D$ (B2). Draw the line DA and write $A\gamma$ for the ray on this line which starts at A and is on the opposite side from A as D. Since $A\alpha, B\beta$ are limiting parallel and not coterminal, the point of intersection C does not lie on the rays $A\alpha, B\beta$. Therefore, C is on the opposite side from AB as the rays $A\alpha$ and $B\beta$. Since $B * C * D$, it thus follows that D is on the opposite side from AB as the ray $B\beta$ and that D is also on the opposite side from AC as B. Therefore, D does not lie inside the angle $\angle BA\alpha$ while the ray $A\gamma$ does. Since $A\alpha, B\beta$ are limiting parallel and not coterminal, the ray $A\gamma$ shall meet $B\beta$ in a point E. Now, there exist two distinct lines going through the points D and E, one passing A and the other containing B. This contradicts axiom (I1). $\square$

Let us review the ideas we have laid out up until this point. On any Hilbert plane, a relation called being limiting parallel, which contains the relation of being coterminal, can be defined on the set of all rays. Furthermore, this is an equivalence relation and the existence of limiting parallel rays are secured in Euclidean and hyperbolic geometry. The hyperbolic case goes a step further and asserts that rays sharing a starting point and limiting parallel to rays on the same line cannot be collinear. In particular, this implies that hyperbolic geometry is non-Euclidean, since the rays from (L) lie on distinct lines which are parallel to l.

4.3 Angle of Parallelism

Definition 4.3.1. For any segment AB, consider the perpendicular to AB at B. Choosing a ray Bb on this line, let Aa be the limiting parallel ray to Bb. We define the congruence class represented by $\angle BAa$ to be the **angle of parallelism** of the congruence class represented by AB and denote it by $\Pi(AB)$.

Since we are trying to define a map Π on a set of equivalence classes, we will have to show that the value of $\Pi(AB)$ is independent of the choice of representation made. To demonstrate this, we will prove the following results.

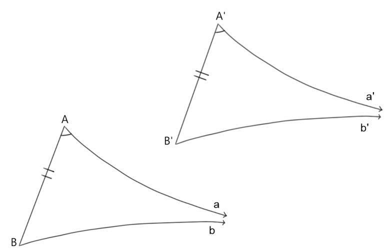


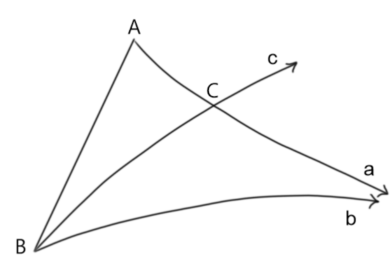
Figure 6: Two limit triangles with one equal side and one equal angle.

Theorem 4.3.2. *Let A, B, A', B' be some points such that $A \neq B$ and $A' \neq B'$. Also let $Aa, Bb, A'a', B'b'$ be some rays such that $Aa \parallel\!| Bb$ and $A'a' \parallel\!| B'b'$. If $\angle BAa \cong \angle B'A'a'$ and $AB \cong A'B'$, then also $\angle ABb \cong \angle A'B'b'$*

Proof. Assume the contrary, i.e. $\angle ABb \ncong \angle A'B'b'$. By theorem 3.3.4, we either have $\angle ABb < \angle A'B'b'$ or $\angle A'B'b' < \angle ABb$. Without loss of generality, we can limit ourselves to the case that $\angle A'B'b'$ is smaller. Using axiom (C4), let Bc be the ray on the same side of the line AB as Bb with the property that $\angle ABc \cong \angle A'B'b'$. Since $\angle ABc < \angle BAa$, the ray Bc is inside $\angle ABb$ and therefore, Bc will meet Aa in a point C. With axiom (C1), let C' be the point on the ray $A'a'$ such that $A'C' \cong AC$. By the side-angle-side axiom (C6) it follows that the triangles ABC and $A'B'C'$ are congruent and therefore, we obtain

$$\angle A'B'b' \cong \angle ABC \cong \angle A'B'C'.$$

If C' lies on the same side of the line $A'B'$ as $B'b'$, then by the uniqueness part of axiom (C4), it would follow that $\overrightarrow{B'C'} = B'b'$. Furthermore, $B'b'$ would meet $A'a'$ in the point C', which is in contradiction with the fact that $A'a' \parallel\!| B'b'$. Therefore, the theorem would be proven once we demonstrate that $B'b', C'$ lie on the same side of $A'B'$.



Now, assume they do not, let $D' \neq B'$ be any point on $B'b'$ and let E' be the point of intersection of $C'D'$ and $A'B'$. Note that such a point of intersection exists since C', D' lie on different sides of $A'B'$. Next, choose a point F' such that $C' * F' * E'$ (lemma 3.2.3). The ray $\overrightarrow{A'F'}$ is then inside the angle $\angle B'A'a'$ and as such meets $B'b'$ in some point G'. Since the line $A'F'$ already meets $D'C'$ in F' and $A'B'$ in A', we know that G' cannot equal B' or D'. Moreover, since $G' \in \overrightarrow{B'D'}$, the following two cases are left to consider.

1. The point G' lies between B' and D'. By Pasch' axiom (B4), the line $A'F'$ should then either meet the lines $A'B'$ or $C'D'$. However, $A'F'$ already meets these lines in the points A' and F'. This case is thus impossible.

2. The point D' lies between B' and G'. By Pasch' axiom (B4), the line $D'E'$ can only meet two of the three side of the triangle $G'B'A'$. Nonetheless, it also meets all three sides in points D', E' and F'. Therefore, this case is impossible too.

We conclude by ways of contradiction that the point C' and ray $A'a'$ lie on the same side of $A'B'$. □

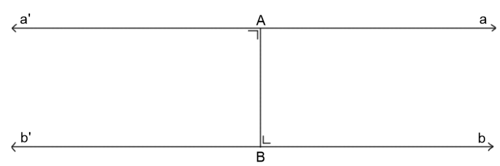
Remark 4.3.3. The figure consisting of the segment AB and the limiting parallel rays Aa, Bb is sometimes referred to as a *limit triangle*. The above theorem gives a sufficient condition for two limit triangles to be congruent.

Corollary 4.3.4. *The map Π, which sends lengths to their angles of parallelism, is well-defined.*

Proof. Let AB and $A'B'$ be any pair of segments such that $AB \cong A'B'$. Drop perpendicular rays $B\beta$ and $B'\beta'$ and let $A\alpha$ be limiting parallel to $B\beta$ and $A'\alpha'$ be limiting parallel to $B'\beta'$. Then, from theorem 4.3.2 it follows that $\angle BA\alpha \cong \angle B'A'\alpha'$. □

Lemma 4.3.5. *The angle of parallelism is always acute.*

Proof.
Consider any line segment AB and let Aa, Bb be perpendicular rays to AB which lie on the same side. From Euclid [1, Prop. 27] we know that the lines carrying Aa, Bb never meet. Therefore, the angle of parallelism is at most a right angle.



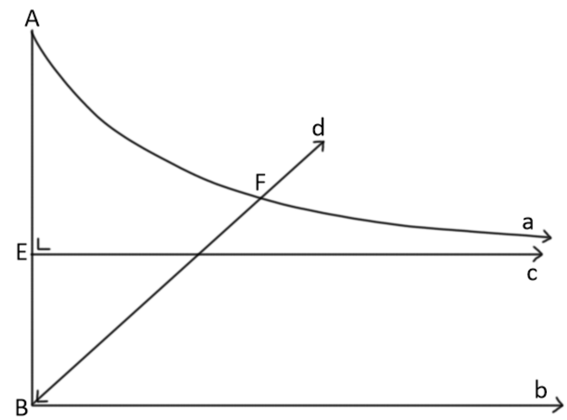
Now assume that $\Pi(AB)$ is a right angle and therefore, $Aa \parallel\!| Bb$. Let Bb' be the other ray from B on the line carrying Bb. Then, by the hyperbolic axiom (L), there exists a ray Aa' emanating from the point A which will be limiting parallel to Bb'. Moreover, (L) assures us that Aa' and Aa do not lie on the same line. Therefore, $\angle BAa'$ is not a right angle. However, by theorem 4.3.2 we also have that $\angle BAa' \cong \angle BAa$, which makes $\angle BAa'$ a right angle by definition. The case that $\Pi(AB)$ is a right angle is shown to be contradictory and so, $\Pi(AB)$ is acute.

□

Lemma 4.3.6. *The angle of parallelism is a decreasing function. That is, if AB, CD are two line segments such that $AB > CD$, we have that $\Pi(AB) < \Pi(CD)$.*

Proof.

Let E be the point on line segment AB such that $AE \cong CD$ and draw perpendicular rays Ec and Bb to AB which lie on the same side of AB. Also let Aa be the ray limiting parallel to Bb. Since Ec, Bb share a common perpendicular, they are parallel. I claim that Aa and Ec do meet in a point.



To see this, assume they do not and consider any ray Bd in the interior of $\angle ABb$. Then Bd will have to meet Aa in some point F since $Aa \mid\mid\mid Bb$. Because Aa and Ec do not intersect, all points on Aa are on the same side of Ec as the point A. Since AB intersects Ec in E, this implies that the rays Aa, Bb are on different sides from AB and therefore, the segment BF intersects Ec in some point. Hence, $Ce \mid\mid\mid Bb$. However, this is impossible since the angle of parallelism is always acute and Ce, Bb share a common perpendicular. So, Ec and Aa do meet. By the crossbar theorem, any ray in the interior of $\angle BAa$ should then also meet Ec and therefore, the ray limiting parallel to Ec is certainly not contained in the angle BAa. Because we also know that this limiting parallel ray to Ec is not equal to Aa, we conclude that $\Pi(CD) > \Pi(AB)$. $\square$

Note that in particular, it follows from lemma 4.3.6 that the map Π is injective. A natural question to ask next is what the image of Π would be. From lemma 4.3.5 we know that this is at most the set of all acute angles. It turns out that this is precisely the image of Π but a synthetic proof is far from simple. The interested reader could consult [2, p.380] or [3, p.224, Appendix III]. Moreover we will give an analytic proof in section 7.

5 Poincaré's Upper Half Plane

Historically, it was János Bolyai and Nikolai Ivanovitch Lobachevsky who first proved the consistency of hyperbolic geometry [5, p. 149]. From then on, mathematicians have looked for hyperbolic models which could lie in $\mathbb{R}^n$, resulting in analytic hyperbolic geometry. Today, modern textbooks often present between one and four standard models: the Beltrami-Klein disk, the hyperboloid, Poincaré's disk and Poincaré's upper half plane. Naturally, these models are isomorphic, so geometric properties of a particular model carry over to all of them. Moreover, working inside a model can be advantageous for we can visualize the hyperbolic plane and reinterpret geometric problems as algebraic ones. Nonetheless, by working inside a model we are also restricting ourselves. Had Lobachevsky and Bolyai stayed inside models of the Euclidean plane, they would have never discovered the hyperbolic plane. In fact, it turns out that there exist non-isomorphic models which satisfy the axioms of hyperbolic geometry we have discussed [2]. Some of these differences are due to the fact that we have not addressed any questions of continuity. For example, the Poincaré upper half plane generally consists out of points $(x, y) \in \mathbb{R} \times (0, \infty)$ with real coordinates while a model consisting out of points $(x, y) \in E \times E \cap (0, \infty)$ in the Euclidean field would also satisfy our axioms. Furthermore, there are the so-called non-Archimedean geometries which contain line segments of infinite and infinitesimal length.

From this points forward we shall restrict ourselves to the upper half-plane model of Poincaré, as can also be found in [5] or [6]. In this chapter, we shall show how Hilbert's notions of incidence, betweenness, and segment and angle congruence are defined in this model. Furthermore, we will demonstrate that this model indeed satisfies the hyperbolic axiom. As a consequence, assuming our model has no logical contradictions, the axiomatic system of the hyperbolic plane is consistent and in particular, the parallel postulate does not follow from the other axioms.

5.1 Hilbert's Axioms

Definition 5.1.1 (Poincaré Upper Half Plane)**.** The points of the Poincaré upper half plane are the complex numbers with positive imaginary part:

$$\mathbb{H}^2 := \{z \in \mathbb{C} : \text{Im } z > 0\}.$$

Its lines or geodesics are the vertical half lines inside $\mathbb{C}$ and the half circles with origins lying on the real axis. That is, lines of $\mathbb{H}^2$ are given by the subsets

$$L_x := \{x + it : t > 0\}, \qquad \Gamma_{(c,r)} := \{c + re^{i\theta} : \theta \in (0, \pi)\},$$

for any $x, c, r \in \mathbb{R}$ with $r > 0$.

Definition 5.1.2. We define the ternary relation $*$ on elements of $\mathbb{H}^2$ as follows. For three distinct points $z_1, z_2, z_2 \in \mathbb{H}^2$ lying on a line we write $z_1 * z_2 * z_3$ if

a. the points lie on a vertical line and they satisfy

$$\text{Im } z_1 < \text{Im } z_2 < \text{Im } z_3, \qquad \text{or} \qquad \text{Im } z_1 > \text{Im } z_2 > \text{Im } z_3;$$

b. or, they lie on a half circle $\Gamma_{(c,r)}$ so that they can be written as $z_i = c + re^{i\theta_j}, \theta_j \in (0, \pi)$ for $j = 1, 2, 3$, and they satisfy

$$\theta_1 < \theta_2 < \theta_3, \qquad \text{or} \qquad \theta_1 > \theta_2 > \theta_3.$$

A useful application of the relation of betweenness is the definition of rays. Recall that rays are defined by a pair of distinct points A, B as the subset of all points C on the line AB such that A is not in between B and C. In the Poincaré model, this means that for a point $x + iy \in \mathbb{H}^2$ and its vertical line L_x that the rays through it are given by

$$\{x + it : t \geq y\}, \qquad \{x + it : 0 < t \leq y\}.$$

while for a point $c + re^{i\theta} \in \mathbb{H}^2$ with half-circle $\Gamma_{(c,r)}$, the rays are given by

$$\{c + re^{it} : 0 < t \leq \theta\}, \qquad \{c + re^{it} : \theta \leq t < \pi\}.$$

To define congruence of line segments in $\mathbb{H}^2$, we introduce a metric and let two line segments be congruent if the distance from the end points are the same. For this, recall from calculus that the arc length of some injective and continuously differentiable path $\gamma : [a, b] \to \mathbb{C}$ in the Euclidean plane is defined as

$$\int_\gamma |dz| := \int_a^b |\gamma'(t)|\, dt.$$

For the hyperbolic case, we want the distance between any two points A, B be given by a similar integral over the path γ which parametrizes the line segment AB. Traditional arc length will not do the trick however, since that will naturally inherit distance from the Euclidean plane. We would expect the hyperbolic plane to have infinite distance in any direction. This means that as we get closer to the real axis, the distance in $\mathbb{C}$ will get smaller while the distance in $\mathbb{H}^2$ should not be bounded. A way of ensuring this is adjusting the integrand in the formula for arc length so that getting closer to the real axis will ensure that distance becomes greater.

Definition 5.1.3. For any two distinct hyperbolic points $z, w \in \mathbb{H}^2$ let $\gamma : [a, b] \to \mathbb{H}^2$ be an injective C^1 path with derivative nowhere zero which parametrizes the line segment from z to w. The latter means that $\gamma(a) = z$, $\gamma(b) = w$ and the image of γ is a line in $\mathbb{H}^2$. Then, we define the distance between these two points to be

$$d_H(z, w) := \int_b^a \frac{|\gamma'(t)|}{\text{Im } \gamma(t)}\, dt.$$

Moreover, we define $d_H(z, z) = 0$ for any $z \in \mathbb{H}^2$ and call two line segments congruent if the end points have the same distance.

Note that our definition lets us choose what parametrization we use for the line segment, but it turns out that the choice we make does not influence the distance and so, the map d_H is well-defined. To see this, let γ, η be two parametrizations of the same line segment, both injective C^1 paths with non-zero derivative. Then, by the restrictions we laid upon these maps, there exists a C^1 diffeomorphism $\zeta = \gamma^{-1} \circ \eta$ which changes the coordinates. Then using the substitution theorem of integration one could prove that this change of coordinates does not change the value of the integral.

The critical reader might also note that our definition of congruent line segments would not make sense if d_H were not symmetrical. Otherwise, choosing the order of the end points of a line segment would influence what distance it would have. However, this small detail is solved by the following lemma.

Lemma 5.1.4. *The map d_H is symmetric and therefore, congruence of line segments is well-defined.*

Proof. Note that if $\gamma : [a, b] \to \mathbb{H}^2$ is a path from $z \in \mathbb{H}^2$ to $w \in \mathbb{H}^2$, then

$$\eta : [a, b] \to \mathbb{H}^2, t \mapsto \gamma(b + t(a - b))$$

is a path from w to z. By the substitution theorem of integration, we then obtain

$$\begin{aligned}\int_a^b \frac{|\eta'(t)|}{\text{Im } \eta(t)}\, dt &= \int_a^b \frac{|\gamma'(b + t(a - b))|}{\text{Im } \gamma(b + t(a - b))}(b - a)\, dt \\ &= \int_a^b \frac{|\gamma'(t)|}{\text{Im } \gamma(t)}\, dt.\end{aligned}$$

Therefore, $d_H(z, w) = d_H(w, z)$ for any $z, w \in \mathbb{H}^2$.

□

It can be shown that the map d_H is a metric on the hyperbolic plane $\mathbb{H}^2$. Although I have refrained from a proof here, the interested reader could resort to [6, Ch. 3] or [5, §4.4].

Evaluating the integral, one can find the following formulae for the distance function. Again, the proof is omitted since these formulae follow simply by computing the integral.

Lemma 5.1.5. *The map d_H for any distinct points $z, w \in \mathbb{H}^2$, can be determined as follows:*

1. *If z, w both lie on the same vertical line L_x, then*
$$d_H(z,w) = \left|\log \frac{Im\ z}{Im\ w}\right|.$$
2. *If z, w both lie on a half circle $\Gamma_{(c,r)}$ and we can write these as $z = c + re^{i\theta}, w = c + re^{i\varphi}$ for some $\theta, \varphi \in (0, \pi)$, then*
$$d_H(z,w) = \left|\log\left(\frac{\tan\frac{\theta}{2}}{\tan\frac{\varphi}{2}}\right)\right|.$$

The final notion we need to turn the set $\mathbb{H}^2$ into a Hilbert plane is that of angle congruence. Similar to our approach of distance, we will define a measure of angles on $\mathbb{H}^2$ and define any pair of angles to be congruent if their angle measure is equivalent. We will be using the usual radian measure of angles in the Euclidean plane.

Definition 5.1.6. Angles in $\mathbb{H}^2$ are the usual angles of the Cartesian plane. This means that the hyperbolic angle measure between two semi-circles is equal to the angle measure made by their tangent lines and the angle of a semi-circle and a vertical ray is the angle made by the tangent of the semi-circle and the vertical ray. Two angles are thus congruent if their angle measures are equal.

5.2 Non-Euclidean Properties

In order to show that the upper half-plane $\mathbb{H}^2$ is truly something different from the Euclidean plane, we will prove that it satisfies the hyperbolic axiom (L). As a start, it can be shown without too much difficulty that limiting parallel rays can be characterized as follows.

Lemma 5.2.1. *Two rays are limiting parallel in the Poincaré model $\mathbb{H}^2$ if and only if they are vertical and not bounded from above, or their topological closures in $\mathbb{C}$ contain the same point on the real axis.*

Recall that we have defined an end to be an equivalence class of limiting parallel rays and say that a ray α has end ω if $\alpha \in \omega$. Lemma 5.2.1 demonstrates that rays in $\mathbb{H}^2$ have the same end if and only if they converge to the same point on the real axis or if they are vertical rays unbounded from above. There thus exists a natural correspondence between the ends and the set $\mathbb{R} \cup \{\infty\}$, where we map each end to the point on the real axis it converges to ór to ∞ if it is unbounded from above.

Theorem 5.2.2. *The Poincaré upper half-plane model $\mathbb{H}^2$ along with the notions of incidence, betweenness, congruence of segments and congruence of angles satisfies the hyperbolic axiom (L).*

Proof. Consider any hyperbolic line l and some point $z \in \mathbb{H}^2$ not on l. First of all, we presume that l is a vertical line and so, it can be written as
$$l = \{x + it : t > 0\},$$
for some real $x \in \mathbb{R}$. By lemma 5.2.1, the ray
$$\{\text{Re } z + it : t > \text{Im } z\},$$
which starts at z is limiting parallel some rays laying on l. Moreover, consider the half-circle $\Gamma_{(c,r)}$ with center c and radius r given by
$$c = \frac{|z|^2 - x^2}{2(\text{Re } z - x)}, \qquad r = |z - c|.$$

By definition of the radius r, the point z has distance r to c and therefore, $z \in \Gamma_{(c,r)}$. Furthermore, it can also be shown that $|x - c| = r$, so that the closure of the half circle $\Gamma_{(c,r)}$ contains the point x. Therefore, one of the rays from z on the line $\Gamma_{(c,r)}$ contains x in its closure and therefore, it is limiting parallel to some rays laying on l. Furthermore, the rays lying on the half-circle $\Gamma_{(c,r)}$ do no lie on the same line as the vertical ray through z does.

Next, we presume that l is a half-circle. So, there exists a center $c \in \mathbb{R}$ and radius $r > 0$ such that

$$l = \{c + re^{it} : t \in (0, \pi)\}.$$

The points of intersections the closure of l and the real axis are given by $c - r$ and $c + r$. As before, we can choose centers $c_1, c_2 \in \mathbb{R}$ and radii r_1, r_2 such that $c - r, z$ lie on $\overline{\Gamma}_{(c_1,r_1)}$ and $c + r, z$ lie on $\overline{\Gamma}_{(c_2,r_2)}$. Moreover, the half-circles $\Gamma_{(c_1,r_1)}, \Gamma_{(c_2,r_2)}$ are not the same. If they were, then the Euclidean circle carrying them would contain both the points $c - r$ and $c + r$ and since the centers c_1, c_2 lie on the real axis, this fixes $c = c_1 = c_2$ and $r = r_1 = r_2$. However, the point z does not lie on the half-circle $\Gamma_{(c,r)}$. Choosing appropriate rays on $\Gamma_{(c_1,r_1)}, \Gamma_{(c_2,r_2)}$ this thus proves the result by lemma 5.2.1. $\square$

6 Hyperbolic Trigonometry

Contrary to Euclidean geometry, there exist two types of triangles in hyperbolic space. The first are the ordinary triangles which only meet at points, the other type are formed by limiting parallel rays. For such triangles, we say that the sides meet at infinity and we call them limit triangles. A specifically interesting case is when one of the angles of a limit triangle is right. We have used this figure before to define the angle of parallelism Π, which gives the relation between the acute angle and the finite side of the right-angled limit triangle. So far, we have shown that Π is well-defined, strictly decreasing and only maps to acute angles. Also, we have defined Π only in terms of the congruence classes of segments and angles. However, because of the distance function in Poincaré's model and radian measure, there exists a natural correspondence between these congruence classes and $\mathbb{R}$. Hence, we can also consider Π as a function from the image of d_H, which is $(0, \infty)$, to all radians $(0, \frac{\pi}{2})$ of acute angles.

In this chapter, trigonometrical identities for right-angled triangles, both finite and infinite, are derived. These are of special importance to the constructions presented in chapter 7 and the algebraic analysis of chapter 8.

6.1 Limit Right Triangles

Perhaps the most important formula of hyperbolic trigonometry is the following due to Bolyai. It gives a direct expression for the angle of parallelism and all other identities presented this chapter follow from this single result.

Theorem 6.1.1 (Bolyai's Formula)**.** *For any $t > 0$, we have*

$$e^{-t} = \tan \frac{\Pi(t)}{2}.$$

Proof.
Consider the ray from the point i which lies on the unit circle and has positive real coordinates:

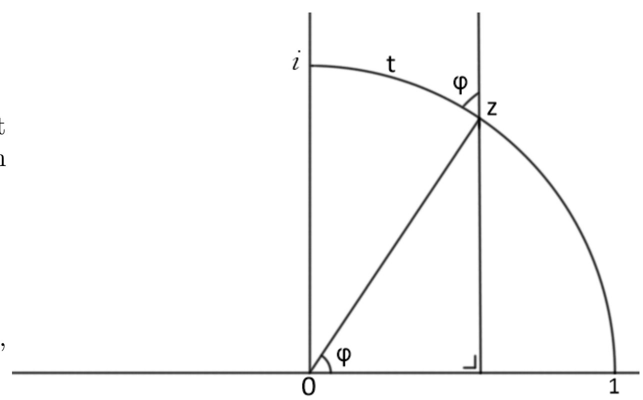
$$\left\{ e^{i\theta} : \theta \in \left(0, \frac{\pi}{2}\right] \right\}.$$

We let $z = e^{i\varphi}$ be the unique point on there such that $d_H(i, z) = t$. The existence of z is ensured by axiom C1. Now,

$$t = \left| \log \left(\frac{\tan \frac{\pi}{2}}{\tan \frac{\varphi}{2}} \right) \right| = \left| \log \left(\tan \frac{\varphi}{2} \right) \right|.$$

Because we have chosen z in the first quadrant of $\mathbb{C}$, we know that $\varphi < \frac{\pi}{2}$ and so, $\tan \frac{\varphi}{2} < 1$. Hence,

$$\tan \frac{\varphi}{2} = e^{-t}.$$



Now consider the Euclidean triangle formed by the vertical line through z, the real axis and the line connecting the points with complex coordinates 0 and z. The hyperbolic angle between the half circle through z and i and the vertical line through z is given by the Euclidean angle between the vertical line through z and the line tangent to the circle. The tangent is perpendicular to the line through the origin 0 and the point z. Since the angles of Euclidean triangles add up to 2π radians, it can be shown that the angle made by the half circle and the vertical line through z is equal to φ. Therefore, $\Pi(t) = \varphi$. □

An alternative proof is given in [7, Thm. 7.2], where the result is shown in the case of Poincaré's disk model. Rewriting Bolyai's formula, one can obtain the following.

Theorem 6.1.2. *Let $t > 0$. Then, the following identities hold for the right-angled limit triangles with side length t,*

$$\tanh t = \cos \Pi(t), \qquad \sinh t = \cot \Pi(t), \qquad \cosh t = \csc \Pi(t).$$

Proof. The identities follow from simple algebraic manipulations of Bolyai's formula. For example, for any $t > 0$ and $\varphi = \Pi(t)$ one has

$$\begin{aligned}\cosh t &= \frac{e^t + e^{-t}}{2} = \frac{\frac{1}{\tanh \frac{\varphi}{2}} + \tanh \frac{\varphi}{2}}{2} \\ &= \frac{\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}}{2 \cos \frac{\varphi}{2} \sin \frac{\varphi}{2}} = \csc \varphi.\end{aligned}$$

The remaining two derivations are left to the reader. □

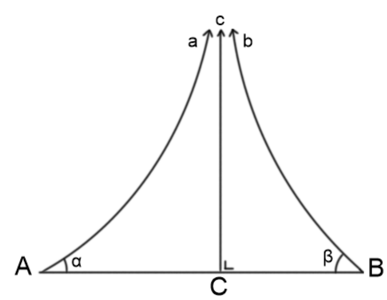
Next, we can easily generalize the above results to limit triangles which do not necessarily have one right angle.

Lemma 6.1.3. *Let A, B be any pair of points with limiting parallel rays Aa and Bb. Letting α, β be the radian measure of the angles $\angle BAa, \angle ABb$ respectively and $t = d_H(A, B)$, we have*

$$e^{-t} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

Proof.

Let C be the point on the ray $\overrightarrow{AB}$ such that α is the angle of parallelism of the segment $\overline{AB}$. Also let Cc be the ray limiting parallel to Aa. As a consequence, angle $\angle ACc$ is right. There now exist three possibilities.



1. First, assume that $\overline{AB} \cong \overline{AC}$. Because of theorem 4.3.2, the angles $\angle ABb$ and $\angle ACc$ are congruent. Hence, $\beta = \frac{\pi}{2}$ and the theorem follows from Bolyai's formula.

2. Second, consider the case that $\overline{AB} < \overline{AC}$. Then, $A * B * C$ and by axiom C3 it follows that

$$|AB| + |BC| = |AC|.$$

   This implies that

$$\begin{aligned}e^{-t} &= e^{-d_H(A,B)} = e^{-(d_H(A,C) - d_H(B,C))} \\ &= e^{-d_H(A,C)} \cdot e^{d_H(B,C)}.\end{aligned}$$

   Since $\alpha = \Pi(AC)$ and $\pi - \beta = \Pi(BC)$, it follows from Bolyai's formula that

$$e^{-t} = \tan \frac{\alpha}{2} \cdot \frac{1}{\tan\left(\frac{\pi - \beta}{2}\right)} = \tan \frac{\alpha}{2} \cdot \tan \frac{\beta}{2}.$$

3. Lastly, we consider the case that $\overline{AC} < \overline{AB}$. Then, $A * C * B$ and by axiom C3 it follows that

$$|AC| + |CB| = |AB|.$$

   Therefore,

$$e^{-t} = e^{-d_H(A,C)} \cdot e^{-d_H(C,B)} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

□

6.2 Ordinary Right Triangles

So far, we have used the upper half-plane to derive Bolyai's formula and applied this to generate trigonometric identities for particular limit triangles. For our final result this chapter, we shall give hyperbolic variants for the Pythagorean theorem and the standard formulae for a right-angled triangle in Euclidean space. These results can be found in [7, Thm. 10.3], where Marvin Greenberg derives the identities by using Poincaré's disk model and Euclidean trigonometry. The approach presented here does not require the use of a model as long as Bolyai's formula has been proven. As the proof requires a long and tedious record of algebraic manipulations, we will restrict to a sketch of the proof only.

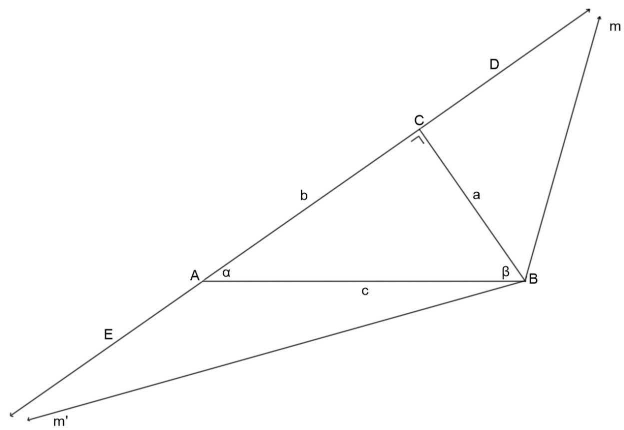


Figure 7: A right-angled triangle ABC as intersection of two limit triangles.

We start with a triangle ABC with a right angle at vertex C. We let a, b, c be the lengths of the sides opposite to the vertices A, B, C and α, β the angles at the vertices A, B. Next, extend the ray $\overrightarrow{AC}$, letting D be some point on this ray such that $A * C * D$. Also draw the ray Bm limiting parallel to $\overrightarrow{AC}$. Because $\angle ACB$ is a right angle, its adjacent angle $\angle DCB$ is also right by definition. Therefore, $\Pi(a) = \angle CBm$. Moreover, by applying lemma 6.1.3 to the limit triangle formed by the segment AB and limiting parallel rays $\overrightarrow{AC}, Bm$, it follows that

$$e^{-c} = \tan\frac{\alpha}{2}\tan\frac{\beta + \Pi(a)}{2}. \tag{6.1}$$

Furthermore, we also extend the ray $\overrightarrow{CA}$, letting E be any point on it such that $C*A*E$ and draw the ray Bm' limiting parallel to this ray. Thus constructed is another limit triangle CBm'. Therefore, $\angle CBm' = \Pi(BC)$ and $\angle ABm' = \Pi(a) - \beta$. Applying lemma 6.1.3 to the triangle ABm', we then obtain

$$e^{-c} = \tan\frac{\pi - \alpha}{2}\tan\frac{\Pi(a) - \beta}{2}. \tag{6.2}$$

Combining equations 6.1 and 6.2, we gain an equality depending exclusively on the values of one short side and the two non-right angles:

$$\tan\frac{\alpha}{2}\tan\frac{\beta + \Pi(a)}{2} = \tan\frac{\pi - \alpha}{2}\tan\frac{\Pi(a) - \beta}{2}. \tag{6.3}$$

After a lengthy and laborious account of rearranging terms, the above equation can be simplified. Moreover, this can then be substituted into equation 6.1 to gain an expression depending exclusively on c, α and β. Repeating this process a couple of times, one finds a list of identities summarized by the following theorem.

Theorem 6.2.1. *Let ABC be a triangle with $\angle ACB$ a right-angle, $\angle BAC = \alpha, \angle ABC = \beta$ and $\overline{BC} =$*

$a, \overline{AC} = b, \overline{AB} = c$. *Then, the following identities hold in the hyperbolic plane:*

$$\cosh c = \cosh a \cdot \cosh b, \qquad \cosh c = \cot\alpha \cot\beta, \qquad \cosh a = \frac{\cos\alpha}{\sin\beta},$$
$$\sin\alpha = \frac{\sinh a}{\sinh c}, \qquad \cos\alpha = \frac{\tanh b}{\tanh c}, \qquad \tan\alpha = \frac{\tanh a}{\sinh b}.$$

Equation

$$\cosh c = \cosh a \cdot \cosh b \tag{6.4}$$

can be seen as a substitute for the Pythagorean theorem in Euclidean space, since it expresses the hypotenuse of a right triangle in terms of its short sides. Besides this point, however, there is not much room for comparison. For the ancient Greeks, the Pythagorean theorem had a clear geometric interpretation. In Euclid's first book, the second to last proposition states that the area of the square with side equal to the hypotenuse of a right triangle is equal to the sum of the areas of the squares with sides equal to the short sides of said triangle [1, p. 349]. In the Hyperbolic case, such an interpretation does not seem to be possible, or at least it is not obvious. Note also that in our proof we relied on a model to demonstrate Bolyai's formula, while Euclid's proof of Pythagoras is synthetic.

Furthermore, equations

$$\sin\alpha = \frac{\sinh a}{\sinh c}, \qquad \cos\alpha = \frac{\tanh b}{\tanh c}, \qquad \tan\alpha = \frac{\tanh a}{\sinh b}$$

can be seen as the hyperbolic equivalent of the Euclidean formulae expressing the sine, cosine and tangens of an angle in terms of the adjacent and opposite sides, and the hypotenuse of a right-angled triangle. However, the equations

$$\cosh c = \cot\alpha \cot\beta, \qquad \cosh a = \frac{\cos\alpha}{\sin\beta}$$

do not have a Euclidean equivalent since the angles of a Euclidean triangle do not determine the lengths of the sides. As a consequence of these identities, any two triangles with equal angles are congruent in the hyperbolic plane.

Finally, there also exist trigonometrical identities for general triangles, including hyperbolic substitutes for the sine and cosines rules. These can be derived by dropping the perpendicular from any vertex to the opposite side, dividing the triangle in two right-angled ones. If interested, the reader may look these formulae up in [7, Thm. 10.4] or [8, p. 33].

7 Hyperbolic Constructions

Most of the earlier propositions of Euclid do not require the parallel postulate. Theorems of this kind are therefore true in any Hilbert plane and as a result, elementary constructions like bisecting angles, dropping perpendicular lines and adding segments can be carried out in any Hilbert plane. In this chapter, we will explore compass and straightedge constructions exclusive to the Hyperbolic plane.

As was the case in previous chapters, limiting parallel rays have a central role to play. A natural first question would be whether these asymptotic rays could be constructed from our familiar tools of geometry. An answer was provided by János Bolyai, who showed that given a segment t, one can construct the angle of parallelism $\Pi(t)$. Moreover, a construction by Bonola shows that the reverse is also possible. These will form the foundation for most constructions of the hyperbolic plane. In the second paragraph, we will give different constructions for a right triangle. These will prove to be of great importance for the algebraic interpretation we give in chapter 8.

7.1 Constructing Limiting Parallel Rays

The following construction, due to Bolyai, can also be found in [3, p. 216, Appendix III].

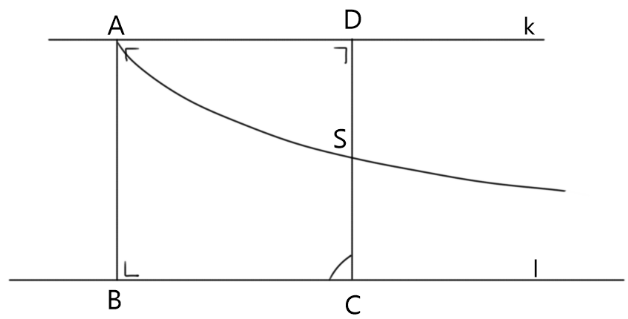


Figure 8: Bolyai's construction of the angle of parallelism to a given line segment $\overline{AB}$.

Construction 7.1.1 (Bolyai)**.** *Given a segment $\overline{AB}$, to construct the angle of parallelism* $\Pi(\overline{AB})$*.*

Proof. Drop perpendicular lines k and l at A and B respectively. Then, choose any point C on the line l and drop the perpendicular from C to k. We will call the point of intersection D. Now, draw the circle with centre point A and radius $\overline{BC}$. This circle should intersect the segment CD in a point S. Moreover, the ray $\overrightarrow{AS}$ should be limiting parallel to the ray $\overrightarrow{BC}$ and therefore, the angle $\angle BAS$ is the required angle of parallelism to $\overline{AB}$.

We will now prove the correctness of the construction. Let Aa be the line limiting parallel to the ray $\overrightarrow{BC}$ and propagating from A. By lemma 4.3.5 we know that $\angle BAa < \frac{\pi}{2}$ and therefore, the ray Aa lies inside the angles $\angle BAD$. By the crossbar theorem (3.2.9), the ray Aa shall have to meet segment $\overline{AD}$ and by Pasch' axiom will Aa meet the segment $\overline{CD}$ in some point S'. Using hyperbolic trigonometry, we will show that $AS' \cong BC$ and therefore, the circle with center A and radius $\overline{BC}$ does meet the segment $\overline{CD}$ in point $S = S'$ and the rays $\overrightarrow{AS}$ and Aa are equal.

Now let ρ be the radian measure of the angle $\angle BAS'$. Then, by theorem 6.2.1 we know that

$$\sin\rho = \cos\left(\frac{\pi}{2} - \rho\right) = \frac{\tanh AD}{\tanh AS'}.$$

Since $\Pi(AB) = \rho$, we can rewrite the above identity as

$$\tanh AS' = \frac{\tanh AD}{\sin\rho} = \tanh AD \cdot \cosh AB. \tag{7.1}$$

Now, also consider the right-angled triangles ABC and ADC. We write φ for the radian measure of $\angle BAC$. Then, by our trigonometrical identities we have that

$$\cos\varphi = \frac{\tanh AB}{\tanh AC}, \qquad \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{\tanh AD}{\tanh AC}.$$

Dividing one identity by the other we obtain

$$\tan\varphi = \frac{\tanh AD}{\tanh AB}. \tag{7.2}$$

Another identity from theorem 6.2.1 gives us

$$\tan\varphi = \frac{\tanh BC}{\sinh AB}$$

and so, combining this with (7.2) we note that

$$\tanh BC = \tanh AD \cdot \cosh AB. \tag{7.3}$$

By identities (7.1) and (7.3) we see that $\tanh AS' = \tanh BC$ and since the hyperbolic tangent is an injective map, we conclude that $\overline{BC} \cong \overline{AS'}$. □

In Bolyai's construction, we have been given a line segment $\overline{AB}$ and created its angle of parallelism. However, we could also use it, given a ray Pm and a point A not on the line containing Pm, to construct a ray limiting parallel to Pm and going through A. For this, we can drop the perpendicular from A onto the line containing Pm and call the point of intersection B. By constructing the angle of parallelism $\Pi(AB)$ on the appropriate side of AB we thus gain a ray limiting parallel to Pm and containing the point A.

Until now, limiting parallel rays have played a central role in our study of the hyperbolic plane and we will witness that it remains this position. Therefore, the importance of Bolyai's construction cannot be overstated. Moreover, in most other constructions unique to the hyperbolic plane we will see it come back in some shape or form.

As we have shown how to obtain the angle of parallelism given any finite segment, the question arises how to perform the reverse. An answer is provided by Bonola [3, p. 106], who came up with the following construction.

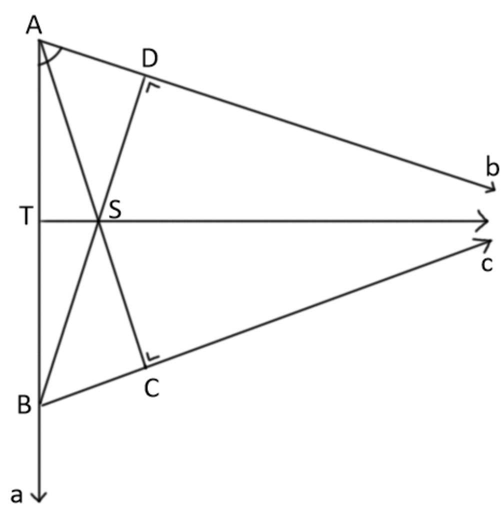


Figure 9: Bonola's construction of the length of parallelism given an angle $\angle aAb$.

Construction 7.1.2 (Bonola)**.** *Given an acute angle $\angle aAb$, to construct the length of parallelism $\Pi^{-1}(\angle aAb)$.*

Proof. Let $\angle aAb$ be any angle. Choose a point B on one arm, say Aa, in such a way that the angle made by the ray through B and limiting parallel to Ab makes an acute angle with Aa. Using Bolyai's construction, draw the ray Bc limiting parallel to Ab and drop perpendicular lines AC onto Bc and BD onto Ab.

The first claim is that the segments $\overline{AC}$ and $\overline{BD}$ intersect in a point S. This is satisfied when the line AC is in the interior of angle $\angle aAb$ and BD is in the interior of angle $\angle ABc$. If this is the case, then by the crossbar theorem (3.2.9) the line AC will have to intersect the line opposite to A in the triangle ABD. Therefore, we will show that AC is in the interior of $\angle aAb$. The proof that BD is in the interior of $\angle ABc$ turns out to be very similar.

Assume to the contrary that AC is not in the interior of $\angle aAb$. Because $Ab \parallel\!| Bc$, the point C will have to be on the same side from Ab as the point B. Otherwise, the lines carrying the rays Ab, Bc would no longer be parallel. So, the point C is on the opposite side from the line AB to the ray Ab. Since angle $\angle ABc$ was acute, its adjacent angle $\angle ABC$ is obtuse and combining with the right angle $\angle ACB$, the sum of angles of the triangle ABC are greater than π radians. This is impossible in the hyperbolic plane and so, our assumption that AC is in the exterior of $\angle aAb$ has led to a contradiction.

Now that the line segments AC, BD meet in a point S, we can drop the perpendicular ST onto Aa. The ray $\overrightarrow{TS}$ will be asymptotic to Ab and therefore, $\overline{AT}$ is the desired line segment. We will prove this fact by comparing the trigonometric identities of various right triangles.

First, let $\rho = \angle aAb, \varphi = \angle BAC$ and t be the length of segment AT. From the right triangles ABC and AST follow

$$\cos\varphi = \frac{\tanh AC}{\tanh AB}, \qquad \cos\varphi = \frac{\tanh t}{\tanh AS}. \tag{7.4}$$

Moreover, by construction $\angle CAD = \rho - \varphi$ is the angle of parallelism of segment $\overline{AC}$ and an angle within the right triangle ASD. Therefore,

$$\tanh AC = \cos(\rho - \varphi) = \frac{\tanh AD}{\tanh AS}. \tag{7.5}$$

Substituting equation (7.5) into equations (7.4) it follows that

$$\frac{\tanh AD}{\tanh AB} = \tanh t. \tag{7.6}$$

From right triangle ABD we moreover have

$$\cos\rho = \frac{\tanh AD}{\tanh AB}. \tag{7.7}$$

Therefore, combining equations (7.6) and (7.7) we conclude that

$$\cos\rho = \tanh t = \cos\Pi(t). \tag{7.8}$$

Since the angles $\rho, \Pi(t)$ are both acute it therefore holds that $\rho = \Pi(t)$ and by definition of the angle of parallelism, $Ab \parallel\!| \overrightarrow{ST}$. $\square$

Corollary 7.1.3. *A length $t \in (0, \infty)$ is constructible if and only if its angle of parallelism $\Pi(t)$ is constructible.*

7.2 Constructing Right Triangles

In a sense, the figures constructed by Bonola and Bolyai are right-angled triangles with one vertex at infinity and corollary 7.1.3 gives sufficient and necessary conditions for these triangles to be constructible. Namely, a limit triangle having a right-angle is constructible if and only if its acute angle, or equivalently, the length of the finite segment is constructible. In the case of an ordinary right-angled triangle in $\mathbb{H}^2$, what conditions would we then have to impose on the angles and sides such that it is constructible? Loosely speaking, it turns out that given any two quantities, we can fully construct the right-angled triangle. To make this more precise, we introduce the following theorem.

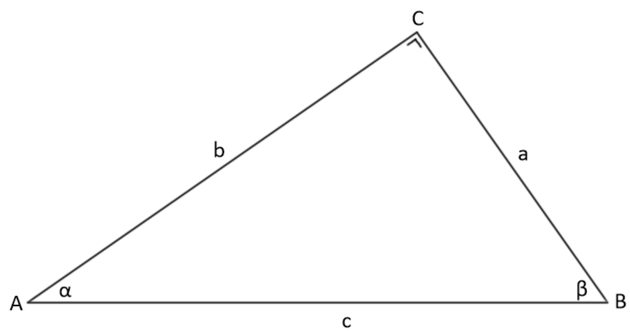


Figure 10: The right triangle of theorem 7.2.1.

Theorem 7.2.1. *Consider a right triangle in the hyperbolic plane with hypotenuse c, short sides a, b and angle α, opposite to a, and angle β opposite to b. Then, such a triangle may be constructed from any two of the values a, b, c, α, β. That is, the triangle is constructible if*

(i) sides a, b are constructible;

(ii) sides a, c are constructible and $a < c$;

(iii) angle α is constructible and length b is constructible with $\Pi(b) > \alpha$;

(iv) angle α is acute and α, c are constructible;

(v) angles α, β are constructible and satisfy $\alpha + \beta < \frac{\pi}{2}$;

(vi) angle α and side length a are constructible.

The first four assertions are arguably the simplest and can be done by constructions which work in both the Euclidean and Hyperbolic plane. For (i), simply construct a right angle, cut off appropriate lengths from both arms and connect the two points thus obtained. For (ii), we can construct a right angle at a point C and cut off a length of a from one of its arms, obtaining a point A. Drawing a circle at A with radius c, we know that the point C will be inside the circle since $c > a$. Therefore, the other arm of the right angle will have to intersect the circle in some point B, giving us the desired triangle ABC. Thirdly, for (iii) we can again cut length b off one arm of an angle with size α and construct a perpendicular at the point thus obtained. The condition that $\Pi(b) > \alpha$ will assure that the perpendicular and other arm of α will meet. Lastly, construction (iv) is similar to (iii) and left to the reader.

The remaining three constructions will require more thought, the first of which was found in [8, p. 34].

Construction 7.2.2. To construct a right triangle in $\mathbb{H}^2$ given two angles $\alpha, \beta > 0$ such that $\alpha + \beta < \frac{\pi}{2}$.

Proof. Since $\beta < \frac{\pi}{2}$, we can subtract β from $\frac{\pi}{2}$ and using Bonola's construction, we can find segments z, t with $\Pi(z) = \alpha$ and $\Pi(t) = \frac{\pi}{2} - \beta$. Because $\alpha < \frac{\pi}{2} - \beta$ and Π is a decreasing map (4.3.6), it follows that $z > t$. So, by 7.2.1ii we can construct a triangle having hypotenuse z and short side t. Let b be the other side of this triangle. The triangle with angle α and adjacent side b, constructible by 7.2.1iii, is the desired triangle.

First of all, note that by the hyperbolic Pythagorean theorem,

$$\cosh z = \cosh t \cdot \cosh b.$$

Moreover, we know that $\csc \Pi(x) = \cosh x$, so it follows that

$$\cosh b = \frac{\sin\left(\frac{\pi}{2} - \beta\right)}{\sin \alpha} = \frac{\cos \beta}{\sin \alpha}.$$

In particular,

$$\frac{\sin \alpha}{\sin \Pi(b)} = \cosh b \cdot \sin \alpha = \cos \beta \in (0, 1),$$

and hence, $\sin \alpha < \sin \Pi(b)$. Since the sinus is an increasing function on the interval $(0, \frac{\pi}{2})$, this implies that $\alpha < \Pi(b)$ and so we can indeed use construction 7.2.1c. Now let γ be the other angle of the right triangle obtained from α and b. Then, one of the trigonometric identities tells us that

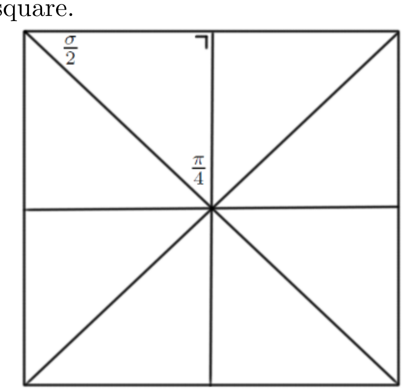
$$\cosh b = \frac{\cos \gamma}{\sin \alpha}.$$

Therefore, we conclude that $\cos \gamma = \cos \beta$, implying $\gamma = \beta$. □

A direct application is the following construction of a hyperbolic square.

Construction 7.2.3. *Given an acute angle $\sigma \in (0, \frac{\pi}{2})$, to make a figure of four vertices such that all inner angles have size σ.*

Proof. Bisect σ and construct a right triangle with angles $\frac{\sigma}{2}, \frac{\pi}{4}, \frac{\pi}{2}$. Now copy this triangle 7 times such that it forms the square with angle σ. □



Construction 7.2.4. *To construct a right triangle in $\mathbb{H}^2$ given one acute angle α and its opposite side a.*

Proof. Since α is acute, we can construct length t such that $\Pi(t) = \frac{\pi}{2} - \alpha$. Now draw the right triangle with short sides a, t and let z be the length of the hypotenuse. Then, z is greater than t and since Π is a strictly increasing function, this implies that $\Pi(t) > \Pi(z)$ and as such, $\alpha + \Pi(z) < \frac{\pi}{2}$. Now we could use construction 7.2.2 to create a right triangle with angles $\alpha, \beta, \frac{\pi}{2}$, where $\beta = \Pi(z)$. The side opposite to angle α, which we will call a', will have length a.

To prove the last assertion, we note from the first triangle that

$$\cosh z = \cosh a \cdot \cosh t.$$

Since $\Pi(z) = \beta$ and $\Pi(t) = \frac{\pi}{2} - \alpha$, we can rewrite this expression as

$$\cosh a = \csc\left(\frac{\pi}{2} - \alpha\right) \csc(\beta) = \frac{\cos \alpha}{\sin \beta}.$$

From 6.2.1 we conclude that this is also the expression for the opposite side a' from α. It thus follows that $\cosh a = \cosh a'$ and since the hyperbolic cosine is injective on $[0, \infty)$, we conclude that $a = a'$. □

Therefore, above mentioned constructions have proven theorem 7.2.1 and we conclude that a right-angled triangle is constructible in the Hyperbolic plane if two of the angles or sides are given. We end this chapter with a construction of an isosceles right triangle where only the hypotenuse is given. One can find this construction without proof in [9, p. 57].

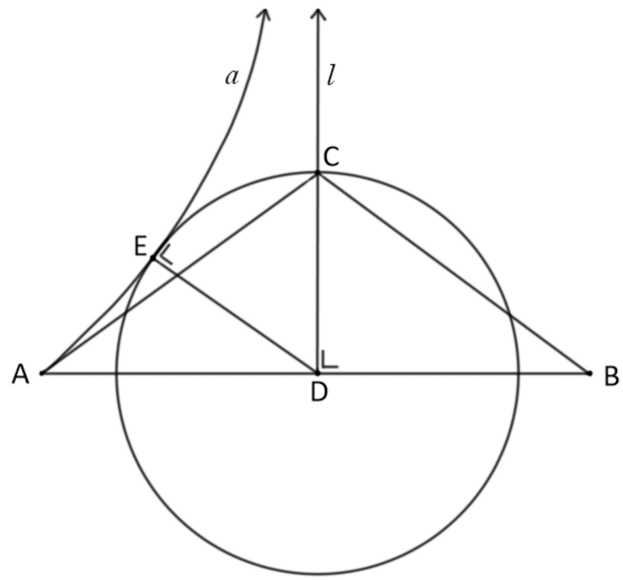


Figure 11: Construction of a right, isosceles triangle ABC.

Construction 7.2.5. *To construct a right triangle with equal short sides, given hypotenuse of length c.*

Proof. Let $\overline{AB}$ be a segment of length c. Construct the perpendicular bisector l to the segment $\overline{AB}$, which intersects $\overline{AB}$ in a point D. Using Bolyai's construction, let Aa be a ray limiting parallel to one ray on l and drop the perpendicular from D onto Aa, resulting in a point E. Drawing the circle with center D and radius $\overline{DE}$, it will intersect l in some point C. We will prove that the triangle ABC is the desired figure.

By construction, the triangles ADC and BDC share a common side $\overline{CD}$, have congruent right angles $\angle ADC \cong \angle BDC$ and have congruent sides $\overline{AD} \cong \overline{BD}$. Therefore, by the side-angle-side axiom (C6), it follows that the triangles ADC and BDC are congruent. In particular, segments $\overline{AC}$ and $\overline{BC}$ have equal length. It only remains to show that angle $\angle ACB$ is right.

Since rays Aa and $\overrightarrow{DC}$ are limiting parallel and $\angle DEa$ is a right angle, we have that

$$\Pi(CD) = \Pi(DE) = \angle CDE.$$

Therefore, by one of the trigonometric identities we obtain

$$\sin \angle CDE = \frac{1}{\cosh DE} = \frac{1}{\cosh CD}. \tag{7.9}$$

Moreover, from the right-angled triangle ADE we obtain

$$\cos \angle ADE = \frac{\tanh DE}{\tanh AD}. \tag{7.10}$$

Moreover, since $\angle ADE + \angle CDE = \frac{\pi}{2}$ we have that $\sin \angle CDE = \cos \angle ADE$. Therefore, by combining formulae (7.9) and (7.10) it follows that

$$\frac{1}{\cosh DE} = \frac{\tanh DE}{\tanh AD}, \tag{7.11}$$

$$\tanh AD = \sinh DE = \sinh CD. \tag{7.12}$$

Because triangle ADC is right-angled, it is hence true that

$$\tan \angle ACD = \frac{\tanh AD}{\sinh CD} = 1. \tag{7.13}$$

Therefore, $\angle ACD = \frac{\pi}{4}$ and since angles $\angle ACD$ and $\angle BCD$ are congruent, we conclude that $\angle ACB = \frac{\pi}{2}$. $\square$

8 Algebraic Analysis of Hyperbolic Constructions

Analogous to the Euclidean results presented in paragraph 2.4, this chapter will offer an algebraic description for the constructibility of lengths and angles in the hyperbolic plane. This description is based on the following important theorem and its corollary. Note first that we also call length $-x \in \mathbb{R}$ constructible if x is and consider 0 to be constructible too. Furthermore, recall that $\alpha \in \mathbb{R}$ is a constructible angle if $\alpha = 0$ or there exists an angle $\angle ABC$ with radian measure α modulo π. These are the same conventions as was the case for the Euclidean plane in paragraph 2.4.

Theorem 8.0.1. *(Mordukhai-Boltovskoi) Length x is constructible in $\mathbb{H}^2$ if and only if $\sinh x, \cosh x$ or $\tanh x$ are constructible in $\mathbb{E}^2$, these conditions being equivalent.*

Theorem 8.0.2. *An angle α is constructible in $\mathbb{H}^2$ if and only if it is constructible in $\mathbb{E}^2$.*

Proof. First, assume that α is a constructible angle of $\mathbb{E}^2$. If $\alpha = 0$, the result follows trivially. For $\alpha \neq 0$, we need to show that there exists some constructible angle in the hyperbolic plane with radian measure α modulo π. By producing perpendicular lines, it is always possible to construct the angle $\frac{\pi}{2}$. Therefore, we can assume that $\alpha \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ and consequently, $\sin \alpha \in (0, 1)$. Since α is a constructible angle in $\mathbb{E}^2$, the length $\sin \alpha$ is constructible in $\mathbb{E}^2$. So, by Mordukhai-Boltovskoi's theorem we can construct a length a in $\mathbb{H}^2$ satisfying $\sinh a = \sin \alpha$. Moreover, we will see in lemma 8.1.1 that a segment of length c satisfying $\sinh c = 1$ is constructible in the hyperbolic plane. Since the hyperbolic sine is an increasing function, we have that $a < c$ and by theorem 7.2.1, it is thus possible to construct a right-angled triangle with hypotenuse c and short side a. Moreover, the angle $\alpha' \in (0, \frac{\pi}{2})$ opposite to a satisfies

$$\sin \alpha' = \frac{\sinh a}{\sinh c} = \sin \alpha.$$

Therefore, we either have that $\alpha' = \alpha$ or $\alpha' = \frac{\pi}{2} - \alpha$. As such, we can conclude that α is constructible in the hyperbolic plane.

Next, we assume that α is a constructible angle of $\mathbb{H}^2$. Again, we can assume that $\alpha \in (0, \pi) \setminus \{\frac{\pi}{2}\}$. By extending one arm of the angle, it is possible to construct the angle $\frac{\pi}{2} - \alpha$. Therefore, we can assume that α is acute. By theorem 7.2.1, a right-angled triangle with hypotenuse c, which satisfies $\sinh c = 1$, and one angle equal to α is constructible. Moreover, the side a opposite to α is given by

$$\sinh a = \frac{\sinh a}{\sinh c} = \sin \alpha.$$

Because a has been shown to be constructible in $\mathbb{H}^2$, it thus follows from Mordukhai-Boltovskoi's theorem that $\sin \alpha$ is a constructible length in $\mathbb{E}^2$. Since this is equivalent with α being a constructible angle in $\mathbb{E}^2$, the proof is concluded. $\square$

For theorem 8.0.1, we will give an altered proof presented by Robert R. Curtis [9, §7], who on his turn credits Mordukhai-Boltovskoi [10]. In fact, we have already given parts of the proof in the constructions of right triangles in the previous chapter. Before we prove this important result, we give two interesting applications. First and foremost, recall that trisecting any given angle was one of the impossible classical problems. Mordukhai-Boltovskoi's theorem not only shows that in this regard, $\mathbb{H}^2$ and $\mathbb{E}^2$ are equal, but also that in the hyperbolic case, attempting to trisect a line segment can lead to complications too.

Corollary 8.0.3. *Generally, it is not possible to trisect an angle nor a segment in $\mathbb{H}^2$ using compass and straightedge constructions.*

Proof. Since constructibility of angles in $\mathbb{H}^2$ and $\mathbb{E}^2$ are equivalent, the first assertion follows from the impossibility of the problem in the Euclidean plane.

If x is a length constructible in $\mathbb{H}^2$, then $\cosh x, \sinh x \in E$ and so,

$$e^x = \cosh x + \sinh x \in E.$$

Therefore, for $\frac{x}{3}$ to be constructible in $\mathbb{H}^2$ would require that $e^{\frac{x}{3}} \in E$. However, $e^{\frac{x}{3}}$ is a zero from the cubic equation

$$y^3 - e^x = 0.$$

By choosing a length x such that the cubic polynomial is irreducible over $\mathbb{Q}$, we find that $\mathbb{Q}(e^{\frac{x}{3}})$ has degree 3 over $\mathbb{Q}$ and therefore, $e^{\frac{x}{3}} \notin E$. For example, choosing $e^x = 2$ implies that $\sqrt[3]{2} \notin E$.

□

The second direct corollary is that constructibility of regular polygons in $\mathbb{H}^2$ and $\mathbb{E}^2$ are equivalent. Indeed, it can be shown in both planes that a polygon with $n \geq 3$ sides is constructible if and only if the angle $\frac{2\pi}{n}$ is constructible. We shall shortly demonstrate this fact. If we start with the regular polygon of n sides, then we can draw lines from each vertex to their opposite vertices or from each vertex to the center of the opposite segment in the case of odd n. The lines from two adjacent vertices meet in the center of the polygon where they make an angle of $\frac{2\pi}{n}$ radians. This follows from the fact that we have divided the regular polygon in n congruent triangles. When we start with an angle $\frac{2\pi}{n}$, we can repeat its construction around a point until we have the same angle n times around a point O. We now draw a circle with center O. This circle will intersect the arms of the angles we have constructed and by connecting these intersection points, we have found a regular polygon of n sides.

So far, we have found that two of the four classical construction problems behave the same in Euclidean and Hyperbolic planes. Moreover, the interested reader could resort to an article by R. Curtis [9] on doubling a cube. The problem of squaring the circle does have a surprising answer in hyperbolic geometry first found by Bolyai and we shall discuss this problem in its full extent in 9.

8.1 Proof of Mordukhai-Boltovskoi's Theorem (1)

As mentioned before, theorem 8.0.1 was first discovered by Russian mathematician Mordukhai-Boltovskoi. According to [8] and [9] he presented the original proof of the theorem in [10]. However, I could not find a copy of Mordukhai-Boltovskoi's article, let alone one translated from the original Russian. The proof I present here is strongly inspired by the one presented by Curtis in [9], which in turn had his based on the original proof in [10].

To make things more readable, we shall divide the proof in two parts, both covering one side of the implication. In this paragraph, we shall show that if x is a constructible length by Euclidean tools, then $\sinh x, \cosh x, \tanh x$ will be constructible lengths in $\mathbb{H}^2$.

Now consider the set H of all real numbers y such that arcsinh$y$ is a constructible length in $\mathbb{H}^2$. The argument thus boils down to demonstrating that H is equal to the Euclidean field E, consisting of all constructible lengths of $\mathbb{E}^2$. Since E is the smallest field containing $\mathbb{Q}$ and closed under taking square roots of positive elements (see also §2.4), to show that $E \subset H$, it suffices to show that H is also a field containing $\mathbb{Q}$ and closed under taking square roots. We shall demonstrate this through a series of lemma's.

Lemma 8.1.1. *We can construct Schweikart's constant p, the unique length in $\mathbb{H}^2$ that satisfies*

$$\sinh p = 1, \text{ and } \cosh p = \sqrt{2}.$$

Proof. By bisecting a right angle, we can construct an angle of radian measure $\frac{\pi}{4}$ and by Bonola's construction (7.1.2), we can find a length p such that $\Pi(p) = \frac{\pi}{4}$. By the trigonometric identities of Π (6.1.2), this length p satisfies

$$\sinh p = 1 \text{ and } \cosh p = \sqrt{2}.$$

□

Lemma 8.1.2. *The set H is closed under taking reciprocals; for each non-zero $x \in H$ we have $\frac{1}{x} \in H$.*

Proof. Given a length $x \neq 0$, we ought to construct a length x' such that

$$\sinh x' = \frac{1}{\sinh x}.$$

Construct the angle of parallelism $\alpha = \Pi(x)$ by 7.1.1 and draw the angle $\alpha' = \frac{\pi}{2} - \alpha$ by letting the angle α be contained within a right angle. Now let x' be the length characterized by $\Pi(x') = \frac{\pi}{2} - \alpha$, which is constructible because of Bonola's construction (7.1.2). Then,

$$\sinh x' = \cot\left(\frac{\pi}{2} - \alpha\right) = \tan(\alpha) = \frac{1}{\sinh x}.$$

□

Lemma 8.1.3. *For each constructible length $x > 0$, we can construct a length $y > 0$ such that*

$$\sinh y = \cosh x$$

and reversely, we can construct for each length $x > 0$ with $\sinh x > 1$ a length $y > 0$ such that

$$\cosh y = \sinh x.$$

Proof. First, we construct a length $y > 0$, given a length $x > 0$, such that

$$\sinh y = \cosh x.$$

Consider the right-angled triangle formed with hypotenuse p and angle $\alpha = \Pi(x)$. This triangle is constructible by 7.2.1 and 7.1.1. Due to the trigonometric identities, the following relationship holds concerning the length y' opposite to α,

$$\frac{1}{\cosh x} = \sin\alpha = \frac{\sinh y'}{\sinh p} = \sinh y'.$$

Using lemma 8.1.2 we thus find y such that $\sinh y = \frac{1}{\sinh y'} = \cosh x$.

Secondly, we show the reverse. So, let $x > 0$ be a length with $\sinh x > 1$. Because the hyperbolic sine is a strictly increasing function, it follows from $\sinh x > 1$ that $x > p$ and so, we can construct a right triangle with hypotenuse x and one short side p (7.2.1). Moreover, the angle α opposite to p satisfies

$$\sin\alpha = \frac{\sinh p}{\sinh x} = \frac{1}{\sinh x}$$

and the length y characterized by $\Pi(y) = \alpha$ thus gives

$$\cosh y = \frac{1}{\sin\alpha} = \sinh x.$$

Since α is constructible, Bonola's construction 7.1.2 shows that y is constructible and hence, the proof is concluded. □

Lemma 8.1.4. *The set H is closed under multiplication. For each $x, y \in H$ we have $xy \in H$.*

Proof. Given $\sinh x, \sinh y \in H$, if one were 0, then the product would again be equal to 0 and by convention, $0 = \sinh 0 \in H$. So, we can assume that we can construct segments of length $|x|, |y|$. There are three different cases to be considered

1. First, we assume that $\sinh|x|, \sinh|y| > 1$. Then, by lemma 8.1.3 we can construct segments a, b such that
$$\cosh a = \sinh|x|, \qquad \cosh b = \sinh|y|.$$
Now construct a right triangle with short sides a, b. By the hyperbolic alternative to Pythagoras' theorem, we obtain for the hypotenuse z of this triangle
$$\cosh z = \cosh a \cosh b = \sinh|x| \sinh|y|.$$
Again by lemma 8.1.3, it then follows that $\sinh|x| \sinh|y| \in H$.

2. For the second case we let $\sinh|x| < 1 < \sinh|y|$. We can already construct p such that $\sinh p = 1$ and so, we can assume that $\sinh|y| \sinh|x| \neq 1$. Using lemmas 8.1.2 and 8.1.3, we then construct segments with lengths a, b such that
$$\cosh a = \frac{1}{\sinh|x|}, \qquad \cosh b = \sinh|y|.$$
Now let $u = \max\{a, b\}$ and $v = \min\{a, b\}$. Since $\sinh|x| \sinh|y| \neq 1$ we have that $u \neq v$ and so, we can construct a right triangle with hypotenuse u and short side v. The other side w of the triangle will satisfy
$$\cosh w = \frac{\cosh u}{\cosh v} = \begin{cases} \sinh|x| \sinh|y|, & \text{if } a > b \\ \frac{1}{\sinh|x| \sinh|y|}, & \text{if } a < b. \end{cases}$$
In either case, it follows from lemma's 8.1.3 and 8.1.2 that we can construct a length z such that $\sinh z = \sinh|x| \sinh|y|$.

3. Lastly, we recognize the case when $\sinh|x|, \sinh|y| < 1$. By lemma 8.1.2, we can translate this problem into the first case and so, this follows from earlier steps.

By considering all three cases, we conclude that $\sinh|x| \sinh|y| \in H$ if $\sinh x, \sinh y \in H$. Moreover, the hyperbolic sine is an odd function and so we can write
$$\sinh x \sinh y = \lambda \cdot \sinh|x| \sinh|y|$$
for $\lambda = \pm 1$. By convention we agreed that $-a \in H$ if $a \in H$ and so, we conclude that $\sinh x \sinh y \in H$ if $\sinh x, \sinh y \in H$. $\square$

Combining lemma's 8.1.2 and 8.1.4 we conclude in particular that we can divide any two non-zero elements of H.

Lemma 8.1.5. *The set H is closed under addition and subtraction.*

Proof. We show that for any $\sinh x, \sinh y \in H$ it follows that
$$\sinh x \pm \sinh y \in H.$$
For this, we shall use the identity
$$\sinh x \pm \sinh y = 2 \sinh \frac{x \pm y}{2} \cosh \frac{x \pm y}{4}. \tag{8.1}$$
For x, y to be constructible it means that $|x|, |y|$ are lengths of segments that can be constructed via compass and straightedge in $\mathbb{H}^2$, or that one of them is equal to 0. Since $\sinh 0 = 0$, the case for 0 is trivial and we can assume that $x, y \neq 0$. By adding or subtracting line segments, we can construct a segment of length $|x \pm y|$ and we can then bisect this segment. So, $\frac{x \pm y}{2}$ is constructible in $\mathbb{H}^2$. Moreover, note that $\cosh^2 p = 2$ and so, $2 \in \mathbb{H}$. Therefore, $\sinh x \pm \sinh y \in H$ follows from equation 8.1 and the previous lemma's. $\square$

Lemma 8.1.6. *The set H is closed under taking square roots of positive elements*

Proof. Given a positive $\sinh x \in \mathbb{H}$ we thus want to show that there exists some constructible segment of length y such that

$$\sinh^2 y = \sinh x.$$

We can assume that $\sinh x > 1$. Otherwise, we can take its reciprocal $\sinh x' = \frac{1}{\sinh x}$, solve the problem to find y' with $\sinh^2 y' = \sinh x'$ and take its reciprocal $\sinh y = \frac{1}{\sinh y'}$. With $\sinh x > 1$ we can construct a length a such that $\cosh a = \sinh x$ and then use construction 7.2.5 to form a right isosceles triangle with hypotenuse a. Therefore, for the length b of the short sides we have

$$\cosh^2 b = \cosh a.$$

Constructing y such that $\sinh y = \cosh b$, we conclude that $\sinh^2 y = \sinh x$. □

Corollary 8.1.7. *Combining previous lemma's, we conclude that $E \subset H$; if $\sinh x$ is a constructible length in $\mathbb{E}^2$, then x is a constructible length in $\mathbb{H}^2$. Moreover, $\sinh x \in E$ is equivalent with $\cosh x \in E$, which in turn is equivalent with $\tanh x \in E$ (as long as $\tanh x$ exists).*

Proof. Because Schweikart's constant is constructible, we know that $1 \in H$. In particular, it cannot be the trivial field $\{0\}$. Because H is also closed under multiplication, division, addition and subtraction we know it must be a field. Because $H \subset \mathbb{R}$, it has characteristic 0 and so H must at least contain the rationals. Furthermore, H is closed under taking square roots of positive elements and therefore, $E \subset H$.

Now let $x \in \mathbb{R}$. Because of the identity

$$\cosh^2 x - \sinh^2 x = 1$$

we can always obtain $\cosh x$ from $\sinh x$ by operations legal in E, and vice versa. Moreover, from the identities

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \sinh^2 x = \frac{\tanh^2 x}{1 - \tanh^2 x}$$

we see that we can also obtain $\tanh x$ from $\sinh x$ and vice versa from operations in E. Therefore, $\sinh x \in E$ iff. $\cosh \in E$ iff. $\tanh \in E$ (as long as $\cosh x \neq 0$). □

8.2 Proof of Mordukhai-Boltovskoi's Theorem (2)

In this section, we will be concerned with proving the second implication of Mordukhai-Boltovskoi's theorem: if $x \in \mathbb{R}$ is constructible in $\mathbb{H}^2$, then $\sinh x$ is constructible in $\mathbb{E}^2$. To make this fact apparent, we will be using Poincaré's half-plane model to compare constructions in $\mathbb{H}^2$ with those of $\mathbb{E}^2$. In particular, we will be proving the following theorem.

Theorem 8.2.1. *If a point A inside Poincaré's half-plane model is constructible by hyperbolic compass and straightedge then it is also constructible by Euclidean compass and straightedge in the Euclidean plane where Poincaré's model resides.*

Curtis proves this theorem in the case of the Klein model [9, §6] but its proof can easily be adapted to Poincaré's model. We base our proof on the Euclidean constructions of a hyperbolic line and a hyperbolic circle. The hyperbolic line goes as follows:

Construction 8.2.2. Given two points A, B in the upper half-plane, to construct the hyperbolic line through them using the Euclidean compass and straightedge.

Proof. If A, B lie on a vertical line, then with the straightedge we can directly draw the line through it. For the remainder of the proof, we thus assume that the Euclidean line AB is not perpendicular to the horizontal axis. Construct the center C of the line segment AB and let m be the perpendicular through C on AB. If AB is parallel to the horizontal axis, then m has to intersect the horizontal axis because there can be only one parallel through the point C. If AB is not parallel, then the line AB has to meet the horizontal axis in a non-right angle. So, the internal angles made by AB through the line m and the horizontal axis do not add up to two right angles. By Euclid's fifth postulate, the line AB and the horizontal axis thus have to intersect. Call the point of intersection M and draw the circle γ with center M through the point A. Since the triangles ACM and BCM both have a right angle and two congruent sides, they are congruent and therefore, the intersection of γ with the upper half-plane is the hyperbolic line through A and B. □

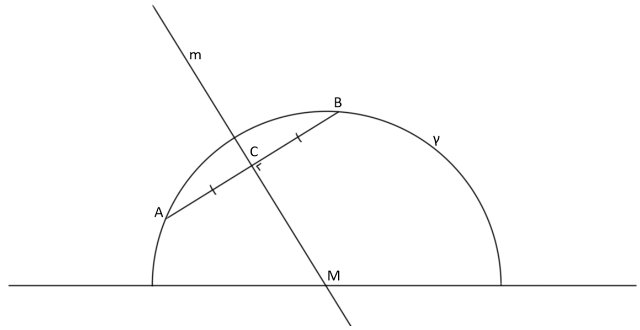


Figure 12: Construction of the hyperbolic line AB by Euclidean tools.

Before we can give the construction of the hyperbolic circle, we will need the following result, which describes the model circle in Euclidean terms.

Theorem 8.2.3. *Let M be a point in the Poincaré upper half-plane with imaginary coordinate h and let $t > 0$ be any length. Also consider the point M' which has the same real coordinate as M and has imaginary coordinate $h \cosh t$. Then, the hyperbolic circle with center M and radius t coincides with the Euclidean circle with center M' and radius $h \sinh t$.*

Proof. With the aim of avoiding confusion, we will denote the Euclidean distance by d while d_H stands for the hyperbolic distance.

Let A be any point on the Euclidean circle with center M' and radius $h \sinh t$. If A is a point on the vertical line MM', then its imaginary coordinate would either be

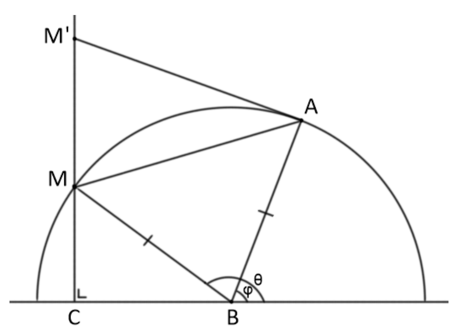
$$h \cosh t + h \sinh t = he^t \quad \text{or} \quad h \cosh t - h \sinh t = he^{-t}.$$

In either case,

$$d_H(M, A) = \left| \log \left(\frac{he^{\pm t}}{h} \right) \right| = t.$$

So, if A would lie on the vertical line MM', then indeed it would also lie on the hyperbolic circle with center M and radius t. From now on, we assume that A does not lie on the vertical line. Furthermore, we assume that Re $A >$ Re M, since the case that Re $A <$ Re M is very similar.

By projecting M onto the real axis, we gain the point C. Also let B be the center of the half-circle through A and M, centered at the real axis. Denote by θ the radian measure of the angle made by BM with the positive real axis and φ the angle made by AB and the positive real axis. Since the hyperbolic distance from A to M is determined by the angles θ, φ, we seek to express t in terms of θ, φ.



For this, note that the Euclidean angle $\angle CBM$ has measure $\pi - \theta$ and from the right triangle BCM we thus obtain

$$d(B, M) = \frac{d(C, M)}{\sin(\pi - \theta)} = \frac{h}{\sin \theta}. \tag{8.2}$$

Moreover, because A, M lie on the half-circle with center B, we have that $d(B, M) = d(B, A)$. Note also that angle $\angle ABM$ has measure $\theta - \varphi$. Applying the cosine rule to the triangle ABM, it thus follows that

$$d(A, M)^2 = 2\frac{h^2}{\sin^2\theta} - 2\frac{h^2}{\sin^2\theta}\cos(\theta - \varphi) \tag{8.3}$$

$$= \frac{4h^2 \sin^2\left(\frac{\theta-\varphi}{2}\right)}{\sin^2\theta}. \tag{8.4}$$

Since the Euclidean triangle ABM is isosceles and angle ABM has measure $\theta - \varphi$, the remaining base angles will have measure $\frac{\pi+\varphi-\theta}{2}$. Also, we have that $\angle BCM$ is right and $\angle CBM$ has measure $\pi - \theta$, from which it follows that angle $\angle BMC$ equals $\theta - \frac{\pi}{2}$. We deduce from this that angle $\angle AMM'$ has measure

$$\pi - \left(\theta - \frac{\pi}{2}\right) - \left(\frac{\pi + \varphi - \theta}{2}\right) = \pi - \frac{\theta + \varphi}{2}.$$

By the cosine rule, we can now deduce that

$$d(A, M')^2 = d(M, M')^2 + d(A, M)^2 - 2d(M, M') \cdot d(A, M)\cos\angle AMM', \tag{8.5}$$

$$h^2\cosh^2 t = (h\cosh t - h)^2 + \frac{4h^2\sin^2\left(\frac{\theta-\varphi}{2}\right)}{\sin^2\theta} - 2(h\cosh t - h)\frac{2h\sin\left(\frac{\theta-\varphi}{2}\right)}{\sin\theta}\cos\left(\pi - \frac{\theta+\varphi}{2}\right). \tag{8.6}$$

Using some trigonometric identities, one could show that the following hold

$$2\sin^2\left(\frac{\theta-\varphi}{2}\right) = 1 - \cos\theta\cos\varphi - \sin\theta\sin\varphi, \qquad 2\sin\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right) = \sin\theta - \sin\varphi. \tag{8.7}$$

Substituting these into (8.6) and performing more algebraic manipulations, we deduce that (8.6) is equivalent to

$$\cosh t = \frac{1 - \cosh\theta\cos\varphi}{\sin\theta\sin\varphi}. \tag{8.8}$$

Substituting the double angle formulae, we further deduce that

$$\cosh t = \frac{1}{2}\left(\cot\frac{\theta}{2}\tan\frac{\varphi}{2} + \tan\frac{\theta}{2}\cot\frac{\varphi}{2}\right) \tag{8.9}$$

$$= \cosh(d_H(A, M)). \tag{8.10}$$

Because the hyperbolic cosine is injective on the interval $[0, \infty)$, we conclude that $t = d_H(A, M)$ and therefore, A lies on the hyperbolic circle with center M and radius t. This proves that the Euclidean circle is contained within the Hyperbolic one. In order to prove the reverse, note that our reasoning has shown formulae 8.6 and 8.10 to be equivalent. Therefore, starting from the assumption that 8.10 holds, it follows that 8.6 holds and because of the cosine rule, this implies that $d(A, M') = h\cosh t$. $\square$

A direct corollary from theorem 8.2.3 is the following result, which can be proven by applying the sine or cosine rule to the Euclidean triangle AMM'. Since this proof is very similar to theorem 8.2.3, it is omitted.

Corollary 8.2.4. *Let M, A be any pair of points in the upper half-plane such that A does not lie on the vertical line through M and let M' be the center of the Euclidean circle which corresponds with the hyperbolic circle with center M and that passes A. Then, the Euclidean line AM' is tangent to the half-circle through A and M.*

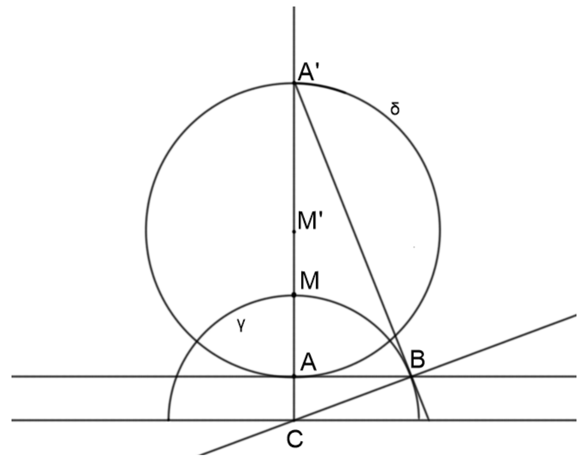
Finally, the necessary results have been acquired to construct the hyperbolic circle.

Construction 8.2.5. Given two points M, A in the upper half-plane, to construct the hyperbolic circle with center M that goes through A using the Euclidean compass and straightedge.

Proof. Let h be the imaginary coordinate of M and $t = d_H(A, M)$.

First off, let us consider the case that A, M lie on the same vertical line. Then, A will either have imaginary coordinate he^t or he^{-t}. By theorem 8.2.3, the Euclidean center M' of the desired circle has real coordinate equal to M and imaginary coordinate $h \cosh t = \frac{1}{2}(he^t + he^{-t})$. Therefore, we would like to construct from the point A with imaginary coordinate $he^{\pm t}$ the point A' with imaginary coordinate $he^{\mp t}$ and take the center of the segment $\overline{AA'}$. We can achieve this by inverting the point A through the circle which passes M and with center the intersection of the vertical line through M and the real axis.

For this, we first assume that A is below the center point M. That it, A lies between M and the point of intersection of AM and the real axis. Then, draw the line through A and M and call C the intersection of this line with the real axis. With the compass, draw the Euclidean circle γ with center C and that passes M. Also drop the perpendicular from A onto the vertical line AM. Since A is below the point M and still in the upper half plane, the point A is contained in the interior of the circle γ. By theorem 3.5.2, the horizontal line through A and the circle γ will intersect in a point B. By drawing the line CB and the perpendicular of CB through B, we have constructed a tangent line to the circle γ passing B. Since $\angle BAC$ is right, angle $\angle ACB$ must be acute.



From the parallel postulate it thus follows that the line perpendicular to BC must meet the vertical line AC in a point A'. Since the sum of angles of Euclidean triangles always add up to π radians, it can be shown that $\angle ACB$ and $\angle ABA'$ are congruent. Therefore, triangles $AA'B$ and ABC are similar. Moreover, from the Pythagorean theorem it follows that $d(A, B) = h\sqrt{1 - e^{-2t}}$. Since triangles $ABC, AA'B$ are similar, it follows that

$$\frac{d(A, A')}{d(A, B)} = \frac{d(A, B)}{d(A, C)}$$
$$d(A, A') = \frac{h^2(1 - e^{-2t})}{he^{-t}} = h(e^t + e^{-t}).$$

Therefore, point A' has imaginary coordinate he^t. We now construct the center M' of the segment $\overline{AA'}$ and draw the Euclidean circle δ with center M' that passes A. Because of theorem 8.2.3, circle δ corresponds with the Hyperbolic circle with center M and radius $d_H(A, M)$.

For the case that A lies above M, we can use proposition 17 from Euclid's third book to draw a tangent line AB to the circle γ. If B is the intersection of this line with the circle γ, then drop the perpendicular from B onto AM, and call its intersection A'. Again, let M' be the center point of the segment $\overline{AA'}$ and draw the circle with center M' through A.

Secondly, we will consider the case that A does not lie on the vertical line through M. Then, we construct the hyperbolic line AM as in 8.2.2, which also gives us the center B of this half-circle. Now draw the perpendicular m to BA at A. By corollary 8.2.4 and the fact that perpendicular lines passing a point are unique, it follows that the line m intersects the vertical line AM in the point M', which is the Euclidean center of the circle. □

From the constructions 8.2.2 and 8.2.5, we note that any hyperbolically constructible point in Poincaré's model is also constructible by Euclidean instruments. Now consider the point A with complex coordinate i and the point B with coordinate ie^t, where $t > 0$ is any positive length. The hyperbolic distance between A and B is given by $d_H(A, B) = t$, while the Euclidean distance is given by $e^t - 1$. So, if t is a constructible length in $\mathbb{H}^2$, then $e^t - 1$ is a constructible length in $\mathbb{E}^2$. In other words, $e^t - 1$ is contained inside the Euclidean field

E, which is closed under addition, division and it contains the integers. Therefore, $\sinh t = \frac{1}{2}\left(e^t + e^{-t}\right) \in E$. So, we conclude that the second implication of Mordukhai-Boltovskoi's theorem also holds.

9 Squaring the Circle

To square a circle means to construct a circle and a square with equal area. Constructibility of the square means that we can find by straightedge and compass its four vertices while call a circle constructible if its radius is. The area of the square is given by the defect of the angles, which is the difference between the sum of angles the square would have in the Euclidean plane versus the sum of the angles it has in the hyperbolic plane. See also [2, §36] for a derivation of the area function. In particular, a hyperbolic square with internal angle σ has area $2\pi - 4\sigma$. As for the circle, we define its area to be the limit of the area of the regular n-gons contained in it, as we let $n \to \infty$. It turns out that this limit exists [7, Thm. 10.7] and with radius r, the area is given by

$$4\pi \sinh^2\left(\frac{r}{2}\right).$$

When comparing these two quantities, a first observation could be that the area of a square is bounded by 2π while that of the circle is unbounded. So, at least not all squares can be constructible with area equal to a constructible circle. Nonetheless, choosing $\sigma = \frac{\pi}{4}$, a square having area π is constructible and since $\frac{1}{2} \in E$, the same goes for the circle. The constructibility of the square follows from bisecting a right angle and 7.2.3, while the constructibility of the circle follows from Mordukhai-Boltovskoi's theorem 8.0.1.

Therefore, to the question whether squaring the circle is possible in hyperbolic geometry, the answer is: sometimes. Moreover, it turns out that we can characterize all possible constructions in a surprising way, linking this problem to the construction regular polygons. For this, we follow a slightly altered proof found in [8], starting with the following observation.

Lemma 9.0.1. *Let $x > 0$. A hyperbolic square of area πx is constructible if and only if πx is a constructible angle, while a hyperbolic circle of area πx is constructible if and only if it is possible to construct the length x in E^2.*

Proof. First, we consider a square with area πx and which has internal angle σ. Then,

$$2\pi - 4\sigma = \pi x.$$

If this square is constructible, then certainly its angle is and by 7.2.3 we know that constructibility of σ implies constructibility of the square. Moreover, given σ, we can double it with compass and straightedge twice to gain the angle angle $2\pi - 4\sigma$. Similarly, we can obtain σ from $2\pi - 4\sigma$. Therefore, the square is constructible if and only if its area is a constructible angle.

Next, we consider a circle with radius r and area πx. This circle is constructible if and only if its radius r is. Moreover, we can always double or halve a segment with the ruler and compass, so constructibility of the circle is equivalent to that of the length $\frac{r}{2}$. By Mordukhai-Boltovskoi 8.0.1, this means that $\sinh \frac{r}{2} \in E$. Since E is a field closed under taking square roots of positive elements (2.4.1), this in turn holds if and only if $4 \sinh^2 \frac{r}{2} \in E$. By recalling that the circle area is given by

$$\pi x = 4\pi \sinh^2 \frac{r}{2},$$

we can conclude that the circle is constructible in $\mathbb{H}^2$ if and only if the length x is in E^2. □

By this lemma, we have moved our geometrical problem to the realm of number theory. As for the rest of the proof, we will need the following theorem.

Theorem 9.0.2. *(Gelfond-Schneider) If φ, χ are non-zero algebraic numbers, $\varphi \neq 1$ and $\chi \notin \mathbb{Q}$, then any value of φ^χ is transcendental.*

Note that algebraic numbers can be complex. In that case, the exponential φ^χ is defined as $e^{\chi \log \varphi}$ and so, this is multivalued. We continue to our final result.

Theorem 9.0.3. *Let a square with corner angles σ and circle in $\mathbb{H}^2$ have equal area. Then, both are constructible if and only if $\sigma \in (0, \frac{\pi}{2})$ and σ is an integer multiple of $\frac{2\pi}{n}$, where n is the number of sides of a regular polygon which can be constructed with compass and straightedge.*

Proof. First, assume that both the square and circle are constructible. Then, by lemma 9.0.1 we know that the shared area πx is a constructible angle and $x \in E$. This means that $\cos \pi x, \sin \pi x \in E$ and so also, $e^{i\pi x} \in E(i)$. Moreover, since $e^{i\pi} = -1$, one possible value of $(-1)^x$ is $e^{i\pi x}$. Any number $y \in E(i)$ is certainly algebraic and therefore, we have that $x, -1$ are both non-zero algebraic, $-1 \neq 1$ and one value of $(-1)^x$ is also algebraic. By the contrapositive of Gelfond-Schneider's theorem, it follows that $x \in \mathbb{Q}$ so we can write $x = \frac{p}{q}$ with $gcd(p, q) = 1$. Using Bézout's identity, there exist integers $a, b \in \mathbb{Z}$ such that

$$ap + bq = 1.$$

Now multiplying by $\frac{\pi}{q}$ we obtain

$$ax + b\pi = \frac{\pi}{q}.$$

Since we can always add or subtract angles using compass and straightedge, x, π are constructible angles and a, b are integers, it follows that $ax + b\pi = \frac{\pi}{q}$ is constructible. Recall from paragraph 2.4 that a polygon of $2q$ sides is constructible iff the angle $\frac{2\pi}{2q}$ is constructible, which is equivalent to q having prime factorization

$$q = 2^l F_1 F_2 \cdots F_r,$$

where $F_1, F_2, .., F_r$ are distinct Fermat primes. Since $2\pi - 4\sigma = \pi x$, this means for σ that

$$\sigma = \frac{2\pi - \pi x}{4} = (2q - p) \cdot \frac{2\pi}{8q}.$$

Letting $n := 8q$, we have that σ is an integer multiple of $\frac{2\pi}{n}$ and that n has prime factorization

$$n = 2^{l+3} F_1 F_2 \cdots F_r$$

Hence, the polygon of n sides is also constructible.

Conversely, assume that σ is an acute angle and an integer multiple of $\frac{2\pi}{n}$ with n the number of sides of a constructible regular polygon. That is, $\sigma = k \cdot \frac{2\pi}{n}$ for some $k \in \mathbb{N}$. Then, the angle σ is certainly constructible by adding $\frac{2\pi}{n}$ enough times to itself. Moreover, as $\frac{2k}{n}$ is rational, it certainly exists inside the field E. Therefore, by lemma 9.0.1 we conclude that the square and circle of equal area are constructible. $\square$

10 Conclusion

Although we cannot use a physical straightedge and compass to draw on a hyperbolic surface, the constructions with these tools could be defined abstractly in terms of circles and straight lines. Moreover, Mordukhai-Boltovskoi's theorem and the squaring of a circle have shown the hyperbolic construction game to be similar to, but distinct from the Euclidean case. This first result proves that the functions $\sinh, \cosh, \tanh$ give a correspondence between the constructible hyperbolic and Euclidean lengths, and that the constructible hyperbolic and Euclidean angles coincide. Additionally, we have seen how these functions are used in hyperbolic trigonometry, which served to prove the constructions exclusive to the hyperbolic plane. Of these, the most important were Bolyai's and Bonola's angle of parallelism constructions.

Were the link between Euclidean and hyperbolic constructions provided by Mordukhai-Boltovskoi not surprising enough, then there was also the problem of squaring the circle. Not only do we conclude on the note, that this is at times plausible in the hyperbolic plane, but that it is also strongly connected to the Euclidean problem of constructing regular polygons.

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