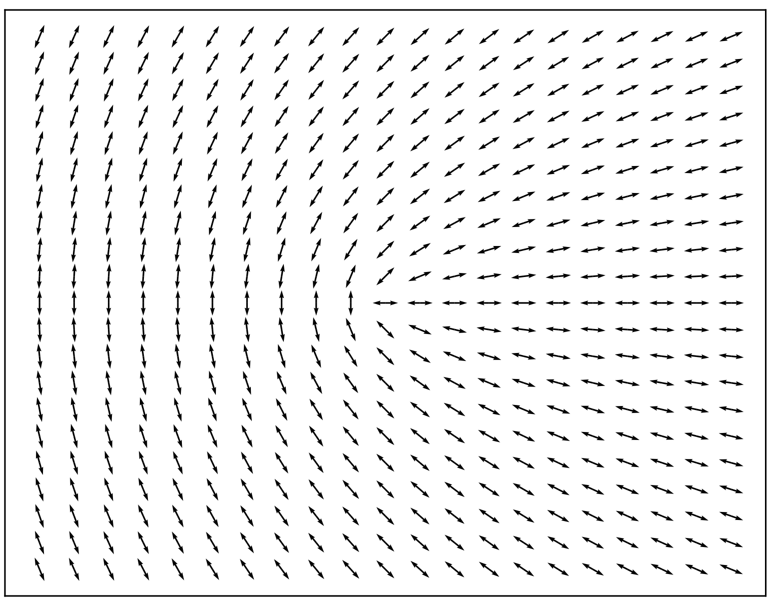
Faculteit Bètawetenschappen

Applications of homotopy theory to topological solitons

Bachelor Thesis

Samuel Klumpers

Natuur- en Sterrenkunde // Wiskunde



Supervisors:

dr. L. Fritz

Institute for Theoretical Physics

dr. G.R. Cavalcanti

Mathematical Institute

June 16, 2021

Abstract

This thesis is an introduction to topological the topological theory defects and solitons, aimed to bring the mathematical and physical perspectives closer together.

The ordered media in which defects arise are introduced, and their configurations are described using homotopy theory. This description is reformulated in terms of symmetry groups, isotropy groups and homogeneous spaces, of which the homotopy groups are calculated using the long exact sequence of homotopy groups for fibrations. Finally, an overview of the applications of this method to a variety of systems is given, and the secondary phenomena arising due to the presence of defects or solitons are described, concluding with the complete classification of defects in $SO(3)$ symmetry broken media in 3 dimensions.

Contents

Introduction

The aim of this thesis is to present the topological theory of defects arising in ordered media in a mathematical setting. These defects have been well-studied in the context of topological quantum field theories, which is strongly influenced by homotopy theory. In these field theories, they are known as topological solitons and exist for example in the form of vortices in the theory of type-II superconductors.

The main motivation is to provide a rederivation and generalization of the results presented in [1] by means of fibrations. We will start by introducing the framework of order-parameter theory, from which we motivate the study and classification of topological defects. We then present a classification scheme of defects, and reflect on the similarities and differences with respect to the classification of [1]. In order to derive the long exact sequence of homotopy groups for fibrations, we introduce the basics of homotopy theory. Together with the formulation of coset spaces of Lie groups as principal bundles, we derive the fundamental theorem on fundamental groups and the long exact sequence of homotopy groups of order-parameter spaces as given in [1]. A minor diversion into miscellaneous phenomena and generalized defects is presented, and to conclude we apply the results to the ordered media with broken $SO(3)$ symmetry.

Basic knowledge in classical and statistical physics, in group theory, and in topology is assumed, however, most of the necessary terminology and used facts are mentioned in passing.

Part I

Ordered Media

In Section I.1 of this part, we divert into an introduction to classical physics, which leads us into statistical physics. There, we outline discuss the necessary theories for the description of phase transitions. From phase transitions we motivate the definition of order and order-parameters. Using the order-parameters we can model certain types of phase transitions. We demonstrate the application to the planar ferromagnet, which is generalized to other ordered media.

In Section I.2 we describe how defects can arise in ordered media, and apply this to the example of the planar ferromagnet. This example illustrates a topological invariant that prevents a connection between two states, and will inspire the more topological discussion of defects later.

I.1 From classical physics to symmetry breaking

The goal of this section is to give a description of defects in ordered media, starting from simpler classical systems. A crystal is such an ordered medium, and has a highly ordered structure. However, a crystal can still have imperfections. We say that defects are the generalization of the crystal imperfections, and from this picture we see that before we can describe defects, we first need to describe "generalized perfection". The order of a medium is precisely the generalization of perfection. We conclude that the discussion of defects is inseparable from the discussion of order.

I.1.1 Classical physics

We enter a lengthy diversion into the physics that ultimately lead to order-parameter theory, the study of ordered media. This introduction into the language of physics is heavily guided by simple examples. Examples of interesting things are the solar system, a glass of water, the air in your room and a nonspecific crystal. Of these, the water in the glass and the nonspecific crystal are most relevant, because they are more ordered. We say that they are condensed, because they are composed of many particles and are cohesive [1], unlike the solar system, which has few components[2], and the air, which is a gas and not very cohesive.

To describe the behaviour of these examples, we need Hamiltonian mechanics. The first example, the solar system, is completely described by coordinates and momenta of the sun and planets, together with the Hamiltonian that describes the total energy of a given state. In this case it is the sum of the kinetic and gravitational energy. We say that the coordinates and momenta take values in the phase space, but the form and description of the phase space depend on the problem considered. If we call the phase space X, and the Hamiltonian H, then a state x in X has an energy assigned by H. Energy is a real number, so we say that the Hamiltonian is a map $H : X \to \mathbb{R}$ and the energy of x is $H(x)$.

[1] To be precise, the correlations of particle-particle distances become peaked at certain lengths, as opposed to being roughly uniform like in a gas.

[2] When the sun and each planet are viewed a single point mass.

I.1.2 Statistical physics

The other three examples are many-particle problems, which we examine through statistical physics. Statistical physics studies systems of 10^{26} interacting particles by means of probability theory and certain approximations. Central to statistical physics are phase transitions; when cooling down materials, gases condense to liquids and liquids freeze to crystals. The temperatures at which phase transitions occur are called critical temperatures. To talk about freezing water, we need to know what temperature is and what happens when a collection of particles is cooled down.

Before describe temperature, we need to be able to describe collections of particles and their energies. We say that a microcanonical ensemble is a collection of N particles confined to a volume V with a fixed energy E. The microcanonical ensemble is also called the NVE ensemble, and N, V and E are called the macroscopic variables. A specific state of the collection is called a microstate, and the number of microstates of energy E is denoted $W = |\{x \in X | H(x) = E\}|$. The microcanonical ensemble assumes[3] that the probability to encounter an arbitrary microstate x with $H(x) = E$ is $P(x) = 1/W$, and the probability to encounter a microstate x with $H(x) \neq E$ is $P(x) = 0$.

To define a temperature on a system X with N particles and volume V, we attach a large heat bath. Writing E_{bath} for the energy of the bath and fixing the total energy $E_{\text{tot}} = E_{\text{bath}} + E$ turns this combination into a microcanonical ensemble. We say that the original system is a canonical ensemble with temperature T, also called the NVT ensemble. We can express W_{tot} as the sum of $W_{\text{bath}}(E_{\text{tot}} - H(x))$ over all microstates x of X. Assuming $E \ll E_{\text{tot}}$, we find $P(x) = \frac{1}{Z}\exp(-\beta H(x))$. Here Z is the partition function $\sum_x \exp(-\beta H(x))$. This defines the temperature as $T = \frac{1}{k\beta}$, where k is the Boltzmann constant. The free energy F is defined as $-\frac{1}{\beta}\log Z$, so that $P(x) = \exp(\beta(F - H(x)))$.

Macroscopic behaviour of an ensemble must certainly emerge from many particles and be described by averages. Given a map ϕ from the phase space X to any vector space Y, we define its thermodynamic average as $\langle\phi\rangle = \sum_{x\in X}\phi(x)P(x)$. For a continuous system we write $f_X(x)$ instead of $P(x)$ for the probability density, and the thermodynamic average becomes an integral over X. Examples of thermodynamic averages include the average energy $U = \langle E\rangle$, the entropy $S = -k\langle\log f_X(x)\rangle$ and the pressure $p = -\frac{\partial F}{\partial V}$. The free energy can be written in terms of averages as $F = U - TS$.

I.1.3 Phase transitions

With the free energy and the thermodynamic averages, we can get back to the discussion of phase transitions. Consider the free energy F as a function of the temperature T. We then say that a phase transition occurs at those T where some derivative of F becomes singular.

We describe phase transitions by their order-parameters, and the order-parameter of a condensing gas is evidently the density. In the example of the gas, the density relates to the pressure, which in turn determines the energy. The free energy is also a function of the order-parameter.

Now we can apply Landau theory: assuming that the free energy F is an analytic function of the order-parameter η and has the same symmetries as the Hamiltonian, we can write down a phenomenological[4] expansion of F in terms of η, near the critical temperature T_c. In most systems we consider, we find that F has a symmetry that requires it to actually be a function of η^2, so the expansion becomes:

$$F(T,\eta) - F_0 = a(T)\eta^2 + \frac{b(T)}{2}\eta^4.$$

[3]This follows from the postulate of a priori equal probabilities.

[4]Phenomenological here means that we assume the expansion is correct, based on empirical evidence, or as is also the case here, is flexible enough to predict what we observe and more.

The order-parameter can be chosen to be 0 at T_c, so the higher terms are negligible. The coefficient $b(T)$ should be positive, as the system is assumed to be stable for finite $|\eta|$. For a phase transition to occur, the coefficient $a(T)$ should change sign around T_c. For simplicity, we approximate that $a(T) = a_0(T - T_c)$ and $b(T) = b_0$, for constants $a_0, b_0 > 0$. Minimizing F then gives that $\eta = 0$ for $T \geq T_c$, and $\eta = \sqrt{-\frac{a_0}{b_0}(T - T_c)}$ for $T < T_c$.

I.1.4 Order-parameters

We can generalize this discussion to general ordered media, using the example of a magnet. There are however some differences we must take into consideration. Namely, a crystal and especially a magnet has many particles that are all at fixed locations. These locations are described by the crystal lattice, which is a specific manifold.

> **Definition 1** A topological n-manifold is a second-countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$.

If the crystal has N identical particles, the total phase space X can be written as X_1^N, the product of N factors of the phase spaces X_1 of the individual particles. Choosing a manifold M to represent the lattice illustrates the correspondence between X_1^N and X_1^M, where the latter is the space of functions $M \to X_1$. Furthermore, we can replace the discrete lattice with a connected manifold and instead consider continuous functions from M to X_1.[5] A state f is then a function $f : M \to X_1$ and is also called a configuration.

We can now describe the ferromagnetic medium and its Curie temperature T_c by the same method as a gas. When heated above its Curie temperature, the medium demagnetizes, and cooling it down below the Curie temperature remagnetizes it again. Take for example the planar ferromagnet, which has spins $s : M \to S^1$ as configurations. The ferromagnetic Hamiltonian is $H = -\sum_{\langle i,j \rangle} s(i) \cdot s(j)$, where $\langle i, j \rangle$ are the pairs of nearest neighbours, and $s(i) \cdot s(j)$ is the dot product on $S^1 \subset \mathbb{R}^2$. The order-parameter is then the local magnetization $m = s$. Contrary to the density, the magnetization is a local order-parameter. Landau theory again provides an expansion of the free energy

$$F = \int_M \left[a\langle m(x) \rangle^2 + b\langle m(x) \rangle^4 + c(\mathrm{d}\langle m(x) \rangle)^2 \right] \mathrm{d}x$$

which now includes a divergence term $\mathrm{d}\langle m(x) \rangle$ to describe the effect of spatial variations in m. The coefficient c should be positive, and considerations for a and b similar to the previous argument must be made. Assuming m is uniform, so that $\mathrm{d}m(x) = 0$, minimizing F again gives a form of $\langle m(x) \rangle$ analogous to that for the global order-parameters η.

If m goes from being on average 0 to nonzero, a symmetry is broken in an ambiguous way, and we must change our description. However, in practice, a system is never in a perfect disordered phase and has a miniscule bias $0 < |\langle m \rangle| \ll 1$. In theory, we resolve this by introducing a conjugate field $h : M \to R$. The conjugate field defines an external term $H_{\mathrm{ext}} = -\int_M f(x) \cdot h(x)\, \mathrm{d}x$, which is included in the Hamiltonian $H' = H + H_{\mathrm{ext}}$. Specifying any $h \neq 0$ fixes the direction of $\langle m \rangle$ after the phase transition, but taking the limit $h \to 0$ brings us back to the original system. This fixes a preferred direction of $\langle m \rangle$ without disturbing properties of the original system.

The planar ferromagnet is the prototype of symmetry breaking, and as a model is known as the $O(2)$-model, precisely because it breaks $O(2)$ symmetry. Namely, if we take $S^1 \subset \mathbb{R}^2$ as the unit circle at the origin, the Hamiltonian is invariant under the action of the orthogonal group $O(2)$

[5] This requires the assumption, for the microscopic length scale λ, the maps $f : M \to X$ only have spatial variations with wavelengths larger than λ.

rotating all the spins around the origin.[6] At $T > T_c$, the average magnetization $\langle m \rangle$ is 0, which is invariant under $O(2)$ as well. However, when $T < T_c$, the average $\langle m \rangle$ becomes nonzero, and is evidently no longer invariant under $O(2)$. We conclude that the ferromagnetic phase transition breaks the $O(2)$ symmetry of the Hamiltonian.

Now that we know about general ordered media, we can return to the discussion of the deviation from order and introduce the defects.

I.2 Defects

From the discussion of ordered media, we conclude that, at low temperatures, the configurations of ordered media tend to be uniform. However, when a medium is cooled down and condenses to uniformity through a process of nucleation, there exist multiple nucleation sites. In this case, there can form two locally uniform regions in the medium that disagree at a shared boundary.

We say that a medium is ordered when it approximates local uniformity. Strictly speaking, a configuration $f : M \to X$ is ordered if, given any $\delta > 0$, there is an open cover $\{U_\alpha\}$ of the medium M, such that for each open U_α there is a constant C, so that $|f(x) - C| < \delta$ for all x in U_α. Conclude that, if f has a discontinuity, it is certainly not ordered.

We call such a discontinuity a defect, and a configuration with a defect is called defective. Note that if P is the defective region of $f : M \to X$, then f restricts to a continuous map on $M - P$, which is a fact we will use extensively later. In mild abuse of terminology, we may call P the defect of f.

Recall the planar ferromagnet, its disordered phase is characterized by the splitting into many Weiss domains. Such domains are islands of local uniformity that disagree at their boundaries.

Consider two defective states. The first is illustrated in Figure 1a, and is called the island. It is uniform everywhere with a single Weiss domain, which is one quarter rotated relative to uniformity. The second is illustrated in Figure 1b, and is called the vortex. It has all spins pointing outward from the origin.

The differences are evident, the island is only locally non-uniform, while the vortex has a defect that sends small variations through the entire medium. We can summarize this as the island being fundamentally closer to uniformity than the vortex, but we want can formalize this idea.

For this we resort to a bit of physical intuition, where we think of distances between configurations in terms of thermal fluctuations. We model a thermal fluctuation at low temperature as the evolution of a single small[7] patch of the medium. In other words, given a small patch U, the configuration $f : M \to X$ may evolve to $f' : M \to X$, as long as f' extends the restriction $f|_{M-U}$ to M. When discussing the distance to uniformity, we disregard everything but the defects, and say that f is close to uniformity if it can fluctuate to uniformity.

Indeed, the island fluctuates to uniformity by extending the configuration to a constant on the rotated domain. However, similar fluctuations on the vortex seem to leave a discontinuity somewhere. This leads us to ask what obstructs the fluctuations to order in the vortex. It turns out that the answer lies in the topology of the medium away from the defect $M - P$, the order-parameter space X, and the configuration $f : M \to X$.

Recall that for the vortex of the planar ferromagnet, the medium is $M = \mathbb{R}^2$, the defect is $P = \{(0,0)\}$, the order-parameter is $X = S^1 \subset \mathbb{R}^2$, and the configuration can be written as $f : M - P \to X, (x, y) \mapsto (x, y)/|(x, y)|$.

[6] The group $O(n)$ is the group of n-by-n matrices preserving the inner product on $\mathbb{R}^n$ after all.
[7] Small means bounded.

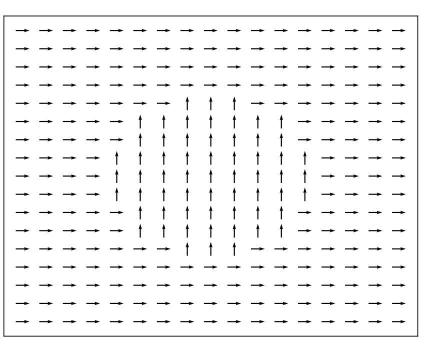
Given a bounded open U that lies in a ball of radius R around the origin, parametrize a circle in $M - U$ by $\gamma : I \to M - U, t \mapsto (2R\cos(2\pi t), 2R\sin(2\pi t))$, and remark that it winds once around the origin. Also remark that $\gamma(2t)$ winds twice around the origin. Extend this to all closed curves in the plane, by counting each counterclockwise loop as $+1$ and each clockwise loop as -1.

We say that the sum of this sequence of ± 1 is the winding number of the curve, and claim that this winding number is constant under deformations of curves that don't pass through the origin.

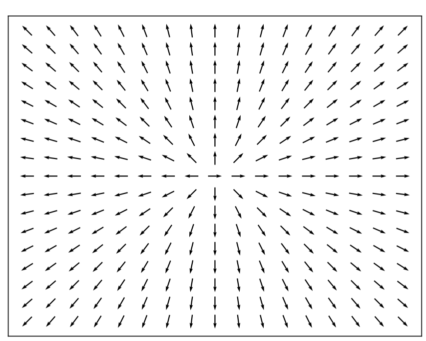
Now, we can reason about the vortex by observations on winding numbers. Namely, we can include γ into M, which makes it possible to deform γ to a point. Note that composition of a loop γ with f forms a loop $f\gamma : I \to X$.

Suppose that f continuously extends to $f' : M \to X$. Then $f\gamma = f'\gamma$, and deforming γ to a point deforms $f'\gamma$ to a point. Remark that a point has winding number 0, while $f\gamma$ has winding number 1. This is a contradiction, so we conclude that f cannot extend continuously to M.

Returning to the original description, there is no f' that continuously extends f from $M - U$ to M, and the obstruction is the nonzero winding number of $f\gamma$. The same observations can be applied to a variety of systems, and form the foundation for the concept of local surgeries.



(a) The island



(b) The vortex

Part II

Classification of Defects

We would like to generalize the argument for vortices, described in Section I.2, to a classification that applies to more general ordered media. The goal is to associate topological invariants to the configurations of a given parameter in a given space.

Two key observations are that closed curves in the medium define closed curves in order-parameter space, and that configurations themselves act as invariants under continuous transformations.

In this part we will first examine how continuous transformations partition the configurations into equivalence classes. This is done in Section II.1. In Section II.1.1 we shall describe how images under configurations of closed curves in physical space can obstruct extensions to other configurations, which is the direct generalization of the argument in Section I.2. The equivalence of the two schemes for a certain class of defects is shown in Appendix A.

II.1 Classifying configurations

In Section I.2 we illustrated how, under certain assumptions, a configuration f and a loop γ around a point p can induce loops $f\gamma$ that obstruct continuous extensions of f to include p. In this section we will present an alternative concept that more easily generalizes to higher dimensions and different shapes.

For this, it is essential that the configurations we consider are continuous. Recall the remark made earlier: if a configuration $f : M \to X$ is discontinuous in a subset P, then its restriction $f : M - P \to X$ is a continuous configuration.

The other way around, to investigate defects of a given shape P, it suffices to classify the continuous maps $M - P \to X$.

To model the defects of simple shapes, we introduce the following spaces.

Definition 2 A *k-flat* is a k-dimensional subspace of a Euclidean space that is itself a Euclidean space. The *standard* k-flat is the subspace $L_{n,k} = \{0\}^{n-k} \times \mathbb{R}^k$, and we write $M_{n,k} = \mathbb{R}^n - L_{n,k}$.

Lemma 1 *The space $M_{n,k}$ is homotopy equivalent to S^{n-k-1}.*

Proof. Write $M_{n,k} = \mathbb{R}^n - \{0\}^{n-k} \times \mathbb{R}^k$ as $(\mathbb{R}^{n-k} - \{0\}^{n-k}) \times \mathbb{R}^k$. Since $\mathbb{R}$ is contractible, $M_{n,k}$ is homotopy equivalent to $\mathbb{R}^{n-k} - \{0\}^{n-k}$. In turn $\mathbb{R}^{n-k} - \{0\}^{n-k}$ deformation retracts onto S^{n-k-1}. ■

Resorting to physical intuition again, we can model the equivalence of configurations by letting them continuously transform over time. We choose this notion to model defects, as, opposed to the arguments in Section I.2, it is more clearly independent of the choice of preferred points or loops.

Furthermore, it intuitively is an equivalence relation. Namely, a constant transformation illustrates reflexivity, reversing a transformation gives symmetry, and appending two transformations shows transitivity. Let us formalize the idea of continuous transformations and confirm our intuition.

Definition 3 A *homotopy* is a continuous map $H : X \times I \to Y$, where $I = [0, 1]$ is the closed unit interval.

We say that a homotopy $H : X \times I \to Y$ is *relative* to a subspace $A \subset X$ if $H(a,t) = H(a,0)$ for all $a \in A$ and $t \in I$.

We say that two maps $f, g : X \to Y$ are *homotopic* if there exists a homotopy $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, and denote this $f \sim g$.

Lemma 2 *Homotopy is an equivalence relation.*

Proof. Given $f : X \to Y$, define the homotopy $H : X \times I \to Y, (x,t) \mapsto f(x)$. Then, $H(x, 0) = H(x, 1) = f(x)$ so that f is homotopic to itself. Conclude that homotopy is reflexive.

Suppose f is homotopic to $g : X \to Y$, so there is a homotopy H from f to g. Define the homotopy $H^{-1} : X \times I \to Y, (x,t) \mapsto H(x, 1-t)$, so g is homotopic to f. Conclude that homotopy is symmetric.

Suppose g is also homotopic to $h : X \to Y$, so there is a homotopy H' from g to h. Define the homotopy $H'' : X \times I \to Y$ by

$$H''(x,t) = \begin{cases} H(x, 2t), & t \leq 1/2 \\ H'(x, 2t-1), & t > 1/2. \end{cases}$$

Then f is homotopic to h, and conclude that homotopy is transitive. ■

From the fact that homotopy is an equivalence relation, it follows that we can define quotients of sets of functions up to homotopy. Specifically, the homotopies then partition the set configurations $M \to X$ into homotopy classes.

Definition 4 We denote $C(X, Y)$ for the *space of continuous maps* from $X \to Y$.

The *set of homotopy classes* is the quotient of $C(X, Y)$ by the relation of homotopy, and is denoted $[X, Y]$.

We say that two configurations in the same class are connected by a homotopy, and the other way around, different classes are necessarily disconnected with respect to homotopy.

The statement that all continuous transformations of a configuration, equivalently all homotopic configurations, are equivalent up to defects, is summarized in the following claim.

Axiom 1 The defects on a medium M with order-parameter X are the homotopy classes $[M, X]$.

We will then refer to the purely mathematical defects derived from $[M, X]$ as 'the' defects. However, as is done in Section II.1.1, there are non-equivalent classifications and different ways to look at defects. For example, instead of considering the defect as arising from the properties of a configuration on a large scale, we can obtain similar information from only regions close to the defect.

Furthermore, the distinction between two defects suggests a form of topological stability, but naturally, this topological stability is that of stability up to homotopies and does not necessarily imply physical stability. Precisely, topological stability could only hold if the system has infinitely many components and is actually a continuum in the first place. How stable the defects are in practice follows from considerations of energetic stability.

The other way around, continuous transformations are not necessarily physically feasible, as demonstrated for example by the metastability of the non-unique ground states of the ferromagnet. This does underline what topological stability does tell us, namely that if the physical processes are a subset of the continuous transforms, then topologically distinct defects are certainly physically distinct.

One may ignore metastability and still ask whether continuous transforms should be feasible at all, as they do not necessarily provide differentiability over time or in intermediate states. Certainly, if the states of a system are given as solutions of a system of differential equations of order $k+1$, a transform that is C^k is evidently not feasible. It, however, turns out that continuous maps $X \to Y$ can be approximated by homotopic smooth maps $X \to Y$. Furthermore, any homotopy between smooth maps, can be approximated by a smooth homotopy[8], that is, a smooth function $X \times I \to Y$, while preserving its value on $X \times \partial I \to Y$.

It follows that the property of configurations being homotopic is equivalent to being smoothly homotopic.

II.1.1 Local surgery

For historical reasons, we also present the direct formalization of the classification by obstructions of induced loops. The formalization and the concept of local surgery is introduced in [1].

The intuition behind a local surgery is that thermal fluctuations drive the equivalence of configurations, and if the defective part of a configuration can fluctuate to the defective part of another, the defects are equivalent. We must keep in mind that only applies to the defects, and two configurations connected by a local surgery are only locally equivalent on the domain of the surgery.

Local surgery has some downsides, as it is defined in terms of basepoints and based loops, so that one must confirm that the phenomena they wish to describe are indeed independent of those choices. Unlike the homotopy picture, it does allow us to consider some phenomena which are filtered away by the nature of homotopy, such as the crossing of lines.

Let us formalize the concept of a local surgery.

Definition 5 Let f and g be continuous maps $f, g : M \to X$, and let U and V be open subsets such that $U \subset V \subset M$.

A *local surgery* from f to g on U in V is a continuous map $h : M \to X$ such that

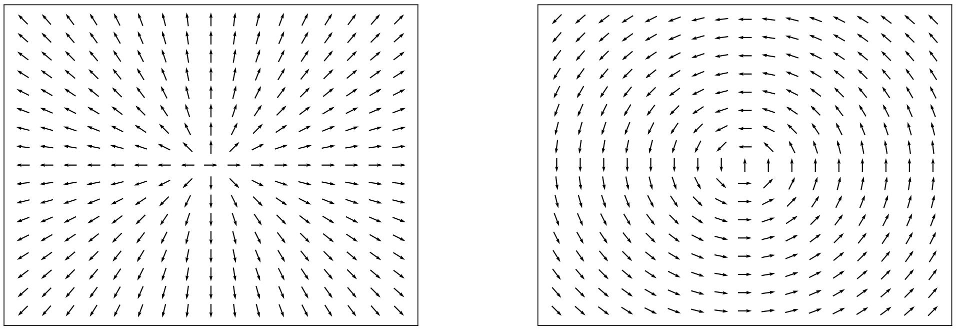
$$h|_{M-V} = f$$

$$h|_U = g$$

We write $\eta(f, g; U, V)$ if there is a local surgery from f to g on U in V.

Unlike the case of homotopy, the conditions in this definition are not strong enough to ensure that local surgery is an equivalence relation. A class of spaces on which the local surgery is equivalent to the homotopy classification is described in Appendix A.

[8]This follows from the Whitney embedding theorem.



(a) The θ vortex

(b) The $\theta + \pi/2$ vortex

Figure 2

Example 1 Let us present the application of the concepts of homotopy and local surgery in a simple example.

Let $f_1 : (r, \theta) \to \theta$, $f_2 : (r, \theta) \to \theta + \pi/2$ and $g : (r, \theta) \to 2\theta$ be states of the planar ferromagnet, as in Figure 2 and Figure 3.

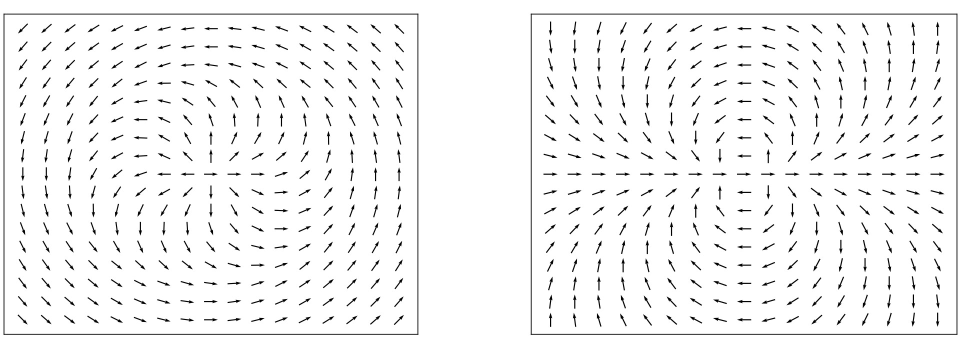
To confirm that f_1 and f_2 are the same defect, we can construct the homotopy $h_t : r, \theta \mapsto \theta + t\pi/2$ from f_1 to f_2.

We also see that f_1 is locally equivalent to f_2, as we can construct a surgery $h|_{r<r_1} = f_1$, $h|_{r>r_2} = f_2$ and otherwise

$$h(r, \theta) = \theta + \frac{\pi}{2}\left(\frac{r - r_1}{r_2 - r_1}\right).$$

However, intuitively, f_1 and f_2 are different from the configuration g. We may prove that no homotopy exists from f_1 to g by looking at the degrees of the mappings.

However, the more general methods introduced in Part III and Part IV will let us compute the homotopy classes more directly, from which we would conclude that indeed $[f_1] = [f_2] \neq [g]$.



(a) The interpolation from θ to $\theta + \pi/2$

(b) The 2θ vortex

Figure 3

Part III

Homotopy Theory

We presented a classification scheme for defects in Part II which associates a homotopy class to each defect. In order to learn about the homotopy classes, we first learn more about homotopy in general.

Therefore, we will give an elementary introduction into homotopy theory. For this, we largely follow parts of the treatment of [2], with a few corollaries and relevant examples added in.

In Section III.1 we introduce the basic terms, concepts and methods of homotopy theory. On top of the basic methods, we provide the tools for the calculation of specific homotopy classes in Section III.2 and Section III.3.

III.1 Homotopy groups

In this section we will give an introduction into homotopy theory, which we will then use to shine a light on the properties of homotopy classes.

Let us first describe some relations that can arise between topological spaces that we will find useful to compare their homotopy related properties.

Definition 6 A *retraction* of a space X to a subspace A is a continuous map $r : X \to A$ such that $r|_A = 1_A$.

A *weak deformation retraction* is a retraction $r : X \to A$ such that ιr is homotopic to 1_X, where $\iota : A \to X$ is the inclusion.

A *strong deformation retraction* is a weak deformation retraction for which ιr is homotopic to 1_X relative to A.

A *homotopy equivalence* is a continuous map $f : X \to Y$ for which there is a continuous map $g : Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$. We write $X \equiv Y$ if there exists a homotopy equivalence $f : X \to Y$.

We can see that a deformation retraction is a special homotopy equivalence. Note that for a CW-complex, as defined later, a weak deformation retraction is also a strong deformation retraction.

Example 2 Take the punctured plane $X = \mathbb{R}^2 - \{0\}$. We can write down a retraction $r : X \to S^1, x \mapsto \frac{x}{|x|}$. We can turn this into a deformation retraction by writing $r_\lambda : x \mapsto (1-\lambda)x + \lambda\frac{x}{|x|}$.

Recall that, given a space Y, we found in our earlier arguments that it was useful to map loops into Y, as a method of studying properties of Y. This motivates the loop space $\Omega(Y)$, defined as $C(S^1, Y)$.

However, we will find later that some properties of Y are described by the combination laws of loops in Y. We define this combination in terms of paths, since a loop γ is simply a path with $\gamma(0) = \gamma(1)$. Given two paths $\gamma, \gamma' : I \to X$, we can only concatenate them to a path $\gamma'\gamma : I \to X$ if $\gamma(1) = \gamma'(0)$. Likewise, loops can only be combined if they share their basepoint.

We see that it is equivalent and more concise to define a variant of spaces in which the loops are fixed at a point to begin with.

Definition 7 A *pointed space* (X, x) is a topological space X with a distinguished point $x \in X$.

Given a pointed space (Y, y), a *pointed map* is a map $f : X \to Y$ that respects the basepoints, that is, $f(x) = y$. In this case the space maps of pointed maps $X \to Y$ is also denoted $C(X, Y)$.

If $A \subset X$, we write $f : (X, A) \to (Y, y)$ for maps equivalent to pointed maps $f : (X/A, a) \to (Y, y)$. Such a map is also referred to as the map f from X to Y, relative to A.

Definition 8 Let (X, x) and (Y, y) be pointed spaces.

A *pointed homotopy* is a homotopy $H : X \times I \to Y$ such that $H(x, t) = y$ for all t.

Like for the usual topological spaces, we can define the pointed analogues of homotopy classes.

Definition 9 The sets of pointed or relative homotopy classes are denoted $[X, Y]$, when X and Y are pointed or relative spaces.

If it is necessary to distinguish pointed homotopy as opposed to the usual homotopy, the latter is referred to as *free* homotopy and the set of pointed homotopy classes may also be denoted as $\langle X, Y \rangle$.

Remark 1 The *set of path components* of X is equivalent to $[\{*\}, X]$, and is denoted $\pi_0(X)$.

The set of homotopy classes $[X, Y]$ is equivalent to the set of path components of $C(X, Y)$, so we write $[X, Y] = \pi_0(C(X, Y))$.

Using the pointed spaces, we can make the notion of combining loops more precise and general.

Definition 10 The *n-th homotopy group* $\pi_n(X, x)$ of a pointed space (X, x) is the group $([(I^n, \partial I^n), X], \cdot)$.

The underlying set $[(I^n, \partial I^n), X]$ is the set of homotopy classes of relative maps $(I^n, \partial I^n) \to (X, x)$.

The binary operation is defined as $[g] \cdot [f] = [g \cdot f]$, where

$$(g \cdot f)(x_1, x_2, \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n), & x_1 \le 1/2 \\ g(2x_1 - 1, x_2, \ldots, x_n), & x_1 > 1/2 \end{cases}.$$

For clarity, if $n = 1$, we may juxtapose $[g][f]$ instead of writing $[g] \cdot [f]$. If $n > 1$, we may write $[g] + [f]$ instead. This notation also extends to define gf and $g + f$ on the representatives.

We can prove some basic properties of $\pi_n(X, x)$.

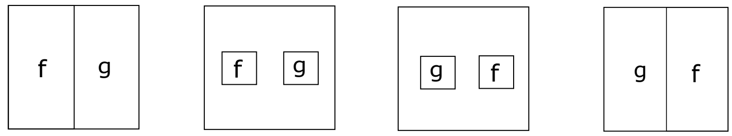


Figure 4: π_n is abelian for $n > 1$.

Lemma 3 *Let (X, x) be a pointed space and let $f, g : (I^n, \partial I^n) \to (X, x)$ be relative maps. Let $s : I^n \to I^n$ be a reparametrization, that is, $s|_{\partial I^n} = 1_{\partial I^n}$.*

Then, maps are homotopic to their reparametrizations, $f \circ s \sim f$.

If $n > 1$, the group $\pi_n(X, x)$ is abelian

$$g + f \sim f + g.$$

The identity in $\pi_n(X, x)$ is the constant map $x : I^n \to X, y \mapsto x$

$$f \cdot x \sim f$$

The inverse of f is the map

$$f^{-1}(s_1, s_2, \dots) = f(1 - s_1, s_2, \dots).$$

Proof. Since $s : (I^n, \partial I^n) \to (I^n, \partial I^n)$ is relative and I^n is convex, there is the linear homotopy H from s to 1_{I^n}. Precomposing makes fH the homotopy from $f \circ s$ to f.

The proof of $f + g \sim g + f$ is sketched in Figure 4.

In words, one takes $f + g$ and shrinks the domains of f and g to smaller cubes surrounded by the constant map x. Then they can be slid around each other, and expand back to $g + f$.

This confirms π_n is abelian, which why we use additive notation for the operation on π_n.

The concatenation $f \cdot x$ is a reparametrization of f, thus $f \cdot x \sim f$.

The homotopy from ff^{-1} to x can be explicitly constructed as

$$H(t_1, \dots, t_n, s) = \begin{cases} f(2t_1(1-s), t_2, \dots, t_n), & t_1 \le 1/2 \\ f(2(1-t_1)(1-s), t_2, \dots, t_n) & t_1 > 1/2 \end{cases}$$

■

Let us introduce some more operations on homotopy classes. Note that these apply to free or pointed classes, and as a special case the homotopy groups alike. Specifically, we can use these to turn a relation between X and Y into a relation between homotopy classes to or from X and Y.

Definition 11 The *pushforward* of a continuous map $f : X \to Y$ is the map

$$f_* : [Z, X] \to [Z, Y], [\gamma] \mapsto [f\gamma]$$

and the *pullback* is the map

$$f^* : [Y, Z] \to [X, Z], [\gamma] \mapsto [\gamma f].$$

Lemma 4 *The pushforward and pullback are well-defined.*

That is, given spaces X, Y, Z and a map $f : X \to Y$, it holds that

$$[\gamma] = [\gamma'] \in [Z, X] \implies f_*[\gamma] = f_*[\gamma']$$

$$[\gamma] = [\gamma'] \in [Y, Z] \implies f^*[\gamma] = f^*[\gamma'].$$

Given a space W and a map $g : Y \to W$, then

$$(gf)_* = g_* f_*$$

$$(gf)^* = f^* g^*.$$

Furthermore, the identity on X induces the identity on the sets of homotopy classes,

$$(1_X)_* = 1_{[Z,X]}$$

$$(1_X)^* = 1_{[X,Z]}.$$

And finally, f_* and f^* are homotopy invariant, that is $f \sim g$ implies $f_* = g_*$ and $f^* = g^*$.

Proof. If H is the homotopy from γ to γ', then fH is the homotopy from $f\gamma$ to $f\gamma'$, and Hf is the homotopy from γf to $\gamma' f$. Thus, f_* and f^* are well-defined.

We confirm that $(gf)\gamma = g(f\gamma)$ and $\gamma(gf) = (\gamma g)f$, so that $(gf)_* = g_* f_*$ and $(gf)^* = f^* g^*$.

Noting that $1_X \gamma = \gamma : Z \to X$ and $\gamma' 1_X = \gamma' : X \to Z$ confirms that $(1_X)_*[\gamma] = [\gamma]$ and $(1_X)^*[\gamma'] = [\gamma']$.

Let H be the homotopy from f to $g : X \to Y$. For $\gamma : Z \to X$ and $\gamma' : Y \to Z$, the map $H\gamma$ is the homotopy from $f\gamma$ to $g\gamma$, and $\gamma' H$ is the homotopy from $\gamma' f$ to $\gamma' g$. ■

With the pushforwards and pullbacks, we can finally demonstrate that homotopy equivalent spaces are, indeed, completely equivalent when viewed through homotopies.

Lemma 5 *For two homotopy equivalent spaces $X \equiv Y$, and a third space Z, the following holds*

$$[Z, X] = [Z, Y]$$

$$[X, Z] = [Y, Z]$$

Proof. Because $X \equiv Y$, there is a pair of maps $f : X \to Y$ and $g : Y \to X$ for which $fg \sim 1_Y$ and $gf \sim 1_X$.

Applying homotopy invariance of pushforwards, we get that $(gf)_* = g_* f_* = 1_{[Z,X]}$ and $(fg)_* = f_* g_* = 1_{[Z,Y]}$. It follows that f_* is both injective and surjective, so that it is an isomorphism $f_* : [X, Z] \to [Y, Z]$.

Similarly, we find that $(fg)^* = g^* f^* = 1_{[Y,Z]}$ and $(gf)^* = f^* g^* = 1_{[X,Z]}$, from which again f^* is an isomorphism $f^* : [Y, Z] \to [X, Z]$. ■

III.2 Computing the fundamental group

We argued in Section III.1 and demonstrated already in Section I.2 that the homotopy classes and groups contain a wealth of information about their spaces. In retrospect, the obstructions for homotopies and local surgeries in the planar ferromagnet are precisely the classes of the fundamental group $\pi_1(S^1)$ of the circle.

It, however, remains for us to compute $\pi_1(S^1)$ and prove that it is indeed non-trivial. For this we will need the following tool.

Definition 12 A *covering map* p is a continuous map from a space C to a space X, such that each point of X is *evenly covered*.

A point $x \in X$ is evenly covered when there is an open neighbourhood U of x, such that $p^{-1}(U)$ is a disjoint union of opens U_i, where each U_i is homeomorphic to U by the restriction $p|_{U_i}$.

We say that C is a *covering space* of X if there is a covering map $p : C \to X$.

Let Z be a space, and let $f : Z \to X$ and $\tilde{f} : Z \to C$ be maps. We say that $\tilde{f}$ lifts f if $p\tilde{f} = f$.

The *universal cover* $\tilde{X}$ of X is the unique simply connected covering space of X, up to isomorphism.

To compute $\pi_1(S^1)$, we take the universal cover $p : \mathbb{R} \to S^1, t \mapsto \exp(2\pi i t)$. The even covering property of p is illustrated in Figure 5, intuitively, closed loops in S^1 correspond to paths of integer lengths in $\mathbb{R}$.

This is formalized by the homotopy lifting property, proven in [2].

Lemma 6 *The covering map $p : \mathbb{R} \to S^1, t \mapsto \exp(2\pi i t)$ has the homotopy lifting property for paths.*

Given any homotopy $\gamma : I \times I \to S^1$. Given any real number r and any map $\tilde{\gamma}_0 : I \to \mathbb{R}$ lifting $\gamma_0 : I \to S^1$.

There exists a homotopy $\tilde{\gamma} : I \times I \to \mathbb{R}$ lifting γ which also satisfies $\tilde{\gamma}|_{I \times \{0\}} = \tilde{\gamma}_0$.

This completes our list of ingredients to prove that $\pi_1(S^1)$ is indeed the additive group of winding numbers.

Lemma 7 *The fundamental group $\pi_1(S^1, 1)$ of the circle S^1 is the infinite cyclic group $\mathbb{Z}$.*

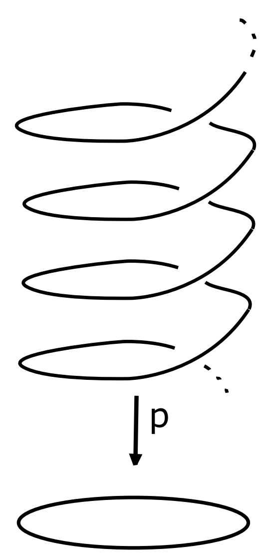


Figure 5: $\mathbb{R}$ as a helix lying above S^1.

Proof. Define $\omega_n(s) = e^{2\pi s}$, and a lift $\tilde{\omega}_n(s) = ns$.

Take any loop $\gamma : I \to S^1$ at 1 and a lift $\tilde{\gamma} : I \to \mathbb{R}$ satisfying $\tilde{\gamma}(0) = 0$. Since $\gamma(0) = \gamma(1) = 0$ and $\tilde{\gamma}(0) = 0$, we find that $\tilde{\gamma}(1)$ is an integer n.

Since $\mathbb{R}$ is convex, we can take the linear homotopy H between $\tilde{\gamma}$ and $\tilde{\omega}_n$. Then pH is the homotopy between γ and ω_n. Conclude that all loops at 1 in S^1 are homotopic to ω_n for some n.

Suppose that there is a homotopy γ_t from ω_n to ω_m, with $\gamma_t(0) = \gamma_t(1) = 1$. Lift the homotopy to $\tilde{\gamma}_t$ with $\tilde{\gamma}_t(0) = 0$.

Then $\tilde{\gamma}_0 = \tilde{\omega_n}$ and $\tilde{\gamma}_1 = \tilde{\omega_m}$. Because γ_t is 1 at the endpoints, the homotopy is constant at the endpoint, that is, $\tilde{\gamma}_t(1)$ is constant. As $\tilde{\gamma}_0(1) = n$ and $\tilde{\gamma}_1(1) = m$, conclude that $n = m$.

We see that $\omega_n \cdot \omega_m$ lifts to $\tilde{\omega}_{n+m}$, so $\omega_n \cdot \omega_m$ is homotopic to ω_{n+m}.

It follows that $\omega_n = \omega_1^n$, so we conclude that $\pi_1(S^1)$ is the infinite cyclic group $\mathbb{Z}$. ∎

Let us summarize what we know about fundamental groups now.

Example 3 Since $\mathbb{R}$ deformation retracts to a point, we have $\pi_1(\mathbb{R}) = \langle S^1, \mathbb{R} \rangle = \langle S^1, \{0\} \rangle = 0$.

On the sphere S^2, any loop contracts to a point so $\pi_1(S^2) = 0$.

We showed that $\pi_1(S^1) = \mathbb{Z}$.

Because $f : Z \to X \times Y$ factors into $f_1 : Z \to X$ and $f_2 : Z \to Y$, we have that maps $S^1 \to X \times Y$ and homotopies $S^1 \times I \to X \times Y$ factor as well. Hence, $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

This gives for the torus $T^2 = S^1 \times S^1$ that $\pi_1(T^2) = \mathbb{Z}^2$.

III.3 Free homotopy

In Section III.2 we demonstrated that we can compute the pointed classes $\langle M, X \rangle$ when $M = S^1$ or if either M or X is contractible.

However, in our classification we claim that the defects correspond to the classes $[M, X]$, rather than the pointed ones. In this section, we set out to compute the classes $[M, X]$, provided we know the pointed classes $\langle M, X \rangle$ and the fundamental group $\pi_1(X)$.

For this section we follow [2] sections 4.1 and 4.A., and the figures in this section are reconstructed from those in [2], with minor modifications.

III.3.1 Basepoints

We realize that the definition of pointed homotopy indeed depends on a choice of basepoint, so before we can discuss free homotopy, we will have to ensure that certain properties are indeed invariant under a change of basepoint. Therefore, in this section, we will prove that the n-th homotopy groups of a path connected space X at distinct basepoints x_1, x_2 are isomorphic $\pi_n(X, x_1) = \pi_n(X, x_2)$.

First we need to define the change of basepoint map, which we will use to transport cubes from one point to another.

Definition 13 Let $\gamma : I \to X$ be a path from x_1 to x_2, and let $f : (I^n, \partial I^n) \to (X, x_2)$ be a map. Further, let C be a smaller concentric cube in I^n, and let $\phi_1 : C \to I^n$ and $\phi_2 : I^n - C \to \partial I^n \times I$ be homeomorphisms.

Define the change of basepoint $\beta_\gamma : f \mapsto \gamma f$ by the map $\gamma f : (I^n, \partial I^n) \to (X, x_1)$ such that

$$\begin{cases} \gamma f(x) = f(\phi_1(x)), & x \in C \\ \gamma f(x) = \gamma(p_2\phi_2(x)), & x \notin C \end{cases}$$

where $p_2 : I^n \times I \to I$ is the projection on the second factor.

The map γf is illustrated in Figure 6. Such a change of basepoint can be visualized as attaching a string to the base of the sphere, and fixing the string in place elsewhere, much like securing a helium balloon.

Lemma 8 *Let $\gamma, \eta : I \to X$ be paths from x_2 to x_3 and x_1 to x_2 respectively. Let $x : I \to X$ denote the constant path $t \mapsto x$.*

The change of basepoint β is associative and has an identity

- $(\gamma\eta)f \sim \gamma(\eta f)$
- $xf \sim f$

and is a homomorphism $\beta_\gamma : \pi_n(X, x_3) \to \pi_n(X, x_2)$

$$\gamma(f + g) \sim \gamma f + \gamma g$$

Proof. Because $\gamma(\eta f)$ and $(\gamma\eta)f$ are reparametrizations of each other, they are also homotopic. In the same fashion, xf is homotopic to f.

To show $\gamma(f + g) = \gamma f + \gamma g$, take $\gamma(f + 0) + \gamma(0 + g)$ and write down the homotopy

$$h_t(s_1, \dots) = \gamma(f + 0)((2 - t)s_1, s_2, \dots), \quad s_1 \in [0, 1/2]$$

$$h_t(s_1, \dots) = \gamma(0 + g)((2 - t)s_1 + t - 1, s_2, \dots), \quad s_1 \in [1/2, 1]$$

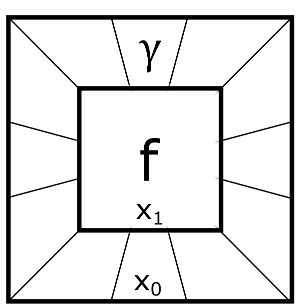


Figure 6: γf is defined by f on the smaller cube and γ along each radial line.

This is the explicit homotopy $\gamma(f+g) = \gamma(f+0) + \gamma(0+g) = \gamma f + \gamma g$. ■

The homotopy from $\gamma(f+g)$ to $\gamma f + \gamma g$ is portrayed in Figure 7.

Theorem 1 *Let γ be a path from x_2 to x_1.*

The evaluation map $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_2), f \mapsto \gamma f$ is an isomorphism.

Proof. Remark that $\beta_\gamma(x_1) \sim x_2$. By Lemma 8, $\beta_\gamma(f+g) = \beta_\gamma(f) + \beta_\gamma(g)$, so it is a homomorphism.

Also, β_γ has an inverse $\beta_{\gamma^{-1}}$, since $\gamma(\gamma^{-1} f) = xf = f$.

Conclude that β_γ is an isomorphism. ■

We confirm that all n-th homotopy groups of a path-connected space are isomorphic.

Using the evaluation map β_γ, we can take $[\gamma] \in \pi_1(X)$ and investigate the relation between $\pi_n(X)$ and $[S^n, X]$ in terms of the orbits of the automorphism $\beta_\gamma : \pi_n(X) \to \pi_n(X)$. In Section III.3.2 we will explore the generalization of this idea.

III.3.2 Free homotopy as action of π_1

Evidently, attaching loops is not limited to spheres and applies to other pointed spaces as well. However, the parametrization of the loops does not generalize, as for the spheres it relied on the parametrization of the spheres in terms of cubes instead.

Let us define the following action to model free homotopies of arbitrary pointed spaces.

Definition 14 Let γ be a loop in X at x_0 and let $f : Z \times I \to X$ be a homotopy from f_0 to f_1 such that $f_s(z_0) = \gamma(s)$.

Define the right-action of $\pi_1(X, x_0)$ on $\langle Z, X \rangle$ by

$$[f_0][\gamma] = [f_1]$$

This action can be visualized as dragging the space Z around the given loop γ, where Z is potentially deformed in the process.

Now, we can get a result that directly relates the free homotopy $[Z, X]$ classes to the orbits of the pointed classes $\langle Z, X \rangle$. This result, however, relies on the space Z having the homotopy extension property.

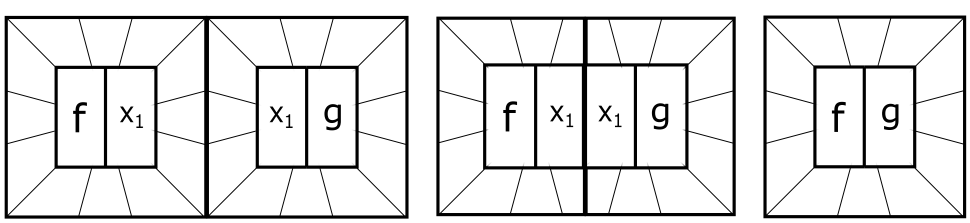


Figure 7: The sequence of deformations for $\gamma(f+g) = \gamma f + \gamma g$.

It turns out that most spaces we consider are a type of space known as a CW-complex. In particular, CW-complexes have the homotopy extension property.

Definition 15 An *n-cell* is a copy of the open n-disk $D^n - \partial D^n$.

A *CW-complex* (closure-finite weak topology) is defined inductively as follows:

1. A 0-dimensional CW-complex is a set of zero or more points.
2. An n-dimensional CW-complex is a k-dimensional CW-complex, $k < n$, with one or more n-cells adjoined.

The proof of the homotopy extension property for CW-complexes, is omitted and may be found in [2].

We can now prove our claim that the orbits of the action of π_1 on the pointed classes $\langle Z, X\rangle$ correspond to the free homotopy classes $[Z, X]$.

Theorem 2 *If (Z, z_0) is a CW-complex and X path connected, then the forgetful map induces a bijection $F : \langle Z, X\rangle/\pi_1(X, x_0) \to [Z, X], [\gamma] \mapsto [\gamma]$.*

Proof. Since X is path-connected any $f : Z \to X$ is homotopic to f' with $f'(z_0) = x_0$ by homotopy extension, thus F is surjective.

If $f_0, f_1 : (Z, z_0) \to (X, x_0)$ are homotopic via f_s then by definition $[f_1] = [f_0][\gamma]$ where $\gamma(s) = f_s(z_0)$, thus F is injective. ■

Corollary 1 *If we choose $Z = S^1$, we recover that the action of $\pi_1(X, x)$ on itself is conjugation. Hence, the homotopy classes $[S^1, X]$ correspond to the conjugacy classes of $\pi_1(X, x)$.*

Note that if $\pi_1(X) = 0$, then the action of π_1 is clearly trivial and $\langle Z, X\rangle \cong [Z, X]$. In general if the action of π_1 on π_n is trivial, we say X is n-simple.

In fact, [2] shows that H-spaces are n-simple for all n. Recall that H-spaces are topological magmas with identity elements.

Lemma 9 *If (X, e) is a connected H-space, the action of $\pi_1(X, e)$ on $\langle Z, X\rangle$ is trivial for all (Z, z_0).*

Proof. Take a map $f : (Z, z_0) \to (X, e)$, and a loop γ at e in X.

Let $f_s(z) = f(z)\gamma(s)$ be the homotopy from f to itself. Because $f_s(z_0) = \gamma(s)$, we get that $f_1 = \gamma f = f$. ■

Part IV

Methods of Computation

With the methodology of Part III we can compute part of the classification in Part II. We wish to expand our methods to allow us to compute the second homotopy group, as this provides a coarse classification of point defects.

In this part we present a set of powerful theorems, and formulate the subclass of order-parameters on which these can be applied to compute higher homotopy groups.

The concept of a fibration is presented in Section IV.1 along with the derivation of the long exact sequence for homotopy groups from the Puppe sequence. In Section IV.2 we describe the case where the order-parameter space is a homogeneous or coset space, and we follow [3] section 6 to support the claim that lets us apply all results for fibrations to those order-parameters.

The Seifert-Van Kampen theorem, which computes the fundamental groups of decomposable spaces, is described in Section IV.3 and the proof by [2] is paraphrased. In Section IV.4 we introduce the homology groups and their simplicial variants, together with the Hurewicz theorem, which relates homology and homotopy.

IV.1 Fibrations

Like the covering maps, fibrations are special maps from some "larger" space to another space, with specifically chosen properties to allow computations of π_1 and sometimes even π_n. The strength of a fibration comes, like universal covers, from the fact that they satisfy the homotopy lifting property.

Definition 16 We say that a continuous map $p : E \to B$ has the *homotopy lifting property* with respect to X if

- for any $f : X \times I \to B$
- for any $\tilde{f}_0 : X \to E$ with $f_0 = p\tilde{f}_0$

there exists a homotopy lift $\tilde{f} : X \times I \to E$ such that $f = p\tilde{f}$ and $\tilde{f}_0 = \tilde{f}|_{X\times\{0\}}$.

Definition 17 A *fibration* is a continuous surjection p from the total space E to the base space B, that satisfies the homotopy lifting property with respect to any space.

A fibration is denoted $F \xrightarrow{\iota} E \xrightarrow{p} B$, where some $b_0 \in B$ is chosen and $F = p^{-1}(b_0)$ is called the fibre.

A *Serre fibration* is a fibration that satisfies the homotopy lifting property with respect to all cubes I^n.

Indeed, we see already that a universal cover with the homotopy lifting property is as good as a fibration, and we can repeat the same steps we used to compute $\pi_1(S^1)$ to compute $\pi_1(B)$ in terms of the fibres.

Using a construction known as the Puppe sequence, we can derive an even stronger result for fibrations. This theorem and proof are given in [4] as Theorem 11.48.

Theorem 3 *Given a fibration $F \to E \to B$ with*

$$b_0 \in B, f_0 \in F = p^{-1}(b_0), e_0 = \iota(f_0)$$

there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(F) \to \pi_0(E)$$

where the maps are $\iota_* : \pi_n(F) \to \pi_n(E)$, $p_* : \pi_n(E) \to \pi_n(B)$, and $\beta_n : \pi_n(B) \to \pi_{n-1}(F)$.

The map β_n is called the connecting morphism.

Proof. If $f : X \to Y$ is a pointed map, the Puppe sequence is the long exact sequence

$$\cdots \to \Omega^2(Mf) \to \Omega^2 X \to \Omega^2 Y \to \Omega(Mf) \to \Omega X \to \Omega Y \to Mf \to X \to Y$$

where Ω is the loop space, and Mf is the mapping fibre of f.

If f is a fibration $p : E \to B$, then Mp and F have the same homotopy type. Hence, $\pi_n(Mp) = \langle S^n, Mp \rangle = \langle S^n, F \rangle = \pi_n(F)$.

Then use that in pointed spaces $\pi_0 \Omega(X) = \pi_1(X)$, and $\pi_1(\Omega^n X) = \pi_0(\Omega^{n+1} X) = \pi_{n+1}(X)$.

Applying π_0 to the Puppe sequence of p then gives the wanted long exact sequence. ∎

Let us present the fibrations and their power in a simple and less simple example.

Example 4 Take the product space $E = B \times F$ and the projection $p : (b, f) \mapsto b$.

Trivially, p satisfies the homotopy lifting property. Note that as β is trivial, we also recover that $\pi_n(E) = \pi_n(B) \times \pi_n(F)$[9].

This fibration is called the trivial fibration, or the trivial fibre bundle.

Example 5 The Hopf fibration is a special fibration $S^1 \to S^3 \to S^2$, constructed as follows.

First identify $\mathbb{R}^4 \cong \mathbb{C}^2$ and $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$.

The Hopf map $p : \mathbb{R}^4 \to \mathbb{R}^3$ is explicitly given as $p(z_0, z_1) = (2z_0 z_1^*, |z_0|^2 - |z_1|^2)$.

By letting $|z_0|^2 + |z_1|^2 = 1$, one finds $|p(z_0, z_1)| = 1$, thus p restricts to $S^3 \to S^2$ and can be proven to be a fibration.

Applying the exact sequence with the fact that $\pi_n(S^1) = 0$ for $n > 1$, we find that $\pi_n(S^3) = \pi_n(S^2)$ for $n > 1$.

[9] Note that swapping B and F gives the opposite sequence, and we can easily check that both split.

IV.2 Homogeneous spaces

Many physical systems have some symmetries, and much of physics is also motivated by understanding and exploiting those symmetries. Most systems have order-parameters in 3 dimensions after all, and as a result are symmetric under $SO(3)$ in their disordered phase.

The class of systems that breaks $SO(3)$ symmetry in their phase transitions then has an ordered phase which is only symmetric under some subgroup of $SO(3)$. This subgroup will be referred to as the isotropy group of the medium, and it is precisely the stabilizer of a reference point in the order-parameter under the action of $SO(3)$.

In this section, we will demonstrate how this observation simplifies the description of the order-parameter to that of its symmetry and isotropy groups. From there, we will relate these in a fibration with which we can compute the groups π_n of the order-parameter in a simpler way.

IV.2.1 Coset spaces

The statement that a certain order-parameter X has an $SO(3)$ symmetry is equivalent to the claim that X is a homogeneous space of $SO(3)$. Definitions and properties of group actions, homogeneous spaces and coset spaces can be found in Appendix B.

In this section we will illustrate the link between homogeneous spaces and coset spaces. Let us refresh the concepts of quotient maps and spaces, to allow us to conclude that a coset space is a special quotient space.

> **Definition 18** A *quotient map* is a surjective map $q : X \to Y$ such that the topology of Y is $\tau_Y = \{U \subset Y | q^{-1}(U) \in \tau_X\}$. We say that Y is a *quotient space* of X.
>
> The quotient of X by an equivalence relation $\sim$ is the quotient space of the quotient map $q : X \to X/\!\sim, x \mapsto [x]$, and is denoted $X/\sim$.
>
> The quotient of X by a subset A is the quotient space $X/\!\sim$ by relation $x \sim y \iff x, y \in A \vee x = y$, and is denoted X/A.

Considering a transitive action of G on X, it is evident that fixing some reference point $x_0 \in X$, all other points $x \in X$ are just as well described by the element of g relating them to x_0. However, the action is not necessarily free, and there may be many g bringing x_0 to x.

Similarly, the isotropy group G_{x_0} describes all the elements bringing x_0 to itself, and intuitively, the isotropy of any x is just the isotropy of x_0 "transported" to x. This suggests that we may view X as G modulo G_{x_0}.

To make this more precise, we first need to following lemma:

> **Lemma 10** *Let G be a compact Lie group with a transitive action a topological space X.*
>
> *Then the evaluation map $\phi_x : G \to X, g \mapsto gx$ is open for all $x \in X$.*

With this we can confirm the relation between homogeneous spaces and coset spaces.

Lemma 11 *Let G be a compact Lie group equipped with a transitive action on a space X.*

Given a reference point $x_0 \in X$, take the stabilizer $H = G_{x_0}$. Take the coset space G/H with the quotient map $q : G \to G/H$.

Then X is homeomorphic to G/H.

Proof. Define the map $f : G/H \to X, [g] \mapsto gx_0$.

Suppose that $[g] = [g']$, then $g \in g'H$, so that there is an $h \in H$ such that $g = g'h$. Then, because H is the stabilizer, $hx_0 = x_0$ so $gx_0 = g'x_0$. Conclude that f is well-defined.

Suppose that $f([g]) = f([g'])$, that is $gx_0 = g'x_0$. Then $g^{-1}g'x_0 = x_0$, so $g^{-1}g' \in H$. Hence, $g' \in gH$, so $[g] = [g']$. Conclude that f is injective.

Given $x \in X$, transitivity of the action gives a $g \in G$ with $gx_0 = x$. Then immediately $f([g]) = x$. Conclude that f is surjective, and bijective.

Define the evaluation $\phi_x : G \to X, g \mapsto gx$.

Given an open $U \subset X$, let $V = f^{-1}(U) = \{[g] | g \in G, gx_0 = x\} = q(\phi_{x_0}^{-1}(U))$. Note that q is trivially an open map because of the quotient topology. Because the action is continuous, we find that V is open. Conclude that f is continuous.

Given an open $V \subset G/H$, let $U = f(V) = \{gx_0 | g \in [g] \in V\} = \phi_{x_0}(q^{-1}(V))$. Because q is continuous and ϕ_{x_0} is an open map, we find that U is open.

Conclude that f is an open map, and also a homeomorphism. ■

Indeed, any homogeneous space X is the quotient space of G known as the coset space. The other way, we may ask what kind of space the coset space of a given group G and subgroup H is.

Lemma 12 *Given a compact Lie group G and a closed Lie subgroup H, the coset space G/H is a smooth manifold.*

Proof. Let H act on G by left-translation. The quotient of G by this action is the coset space G/H.

As a Lie group, G is also a smooth manifold. Because H is closed, left-translation is smooth, free, and proper.

The quotient manifold theorem then gives that the quotient G/H is a smooth manifold and $q : G \to G/H$ a submersion. ■

Let us give a concrete example.

Example 6 Take the special orthogonal group $SO(3)$ with its usual action on S^2.

The stabilizer of any $p \in S^2$ is $SO(2)$, so we conclude that $S^2 \cong SO(3)/SO(2)$.

IV.2.2 Fibre bundles

Given a homogeneous space X, we get a group G acting on X, and a stabilizer H of $x \in X$, so that $X \cong G/H$. Now the canonical map $q : G \to G/H$ has fibres $q^{-1}([g]) = gH$, which are all homeomorphic to H.

This suggests that $H \to G \to G/H$ might be a fibration. We will follow [3] for this section.

Let us define an intermediate step towards making $q : G \to G/H$ into a fibration.

Definition 19 A *principal G-bundle* is a map $p : E \to B$ together with an action of G on E such that

1. the action *preserves the fibres*, that is $y \in p^{-1}(b) \implies gy \in p^{-1}(b)$,
2. for any $x \in p^{-1}(b)$, the map $\psi_x : G \to p^{-1}(b) : g \mapsto gx$ is a homeomorphism,
3. p has *enough local sections*.

A *local section* is a continuous map $s : U \to p^{-1}(U)$ such that $ps = 1_U$. The map p has enough local sections when each $x \in B$ has an open neighbourhood U and a local section $s : U \to p^{-1}(U)$.

Note that p preserving the fibres is equivalent to p being invariant under G, namely both state $p(gy) = y$.

We can indeed show that a coset space is a principal bundle.

Theorem 4 *Let G be a compact Lie group and H be a closed Lie subgroup of G.*

The canonical map $q : G \to G/H$ is a principal H-bundle.

Proof. The action $\phi : G \times H \to G, (g, h) \mapsto hg$ preserves the fibres, because $y \in q^{-1}(x) = xH$ implies $hy \in xH$ for all $h \in H$.

Further, the map $H \to q^{-1}(x), h \mapsto hy$ is a homeomorphism for all $y \in xH$.

By the quotient manifold theorem, the map $q : G \to G/H$ is a submersion, and because G is compact, q is also proper. By the Ehresmann's theorem q then has enough local sections, so we conclude that q is a principal H-bundle. ■

However, the principal bundle is in fact a special case of a more general type of bundle, known as the fibre bundle.

Definition 20 A *fibre bundle* is a continuous surjection $p : E \to B$ that has a *local trivialization* at any $b \in B$.

A local trivialization is a homeomorphism $\phi_U : U \times F \to p^{-1}(U)$ such that

$$\pi = p\phi_U$$

where $\pi : U \times F \to U$ is the projection.

We can visualize a fibre bundle by its name, and thinking of how the fibres of the bundle all project down onto points of the base space. A more vivid and perhaps more well-known example is the Möbius band.

Example 7 A Möbius band M, defined as a square with one pair of opposite sides identified in equal orientation, is a bundle (M, S^1, p, I).

The base is the S^1 running vertically along the middle, and the fibres are the horizontal intervals.

Let us confirm that principal bundles, and specifically coset spaces, are indeed special fibre bundles.

Theorem 5 *A principal G-bundle is a fibre bundle with fibre G.*

Proof. For any $b \in B$ take a local section $s : U \to p^{-1}(U)$ with $b \in U$.

Then define $\phi_U : U \times G \to p^{-1}(U), (x, g) \mapsto g \cdot s(x)$, and note that it is continuous.

We see that $p\phi_U(x, g) = p(g \cdot s(x)) = x$, as the action preserves the fibres. This confirms that $\pi = p\phi_U : U \times G \to U$ is the projection.

Recall that $\psi_x : G \to p^{-1}(U), g \mapsto g \cdot x$ is a homeomorphism. Define $\delta_U : p^{-1}(U) \to U \times G, x \mapsto (p(x), \psi_x^{-1}(x))$, and note that it is continuous.

Then $\delta_U\phi_U(x) = \delta(g \cdot s(x)) = (p(g \cdot s(x)), \psi_{g \cdot s(x)}(g \cdot s(x))) = (x, g)$. Thus, δ_U is the inverse of ϕ_U, so we conclude that ϕ_U is a homeomorphism.

As this gives for any $b \in B$ a homeomorphism ϕ_U with $\pi = p\phi_U$, we see that p satisfies local trivialization. Conclude that p is a fibre bundle. ∎

Furthermore, the fibre bundle we introduced is a special case of a fibration we defined and used in Section IV.1.

Conjecture 1 *Any fibre bundle is a Serre fibration.*

The proof is given in [2], in essence, the proof of the following relies on repeated local trivialization to inductively construct homotopy lifts.

We see that indeed $H \to G \to G/H$ is an H-bundle, a fibre bundle and a fibration.

With this information, we can now apply the long exact sequence for homotopy groups to the homotopy groups of coset spaces.

Corollary 2 *If G is a Lie group and H closed subgroup, the fibration $q : G \to G/H$ gives the long exact sequence*

$$\cdots \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \cdots \to \pi_0(H) \to \pi_0(G)$$

where $g_0H \in G/H$, $g_0 \in g_0H = p^{-1}(g_0H)$ and $g_0 = i(g_0)$.

Note, however, that $\pi_0(H)$ is the set of path components of H, and is not necessarily a group. Fortunately, we can recover the group structure because H is a group.

Remark 2 Recall that $\pi_0(X) = \langle S^0, X\rangle$ and $S^0 = \{*, 1\}$. Now a homotopy $H : S^0 \times I \to X$ then maps the basepoint to $H(*, t) = x$, while $H(1, t)$ is a path in X.

Hence, two maps $f, g : S^0 \to X$ are homotopic if and only if there is a path γ from $f(1)$ to $g(1)$. That is, if we let the class of x be $[x] = \{\gamma(1)|\gamma \in X^I, \gamma(1) = x\}$, then $\pi_0(X)$ is the quotient set $\{[x]|x \in X\}$.

In particular, for a topological group H, we write H_0 for the connected component of the identity, that is $H_0 = [e]$.

Since conjugation is continuous, and $heh^{-1} = e$ we see that H_0 is a normal subgroup of H.

Let us recover the group structure on $\pi_1(G/H)$.

Theorem 6 *Let G be a simply connected Lie group and H be is a closed Lie subgroup of G.*

Then $\pi_1(G/H, H)$ is isomorphic to H/H_0.

Proof. Use that $q : G \to G/H$ is a fibration.

Define the map
$$\phi : \pi_1(G/H, H) \to H/H_0, [\gamma] \mapsto \tilde{\gamma}(1)H_0$$
where $\tilde{\gamma}$ is the lift of γ for which $\tilde{\gamma}(0) = e$.

Suppose $\phi[\alpha] = \phi[\beta]$. Then $\tilde{\alpha}(0) = \tilde{\beta}(0) = e$, $\tilde{\alpha}(1) = a$, and $\tilde{\beta}(1) = b$. Here $aH_0 = bH_0$, so $a \in bH_0$.

As G is simply connected, there is a homotopy $\tilde{\eta}$ from $\tilde{\alpha}$ to $\tilde{\beta}$, relative to H_0 and aH_0. Let $\eta = p\tilde{\eta}$ be the homotopy from α to β, relative to H. Then $[\alpha] = [\beta]$, so ϕ is injective.

Suppose we have a coset hH_0. Choose a path $\tilde{\gamma}$ in G, from e to h. Then for $\gamma = p\tilde{\gamma}$ is a loop for which $\phi[\gamma] = hH_0$. Conclude that ϕ is a surjection, and also a bijection.

Take $\alpha\beta \in \pi_1(G/H, H)$. Lift α and β to $\tilde{\alpha}$ and $\tilde{\beta}$ with again $\tilde{\alpha}(0) = \tilde{\beta}(0) = e$. The concatenation $\tilde{\alpha}\tilde{\beta}$ then lifts $\alpha\beta$ and has $\tilde{\alpha}\tilde{\beta}(0) = e$, and $\tilde{\alpha}\tilde{\beta}(1) = ab$. Conclude that $\phi([\alpha][\beta]) = \phi([\alpha])\phi([\beta])$, thus ϕ is an isomorphism. ∎

Using this machinery we can compute some homotopy groups of homogeneous spaces X. However, there are some results that let us choose the group G acting on X to have trivial groups $\pi_1(G) = \pi_2(G) = 0$.

Conjecture 2 *For a path-connected Lie group G, there is always a universal covering group $\tilde{G}$, and $G/H \cong \tilde{G}/\tilde{H}$, where $\tilde{H} = p^{-1}(H)$.*

Remark 3 A path-connected Lie group G is in particular locally path-connected and semilocally simply connected, from which it follows that there is a universal cover $\tilde{G}$.

The homeomorphism between the coset spaces is simply the induced map of the covering map.

Conjecture 3 *Any connected Lie group G deformation retracts onto a maximal compact subgroup G'.*

Remark 4 A theorem due to Malcev gives the stronger result that G is diffeomorphic to $G' \times \mathbb{R}^n$ where G' is a maximal compact subgroup.

A theorem due to Cartan gives the following.

Conjecture 4 *If G is a compact Lie group, then $\pi_2(G) = 0$ and $\pi_3(G) = \mathbb{Z}^r$.*

We can apply the triviality of the homotopy groups to find the second homotopy group of the homogeneous space.

Corollary 3 *Since $\pi_1(G) = \pi_2(G) = 0$, our fibration $H \to G \to G/H$ gives us the sequence*

$$\pi_2(G) = 0 \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) = 0$$

which implies an isomorphism $\pi_2(G/H) = \pi_1(H)$.

When $\pi_k(H) = 0$ for $k > 0$, we see that $\pi_k(G/H) = \pi_k(G)$.

IV.3 The Seifert-Van Kampen theorem

The Seifert-Van Kampen theorem is a powerful theorem, that simplifies the computation of π_1 of spaces that are nicely decomposable.

Unfortunately, the proof of the theorem for fundamental groups is mildly tedious and no more insightful than a sketch of the proof. A complete proof can be found in [2], but we will only give the idea behind it.

There is also a variant for fundamental groupoids, which lies closer to abstract nonsense than homotopy, and there are also higher homotopy Van Kampen-type theorems, which are esoteric at best.

We will only need the fundamental group variant, but one may keep in mind that the groupoid variant gives an almost equivalent statement in the category of groupoids instead.

Theorem 7 *Let X be the union $A \cup B$ of path connected opens A and B, so that $A \cap B$ is path connect and contains the basepoint.*

The inclusions

$$i_1 : A \cap B \to A, i_2 : A \cap B \to B, j_1 : A \to X, j_2 : B \to X$$

have induced maps $\pi_1(i_1), \pi_1(i_2), \pi_1(j_1), \pi_1(j_2)$ that form a pushout square.

That is, $\pi_1(X)$ is the amalgamated free product $\pi_1(A) *_{\pi_1(A\cap B)} \pi_1(B)$.

Corollary 4 *An equivalent result is that $\Phi : \pi_1(A) * \pi_1(B) \to \pi_1(X)$ is surjective with kernel $N = \ker(\Phi)$.*

Then N is generated by elements of the form $i_1(\omega)i_2(\omega)^{-1}$ for $\omega \in \pi_1(A \cap B)$. And hence, $\pi_1(X)$ and $\pi_1(A) * \pi_1(B)/N$ are isomorphic.

The surjectivity of Φ is equivalent to factorizing loops in X into loops in A and B. [2] treats the general case where $X = \bigcup_\alpha U_\alpha$, and to show N is the kernel of Φ, he defines two moves on factorizations of loops in $\pi_1(X)$.

The first move accumulates loops in the same factor $\pi_1(U_\alpha)$, while the second sorts the loops in $\pi_1(U_\alpha \cap U_\beta)$ to $\pi_1(U_\alpha)$. In particular, if any two factorizations of any $[f] \in \pi_1(X)$ are equivalent under these moves, then $\Phi : *_\alpha \pi_1(U_\alpha)/N \to \pi_1(X)$ is injective.

To show that two factorizations are equivalent, he takes the homotopy $F : I \times I \to X$ between the two. Then, the rectangle $I \times I$ is partitioned into smaller rectangles, each mapping to a single U_α.

The moves can be explicitly performed on this partition for any pair of factorizations, which completes the proof.

Corollary 5 *In particular, if $\pi_1(A \cap B) = 0$, then $N = \{e\}$ and it follows that the fundamental group of a wedge sum $\bigvee_\alpha X_\alpha$ is $*_\alpha \pi_1(X_\alpha)$.*

Example 8 As $\pi_1(S^1) = \mathbb{Z}$, it follows that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

Similarly, because $\pi_1(S^2) = 0$, the sphere with two distinct points identified $W = S^2/\{a, b\}$ is homotopy equivalent to $S^2 \vee S^1$, hence $\pi_1(W) = \mathbb{Z}$.

IV.4 Simplicial Homology

In the simplest terms, the n-th homotopy groups measure the number of ways one can fit an n-sphere into a space X.

This is roughly the motivation for the homology groups, which instead measure n-dimensional holes in X. Homology comes in many variants, of which we will introduce simplicial homology, as it is most readily computable provided we can triangulate our spaces.

Given the homology groups of a space X, we may apply the Hurewicz theorem, which links the homology and homotopy groups. In particular, if $n > 1$ and X is $(n-1)$-connected, the n-th groups are isomorphic $\pi_n(X) = H_n(X)$.

IV.4.1 Homology groups

We abridge the treatment in [2]. All variants of homology are pointed around the (co)chain complex $C(X)$ of X.

Definition 21 A *chain complex* $C(X)$ of X is a sequence of abelian groups (or modules) C_n connected by *boundary operators* ∂_n, and is denoted

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

The boundary operators are homomorphisms such that $\partial_n \partial_{n+1} = 0$, equivalently $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$.

The *boundaries* are the elements of $\mathrm{im}(\partial_{n+1})$, and the *cycles* are the elements of $\ker\partial_n$. We say that two cycles are *homologous* when their difference is a boundary.

The homology of cycles is an equivalence relation, so homology gives rise to homology classes of cycles.

Definition 22 The *n-th homology group* is the quotient group $H_n(X) = \ker(\partial_n)/\mathrm{im}\partial_{n+1}$.

We will use the variant of homology defined on simplicial complexes. [10]

Definition 23 The n-th homology group of a simplicial complex S is $H_n(S) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$.

The n-th chain group C_n is the free abelian group on the set of n-simplices in S, equivalently it is the abelian group on n-chains or formal sums

$$\sum_{i=1}^{N} c_i \sigma_i$$

where c_i are integers and σ_i are n-simplices.

The boundary operator $\partial_n : C_n \to C_{n-1}$ is defined on a simplex $\sigma = (v_0, \ldots, v_k)$ by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, v_n).$$

To practically compute n-th homology group $H_n(X)$ of a smooth manifold, we can triangulate it.

Definition 24 A *triangulation* of a space X is a simplicial complex K with a homeomorphism $h : K \to X$.

Then it follows that $H_n(X) = H_n(K)$, where the latter can be computed by simplicial homology.

Example 9 Let $X = S^2 \vee S^2$. Indeed, X is triangulated by K, the complex of two tetrahedra joined at a vertex.

After a long calculation, we may find that $H_0(X) = \mathbb{Z}$ and $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$.

[10] The cellular variant, adapted to CW-complexes, indeed applies more directly to the spaces we are interested in. Manually computing the simplicial homology groups of the tetrahedron is a nice exercise. Another exercise is to find a triangulation of the tetrahedron with only one face. This should also elucidate why cellular homology is more suitable to apply to a sphere.

Note that the same computation can be easily generalized by using the Mayer-Vietoris sequence.

IV.4.2 The Hurewicz theorem

The absolute version of the Hurewicz theorem relates elements f of $\pi_n(X)$ to elements of $H_n(X)$ by pushing forward generators of $H_n(S^n)$ by f.

A proof of the Hurewicz theorem is given in [2] Section 4.2.

Conjecture 5 *For a space X with $\pi_0(X) = 0$ and $n > 0$ the map*

$$h_* : \pi_n(X) \to H_n(X), [f] \mapsto [f_*(u_n)]$$

is a homomorphism, where $f_* : H_n(S^n) \to H_n(X)$ is the homology pushforward and u_n is a generator of $H_n(S^n)$.

If $n = 1$, then h_* induces the isomorphism

$$\tilde{h}_* : \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X).$$

If $n > 1$ and $\pi_k(X) = 0$ for $k < n$, then $h_* : \pi_n(X) \to H_n(X)$ is an isomorphism and $h_* : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism.

We can apply the Hurewicz theorem to get the familiar result of homotopy groups of spheres.

Corollary 6 *Because $\pi_k(S^n) = 0$ for $k < n$, the Hurewicz theorem gives that $\pi_n(S^n) = H_n(S^n) = \mathbb{Z}$.*

Example 10 Let $X = S^2 \vee S^2$.

We know that $\pi_1(S^2) = 0$, so by the Seifert-Van Kampen theorem $\pi_1(X) = 0$.

Hence, the Hurewicz theorem gives that $\pi_2(X) = H_2(X) = \mathbb{Z}^2$.

Part V

Secondary Phenomena

Using the theorems from the previous parts, we can classify most defects of many order-parameters, if not all naturally occurring defects. In this discussion, we considered only the isolated defects, while in practice it is not uncommon to encounter defects in pairs or larger groups.

This part presents the generalization of the description of specific spaces and single defects to different realizations of distinct homotopy classes and multiple defect states.

In Section V.1 we motivate the interest in ring defects, as studied in [5], and the non-defective states distinct from the ground state, known as solitons. When two line defects meet, they may interact and entangle in ways other defects cannot. This is described in Section V.2 following the treatment of [1]. In Section V.3 we illustrate how the motion of a defect can generate solitons in its medium. Similarly, in Section V.4 the effect of winding defects around π_1 type defects is examined and generalized to the case of arbitrary numbers of defects.

V.1 Generalized defects

Until now, we have only demonstrated defects arising in $M_{n,k}$, but in fact we can find defects corresponding to the classes $[M, X]$ for any M. Note that our usage of the term "defects" suggests that something is "defective" in M, like the vortices in $M_{2,0}$ which are broken when the medium is considered to be part of $\mathbb{R}^2$. In general, there can be different kinds of configurations that arise on M that are homotopically distinct, when M is not considered as a subspace. Such structures are also known as solitons. Other literature also refers to some kinds of solitons as textures.

For example, it is possible to make a ferromagnetic sphere, when modelled by the $O(3)$ model on S^2, we would find solitons $[S^2, S^2] = \mathbb{Z}$. Solitons with different integers are not homotopic, but also do not contain discontinuities.

However, this also highlights their dubious stability in some cases. The sphere S^2 is compact, thus all solitons can be brought to uniformity by a local surgery. They are still metastable, and become exponentially "more stable" proportional to the size of the sphere.

Let us give two cases of these topological solitons that open the door to new theories.

V.1.1 Ring defects

Instead of taking a line or a point defect, we can go halfway and take a ring defect. Locally a ring looks like a line, while it fits inside a sphere like a point. A detailed discussion of ring defects can be found in [5].

Let us model a ring defect in $\mathbb{R}^3$. Embed a circle into $\mathbb{R}^3$ via $a : S^1 \hookrightarrow \mathbb{R}^3$ and let the defective region be $P = \mathrm{im}(a)$. The space modelling the ring defects is then $M = \mathbb{R}^3 - P$.

We see that M deformation retracts to a sphere where two points x_N, x_S are connected, which is homotopy equivalent to $S^2/{\sim}$ with $x_N \sim x_S$. Then, $S^2/{\sim}$ is homotopy equivalent to $S^1 \vee S^2$.

To classify the ring defects, we have to calculate $[S^1 \vee S^2, X]$, but unfortunately our theory is not adapted to calculate $[Z, X]$ for $Z \neq S^n$. Intuitively, as a ring defect is halfway between a point and a line, the classes $\langle S^1 \vee S^2, X\rangle$ are some kind of product of $\pi_1(X)$ and $\pi_2(X)$.

The confirmation of this intuition and the computation of $[S^1 \vee S^2, X]$ is demonstrated in [5], where it is denoted $\tau_2(X)$. Here $\tau_2(X)$ is computed by constructing sequences on a space of functions $W \to X$ where $W \equiv S^1 \vee S^2$ directly.

Alternatively, classes $[M, X]$, and in particular $M = S^1 \vee S^2$ as $\dim(M) = 2$, may be computed more algebraically by using a theorem due to J.H.C Whitehead, given as Theorem 1.1 in [6].

V.1.2 Solitons

Let us demonstrate that the homotopy classes $[S^3, X]$, which seem unrelated to the 3 dimensional situation, can classify solitons in M.

Shortly put, a defect and the local uniformity of media constrain the behaviour away from the defect. Dually, a constraint outside some region and the local uniformity, may let a distinct stable configuration arise inside: the soliton.

Often, and as we will show later, soliton form naturally in the wake of a moving defect. We define the spaces modelling the boundary conditions as follows.

Definition 25 Let $C_{n,k} = [-1,1]^k \times \mathbb{R}^{n-k}$ the long box.

Let $R_{n,k} = \overline{\mathbb{R}^n - C_{n,k}}$, the exterior of the box.

Define $S_{n,k} = \mathbb{R}^n / R_{n,k}$ as the quotient space with quotient map $q : \mathbb{R}^n \to S_{n,k}$. The boundary condition is represented by the image $q(R_{n,k})$. Let the image be $*$ and consider $(S_{n,k}, *)$ to be pointed.

Define $V_{n,k}(X) = \{f \in C(\mathbb{R}^n, X) : f|_{R_{n,k}} = c\}$, the constrained space of maps, for a fixed c.

We claim that the homotopy classes of constrained maps $f|_{R_{n,k}} = c$ correspond to solitons. For this, we first simplify the constrained map space $V_{n,k}(X) \cong C(S_{n,k}, X)$, so that its homotopy classes simplify to $\pi_0(V_{n,k}(X)) \cong \langle S_{n,k}, X\rangle$. Then finding a homotopy equivalent space for $S_{n,k}$ completes the calculation.

Lemma 13 *The map space $V_{n,k}(X)$ is homeomorphic to $C(S_{n,k}, X)$.*

The homotopy classes $\pi_0(V_{n,k}(X)) \cong \langle S_{n,k}, X\rangle$.

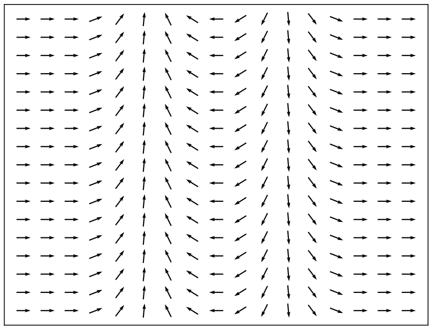
The space $S_{n,k}$ is homotopy equivalent to S^k.

Proof. We see that a map $f : (S_{n,k}, *) \to (X, c)$ corresponds bijectively to a map $f' : \mathbb{R}^n \to X, x \mapsto fq$, then it holds $f'|_{R_{n,k}} = c$. Hence, $V_{n,k}(X) \cong C(S_{n,k}, X)$.

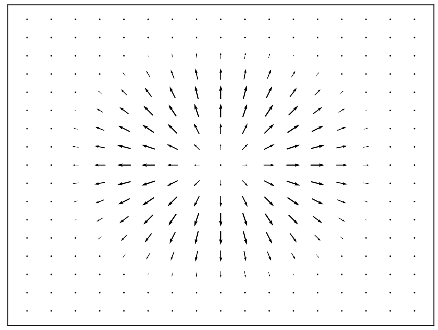
The connected components $\pi_0(V_{n,k}(X))$ are then the homotopy classes $\langle S_{n,k}, X\rangle$, which are pointed as $(S_{n,k}, *)$ is a pointed space.

We can simplify $S_{n,k}$ to $C_{n,k}/\partial C_{n,k}$. The boundary is then $\partial C_{n,k} = \partial([-1,1]^k \times \mathbb{R}^{n-k}) \cong (\partial[-1,1]^k) \times \mathbb{R}^{n-k}$.

Let us construct a deformation retraction from $[-1,1]^k \times \mathbb{R}^{n-k} / \partial([-1,1]^k \times \mathbb{R}^{n-k})$ to $[-1,1]^k / \partial[-1,1]^k$. Write $\vec{x} \in [-1,1]^k$ and $\vec{y} \in \mathbb{R}^{n-k}$.



(a) The $O(2)$ soliton.



(b) The $O(3)$ soliton, note that the arrows in the centre are pointing up, while those at the edges are pointing down.

Figure 8

Define the retraction as

$$r_\lambda : C_{n,k}/\partial C_{n,k} \to C_{n,k}/\partial C_{n,k}$$
$$[(\vec{x}, \vec{y})] \mapsto [(\vec{x}, (1-\lambda)\vec{y})]$$

Because $r_\lambda(\partial C_{n,k}) \subset \partial C_{n,k}$ we see r_λ is indeed well-defined. Embedding $\iota : [-1,1]^k/\partial[-1,1]^k \to C_{n,k}/\partial C_{n,k}, [\vec{x}] \mapsto [(\vec{x}, 0)]$ shows that r_1 retracts $C_{n,k}/\partial C_{n,k}$ to $[-1,1]^k/\partial[-1,1]^k \subset C_{n,k}/\partial C_{n,k}$.

Conclude that $C_{n,k}/\partial C_{n,k} \equiv [-1,1]^k/\partial[-1,1]^k$.

We can then use that $[-1,1]^k/\partial[-1,1]^k$ is homeomorphic to S^k. Conclude that $S_{n,k} \equiv S^k$. ■

With these observations we classify the solitons.

Theorem 8 *The solitons of parameter X with boundary condition $R_{n,k}$ correspond to $\pi_k(X)$.*

Equivalently $\pi_0(V_{n,k}(X)) \cong \pi_k(X)$.

Proof. This is a corollary of Lemma 13.

Namely, the maps $f : \mathbb{R}^n \to X$ with boundary condition $f|_{R_{n,k}} = c$ correspond to maps from $S_{n,k}$ to X. This map space is $V_{n,k}(X)$.

Then because $V_{n,k}(X) \cong C(S_{n,k}, X)$, it follows that $\pi_0(V_{n,k}(X)) \cong \langle S_{n,k}, X\rangle$.

Finally, using that $S_{n,k} \equiv S^k$ we get $\langle S_{n,k}, X\rangle \cong \pi_k(X)$. ■

One may question the relevance of cubical solitons or π_3 in this picture, when one may be lead to think that π_3 is 0 for most 1 or 2-dimensional spaces, like $S^1, T^2, S^1 \vee S^1$.

Surprisingly, the same is not true for S^2. In fact the Hopf fibration shows that $\pi_3(S^2) = \mathbb{Z}$ and more generally $\pi_n(S^2) = \pi_n(S^3)$ for $n > 2$.

Consequently, the solitons associated with π_3 in the $O(3)$ model are also known as the Hopf solitons, or, the Hopfions.

V.1.2.1 Stability

The physical stability of solitons is rather subtle in two ways. First, if we disregard the boundary conditions, there are homotopies moving the non-uniform part out of the medium[11], or zooming in to a constant patch. Hence, for the solitons to be stable, there needs to be a restoring force that compresses the non-uniformity to a small region, like $[-1,1]^k$.

Second, we need to be careful with local surgery on solitons, while we can replace $[-1,1]$ with larger intervals, no matter how large the replacement, there is the local surgery $\eta(f, c; C_{n,k}, C_{n,k})$, where c is the constant configuration.

However, if we do preserve the boundary condition, there is no homotopy from f to c. The barrier preventing such a homotopy is not "strong".

We can take a "homotopy" of f on $[0,1)$, such that $f_t \to c$ as $t \to 1$, effectively contracting $C_{n,k}$ to a point. During this contraction, f_t goes towards a singularity at 0.

Since the volume of $C_{n,k}$ scales as r^k but $\mathrm{d}f$ scales with r^{-1}, we see $E = \int_{C_{n,k}} |\,\mathrm{d}f|^2 \,\mathrm{d}x \sim r^{k-2}$. Thus, if $k > 2$, we see that the energy E goes to 0 as r goes to 0. If $k = 2$ then E is constant.

By shrinking $C_{n,k}$ we in fact obliterate the singularity at the last moment and end up in uniformity. [1]

However, we can prevent $C_{n,k}$ from shrinking completely by introducing higher terms $\mathrm{d}^p f$ in the local energy, so that if $p > k$ the energy E diverges as r goes to 0.

V.2 Entanglement of line defects

When we consider two point defects, we can look at how the combine when brought close together. Lines have the property that they, unlike points, can form knots and braids, or get stuck in some other way.

Consider in $\mathbb{R}^3$ two line defects represented by the pushforwards of $\pi_1(M, x)$ to α and β in $\pi_1(X)$.

Let us attempt to move one across the other continuously. Keep β still, and move α to the other side, creating a loop pinched around β as shown in Figure 10.

Clearly the types of the lines are the same as before, thus we only need to look at the pinched segment γ. Draw a loop around γ so that we can push it forward to X and find its type. Figure 10 illustrates that $\gamma = [\alpha, \beta]$.

[11] Out of the medium is the boundary of $\mathbb{R}^n$, so to speak.

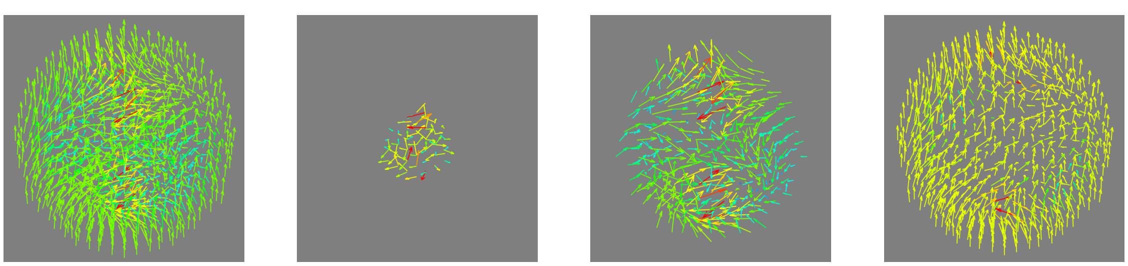


Figure 9: Left: the 1-Hopfion. The right three figures are from left to right the one-third radial slices of the Hopfion.

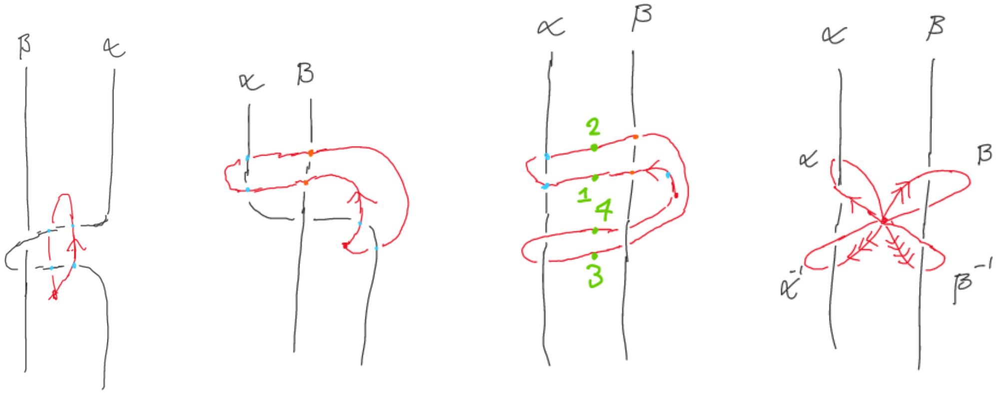


Figure 10: Illustrating the loop pinched around β, and the class γ around that loop.

From this, we observe that if the medium is abelian $\gamma \sim 0$ and lines can pass through each other without leaving a trace.

One can demonstrate that such a connecting line segment contributes energy proportional to the separation of the lines, preventing them from moving far away.

V.3 Formation of solitons

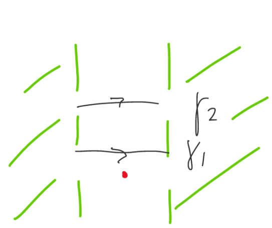
When we introduced solitons, we claimed they could be generated by moving defects; we illustrate this process in 2 dimensions, and generalize it to higher dimensions. The situation in $M_{2,1}$ is shown in Figure 11.

Let the initial configuration be $f : M \to X$ and take the paths shown, so that $f\gamma_1 \sim f\gamma_2 \sim e$. These paths are indeed maps $S^1 \to M$, since the endpoints touch $R_{2,1}$ thus go to the same basepoint.

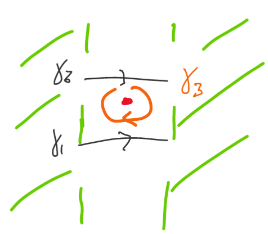
Now, we suppose that moving the defect above γ_1 gives us the configuration $f' : M \to X$, which satisfies $f'\gamma_2 \sim e$.

Draw a loop γ_3 around the defect. We see that γ_3 is homotopic to $\gamma_1^{-1}\gamma_2$. As $f'\gamma_2 \sim e$, we see that $f'(\gamma_1^{-1}\gamma_2) \sim f'\gamma_1^{-1} \sim f'\gamma_3$.

Conclude that moving the defect across γ_1 turned $f\gamma_1 \sim e$ into $f'\gamma_1 \sim f'\gamma_3^{-1}$.

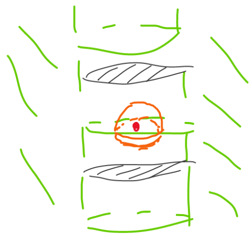


(a) The initial configuration.



(b) The configuration after the defect crosses γ_1.

Figure 11



(a) After crossing the lower sphere

(b)

Figure 12: Example for $\mathbb{R}^3$ and $C_{3,2}$

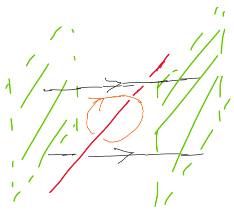


Figure 13: The formation of a plane soliton in $\mathbb{R}^3$ and $C_{3,1}$

If we brought in the defect from infinity and moved it towards the opposite side, we would see that all γ would have $f'\gamma \sim \tau^{-1}(f\gamma)$, writing τ for the type of defect in π_1. Thus, this process turns the $\gamma \in \pi_1(X)$ soliton into the $\tau^{-1}\gamma \in \pi_1(X)$ soliton.

This generalizes to $M_{n,k}$ by instead moving a $(n-k-1)$-flat defect parallel to $C_{n,k}$ across M. Again this time the maps $(I^k, \partial I^k) \to (D^k_\epsilon, *)$ are maps $S^k \to M$. See Figure 12 and Figure 13.

V.4 Winding of defects

The solitons suggest that moving defects along non-trivial paths may generate some topologically stable quantities. Such non-trivial paths may also be loops around other defects, as we consider here.

V.4.1 Double exchange

Fix a basepoint x and consider the case of an $n-2$-flat defect, corresponding to $\gamma \in \pi_1(X)$, and an arbitrary defect corresponding to $\alpha \in \pi_n(X)$.

Take a loop around the γ defect, and move the α defect along this loop. Pulling the sphere fixed at x along with α gives the shape in Figure 14.

This shape is homeomorphic to the original sphere with the loop around γ attached. Since this shape pushes forward to α, we conclude that the spherical part pushes forward to $\gamma\alpha$.

We see that the action of winding α around γ is the same as the action of π_1 on π_n. Hence, one form of a defect can be brought to any other form by winding it around the appropriate γ.

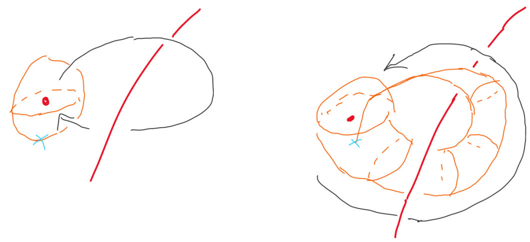


Figure 14: Moving α around γ.

V.4.2 Braiding

When we have multiple defects corresponding to elements of π_1, there are more ways to braid defects.

The braid group on k strands B_k gives us a way to represent all braids. We may then consider the action of B_k on our configuration, by shuffling the defects in accordance with the crossings in the braid.

The action of a crossing is described that of the Π_1 action on π_1. This action is not well-defined as it depends on the representative of the defect. For example, if $X = S^1$, we may have a loop around a vortex in which the first half is null homotopic as a path, while the second half completes the loop in S^1.

We conclude that with homotopy groups we can only describe the situation where the defects end where they start. This is captured by the pure braid group P_k, which is the kernel of the map $\phi : B_k \to S_k, \sigma_i \mapsto \tau_i$.

It must be noted that the pure braid group still allows for braids that aren't clearly composed of closed loops. [7] gives that P_k is generated by the pure crossings $A_{i,j}$. As a consequence, we get that a pure braid decomposes into a product of closed loops. Thus, any pure braid can be written in terms of $A_{i,j}$ of which the actions are again those of the winding of 2 defects around each other: conjugation.

Part VI

Defects in $SO(3)$ symmetry-broken media

With Part III, Part IV and Part V we can essentially compute all classes of simple defects, solitons, and some other phenomena. And so we will.

In this part we focus on the $n = 3$ case of $SO(n)$, and we apply the knowledge our ancestors used long ago to explain the elements of matter.

We can directly describe all order-parameters of broken $SO(3)$ symmetry, since we know all the closed subgroups and hence all possible isotropies after symmetry breaking.

Theorem 9 *The closed subgroups of $SO(3)$ are:*

- $\mathbb{Z}_n$
- D_n *(sometimes D_{2n})*
- A_4
- S_4
- A_5
- $SO(2)$
- D_∞

Proof. The finite subgroups of $SO(3)$ are classified in [8].

We confirm that $SO(2)$ as the rotation around a nonzero vector v and the infinite dihedral group D_∞, as the generalized dihedral group of $SO(2)$, are both closed subgroups. We claim that $SO(2)$ and D_∞ are the only infinite, closed and proper subgroups of $SO(3)$.

Let $H \subset SO(3)$ be infinite, and assume it is closed. Because $SO(3)$ is compact, H is also compact.

Then H cannot be discrete, because then it has an infinite open cover with no finite subcover.

Because H is not discrete, it has a nontrivial connected component. Define the absolute angle of rotation map $\alpha : SO(3) \mapsto [0, \pi]$.

As H has a nontrivial connected component, $\alpha(H)$ does too. The irrationals are dense in $[0, \pi]$, so H contains a rotation by r/π where r is irrational. The multiples of r/π, modulo π, are dense in $[0, \pi]$. Hence, this rotation in H is dense in all rotations around an axis.

Conclude that H has contains $SO(2)$. We say that this subset $SO(2)$ are the rotations around some vector v.

If H contains an element that does not fix v, then we may iteratively construct a subset homeomorphic to $SO(2)$ of rotations around an orthogonal vector w. Then, $H = SO(3)$.

Conclude that the only infinite closed proper subgroups are isomorphic to $SO(2)$ and D_∞. ■

Using these subgroups, we can apply Corollary 3 to find some homotopy groups of the order-parameter.

First we can compute the higher homotopy groups in the simpler cases. In particular, for the finite subgroups of $SO(3)$, we see that $\pi_k(SO(3)/H) = \pi_k(SO(3))$ when $k > 1$.

We can use the fact that S^2 is a homogeneous space of $SO(3)$ to compute homotopy groups of $SO(3)$.

Theorem 10

$$\pi_k(SO(3)) = \pi_n(S^2), n > 2$$

Proof. Because the coset space $SO(3)/SO(2)$ is homeomorphic to S^2, we can apply Corollary 2 to the long exact sequence.

However, we have $\pi_n(SO(2)) = 0$ for $n > 1$, so the sequence breaks into sequences

$$0 \to \pi_n(SO(3)) \to \pi_n(S^2) \to 0$$

for $n > 2$, which confirms the wanted result. ∎

Note that $SO(3)$ is also a compact Lie group, so it has $\pi_2(SO(3)) = 0$.

Because $SO(3)$ is not simply connected, before we can compute the fundamental groups, we first need to lift the subgroups to the universal cover of $SO(3)$.

Example 11 The universal cover of $SO(3)$ is $SU(2)$ and can be explicitly constructed as follows[12].

Write $g(\phi, \theta, \psi) \in SO(3)$, and define the covering map as

$$p : \pm \begin{pmatrix} \cos\frac{\theta}{2} e^{i\frac{\phi+\psi}{2}} & i\sin\frac{\theta}{2} e^{i\frac{\phi-\psi}{2}} \\ i\sin\frac{\theta}{2} e^{-i\frac{\phi-\psi}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\phi+\psi}{2}} \end{pmatrix} \mapsto g(\phi, \theta, \psi).$$

Corollary 7 *We attempt to compute $\pi_1(SO(3)/H) = \tilde{H}/\tilde{H}_0$.*

Using the universal cover $SU(2) \to SO(3)$, we find the following pre-images of the finite subgroups[a]

H	$\mathbb{Z}_n$	D_n	A_4	S_4	A_5
$\tilde{H}$	$\mathbb{Z}_{2n}$	Dic_n	$2T$	$2O$	$2I.$

The names are rather uninformative, as $A_4 \cong T$, $S_4 \cong O$ and $A_5 \cong I$.

The infinite groups lift to themselves

$$\widetilde{SO(2)} \cong SO(2), \quad \widetilde{D_\infty} \cong D_\infty.$$

Unlike the discrete groups that are their path components themselves, $SO(2)$ only has 1 component, while D_∞ has 2.

[a]These may be found in most listings of small groups.

To describe the flat defects, we need to compute $[S^n, SO(3)/H]$ now. Let us first describe the line defects in 3 dimensions, as they are the $n = 1$ case.

[12]Usually $Spin(n)$ is the universal cover of $SO(n)$, but $Spin(3)$ is exceptionally isomorphic to $SU(2)$

Corollary 8 *In $M_{3,1}$, the line defects correspond to the homotopy classes $[M_{3,1}, SO(3)/H] \cong [S^1, SO(3)/H]$.*

Use that set $[S^1, SO(3)/H]$ corresponds to the conjugacy classes of $\pi_1(SO(3)/H)$.

Summarizing the numbers conjugacy classes of the finite H

H	$\mathbb{Z}_n$	D_n	A_4	S_4	A_5
$[S^1, SO(3)/H]$	$2n$	$n+3$	7	8	9

The numbers of conjugacy classes of $\pi_1(SO(3)/H)$ for $SO(2)$ and D_∞ are then respectively 1 and 2.

To compute $[S^n, SO(3)/H]$ for $n > 1$, we need to compute the action of π_1 on π_n. Fortunately, this action greatly simplifies in some cases, and is for example trivial for all H-spaces.

Corollary 9 *Let H be a finite subgroup of $SO(3)$.*

We simplify $\langle S^n, SO(3)/H\rangle = \pi_n(SO(3)/H) = \pi_n(SO(3)) = \langle S^n, SO(3)\rangle$, for $n > 2$. Because $SO(3)$ is a topological group, the action of $\pi_1(SO(3)/H)$ on $\langle S^n, SO(3)/H\rangle$ is trivial.

Conclude that $[S^n, SO(3)/H] = \pi_n(SO(3)) = \pi_n(S^2)$, for $n > 2$.

Note that, to us, this is only relevant for $M_{n,k}$ with $n-k-1 > 2$, that is the k-flats in $4+k$ or more dimensions.

The point defects in 3 dimensions are classified by $[S^2, SO(3)/H]$, and are again given by Corollary 3.

Corollary 10 *Using the universal cover, we get $\pi_2(SO(3)/H) = \pi_2(SU(2)/\tilde{H})$. Then $\pi_2(SU(2)/\tilde{H}) = \pi_1(\tilde{H})$ by Corollary 3.*

Conclude that the finite subgroups H have no point defects.

For $SO(2)$, we have $\pi_2(SO(3)/SO(2)) = \pi_1(SO(2)) = \mathbb{Z}$, and similarly for D_∞ we have $\pi_2(SO(3)/D_\infty) = \pi_1(D_\infty) = \mathbb{Z}$.

As $\pi_1(SO(3)/SO(2)) = 0$, we get that $[S^2, SO(3)/SO(2)] \cong \mathbb{Z}$.

For D_∞, $\pi_1(SO(3)/D_\infty) = \mathbb{Z}_2$. Because the action is nontrivial, we get that the classes $[S^2, SO(3)/D_\infty] \cong \mathbb{N}$ correspond to pairs of n and $-n$ in $\mathbb{Z}$.

We write $X = SO(3)/H$. This table summarizes the defects and solitons, as the numbers of classes $[M, X]$ or as groups. Recall that $[S^{n-k-1}, X]$ represents the k-flat defects in n dimensions, and $\pi_n(X)$ represents the $n-1$ solitons.[13]

	$\pi_1(X)$	$\pi_2(X)$	$\pi_3(X)$	$\lvert[S^1, X]\rvert$	$\lvert[S^2, X]\rvert$
$\mathbb{Z}_n$	$\mathbb{Z}_{2n}$	0	$\mathbb{Z}$	$2n$	0
D_n	Dic_n	0	$\mathbb{Z}$	$n+3$	0
A_4	$2T$	0	$\mathbb{Z}$	7	0
S_4	$2O$	0	$\mathbb{Z}$	8	0
A_5	$2I$	0	$\mathbb{Z}$	9	0
$SO(2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	1	$\lvert\mathbb{Z}\rvert$
D_∞	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	2	$\lvert\mathbb{Z}\rvert$

[13]Note that π_2 for $SO(2)$ and D_∞ are computed by computing $\pi_2(S^2)$ and $\pi_2(\mathbb{RP}^2)$.

Appendix

A Flat defects

This appendix assumes knowledge of homotopy theory to present more general results. If one is unfamiliar with homotopy equivalences and deformation retractions, refer to Section III.1.

In this appendix we will show that for sufficiently simple spaces, the description of defects as homotopy classes and local surgery classes coincide. Specifically, the Euclidean spaces with flats removed are simple enough for this purpose.

A.1 Homotopy groups and surgery

Let us establish that the classes induced by local surgeries correspond to the free homotopy classes $[M, X]$ when M is a flat-removed space $M_{n,k}$.

We will prove the retraction homotopy lemma.

Lemma 14 *Given topological spaces X, $Y \subset X$, and Z.*

Suppose there is a deformation retraction $r : X \to Y$, with inclusion $\iota : Y \to X$.

Let $a : X \to Z$ and $b : X \to Z$ be continuous maps, and let $a' = a\iota : Y \to Z$ and $b' = b\iota : Y \to Z$.

Then a and b are homotopic if a' and b' are homotopic.

Proof. Let $H_0 : X \times I \to X$ be the homotopy from ιr to 1_X.

Push this homotopy forward to $H_1 = aH_0$ and $H_3 = bH_0$, so $H_1, H_3 : X \times I \to Z$. Conclude that a is homotopic to $a'r$, and b is homotopic to $b'r$.

Let $H_2 : Y \times I \to Z$ be the homotopy from a' to b'. Pulling the homotopy H_2 back to $H_2' : X \times I \to Z, (x, t) \mapsto H_2(r(x), t)$ gives that $a'r$ and $b'r$ are homotopic.

Conclude that a and b are homotopic. ∎

We can now prove a general result that relates the pushforwards of configurations to their surgeries.

Theorem 11 *Given a medium M and an embedding $a : K \hookrightarrow M$ of $K = S^n \times Y \times I$, where Y is contractible.*

Write $a_t : S^n \times Y \to M, (s, y) \mapsto (s, y, t)$. Require that $\partial \mathrm{im}(a) = \mathrm{im}(a_0) \sqcup \mathrm{im}(a_1)$.

Given a closed subset $U \subset M$ such that $\partial U = \mathrm{im}(a_1)$. Let $V = U \cup \mathrm{im}(a)$.

Then two configurations $f, g : M \to X$ have equal pushforwards on $\mathrm{im}(a_0)$ if and only if there is a surgery between them on U, V, that is

$$(fa_0)_* = (ga_0)_* \iff \eta(f, g; U, V)$$

Proof. Note that $\partial V = \mathrm{im}(a_0)$.

Suppose $(fa_0)_* = (ga_0)_*$. Let $r : S^n \times Y \to S^n$ be the contraction of Y, and let $\iota : S^n \to S^n \times Y$ be the inclusion so that $r\iota$ is homotopic to 1_{S^n}. Conclude that ι_* is bijective.

Note that fa_0, ga_0 are maps $S^n \times Y \to X$. We see that $(fa_0\iota)_* = (ga_0\iota)_*$, but $fa_0\iota, ga_0\iota$ are maps $S^n \to X$, so they are also homotopic.

The retraction homotopy lemma gives that fa_0 and ga_0 are homotopic. Because a_t is the homotopy from a_0 to a_1, we see that fa_0 and ga_1 are homotopic.

Let $H : S^n \times Y \times I$ be the homotopy from fa_0 to ga_1.

Define the local surgery for $\eta(f, g; U, V)$ as

$$h(x) = \begin{cases} g(x), & x \in U \\ f(x), & x \notin V \\ H(a^{-1}(x)), & \text{otherwise} \end{cases}$$

Conclude that the equality of pushforwards implies the existence of a surgery.

Next, suppose h is the surgery for $\eta(f, g; U, V)$. We see that $fa_0 = ha_0$ and $ga_1 = ha_1$. Then ha_t is the homotopy from fa_0 to ga_1.

Further, ga_t is the homotopy from ga_0 to ga_1. Then fa_0 is homotopic to ga_0, and because of the homotopy invariance of the pushforward $(fa_0)_* = (ga_0)_*$.

Conclude that the surgery implies equality of pushforwards. ■

Corollary 11 *We see that on $M_{n,k}$, the only nontrivial domains for surgery are embeddings of $K = S^{n-k-1} \times (0,1)^k \times I$ around the removed flat $L_{n,k}$.*

If a is the embedding, then a_0 induces the map $[M_{n,k}, X] \to [S^{n-k-1} \times (0,1)^k, X] \cong [S^{n-k-1}, X]$.

Then equality of pushforwards $f_* = g_*$ is equivalent to being freely homotopic, that is $[f] = [g]$ in $[M_{n,k}, X]$.

Hence, we state k-flat defects in n dimensions are classified up to local surgery by $[S^{n-k-1}, X]$.

B Group theory

In this appendix we give a short introduction into the relevant group theory, namely group actions and long exact sequences.

Definition 26 An *automorphism* is an isomorphism $f : X \to X$. The set of automorphisms is denoted $\mathrm{Aut}X$.

Given a map $h : A \to C^B$, we write $ac = h(a)(c)$ for the evaluation of a.

An *action* of a group G on a set X is a map $\phi : G \to \mathrm{Aut}X$ such that

$$ex = x$$

$$g(hx) = (gh)x$$

The *orbit* Gx of x is the set $G \cdot x = \{gx | g \in G\}$. An action is *transitive* when $G \cdot x = X$ for all x.

The *action quotient* of X by G is the set of orbits $X/G = \{Gx | x \in X\}$.

A *homogeneous space* of G is a manifold X together with a transitive action of G on X.

The *stabilizer* G_x of x is the set $\{g \in G | gx = x\}$.

Note that currying gives the described correspondence between maps $G \to X^X$ and maps $G \times X \to X$.

Definition 27 Given a group G and a subgroup H, the coset space G/H is defined as the action quotient of left-translation of G by H.

That is, G/H is the quotient space with quotient map $q : G \to G/H, g \mapsto gH$. The topology of G/H is defined as the quotient topology.

Lemma 15 *If G is a group with an action on X, then*

1. *the stabilizer G_x is a subgroup of G,*
2. *stabilizers of the same orbit are conjugate, that is, $y = hx$ implies $G_y = hG_xh^{-1}$,*
3. *conjugation $\phi_h : g \mapsto hgh^{-1}$ is an isomorphism,*
4. *stabilizers of the same orbit are isomorphic $G_x \cong G_y$.*

Proof. Take $g, h \in H$. Then $hx = x$, so $h^{-1}x = h^{-1}hx = x$ and $gh^{-1}x = x$ thus $gh^{-1} \in H$.

If $y = hx$ then

$$G_y = \{g \in G | gy = y\} = \{g \in G | ghx = hx\} = \{g \in G | h^{-1}ghx = x\} = \{hgh^{-1} \in G | gh = x\} = hG_xh^{-1}$$

Conjugation is a homomorphism since $hxy^{-1}h^{-1} = (hxh^{-1})(hyh^{-1})^{-1}$. The inverse is $\phi_h^{-1} = \phi_{h^{-1}}$ so ϕ_h is an isomorphism.

Since ϕ_h restricts to $G_x \to G_y$, conclude that G_x is isomorphic for all x. In fact if G is also a topological or Lie group, conjugation is also continuous or smooth and so a homeomorphism or diffeomorphism. ■

Definition 28 A *long exact sequence* is a sequence of groups G_n and homomorphisms f_n such that the image of f_{n+1} is the kernel of f_n.

$$\cdots \to G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_{n+1}} G_{n-1} \to \cdots$$

C Variational calculus

We usually take for granted that the entropy $-E[\log f]$ of a density $f : X \to \mathbb{R}$ is maximal when f is constant. Let us confirm this fact.

Maximizing $-E[\log f]$ is equivalent to minimizing $J[f] = \int_X L(f)\,\mathrm{d}x$ where $L(f) = \log f$. Because f is a probability density, we impose the constraint $\int_X M(f)\,\mathrm{d}x = 1$, where $M(f) = f$.

Then from variational calculus and the Euler-Lagrange equation, we find that $J[f]$ is minimal for

$$\frac{\partial L - \lambda M}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L - \lambda M}{\partial f'}\right) = 0.$$

Evaluating gives $\frac{\partial L - \lambda M}{\partial f} = -\frac{1}{f} - \lambda$ and $\frac{\partial L - \lambda M}{\partial f'} = 0$. Hence, $f = -\frac{1}{\lambda}$ so f is constant.

D Bibliography

References

[1] N. D. Mermin. "The topological theory of defects in ordered media". In: *Rev. Mod. Phys.* 51 (3 July 1979), pp. 591–648. DOI: 10.1103/RevModPhys.51.591. URL: https://link.aps.org/doi/10.1103/RevModPhys.51.591.

[2] Allen Hatcher. *Algebraic topology*. URL: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.

[3] Moritz Groth. *Lecture notes on homotopy theory*. URL: https://webspace.science.uu.nl/~caval101/homepage/Geometry_and_Topology_2016_files/Groth%20-%20lecture%20notes%20on%20homotopy%20theory.pdf.

[4] Joseph J. Rotman. *An Introduction to Algebraic Topology*. Springer New York, 1988. DOI: 10.1007/978-1-4612-4576-6. URL: https://doi.org/10.1007/978-1-4612-4576-6.

[5] H. Nakanishi, K. Hayashi, and H. Mori. "Topological classification of unknotted ring defects". In: *Commun. Math. Phys.* (1988). URL: https://doi.org/10.1007/BF01223590.

[6] G. Ellis. "Homotopy classification the J.H.C. Whitehead way." In: 2013.

[7] Christian Kassel and Vladimir Turaev. *Braid Groups*. 2008th ed. Springer, 2007. ISBN: 9780387338415.

[8] Felix Klein. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. ger. Leipzig: Teubner, 1884. URL: http://eudml.org/doc/203220.