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Clifford algebras and their application in the Dirac equation

Bachelor Thesis

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Mathematics and Physics

$$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\Psi = 0$$

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Abstract

The aim of this thesis will be to study the Clifford algebras that appear in the derivation of the Dirac equation and investigate alternative formulations of the Dirac equation using (complex) quaternions. To this end, we will first look at the symmetries of the Dirac equation and some of the additional insights that follow from the Dirac equation. We will also give a derivation of the Dirac equation starting from the Schrödinger equation, in which we will come across the gamma matrices. These gamma matrices are a representation of a Clifford algebra. We then give a mathematical description of Clifford algebras and explore some of the properties of Clifford algebras. We eventually classify the universal Clifford algebras over regular quadratic spaces for all possible dimensions and find that the Clifford algebra mostly used as the algebra of space-time is actually isomorphic to a matrix algebra with entries from the quaternions. In the last part, we therefore look at different (complex) quaternionic formulations of the Dirac equation and some of the operators and operations in these formalisms. We end with the conclusion that even though some of these (complex) quaternionic formulations are very elegant, it is highly unlikely that it will yield any new physical results and that there is no real reason to prefer a (complex) quaternionic formulation of the Dirac equation over the standard complex formulation.

Contents

1 Introduction

In his paper "The Quantum Theory of the Electron. Part I", published on the first of February 1928 [10], Paul Dirac formulated a first-order relativistic wave equation This equation is now called the Dirac equation. The Dirac equation was the first attempt to unify the theory of relativity and quantum mechanics. The equation turned out to have a great deal more hidden inside of it than Dirac expected. Examples of the additional insights that followed from the Dirac equation are an explanation of spin and the prediction of the existence of antimatter. These additional insights caused the Dirac equation to become one of the most important equations in physics. When writing the Dirac equation in its covariant notation

$$(i\hbar\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \tag{1.1}$$

one stumbles upon the gamma matrices, or as they are also called: the Dirac matrices. These gamma matrices are a set of 4×4-matrices with entries from the set of complex numbers that satisfy specific anticommutation relations. The anticommutation relations between these gamma matrices are the defining relations of a 4-dimensional Clifford algebra with metric signature $(+ - --)$. Clifford algebras are associative algebras generated by a vector space with a quadratic form. The concept of Clifford algebra already existed for quite some time before Dirac derived his equation. It was discovered in 1878 by the mathematician William Kingdon Clifford (1845-1879), who only had two sources for his paper: Grassmann's algebra and Hamilton's quaternions. In this thesis we will not pay a lot of attention to Grasmann's algebra, but the quaternions will definitely play an important role.

The Irish mathematician William Rowan Hamilton is of course most known for his formulation of classical mechanics known as Hamiltonian mechanics. By the time he formulated Hamiltonian mechanics he had realized that multiplication by a complex number of absolute value one is equivalent to a rotation in the complex 2-plane $\mathbb{C}$. He wanted to extend this concept for the geometry of 3-space, but the attempts that he made to find a three-dimensional generalization of the complex numbers were unsuccesful. That is why he turned to the idea of generalizing $\mathbb{C}$ for four dimensions and thus, in 1843, came up with the idea of quaternions. [1] The quaternions can be applied to describe rotations in three-dimensional space, but in this thesis we will come across the quaternions when classifying Clifford algebras. [33]

The fact that we stumble across quaternions when classifying Clifford algebras gave us the idea of looking into the possibility of finding a formulation of the Dirac equation using quaternions. However, using quaternions with real coefficients is very limiting to our study of formulating the Dirac equation in a different way. That is why we will also look at the formulation of the Dirac equation using complex quaternions. The application of (complex) quaternions in physics has been an interesting topic for physicists for a very long time. As early as 1912, Arthur Conway [6] was interested in the application of (complex) quaternions to the special theory of relativity. In fact, even Dirac [8] looked into the applications of quaternions in physics and even found a relation between Lorentz transformations and quaternions.

The aim of this thesis will be to study Clifford algebras and their properties to then use this knowledge to get a better understanding of the Dirac equation and investigate the elegance of, and the physical results that follow from, a (complex) quaternionic formulation of the Dirac equation. To this end, we will first take a look at the derivation of the Dirac equation along with its symmetries and some of the additional insights that sprung from the equation. These additional insights include an explanation of spin and the prediction of antimatter. After seeing how we naturally stumbled upon Clifford algebras in the derivation of the Dirac equation, we will give a mathematical definition of Clifford algebras and explore its properties in the third chapter. In the third chapter we will also apply our new knowledge to get a better understanding of the Dirac equation. The main result of the third chapter will be the classification of the universal Clifford algebras over regular quadratic spaces for all possible dimensions. In chapter 4 we will investigate the possibility of writing the Dirac equation using real quaternions and complex quaternions after coming across them when classifying Clifford algebras. We will end with a conclusion in which we will discuss the outcomes of formulating the Dirac equation using (complex) quaternions.

[1] Sir William Rowan Hamilton eventually published his ideas in 1853 in "Lectures on quaternions" [14].

2 The Dirac Equation

In the early 20th century an effort was made to combine the findings on quantum mechanics with the theory of relativity that Einstein had just published. The general problem of forming a relativistic wave equation for a free particle appeared to be particularly interesting. As a starting point for such a relativistic wave equation the *Schrödinger equation* seemed to be the most logical choice as everything that has to do with time development follows from this fundamental equation. For this chapter we will follow chapter 8 of the book "Modern Quantum Mechanics" [26]. The Schrödinger equation states that a state $|\psi(t)\rangle$ evolves in time according to the following equation.

Definition 2.0.1 (The Schrödinger equation)**.**

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

To incorporate special relativity into this equation we insert the expression for the *Hamiltonian* of a free particle

$$H = \sqrt{c^2p^2 + m^2c^4} \tag{2.1}$$

Substituting this expression into the Schrödinger equation gives us

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \sqrt{c^2p^2 + m^2c^4}\,|\psi(t)\rangle \tag{2.2}$$

The problem with this equation, however, is that it does not put time and space on "equal footing". To see this we need to take the taylor expansion of the Hamiltonian.

$$\begin{aligned} H &= \sqrt{c^2p^2 + m^2c^4} = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \\ &= mc^2\left(1 + \frac{p^2}{2m^2c^2} - \frac{p^4}{8m^4c^4} + ...\right) \end{aligned} \tag{2.3}$$

By making the substitution $p^2 \to -\hbar^2\nabla^2$ we see that in Equation 2.3 this would result in an infinite series of increasing spatial derivatives, but only one time derivative. Space and time are thus treated asymmetrically. To get an equation that is of the same order in both space and time we take a look at the square of the Hamiltonian. We start with the Schrödinger equation, multiply it on both sides with $\frac{i}{\hbar}$ and take the time derivative.

$$-\frac{\partial^2}{\partial t^2}|\psi(t)\rangle = \frac{i}{\hbar}\frac{\partial}{\partial t}H|\psi(t)\rangle = \frac{1}{\hbar^2}H^2|\psi(t)\rangle \tag{2.4}$$

If we now plug in the square of the Hamiltonian ($H^2 = c^2p^2 + m^2c^4$) we get

$$-\frac{\partial^2}{\partial t^2}|\psi(t)\rangle = \left(\frac{c^2p^2}{\hbar^2} + \frac{m^2c^4}{\hbar^2}\right)|\psi(t)\rangle \tag{2.5}$$

If we now use that $\Psi(\mathbf{x}, t) \equiv \langle\mathbf{x}|\psi(t)\rangle$ and $\langle\mathbf{x}|p^2|\psi(t)\rangle = -\hbar^2\nabla^2\Psi(\mathbf{x}, t)$, we get the *Klein-Gordon equation.*

Definition 2.0.2 (Klein-Gordon equation)**.**

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\Psi(\mathbf{x}, t) = 0$$

The Klein-Gordon equation looks a lot like a classical wave equation except for the $\left(\frac{mc}{\hbar}\right)^2$ term. This term actually introduces a length scale $\hbar/mc$ called the *Compton wavelength.* A very desirable property of the Klein-Gordon equation is that it is *Lorentz covariant.* To show that this equation has this property we introduce the relativistic covariant notation. In this notation the Greek indices run 0,1,2,3 and the Latin indices run 1,2,3. If an index is ever repeated in an expression then summation over that index is implied. We define a contravariant vector as $a^\mu \equiv (a^0, \mathbf{a})$ and a covariant vector as $a_\mu = \eta_{\mu\nu}a^\nu$ where $\eta_{\mu\nu}$ is the minkowski tensor ($\eta_{00} = +1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$ and all the other elements are zero). This means that

$a_\mu = (a^0, -\boldsymbol{a})$. The inner product of two four vector we define as $a^\mu b_\mu = a^0 b^0 - \mathbf{a}\cdot\mathbf{b}$. The inner product is thus always with a contravariant and a covariant vector. A property of the inner product of four vectors is that they are Lorentz invariant. This means that $a^\mu b_\mu$ will be the same in every reference frame. For a space-time position four-vector $x^\mu = (ct, \mathbf{x})$ we have that the four-gradient is given by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{2.6}$$

which we call a covariant vector operator.

Intermezzo: Relativistic covariance
Given any dynamical equation of the form

$$L\phi = 0 \tag{2.7}$$

where L is a linear operator. We say that this equation is covariant under a given transformation if

$$L'\phi' = 0 \tag{2.8}$$

where ϕ' represents the transformed wavefunction and L' the transformed operator. To see if an equation is relativistically covariant, we have to check that every quantity has the correct transformation properties under a Lorentz transformation. A Lorentz transformation of spacetime coordinates can be written in the form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.9}$$

where $\Lambda^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}$. Note that scalars are invariant under Lorentz transformations. An example of a Lorentz transformation is a *Lorentz boost*. A Lorentz boost is a Lorentz transformation that does not involve rotation. A Lorentz boost in the x-direction where the relative velocity between the two frames is v, looks as follows:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \alpha & -\beta\alpha & 0 & 0 \\ -\beta\alpha & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.10}$$

where $\beta = \frac{v}{c}$ and $\alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. For more information on Lorentz transformations, see section 3.1 of [7]. For general Lorentz transformations, we have that $x'^\mu x'_\mu = \Lambda^\mu{}_\nu \Lambda_\mu{}^\alpha x^\nu x_\alpha \equiv x^\nu x_\nu$ so that

$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\alpha = \Lambda^{T\alpha}{}_\mu \Lambda^\mu{}_\nu = \delta^\alpha_\nu \tag{2.11}$$

From this it follows that $\Lambda^{T\alpha}{}_\mu = \Lambda^{-1\alpha}{}_\mu$. With the Minkowski metric $\eta^{\mu\nu}$ (see Equation 2.27), we can write Equation 2.11 in the following way

$$\Lambda_{\mu\alpha}\Lambda^\mu{}_\nu = \Lambda_{\mu\alpha}\eta^{\mu\beta}\Lambda_{\beta\nu} = \eta_{\alpha\nu} \tag{2.12}$$

where $\eta_{\alpha\nu} = \eta^{\alpha\nu}$. The above equation is frequently used as the definition of a Lorentz transformation. Let us take a look at the four-gradient under a Lorentz transformation.

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha} = (\Lambda^{-1})^\alpha{}_\mu \frac{\partial}{\partial x^\alpha} = \Lambda_\mu{}^\alpha \partial_\alpha \tag{2.13}$$

What we see is that the four-gradient behaves like a four-vector under a Lorentz transformation so that we have that $\partial'^\mu \partial'_\mu = \partial^\mu \partial_\mu$. [5]

Let us now write the Klein-Gordon equation in this relativistic covariant notation

$$\left[\partial_\mu \partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\Psi(\boldsymbol{x}, t) = 0 \tag{2.14}$$

Since $\partial'^{\mu}\partial'_{\mu} = \partial^{\mu}\partial_{\mu}$ and $(\frac{mc}{\hbar})^2$ is just a scalar, we see that this equation is Lorentz covariant. Another desirable property of the Klein-Gordon equation is that it gives us the solutions that we expect for a free particle with mass m. The solution that we expect is namely

$$\Psi(\mathbf{x},t) = Ae^{-\frac{i}{\hbar}(Et-\mathbf{p}\cdot\mathbf{x})} = Ae^{-\frac{i}{\hbar}p^{\mu}x_{\mu}} \tag{2.15}$$

with $p^{\mu} = (E, \mathbf{p})$ the four-momentum. If we plug this solution into Equation 2.14 we get

$$-\frac{1}{\hbar^2}p^{\mu}p_{\mu} + \left(\frac{mc}{\hbar}\right)^2 = \frac{1}{\hbar^2}\left(-\frac{E^2}{c^2} + p^2 + m^2c^2\right) = 0 \tag{2.16}$$

Which gives us the following *dispersion relation*:

$$E = \pm\sqrt{p^2c^2 + m^2c^4} \equiv \pm E_p \tag{2.17}$$

The positive eigenvalue for the energy we expect, but the negative energy eigenvalue that we get seems to be physically impossible. These negative energy solutions appear because the Klein-Gordon equation is second-order in time derivatives, contrary to the Schrödinger equation. At first these negative eigenvalues were a big problem in the development of relativistic quantum mechanics. However, nowadays we have an explanation for these negative eigenvalues. We will discuss this explanation in section 2.4.

The Klein-Gordon equation was eventually left behind due to several reasons. One of them was that when we go as far as to solve the Klein-Gordon equation for an atomic system, the results do not compare well with experiments if that atomic system has so-called *spin*. Another problem with the Klein-Gordon equation is that the expression for the *probability density* following from the Klein-Gordon equation is not positive definite. We will discuss this problem with the probability density in section 2.1. These two problems were the main reasons for the Klein-Gordon equation to be left behind and meant that a new approach was necessary. In comes Dirac who made a leap of faith to create a wave equation linear in space. The idea with which Dirac started was to take the square root of $c^2\mathbf{p}^2 + m^2c^4$.

$$c^2\mathbf{p}^2 + m^2c^4 = (c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta mc^2)^2 \quad = (c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2)^2 \tag{2.18}$$

To get all the cross terms on the right side to vanish we find that

$$\alpha_i\alpha_j + \alpha_j\alpha_i \equiv \{\alpha_i, \alpha_j\} = 0 \tag{2.19a}$$

$$\alpha_i\beta + \beta\alpha_i \equiv \{\alpha_i, \beta\} = 0 \tag{2.19b}$$

and

$$\alpha_i^2 = \beta^2 = 1 \tag{2.20}$$

We also require the α_i for $i = 1, 2, 3$ and β to be *Hermitian* since the Hamiltonian must be Hermitian

$$\boldsymbol{\alpha}^{\dagger} \equiv (\boldsymbol{\alpha}^*)^T = \boldsymbol{\alpha} \text{ and } \beta^{\dagger} \equiv (\beta^*)^T = \beta \tag{2.21}$$

where * indicates taking the complex conjugate and ^{T} taking the transpose. From these relations we find that α and β are not numbers, but matrices. We see that these matrices must be traceless (because of the first two properties) and have eigenvalues ± 1 (as property three implies that the matrices are involutory). The fact that these matrices are Hermitian also implies that they have to be square. We also know that they have to be at least 4×4 matrices. This is because 2×2 matrices are not big enough. We can see this by taking a look at the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.22}$$

These Pauli matrices have the following relations

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{2.23a}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{2.23b}$$

The set of these three Pauli matrices together with the identity matrix form a complete set. However, we find that $\{\sigma_k, I\} = 2\sigma_k$ and thus that this set is not large enough to satisfy the properties of Equation 2.19 and 2.20. This implies that the wave function will have four components, a lot more than the one component that it has in the Schrödinger equation. The matrices $\boldsymbol{\alpha}$ and β are not unique. We see that $\alpha \to S^\dagger \alpha S$ and $\beta \to S^\dagger \beta S$ also satisfy Equation 2.19 and 2.20 if S is unitary ($S^\dagger S = I$). For α and β the following matrices are chosen

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.24}$$

We have written the 4×4 matrices as 2×2 matrices of 2×2 matrices. With these expressions we arrive at the *Dirac equation*.

Definition 2.0.3 (Dirac equation)**.**

$$i\hbar \frac{\partial \left|\psi(t)\right\rangle}{\partial t} = (c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2) \left|\psi(t)\right\rangle$$

where we call the term $c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2$ the *Dirac Hamiltonian*.

We would like to write the Dirac equation in covariant notation. To do so, we introduce the *gamma matrices*, also known as the *Dirac matrices*.

Definition 2.0.4 (Gamma matrices)**.**

$$\gamma^0 = \beta,\ \gamma^i = \gamma^0 \alpha_i$$

With a little algebra, the Dirac equation can then be written in the following form

$$(i\hbar\gamma^\mu \partial_\mu - mc)\Psi(\boldsymbol{x}, t) = 0 \tag{2.25}$$

The gamma matrices in this equation have the following properties

$$(\gamma^0)^2 = 1 \tag{2.26a}$$

$$(\gamma^i)^2 = -1 \qquad i = 1, 2, 3 \tag{2.26b}$$

$$\text{and} \qquad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \tag{2.26c}$$

Or shortly written as

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4 \tag{2.27}$$

With $\eta^{\mu\nu}$ the Minkowski metric.

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.28}$$

The proof that the Dirac equation is in fact Lorentz covariant is actually quite lengthy. We have therefore chosen to leave it out of this thesis, but if one is interested in the proof, see section 3.2 of [7]. The *standard representation* of the gamma matrices is the following

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \text{ and } \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \tag{2.29}$$

Another representation that is frequently used is the so-called *Weyl representation*. For this representation we have that

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{2.30}$$

and that the other γ^i are equal to the γ^i in the standard representation. In the rest of this thesis we will be referring to the standard representation of the gamma matrices when we mention the complex gamma matrices. The relations between these gamma matrices defined in Equation 2.26 are the defining relations of a so-called *Clifford algebra* over a 4-dimensional space with metric signature $(+ - --)$.The specific Clifford

algebra that is used in the Dirac equation is called the *Dirac algebra*. This Dirac algebra was more of a sideproduct of the creation of the Dirac equation, but turned out to be a large contributor to the field of geometric algebra and eventually also the quantum field theory.
To simplify expressions from now on, we will switch to the *natural units*. This means that we set $\hbar = c = 1$. Setting $c = 1$ implies that time (=distance/c) is measured in units of length and that momentum and mass is measured in units of energy. Also setting $\hbar = 1$ means that we tie up the units of length and energy together.

2.1 Conserved Current

A very important property of the Schrödinger equation is that it implies that probability is conserved. The probability density derived from the Schrödinger equation is defined as follows

$$\rho(\mathbf{x}, t) \equiv \psi^* \psi \tag{2.31}$$

Note that this probability density is positive definite. For the probability to be conserved, the probability flux has to obey the *continuity equation*.

Definition 2.1.1 (The continuity equation)**.**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

In this equation the *probability flux* $\boldsymbol{j}$ is defined as

$$\begin{aligned} \mathbf{j}(\mathbf{x}, t) &= -\left(\frac{i}{2m}\right)[\psi^* \nabla \psi - (\nabla \psi^*)\psi] \\ &= \left(\frac{1}{m}\right) Im(\psi^* \nabla \psi) \end{aligned} \tag{2.32}$$

2.1.1 Klein-Gordon equation

One would like to identify analogous expressions using the Klein-Gordon equation. The form of the continuity equation suggests that there exists a four-vector current j^μ with the property that $\partial_\mu j^\mu = 0$. The probablity density will then be defined as $\rho \equiv j^0$. If we follow Equation 2.32, we can write

$$j^\mu = \frac{i}{2m}(\Psi^* \partial^\mu \Psi - (\partial^\mu \Psi)^* \Psi) \tag{2.33}$$

It is then easily seen that $\partial_\mu j^\mu = 0$. The probability density that we then find is

$$\rho(\boldsymbol{x}, t) = j^0(\boldsymbol{x}, t) = \frac{i}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial t} - \left(\frac{\partial \Psi}{\partial t}\right)^* \Psi\right) \tag{2.34}$$

Since the Klein-Gordon equation is a second order equation in time, the initial conditions of both Ψ and $\frac{\partial \Psi}{\partial t}$ have to be specified. This extra freedom in the choice of initial conditions means that the probability density that we have defined can take on negative values. The standard probabilistic interpretation of the wave function is therefore impossible. This caused a big problem in the development of relativistic quantum mechanics and was one of the reasons for the Klein-Gordon equation to be left behind. Eventually a consistent physical interpretation was found. However, we will not discuss this interpretation in this thesis. If one is interested in this interpretation though, see [32].

2.1.2 Dirac equation

We can show that for the Dirac equation the quantity $\rho = \Psi^\dagger \Psi$ can be interpreted as a probability density. First, we note that this quantity can be interpreted as the sum of the squared magnitudes of all four components of $\Psi(\mathbf{x}, t)$ and is thus positive definite. We want to show that this probability density also satisfies the continuity equation with $\boldsymbol{j} = \Psi^\dagger \boldsymbol{\alpha} \Psi$. We therefore take the Hermitian conjugate of the Dirac equation

$$i\frac{\partial \Psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \Psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \Psi^\dagger}{\partial z}\alpha_z^\dagger + m\Psi^\dagger \beta^\dagger = -i\frac{\partial \Psi^\dagger}{\partial t} \tag{2.35}$$

If we now multiply the Dirac equation from the left by $\Psi^\dagger$ and subtract from that Equation 2.35 multiplied from the right by Ψ, we find that

$$\Psi^\dagger\left(\alpha_x\frac{\partial\Psi}{\partial x}+\alpha_y\frac{\partial\Psi}{\partial y}+\alpha_z\frac{\partial\Psi}{\partial z}\right)+\left(\frac{\partial\Psi^\dagger}{\partial x}\alpha_x^\dagger+\frac{\partial\Psi^\dagger}{\partial y}\alpha_y^\dagger+\frac{\partial\Psi^\dagger}{\partial z}\alpha_z^\dagger\right)\Psi+\Psi^\dagger\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi^\dagger}{\partial t}\Psi=0 \tag{2.36}$$

which can be compactly written as

$$\nabla\cdot(\Psi^\dagger\boldsymbol{\alpha}\Psi)+\frac{\Psi^\dagger\Psi}{\partial t}=0 \tag{2.37}$$

In this equation we can recognize ρ and $\boldsymbol{j}$. From this we can conclude that ρ can only change if there is a flow into or out of the system and is therefore a conserved quantity. The fact that the Dirac equation provided a probability density and was consistent with experimental findings meant that it was from then on accepted as the correct equation for describing relativistic quantum mechanics.
The expression for the current can be simplified even more by introducing the *adjoint* of the four-component wave function $\overline{\Psi}\equiv\Psi^\dagger\gamma^0$. We find that $\rho=\Psi^\dagger\Psi=\Psi^\dagger\gamma^0\gamma^0\Psi=\overline{\Psi}\gamma^0\Psi$ and $\boldsymbol{j}=\Psi^\dagger\gamma^0\gamma^0\boldsymbol{\alpha}\Psi=\overline{\Psi}\gamma^0\boldsymbol{\alpha}\Psi=\overline{\Psi}\boldsymbol{\gamma}\Psi$. With this new notation we can write the continuity equation as

$$\frac{\partial}{\partial t}\left(\overline{\Psi}\gamma^0\Psi\right)+\nabla\cdot\left(\overline{\Psi}\boldsymbol{\gamma}\Psi\right)=\partial_\mu j^\mu=0 \tag{2.38}$$

with j^μ being the following four-vector current

$$j^\mu=\overline{\Psi}\gamma^\mu\Psi \tag{2.39}$$

2.2 Free-particle solutions

As we already know, the wave function in the Dirac equation has four components. This is twice as many components as the wave function in the Klein-Gordon equation. We will eventually see that these additional degrees of freedom have to do with the quantity spin $\frac{1}{2}$. We call the four-component wave function $\Psi(\boldsymbol{x},t)$ a *spinor*. Let us first take a look at the solutions of the Dirac equation for free particles ($\boldsymbol{p}=0$). The Dirac equation then boils down to $i\partial_t\Psi=\beta m\Psi$. The four independent solutions for $\Psi(\boldsymbol{x},t)$ are then

$$\Psi_1=e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\Psi_2=e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\Psi_3=e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\text{ and }\Psi_4=e^{+imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{2.40}$$

We see that the lower half of the wave function corresponds to negative energy and the upper half to positive energy. Both the upper half and the lower half of course consist of two components. It seems tempting to call one component "spin up" and the other one "spin down". This interpretation is in fact correct, but to prove this statement we have to take a look at the free-particle solutions with nonzero momentum $\boldsymbol{p}=p\hat{\boldsymbol{z}}$. The particle will thus be moving freely in the z-direction. The eigenvalue problem that we have to solve in this case is the following $H\Psi=E\Psi$ for $H=\alpha_z p+\beta m$. We can write this equation out.

$$\begin{pmatrix}m&0&p&0\\0&m&0&-p\\p&0&-m&0\\0&-p&0&-m\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=E\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} \tag{2.41}$$

For this eigenvalue problem we again find that $E=\pm E_p$. Notice that the equations for u_1 and u_3 and the equations for u_2 and u_4 are coupled. For $E=\pm E_p$ we can set either $u_1=1$ (and $u_2=u_4=0$), in which case $u_3=+\frac{p}{E_p+m}$, or $u_2=1$ (and $u_1=u_3=0$), in which case $u_4=-\frac{p}{E_p+m}$. In the nonrelativistic case ($p\ll E_p$) we see that the upper components dominate. For $E=-E_p$ the nonzero components are either $u_3=1$ and $u_1=-\frac{p}{E_p+m}$ or $u_4=1$ and $u_2=\frac{p}{E_p+m}$. In this case, the lower components dominate nonrelativistically.
Let us now introduce the *spin operator* $\boldsymbol{S}=\frac{1}{2}\sum$ with $\sum=(\sum_1,\sum_2,\sum_3)$ via

$$\textstyle\sum_i=\begin{pmatrix}\sigma_i&0\\0&\sigma_i\end{pmatrix} \tag{2.42}$$

with σ_i being the corresponding Pauli matrices. The projection of the spin onto the direction of momentum that this spin operator projects out, is positive for the positive-energy solution with $u_1 \neq 0$ and negative for the positive-energy solution with $u_2 \neq 0$. An analogous result is found for the negative-energy solutions. From this we may conclude that the free-particle solutions do in fact behave according to our spin up/down conjecture. The Dirac equation thus describes spin $\frac{1}{2}$ particles.
If we put all this together, we have that the positive-energy solutions are

$$u_R^{(+)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{pmatrix} \text{ and } u_L^{(+)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p}{E_p+m} \end{pmatrix} \tag{2.43}$$

where the subscript R (L) denotes the so-called *right (left) handedness*. Right (left) handedness means that the spin operator projects out a positive (negative) projection of the spin onto the direction of momentum for that solution. For the negative-energy solutions we have that

$$u_R^{(-)}(p) = \begin{pmatrix} \frac{-p}{E_p+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_L^{(-)}(p) = \begin{pmatrix} 0 \\ \frac{p}{E_p+m} \\ 0 \\ 1 \end{pmatrix} \tag{2.44}$$

The free-particle wave functions are then formed by including the normalization factor and also the term $e^{-ip_\mu x^\mu}$.

2.3 Symmetries of the Dirac equation

Let us from now on consider situations in which there is a spin $\frac{1}{2}$ particle sitting in an *external potential*. This means that the Hamiltonian will become

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m + V(\boldsymbol{x}) \tag{2.45}$$

for some potential energy function $V(\boldsymbol{x})$.

2.3.1 Angular momentum

When we consider a system with a Hamiltonian that contains a *central potential* (the system is spherically symmetrical), we know that the Hamiltonian will commute with the *orbital-angular-momentum operator* $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$. The so-called *Heisenberg picture* tells us that in this system the orbital angular momentum is a constant of motion. The Heisenberg picture namely states that operators incorporate a time-dependency while the state vectors are time-independent. This picture stands in contrast with the *Schrödinger picture*, in which the operators are time-independent and the state vectors have a time dependency. In the Heisenberg picture the observables A satisfy

$$\frac{d}{dt} A_H(t) = i[H_H, A_H(t)] + \left(\frac{\partial A_S}{\partial t}\right)_H \tag{2.46}$$

where the subscripts H and S denote the observables in the Heisenberg and Schrödinger picture respectively. Let us now take a look at the commutation relation between the free Dirac Hamiltonian and the orbital angular momentum $[H, \boldsymbol{L}]$. To do so, we need to check the relations $[\beta, \boldsymbol{L}]$ and $[\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i]$. We see that $[\beta, \boldsymbol{L}]$ is not that interesting, as it is clearly equal to zero. However, $[\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i]$ is more interesting:

$$\begin{aligned} [\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i] &= [\alpha_l p_l, \epsilon_{ijk} x_j p_k] \\ &= \epsilon_{ijk} \alpha_l [p_l, x_j p_k] \end{aligned} \tag{2.47}$$

Knowing that $[x_i, p_j] = i\delta_{ij}$, we find

$$\begin{aligned} [\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i] &= \epsilon_{ijk} \alpha_l (p_l x_j p_k - x_j p_k p_l) \\ &= \epsilon_{ijk} \alpha_l (p_l x_j p_k - i\delta_{jl} p_k + p_l x_j p_k) \\ &= -i\epsilon_{ijk} \alpha_j p_k \neq 0 \end{aligned} \tag{2.48}$$

From this it follows that the orbital angular momentum $\boldsymbol{L}$ will not be a constant of motion for spin $\frac{1}{2}$ particles that are free or find themselves in an external potential. However, recall that we also have the spin operator $\boldsymbol{S} = \frac{1}{2}\sum$. To find the commutation relation between this spin operator and the Dirac Hamiltonian we take a look at the relations $[\beta, \sum_i]$ and $[\boldsymbol{\alpha}\cdot\boldsymbol{p}, \sum_j]$. We again see that $[\beta, \sum_i]$ is not that interesting, as it is also equal to zero. The expression that we get for $[\boldsymbol{\alpha}\cdot\boldsymbol{p}, \sum_j]$ is more interesting though:

$$\begin{aligned}[\boldsymbol{\alpha}\cdot\boldsymbol{p}, \textstyle\sum_j] &= [\alpha_i p_i, \textstyle\sum_j] \\ &= \alpha_i p_i \textstyle\sum_j + [\alpha_i, \textstyle\sum_j] p_i - \alpha_i \textstyle\sum_j p_i \\ &= [\alpha_i, \textstyle\sum_j] p_i\end{aligned} \tag{2.49}$$

Using the commutation relations of the Pauli matrices (Equation 2.23), we get

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}, \textstyle\sum_j] = 2i\epsilon_{ijk}\alpha_k p_i \tag{2.50}$$

So even though both $\boldsymbol{S}$ and $\boldsymbol{L}$ do not commute with the Hamiltonian, we see that the vector operator $\boldsymbol{J} \equiv \boldsymbol{L} + \boldsymbol{S}$ does commute with the Hamiltonian

$$[H, J_i] = [H, L_i] + \frac{1}{2}[H, \textstyle\sum_i] = i\epsilon_{ijk}\alpha_k p_i - i\epsilon_{ijk}\alpha_k p_i = 0 \tag{2.51}$$

We call this vector operator $\boldsymbol{J}$ the *total angular momentum*. From the fact that the total angular momentum operator commutes with the Dirac Hamiltonian, we may conclude that the total angular momentum is conserved for a system containing spin $\frac{1}{2}$ particles.

2.3.2 Parity

Let us now consider a few new operators. The first operator that we will be discussing is the *parity operator*. When we look at the parity operator π we look at a transformation on state kets. A property of this parity operator is that it changes the sign of $\boldsymbol{x}$. This means that $\boldsymbol{x} \to -\boldsymbol{x}$ and thus that $\boldsymbol{p} \to -\boldsymbol{p}$. If we now consider the case where $V(\boldsymbol{x}) = V(|\boldsymbol{x}|)$, then we expect the solutions to the Dirac equation to obey $\Psi(-\boldsymbol{x}) = \pm\Psi(\boldsymbol{x})$. However, due to the fact that the Dirac Hamiltonian changes under the operation $\boldsymbol{p} \to -\boldsymbol{p}$, this does not seem to hold. In this consideration we have not taken into account the effect of the parity transformation on the spinors though. To incorporate this effect we need to change our parity operator. Since our parity operator is a *unitary operator* (applying the parity operator twice to an object leaves it unchanged), it must contain a unitary operator U_p. This unitary operator is a 4×4-matrix which will leave the Hamiltonian invariant under a parity transformation. This gives us the following parity operator

$$\mathcal{P} \equiv \pi U_p \tag{2.52}$$

The following relations for π must hold

$$\pi^\dagger \boldsymbol{x} \pi = -\boldsymbol{x} \tag{2.53a}$$

$$\pi^\dagger \boldsymbol{p} \pi = -\boldsymbol{p} \tag{2.53b}$$

This means that our matrix U_p must have the following properties

$$U_P \boldsymbol{\alpha} U_p^\dagger = -\boldsymbol{\alpha} \tag{2.54a}$$

$$\text{and} \qquad U_p \beta U_p^\dagger = \beta \tag{2.54b}$$

$$\text{in addition to} \qquad U_p^2 = 1 \tag{2.54c}$$

From this it follows that $U_p = \beta = \beta^\dagger = \gamma^0$. This means that a parity transformation will not only consist of $\boldsymbol{x} \to -\boldsymbol{x}$, but also a multiplication on the left and right of β. This means that the parity transformation of our solutions $\Psi(\boldsymbol{x})$ will give us $\beta\Psi(-\boldsymbol{x})$.

2.3.3 Charge conjugation

Let us now add *electromagnetic interactions* into the Dirac Hamiltonian. We may assume for our particle that it has an electric charge of $e < 0$. In a classical Hamiltonian we make the substitutions $E \to E - e\Phi$ and $\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}$ where Φ is the *scalar potential* and $\boldsymbol{A}$ is the *vector potential.* The procedure that we use here to couple to the electromagnetic field is called *minimal coupling* (see section 3.9 of [7]). This minimal coupling is an effect of Lorentz invariance. There has also been done a lot of research on the Lorentz violating non-minimal coupling in the context of the Dirac equation. We will not go deeper into this topic in this thesis, but for further reading on this topic, see [1]. Let us write the minimal coupling using covariant notation

$$p^\mu \to p^\mu - eA^\mu \tag{2.55}$$

where $A^\mu = (\Phi, \boldsymbol{A})$. If we plug this into our covariant form of the Dirac equation 2.25, we get

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\Psi(x^\mu) = 0 \tag{2.56}$$

We now introduce a term called the *antiparticle.* An antiparticle is an object whose wave function behaves just like the one for a 'normal' particle but with opposite electric charge. To find the corresponding antiparticle wave equation in relation to the original wave equation we thus need an equation where $e \to -e$. To do this we first take the complex conjugate of Equation 2.56

$$\left[-i(\gamma^\mu)^*\partial_\mu - e(\gamma^\mu)^* A_\mu - m\right]\Psi^*(x^\mu) = 0 \tag{2.57}$$

We see that the relative sign between the first two terms has now changed. To change the sign of the first two terms relative to the third we define a matrix C with the property that

$$C(\gamma^\mu)^* C^{-1} = -\gamma^\mu \tag{2.58}$$

If we now plug in $1 = C^{-1}C$ before the wave equation in Equation 2.57 and multiply with matrix C from the left we get

$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m)C\Psi^*(x^\mu) = 0 \tag{2.59}$$

We see that only the sign in front of the e has changed in comparison with Equation 2.56. The wave function $C\Psi^*(\boldsymbol{x}, t)$ thus satisfies the antiparticle wave equation 2.59 if $\Psi(\boldsymbol{x}, t)$ satisfies the normal wave equation.
We now want to find an expression for this matrix C. As γ^2 is the only gamma matrix with imaginary elements and given that $(\gamma^2)^* = -\gamma^2$, we find that

$$C = i\gamma^2 \tag{2.60}$$

It turns out to be more useful to write the wave function in terms of $\overline{\Psi} = (\Psi^*)^T\gamma^0$. If we plug this into our antiparticle wave equation we find that

$$C\Psi^*(\boldsymbol{x}, t) = i\gamma^2(\overline{\Psi}\gamma^0)^T = U_c(\overline{\Psi})^T \tag{2.61}$$

Where U_c is the unitary matrix given by

$$U_c \equiv i\gamma^2\gamma^0 \tag{2.62}$$

We can now define the so-called *charge conjugation operator* $\mathcal{C}$, which satisfies

$$\mathcal{C}\Psi(\boldsymbol{x}, t) = U_c(\overline{\Psi})^T \tag{2.63}$$

The change of this charge conjugation operator $\mathcal{C}$ to the space-time part of the free particle wave function is to effectively take $t \to -t$ and $\boldsymbol{x} \to -\boldsymbol{x}$.

2.3.4 Time reversal

We will now introduce an operator called the *time reversal operator.* The function of this time reversal operator is to, like the name says, reverse the time (or, more properly, to reverse motion). This means that for a time reversal operator Θ the following relations hold

$$\Theta\boldsymbol{x}\Theta^{-1} = \boldsymbol{x} \tag{2.64a}$$

$$\Theta\boldsymbol{p}\Theta^{-1} = -\boldsymbol{p} \tag{2.64b}$$

With time reversal we thus do not reverse the position, but the velocity.
Let us now introduce a theorem called Wigner's theorem. [24] This theorem states that any symmetry transformation is represented by a unitary or antiunitary transformation. This thus tells us that the time reversal operator is anti-unitary and can be written as

$$\Theta = UK \tag{2.65}$$

where U is a unitary operator and K is an operator that takes the complex conjugate of any complex numbers that follow it. Using this recipe, we can define a time reversal operator $\mathcal{T}$ for the Dirac equation

$$\mathcal{T} = U_T K \tag{2.66}$$

where U_T is a unitary matrix. To identify the matrix U_T we take a look at the Schrödinger equation and insert the Dirac Hamiltonian using the gamma matrices

$$i\partial_t \Psi(\boldsymbol{x}, t) = \left[-i\gamma^0 \boldsymbol{\gamma} \cdot \nabla + \gamma^0 m\right] \Psi(\boldsymbol{x}, t) \tag{2.67}$$

Using the same method as with the charge conjugation operator, we plug in $1 = \mathcal{T}^{-1}\mathcal{T}$ before the wave function on both sides and multiply from the left by $\mathcal{T}$. The left side then becomes

$$\begin{aligned} \mathcal{T}(i\partial_t)\mathcal{T}^{-1}\mathcal{T}\Psi(\boldsymbol{x}, t) &= U_T K(i\partial_t) K U_T^{-1} U_T \Psi^*(\boldsymbol{x}, t) \\ &= -i\partial_t U_T \Psi^*(\boldsymbol{x}, t) \\ &= i\partial_{-t}\left[U_T \Psi^*(\boldsymbol{x}, t)\right] \end{aligned} \tag{2.68}$$

Note that in the last line, the sign of the time in the derivative is reversed. As the time-reversed form of Equation 2.67 is given by

$$i\partial_{-t}\left[U_T \Psi^*(\boldsymbol{x}, t)\right] = \left(-i\gamma^0 \boldsymbol{\gamma} \cdot \nabla + \gamma^0 m\right)\left[U_T \Psi^*(\boldsymbol{x}, t)\right] \tag{2.69}$$

we see that for $[U_T \Psi^*(\boldsymbol{x}, t)]$ to satify this equation we get the following relations

$$\mathcal{T}\left(i\gamma^0 \boldsymbol{\gamma}\right)\mathcal{T}^{-1} = i\gamma^0 \boldsymbol{\gamma} \tag{2.70a}$$
$$\mathcal{T}\left(\gamma^0\right)\mathcal{T}^{-1} = \gamma^0 \tag{2.70b}$$

To turn these into relations that contain U_T, we multiply both equations by $\mathcal{T}^{-1}$ from the left and by $\mathcal{T}$ from the right

$$i\gamma^0 \boldsymbol{\gamma} = K U_T^{-1}\left(i\gamma^0 \boldsymbol{\gamma}\right) U_T K \tag{2.71a}$$
$$\gamma^0 = K U_T^{-1}\left(\gamma^0\right) U_T K \tag{2.71b}$$

Next we multiply by K from the left and the right and insert $U_T U_T^{-1}$ in between the γ matrices in the first equation

$$K\left(i\gamma^0\right) U_T U_T^{-1}(\boldsymbol{\gamma}) K = U_T^{-1}\left(i\gamma^0\right) U_T U_T^{-1}(\boldsymbol{\gamma}) U_T \tag{2.72a}$$
$$K\left(\gamma^0\right) K = U_T^{-1}\left(\gamma^0\right) U_T \tag{2.72b}$$

If we now use the fact that $\left(\gamma^0\right)^* = \gamma^0$ and plug the second equation into the first, we find

$$U_T^{-1}(\boldsymbol{\gamma}) U_T = -\boldsymbol{\gamma}^* \tag{2.73a}$$
$$U_T^{-1}\left(\gamma^0\right) U_T = \gamma^0 \tag{2.73b}$$

We know that γ^2 is the only imaginary gamma matrix. This means that on the right side of both equations in Equation 2.73 we only have a minus sign in front of γ^1 and γ^3. From this we can conclude that

$$U_T = \gamma^1 \gamma^3 \tag{2.74}$$

2.3.5 CPT

Now that we have introduced the operators $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ we can look at their combination $\mathcal{CPT}$. If we let $\mathcal{CPT}$ operate on the Dirac wave function we find

$$\begin{aligned}\mathcal{CPT}\Psi(\boldsymbol{x},t) &= i\gamma^2\left[\mathcal{PT}\Psi(\boldsymbol{x},t)\right]^* \\ &= i\gamma^2\gamma^0\left[\mathcal{T}\Psi(-\boldsymbol{x},t)\right]^* \\ &= i\gamma^2\gamma^0\gamma^1\gamma^3\Psi(-\boldsymbol{x},t) \\ &= i\gamma^0\gamma^1\gamma^2\gamma^3\Psi(-\boldsymbol{x},t)\end{aligned} \tag{2.75}$$

This combination of gamma matrices brings us at the so-called *fifth gamma matrix.*

Definition 2.3.1 (Fifth Gamma matrix)**.**

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

In matrix form this gamma matrix is given by

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.76}$$

The concept of $\mathcal{CPT}$ invariance actually has very interesting consequences. The implication of the $\mathcal{CPT}$ symmetry is that there is a mirror-image of our universe. This mirror image consists of all the objects in our universe but with its position reflected through an arbitrary point, its momenta reversed and with its matter replaced by antimatter. This $\mathcal{CPT}$ symmetry holds for any Lorentz invariant local quantum field with a Hamiltonian that is Hermitian. For a relatively recent article on the $\mathcal{CPT}$ symmetry in our universe, see [4].

2.4 Dirac's interpretation of negative energies

For this section we will follow section 20.3 of [30]. The problem with the union of relativity and quantum mechanics is that relativity allows particles to be produced if there is enough energy while quantum mechanics is mostly based on the conservation of probability. This means that a relativistic system that starts of with one particle can end up in a state with 10 particles. Nevertheless, the Dirac theory appears to be a single-particle theory and does not seem to really be influenced by this problem. The fact is that this phenomenon that relativity brings with it does occur in the Dirac theory, but in the negative energy solutions. To see this we take a look at the free-particle Dirac equation with natural units

$$i\frac{\partial\Psi}{\partial t} = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\Psi \tag{2.77}$$

Let us now consider the plane wave solutions

$$\Psi = w(\boldsymbol{p})e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)} \tag{2.78}$$

we find that they satisfy

$$Ew = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)w \tag{2.79}$$

To write the Dirac equation in a more compact way we divide the four-component spinor Ψ into two two-component spinors χ and Φ

$$\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix} \tag{2.80}$$

With this notation we find that Equation 2.79 becomes

$$\begin{pmatrix} E-m & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & E+m \end{pmatrix}\begin{pmatrix} \chi \\ \Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.81}$$

If we now consider a particle at rest ($\boldsymbol{p} = 0$) we find that the equation for χ becomes

$$(E - m)\chi = 0 \tag{2.82}$$

which means that

$$E = m \tag{2.83}$$

This is of course the result that we expect: a positive-energy solution. But if we now take a look at the equation for Φ

$$(E + m)\Phi = 0 \tag{2.84}$$

we find that

$$E = -m \tag{2.85}$$

This solution of course seems odd. The energy of a particle always has to have a positive value. Note that by dividing Ψ into two two-component spinors we split up Ψ into a positive enrgy part χ and a negative energy part Φ. The two components of χ and Φ correspond to the two different spin orientations (up and down). If we now take a look at the case when $\boldsymbol{p} \neq 0$ we find that

$$\chi = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E - m}\Phi \tag{2.86}$$

$$\Phi = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E + m}\chi \tag{2.87}$$

Which we only find to be consistent if

$$\frac{p^2}{E^2 - m^2} = 1 \tag{2.88}$$

which gives us

$$E = \pm\sqrt{p^2 + m^2} \tag{2.89}$$

We see that we retrieve exactly the same dispersion relation. To explain these negative energy solutions, Dirac made use of the *Pauli exclusion principle*. The Pauli exclusion principle states that two or more identical spin $\frac{1}{2}$ particles cannot share a single energy state within an atom. Dirac postulated a sea of negative-energy electrons in which all the negative-energy states were occupied. The left picture in figure 1 shows a depiction of this so-called *Dirac sea*. Due to the exclusion principle this meant that there was no way for the electrons to fall into that negative energy sea. However, this meant that high energy photons would be able to bring a negative-energy electron into a positive-energy state. Since this negative-energy electron would come out of the unobservable Dirac sea, it would leave an observable hole. This hole will have an opposite charge to the electron and a positive energy. This particle is called a *positron*. In the right picture of figure 1 the discovery of the positron in 1933 by Carl Anderson is shown. When an electron and a positron collide both particles will disappear and an energy of 2m will be let out into the world in the form of photons.

With Equation 2.81 we have found an equation where Φ and χ are coupled. We can use this equation to look at what happens to the Dirac equation in the nonrelativistic case. At nonrelativistic (positive) energies $E = K + m$ we have that $K \ll m$. This means that Equation 2.87 becomes

$$\Phi \approx \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{2m}\chi \tag{2.90}$$

If we plug this into Equation 2.86, we get

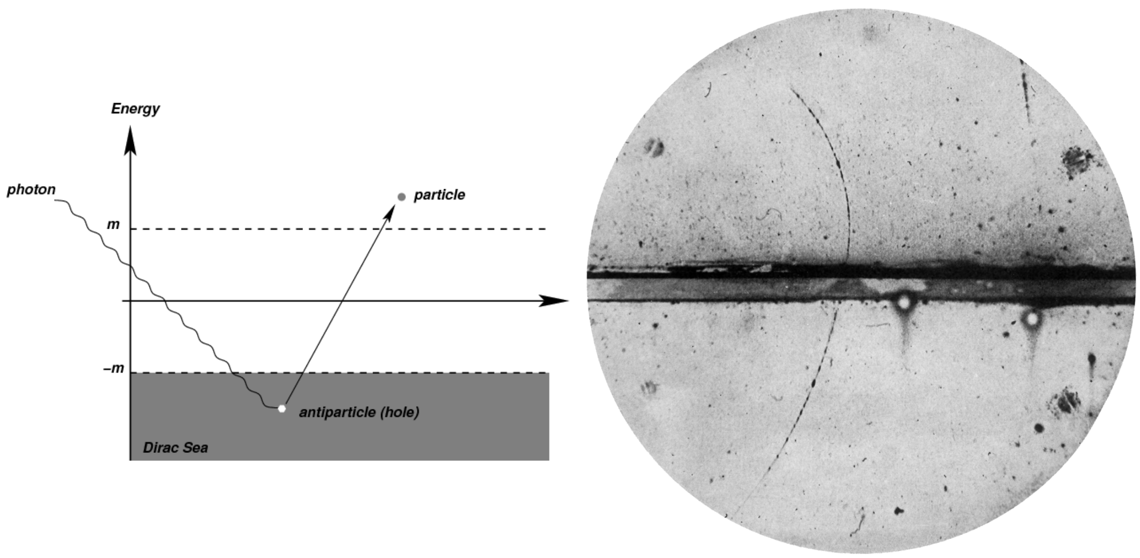
$$\frac{(\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{2m}\chi = \frac{\boldsymbol{p}^2}{2m}\chi = K\chi \tag{2.91}$$

In this equation we recognize the Hamiltonian of a free particle $H = \frac{\boldsymbol{p}^2}{2m}$. The Dirac equation thus reduces nonrelativistically to

$$H\Phi = K\Phi \tag{2.92}$$

This equation is called the *time-independent Schrödinger equation* with energy eigenvalue K. The time-independent Schrödinger equation is used when the Hamiltonian is not dependent on time explicitly. However, even in this case the wave function has a time dependency. In the language of linear algebra, we call the time-independent Schrödinger equation an eigenvalue equation. In that sense, the wave function is an eigenfunction of the Hamilton operator with corresponding energy eigenvalues.

Figure 1: The figure on the left, taken from [2], shows a depiction of the Dirac sea and how a high energy photon can make a particle from an antiparticle. The figure on the right, taken from [3], shows a cloud chamber picture of the first ever positron detected. The curvature of the path of the positron is due to the presence of a magnetic field. The curvature above the lead plate is larger than below as the positron lost energy while traversing it.



3 Clifford algebra

While deriving the Dirac equation we stumbled upon the gamma matrices. The relations between these gamma matrices were the defining relations of a Clifford algebra over a 4-dimensional space. Clifford algebras actually already existed a pretty long time before Dirac posed his relativistic wave equation. However, it only gained the interest of physicists after Dirac published the article "The Quantum Theory of the Electron. Part I" on the first of February 1928 [10] and its sequel one month later. In this chapter we will look at some interesting properties of the Clifford algebras and eventually classify them. To do so, we start by giving the definitions needed to explain Clifford algebras. For this chapter we will mostly follow the book "Clifford Algebras: An Introduction" [12].

3.1 Preliminaries

Let us first give the necessary definitions and some theorems that will help in the study of Clifford algebras. For this section we will follow chapters 2, 3 and 4 and section 1.3 of [12]. Let K denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let E denote a real vector space.

Definition 3.1.1. A finite-dimensional (associative) *algebra* A over K is a finite-dimensional vector space over K equiped with a binary operation $A \times A \to A$, called multiplication $(a, b) \mapsto ab$. This mapping must satisfy

- Associativity: $(ab)c = a(bc)$
- Left distributivity: $a(b + c) = ab + ac$
- Right distributivity: $(a + b)c = ac + bc$
- Scalar compatibility: $\lambda(ab) = (\lambda a)b = a(\lambda b)$

for $\lambda \in K$ and $a, b, c \in A$.

A linear subspace B of an algebra A is a *subalgebra* of A if $b_1b_2 \in B$ whenever $b_1, b_2 \in B$. A subalgebra J of A is a *left ideal* if $aj \in J$ whenever $a \in A$ and $j \in J$. A *right ideal* is defined in a similar way. A subalgebra is then called an *ideal* if it is both a right-ideal and a left-ideal. An ideal J in A is *proper* if J is a proper subset of A. An algebra is then *simple* if the only proper ideal in A is the *trivial ideal* $\{0\}$. Another property that an algebra can have is that can be *unital*. This is when an algebra contains an *identity element* 1, which satisfies $1a = a1 = a$ for all $a \in A$. We also have that an algebra is commutative if $ab = ba$ for all $a, b \in A$. An important example of a commutative subalgebra is the so-called *centre* $Z(A)$ of an algebra A. The centre is defined as

$$Z(A) = \{a \in A : ab = ba \text{ for all } b \in A\} \tag{3.1}$$

The centre $Z(A)$ is then of course unital if A is unital. Let us now take a look at mappings between algebras.

Definition 3.1.2. A mapping ϕ from an algebra A over K to an algebra B over K is called an *algebra homomorphism* if it is linear and if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$.

An algebra homomorphism is called a *unital homomorphism* if it is between two unital algebras and if it satisfies $\phi(1_A) = 1_B$, where 1_A is the identity element of A and 1_B is the identity element of B. If an algebra homomorphism is bijective we call it an *algebra isomorphism*. An algebra homomorphism of an algebra into itself is called an *endomorphism*. To give you an idea of what an algebra is, we will give some examples:

- The vector space $L(E)$ of all endomorphisms of the vector space E over K becomes a unital algebra when multiplication is defined to be the composition of mappings. The identity element of this unital algebra is then the identity mapping I.
- Suppose $dim(E) = d$. If $T \in L(E)$, then T can be represented by a matrix (t_{ij}). The mapping $T \mapsto (t_{ij})$ is then an algebra isomorphism of $L(E)$ onto the algebra $M_d(K)$ of $d \times d$ matrices. The composition of this algebra of $d \times d$ matrices is defined as matrix multiplication.

A unital homomorphism from a unital algebra into $M_d(K)$ or $L(E)$ is called a *representation* of A. A linear subspace F of E is π-*invariant* if $\pi(a)(F) \subseteq F$ for each $a \in A$. The restriction of π to this π-invariant subspace is then called a *subrepresentation* of π. A representation is called *irreducible* if $\{0\}$ and E are the only π-invariant subspaces of E. If a unital homomorphism is also injective, then the representation is called *faithful*. A faithful representation of A onto a subalgebra of $M_d(K)$ is a unital isomorphism; the elements of A are then represented as matrices. An example of a representation of elements of a Clifford algebra is the gamma matrices that we found in the derivation of the Dirac equation in chapter 2. An important example of a faithful representation of a unital algebra A into $L(A)$ is the *left regular representation* ($l : A \to L(A)$). The left regular representation is given by setting $l(a)(b) = ab$. The interesting thing about the left regular representation l of A is that it considers A as a so-called *left A-module*.

Definition 3.1.3. A *left A-module* M is a real vector space M together with a multiplication mapping $(a, m) \mapsto am$ from $A \times M$ to M that satisfies

- $(\lambda_1 a_1 + \lambda_2 a_2)m = \lambda_1(a_1 m) + \lambda_2(a_2 m)$
- $a(\mu_1 m_1 + \mu_2 m_2) = \mu_1(am_1) + \mu_2(am_2)$
- $(ab)m = a(bm)$
- $1_A m = m$

for all $a, a_1, a_2, b \in A$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$, $m \in M$.

Another very important example of an algebra is the *quaternions*. Like $\mathbb{R}$ and $\mathbb{C}$, the algebra $\mathbb{H}$ of quaternions is a *division algebra*. An algebra is a division algebra if and only if that algebra has no non-zero *zero divisors*. A zero divisor a is an element of the algebra for which there exists a non-zero element b in the algebra with $ab = ba = 0$. The fact that a division algebra has no non-zero zero divisors means that all non-zero elements in a division algebra have a multiplicative inverse. The only thing that is different about $\mathbb{H}$ compared to $\mathbb{R}$ and $\mathbb{C}$ is that $\mathbb{H}$ is not commutative. We therefore call $\mathbb{H}$ a non-commutative finite-dimensional real division algebra. This algebra was invented by the famous mathematician and physicist Sir William Rowan Hamilton in 1843. [33] We will construct the algebra by using the so-called *associate Pauli matrices* τ_0, τ_1, τ_2 and τ_3. These associate Pauli matrices are defined as $\tau_0 = I_2, \tau_1 = i\sigma_1, \tau_2 = -i\sigma_2$ and $\tau_3 = i\sigma_3$ with σ_1, σ_2 and σ_3 the Pauli matrices as defined in chapter 2. These associate Pauli matrices form a linearly independent subset of the eight-dimensional real algebra $M_2(\mathbb{C})$. The linear span of these matrices is denoted by H and is four-dimensional. If $h = a\tau_0 + b\tau_1 + c\tau_2 + d\tau_3 \in H$, then

$$h = \begin{pmatrix} a + id & c + ib \\ -c + ib & a - id \end{pmatrix} \tag{3.2}$$

This means that

$$H = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z, w \in \mathbb{C} \right\} \tag{3.3}$$

so that H is a four-dimensional unital subalgebra of $M_2(\mathbb{C})$. The algebra $\mathbb{H}$ of quaternions is defined to be any real algebra that is isomorphic as an algebra to H. Let $\phi : H \to \mathbb{H}$ be such an isomorphism. We set

$$1 = \phi(\tau_0), \mathcal{I} = \phi(\tau_1), \mathcal{J} = \phi(\tau_2) \text{ and } \mathcal{K} = \phi(\tau_3) \tag{3.4}$$

These elements of $\mathbb{H}$ satisfy the following relations

$$\begin{array}{ccc} \mathcal{IJ} = \mathcal{K} & \mathcal{JK} = \mathcal{I} & \mathcal{KI} = \mathcal{J} \\ \mathcal{JI} = -\mathcal{K} & \mathcal{KJ} = -\mathcal{I} & \mathcal{IK} = -\mathcal{J} \\ \mathcal{I}^2 = -1 & \mathcal{J}^2 = -1 & \mathcal{K}^2 = -1 \end{array} \tag{3.5}$$

Let us now take a look at mappings between vector spaces. Of particular interest are the mappings that are *linear*.

Definition 3.1.4. Let $E_1, ..., E_k$ and F be vector spaces over K. A mapping $T : E_1 \times ... \times E_k \to F$ is called *multilinear*, or *k-linear*, if it is linear in each variable:

$$T(x_1, ..., x_{j-1}, \alpha x_j + \beta y_j, x_{j+1}, ..., x_k) = \alpha T(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_k) + \beta T(x_1, ..., x_{j-1}, y_j, x_{j+1}, ..., x_k)$$

for $\alpha, \beta \in K$, $x_j, y_j \in E_j$ and $1 \leq j \leq k$.

These k-linear mappings form, under pointwise addition, a vector space. We denote this vector space by $M(E_1, ..., E_k; F)$. We write $M^k(E; F)$ when $E_1 = ... = E_k = E$ and $M(E_1, ..., E_k)$ for $M(E_1, ..., E_k; K)$. Elements of $M(E_1, ..., E_k; F)$ are called multilinear forms or k-linear forms. An important property of k-linear forms is *symmetry*.

Definition 3.1.5. A k-linear mapping T is *symmetric* if

$$T(x_1, ..., x_k) = T(x_{\sigma(1)}, ..., x_{\sigma(k)}) \text{ for each } \sigma \in \Sigma_k$$

with Σ_k the group of permutations of $\{1, ..., k\}$.

The set $S^k(E; F)$ of symmetric k-linear mapping $s : E^k \to F$ is a linear subspace of $M^k(E; F)$. We denote $S^k(E; K)$ by $S^k(E)$. For a 2-linear mapping we have an explicit name, namely a *bilinear mapping*. The vector space of bilinear mappings from $E_1 \times E_2$ into F is denoted by $B(E_1, E_2; F)$ and elements of $B(E_1, E_2; F)$ are called *bilinear forms*. Bilinear forms and especially symmetric bilinear forms will play a very important role in the study of Clifford algebras. An important property of a bilinear form is its *rank*.

Definition 3.1.6. Suppose $b \in B(E_1, E_2; F), e_1 \in E_1$ and $e_2 \in E_2$. Let $l_b(e_1) : E_2 \to F$ be defined by $l_b(e_1)(e_2) = b(e_1, e_2)$. Then $l_b : E_1 \to L(E_2, F)$ is linear and we define the *rank* of b to be the dimension of the image of l_b, which we call the rank of l_b.

The bilinear form b is then called *non-singular* if $rank(l_b) = dim(E_1) = dim(E_2)$.

3.1.1 The algebra of tensors

To get a better understanding of Clifford algebras and their properties we have to define *tensors* and their corresponding *tensor algebra*. For this subsection we will follow chapter 3 and section 1.3 of [12]. Before we can define what a tensor is, we have to define what a *linear functional* is.

Definition 3.1.7. Suppose K is a one-dimensional vector space over K. A *linear functional* on the vector space E is then a linear mapping from E into K.

The set of linear functionals on E is denoted by $L(E; K)$ or E' and is called the *dual space* of E. Suppose $(e_1, ..., e_d)$ is a basis for E. If $x = \sum_{i=1}^{d} x_i e_i$, let $\phi_i(x) = x_i$ for $1 \leq i \leq d$. Then $\phi_i \in E'$ and $(\phi_1, ..., \phi_d)$ is a basis for E'. We call this basis the *dual basis* to $(e_1, ..., e_d)$.

Definition 3.1.8. Suppose that A is a subset of E. Then the *annihilator* $A^\perp$ in E' of A is the set

$$A^\perp = \{\phi \in E' : \phi(a) = 0 \text{ for all } a \in A\}$$

We see that $A^\perp$ is a linear subspace of E'. In the same way, suppose that B is a subset of E'. Then the *annihilator* $B^\perp$ in E of B is the set

$$B^\perp = \{x \in E : \phi(x) = 0 \text{ for all } \phi \in B\}.$$

We see that $A^{\perp\perp} = span(A)$ and $B^{\perp\perp} = span(B)$. If F is a linear subspace we also have that $dim(F) + dim(F^\perp) = dim(E)$. We are now ready to define tensors.

Definition 3.1.9. Suppose that $(x_1, ..., x_k) \in E_1 \times ... \times E_k$. The *evaluation mapping* $m \mapsto m(x_1, ..., x_k)$ from $M(E_1, ..., E_k)$ into K is a linear functional on $M(E_1, ..., E_k)$. We denote this linear functional by $x_1 \otimes ... \otimes x_k$ and call it an *elementary tensor*. We denote the linear span of all such elementary tensors by $E_1 \otimes ... \otimes E_k$ and call it the *tensor product* of $(E_1, ..., E_k)$.

We denote k copies of the vector space E by $\otimes^k E$. We show next that the elementary tensors span the dual space $M'(E_1,...,E_k)$ of $M(E_1,...,E_k)$.

Proposition 3.1.10. $E_1 \otimes ... \otimes E_k = M'(E_1,...,E_k)$.

Proof. If $m \in (E_1 \otimes ... \otimes E_k)^\perp$, then $(x_1 \otimes ... \otimes x_k)(m) = m(x_1,...,x_k) = 0$ for all $(x_1,...,x_k) \in E_1 \times ... \times E_k$. This means that $m = 0$, implying that for all $\phi \in M'(E_1,...,E_k)$ we have that $\phi(m) = 0$. From this it follows that $(E_1 \otimes ... \otimes E_k)^{\perp\perp} = M'(E_1,...,E_k)$ and thus that $E_1 \otimes ... \otimes E_k = M'(E_1,...,E_k)$. □

We can obtain a basis for $E_1 \otimes ... \otimes E_k$ by taking a basis of the spaces $E_1,...,E_k$ and taking all tensor products of the form $e_1 \otimes ... \otimes e_k$, where each e_j is an element of the basis of E_j for $1 \leq j \leq k$. From this and the proposition above we may then conclude that $dim(E_1 \otimes ... \otimes E_k) = \prod_{i=1}^k dim(E_i)$. The following proposition will help us in proving other theorems about Clifford algebras and in eventually classifying Clifford algebras.

Proposition 3.1.11. *The mapping $\otimes^k : E_1 \times ... \times E_k \to E_1 \otimes ... \otimes E_k$ defined by $\otimes^k(x_1,...,x_k) = x_1 \otimes ... \otimes x_k$ is k-linear. If $m \in M(E_1,...,E_k;F)$ there exists a unique linear mapping $L(m) : E_1 \otimes ... \otimes E_k \to F$ such that $m = L(m) \circ \otimes^k$. The mapping $L : m \mapsto L(m)$ is an isomorphism between $M(E_1,...,E_k;F)$ and $L(E_1 \otimes ... \otimes E_k;F)$.*

Proof. As we know, an elementary tensor denotes a linear functional on $M(E_1,...,E_k)$. This means that

$$(x_1 \otimes ... \otimes (\alpha x_j + \beta y_j) \otimes ... \otimes x_k)(m) = m(x_1,...,(\alpha x_j + \beta y_j),...,x_k)$$

From the linearity of m we may then conclude that

$$\begin{aligned} m(x_1,...,(\alpha x_j + \beta y_j),...,x_k) &= \alpha m(x_1,...,x_j,...x_k) + \beta m(x_1,...,y_j,...,x_k) \\ &= \alpha(x_1 \otimes ... \otimes x_j \otimes ... \otimes x_k)(m) + \beta(x_1 \otimes ... \otimes y_j \otimes ... \otimes x_k)(m) \end{aligned}$$

The mapping $\otimes^k$ is thus a multilinear mapping. We see that the mapping $T : L(m) \mapsto L(m) \circ \otimes^k$ from $L(E_1 \otimes ... \otimes E_k;F)$ into $M(E_1,...,E_k;F)$ is a linear mapping. To check if this mapping is an isomorphism we have to check if it is injective and surjective. If $L(m) \circ \otimes^k = 0$, then $L(m)(x_1 \otimes ... \otimes x_k) = 0$ for all $x_1 \otimes ... \otimes x_k \in E_1 \otimes ... \otimes E_k$. We may assume that for all $i \in \{1,...,k\}$ the $E_i \neq 0$, meaning that $E_1 \otimes ... \otimes E_k$ must contain elements that are not equal to zero. From this it follows that $L(m)$ must be equal to the zero mapping, implying that $ker(T) = 0$. This then means that T is injective. As $dim(E_1 \otimes ... \otimes E_k) = \prod_{i=1}^k E_i = dim(E_1 \times ... \times E_k)$, we know that both $L(E_1 \otimes ... \otimes E_k;F)$ and $M(E_1,...,E_k;F)$ have dimension $(\prod_{i=1}^k E_i)dim(F)$. From this we may conclude that T is also surjective and thus an isomorphism. The inverse $T^{-1} : m \mapsto L(m)$ must then also be an isomorphism. The uniqueness of $L(m)$ follows from the fact that the inverse T^{-1} is injective. □

From this proposition we may conclude that multilinear mappings and linear mappings can be used interchangeably.
On a vector space we can define an algebra with multiplication being the tensor product. We call this algebra the *tensor algebra*. To give a formal definition of this algebra we first have to proof the following corollary.

Corollary 3.1.12. $(E_1 \otimes ... \otimes E_k) \otimes (E_{k+1} \otimes ... \otimes E_l) \cong E_1 \otimes ... \otimes E_l$

Proof. From Proposition 3.1.11 it follows that

$$(E_1 \otimes ... \otimes E_k) \otimes (E_{k+1} \otimes ... \otimes E_l) = B'(E_1 \otimes ... \otimes E_k, E_{k+1} \otimes ... \otimes E_l)$$

Let us now define the linear mapping $T : B(E_1 \otimes ... \otimes E_k, E_{k+1} \otimes ... \otimes E_l) \to M(E_1,...,E_l)$ by $b \mapsto m$ where $m : E_1 \times ... \times E_l \to K$ is defined by $(x_1,...,x_l) \mapsto b(x_1 \otimes ... \otimes x_k, x_{k+1} \otimes ... \otimes x_l)$. We immediately see that $T(ab) = T(a)T(b)$ for $a,b \in B(E_1 \otimes ... \otimes E_k, E_{k+1} \otimes ... \otimes E_l)$, implying that T is an algebra homomorphism. If $T(b) = 0$ then $b(x_1 \otimes ... \otimes x_k, x_{k+1} \otimes ... \otimes x_l) = 0$ for all $(x_1,...,x_l) \in E_1 \times ... \times E_l$. This then means that b is the zero map and thus that $ker(T) = \{0\}$. Therefore T is injective. We also have that $dim(B(E_1 \otimes ... \otimes E_k, E_{k+1} \otimes ... \otimes E_l)) = \prod_{i=1}^l dim(E_i) = dim(M(E_1,...,E_l))$, implying that T is an isomorphism. Now that we have proven that $B(E_1 \otimes ... \otimes E_k, E_{k+1} \otimes ... \otimes E_l) \cong M(E_1, ..., E_l)$, we may conclude that $B'(E_1 \otimes ... \otimes E_k, E_{k+1} \otimes ... \otimes E_l) \cong M'(E_1, ..., E_l)$. This means that

$$\begin{aligned}(E_1 \otimes ... \otimes E_k) \otimes (E_{k+1} \otimes ... \otimes E_l) &\cong M'(E_1, ..., E_l)\\ &\cong E_1 \otimes ... \otimes E_l\end{aligned}$$

□

Using this corollary, we can give a formal definition of the tensor algebra.

Definition 3.1.13. Let E be a vector space on K. If we define the vector multiplication on the following infinite direct sum

$$\otimes^* E = K \oplus E \oplus (E \otimes E) \oplus ... \oplus \otimes^k E \oplus ...$$

to be $\otimes$, then it follows from the preceding corollary that $\otimes^* E$ is an infinite-dimensional unital associative algebra. We call this the *tensor algebra* of E.

We have that $(\otimes^k E) \otimes (\otimes^l E) \subseteq \otimes^{k+l} E$. The tensor algebra is, in general, not commutative and has 1_K as its unique identity element. We will use the tensor algebra later on to give an alternative definition of a Clifford algebra. The tensor algebra has the following universal property.

Theorem 3.1.14. *Suppose that $T : E \to A$ is a linear mapping from a vector space E into a unital algebra A. Then T extends uniquely to a unital algebra homomorphism $\widetilde{T} : \otimes^* E \to A$.*

Proof. Let $(e_1, ..., e_d)$ be a basis for E. If $e_{i_1} \otimes ... \otimes e_{i_k}$ is a basis vector for $\otimes^k E$, define

$$\widetilde{T}(e_{i_1} \otimes ... \otimes e_{i_k}) = T(e_{i_1})T(e_{i_2})...T(e_{i_k})$$

If we extend this linearly, we get an algebra homomorphism of $\otimes^k E$ into A. The required map is the infinite sum of all these maps, which means that it is clearly a homomorphism extending T. Since E generates $\otimes^k E$ we also know that $\widetilde{T}$ is unique. □

Besides from taking the tensor product of vector spaces we can also take the tensor product of algebras.

Definition 3.1.15. Suppose A and B are unital algebras. The vector space tensor product $A \otimes B$ with multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

is a unital associative algebra with identity element $1_A \otimes 1_B$.

We also see that

$$ab = (a \otimes 1_B)(1_A \otimes b) = a \otimes b = (1_A \otimes b)(a \otimes 1_B) = ba \tag{3.6}$$

so the elements taken from A and B commute. To see if an algebra is isomorphic to the tensor product of two algebras we use the following proposition and corollary. These will turn out to be very useful when classifying Clifford algebras for all possible dimensions.

Proposition 3.1.16. *Suppose that F and G are subalgebras of a finite dimensional unital algebra A which generate A. If F and G commute, $fg = gf$ for $f \in F$ and $g \in G$, then there exists a unique algebra homomorphism $\phi : F \otimes G \to A$ which satisfies $\phi(f \otimes g) = fg$ for $(f, g) \in F \times G$.*

Proof. Define a mapping $\theta : F \times G \to A$ by setting $\theta(f, g) = fg$. Then θ is a bilinear mapping. From Proposition 3.1.11 we may then conclude that there exists a unique linear mapping $\phi(f \otimes g) = fg$ for $(f, g) \in F \times G$. What we see is that

$$\begin{aligned}\phi(f_1 \otimes g_1)\phi(f_2 \otimes g_2) &= (f_1 g_1)(f_2 g_2)\\ &= (f_1 f_2)(g_1 g_2)\\ &= \phi(f_1 f_2 \otimes g_1 g_2)\end{aligned}$$

This means that ϕ is an algebra homomorphism. □

Corollary 3.1.17. *The mapping ϕ is an isomorphism if and only if $dim(A) = (dim(F))(dim(G))$*

Proof. We know that $dim(F \otimes G) = (dim(F))(dim(G))$. Since $F \cup G$ generates A, we know that ϕ is surjective. This means that ϕ is an isomorphism if and only if $dim(A) = (dim(F))(dim(G))$. $\square$

We can immediately make use of the previous corollary in proving the following proposition. This proposition will also turn out to be very useful when we will eventually classify Clifford algebras.

Proposition 3.1.18. *Suppose that A is a real finite-dimensional unital algebra with identity element I. Consider $\mathbb{R}^d$ with coordinate-wise multiplication:*

$$(x_1, ..., x_d)(y_1, ..., y_d) = (x_1 y_1, ..., x_d y_d)$$

Then:

- $M_d(\mathbb{R}) \otimes A \cong M_d(A)$.
- $\mathbb{C} \otimes \mathbb{H} \cong M_2(\mathbb{C})$.
- $\mathbb{H} \otimes \mathbb{H} \cong M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \cong M_4(\mathbb{R})$.

Proof.

- Let
$$F = \{(x_{ij}I) : x_{ij} \in \mathbb{R},\ for\ 1 \leq i, j \leq d\}$$
$$\text{and } G = \{diag(a, ..., a) : a \in A\}$$

  We see that $(f_{ij}I)(h_{ij}I) = (\sum_{k=1}^{d} f_{ik}h_{kj}I) \in F$ if $(f_{ij}I), (h_{ij}I) \in F$ and $diag(a_1, ..., a_1)diag(a_2, ..., a_2) = diag(a_1a_2, ..., a_1a_2) \in G$ if $diag(a_1, ..., a_1), diag(a_2, ..., a_2) \in G$. Also, both $I_d \in F$ and $I_d \in G$. This means that both F and G are unital subalgebras of $M_d(A)$. If $(x_{ij}I) \in F$ and $diag(a, ..., a) \in G$, then $(x_{ij}I)diag(a, ..., a) = (ax_{ij}I) = diag(a, ..., a)(x_{ij}I)$ meaning that F and G commute. We also have that

$$\begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix} = diag(a_{11}, ..., a_{11}) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + ... + diag(a_{dd}, ..., a_{dd}) \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

  Together with the fact that $F \cong M_d(\mathbb{R})$, $G \cong A$ and that $dim(M_d(A)) = d^2 dim(A) = dim(F)dim(G)$, we may conclude that $M_d(\mathbb{R}) \otimes A \cong M_d(A)$

- Let $F = diag\{(z, z) : z \in \mathbb{C}\}$ and $G = H$, the subalgebra of $M_2(\mathbb{C})$ spanned by the associate Pauli matrices that we used to define the quaternions. From the definition of the algebra of quaternions it follows that $G \cong \mathbb{H}$. It is also easy to see that $F \cong \mathbb{C}$. From the fact that F is a set of diagonal matrices just like G in the item above, we conclude that F and G from this item commute as well. The subalgebra of $M_2(\mathbb{C})$ generated by $F \cup G$ contains a linearly independent set of 8 matrices $\{\pm I_2, \pm\sigma_1, \pm i\sigma_2, \pm\sigma_3\}$. Together with the fact that $dim(M_2(\mathbb{C})) = 8 = 2 \cdot 4 = (dim(F))(dim(G))$ we may conclude that $M_2(\mathbb{C})$ is generated by $F \cup G$ and thus $\mathbb{C} \otimes \mathbb{H} \cong M_2(\mathbb{C})$.

- Define the injective linear mappings $\theta_F : \mathbb{H} \to M_2(\mathbb{R}) \otimes M_2(\mathbb{R})$ and $\theta_G : \mathbb{H} \to M_2(\mathbb{R}) \otimes M_2(\mathbb{R})$ by setting

$$\theta_F(1) = I \otimes I,\ \theta_F(\mathbf{i}) = -i\sigma_1 \otimes \sigma_2,\ \theta_F(\mathbf{j}) = i\sigma_2 \otimes I,\ \theta_F(\mathbf{k}) = -i\sigma_3 \otimes \sigma_2$$
$$\theta_G(1) = I \otimes I,\ \theta_G(\mathbf{i}) = -i\sigma_2 \otimes \sigma_1,\ \theta_G(\mathbf{j}) = iI \otimes \sigma_2,\ \theta_G(\mathbf{k}) = -i\sigma_2 \otimes \sigma_3$$

  and extending by linearity. Now let F and G be the images of θ_F and θ_G respectively. It can be checked that F and G commute and we see that $dim(M_2(\mathbb{R}) \otimes M_2(\mathbb{R})) = 16 = 4 \cdot 4 = (dim(F))(dim(G))$. The set $\{\theta_F(u)\theta_G(v) : u, v \in \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}\}$ is a linear independent set containing 16 elements in $M_2(\mathbb{R}) \otimes M_2(\mathbb{R})$. From this we may conclude that $M_2(\mathbb{R}) \otimes M_2(\mathbb{R})$ is generated by $F \cup G$. This means that $\mathbb{H} \otimes \mathbb{H} \cong M_2(\mathbb{R}) \otimes M_2(\mathbb{R})$ and thus also that $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$.

$\square$

3.1.2 Quadratic forms

Before we can give a definition of a Clifford algebra, we first have to define what is called a *quadratic form*. For this subsection we will follow chapter 4 of [12]. Readers are probably familiar with Euclidean spaces. On these finite-dimensional real vector spaces an inner product is defined. An inner product is actually a special case of a quadratic form.

Definition 3.1.19. A real-valued function on E is called a *quadratic form* on E if there exists a symmetric bilinear form b on E such that $q(x) = b(x, x)$ for all $x \in E$

A vector space equipped with a quadratic form q is called a *quadratic space* (E, q). When this quadratic form is positive definite ($q(x) > 0$ for all nonzero $x \in E$), the associated bilinear form is called an inner product on E. In that case, E is an *inner-product space.* If $(e_1, ..., e_d)$ is a basis for the quadratic space (E, q) and $B = (b_{ij})$ the matrix representing the associated bilinear form b, then

$$q(x) = \sum_{i=1}^{d}\sum_{j=1}^{d} b_{ij}x_ix_j \tag{3.7}$$

Each symmetric bilinear form on E defines a quadratic form on E.

Proposition 3.1.20. *Distinct symmetric bilinear forms define distinct quadratic forms*

Proof. Suppose that $q(x) = b(x, x)$. Then

$$\begin{aligned} q(x + y) &= b(x + y, x + y) \\ &= b(x, x) + b(x, y) + b(y, x) + b(y, y) \\ &= q(x) + q(y) + 2b(x, y) \end{aligned}$$

This means that $b(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$. We call this equation the *polarization formula.* We see that q determines b uniquely via this polarization formula. □

A quadratic space is called *regular* if the bilinear form associated with the quadratic form is non-singular. On a regular quadratic space we can define *orthogonality.*

Definition 3.1.21. Suppose (E, q) is a regular quadratic space with associated bilinear form b. If $x, y \in E$, we say that x and y are *orthogonal* if $b(x, y) = 0$. We then write $x \perp y$.

Since b is symmetric, we have that $x \perp y$ is equivalent to $y \perp x$. If A is a subset of E, we define the orthogonal set $A^\perp$ by

$$A^\perp = \{x : x \perp a \; \forall a \in A\} \tag{3.8}$$

The following theorem tells us that the matrix that represents the bilinear form can always be written as a diagonal matrix.

Theorem 3.1.22. *Suppose that b is a symmetric bilinear form on a real vector space E. Let r be the rank of b. There exists a basis $(e_1, ..., e_d)$ and non-negative integers p and m, with $p + m = r$, such that if b is represented by the matrix $B = (b_{ij})$, then*

$$\begin{aligned} &b_{ii} = 1 \text{ for } 1 \leq i \leq p, \\ &b_{ii} = -1 \text{ for } p + 1 \leq i \leq p + m \text{ and} \\ &b_{ij} = 0 \text{ otherwise} \end{aligned}$$

A basis that satisfies this condition is called a standard orthogonal basis. If $m = 0$ and $r = d$, then the condition becomes $b_{ii} = 1$ for $1 \leq i \leq d$, the basis is then called an orthonormal basis.

Proof. The proof of this theorem will be done by induction on d, the dimension of E. This means that we first need to take a look at the case for $d = 0$. When $d = 0$, E only consists of its zero element so the statement is obviously true. Let us now assume that the theorem holds for all spaces of dimension less than d and all symmetric bilinear forms on them. Suppose (E, q) is a quadratic space of dimension d with associated bilinear form b. We will consider three possible cases

- The first possible case is that $q(x) = 0$ for all $x \in E$. As we have seen before, q uniquely determines b. This means that in this case, $b(x, y) = 0$ for all $x, y \in E$ and thus $p = m = r = 0$. Therefore any basis will satisfy the conclusions of the theorem.
- The second case that we will consider is that there exists $x \in E$ with $q(x) > 0$. In this case we set $e_1 = \frac{x}{\sqrt{q(x)}}$, in order that $b(e_1, e_1) = \frac{1}{\sqrt{q(x)}^2} b(x, x) = \frac{q(x)}{q(x)} = 1$. This then means that $b_{11} = 1$.
- The third case is that $q(x) \leq 0$ for all $x \in E$. This case must be different from the first so there exists $x \in E$ with $q(x) < 0$. In this case we set $e_1 = \frac{x}{\sqrt{-q(x)}}$ so that $b(e_1, e_1) = \frac{1}{\sqrt{-q(x)}^2} b(x, x) = \frac{q(x)}{-q(x)} = -1$. This then means that $b_{11} = -1$.

Let $F = \{e_1\}^{\perp}$ in both of the last two cases. To see what dimension $F^{\perp}$ has, we take a look at the isomorphism l_b from E onto E'. We have that $l_b(x)(y) = 0$ if and only if $b(x, y) = 0$. This means that we may use the results that we found for the dual spaces. We may thus state that $dim(F) = d - 1$. We also have that $span(e_1) \cap F = \{0\}$ and $E = span(e_1) + F$ so that $E = span(e_1) \oplus F$. The restriction of b to F is still a symmetric bilinear form and due to the inductive hypothesis there is a standard orthogonal basis $(e_2, ..., e_d)$ for F. If $j > 1$, then $e_j \in F$ and due to the definition of F we then have that $b(e_j, e_1) = b(e_1, e_j) = 0$. In both of the last two cases we then have that $(e_1, ..., e_d)$ is a standard orthogonal basis for E. From the fact that this basis is orthogonal it immediately follows that the rank of b is equal to $p + m$. □

If $(e_1, ..., e_d)$ is such a standard orthogonal basis, $x = \sum_{i=1}^{d} x_i e_i$ and $y = \sum_{i=1}^{d} y_i e_i$, then

$$b(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+m} x_i y_i \text{ and } q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+m} x_i^2 \tag{3.9}$$

Theorem 3.1.23 (Sylvester's law of inertia)**.** *Suppose that* $(e_1, ..., e_d)$ *with parameter* (p, m) *and* $(f_1, ..., f_d)$ *with parameter* (p', m') *are standard orthogonal bases for a quadratic space* (E, q)*. Then* $p = p'$ *and* $m = m'$.

Proof. Let $U_1 = span\{e_1, ..., e_p\}$ and $W_1 = span\{f_{p'+1}, ..., f_d\}$. The restriction of q to U_1 is then positive definite and the restriction of q to W_1 is negative semi-definite, $q(w) \leq 0$ if $w \in W_1$. This means that $U_1 \cap W_1 = \{0\}$ and thus

$$p + (d - p') = dim(U_1) + dim(W_1) \leq d = dim(E)$$

We can rewrite this into $p \leq p'$.
Now let $U_2 = span\{e_{p+1}, ..., e_d\}$ and $W_2 = span\{f_1, ..., f_{p'}\}$. The restriction of q to U_2 is then negative semi-definite and the restriction of q to W_2 is positive definite. This means that $U_2 \cap W_2 = \{0\}$ and thus

$$p' + (d - p) = dim(U_2) + dim(W_2) \leq d = dim(E)$$

We can rewrite this into $p' \leq p$. We now have that $p' \leq p$ and that $p \leq p'$, which means that $p = p'$. From the fact that $p + m = r = p' + m'$ it now also follows that $m = m'$. □

We call (p, m) the *signature* of q. Sylvester's law of inertia showed us that this signature is invariant under basis transformation. There are certain spaces that have a distinct signature, in the following table we will name a few.

Table 1: Spaces with a distinct signature

$p = d, m = 0$	Euclidean space
$min(p, m) \geq 0$	Minkowski space
$p = d - 1, m = 1$	Lorentz space
$p = m, d = 2p$	Hyperbolic space

Furthermore, suppose p and m are non-negative integers with $p+m=d$. Let $\mathbb{R}_{p,m}$ denote $\mathbb{R}^d$ equipped with the quadratic form

$$q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+m} x_i^2 \tag{3.10}$$

We call $\mathbb{R}_{p,m}$ the *standard regular quadratic space* with dimension d and signature (p, m).

3.2 Clifford algebras

As explained in [33], Clifford formed what are now called Clifford algebras in 1878, even though he referred to them as geometric algebras. He did this by expanding on Grassman's algebraic work and looking at the quaternions of Hamilton. The development of the geometric algebra would lead to 20th century mathematicians to formalize and explore the properties of Clifford algebras. As almost two centuries of formalizing and exploring the properties of Clifford algebras has passed, the theory around Clifford algebra has become quite advanced. In this section we will give a formal definition of a Clifford algebra and deal with some of its properties. It is of course impossible to deal with all the properties of the Clifford algebras in this section, so we will only focus on the properties that are needed for a better understanding of the gamma matrices and the ones that are needed to classify Clifford algebras for all possible dimensions. In this section we shall suppose that (E, q) is a d-dimensional real vector space E with a quadratic form q, associated bilinear form b and standard orthogonal basis $(e_1, ..., e_d)$. For this section we will follow chapters 2, 3, 4 and 5 of [12].

Definition 3.2.1. Suppose that A is a unital algebra. A *Clifford mapping* j is an injective linear mapping $j : E \to A$ that satisfies

- $1 \notin j(E)$
- $(j(x))^2 = -q(x)1 = -q(x)$ for all $x \in E$

If j also has the property that $j(E)$ generates A, then A together with the mapping j is called a *Clifford algebra* for (E, q).

In this unital algebra A we can identify $\mathbb{R}$ with $span(1)$ and call these the *scalars* in A. We can also identify E with $j(E)$ making E a linear subspace of A. We call the elements of E the *vectors* in A. If j is a Clifford mapping and $x, y \in E$, then we have that

$$\begin{aligned} j(x)j(y) + j(y)j(x) &= j(x+y)^2 - j(x)^2 - j(y)^2 \\ &= (-q(x+y) + q(x) + q(y))1 \\ &= -2b(x,y)1 \end{aligned} \tag{3.11}$$

In particular we have that $j(x)j(y) = -j(y)j(x)$ if $x \perp y$. The Minkowski metric that we used to describe the relations between the gamma matrices from chapter 2 is actually an example of a bilinear form. We see that if we plug in $b(x, y) = -\eta(x, y)$, we retrieve Equation 2.27 that we found in the derivation of the Dirac equation. In proving that an algebra is a Clifford algebra, the following elementary result will be very useful.

Theorem 3.2.2. *Suppose that $a_1, ..., a_d$ are elements of a unital algebra A and $(e_1, ..., e_d)$ a standard orthogonal basis for the quadratic space (E, q). Then there exists a unique Clifford mapping $j : (E, q) \to A$ that satisfies $j(e_i) = a_i$ for $1 \leq i \leq d$ if and only if*

$$\begin{aligned} a_i^2 &= -q(e_i) \; for \; 1 \leq i \leq d, \\ a_i a_j + a_j a_i &= 0 \; for \; 1 \leq i < j \leq d, \\ and \; 1 &\notin span(a_1, ..., a_d) \end{aligned}$$

Proof. First we will prove that the conditions follow from the fact that j is a Clifford mapping with $j(e_i) = a_i$ for $1 \leq i \leq d$. Since j is a Clifford mapping it immediately follows that $1 \notin span(a_1, ..., a_d)$, $a_i^2 = j(e_i)^2 = -q(e_i)$ and $a_i a_j + a_j a_i = j(e_i)j(e_j) + j(e_j)j(e_i) = 0$ for $i \neq j$, as $e_i \perp e_j$.

To prove the other way around we consider $x = x_1e_1 + ... + x_de_d \in E$ and set $j(x) = x_1a_1 + ... + x_da_d$. We then see that j is an injective linear mapping from E into A with $1 \notin j(E)$. We also see that

$$j(x)^2 = \sum_{i=1}^{d} x_i^2 a_i^2 + \sum_{1 \leq i < j \leq d} x_i x_j (a_i a_j + a_j a_i) = -\sum_{j=1}^{d} x_i^2 q(e_i) = -q(\sum_{i=1}^{d} x_i e_i) = -q(x)$$

This means that $j : (E, q) \to A$ is a Clifford mapping. The uniqueness of j follows from the fact that $E = span(e_1, ..., e_d)$. □

One of the most important properties that a Clifford algebra can have is that it can be *universal*. To define this property we first have to define what is called an *isometry*.

Definition 3.2.3. Let us consider a linear mapping T from the quadratic space (E_1, q_1) to (E_2, q_2). This linear mapping T is an *isometry* if it satisfies

- T is injective
- $q_2(T(x)) = q_1(x)$ for all $x \in E_1$

A Clifford algebra $A(E, q)$ is then said to be universal if whenever $T \in L(E, F)$ is an isometry of (E, q) into (F, r) with $B(F, r)$ a Clifford algebra for (F, r), then T extends to an algebra homomorphism $\tilde{T} : A(E, q) \to B(F, r)$. This is illustrated in the following diagram.

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, "\subset"'] & F \arrow[d, "\subset"] \\
A(E,q) \arrow[r, dashed, "\tilde{T}"] & B(F,r)
\end{tikzcd}$$

A universal Clifford algebra for (E, q) is denoted by $A(E, q)$. Since $A(E, q)$ is generated by E, we know that $\tilde{T}$ is unique. If we use the identity mapping as the isometry, we see that a universal Clifford algebra for (E, q) is unique.

Let us now take a look at a basis for Clifford algebras. Suppose $(e_1, ..., e_d)$ is a standard orthogonal basis for the quadratic space (E, q). We set $\Omega = \Omega_d = \{1, ..., d\}$. If $C = \{i_1, ..., i_k\}$ with $1 \leq i_1 < ... < i_k \leq d$, then we define the element e_C to be the product $e_C = e_{i_1} ... e_{i_k}$, which is taken inside the Clifford algebra. We set $e_\emptyset = 1$. Note that e_C will depend on the ordering of $\{1, ..., d\}$ if $|C| > 1$. The element $e_\Omega = e_1 ... e_d$, which we call the *volume element*, will be particularly important. From chapter 2 we recognize $-i\gamma^5$ as the volume element of the algebra of space-time. Note that $e_\Omega^2 = (-1)^{\sum_{i=1}^{d-1} i} \prod_{j=1}^{d} q(e_j)$. This means that $e_\Omega^2 = (-1)^\eta$ with $\eta = \frac{1}{2}d(d-1) + p$. To see when e_Ω^2 is either 1 or -1 we look at two cases:

- Suppose that $d = 2k$ is even and that $p = k + t$ and $m = k - t$. Then $\eta = k(2k-1) + k + t = 2k^2 + t$ so $\eta \pmod 2 \equiv t \pmod 2$. If $p - m \equiv 0 \pmod 4$, then $p - m = k + t - (k - t) = 2t = 4l$ for some $l \in K$. This means that then $t = 2l$, implying that $e_\Omega^2 = 1$. If $p - m \equiv 2 \pmod 4$, then $p - m = 2t = 4l + 2$ for some $l \in K$. This means that in that case $t = 2l + 1$, which means that $e_\Omega^2 = -1$.
- Now suppose that $d = 2k+1$ is odd and $p = k+t$ and $m = k-t+1$. Then $\eta = k(2k+1)+k+t = 2k(k+1)+t$, so again $\eta \pmod 2 \equiv t \pmod 2$. If $p-m \equiv 1 \pmod 4$, then $p-m = k+t-(k-t+1) = 2t-1 = 4l+1$ for some $l \in K$. This means that we then have that $t = 2l + 1$ so $e_\Omega^2 = -1$. If $p - m \equiv 3 \pmod 4$, then $p - m = 2t - 1 = 4l + 3$ for some $l \in K$. This means that in that case $t = 2(l + 1)$, implying that $e_\Omega^2 = 1$.

Theorem 3.2.4. *Let $P = \{e_C : C \subseteq \Omega\}$. Suppose that A is a Clifford algebra for (E, q), then $A = span(P)$. If P is linearly independent, then A is universal and the elements of P form a basis for A.*

Proof. What we know is that a Clifford algebra A for (E,q) is generated by all the elements of the basis of (E,q). This Clifford algebra A is therefore the span of all the infinite amount of possible products between these elements. We thus need to check if the set P contains enough elements to span A. We know that e_i^2 is either 0, 1 or -1 and that $e_ie_j = -e_je_i$ for $i \neq j$. From this it follows that if $e_C, e_D \in P$, then either $e_Ce_D = 0$ or $e_Ce_D = \pm e_{C\Delta D}$, where $C\Delta D = (C \setminus D) \cup (D \setminus C)$. This means that the product of two elements of P will end up in either P or $-P$ so we may conclude that P is large enough to span A.
Suppose P is linearly independent and let $T \in L(E;F)$ be an isometry of (E,q) into (F,r) with $B(F,r)$ a Clifford algebra for (F,r). We can extend T into an algebra homomorphism $\widetilde{T} : A \to B$ by setting

$$\widetilde{T}(e_{i_1}...e_{i_k}) = T(e_{i_1})...T(e_{i_k})$$

and then extending this by linearity. From this we may conclude that A is then universal. The fact that elements of P form a basis for A immediately follows from the fact that $A = span(P)$ and that P is linearly independent. ☐

Corollary 3.2.5. *If $dim(A) = 2^d$, then A is universal.*

Proof. From Theorem 3.2.4 we know that $A = span(P)$. This means that the dimension of A must be equal to the amount of elements in P minus the amount of elements in P that can be written as the linear combination of other elements in P. If we take $e_C \in P$, we have for every element of Ω that it is either in $C = \{i_1, ..., i_k\}$ or not in C. This means that there 2^d different ways to define e_C. The amount of elements in P is thus equal to 2^d. We therefore find that P must be linearly independent if $dim(A) = 2^d$ and from Theorem 3.2.4 we may then conclude that A is universal. ☐

3.2.1 Existence of a universal Clifford algebra

We want to show that for every quadratic space there exists a universal Clifford algebra. To do so, we first have to define what is called a $\mathbb{Z}_2$-*graded algebra* or *super-algebra*. For this subsection we will follow chapters 2, 3 and 5 of [12].

Definition 3.2.6. A *super-algebra* is a unital algebra A that can be decomposed into a so-called *even* A^+ and an *odd* part A^-, $A = A^+ \oplus A^-$. For these even and odd parts the following must hold

$$A^+A^+ \subseteq A^+, A^-A^- \subseteq A^+, A^-A^+ \subseteq A^- \text{ and } A^-A^+ \subseteq A^-$$

To define this even and odd part we will take a look at *involutions*.

Definition 3.2.7. An *involution* is a function f that is its own inverse, i.e. $f(f(x)) = x$ for all x in the domain of f.

If θ is an involution of a unital algebra A onto itself, then we find that $p = (I + \theta)/2$ is an *idempotent*. This means that $p^2 = p$. If $a \in A$, then $a = \frac{I(a)+\theta(a)}{2} + \frac{I(a)-\theta(a)}{2}$ so that $A = p(A) + (I - p)(A)$. We also have that $p(A) \cap (I - p)(A) = \{0\}$. Using this we can write $A = A^+ \oplus A^-$, where $A^+ = p(A)$ and $A^- = (I - p)(A)$. We then have that $p(A)$ is a subalgebra of A and

$$A^+ = \{a \in A : \theta(a) = a\} \text{ and } A^- = \{a \in A : \theta(a) = -a\} \tag{3.12}$$

These even and odd parts satisfy the relations in Definition 3.2.6. If $a \in A^+ \cup A^-$ we say that a is *homogeneous*. When A and B are super-algebras we can define another law of multiplication called the *graded tensor product*.

Definition 3.2.8. Let $A = A^+ \oplus A^-$ and $B = B^+ \oplus B^-$, then the *graded tensor product* is defined as

$$(a_1 \otimes b_1)_g(a_2 \otimes b_2) = \begin{cases} -a_1a_2 \otimes b_1b_2 & \text{if } b_1 \in B^- \text{ and } a_2 \in A^- \\ a_1a_2 \otimes b_1b_2 & \text{otherwise} \end{cases}$$

for a_1, a_2 homogeneous in A and b_1, b_2 homogeneous in B.

We write $A \otimes_g B$ for $A \otimes B$ with the graded tensor product as the law of multiplication and write $a \otimes_g b$ for the elementary tensors in it.

Theorem 3.2.9. *Suppose that the quadratic space (E, q) is the orthogonal direct sum $(E_1, q_1) \oplus (E_2, q_2)$ of quadratic spaces where q_1 is the restriction of q to E_1 and q_2 the restriction of q to E_2. If, also, there exists a universal Clifford algebra $A(E_1, q_1)$ for (E_1, q_1) and a universal Clifford algebra $A(E_2, q_2)$ for (E_2, q_2), then $G \cong A(E_1, q_1) \otimes_g A(E_2, q_2)$ is a universal Clifford algebra.*

Proof. For $x_1 \in E_1$ and $x_2 \in E_2$ define the linear mapping j from E to G by $j(x_1+x_2) = (x_1 \otimes_g 1)+(1 \otimes_g x_2)$. We see that j is an injective linear mapping with $1 \notin j(E)$. As $x_1 \in A^-(E_1, q_1)$ and $x_2 \in A^-(E_2, q_2)$ we have that

$$\begin{aligned} j(x_1 + x_2)^2 &= ((x_1 \otimes_g 1) + (1 \otimes_g x_2))_g((x_1 \otimes_g 1) + (1 \otimes_g x_2)) \\ &= (x_1 \otimes_g 1)_g(x_1 \otimes_g 1) + (x_1 \otimes_g 1)_g(1 \otimes_g x_2) + (1 \otimes_g x_2)_g(x_1 \otimes_g 1) + (1 \otimes_g x_2)_g(1 \otimes_g x_2) \\ &= -q_1(x_1) + (x_1 \otimes_g x_2) - (x_1 \otimes_g x_2) - q_2(x_2) = -q(x) \end{aligned}$$

So j is a Clifford mapping from (E, q) into G. Furthermore, G is generated by elements of the form $x \otimes_g y = (x \otimes_g 1)(1 \otimes_g y)$. Since $j(x_1 + 0) = x_1 \otimes_g 1$ and $j(0 + x_2) = 1 \otimes_g x_2$, we find that $j(x_1 + 0)j(0 + x_2) = x_1 \otimes_g x_2$ and thus that G is generated by $j(E_1 + E_2)$. This means that G is a Clifford algebra for (E, q). We also have that

$$dim(A) = dim(A(E_1, q_1))dim(A(E_2, q_2)) = 2^{d_1}2^{d_2} = 2^d$$

so from Corollary 3.2.5 we may then conclude that G is a universal Clifford algebra for (E, q). □

The theorem above is not used very often, as the the graded tensor product is not very practical in solving problems. However, it is very useful in proving the next important theorem.

Theorem 3.2.10. *If (E, q) is a quadratic space, then there exists a universal Clifford algebra $A(E, q)$.*

Proof. We will prove this theorem using induction. First we take a look at the case $d = 1$. In this case there are three possibilities

- The basis element e_1 satisfies $q(e_1) = 0$. The universal Clifford algebra in this case is actually the so-called *exterior algebra* introduced by Hermann Grassmann. We will not explore the exterior algebra in this thesis, but if one is interested in the mathematical description of this algebra, see section 3.4 of [12].

- Another possibility is that the basis element e_1 satisfies $q(e_1) = 1$. In this case, let $A = \mathbb{C}$ and $j(\lambda e_1) = \lambda i$. We see that $(j(\lambda e_1))^2 = -\lambda^2 = -q(\lambda e_1)$ and that $1 \notin j(E)$, implying that j is a Clifford mapping. As $\mathbb{C}$ is generated by $\{1, i\}$, we see that $j(E)$ generates $\mathbb{C}$. This means that $\mathbb{C}$ is a Clifford algebra for (E, q). Since $dim(\mathbb{C}) = 2$, we may conclude that this Clifford algebra is universal.

- The last possibility is that e_1 satisfies $q(e_1) = -1$. Let $A = \mathbb{R}^2$ with coordinate-wise multiplication and $j(\lambda e_1) = (\lambda, -\lambda)$. We see that $1 \notin j(E)$ and that $(j(\lambda e_1))^2 = \lambda^2(1, 1) = -q(\lambda e_1)1$. This means that j is a Clifford mapping of E into $\mathbb{R}^2$. Furthermore, for every $(a, b) \in \mathbb{R}^2$ we have that

$$\begin{aligned} (max(a,b) - \frac{|a-b|}{2})j(\lambda e_1)j(\frac{1}{\lambda}e_1) + j(\frac{|a-b|}{2}e_1) &= (max(a,b) - \frac{|a-b|}{2}, max(a,b) - \frac{|a-b|}{2}) \\ &\quad + (\frac{|a-b|}{2}, -\frac{|a-b|}{2}) \\ &= (max(a,b), max(a,b) - |a-b|) = (a,b) \end{aligned}$$

  if $a \geq b$ and

$$\begin{aligned} (min(a,b) + \frac{|a-b|}{2})j(\lambda e_1)j(\frac{1}{\lambda}e_1) - j(\frac{|a-b|}{2}e_1) &= (min(a,b) + \frac{|a-b|}{2}, min(a,b) + \frac{|a-b|}{2}) \\ &\quad - (\frac{|a-b|}{2}, -\frac{|a-b|}{2}) \\ &= (min(a,b), min(a,b) + |a-b|) = (a,b) \end{aligned}$$

if $b > a$. This means that $j(E)$ generates $\mathbb{R}^2$ and thus that $\mathbb{R}^2$ is a Clifford algebra for (E, q). Since $dim(\mathbb{R}^2) = 2$, we may conclude that $\mathbb{R}^2$ is a universal Clifford algebra for (E, q).

Assume now that the result is true for all quadratic spaces with a dimension less than $dim(E) = d$, with $d > 1$. Because $d > 1$ we can write E as an orthogonal direct sum $E_1 \oplus E_2$, with $dim(E_1) = d_1 < d$ and $dim(E_2) = d_2 < d$. By the inductive hypothesis there exist universal Clifford algebras $A(E_1, q_1)$ and $A(E_2, q_2)$, with q_1 and q_2 the restrictions of q to E_1 and E_2 respectively. From Theorem 3.2.9 we may then conclude that the graded product $A(E_1, q_1) \otimes_g A(E_2, q_2)$ is a universal Clifford algebra for (E, q). □

Corollary 3.2.11. *A Clifford algebra A of the quadratic space (E, q) is universal if and only if $dim(A) = 2^d$.*

Proof. As we have already proven that $dim(A) = 2^d$ implies that A is universal, we only have to prove it the other way around. We will prove the other way around using induction. We start with the case where $d = 1$. In this case we see that a universal Clifford algebra of (E, q) is either $\mathbb{C}$, $\mathbb{R}^2$ or the exterior algebra. In all these three cases the dimension of the Clifford algebra is 2 [12] so for $d = 1$ the theorem holds.
Suppose now that (E, q) is a quadratic space with dimension d. We can then write E as an orthogonal direct sum $E_1 \oplus E_2$, with $dim(E_1) = d_1 < d$ and $dim(E_2) = d_2 < d$. We now know that for these quadratic spaces there exists universal Clifford algebras $A(E_1, q_1)$ and $A(E_2, q_2)$. From Theorem 3.2.9 we may then conclude that $A(E, q) \cong A(E_1, q_1) \otimes_g A(E_2, q_2)$. From the inductive hypothesis it then follows that $dim(A(E, q)) = 2^{d_1} 2^{d_2} = 2^d$. □

A universal Clifford algebra for a standard regular quadratic space with signature (p, m) will be denoted by $A_{p,m}$. An interesting property of universal Clifford algebras is that we can consider them as a quotient of $\otimes^* E$. To do so, we have to define the ideal I_q. This is the ideal in $\otimes^* E$ generated by all elements of the form

$$x \otimes x + q(x)1 \text{ for all } x \in E \tag{3.13}$$

With this ideal we can give the following alternative definition of a universal Clifford algebra $A(E, q)$.

Proposition 3.2.12. *A universal Clifford algebra A for a quadratic space (E, q) can be defined as*

$$A(E, q) = (\otimes^* E)/I_q$$

Proof. Let $C(E, q) = (\otimes^* E)/I_q$ be the quotient algebra and $\pi : \otimes^* E \to C(E, q)$ the quotient mapping. Also define i to be the inclusion mapping from E into $\otimes^* E$ and let $j_B = \pi \circ i$. We see that

$$(j_B(x))^2 = \pi(i(x)i(x)) = \pi(x \otimes x) = \pi(x \otimes x + q(x)1) - \pi(q(x)1) = -q(x)$$

We also have that $\pi^{-1}(1) = 1 + I_q$. From the fact that $I_q \cap E = \{0\}$ it then follows that $i(E) \cap \pi^{-1}(1) = \{0\}$ and thus that $1 \notin j_B(E)$. This means that j_B is a Clifford mapping from E into $C(E, q)$ and thus that $C(E, q)$ is a Clifford algebra for (E, q).
We will now prove that $C(E, q)$ is a universal Clifford algebra for (E, q). Suppose that we have an isometry T from E into another quadratic space (F, r) and that $B(F, r)$ is a Clifford algebra for (F, r). Denote the Clifford mapping from (F, r) into $B(F, r)$ by j_A. Since $f = j_A \circ T$ is a linear mapping from E into the unital algebra $B(F, r)$, we may conclude from Theorem 3.1.14 that f extends uniquely to a unital algebra homomorphism $\tilde{f} : \otimes^* E \to B(F, r)$. The way we defined this extension in the proof of Theorem 3.1.14 is the following

$$\tilde{f}(e_{i_1} \otimes ... \otimes e_{i_k}) = f(e_{i_1})...f(e_{i_k})$$

From the fact that T is an isometry it then follows that

$$\tilde{f}(x \otimes x + q(x)) = f(x)f(x) + q(x) = j_A^2(T(x)) + r(T(x)) = 0$$

We thus see that $I_q \subseteq ker(\tilde{f})$. From the *universal property of the quotient* (see Proposition 2.1.26 of [23]) it then follows that there exists a unique algebra homomorphism $\tilde{f}' : C(E, q) \to B(F, r)$ such that $\tilde{f}' \circ \pi = \tilde{f}$. This means that every isometry of (E, q) into another quadratic space (F, r) extends to an algebra homomorphism of $C(E, q)$ into $B(F, r)$, with $B(F, r)$ a Clifford algebra for (F, r). We may therefore conclude that $C(E, q)$ is a universal Clifford algebra for (E, q). □

Let us now take a look at the concept of even and odd algebras for universal Clifford algebras $A(E,q)$. Let $m(x) = -x$ for $x \in E$. The mapping m is an isometry of E onto itself ($q(-x) = (-1)^2 q(x) = q(x)$). As $A(E,q)$ is universal, m extends to an algebra homomorphism $\widetilde{m} : A(E,q) \to A(E,q)$ in a natural way by defining:

$$\widetilde{m}(e_{i_1}...e_{i_k}) = m(e_{i_1})...m(e_{i_k})$$

and then extending this by linearity. We write a' for $\widetilde{m}(a)$. We know that every element of $A(E,q)$ can be written in a unique way as a finite linear combination of the elements of the basis for $A(E,q)$. From the definition of $\widetilde{m}$ it then follows that $a = b$ if $b' = a'$ for $b', a' \in A(E,q)$. The mapping $a \mapsto a'$ is therefore injective and thus an isomorphism of $A(E,q)$ onto itself. Note that $e'_C = (-1)^{|C|} e_C$ and that $a'' = a$. We thus have that the mapping $\widetilde{m}$ is an involution. We call this involution the *principal involution*. Let us now set

$$A^+ = \{a : a = a'\} \text{ and } A^- = \{a : a = -a'\} \tag{3.14}$$

We then have that $A = A^+ \oplus A^-$ and that A^+ is a subalgebra of A. We call this subalgebra the *even Clifford algebra*. Furthermore,

$$A^+A^- = A^-A^- = A^+ \text{ and } A^+A^- = A^-A^+ = A^- \tag{3.15}$$

This means that every universal Clifford algebra can be decomposed into an even and an odd part and is thus a super-algebra. We also see that $e_C \in A^+$ if $|C|$ is even and that $e_C \in A^-$ if $|C|$ is odd. This means that all elements of the basis $(e_1, ..., e_d)$ are in A^-, implying that $j(E) \subseteq A^-$.

3.2.2 Simplicity

Even though the most interesting Clifford algebras are universal, it also interesting to look at the cases in which we encounter non-universal Clifford algebras. For this subsection we will follow section 5.5 of [12]. If (E,q) is not regular, then $e_\Omega^2 = 0$. We then have that $A(E,q)e_\Omega$ is an ideal in $A(E,q)$ and because $1_A \notin A(E,q)e_\Omega$ we see that this ideal is actually a proper ideal. Let $B(E,q) = A(E,q)/A(E,q)e_\Omega$ and $q : A(E,q) \to B(E,q)$ the quotient mapping. If $dim(E) > 1$, then there exists a Clifford mapping j from (E,q) into $A(E,q)$. Now define $j_B = q \circ j$. For $a \in A(E,q)$ we have that $1 + ae_\Omega \notin j(E)$. This means that $1 \notin j_B(E)$. We also have that

$$(j_B(x))^2 = \pi(j(x))\pi(j(x)) = \pi(j(x)^2) = -q(x) \tag{3.16}$$

This means that j_B is a Clifford mapping from (E,q) into $B(E,q)$ and thus that $B(E,q)$ is a Clifford algebra for (E,q). We also have that $dim(B(E,q)) < 2^d$, implying that $B(E,q)$ is a non-universal Clifford algebra.
Now that we know that there exist non-universal Clifford algebras for (E,q) when (E,q) is not regular, let us look at the cases in which there are non-universal Clifford algebras when (E,q) is regular and has signature (p,m). Suppose A is a Clifford algebra for (E,q), where $dim(E,q) > 0$. We then we have the following diagram.

$$\begin{tikzcd} E \arrow[r, "Id"] \arrow[d, "\subset"'] & E \arrow[d, "\subset"] \\ A(E,q) \arrow[r, dashed, "\tilde{Id}"] & A \end{tikzcd}$$

As $A(E,q)$ is universal, we know that $\tilde{Id}$ is an algebra homomorphism of $A(E,q)$ into A. Let us take a look at $ker(\tilde{Id})$. If we take $a \in A(E,q)$ and $b \in ker(\tilde{Id})$, then $\tilde{Id}(ab) = \tilde{Id}(ba) = \tilde{Id}(b)\tilde{Id}(a) = 0$, implying that $ab, ba \in ker(\tilde{Id})$. This means that $ker(\tilde{Id})$ is an ideal in $A(E,q)$. If $A(E,q)$ is simple, then we know that either $ker(\tilde{Id}) = 0$ or $ker(\tilde{Id}) = A(E,q)$. From the assumption that $(E,q) \neq \{0\}$ it follows that $ker(\tilde{Id}) = A(E,q)$ is not possible. We therefore must have that $ker(\tilde{Id}) = 0$ and thus that $\tilde{Id}$ is injective. From this it follows that $dim(A(E,q)) \leq dim(A)$. From the proof of Corollary 3.2.5 we may conclude that $dim(A(E,q)) \geq A$, which means that $dim(A(E,q)) = dim(A)$. This means that $\tilde{Id}$ is an isomorphism if $A(E,q)$ is simple and thus that every Clifford algebra for (E,q) is universal.
Let us now take a look at the cases in which the universal Clifford algebra $A(E,q)$ for the regular quadratic

space (E, q) is simple. Since a universal Clifford algebra for every quadratic space is unique, we have that $(A, q) \cong A_{p,m}$. It is therefore sufficient to only consider the algebras $A_{p,m}$.

Theorem 3.2.13. *If $p - m \not\equiv 3 \pmod 4$, then $A_{p,m}$ is simple. This means that all Clifford algebras for $\mathbb{R}_{p,m}$ are universal.*

Proof. Suppose J is a proper ideal of $A_{p,m}$. As J is a proper subset of $A_{p,m}$, it must have a dimension of at most $2^d - 1$. This means that all $A_{p,m}$ with $dim(A_{p,m}) < 2$ are simple. For the rest of our proof we may thus assume that $d \geq 1$. Suppose I is a non-zero ideal in $A_{p,m}$. The technique that we are going to use for this proof is proof by contradiction. Let x be a non-zero element in I with a minimal number of non-zero coefficients in its expansion with respect to the basis $\{e_C : C \subseteq \Omega\}$. By multiplying and scaling we can write

$$x = 1 + \sum_{C \in R} \lambda_C e_C$$

where R is a set of nonempty subsets of Ω. With our contradiction we will prove that R is either the empty set or only contains Ω. Suppose that $B \in R$ and that $B \neq \Omega$, this means that there exist $i \in B$ and $j \notin B$. Then

$$\begin{aligned} e_i e_j x e_i e_j &= e_i e_j e_i e_j + \sum_{C \in R} e_i e_j \lambda_C e_C e_i e_j \\ &= -q(e_i)q(e_j) + \sum_{C \in R} \mu_C e_C \end{aligned}$$

where $\mu_C = \pm q(e_i)q(e_j)\lambda_C$. However, as $e_i e_j = -e_j e_i$ for $i \neq j$, we have that

$$e_i e_j e_B e_i e_j = (-1)^{|B|} q(e_i) e_j e_B e_j$$

with $|B|$ the amount of elements in B. This means that for μ_B we find that $\mu_B = (-1)^{2|B|} q(e_i)q(e_j)\lambda_B = q(e_i)q(e_j)\lambda_B$. As $q(e_i), q(e_j), e_i, e_j \in A_{p,m}$, we know that $q(e_i)q(e_j)x \in I$ and $e_i e_j x e_i e_j \in I$. Together with the fact that I is closed under addition, we have that $q(e_i)q(e_j)x - e_i e_j x e_i e_j \in I$. If we try to write the expansion of $q(e_i)q(e_j)x - e_i e_j x e_i e_j$ with respect to the basis $\{e_C : C \subseteq \Omega\}$, we find that

$$\begin{aligned} q(e_i)q(e_j)x - e_i e_j x e_i e_j &= q(e_i)q(e_j) - (-q(e_i)q(e_j)) + \sum_{C \in R} q(e_i)q(e_j)\lambda_C e_C - (\sum_{C \in R} \pm q(e_i)q(e_j)\lambda_C e_C) \\ &= 2q(e_i)q(e_j) + \sum_{C \in R \setminus B} (q(e_i)q(e_j)\lambda_C e_C - \pm q(e_i)q(e_j)\lambda_C e_C) + q(e_i)q(e_j)\lambda_B e_B - q(e_i)q(e_j)\lambda_B e_B \\ &= 2q(e_i)q(e_j) + \sum_{C \in R \setminus B} (q(e_i)q(e_j)\lambda_C e_C - \pm q(e_i)q(e_j)\lambda_C e_C) \end{aligned}$$

What we see is that $q(e_i)q(e_j)x - e_i e_j x e_i e_j$ is a non-zero element of I with fewer non-zero terms than x, which of course gives a contradiction. This means that we can write $x = 1 + \lambda_\Omega e_\Omega$. Let us now consider the two possible cases.

- Suppose that d is even. Then $e_1 x e_1 = -q(e_1)1 + (-1)^d q(e_1)\lambda_\Omega e_\Omega = -q(e_1)(1 - \lambda_\Omega e_\Omega)$. This means that $q(e_1)x - e_1 x e_1 = 2q(e_1)1 \in I$. We know that $q(e_1) \neq 0$, so $I = A_{p,m}$.

- Suppose that $p - m \equiv 1 \pmod 4$. In that case $e_\Omega^2 = -1$ meaning that $x(1 - \lambda_\Omega e_\Omega) = (1 + \lambda_\Omega^2)1$. As $\lambda_\Omega^2 \neq -1$, we may conclude that $I = A_{p,m}$.

In both cases we see that $A_{p,m}$ is simple. Therefore when $p - m \not\equiv 3 \pmod 4$, all the Clifford algebras for $\mathbb{R}_{p,m}$ are universal. $\square$

We can find corresponding results for even Clifford algebras if we combine the above theorem and Theorem 3.3.4, which we will touch upon later.

Corollary 3.2.14. *If $p \not\equiv m \pmod 4$, then $A^+_{p,m}$ is simple.*

Let us now take a look at the case where $p - m \equiv 3 \pmod 4$.

Theorem 3.2.15. *If $p - m \equiv 3$ (mod 4), then $A_{p,m}$ is not simple and*

$$A_{p,m} \cong A^+_{p,m} \oplus A^+_{p,m}$$

There exists a non-universal Clifford algebra $B_{p,m}$ for $\mathbb{R}_{p,m}$, and any such algebra is isomorphic to $A^+_{p,m}$.

Proof. Let us denote $A_{p,m}$ by A in this proof. When $p - m \equiv 3 \pmod 4$ we have that $e_\Omega^2 = 1$. From this it follows that $f = \frac{1}{2}(1 + e_\Omega)$ and $g = \frac{1}{2}(1 - e_\Omega)$ are idempotents satisfying $f + g = 1$ and $fg = 0$. We thus have that $a = a(\frac{1}{2}(1 + e_\Omega) + \frac{1}{2}(1 - e_\Omega))$ if $a \in A$. We also have that $Af \cap Ag = \{0\}$. This means that A can be written as $Af \oplus Ag$, where Af and Ag are unital algebras with identity elements f and g respectively.
Let us define the mapping $m_f : a \mapsto af$ from A into Af. We see that this mapping is an algebra homomorphism and since $a = af + ag$ we also see that Ag is its null-space. Let j be the Clifford mapping from $\mathbb{R}_{p,m}$ into A. Define $l : \mathbb{R}_{p,m} \to Af$ to be the mapping $x \mapsto j(x)f$. Since Ag is the null-space of m_f, we have that $l^{-1}(f) = 1 + Ag$. From the fact that $1 + Ag \cap j(\mathbb{R}_{p,m}) = \{0\}$ it then follows that $f \notin l(\mathbb{R}_{p,m})$. We also have that

$$(j(x)f)^2 = j(x)^2 f^2 = -q(x)f$$

This means that l is a Clifford mapping. We also have that $j(\mathbb{R}_{p,m})$ generates Af so that Af is a Clifford algebra for $\mathbb{R}_{p,m}$. To see if Af is a universal Clifford algebra for $\mathbb{R}_{p,m}$ we have to take a look at the dimension of Af.
Since $d = p + m$ is odd, we have that $ae_\Omega \in A^+$ if $a \in A^-$ and $ae_\Omega \in A^-$ if $a \in A^+$. In both cases we see that $\frac{1}{2}(a' + (ae_\Omega)') \in Ag$. This means that $\widetilde{m}|_{Af}$ is an algebra homomorphism from Af into Ag, where $\widetilde{m}$ is the extension of the principal involution. What we see is that $\widetilde{m}|_{Af}^{-1}(0) = 0' = 0$. We thus have that $ker(\widetilde{m}|_{Af}) = 0$, implying that $\widetilde{m}|_{Af}$ is injective. Suppose $l \in Ag$ and $l \neq 0$. We then have that $l = \frac{1}{2}(a - ae_\Omega)$ for some a that is either an element of A^+ or A^-. In both cases we have that $\widetilde{m}|_{Af}^{-1}(l) = l' = \frac{1}{2}(a' - (ae_\Omega)') \in Af$. We may thus conclude that $\widetilde{m}|_{Af}$ is also surjective and therefore an isomorphism. The dimension of Af is thus equal to the dimension of Ag. Together with the fact that $dim(A) = dim(Af) + dim(Ag)$ we find that $dim(Af) = 2^{d-1}$. This means that Af is a non-universal Clifford algebra for $\mathbb{R}_{p,m}$.
Let us now prove that this non-universal Clifford algebra is isomorphic to A^+. Since $d = p + m$ is odd, we have that $ae_\Omega \in A^-$ if $a \in A^+$. Every element of A can be decomposed into an even and an odd part $a = a^+ + a^- = \frac{a+a'}{2} + \frac{a-a'}{2}$. If we take $a \in A^+$ we thus find that $(af)^+ = \frac{\frac{1}{2}(a+ae_\Omega)+\frac{1}{2}(a'+(ae_\Omega)')}{2} = \frac{a}{2}$. From this we may conclude that the restriction of m_f to A^+ is one-to-one. Since both Af and A^+ have half the dimension of A, we may conclude that the restriction of m_f to A^+ is also surjective and thus an isomorphism. In the same way we can find that the restriction of $m_g : a \mapsto ag$ to A^+ is an isomorphism. This means that $A(E, q) \cong Af \oplus Ag \cong A^+ \oplus A^+$.
Let us now prove that any non-universal Clifford algebra for $\mathbb{R}_{p,m}$ is isomorphic to $A^+_{p,m}$. Suppose that B is a Clifford algebra for $(E, q) \cong \mathbb{R}_{p,m}$ where $p + m > 0$. Then we have the following diagram

$$\begin{tikzcd} E \arrow[r, "Id"] \arrow[d, "\subset"'] & E \arrow[d, "\subset"] \\ Af \oplus Ag \arrow[r, dashed, "\tilde{Id}"] & B \end{tikzcd}$$

We have that $Af \cong A^+(E, q)$ and from Corollary 3.2.14 we know that $A^+(E, q)$ is simple. This means that either $ker(\tilde{Id}|_{Af}) = 0$ or $ker(\tilde{Id}|_{Af}) = Af$, the same holds for Ag. From the assumption that $E \neq \{0\}$ it follows that it is not possible that both $ker(\tilde{Id}|_{Af}) = Af$ and $ker(\tilde{Id}|_{Ag}) = Ag$. If either $ker(\tilde{Id}|_{Af}) = Af$ or $ker(\tilde{Id}|_{Ag}) = Ag$, then we have that $B \cong Af \cong Ag \cong A^+$ (where we have used that $B \neq \{0\}$). This means that in that case B is a non-universal Clifford algebra that is isomorphic to A^+. If both $ker(\tilde{Id}|_{Af}) = 0$ and $ker(\tilde{Id}|_{Ag}) = 0$, then from the rank-nullity theorem (see Theorem 2.3 of [11]) it follows that $dim(B) = 2^d$ and thus that B is universal. From this we may conclude that there exists a non-universal Clifford algebra for $\mathbb{R}_{p,m}$ if $p - m \equiv 3 \pmod 4$ and that this non-universal Clifford algebra is isomorphic to $A^+_{p,m}$. $\square$

In the case where (E, q) is regular, we thus see that there are only non-universal Clifford algebras for (E, q) possible if $p - m \equiv 3 (\text{mod } 4)$. In this case, we actually see that a universal Clifford algebra for (E, q) is isomorphic to the direct sum of these non-universal Clifford algebras. We will see that this plays an important role in classifying Clifford algebras for $p - m \equiv 3 (\text{mod } 4)$.

3.3 Classifying Clifford algebras

To get a better understanding of where the gamma matrices and the other representations of Clifford algebras come from, we will classify the Clifford algebras for all possible dimensions of (E, q). For this section we will follow chapter 6 of [12]. From now on, we will only consider Clifford algebras for regular quadratic spaces. This means that the square of the volume element is equal to either 1 or -1. Let us first have a look at the Clifford algebras $A(E, q)$ with $dim(E) = 2$ before we classify Clifford algebras for higher dimensions.

The algebra $A_{2,0}$
Let us define the mapping $j : \mathbb{R}^2 \to \mathbb{H}$ by $j(x_1e_1 + x_2e_2) = x_1\mathbf{i} + x_2\mathbf{j}$. We then see that $1 \notin j(\mathbb{R}^2)$, that j is injective and that

$$\begin{aligned} j(x_1e_1 + x_2e_2)^2 &= x_1^2\mathbf{i}^2 + x_1x_2\mathbf{ij} + x_2x_1\mathbf{ji} + x_2^2\mathbf{j}^2 \\ &= -x_1^2 + x_1x_2\mathbf{k} - x_1x_2\mathbf{k} - x_2^2 \\ &= -x_1^2 - x_2^2 \end{aligned} \tag{3.17}$$

This means that j is a Clifford mapping. This Clifford mapping extends to an algebra homomorphism of $A_{2,0}$ into $\mathbb{H}$. We also have that $2 - 0 \not\equiv 3$ (mod 4), which means that $A_{2,0}$ is simple. Together with the fact that $dim(A_{2,0}) = dim(\mathbb{H}) = 2$ we may conclude that $A_{2,0} \cong \mathbb{H}$.

The algebra $A_{1,1}$
Let us define the mapping $j : \mathbb{R}_{1,1} \to M_2(\mathbb{R})$ by

$$j(x_1e_1 + x_2e_2) = \begin{pmatrix} 0 & -x_1 + x_2 \\ x_1 + x_2 & 0 \end{pmatrix} \tag{3.18}$$

We see that $1 \notin j(\mathbb{R}_{1,1})$, that j is injective and that

$$j(x_1e_1 + x_2e_2)^2 = \begin{pmatrix} -x_1^2 + x_2^2 & 0 \\ 0 & -x_1^2 + x_2^2 \end{pmatrix} \tag{3.19}$$

This means that j is a Clifford mapping, which extends to an algebra homomorphism of $A_{1,1,}$ into $M_2(\mathbb{R})$. As $1 - 1 \not\equiv 3$ (mod 4), we know that $A_{1,1}$ is simple. We also have that $dim(A_{1,1}) = dim(M_2(\mathbb{R})) = 4$ so we may conclude that $A_{1,1} \cong M_2(\mathbb{R})$.

The algebra $A_{0,2}$
Let $j : \mathbb{R}_{0,2} \to M_2(\mathbb{R})$ be defined by

$$j(x_1e_1 + x_2e_2) = \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix} \tag{3.20}$$

We see that $1 \notin j(\mathbb{R}_{0,2})$, that j is injective and that

$$j(x_1e_1 + x_2e_2)^2 = \begin{pmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{pmatrix} \tag{3.21}$$

This means that j is a Clifford mapping and extends to an algebra homomorphism of $A_{0,2}$ into $M_2(\mathbb{R})$. As $0 - 2 \not\equiv 3$ (mod 4), we know that $A_{0,2}$ is simple. We also have that $dim(A_{0,2}) = dim(M_2(\mathbb{R})) = 4$ so we may conclude that $A_{0,2} \cong M_2(\mathbb{R})$.

To classify Clifford algebras for a regular quadratic space (E, q) wih an even dimension higher than 2, we use Clifford's theorem.

Theorem 3.3.1 (Clifford's theorem)**.** *Suppose that (E, q) is a regular quadratic space of even dimension 2k and that E is an orthogonal direct sum $F \oplus G$, where F and G are regular subspaces of dimensions $2k-2$ and 2 respectively. Let ω_F be the volume element in the subalgebra $A_F = A(F, q)$ of $A = A(E, q)$, and let (g_1, g_2) be an orthogonal basis for G. Let $c_1 = \omega_F g_1$ and $c_2 = \omega_F g_2$ and let C be the subalgebra generated by $\{c_1, c_2\}$. Then $dim(C) = 4$ and A_F and C commute. This means that $A \cong A_F \otimes C$. We can then distinguish between two cases*

- *If, in addition, G is hyperbolic or $g_1^2 = g_2^2 = \omega_F^2$, then $C \cong M_2(\mathbb{R})$.*
- *If, instead, $g_1^2 = g_2^2 = -\omega_F^2$, then $C \cong \mathbb{H}$.*

Proof. Since $dim(F)$ is even, we see that $\omega_F g_i = c_i = g_i \omega_F$ for $i = 1, 2$. We also have that $c_i f_j = f_j c_i$ for $i = 1, 2$ if f_j is an element of the orthogonal basis for F. This means that $c_i x = x c_i$ for $i = 1, 2$ and $x \in F$. We thus have that A_F and C commute. To see what the dimension of C is, we look at all the possible combinations of the two elements c_i. First, we see that $c_i^2 = \omega_F^2 g_i^2 = \pm 1$ for $i = 1, 2$. Next we find that

$$c_1 c_2 = \omega_F g_1 \omega_F g_2 = \omega_F^2 g_1 g_2 = -\omega_F g_2 \omega_F g_1 = -c_2 c_1$$

and that

$$(c_1 c_2)^2 = \omega_F^4 g_1 g_2 g_1 g_2 = -g_1^2 g_2^2 = \pm 1$$

We may thus conclude that C is four-dimensional. From Corollary 3.1.17 it then follows that $A \cong A_F \otimes C$. We also have that $\lambda_1 c_1 + \lambda_2 c_2 = \omega_F(\lambda_1 g_1 + \lambda_2 g_2)$ for $\lambda_1, \lambda_2 \in K$. Together with the fact that $\frac{1}{\omega_F}$ is clearly not in $span(g_1, g_2)$, we find that $1 \notin span(c_1, c_2)$. We also have that $c_1 c_2 + c_2 c_1 = 0$ and that $c_i^2 = \pm 1$ for $i = 1, 2$. From Theorem 3.2.2 it then follows that C is a Clifford algebra for G and from the fact that $dim(C) = 4$ it follows that C is a universal Clifford algebra. Using what we have found on universal Clifford algebras $A(E, q)$ with $dim(E) = 2$ we can distinguish between three cases.

- If G is hyperbolic, then $c_1^2 = -c_2^2$ and $(c_1 c_2)^2 = -c_1^2 c_2^2 = 1$ so that $C \cong M_2(\mathbb{R})$.
- If $g_1^2 = g_2^2 = \omega_F^2$, then $c_1^2 = c_2^2 = 1$ and $(c_1 c_2)^2 = -1$. This then means that $C \cong M_2(\mathbb{R})$ as well.
- If $g_1^2 = g_2^2 = -\omega_F^2$, then $c_1^2 = c_2^2 = -1$ and $(c_1 c_2)^2 = -1$. This means that $C \cong \mathbb{H}$. □

The corollaries of this theorem are what we actually use to classify Clifford algebras.

Corollary 3.3.2. *Suppose $d = p + m = 2k$.*

- *If $p - m \equiv 2$ or 4 (mod 8), then $A_{p,m} \cong M_{2^{k-1}}(\mathbb{H})$*
- *If $p - m \equiv 0$ or 6 (mod 8), then $A_{p,m} \cong M_{2^k}(\mathbb{R})$*

Proof. If we combine Clifford's theorem with Proposition 3.1.18 we find that $A_{p+1,m+1} \cong A_{p,m} \otimes M_2(\mathbb{R}) \cong M_2(A_{p,m})$. From this it follows that $A_{p+j,m+j} \cong A_{p,m} \otimes M_{2^j}(\mathbb{R}) \cong M_{2^j}(A_{p,m})$. We know that $M_{2^j}(M_{2^{k-1}}(\mathbb{H})) \cong \mathbb{H} \otimes M_{2^{k-1}}(\mathbb{R}) \otimes M_{2^j}(\mathbb{R}) \cong M_{2^{k+j-1}}(\mathbb{H})$ and that $M_{2^j}(M_{2^k}(\mathbb{R})) \cong M_{2^j}(\mathbb{R}) \otimes M_{2^k}(\mathbb{R}) \cong M_{2^{k+j}}(\mathbb{R})$. It is therefore sufficient to prove the result when $m = 0$ or $p = 0$. Suppose that $m = 0$. We will prove the result using induction. We know that $A_{2,0} \cong H$. Now suppose that the corollary holds for all even $d = 2k$ with $d \leq 8j + 2$. This means that we assume that $A_{8j+2} \cong M_{2^{4j}}(\mathbb{H})$. Keeping in mind that $\omega_F^2 = -1$ if $p - m \equiv 0$ (mod 4) and that $\omega_F^2 = 1$ if $p - m \equiv 2$ (mod 4) we may conclude from Clifford's theorem that

- $A_{8j+4} \cong M_{2^{4j}}(\mathbb{H}) \otimes M_2(\mathbb{R}) \cong M_{2^{4j+1}}(\mathbb{H})$
- $A_{8j+6} \cong M_{2^{4j+1}}(\mathbb{H}) \otimes \mathbb{H}$
  Using Proposition 3.1.18 we find that
  $A_{8j+6} \cong M_{2^{4j+1}}(\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \cong M_{2^{4j+1}}(\mathbb{R}) \otimes M_4(\mathbb{R}) \cong M_{2^{4j+3}}(\mathbb{R})$
- $A_{8j+8} \cong M_{2^{4j+3}}(\mathbb{R}) \otimes M_2(\mathbb{R}) \cong M_{2^{4j+4}}(\mathbb{R})$
- $A_{8j+10} \cong M_{2^{4j+4}}(\mathbb{R}) \otimes \mathbb{H} \cong M_{2^{4j+4}}(\mathbb{H})$

Now suppose that $p = 0$. We will again prove the result using induction. We know that $A_{0,2} \cong M_2(\mathbb{R})$. Suppose that the corollary holds for all even $d = 2k$ with $d \leq 8j + 2$. This means that we assume that $A_{0,8j+2} \cong M_{2^{4j+1}}(\mathbb{R})$. From Clifford's theorem it then follows that

- $A_{0,8j+4} \cong M_{2^{4j+1}}(\mathbb{R}) \otimes \mathbb{H} \cong M_{2^{4j+1}}(\mathbb{H})$
- $A_{0,8j+6} \cong M_{2^{4j+1}}(\mathbb{H}) \otimes M_2(\mathbb{R}) \cong \mathbb{H} \otimes M_{2^{4j+1}}(\mathbb{R}) \otimes M_2(\mathbb{R}) \cong M_{2^{4j+2}}(\mathbb{H})$
- $A_{0,8j+8} \cong M_{2^{4j+2}}(\mathbb{H}) \otimes \mathbb{H} \cong M_{2^{4j+4}}(\mathbb{R})$
- $A_{0,8j+10} \cong M_{2^{4j+4}}(\mathbb{R}) \otimes M_2(\mathbb{R}) \cong M_{2^{4j+5}}(\mathbb{R})$ □

With Clifford's theorem we have only classified Clifford algebras for quadratic spaces with an even dimension. Let us now take a look at Clifford algebras for quadratic spaces with an odd dimension. We will start by looking at the case where $p - m \equiv 1 (\text{mod } 4)$.

Proposition 3.3.3. *If $d = p + m = 2k + 1$ and $p - m \equiv 1$ (mod 4), then $A_{p,m} \cong M_{2^k}(\mathbb{C})$*

Proof. Let F be a regular subspace of $\mathbb{R}_{p,m}$ of dimension $2k$ and e_Ω be the volume element of $A_{p,m}$. Since $p - m \equiv 1$ (mod 4), we have that $e_\Omega^2 = -1$. We see that $C = \mathit{span}(1, e_\Omega)$ is a subalgebra of $A_{p,m}$. We also have that $\mathbb{C} = \mathit{span}(1, i)$ with $i^2 = -1$. This means that $C \cong \mathbb{C}$. As the dimension of $\mathbb{R}_{p,m}$ is odd, we know that $A(F, q)$ and C commute. We also know that $A(F, q) = \mathit{span}(P)$ with $P = \{e_c : C \subseteq \Omega_{2k}\}$ and $\Omega_{2k} = \{1, ..., 2k\}$. Together with the fact that C contains the volume element of $A_{p,m}$, we may conclude that $A(F, q)$ and C generate $A_{p,m}$. We also have that $\mathit{dim}(A_{p,m}) = 2^{2k+1} = 2^{2k} \cdot 2 = \mathit{dim}(A(F, q))\mathit{dim}(C)$. It thus follows from Corollary 3.1.17 that $A_{p,m} \cong A(F, q) \otimes C \cong A(F, q) \otimes \mathbb{C}$.
As $p - m \equiv 1$ (mod 4), there are two possibilities for $A(F, q)$:

- $A(F, q) \cong M_{2^k}(\mathbb{R})$. In this case it is clear to see that $A_{p,m} \cong M_{2^k}(\mathbb{R}) \otimes \mathbb{C} \cong M_{2^k}(\mathbb{C})$.
- $A(F, q) \cong M_{2^{k-1}}(\mathbb{H})$. In this case we have that $A_{p,m} \cong M_{2^{k-1}}(\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{C}$. This means that from Proposition 3.1.18 it follows that $A_{p,m} \cong M_{2^{k-1}}(\mathbb{R}) \otimes M_2(\mathbb{C}) \cong M_{2^k}(\mathbb{C})$. □

Recall from the previous section that for the universal Clifford algebras $A_{p,m}$ over the regular quadratic spaces with $p - m \equiv 3$ (mod 4), we proved that they were isomorphic to the direct sum of two copies of their even Clifford algebra $A^+_{p,m}$. To classify these Clifford algebras we thus have to first prove the following theorem regarding even Clifford algebras.

Theorem 3.3.4. *$A^+_{p+1,m} \cong A_{p,m}$ and $A^+_{p,m+1} \cong A_{m,p}$*

Proof. Let $(e_1, ..., e_{p+m+1})$ be the standard orthogonal basis for $\mathbb{R}_{p+1,m}$ and $(f_1, ..., f_{p+m})$ the standard orthogonal basis for $\mathbb{R}_{p,m}$. Since $q(e_1) = 1$, we have that $e_{i+1}e_{j+1} = (e_1e_{i+1})(e_1e_{j+1})$. The basis of $A^+_{p+1,m}$ consists of all the products consisting of an even amount of elements from $(e_1, ..., e_{p+m+1})$. This means that $\{e_1e_{j+1} : 1 \leq j \leq p + m\}$ generates $A^+_{p+1,m}$. Let $f : \mathbb{R}_{p,m} \to A^+_{p+1,m}$ be the mapping defined by $f_j \mapsto e_1e_{j+1}$ for $1 \leq j \leq p + m$. We see that

$$(e_1e_{j+1})^2 = -e_1^2e_{j+1}^2 = -q(e_{j+1}) = -q(f_j)$$

We also have that $1 \notin f(\mathbb{R}_{p,m})$ and that f is injective. This means that we can extend f by linearity into a Clifford mapping from $\mathbb{R}_{p,m}$ into $A^+_{p+1,m}$. This Clifford mapping can be extended to an algebra homomorphism of $A_{p,m}$ into $A^+_{p+1,m}$. We also have that $\mathit{dim}(A^+_{p+1,m}) = 2^{p+m+1}/2 = 2^{p+m} = \mathit{dim}(A_{p,m})$. Together with the fact that the elements of the set $\{e_1e_{j+1} : 1 \leq j \leq p + m\}$ generate $A^+_{p+1,m}$ and thus that this algebra homomorphism is surjective, we may conclude that this algebra homomorphism is actually an algebra isomorphism.
Now let $(e_1, ..., e_{p+m+1})$ be the standard orthogonal basis for $\mathbb{R}_{p,m+1}$ and $(g_1, ..., g_{p+m})$ the standard orthogonal basis for $\mathbb{R}_{m,p}$. Since $(e_je_{p+m+1})(e_ie_{p+m+1}) = -e_je_i$, we have that $\{e_je_{p+m+1} : 1 \leq j \leq d\}$ generates $A^+_{p,m+1}$ in the same way as above. Let $g : \mathbb{R}_{m,p} \to A^+_{p,m+1}$ be the mapping $g_j \mapsto e_{p+m+1-j}e_{p+m+1}$. We see that

$$(e_{p+m+1-j}e_{p+m+1})^2 = -e_{p+m+1-j}^2e_{p+m+1}^2 = -q(g_j)$$

We also see that $1 \notin g(\mathbb{R}_{m,p})$ and that g is injective. We can again extend this map by linearity to create a Clifford mapping from $\mathbb{R}_{m,p}$ into $A^+_{p,m+1}$. Which, again, can be extended to an algebra homomorphism of $A_{m,p}$ into $A^+_{p,m+1}$. In the same way as above, $A_{m,p}$ and $A^+_{p+1,m}$ both have dimension 2^{p+m}. Together with the fact that the elements of the set $\{e_j e_{p+m+1} : 1 \leq j \leq d\}$ generate $A^+_{p,m+1}$, we may conclude that $A_{m,p} \cong A^+_{p+1,m}$. □

Remark 3.3.5. From Theorem 3.3.4 it follows that $A_{p,m+1} \cong A^+_{p+1,m+1} \cong A_{m,p+1}$. However, we also have that $A^+_{p,m+1} \cong A_{m,p}$ and that $A^+_{p+1,m} \cong A_{p,m}$. In general, $A_{m,p}$ and $A_{p,m}$ are not isomorphic to each other. For example, when $p + m = 2k$ and $p - m \equiv 2 \pmod 8$, then $A_{p,m} \cong M_{2^{k-1}}(\mathbb{H})$ and $A_{m,p} \cong M_{2^k}(\mathbb{R})$. So, in general $A^+_{p,m+1}$ and $A^+_{p+1,m}$ are not isomorphic to each other. This means that most of the time, $A_{p,m+1}$ and $A_{m,p+1}$ decompose into a different direct sum of an even and an odd Clifford algebra. The isomorphisms between algebras following from Theorem 3.3.4 thus do not always imply that these algebras are also isomorphic as super-algebras.

Using Theorem 3.3.4, we can classify the Clifford algebras $A_{p,m}$ for the case where $p - m \equiv 3 \pmod 4$.

Corollary 3.3.6. *Suppose that $d = p + m = 2k + 1$*

- *If $p - m \equiv 3$ (mod 8), then $A_{p,m} \cong M_{2^{k-1}}(\mathbb{H}) \oplus M_{2^{k-1}}(\mathbb{H})$*
- *If $p - m \equiv 7$ (mod 8), then $A_{p,m} \cong M_{2^k}(\mathbb{R}) \oplus M_{2^k}(\mathbb{R})$*

Proof. In both cases we have that $p - m \equiv 3 \pmod 4$. This means that from Theorem 3.2.15 we may conclude that $A_{p,m} \cong A^+_{p,m} \oplus A^+_{p,m}$. If $p > 0$, then from the Theorem above it follows that $A^+_{p,m} \cong A_{p-1,m}$. In that case, we can distinguish between two cases:

- If $p - m \equiv 3 \pmod 8$, then $p - 1 - m \equiv 2 \pmod 8$, implying that $A_{p-1,m} \cong M_{2^{k-1}}(\mathbb{H})$. This then means that $A_{p,m} \cong M_{2^{k-1}}(\mathbb{H}) \oplus M_{2^{k-1}}(\mathbb{H})$.
- If $p - m \equiv 7 \pmod 8$, then $p - 1 - m \equiv 6 \pmod 8$, which means that $A_{p-1,m} \cong M_{2^k}(\mathbb{R})$. From this it follows that $A_{p,m} \cong M_{2^k}(\mathbb{R}) \oplus M_{2^k}(\mathbb{R})$

If $p = 0$, then from the Theorem above it follows that $A^+_{0,m} \cong A_{m-1,0}$. In this case we can also distinguish between two cases:

- If $-m \equiv 3 \pmod 8$, then $m - 1 \equiv 4 \pmod 8$, so that $A_{m-1,0} \cong M_{2^{k-1}}(\mathbb{H})$. In this case we thus have that $A_{p,m} \cong M_{2^{k-1}}(\mathbb{H}) \oplus M_{2^{k-1}}(\mathbb{H})$.
- If $-m \equiv 7 \pmod 8$, then $m - 1 \equiv 0 \pmod 8$, implying that $A_{m-1,0} \cong M_{2^k}(\mathbb{R})$. From this it follows that $A_{p,m} \cong M_{2^k}(\mathbb{R}) \oplus M_{2^k}(\mathbb{R})$ □

Essentially, we have now already classified the Clifford algebras for all possible dimensions over regular quadratic spaces. However, an attentive reader might have already noticed that there is a form of periodicty hidden in the classification of the Clifford algebras. This periodicity is summarized in Cartan's periodicty law, named after the French mathematician Élie Cartan. Before we can take a look at this law, we first have to prove the following proposition.

Proposition 3.3.7. *There is an isomorphism of $A_{p,m+4}$ onto $A_{p+4,m}$.*

Proof. Let $(e_1, ..., e_d)$ denote a standard orthogonal basis for $\mathbb{R}_{p+4,m}$ and $(g_1, ..., g_d)$ a standard orthogonal basis for $\mathbb{R}_{p,m+4}$, with $d = p + m + 4$. Let $f = e_{p+1}e_{p+2}e_{p+3}e_{p+4} \in A_{p+4,m}$ and define the mapping $\pi : \mathbb{R}_{\shortmid,>+\not\trianglerighteq} \to A_{p+4,m}$ by setting

$$\pi(g_j) = \begin{cases} e_j f \text{ for } p+1 \leq j \leq p+4 \\ e_j \text{ for } 1 \leq j \leq p \text{ and for } p+5 \leq j \leq d \end{cases}$$

and then extending by linearity. Keeping in mind that $f^2 = e_{p+1}^2 e_{p+2}^2 e_{p+3}^2 e_{p+4}^2 = 1$, we see that

$$\pi(g_j)^2 = e_j^2 = -1 \text{ for } 1 \leq j \leq p,$$
$$\pi(g_j)^2 = -e_j^2 f^2 = 1 \text{ for } p+1 \leq j \leq p+4,$$
$$\pi(g_j)^2 = e_j^2 = 1 \text{ for } p+5 \leq j \leq d.$$

We also have that $\pi(g_j)\pi(g_k) = -\pi(g_k)\pi(g_j)$, implying that the mapping π can be extended linearly into a Clifford mapping from $\mathbb{R}_{p,m+4}$ into $A_{p+4,m}$. This Clifford mapping then extends to an algebra homomorphism from $A_{p,m+4}$ into $A_{p+4,m}$. From the fact that the set $\{\pi(g_j) : 1 \leq j \leq d\}$ generates $A_{p+4,m}$ it follows that this algebra homomorphism is actually surjective. Together with the fact that $dim(A_{p,m+4}) = dim(A_{p+4,m}) = 2^{p+m+4}$ we may conclude that this homomorphism is actually an isomorphism. □

Let us now take a look at Cartan's periodicity law.

Theorem 3.3.8 (Cartan's periodicity law). *There are isomorphisms between the three algebras $A_{p+8,m}$, $A_{p,m+8}$ and $M_{16}(A_{p,m})$.*

Proof. From Proposition 3.3.7 it follows that $A_{p+8,m} \cong A_{p+4,m+4} \cong A_{p,m+8}$. From Clifford's theorem it follows that $A_{p+4,m+4} \cong A_{p,m} \otimes M_{16}(\mathbb{R})$ and from Proposition 3.1.18 it then follows that $A_{p+4,m+4} \cong M_{16}(A_{p,m})$. □

This theorem essentially tells us that it is sufficient to classify the Clifford algebras for $0 \leq p, m \leq 7$ as the classification of the Clifford algebras for other dimensions follow from these classifications. In this table we write A^2 for $A \oplus A$, where A denotes an algebra.

Table 2: Universal Clifford algebras

		p	$\rightarrow$						
		0	1	2	3	4	5	6	7
m	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}^2$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})^2$
$\downarrow$	1	$\mathbb{R}^2$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})^2$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$
	2	$M_2(\mathbb{R})$	$M_2(\mathbb{R})^2$	$M_4(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H})^2$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$
	3	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$	$M_4(\mathbb{R})^2$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H})^2$	$M_{16}(\mathbb{H})$
	4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})^2$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	$M_{16}(\mathbb{H})^2$
	5	$M_2(\mathbb{H})^2$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})^2$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$
	6	$M_4(\mathbb{H})$	$M_4(\mathbb{H})^2$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R})^2$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{C})$
	7	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H})^2$	$M_{16}(\mathbb{H})$	$M_{32}(\mathbb{C})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{R})^2$	$M_{128}(\mathbb{R})$

The periodicity of 8 that we see in this table is an example of the so-called *Bott periodicity*. Bott periodicity has applications in many areas of mathematics, from algebraic topology to functional analysis. However, Bott periodicity lies outside the topic of this thesis so we will not go further into it. If one does want to know more about the link between the Bott periodicity and Clifford algebras see [31].

3.4 Spinors

Now that we have classified the universal Clifford algebras for all possible dimensions of (E, q), let us take a look at what these Clifford algebras act on. For this section we will follow section 7.1 of [12]. When $A_{p,m}$ is simple, we have seen in the section before that it can be represented as $M_k(\mathbb{D})$, with $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Simple Clifford algebras can thus be identified as matrix algebras. On the matrix algebra $M_k(\mathbb{D})$ we have the natural representation ρ of $M_k(\mathbb{D})$ into $\mathbb{D}^k$.

Proposition 3.4.1. *The natural representation ρ of $M_k(\mathbb{D})$ into $\mathbb{D}^k$ is irreducible.*

Proof. We will prove this proposition by contradiction. Suppose that the natural representation ρ is reducible and that it thus has sub-representations. This means that there exists $E_1, E_2 \in \mathbb{D}^k$ with $E_1, E_2 \neq 0$ such that $E_1 \oplus E_2 = \mathbb{D}^k$. For E_1 to be a sub-representation we must have that $Av \in E_1$ if $v \in E_1$ and $A \in M_k(\mathbb{D})$. Suppose that $e_1 \neq 0$ is a basis element of E_1 and $e_2 \neq 0$ a basis element of E_2. We can pick these basis elements to be nonzero because $E_1, E_2 \neq 0$. We know that there exists a transformation matrix $A \in M_k(\mathbb{D})$ such that $Ae_1 = e_2$. Since $E_1 \cap E_2 = 0$, this must then mean that either $e_1 = 0$ or $e_2 = 0$. This is in contradiction with the fact that we could pick two basis elements out of E_1 and E_2 that were not equal to zero. From this contradiction it follows that the natural representation is irreducible. $\square$

There is actually an even stronger theorem that says that any irreducible representation of $M_k(\mathbb{D})$ is isomorphic to this natural representation. However, the proof of this statement is actually pretty complicated and we will therefore not include it in this thesis. If one is interested though, the proof can be deduced from theorem 4.3 and 4.4 on page 653 of [16]. If $A_{p,m}$ is simple we can thus consider $A_{p,m}$ acting on the left $\mathbb{D}$-module $\mathbb{D}^k$. A left $\mathbb{R}$-module is just a real vector space, a left $\mathbb{C}$-module is a complex vector space and a left $\mathbb{H}$-module is called a vector space over $\mathbb{H}$. Using this theorem we can give the following definition.

Definition 3.4.2. If $A_{p,m}$ is simple and thus isomorphic to a matrix algebra $M_k(\mathbb{D})$, then $\mathbb{D}^k$ represents the so-called *spinor space* of $A_{p,m}$. We call the elements of $\mathbb{D}^k$ the *spinors*.

Let us take a look at the dimension of the spinor spaces. If $A_{p,m} \cong M_k(\mathbb{R})$, then

$$dim(A_{p,m}) = 2^d = dim(M_k(\mathbb{R})) = k^2 \tag{3.22}$$

This means that $d = 2t$ is even and that $k = 2^t$. The real dimension of a spinor space for $A_{p,m}$ is then $2^{\frac{d}{2}}$.
If $A_{p,m} \cong M_k(\mathbb{C})$, then

$$dim_{\mathbb{R}}(A_{p,m}) = 2^d = dim_{\mathbb{R}}(M_k(\mathbb{C})) = 2k^2 \tag{3.23}$$

This means that $d = 2t + 1$ is odd and that $k = 2^t$. The complex dimension of a spinor space for $A_{p,m}$ is then $2^{\frac{d-1}{2}}$, and the real dimension is $2^{\frac{d+1}{2}}$.
If $A_{p,m} \cong M_k(\mathbb{H})$, then

$$dim_{\mathbb{R}}(A_{p,m}) = 2^d = dim_{\mathbb{R}}(M_k(\mathbb{H})) = 4k^2 \tag{3.24}$$

This means that $d = 2t$ is even and that $k = 2^{t-1}$. We then have that the quaternionic dimension of a spinor space for $A_{p,m}$ is $2^{\frac{d-2}{2}}$, and that the real dimension is $2^{\frac{d+2}{2}}$.
When $A_{p,m}$ is not simple, we saw that it was isomorphic to $M_k(\mathbb{D}) \oplus M_k(\mathbb{D})$. We then have exactly two equivalence classes of irreducible representations. They are given by

$$\rho_1(\phi_1, \phi_2) = \rho(\phi_1) \text{ and } \rho_2(\phi_1, \phi_2) = \rho(\phi_2) \tag{3.25}$$

where $\phi_1, \phi_2 \in M_k(\mathbb{D})$. If $A_{p,m}$ is not simple we will consider the irreducible representation of the non-universal Clifford algebra $M_k(\mathbb{D})$ and call $\mathbb{D}^k$ the *semi-spinor space*. The elements of $\mathbb{D}^k$ we call the *semi-spinors*. Table 3 shows the results of this section. The square brackets in this table indicate that we are dealing with semi-spinor spaces instead of spinor spaces.

Table 3: Spinor and semi-spinor spaces

		p	→						
		0	1	2	3	4	5	6	7
m	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$[\mathbb{H}]$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$[\mathbb{R}^8]$
↓	1	$[\mathbb{R}]$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$[\mathbb{H}^2]$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$
	2	$\mathbb{R}^2$	$[\mathbb{R}^2]$	$\mathbb{R}^4$	$\mathbb{C}^4$	$\mathbb{H}^4$	$[\mathbb{H}^4]$	$\mathbb{H}^8$	$\mathbb{C}^{16}$
	3	$\mathbb{C}^2$	$\mathbb{R}^4$	$[\mathbb{R}^4]$	$\mathbb{R}^8$	$\mathbb{C}^8$	$\mathbb{H}^8$	$[\mathbb{H}^8]$	$\mathbb{H}^{16}$
	4	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$[\mathbb{R}^8]$	$\mathbb{R}^{16}$	$\mathbb{C}^{16}$	$\mathbb{H}^{16}$	$[\mathbb{H}^{16}]$
	5	$[\mathbb{H}^2]$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$	$[\mathbb{R}^{16}]$	$\mathbb{R}^{32}$	$\mathbb{C}^{32}$	$\mathbb{H}^{32}$
	6	$\mathbb{H}^4$	$[\mathbb{H}^4]$	$\mathbb{H}^8$	$\mathbb{C}^{16}$	$\mathbb{R}^{32}$	$[\mathbb{R}^{32}]$	$\mathbb{R}^{64}$	$\mathbb{C}^{64}$
	7	$\mathbb{C}^8$	$\mathbb{H}^8$	$[\mathbb{H}^8]$	$\mathbb{H}^{16}$	$\mathbb{C}^{32}$	$\mathbb{R}^{64}$	$[\mathbb{R}^{64}]$	$\mathbb{R}^{128}$

3.5 The Dirac equation

In the previous sections we have learned more about Clifford algebras. Let us use this knowledge to get a better understanding of the Dirac equation. For this section we will follow chapter 9 of [12]. As one might know, we use $\mathbb{R}_{3,1}$ to describe space-time in physics. If we take a look at the table in which we classified the universal Clifford algebras over regular quadratic spaces for all dimensions, we see that $A_{3,1}$ is isomorphic to $M_2(\mathbb{H})$. However, the representations of the gamma matrices that we have seen so far are in $M_4(\mathbb{C})$. The reason why is because in normal quantum mechanics the wave function takes on complex values. We therefore want our spinor space to be a complex vector space. The Clifford algebra that we use, thus has to be isomorphic to a matrix algebra that has entries from the complex numbers. Physicists therefore chose the smallest Clifford algebra that is isomorphic to a matrix algebra with entries from the complex numbers and decided to represent $A_{3,1}$ as a subalgebra of this algebra. To see how this works, we will explicitly show how $M_4(\mathbb{C})$ is a Clifford algebra for $\mathbb{R}_{3,1}$. Let us write γ^4 for γ^0 for this mapping only. Define the injective linear map $\gamma : \mathbb{R}_{3,1} \to M_4(\mathbb{C})$ by

$$\gamma(x_1e_1 + ... + x_4e_4) = x_1\gamma^1 + ... + x_4\gamma^4 \tag{3.26}$$

We see that $1 \notin \gamma(\mathbb{R}_{3,1})$ and that $\gamma(x_1e_1+...+x_4e_4)^2 = x_4^2 - x_1^2 - x_2^2 - x_3^2 = -q(x_1e_1+...+x_4e_4)$. This means that γ is a Clifford mapping of $\mathbb{R}_{3,1}$ into $M_4(\mathbb{C})$. This Cliffford mapping extends to an algebra homomorphism of $A_{3,1}$ into $M_4(\mathbb{C})$. The image $\Gamma_{3,1} = \gamma(A_{3,1})$ we call the *Dirac algebra*. To see what the Dirac equation looks like, we define the so-called *Dirac operator*.

Definition 3.5.1. Suppose that U is an open subset of $\mathbb{R}_{p,m}$, that F is a finite-dimensional left $A_{p,m}$-module and that f is a continuously differentiable function from U into F. We can then define the *Dirac operator* D_q as

$$D_q f(x) = \sum_{j=1}^{d} q(e_j)e_j \frac{\partial f}{\partial x_j}$$

The Dirac operator $D_{3,1}$ for the algebra of space-time $\mathbb{R}_{3,1}$ reads

$$D_{3,1} = -\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x_1} + \gamma^2\frac{\partial}{\partial x_2} + \gamma^3\frac{\partial}{\partial x_3} \tag{3.27}$$

so that that the Dirac equation becomes

$$D_{3,1}\Psi(\mathbf{x},t) = \left(-\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x_1} + \gamma^2\frac{\partial}{\partial x_2} + \gamma^3\frac{\partial}{\partial x_3}\right)\Psi(\mathbf{x},t) = im\Psi(\mathbf{x},t) \tag{3.28}$$

where we consider $A_{3,1}$ to be acting on $\mathbb{C}^4$.

In the standard representation we have chosen to represent $A_{3,1}$ as a subalgebra of $A_{3,2}$. One might ask oneself; what happens if we do not restrict ourselves to only working with wave functions that take on complex values. In the next chapter we will explore this possibility. We will take a look at wave functions that can take on quaternion values and explore how the gamma matrices will look if we used Clifford algebras that are isomorphic to quaternionic algebras. Maybe working with Clifford algebras that are not ‘too big’ will even provide us with new physical results.

4 A quaternionic formulation of the Dirac equation

The Dirac equation as we know it is formulated using the gamma matrices in $M_4(\mathbb{C})$. However, in the previous chapter we found that the algebra usually used to describe space-time ($A_{3,1}$) is actually isomorphic to $M_2(\mathbb{H})$. One of the reasons why we use the matrix algebra $M_4(\mathbb{C})$ is because in normal quantum mechanics we expect the wave function to take on complex values. Using $M_4(\mathbb{C})$ also has the advantage that we can use the rich theory of complex analysis [2]. Quaternions on the other hand, are relatively hard to work with due to their noncommutativity. However, this challenge did not stop physicists to attempt to write a quaternionic version of quantum mechanics. This included the attempt to formulate the Dirac equation using quaternions. In the following chapter we will take a look at some of the formulations of the Dirac equation using real and complex quaternions. We will also take a look at some of the operators and operations in these formalisms. Before we can take a look at these quaternionic formulations, we have to expand our knowledge on quaternionic algebra.

4.1 Quaternion algebra

For this section we will follow [21]. In chapter 3 we already encountered the quaternions while classifying Clifford algebras. The quaternions are actually an extension of the complex numbers in such a way that besides the complex unit i that squares to -1, we also have the elements $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ that square to -1. To distinguish between when we are working with quaternions and when we are working with complex numbers, we will denote the complex unit using i and the quaternion units using $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$.

Definition 4.1.1. A *real quaternion* is of the form

$$q = a_0 + a_1\mathcal{I} + a_2\mathcal{J} + a_3\mathcal{K}$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ satisfy the relations defined in section 3.1.

The real number a_0 we call the *scalar part* of the quaternion and $a_1\mathcal{I} + a_2\mathcal{J} + a_3\mathcal{K}$ the *vector part* of q. Let us denote the vector of all the bais elements of the vector part of q as follows $\boldsymbol{h} \equiv (\mathcal{I}, \mathcal{J}, \mathcal{K})$. Note that we can identify the space-time point (ct, x, y, z) by the quaternion

$$q = ct + \mathcal{I}x + \mathcal{J}y + \mathcal{K}z \tag{4.1}$$

Analogous to the complex conjugate

$$i^* = -i \tag{4.2}$$

we also have a conjugation operation on quaternions.

Definition 4.1.2. The conjugation operation on quaternions we call the *quaternion conjugation.* The quaternion conjugation is denoted by $^{@}$ and has the following property

$$1^{@} = 1, \mathcal{I}^{@} = -\mathcal{I}, \mathcal{J}^{@} = -\mathcal{J} \text{ and } \mathcal{K}^{@} = -\mathcal{K}$$

The quaternion conjugate of a real quaternion q has the following form

$$q^{@} = a_0 - a_1\mathcal{I} - a_2\mathcal{J} - a_3\mathcal{K} \tag{4.3}$$

Note that for this conjugation we have that

$$(qp)^{@} = p^{@}q^{@} \tag{4.4}$$

In this chapter we will also look at formulations of the Dirac equation using so-called *complex quaternions.* Complex quaternions are an extension of the quaternions where instead of using real coefficients we will use complex ones. A complex quaternion will thus come from the tensor product space $\mathbb{C} \otimes \mathbb{H}$.

[2] If one is looking to learn more about the rich theory of complex analysis, see [17]

Definition 4.1.3. A *complex quaternion* has the following form

$$q_c = c_0 + c_1\mathcal{I} + c_2\mathcal{J} + c_3\mathcal{K}$$

where $c_0, c_1, c_2, c_3 \in \mathbb{C}$ and the imaginary unit i commutes with the quaternionic imaginary units $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$.

Recall from section 3.1 that an algebra is a division algebra if there are no non-zero zero divisors for that algebra. For $1 - i\mathcal{I}, 1 + i\mathcal{I} \in \mathbb{C} \otimes \mathbb{H}$ we have that

$$(1 - i\mathcal{I})(1 + i\mathcal{I}) = 1 - (-1)(-1) = 0 \tag{4.5}$$

We thus see that $\mathbb{C} \otimes \mathbb{H}$ is not a division algebra. This means that for complex quaternions we may not assume that every non-zero element has a multiplicative inverse.
On complex quaternions we have three different ways to define conjugation operations:

$$\begin{aligned} q_c^{\bullet} &= c_0^* + \boldsymbol{h} \cdot \boldsymbol{c}^* \\ q_c^{\star} &= c_0 - \boldsymbol{h} \cdot \boldsymbol{c} \\ q_c^{@} &= c_0^* - \boldsymbol{h} \cdot \boldsymbol{c}^* \end{aligned} \tag{4.6}$$

where * denotes the standard complex conjugation ($i \to -i$) and $\boldsymbol{c} = (c_1, c_2, c_3)$. Note that

$$\begin{aligned} (q_c p_c)^{\bullet} &= q_c^{\bullet} p_c^{\bullet} \\ (q_c p_c)^{\star} &= p_c^{\star} q_c^{\star} \end{aligned} \tag{4.7}$$

and consequently

$$(q_c p_c)^{@} = (q_c p_c)^{\bullet\star} = (q_c^{\bullet} p_c^{\bullet})^{\star} = p_c^{\bullet\star} q_c^{\bullet\star} = p_c^{@} q_c^{@} \tag{4.8}$$

In $\mathbb{C} \otimes \mathbb{H}$ we can either choose i or $\mathcal{I}$ to represent the imaginary unit. We therefore denote the set of complex numbers by $\mathbb{C}(1, i)$ when we are using i as the imaginary unit and by $\mathbb{C}(1, \mathcal{I})$ when we are using $\mathcal{I}$ as the imaginary unit. With a little algebra we can also write a complex quaternion in the following way

$$q_c = b_0 + b_1\mathcal{J} + i(b_2 + b_3\mathcal{J}) \tag{4.9}$$

where $b_0, b_1, b_2, b_3 \in \mathbb{C}(1, \mathcal{I})$.
Due to the noncommutative nature of the quaternions we must consider left/right-actions. We therefore introduce *barred operators* to distinguish between left- or right-multiplication.

Definition 4.1.4. Suppose that q_c, p_c and r_c are (complex) quaternions. The action of the *barred operator* $q_c|p_c$ on the (complex) quaternion r_c is defined as follows

$$(q_c|p_c)r_c \equiv q_c r_c p_c$$

We thus write

$$1|\mathcal{I},\ 1|\mathcal{J} \text{ and } 1|\mathcal{K} \tag{4.10}$$

to denote the right multiplication of $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$.
To find the most general transformation on quaternions we will make use of lemma 2.1 of [27]. The A^o that they use here is the *opposite algebra*. The opposite algebra has the same additive structure as A but the product is defined by setting $a \cdot b = ba$, where ba refers to the product in A. In this lemma they also use the term *central simple algebra*. A central simple algebra is an algebra over the field F of which the center is exactly F. The algebras that we have looked at are all over $\mathbb{R}$ and from the commutation relations of the basis elements of $\mathbb{H}$ we may conclude that the centre of $\mathbb{H}$ is exactly $\mathbb{R}$. We thus have that $\mathbb{H}$ is a central simple algebra. From the fact that $\mathbb{H} = span(1, \mathcal{I}, \mathcal{J}, \mathcal{K})$ it then follows that the most general transformation on a quaternion is of the following form:

$$q_0 + q_1|\mathcal{I} + q_2|\mathcal{J} + q_3|\mathcal{K} \tag{4.11}$$

where q_0, q_1, q_2 and q_3 are quaternions. We call these transformations *real linear* barred operators. A subset of the set of real linear barred operators is the set of *complex linear* barred operators.

Definition 4.1.5. *Complex linear* barred operators on quaternions are of the following form:

$$q_0 + q_1|\mathcal{I}$$

where q_0 and q_1 are quaternions.

These complex linear barred operators on quaternions are characterized by four complex numbers from $\mathbb{C}(1, \mathcal{I})$.
To find the form of the general transformations on complex quaternions we make use of lemma 2.2.2 from [13]. From this lemma it follows that the algebra of complex quaternions $\mathbb{C} \otimes \mathbb{H}$ is a central simple algebra over $\mathbb{C}$. Where we have used that $\mathbb{C}$ is an extension of $\mathbb{R}$. This means that to find the most general transformation on complex quaternions, we may also use lemma 2.1 of [27]. The most general transformation on a complex quaternion that follows from this lemma is of the following form:

$$\mathcal{O}_c^i \equiv q_c + p_c|\mathcal{I} + r_c|\mathcal{J} + s_c|\mathcal{K} \tag{4.12}$$

where q_c, p_c, r_c and s_c are all complex quaternions. We call these general transformations *i-complex linear* barred operators. Multiplication of two i-complex linear barred operators $\mathcal{O}_c^{i,1}$ and $\mathcal{O}_c^{i,2}$ in terms of complexified quaternions is defined as

$$\begin{aligned}
\mathcal{O}_c^{i,1}\mathcal{O}_c^{i,2} = \; & q_{c,1}q_{c,2} - p_{c,1}p_{c,2} - r_{c,1}r_{c,2} - s_{c,1}s_{c,2} \\
& + (q_{c,1}p_{c,2} + p_{c,1}q_{c,2} - r_{c,1}s_{c,2} + s_{c,1}r_{c,2})|\mathcal{I} \\
& + (q_{c,1}r_{c,2} + r_{c,1}q_{c,2} - s_{c,1}p_{c,2} + p_{c,1}s_{c,2})|\mathcal{J} \\
& + (q_{c,1}s_{c,2} + s_{c,1}q_{c,2} - p_{c,1}r_{c,2} + r_{c,1}p_{c,2})|\mathcal{K}
\end{aligned} \tag{4.13}$$

And the conjugation operations are defined as follows

$$\begin{aligned}
\mathcal{O}_c^{i\bullet} &\equiv q_c^\bullet + p_c^\bullet|\mathcal{I} + r_c^\bullet|\mathcal{J} + s_c^\bullet|\mathcal{K} \\
\mathcal{O}_c^{i\star} &\equiv q_c^\star - p_c^\star|\mathcal{I} - r_c^\star|\mathcal{J} - s_c^\star|\mathcal{K} \\
\mathcal{O}_c^{i@} &\equiv q_c^@ - p_c^@|\mathcal{I} - r_c^@|\mathcal{J} - s_c^@|\mathcal{K}
\end{aligned} \tag{4.14}$$

A transformation like the one in Equation 4.12 is characterized by 16 i-complex parameters. A subset of the set of i-complex linear barred operators is the set of $\mathcal{I}$*-complex linear* barred operators.

Definition 4.1.6. $\mathcal{I}$*-complex linear* barred operators on complex quaternions are of the form:

$$\mathcal{O}_c^{\mathcal{I}} \equiv q_c + p_c|\mathcal{I}$$

where q_c and p_c are complex quaternions.

What we see is that this barred operator $\mathcal{O}_c^i$ is characterized by 8 $\mathcal{I}$-complex numbers. This means that we cannot relate these operators to 4×4 complex matrices and it thus seems that we cannot express the Dirac algebra using these operators. However, later on in this paper we will see that we can still express the Dirac equation with these kind of linear barred operators using a 'special' trick.

4.2 Translations from (complex) quaternions to complex matrices

For this section we will also follow [21]. As we saw, complex linear barred operators on quaternions were characterized by four complex numbers from $\mathbb{C}(1, \mathcal{I})$ and i-complex linear barred operators were characterized by 16 i-complex parameters. This suggests a possible identification between 2×2 complex matrices and complex linear barred operators on quaternions and between 4×4 complex matrices and i-complex linear barred operators on complex quaternions.

4.2.1 2×2 complex matrix identification

Let us first take a look at the correspondence between complex linear barred operators on quaternions and 2×2 matrices. To do so we write a real quaternionic state in the following way

$$q = z_1 + z_2\mathcal{J} \tag{4.15}$$

where z_1 and z_2 are elements of $\mathbb{C}(1, \mathcal{I})$. In this way, we can introduce the so-called *symplectic* complex representation by a column matrix of a real quaternionic state

$$q \leftrightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{4.16}$$

where the $\mathcal{I}$ in the expressions for z_1 and z_2 has now become an i. An operator representation of $1, \mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ that is consistent with the previous identification is the following

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \mathcal{I} \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3, \; \mathcal{J} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \text{ and } \mathcal{K} \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \tag{4.17}$$

where the σ_i represent the corresponding Pauli matrices defined in chapter 2. This representation has actually been known since the discovery of the quaternions and permits any real quaternionic number to be translated into a 2×2 complex matrix. However, the other way around not necessarily. There are eight real numbers necessary to define a 2×2 complex matrix while there are only 4 real numbers necessary to define a real quaternion. Complex linear barred operators complete the identification. With the representation of the barred quaternionic imaginary unit

$$1|\mathcal{I} \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{4.18}$$

we can add four additional degrees of freedom. Namely, by matrix multiplication of the four following corresponding matrices

$$1|\mathcal{I}, \; \mathcal{I}|\mathcal{I}, \; \mathcal{J}|\mathcal{I} \;\; \mathcal{K}|\mathcal{I} \tag{4.19}$$

In that way we get 8 linearly independent 2×2 complex matrices from the 8 linearly independent complex linear barred operators on quaternions. With these additional degrees of freedom we thus have a set of rules for translating from 2×2 complex matrices to complex linear barred operators on quaternions and vice versa.

4.2.2 4×4 complex matrix identification

Let us now take a look at the correspondence between i-complex linear barred operators on complex quaternions and 4×4 complex matrices. In analogy to the symplectic complex representation for a real quaternionic state, we introduce for a complexified quaternionic state

$$q_c = c_0 + c_1\mathcal{I} + c_2\mathcal{J} + c_3\mathcal{K}, \; c_0, c_1, c_2, c_3 \in \mathbb{C}(1, i) \tag{4.20}$$

the symplectic i-complex representation by the following four-dimensional column matrix:

$$q_c \leftrightarrow \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{4.21}$$

The representation for $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ that is consistent with the above identification is the following

$$\mathcal{I} \leftrightarrow \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \; \mathcal{J} \leftrightarrow \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \text{ and } \mathcal{K} \leftrightarrow \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \tag{4.22}$$

We also identify the imaginary unit i with the matrix iI_4. This is, of course, not enough to translate any 4×4 complex matrix into an i-complex linear barred operator. For that, we must also give a representation for the right-action of all the quaternionic imaginary units

$$1|\mathcal{I} \leftrightarrow \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \; 1|\mathcal{J} \leftrightarrow \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \text{ and } 1|\mathcal{K} \leftrightarrow \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \tag{4.23}$$

Since $[\boldsymbol{h}, 1|\boldsymbol{h}] = 0$ we can construct products of the above identifications via left or right multiplication. In that way, we can create a set of rules for translating from 4×4 complex matrices to i-complex linear barred operators and vice versa.

4.3 Real quaternionic formulation of the Dirac equation

For this section we will follow [25]. We begin by stating the Dirac equation in the standard form:

$$i\frac{\partial \Psi}{\partial t} = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\Psi \tag{4.24}$$

with $\boldsymbol{p} = -i\nabla$ and Ψ as a 4×1 complex column matrix. In Equation 4.24 the norm of Ψ is not conserved (see [25]). This means that we have to modify Equation 4.24. The first modification that we make is writing the Dirac equation in the following way

$$\frac{\partial \Psi}{\partial t}\mathcal{I} = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\Psi \tag{4.25}$$

where now Ψ is a real quaternionic column matrix of which we will deduce the dimension later on. Since we have placed the $\mathcal{I}$ on the right of $\frac{\partial \Psi}{\partial t}$, we see that with the old definition of the momentum operator time and space are not treated in the same way. To ensure that Equation 4.25 is relativistically covariant we thus have to modify the action of our momentum operator. The modification that we make is the following

$$\boldsymbol{p}\Psi \equiv -\nabla \Psi \mathcal{I} \tag{4.26}$$

This means that we define the momentum operator as

$$\boldsymbol{p} = -\nabla|\mathcal{I} \tag{4.27}$$

From chapter 2 we know that the $\boldsymbol{\alpha}$ and β in Equation 4.25 have to satisfy the following relations:

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= 0 \\ \{\alpha_i, \beta\} &= 0 \\ \alpha_i^2 = \beta^2 &= 1 \end{aligned} \tag{4.28}$$

for $i, j = 1, 2, 3$. Recall that the α_i and β matrices also had to be Hermitian. In the quaternion formulation this means that the α_i and β matrices have to be *quaternion Hermitian*. The α_i and β matrices thus have to satisfy

$$\begin{aligned} \alpha_i^{@} &= \alpha_i \\ \beta^{@} &= \beta \end{aligned} \tag{4.29}$$

where $i = 1, 2, 3$. Using quaternions, we can give a matrix representation for the α_i and β that satisfies the relations in 4.28 and 4.29:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \alpha_1 = \mathcal{I}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \alpha_2 = \mathcal{J}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \alpha_3 = \mathcal{K}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.30}$$

We can write this equation in covariant form by introducing the γ matrices in the usual way

$$\gamma^0 = \beta,\ \gamma^i = \gamma^0\alpha_i \tag{4.31}$$

The Dirac equation then becomes

$$\gamma^\mu \partial_\mu \Psi \mathcal{I} = m\Psi \tag{4.32}$$

where we see that Ψ is a 2×1 matrix with entries from $\mathbb{H}$. In this equation the gamma matrices again satisfy the defining relations of a Clifford algebra over a 4-dimensional space with metric signature $(+ - --)$.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{4.33}$$

We also have that

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu @} \tag{4.34}$$

A representation of the gamma matrices in $M_2(\mathbb{H})$ is then given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^1 = \mathcal{I}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^2 = \mathcal{J}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma^3 = \mathcal{K}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.35}$$

or compactly written as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{\gamma} = \boldsymbol{h}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.36}$$

Now that we have found this representation of the gamma matrices using real quaternions, let us compare this representation to the representation that we get when we use the translation rules from section 4.2. The translation from the standard gamma matrices to 2×2 real quaternionic matrices yields the following matrices

$$\gamma_t^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma_t^1 = \begin{pmatrix} 0 & \mathcal{K}|\mathcal{I} \\ -\mathcal{K}|\mathcal{I} & 0 \end{pmatrix}, \ \gamma_t^2 = \begin{pmatrix} 0 & \mathcal{J}|\mathcal{I} \\ -\mathcal{J}|\mathcal{I} & 0 \end{pmatrix} \gamma_t^3 = \begin{pmatrix} 0 & -\mathcal{I}|\mathcal{I} \\ \mathcal{I}|\mathcal{I} & 0 \end{pmatrix} \tag{4.37}$$

At first sight these matrices do not seem the same. However, there exists a similarity transformation which transforms the matrices given by the translation into the ones that we got earlier [20]

$$S\gamma_t^\mu S^{-1} = \gamma^\mu \tag{4.38}$$

where

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\mathcal{J} & 0 \\ 0 & (1+\mathcal{J})|\mathcal{I} \end{pmatrix} \tag{4.39}$$

Let us now look at how some of the objects that we defined using the standard complex gamma matrices look in a real quaternionic formalism. From now on we will use the gamma matrices from the paper written by Rotelli [25]. From our expressions for $\boldsymbol{\alpha}$ and β we find that

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m = \begin{pmatrix} m & \boldsymbol{h} \cdot \boldsymbol{p} \\ -\boldsymbol{h} \cdot \boldsymbol{p} & -m \end{pmatrix} \tag{4.40}$$

The spin operator in this quaternionic formulation of the Dirac equation is then defined as follows

$$\boldsymbol{S} \equiv \frac{1}{2}\boldsymbol{h}|\mathcal{I} \tag{4.41}$$

Due to the new definition of $\boldsymbol{p}$, we also have that $[H, \boldsymbol{p}] = 0$. This means that there exist plane wave solutions. These plane wave solutions are of the form

$$\Psi(\boldsymbol{x}, t) = ue^{-ip_\mu x^\mu} \tag{4.42}$$

Rotelli gives the following four complex-orthogonal solutions for u:

$$u(\boldsymbol{p}) = \begin{pmatrix} 1 \\ \frac{-\boldsymbol{h}\cdot\boldsymbol{p}}{E_p+m} \end{pmatrix}\mathcal{J} \text{ and } u(\boldsymbol{p}) = \begin{pmatrix} 1 \\ \frac{-\boldsymbol{h}\cdot\boldsymbol{p}}{E_p+m} \end{pmatrix} \text{ for } E = +E_p, \tag{4.43}$$

$$u(\boldsymbol{p}) = \begin{pmatrix} \frac{-\boldsymbol{h}\cdot\boldsymbol{p}}{-E_p+m} \\ 1 \end{pmatrix}\mathcal{J} \text{ and } u(\boldsymbol{p}) = \begin{pmatrix} \frac{-\boldsymbol{h}\cdot\boldsymbol{p}}{-E_p+m} \\ 1 \end{pmatrix} \text{ for } E = -E_p, \tag{4.44}$$

where we have left out the normalization factor. The solutions of the Dirac equation are then given by the corresponding $\Psi(\boldsymbol{x}, t)$. We can also define a conserved current in this representation. To do so, we make a slight modification in the definition of $\overline{\Psi}$. Instead of defining it like in section 2.1, we define it as

$$\overline{\Psi} = \Psi^{@}\gamma^0 \tag{4.45}$$

To prove that $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ is now our conserved current we take a look at Equation 4.32

$$\gamma^0 \frac{\partial \Psi}{\partial t}\mathcal{I} + \boldsymbol{\gamma} \cdot \nabla\Psi\mathcal{I} = m\Psi \tag{4.46}$$

and take the adjoint of this equation

$$-\mathcal{I}\frac{\partial \Psi^@}{\partial t}\gamma^0 + \mathcal{I}\nabla\Psi^@ \cdot \boldsymbol{\gamma} = m\Psi^@ \tag{4.47}$$

Let us now multiply Equation 4.46 from the left by $\overline{\Psi}$ and Equation 4.47 from the right by $\gamma^0\Psi$. Equation 4.46 then becomes

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi\mathcal{I} = m\overline{\Psi}\Psi \tag{4.48}$$

and Equation 4.47 becomes

$$\mathcal{I}(\partial_\mu\overline{\Psi})\gamma^\mu\Psi = -m\overline{\Psi}\Psi \tag{4.49}$$

By observing that the term on the right hand side of both of these equations is a real number, we conclude that the terms $\overline{\Psi}\gamma^\mu\partial_\mu\Psi$ and $(\partial_\mu\overline{\Psi})\gamma^\mu\Psi$ are quaternions ($q = c_0 + c_1\mathcal{I} + c_2\mathcal{J} + c_3\mathcal{K}$) that only have a nonzero coefficient in front of $\mathcal{I}$. This means that $\overline{\Psi}\gamma^\mu\partial_\mu\Psi$ and $(\partial_\mu\overline{\Psi})\gamma^\mu\Psi$ commute with $\mathcal{I}$. If we now add them, we find that

$$\partial_\mu(\overline{\Psi}\gamma^\mu\Psi) = 0 \tag{4.50}$$

We may therefore conclude that in this formalism, $\overline{\Psi}\gamma^\mu\Psi$ is the conserved current.
A particularly interesting thing that Rotelli noticed was that in order to derive the real quaternionic Dirac equation he had to admit the axiom that all scalar products are complex, even if wave functions and operators were quaternions. One may note that in a way this is analogous to the passage from classical mechanics (essentially based on the reals) to quantum mechanics (based on the complex numbers) where observables are required to have real eigenvalues.

4.4 Complex quaternionic formulation of the Dirac equation

In section 4.1 we found that there are two types of complex linear operators on complex quaternions. We have the i-complex linear operators and the $\mathcal{I}$-complex linear operators. We can use both of these to formulate a complex quaternionic Dirac equation.

4.4.1 i-complex linear formulation

First we will take a look at the complex quaternionic formulation of the Dirac equation using i-complex geometry. For this section we will follow [21] and [19]. We can obtain a one-component complexified quaternionic Dirac equation using the translation that we found in section 4.2. The complex quaternionic representation for the standard γ^μ-matrices that we find in this way is the following

$$\gamma^0 = -\mathcal{I}|\mathcal{I} \text{ and } \boldsymbol{\gamma} = -(\mathcal{K}, i\mathcal{I}|\mathcal{J}, \mathcal{J}) \tag{4.51}$$

Even though this translation between 4×4 complex matrices and i-complex linear barred operators is interesting itself, we will not use the complex quaternionic representation of the γ^μ-matrices that we get from it. This representation is namely not a very elegant complex quaternionic representation. We will therefore use the following (more elegant) complex quaternionic representation for the rest of this section.

$$\gamma^0 = \mathcal{I}|\mathcal{I} \text{ and } \boldsymbol{\gamma} = i\boldsymbol{h}|\mathcal{J} \tag{4.52}$$

These matrices satisfy the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{4.53}$$

and

$$\gamma^{0@} = \gamma^0, \ \boldsymbol{\gamma}^@ = -\boldsymbol{\gamma} \tag{4.54}$$

The one-component Dirac equation then reads

$$i\gamma^\mu\partial_\mu\Psi(x) = m\Psi(x) \tag{4.55}$$

where $\Psi(x)$ is a complex quaternion. Note that when we are working with i-complex geometry, we use the original definition of the momentum operator ($\boldsymbol{p} = i\nabla$). Even though this one-component wave equation is quite elegant, it does come with some problems. For example, the spinor structures are complicated and the $\mathcal{CPT}$ interpretation is unclear. Therefore there is no real reason to prefer this formulation over the complex formulation. However, there is one interesting thing that follows from this interpretation. As we know from Proposition 3.1.18, the algebra of complex quaternions $\mathbb{C} \otimes \mathbb{H}$ is isomorphic to $M_2(\mathbb{C})$. This would suggest that we can write the Dirac equation based on the Clifford algebra $A_{0,3}$. This Clifford algebra is called the *Pauli algebra* and is generated by the three Pauli matrices. Table 4 shows the basis of the matrix algebra $M_2(\mathbb{C})$ together with the corresponding basis of the complexified quaternion algebra.

Table 4: Basis of $M_2(\mathbb{C})$ and $\mathbb{C} \otimes \mathbb{H}$

$M_2(\mathbb{C})$	$\mathbb{C} \otimes \mathbb{H}$
I_2	1
$\sigma_1, \sigma_2, \sigma_3$	$i\mathcal{I}, i\mathcal{J}, i\mathcal{K}$
$\sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_1$	$\mathcal{I}, \mathcal{J}, \mathcal{K}$
$\sigma_1\sigma_2\sigma_3$	i

We can try to obtain a formulation of the Dirac equation using the matrix algebra $M_2(\mathbb{C})$ by translation from our complex quaternionic version. The spinors in our new formulation will be 2×2 complex matrices. The most general transformation on the 4-dimensional complex column matrix

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{4.56}$$

is of course performed by a complex 4×4 matrix. Which, as we know, has 16 complex parameters. Let us now rewrite the previous 4-dimensional column matrix by a 2×2 complex matrix.

$$\Psi = \begin{pmatrix} \psi_a & \psi_b \\ \psi_c & \psi_d \end{pmatrix}, \;\; \psi_a = \psi_1 - i\psi_2,\; \psi_b = -\psi_4 + i\psi_3,\; \psi_c = \psi_4 + i\psi_3,\; \psi_d = \psi_1 + i\psi_2 \tag{4.57}$$

This matrix can be rewritten as

$$\Psi = \psi_1 I_2 + \sigma_2\sigma_1\psi_2 + \sigma_2\sigma_3\psi_3 + \sigma_3\sigma_1\psi_4 \tag{4.58}$$

From our expression for the most general transformation on complex quaternions, we find that the most general transformation on our "new" spinors is given by

$$M_0 + M_1|\sigma_1 + M_2|\sigma_2 + M_3|\sigma_3 \tag{4.59}$$

where the M_0, M_1, M_2 and M_3 are 2×2 complex matrices. As you can see, this general transformation also has 16 complex parameters. We can therefore translate from i-complex linear barred complexified quaternionic operators to these general transformations containing 2×2 complex matrices and barred operators. An explicit representation of the gamma matrices using 2×2 complex matrices is given by

$$\gamma^0 = I_2|\sigma_1,\; \gamma^1 = -\sigma_{123}\sigma_1|\sigma_2,\; \gamma^2 = -\sigma_{123}\sigma_2|\sigma_2 \text{ and } \gamma^3 = -\sigma_{123}\sigma_3|\sigma_2 \tag{4.60}$$

where $\sigma_{123} = \sigma_1\sigma_2\sigma_3$. As you can see, the explicit expressions for these gamma matrices contain barred operators. We can therefore not say with certainty that these matrices are elements of $M_2(\mathbb{C})$. If we have formulated the Dirac equation using Pauli algebra here is thus open to question.

4.4.2 $\mathcal{I}$-complex linear formulation

For this section we will follow [21] and [22]. When working with $\mathcal{I}$-complex geometry we again use the following definition for the momentum operator

$$\boldsymbol{p} = -\nabla|\mathcal{I} \tag{4.61}$$

As we saw, an $\mathcal{I}$-complex linear barred operator on the complexified quaternionic spinor was characterized by only 8 $\mathcal{I}$-complex parameters instead of 16.

$$\mathcal{O}_c^{\mathcal{I}} \equiv q_c + p_c|\mathcal{I} \tag{4.62}$$

where q_c and p_c are complex quaternions. This would already suggest that we will find difficulties in formulating the Dirac equation using $\mathcal{I}$-complex geometry. We do indeed find these difficulties when looking for the γ^μ-matrices satisfying the Dirac algebra. There are actually not that many problems with finding the $\boldsymbol{\gamma}$-matrices, in fact we find as a suitable choice

$$\boldsymbol{\gamma} = \boldsymbol{h} \equiv (\mathcal{I}, \mathcal{J}, \mathcal{K}),\ \{h^m, h^n\} = 2\eta^{mn}\ (m, n = 1, 2, 3),\ \boldsymbol{h}^{@} = -\boldsymbol{h} \tag{4.63}$$

Nevertheless, we cannot seem to find a quaternionic number that anticommutes with $\boldsymbol{h}$ and therefore cannot give a complexified quaternionic representation of the γ^0-matrix. However, there is a trick that we can use to solve this problem. Let us take a look at the action of the standard γ^0-matrix on the complex spinor $\Psi \in \mathbb{C}^4$

$$\gamma^0\Psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{4.64}$$

For the complex quaternions

$$\psi_1 + \mathcal{J}\psi_2 + i(\psi_3 + \mathcal{J}\psi_4),\ \psi_{1,2,3,4} \in \mathbb{C}(1, \mathcal{I}) \tag{4.65}$$

to correspond with the complex spinor $\psi \in \mathbb{C}^4$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix} \tag{4.66}$$

we have have to find an operation that performs the following translation

$$\psi_1 + \mathcal{J}\psi_2 + i(\psi_3 + \mathcal{J}\psi_4) \to \psi_1 + \mathcal{J}\psi_2 - i(\psi_3 + \mathcal{J}\psi_4) \tag{4.67}$$

The solution is then of course the $\bullet$-involution, $\Psi \to \Psi^\bullet$. Our Dirac equation

$$\left(\partial_t + \gamma^0\boldsymbol{\gamma}\cdot\nabla\right)\Psi(x)\mathcal{I} = m\gamma^0\Psi(x) \tag{4.68}$$

can then be rewritten as

$$\left(\partial_t + i\boldsymbol{h}\cdot\nabla\right)\Psi(x)\mathcal{I} = m\Psi^\bullet(x) \tag{4.69}$$

On the left hand side, you can see that we substituted γ^0 for i. To justify this choice we will show that this new Dirac equation reduces to the Klein-Gordon equation. Let us therefore write Equation 4.69 as

$$D\Psi = m\psi^\bullet \tag{4.70}$$

where

$$D \equiv \left(\partial_t + i\boldsymbol{h}\cdot\nabla\right)|\mathcal{I} \tag{4.71}$$

If we now multiply Equation 4.70 from the left with the barred operator

$$D^\bullet \equiv \left(\partial_t - i\boldsymbol{h}\cdot\nabla\right)|\mathcal{I} \tag{4.72}$$

we get

$$D^\bullet D\Psi = -\left(\partial_t^2 - \nabla^2\right)\Psi = mD^\bullet\Psi^\bullet \tag{4.73}$$

Note that $D^\bullet\Psi^\bullet = m\Psi$, so that 4.73 gives us the Klein-Gordon equation

$$(\partial_\mu\partial^\mu + m^2)\Psi = 0 \tag{4.74}$$

Let us try to find the solutions for this Dirac equation. Instead of solving four coupled equations directly, we will make the following ansatz

$$\Psi = \psi_{\boldsymbol{p}} e^{-\mathcal{I}p_\mu x^\mu} \tag{4.75}$$

When we insert this function into Equation 4.69 we find

$$(E - \boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p})\,\psi_{\boldsymbol{p}} = m\psi_{\boldsymbol{p}}^\bullet \tag{4.76}$$

If we then pose

$$\psi_{\boldsymbol{p}} = u(\boldsymbol{p}) + v(\boldsymbol{p}),\ u(\boldsymbol{p}) \in \mathbb{H} \text{ and } v(\boldsymbol{p}) \in i\mathbb{H}, \tag{4.77}$$

then the Dirac equation becomes

$$(\partial_t + \boldsymbol{i}\boldsymbol{h}\cdot\nabla)\,(u(\boldsymbol{p}) + v(\boldsymbol{p}))e^{-\mathcal{I}p_\mu x^\mu}\mathcal{I} = m(u(\boldsymbol{p}) - v(\boldsymbol{p}))e^{-\mathcal{I}p_\mu x^\mu} \tag{4.78}$$

so that

$$(E - \boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p})\,(u(\boldsymbol{p}) + v(\boldsymbol{p})) = m(u(\boldsymbol{p}) - v(\boldsymbol{p})) \tag{4.79}$$

From the following two coupled equations

$$\begin{aligned} Eu(\boldsymbol{p}) - \boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p}v(\boldsymbol{p}) &= mu(\boldsymbol{p}) \\ Ev(\boldsymbol{p}) - \boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p}u(\boldsymbol{p}) &= -mv(\boldsymbol{p}) \end{aligned} \tag{4.80}$$

we can then derive the complexified quaternionic solutions to the Dirac equation. The positive-energy solutions that we find are

$$\psi_{\boldsymbol{p}}^+ = 1 + \frac{\boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p}}{|E| + m} \text{ and } \psi_{\boldsymbol{p}}^+ = \left(1 + \frac{\boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p}}{|E| + m}\right)\mathcal{J} \tag{4.81}$$

and the negative-energy solutions that we find are

$$\psi_{\boldsymbol{p}}^- = \left(1 + \frac{\boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p}}{|E| + m}\right) i \text{ and } \psi_{\boldsymbol{p}}^- = \left(1 + \frac{\boldsymbol{i}\boldsymbol{h}\cdot\boldsymbol{p}}{|E| + m}\right) i\mathcal{J} \tag{4.82}$$

where we have left out the normalization factor. We can also give an explicit form of the spin operator in this framework, namely

$$S_x = -\frac{\mathcal{I}|\mathcal{I}}{2} \tag{4.83}$$

For $\boldsymbol{p} = (p_x, 0, 0)$ we find that our four solutions, again, correspond to particles with $S_x = \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$ respectively. We have chosen our polarization direction along the x-axis because we associate the imaginary unit $\mathcal{I}$ with p_x.

The interesting thing about this formulation of the Dirac equation is that it has a very elegant $\mathcal{CPT}$ interpretation. Let us recall the $\mathcal{CPT}$ operations from section 2.3

$$\begin{aligned} \Psi &\equiv \Psi(x), \\ \Psi_P &\equiv \gamma^0\Psi(-x), \\ \Psi_C &\equiv i\gamma^2(\overline{\Psi})^T, \\ \Psi_T &\equiv \gamma^1\gamma^3\Psi^* \end{aligned} \tag{4.84}$$

where in Equation 4.84 we have of course used the standard complex representation to represent the γ^μ-matrices. Let us start with formulating the parity transformation in our new formalism. If we start with Equation 4.69 and perform the required coordinates transformation

$$\boldsymbol{x} \to -\boldsymbol{x} \tag{4.85}$$

then we obtain the following transformed Dirac equation

$$(\partial_t - i\boldsymbol{h} \cdot \nabla)\, \Psi_P \mathcal{I} = m\Psi_P^{\bullet} \tag{4.86}$$

It now becomes very easy to find the relation between the transformed wave function Ψ_P and the initial wave function Ψ. The $\bullet$-involution transforms the original Dirac equation as follows

$$(\partial_t - i\boldsymbol{h} \cdot \nabla)\, \Psi^{\bullet} \mathcal{I} = m\Psi \tag{4.87}$$

If we compare this equation with Equation 4.86, we immediately find that

$$\Psi_P \equiv \Psi^{\bullet} \tag{4.88}$$

Let us now take a look at the charge conjugation operation. To discuss this operation we also have to introduce an external potential $(\Phi, \boldsymbol{A})$, just like in section 2.3.3. We do this in the following way

$$\begin{aligned} \partial_t|\mathcal{I} &\to \partial_t|\mathcal{I} + e\Phi, \\ \nabla|\mathcal{I} &\to \nabla|\mathcal{I} - e\boldsymbol{A} \end{aligned} \tag{4.89}$$

The Dirac equation then reads

$$\left[\partial_t|\mathcal{I} + e\Phi + i\boldsymbol{h} \cdot (\nabla|\mathcal{I} - e\boldsymbol{A})\right] \Psi = m\Psi^{\bullet} \tag{4.90}$$

If we now perform the change $e \to -e$, the Dirac equation becomes

$$\left[\partial_t|\mathcal{I} - e\Phi + i\boldsymbol{h} \cdot (\nabla|\mathcal{I} + e\boldsymbol{A})\right] \Psi_C = m\Psi_C^{\bullet} \tag{4.91}$$

Let us now search for the charge conjugated wave function. By multiplying Equation 4.90 by i

$$\left[\partial_t|\mathcal{I} + e\Phi + i\boldsymbol{h} \cdot (\nabla|\mathcal{I} - e\boldsymbol{A})\right] (i\Psi) = -m(i\Psi)^{\bullet} \tag{4.92}$$

and from the right by $\mathcal{J}$, we get

$$\left[-\partial_t|\mathcal{I} + e\Phi + i\boldsymbol{h} \cdot (-\nabla|\mathcal{I} - e\boldsymbol{A})\right] (i\Psi\mathcal{J}) = -m(i\Psi\mathcal{J})^{\bullet} \tag{4.93}$$

We can rewrite this equation as

$$\left[\partial_t|\mathcal{I} - e\Phi + i\boldsymbol{h} \cdot (\nabla|\mathcal{I} + e\boldsymbol{A})\right] (i\Psi\mathcal{J}) = m(i\Psi\mathcal{J})^{\bullet} \tag{4.94}$$

from which we can conclude that

$$\Psi_C \equiv i\Psi\mathcal{J} \tag{4.95}$$

Let us now look at the last operation, the time reversal operation. In the presence of an external potential, the motion can only be reversed by the following transformation (See chapter 11.4.1 of [29])

$$\Phi \to \Phi \text{ and } \boldsymbol{A} \to -\boldsymbol{A} \tag{4.96}$$

Applying this transformation on the Dirac equation gives us

$$\left[-\partial_t|\mathcal{I} + e\Phi + i\boldsymbol{h} \cdot (\nabla|\mathcal{I} + e\boldsymbol{A})\right] \Psi_T = m\Psi_T^{\bullet} \tag{4.97}$$

If we multiply Equation 4.90 from the right by $\mathcal{J}$, we get

$$\left[-\partial_t|\mathcal{I} + e\Phi + i\boldsymbol{h} \cdot (-\nabla|\mathcal{I} - e\boldsymbol{A})\right] (\Psi\mathcal{J}) = m(\Psi\mathcal{J})^{\bullet} \tag{4.98}$$

Applying the $\bullet$-involution to this equation gives us

$$\left[-\partial_t|\mathcal{I} + e\Phi + i\boldsymbol{h} \cdot (\nabla|\mathcal{I} + e\boldsymbol{A})\right] (\Psi\mathcal{J})^{\bullet} = m(\Psi\mathcal{J}) \tag{4.99}$$

It is now easily seen that

$$\Psi_T = \Psi^{\bullet}\mathcal{J} \tag{4.100}$$

As you can see, the $\mathcal{C}$,$\mathcal{P}$ and $\mathcal{T}$ operations are very elegant. One may even prefer the $\mathcal{I}$-complex formulation over the standard complex formulation when working with these operations.

4.5 Another complex quaternionic formulation of the Dirac equation

Another formulation of the Dirac equation using complex quaternions that differs from the two above is one given in [28]. To give this formulation we define the *contravariant differentiation operator* D by

$$D \equiv i\frac{\partial}{\partial t} - \mathcal{I}\frac{\partial}{\partial x} - \mathcal{J}\frac{\partial}{\partial y} - \mathcal{K}\frac{\partial}{\partial z} = i\partial_t - \mathcal{I}\partial_x - \mathcal{J}\partial_y - \mathcal{K}\partial_z \tag{4.101}$$

The Dirac equation stated in [28] is the following

$$\begin{pmatrix} -m & D \\ D^{@} & -m \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \tag{4.102}$$

To link this Dirac equation to Table 2 from section 3.3 we want to write this equation in the following form

$$i\partial_t\Psi = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\Psi \tag{4.103}$$

where $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. The first step in doing so, is writing Equation 4.102 as

$$i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_t\Psi = \left(\mathcal{I}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial_x + \mathcal{J}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial_y + \mathcal{K}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial_z + m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\Psi \tag{4.104}$$

If we now write this as

$$i\partial_t\Psi = \left(\mathcal{I}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_x + \mathcal{J}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_y + \mathcal{K}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_z + m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\Psi \tag{4.105}$$

we can recognize the α_i and β matrices with which we retrieve Equation 4.103. Namely

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \boldsymbol{\alpha} = \boldsymbol{h}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.106}$$

Recall that the gamma matrices are defined as $\gamma^0 = \beta$ and $\gamma^i = \gamma^0\alpha_i$. This definition gives us the following gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \boldsymbol{\gamma} = \boldsymbol{h}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.107}$$

These gamma matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ so that we can write the Dirac equation in its covariant form

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0 \tag{4.108}$$

As you can see, the gamma matrices are all elements of $M_2(\mathbb{C}\otimes\mathbb{H})$. However, from Proposition 3.1.18 we know that $\mathbb{C}\otimes\mathbb{H} \cong M_2(\mathbb{C})$. This means that $M_2(\mathbb{C}\otimes\mathbb{H}) \cong M_2(\mathbb{C})\otimes M_2(\mathbb{C}) \cong M_4(\mathbb{C})$. By writing the Dirac equation this way we are thus again representing $A_{3,1}$ as a subalgebra of $A_{3,2}$ and essentially using the same Clifford algebra that is "too large". We can even retrieve the Weyl representation of the complex gamma matrices when we perform the following translation

$$1 \to I_2, \mathcal{I} \to i\sigma_1, \mathcal{J} \to i\sigma_2 \text{ and } \mathcal{K} \to i\sigma_3 \tag{4.109}$$

Even though we are essentially not using a different Clifford algebra, let us still take a look at what some of the operators and solutions look like in this formalism. The spin operator in this formulation of the Dirac equation is defined as follows

$$\boldsymbol{S} = (S_x, S_y, S_z) \equiv \frac{i}{2}\boldsymbol{h} \tag{4.110}$$

It can be verified that these expressions for the components of the spin operator satisfy the following algebra

$$[S_x, S_y] = iS_z,\ [S_y, S_z] = iS_x \text{ and } [S_z, S_x] = iS_y \tag{4.111}$$

If we then choose the z-axis as the spin quantization direction we find that the solutions for this formulation of the Dirac equation (which are also eigenstates of S_z with eigenvalues $m_z = \pm\frac{1}{2}$) are of the form [28]

$$\begin{aligned} m_z = \frac{1}{2} &: \ (1 + i\mathcal{K})\Psi_0, \ (\mathcal{I} + i\mathcal{J})\Psi_0 \\ m_z = -\frac{1}{2} &: \ (i\mathcal{I} + \mathcal{J})\Psi_0, \ (-i - \mathcal{K})\Psi_0 \end{aligned} \tag{4.112}$$

where

$$\Psi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-imt} \tag{4.113}$$

It can be checked that the following two subspaces

$$span_{\mathbb{C}}(1 + i\mathcal{K})\Psi_0, (i\mathcal{I} + \mathcal{J})\Psi_0 \text{ and } span_{\mathbb{C}}(\mathcal{I} + i\mathcal{J})\Psi_0, (-i - \mathcal{K})\Psi_0 \tag{4.114}$$

are closed under the spin algebra 4.111. We call these two subspaces the *spin eigenspaces*. Note that we can convert the eigenvectors of the same eigenvalues in 4.112 by right multiplication with the quaternionic basis elements $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$:

$$(1 + i\mathcal{K})\Psi_0\mathcal{I} = (\mathcal{I} + i\mathcal{J})\Psi_0 \text{ and } (i\mathcal{I} + \mathcal{J})\Psi_0\mathcal{I} = (-i - \mathcal{K})\Psi_0 \tag{4.115}$$

Right multiplication with the other basis elements $\mathcal{J}$ and $\mathcal{K}$ does not yield independent states. We thus see that we can connect the two spin eigenspaces in 4.114 by right multiplication with $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$.
In this complex quaternionic formulation of the Dirac equation a doubling of solutions occurs. This is evident from the existence of two closed spin eigenspaces instead of one. These two spin eigenspaces are connected by right multiplication of the quaternionic basis elements $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$. We can generalize these transformations to

$$\Psi \to \Psi' = \Psi e^{-n\beta} \tag{4.116}$$

where $n = n_x\mathcal{I} + n_y\mathcal{J} + n_z\mathcal{K}$ is a complex quaternion and $\beta \in \mathbb{R}$. In [28], it is shown that this transformation leaves the *Lagrangian density* invariant and that 4.116 represents a *global* $SU(2)$ *gauge symmetry*. In this thesis we will not go into the details of these derivations but just state the conclusions. This symmetry namely connects the two spin eigenspaces while leaving the spin eigenvalues invariant. If we thus identify the states connected by gauge transformations 4.116 with each other, the number of solutions for our complex quaternionic formulation of the Dirac equation reduces back to the original amount.
If we take a look at the solutions that we find for our complex quaternionic formulation of the Dirac equation, we see that the following states

$$\Psi_0, \mathcal{I}\Psi_0, \mathcal{J}\Psi_0, \mathcal{K}\Psi_0 \tag{4.117}$$

actually form a basis of the subspace of particle solutions. Note that the elements of this basis mix the two eigenspaces 4.114:

$$\Psi_0 = \frac{1}{2}(1 + i\mathcal{K})\Psi_0 + \frac{i}{2}(-i - \mathcal{K})\Psi_0 \tag{4.118}$$

Finding this basis does not specifically lead us to any new physical results, but it is still an elegant peculiarity of this complex quaternionic formulation of the Dirac equation.

5 Conclusion

After first looking at the derivation of the Dirac equation and investigating its symmetries and some of the insights that followed from it, we looked at the Clifford algebras of which we found a representation in the derivation of the Dirac equation. We eventually classified universal Clifford algebras for regular quadratic spaces of all possible dimensions and saw that it would be interesting to take a look at the formulations of the Dirac equation using quaternions.

In the last chapter we have looked at certain different formulations of the Dirac equation using (complex) quaternions and looked at certain aspects of the Dirac theory in these new formalisms. Especially the one-component Dirac equation using $\mathcal{I}$-complex geometry returned a very elegant Dirac equation and $\mathcal{CPT}$ interpretation. However, elegance seems to be the only actual result of formulating the Dirac equation using (complex) quaternions. It is highly unlikely that these formulations of the Dirac equation will yield any new physical results. At first it was thought that the doubling of solutions that occurred in the complex quaternionic Dirac equation that James Edmonds [15] formulated would allow for more degrees of freedom and in that way produce 'new physics'. However, De Leo later showed [18] that the doubling of solutions is strictly connected with the use of reducible matrices.. Also Schuricht and Greiter showed [28] that the doubling of solutions that occurred in their complex quaternionic formulation of the Dirac equation could be attributed to different gauge choices of a global $SU(2)$ gauge symmetry. All in all, we can say that formulating the Dirac equation using (complex) quaternions has not been particularly useful. Finding new physical results from these new formulations turned out to be highly unlikely and the anticommutativity of the quaternions makes doing calculations with quaternions rather cumbersome. Together with the underdevelopedness of the system of quaternions in comparison with the notation usually used in quantum mechanics means that there is no real reason to prefer the (complex) quaternionic formulation of the Dirac equation over the standard complex one. Even though the standard complex notation essentially makes use of a Clifford algebra that is 'too big'. Of course this result is not what the mathematicians and physicists hoped for when they started with the idea of formulating the Dirac equation using complex quaternions, but the elegance of the one-component Dirac equation is still an impressive result in itself. So to end on a positive quote by Dirac himself: "If one is working from the point of view of getting beauty into one's equation, ... one is on a sure line of progress." [9].

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