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Regular Homotopies and the Sphere Eversion

Bachelor Thesis

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June 13, 2021

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1 Introduction

In the 1940's, Hassler Whitney initiated the study on immersions. Recall that an immersion is defined as follows:

Definition 1.0.1 (Immersion)**.** Given two smooth manifolds V, W, a function $f : V \to W$ is an immersion if and only if its derivative at v, which is the linear map $(df)_v : T_vV \to T_wW$ is injective for every $v \in V$.

There is one main question when talking about immersions. This question is about the classification of immersions. The classification is given as follows: two immersions are equivalent, or of the same class, if there exists a regular homotopy connecting the two.

Definition 1.0.2 (Regular homotopy)**.** Given two smooth manifolds V, W and the unit interval I, a regular homotopy is a smooth map $H : V \times I \to W$ such that $H(v, i) = h_i(v)$, where $h_i : V \to W$, is an immersion for all $i \in I$.

Whitney showed in [Whi55] that, given a manifold V, there exist an abundance of immersions of V into $\mathbb{R}^m$, given that $m \geq 2\dim(V)$. Ralph Cohen refined this in his paper [Coh85], showing that given a smooth manifold V with dimension n, there exists an immersion $f : V \to \mathbb{R}^{n-a(n)}$, where $a(n)$ is the number of 1s in the binary representation of n.

In Section 3, we will prove the Whitney immersion theorem. In the same section, we will prove the parametric version of the same theorem. This tells us that for a submanifold V of dimension n, the space of regular homotopies of V into R^m, with $m \geq 2n + 1$, lies dense in the space of homotopies. This gives us that, if we find a homotopy between two immersions, there will also exist a regular homotopy between the two. Additionally, the parametric version talks about 'higher' homotopies, meaning homotopies parameterized over I^k for arbitrary k. In order for us to prove these theorems, we need the Thom Transversality theorem, which we will prove in Section 3. This theorem, named after the French mathematician René Thom, tells us that transversality is a generic property. Then by turning the immersion requirement into a statement about transversality, we can show that being an immersion is a generic property as well, given a big enough codimension.

Before proving his immersion theorem, Whitney found a classification of the immersions from S^1 to $\mathbb{R}^2$, showing that the set of equivalence classes is exactly $\mathbb{Z}$, where each immersion corresponds with its turning number. This result, which was from his paper [Whi37], is known as the Whitney-Graustein theorem, seen as part of the proof of the theorem was given to Whitney by Graustein.

Smale extended this result, by giving the classifications of immersions from S^1 to any C^2 manifold with dimension greater or equal to 2. He then went on to give classification of the immersions from S^2 to any $\mathbb{R}^m$ where $m > 2$ in his paper [Sma59]. One surprising result he found is Smale's sphere eversion, which tells us that the canonical embedding of the sphere into $\mathbb{R}^3$ is equivalent to the inverse embedding, that sends all points on the sphere to their opposites in $\mathbb{R}^3$. We will prove this result in Section 5. In order for us to complete this proof, we will need to introduce a new tool. This tool is the Holonomic Approximation theorem, which we will prove in Section 4. This theorem has many useful applications, one of which being a way to construct regular homotopies.

Both the Whitney immersion theorem and the Holonomic Approximation theorem require the use of jet spaces. This is why in Section 2 we will introduce the concept of jet spaces, along with some operations on them. We will also introduce the Whitney C^∞ topology on the space of smooth functions that is induced by the jet spaces.

In this thesis, the notation $Op(A)$ represents an open neighborhood of A that retracts back to A.

2 Jet spaces

When talking about maps between Euclidean spaces $\mathbb{R}^n, \mathbb{R}^m$, it is possible to visualise a map $f : \mathbb{R}^n \to \mathbb{R}^m$ as a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$. We call the image of this function the graph of f. Since the image of $\tilde{f}$ is a subset of $\mathbb{R}^n \times \mathbb{R}^m$, at each point we can look at the tangent space of this subset. This tangent space corresponds to the value and the derivative of f at that point. In a similar way, we can represent the value, first derivative and second derivative of f at a point by a paraboloid through that point and so on. This way of looking at a function and its derivative can be generalised to non-Euclidean smooth manifolds through the concept of jet spaces. In this section we will introduce this concept along with some of its operations. This will provide us with the language we need in subsequent sections. We will show that these jet spaces are smooth manifolds, and that they induce a topology on the space of smooth functions.

This chapter will be following chapters 2 to 3 of [GG12].

2.1 Jet Spaces

Before we define the jet space, we need a general concept of functions 'touching' each other in a point. This can be seen as the functions intersecting whilst also having the same (higher order) derivative. We formalise this in 'r-th order contact'.

Definition 2.1.1 (r-th order contact)**.** Given two smooth manifolds V, W and two smooth functions $f, g : V \to W$ such that $f(v) = g(v) = w$, we say that f and g have first order contact at v if $(df)_v = (dg)_v$. Inductively, we say that f and g have r-th order contact at v if df and dg have $(r-1)$-th order contact as functions from $V \to Hom(TV, TW)$ at v. This gives us an equivalence relation $\sim_r$ at v. We denote $j^r f(v)$ as the $\sim_r$ at v equivalence class of f.

Definition 2.1.2 (The r-jet space)**.** Denote $J^r(V, W)_{v,w}$ as the set of r-th order contact at v equivalence classes of functions f with $f(v) = w$, elements σ of this set have source $\alpha(\sigma) = v$ and target $\beta(\sigma) = w$. Then we define the r-jet over V and W as $J^r(V, W) := \cup_{v \in V, w \in W} J^r(V, W)_{v,w}$.

It turns out that this definition of r-th order contact is equivalent to a condition that is much easier to work with.

Lemma 2.1.3. *Given two smooth functions $f, g : V \to \mathbb{R}^m$ for $V \subset \mathbb{R}^n$, $f \sim_r g$ at v if and only if all nth order partial derivatives of f and g at v are the same for $n \leq r$.*

Proof. We will perform this proof by induction on r. We see that for $r = 1$, $f \sim_1 g$ iff $(df)_v = (dg)_v$ and we note that $(df)_g$ contains all first order partial derivatives of f at v.

Now suppose the claim is true for $r-1$. Then $f \sim_r g$ iff $(df)_v \sim_{r-1} (dg)_g$ iff all partial derivatives of order up to $r-1$ of df and dg agree on T_vV, since we assumed our result for $r-1$. Now we now that (df) can be seen as a function with $n \times m$ components, each of these being a partial derivative of f.

In particular, let $f_{i,j} = \frac{\partial f_i}{\partial x_j}$. Then $(df)(x) = (f(x), f_{1,1}(x), f_{1,2}(x)..., f_{n,n}(x))$. Thus any partial derivative of a component of (df) can be seen as a higher-order partial derivative of f.

Recall that, by our assumption, all partial derivatives at v of order up to $(r-1)$ of df correspond to those of dg. Then we see that all partial derivatives of order less or equal to r at v of f must correspond to those of g. □

In Euclidean space, the truncated Taylor series of a function around a point contains information about the derivatives of that function at that point. We can use this concept to represent $J^r(\mathbb{R}^n, \mathbb{R}^m)$ in a different, more familiar way. Let $B^r_{n,m}$ be the space of functions $p : \mathbb{R}^n \to \mathbb{R}^m$ such that each component function p_i is a polynomial in n variables of degree up to r. Then there exists a bijection between $J^r(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathbb{R}^n \times \mathbb{R}^m \times B^r_{n,m}$. We will first give this function and then prove that it is indeed bijective.

Definition 2.1.4. Given $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$, let $B^r_{n,m}$ be the space of functions $p : \mathbb{R}^n \to \mathbb{R}^m$ such that each component p_i of p is a polynomial in n variables with degree up to r. Then we define

$$T^r : C^\infty(V, W) \times V \to B^r_{n,m}$$

component by component:

$$(T^r(f,v))_k = \sum_{1\le|\alpha|\le r} \frac{\partial^{|\alpha|} f_k(v)}{\partial x^\alpha} x^\alpha \in B^r_{n,1},$$

where α is a multi-index. Then we define

$$T^r_{V,W} : J^r(V,W) \to V \times W \times B^r_{n,m}$$

as follows:

$$T^r_{V,W}(\sigma) := (v, w, T^r(f,v)).$$

Where f represents σ.

Lemma 2.1.5. *$T^r_{V,W}$ is a bijection.*

Proof. Injectivity follows from Lemma 2.1.3 and the fact that all partial derivatives of f are to be found in $T^r(f)$. For surjectivity, suppose $(v,w,p) \in V \times W \times B^r_{n,m}$. Then we take $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x) = w + p(x-v)$, which implies that $f(v) = w$, and $T^r f(v) = p$. Now we set $\sigma = j^r f(v) = \sigma$, and since the source and the target of σ lay in V, W respectively, we can see that $\sigma \in J^r(V,W)$ and by construction $T^r_{V,W}(\sigma) = (v,w,p)$. Thus $T^r_{V,W}$ is indeed bijective. □

Definition 2.1.6 (Fibration)**.** A topological map $p : X \to Y$ is called a fibration over Y if p is a surjective function that satisfies the homotopy lifting property. That is to say, for each homotopy $F : Z \times I \to Y$ where I is the unit interval, and a given starting lift $F_0 : Z \to X$ such that $p \circ F_0(z) = F(z,0)$, there exists a unique $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z,0) = F_0(z)$ and $p \circ \tilde{F} = F$. We call p the projection.

Definition 2.1.7 (Section)**.** Given a map $p : X \to Y$. Then a function $f : A \subset Y \to X$ is called a section of p if and only if $p \circ f = \mathrm{id}_A$.

Definition 2.1.8 (Holonomic Sections)**.** Given V, W smooth manifolds. Then a section $F : V \to J^r(V,W)$ with respect to the source map $\alpha : J^r(V,W) \to V$ is called holonomic if there exists some $f \in C^\infty(V,W)$ such that $j^r f = F$.

Note that an arbitrary section $F : V \to J^r(V,W)$ is not necessarily holonomic. The only condition on such a section is that for all $v \in V$, $F(v)$ is some σ such that $\alpha(\sigma) = v$, and that the section is smooth. Thus when we look at $f = \beta \circ F : V \to W$, it is not necessarily the case that $j^r f = F$. With holonomic section however, we do have this property.

As an example, we can take $G : \mathbb{R}^1 \to J^r(\mathbb{R}^1, \mathbb{R}^1) = \mathbb{R}^1 \times \mathbb{R}^1 \times \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^1)$, such that $G(x) = (x, x^2, 0)$. It is clear that with $f = \alpha \circ G : \mathbb{R}^1 \to \mathbb{R}^1$ such that $f(x) = x^2$, $j^r f(x) = (x, x^2, 2x) \neq G(x)$.

In the next section we will show that $J^r(V,W)$ carries a manifold structure, thus inherits a topology by virtue of being a manifold. Then we can see that the projections $\alpha : J^r(V,W) \to V$ and $\beta : J^r(V,W) \to W$ are fibrations over V and W. In this way we can identify every function $f : V \to W$ with a section $j^r f : V \to J^r(V,W)$ with respect to α.

2.2 The jet space as a manifold

In this subsection, we will work towards showing that the jet space carries a natural manifold structure. Before we can get to this result, we need to define a few operations that allow us to move between different jet spaces.

In order to define these operations we will need the following lemma on Euclidean spaces, that we can use in more general smooth manifolds by using local charts.

Lemma 2.2.1. *Let V and W be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $f_1, f_2 : V \to W$ such that $f_1 \sim_r f_2$ at v, and $g_1, g_2 : W \to \mathbb{R}$ such that $g_1 \sim_r g_2$ at $f_1(v)$. Then $g_1 \circ f_1 \sim_r f_2 \circ g_2$ at v.*

Proof. We will again use induction on r. For $r = 1$, we see that by the chain rule:

$$d(g_1 \circ f_1)_v = d(g_1)_{f_1(v)} d(f_1)_v = d(g_2)_{f_2(v)} d(f_2)_v = d(g_2 \circ f_2)_v$$

Now assume it holds for $r - 1$. Then, if $f_1 \sim_r f_2$ at v, we know by definition of r-th order contact that $d(f_1) \sim_{r-1} d(f_2)$ at v, and similarly for g_1, g_2 at $f_1(v)$. Then by the induction hypothesis, we can see that

$$d(g_1)_{f_1(v)} \cdot d(f_1)_v \sim_{r-1} d(g_2)_{f_1(v)} \cdot d(f_2)_v$$

at v, thus, again by the definition of r-th order contact and by observing that $d(g \circ f) = (dg) \cdot (df)$, we see that indeed $g_1 \circ f_1 \sim_r g_2 \circ f_2$ at v. □

Now we can define two important operations that allow us to make maps between two jet spaces. These operations are the push-forward and the pull-back.

Definition 2.2.2 (The push-forward)**.** Let V, W, U be smooth manifolds and $h : W \to U$ a smooth function. Then the push-forward operation $h_* : J^r(V, W) \to J^r(V, U)$ along h is defined as follows: let f represent some $\sigma \in J^r(V, W)$ at some point v. Then the push-forward of σ along h is $h_\star(\sigma) = j^r(h \circ f)(v)$.

Now since we chose a representative f of σ, it is necessary to check that the image of σ does not depend on the f we have chosen.

Lemma 2.2.3. *The push-forward h_* is well defined.*

Proof. We can see this by looking at the local case and taking local charts ϕ, ψ, θ on V', W', U' respectively such that V' is a nbhd of $v \in V$, W' is a nbhd of $f(v) \in W$ and U' is a nbhd of $g \circ f(h) \in U$, such that $\psi \circ f \circ \phi^{-1} : \phi(V') \to \mathbb{R}^m$ and $\theta \circ g \circ \psi^{-1} : \psi(W') \to \mathbb{R}^q$. We can restrict V' to V'' such that $\psi \circ f(V'') \subset W'$ Then we can apply Lemma 2.2.1 by noting that $\theta \circ g \circ \psi^{-1}$ can be seen as a set of functions $\psi(W') \to \mathbb{R}$. □

Remark 2.2.4. Let U, V, W be smooth manifolds, $g : U \to V$ and $h : V \to W$. Then we can see by the definition of the push-forward operation that $h_* \circ g_* = (h \circ g)_*$.

In a similar way, we have:

Definition 2.2.5 (The pull-back)**.** Let V, W, U be smooth manifolds and $h : U \to V$ a diffeomorphism. Then the pullback operation $h^* : J^r(V, W) \to J^r(U, W)$ along h is defined as follows: let f again represent some $\sigma \in J^r(V, W)$ at v. Then the pull-back of σ along h is $h^*(\sigma) = j^r(f \circ h)(h^{-1}(v))$.

The check that this is well-defined is analogous to that of the push-forward.

Remark 2.2.6. As with the push-forward operation, we note that for U, V, W smooth manifolds, $g : U \to V$ and $h : V \to W$ diffeomorphisms, $g^* \circ h^* = (h \circ g)^*$. Since $(id_U)^* = id_{J^r(U,X)}$ for any smooth manifold X, we can see that g^* is a bijection with inverse $(g^{-1})^*$.

With these tools we are almost ready to prove that the jet space is a smooth manifold. Since in the Euclidean case, we can see $J^r(V, W)$ as $V \times W \times B^r_{n,m}$ and the latter is a manifold by virtue of being a product of manifolds, we can talk about smooth maps.

Lemma 2.2.7. *Given $V, V' \subset \mathbb{R}^n$ and $W, W' \subset \mathbb{R}^m$ open, $h : V \to V'$ a smooth mapping and $g : W \to W'$ a diffeomorphism, then*

$$T^r_{V',W'}(g^{-1})^* h_*(T^r_{V,W})^{-1} : V \times W \times B^k_{n,m} \to V' \times W' \times B^k_{n,m}$$

is a smooth mapping.

Proof. Suppose $D = (v, w, p) \in V \times W \times B^k_{n,m}$. Then let $f(x) = w + p(x - v)$ such that $f(v) = w$, and let $\sigma = j^r f(v) \in J^r(\mathbb{R}^n, \mathbb{R}^m)$. Then we can see that $T^r_{V,W}(\sigma) = D$. Additionally, because the target of σ lies in W and the source lies in V, we can see that $\sigma \in J^r(V, W)$.
Then we see that $T^r_{V',W'}(g^{-1})^* h_*(T^r_{V,W})^{-1}(D) = T^r_{V',W'}(g^{-1})^* h_*(J^r(f)(v)) = T^r_{V',W'} J^r(h \circ f \circ g^{-1})(g(v))$. The source of $J^r(h \circ f \circ g^{-1})(g(v))$ is $g(v)$, which varies smoothly on the source v of D and the target is

$h \circ f(v)$, which varies smoothly on the source v of D and the function f. We know that the polynomial of $T^r_{V',W'}(g^{-1})^* h_*(J^r(f)(v)) = T^r_{V,W} j^r(h \circ f \circ g^{-1})(g(v))$ is given by the partial derivatives of $h \circ f \circ g^{-1}$. Since h and g do not vary, we can see by the chain rule that the partial derivatives vary smoothly on f. We now only need to show that f varies smoothly on D. Note that since any polynomial p is smooth, this is indeed the case by construction of f. Thus indeed $T^r_{V',W'}(g^{-1})^* h_*(T^r_{V,W})^{-1}$ is smooth. □

Theorem 2.2.8. *Given V, W two smooth manifolds with dimensions n, m respectively. Then*

1. *$J^r(V,W)$ can be equipped with a smooth manifold structure with*
$$\dim J^k(V,W) = m + n + \dim(B^r_{n,m}).$$
2. *The source map $\alpha : J^r(V,W) \to V$, the target map $\beta : J^r(V,W) \to W$ and their product $\alpha \times \beta : J^r(V,W) \to V \times W$ are submersions.*
3. *The push-forward operation along a smooth map is smooth.*
4. *The pullback operation along a diffeomorphism is a diffeomorphism.*
5. *For any smooth function f, the function $j^r f : V \to J^r(V,W)$ is smooth.*

Proof. We will prove each claim separately.

1. Take $X \subset V, Y \subset W$ two domains of charts ϕ, ψ, and X', Y' their respective images. Then we have $\phi_* \circ (\psi^{-1})^* =: J^r(X,Y) \to J^r(X',Y')$, and $\tau_{X,Y} := T^r_{X',Y'} \circ \phi_*(\psi^{-1})^* : J^r(X,Y) \to X' \times Y' \times B^r_{n,m}$. We give $J^r(V,W)$ the manifold structure induced by the charts $\tau_{X,Y}$. We do still need to verify that the τ charts behave properly on overlaps. Let $\phi_1, \psi_1, X_1, Y_1, X'_1, Y'_1$ be the charts and sets of another chart τ_{X_1,Y_1}. Then we can see that
$$\tau_{X_1,Y_1} \circ (\tau_{X,Y})^{-1} = T^r_{X',Y'}(\phi_1)_*(\psi_1^{-1})^* \psi^* (\phi^{-1})_* (T^r_{X,Y})^{-1}$$
and due to the properties of the pullback and push-forward operations, we can rewrite this to see:
$$\tau_{X_1,Y_1} \circ (\tau_{X,Y})^{-1} = T^r_{X',Y'}(\psi_1^{-1}\psi)^*(\phi_1\phi^{-1})_*(T^r_{X,Y})^{-1}$$
which is smooth by Lemma 1.1.10.
2. We note that the projection $p_{X'} : X' \times Y' \times B^r_{n,m} \to X'$ is a smooth submersion, as is the projection $p_{Y'} : X' \times Y' \times B^r_{n,m} \to X'$. Then locally, $\alpha = \phi^{-1} \circ p_{X'} \circ \tau_{X,Y}$ and $\beta = \psi^{-1} \circ p_{Y'} \circ \tau_{X,Y}$. Now since $\tau_{X,Y}$, ϕ and ψ are charts, thus local diffeomorphisms, we can see that α and β are indeed submersions.
3. Suppose $g : W \to Z$ is smooth. Let f represent $\sigma \in J^r(V,W)$, and $D = \tau_{X,Y}(\sigma)$, then $\tau_{X',Z'} g_* \tau^{-1}_{X,Y}(D) = \tau_{X',Z'} j^r(g \circ f)(v)$ where X, X' chart domains in V, and Y, Z' chart domains in W, Z respectively. Since τ is a chart and g is smooth, we can see that $g_* : J^r(V,W) \to J^r(V,Z)$ is smooth.
4. In a similar way as in (2), it follows that for $h : V \to Z$ a diffeomorphism, $h^* : J^r(V,W) \to J^r(Z,W)$ is smooth. It is also clear to see that the pullback along h^{-1} is smooth and the inverse of the pullback along h.
5. Given $f : \mathbb{R}^n \to \mathbb{R}^m$ a smooth function, we look at $j^r f : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times B^r_{n,m}$. Let $T^r f(v)$ give the polynomial $p \in B^r_{n,m}$ such that $f \sim_r (f(v) + p(x - v))$. Then $T^r_{\mathbb{R}^n,\mathbb{R}^m} j^r f(x) = (x, f(x), T^r f(x))$. We note that $T^r f(v)$ is given by the partial derivatives of f at v. Since f is smooth, all of these partial derivatives are smooth, thus $T^r f(v)$ varies smoothly over v. Thus $j^r f$ is smooth in the local situation and since smoothness is a local property, we can conclude that for $g : V \to W$ a smooth function between smooth manifolds, $j^r g$ is smooth.

□

Remark 2.2.9. Given V, W smooth manifolds. We define $\pi^l_k : J^l(V,W) \to J^k(V,W)$ with $k \leq l$ to be the projection that, with f a smooth function representing $\sigma \in J^l(V,W)$, acts as follows: $\pi^l_k(\sigma) = j^k f(\alpha(\sigma))$. We note that since $k \leq l$, and $g \sim_l f$ if and only if all partial derivatives up to order l coincide, we see that π^l_k is not dependent on the choice of representation. Furthermore, since f is smooth, and π^l_k only relies on the derivatives of f, we can see that π^l_k is smooth as well.

2.3 The Whitney Topology

Now that we have defined our jet spaces as manifolds, they inherit the topology induced by the charts. We will use this topology to define topologies on the set of smooth functions between two smooth manifolds. This will be useful for phrasing questions about the set of smooth functions.

Definition 2.3.1 (The Whitney C^k topology)**.** Let V, W be smooth manifolds and $C^\infty(V, W)$ be the set of smooth functions between V and W. Let U be an open subset of $J^r(V, W)$, then we define:

$$M(U) := \{f \in C^\infty(V, W) | \operatorname{Im}(j^r f) \subset U\}$$

With this, we can define the Whitney C^r topology on $C^\infty(V, W)$ by taking the set of all $M(U)$ as a basis. We denote with $\mathcal{T}_r$ the set of opens in the Whitney C^r topology.

Definition 2.3.2 (The Whitney C^∞ topology)**.** Let $\mathcal{T} = \cup_{r \in \mathbb{N}} \mathcal{T}_r$. We note that for $k \leq l$, $\mathcal{T}_k \subset \mathcal{T}_l$, since for $U \subset J^k(V, W)$, we can take $U' = (\pi^l_k)^{-1}(U)$ such that $M(U) = M(U')$. Then we can see that $\mathcal{T}$ is a basis, and we define the Whitney C^∞ topology on $C^\infty(V, W)$ to be the topology induced by this basis.

An interesting property of $C^\infty(V, W)$ equipped with the Whitney Topology is that it is a Baire space. This intuitively means that all open dense sets under this topology overlap, such that the intersection of two or more open dense sets is still dense.

Definition 2.3.3. Let X be a topological space, then a subset Y of X is called a residual set if there exist countably many dense sets $Y_i \subset X$ such that $Y = \bigcap Y_i$. X is called a Baire space if all residual subsets are dense.

Lemma 2.3.4. *Let V, W be smooth manifolds, $C^\infty(V, W)$ is a Baire space when equipped with the C^∞ topology.*

This Lemma is proven by taking a residual subset and an open subset of $C^\infty(V, W)$ and constructing a function that lives in both. This is done by constructing a sequence of functions in the open subset that is Cauchy, and where each subsequent function lives in an increasing intersection of the dense subsets that form our residual subset. For details, see Proposition 3.3 of [GG12].
Having equipped the space of smooth functions with a topology, we are now able to talk about continuous functions on this space. The following lemmas will show the continuity of some useful maps.

Lemma 2.3.5. *Let V and W be smooth manifolds, the mapping $j^r : C^\infty(V, W) \to C^\infty(X, J^r(X, Y))$ is continuous in the Whitney C^∞ topology.*

Proof. Let X be an open subset of $J^l(V, J^r(V, W))$, then $M(X)$ is an open in $C^\infty(V, J^r(V, W))$. It is enough to show that $(j^r)^{-1}(M(X))$ is open in $C^\infty(V, W)$.

Note that $(j^r)^{-1}(M(X))$ is the set of f such that $\operatorname{Im}(j^l(j^r(f))) \subset X$. We take $p : J^r(V, W) \to J^l(V, J^r(V, W))$ to be the function that for f a representative of σ, is defined as $p(\sigma) = j^l(j^r f(\alpha(\sigma)))$. This does not depend on choice of representation, due to our evaluation at $\alpha(\sigma)$.

We can thus see that p only depends on the partial derivatives of f at $\alpha(\sigma)$ and is thus smooth.

Let $Y \subset J^r(V, W)$ be the set such that $Y = p^{-1}(X)$, then Y is open, thus $M(Y)$ is open. We claim that $M(Y) = (j^r)^{-1}(M(X))$.

We note that since $j^r(f(v)) \in Y$ for all $f \in M(Y)$ and for all $v \in V$, $j^l(j^r)(f(v)) \in X$ for all $f \in M(Y)$ and $v \in V$, thus for all $f \in M(Y))$, $j^r(f) \in M(X)$. Moreover, if $j^r(f) \in M(X)$, then $j^l(j^r(f))(V) \subset X$, thus $j^r(f) \subset Y$ and $f \in M(Y)$. We then conclude that $M(Y) = (j^r)^{-1}(M(X))$, thus j^r is continuous. □

Lemma 2.3.6. *Let V, W, X be smooth manifolds and $\phi : W \to X$ be smooth, then the push-forward $\phi_\star : C^\infty(V, W) \to C^\infty(V, X)$ along ϕ is continuous in the Whitney C^∞ topology.*

Proof. Let U be an open in $J^r(V, X)$, then $M(U)$ is an open in $C^\infty(V, X)$. Recall that the push-forward ϕ_* on jet spaces is smooth, then $S = (\phi_*)^{-1}(U)$ is open in $J^r(V, W)$, and $M(S) = (\phi_*)^{-1}(U)$ is open in $C^\infty(V, W)$ □

Lemma 2.3.7. *Let V, W, X be smooth manifolds and $\phi : V \to X$ be smooth, then the pull-back $\phi^\star : C^\infty(X, W) \to C^\infty(V, W)$ along ϕ is continuous in the Whitney C^∞ topology.*

The proof of this is analogous to that of the push-forward.

3 The Thom Transversality theorem

In this section we will be introducing the concept of transversality. Transversality is a condition that tells us how two submanifolds, in our case a submanifold and the image of a function, intersect. The condition looks at the tangent spaces of the two submanifolds and tells us that the overlap between the two is minimized through optimal usage of the dimensions. This condition is generic, so when taking two arbitrary submanifolds, it is likely that they are transversal. The Thom transversality theorem intuitively tells us that two subsets that intersect can be perturbed slightly such that the intersection becomes 'optimised'. We can see functions as subsets of the jet space, by seeing $f : V \to W$ as $j^r f(V) \subset J^r(V, W)$. Certain conditions on functions can be translated to submanifolds of $J^r(V, W)$. We can then use the Thom Transversality Theorem to perturb f slightly, so that it avoids these submanifolds as much as possible.

After introducing transversality, we will work towards proving the Thom Transversality theorem. We will ultimately use this theorem to prove Whitney's immersion theorem and its parametric extension and explain what this result tells us about regular homotopies.

This section will follow chapter 4 of [GG12].

3.1 Transversality

Definition 3.1.1 (Transverse intersection)**.** Let V, W be smooth manifolds and $f : V \to W$ a smooth mapping. Let X be a submanifold of W and v a point in V. Then we say that f intersects X transversely at v (denoted by $f \pitchfork X$ at v) if either

1. $f(v) \notin X$ or
2. $f(v) \in W$ and $T_{f(v)}X + d(f)_v T_v V = T_{f(v)}W$

If A is a subset of V, we say that f intersects X transversely on A (denote $f \pitchfork X$ on A) if $f \pitchfork X$ at a for all $a \in A$. We say that f intersects X transversely (denote $f \pitchfork X$) if f intersects X transversely on V. Finally, if B is a subset of W, we say that f intersects X transversely on B (denote $f \pitchfork X$ on B) if $f \pitchfork X$ at v for all $v \in f^{-1}(B)$.

Remark 3.1.2. If $\dim(X) + \dim(V) < \dim(W)$, we note that it is impossible to have that $T_{f(v)}X + d(f)_v T_v V = T_{f(v)}W$, thus in this case we have that $f \pitchfork X$ at v if and only if $f(v) \notin X$, and by extension, $f \pitchfork X$ if and only if $\mathrm{Im}(f) \cap X = \emptyset$.

For additional intuition about transversality, see Figure 1

Before working towards proving the Thom Transversality Theorem, we will prove a smaller result, theorem 3.1.5. This result shares similarities with the submersion theorem, in fact the submersion theorem is used in its proof. In order to apply the submersion theorem we first need to state the following lemma.

Lemma 3.1.3. *Let V, W be smooth manifolds, X a submanifold of W and $f : V \to W$ smooth. Let $v \in V$ such that $f(v) = w \in X$. Suppose U is a neighborhood of w, and $\phi : U \to \mathbb{R}^k$ a submersion, where k is the co-dimension of X, such that $X \cap U = \phi^{-1}(0)$, then $f \pitchfork X$ at v iff $\phi \circ f$ is a submersion at v.*

Proof. Since $f(v) \in X$ and $\phi(X) = \{0\}$, we note that $\phi \circ f(v) = 0$. Moreover, we note that $\mathrm{Ker}\, d(\phi)_w = T_w X$, since $\phi^{-1}(0) = X \cap U$. Now we note that $f \pitchfork X$ at v iff $T_w W = T_w X + d(f)_v(T_v V) = \mathrm{Ker}\, d(\phi)_w + d(f)_v T_v V$. Since ϕ is a submersion, we know that $d(\phi)_w$ is a surjection. Then $d(\phi \circ f)_v$ is a surjection if and only if the last equality holds. □

Remark 3.1.4. Because X is a submanifold, we can always find a U and a ϕ to meet the criteria: Let U be a chart nbhd of w with chart $\psi : U \to \mathbb{R}^m$ where $m = \dim(W)$. Then $\dim(X) = \dim(\psi(X))$, and we can decompose $\mathbb{R}^m$ to $\mathbb{R}^k \times \mathbb{R}^{m-k}$ in such a way that $X \cap U = \psi^{-1}(0 \times \mathbb{R}^{m-k})$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^k$ be the projection to the first k factors, then $\phi = \pi \circ \psi$.

Theorem 3.1.5. *Let V, W be smooth manifolds with X a submanifold of W. Let $f : V \to W$ and assume $f \pitchfork X$. Then $f^{-1}(X)$ is a submanifold of V, with* $\mathrm{codim}(f^{-1}(W)) = \mathrm{codim}(W)$.

Proof. Being a submanifold is a local requirement. Thus it is enough to show that for every point $v \in f^{-1}(X)$, there exists an open nbhd V of v such that $f^{-1}(W) \cap V$ is a submanifold. Let U and ψ be chosen as in Remark 1.3.4, and $S \subset V$ an open such that $f(S) \subset U$. Then $\psi \circ f$ is a submersion at p by Lemma 3.1.3. In particular, we can see that f is a submersion at every point in S. Then $S \cap f^{-1}(X) = (\phi \circ f|_S)^{-1}(0)$ is a submanifold with co-dimension equal to the co-dimension of X by the submersion theorem. Thus $f^{-1}(X)$ is locally a submanifold and therefore globally as well. □

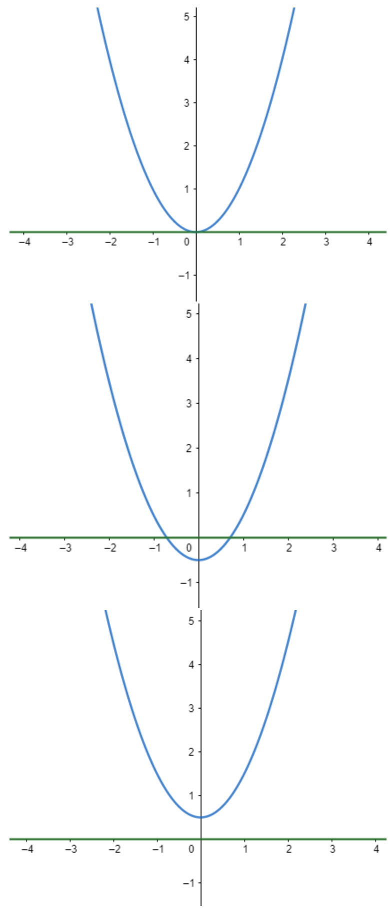


Figure 1: The parabola does not intersect the 0 line transversely, but it does when translated up or down.

3.2 The Thom Transversality Theorem

We will now work towards proving the Thom Transversality theorem. Before being able to prove the theorem we will prove two lemmas. The main theorem will rely on constructing a parameterized family of functions and uses the first of the two lemmas to show that the set of values that parameterise functions that intersect a given submanifold transversely lies dense in the parameterisation space. This will allow us to construct a convergent sequence to any function with functions that intersect the submanifold transversely, thus showing the set of functions that intersect the submanifold transversely to be dense. The second lemma will be used to show that the set of functions that intersect the submanifold transversely is open.

Lemma 3.2.1. *Let V, W, B be smooth manifolds with X a submanifold of W. Let $j : B \to C^\infty(V, W)$ be a not necessarily continuous mapping and define $\Phi : V \times B \to W$ by $\Phi(v, b) = j(b)(v)$. If Φ is smooth and $\Phi \pitchfork X$, then the set $\{b \in B | j(b) \pitchfork X\}$ is dense in B.*

Proof. Let $X' = \Phi^{-1}(X) \subset V \times B$ be a given subset and $\pi : X' \to B$ be the projection induced by X' being a subset of $V \times B$. By Theorem 3.1.5, $\Phi \pitchfork X$ implies that X' is a submanifold of $V \times B$.

Suppose $b \notin \pi(X')$, then $j(b)(v) \notin X$ for all $v \in V$, thus $j(b) \pitchfork W$.

If $\dim(X') < \dim(B)$, we know that $\pi(X')$ has measure 0 in B, thus $B - \pi(X')$ is dense in B. Then the set $\{b \in B | j(b) \pitchfork X\}$ is indeed dense in B.

Now suppose $\dim(X') \geq \dim(B)$. We shall proceed by proving that if b is a regular value of π, $j(b) \pitchfork X$. This will complete the proof, since by Sard's Theorem, the set of critical values of a smooth function has measure 0 in the domain of the function and since π is smooth, the set of critical values of π has measure 0 and the complement of this set consists of regular values of π and points outside the image of π, in both cases $j(b) \pitchfork X$.

Let b be a regular value of π. If $(v, b) \notin X'$, then $j(b)(v) \notin X$ thus $j(b) \pitchfork X$ at v. Thus we may assume that $(v, b) \in X'$. Now because b is a regular value of π, and $\dim(X') \geq \dim(B)$ we have $T_b B = d(\pi)_{(v,b)} T_{(v,b)} X'$, and thus

$$T_{(v,b)}(V \times B) = T_{(v,b)} X' + T_{(v,b)}(V \times \{b\}).$$

Applying $d(\Phi)_{(v,b)}$ on both sides gives us:

$$d(\Phi)_{(v,b)} T_{(v,b)}(V \times B) = T_{j(b)(v)} X + d(j(b))_v T_v V.$$

Now $\Phi \pitchfork X$, thus

$$T_{\Phi(v,b)} W = T_{\Phi(v,b)} X + d(\Phi)_{(v,b)}(T_{(v,b)}(V \times B)).$$

Now we note that $\Phi(v, b) = j(b)(v)$, thus we can combine these two to see that

$$T_{j(b)(v)} W = T_{j(b)(v)} X + d(j(b))_v T_v V.$$

Thus $j(b) \pitchfork X$ at x. □

Lemma 3.2.2. *Let V, W be smooth manifolds with $X \subset W$ a submanifold. Let*

$$T_X = \{f \in C^\infty(V, W) | f \pitchfork X\}.$$

Then T_X is an open subset of $C^\infty(V, W)$ in the Whitney C^∞ topology if X is closed in W.

Proof. We take Y a subset of $J^r(V, W)$ such that $Y := \{\sigma \in J^r(V, W) | \beta(\sigma) \notin X \text{ or } f \pitchfork X \text{ at } \alpha(\sigma)\}$, with f representing σ. Let v be $v = \alpha(\sigma)$. It is clear that if $f \in M(Y)$, then $f \pitchfork X$ and vice versa, thus $M(Y) = T_X$. Proving that Y is open is now sufficient to complete the proof.

Let $Z = J^r(V, W) - Y$ be the complement of Y. We will show that Z is closed, making Y open.

Let $\sigma_1, \sigma_2, ...$ be a sequence in Z converging to some $\sigma \in J^r(V, W)$. We will show that $\sigma \in Z$. For this, take p, q to be the source and the target of σ respectively, and p_i, q_i the source and target of σ_i for all i. Then the sequence of p_i must converge to p and the sequence of q_i to q.

Now since $\sigma_i \notin Y$, we must have that $q_i \in X$ for all i. We assumed X to be closed and since the sequence of q_i converges to q, q must be in X as well.

Let $f : V \to W$ represent σ. We chose V' a chart nbhd of p and W' a chart nbhd of q with surjective charts ϕ, ψ respectively, such that $f(V') \subset W'$. Then we take $X' := X \cap W'$. Let n be the dimension of V, m that of W and k that of X, then these dimensions also apply to V', W', X', by virtue of V', W' being open sets.

By choosing a coordinate basis we may assume that $\psi(X') = \mathbb{R}^k \subset \mathbb{R}^m$. Then we can see $\psi(V') = \mathbb{R}^k \times \mathbb{R}^{m-k}$. Let $\rho : \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^{m-k}$ be the canonical projection, which is surjective. Then by Lemma 3.1.3, $f \pitchfork W$ locally if and only if $\rho \circ \psi \circ f : V \to \mathbb{R}^{m-k}$ is a submersion, if and only if $g = \rho \circ \psi \circ f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^{m-k}$ is a submersion.

Let

$$F = \{A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}) | \text{ rank } A < m - k\}$$

be the set of non-surjective linear functions between $\mathbb{R}^n$ and $\mathbb{R}^{m-k}$. We take the mapping

$$\mu : \mathbb{R}^n \times \mathbb{R}^k \times \text{ Hom } (\mathbb{R}^n, \mathbb{R}^m) \subset J^r(\mathbb{R}^n, \mathbb{R}^m) \to \text{ Hom } (\mathbb{R}^n, \mathbb{R}^m)$$

given by $(x, y, B) \mapsto \rho \circ B$. We note that if $\sigma \in Z$ with source and target in V', W' respectively, we take $\sigma' = \phi^*\psi_*(\sigma)$ and we see that $\mu(\sigma') \in F$, in fact, locally we can see that $\mu^{-1}(F) = \phi^*\psi_*(Z)$. Now since F is closed and all our functions are continuous, Z is locally closed at every point, thus Z is closed and Y is open. □

We are now ready to prove the main theorem.

Theorem 3.2.3 (Thom Transversality Theorem)**.** *Let V and W be smooth manifolds and X a submanifold of $J^r(V, W)$. We define*

$$T_X := \{f \in C^\infty(V, W) | f \pitchfork X\}.$$

Then T_X is a residual subset of $C^\infty(V, W)$. In addition, if X is closed, then T_X is open.

Proof. We will construct a countable set of subsets of $C^\infty(V, W)$ such that T_X is their intersection and we will then prove that these subsets are open and dense to conclude that T_X is a residual set. First, let $X_1, X_2, ...$ be a countable sequence of sets such that they cover X, and for each X_i :

1. The closure of X_i in $J^r(V, W)$ is contained in X,
2. $\bar{X}_i$ is compact,
3. there exists coordinate neighborhoods $Y_i \subset V$ and $Z_i \subset W$ such that $\pi(\bar{X}_i) \subset Y_i \times Z_i$, where $\pi : J^r(V, W) \to V \times W$ is the canonical projection,
4. $\bar{Y}_i$ is compact.

We note that around every $x \in X$ we can choose a neighborhood that satisfies these conditions and X is second-countable, so we can chose such a covering.

Now let

$$T_{X_i} = \{f \in C^\infty(V, W) | j^r f \pitchfork X \text{ on } \bar{X}_i\}.$$

Such that $T_X = \bigcap_{i=1}^\infty T_{X_i}$.

Define $T_i = \{g \in C^\infty(V, J^r(V, W)) | g \pitchfork X \text{ on } \bar{X}_i\}$ such that $T_{X_i} = (j^r)^{-1}(T_i)$. By using Lemma 3.2.2 and noting that $J^r(V, W)$ is a smooth manifold, we can see that T_i is open. Then since j^r is a continuous function, T_{X_i} is open as well.

Now we need to prove that T_{X_i} is dense. Let $\phi : Y_i \to \mathbb{R}^n$ and $\psi : Z_i \to \mathbb{R}^m$ be charts, and $\rho : \mathbb{R}^n \to [0, 1]$ and $\rho' : \mathbb{R}^m \to [0, 1]$, such that:

$$\rho = \begin{cases} 1 \text{ on a nbhd of } \phi(\alpha(\bar{X}_i)) \\ 0 \text{ off } \phi(Y_i) \end{cases}$$

and

$$\rho' = \begin{cases} 1 \text{ on a nbhd of } \psi(\beta(\bar{X}_i)) \\ 0 \text{ off } \psi(Z_i) \end{cases}$$

where n, m are the dimensions of V, W respectively, α is the source map and β is the target map. Recall that since $\pi(X_i) \subset Y_i \times Z_i$, $\psi \circ \alpha$ and $\phi \circ \beta$ are well defined functions on $\bar{X}_r$ and because $\bar{X}_r$ is compact, it is possible to find such functions ρ, ρ'.

Now let $f \in C^\infty(V, W)$ be some function. We will aim to construct a sequence of functions in T_{X_i} that converges to f. Let

$$\epsilon = \frac{1}{2} \min\{d(\text{supp}(\rho'), \mathbb{R}^m - \psi(Z_i)), d(\psi(\beta(\bar{X}_i)), (\rho')^{-1}([0, 1)))\}$$

where d is the standard metric between subsets, such that for A, B subsets of some metric space (C, d_C), $d(A, B) := \min_{a\in A, b\in B} d_C(a, b)$. Then we set

$$B = \{b \in B^r_{n,m} | \, |b(x)| < \epsilon \, \forall x \in \text{supp } \rho\}.$$

Define $g : B \to C^\infty(V, W)$ such that $b \mapsto g_b$, where

$$g_b(v) = \begin{cases} f(x) & \text{if } v \notin Y_i \text{ or } f(v) \notin Z_i \\ \psi^{-1}(\psi(f(v)) + \rho(\phi(v)) \cdot \rho'(\psi(f(v))) \cdot b(\phi(v))) & \text{otherwise} \end{cases}.$$

With this we can define the function: $\Phi : V \times B \to J^r(V, W)$ as follows: $\Phi(v, b) = j^r g_b(v)$.

Claim 1. If $\Phi(v, b) \in \bar{X}_i$, Φ is a local diffeormorphism.

If we assume this claim, we can complete the proof as follows: since Φ is a local diffeomorphism at (v, b) when $\Phi(v, b) \in \bar{X}_i$, its derivative is surjective, and thus trivially $\Phi \pitchfork X$ on $\bar{X}_i$. We can now apply Lemma 3.2.1 to see that the set $\{b \in B | b \pitchfork X_i\}$ is dense in B.

We note that $g_0 = f$. Then take a sequence $b_1, b_2, ...$ in B that converges to 0. We know that $j^r g_{b_i} \pitchfork X$ on X_i, thus $g_{b_i} \in T_{X_i}$. We can see that g is smooth by construction, thus since $\lim_{i\to\infty} b_i = 0$, $\lim_{i\to\infty} g_{b_i} = g_0 = f$. Thus we have constructed a sequence in T_{X_i} that converges to an arbitrary f, thus T_{X_i} is dense.

Now all that is left to do is to prove claim 1. By assumption, $\Phi(v, b) \in \bar{X}_i$, thus $v \in \alpha(\bar{X}^i)$ and $g_b(x) \in \beta(\bar{X}_i)$. Now by the construction of g_b, we can see that

$$|\psi(g_b) - \psi(f)| = |\rho(\phi(v)) \cdot \rho'(\psi(f(v))) \cdot b(\phi(v))| \leq |b(\phi(v))| \leq \epsilon.$$

Now since $\psi(g_b(x)) \in \psi(\beta(\bar{X}_i))$ and $\epsilon \leq d(\psi(\beta(\bar{X}_i)), (\rho')^{-1}([0, 1))$, we can see that $\rho'(\psi(f(v))) = 1$. Note that this still holds in a small neighborhood of v and a small neighborhood of b. Then in a small neighborhood of (v, b), $g_{b'}(v') = \psi^{-1}(\psi(f)(v') + b'(\phi(v)))$. Recall that locally $J^r(V, W)$ is diffeomorphic to $V \times W \times B^r_{n,m}$, then it is clear to see that Φ is a local diffeomorphism.

Remark 3.2.4. It is important to note that when we have a function f, we only need to perturb it in the areas where it does not already intersect X transversely. We can see this by noting that $g_b(v)$ does nothing to f outside of a local neighbourhood.

□

3.3 The Whitney's Immersion Theorem

In this section we will work towards the Whitney immersion theorem. This theorem tells us that for any smooth manifold V of dimension n, the the set of functions $f \in C^\infty(V, \mathbb{R}^{2n})$ that are immersions is dense in the Whitney C^∞ topology. We can then expand the theorem to show that, given similar restrictions, the set of parameterized families of immersions lies dense in the set of parameterized families of smooth functions. In particular, given these dimensional restrictions, the set of regular homotopies lies dense in the set of homotopies.

The main bulk of this result is translating the condition of being an immersion into a statement about transversality. This will require a clever rewriting of the immersion requirement and some linear algebra.

Definition 3.3.1. Let V, W be two smooth manifolds and $\sigma \in J^1(V, W)$. Let f represent σ. Then $\operatorname{rank} \sigma := \operatorname{rank} d(f)_{\alpha(\sigma)}$. Let $l = \min(\dim V, \dim W)$, then $\operatorname{corank}(\sigma) := l - \operatorname{rank}(\sigma)$. We now define the following sets:

$$S_r(V, W) := \{\sigma \in J^1(V, W) | \operatorname{corank} \sigma = r\}.$$

Lemma 3.3.2. *Let V, W be two smooth manifolds and $f : V \to W$ smooth. Then f is an immersion if and only if $j^1 f(V) \cap (\bigcup_{r\neq 0} S_r(V, W)) = \emptyset$.*

Proof. We note that f is an immersion if f has maximal rank and $\dim V \leq \dim W$, in other words, if df has co-rank 0, thus if $f \in M(S_0(V,W))$. Now since $S_i(V,W) \cap S_j(V,W) = \emptyset$ for $i \neq j$, this is equivalent to the statement that $j^1 f(v) \in S_0(V,W)$ for all $v \in V$, thus indeed f is an immersion if and only if $j^1 f(V) \cap (\bigcup_{r \neq 0} S_r(V,W)) = \emptyset$. □

Now we will show that these S_i are submanifolds of $J^1(V,W)$ and compute their dimension. Recall that if the codimension of a submanifold of the codomain is bigger then the dimension of the domain, the only way for a function to intersect the submanifold transversely is by not intersecting it at all. Recall that $\operatorname{corank} S = \min(n,m) - \operatorname{rank} S$.

Definition 3.3.3. Let X, Y be vector spaces of dimension n, m respectively. Then we define the set $L^r(X,Y) := \{S \in \operatorname{Hom}(X,Y) | \operatorname{corank} S = r\}$.

Lemma 3.3.4. *Let S be an $m \times n$ matrix where $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is some $k \times k$ invertible matrix. Then* $\operatorname{rank} S = k$ *iff* $D - CA^{-1}B = 0$.

Proof. We note that since A is a $k \times k$ matrix, C is a $m - k \times k$ matrix and thus $-CA^{-1}$ is a $m - k \times k$ matrix, which makes

$$T = \begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{m-k} \end{pmatrix}$$

an invertible $m \times m$ matrix by virtue of being a lower-triangle matrix with non-zero entries on the diagonal. Then

$$\operatorname{rank} S = \operatorname{rank} TS = \operatorname{rank} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

This last matrix has rank k if and only if $D - CA^{-1}B = 0$. □

We can now prove the following lemma that shows us that S_i is indeed a submanifold in the Euclidean case. We will use this to prove the same result for the general case.

Lemma 3.3.5. *Let X, Y be two vector spaces of dimension n, m respectively. Then $L^r(X,Y)$ is a submanifold of* $\operatorname{Hom}(X,Y)$ *with* $\operatorname{codim}(L^r(X,Y)) = (|m-n| + r)r$.

Proof. Let $S \in L^r_{(X,Y)}$, and choose a coordinate system such that S is in the form as in Lemma 3.3.4 with A being a $k \times k$ invertible matrix. Here $k = \operatorname{rank} S = \min(n,m) - r$. Then chose a neighborhood M of S such that for all $S' \in M$, the top-left $k \times k$ sub-matrix is invertible. We define the function $f : U \to \operatorname{Hom}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k})$ such that $S' \mapsto D' - C'A'^{-1}B'$, with A', B', C', D' as in 3.3.4 We claim that f is a submersion. We can see this by observing that if we take A', B', C' constant, $g : \operatorname{Hom}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k}) \to \operatorname{Hom}(\mathbb{R}^{n-k}, \mathbb{R}^{m-k})$, $D' \mapsto D' - C'A'^{-1}B'$ is a diffeomorphism, and it is clear to see that $\operatorname{Im}(d(g)_{D'}) \subset \operatorname{Im}(d(f)_{S'})$.
We note that by 3.3.4, $f^{-1}(0) = L^r(X,Y) \cap U$, which is a submanifold because f is a submersion. Additionally, $\operatorname{codim} L^r = \dim \operatorname{Hom}(\mathbb{R}^{n-r} \times \mathbb{R}^{m-r}) = (n-k)(m-k) = (n - \min(n,m) + r)(m - \min(n,m) + r) = r(|m-n| + r)$. □

Theorem 3.3.6. *Let V, W be smooth manifolds of dimension n, m respectively, then $S_r(V,W)$ is a submanifold of $J^1(V,W)$ with co-dimension $r(|m-n| + r)$.*

Proof. We note that as proven previously, $J^1(V,W)$ locally looks like $V \times W \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. Then under this local view, we can see that $S_r(V,W)$ looks like $V \times W \times L^r(\mathbb{R}^n, \mathbb{R}^m)$. Then by Lemma 3.3.5, S_r is everywhere locally and thus globally a submanifold of $J^1(V,W)$ with co-dimension $r(|m-n| + r)$. □

Corollary 3.3.7. *For V, W two smooth manifolds, the set of immersions from V to W is open in the Whitney C^∞ topology.*

Proof. We note that Theorem 3.3.6 tells us that S_0 is a submanifold with co-dimension 0, thus S_0 is open in $J^1(V,W)$. As noted before, the set of immersions is exactly $M(S_0)$ and thus open. □

Theorem 3.3.8 (Whitney Immersion Theorem). *Let V, W be smooth manifolds with* $\dim W \geq 2 \cdot \dim V$. *Then the set $Im(V,W)$ of immersions from V to W is open and dense in the Whitney C^∞ topology.*

Proof. We note that $Imm(V,W)$ is open by Corollary 3.3.7. In this proof, we notate $S_r = S_r(V,W)$. To show that it is dense, we will use the Thom Transversality Theorem. We note that for $r \geq 1$, $\operatorname{codim} S_r = r(|n-m|+r) \geq |n-m|+1 \geq |n-2n|+1 = n+1$. Now since $\dim V < \operatorname{codim} S_r$, we have that for $f : V \to W$, $j^1 f \pitchfork S_r$ if and only if $j^1(f) \cap S_r = \emptyset$. By the Thom Transversality Theorem, the set of functions f for which $j^1 f \pitchfork S_r$ is dense in the Whitney C^∞ topology. Then the set $\{f \in C^\infty(V,W) | j^1 f(X) \cap (\bigcup_{r \neq 0} S_r) = \emptyset\}$ is a residual subset of $C^\infty(V,W)$ and thus dense. □

We can generalize this result slightly to make it say something about parameterized functions. This will give us that the set of regular homotopies lies dense in the set of smooth homotopies. We will do this in the following theorem.

Theorem 3.3.9. *Let V, W be two smooth manifolds of dimension n, m respectively such that $m \geq 2n+k$ and I^k the k dimensional unit cube. Then the set of smooth functions $F : V \times I^n \to W$ such that $F(v,i) = f_i(v)$, where $f_i : V \to W$ is an immersion, is dense in $C^\infty(V \times I^k, W)$.*

Proof. Let $\sigma \in J^1(V \times I^k, W)$, then we can write $\sigma = ((v,i), w, L)$, where $L \in \operatorname{Hom}(T_{(v,i)}(V \times I^k), T_w W)$. Let $p : J^1(V \times I^k, W) \to J^1(V,W)$ such that $p((v,i), w, L) \mapsto p(v, w, L')$, where $L' = LT$ with $T = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$ a $(n+k) \times n$ matrix so that $L' \in \operatorname{Hom}(T_v V, T_w W)$. It is clear that p is a surjection, and by restricting p to the submanifold where i and the $k \times m$ right sub-matrix of L are constant, denoted by $\tilde{i}$ and $\tilde{A}$ respectively, such that $L = \begin{pmatrix} L' & \tilde{A} \end{pmatrix}$ we can see that $p'((v,\tilde{i}), w, \begin{pmatrix} L' & \tilde{A} \end{pmatrix}) = (v, w, L')$ is a diffeomorphism, thus p is a submersion.
Then we look at $S_r(V,W)$. We note that as we saw, $\operatorname{codim} S^r = r(|n-m|+r) \geq |n-m|+1 \geq 2n+k-n+1 = n+k+1$. By virtue of p being a submersion, $S'_r(V \times I^k, W) := p^{-1}(S^r(V,W))$ has the same co-dimension. We claim that if $j^1 F(V \times I^k) \cap (\bigcup_{r \neq 0} S'_r(V \times I^k, W)) = \emptyset$, F can be seen as a parameterized family of immersions.
To see this, we note that if F satisfies this requirement, we can see that $p(j^1 F(V \times I)) \subset S_0$, thus for all i, $F(v,i) = f_i(v)$ with f_i an immersion.
Now note that for all S'_r, their co-dimension is greater than $2n+k$, which is the dimension of $V \times I^k$. Then $j^1 F(V \times I^k) \cap S'_r = \emptyset$ if and only if $j^1 F(V \times I^k) \pitchfork S'_r$. These sets are dense by the Thom Transversality Theorem, thus we are done. □

As said before, this implies that the set of regular homotopies from V to $\mathbb{R}^m$ where $m \geq 2\dim(V)+1$ lies dense in the set of homotopies. Then, if we find a homotopy between two immersions, we are able to perturb it slightly into a regular homotopy between the two immersions We can guarantee that the beginning and ending of the homotopy do not change by Remark 3.2.4. Now since all Euclidean spaces are null homotopic, we can see that if $m \geq 2\dim(V)+1$, there is exactly one class of immersions from V to $\mathbb{R}^m$.

4 Holonomic Approximation

We concluded last chapter by showing that when the codomain of a function is big enough, we can easily find immersions and even regular homotopies. However, as we will show in this chapter, there is a way to construct these type of functions with fewer restriction on the dimension of the target. In this chapter the concept of holonomic section will be introduced and we will work towards proving the Holonomic Approximation Theorem. This chapter will follow chapter 3 of [EM02].

4.1 Introduction of the Parametric Holonomic Approximation Theorem

In this section, we will state the parametric holonomic approximation theorem. Given smooth manifolds V, W, this theorem allows us to construct families $V \times I^k \to W$ with certain requirements by first constructing a non-holonomic family of sections $F : V \times I^K \to J^r(V, W)$, then approximating it with a family $\tilde{F} : V \times I^k \to J^r(V, W)$, where $\tilde{F_i} = \tilde{F}(-, i)$ is a holonomic section, this step requires the Holonomic approximation theorem. This family induces another family f that satisfies our requirements by taking $f = \beta \circ \tilde{F}$ such that $j^r f = \tilde{F}$. Note that for $k = 1$, we are talking about homotopies.

Before stating the theorem, there is one more definition we need to introduce.

Definition 4.1.1 (Polyhedron)**.** Given V a smooth manifold, recall that V admits a triangulation. Let T be the set of simplices together with their given boundaries. Then a subset $A \subset V$ that is induced by a subtriangulation $Q \subset T$ is called a polyhedron of V.

Theorem 4.1.2 (Parametric Holonomic Approximation Theorem)**.** *Let V, W be two smooth manifolds, $A \subset V$ a polyhedron of positive co-dimension and $Op(A)$ an open neighborhood of A. Furthermore, let*

$$F_z : Op(A) \to J^r(V, W)$$

be a family of sections parametrized by a unit cube I^k, such that the sections F_z are holonomic for all z in an open neighborhood of ∂I^k. Then for arbitrarily small, positive δ, ϵ there exists a family of diffeotopies

$$h_z^\tau : V \to V,$$

where $\tau \in [0, 1]$ $z \in I^k$ such that $d(h_z^1, \mathrm{id}_V) < \delta$, where d is a metric on functions, and a family of holonomic sections

$$\tilde{F}_z : Op(h_z^1(A)) \to J^r(V, W)$$

where $Op(h_z^1(A) \subset Op(A))$, such that

- *$h_z^\tau = \mathrm{id}_V$ and $\tilde{F}_z = F_z$ for all $z \in Op(\partial I^k)$*
- *$d(\tilde{F}_z, F_z|_{Op(h_z^1(\tilde{A}))}) < \epsilon$ for all $z \in I^k$.*

We will prove this theorem by noting that since A is a polyhedron, we can use induction over the skeleton of A, i.e. the simplicies that build up A. This allows us to reduce the proof to proving the parametric version of the following theorem:

Theorem 4.1.3 (Holonomic approximation over a cube)**.** *Given $I^k \subset \mathbb{R}^n$ with $k < n$, for any section*

$$F : Op(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

that is holonomic over $Op(\partial I^k)$ and ϵ, δ arbitrarily small, there exists a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \phi(x_1, ..., x_n))$ such that $d(h, \mathrm{id}_{\mathbb{R}^n}) < \delta$ and a holonomic section

$$\tilde{F} : Op(h(I^k)) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

such that $h = \mathrm{id}_{\mathbb{R}^n}$ and $\tilde{F} = F$ on $Op(I^k)$ and $d(\tilde{F}, F|_{Op(h(I^k))}) < \epsilon$.

This result implies the the parametric version of this result, by taking a family of sections $F_z : I^k \to \mathbb{R}_m$ with $z \in I^l$ and noting that we can see this family as a function $F' : I^k \times I^l \to \mathbb{R}^m$ and taking $I^k \times I^l = I^{k \times l}$. For a detailed proof, see Theorem 3.7.1 of [EM02].
The core of this proof is an inductive step, we will proof this step in a lemma we will call the inductive lemma, for which we need the notion of fiberwise holonomic sections.

4.2 Fiberwise Holonomic Sections

In this section we will introduce some concepts that allow us to state the inductive lemma. These definitions work towards being able to formally state whether a function is holonomic in a certain direction. The inductive lemma then tells us we are able to make the function holonomic in additional directions.

Definition 4.2.1. Let V, W be smooth manifolds, $A \subset V$ be an arbitrary subset and $F : A \to J^r(V, W)$ be a section, we say that F is holonomic over A if there exists a holonomic extension $\tilde{F} : Op(A) \to J^r(V, W)$.

We note that given two holonomic extensions G, H with domains B, C, we can look at their shared domain $B \cap C$. We will say that two functions are equal if they agree over their shared domain. We know that they already agree over $A \subset B \cap C$. Now assuming that $B \cap C \backslash A$ is small enough, we can see that G, H must be close to each other. Indeed we may assume that locally their images share a chart neighborhood. Then in this chart neighborhood we can simply use linear interpolation to connect the two functions.

Definition 4.2.2. Given V, W two smooth manifolds and $A \subset V$ an arbitrary subset, a section $F : V \to J^r(V, W)$ is called holonomic over A if $F|_A$ is holonomic. Given a fibration $\pi : V \to B$, $F : V \to J^r(V, W)$ is called fiberwise holonomic if there exists a continuous family of extensions

$$\tilde{F}_b : Op(\pi^{-1}(b)) \to J^r(V, W)$$

with $b \in B$ such that for each $b \in B$, $\tilde{F}_b$ coincides with F on $\pi^{-1}(b)$.

We note that any section $F : V \to J^r(V, W)$ is holonomic over any point $v \in V$, we can see this by taking some representative f of $\sigma = F(v)$ and noting that $j^r f$ is an extension of $F|v$.

Remark 4.2.3. Let V, W be two smooth manifolds and $B \subset A \subset V$. Suppose there is a section $F : Op(A) \to J^r(V, W)$ that is holonomic over $Op(B)$. Then there exists a family of sections

$$\tilde{F}_v : Op(v) \to J^r(V, W)$$

with $v \in A$ such that $\tilde{F}_v(v) = F(v)$ for all $v \in A$ and $\tilde{F}_v = F|_{Op(v)}$ for all $v \in B$.

This follows from the fact that the space of holonomic extensions is contractible.

4.3 The inductive Lemma

We will now prove the inductive lemma. This lemma is the core of holonomic approximation and contains most of the technical work. As its name implies, this lemma leads to the holonomic approximation over a cube by induction. The idea is to take a section F and approximate it with another function $\tilde{F}$ that is holonomic over a specific fiber, in this case a fiber will correspond to a direction. It will show that this can be done without disturbing holonomy over a different fibre, allowing us to approximate F holonomically fibre by fibre.

Definition 4.3.1 ($\mathcal{N}_\delta$ neighborhood)**.** Let d_∞ be the maximal coordinate distance metric on a Euclidean space, such that $d_\infty((x1, .., x_n); (y_1, ..., y_n)) = \max_{0<i\le n}(|x_i - y_i|)$. Then given a subset $A \subset \mathbb{R}^n$ and $\delta > 0$ we will write

$$\mathcal{N}_\delta(A) := \{x \in \mathbb{R}^n | d_\infty(x, A) < \delta\}.$$

In the Inductive lemma, we will use the following notation:
We write $\pi_s : \mathbb{R}^n \to \mathbb{R}^s$ as the standard projection to the first s coordinates. For $y = (z, t) \in I^{k-l} = I^{k-l-1} \times I \subset I^k \subset \mathbb{R}^n$, $I^l \subset \mathbb{R}^n$ such that $I^{k-l} \times I^l = I^k \subset \mathbb{R}^n$, where I is the unit interval, and for some $\delta > 0$ and fixed $0 < \theta < 1$, $\delta_1 = \theta\delta$ we define the following sets:

- $U_\delta(y) := \mathcal{N}_\delta(y \times I^l)$,
- $V_\delta(y) := \mathcal{N}_\delta(y \times \partial I^l)$,
- $A_\delta(y) := (\overline{U_{\delta_1}(y) \backslash V_\delta(y)}) \cap \pi_{k-l}^{-1}(y)$
- $\Omega_\delta(z, N) = Op\left(\left(\bigcup_{i=1}^N A_\delta(z, c_i)\right) \cup (z \times I^{l+1})\right) \setminus \bigcup_{i=1}^N A_\delta(z, c_i)$ where N is some positive integer and $c_i = \frac{2i-1}{2N}$

We note that $V_\delta(y) \subset U_\delta(y)$ and $A_\delta(y) \subset U_\delta(y)$. For more intuition on these spaces, see the following figures.

Figure 2: Figure 3.2 of [EM02], The spaces $U_\delta(y)$ (grey) and $A_\delta(y)$ (black) for n=2, k=1 and l=0.

As we can see in the pictures, $A_\delta(y) \subset U_\delta(y)$ is the set of all points that share the first $k-l$ coordinates with y, except for those close to the edge of $U_\delta(y)$. Then in order for us to construct $\Omega_\delta(z,N)$, we first take the cube in the last $l+1$ coordinates. Then for $c_i = \frac{i-\frac{1}{2}}{N}$, we add N strips $A(z,c_i)$ to this cube we then get something like Figure 3a and Figure 4. Then to get Ω, we take some open around this construct, and remove all the $A(z,c_i)$, to get something that looks like Figure 3b. This is then some open with N strip-shaped holes.

Lemma 4.3.2 (The Inductive Lemma). *Let $I^k \subset \mathbb{R}^n$ be the unit cube in the first k coordinates. Suppose that a section*

$$F : Op(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

is holonomic over $Op(\partial I^k)$ and for a non-negative integer $l < k$, F is fiberwise holonomic with respect to the fibration $\pi_{k-l} : I^k \to I^{k-l}$. This is to say, F is holonomic over the subsets $y \times I^l$ where

$$y = (z,t) \subset I^{k-l} = I^{k-l-1} \times I.$$

In particular, suppose that for a positive δ there exists a family of holonomic sections

$$F_y = j^r f_y : U_\delta(y) \to J^r(U_\delta(y), \mathbb{R}^m),$$

where y as before, such that

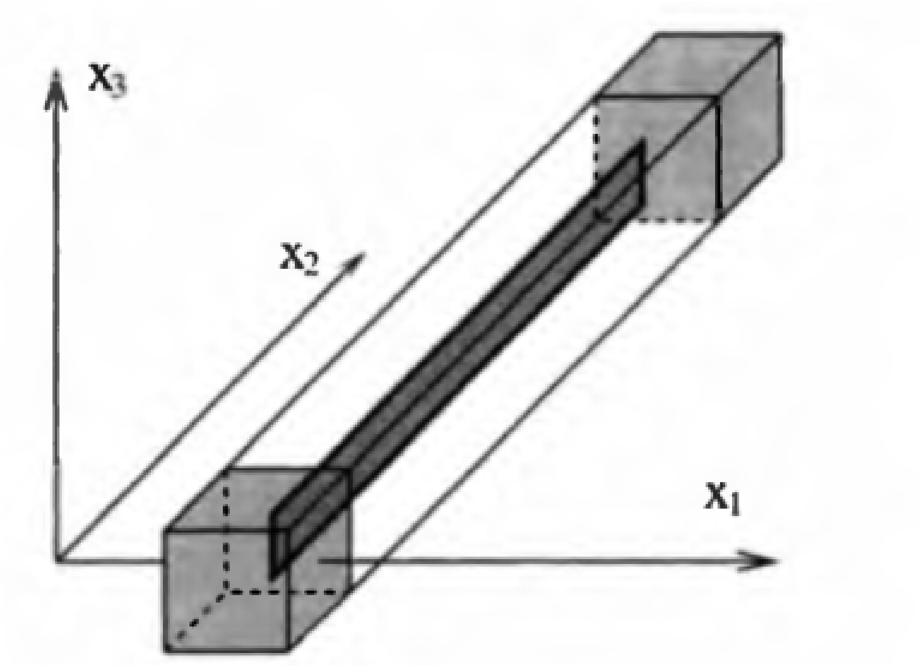
- $F_y|_{(y\times I^l)\cup V_\delta(y)} = F|_{(y\times I^l)\cup V_\delta(y)}$ *for all* $y \in I^{k-l}$
- $F_y = F|_{U_\delta(y)}$ *for* $y \in Op(\partial I^{k-l})$.

Then for an arbitrarily small $\epsilon > 0$ there exists a positive integer N and a family of holonomic sections

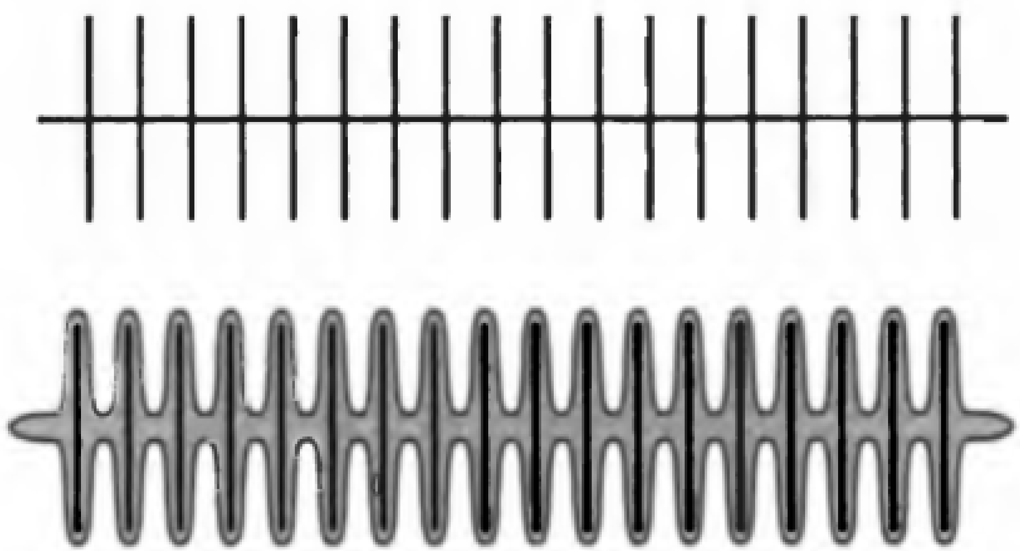
$$\tilde{F}_z : \Omega_\delta(z,N) \to J^r(\mathbb{R}^n, \mathbb{R}^q),$$

such that

- $\tilde{F}_z = F$ *on* $\Omega_\delta(z,N) \cap Op(\partial I^k)$ *and*
- $d(\tilde{F}_z, F|_{\Omega_\delta(z,N)}) < \epsilon$.



(a) Figure 3.3 of [EM02], The spaces $U_\delta(y)$ (white), $V_\delta(y)$ (grey) and $A_\delta(y)$ (black) for n=3, k=2 and l=1.



(b) Figure 3.4 of [EM02], The sets $\left(\bigcup_{i=1}^N A_\delta(z,c_i)\right) \cup (z \times I^{l+1})$ (upper picture) and $\Omega_\delta(z,N)$ (lower picture grey) for $n=2$, $k=1$, $l=0$.

Figure 3

Proof. In this proof, we will consistently write $y=(z,t) \in I^{k-l-1} \times I$. We note that our F is already fiberwise holonomic over the last l directions. The goal is to approximate F in one with a function that is fiberwise holonomic in one additional direction. This direction is the one that varies with t in (z,t).

Recall that we have the family of holonomic functions $F_y = F_{(z,t)}$. We will use these functions to eventually construct our family of holonomic functions $\tilde{F}_z$. The method we will use to construct $\tilde{F}_z$ for a given z is by taking a finite amount of F_{z,c_i}, where $c_i \in I$ for some finite set of i. Then we will perturb these such that on a small set, $\tilde{F}_{z,c_i}$ agrees with $F_{z,c_{i+1}}$. We can then glue F_{z,c_i} and $F_{z,c_{i+1}}$ together for all i to end up with

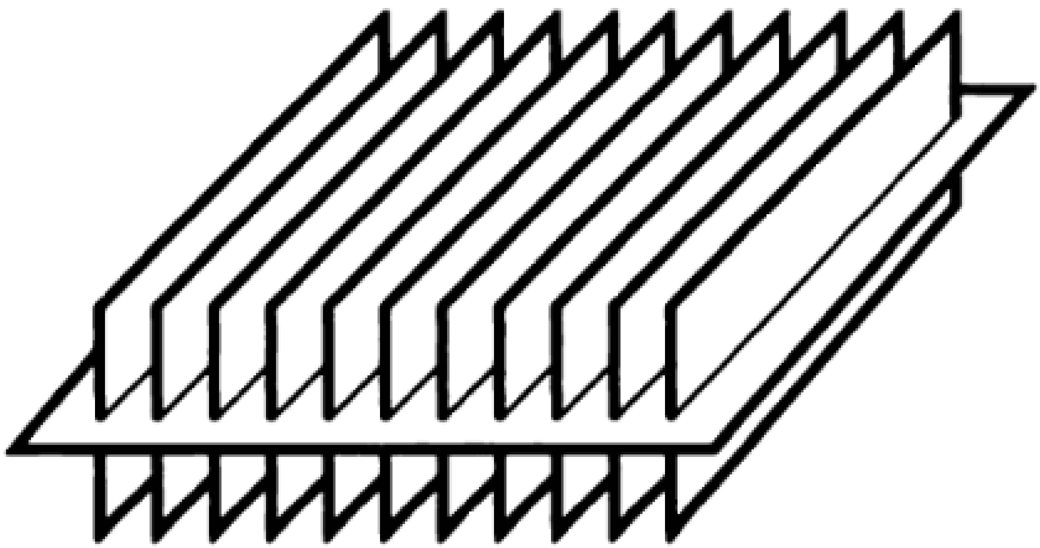


Figure 4: Figure 3.5 of [EM02], $\left(\bigcup_{i=1}^{N} A_\delta(z, c_i)\right) \cup (z \times I^{l+1})$ for $n = 3$, $k = 2$, $l = 1$.

$\tilde{F}_{z,t}$. The perturbation we will use will be given by Claim 1. The key idea here is that we will not touch the function on the boundary of the domain of $F_{z,t}$. This will allow us to combine the perturbed function with the original function. This would enable us to glue $\tilde{F}_{z,c_i}$ to $\tilde{F_{z,c_{i+1}}}$.

In order for us to achieve this, we will need to make small cuts in the domain of each F_{z,c_i} that allow us to perturb these functions without having to worry about continuity. This is how we end up with a domain that looks like $\Omega_\delta(z, N)$.

Now, we note that F agrees with F_y over the boundary of I^k, so we do not need to perturb in that area. Thus for any $(z, t) \in I^{k-l}$, we will create a little buffer. This buffer is provided to us by V_δ.

We define the set

$$W_\delta(z, t) := \left(\overline{U_\delta(z, t)} \backslash U_{\delta_1}(z, t)\right) \cup V_\delta(z, t),$$

where $\delta_1 = \theta\delta$ for some $0 < \theta < 1$.

We can see $W_\delta(z, t)$ as a hollowed out version of $U_\delta(z, t)$ with an extra space around $(z, t) \times (\partial I^k)$ that ensures that our functions will behave properly on the boundary of $(z, t) \times I^k$.

Fixing δ, we will now just write $U(z, t) := U_\delta(z, t)$ and in the same way $V(z, t)$, $A(z, t)$, $W(z, t)$ and $\Omega(z, N)$.

We write $F_{z,t} = F_{z,1}$ for $t > 1$ so that we can talk about $F_{z,t+\sigma}$ without having to worry about edge cases. Then we can see that

$$\lim_{\sigma \to 0} \left(\max_{(z,t) \in I^{k-l-1} \times I, x \in U(z,t+\sigma) \cap U(z,t)} d(F_{z,t+\sigma}(x), F_{z,t}(x)) \right) = 0.$$

Here, d is a metric on the jet space. In other words, for any $\epsilon > 0$ there is a sufficiently small σ such that $d(F_{z,t}(x), F_{z,t+\sigma}(x)) < \epsilon$. This follows from the continuity of the family of functions. We will use this to make our perturbations of $F_{z,t}$.

Claim 1 (The Interpolation Property)**.** For any $\epsilon > 0$ there exists a positive integer N, $\sigma = \frac{1}{N}$ and a family of holonomic sections

$$F^\tau_{z,t} : U(z, t) \to J^r(\mathbb{R}^n, \mathbb{R}^m),$$

where $\tau \in [0, \sigma]$ such that

1. $F^0_{z,t} = F_{z,t}$ for all $(z,t) \in I^{k-l-1} \times I$.
2. $F^\tau_{z,t}|_{W(z,t)} = F_{z,t}|_{W(z,t)}$ for all $(z,t) \in I^{k-l-1} \times I$ and $\tau \in [0,\sigma]$.
3. $d(F^\tau_{z,t}, F_{z,t}) \leq \epsilon$ for all $(z,t) \in I^{k-l-1} \times I$ and $\tau \in [0,\sigma]$.
4. $F^\tau_{z,t}|_{Op(z\times t\times I^l)} = F_{z,t+\tau}|_{Op(z\times t\times I^l)}$ for all $(z,t) \in I^{k-l-1} \times I$ and $\tau \in [0,\sigma]$.

We can see $F^\tau_{z,t}$ as a homotopy between $F^0_{z,t}$ and $F^\sigma_{z,t}$. Because of property (4), this homotopy is how we can connect $F_{z,t}$ with $F_{z,t+\sigma}$, which will ultimately give us our desired $\tilde{F}_z$. It is important to note that property (2) and (3) ensure that $F^\sigma_{z,t}$ will stay close to our original F. This guarantees that our eventual $\tilde{F}_z$ satisfies the conditions:

- $\tilde{F}_z = F$ on $\Omega_\delta(z,N) \cap Op(\partial I^k)$ and
- $d(\tilde{F}_z, F|_{\Omega_\delta(z,N)}) < \epsilon$.

To see that Claim 1 is indeed true, we note that $Op(z\times t\times I^l) \cap W(z,t) \subset V(z,t)$, and because we assumed $F_{z,t}|_{V(z,t)} = F|_{V(z,t)}$, this implies $F_{z,t}|_{V(z,t)\cap V(z,t+\tau)} = F_{z,t+\tau}|_{V(z,t)\cap V(z,t+\tau)}$. Thus requirements (2) and (4) do not contradict each other. Furthermore, requirement (3) can be guaranteed by the limit stated just above the claim and picking N sufficiently large.

The rest of this proof will be using this new family of functions to glue F_{z,c_i} and $F_{z,c_{i+1}}$ together, giving us $\tilde{F}_z$. This will involve cutting the domain of F_{z,c_i} to allow us to merge F_{z,c_i} and F^σ_{z,c_i}. To do this, we will be dividing $U(z,t)$ into small subsets. These subsets are given below.
We will proceed with the family of functions given by the interpolation property that satisfies our given ϵ, then for $i = 0,1,..,N$, set

$$B_{z,i} := z \times i\sigma \times I^l,$$

and for $i = 1,...,N$ we set:

- $F^{old}_{z,i} := F_{z,i\sigma}$
- $F^{new}_{z,i} := F^\sigma_{z,i\sigma}$
- $\tilde{U}_{z,i} := U(z,i\sigma) \cap \pi^{-1}_{k-l}(z \times \Delta_i)$
- $\tilde{U}^-_{z,i} := U(z,i\sigma) \cap \pi^{-1}_{k-l}(z \times \Delta^-_i)$
- $\tilde{U}^+_{z,i} := U(z,i\sigma) \cap \pi^{-1}_{k-l}(z \times \Delta^+_i)$
- $\tilde{U}'_{z,i} := U(z,i\sigma) \cap \pi^{-1}_{k-l}(z \times c_i)$

Here $c_i = i\sigma - \frac{\sigma}{2}$, $\Delta_i = ((i-1)\sigma, i\sigma)$, $\Delta^-_i = ((i-1)\sigma, c_i]$ and $\Delta^+_i = [c_i, i\sigma)$. See also Figure 5.

We note that the set $U'_{z,i}\backslash A(z,i\sigma)$ is contained in $W(z,i\sigma)$. This implies that $F^{new}_{z,i}$ coincides with $F^{old}_{z,i}$ on $U'_{z,i}\backslash A(z,i\sigma)$ by property (2) of the interpolation property.

All of this allows us to define the following family of functions

$$\tilde{F}_z : \bigcup_{i=1}^{N} \left(\tilde{U}_{z,i}\backslash A(z,i\sigma)\right) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

as follows:

$$\tilde{F}_z(x) := \begin{cases} F^{old}_{z,i}(x), & x \in \tilde{U}^-_{z,i} \\ F^{new}_{z,i}(x), & x \in \tilde{U}^+_{z,i} \end{cases}.$$

Note that $\tilde{U}_{z,i} \cap \tilde{U}_{z,j} = \emptyset$, thus $\tilde{F}_z$ is defined on N disjoint sets and is continuous on each disjoint set.

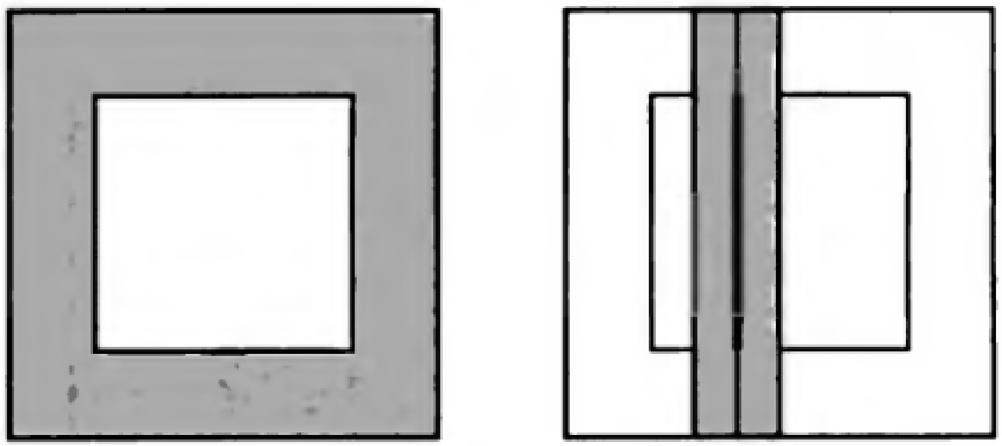


Figure 5: Figure 3.9 of [EM02], The set $W(z, i\sigma) \subset U(z, i\sigma)$ (left picture, grey), the set $\tilde{U}_i$ (right picture, grey), $\tilde{U'}_i$ (right picture, black line), with on either side $\tilde{U}_i^-$ and $\tilde{U}_i^+$, and $A(c_i)$ (right picture, bold line).

Now we can see that $B_{i,z} \subset \overline{\tilde{U}_{z,y}} \cap \overline{\tilde{U}_{z,i+1}}$. Recall that, due to property (4) of Claim 1 $F^{new}_{z,i}$ coincides with $F^{old}_{z,i+1}$ on $Op(B_{z,i})$. Then we can extend the domain of $\tilde{F}_z$ to

$$\bigcup_{i=1}^{N} \left(\tilde{U}_{z,i} \backslash A(z, i\sigma)\right) \cup \bigcup_{i=0}^{N} Op(B_{z,i}) = \Omega(z, N).$$

By combining property (2) of Claim 1 and the assumption that $F_{z,t} = F|_{U_\delta(y)}$ for $y \in Op(\partial I^{k-l})$, we can see that indeed $\tilde{F}_z = F$ on $\Omega_\delta(z, N) \cap Op(\partial I^k)$. Additionally, by property (3) of the interpolation property, we see that $d(\tilde{F}_z, F|_{\Omega_\delta(z,N)}) < \epsilon$. Thus our constructed $\tilde{F}_z$ does indeed satisfy our conditions. □

We can reinterpret this lemma as follows:

Corollary 4.3.3 (Inductive lemma, rephrased). *Let $I^k \subset \mathbb{R}^n$ be the unit cube in the first k coordinates. Suppose that a section*

$$F : Op(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

is holonomic over $Op(\partial I^k)$ and for a non-negative integer $l < k$, F is fiberwise holonomic with respect to the fibration $\pi_{k-l} : I^k \to I^{k-l}$. This is to say, F is holonomic over the subsets $y \times I^l$ where

$$y = (z, t) \subset I^{k-l} = I^{k-l-1} \times I.$$

In particular, suppose that for a positive δ there exists a family of holonomic sections

$$F_y = j^r f_y : U_\delta(y) \to J^r(U_\delta(y), \mathbb{R}^m),$$

where y as before, such that

- $F_y|_{(y \times I^l) \cup V_\delta(y)} = F|_{(y \times I^l) \cup V_\delta(y)}$ *for all* $y \in I^{k-l}$
- $F_y = F|_{U_\delta(y)}$ *for* $y \in Op(\partial I^{k-l})$.

Then there exists a isomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \phi(x_1, ..., x_n))$$

such that $d(h, \mathrm{id}) < \delta$ and a section

$$\tilde{F} : Op(h(I^k)) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

such that

- $h = \text{id}$ *and* $\tilde{F} = F$ *on* $Op(\partial I^k)$*,*
- $d(\tilde{F}, F|_{Op(h(I^k))}) < \epsilon$*,*
- *the section* $\tilde{F}|_{h(I^k)}$ *is fiberwise holonomic with respect to the fibration* π_{k-l-1}*.*

Proof. There exists a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ with $h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \phi(x_1, ..., x_n))$ such that $h = \text{id}$ on $Op(\partial I^k)$ and for each $z \in I^{k-l-1}$ the set $h(z \times I^{l+1})$ is contained in Ω_z. In 6 we can see how this h would be constructed. The section $\tilde{F}_z$ constructed in the inductive lemma is defined on $Op(h(z \times I^{l+1}))$ thus the section

$$\tilde{F}(z,t,x) = \begin{cases} \tilde{F}_z(z,t,x) & (z,t,x) \in (Op(I^k)) \cap (I^{k-l-1} \times \mathbb{R}^{n-k-l+1}) \\ F(z,t,x) & (z,t,x) \in Op(\partial I^k) \end{cases}$$

satisfies the required properties. □

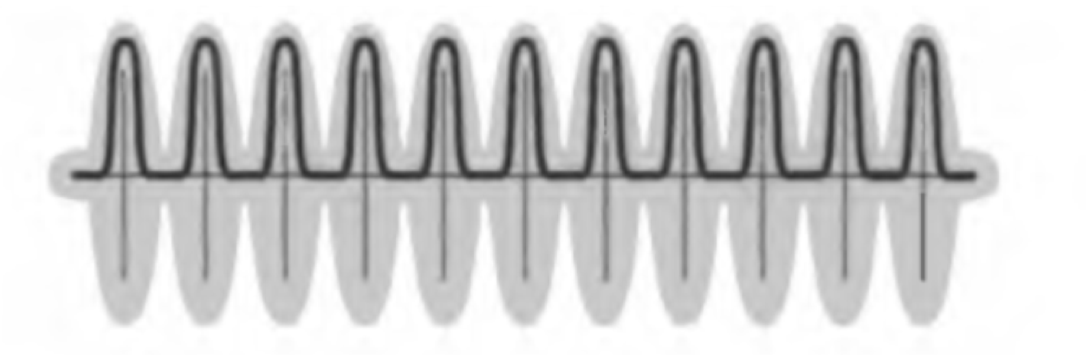


Figure 6: Figure 3.6 of [EM02], The image $h(I)$ for $n = 2, k = 1, l = 0$

Now we are ready to prove 4.1.3.

Proof of theorem 4.1.3. We note that a section F is holonomic if and only if it is fiberwise holonomic with respect to the fiber $\pi_0 : I^k \to I^0 = \{p\}$, since $\pi_0^{-1}(p) = I^k$. Furthermore, since any section is holonomic over any point, it is fibre wise holonomic with respect to the fibre $\pi_k \equiv \text{id} : I^k \to I^k$. We say that F is i-holonomically approximable if there is a diffeomorphism $h^i : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \phi(x_1, ..., x_n))$ and $d(h, \text{id}) < \delta$ and a function $F^i : Op(h(I^k)) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$ such that

- $h^i = \text{id}$ and $F^i = F$ on $Op(\partial I^k)$
- $d(F^i, F|_{Op(h(I^k))}) < \epsilon$
- the section $F^i|_{h(I^k)}$ is fiberwise holonomic with respect to the fibration π_i

Note that since F^0 is fibrewise holonomic with respect to the fibre π_0, F^0 is a holonomic section, thus we need to show that F is 0-holonomically approximable.

We will show that if F is i-holonomically approximable, it is $i-1$ holonomically approximable, and that since F is k-holonomically approximable, induction will then tell us that it is 0-holonomically approximable.

Suppose F is i-holonomically approximable. Then we have h^i and F^i. We would like to apply the inductive lemma on F^i, however, F^i is defined on $Op(h^i(I^k))$ instead of on $Op(I^k)$. To fix this, we will first look at

$$\tilde{F}^i := F^i \circ h^i,$$

assuming that $h_i(Op(I^k)) \subset Op(h_i(I^k))$, this function is indeed defined on $Op(I^k)$.

Due to the properties of h^i, we can see that $\tilde{F^i} = F$ on $Op(\partial I^k)$ and $\tilde{F^i}$ is still fiberwise holonomic with respect to the fibre π_i. Then by the inductive lemma, we can find a diffeomorphism $\tilde{h}^{i-1}$ and a section

$$\tilde{F}^{i-1} : Op(\tilde{h}^{i-1}(I^k)) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$$

that approximates $\tilde{F^i}$ and is fibrewise holonomic over $\pi_{i-1} \circ (\tilde{h}^{i-1})^{-1} : \tilde{h}^{i-1}(I^k) \to I^{i-1}$.

We may assume that $\tilde{F}^{i-1}$ approximates $\tilde{F^i}$ sufficiently close, such that $F^{i-1} = F^{\tilde{i-1}} \circ (h^i){-1}$ approximates F^i sufficiently close. Then with $h^{i-1} = h^i \circ \tilde{h}^{i-1}$, we can see that $F^{i-1} : Op(h^{i-1}(I^k)) \to J^r(\mathbb{R}^n, \mathbb{R}^M)$ satisfies all the required conditions, thus our induction is complete. □

Recall that we have shown that the Parametric Holonomic Approximation Theorem follows from the parametric holonomic approximation over a cube which in turn follows from the holonomic approximation over a cube, thus we have successfully shown the Parametric Holonomic Approximation Theorem to be true.

5 The sphere eversion

One of the interesting applications of the Parametric Holonomic Approximation Theorem is Smale's sphere eversion. In [Sma59], Smale set out to give a classification of all immersions of the 2-sphere into $\mathbb{R}^n$ for $n > 2$. Here, two immersions were seen as equivalent if there existed a regular homotopy between the two.

Smale found that under this classification, in $\mathbb{R}^3$ the regular inclusion was equivalent to one that turned the sphere 'inside-out'. In this chapter we will provide a proof that these two functions can indeed be connected by a regular homotopy, using the Parametric Holonomic Approximation Theorem.

In order for us to use the Parametric Holonomic Approximation Theorem on a homotopy on the sphere, we will need to define our original function on such a manifold V, that the sphere is a polyhedron of V. To achieve this, we choose V to be the thickened sphere

$$V := \{x \in \mathbb{R}^3 | (1-\delta) < ||x|| < (1+\delta)\}.$$

Then let i_V be the inclusion of V into $\mathbb{R}^3$,

$$\text{inv} : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$$

such that

$$\text{inv}(x) = \frac{x}{||x||}$$

and

$$r : \mathbb{R}^3 \to \mathbb{R}^3$$

such that

$$r(x) = -x.$$

Theorem 5.0.1 (Smale's sphere eversion). *The function $r \circ \text{inv} \circ i_V|_{S^2} : S^2 \to \mathbb{R}^3$ is regularly homotopic to $i_V|_{S^2}$.*

Proof. We will write $f_0 = i_V$ and $f_1 = r \circ \text{inv}\, i_V$. We note that f_0 and f_1 are both immersions. We note that $df_0, df_1 : V \to J^1(V, \mathbb{R}^n)$ can be seen as sections of the 1-jet. For $v \in V$, we can chose a basis of T_vV by taking v as a vector in $\mathbb{R}^3 = T_vV$ and b_1, b_2 two vectors that make v, b_1, b_2 into an orthogonal basis of T_vV, then it is clear to see that $(df_0)_v$ sends the base vectors v, b_1, b_2 to v, b_1, b_2. Additionally $d(\text{inv} \circ i_V)_v$ sends v to $-v$, and does nothing to b_1, b_2, and $(dr)_v$ sends any tangent vector x to $-x$. Then we see that $(df_1)_v$ sends the base vectors v, b_1, b_2 to $v, -b_1, -b_2$ respectively. We thus see that $(df_1)_v$ rotates the tangent plane around the vector v over π radials.
We now define

$$F_t : V \to J^1(V, \mathbb{R}^3) = V \times \mathbb{R}^3 \times \text{Hom}(\mathbb{R}^3, \mathbb{R}^3)$$

as a family of sections parameterized by $t \in I$, such that

$$F_t(v) = (v, f_0(v) + t(f_1(v) - f_0(v)), M_{v,i}),$$

where $M_{v,i}$ is the rotation around the vector v by $i\pi$ radials.
We can slightly reparameterize this function such that is constant in t for $t \in Op(\partial I)$.
We note that S^2 is a subpolyhedron of V, thus by Theorem 4.1.2 there exists a family of diffeomorphisms

$$h_t : V \to V$$

and a family of holonomic sections

$$\tilde{F}_t : Op(h_t(S^2)) \to J^1(V, \mathbb{R}^3)$$

such that

1. $h_t = \text{id}$ and $\tilde{F}_t = F_t$ for $t \in Op(\partial I)$ and
2. $d(\tilde{F}_t, F_t|_{Op(h_t(S^2))}) < \epsilon$ for arbitrary ϵ and all t.

Since we can find this $\tilde{F}_t$ sufficiently close to F_t, we can guarantee that $\tilde{f}_t$ is an immersion, where $j^1\tilde{f}_t = \tilde{F}_t$. Now since h_t is a diffeomorphism, $f'_t := \tilde{f}_t \circ h_t|_{S^2} : S^2 \to \mathbb{R}^3$ is also an immersion. Additionally, due to (1) we see that $f'_0 = f_0|_{S^2}$ and $f'_1 = f_1|_{S^2}$, thus f'_t is a regular homotopy between $f_0|_{S^2}$ and $f_1|_{S^2}$. □

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