Utrecht University

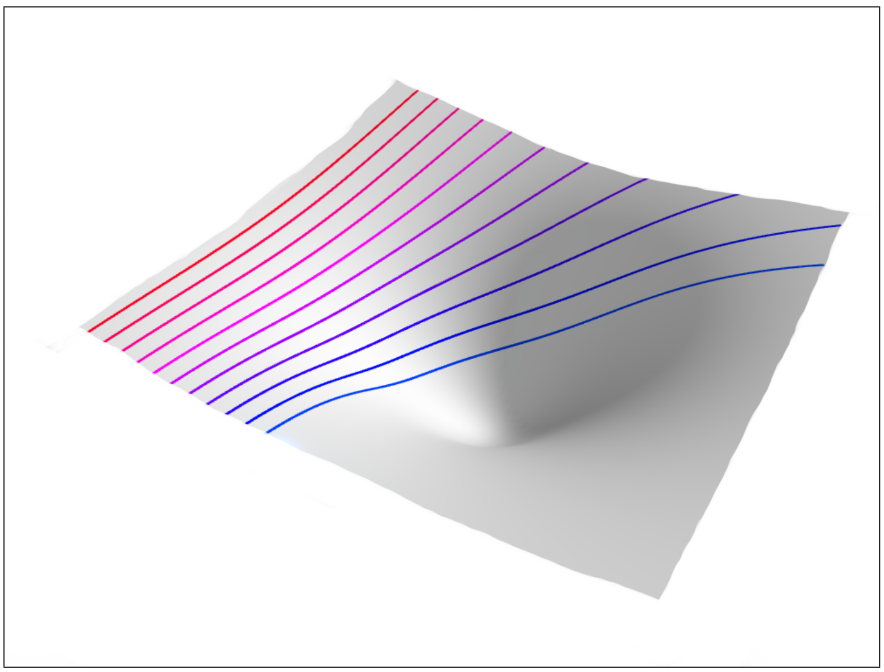
Faculty of Science

Geodesics in semi-Riemannian Geometry and links to General Relativity

Bachelor Thesis

Hanneke Schroten

Mathematics and Physics



Supervisors:

Dr. Álvaro del Pino Gomez Supervisor

Mathematical Institute, Utrecht University

Dr. Thomas W. Grimm Supervisor

Institute for Theoretical Physics, Utrecht University

Msc. Jeroen Monnee Advisor

Institute for Theoretical Physics, Utrecht University

June 2021

Abstract

Gravity, as a fundamental force of nature, is very well described by General Relativity. In General Relativity, our universe is thought of as a spacetime manifold. Differential geometry gives us a lot of tools to study all manifolds, including spacetime. In this thesis, we develop tools to study geodesics, the generalization of straight, unaccelerated curves, that determine paths of freely falling particles in spacetime. Those tools include manifolds, tangent spaces, tensor bundles, the metric tensor, linear connections and Christoffel symbols. We explain how these tools from Differential Geometry play a fundamental role in the theory of General Relativity. Lastly, we go through the geodesic equations for Minkowski, Schwarzschild and Misner spacetime, providing solutions for Minkowski and Misner spacetime.

Front page: Picture made using [1]. A curved surface changes the geometry. A much used visualisation for how mass curves spacetime: The surface represents spacetime where a mass is placed in the centre. Depicted are some geodesics. Initially parallel but differently affected by the curvature, they represent free falling particles affected by the curvature of spacetime.

Specifics for surface used: $-.3/(.3 + (x - 0.1) * (x - 0.1) + y * y)$ with $-1 < x < 1$ and $-1 < y < 1$.

Contents

1 Introduction

Over the centuries, physicists have been trying to understand the nature of gravity, as it is a fundamental force in our universe. This search eventually lead us to presently explain gravity using the theory of General Relativity. In this theory our universe is represented by a *spacetime*, a *Lorentz manifold* on which each point represents an *event*, so a time and place. General Relativity tells us curvature of this spacetime is linked to the gravity we experience. We are interested as to what that curvature actually is and especially how we would go about measuring it. After all we are bound to spacetime and so cannot view it from the outside. This also raises the question if there even are straight lines in our universe. These straight lines are fundamental in measuring distance, what we would still like to be able to do in this spacetime model. We will develop mathematical tools to tackle these questions and discuss some physical relevance and applications.

We will first look into the mathematical concepts of *manifolds* and *geometry*, that describe space and its properties in a very abstract way. When describing the space around us, or any object in space, we use coordinates, a set of numbers to specify the position and or time. Sometimes on a bigger scale these coordinates are not suitable everywhere: longitude and latitude work well within The Netherlands, but if we are close to the poles these coordinates behave very weirdly (longitude runs very quickly and is undefined at the poles).

We will build on geometric tools that are coordinate independent in chapter 2, which is also very relevant for physicists. This motivates wanting to find coordinate independent tools, as we expect physical phenomena not to depend on our choice of coordinates. The main motivation in this thesis is, however, not Earth's atmosphere, but *spacetime* as described by General Relativity. Here, being able to describe non-Euclidean spaces is crucial, as there is no physical Euclidean ambient space in which spacetime lies. Therefore, we need to be able to describe properties of spacetime as intrinsic properties. An important example of such a property is *curvature*, as it is linked to gravity, which we want to be able to describe. We will see more about this in chapter 4.

Next in line for our mathematical tools, we will describe curves with constant velocity, called *geodesics*. We will see they are a generalization of straight lines on manifolds, as curves that are not accelerated, and explain why they are useful in our mathematical framework in chapter 3. In this chapter we will mostly focus on Riemannian Geometry. After developing these mathematical tools, we will explain how they form the basis of General Relativity in chapter 4, and what else we need to build the theory of General Relativity. Next, in chapter 5, we will explain how some fundamental ideas in physics can be translated to the abstract mathematical framework, with which General Relativity can describe our reality. We will look into the semi-Riemannian Geometry and Lorentz manifolds. We will also explain the physical relevance of geodesics in spacetime, as paths of freely falling objects that only experience gravitational effects. In chapter 6, we will calculate the geodesic equations for Minkowski, Schwarzschild and Misner spacetimes, a few explicit examples of Lorentz manifolds. Lastly we will discuss our results and give some interesting ideas one could explore after understanding the concepts explained in this thesis, in chapter 7.

The main references we use for this thesis are

- *Introduction to Smooth Manifolds* by Lee [2], which develops the tools in the theory of manifolds needed to describe a metric.
- *Riemannian Manifolds* by Lee [3], which develops tools to understand more about geometry on manifolds, and describe Riemannian geodesics.
- *Semi-Riemannian Geometry* by O'Neill [4], which develops tools for the geometry used in General Relativity, and explains some spacetimes and their geodesics. This text is very suitable for mathematicians wanting to understand more about relativity.
- *An Introduction to General Relativity* by Carroll [5], where almost everything we describe is written, in a more suitable text for physics students.
- *The Large Scale Structure of Space-Time* by Hawking and Ellis [6], which describes a lot of what we

will discuss and stresses the physical relevance and limitations for experiments to verify whether the framework of manifolds and spacetime describes our reality.

- *General Relativity* by Wald [7], a physics book which is more mathematically formal, but includes subjects that require much more study beyond this thesis.

In the present age, online are also a lot of other references than books. One has to be careful of the exactness of online sources as Wikipedia and Youtube, but they can be very helpful in studying these subjects. It is best to verify the online statements using literature. For example at [8] a lot of the concepts used are named and you can easily navigate to pages that explain them. On Youtube I would like to point out two video series, based in physics, that explain these concepts. They are thoroughly explained in [9], and more example-driven in [10].

2 Manifolds and Geometry

The concepts *manifold* and *geometry* talk about space and its properties in a very abstract way. Before we dive into the definitions, theorems and applications let us start with what we are about to consider, with as little technicalities as possible.

We describe the world around us using coordinates, a set of numbers to specify the position and or time of objects. Usually, these systems are considered **Euclidean**, meaning we can describe all positions with real numbers on perpendicular axes, for example the two-dimensional (x, y)-grid, that we used to plot functions in high school. Sometimes, however, this view gets us into trouble: in the introduction we saw longitude and latitude are not suitable coordinates for the whole world. We get into trouble at the poles, see the left hand side of Figure 1. If we consider a calculation for which we need one, everywhere well-defined, coordinate system, we can consider our Earth in its three dimensional ambient space. We can, however, not do that for spacetime, so we do not want to rely on ambient spaces so much.

In this chapter we will work on a lot of tools that we are already familiar with in the Euclidean setting. We see how they can be defined and worked with in systems for which one coordinate system is not enough, without needing to turn to an ambient Euclidean space. We can even do this without needing to write down the coordinates explicitly, which ensures we talk about intrinsic properties, and not properties that somehow emerged from our choice of coordinates.

We will first explain tools for describing spaces without the ambient space, that we do by looking into manifolds. We will need these tools to describe geodesics in chapter 3. To describe curvature we need to be able to express distances and angles, which are defined by a metric. There are different kinds of metrics, the most intuitive is called a *Riemannian metric*. However, General Relativity uses a *semi-Riemannian metric*, sometimes called *pseudo-Riemannian metric*. We will first look into Riemannian systems, physically that is a system with only "spatial" dimensions, but as usual in mathematical texts we will consider very general cases of arbitrary spatial dimensions.

For references during this chapter we use throughout *Introduction to Smooth Manifolds* by Lee [2], mostly chapters 1 to 3 and 8 to 14. For a reference with more physical interpretations and notations we refer to *An introduction to General Relativity: Spacetime and Geometry* by Carroll [5], chapters 1 to 3.

2.1 Manifolds

The intuition that inspires our manifold definition is what we considered before: The Netherlands as small part of Earth's surface, so that we only need two coordinates instead of the three we would use in the Euclidean ambient space. Small parts of a manifold are Euclidean, but the whole manifold need not be.

Definition 2.1.1 (Manifold, conceptual)**.** A **manifold** is a space that looks locally like Euclidean space.

Definition 2.1.2 (Manifold, exact mathematical)**.** Let M be a topological space, then it is an n-dimensional **topological manifold** if

- M is *Hausdorff* (i.e. for $p, q \in M, p \neq q$ we have $U, V \subseteq M$ opens, $p \in U$, $q \in V$ with $U \cap V = \emptyset$;
- M is *second countable*, so the topology of M has a countable basis;
- M is *locally Euclidean of dimension n*, so for each $p \in M$ we can find a neighbourhood homeomorphic to an open in $\mathbb{R}^n$: $p \in U \subseteq M$ open, $\hat{U} \subseteq \mathbb{R}^n$ open, $\varphi\colon U \to \hat{U}$ homeomorphism, also called a **(coordinate) chart**.

To avoid a lengthy explanation about topology (where the first two properties of the exact definition would become clearer) see [2, pp. 2–10]. The important part of the definition we focus on (as made clear by definition

2.1.1) is the third point of 2.1.2. The beauty of manifolds is that they are more general than Euclidean space, but we can use a lot of Euclidean concepts and intuition on manifolds. Sometimes, as we will see, this requires some work, but many ideas from Euclidean space translate to manifolds.

The coordinates on our manifold are given by coordinates of the image of the homeomorphisms from definition 2.1.2. These coordinates are not unique, for example take $\mathbb{R}^2$. The most obvious Euclidean way of giving coordinates to points is by defining an origin and some x and y directions, where the point $(0, 0)$ is the origin. There are other possibilities to assign coordinates. With the same choice of origin we could also choose polar coordinates, where we measure the distance to the origin r, and measure the angle θ the line to the origin makes with respect to one special radial line (usually the x-axis as previously chosen). These are coordinates for parts of $\mathbb{R}^2$ where we avoid the origin and special radial line, as we can plot them in a (r, θ)-plane for $r > 0$ and $0 < \theta < 2\pi$, see Figure 1.

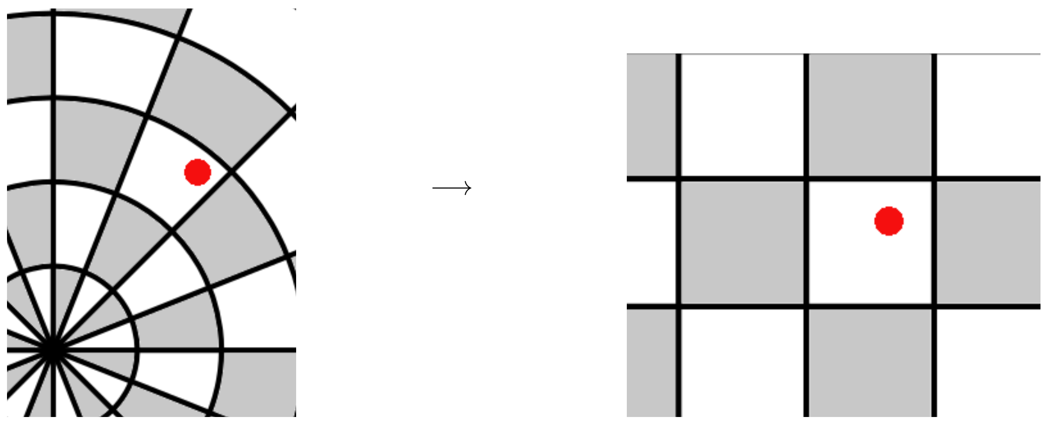


Figure 1: Polar coordinates as a coordinate chart to Euclidean space.

This brings us to the problems for manifolds: usually one coordinate chart is not sufficient to cover the whole manifold. In the case of $\mathbb{R}^2$ with polar coordinates we would need two charts, that have different $r = 0$ points and $\theta = 0$ half-lines. In general each point on a manifold has a neighbourhood which has a coordinate chart. However, for our purposes, we demand the coordinate charts to behave nicely.

Definition 2.1.3 (Atlas)**.** Given ϕ_i, a coordinate chart on M, its **atlas** is a collection of all coordinate charts smoothly compatible with ϕ_i, such that, for ϕ_j, another coordinate chart of$M$ in the atlas, agrees with it: This means the **transition map**, $\phi_i \circ \phi_j^{-1}$ is smooth[1].

Definition 2.1.4 (Smooth manifold)**.** An n-dimensional **smooth manifold** M is a topological manifold (definition 2.1.2), together with a **smooth structure**. A smooth structure is an equivalence class of atlases, under the equivalence relation of smooth diffeomorphisms[2]. This means all transition maps are smooth diffeomorphisms.

In general, a topological manifold need not admit any smooth structure: the transition maps need not be smoothly diffeomorphic at all. A topological manifold may also admit more than one smooth structure. That means the same topological manifold could be endowed with two (or more) 'well-behaved' sets of coordinate charts, that are not well-behaved with respect to each other. The different structures, on one topological manifold, define different smooth manifolds.

In this thesis we will consider only smooth manifolds.

In the rest of this thesis (mostly this chapter), we will sometimes use maps between two different manifolds, and more commonly use functions from a manifold to $\mathbb{R}$. Manifolds will usually be denoted M or N. To distinguish the two types of maps, we will use capitals for maps between manifolds, e.g. $F, G \colon M \to N$, and

[1] Smooth is to have n-th derivative continuous for all $n \in \mathbb{N}$.
[2] Meaning the diffeomorphism and its inverse are both smooth.

small letters for maps to numbers, e.g. $f, g\colon M \to \mathbb{R}$. The latter can also be denoted $f, g \in C^\infty(M)$, because we assume maps to be smooth[3].

Throughout this thesis, maps and manifolds are assumed smooth, unless explicitly stated otherwise. The notion of smoothness is defined by taking the coordinates such that we talk about maps between Euclidean spaces[4].

2.2 Tangent and Cotangent space

We have already seen manifolds are locally Euclidean. Now, we want to use some tools from Euclidean space on our manifold. One nice aspect of Euclidean space is the linear algebra we can use, being able to express a point using coordinates, but also describing vectors, since Euclidean space is a vector space.

We introduce two vector spaces related to a manifold, but first give some motivation for this structure. We assume our manifold has some function $f\colon M \to \mathbb{R}$. A physical situation we could keep in mind is where we have the Earth's surface as our manifold M and a function f we could measure on each point, say hours of the sunlight during a year. If we want to find someplace sunnier, we need to determine which way to travel for more sunlight hours, so a higher f. We could just go somewhere and see if it is sunnier, but we risk going somewhere less sunny. We want to know in which direction the sun shines more, so we take a directional derivative of f. This examples demonstrate two important features of the vector space we will later define: Firstly, elements of this vector space are directions on our manifold. Secondly, these vectors can be applied to real-valued functions.

We now give some definitions, before going back to a less abstract way of viewing these derivations and tangent space.

Definition 2.2.1 (Derivations as tangent vectors)**.** A **derivation** X_p on M at point p is a linear map $C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz product rule: $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$.

Definition 2.2.2 (Tangent space)**.** All derivations or tangent vectors form a vector space. The **tangent space** of M at p, denoted T_pM is the vector space of all derivations of M at p. A basis of the tangent space is notated $(\partial_1, \dots, \partial_n)$, using a local coordinate system $(x_1, \dots, x_n)$ in a neighbourhood of p. Note that the coordinate representations of the tangent vector thus depends on the choice of coordinates on the manifold. We will denote tangent vectors in different ways, depending on how much we want to stress at what point they start. We will use X, v if the point is not relevant to note explicitly, X_p, v_p for some relevance and (p, v) if we want to stress v is a tangent vector at p explicitly.

Definition 2.2.3 (Vector field)**.** A **(rough) vector field** X is a choice of tangent vector for any point on the manifold, $X_p \in T_pM$.

Before we can make this tangent space a bit more intuitive with a visual representation, we will look at maps between manifolds. Let M and N be manifolds, $F\colon M \to N$ a map between them. It would be nice to be able to construct a map between T_pM and $T_{F(p)}N$ using F and we will do exactly that.

Definition 2.2.4 (Differential)**.** Given a function $F\colon M \to N$ there exists a linear map, $dF_p\colon T_pM \to T_{F(p)}N$ called the **differential of** F **at** p. For $v \in T_pM$ and $g \in C^\infty(N)$ it acts as

$$(dF_p(v))\,(g) = v(g \circ F). \tag{2.1}$$

Now we can go on to visualise our tangent space. For this, we consider a curve on our manifold, that could, for example, represent the trajectory of a particle. The curve's derivative measures its velocity. Let $\gamma\colon \mathbb{R} \to M$

[3] By this we mean $\left(f \circ \phi_i^{-1}\right)_j$, the j-th coordinate of the map from $\mathbb{R}^n$ to $\mathbb{R}$ describing f in a neighbourhood $U_i \subset M$, is smooth in the classical sense that the n-th derivative is continuous, for any $n \in \mathbb{N}$.

[4] We state here explicitly, for a smooth map $F\colon M \to N$, this means $\phi_{N_j} \circ F \circ \phi_{M_i}^{-1}$ is smooth for $\phi_{N_j} : U_j \subset N \to \mathbb{R}^n$,where $\phi_{M_i} : U_i \subset M \to \mathbb{R}^m$ are the appropriate coordinate charts.

be a curve, such that $\gamma(0) = p \in M$. Around p we can choose coordinates, and we can think of derivations as the velocity of the curve at p. We have seen a basis for the tangent space[5] is given by the coordinates, and in that basis we can express the tangent vector represented by this curve:

$$\gamma'(0) = (\gamma_1'(0), \dots, \gamma_n'(0))\,. \tag{2.2}$$

This way, we can give a curve for any derivation at p, and see why the basis of the tangent space is linked to the coordinate basis. Note actually there are many possible curves for one vector, so to be able to state an actual isomorphism between curves and vectors this is not enough. To do that we need to define an equivalence relation: if for each $f\colon M \to \mathbb{R}$ we have $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ that means $\gamma_1 \sim \gamma_2$ [2, p. 72]. Derivations are then given by $[\gamma]$ and denoted $\gamma'(0)$. This also gives an intuition about why we require derivations to adhere to the Leibniz product rule, because of the usual product rule of derivatives

$$\begin{aligned}
(fg \circ \gamma)'(0) &= (fg)'(\gamma(0)) \cdot \gamma'(0) && (2.3)\\
&= (f'g + fg')(p)\gamma'(0) && (2.4)\\
&= g'(p)\gamma'(0)g(p) + f(p)g'(p)\gamma'(0). && (2.5)
\end{aligned}$$

We have stated derivations form a vector space, called the tangent space. Actually, in this viewpoint of velocities of curves, that structure is not too obvious, as, in general, we cannot intuitively add two curves to get the sum of the velocities. We can only do that if our manifold has a vector space structure, for example $\mathbb{R}^n$ is also a vector space. We can see scalar multiples as speeding up our curve (i.e. replacing $\gamma(\lambda)$ with $\gamma(\mu\lambda)$ for $\mu \in \mathbb{R}$). There are also other ways of viewing the tangent space, see [2, pp. 71–73], but we will use curves throughout this thesis.

We next look at some visual examples, to test how a tangent space should work. Let us look at the sphere, $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Intuitively we would say the tangent space at some point p would be a plane in R^3, perpendicular to the radius of the sphere, touching the sphere only at that point. Indeed, possible velocities of curves through our point, form that plane. Similarly all manifolds M of dimension n have at some point p a tangent space that is also n dimensional.

The intuitive picture however is not always suitable, see figure 2: If we go back to the sphere, we can take some other point q on the sphere, not too far away and not too close to p. The tangent spaces of p and q are both two-dimensional, but where in our picture they intersect somewhere, in reality $T_pM \cap T_qM = \emptyset$, as the tangent vectors do not start at the same point and therefore are not the same vectors. If we consider our curves it becomes clear there is no overlap between the vector spaces.

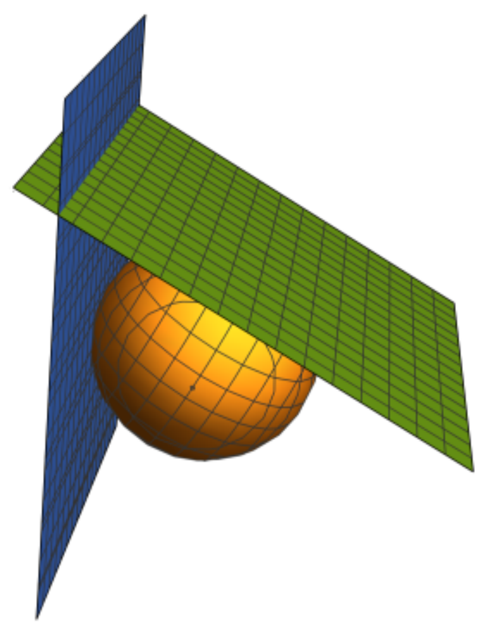


Figure 2: The tangent spaces of two points seem to intersect.

So that means by just defining a tangent space on our manifold, we now have some infinite number of vector spaces, one vector space for each point on the manifold. This is nice for each point, but moving on our manifold (as we did on the sphere in Figure 2) takes us to another vector space which, so far, has nothing to do with the vector space we started at. In particular, the velocity

[5] We have denoted this basis as ∂_i, but this point of view gives us motivation for the notation $\frac{\partial}{\partial x^i}$, as that is the 'operation' a tangent vector does on a function. We stick with the ∂_i notation as it limits the amount of ink or chalk one uses.

of a curve at one point has no relation (yet) to it's velocity at another point. We will glue all of these spaces together in the tangent bundle in section 2.3.

Throughout this thesis (and other texts on these subjects) it will become more and more necessary but difficult to keep track of what spaces our symbols live in and what they do. Now that we have more of a view for what a tangent vector is, we will explain one equation in detail, namely (2.1). First of all, we note it relates two real numbers:

$$\underbrace{\mathbb{R} \ni (dF_p(v))\,(g)}_{} = \underbrace{v(g \circ F) \in \mathbb{R}}_{} \tag{2.6}$$

We define dF_p by how it acts on a vector v tangent to M, and how the result, $dF_p(v)$, a tangent vector on N, acts on a function g.

$v \in T_pM$ so it assigns a number to $f \in C^\infty(M)$ $g \circ F$ is such a function, as $(g \circ F)(p) = g(F(p))$, $g \in C^\infty(N)$ and $F(p) \in N$.

Definition 2.2.5 (Cotangent space)**.** We define the **cotangent space** of M at p as the dual[6] of the tangent space and thus denote it T_p^*M. This vector space has by definition a relation with the tangent space, it consists of linear maps, called **covectors**, $\omega\colon T_pM \to \mathbb{R}$.

An natural basis for the cotangent space is the dual basis, which we denote as $\{dx^i\}_i \subset T_p^*M$. Dual basis meaning $dx^i\partial_j = \delta^i_j$ for all $i, j \in \{1, 2, \ldots, n\}$, where $delta^i_j$ denotes the Kronecker delta.

Definition 2.2.6 (Covector field)**.** Similar to the definition 2.2.3 we can define a **(rough) covector field** ω as a choice of covector for any point on the manifold, $\omega_p \in T_p^*M$.

We can also define the analogue of the differential (definition 2.2.4) for cotangent spaces.

Definition 2.2.7 (Pullback)**.** Given a function $F\colon M \to N$ and $p \in M$ we define the **pullback of** F **at** p as

$$dF_p^*\colon T^*_{F(p)}N \to T_p^*M \tag{2.7}$$

For $\omega \in T^*_{F(p)}N$ and $v \in T_pM$ it acts as[7]

$$dF_p^*(\omega)(v) = \omega(dF_p(v)). \tag{2.8}$$

Using equation (2.8) we can even pullback whole covector fields pointwise[8]. For ω a covector field on M, $F\colon M \to N$ and $v \in T_pM$ we can define $F^*\omega$ pointwise as

$$(F^*\omega)_p v = \omega_{F(p)}(dF_p(v)). \tag{2.9}$$

[6] The dual of a vector space V is defined as the space of all real-valued linear maps from V to $\mathbb{R}$. We only consider real vector spaces in this thesis but these definitions are the same if you read 'complex' and $\mathbb{C}$ instead of 'real' and $\mathbb{R}$.

[7] The reader is encouraged to do the an analysis of this equation, similar to what we did in (2.6), note we again equate two real numbers, and keep in mind the definition of covectors (definition 2.2.5).

[8] We would in the same way want to be able to push vector field forwards, but this is only possible for F diffeomorphisms as we would define $(F_*X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right)$ [2, p. 183].

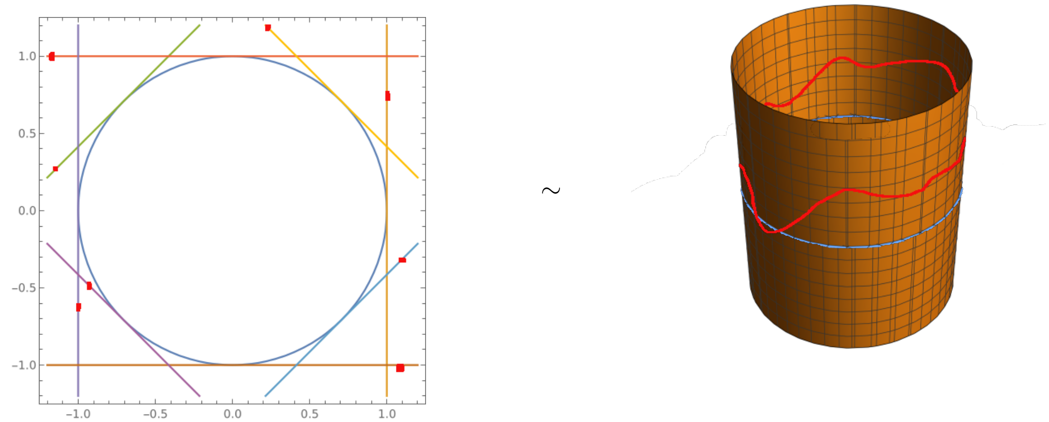


Figure 3: Tangent bundle of the circle as (infinite) cylinder with a section in red.

2.3 Tangent bundle

Next we glue all tangent spaces together, creating a new space that has a natural projection to the original manifold M.

Definition 2.3.1 (Tangent bundle)**.** We define the **tangent bundle of** M as

$$TM = \{(p, v) | p \in M, v \in T_pM\}, \tag{2.10}$$

which has a natural projection

$$\begin{aligned} \pi\colon TM &\to M && (2.11)\\ (p, v) &\mapsto p && (2.12) \end{aligned}$$

Where we define a **fiber** of p to be $\pi^{-1}(p)$, which turns out to be the tangent space T_pM.

For example we consider $\mathbb{S}^1$ and its tangent bundle, which can be seen as the cylinder $\mathbb{S}^1 \times \mathbb{R}$, see figure 3. Actually, any tangent bundle is itself a $2n$-dimensional manifold [2, p. 66].

Definition 2.3.2 (Cotangent bundle)**.** We define the **cotangent bundle of** M as

$$T^*M = \{(p, \omega) | p \in M, \omega \in T_p^*M\}, \tag{2.13}$$

which has a natural projection

$$\begin{aligned} \pi\colon T^*M &\to M && (2.14)\\ (p, \omega) &\mapsto p && (2.15) \end{aligned}$$

The **fiber** of p, $\pi^{-1}(p)$, is the cotangent space T_p^*M.

Definition 2.3.3 (Section)**.** Let E denote either TM or T^*M. A **(rough) section** is a map $\sigma\colon M \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_M$, see figure 3 for an example. A section corresponds to choosing a (co)vector field, and it is here we can finally define smoothness of a field, a (co)vector field is smooth if it corresponds to a smooth section[9].

Definition 2.3.4 (Frame)**.** A local basis in TM or T^*M is called a **frame**, this means for some neighbourhood U of p in M a tuple of sections $(\sigma_1, \dots, \sigma_n)$ which for all $q \in U \subset M$ is a basis of the fiber of q, $\sigma^{-1}(q)$. A special frame is the **coordinate frame**, $\{\partial_i\}_i$ for the tangent bundle and $\{dx_j\}_j$ for the cotangent bundle.

[9]Smooth meaning the coordinate representations are smooth. This defines a local smooth structure, for example $T\mathbb{R}^n = \mathbb{R}^2 n$, for Euclidean spaces, and locally the same holds for any tangent bundle.

We will now consider curves on a manifold. Curves play a very important physical role, as they can describe a moving particle. So for mechanics on a manifold we would expect to find the laws of nature to be expressed in terms of curves.

Definition 2.3.5 (Curve)**.** A **smooth curve** is a function $\gamma \colon I \to M$, where $I \subseteq \mathbb{R}$ is a closed interval[10]. Apart from completely smooth curves, we will see it is also very useful to define a **piecewise smooth curve** as $\tilde{\gamma} \colon I \to M$ (continuous), such that there is a finite partition of I, into intervals I_n such that $\tilde{\gamma}|_{I_n}$ is a smooth curve for each n.

Before we go on, we note that when we do calculations where we express terms in local coordinates, we will denote them with indices, some will be upper indices and some will be lower indices. This is to ensure we can write expressions in a compact way, using **(Einstein) summation convention**. The convention is: if the same index appears once as a lower index and once as an upper index, we sum over all possibilities of that index.

An important application of smooth covector fields is being able to define a **line integral** without needing to explicitly use the coordinate system (and thus not choosing any preferred coordinate system). As said, we will make use of (2.9). We will later on use this to actually define our distance function in section 2.5. On a manifold M with local coordinates $(x_i)_i$ we choose a curve $\gamma(\lambda) = (\gamma^1(\lambda), \dots, \gamma^n(\lambda))$ and a covector field $\omega = \omega_j dx^j$ (sum over j). We denote $\omega_{\gamma(\lambda)} =: \omega(\lambda) = \omega_j(\lambda) dx^j$. We calculate $\gamma^* \omega$, noting $\omega_j(\lambda)$ and $\gamma^i(\lambda)$ are just real numbers,

$$\begin{aligned} \gamma^* \omega &= \omega_j(\lambda) dx^j d\gamma(\lambda) = \omega_j(\lambda) dx^j \gamma^i(\lambda) \partial_i d\lambda \\ &= \omega_i(\lambda) \gamma^i(\lambda) d\lambda = f(\lambda) d\lambda, \qquad \text{where} \qquad f(\lambda) = \omega_i(\lambda)\gamma^i(\lambda). \end{aligned} \tag{2.16}$$

Definition 2.3.6 (Line integral)**.** Let $\gamma \colon [a,b] \to M$ be a smooth curve and ω a smooth covector field on M.

$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega = \int_a^b f(\lambda) d\lambda \tag{2.17}$$

Where $f \colon [a,b] \to \mathbb{R}$ is calculated as in (2.16). We can even define line integrals on **piecewise smooth** curves $\tilde{\gamma}$, meaning $\tilde{\gamma} \colon [a,b] \to M$ is smooth on each subinterval of a finite division of the interval $[a,b] = [a_0, a_1] \cup \cdots \cup [a_{n-1}, a_n]$

$$\int_{\tilde{\gamma}} \omega = \sum_{i=1}^n \int_{[a_{i-1}, a_i]} \tilde{\gamma}^* \omega \tag{2.18}$$

[10] This definition could include open intervals, but to avoid problems for later definitions involving curves we define curves with closed intervals.

2.4 Tensors

In this section, we will combine the tangent space and cotangent space and look at the space where combinations of tangent and cotangent vectors live in. We have seen we can define a *line integral*, and we might wonder if we are able to do *area integrals* or *volume integrals* on our manifold. This is in fact possible but we need a combination of covector fields to do that.

Definition 2.4.1 (Tensor product space)**.** For any vector space V with dual V^* and natural numbers k, l we can construct a (k,l)**-tensor**[11], which is a multilinear[12] function

$$\underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{l \text{ times}} \to \mathbb{R}$$

We can identify this with the (k,l)-tensor being an element of

$$\underbrace{V \otimes \ldots \otimes V}_{k \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{l \text{ times}}, \tag{2.19}$$

where $\otimes$ denotes the **tensor product**[13]

Next we apply this to our known vector space and its dual, the tangent and cotangent spaces.

Definition 2.4.2 (Tensor bundle and field)**.** We define the (k,l) **tensor bundle of** M as

$$\mathfrak{T}_{(k,l)}M = \{(p,T) | p \in M, T \text{ a } (k,l) \text{ tensor of } T_pM\}, \tag{2.20}$$

which has a natural projection

$$\pi \colon \mathfrak{T}_{(k,l)}M \to M \tag{2.21}$$

$$(p,T) \mapsto p \tag{2.22}$$

Where we define a **fiber** of p to be $\pi^{-1}(p)$. A **tensor field** is a smooth section of this tensor bundle, in the sense of 2.3.3, where we take $E = \mathfrak{T}_{(k,l)}M$.

For integrals we stated we needed a combination of covectors, and actually these $(k,0)$-tensors are also called k-forms. There is one very important tensor which has our particular interest:

Definition 2.4.3 (Metric tensor)**.** Let g be a $(2,0)$-tensor field, such that, for $p \in M$ and $v, w \in T_pM$ we have

$$g((p,v),(p,w)) = g((p,w),(p,v)), \qquad \text{meaning } g \text{ is } \textbf{(symmetric)}\ , \tag{2.23}$$

then g is a **metric tensor**. Whenever $g(v,w) = 0$ we say v and w are **orthogonal**. The pair (M,g) is called a **metric manifold** and is often just denoted M after g is given.

With this seemingly very specific example we will be able to define the geometry of a manifold and a lot of interesting properties.

2.5 Riemannian Geometry

Now you might be tempted to say, "we're there, we have a metric defined on our manifold!", and ask how to calculate distances with the metric. However, this is a metric *tensor* on each *tangent space* of the manifold, so we are not quite there yet. We will, however, use this section to explain how we can use the metric tensor to construct a distance function on a manifold.

We start by defining special maps between different metric manifolds.

[11] Note the roles of k and l are not strict and may be reversed from this usage in other literature.
[12] A multilinear function $f(v_1, \ldots, v_n)$ is linear in any of its entries v_i if you keep all other entries fixed.
[13] See also chapter 12 of [2].

Definition 2.5.1 (Isometry)**.** Let (M, g_M) and (N, g_N) be metric manifolds. An **isometry** from M to N is a diffeomorphism $\varphi\colon M \to N$ which preserves the metric tensor, meaning for $v, w \in T_pM$ we have

$$g_N(d\varphi(v), d\varphi(w)) = g_M(v, w). \tag{2.24}$$

This is also denoted $\varphi^*(g_N) = g_M$.

Definition 2.5.2 (Riemannian metric)**.** A **Riemannian metric tensor** is a symmetric $(2, 0)$-tensor field on a manifold which is positive definite at every point, meaning for each $p \in M$ and $v, w \in T_pM$ we have

$$g((p, v), (p, v)) > 0 \text{ for } v \neq 0 \text{ and } g((p, 0), (p, 0)) = 0. \qquad \textbf{(positive definite)} \tag{2.25}$$

A manifold together with a Riemannian metric tensor is called a **Riemannian manifold**.

Because this defines an inner product on each tangent space we could write[14] $g((p, v), (p, w)) = \langle v, w\rangle_g$ and $\langle v, v\rangle_g = |v|_g^2$.

Definition 2.5.3 (Riemmanian Geometry)**.** Just as our Euclidean intuition tells us we can use this inner product to define **angles** between vectors as the unique angle θ between 0 and π such that

$$\cos\theta = \langle v_1, v_2\rangle / (|v_1|_g\, |v_2|_g). \tag{2.26}$$

We have already seen what orthogonality means in definition 2.4.3. We can also define the **length of a vector** v as $|v|_g$.

For smooth local coordinates on our manifolds around p we can express our metric tensor with values[15]:

$$g_p = g_{ij}dx^i \otimes dx^j = g_{ij}(1/2dx^i \otimes dx^j + 1/2dx^j \otimes dx^i) = g_{ij}dx^i \odot dx^j, \tag{2.27}$$

where we can write the right-hand side because of the symmetry of the metric tensor. Note in this thesis we will use notations $dx^idx^j := dx^i \odot dx^j$ and $\left(dx^i\right)^2 := dx^i \odot dx^i$.

The Riemannian metric provides a natural isomorphism between the tangent bundle and the cotangent bundle.

Definition 2.5.4 (Musical isomorphisms)**.** Given a Riemannian metric tensor g we can assign a covector to tangent vector v_p

$$\begin{aligned} &\hat{g}\colon T_pM \to T_p^*M \\ &v_p \mapsto \omega_p := \hat{g}(v_p) : \\ &\hat{g}(v_p)(w) = g((p, v), (p, w)) \text{ for } w \in T_pM. \end{aligned} \tag{2.28}$$

Since g is positive definite, $\hat{g}$ is injective[16]. By equal dimensions of the tangent and cotangent bundle, $\hat{g}$ is an isomorphism. When chosen a coordinate system[17] we can express this isomorphism both ways. Let $g = g_{ij}dx^idx^j$, $X = X^i\partial_i$, then

$$\hat{g}(X) = g_{ij}X^idx^j \text{ usually written as} \tag{2.29}$$

$$\hat{g}(X) = X_jdx^j \text{ with } X_j = g_{ij}X^i \tag{2.30}$$

The usual notation inspires the term **lowering an index** from vector field X to covector field $\hat{g}(X)$, also denoted $X^\flat$. Because $\hat{g}$ is an isomorphism, it is invertible at each point. The matrix notation[18] for the inverse

[14] Note we will not use this notation if g is a semi-Riemannian metric.

[15] Note we use the Einstein summation convention again, as explained on page 9.

[16] Since if $\hat{g}(v_p)(w) = 0$ holds for any w it does especially for $w = v_p$, and that means $v_p = 0$. Note this result also holds for g being *non-degenerate*, meaning this holds for a semi-Riemannian metric, described in section 2.7, as well.

[17] Whenever we write expressions with coordinates we mostly use Einstein summation convention for shorter expressions.

[18] The matrix notation of $\hat{g}$ is g_{ij} and we do not worry about which order the indices have as g and thus its inverse are symmetric.

is g^{ij}, and it can be used to **raise an index** from a covector field ω to a vector field

$$\hat{g}^{-1}(\omega) = g^{ij}\omega_j\partial_i \tag{2.31}$$
$$= \omega^i\partial_i \text{ with } \omega^i = g^{ij}w_j \tag{2.32}$$
$$= \omega^\sharp. \tag{2.33}$$

Together these are called the **musical isomorphisms**[19].

The subject of *Riemannian Geometry* studies Riemannian manifolds:

Definition 2.5.5 (Riemannian Geometry)**.** The study of properties of Riemannian manifolds that are invariant under isometries is called **Riemannian geometry** [2, p. 332].

2.6 Riemannian distance function

For a long time our goal has been getting a distance function on our manifold. We have already come as far as defining some norm *infinitesimally* (namely our Riemannian metric on T_pM). We have also been able to glue all tangent spaces together in the tangent bundle TM. Now, we need some way to glue all the norms to a global distance, and for that we laid the groundwork in our tangent space representation as derivatives of curves (section 2.2) and line integral over curves 2.3.6.

Let us fix $p, q \in M$, between which we want to know the distance. We choose a path $\gamma\colon [a, b] \to M$ between p and q, that we allow to be only piecewise smooth[20]. So $\gamma(a) = p, \gamma(b) = q$. Let the set of all such piecewise smooth paths between p and q be denoted by $\mathcal{P}(p, q)$.

Definition 2.6.1 (Length of a curve)**.** We define the **length** of γ to be

$$L_g(\gamma) = \int_{[a,b]} |\gamma'(\lambda)|_g \, d\lambda. \tag{2.34}$$

Of course γ does not have to be the 'shortest' path from p to q, but we are now ready to define our **distance function** on M as the 'shortest length':

Definition 2.6.2 (Riemannian distance function)**.** For $p, q \in M$ we define the distance between p and q,

$$d_g(p, q) = \inf_{\gamma\in\mathcal{P}(p,q)} L_g(\gamma) \tag{2.35}$$

Now our metric or manifold does not have to be very similar to Euclidean space between p and q, therefore it might not be as straightforward to draw the shortest path. We will see how *geodesics* come into the picture shortly, in chapter 3.

We might wonder how to construct such a metric. We will assume Euclidean space with its metric $\bar{g}$ as the regular inner product on Euclidean space. We will give two examples, using pullbacks of metrics described in (2.24).

Example 2.6.3 (Embedded submanifold)**.** If we have a natural placing[21] of our manifold in a Euclidean space (we called it the *ambient space* before), we can pull back the metric from the ambient space[2, p. 333]. These metrics are called **standard** metrics[22] as they come directly from the Euclidean metric. Because of that they are not more exciting or different from what we could already do in Euclidean spaces. △

[19] This term is because of the ♭ and ♯ notation in music where a note is lowered or raised.
[20] Note here we are going to use the derivative of γ so it was necessary to use closed intervals in definition 2.3.6.
[21] The exact definition of an embedded submanifold can be found in Chapter 4 of [2].
[22] There is even a theorem, due to Nash, that states any metric is the standard metric coming from some embedding of the manifold in Euclidean space [11, 12].

Example 2.6.4 (Using coordinate charts)**.** As discussed in section 2.1, once given a manifold we have coordinate charts. For M compact[23], we can pull back the Euclidean metric via these charts and patch them together with a partition of unity. In fact, this proofs at least any compact manifold can be endowed with a Riemannian metric [2, p. 329]. △

2.7 Semi-Riemannian Geometry

In semi-Riemannian Geometry we use a semi-Riemannian (or pseudo-Riemannian) metric, which differs from the Riemannian metric of definition 2.5.2:

Definition 2.7.1 (Semi-Riemannian metric)**.** A **semi-Riemannian metric tensor** is a smooth, symmetric $(2,0)$-tensorfield on a manifold which is non-degenerate at every point $p \in M$, meaning for each $v \in T_pM$, $v \neq 0$, we can find some $w \in T_pM$, such that

$$g((p,v),(p,w) \neq 0. \qquad \textbf{(non-degenerate)} \tag{2.36}$$

In semi-Riemannian geometry can we define 'lengths' of paths, but this is not an intuitive length, since we do not necessarily have $g_p(v,v) > 0$ for nonzero vectors v, as the metric is not positive definite. That means there could be paths of zero or negative length, also meaning the triangle inequality does not hold. We can define[24] three different kinds of tangent vectors

$$\boxed{g(v,v) = c \text{ where } \begin{cases} c & <0 \quad v \text{ is } \textbf{timelike} \\ c & =0 \quad v \text{ is } \textbf{lightlike} \text{ (also called } \textbf{null}\text{)} \\ c & >0 \quad v \text{ is } \textbf{spacelike}. \end{cases}} \tag{2.37}$$

For semi-Riemannian metric tensors there are a lot of choices, but we can classify them using Sylvester's law of inertia, stating we can choose a basis of the tangent space in which the metric becomes a diagonal matrix with only +1 and -1 entries[2, p. 343]. The **signature** of a metric, (r,s), consists of two numbers that count the amount of positive and negative entries the metric has in that diagonal form. Notations vary as to whether we first notate the number of negative or positive ones, we will first denote the negative counter. We will focus on metrics that have a single negative entry, as those are the ones with which we can describe our universe with the General Relativity setting. An n dimensional manifolds having a metric with $(1,n-1)$ signature is called a **Lorentz manifold**.

An explicit and important example of a Lorentz manifold is **Minkowski spacetime** (see section 6.1), which has metric

$$g_{ij} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{2.38}$$

Physicist usually use a formula notation for this:

$$g = -(dx^1)^2 + \sum_{i=2}^{n}(dx^i)^2 = -(dt)^2 + \sum_{i=2}^{n}(dx^i)^2 \tag{2.39}$$

using $x^1 =: t$ as time coordinate.

We will see that this metric can be connected with an actual physical quantity, the *proper time*, as we will explain in section 5.2.

[23] Meaning it has a finite open cover.

[24] Note the terminology is heavily suggesting usage in General Relativity, which we will go into in chapter 5.

3 Geodesics

Now that we have the theoretical framework of manifolds, metrics and a distance function, we go on to *geodesics*. We have defined lengths of curves an a lot of linear algebra on our manifold so far.In this chapter we will generalise the Euclidean idea of *straight lines*, that is, a line with constant direction[25]. We will see we need more tools to generalize that idea in our manifold setting.

In Euclidean geometry, vectors live inside the space itself. That means a straight line is just a curve following a vector: $\gamma = \lambda\vec{v} + w$ for $\lambda \in \mathbb{R}$ and $\vec{v}, w \in \mathbb{R}^n$. On manifolds, we have seen, our vectors live in the tangent spaces and not on the manifold itself. These tangent spaces are pointwise defined, and although we have defined how vectors in the tangent bundle can vary smoothly (see definition 2.3.3), we have no notion of a 'constant' section other than the zero section. We will see how we can transport a vector in the tangent bundle, keeping it 'constant', after which we define geodesics. We give a differential equation, known as the geodesic equation, that one can solve to find geodesics, or one can use to proof a curve is a geodesic. We also give some tools used in differential geometry, for which geodesics play a crucial role.

For references during this chapter one can best turn to chapters 3-6 of [3], although it is written with Riemannian Geometry in mind, what translates to semi-Riemannian Geometry is stated. For a physics reference one can use chapters 3 of [4, 5, 7], those chapters are immediately applicable (or already applied) to semi-Riemannian Geometry, as they are written to introduce a reader into General Relativity.

3.1 The problem of *straightness*

As people with some mathematical background, we should have a Pavlovian reaction to reading the word *constant*, and the notion of *zero derivative* should immediately come to mind. This is exactly what we want to do, but again we need some more work for the general picture.

We could just pick a curve and see if its second derivative is constant, as then the speed is constant in every direction. Consider the following curves in $\mathbb{R}^2$:

$$\gamma_1(\lambda) = (\sin(\lambda), \cos(\lambda)), \gamma_2(\lambda) = (\lambda, 3/5 + \lambda/2) \tag{3.1}$$

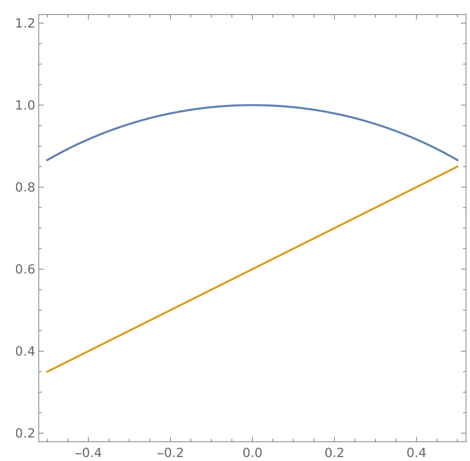


Figure 4: The curves from (3.1), γ_1 in blue and γ_2 in orange.

[25] Another way to characterize a straight line in Euclidean space is that it is length-minimizing. However, this is not a strict local notion, so it will be harder to generalize. But as it turns out, if we generalize the constant direction curves, that curve will be a critical point of the length.

We determine the second derivative of the coordinates:

$$
\begin{aligned}
\gamma_1 &= (\sin(\lambda), \cos(\lambda)) & \gamma_2 &= (\lambda, 3/5 + \lambda/2) \\
\gamma_1' &= (\cos(\lambda), -\sin(\lambda)) & \gamma_2' &= (1, 1/2) \\
\gamma_1'' &= (\sin(\lambda), -\cos(\lambda)) & \gamma_2'' &= (0, 0)
\end{aligned} \tag{3.2}
$$

According to (3.2) we can state γ_1 is not a straight line and γ_2 is. However, we could also change our coordinates and consider these lines in polar coordinates[26] (r, θ). We will not give an expression of γ_2 in these coordinates[27].

$$\gamma_1(\lambda) = (1, \lambda) \tag{3.3}$$

We determine the second derivative of the coordinates:

$$
\begin{aligned}
\gamma_1 &= (1, \lambda) \\
\gamma_1' &= (0, 1) \\
\gamma_1'' &= (0, 0)
\end{aligned} \tag{3.4}
$$

and see that, according to (3.4), γ_1 is a straight line. It is left to the reader to ascertain γ_2 is not a straight line.

The goal of these calculations is to motivate the new concept next section. We need a way to determine 'straightness' of a curve, that does not depend on choice of coordinates. After all, $\mathbb{R}^2$ stays the same, whatever coordinate system we choose to describe locations on it. We would like 'straightness' to be an intrinsic property of a curve, instead of emerging from our choice of coordinates to describe it. For example, a physical object that accelerates, will experience forces. Which coordinates we choose to describe the object, does not change the forces experienced by that object. We will see this is a fundamental notion in Physics and General Relativity, where indeed the geodesics (which we will define later) play an important role, see 5.3.

[26]Since we stay away from the origin we only need one chart centred at the origin.
[27]It involves arctans. One only needs to convince oneself that in these coordinates, γ_2'' will not vanish.

3.2 Connections and Christoffel symbols

Let us state what we want, in terms of the coordinate independent definitions we have already established. We want a constant speed vector, meaning if we have some $(p, v) \in TM$ we want some $(q, w) \in TM$ for a q close to p and somehow w must be 'constant' with respect to v. Somehow we need to relate two tangent vectors at different points on our manifold to judge whether it stayed 'constant'. In particular, we want this for tangent vectors of a curve, to check whether it is 'straight'. Clearly, the idea of a tangent bundle and sections is not yet enough and we need some more tools for manifolds to come up with this.

Definition 3.2.1 (Linear connection)**.** For M a manifold and $\mathfrak{X}(M)$ the set of vector fields on M, a **linear connection on** M, or connection for short, is a map

$$\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \tag{3.5}$$

$$(X, Y) \mapsto \nabla_X Y, \tag{3.6}$$

satisfying the following:

1. linearity over $\mathbb{C}^\infty(M)$ in X:
$$\nabla_{\left(f_1 X + f_2 \tilde{X}\right)} Y = f_1 \nabla_X Y + f_2 \nabla_{\tilde{X}} Y; \tag{3.7}$$
2. linearity over $\mathbb{R}$ in Y:
$$\nabla_X \left(\lambda Y + \mu \tilde{Y}\right) = \lambda \nabla_X Y + \mu \nabla_X \tilde{Y}; \tag{3.8}$$
3. Leibniz product rule[28] in Y, for $f \in C^\infty$:
$$\nabla_X f Y = f \nabla_X Y + (Xf)\, Y. \tag{3.9}$$

We want to pick a special linear connection, as the definition makes clear there could be several. On top of that, we need some way of calculating the image of two vector fields under the linear connection. We will later see that a connection is the tool we need to translate a vector over a given curve, that translation is called *transporting linearly.* First, as we said we want to compute the connection on two given vector fields. For that we need a coordinate system $\{x^i\}_i$ and as usual that probably means we will use the Einstein summation convention.

Definition 3.2.2 (Christoffel symbols)**.** Given a manifold M, together with a connection ∇, we can compute the image of the connection. Given the local coordinate frame[29] $\{\partial_i\}_i$ for $U \subset M$, we can express

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \tag{3.10}$$

For M having dimension n, we see a linear connection defines n^3 functions on a neighbourhood $U \subset M$, $\Gamma^k_{ij} \colon U \to \mathbb{R}$, but usually we define[30] the connection by explicitly giving the so called **Christoffel symbols** Γ^k_{ij} for the coordinate frame.

Expressing connections in their Christoffel symbols makes clear there is, in general, a lot of choice in choosing a connection. Using a smooth partition of unity, subordinate to domains of coordinate charts, to patch these locally defined connections together, we can actually define a global connection. See also lemma 4.4 and proposition 4.5 in [3, p. 52], of which the latter ensures every manifold admits at least one linear connection.

Now that we have this tool, we want to find some intuitive criteria to narrow it down to preferably one 'obvious' connection. It would be very nice to have that one 'obvisous' connection to be 'nicer' than others on a given manifold.

[28] Note because we only ask $\mathbb{R}$-linearity and the product rule in Y, and not $\mathbb{C}^\infty(M)$-linearity in Y, it follows ∇ is NOT a tensor field.

[29] This can be done for any local frame, but we will make use of Christoffel symbols that are defined for the coordinate frames.

[30] They uniquely determine the connection as locally on U we can express $X = X^i \partial_i$ and $Y = Y^j \partial_j$, from the linearity and product rule that define the connection then follows $\nabla_X Y = \left(XY^k + X^i Y^j \Gamma^k_{ij}\right) \partial_k$.

It turns out if we have a metric manifold, a connection can be *compatible* with the metric. Before we can give the definition we need to know a linear connection can be extended to work on functions as well[31]. If f is a smooth function, and X a vector field, on M, we define the extended connection to act as

$$\nabla_X f = Xf. \tag{3.11}$$

Definition 3.2.3 (Compatibility with the metric)**.** Given a manifold M and metric g on it, a connection ∇ is called **compatible** with g if for any vector fields X, Y, Z we have[32]

$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{3.12}$$

Unfortunately for us, this extra restriction on the connection does not narrow it down to a unique connection yet, so we will also demand another special property:

Definition 3.2.4 (Symmetry)**.** A linear connection ∇ is **symmetric** or **torsion-free**[33] if

$$\nabla_X Y - \nabla_Y X \equiv [X,Y] := XY - YX \tag{3.13}$$

Theorem 3.2.5 (Fundamental Lemma)**.** *Given a manifold M and metric g, there is a unique connection ∇ compatible with g and symmetric. For g Riemannian this is called the* ***Riemannian connection*** *or* ***Levi-Civita connection*** *of g, the latter name is also given to ∇ if g is semi-Riemannian [5, p. 100], [3, p. 68].*

Proof. As we said, we need symmetry (3.13) and compatibility with g (3.12). In this proof, we will also give a formula to calculate the Christoffel symbols explicitly for a given metric. This will be very useful in calculations, for example those we will do in chapter 6.

Let X, Y, Z be vector fields on M. First compatibility, later symmetry, and finally linearity of g yields

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla_Z X + [X,Z]) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_Z X) + g(Y, [X,Z]) \qquad (3.14) \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) = g(\nabla_Y Z, X) + g(Z, \nabla_X Y + [Y,X]) \\ &= g(\nabla_Y Z, X) + g(Z, \nabla_X Y) + g(Z, [Y,X]) \qquad (3.15) \\ Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = g(\nabla_Z X, Y) + g(X, \nabla_Y Z + [Z,Y]) \\ &= g(\nabla_Z X, Y) + g(X, \nabla_Y Z) + g(X, [Z,Y]) \qquad (3.16) \end{aligned}$$

Making a linear combination (adding the first two expressions and subtracting the last) yields

$$\begin{aligned} &Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) = \\ &\qquad 2g(\nabla_X Y, Z) + g(Y,[X,Z]) + g(Z,[Y,X]) - g(X,[Z,Y]) \end{aligned} \tag{3.17}$$

We extract an expression for $g(\nabla_X Y, Z)$ and get

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])\right). \tag{3.18}$$

Since we assumed symmetry and compatibility with g, this determines a unique connection.

Equation (3.18) also gives us an existence if we express it in local coordinates, which we will use in chapter 6. An important feature of coordinate vector fields ∂_i is that $[\partial_i, \partial_j] = 0$. For a function $f\colon M \to \mathbb{R}$ we have

$$[\partial_i, \partial_j]f = \partial_i\partial_j f - \partial_j\partial_i f = \partial_i\partial_j f - \partial_j\partial_i f = \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} = 0. \tag{3.19}$$

[31] See [3, p. 49] or [7, p. 30].
[32] Note we apply ∇ to a smooth function as defined (3.11). Also note equation (3.12) is a pointwise definition, since g is defined in terms of tangent vectors.
[33] The torsion would be $\nabla_X Y - \nabla_Y X - [X,Y]$, what we set to zero by demanding (3.13).

On top of that fact, we also use (2.27) stating $g(\partial_i, \partial_j) = g_{ij}$, and (3.10) stating $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$. Using (3.18) for $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$, applying linearity of g, multiplying by the inverse matrix element and applying symmetry of g yields

$$g(\nabla_{\partial_i}\partial_j, \partial_l) = \frac{1}{2}\left(\partial_i g(\partial_j, \partial_l) + \partial_j g(\partial_l, \partial_i) - \partial_l g(\partial_i, \partial_j)\right) \tag{3.20}$$

$$g(\Gamma^m_{ij}\partial_m, \partial_l) = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right) \tag{3.21}$$

$$\Gamma^m_{ij} g_{ml} = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right) \tag{3.22}$$

$$\Gamma^m_{ij} g_{ml} g^{lk} = g^{lk}\frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right) \tag{3.23}$$

$$\Gamma^m_{ij}\delta^k_m = g^{lk}\frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right). \tag{3.24}$$

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{lk}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right).} \tag{3.25}$$

Note (3.25) ensures $\Gamma^k_{ij} = \Gamma^k_{ji}$. See [3, p. 70] for the proof that these Christoffel symbols indeed determine a symmetric connection that is compatible with the metric g. □

Example 3.2.6 (Euclidean space)**.** As an example, we go back to Euclidean space, where the standard connection is denoted $\bar{\nabla}$. We calculate the Christoffel symbols, where we use the Euclidean metric, that is defined by $g = \mathrm{id}$, meaning $|\partial_i|_g = 1$. Since $g_{ij} = g^{ij} = 0, 1$, we can see equation (3.25) yields zero, since the entries of g do not depend on any coordinate. So $\bar{\nabla}$ is defined by $\Gamma^k_{ij} \equiv 0$ for all indices. We can now also calculate the image of the connection map for $X = X^i\partial_i$, $Y = Y^j\partial_j$

$$\begin{aligned}
\bar{\nabla}_X Y &= \bar{\nabla}_{X^i\partial_i}\left(Y^j\partial_j\right)\\
&= X^i\bar{\nabla}_{\partial_i} Y^j\partial_j\\
&= X^i Y^j\bar{\nabla}_{\partial_i}\partial_j + X^i\left(\partial_i Y^j\right)\partial_j\\
&= X^i Y^j\Gamma^k_{ij}\partial_k + \left(X^i\partial_i Y^j\right)\partial_j\\
\bar{\nabla}_X Y &= 0 + \left(XY^j\right)\partial_j = \left(XY^j\right)\partial_j = XY,
\end{aligned} \tag{3.26}$$

where we used (3.7) and (3.9). Indeed, the Euclidean connection can be defined by (3.26)[3, p. 52]. △

3.3 Parallel transport and the Geodesic equation

Now that we have a connection we will go on to how we actually go about comparing the velocity of a curve at different points, so that we can define the acceleration, see Figure 5.

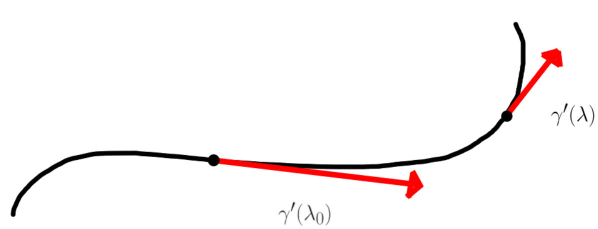


Figure 5: A curve γ with its velocity at two different points. We want to express its acceleration, so we need to compare the two velocities. Figure inspired by Figure 4.3 in [3, p. 49].

We will talk about a vector field on a curve $\gamma(I) \subset M$, so for any λ we have a tangent vector in $T_{\gamma(\lambda)}M$, that we denote $V(\lambda)$.

Theorem 3.3.1 (Covariant derivative). *For ∇ a linear connection on M and $\gamma\colon I \to \mathbb{R}$ a curve, there is a unique operator*

$$D_\lambda\colon \mathfrak{X}(\gamma(I)) \to \mathfrak{X}(\gamma(I)), \tag{3.27}$$

such that for V, W vector fields on $\gamma(I)$, the following statements hold:

- *linearity:*
$$D_\lambda(\mu_1 V + \mu_2 W) = \mu_1 D_\lambda(V) + \mu_2 D_\lambda(W) \qquad \text{for } \mu_1, \mu_2 \in \mathbb{R}, \tag{3.28}$$
- *Leibniz product rule:*
$$D_\lambda(fV) = f'V + fD_\lambda V \qquad \text{for } f \in C^\infty(I), \tag{3.29}$$
- *extension: if V is smoothly extendible to $\tilde{V}$ on a neighbourhood of $\gamma(I)$ in M,*
$$D_\lambda V(\lambda) = \nabla_{\gamma'(\lambda)}\tilde{V}. \tag{3.30}$$

Then $D_\lambda V$ is called the ***covariant derivative*** *of V, a vector field along γ.*

Proof. We follow the proof given in lemma's 4.1 and 4.9 in [3]. Let M be a manifold with linear connection ∇, on which $\gamma(I)$ is a curve. Let V be a vector field on $\gamma(I)$. We prove uniqueness and existence of the covariant derivative of V.

Firstly, since I is a closed interval, we can extend γ to an open interval[3, p. 57] $\tilde{I}$, such that for some $\varepsilon > 0$ we have $(\lambda_0 - \varepsilon, \lambda_o + \varepsilon) \subset \tilde{I}$ for any $\lambda_0 \in I$. The value of $D_\lambda V$ at λ_0 is determined by the values of V on the small interval $(\lambda_0 - \varepsilon, \lambda_o + \varepsilon)$. We show this by taking V, $\tilde{V}$ to have the same value on the described neighbourhood of λ_0, let $W = V - \tilde{V}$. Now let φ be a smooth bump function on M, such that $\varphi = 0$ everywhere but a small neighbourhood U of $\gamma(\lambda_0)$, such that $\varphi W \equiv 0$, and $\varphi(\gamma(\lambda_0)) = 1$. We have

$$D_\lambda(\varphi W) = D_\lambda\left(0\varphi W\right) = 0D_\lambda(\varphi W) = 0, \tag{3.31}$$

by linearity. By the product rule we have

$$D_\lambda(\varphi W) = \varphi' W + \varphi D_\lambda W, \tag{3.32}$$

For which the left-hand side is zero as in (3.31). On the right-hand side, the first term is zero as W vanishes on the support of φ and thus the support of φ'. Evaluation at $\gamma(\lambda_0)$, where $\varphi(\gamma(\lambda_0)) = 1$ yields $D_\lambda W = 0$, or

$$D_\lambda V(\lambda_0) = D_\lambda \tilde{V}(\lambda_0). \tag{3.33}$$

This means the covariant derivative only depends of the values of V at a small interval around the evaluated point.

Next, we choose coordinates near $\gamma(\lambda_0)$ and write $V(\lambda) = V^i(\lambda)\partial_i$, for a neighbourhood of $\gamma(\lambda)$. Since ∂_i is vector field on the neighbourhood, and not just defined on γ, we can write (3.30), (3.9)

$$\begin{aligned} D_\lambda V &= \nabla_{\gamma'(\lambda)} V^i(\lambda)\partial_i && (3.34)\\ &= V^i(\lambda)\nabla_{\gamma'(\lambda)}\partial_i + \left(\gamma'(\lambda)V^i(\lambda)\right)\partial_i. && (3.35)\end{aligned}$$

Since $\gamma'(\lambda)V^i(\lambda) = (V')^i(\lambda)$ and $\nabla_{\gamma'(\lambda)}\partial_i = (\gamma'(\lambda))^j(\lambda)\Gamma^k_{ij}\partial_k$, it follows

$$D_\lambda V(\lambda_0) = \Big((V')^k(\lambda_0) + V^i(\lambda_0)\,(\gamma')^j(\lambda_0)\Gamma^k_{ij}(\gamma(\lambda_0))\Big)\,\partial_k. \tag{3.36}$$

Now, we want to proof existence, of covariant derivatives. We can actually define $D_t V$ by (3.36) for domains of coordinate charts, uniqueness making sure they agree when those maps overlap. One can show defining a covariant derivative like that makes sure it has the required properties (3.28), (3.29), (3.30). □

In particular, this covariant derivative allows us to compare a curve's velocity at diferent points:

Definition 3.3.2 (Acceleration of a curve)**.** Let M be a manifold with ∇ linear connection, and γ a curve in M. We define the **acceleration** of γ as $D_\lambda \gamma'$.

We can actually give a formula, using (3.36). Let $x(\lambda) = \big(x^1(\lambda), \dots, x^n(\lambda)\big)$ describe a curve in local coordinates, then

$$D_\lambda x'(\lambda) = \Big(\left(x''\right)^k(\lambda) + \left(x'\right)^i(\lambda)\left(x'\right)^j(\lambda)\Gamma^k_{ij}(x(\lambda)) \Big)\partial_k. \tag{3.37}$$

Now we can finally define the main object of interest in this thesis.

Definition 3.3.3 (Geodesics)**.** On a manifold M with linear connection ∇, the curve γ is called a **geodesic** if it has zero acceleration everywhere[34]. In terms of (local) coordinates (x^i), a curve $x(\lambda) = \big(x^1(\lambda), \dots, x^n(\lambda)\big)$ is a geodesic if (3.37) vanishes, thus yielding the **geodesic equation** for each realvalued function that represents the coordinates $x^i(\lambda)$:

$$\boxed{(x'')^k(\lambda) + (x')^i(\lambda)(x')^j(\lambda)\Gamma^k_{ij}(x(\lambda)) = 0.} \tag{3.38}$$

Since we can make this expression into a $2n$-dimensional ordinary differential equation[35], it follows locally there exist geodesics. We call a geodesic a **maximal geodesic** if it cannot be extended to a bigger interval for λ and usually when the word *geodesic* is mentioned the writer means a maximal geodesic. We call a manifold **geodesically complete**[36] if all maximal geodesics are defined on the whole real line.

Example 3.3.4 (Euclidean space)**.** Earlier in section 3.1, we saw two lines of which we could not determine the *straightness*. We turn back to them to see which one is a geodesic. In $\mathbb{R}^2$, with local coordinates (x, y), we have seen the Euclidean connection is determined by $\bar{\nabla}_X Y = XY$ and $\Gamma^k_{ij} = 0$. Note the Christoffel symbols depend on the coordinate system used. Firstly, we calculated the coordinate representation and their derivatives for $\gamma_1 = (\sin(\lambda), \cos(\lambda))$,

$$\begin{aligned} x(\lambda) &= \sin(\lambda) & y(\lambda) &= \cos(\lambda) \\ x'(\lambda) &= \cos(\lambda) & y'(\lambda) &= -\sin(\lambda) \\ x''(\lambda) &= -\sin(\lambda) & y''(\lambda) &= -\cos(\lambda) \end{aligned}$$

We determine the acceleration, $D_\lambda \gamma_1'$

$$\begin{aligned} D_\lambda \gamma_1'(\lambda) &= \Big(\left(x''\right)^k(\lambda) + \left(x'\right)^i(\lambda)\left(x'\right)^j(\lambda)\Gamma^k_{ij}(x(\lambda)) \Big)\partial_k \\ &= \left(\left(x''\right)(\lambda) + 0\right)\partial_x + \left(\left(y''\right)(\lambda) + 0\right)\partial_y \qquad (3.39) \\ &= -\sin(\lambda)\partial_x - \cos(\lambda)\partial_y, \qquad (3.40) \end{aligned}$$

which we see does not vanish for all λ, hence γ_1 is no geodesic with respect to the Euclidean connection. For γ_2 we have seen second derivatives vanish (3.4), hence $D_\lambda \gamma_2' \equiv 0$. That means we can now determine which lines are geodesics. Of course, we could look at these curves in polar coordinates, but it is important to note that the polar coordinate frame $\partial_r, \partial_\theta$ will yield different Γ^k_{ij} for the Euclidean connection $\bar{\nabla}$, thus we cannot simply put in the polar coordinates, but need to calculate the Christoffel symbols in this coordinate frame. However, it does not matter in which coordinate system we compute the geodesic equation, as it depends only on the connection, which is independent of the coordinates. It suffices to check the geodesic equation in only one coordinate system, and the computations are most straightforward for Euclidean coordinates when using the Euclidean connection. △

Geodesics are very important in the physical interpretations, which we will turn to shortly in chapter 5. Acceleration is a fundamental part of mechanics and geodesics can be used to describe trajectories of particles. We work on giving explicit equations for the geodesics in chapter 6. One obvious geodesic is the constant curve, $x(\lambda) = x_0$ for all λ, where $x_0 \in M$.

---

[34] Since accelearation depends on the connection, so do the geodesics and they are sometimes explicitly called *geodesics with respect to the connection*.

[35] This is done by defining $y := x'$ and writing (3.38) (ODE) as a first order differential equation for (x, y).

[36] In particular, all compact manifolds are geodesically complete [3, p. 111].

3.4 Applications of geodesics

We have seen that geodesics are a solution to an ODE, and we can define a geodesic for each initial point $p \in M$ and initial velocity $v \in T_pM$. It is very important to be able to describe how the geodesic would change if we were to change the initial point or velocity, since it would make us better understand the geometry of the manifolds, but this is also very physically motivated. If geodesics are particle trajectories, we could be interested in how small differences in initial conditions (point and direction) affect the trajectory. This could be applied, for example, if we study a cloud of particles and how that cloud moves on the manifold, as each particle in the cloud has slightly different initial points and velocities. As we have seen, geodesics are only locally defined. Thus, there is in general no way of defining a 'minimal' interval I, such that all maximal geodesics are defined on I[37].

Definition 3.4.1 (Exponential map)**.** Let $\gamma_{(p,v)}$ be the geodesic satisfying $\gamma_{(p,v)}(0) = p$, and $\gamma'_{(p,v)}(0) = v$. We define

$$\mathcal{E} := \{(p, v) \in TM | \gamma_{(p,v)} \text{ is defined on an interval containing } [0, 1]\}.$$

The **exponential map** $\exp\colon \mathcal{E} \to M$ is defined

$$\exp((p, v)) = \gamma_{(p,v)}(1). \tag{3.41}$$

We can restrict the map to a single point q on M by $\exp|_q\colon \mathcal{E}_q \to M$, where $\mathcal{E}_q := \mathcal{E} \cap T_qM$.

$$\exp|_q(v) = \gamma_{(q,v)}(1). \tag{3.42}$$

Using that T_pM is a vector space we can trace out the maximal geodesic by

$$\exp_p(\lambda v) = \gamma_{(p,v)}(\lambda), \tag{3.43}$$

for any $\lambda \in \mathbb{R}$ such that both sides are defined [3, p. 73]. But we can do even more interesting things with this map, since it is a smooth map from an open in T_pM to M[3, pp. 73–75]. If we restrict ourselves to neighbourhoods small enough, the exponential map gives a diffeomorphism [3, p. 76]

$$T_pM \supset \mathcal{V} \to \mathcal{N} \subset M, \tag{3.44}$$

where $0 \in \mathcal{V}$ and $p \in \mathcal{N}$.

Definition 3.4.2 (Normal neighbourhood)**.** For a manifold M with linear connection ∇, where $p \in M$, we have seen p has a neighbourhood $\mathcal{N}$ diffeomorphic to a subset of T_pM, containing the origin, (3.44). This neighbourhood $\mathcal{N}$ is called a **normal neighbourhood**. In a normal neighbourhood we can use coordinates with respect to the image basis: $\exp_p(\partial_i)$, meaning geodesics through p[38] inside $\mathcal{N}$ have the form $\gamma_{(p,v)}(\lambda) = \lambda(\exp_p)_*(v)$. That means geodesics through p can, in $\mathcal{N}$, be described as radial lines, as can be seen in Figure 6. Note geodesics in $\mathcal{N}$ that do not pass through p can look completely different, though. These coordinates are called **normal coordinates centred at** p. The image of a ball[39] around the origin in T_pM, $\exp_p(B_\varepsilon(0))$ is called a **geodesic ball** in M.

In normal coordinates a Riemannian manifold behaves as much as Euclidean space as they can: Geodesics through the origin are straight lines; The metric at the origin reduces to the flat metric[40] $g_{ij} = \delta_{ij}$, meaning the Christoffel symbols are zero at the origin as well.

Example 3.4.3 (Euclidean space)**.** For $M = \mathbb{R}^n$, with coordinates x^i, and $p = (p^1, \dots, p^n) \in M$, we can take the whole of $\mathbb{R}^n$ as normal neighbourhood, where we have normal coordinates $x^i - p^i$. △

[37] Except for the **geodesically complete** manifolds, where $I = \mathbb{R}$.

[38] Note geodesics that do not pass through p may have an entirely different form than described for those through p.

[39] We use the metric g on M to define the length of vectors.

[40] For semi-Riemannian the flat metric is the Minkowski metric, which we will see later. There $g_{11} = -1$ and for other i, j we have $g_{ij} = \delta_{ij}$.

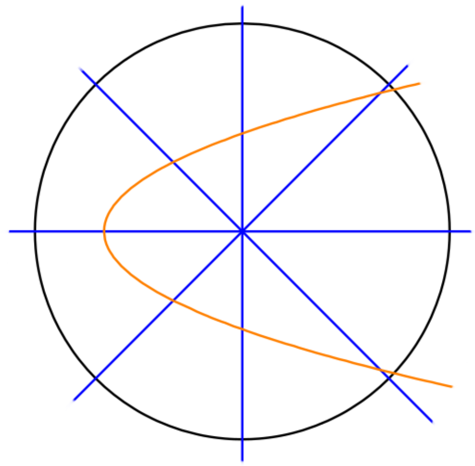


Figure 6: Geodesics through p (blue lines through center) in a normal neighbourhood $\mathcal{N}$, represented by the black circle. Note geodesics inside the circle, but not through p are not, generally, straight lines as plotted for normal coordinates, such as the orange line not passing through p.

In Riemannian manifolds, geodesics have a very interesting property, namely they are precisely the paths that locally minimize length. [3, Theorem 6.6]. We can only state this locally, since globally this is not necessarily true.

Example 3.4.4 (Sphere in Euclidean space)**.** For example, on the sphere $\mathbb{S}^2$, the geodesics are the *great circles*, intersections of the sphere with planes through the origin, as seen in $\mathbb{R}^3$. For two points on the sphere that are not antipodal[41], there are two geodesic curves that connect them, but since they are not antipodal, one of them is shorter than the other. △

One could even define geodesics by being locally length-minimizing and having unit speed [13, definition B.2]. But one can expand on this statement, because a weaker statement is also true for semi-Riemannian geodesics, as any geodesic is a critical point of the length[42]. This means if we change our curve a little bit, the length changes only a little bit.

[41] On $\mathbb{S}^2$ centred on the origin, p and q being antipodal means $p = -q$.

[42] We will later see what 'length' means for semi-Riemannian curves. In any case, the length involves some function of $g(\gamma', \gamma')$, integrated over the curve γ.

4 Towards General Relativity

Now that we have developed a lot of mathematical tools to deal with manifolds, it is time to turn to physics.

We will make use, of course, of the tools we developed in the last chapters, but mostly take them as a given to see how we can use them to describe the world around us. And, perhaps even more importantly, make predictions by calculations. But before we can do that, we need to know how these tools help us to describe our reality, which we will do in this chapter.

General Relativity describes our universe, using a few principle assumptions. We will go through them, inspired by the explanations in [4, pp. 332–337].

But before we do that, we need the most important framework, whose name has already been mentioned a few times. A **spacetime** is a connected $n+1$-dimensional Lorentz manifold[43], that is inextendible, i.e. there is no isometric embedding into a bigger spacetime [6, section 3.1].

The fundamental assumption in General Relativity is that we can describe our reality with a (special) spacetime. As any theory, General Relativity needs to be verified by experiments. Its description of our universe, as spacetime manifold, seems to hold up very well when considering larger distances, but on scales smaller than 10^{-17}m it has not been very well confirmed. This description also holds up very well for densities less than 10^{61}kg m^{-1}, but for higher densities, it has not been verified [6]. In fact, the scientific community seems to agree there is a lot of work still to be done on the quantum scale for the model of spacetime used in General Relativity. However, for normal cosmology scales the description works very well and General Relativity's spacetime is a valuable and much used tool since its acceptance, around 1919 [14].

We will go through exactly how the spacetime manifold can describe physics, and in the next chapter, we explain how we can translate very important physical notions into the mathematical language that describes the spacetime manifolds.

As references during this chapter one can use [4], of which we follow the beginning of chapter 12. One can also turn to chapter 4 of [5], or chapter 4 of [7].

4.1 Special Relativity

Of course, as the names suggest, General Relativity yields Special Relativity in some cases. This is the case for the flat Minkowski spacetime. The link between the two Relativities is that Special Relativity does not account for gravitational effects. This means we cannot describe systems in which gravity varies, such as our universe on bigger scale where we have different stars and planets. What it does mean, is that General Relativity describes spacetimes, meaning every tangent space can be seen as Minkowski spacetime and via the exponential map (described in section 3.4) we can compare a small neighbourhood of a spacetime to Minkowski spacetime.

The usage of the term **small neighbourhood** is a delicate matter, in mathematics it is just a 'small enough' region for the desired result to hold. However now that we are talking about 'real' spacetime we need to be more careful, although the direction 'small enough' for results to hold will of course be the goal of the definition. A small neighbourhood means a small enough region such that an approximation of a homogeneous gravitational field is valid.

[43] Actually we consider two manifolds that are isometric as the same spacetime, so it is actually an equivalence class of metric manifolds.

4.2 Gravity

The main force of interest in General Relativity is gravity. That is not to say we cannot use the framework to describe other forces, but there might be better models for those forces. The idea behind this is that gravity has a very long range. To specify this, electrical charges come in two types, negative and positive. Electrical charges do not only attract but also repel, and therefore on the larger scale we do rarely see their effects. The same is true for colour charges for the strong force, they come in different types and the effects even out on the larger scale. Gravity however only attracts. There are no 'negative mass' objects to cancel the effects on larger scales. In cosmology, the structure of our universe can be described using gravity. But as we shall see next, in General Relativity gravity is given a totally different meaning to what we are used to, namely more of an emergent phenomenon instead of a fundamental force.

4.3 Free fall and the Equivalence principle

The basic idea behind General Relativity is the idea that an observer cannot really distinguish acceleration from gravity, or lack thereof. See also Figure 7, where we visualised the two types of distinctions described in this section. Even in high school, we learn that we experience forces. For acceleration we experience $F = ma$, for gravity this is $F = mg$, where F is the force, m the mass of the observer, a the acceleration and g the local gravity constant[44]. Einstein argued that, for example in a sealed box, an observer could not feel the difference between gravity or acceleration. Or even feel the difference between free falling down to earth, or being in outer space where no gravitational effects are felt. He concluded that these situations actually were no different from each other inside the box of the observer, as who is to say their reference system is better?[45] In fact, both of the observers that experience no acceleration with respect to the boxes, have geodesic trajectories (see section 5.3) in spacetime and are called an **inertial observer**. A consequence of this is that gravity can be seen as the curvature of spacetime, changing the geodesics so they are no longer what we perceive as straight lines, but could, for example, be an orbit around a star.

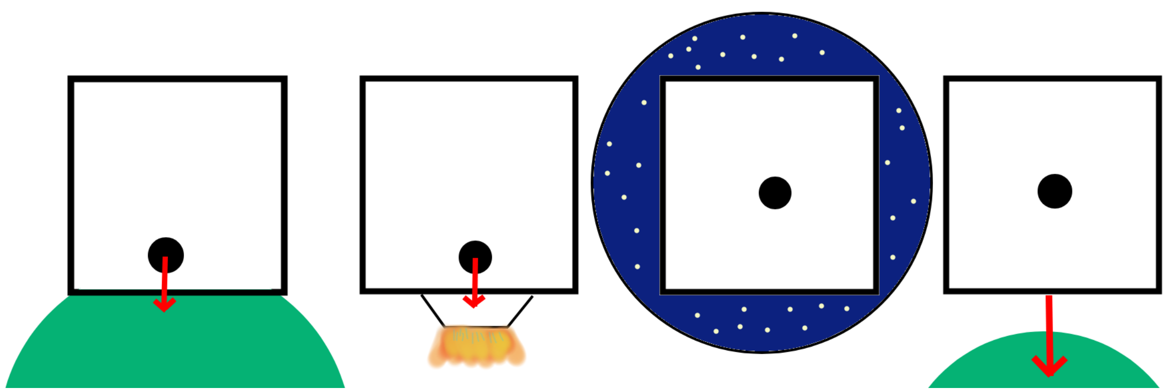


Figure 7: The thought experiment of the equivalence principle. On the left, the box is placed on Earth, so a particle inside a closed box is pulled to the ground $F = mg$. However, in the box next to it, accelerated by a rocket, a particle is also pulled to the ground $F = ma$. One could not determine which situation one is in if the box is sealed and kept in the same condition.
On the right, we have two boxes in which the particle is floating in the box, i.e. not accelerating or seemingly affected by any force. This can be the case either because the box is floating in outer space and is not experiencing gravity, but can also be a box that is freely falling towards Earth. Again, one could not determine which of these situations one is in. Both boxes are inertial frames. In section 5.3 we will argue they are both in a geodesic trajectory through spacetime. Figure inspired by [15].

[44] Of course we can write this out in terms of the mass of the earth/object and distance between the observer and (centre of mass of) the object.

[45] Note this is a fundamental part of special relativity: any inertial observer is as good as the next inertial observer.

4.4 Curvature and Gravity

So far, we have seen components of General Relativity that do not seem to be linked. It would be really great if we have some way to predict the curvature of the spacetime that describes our universe. And in fact, that is exactly what Einstein made possible, as the Einstein equation relates the energy at a point in spacetime to its curvature. When we say energy, that includes energy of matter[46].

Before we can actually give the Einstein equation, we have to note that, at this point, we cannot yet understand it fully. However, we want to give the foundations of General Relativity, and the Einstein equation connects the theory with our reality. A reader new to this subject will have to turn to other literature to read up on this subject, but this section gives one an idea about the bigger picture. However, we will not go into this deeper and one can understand the rest of this thesis without knowing the Einstein equation or its constituents.

To verify any theory, it needs to make predictions one can test. Therefore, we need a way to calculate what the theory predicts. For General Relativity, that is where we need the Einstein equation. We need two more tools to understand the equation, one from differential geometry and one from physics. For differential geometry, we have only seen how a metric can cause something we would want to call *curvature*. In fact, we can actually quantify curvature by the **Riemann tensor**, its contraction, the **Ricci tensor**, of which the **Ricci scalar** is the trace. In [3] or in [5] you can read up on what those are and how to calculate them. We also need a physical tool we have not mentioned so far, namely the **energy-momentum tensor**, more can be found on this tensor in [5]. Now, we can give the Einstein equation

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij}, \tag{4.1}$$

where R_{ij} is the Ricci tensor, R the Ricci scalar and g_{ij} the metric describing our spacetime, together they describe our spacetime on the left-hand side of the equation. On the right-hand side, G is the gravitational constant and T_{ij} the energy-momentum tensor, so the right-hand side of (4.1) describes the content of the spacetime. Together this formula describes how the content of spacetime influences the curvature. It links the mathematical description of curvature and metrics to the physical description of the universe with its content. In reality one has to make approximations to solve this equation, as curvature affects the movement of content of spacetime, and this flow of energy influences the curvature, again affecting the content...

For the special case where we consider actual empty space (defined as $T_{ij} = 0$), we can rewrite to $R_{ij} = 0$, as the Ricci scalar is the trace of the Ricci tensor. A manifold with no curvature is called flat, for example, Minkowski spacetime is flat.

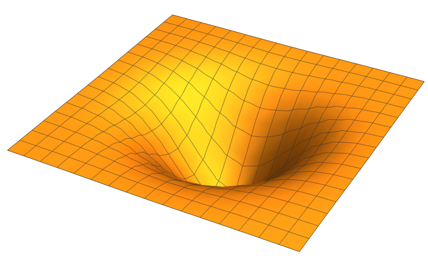


Figure 8: A much used visualisation of spacetime being curved by mass. Imagine the surface representing spacetime. If a mass is placed in the middle, its energy-momentum curves the spacetime, such that the squares (that can be seen as visualisation of the metric) are deformed. That deformation in turn, changes trajectories of other particles, bending their paths. One could call that gravity, meaning gravity is actually an emergent phenomenon caused by spacetime curvature. This picture is similar to the one on the front page, where we have drawn trajectories of some (test)particles on such a surface.

[46] As the famous $E = mc^2$ tells us those are linked.

5 Differential Geometry in Physics

At this point, we know how to use a lot of mathematical tools. We have also discussed how Differential Geometry is a fundament of General Relativity. In this chapter, we prepare our tools specifically for usage in the spacetime setting of General Relativity.

We will see how ideas about causality translate to Lorentz manifolds first. Then, we go deeper into the semi-Riemannian Geometry used to describe spacetime. Finally, we will look into the special role that geodesics play in this framework.

After this chapter, we will go on to calculating geodesics in a few different spacetimes.

For this chapter, one not only has literature to turn to for references, but there are also some learning lists on Youtube that explain the material in a totally different (albeit somewhat less formal) way. One could look at section 2.7 of [5], or turn to [4, 6]. On Youtube, [9, 10] can provide some context.

5.1 Terminology in Physics

First of all we will link mathematics to physics, for that we will need a few more definitions, but these are more physically intuitive than those we have seen before, although we do make use of the ones previously given. Any point on a spacetime is called an **event**, its coordinates expressed in 1 temporal coordinate and n spatial coordinates. It is important to note the convention we use in this thesis (a minus sign for the temporal coordinate in the metric) is not universal, in some literature the metric is defined with a minus sign extra[47].

With this abstract viewpoint it is very important to keep in mind a fundamental physical importance: *causality*, the order of events. We will give some definitions to help us consider causality in this abstract manifold setting of the world around us. We define **purely spacelike, timelike and lightlike (or null) paths** as paths that have derivatives $\gamma'(\lambda)$ that satisfy their respective parts of (2.37). That means $|\gamma'(\lambda)|_g$ is positive, negative and zero respectively[48].Furthermore we want to say something about causality and thus we need a term for paths information can follow, we call those **causal curves** and they are everywhere timelike or lightlike. A very important feature in General Relativity is this: *information cannot follow a spacelike path.* We probably know of this as the statement 'nothing can travel faster than the speed of light'. As a tool, we often use **lightcones**, the collection of all lightlike vectors from a point. Inside the lightcone all vectors are timelike, outside of it all vectors are spacelike, as can be seen in Figure 9.

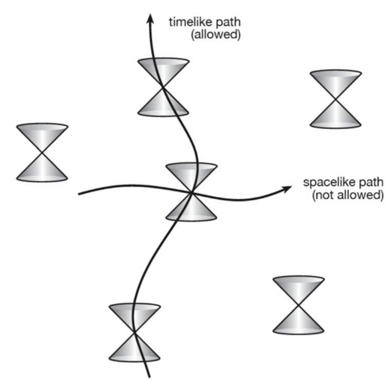


Figure 9: Lightcones in flat spacetime. A timelike path going upwards, a spacelike path from left to right. Taken from [16].

Considering a collection of events, that could just be a single event, $S \subset M$, we define its **causal past and future**, notated $J^-(S)$, $J^+(S)$ respectively. The causal past and future are the events that can be reached from S by taking a past or future directed[49] causal curve. These are also the events that can influence S or that S has influence over, since signals travel along causal curves. We make the distinction with **chronological past and future**, denoted $I^-(S)$ and $I^+(S)$, which are the events that can be reached from S by taking a timelike curve. Timelike curves are paths that particles with mass can follow, so these are all the events an object in S could come from and go to. A subset S is called **achronal** if no two events on S can connect through a timelike curve. For closed achronal sets we can define the **past and future**

[47] That convention can be explained by (5.1), which would no longer need a minus sign.
[48] In literature where minus signs in the metric are flipped this terminology is also flipped.
[49] So the time coordinate decreasing or increasing on the path.

domains of dependence, denoted $D^-(S)$ and $D^+(S)$, as the set of all events p such that all future-moving or past-moving maximal[50] causal curve through p also has a point in S. The union of them is called the domain of dependence and denoted $D(S)$. These domains of dependence can have a boundary, called the **past and future Cauchy Horizon**, denoted $H^-(S)$ and $H^+(S)$.

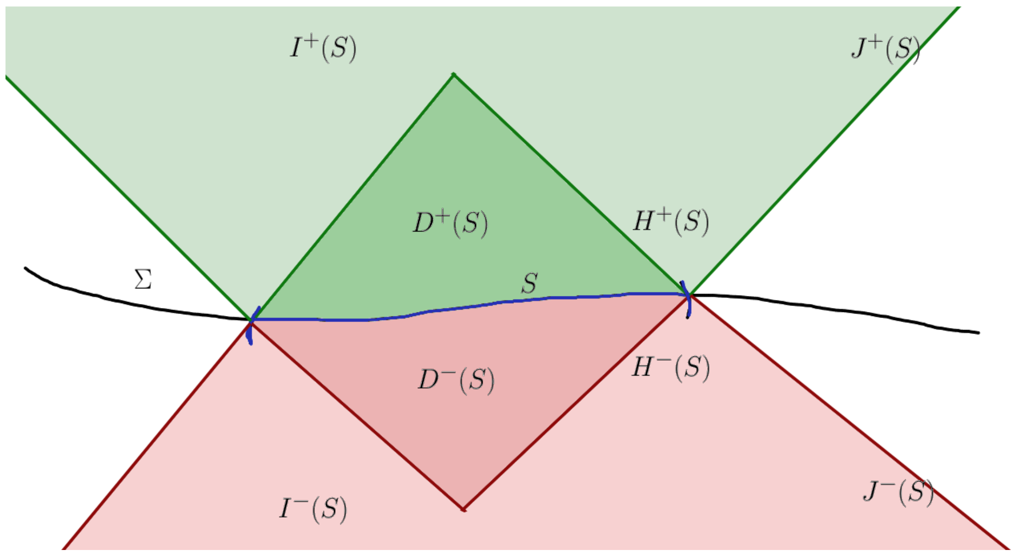


Figure 10: A picture of the causality terms defined in section 5.1. We have Σ in black, an achronal subset of a flat spacetime. We take a region $S \subset \Sigma$ in blue. From the endpoints of S we draw the lightcones, the future in shades of green, the past in shades of red. The domains of dependence ($D(S)$) are the inner most green and red parts, with the cauchy horizons $H(S)$ as boundaries. The causal past and future $J(S)$ are the whole coloured region, including the boundary. The chronological past and future ($I(S)$) do not include the outer most boundaries.

The domain of dependence is useful because by definition it is all the events that influence or can be influenced by S, thus information about S would give us information about the domain of dependence, so this allows us to define and solve *initial value problems* that play a vital role throughout physics. This inspires us to give a special name to sets[51], **Cauchy surfaces**, that have the whole of spacetime as a domain of dependence. Since a spacetimes need not even have Cauchy surface, the ones that allow them are special and called **globally hyperbolic**. We call any closed achronal set without boundary a **partial Cauchy surface**, which could fail to be an actual Cauchy surface (meaning its Cauchy horizon is nonempty) because it is in some way 'badly chosen', or because the spacetime is not globally hyperbolic.

There are no real criteria for 'badness' of choice, hence the quotation marks, but we give an example in Figure 11.

As an example of a spacetime that is not globally hyperbolic, we will consider **Misner spacetime**, which we will return to in section 6.3. This is a 2-dimensional Lorentzian manifold $\mathbb{R} \times \mathbb{S}^1 = \{(t, \varphi)\}$ with metric $g = -2dtd\varphi + td\varphi^2$. This is a spacetime in which there exist **closed timelike curves**, which arises from the fact all the 'rules' are local rules. We only defined the lightcones and what a timelike curve is locally. There is, however, nothing in these definitions that forbids the spacetime from 'bending' so much that globally weird things happen. In the case of Misner spacetime, for $t > 0$, there exist closed timelike curves. We go into this in the next chapter, section 6.3.

[50] That means it cannot be defined for a bigger interval.
[51] The name surfaces is because they are $n-1$ dimensional, but they are not necessarily two-dimensional objects.

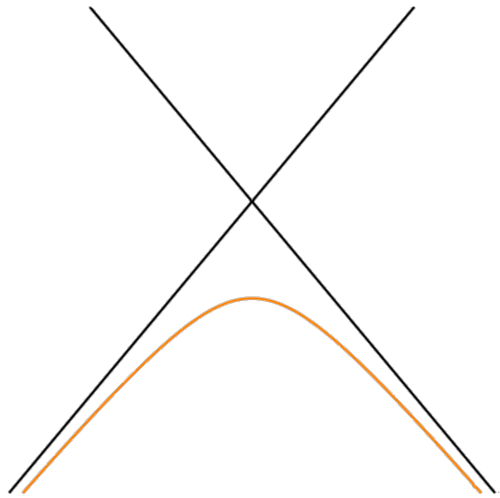


Figure 11: An example of a partial Cauchy surface that is 'badly chosen'. A lightcone in black with a partial Cauchy surface in orange, so completely in the past of one event. One can see the partial Cauchy surface does not have the whole spacetime as a domain of dependence. However, this spacetime could perfectly well be globally hyperbolic. For example, this picture could be drawn in flat 2-dimensional spacetime, where a horizontal line is a Cauchy surface. Figure inspired by Figure 2.24 in [5, p. 80].

We need all these formal definitions since there is in general no cover for our spacetime with only a single coordinate chart to Minkowski spacetime. Spacetime in general is *not* merely a cartesian product of some threedimensional manifold (space) with $\mathbb{R}$ (time) and with a product metric. This is hard to grasp, as we tend to think of time as something universal, that runs the same everywhere. In fact, time can also be curved so there is in general no universal concept of time.

5.2 Semi-Riemannian distance

In chapter 2 we have looked into the Riemannian distance function (section 2.6). We will look into the analogues of that section in semi-Riemannian geometry. We start with the semi-Riemannian metric g on (the tangent bundle of) M (see 2.7.1) and fix $p, q \in M$. These points we will now call events, as we are interested in spacetime, so M is a Lorentz-manifold. In fact, we are most interested in particle trajectories, so we will focus on those.

If we want to consider particle trajectories between p and q we look at timelike paths and points that can be connected with those. We let $\gamma\colon I \to M$ be some piecewise smooth timelike curve between p and q. The **proper time**, τ, of γ is

$$\tau(\gamma) = \int_I \sqrt{-g(\gamma'(\lambda), \gamma'(\lambda)}d\lambda. \tag{5.1}$$

In this case it is good to look at definition 2.6.2 and think about whether that would be an interesting thing to define here. Because our metric is not positive definite and (5.1) does not hold for spacelike paths, we have to be careful what the infimum means.

As an example, we could try to approximate a timelike curve with lightlike segments patched together (see Figure 12).

We could also consider the famous twin paradox [17]: two observers that have taken different paths through spacetime between the same events do not experience the path to the endpoint in the same way. In fact, this proper time is exactly the 'experienced' time interval of a path in spacetime. The minus sign in (5.1) is also why some authors decide to define their metric as $-g$, to make the proper time more intuitively follow from the metric without the need of a minus sign in (5.1).

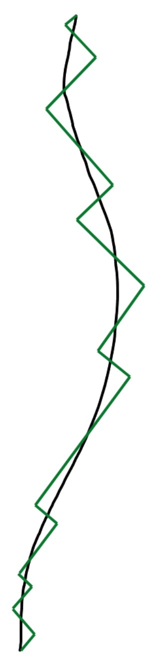


Figure 12: A timelike curve could be approximated with timelike segments, which all have zero length. Figure inspired by Figure 3.3 in [5, p. 110].

5.3 Geodesics in physics

Out of all (timelike) paths joining p and q, there are special paths. These are paths of 'stationary' proper time, which are a critical point of the proper time (5.1) and they are called **geodesics**[18]. Indeed this is Theorem 6.6 from [3] in the Riemannian setting, where we can no longer state minimization but keep geodesics as critical points. In particular, even in Minkowski there is a caveat, geodesics described with $t = \lambda$ have maximum proper time. In fact, with the proper parametrization geodesics maximize the proper time in spacetime[7, p. 60]. In that case we can describe the spatial coordinates as a function of t. But when describing a geodesic with parameter λ, varying the 'temporal speed', i.e. taking $t = \mu\lambda$ with $\mu \neq 1$, can cause the proper time to decrease. In those cases the geodesic does not have maximum proper time.

By definition, geodesics have constant speed, which means geodesics of a time-like vector are everywhere time-like, and the same holds for space-like and light-like vector geodesics [4, p. 69]. We can also construct lightcones with light-like geodesics [4, p. 153].

But these are not the only properties of geodesics. A very important one is that a time-like geodesic describes how a *test particle* behaves in the absence of forces (except gravity, which is our interest and accounted for by the geometry of the spacetime). A **test particle** is a virtual particle that itself does not affect spacetime, and it is an approximation of a small[52] physical particle [5, p. 108]. This is mainly why we are very interested in solving the geodesic equations, as we for example do in rocket science. Since a rocket's mass is small in comparison to planets it is very well approximated by a test particle. And since it is only able to generate acceleration very briefly (for getting into a desired path and getting out of it at the destination), mostly its journey is governed by gravity. In fact, calculating geodesics is what we will do next, although it might not come as a surprise that finding a general solution is not always easy to do. We are however able to state the differential equations one would need to solve for an initial value problem. One could give those to a computer to numerically approximate the geodesic.

[52] I.e. its mass is small compared to masses that curve the spacetime.

6 Different spacetimes and their geodesic equations

In this chapter we will go into explicit spacetimes and calculate the geodesic equation. As we have seen in section 5.3, timelike geodesics describe particles only influenced by gravity. So solving the equation allows us to predict their trajectories. As argued in chapter 4, on cosmological scales, taking only gravity into account is usually a good approximation. We can use geodesics to predict a path of an asteroid, or compute when and where would be best to launch a rocket[53] with a certain location.

Of course, most of these situation are too complicated to compute by hand, and a computer is needed. In this chapter we will go into some simple cases, that are used as approximations. *Minkowski spacetime* (6.1) is used to approximate parts of the universe where bodies are very far away, thus gravity is negligible there. We give the general solution to geodesics in Minowski spacetime. *Schwarzschild spacetime* (6.2) is used to approximate parts of the universe where one symmetrical, massive object governs, so other objects with energy or asymmetries are not taken into account. We give examples of geodesics, but the general solution is not as easily written down as for the other spacetimes we consider, so we do not try to state it. Finally, *Misner spacetime* (6.3) is used as a counterexample to a lot of intuition. Causality cannot be well-defined, but that does not mean Misner spacetime is no spacetime. We give the general solution to the geodesic equation, and see some weirdly behaved geodesics.

Note the constant geodesic, $x(\lambda) = x_0$ is a solution of any geodesic equation, but not an interesting physical solution, as that is not a trajectory.

6.1 Minkowski spacetime

Minkowski spacetime is the simplest of spacetimes, it represents an empty universe where the rules of general relativity do not imply anything else than special relativity. The representation of the metric we will do in two different ways, namely the matrix form g_{ij} such that $g(v,w) = v^T gw$ in coordinate representation[54]. The other representation is very common in physics, describing an infinitesimal area with tensor product of covectors, as in (2.27), which we will give first. We will consider two different coordinate systems for Minkowski spacetime, and see that they give the same answers, but for ease of calculations it is good to think about what coordinate system to use.

6.1.1 Euclidean coordinates x^1, x^2, x^3, x^4

For **Euclidean coordinates** the Minkowski metric looks as following

$$ds^2 = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \qquad \text{for } (x^1, x^2, x^3, x^4) \in \mathbb{R}^4. \tag{6.1}$$

We can also give this metric in matrix form

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

We now derive the geodesic equation, using (3.25)

$$\Gamma^k_{ij} = \frac{1}{2} g^{lk} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \tag{6.3}$$

[53]Rockets can only take so much fuel with them, so fuel is only used to get in and out of orbit, the major part of its trajectory is governed by gravity.

[54]Note a metric is symmetric by definition so we could interchange v and w and get the same result.

We will denote the basis of our tangent spaces as $(\partial_1, \partial_2, \partial_3, \partial_4)$. Firstly, since we need the inverse metric matrix we will give that

$$g^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.4}$$

We see only elements on the diagonal have nonzero value, so our sum over l will disappear. More importantly we note all entries are constants, which means $\partial_i g_{jk}$ will turn out te be zero for all i, j, k. This means all Christoffel symbols are equal to zero, reducing the geodesic equation (3.38) to

$$(x'')^k (\lambda) = 0 \tag{6.5}$$

This equation we can solve explicitly, for initial point $b \in \mathbb{R}^4$ and initial velocity $a \in \mathbb{R}^4$ the geodesic will look like

$$x(\lambda) = a\lambda + b \tag{6.6}$$

6.1.2 Spherical coordinates t, r, θ, ϕ

For point suitable for sperical coordinates[55] we can also describe Minkowski space and its geodesics in spherical coordinates[56]. We use coordinates t, r, θ, ϕ, related to Eulidean coordinates as $x^1 = t$, $x^2 = r\sin\theta\sin\phi$, $x^3 = r\sin\theta\cos\phi$, $x^4 = r\cos\theta$. The metric looks like

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad \text{or} \qquad g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{6.7}$$

We will again solve the Christoffel symbols, for which we need the inverse matrix

$$g^{ij} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{6.8}$$

As for Euclidean coordinates, the sum over l will disappear since this is a diagonal matrix, but the entries are not constants so there seems to be some actual calculations to be done. We will denote the Christoffel symbols and elements of the tangent spaces by indices t, r, θ, ϕ.

Firstly we note $\Gamma^i_{ii} = 0$ since $g_i i$ does not depend on the ith coordinate. Next we note the independence of entries in g, since $g_{\theta\theta}$ depends only on r, we need to calculate Christoffel symbols with two entries θ and one r. Keep in mind we have $\Gamma^k_{ij} = \Gamma^k_{ji}$. We end up with

$$\Gamma^r_{\theta\theta} = -r \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}. \tag{6.9}$$

Also $g_{\phi\phi}$ depends only on r and θ, so we need to calculate Christoffel symbols with two entries ϕ and one r or θ, yielding

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta \qquad \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = r\sin^2\theta \tag{6.10}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta) \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} = \cot\theta. \tag{6.11}$$

[55] We need at least two neighbourhoods with these coordinates to cover the whole manifold, in those neighbourhoods our metric should be smooth and invertible.

[56] We will use the physical convention of denoting the polar angle as θ.

All other Christoffel symbols are zero. The geodesic equations for the spherical Minkowski spacetime are

$$
\begin{aligned}
&\left(x''\right)^k(\lambda) + \left(x'\right)^i(\lambda)\left(x'\right)^j(\lambda)\Gamma^k_{ij}(x(\lambda)) = 0 \\
&t''(\lambda) = 0 && (6.12)\\
&r''(\lambda) = r(\lambda)\left[\left(\theta'(\lambda)\right)^2 + \sin^2\left(\theta(\lambda)\right)\left(\phi'(\lambda)\right)^2\right] && (6.13)\\
&\theta''(\lambda) = \frac{1}{2}\sin\left(2\theta\right)\left(\phi'(\lambda)\right)^2 - 2\frac{r'(\lambda)\theta'(\lambda)}{r(\lambda)} && (6.14)\\
&\phi''(\lambda) = -2r(\lambda)\sin^2\left(\theta(\lambda)\right)\phi'(\lambda)r'(\lambda) - 2\cot\left(\theta(\lambda)\right)\phi'(\lambda)\theta'(\lambda). && (6.15)
\end{aligned}
$$

We will not solve this in general, but note a few possible solutions. First note we can take a path parametrized by t instead of λ as t does not accelerate. Note also that a constant point[57], so $(r, \theta, \phi)(t) = (r_0, \theta_0, \phi_0)$ is a geodesic. Any path keeping the angles θ, ϕ constant and setting $r(t) = r_0 + at$ with $a \in \mathbb{R}$ is also a geodesic. The last one should give you an indication the geodesics here represent the same physical paths as in the euclidean coordinates, where straight lines also solved the geodesic equation.

6.1.3 Lightcone coordinates v, w, θ, ϕ

We also consider a coordinate system that is convenient with our metric, the **lightcone coordinates**, where $v = t + r$, $w = t - r$ and θ, ϕ are as above. We use (v, w, θ, ϕ) as basis. Note again these coordinates do not cover the whole of Minkowski space, we should especially stay away from events with $r = 0$, so $v = w$, and events with $\theta = 0$. Note we use the convention $c = 1$.

$$
ds^2 = -dvdw + \frac{1}{4}(v-w)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad \text{or} \qquad g_{ij} = \frac{1}{4}\begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & (v-w)^2 & 0 \\ 0 & 0 & 0 & (v-w)^2\sin^2\theta \end{pmatrix} \tag{6.16}
$$

These coordinates make it easy to consider the null paths, as a path with constant v or w means a null path. This means we are directly describing a lightcone with $vw = 0$, which in other coordinate systems would be a less simple equations. Next we calculate the inverse metric matrix

$$
g^{ij} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 4(v-w)^{-2} & 0 \\ 0 & 0 & 0 & 4(v-w)^{-2}\sin^{-2}\theta \end{pmatrix}. \tag{6.17}
$$

Now we calculate the Christoffel symbols. Since the inverse metric still has only one entry in each row and column the sum over l still vanishes for all but one coordinate. Looking at the dependencies of the entries on the coordinates themselves, we should only consider triplets i, j, k where at least one of them is either θ or ϕ. For $k = v$, which forces $l = w$ we get as non-zero Christoffel symbols

$$
\Gamma^v_{\theta\theta} = -\frac{1}{4}(v-w) \qquad \Gamma^v_{\phi\phi} = -\frac{1}{4}(v-w)\sin^2\theta. \tag{6.18}
$$

For $k = w$, which forces $l = v$ we get as non-zero Christoffel symbols

$$
\Gamma^w_{\theta\theta} = \frac{1}{4}(v-w) \qquad \Gamma^w_{\phi\phi} = \frac{1}{4}(v-w)\sin^2\theta. \tag{6.19}
$$

For $k = \theta = l$ we get as non-zero Christoffel symbols

$$
\Gamma^\theta_{\theta v} = \frac{1}{2}(v-w)^{-1} \qquad \Gamma^\theta_{\theta w} = -\frac{1}{2}(v-w)^{-1} \qquad \Gamma^\theta_{\phi\phi} = \sin\left(2\theta\right). \tag{6.20}
$$

[57] We do not mean the constant geodesic because in this case t does change so it is a trajectory through spacetime.

For $k = \phi = l$ we get as non-zero Christoffel symbols

$$\Gamma^{\phi}_{\phi v} = -\frac{1}{2}\frac{1}{v-w} \qquad \Gamma^{\phi}_{\phi w} = \frac{1}{2}\frac{1}{v-w}. \tag{6.21}$$

Now we can use those to fill in the geodesic equations

$$(x'')^k(\lambda) + (x')^i(\lambda)(x')^j(\lambda)\Gamma^k_{ij}(x(\lambda)) = 0$$

$$v''(\lambda) = \frac{1}{4}(v(\lambda) - w(\lambda))\left[(\theta'(\lambda))^2 + \sin^2(\theta(\lambda))(\phi'(\lambda))^2\right] \tag{6.22}$$

$$w''(\lambda) = -\frac{1}{4}(v(\lambda) - w(\lambda))\left[(\theta'(\lambda))^2 + \sin^2(\theta(\lambda))(\phi'(\lambda))^2\right] \tag{6.23}$$

$$\theta''(\lambda) = -\frac{1}{2}\left(\frac{1}{v(\lambda) - w(\lambda)}\right)\theta'(\lambda)\left[v'(\lambda) - w'(\lambda)\right] - \sin(2\theta(\lambda))(\phi'(\lambda)) \tag{6.24}$$

$$\phi''(\lambda) = \frac{1}{2}\frac{1}{v(\lambda) - w(\lambda)}(v'(\lambda) - w'(\lambda)). \tag{6.25}$$

Again some easily seen solutions are those who keep θ, ϕ constant, again the straight lines. In fact if we assume radial symmetry we reduce down to

$$w''(\lambda) = 0 \tag{6.26}$$

$$v''(\lambda) = 0 \tag{6.27}$$

$$\tag{6.28}$$

Meaning v and w are both of the form $a\lambda + b$ for $a, b \in \mathbb{R}$ one pair determined by the initial value of v, one pair by the initial value of w. Note in particular $v = 0$ and $w = 0$ are solutions, hence light paths are a solution to the geodesic equation.

6.2 Schwarzschild spacetime

Schwarzschild spacetime is used to describe spacetime in the neighbourhood of a mass, usually a massive object such as a star or black hole. The mass of the object is $M > 0$ and the radius of the object is denoted $r^* > 0$. We will furthermore use the gravitational constant G, which also has positive value and makes sure the result has the right units[19]. We will use the **Schwarschild function**

$$h(r) = 1 - \frac{2GM}{r}. \tag{6.29}$$

It is important to note that this function should be treated with care, as it is not defined for $r = 0$ and vanished for $r = 2GM$.Using the Schwarzschild approach is usually done wanting to know something of the system outside of the object, so for $r > r^*$. For stars and most objects we have come across in our universe we have $r^* > 2GM$, so we do not have to worry about h vanishing or not being defined. There is however, a very special exception, those are black holes. Black holes have all their mass at a single point, and we describe them by setting $r^* = 0$ [4, p. 367]. The Schwarzschild spacetime we will describe in spherical coordinates (t, r, θ, ϕ), with metric

$$ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad \text{or} \quad g_{ij} = \begin{pmatrix} -h(r) & 0 & 0 & 0 \\ 0 & h^{-1}(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{6.30}$$

Actually this metric describes two different disjoined systems, one for $r > 2GM$ and one[58] for $0 < r < 2GM$, but their metrics are the same, as are the inverse metrics

$$g^{-1} = g^{ij} = \begin{pmatrix} -h^{-1}(r) & 0 & 0 & 0 \\ 0 & h(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}. \tag{6.31}$$

We have a diagonal matrix so our Christoffel symbols equation will have $k = l$. We can use the diagonality and dependence of the entries on only r, θ to reduce our calculations and end up with non-zero Christoffel symbols

$$\Gamma^r_{\phi\phi} = -h(r) r \sin^2\theta \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{GM}{r^2} h^{-1}(r) \tag{6.32}$$

$$\Gamma^r_{\theta\theta} = -h(r) r \qquad \Gamma^\theta_{\phi\phi} = -\frac{1}{2}\sin(2\theta) = -\sin\theta\cos\theta \tag{6.33}$$

$$\Gamma^r_{rr} = -h^{-1}(r)\frac{GM}{r^2} \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{r\phi} = \frac{1}{r} \tag{6.34}$$

$$\Gamma^r_{tt} = h(r)\frac{GM}{r^2} \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta = \frac{\cos\theta}{\sin\theta}. \tag{6.35}$$

These lead to the system of geodesic equations

$$t''(\lambda) = -h^{-1}(\lambda)\frac{2GM}{r^2(\lambda)} t'(\lambda) r'(\lambda) = -\frac{2GM}{r(\lambda)\left(r(\lambda) - 2GM\right)} r'(\lambda) t'(\lambda) \tag{6.36}$$

$$r''(\lambda) = -h^{-1}(\lambda) GM \left(\frac{r'(\lambda)}{r(\lambda)}\right)^2 + h(\lambda)\left[-\frac{GM}{r^2(\lambda)}\left(t'(\lambda)\right)^2 + r(\lambda)\left(\left(\theta'(\lambda)\right)^2 + \sin^2\left(\theta(\lambda)\right)\left(\phi'(\lambda)\right)^2\right)\right] \tag{6.37}$$

$$\theta''(\lambda) = \frac{1}{2}\sin\left(2\theta(\lambda)\right)\left(\phi'(\lambda)\right)^2 - 2\frac{r'(\lambda)}{r(\lambda)}\theta'(\lambda) \tag{6.38}$$

$$\phi''(\lambda) = -2\phi'(\lambda)\left[\frac{r'(\lambda)}{r(\lambda)} + \cot\left(\theta(\lambda)\right)\theta'(\lambda)\right], \tag{6.39}$$

where we use $h(\lambda) = h(r(\lambda))$ as in (6.29).

In this case you can make the remark that the metric has spherical symmetry and symmetry by time translation. By Noether's principle, these symmetries correspond to a conserved quantity. The symmetries can be described by **Killing vectors**, and the conserved quantities are constant of motions, related to constants such as energy and angular momentum. These constants allow us to simplify the calculations of the differential equations. More on how to do that can be found in [5, pp. 205–212] or [4, pp. 372–384].

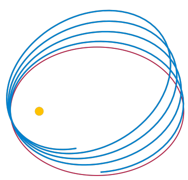
The results obtained from the differential equations are very often applicable, as they describe a lot of cosmological situations where one object is very massive with respect to the others. The Schwarzschild timelike geodesics come in different types: bound orbits, where a test particle stays close to the massive object and r is constant or fluctuating between two finite values, and unbound orbits, where r varies enormously and escapes to infinity. In extreme cases also lightlike geodesics can form a closed orbit, as is the case with black holes and their horizon, but usually lightlike geodesics are unbound orbits.

An important phenomenon explained by this metric is the **progression of parahelion**[59], depicted in Figure 13a. Most objects in orbit have not a perfectly round orbit, but an elliptical orbit with the massive object it is bound as focus of the ellipse. However, as has been observed for Mercury's orbit[60] that the elliptic orbit does not stay the same, but progresses in time, something that cannot properly be predicted or explained without the use of General Relativity.

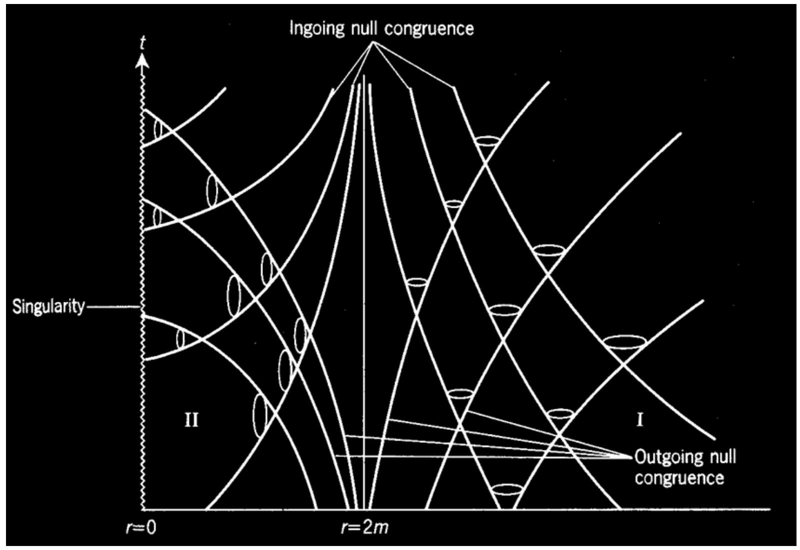
[58] Note that the metric for the later region still assumes a mass of M at the center of mass, so we have to be careful not to cross r^*.

[59] Or procession of elliptical orbits in general, as it describes orbits in general and not just around the sun.

[60] First recognised as a flaw in the current theory in 1859 by Urbain Le Verrier [20].



(a) Procession of parehelion.
Picture from [21].



(b) Lightlike geodesics in Schwarzschild spacetime describing a black hole (singularity at $r = 0$), θ, ϕ constant. Picture from [22], similar to Figure 7 in [4, p. 153]. Null congruence are light geodesics, so we have light geodesics that diverge from the horizon (outgoing null congruence), and light geodesics that converge to the horizon (ingoing null congruence).

Figure 13: Visualisation of Schwarschild spacetime.
13a: A schematic visualisation for the precession of the parahelion.
13b: Lightlike geodesics, that define lightcones where they cross. We see both the inside and outside region of a black hole as described by Schwarzschild spacetime.

If we assume θ and ϕ to be constant, we can reduce to two dimensions, as we did in Figure 13b.

The problem with these coordinates when looking at a black hole, is that we cannot determine what happens if a particle crosses the boundary of $r = 2GM$. A spacetime without that problem that is still able to describe a black hole is called **Kruskal-Szekeres spacetime**. More on that can be found at [4, pp. 386–398], [5, pp. 222–229] or a very brief overview at [23, pp. 94–96].

6.3 Misner spacetime

Next we look at a two dimensional spacetime , so it has only consider one temporal dimension and one spatial dimension. This spacetime has very weird properties, we will elaborate on that after we solved the geodesic equation. We will build Misner space by taking the infinite cylinder $M = \mathbb{R} \times \mathbb{S}^1 \ni (t, \phi)$, so $\phi \in [0, 2\pi)$ and define the metric as

$$ds^2 = -2dtd\phi + td\phi^2 \quad \text{or} \quad g_{ij} = \begin{pmatrix} 0 & -1 \\ -1 & t \end{pmatrix}. \tag{6.40}$$

We will give the inverse metric matrix

$$g^{-1} = g^{ij} = \begin{pmatrix} -t & -1 \\ -1 & 0 \end{pmatrix}, \tag{6.41}$$

and non-zero Christoffel symbols, using the fact that there is only one non-constant entry in the metric

$$\Gamma^t_{\phi\phi} = \frac{1}{2}t \qquad\qquad \Gamma^\phi_{\phi\phi} = \frac{1}{2} \tag{6.42}$$

$$\Gamma^t_{\phi t} = \Gamma^t_{t\phi} = -\frac{1}{2}. \tag{6.43}$$

Using those we can write down the geodesic equations for Misner space

$$\begin{cases} t''(\lambda) & = \left(t'(\lambda) - \frac{1}{2}t(\lambda)\phi'(\lambda)\right)\phi'(\lambda) \\ \phi''(\lambda) & = -\frac{1}{2}\left(\phi'(\lambda)\right)^2. \end{cases} \tag{6.44}$$

To solve this system we should first look at the last equation, noting ϕ is not just any real number but determines the angle, we will thus look at $\phi \in (0, 2\pi)$, later we will try to see what happens for $\phi = 0$, but we could always use two covering maps where one has $(t, \phi) \in \mathbb{R} \times (0, 2\pi)$. The general solution for ϕ is

$$\phi(\lambda) = \phi_0 \qquad \text{for } \phi_0 \in (0, 2\pi), \text{ or} \tag{6.45}$$

$$\phi(\lambda) = 2\log(a + \lambda) + b \qquad \text{with } a, b \in \mathbb{R}. \tag{6.46}$$

This enables us to rewrite the geodesic equation for t as

$$t''(\lambda) = \frac{2}{a + \lambda}\left(t'(\lambda) - t\frac{1}{a + \lambda}\right), \tag{6.47}$$

with solution

$$t(\lambda) = A(a + \lambda)^2 + B(a + \lambda) \qquad \text{with } A, B \in \mathbb{R}. \tag{6.48}$$

Note this solution exists for $a + \lambda > 0$ and without loss of generality we can set $a = 0$, yielding the general solution for the geodesic equation in Misner space (6.44) as

$$\begin{cases} \phi(\lambda) = 2\log(\lambda) + b \quad \text{or } \phi_0 \\ t(\lambda) = A\lambda^2 + B\lambda, \end{cases} \tag{6.49}$$

for $b \in \mathbb{R}$ that shifts the position of ϕ and $A, B \in \mathbb{R}$ that shift and scale the t coordinate. Actually, A has a special role as we will see next.

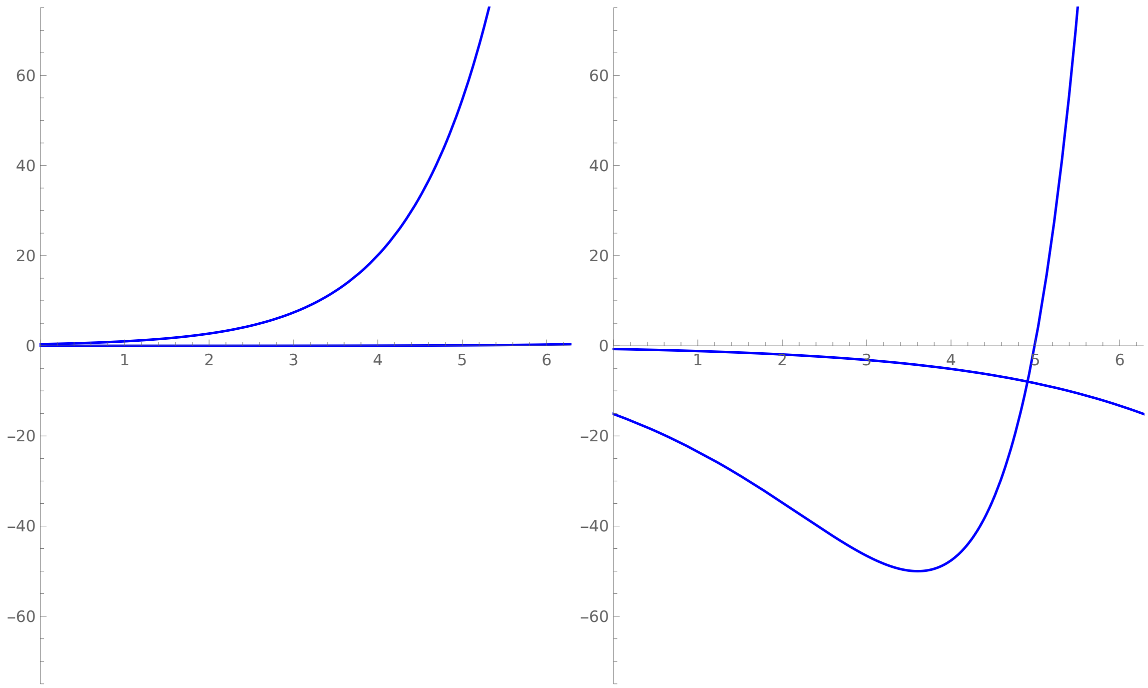
Before we go on to visualising these geodesics, we want to determine when this geodesic is spacelike, timelike or lightlike. So we calculate $g(\gamma'(\lambda), \gamma'(\lambda))$ for $\gamma'(\lambda) = (\phi'(\lambda), t'(\lambda))$.

$$g(\gamma'(\lambda), \gamma'(\lambda)) = \left(\gamma'(\lambda)\right)^T g\gamma'(\lambda) \tag{6.50}$$

$$= \begin{pmatrix} t'(\lambda) & \phi'(\lambda) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & t \end{pmatrix} \begin{pmatrix} t'(\lambda) \\ \phi'(\lambda) \end{pmatrix} \tag{6.51}$$

$$= -4A \tag{6.52}$$

This means for $A < 0$ we have $g(\gamma'(\lambda), \gamma'(\lambda) > 0$ so the geodesic is spacelike. For $A = 0$ our geodesic γ is lightlike, and for $A > 0$ we describe timelike geodesics.



(a) Timelike geodesic for $A = 1$, $B = 0$ and $b = 1$.

(b) Timelike geodesic for $A = 0.5$, $B = -13$ and $b = -1$.

Figure 14: Parts of two timelike geodesics in Misner spacetime plotted. Misner coordinate t on the vertical axis, coordinate ϕ on the horizontal axis (the identification of ϕ at 0 and 2π is taken into account once).

We have made a visual representation[61] of the causal structure in Misner space. In Figure 14, two timelike geodesics are plotted. In Figure 15, some lightlike geodesics are plotted. Note we have not accounted for the full periodicity of our space[62] for all geodesics, as they mostly converge to the $t = 0$ line, the plots would become very cluttered close to that line. In appendix A, one can find the exact code used to plot these figures, and thus which periodicities are taken into account.

Note in particular, as time increases the causal character of ∂_ϕ goes from timelike to spacelike[63].

In fact, this Misner space is invented as sort of counterexample to our intuition[64], as it allows closed, **timelike loops** at $t > 0$, meaning a timelike path that goes through the same event twice. The existence of closed timelike curves mean we have a Cauchy horizon and our space is not globally hyperbolic [5, p. 81].

In particular, causality is not well-behaved. We can see a timelike geodesic, standing for a particle not experiencing forces other than gravity, can be at two places at once, or pass through the same event more than once

[61] See also Appendix A for the source code of these pictures.
[62] Here, the periodicity means the identification of (ϕ, t) and $(\phi + 2\pi, t)$.
[63] We can also calculate this, $g(\partial_\phi, \partial_\phi) = t$.
[64] It is a simplification of Taub-NUT space, described in an aptly named article by Misner himself [24].

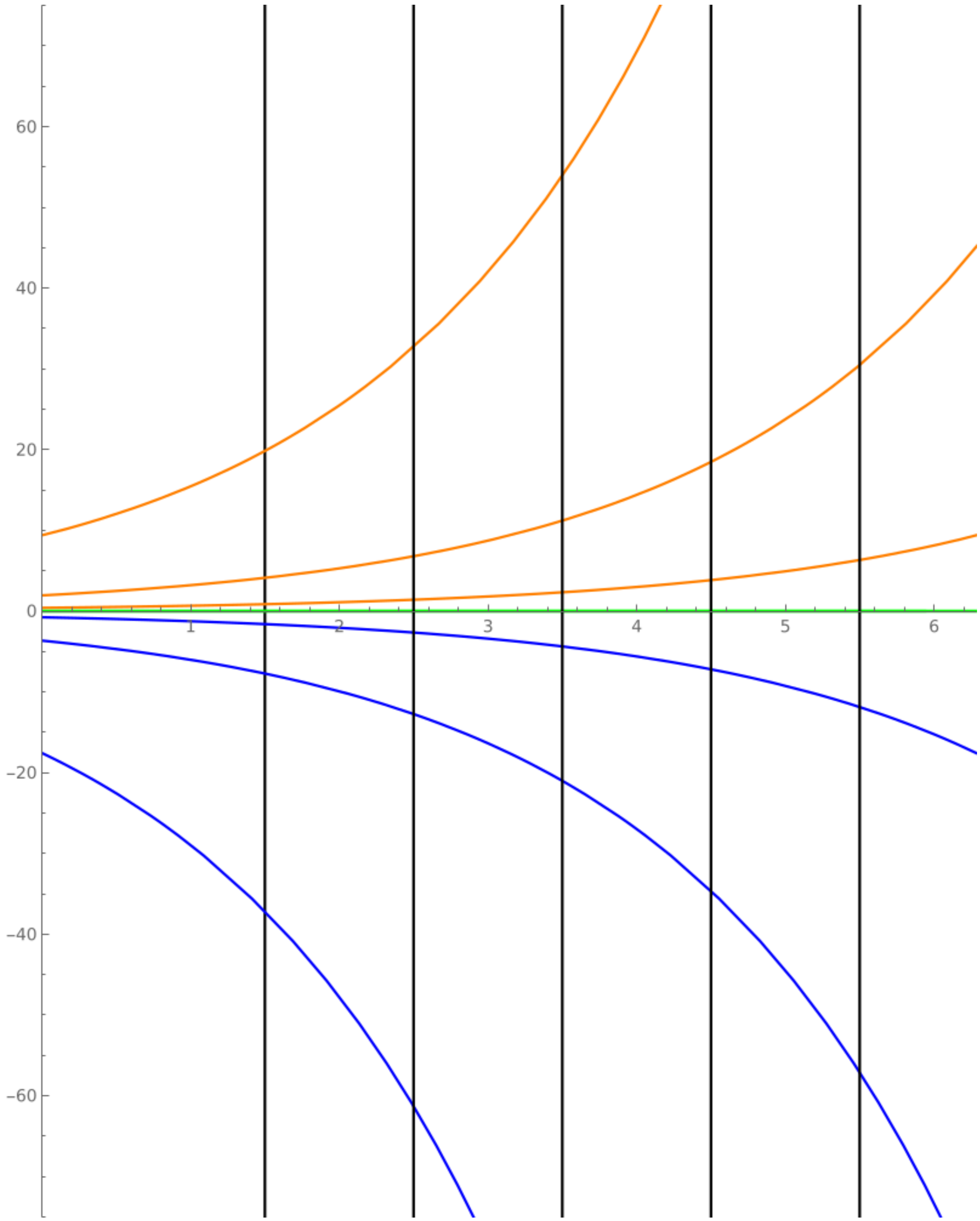


Figure 15: Some lightlike geodesics in Misner space plotted by setting $A = 0$. Coordinate t on the vertical axis, coordinate ϕ on the horizontal axis, the identification of ϕ at 0 and 2π is once taken into account for the most upper orange and lowest blue line. Blue denotes $B < 0$ (lines at $t < 0$), green $B = 0$ (horizontal line at $t = 0$), orange $B > 0$ (lines at $t > 0$). Black denotes ϕ is constant (vertical lines).

7 Conclusions

In this thesis, we have seen some tools from Differential Geometry and how they are used in General Relativity to explain gravity. We explained the concept of a manifold, which is a locally Euclidean (or Lorentzian) space. This allows us to apply principles from linear algebra and geometry.

Using tangent spaces we can describe velocities on a manifold, and even some notion of speed and orthogonality when given a metric. We have seen how a manifold inherits a lot of properties from being locally Euclidean, for example an intuitive metric and connection. That connection allows us to transport velocities over a trajectory. In turn that enables us to define the notion of acceleration, thus a notion of constant velocity. Those non-accelerating trajectories are described by the geodesic equation, for which we gave an explicit differential equation in terms of Christoffel symbols.

Later on, we described how these tools stand at the basis of General Relativity. We saw our universe can be seen as a Lorentzian manifold, where we explained notions of causality in this mathematical framework. Using our tools, we gave the differential equations needed to solve to describe a freely falling trajectory. We even showed solutions to some of them.

Of course there is much more to look into, we will mention a few interesting subjects one could study after this thesis.

There are more spacetimes, that we have not described, for which one can find and solve the geodesic equations. The Misner spacetime we have seen is actually a 2-dimensional version of Taub-NUT space [24], which is a 4-dimensional spacetime, but with very strange properties, hence the title of the publication by Misner. This spacetime clashes with fundamental ideas about causality, that we do not observe in our own universe.

When given a manifold with boundary, geodesics abruptly 'end' on the boundary. However, one can define rules to overcome those endings by considering **billiards**. Essentially one defines 'bouncing' conditions on the boundary, so we glue two geodesic paths together at the boundary. Especially in semi-Riemannian settings there is interesting behaviour, as the boundary itself could be space-, time- or lightlike, with consequences for the billiards, see also [25]. We need **symplectic geometry** to describe their properties correctly, an introduction to the matter could be [26]. It would be very interesting to make a visualisation of geodesics and even billiards with spacelike, timelike and lightlike boundaries, using Mathematica, Python or another computer program to numerically approximate billiards.

On the curvature of manifolds there is a lot more to be discussed, there are tools to measure curvature, for example the Riemann tensor or Ricci scalar described in [3–5] that we briefly mentioned in chapter 4.

So far we have only given the starting points for General Relativity, but there is much more to be said about the theory, as for example in [6].

One of the current open questions in physics is how General Relativity and Quantum Mechanics could be united in one framework. Right now we use two different theories, that both predict a big part of our reality very well, as well as being aesthetically pleasing theories. In this thesis, we have seen the beauty of manifolds, and how they help us to describe particles in gravitational fields. However, it is not yet possible to describe both theories within one framework. Uniting theories has been managed before, for example with mechanics, where special relativity at slow speeds gives the same predictions as classical mechanics, or with electromagnetism, which describes both electric and magnetic forces and phenomena. Such a bigger framework would allow us to describe systems where both gravity and quantum effects play big roles, for example black holes or neutron stars. Naturally, such a framework has been a goal for a long time in physics research, and has already been given names, such as Theory of Everything (ToE), final theory, ultimate theory, or master theory [27]. However, if at all possible to construct such a framework, it is probably still far away from being constructed. String theory is a possible way forward to such a Theory of Everything.

8 Acknowledgements

Although there were certainly hard times during the project of my thesis, I have very much enjoyed myself. It has not been easy, especially since during the process I only made a few trips to the library for books and otherwise sat at home, luckily sometimes with virtual company. I want to thank Dr. Álvaro del Pino Gomez for the lengthy and interesting discussions, for his enthusiasm, for always lining up the next object of interest and for the feedback on my texts and presentation, investing a lot of his time in this project. I want to thank Dr. Thomas Grimm for his supervision and keeping track of the goal of the project, as well as easing the difficulties of writing a thesis for two departments with different conventions. I want to thank both my supervisors for giving me a glimpse into the reality of academia and how scientists of different disciplines can interact. They enabled me to see science 'in action' during conversations I could not yet completely understand, but enjoyed nonetheless. I want to thank MSc. Jeroen Monnee for his practical advise on writing and structuring my thesis and project, both verbally and via his own bachelor thesis, and helping me grasp the abstract concepts in differential geometry and their usage and significance in physics. I am hoping to meet you all again in person soon.

I would also like to thank my significant other, friends, family, fellow students and other university staff who supported me, both in real life and online, to focus, to stay motivated and to keep calm, as well as trying to grasp what I was writing about and allowing me to try and explain.

A Mathematica code used for figures

Figure 1, left: ContourPlot[{$x^2+y^2 == 1$, $x^2+y^2 == 4$, $x^2+y^2 == 9$, $x^2+y^2 == 16$, $x^2+y^2 == 25$, $x^2+y^2 == 36$, $x^2+y^2 == 49$, $y == -x$, $y == x$, $y == 0$, $x == 0$, $y == -2.5x$, $y == 2.5x$, $y == x/2.5$, $y == -x/2.5$}, {x, -1.5, 5.5}, {y, -1.5, 5.5}, ContourStyle→Directive[RGBColor[0.,0.,0.], Opacity[1.], AbsoluteThickness[3.]]]

Figure 1, right: ContourPlot[{$x == 1$, $x == 2$, $x == 0$, $x == -1$, $x == -2$, $y == 0$, $y == 1$, $y == -1$, $y == 2$, $y == -2$}, {x, -2.5, 2.5}, {y, -2.5, 2.5}, ContourStyle→Directive[RGBolor[0.,0.,0.], Opacity[1.], AbsoluteThickness[3.]]]

Figure 2: ContourPlot3D[{$x^2+y^2+z^2 == 1$, $y == 2x-\sqrt{5}$, $y == 1$}, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}, Boxed→False]

Figure 3, left: ContourPlot[{$x^2+y^2 == 1$, $x == 1$, $x == y-\sqrt{2}$, $y == 1$, $x == -1$, $y == -1$, $x == y+\sqrt{2}$, $x == -y+\sqrt{2}$, $x == -y-\sqrt{2}$}, {x, -1.2, 1.2}, {y, -1.2, 1.2}]

Figure 3, right: ContourPlot3D[{$x^2+y^2 == 1$}, {x, -1.2, 1.2}, {y, -1.2, 1.2}, {z, -3, 3}, Boxed→False]

Figure 4: ContourPlot[{$\sqrt{x^2+y^2} == 1$, $y == 3/5+x/2$}, {x, $-1/2$, $1/2$}, {y, $1/5$, $6/5$}]

Figure 6: ContourPlot[{$x^2+y^2 == 9$, $y == -x$, $y == x$, $y == 0$, $x == 0$, $x+2 == y^2$}, {x, -3.5, 3.5}, {y, -3.5, 3.5}, ContourStyle → {Black, Blue, Blue, Blue, Blue, Orange}]

Figure 8: Plot3D[{−Exp[$-\left(x^2+y^2\right)/20$]}, {x, -10, 10}, {y, -10, 10}, Axes→False, Boxed→False, ImageSize→Full]

Figure 11: ContourPlot[{Abs[r] ==Abs[z], −Abs[z] = $1/(z+r)+r$}, {r, -3, 3}, {z, -3, 2}, ContourStyle → {Black, Orange}]

Figure 14: Manipulate[ParametricPlot[{{2Log[λ] + b, $A(\lambda)^2+B(\lambda)$}, {2Log[λ] + $b+2\pi$, $A(\lambda)^2+B(\lambda)$}}, {λ, 0, 50}, PlotRange→{{0,2π}, {-75, 75}}, AspectRatio→Full, PlotStyle→Blue],
{b, $-\pi$, 2π}, {A, 0, 1}, {B, -15, 15}]

Figure 15:
listff={{1.5, -100+50λ}, {$-(4/5)\pi$ + 2Log[λ], -5λ}, {$-(3/5)\pi$ + 2Log[λ], 0}, {$-(2/5)\pi$ + 2Log[λ], 5λ}, {$-(4/5)\pi+2\pi$+2Log[λ], -5λ}, {$-(3/5)\pi+2\pi$+2Log[λ], 0}, {$-(2/5)\pi+2\pi$+2Log[λ], 5λ}, {$(1/5)\pi$+2Log[λ], -5λ}, {$(2/5)\pi$ + 2Log[λ], 0}, {$(3/5)\pi$ + 2Log[λ], 5λ}, {2.5, -100+50λ}, {3.5, -100+50λ}, {4.5, -100+50λ}, {5.5, -100+50λ}}

ParametricPlot[listff, {$\lambda, 0, 50$}, PlotRange→{ {0, 2π},{-75,75}}, AspectRatio→Full, PlotStyle→{Black, Blue, Green, Orange, Blue, Green, Orange, Blue, Green, Orange, Black, Black, Black, Black}, ImageSize→Large]

References

[1] F. Herrmann and Karlsruher Institut für Technologie I. Halupczok. *GeodesicLab*. Interactive model. http://www.physikdidaktik.uni-karlsruhe.de/software/geodesiclab/a3.html, accessed June 3rd, 2021.

[2] John M. Lee. *Graduate texts in Mathematics: Introduction to Smooth Manifolds*. 2nd ed. Springer, 2012. ISBN: 978-1-4419-9981-8.

[3] John M. Lee. *Graduate texts in Mathematics: Riemannian Manifolds: An Introduction to Curvature*. 1st ed. Springer, 1997. ISBN: 978-0-387-98271-7.

[4] Barrett O'Neill. *Semi-Riemannian Geometry With Applications to Relativity*. Academic Press, July 1983. ISBN: 978-0-125-26740-3.

[5] Sean M. Carroll. *An Introduction to General Relativity: Spacetime and Geometry*. 1st ed. Addison Wesly, June 2003. ISBN: 0-8053-8732-3.

[6] S. W. Hawking and George F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973. ISBN: 9780521200165. URL: http://search.ebscohost.com.proxy.library.uu.nl/login.aspx?direct=true&db=nlebk&AN=502384&site=ehost-live.

[7] Robert M. Wald. *General Relativity*. 1st ed. The University of Chicago Press, June 1984. ISBN: 978-0-226-87033-2.

[8] *General Relativity*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/General_relativity, accessed June 2nd, 2021.

[9] *The Maths of General Relativity*. Youtube, ScienceClic English. https://www.youtube.com/watch?v=xodtfM1r9FA&list=PLu7cY2CPiRjVY-VaUZ69bXHZr5QslKbzo, accessed June 3rd, 2021.

[10] *General Relativity Learning Playlist*. Youtube, PBS Space Time. https://www.youtube.com/playlist?list=PLsPUh22kYmNAmjsHke4pd8S9z6m_hVRur, accessed April 14th, 2021.

[11] *Nash embedding theorem*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Nash_embedding_theorem, accessed June 16th, 2021.

[12] Dr. Álvaro del Pino Gomez. consulted 16th of June, 2021.

[13] Françoise Dal'Bo. *Geodesic and Horocyclic Trajectories*. 1st ed. Springer, 2011. ISBN: 978-0-85729-073-1.

[14] *History of General Relativity*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/History_of_general_relativity, accessed May 31st, 2021.

[15] *Principle Of Equivalence*. CheggStudy. https://www.chegg.com/learn/physics/introduction-to-physics/principle-of-equivalence, accessed June 15th, 2021.

[16] Sean Carroll. *Spacetime*. Preposterous Universe Blog. https://www.preposterousuniverse.com/blog/wp-content/uploads/2016/11/figure15.jpg, accessed June 15th, 2021.

[17] Peter Isley. *Twin Paradox*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Twin_paradox, accessed May 21st, 2021.

[18] *Geodesics in General Relativity*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Geodesics_in_general_relativity, accessed June 2nd, 2021.

[19] *Gravitational Constant*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Gravitational_constant, accessed May 29th, 2021.

[20] *Tests of General Relativity*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Tests_of_general_relativity, accessed June 9th, 2021.

[21] CC BY-SA 3.0 I KSmrq. https://commons.wikimedia.org/w/index.php?curid=2291973, accessed June 3rd, 2021.

[22] Eugene Kogan. *General Relativity Theory course by E. Kogan*. Feb. 2018. URL: https://www.researchgate.net/publication/323401370_GENERAL_RELATIVITY_THEORY_Course_by_E_Kogan.

[23] Stefan Vandoren. *Black Holes syllabus*. Apr. 2018.

[24] C.W. Misner. *Taub-NUT space as a counterexample to almost anything*. accessed April 29th, 2021. Nov. 1965. URL: https://ntrs.nasa.gov/citations/19660007407.

[25] Boris Khesin and Serge Tabachnikov. *Pseudo-Riemannian geodesics and billiards*. arXiv:math/0608620v2. Mar. 2007. URL: https://arxiv.org/pdf/math/0608620v2.pdf.

[26] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Springer, 2006. ISBN: 978-3-540-42195-5.

[27] *Theory of Everything*. Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Theory_of_everything, accessed June 2nd, 2021.

List of Mathematical Definitions and Theorems