Extracting Energy and Angular Momentum from a Kerr Black Hole

A physics bachelor thesis

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Abstract

In this thesis we take a look at energy and angular momentum extraction from a rotating black hole. We start off by introducing basic concepts regarding black hole before moving on to Kerr black holes. We compute basic properties of a Kerr black hole such as its the location of its event horizon, the area of the event horizon and we introduce the ergosphere.

Next we move on to the constituting mechanism for energy and angular momentum extraction from a Kerr black hole: the Penrose process. The basic idea of the Penrose process is that it is possible for a particle within the ergosphere to have a decay product which has a negative energy. If such a decay product falls into the black hole, it extracts energy and angular momentum from it.

A more effective energy and angular momentum extraction mechanism is the Blandford-Znajek process. The Blandford-Znajek process is an electromagnetic process and understanding it amounts to understanding three central components of which we only discuss two in detail in this thesis.

Firstly, we need to understand force-free electromagnetism. This can be understood as the physics explaining how the energy and angular momentum is being carried away. This force-free electromagnetism induceses a current in the plasma surrounding the black hole. This current is connected to the electromagnetic fields and carries away the energy and angular momentum.

Secondly we will briefly take a look at the plasma surrounding the black hole. We will answer the question of how it is formed and what properties it has to satisfy.

Lastly, we need to take a look at the mechanism explaining how energy and angular momentum is take away from the black hole. For a positive radially outflowing energy flux measured far away from the black hole, we require a negative radially inflowing energy flux near the black hole. Here we also recognise the role of the Penrose process.

1 Introduction

1.1 Historical Overview

In 1969 last year's Nobel prize winner in physics Roger Penrose proposed a mechanism through which energy and angular momentum could be extracted from the recently discovered rotating or Kerr black holes. This simple observation kick-started a large area of research focussed on the question: how can rotational energy and angular momentum be extracted from a Kerr black hole? This lead to various other mechanisms being proposed that could extract energy, most famously the Hawking radiation.

Sadly thoug, five years after Penrose's discovery, Robert Wald showed that the Penrose process was insufficiently effective to explain various observed phenomena. Originally the hope was that the Penrose process could supply the energy, or at least play a significant role, in the formation of jets and the active nature of active galactic nuclei. Wald showed that this could not bet done by the Penrose process.

The search continued and in 1977 Roger Blandford and Roman Znajek proposed another mechanism for energy extracted apply called the Blandford-Znajek process. This process can be thought of as the electromagnetic equivalence for the Penrose process. They based a lot of their mechanism on pulsar physics and thereby linking the study of energy extraction from a black hole to that of a pulsar.

Even though this is physics from the 70's, it is still an active area of research with papers coming out every day still. It is still hotly contested whether or not the Blandford-Znajek process can supply the energy necessary for various observed phenomena. No analytic solution is also known in the case of fast spinning black holes, a huge omission since most black holes are fast spinning.

The aim of this thesis is not to offer criticism or support for the Blandford-Znajek process. Rather its goal is to offer an in-depth explanation of the Blandford-Znajek process, which we will often refer to as the BZ-process. This has as downside that we do not offer any new result to this field of research. Rather we offer a textbook like introduction to the Blandford Znajek process. This is a fairly self-contained document where we go into a lot of details both regarding computations as well as regarding the physical intuition behind it.

1.2 Structure of this Thesis

To be able to understand the Blandford-Znajek process, we need to understand the basics of black hole physics. This is done in chapter 2. This chapter can viewed as a refresher on this subject and so can easily be skipped by a reader familiar with the subject. In this chapter we give a basic explanation of the formation of black holes before moving on to the mathematics. We introduce the Schwarzschild metric which plays an important role later on in the development of Michel's rotating magnetic monopole, a toy model for the BZ-process. Next we introduce the Kerr metric which describes spacetime around a rotating black hole. We derive the event horizon and the area of such a black hole before introducing the object that allows energy and angular momentum extraction to take place: the ergosphere.

In chapter 3 we aim to clarify the Penrose process. Again the reader familiar with this can skip it safely. In this chapter, we will show that it is possible for the decay product of a particle which decayed after having entered the ergosphere, to have a negative energy. Since the amount of energy extracted comes from the kinetic energy of the black hole, this means that the angular momentum will also decrease. Then we will show that this process in fact increases the area of the black hole, but the amount of energy that can be extracted is limited by the irreducible mass.

Chapters 4 throughout 7 are aimed at understanding the Blandford-Znajek process. In chapter 4 we develop an analogy between the BZ-process and the rotating cylinder immersed in a uniform magnetic field. The main goal of this analogy is to sketch in broad lines what we need to do to understand the Blandford-Znajek process as well as allow us to make some basic estimates.

In chapter 5 we introduce the concept of force-free electromagnetism. Bluntly put, we can understand force-free electromagnetism as the physics describing how the energy and angular momentum gets carried away. We give an in-depth description of the governing equations of force-free electromagnetism and apply it to a class of very important examples before specifying it to the context of the Michel's rotating magnetic monopole. The Michel's monopole will serve as the analytic starting point for the BZ-process.

In chapter 6 we give a briefly discuss how a plasma can form around a black hole. This plasma is essential since it contains the charge carriers necessary for the energy and angular momentum to be carried away from the black hole.

In chapter 7 we go into the mechanics of how energy and angular momentum are extracted. We start by taking a look at the Michel's monopole and we compute how much energy and angular momentum it loses. Then we discuss the physics behind this extraction and we note that this cannot coincide with the physical reality in the case of a Kerr black hole. These problems though are not insurmountable and we discuss how we can fix the situation.

Contents

2 Black Holes

Since black holes will be the central object of study in this thesis, it is worthwhile to start off by giving a description of what black holes are. We will do this in two ways: an astrophysical approach to the formation of black holes as well as a more mathematical description of black holes as singularities occurring in the metric of curved spacetime.

2.1 The Astrophysical Approach

Let us start off by giving a fairly informal description of how black holes are formed. We will limit ourselves to a brief discussion about stellar black holes since they form the largest class of black holes and the process of their formation is well understood. The more interested reader we refer to [4] chapter 2.

To better understand stellar black holes, an important question one must ask is: why do stars not collapse in on themselves? To say it crudely, stars are just a very heavy ball of gas. Gravity should just pull all that gas to the center and the star should be more or less inseparable from a point particle. So the question then becomes: why does that not happen? The answer is actually that this does happen. This is the very basics of a stellar black hole, but this does not happen while the star is 'alive'.

The way to understand this is as follows: A star starts out as a cloud of gas. This gas is attracted due to gravity to its center of mass (of course, assuming that there is no external force acting on this cloud). As this cloud moves towards its center of mass, the pressure around the center of mass starts to increase. This causes the particles (in this phase usually hydrogen) to start interacting with each other to form a heavier particle, a process known as **nuclear fusion**. In this interaction, energy, in the form of a photon, is released. This photon moves away from the center of the star, but it can hit other particles which are pulled to the center by gravity. This being hit by photons is what causes the particles to slow down untill an equilibrium is reached: the gravitational force pulling the particles in is offset by the light emitted due to the burning core of the star. To put it simples: the star stops collapsing.

Of course, this process does not hold indefinitely. After billions of years, the core has exhausted its fuel and gravitational collapse resumes. It is possible that the core of the star reignites. For example, suppose we have a star consisting only out of hydrogen. When its core is burning, what in fact is happening is the conversion of two hydrogen atoms into one helium atom and a photon. After all the hydrogen is converted into helium, it is possible that the gravitational collapse increases the pressure on the stars core to such a level that the helium atoms fuse to form heavier atoms. This interaction also releases energy which is able to counteract the gravitational collapse.

This exhaustion of the fuel and reignition can only happen so many times. An end point of this process is for example the case that the core consists only out of iron. In that case two or more iron atoms interacting to form a heavier atom does not release energy, but actually requires it. So then the burning of the core will not compensate the gravitational collapse. Another end point might be that the mass of the star is not sufficient to raise the pressure to such a level that the core combusts.

The most important thing to note is that in both of these cases the gravitational collapse is not counteracted. The star keeps collapsing in on itself. There are various other fail-safes which might counteract the gravitational collapse of the star. For example, the neutron degeneracy pressure, which is a consequence of the Pauli exclusion principle, can counteract gravitational collapse and leads to the formation of a neutron star.

It is possible though that the star has such a high mass that these various other fail-safes are not sufficient to counter the gravitational collapse. This results all the mass to converge on a single point, called the **singularity**, and a black hole is formed.

2.2 Black Holes: The Mathematical Approach

The astrophysical approach discussed here above is quite informal and serves only to develop an idea of what kind of objects (stellar) black holes are. What is more important for us is the fact that general relativity allows for the existence of black holes.

2.2.1 The Schwarzschild metric

The metric describing curved space around a planet or a star is the **Schwarzschild metric**:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (1)$$

We note that for the spatial components we have switched from Cartesian coordinates to spherical coordinates. We call (t, r, θ, ϕ) the **Schwarzschild coordinates**. Furthermore M is the mass of the black hole.

The first thing to note is that there are a couple of symmetries in this metric. Applying equation (96) one can show that ∂_t is a Killing vector, implying that the metric is time translation invariant. More difficultly, one can show that there exist Killing vectors implying the spherical symmetry of the metric. The easiest one of those is ∂_ϕ.

Next, since the Schwarzschild metric is diagonal, it is quite easy to compute its inverse. This gives us:

$$g^{\mu\nu} = \begin{pmatrix} -(1-\frac{2M}{r})^{-1} & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r}) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2(\theta)} \end{pmatrix} \quad (2)$$

Another point to note is that there are two singularities in this metric, namely at $r = 0$ and at $r = 2M$. Let us first take a look at $r = 2M$. It appears that this is not a real physical singularity, but occurs as a consequence of the choice of coordinates. To make this clear, we transform time in the Schwarzschild coordinates as follows:

$$t = u + r + 2M \ln|\frac{r}{2M} - 1|$$

The coordinates (u, r, θ, ϕ) are called **outgoing Eddington-Finkelstein coordinates**. We can compute its differential:

$$dt = \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial r}dr = du + dr + \frac{1}{\frac{r}{2M} - 1}dr = du + \frac{1}{1 - \frac{2M}{r}}dr$$

If we square this, we get:

$$dt^2 = du^2 + \frac{2}{1 - \frac{2M}{r}}dudr + \frac{1}{(1 - \frac{2M}{r})^2}dr^2$$

Now if we plug this into the Schwarzschild metric, we get:

$$ds^2 = -(1 - \frac{2M}{r})du^2 - 2dudr + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

The first thing to note is that this is still the Schwarzschild metric. So the space it describes, is unchanged. The second thing to note is that there is no singularity at the point $r = 2M$, but something interesting still does happen. If we look at the first component of the metric $(1 - \frac{2M}{r})$, we see that if we have that $r < 2M$, then we have $(1 - \frac{2M}{r}) < 0$. But if we have $r > 2M$, we get that $(1 - \frac{2M}{r}) > 0$. So that metric component changes sings at $r = 2M$.

The consequence of this is actually what allows black holes to exist in general relativity. To make this clear, let us take a look at the light cone $ds^2 = 0$. To simplify computations, we assume $d\theta^2 + sin^2(\theta)d\phi^2 = 0$. This then gives us the following equation:

$$-(1 - \frac{2M}{r})du^2 + 2drdu = 0 \quad (3)$$

So the first solution is straightforward: u is constant. Since we defined $u = t - r - 2M \ln|\frac{r}{2M} - 1|$, if we want it to remain constant as time passes on, then r must become larger. This then implies that any light ray that follows the geodesic described by $u = const$, moves away from the centre of the system. These light rays are called **outgoing radial light rays** since they go away from the black hole. The light rays which move away to the singularity are called **ingoing radial light rays**.

The second solution to the above equation is described by:

$$\frac{du}{dr} = \frac{2}{1 - \frac{2M}{r}}$$

We can solve this differential equation to get:

$$u = 2(r + 2M \ln|\frac{r}{2M} - 1|) + const.$$

If we look back at the equation $\frac{du}{dr}$, what is interesting about this is that if $r < 2M$, then $\frac{du}{dr} < 0$. This implies that if $r < 2M$, the radial light rays are ingoing. So the two solutions to (3) describe ingoing light rays if $r < 2M$. This means that not even light can escape if it gets too close to the singularity. This is the defining property of black holes: If one crosses the **event horizon**, or simply put the **horizon**, one is doomed to remain in the black hole forever.

Due to the above discussed special properties of $r = 2M$, it is also called the **Schwarzschild radius** and it is the event horizon of a Schwarzschild black hole.

2.3 The Kerr Geometry

Schwarzschild black holes are the simplest sort of black holes. They were discovered quite early after Einstein's publication of the theory of general relativity. There is a problem though with Schwarzschild black holes. As described, black holes are remnants of stars. We also know that some stars rotate around an axis in the same way the earth rotates around its axis. We also know that angular momentum is a conserved quantity. Yet, in the description of Schwarzschild black holes, no mention has been made about angular momentum. So does a black hole break the law of conservation of angular momentum or is the Schwarzschild description of black holes insufficient? It appears that the second one is the case. Black holes do rotate and the description of these rotating black holes is given by the **Kerr geometry**.

2.3.1 Rotating Black Hole?

So a first question one can ask is this: What does it mean for a black hole to rotate? As said, a black hole is just the remnant of a star which collapsed into a singularity. This singularity is just a point and it is not well-defined what it means for a point to rotate around an axis. Furthermore, it also appears that we can observe this rotation from outside the black hole. If it were just this singularity spinning, we would never have been able to see it since it is hidden behind the event horizon.

So what does it mean for a black hole to rotate? It is clearly not the same rotation as we see in everyday objects such as a football or the earth. The better way to think about a rotating black hole is that it rotates spacetime around the singularity. So it is not actually the black hole which rotates, but space itself. Mathematically, this is precisely what we will do. We will encode the angular momentum within the metric describing spacetime around this rotating black hole and so in essence say that spacetime is rotating around the black hole.

2.3.2 The Mathematics

The metric describing the spacetime around a rotating black hole is called the **Kerr metric** and it is given by:

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4Mar\sin^2(\theta)}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2 + \frac{2Mra^2\sin^2(\theta)}{\rho^2})\sin^2(\theta)d\phi^2 \tag{4}$$

where M is the mass of the black hole and where a, ρ and Δ are defined as:

$$a = \frac{J}{M} \qquad \rho^2 = r^2 + a^2\cos^2(\theta) \qquad \Delta = r^2 - 2Mr + a^2$$

where J is the angular momentum and a is called the **Kerr parameter**. In this case the coordinate (t, r, θ, ϕ) are called the **Boyer-Lindquist coordinates** and they are the analogue of the Schwarzschild coordinates for a Schwarzschild metric.

The first thing we will show is if the black hole is not rotating, i.e. $J = 0$, then we should recover the Schwarzschild metric. This is in fact the case. For example, the cross term vanishes since $J = 0$ implies $a = 0$. One can do this check for all the other terms and see that we get back the Schwarzschild metric.

The second thing we should take a look at are the symmetries in the metric. We see that the metric does not depend on time nor on the azimuthal angle ϕ. This gives us two Killing vectors ∂_t and ∂_ϕ. But note that this metric is not spherically symmetric. For example, if we replace θ by $\theta' = \theta + \frac{\pi}{2}$, the cross term becomes:

$$\frac{4Mar\sin^2(\theta')}{r^2 + a^2\cos^2(\theta')} = \frac{4Mar\cos^2(\theta)}{r^2 + a^2\sin^2(\theta)} \neq \frac{4Mar\sin^2(\theta)}{\rho^2}$$

where in the first step we applied the equality $\sin(\theta + \frac{\pi}{2}) = -\cos(\theta)$ and $\cos(\theta + \frac{\pi}{2}) = \sin(\theta)$. So this shows that the metric is not spherical symmetric. The best we can do is say that the metric is invariant under the replacement of θ by $\pi - \theta$.

Lastly, we note that the singularities of this metric are given by $\rho^2 = 0$ and $\Delta = 0$. $\rho^2 = 0$ implies that $r = 0$ and $\theta = \frac{\pi}{2}$. The singularity given by $\Delta = 0$ can be solved using the quadratic formula to give $r_\pm = M \pm \sqrt{M^2 - a^2}$. r_+ is the horizon of a Kerr black hole. We note that if $a = 0$, then we get back the Schwarzschild radius. Also note that $r_\pm$ has to be real since it is a length. This limits the values of a to be $a \leq M$.

It appears that $\rho^2 = 0$ describes a real physical singularity, but that $\Delta = 0$ is a coordinate singularity. To make this clear, we can write the Kerr metric using outgoing Eddington-Finkelstein coordinates as Roy Kerr, the one who discovered the metric, did:

$$\begin{aligned} ds^2 = &-(1 - \frac{2Mr}{\rho^2})(du + a\sin^2(\theta)d\phi)^2 \\ &+ 2(du + a\sin^2(\theta)d\phi)(dr + a\sin^2(\theta)d\phi) \\ &+ (r^2 + a^2\cos^2(\theta)(d\theta^2 + \sin^2(\theta)d\phi^2) \end{aligned}$$

Showing that these two metrics coincide requires a lot of algebra and is not very insightful, so for those reasons we will not do it. But as one can see, the only singularity occurring in this way of writing is when $\rho^2 = 0$ and so $\Delta = 0$ is a coordinate singularity.

2.3.3 Angular Velocity at Event Horizon

We will now compute the angular velocity of the rotating black hole at the event horizon. Suppose we have a light ray traveling with four-velocity $u = u^t\partial_t + u^\theta\partial_\theta + u^\phi\partial_\phi$. We have assumed that this light ray is situated at the event horizon r_+ and does not move in radial direction. Since it is a light ray it has to satisfy $ds^2(u, u) = 0$. This gives us:

$$g_{tt}(u^t)^2 + 2g_{\phi t}u^t u^\phi + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2 = 0 \tag{5}$$

Let us first look at $g_{\phi\phi}$. We will rewrite this in a handier way. We note that $r_+^2 = 2M^2 - a^2 + \sqrt{M^2 - a^2}$ and hence we get $r_+^2 + a^2 = 2M^2 + \sqrt{M^2 - a^2} = 2Mr_+$. Using this, we can write:

$$\begin{aligned} g_{\phi\phi} =& (r_+^2 + a^2 + \frac{2Mr_+ a^2 \sin^2(\theta)}{\rho_+^2}) \sin^2(\theta) \\ =& (2Mr_+ + \frac{2Mr_+ a^2 \sin^2(\theta)}{\rho_+^2}) \sin^2(\theta) \\ =& \frac{2Mr_+(\rho_+^2 + a^2 \sin^2(\theta))}{\rho_+^2} \sin^2(\theta) \end{aligned}$$

We note $\rho_+^2 = r_+^2 + a^2 \cos^2(\theta)$. This gives us:

$$g_{\phi\phi} = \frac{2Mr_+(\rho_+^2 + a^2 \sin^2(\theta))}{\rho_+^2} \sin^2(\theta) = \frac{2Mr_+(r_+^2 + a^2)}{\rho_+^2} \sin^2(\theta)$$

Using again $r_+^2 + a^2 = 2Mr_+$ we get:

$$g_{\phi\phi} = (\frac{2Mr_+ \sin(\theta)}{\rho_+})^2$$

It is also an easy check to see:

$$2g_{\phi t} = -\frac{4Mar_+ \sin^2(\theta)}{\rho_+^2} = -(\frac{2Mr_+ \sin(\theta)}{\rho_+})^2 \frac{a}{Mr_+}$$

Now we are also going to rewrite g_{tt}.

$$g_{tt} = -(1 - \frac{2Mr}{\rho^2}) = -(\frac{\rho_+^2 - r_+^2 + r_+^2}{\rho_+^2} - \frac{2Mr}{\rho^2}) = -(\frac{\rho_+^2 - r_+^2}{\rho_+^2} + \frac{r_+^2 - 2Mr}{\rho^2})$$

By definition of ρ_+^2 we get that $\rho_+^2 - r_+^2 = a^2 \cos^2(\theta)$. Also by definition of r_+ we get $r_+^2 - 2Mr = -a^2$. Plugging this in gives us:

$$g_{tt} = -(\frac{a^2 \cos^2(\theta)}{\rho_+^2} + \frac{-a^2}{\rho^2}) = \frac{a^2}{\rho_+^2}(1 - \cos^2(\theta)) = \frac{a^2}{\rho_+^2} \sin^2(\theta)$$

If we multiply this with a usefull chosen 1 we get:

$$g_{tt} = \frac{a^2}{\rho_+^2} \sin^2(\theta) = \frac{a^2}{\rho_+^2} \sin^2(\theta) \frac{4M^2 r_+^2}{4M^2 r_+^2} = (\frac{2Mr_+ \sin(\theta)}{\rho_+})^2 \frac{a^2}{4M^2 r_+^2}$$

Plugging the above calculated values into the equation $g_{tt} + g_{\phi t} u^\phi + g_{\phi\phi}(u^\phi)^2$ we get:

$$\begin{aligned} g_{tt}(u^t)^2 + g_{\phi t} u^t u^\phi + g_{\phi\phi}(u^\phi)^2 &= (\frac{2Mr_+ \sin(\theta)}{\rho_+})^2 (\frac{a^2}{4M^2 r_+^2}(u^t)^2 - \frac{a}{Mr_+} u^t u^\phi + (u^\phi)^2) \\ &= (\frac{2Mr_+ \sin(\theta)}{\rho_+})^2 (u^\phi - \frac{a}{2Mr_+} u^t)^2 \end{aligned}$$

So now looking back at (5) we get:

$$(\frac{2Mr_+ \sin(\theta)}{\rho_+})^2 (u^\phi - \frac{a}{2Mr_+} u^t)^2 + g_{\theta\theta}(u^\theta)^2 = 0$$

Which can only be satisfied if $u^\phi = \frac{a}{2Mr_+} u^t$ and $u^\theta = 0$. The value $\frac{a}{2Mr_+}$ is what we call the angular velocity at the event horizon of a rotating black hole. We denote it as Ω_H:

$$\Omega_H = \frac{a}{2Mr_+} \tag{6}$$

To check that this equation is correct, we show that $M\Omega_H \leq 1$. Note that $M\Omega_H$ is a velocity. Note that $ar_+^{-1} \leq 1$ since $a \leq M$ and $r_+ \geq M$. Hence it follows that $M\Omega_H \leq 1$.

2.3.4 Area of Event Horizon

Let us compute the area of the event horizon of a Kerr black hole. The formula for the area is given by:

$$A = \int_0^{\pi} d\theta \int_0^{2\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}} d\phi$$

$$A = \int_0^{\pi} d\theta \int_0^{2\pi} \sqrt{\rho_+^2 (r_+^2 + a^2 + \frac{2Mr_+ a^2 \sin^2(\theta)}{\rho_+^2}) \sin^2(\theta) d\phi}$$

Since we want to know the area at the event horizon, we had to replace all r in $g_{\theta\theta}$ and $g_{\phi\phi}$ with r_+. We also note that since ρ depended on r, we write ρ_+ to indicate that the r has also been replaced with r_+. Some simple algebra now gives us:

$$A = 2\pi \int_0^{\pi} \sin(\theta) \sqrt{(r_+^2 + a^2)(r_+^2 + a^2 \cos(\theta)) + 2Mr_+ a^2 \sin^2(\theta) d\theta}$$

By definition we have

$$r_+^2 = 2M^2 - a^2 + 2M\sqrt{M^2 - a^2}$$

From which readily follows:

$$r_+^2 + a^2 = 2M^2 + 2M\sqrt{M^2 - a^2} = 2Mr_+$$

Also by the above follows that:

$$r_+^2 + a^2 \cos^2(\theta) = 2M^2 + 2M\sqrt{M^2 - a^2} - a^2(1 - \sin^2(\theta)) = 2Mr_+ - a^2 \sin^2(\theta)$$

From which now follows:

$$(r_+^2 + a^2)(r_+^2 + a^2 \cos(\theta)) = 4M^2 r_+^2 - 2Mr_+ a^2 \sin^2(\theta)$$

Plugging this into our formula for A gives us:

$$A = 2\pi \int_0^{\pi} \sin(\theta) \sqrt{4M^2 r_+^2 - 2Mr_+ a^2 \sin^2(\theta) + 2Mr_+ a^2 \sin^2(\theta) d\theta}$$

$$A = 4\pi M r_+ \int_0^{\pi} \sin(\theta) d\theta$$

Solving the integral now gives the area of the event horizon:

$$A = 8\pi M r_+ \tag{7}$$

2.3.5 The Ergosphere

Something interesting happens in the Kerr metric. Let us take a look at the time component g_{tt} of the metric in Boyer-Lindquist coordinates. We can ask ourselves at what distance r from the centre does that component switch signs. To know this, we have to solve the following equation:

$$-(1 - \frac{2Mr}{\rho^2}) = 0$$

After some algebra and noting that ρ depends on r, we get the following equation:

$$r_E^{\pm} = M \pm \sqrt{M^2 - a^2 \cos^2(\theta)} \tag{8}$$

What one should note is that if $a \neq 0$, then we have $r_E^+ > r_+$. Put in words, the time component of the Kerr metric switches signs outside the horizon of the black hole. The zone between r_+ and r_E^+ is what we call the **ergosphere**.

Some strange things happen due to the presence of the ergosphere. For example: it is impossible to remain stationary within the ergosphere. To make this clear, suppose we have an observer which moves forward through time at speed u^t, but remains stationary in the spatial dimensions. The four-velocity of this observer is given by $u = u^t\partial_t$. In general, we will require a four-velocity u to be normalizable and its normalization is given by $ds^2(u,u) = -1$. From this normalization we get for our stationary observer:

$$-(1 - \frac{2Mr}{\rho^2})(u^t)^2 = -1$$

If we have $r > r_E^+$, then this above equation gives us the value for u^t. The problem arises when $r < r_E^+$. In that case we have that $-(1 - \frac{2Mr}{\rho^2}) > 0$ and the only way this equation can hold is if u^t is imaginary. This would then imply that time is imaginary which is not the case. So a stationary observer cannot exist in the ergosphere.

We can actually claim something more, namely that, when viewed form infinity, an observer within the ergosphere has to rotate in the same direction as the black hole. To make this point clear, suppose our four-velocity is given by $u = u^t\partial_t + u^\phi\partial_\phi$. The normalization condition then gives us:

$$-(1 - \frac{2Mr}{\rho^2})(u^t)^2 - \frac{4Mar\sin^2(\theta)}{\rho^2}u^\phi u^t + (r^2 + a^2 + \frac{2Mra^2\sin^2(\theta)}{\rho^2})\sin^2(\theta)(u^\phi)^2 = -1$$

Now we note that $-(1-\frac{2Mr}{\rho^2})(u^t)^2$ and $(r^2+a^2+\frac{2Mra^2\sin^2(\theta)}{\rho^2})\sin^2(\theta)(u^\phi)^2$ are positive within the ergosphere. So the only way this equation can hold is if $-\frac{4Mar\sin^2(\theta)}{\rho^2}u^\phi$ is negative. This is the case since if it were positive, we would have that all the components on the left hand side of the equation are positive and thus could never sum up to -1. The only two elements in $\frac{4Mar\sin^2(\theta)}{\rho^2}u^\phi$ which can be negative are a and u^ϕ. So the only way for it to be negative, we must have that $au^\phi > 0$. From this last inequality follows that the observer must spin in the same direction as the black hole.

Remark. Since we now have shown that a and u^ϕ both rotate in the same direction, from now on we will use as convention that the black hole always rotates in positive direction. This implies that both a and u^ϕ will be positive. △

So now we have shown that u^ϕ has to rotate in the same direction as the black hole. If we look at fixed values of r and θ, we are actually able to do something more, namely derive boundaries for the value u^ϕ. To that end, suppose we have a light ray which only moves in the direction ∂_ϕ i.e. it has four-velocity $v = v^t\partial_t + \Omega\partial_\phi$. We know that light has to satisfy the condition $ds^2(v,v) = 0$. This now gives us the following equation:

$$g_{\phi\phi}\Omega^2 + g_{\phi t}v^t\Omega + g_{tt}(v^t)^2 = 0$$

Note that this equation gives us that v^t has to be positive. This is so because within the ergosphere we have that $g_{\phi\phi}$, g_{tt}, $\Omega > 0$ and $g_{\phi t}\Omega < 0$ by the above discussion. So now if v^t is negative, all the terms on the left-hand side would be positive and could never sum up to 0. Thus v^t is positive.

We can solve this equation to get:

$$\Omega_\pm = \frac{-v^t g_{\phi t} \pm \sqrt{(g_{\phi t}v^t)^2 - 4g_{\phi\phi}g_{tt}v^t}}{2g_{\phi\phi}} \tag{9}$$

Anything else than a light ray which has four-velocity $u := v^t\partial_t + u^\phi\partial_\phi$ has to satisfy $ds^2(u,u) < 0$ i.e. it has to move in a timelike direction. For simplicity we have assumed that u has the same velocity in the time direction as the light ray. If we calculate $ds^2(u,u)$ we get:

$$ds^2(u,u) = g_{\phi\phi}(u^\phi)^2 + g_{\phi t}u^\phi v^t + g_{tt}(v^t)^2$$

Now since $g_{\phi\phi}$ is positive within the ergosphere, we get that $ds^2(u,u)$ when viewed as function in u^ϕ is an upward facing parabola. Hence we get that $ds^2(u,u)$ is only negative for the values $\Omega_- \leq u^\phi \leq \Omega_+$.

After doing a lot of not very insightful algebra, we can compute the limits of Ω_+ and Ω_- closer to the event horizon. We then get that:

$$\lim_{r \downarrow r_H} \Omega_-(r) = \lim_{r \downarrow r_H} \Omega_+(r) = \Omega_H$$

2.3.6 Energy and Angular Momentum of a Particle in Kerr Spacetime

We will conclude this chapter by computing an explicit formula for the energy and angular momentum of a particle in the Kerr metric. Suppose we have some particle originating from infinity with velocity $u = u^\mu \partial_\mu$. Because of the fact that we know that ∂_t and ∂_ϕ are Killing vectors of the Kerr metric, we are able to calculate the energy and the angular momentum of this incoming particle in the following way:

$$E = -ds^2(\partial_t, u), \qquad L = ds^2(\partial_\phi, u)$$

We have written an appendix proving these formulas.

We will do the algebra in the case of the energy, but then we will just state the formula for angular momentum since the way to derive it, is completely analogous to the case of the energy. So we start by writing:

$$E = -ds^2(\partial_t, u) = -ds^2(\partial_t, u^\mu \partial_\mu) = -u^\mu ds^2(\partial_t, \partial_\mu)$$

In the last step we made use of the fact that any metric is linear. Now we note that $ds^2(\partial_t, \partial_\mu) = 0$ if we have $\mu = r$ or $\mu = \theta$. This is so since there are no cross terms in the Kerr metric in the shape of $dtdr$ or $dtd\theta$. This then gives us:

$$E = -u^t ds^2(\partial_t, \partial_t) - u^\phi ds^2(\partial_t, \partial_\phi)$$

Now we get:

$$ds^2(\partial_t, \partial_t) = -(1 - \frac{2Mr}{\rho^2})$$
$$ds^2(\partial_t, \partial_\phi) = -\frac{2Mar\sin^2(\theta)}{\rho^2}$$

We have to note one small thing, namely that the second equation differs by a factor of $\frac{1}{2}$ from the cross term component in the Kerr metric. The reason for this is because $dtd\phi$ is actually defined as $dtd\phi = \frac{1}{2}(dt \otimes d\phi + d\phi \otimes dt)$. Now we note that $dt \otimes d\phi(\partial_t, \partial_\phi) = dt(\partial_t)d\phi(\partial_\phi) = 1$, but that $d\phi \otimes dt(\partial_t, \partial_\phi) = d\phi(\partial_t)dt(\partial_\phi) = 0$ and hence $dtd\phi(\partial_t, \partial_\phi) = \frac{1}{2}(1+0) = \frac{1}{2}$. So this gives us the following equation for the energy:

$$E = (1 - \frac{2Mr}{\rho^2})u^t + \frac{2Mar\sin^2(\theta)}{\rho^2}u^\phi$$

In an analogous way, one can show that the angular momentum of the particle equals:

$$L = ds^2(\partial_\phi, u) = -\frac{2Mar\sin^2(\theta)}{\rho^2}u^t + (r^2 + a^2 + \frac{2Mra^2\sin^2(\theta)}{\rho^2})\sin^2(\theta)u^\phi$$

3 The Penrose Process

In the previous chapter we have shown that a Kerr black hole has an ergosphere. Within this ergosphere, it is impossible for an observer to remain stationary. The observer must rotate in the rotation direction of the black hole. This though is not the only quirky property of a Kerr black hole. Another property of having an ergosphere is that it allows for the extraction of energy and angular momentum from a black hole through a process called the **Penrose process**. In this chapter we will study this process in more detail.

3.1 Penrose Process in broad strokes

Let us now paint in broad strokes how the Penrose process occurs. Suppose we have an particle entering the ergosphere of a Kerr black hole. After this incoming particle crosses the ergosphere, it splits into two particles. One of these particles falls into the black hole. The other escapes the ergosphere and flies away from the black hole. What makes this process strange is that because the decay happens within the ergosphere, an observer at infinity can observe a negative energy for one of the decay products. If the particle which falls into the black hole has this negative energy, then we must have by conservation of energy that the outgoing particle has a higher energy than the incoming particle. This implies that the Penrose process extracts energy from the black hole. From this extraction of energy follows also the extraction of angular momentum implying that the black hole will spin slower.

3.2 Energy Extraction

To show that a particle can have negative energy, we will derive the **Wald inequalities**. These inequalities give an upper and lower bound to the energy a particle produced from a decay process by another particle, can have. To be more concrete, let us suppose that we have an incoming particle with specific energy E, which is the energy per unit mass. We also suppose that this incoming particle has a four-velocity U^μ and that this incoming particle splits into two particles, one of which has specific energy E' and four-velocity u^μ. So what the Wald inequality give us are limits on E' in terms of E. Let us derive them.

We can choose four orthogonal vector fields $e_{(\alpha)}$, where we have $\alpha \in \{0,1,2,3\}$, which at every point in spacetime form a basis for spacetime. We call four of such vector fields together a **tetrad**. We write $e^\mu_{(\alpha)}$ to mean the μ-th component of $e_{(\alpha)}$. Furthermore, we choose the $e^\mu_{(0)}$ to coincide with U^μ. We can now write the four-velocity u^μ in terms of this tetrad frame as:

$$u^\mu = \gamma(U^\mu - v^{(j)} e^\mu_{(j)})$$

where $v^{(j)}$ is the spatial velocity of the emitted particle and j runs over $\{1,2,3\}$. Furthermore we note $\gamma = (1-|v|^2)^{-1/2}$ is the Lorentz factor and $|v|^2 = ds^2(v^j, v^j)$. This equation can be derived from the normalisation condition $ds^2(u^\mu, u^\mu) = -1$.

We know that the Kerr metric has a timelike Killing vector, namely $\xi = \partial_t$. In fact, this approach works for all the metrics which have a timelike Killing vector, but let us focus on the Kerr metric. We can represent this Killing vector in terms of the chosen tetrad frame as:

$$\xi^\mu = \xi^{(0)} U^\mu + \xi^{(j)} e^\mu_{(j)}$$

Because ξ is the Killing vector of the Kerr metric, we can calculate the energy of the incoming particle in the following way:

$$E = -ds^2(\xi^\mu, U^\nu) = \xi^{(0)} ds^2(U^\mu, U^\nu) - \xi^{(j)} ds^2(e^\mu_{(j)}, U^\nu) = -\xi^{(0)}$$

We note that in the second step, we made use of the fact that the metric is linear and in last step that the tetrad frame is orthogonal.

Next, by the way we have chosen ξ we can compute g_{tt} as:

$$g_{tt} = ds^2(\xi, \xi) = -\xi^2_{(0)} + \xi_{(j)}\xi^{(j)} = -E^2 + |\xi|^2$$

where again in the second step we made use of the linearity and orthogonality of the metric. Lastly, in the same way we computed the specific energy of the incoming particle, we can compute the specific energy of the emitted particle as:

$$E' = ds^2(\xi^i, u^i) = \gamma(\xi_{(0)} + v^{(\alpha)}\xi_{(\alpha)})$$

By using the metric, we can compute the angle between two spacelike vectors as $v^{(\alpha)}\xi_{(\alpha)} = |v||\xi|\cos(\chi)$. Since we also know that $|\xi| = \sqrt{E^2 + g_{tt}}$. If we plug this all in, we get:

$$E' = \gamma(E + |v|\sqrt{E^2 + g_{tt}}\cos(\chi))$$

Lastly, we note that $-1 \leq \cos(\chi) \leq 1$. This gives us the Wald inequalities:

$$\gamma(E - |v|\sqrt{E^2 + g_{tt}}) \leq E' \leq \gamma(E + |v|\sqrt{E^2 + g_{tt}})$$

For the Kerr metric we have $g_{tt} = -(1 - \frac{2Mr}{\rho^2})$. Within the ergosphere we have that $g_{tt} \geq 0$. Since we consider only the region outside the event horizon, it is also bounded by above. This then gives us that the maximum value is reached when $r = r_+$, $a = M$ and $\theta = \frac{\pi}{2}$. In that case, we see that $g_{tt} \leq 1$. This then gives us:

$$\gamma(E - |v|\sqrt{E^2 + 1}) \leq E' \leq \gamma(E + |v|\sqrt{E^2 + 1})$$

Since we require $E' \leq 0$ for the Penrose process, we look at:

$$\gamma(E - |v|\sqrt{E^2 + 1}) \leq E' \leq 0$$

We can rewrite this as:

$$\frac{E^2}{E^2 + 1} \leq \frac{|v|^2}{\gamma^2}$$

Now we note that $\gamma^{-1} \leq 1$. This now gives us:

$$\frac{E^2}{E^2 + 1} \leq \frac{|v|^2}{\gamma^2} \leq |v|^2$$

According to [**Wald**], choosing $E \geq 3^{-1/2}$ is a reasonable choice. So let us plug in $E = 3^{-1/2}$ to get:

$$|v| \geq \frac{1}{2}$$

This is an allowed velocity implying that it is possible for a particle to extract energy from a Kerr black hole through the Penrose process.

3.3 Angular Momentum Extraction

We claimed that the Penrose process also extracts angular momentum from the black hole. To this end, we now suppose we have an observer within the ergosphere which rotates along with the black hole i.e. it has four-velocity $v_{obs} = v^t\partial_t + v^\phi\partial_\phi$. This observer must measure a positive energy for the particle which falls into the black hole. So we get:

$$0 \leq -ds^2(v_{obs}, u^i) = -v^t ds^2(\partial_t, u^i) - v^\phi ds^2(\partial_\phi, u^i) = v^t E_{bh} - v^\phi L_{bh}$$

Where we note that $-ds^2(v_{obs}, u^i)$ is the energy of the particle for the observer within the ergosphere. Furthermore we made use of the fact that $-ds^2(\partial_t, u^i) =: E_{bh}$ is the energy of the particle falling into the black hole as measured by an observer at infinity and $ds^2(\partial_\phi, u^i) =: L_{bh}$ is the angular momentum of a particle as measured by an observer at infinity. From this equation follows:

$$v^t E_{bh} \geq v^\phi L_{bh}$$

As we have shown v^ϕ is assumed to be positive and thus it now follows that if E_{bh} is negative, then L_{bh} must be negative as well. Since angular momentum is a conserved quantity, it follows that the escaping particle has a higher angular momentum then the incoming particle implying that angular momentum has been extracted from the black hole

3.4 The Area Increase Theorem

Now we will show that the Penrose process increases the area of a black hole. As we have shown, the formula for the area of the event horizon of a Kerr black hole is given by $A = 8\pi M r_+$. So we know that the Penrose process changes the energy and angular momentum of a black hole. This then results in a change of mass and of radius r_+. So we now get:

$$\Delta A = 8\pi(r_+\Delta M + M\Delta r_+)$$

What we want to do now is express Δr_+ in terms of changes in the mass and angular momentum. To this end we note that $r_+ = M + \sqrt{M^2 - a^2}$. So now we know that $\Delta r_+ = \frac{\partial r_+}{\partial M}\Delta M + \frac{\partial r_+}{\partial a}\Delta a$. We now get:

$$\Delta r_+ = (1 + \frac{M}{\sqrt{M^2 - a^2}})\Delta M - \frac{a}{\sqrt{M^2 - a^2}}\Delta a$$

So now we have to express Δa in terms of changes in the mass and angular momentum. Using again that $\Delta a = \frac{\partial a}{\partial M}\Delta M + \frac{\partial a}{\partial J}\Delta J$ and plugging in $a = \frac{J}{M}$, we get:

$$\Delta a = -\frac{J}{M^2}\Delta M + \frac{1}{M}\Delta J$$

So plugging now everything into the equation for ΔA gives us:

$$\Delta A = 8\pi\Big(r_+\Delta M + M\big((1 + \frac{M}{\sqrt{M^2 - a^2}})\Delta M - \frac{a}{\sqrt{M^2 - a^2}}(-\frac{J}{M^2}\Delta M + \frac{1}{M}\Delta J)\big)\Big)$$

Organizing the terms gives us:

$$\Delta A = 8\pi\big((r_+ + M + \frac{M^2}{\sqrt{M^2 - a^2}} + \frac{a^2}{\sqrt{M^2 - a^2}})\Delta M - \frac{a}{\sqrt{M^2 - a^2}}\Delta J)\big)$$

If we plug in the value for r_+ and do some algebra, we can rewrite the equation as:

$$\begin{aligned}\Delta A =& 8\pi(\frac{2Mr_+}{\sqrt{M^2 - a^2}}\Delta M - \frac{a}{\sqrt{M^2 - a^2}}\Delta J))\\ =& 8\pi\frac{2Mr_+}{\sqrt{M^2 - a^2}}(\Delta M - \frac{a}{2Mr_+}\Delta J)\\ =& 8\pi\frac{2Mr_+}{\sqrt{M^2 - a^2}}(\Delta M - \Omega_H\Delta J)\end{aligned}$$

Since $E_b h \leq 0$, it follows that the change in the mass of the black hole is negative i.e. $\Delta M \leq 0$. In the previous subsection we have shown that $E_{bh} \geq v^\phi L_{bh}$ for all allowed values of v^ϕ. Thus we now get $E_{bh} \geq \Omega_H L_{bh}$. This implies that the amount on angular momentum extracted from the black hole is more than the amount of energy extracted from it, i.e. $\Delta M \geq \Omega_H\Delta J$. The greater than or equal to sign points towards ΔJ since both of these values are negative. Since then $\Omega_H\Delta J \leq 0$, it follows that $\Delta M - \Omega_H\Delta J \geq 0$. This shows that ΔA is positive since $8\pi\frac{2Mr_+}{\sqrt{M^2-a^2}}$ is also positive. Hence we get that the Penrose process increases the area of a Kerr black hole.

3.5 Irreducible Mass

We define the **irreducible mass** as follows:

$$M_{irr} = \sqrt{\frac{A}{16\pi}}$$

As we have shown here above, the area of a rotating black hole can only remain constant or increase through the Penrose process. By the way we have defined it, this must also hold true for the irreducible mass (hence its name).

We can also express the mass of a black hole in terms of the irreducible mass by:

$$M^2 = M_{irr}^2 + \frac{J^2}{4M_{irr}^2}$$

We will prove this equation now. Let us start from the right-hand side. We have shown in the previous chapter that $A = 8\pi M r_+$. If we plug this into our equation, we get:

$$M_{irr}^2 + \frac{J^2}{4M_{irr}^2} = \frac{A}{16\pi} + \frac{4\pi J}{A} = 2Mr_+ + \frac{J^2}{2Mr_+} = \frac{M^2 r_+^2 + J^2}{2Mr_+}$$

We know that by definition of the Kerr parameter that $J^2 = a^2 M^2$ and hence:

$$\frac{M^2 r_+^2 + J^2}{2Mr_+} = \frac{M^2 r_+^2 + a^2 M^2}{2Mr_+} = M\frac{r_+^2 + a^2}{2r_+}$$

We also know by definition of r_+ that $r_+^2 + a^2 = 2Mr_+$. This gives us:

$$M\frac{r_+^2 + a^2}{2r_+} = M\frac{2Mr_+}{2r_+} = M^2$$

This proves the equation.

As a check that the irreducible mass cannot decrease through the Penrose process, we can take a look at Schwarzschild black hole. We know that a Schwarzschild black hole has no angular momentum and thus follows $M = M_{irr}$. This says that all the mass of a black hole is irreducible. This is precisely what we would expect since for the Penrose process to take place, one needs an ergosphere outside the event horizon of a black hole. For the existence of the ergosphere, the black hole needs to spin which is not the case for a Schwarzschild black hole. So this shows that the Penrose process cannot decrease the mass of a black hole to a quantity lower than the irreducible mass.

4 The Foundations of the Blandford-Znajek Process

The aim of the coming four chapters is to understand the Blandford-Znajek process (henceforth called the BZ-process). In this first chapter, we will describe the BZ-process in very broad strokes and we will develop an analogy between an element of circuit theory and the BZ-process. The three chapters after this one are all dedicated to a specific component of the BZ-process, namely force-free electromagnetism, plasma formation and the extraction of energy and angular momentum from the rotating black hole.

4.1 BZ-process in Broad Strokes

The core claim of the BZ-process is that it is possible to extract energy and angular momentum from a rotating black hole immersed in a magnetic field. The first question that we need to understand is then: how does this magnetic field form? It appears that this field is caused by a current flowing in the accretion disk of the black hole. This immediately gives us our first assumption, namely the black hole needs to have a charged accretion disk.

A magnetic field alone though is not sufficient to explain how the energy and angular momentum are being carried away from the black hole. We need an electric current to be able to explain this extraction. This electric current needs to flow into and out of the black hole. The problem is that the space outside the event horizon is just empty space and it alone would not be able to supply the particles necessary, i.e. electrons and positrons, for this current to exist. Yet because there is a current flowing in the accretion disk, a voltage difference between the accretion disk and a point outside of it exists. If this voltage difference is high enough to accelerate a stray electron into causing an electron-positron cascade, then we would be able to fill up the space around the black hole with a plasma of positrons and electrons. This plasma could act as a conductor and would supply us with the particles necessary for the current through the black hole to exist.

Importantly, we assume that the current threading the black hole is not independent of the magnetic field. What this means is that the energy flux is dominated by the electromagnetic component and we ignore the energetic flux as a result of the plasma. This assumption is fairly complex to understand and we will go into it in more detail later on, but basically we are allowed to make this assumption if the magnetic field is strong enough and the density of the plasma is low enough. We refer to this assumption as the black hole having a force-free magnetosphere. We also note briefly that this force-free assumption lays further conditions on the charge density of the accretion disk.

Lastly we need to understand how energy and angular momentum are being transferred from the black hole to the current threading the black hole. In the next section we understand this transfer as the black hole being a non-perfect conductor and thus having a resistance. Having a resistance means that the black hole converts power into heat. This heat emitted by the black hole can only exist due to the fact that the black hole radiates energy and angular momentum.

This energy and angular momentum transfer process can also be understood mechanically, but we will refrain from this until later on.

4.2 An Estimate for the BZ-process

Let us develop an intuitive understanding of what is happening in the BZ-process. In this subsection we will work out in a bit more detail what is also explained [10] on page 326 and 327.

4.2.1 Setup

For convenience we will work with cylindrical coordinates (ρ, ϕ, z). Suppose we have a cylindrical conductor with radius R and length L rotating around its axis with constant angular velocity Ω. Suppose also that this cylinder is immersed by a homogeneous magnetic field $\vec{B}$ pointing in the z-direction i.e. $\vec{B} = B\vec{e}_z$ where B is the magnetic field strength. We furthermore assume that this field is static. This means that it does not evolve in time. More formally we write $\partial_t B = 0$. Lastly, suppose we have a path $\mathcal{C}$ from the point $z = \frac{1}{2}L$ and $\rho = 0$ to the point $z = z_0$, $\phi = \phi_0$

and $\rho = R$ where z_0 and ϕ_0 are constants. Now since a cylinder is axisymmteric around the z-axis we can rotate our coordinate system such that $\phi_0 = 0$. We have not specified how this path looks like, but, as we will argue below, we are free to choose whichever path we like. The setup then looks as follows:

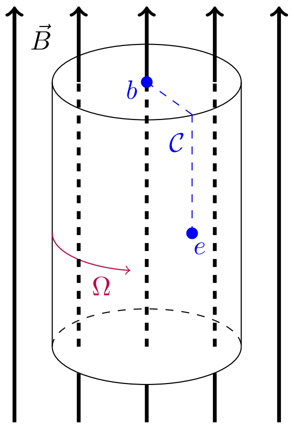

Figure 1: Picture inspired by [10] page 326.

4.2.2 Voltage for a Cylindrical Conductor

What we are interested in now is computing the voltage difference at the endpoints of our curve $\mathcal{C}$. This voltage difference is defined as:

$$V = \frac{1}{q}\int_{\mathcal{C}} \vec{F} \cdot d\vec{l} \tag{10}$$

where $\vec{F}$ is the Lorentz force acting on a charge carrier, for example an electron, within the conductor and is given by:

$$\vec{F}(\rho) = q\vec{v} \times \vec{B} \tag{11}$$

So q in both equations is the charge of the charge carrier. We assume that this charge carrier rotates in the same direction and with the speed proportional to the distance from the rotational axis of the cylinder. To put it formally, it has velocity $\vec{v} = \rho\Omega\vec{e}_\phi$. This then allows us to compute the Lorentz force as:

$$\vec{F}(\rho) = q((\rho\Omega\vec{e}_\phi) \times (B\vec{e}_z)) = q\rho\Omega B\vec{e}_\rho$$

This means that the Lorentz force points in the radial direction. What this also tells us is that the Lorentz force is conservative i.e. its curl is zero. To show this, let us calculate the curl of the Lorentz force:

$$\vec{\nabla} \times \vec{F}(\rho) = \frac{\partial}{\partial\phi}(q\rho\Omega B)\vec{e}_z - \frac{\partial}{\partial z}(q\rho\Omega B)\vec{e}_\phi$$

We note that the magnetic field is uniform and the angular velocity Ω is constant. This should make it clear that the partial derivatives are all equal to 0. This thus shows that the Lorentz force is conservative. This then implies that the voltage difference does not depend on the choice made for the path $\mathcal{C}$ and thus we are free to choose it as easy as possible.
The way in which we have drawn the path shows that we first move in the radial direction and then in the z direction. So we can separate the integral of the voltage as:

$$V = \frac{1}{q}\left(\int_0^R (\vec{F} \cdot \vec{e}_\rho)d\rho + \int_{L/2}^{z_0} (\vec{F} \cdot \vec{e}_z)dz\right)$$

We note that $\vec{F} \cdot \vec{e}_z = 0$, so the voltage only depends on the radial direction. This then gives us:

$$V = \int_0^R \rho\Omega B d\rho = \frac{R^2\Omega B}{2} \tag{12}$$

4.2.3 Voltage for any Axisymmetric Conductor

We will rewrite equation (12) in a way that holds not only for the cylinder, but for any axisymmetric conductor. To this end, we will introduce the magnetic flux Φ_B. The magnetic flux can intuitively be though of as the net magnetic field lines passing through a surface A. With net magnetic field lines we mean to say the number of field lines passing through this surface in one direction minus the number of field lines passing through the same surface in the other direction. For a uniform magnetic field, the magnetic flux is given by:

$$\Phi_B = BA\cos(\theta)$$

where B is the strength of the magnetic field, A is the are of the chosen surface and θ is the angle between the magnetic field $\vec{B}$ and the normal to the surface A.
In our case of the cylinder, we can choose the surface A to be a slice of the cylinder with normal pointing in the z-direction. To put it differently, we can take A to be the disk of radius R parallel to the xy-plane in Cartesian coordinates. The area is then $A = \pi R^2$. The angle θ is equal to zero since the normal of A is pointing in the z-direction and thus parallel to $\vec{B}$. This then gives us for the magnetic flux:

$$\Phi_B = B\pi R^2$$

We can substitute this into the equation (12) to get:

$$V = \frac{\Omega\Phi_B}{2\pi} \tag{13}$$

To show that this equation holds for any axisymmetric conductor, we simply have to note that any such conductor can be approximated by a series of concentric cylinders. To make this clear let us look at a cross section of a axisymmetric conductor at a fixed value $\phi = \phi_0$:

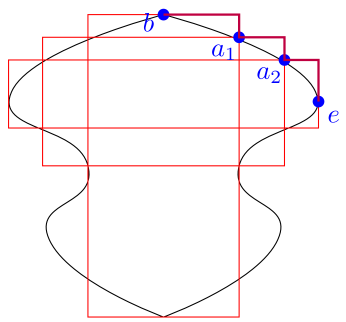

Figure 2: The above cross section is for a fixed value ϕ_0. The axisymmetric conductor is coloured black. The series of concentric cylinders are drawn in red.

We have not drawn it, but we assume that the conductor is rotating with velocity $\vec{v} = \Omega\vec{e}_\phi$ and it is immersed in a uniform magnetic field $\vec{B} = B\vec{e}_z$.

So what we want to do is compute the voltage difference between the points b and e. The problem is that the curve over the surface connecting these two points is not so easy to compute. The way in which we can compute it, is by approximating it with a series of concentric cylinders.

We can compute the voltage difference between points b and a_1, both lying on the same cylinder as shown in the picture. We know the voltage difference between these two points, namely its just $V = \frac{\Omega\Phi_B^1}{2\pi}$, where Φ_B^1 is the magnetic flux through the top of this first cylinder.

Next we can compute the voltage difference between the two points a_1 and a_2. We can again choose a cylinder such that the point a_1 lays on the top of the cylinder and the point a_2 lays on the side as in the picture. Now here we have to note that the voltage difference is not equal to $\frac{\Omega\Phi_B^2}{2\pi}$, where Φ_B^2 is the magnetic flux going through this cylinder. This is so because a_1 does not lay on the rotation axis of the cylinder. There is an easy fix though. The vertical part of the cylinder does not contribute anything to the voltage difference, since the magnetic field is pointing in a direction parallel to the cylinders. So the voltage difference between the rotation axis of the cylinder and the point a_1 is equal to $\frac{\Omega\Phi_B^1}{2\pi}$. We also know that the voltage difference between the rotation axis and the point a_2 is equal to $\frac{\Omega\Phi_B^2}{2\pi}$. This then implies that the voltage difference between the points a_1 and a_2 is $\frac{\Omega(\Phi_B^2-\Phi_B^1)}{2\pi}$.

Analogously we can argue that the voltage difference between a_2 and b is equal to $\frac{\Omega(\Phi_B^3-\Phi_B^2)}{2\pi}$ where Φ_B^3 is the magnetic flux through the third cylinder where a_2 and b both lay on. This allows us to give a crude approximation of the voltage difference between the points a and b, namely just as the sum of these three voltage differences. It is easy to see that this is just $\frac{\Omega\Phi_B^3}{2\pi}$. What should also be clear from the drawing is that if we were to choose more than three cylinders, the magnetic flux Φ_B^n passing through the n-th cylinder would approximate the magnetic flux Φ_B passing through the axisymmetric conductor. So the precise value of the voltage difference between a and b is $\frac{\Omega\Phi_B}{2\pi}$.

4.2.4 The Current

We will now connect an external resistance to our setup in the following way:

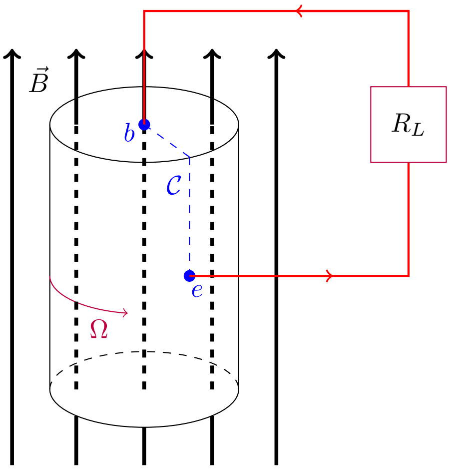

Figure 3: We have connected to our cylindrical conductor at the points b and e an external resistance R_L.

Since there is a voltage difference between the points b and e, we get that at current starts flowing from e to b. Applying Ohm's law we see that the current I that starts flowing equals to:

$$I = \frac{V}{R_C + R_L}$$

Where R_C is the resistance of the conductor and R_L is the resistance of the load. The power this current then supplies to the external load is then maximised when $R_L = R_C$. We can see this by taking the derivative with respect to R_L of the equation $P_L = I^2 R_L$, which is the equation telling us how much power is transferred to the resistance by the current. If we do this, we get:

$$\frac{\partial P_L}{\partial R_L} = \frac{V^2(R_C^2 - R_L^2)}{(R_C + R_L)^4}$$

Setting this equation equal to 0 shows us that the power is maximised when $R_L = R_C$. So now we can calculate the maximum power dissipated by the load as:

$$P_{max} = I^2 R_L = \frac{V^2}{4R_C} = \frac{\Omega^2\Phi_B^2}{16\pi^2 R_C} \tag{14}$$

4.2.5 Relation to Black Holes

We have not mentioned black holes anywhere in the above discussion which is weird since the subject of this thesis is the extraction of energy from a rotating black hole. So how do black holes fit into this discussion? It appears that black holes can have a charge, conduct electric current and even have electrical resistance. Charged black holes are refered to as Kerr-Newman or Reissner-Nordström black holes depending on whether or not they rotate. That is not something we are interested in, so we will leave it be and focus our attention on the other two properties.

To understand how a black hole can conduct a current, suppose we have a stream of negatively charged particles flowing into the black hole on one side and an equally big stream of positively charged particles flowing into the other side. Importantly, an observer sitting at infinity will never see one of those particles cross the event horizon. This is due to the property of the event horizon. Instead what this observer will see is that the charge will spread out over the event horizon creating a surface charge. Since the two streams of particles is equally big, no surface charge will form, but for an observer sitting at infinity will see that the black hole has conducted electricity. This process shows that black holes are electric conductors.

Now since black holes are not perfect conductors, they will have a resistance. We see now that black holes take the role of the external resistance R_L from the previous subsection. The aim of the rest of this subsection will be to guess the resistance of a black hole using a dimensional argument and from that we will estimate how much power is converted into heat by the black hole.

4.2.6 The Resistance of a Black Hole

The dimension of resistance in geometrized units is actually quite simple. Voltage is dimensionless i.e. $[V]_G = 1$ where the brackets with subscript G means the dimension in geometrized units. Current is also dimensionless. We know that current is defined as charge flowing in time i.e. $I = \frac{Q}{t}$. Now both electric charge and time have dimension of length L. Thus it follows that $[I]_G = \frac{[Q]_G}{[t]_G} = 1$. Since resistance is defined as $R = \frac{U}{I}$, we get that resistance is dimensionless i.e. $[R]_G = 1$.

So let us suppose that the resistance of the black hole $R_{BH} = 1$. Is this a reasonable guess? The answer is yes and to make this clear, we have to convert back to SI units. We know that in SI units the dimension of voltage is the volt which can also be written as $[V]_{SI} = \frac{kgm^2}{As^3}$. It is an easy check to see that the dimension of $\frac{c^2}{\sqrt{G\epsilon_0}}$ has the same units as the voltage. This implies that the conversion factor from SI units to geometrized units is

$$[V]_{SI} = \frac{c^2}{\sqrt{G\epsilon_0}}[V]_G \tag{15}$$

Analogously, using the formula $I = \frac{Q}{t}$, one can show that the way to convert current from SI to geometrized units is

$$[I]_{SI} = \sqrt{\frac{\epsilon_0}{G}}c^3[I]_G \tag{16}$$

Applying these two equations, we get:

$$[R]_{SI} = \frac{[V]_{SI}}{[I]_{SI}} = \frac{1}{\epsilon_0 c}[R]_G \tag{17}$$

If we compute $\frac{1}{\epsilon_0 c}$, we get a value of 377Ω. This value does not differ from the value of the impedance of free space. This is what one would expect since a black hole is mostly empty space. All of its mass is centered at the singularity.

4.2.7 Black Hole's Power Loss

So now that we have estimated the value of the resistance of the black hole, we are almost in a position to use equation (14). We only need a value for the magnetic flux Φ_B. To this end suppose the magnetic field is pointing in the same direction as the rotation axis of the black hole. This is a reasonable expectation since the current causing the magnetic field is located in the accretion disk of that black hole and either corotates or counterrotates in a circular loop around the black hole.

As a guess for Φ_B we will look at the magnetic flux threading the equatorial plane of the black hole. The reason that we look at the equatorial plane and not at the surface area of the black hole has to do with how we view the black hole in the context of the axisymmetric space surrounding it. As said before, the black hole conducts a current. This current flows in from the space outside the event horizon. So the space outside the event horizon together with the black hole act as an axisymmetric conductor for this current. So what we are interested in is knowing the magnetic flux through this conductor and not only the magnetic flux through the surface of the black hole, which is zero since it is a closed surface. Hence we look at the magnetic flux threading the equatorial plane of the black hole.

We know that the radius from sigularity to the event horizon of a Kerr black hole equals $r_+ = M + \sqrt{M^2 - a^2}$, where M is the mass of the black hole. Now since the surface we are looking at is just a disk, we get that the magnetic flux through it equals $\Phi_B = \pi r_+^2 B$. Lastly, if we substitute Ω in equation (14) with the rotational velocity Ω_H of the black hole and plug in our estimate for the resistance of the black hole, we get an equation for the power loss:

$$P_{BH} = \frac{\Omega_H^2 r_+^4 B^2}{16} = (\frac{\Omega_H r_+^2 B}{4})^2 \tag{18}$$

Note that this equation is only zero if there is no magnetic field or if the black hole does not rotate, i.e. $\Omega_H = 0$. What this means is that a Kerr black hole, when immersed in a uniform magnetic field, will always lose energy. This energy can only be supplied by the rotational energy of the black hole, meaning that the black hole slows down.

5 Force-Free Electromagnetism

5.1 Force-Free Magnetosphere

Let us dive into a bit more details now. We start off by looking at the most important assumption of the BZ-process: the presence of a **force-free magentosphere**. To this end, we already assume a presence of a conducting plasma around our black hole. In the next section, we will justify the presence of this plasma.

We also want to briefly note that while we apply force-free electromagnetism in the context of the BZ-process, it is in fact a theory which finds its applications in broader relativistic astronomy. Good papers to better understand force-free electromagnetism are [20] and [8].

We also want to say that a lot of work in these coming chapters is based on [8]. What we add to their work is a lot more detail as well as more worked out computations.

5.1.1 Main Equations

There are three central equations: the two Maxwell equations plus the so-called force-free equation. These equations are given by:

$$\nabla_{[\mu}F_{\nu\rho]} = 0 \tag{19}$$

$$\nabla_{\mu}F^{\mu\nu} = 4\pi j^{\nu} \tag{20}$$

$$F_{\mu\nu}j^{\nu} = 0 \tag{21}$$

where $F_{\mu\nu}$ is the Faraday tensor and j^{ν} is the electric four-current. The first two equations are the Maxwell equations and the last one is the force-free equation.

What the equation $F_{\mu\nu}j^{\nu}$ describes is the rate of energy and momentum transfer from the electromagnetic fields to the charge carriers. It is sometimes also called the **Lorentz four-force density**. In the force-free regime, what equation (21) means, is that no energy and momentum transfer takes place between the fields and the charge carriers. Alternatively we can say that the electromagnetic fields do not exert a force on the plasma surrounding the black hole since the charge carriers are situated within the plasma. Hence it is named force-free.

An important question is: when are we allowed to make the force-free assumption? The answer is that we are allowed to assume this if the electromagnetic fields are strong enough compered with the density of the plasma. To make this clear, suppose we have a really dense plasma. It would then not be reasonable to expect that the energy transfer between the fields and this high-density plasma is neglectable since then a lot of interaction is taking place between the fields and the plasma. So one would expect the energy transfer to be significant. Analogously this would hold true in the case of weak fields.

Normally when not assuming the force-free condition, one determines the electric four-current by looking at how the charges carriers are spread in the matter, i.e. determining the charge density, and what the net flow of these charge carriers is, i.e. determining the electric current. Yet in the setting of force-free electromagnetism, this electric four-current depends entirely on the fields. What we mean by this is that the electric four-current is defined by equation (20) and does not impose any boundary conditions on our fields as it does in general electromagnetism. In fact we can use equation (20) to rewrite equation (21) as:

$$F_{\mu\nu}\nabla_{\sigma}F^{\sigma\nu} = 0 \tag{22}$$

5.1.2 Alternative Understanding of the Force-Free Equations

We could have also defined the force-free equations using the stress-energy tensor $T_{\mu\nu}$ which we can write as:

$$T_{\mu\nu} = T_{\mu\nu}^{(plasma)} + T_{\mu\nu}^{(EM)}$$

where $T_{\mu\nu}^{(plasma)}$ is the part of the stress-energy tensor due to plasma and $T_{\mu\nu}^{(EM)}$ is the stress-energy tensor due to the electromagnetic fields. The last one is given by:

$$T_{\mu}^{\nu,(EM)} = F_{\mu\sigma}F^{\nu\sigma} - \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta_{\mu}^{\nu} \tag{23}$$

where $F_{\mu\nu}$ is the faraday tensor and δ_μ^ν is the Kronecker delta.

What a stress-energy tensor describes is the energy and momentum density together with the energy and momentum flux. In the force-free regime, the density of the plasma surrounding the black hole should be sufficiently low for there to be little energy transfer from the electromagnetic fields to the plasma. What this tells us is that the component of the stress-energy tensor due to the plasma can be neglected as it pales in comparison to the stress-energy tensor of the electromagnetic field. From this follows:

$$T_\mu^\nu = T_\mu^{\nu,(EM)}$$

So from now on, we will drop the superscript (EM). What we will aim to prove now is that the force-free equations imply:

$$\nabla_\nu T_\mu^\nu = 0 \tag{24}$$

We start from equation (22). We note that by definition of the covariant derivative we have:

$$\nabla_\sigma (F_{\mu\nu} F^{\sigma\nu}) = (\nabla_\sigma F_{\mu\nu}) F^{\sigma\nu} + F_{\mu\nu}(\nabla_\sigma F^{\sigma\nu})$$

So we can write the Lorentz four-force density, which is the left-hand side of equation (22), as:

$$F_{\mu\nu} \nabla_\sigma F^{\sigma\nu} = \nabla_\sigma (F_{\mu\nu} F^{\sigma\nu}) - (\nabla_\sigma F_{\mu\nu}) F^{\sigma\nu}$$

We can rewrite the last component on the right-hand side as:

$$(\nabla_\sigma F_{\mu\nu}) F^{\sigma\nu} = \frac{1}{2} F^{\sigma\nu} (\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu}) \tag{25}$$

where we made use of the antisymmetry of the Faraday tensor. Now we note that equation (19) can be written as:

$$0 = \nabla_{[\mu} F_{\nu\sigma]} = 2\nabla_\mu F_{\nu\sigma} + 2\nabla_\nu F_{\sigma\mu} + 2\nabla_\sigma F_{\mu\nu}$$

Hence we get:

$$\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} = -\nabla_\mu F_{\nu\sigma}$$

If we plug this into equation (25) we get:

$$(\nabla_\sigma F_{\mu\nu}) F^{\sigma\nu} = -\frac{1}{2} F^{\sigma\nu} \nabla_\mu F_{\nu\sigma} = \frac{1}{2} F^{\nu\sigma} \nabla_\mu F_{\nu\sigma}$$

Now we use the same trick to note that:

$$\nabla_\mu (F^{\nu\sigma} F_{\nu\sigma}) = (\nabla_\mu F^{\nu\sigma}) F_{\nu\sigma} + (\nabla_\mu F_{\nu\sigma}) F^{\nu\sigma} = 2(\nabla_\mu F_{\nu\sigma}) F^{\nu\sigma}$$

Thus we get:

$$(\nabla_\sigma F_{\mu\nu}) F^{\sigma\nu} = \frac{1}{4} \nabla_\mu (F^{\nu\sigma} F_{\nu\sigma}) = \nabla_\lambda (\frac{1}{4} \delta_\mu^\lambda F^{\nu\sigma} F_{\nu\sigma})$$

Now if we plug this into equation (5.1.2) we get:

$$F_{\mu\nu} \nabla_\sigma F^{\sigma\nu} = \nabla_\lambda (F_{\mu\nu} F^{\lambda\nu}) - \nabla_\lambda (\frac{1}{4} \delta_\mu^\lambda F^{\nu\sigma} F_{\nu\sigma}) = \nabla_\lambda T_\mu^\lambda$$

where in the last step we used the linearity of the covariant derivative and equation (23). Now since the left-hand side of the equation is zero by equation (22), equation (24) follows.

By definition of equation (24), the left-hand side of the equation is a measure for how much energy is being extracted from or added to the electromagnetic fields. Since this equals zero, there is no energy transfer. This coincides with the aforementioned interpretation of the force-free condition.

5.1.3 Degeneracy

It appears that any force-free electromagnetic fields are **degenerate**. This means that they satisfy $F_{[\mu\nu}F_{\sigma\rho]} = 0$ where the square brackets denote the antisymmetric part of the tensor $F_{\mu\nu}F_{\sigma\rho}$. To show this, we need the following formula:

$$F_{[\mu\nu}F_{\sigma\rho]} = \frac{1}{3}(F_{\mu\nu}F_{\sigma\rho} + F_{\mu\rho}F_{\sigma\nu} + F_{\mu\sigma}F_{\nu\rho}) \tag{26}$$

One can derive this formula using the definition of the antisymmetric part of a tensor, a lot of algebra and noting that the Faraday tensor is antisymmetric. What this formula tells us is that if two indices are the same, then $F_{[\mu\nu}F_{\sigma\rho]} = 0$. To make this clear, let us look at:

$$F_{[\mu\mu}F_{\sigma\rho]} = \frac{1}{3}(F_{\mu\mu}F_{\sigma\rho} + F_{\mu\rho}F_{\sigma\mu} + F_{\mu\sigma}F_{\mu\rho}) = \frac{1}{3}(0 + F_{\mu\rho}F_{\sigma\mu} - F_{\sigma\mu}F_{\mu\rho}) = 0$$

Using formula (26), we see that $F_{[\mu\mu}F_{\sigma\rho]} = F_{[\mu\sigma}F_{\mu\rho]}$. Thus we conclude that if any of the two indices are the same, $F_{[\mu\nu}F_{\sigma\rho]} = 0$ holds.

It should be noted that due to the force-free property and equation (26), we have that $F_{[\mu\nu}F_{\sigma\rho]}j^\rho = 0$. Let μ, ν, σ and λ be four distinct indices. Since we work in four-dimensional spacetime, we see that the sum $F_{[\mu\nu}F_{\sigma\rho]}j^\rho$ has four terms and thus follows that ρ takes on the indices μ, ν, σ and λ. If ρ is equal to one of the indices μ, ν or σ, we have that $F_{[\mu\nu}F_{\sigma\rho]} = 0$. So the only non-zero term is $\rho = \lambda$. This gives us:

$$0 = F_{[\mu\nu}F_{\sigma\rho]}j^\rho = F_{[\mu\nu}F_{\sigma\lambda]}j^\lambda$$

where λ here is not summed over, but is a fixed index. Since by equation (20) j^ν is totally determined by $F_{\mu\nu}$, the only way the above equation can be zero is if $F_{[\mu\nu}F_{\sigma\lambda]} = 0$. This means that $F_{[\mu\nu}F_{\sigma\lambda]}$ is zero for all the terms, showing that a force-free field is degenerate.

Note that we did not really need the fact that $F_{\mu\nu}j^\nu = 0$. It would have been sufficient if there was a non-zero vector field v^ν such that $F_{\mu\nu}v^\nu = 0$. This tells us that there are degenerate, but non-force-free electromagnetic fields.

We can write the degeneracy condition down in flat spacetime. If we do this, we would get:

$$\vec{E} \cdot \vec{B} = 0 \tag{27}$$

$$\rho\vec{E} + \vec{B} \times \vec{j} = 0 \tag{28}$$

The second equation follows trivially from the definition of the electric four-current and the definition of the Faraday tensor. To show the first equation, we have to rewrite equation (26). To this end we look at the Hodge dual of the Faraday tensor $\star F^{\mu\nu} := \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, where ϵ is the Levi-Civita symbol. We can look at the following Lorentz invariant quantity

$$F_{\mu\nu} \star F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

Thus by property of the Levi-Civita symbol, we get:

$$F_{\mu\nu}F_{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}F_{\alpha\beta} \star F^{\alpha\beta}$$

We know that $F_{\alpha\beta} \star F^{\alpha\beta} = 4\vec{B} \cdot \vec{E}$. This then allows us to write equation (**??**) as:

$$F_{[\mu\nu}F_{\sigma\rho]} = \frac{1}{3}(\epsilon_{\mu\nu\sigma\rho} + \epsilon_{\mu\rho\sigma\nu} + \epsilon_{\mu\sigma\nu\rho})F_{\alpha\beta} \star F^{\alpha\beta} = -\frac{4}{3}\epsilon_{\mu\nu\sigma\rho}(\vec{B} \cdot \vec{E})$$

where in the last step we used the fact that $\epsilon_{\mu\rho\sigma\nu} = \epsilon_{\mu\sigma\nu\rho} = -\epsilon_{\mu\nu\sigma\rho}$. Hence we get that the property $F_{[\mu\nu}F_{\sigma\rho]} = 0$ reduces to $\vec{B} \cdot \vec{E} = 0$.

5.1.4 Understanding Degeneracy

As said, in tensor notation the degeneracy condition is given by $F_{[\mu\nu}F_{\sigma\rho]} = 0$. Note that in the derivation of this formula from the force-free condition, we had to make use of the fact that $F_{\mu\nu}j^\nu = 0$. As also stated, we did not necessarily need that it sums to zero with the electric four-current, but only that there exists a non-zero vector field v^μ such that $F_{\mu\nu}v^\nu = 0$. We could have taken this v^μ to be the four-velocity of an observer. Then we can understand $F_{\mu\nu}v^\nu = 0$ to mean that observer does not observe an electric field. To make this clear, just plug in $\mu = t$.

In conclusion, we can understand the degeneracy condition to imply the existence of an observer in whose rest frame no electric field is observed.

5.1.5 Alternative Formulation Main Equations

Up until now, we have made use of tensorial notations. To understand the BZ-process, it appears to be easier to formulate our equations using exterior calculus. We know how the Maxwell equations look like using exterior calculus:

$$dF = 0 \tag{29}$$

$$d \star F = J \tag{30}$$

Here F is the **Faraday 2-form**. Given a coordinate basis x^μ, we can write the Faraday 2-form as $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$, where $F_{\mu\nu}$ is the Faraday tensor. Furthermore J is the current 3-form. It is related to the electric four-current by $J_{\mu\nu\rho} = j^\sigma \tilde{\epsilon}_{\sigma\mu\nu\rho}$. Here $\tilde{\epsilon}_{\sigma\mu\nu\rho}$ is the Riemannian volume element defined by $\tilde{\epsilon}_{\sigma\mu\nu\rho} = \sqrt{|g|}\epsilon_{\sigma\mu\nu\rho}$, where g is the determinant of the metric and $\epsilon_{\sigma\mu\nu\rho}$ is the Levi-Civita symbol. Note that in Minkowski spacetime, the Riemannian volume element is the Levi-Civita symbol.

So what remains to do, is determine how to write the force-free condition using the language of exterior calculus. To ths end, let us rewrite the force-free equation (21) using the current 3-form. We note that $j^\nu = \frac{1}{3!}\tilde{\epsilon}^{\nu\sigma\rho\lambda}J_{\sigma\rho\lambda}$. This gives us:

$$F_{\mu\nu}j^\nu = \frac{1}{3!}\tilde{\epsilon}^{\nu\sigma\rho\lambda}F_{\mu\nu}J_{\sigma\rho\lambda}$$

A tedious algebraic computation shows us that $\tilde{\epsilon}^{\nu\sigma\rho\lambda}F_{\mu\nu}J_{\sigma\rho\lambda} = \sqrt{|g|}F_{\mu[\nu}J_{\sigma\rho\lambda]}$. It follows thus that if $F_{\mu\nu}j^\nu = 0$, then we have $F_{\mu[\nu}J_{\sigma\rho\lambda]} = 0$.

The next step is to look at how to write the degeneracy condition using exterior calculus. For this we look at $F \wedge F$. By definition, the wedge product of two 2-forms is a 4-form whose components are written as:

$$(F \wedge F)_{\mu\nu\rho\sigma} = 6F_{[\mu\nu}F_{\rho\sigma]}$$

The number 6 in front has to do with the fact that we are taking the wedge product of two 2-forms. If we were to take the wedge product of a k-form and a l-form, the numer in front would be $\frac{(k+l)!}{k!l!}$. From this we readily see that the degeneracy condition $F_{[\mu\nu}F_{\rho\sigma]} = 0$ implies that $F \wedge F = 0$.

There is a mathematical theorem that states that any 2-form ω in four-dimensional space whose wedge with itself vanishes, i.e. $\omega \wedge \omega = 0$, can be decomposed as a wedge product of two linearly independent one-forms. Since this is true for the Faraday 2-form, we can write it as $F = \alpha \wedge \beta$ where α and β are one-forms.

Using the fact that we can decompose F in this way, let us write $F_{\mu\nu}$ in another way:

$$F_{\mu\nu} = (\alpha \wedge \beta)_{\mu\nu} = \frac{2!}{1!1!}\alpha_{[\mu}\beta_{\nu]} = (\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)$$

where we note that $\alpha_{[\mu}\beta_{\nu]} = \frac{1}{2}(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)$. Thus we can write:

$$F_{\mu[\nu}J_{\sigma\rho\lambda]} = \alpha_\mu\beta_{[\nu}J_{\sigma\rho\lambda]} - \beta_\mu\alpha_{[\nu}J_{\sigma\rho\lambda]}$$

Note that $(\beta \wedge J)_{\nu\sigma\rho\lambda} = 4\beta_{[\nu}J_{\sigma\rho\lambda]}$ and the same holds true with α. This gives us:

$$F_{\mu[\nu}J_{\sigma\rho\lambda]} = \frac{1}{4}\alpha_\mu(\beta \wedge J)_{\nu\sigma\rho\lambda} - \frac{1}{4}\beta_\mu(\alpha \wedge J)_{\nu\sigma\rho\lambda}$$

We know that $F_{\mu[\nu}J_{\sigma\rho\lambda]} = 0$. We also know that α and β are linearly independent. So the only way the above equation can be zero is if

$$\alpha \wedge J = \beta \wedge J = 0 \tag{31}$$

This gives us the force-free equation in terms of the wedge product.

5.1.6 Euler Potentials

Another formulation for the Faraday two-form is possible by using two scalar functions called **Euler potentials**. The advantage of this formulation is that it allows for a much more geometrical interpretation of BZ-process.

Using the Euler potentials ϕ_1 and ϕ_2, we can write:

$$F = d\phi_1 \wedge d\phi_2 \tag{32}$$

Proving this exactly is outside the scope of this thesis. One needs to use a version of Frobenius theorem for differential forms in addition to using equation (**??**) and the fact that the Faraday two-form is decomposable.

We will offer a more intuitive approach as to what this means. For this we will be looking at an example due to [20]. Suppose we live in Minkowski spacetime and we use Cartesian coordinates. Suppose furthermore that our Faraday tensor is given by:

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ E & B & 0 & 0 \end{pmatrix} \tag{33}$$

Where we assume B and E to be constant. This is a rank two matrix and it has two annihilators given by:

$$\xi^{(1)}_\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi^{(2)}_\mu = \begin{pmatrix} -B \\ E \\ 0 \\ 0 \end{pmatrix}$$

Note also that any linear combination of these two annihilators, gives an annihilator. Hence we get a two-dimensional linear subspace dimension of annihilators for $F_{\mu\nu}$. This is precisely what it means for our Faraday two-form to be written in the way of equation (32). At every point on our manifold, there exists a two-dimensional linear subspace of the tangent space at that point which annihilates the Faraday two-tensor. In the case of the example, this is given by the Euler potentials:

$$\phi_1 = z \qquad \phi_2 = Et + Bx$$

As a simple check, we can plug these values into (32) and show that what we get, coincides with equation (33):

$$F = d\phi_1 \wedge d\phi_2 = dz \wedge d(Et + Bx) = dz \wedge (Edt + Bdx) = -Edt \wedge dz - Bdx \wedge dz$$

5.2 Force-Free in Action

In the previous section we have laid the foundations of force-free electromagnetism. In this one we are going to look at some situations where force-free electromagnetism plays a role. This is important later on when analysing the BZ-process.

5.2.1 Magnetic Monopole

Let us start of with an easy example and look at the magnetic monopole whose Faraday two-form is given by:

$$F = q_m \sin(\theta) d\theta \wedge d\phi = -d(q_m \cos(\theta)) \wedge d\phi \tag{34}$$

where we write q_m to clearly indicate that it is a magnetic charge, in this case the magnetic charge of the monopole. By using the fact that $F_{\theta\phi} = r^2 \sin(\theta) B_r$, where B_r is the magnetic field in the radial direction, it is easy to see that this is indeed the magnetic field created by a monopole. In the second equation it becomes clear that $-q\cos(\theta)$ and ϕ are Euler potentials for our Faraday two-form.

What we want to check now is that this situation is force-free. First of all, let us look at:

$$dF = q_m d(\sin(\theta)) \wedge d\theta \wedge d\phi = q_m \cos(\theta) d\theta \wedge d\theta \wedge d\phi = 0$$

So now we compute the dual $\star F$. To this end we make use of the fact that the Schwarzschild metric is diagonal which implies that the following one forms are orthogonal: $(1-\frac{2M}{r})^{1/2}dt$, $(1-\frac{2M}{r})^{-1/2}dr$, $rd\theta$ and $r\sin(\theta)d\phi$. So let us look at:

$$\begin{aligned}\star(d\theta \wedge d\phi) &= \frac{1}{r^2\sin(\theta)} \star (rd\theta \wedge r\sin(\theta)d\phi) \\ &= \frac{1}{r^2\sin(\theta)}(1-\frac{2M}{r})^{1/2}(1-\frac{2M}{r})^{-1/2} dt \wedge dr \\ &= \frac{1}{r^2\sin(\theta)} dt \wedge dr\end{aligned}$$

where we have chosen the orientation $dt \wedge dr \wedge d\theta \wedge d\phi$ to be positive. Hence we get:

$$\star F = \frac{q_m}{r^2} dt \wedge dr$$

From this follows that the current three-form equals:

$$J = d \star F = -\frac{2q_m}{r^3} dr \wedge dt \wedge dr = 0 \tag{35}$$

Thus the magnetic monopole trivially satisfies the force-free equations.

5.2.2 Poynting Flux

Let us look at another important example. Suppose we are in Schwarzschild spacetime and our Faraday two-form is given by:

$$F = d\xi \wedge du \tag{36}$$

where $\xi = \xi(\theta, \phi, u)$ is a function and u is the retarded time in outgoing Eddington-Finkelstein coordinates. Note that in this case ξ and u are the Euler potentials. From this follows that the space of annihilators is not fixed, but depends on where we are in spacetime.

We can interpret the above equation as an Poynting flux flowing along u. To see this, we simply need to compute the Faraday tensor. To this end, we use the following two facts:

$$d\xi = \frac{\partial \xi}{\partial \theta} d\theta + \frac{\partial \xi}{\partial \phi} d\phi + \frac{\partial \xi}{\partial u} du \tag{37}$$

$$du = dt - (1-\frac{2M}{r})^{-1} dr \tag{38}$$

From this follows:

$$F = \frac{\partial \xi}{\partial \theta} d\theta \wedge dt - \frac{\partial \xi}{\partial \theta}(1-\frac{2M}{r})^{-1} d\theta \wedge dr + \frac{\partial \xi}{\partial \phi} d\phi \wedge dt - \frac{\partial \xi}{\partial \phi}(1-\frac{2M}{r})^{-1} d\phi \wedge dr \tag{39}$$

where we made us of the fact that $du \wedge dr = du \wedge dt = 0$. This gives us for the Faraday tensor in Boyer-Lindquist coordinates:

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{\partial \xi}{\partial \theta} & -\frac{\partial \xi}{\partial \phi} \\ 0 & 0 & \frac{\partial \xi}{\partial \theta}(1-\frac{2M}{r})^{-1} & \frac{\partial \xi}{\partial \phi}(1-\frac{2M}{r})^{-1} \\ \frac{\partial \xi}{\partial \theta} & -\frac{\partial \xi}{\partial \theta}(1-\frac{2M}{r})^{-1} & 0 & 0 \\ \frac{\partial \xi}{\partial \phi} & -\frac{\partial \xi}{\partial \phi}(1-\frac{2M}{r})^{-1} & 0 & 0 \end{pmatrix}$$

This implies that the Poynting vector $\vec{S} \propto \vec{E} \times \vec{B}$ only has a radial component. Since the Poynting vector is lightlike, it must move along the direction u. Now let us show that this setup satisfies all the demands for force-free electromagnetism. It should be clear that $dF = 0$ since by property of the exterior derivative $d \circ d = 0$.

Next we need to compute the current three-form. To this end we have to calculate the Hodge dual of the Faraday two-form. In the same way as before, we can show that:

$$\begin{aligned}
\star(d\theta \wedge dt) &= -\frac{\sin(\theta)}{(1-\frac{2M}{r})} dr \wedge d\phi \\
\star(d\theta \wedge dr) &= \sin(\theta)(1-\frac{2M}{r}) d\phi \wedge dt \\
\star(d\phi \wedge dt) &= -\frac{1}{(1-\frac{2M}{r})\sin(\theta)} d\theta \wedge dr \\
\star(d\phi \wedge dr) &= \frac{(1-\frac{2M}{r})}{\sin(\theta)} dt \wedge d\theta
\end{aligned}$$

where again we have chosen the orientation $dt \wedge dr \wedge d\theta \wedge d\phi$ to be positive.

So now we can compute the Hodge dual of the Faraday two form using equation (39) and the linearity of the Hodge dual:

$$\star F = \frac{\partial \xi}{\partial \theta} \frac{\sin(\theta)}{(1-\frac{2M}{r})} dr \wedge d\phi + \frac{\partial \xi}{\partial \theta} \sin(\theta) d\phi \wedge dt + \frac{\partial \xi}{\partial \phi} \frac{1}{(1-\frac{2M}{r})\sin(\theta)} d\theta \wedge dr + \frac{\partial \xi}{\partial \phi} \frac{1}{\sin(\theta)} dt \wedge d\theta$$

Note that the Hodge dual of the Faraday two-form is only defined outside of the equator plane $\theta = \frac{\pi}{2}$. Now the current three-form equals:

$$J = d \star F = (\frac{\partial \xi}{\partial \theta} \cos(\theta) + \frac{\partial^2 \xi}{\partial \theta^2} \sin(\theta) + \frac{\partial^2 \xi}{\partial \phi^2} \frac{1}{\sin(\theta)}) d\theta \wedge d\phi \wedge du \tag{40}$$

It requires a lot of not so interesting algebra to see. If one desires to compute it, note that $d(\sin(\theta)) = \cos(\theta) d\theta$. Next use equation (37), but replace ξ with $\frac{\partial \xi}{\partial \theta}$ or with $\frac{\partial \xi}{\partial \phi}$. Furthermore use that $d\theta \wedge d\theta = dr \wedge dr = d\phi \wedge d\phi = 0$. Lastly use equation (38).

To check that F satisfies the force-free condition, we simply note that F is decomposable into the two one-forms $d\xi$ and du. Furthermore, using equation (37) we note that:

$$d\xi \wedge J = du \wedge J = 0$$

This is precisely the force-free property.

5.2.3 Superpostition

In general, superposition in the case of the force-free equations is not available to us. This is so because equation (22) is not a linear equation. Yet, if we superimpose the above two solutions for the magnetic monopole and radial Poynting flux, we do get a new force-free solution.

$$F = F^{(mon)} + F^{(Poy)} = q_m \sin(\theta) d\theta \wedge d\phi + d\xi \wedge du \tag{41}$$

To make clear that this solution is force-free, let us revert to the tensor notation. We can define a new Faraday tensor $F_{\mu\nu}$ as the sum of the Faraday tensor $F^{(mon)}_{\mu\nu}$ of the magnetic monopole and the Faraday tensor $F^{(Poy)}_{\mu\nu}$ of the Poynting flux i.e.:

$$F_{\mu\nu} = F^{(mon)}_{\mu\nu} + F^{(Poy)}_{\mu\nu} \tag{42}$$

If we plug this into the equation (22) we get:

$$F_{\mu\nu} \nabla_\sigma F^{\sigma\nu} = F^{(mon)}_{\mu\nu} \nabla_\sigma F^{\sigma\nu}_{(mon)} + F^{(Poy)}_{\mu\nu} \nabla_\sigma F^{\sigma\nu}_{(mon)} + F^{(mon)}_{\mu\nu} \nabla_\sigma F^{\sigma\nu}_{(Poy)} + F^{(Poy)}_{\mu\nu} \nabla_\sigma F^{\sigma\nu}_{(Poy)}$$

where $F^{\sigma\nu}_{(mon)}$ and $F^{\sigma\nu}_{(Poy)}$ are the contravariant forms of $F^{(mon)}_{\sigma\nu}$ and $F^{(Poy)}_{\sigma\nu}$ respectively.

So we know a couple of things. Firstly, both $F^{(Poy)}_{\mu\nu}$ as $F^{(mon)}_{\mu\nu}$ are force-free solutions. This implies by definition that:

$$F^{(mon)}_{\mu\nu}\nabla_\sigma F^{\sigma\nu}_{(mon)} = F^{(Poy)}_{\mu\nu}\nabla_\sigma F^{\sigma\nu}_{(Poy)} = 0$$

Secondly, we know that the current three-form is zero in the case of the magnetic monopole which implies that the electric four-current is zero and thus

$$\nabla_\sigma F^{\sigma\nu}_{(mon)} = 0$$

So to show that this superposition results in a force-free situation, we have to prove that:

$$F^{(mon)}_{\mu\nu}\nabla_\sigma F^{\sigma\nu}_{(Poy)} = 0 \tag{43}$$

First, we are going to make use of equation (20) to write:

$$\nabla_\sigma F^{\sigma\nu}_{(Poy)} = 4\pi j^{\nu}_{(Poy)}$$

Next, by the definition of the current three-form we have that $j^\nu = \frac{1}{3!}\tilde{\epsilon}^{\nu\sigma\rho\lambda}J_{\sigma\rho\lambda}$. This allows us to write:

$$\nabla_\sigma F^{\sigma\nu}_{(Poy)} = \frac{1}{3!}\epsilon^{\nu\sigma\rho\lambda}J^{(Poy)}_{\sigma\rho\lambda}$$

Equation (40) tells us how the current three-form in the Poynting case looks like and which we can rewrite as:

$$J = f(\theta,\phi,u)d\theta\wedge d\phi\wedge dt - \frac{f(\theta,\phi,u)}{1-\frac{2M}{r}}d\theta\wedge d\phi\wedge dr \tag{44}$$

where we define f as:

$$f(\theta,\phi,u) = (\frac{\partial\xi}{\partial\theta}\cos(\theta) + \frac{\partial^2\xi}{\partial\theta^2}\sin(\theta) + \frac{\partial^2\xi}{\partial\phi^2}\frac{1}{\sin(\theta)})$$

It now readily follows that:

$$\nabla_\sigma F^{\sigma t}_{(Poy)} = -\frac{f(\theta,\phi,u)}{1-\frac{2M}{r}}$$
$$\nabla_\sigma F^{\sigma r}_{(Poy)} = -f(\theta,\phi,u)$$
$$\nabla_\sigma F^{\sigma\theta}_{(Poy)} = \nabla_\sigma F^{\sigma\phi}_{(Poy)} = 0$$

What we are interested in is showing equation (43). Note that the only non-zero component of $F^{(mon)}_{\mu\nu}$ has $\nu=\theta$ or $\nu=\phi$. Combining this with the above equations, equation (43) follows readily which shows that this superposition is force-free.

It is also worthwhile to compute the electric and magnetic field explicitly. In spherical coordinates the Faraday tensor is given by:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_r & rE_\theta & r\sin(\theta)E_\phi \\ -E_r & 0 & -rB_\phi & r\sin(\theta)B_\theta \\ -rE_\theta & rB_\phi & 0 & -r^2\sin(\theta)B_r \\ r\sin(\theta)E_\phi & -r\sin(\theta)B_\theta & r^2\sin(\theta)B_r & 0 \end{pmatrix} \tag{45}$$

So let us now rewrite equation (41) using equation (39) to get:

$$F = q_m\sin(\theta)d\theta\wedge d\phi + \frac{\partial\xi}{\partial\theta}d\theta\wedge dt - \frac{\partial\xi}{\partial\theta}(1-\frac{2M}{r})^{-1}d\theta\wedge dr + \frac{\partial\xi}{\partial\phi}d\phi\wedge dt - \frac{\partial\xi}{\partial\phi}(1-\frac{2M}{r})^{-1}d\phi\wedge dr$$

So this gives us for the Faraday tensor:

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{\partial\xi}{\partial\theta} & -\frac{\partial\xi}{\partial\phi} \\ 0 & 0 & \frac{\partial\xi}{\partial\theta}(1-\frac{2M}{r})^{-1} & \frac{\partial\xi}{\partial\phi}(1-\frac{2M}{r})^{-1} \\ \frac{\partial\xi}{\partial\theta} & -\frac{\partial\xi}{\partial\theta}(1-\frac{2M}{r})^{-1} & 0 & q_m\sin(\theta) \\ \frac{\partial\xi}{\partial\phi} & -\frac{\partial\xi}{\partial\phi}(1-\frac{2M}{r})^{-1} & -q_m\sin(\theta) & 0 \end{pmatrix} \tag{46}$$

If we compare equation (45) with equation (46), we can just read of the components of the magnetic and electric field:

$$
\begin{aligned}
E_r &= 0 \\
E_\theta &= -\frac{1}{r}\frac{\partial \xi}{\partial \theta} \\
E_\phi &= -\frac{1}{r\sin(\theta)}\frac{\partial \xi}{\partial \phi} \\
B_r &= -\frac{q_m}{r^2} \\
B_\theta &= \frac{1}{r\sin(\theta)(1-\frac{2M}{r})}\frac{\partial \xi}{\partial \phi} \\
B_\phi &= -\frac{1}{r(1-\frac{2M}{r})}\frac{\partial \xi}{\partial \theta}
\end{aligned} \tag{47}
$$

Note that in the above equations we have assumed $\theta \neq 0$ and $\theta \neq \pi$. If we have $\theta = 0$ or $\theta = \pi$, then $E_\phi = 0$ and $B_\theta = 0$.

5.2.4 Michel's Rotating Magnetic Monopole

A very important example of such a superpositioned solution is the Michel's rotating magnetic monopole since this monopole serves as the starting point for most analytical models of black hole magnetospheres. We suppose that we are in Schwarzschild spacetime. In the case of the Michel's monopole we have $\xi = q_m\Omega\cos(\theta)$. This gives the following Faraday two-form:

$$F = q_m d(\cos(\theta)) \wedge (d\phi - \Omega du) \tag{48}$$

where Ω is the rotation frequency of the monopole. Note that for the above equation we mage use of the fact that $\sin(\theta)d\theta = -d(\cos(\theta))$.

Let us now show that our understanding of degenerate fields is in fact correct. To this end, suppose that we have an observer corotating with the monopole i.e. having a four-velocity $U = \partial_t + \Omega\partial_\phi$. Let us compute:

$$F_{t\mu}U^\mu = F_{tt}U^t + F_{tr}U^r + F_{t\theta}U^\theta + F_{t\phi}U^\phi$$

Note that trivially $F_{tt} = 0$. Next note that $U^r = U^\theta = 0$ since U has no components ∂_r or ∂_θ. Lastly note that $F_{t\phi} = 0$ since F has no two-form of the form $dt \wedge d\phi$. This shows:

$$F_{t\mu}U^\mu = 0$$

This tells us that an observer corotating with the magnetic monopole does not see any electric field. This supports our interpretation of degenerate fields.

We can also compute the current three-form for the Michel's monopole. As argued above, this amounts to computing the current due to the Poynting part of the Faraday tensor as the monopole part does not have a current. We can use equation (40) to see:

$$J = -q_m\Omega\sin(2\theta)d\theta \wedge d\phi \wedge du$$

We can also compute the electric four-current. To this end let us rewrite current three-form as in equation (44):

$$J = -q_m\Omega\sin(2\theta)d\theta \wedge d\phi \wedge dt + \frac{q_m\Omega\sin(2\theta)}{1-\frac{2M}{r}}d\theta \wedge d\phi \wedge dr$$

Using the fact that we can write $j^\rho = \frac{1}{3!}\tilde{\epsilon}^{\rho\mu\nu\sigma}J_{\mu\nu\sigma}$, we are able to compute the components of the electric four-current:

$$j^t = \frac{q_m\Omega\sin(2\theta)}{r^2\sin(\theta)(1-\frac{2M}{r})} = \frac{2q_m\Omega\cos(\theta)}{r^2(1-\frac{2M}{r})} \tag{49}$$

$$j^r = \frac{q_m\Omega\sin(2\theta)}{r^2\sin(\theta)} = \frac{2q_m\Omega\cos(\theta)}{r^2} \tag{50}$$

$$j^\theta = j^\phi = 0 \tag{51}$$

where the determinant of the Schwarzschild metric is $g = -r^4\sin^2(\theta)$. Note also that $\tilde{\epsilon}^{\rho\mu\nu\sigma} = (|g|)^{-1/2}\epsilon^{\rho\mu\nu\sigma}$. Hence the factor $(r^2\sin(\theta))^{-1}$ in the above equations. We furthermore note that by equation (38) we have:

$$du(j^t\partial_t + j^r\partial_r) = dt(j^t\partial_t + j^r\partial_r) - (1 - \frac{2GM}{r})^{-1}dr(j^t\partial_t + j^r\partial_r) = 0$$

This means that the electric four-current flows in radial directions along the lines where u is constant.

We want to note that there is no reason for the magnetic monopole to be a point particle. It is equally valid if it would be a spherical symmetric object. In that case the magnetic charge would be $q_m = B_0R^2$, where B_0 is the magnetic field strength on the star surface and R is the radius of the star.

Again, we can compute the magnetic and electric field using equations (47) to see:

$$\begin{aligned} E_\theta &= \frac{q_m\Omega\sin(\theta)}{r} \\ B_r &= -\frac{q_m}{r^2} \\ B_\phi &= \frac{q_m\Omega\sin(\theta)}{r(1-\frac{2M}{r})} \\ E_r = B_\theta &= E_\phi = 0 \end{aligned} \tag{52}$$

This example will play an important role when analysing the energy and angular momentum extraction mechanism since the rotating black hole can be approximated as a split monopole as Blandford and Znajek did.

6 Plasma Formation

In this chapter, we just want to mention that there is a plasma surrounding the black hole and briefly explain how this plasma is formed. The details are quite scarce and a more in-depth treatment would relate the plasma formation to the force-free condition. The more interested reader we refer to [1] chapters 2 and 3.

Let us return to the example of subsection 4.2. We have discussed that the black hole acts as a resistance. Yet what we have not discussed is what is the equivalent of the axisymmetric conductor within the BZ-process. The answer is the space surrounding the black hole. The problem is that this is empty space and empty space cannot conduct electricity. So something needs to happen before it is able to act as a conductor. This something is the formation of a plasma. Our aim for now is to give an intuitive example as to the formation of this plasma.

Our setup is as follows:

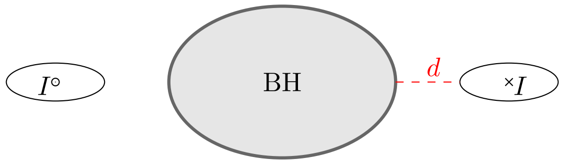

Figure 4: This is a cross section of the setup. In the centre we have the rotating black hole. On both sides we have the accretion disk. In this accretion disk runs in a circular loop around the black hole a current I. In this drawing it is ingoing on the right-hand side and outgoing on the left-hand side. d is the shortest distance between the black hole horizon and the accretion disk.

Suppose that there is no conducting material between the charged accretion disk and the black hole. We would still have a voltage drop described by equation (13). Using $\Omega = \Omega_H$ and our estimate for the magnetic flux $\Phi_B = Br_+^2\pi$ we get:

$$V = \frac{\Omega\Phi_B}{2\pi} = \frac{\Omega_H B r_+^2 \pi}{2\pi} = \frac{1}{2}\Omega_H B r_+^2 \tag{53}$$

Let us now just plug in some values. Suppose our black hole is a supermassive one with mass around $10^9 M_\odot$ which is a reasonable guess. We estimate the magnetic field strength to be $B = 5\cdot 10^3 gauss$. This equals the measured magnetic field strength of the black hole Cygni V404. We also estimate the black hole to be extremely fast rotating i.e. $a = M$. This gives us that $r_+ = M$ and $\Omega_H M = \frac{1}{2}$.

Now we rewrite equation (53) as follows:

$$V = \frac{1}{2}\Omega_H B r_+^2 = \frac{1}{2}(\Omega_H M)BM = \frac{1}{4}BM$$

Let us now make B and M dimensionless by dividing them by $5\cdot 10^3 gauss$ and $10^9 M_\odot$ respectively to get:

$$V = \frac{1}{4}(5\cdot 10^3 gauss)(10^9 M_\odot)(\frac{M}{10^9 M_\odot})(\frac{B}{5\cdot 10^3 gauss}) \approx 10^{42}(\frac{M}{10^9 M_\odot})(\frac{B}{5\cdot 10^3 gauss})$$

There still is a problem though and that is the problem of units. We want our voltage in geometrized units, but the values we have given are in SI-units. The way to convert voltage from SI units to geometrized units is given by equation (15). So this tells us that the voltage in geometrized units is:

$$V^G = \frac{\sqrt{G\epsilon_0}}{c^2}V^{SI} = \frac{\sqrt{G\epsilon_0}}{c^2}10^{42}(\frac{M^{SI}}{10^9 M_\odot^{SI}})(\frac{B^{SI}}{5\cdot 10^3 gauss}) \approx 10^{15}(\frac{M^{SI}}{10^9 M_\odot^{SI}})(\frac{B^{SI}}{5\cdot 10^3 gauss})$$

So if we plug in our values, we get a voltage difference of $V^G = 10^{15}$.

We now claim that the plasma we are going to create exists only out of electrons and positrons. The way they get produced is by a stray electron. This stray electron notices this potential drop and accelerates in the direction of the black hole. The question now becomes: Does the electron accelerate enough for it to emit a photon with enough energy to create an electron-positron pair? For a photon to be able to produce such a pair, it needs an energy of about $1MeV$. So this stray electron needs to accelerate enough to gain $1MeV$ of energy. The defintion of one electronvolt is the kinetic energy gained by an electron after traversing the distance equivalent to a drop of electric potential by one volt. Since the distance between the event horizon and the accretion disk is d and the voltage difference is 10^{15}, we can calculate how distance the electron has to traverse:

$$\frac{10^6}{10^{15}}d = 10^{-9}d$$

So the electron has to traverse a distance of $10^{-9}d$ before gaining eneough energy to emit a photon with enough energy for it to produce an electron-positron pair. This seems like it would be a reasonable distance. Thus the plasma would get formed.

7 Energy and Angular Momentum Extraction in the BZ-process

In chapter 4 we have understood the extraction of energy and angular momentum from a rotating black hole as the heat emitted by a resistance placed in a circuit. This analogy though does not offer us insight into how the black hole's energy is extracted. To put it in terms of that example: how can the black hole act as an resistance? Understanding this is the aim of this subsection. After

7.1 A Mechanical Analogy

Let us start off with another analogy: suppose we have a disk rotating with angular velocity Ω_H and suppose we have an annulus around that disk rotating with angular velocity ω. In terms of the BZ-process, the disk is the black hole and the annulus is the magnetic field. The setup looks as follows:

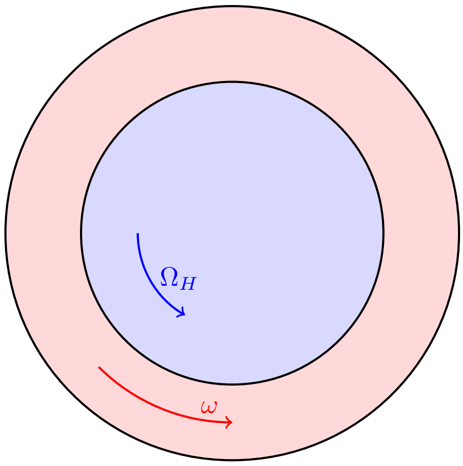

Figure 5: In blue we have drawn the disk. In red we have drawn the annulus surrounding the disk. The disk is rotating with angular velocity Ω_H. The annulus is rotating with angular velocity ω. The rotation direction is pointing outwards of the paper. Note that we have drawn Ω_H and ω both rotating in the same direction. This need not be necessarily the case. It is possible that they rotate in opposite directions.

Suppose a torque is exerted by the disk on the annulus which is proportional to the relative velocity of the two. This gives us:

$$\tau = K(\Omega_H - \omega)$$

By Newton's third law the annulus then exerts a torque in opposite direction i.e. $-\tau$. The power gained by applying a torque is given by the product of the torque applied together with the angular velocity. Thus the power gained by the annulus is:

$$P_A = \tau\omega = K\omega(\Omega_H - \omega)$$

And the power lost by the disk is:

$$P_D = -\tau\Omega_H = -K\Omega_H(\Omega_H - \omega)$$

If $\omega \neq \Omega_H$, there is some energy not accounted for. This energy is converted into heat. By conservation of energy we then have $P_D + P_A + P_H = 0$, where P_H is the amount of power which becomes heat. Thus it follows:

$$P_H = -(P_A + P_D) = K(\Omega_H - \omega)^2$$

Now we see that if $\omega > \Omega_H$, then the annulus loses energy to the disk. This is to be expected since if $\omega > \Omega_H$ the annulus would spin faster than the disk. So the friction between the disk and the annulus causes it to slow down and thus lose rotational kinetic energy. This process would speed up the disk, giving it energy. Note that this process also transfers angular momentum from the annulus to the disk.

If $\Omega_H \geq \omega \geq 0$, then the annulus would gain energy from the disk. The situation is analogous to the case of $\omega > \Omega_H$, but now the disk spins faster than the annulus. Again the angular momentum is also being transferred from the disk to the annulus.

If $\omega < 0$, then both the disk and the annulus would lose energy. All this lost energy would be converted into heat. Note though that also angular momentum is being transferred in this case. To see this, note that if $\omega < 0$, then the disk and the annulus rotate in opposite direction. The friction between them causes the disk to rotate slower. This means that the disk loses angular momentum. Since angular momentum is conserved, the annulus must gain it.

Note that this is not in contradiction with the fact that the annulus also slows down due to the friction between the disk and the annulus. This is so because the annulus was rotating with a negative angular velocity. Hence if it slows down, it implies that it rotates with a less negative angular velocity. Since angular momentum is proportional to the angular velocity, this implies that the angular momentum becomes less negative and thus increases.

7.2 Energy and Angular Momentum Extraction

7.2.1 Conserved Current

For there to be energy and angular momentum extraction from a black hole, there needs to be conserved current of energy and angular momentum. Without it we are not able to talk about conservation of energy and angular momentum and thus our argument and the entire BZ process would fail. So we need a conserved current. To show that we have such a current, we have to proof:

$$\nabla_\nu(\xi_\mu T^{\mu\nu}) = 0 \tag{54}$$

Here ξ is an arbitrary Killing vector and $T^{\mu\nu}$ is the stress-energy tensor. Noether has shown that if this holds true then there is a conserved quantity associated to this Killing vector.

So we have to show that this equation holds true in our case. Since we still work in the force-free regime, we have that $T^{\mu\nu}$ is the electromagnetic stress-energy tensor defined by equation (23), but now we have raised the index μ. Furthermore, we note again that the Kerr metric has two Killing vector fields namely ∂_t and ∂_ϕ. So for the time being we will continue to work with an arbitrary Killing vector fields ξ and later we will plug in one of these two vector fields.

Next by property of the covariant derivative, we can rewrite equation (54) as:

$$\nabla_\nu(\xi_\mu T^{\mu\nu}) = (\nabla_\nu \xi_\mu)T^{\mu\nu} + \xi^\mu(\nabla_\nu T^{\mu\nu}) \tag{55}$$

As we have show equation (24) holds true. This tells us that $\nabla_\nu T^{\mu\nu} = 0$. So we are left to show that the first component is also zero. To this end, we note that any Killing vector field has to satisfy the following equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{56}$$

Now we note that we can rewrite the first component of the left-hand side of equation (55) as:

$$(\nabla_\nu \xi_\mu)T^{\mu\nu} = \frac{1}{2}((\nabla_\nu \xi_\mu)T^{\mu\nu} + (\nabla_\mu \xi_\nu)T^{\nu\mu}) \tag{57}$$

We know that any stress-energy tensor is symmetric i.e. $T^{\mu\nu} = T^{\nu\mu}$. This tells us:

$$(\nabla_\nu \xi_\mu)T^{\mu\nu} = \frac{1}{2}((\nabla_\nu \xi_\mu) + (\nabla_\mu \xi_\nu))T^{\mu\nu} = 0 \tag{58}$$

where in the last step we have used equation (56).

So what we have shown is that there is a conserved current associated to our setup. We can define this current as the energy flux in the case of ∂_t:

$$\mathcal{E}^\nu = -(\partial_t)_\mu T^{\mu\nu} = -T^\nu_t \tag{59}$$

The minus sign arises due to Noether's theorem. For more details one can look at appendix 1.

Furthermore we can define the conserved current as the angular momentum flux in the case of ∂_ϕ:

$$\mathcal{L}^\nu = (\partial_\phi)_\mu T^{\mu\nu} = T^\nu_\phi \tag{60}$$

7.2.2 The Extraction Equations

Now that we have conserved currents, we can measure how much energy and angular momentum is being extracted from the system. The rate of energy extraction is given by:

$$\mathcal{P}_E(t) = -\lim_{r\to\infty} \int\int_A \mathcal{E}^r d\Omega = -\lim_{r\to\infty} \int\int_A T^r_t d\Omega \tag{61}$$

One can interpret this equation as the energy equivalent of the magnetic flux flowing through a surface. T^r_t is the amount of energy flowing in radial direction at some time t. We are only interested in this value since this tells us the amount of energy flowing away from the system or, in the case of the BZ-process, from the black hole. If we integrate this over some surface A, it become the amount of energy flowing through that surface. If we want to know how much energy escapes to infinity, we must take the limit.

Analogously, we can define the amount of angular momentum extracted as:

$$\mathcal{P}_L(t) = \lim_{r\to\infty} \int\int_A \mathcal{L}^r d\Omega = \lim_{r\to\infty} \int\int_A T^r_\phi d\Omega \tag{62}$$

7.3 The Michel's Magnetic Monopole Revisited

7.3.1 The Extraction of Energy from Superposed Situation

Our aim now is to compute the amount of energy extracted from Michel's rotating magnetic monopole. As we have seen in chapter 5 the rotating monopole case arises from the broader case of the superposition of the non-rotating magnetic monopole together with the situation of the Poynting flux i.e. equation (41). So it is in fact worthwhile to compute the energy extracted from this case ans then specify it to the case of the magnetic monopole.

So our starting point are equations (23) and (41). We also note that we are working in Schwarzschild spacetime. We look at:

$$T^r_t = F_{t\sigma}F^{r\sigma} - \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta^r_t = F_{t\sigma}F^{r\sigma} \tag{63}$$

Let $F^{(Poy)}_{\mu\nu}$ and $F^{(mon)}_{\mu\nu}$ be the Faraday tensors as in equation (42). This now gives us:

$$F_{t\sigma}F^{r\sigma} = (F^{(Poy)}_{t\sigma} + F^{(mon)}_{t\sigma})(F^{r\sigma}_{(Poy)} + F^{r\sigma}_{(mon)})$$

As we can see in equation (34), $F^{(mon)}_{\mu\nu}$ is zero if $\mu = t$ or if $\mu = r$. This also holds true for the contravariant form $F^{\mu\nu}_{(mon)}$. Hence this gives us:

$$F_{t\sigma}F^{r\sigma} = F^{(Poy)}_{t\sigma}F^{r\sigma}_{(Poy)} = g^{r\alpha}g^{\sigma\beta}F^{(Poy)}_{t\sigma}F^{(Poy)}_{\alpha\beta} = g^{rr}g^{\sigma\beta}F^{(Poy)}_{t\sigma}F^{(Poy)}_{r\beta}$$

where in the second step we have lowered the indices and in the third step we have made use of the fact that we work with the Schwarzschild metric. It should not come as a surprise that the monopole part does not contribute anything to the amount of energy extracted from the superposed situation since we have shown (equation (35)) that the magnetic monopole does not cause any current flow. In the force-free regime, this current is essential since it carries away the energy and angular momentum. Thus it follows that no energy is being carried away from the magnetic monopole.

By equation (**??**), we can write:

$$F^{(Poy)}_{\mu\nu} = (d\xi \wedge du)_{\mu\nu} = (d\xi_\mu du_\nu - d\xi_\nu du_\mu)$$

In the second step we have made use of the formula for the components of the wedge product. This now gives us:

$$\begin{aligned} g^{rr}g^{\sigma\beta}F^{(Poy)}_{t\sigma}F^{(Poy)}_{r\beta} &= g^{rr}g^{\sigma\beta}(d\xi_t du_\sigma - d\xi_\sigma du_t)(d\xi_r du_\beta - d\xi_\beta du_r) \\ &= g^{rr}g^{\sigma\beta}(d\xi_t d\xi_r du_\sigma du_\beta - d\xi_t d\xi_\beta du_\sigma du_r \\ &\quad - d\xi_\sigma d\xi_r du_t du_\beta + d\xi_\sigma d\xi_\beta du_t du_r) \end{aligned} \tag{64}$$

We will show now that the first three components of this equation are equal to zero. We look at:

$$g^{\sigma\beta}du_\sigma du_\beta = g^{tt}du_t du_t + g^{rr}du_r du_r$$

Using equation (38) we see that $du_t = 1$ and $du_r = -(1-\frac{2M}{r})^{-1}$. Thus it now follows:

$$g^{\sigma\beta}du_\sigma du_\beta = g^{tt}du_t du_t + g^{rr}du_r du_r = -\frac{1}{1-\frac{2M}{r}} + \frac{1-\frac{2M}{r}}{(1-\frac{2M}{r})^2} = 0$$

where we have made use of equation (2) for g^{tt} and g^{rr}.

Next let us look at:

$$g^{\sigma\beta}d\xi_\sigma du_\beta = g^{tt}d\xi_t du_t + g^{rr}d\xi_r du_r + g^{\theta\theta}d\xi_\theta du_\theta + g^{\phi\phi}d\xi_\phi du_\phi \tag{65}$$

As we can see in equation (38) we have that $du_\theta = du_\phi = 0$. So we only have to look at the first two terms. We can use equation (37) together with equation (38) to compute $d\xi_t$ and $d\xi_r$:

$$d\xi = \frac{\partial\xi}{\partial\theta}d\theta + \frac{\partial\xi}{\partial\phi}d\phi + \frac{\partial\xi}{\partial u}du = d\xi = \frac{\partial\xi}{\partial\theta}d\theta + \frac{\partial\xi}{\partial\phi}d\phi + \frac{\partial\xi}{\partial u}dt - \frac{\partial\xi}{\partial u}(1-\frac{2M}{r})^{-1}dr$$

Thus we see that:

$$\begin{aligned} d\xi_t &= \frac{\partial\xi}{\partial u} \\ d\xi_r &= -\frac{1}{1-\frac{2M}{r}}\frac{\partial\xi}{\partial u} \end{aligned}$$

We note again that by equation (38) we have that $du_t = 1$ and $du_r = -(1-\frac{2M}{r})^{-1}$ We can now plug this into equation (65) to get:

$$g^{\sigma\beta}d\xi_\sigma du_\beta = g^{tt}d\xi_t du_t + g^{rr}d\xi_r du_r = -\frac{1}{1-\frac{2M}{r}}\frac{\partial\xi}{\partial u} + (1-\frac{2M}{r})\frac{1}{(1-\frac{2M}{r})^2}\frac{\partial\xi}{\partial u} = 0$$

Hence we get for equation (64):

$$g^{rr}g^{\sigma\beta}F^{(Poy)}_{t\sigma}F^{(Poy)}_{r\beta} = g^{rr}g^{\sigma\beta}d\xi_\sigma d\xi_\beta du_t du_r = -g^{\sigma\beta}d\xi_\sigma d\xi_\beta$$

where in the last step we have filled in the values for g^{rr}, du_t and du_r. If we plug this into equation (63), we now get:

$$T^r_t = -g^{\sigma\beta}d\xi_\sigma d\xi_\beta$$

And thus the amount of energy extracted follows from equation (61)

$$\mathcal{P}_E(t) = \lim_{r\to\infty}\int\int_A g^{\sigma\beta}d\xi_\sigma d\xi_\beta d\Omega \tag{66}$$

7.3.2 The Extraction of Angular Momentum from Superposed Situation

Analogously we can compute the formula for the amount of angular momentum extracted from the superposed situation. We start from equation (62) Again using equation (23) we get:

$$T^r_\phi = F_{\phi\sigma}F^{r\sigma} - \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\delta^r_\phi = F_{\phi\sigma}F^{r\sigma} = g^{rr}g^{\sigma\alpha}F_{\phi\sigma}F_{r\alpha} \tag{67}$$

Let us look at:

$$g^{\sigma\alpha}F_{\phi\sigma}F_{r\alpha} = g^{tt}F_{\phi t}F_{rt} + g^{rr}F_{\phi r}F_{rr} + g^{\theta\theta}F_{\phi\theta}F_{r\theta} + g^{\phi\phi}F_{\phi\phi}F_{r\phi} = g^{tt}F_{\phi t}F_{rt} + g^{\theta\theta}F_{\phi\theta}F_{r\theta}$$

where in the second step we made use of the antisymmetric property of the Faraday tensor. Instead of decomposing the Faraday tensor as we did before, we can just read of the values of the Faraday tensor as given by equation (46). This gives us:

$$g^{\sigma\alpha}F_{\phi\sigma}F_{r\alpha} = g^{tt}F_{\phi t}F_{rt} + g^{\theta\theta}F_{\phi\theta}F_{r\theta} = 0 - \frac{q_m \sin(\theta)}{r^2(1-\frac{2M}{r})}\frac{\partial\xi}{\partial\theta}$$

If we plug this into equation (67) we get:

$$T^r_\phi = g^{rr}g^{\sigma\alpha}F_{\phi\sigma}F_{r\alpha} = -g^{rr}\frac{q_m \sin(\theta)}{r^2(1-\frac{2M}{r})}\frac{\partial\xi}{\partial\theta} = -\frac{q_m \sin(\theta)}{r^2}\frac{\partial\xi}{\partial\theta}$$

Hence we get for the angular momentum extracted:

$$\mathcal{P}_L(t) = -\lim_{r\to\infty}\int\int_A \frac{q_m \sin(\theta)}{r^2}\frac{\partial\xi}{\partial\theta}d\Omega \tag{68}$$

7.3.3 Energy and Angular Momentum Extracted from Michel's Rotating Magnetic Monopole

We can now specify the situation to the Michel's monopole. In that case we know that $\xi = q_m\Omega_m\cos(\theta)$. We have now written Ω_m to distinguish from the $d\Omega$ in equation (61) which stands for an area element and not for the angular velocity. Hence it follows $d\xi = -q_m\Omega_m\sin(\theta)d\theta$. So now we can compute:

$$g^{\sigma\beta}d\xi_\sigma d\xi_\beta = g^{\theta\theta}d\xi_\theta d\xi_\theta = \frac{q_m^2\Omega_m^2\sin^2(\theta)}{r^2}$$

Since for $\sigma \neq \theta$ we have that $d\xi_\sigma = 0$. Furthermore, since our situation is spherical symmetric, we can take A to be the sphere of radius r. If we then work in spherical coordinates, we get that $d\Omega = r^2\sin(\theta)d\theta d\phi$. Hence the amount of energy extracted equals:

$$\begin{aligned}
\mathcal{P}_E(t) &= \lim_{r\to\infty}\int\int_A g^{\sigma\beta}d\xi_\sigma d\xi_\beta d\Omega \\
&= \lim_{r\to\infty}\int_0^{2\pi}d\phi\int_0^{\pi}\frac{q_m^2\Omega_m^2\sin^2(\theta)}{r^2}r^2\sin(\theta)d\theta \\
&= \lim_{r\to\infty}2\pi q_m^2\Omega_m^2\int_0^{\pi}\sin^3(\theta)d\theta \\
&= \frac{8}{3}\pi q_m^2\Omega_m^2
\end{aligned}$$

We want to note now that if we have a spherical symmetric star of radius R, it is conventional to write the magnetic monopole charge q_m in terms of the magnetic field on the surface as $q_m = B_0R^2$ where B_0 is the magnetic field strength on the surface. This is just Gauss's law for magnetism modified by assumption that magnetic monopoles exist. This gives us:

$$\mathcal{P}_E(t) = \frac{8}{3}\pi B_0^2R^4\Omega_m^2 \tag{69}$$

Note that up to a constant, this is precisely the equation we estimated in equation (18) when we replace $R = r_+$ and $\Omega_m = \Omega_H$, the values for a rotating black hole. Later on in this chapter we will argue that this should be expected since we can regard Michel's rotating magnetic monopole as a linear approximation of the BZ-process.

Note also that the formula for the amount of energy extracted is the same as the formula for the amount of angular momentum extracted from the black hole. If we take A to be the sphere and plug in $\partial_\theta\xi = -q_m\Omega_m\sin(\theta)$ and $d\Omega = r^2\sin(\theta)d\theta d\phi$ into equation (62) we get:

$$\mathcal{P}_L(t) = -\lim_{r\to\infty}\int\int_A\frac{q_m\sin(\theta)}{r^2}\frac{\partial\xi}{\partial\theta}d\Omega = \lim_{r\to\infty}\int_0^{2\pi}d\phi\int_0^{\pi}\frac{q_m^2\Omega_m^2\sin^2(\theta)}{r^2}r^2\sin(\theta)d\theta$$

Which we can read of immediately equals to $\frac{8}{3}\pi q_m^2 \Omega_m^2$. That these two values are the same, need not be necessarily the case in an arbitrary force-free situation. But that they are related is to be expected since the energy extracted comes from the rotational kinetic energy of the black hole.

7.3.4 Understanding the Energy and Angular Momentum Extraction

How must we understand the fact that in the force-free regime the Michel's rotating magnetic monopole loses energy and angular momentum? To answer this question we have to analyse how the current flows. First of all, note that outside the monopole due to equation (50), we only have a radial current which is infalling in the northern hemisphere and outgoing in the southern hemisphere. So what we need to understand is what happens to the current within the monopole. Let us first make a picture before we explain what happens.

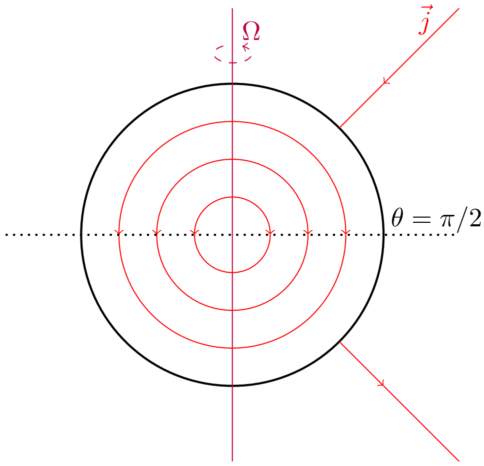

Figure 6: In black we have drawn the Michel's rotating magnetic monopole. The rotation axis is drawn in purple and in red we have drawn the current. This is a cross section where we have fixed $\phi = \phi_0$.

So we must show now that the current in fact does act that way within the monopole. To this end, we assume that the Michel's rotating magnetic monopole is a conductor. This is a reasonable assumption in our case since we will apply it to a black hole which in chapter 4 we have demonstrated is a conductor.

When the current enters the magnetic monopole, the force-free condition breaks down because the density within the monopole (which, remember, is not necessarily a point particle, but can also be a star) is too high to allow for the force-free approximation. So what this means is that the coupling between the magnetic field and the current is lost. The current starts experiencing external forces other than the magnetic field. Specifically it experiences a Lorentz force. We can compute this Lorentz force by noting that the charge carriers within the monopole corotate with it. So they have a velocity given by $\vec{v} = r\Omega\sin(\theta)\vec{e}_\phi$, where we note that we work in spherical coordinates. The equation for the magnetic fields has been given above by equation (52). So this gives us for the Lorentz force:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} = (\frac{qq_m\Omega}{r}\sin(\theta) + \frac{qq_m\Omega}{r}\sin(\theta))\vec{e}_\theta = 2\frac{qq_m\Omega}{r}\sin(\theta)\vec{e}_\theta$$

where q is the charge of the charge carrier within the monopole. What this equation tells us is that all the charges in the in the monopole experience a Lorentz force which forces them to flow to the southern hemisphere of the monopole.

We now know that there is a current flowing from the northern to the southern hemisphere of the magnetic monopole. The problem is that we do not know precisely the shape of it. We can make a couple of approximations. Firstly, we note that the magnetic monopole is by assumption spheric symmetric. This means that the current should not be influenced by the ϕ coordinate. Next, we note that our charge carriers within the monopole move along the $\vec{e}_\phi$ direction and that the force they experience is along the $\vec{e}_\theta$ direction. Hence there should not be a current flowing in the radial direction. Lastly, because the current should flow from the northern to the southern hemisphere, we have to assume that the component of the current parallel to $\vec{e}_\theta$ has to positive. These assumptions allow us to write the current as:

$$\vec{j}_{(mon)}(r,\theta) = f(r,\theta)\vec{e}_\theta + r\Omega\sin(\theta)\vec{e}_\phi$$

Where we assume $f(r,\theta)$ to always be positive and $r\Omega\sin(\theta)$ is the necessary velocity in the $\vec{e}_\phi$-direction of the charge carriers for them to corotate with the monopole.

Again we can compute the Lorentz force a charge carrier flowing along $\vec{j}_{(mon)}$ feels. We again use equation (52) to compute:

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B} = f(r,\theta)\frac{qq_m\Omega\sin(\theta)}{r(1-\frac{2M}{r})}\vec{e}_r + (\frac{qq_m\Omega\sin(\theta)}{r} + \frac{qq_m\Omega}{r}g(r,\theta))\vec{e}_\theta - \frac{q_m}{r^2}f(r,\theta)\vec{e}_\phi$$

where $\vec{v}$ is taken to be $\vec{j}_{(mon)}$ evaluated at some point (r,θ).

Now the most important part is the last component. Note that the force the current experiences in the $\vec{e}_\phi$ direction is against the rotation of the monopole. This force slows down the rotation of the monopole and thus lowers the rotational kinetic energy and angular momentum. To compare it with our mechanical analogy, the current flowing from the north to the south is the annulus and the monopole is the disk.

Now by conservation of energy and angular momentum, it has to be transferred from the monopole to the current. Now note that on the south pole we have a radially outflowing current outside of the monopole. The charges for this radially outflowing current have to be supplied by the monopole. So what this tells us is that the charge carriers of the current flowing over the surface of the monopole are being stripped away from somewhere on the southern hemisphere. Now note that these charge carriers have a higher energy and angular momentum than the charge carriers flowing radially inward from the northern hemisphere. Hence energy and angular momentum are extracted from the monopole.

What should also become clear is that there exists a relation between the amount of energy and angular momentum extracted from the rotating magnetic monopole. This is so because the energy that is being extracted is taken from the rotational kinetic energy of the monopole which is related to the angular momentum.

7.3.5 Problems with Comparison to the BZ-process

Even though this process offer a good insight into how energy and angular momentum can be extracted from a pulsar, it does not coincide with the physical reality of the extraction of energy and angular momentum from a black hole. The problems is twofold.

Firstly, we required the force-free condition to break down within the rotating monopole. We could do that since we assumed that the density of the rotating monopole was too high with respect to the electromagnetic fields. The problem is that the density within the black hole, at least near the event horizon, is not that much higher than the density of the plasma outside of the black hole since most of the mass of the black hole is situated around the singularity. This implies that the force-free condition could still hold true within the black hole.

Secondly, even if the force-free condition breaks down within the black hole, for energy and angular momentum to be extracted we also required particles to escape from the rotating magnetic monopole. This cannot happen in the context of a black hole. When a particle falls within the black hole, it can never escape.

So as shown the Michel's rotating magnetic monopole is a good approximation for the Kerr black hole. This implies that energy is flowing away from the black hole. Yet we have described that the way in which energy is extracted from a Michel's rotating monopole cannot describe the reality in the case of a black hole. These problems though are not insurmountable and the solution for them are related to the Penrose process. Remember that in the Penrose process, a particle which entered the ergosphere and then decayed, could have a decay product that has a negative energy. Something similar also happens when we measure a positive energy flux far away from the black hole. It is compensated by a negative energy flux into the black hole. We will not go into details as to under which conditions this negative energy flux can happen, but the interested reader is referred to [8] chapter 7 and 9.

7.4 Towards the BZ-process

7.4.1 Approximation

We have not yet discussed how this all fits in the broader context of the BZ-process. We will argue now that in the regime of a slowly rotating black hole, i.e. $a \ll M$, the Michel's monopole solution is a good approximation. So we assume that we have a slowly rotating Kerr black hole. We approach the electromagnetic fields around this black hole as fields generated by a non-rotating magnetic monopole i.e. by equation (34). Clearly, this cannot be the case and thus we get a correction term. We assume this correction term to be given by the second term in Michel's rotating magnetic monopole i.e. the second term is equation (48). To put in more concrete terms, the Faraday two-form for this situation is given by equation (48), which we will state here again:

$$F_{BZ} = q_m d(\cos(\theta)) \wedge (d\phi - \Omega du) = -q_m \sin(\theta) d\theta \wedge d\phi + q_m \Omega \sin(\theta) d\theta \wedge du \tag{70}$$

The way we interpret this equation now is different. The background solution is the non-rotating magnetic monopole in Schwarzschild spacetime. The second term is the error due to the rotation of the black hole. For the sake of reference, we also define:

$$\begin{aligned} F^{(0)} &= -q_m \sin(\theta) d\theta \wedge d\phi \\ F^{(1)} &= q_m \Omega \sin(\theta) d\theta \wedge du \end{aligned}$$

Where we interpret $F^{(0)}$ as the background solution and $F^{(1)}$ as correction term. Since we assume this rotation to be small, we regard only components which are at most linear in a. We regard Ω as a function of a, i.e. $\Omega = \Omega(a)$. This means that $F^{(1)}$ is also only a function of a. We also note that there are could be higher order terms in a. We just choose to chop it of at the first order approximation.

Now we note that since we work in Kerr spacetime, the coordinates you see are not Eddington-Finkelstein, but Boyer-Lindquist coordinates. What we are doing now is is just a brute force solution which we expect should correctly describe the situation when the Kerr black hole is slowly rotating because we know that in the slowly rotating limit the Kerr metric becomes the Schwarzschild metric.

Since we work in the Kerr metric, we have to upgrade u from outgoing Eddington-Finkelstein to its equivalent in Kerr spacetime. This is done by defining its differential as:

$$du = dt - \frac{r^2 + a^2}{\Delta} dr$$

7.4.2 Maxwell and Force-Free

As a check that this approximation holds, we need to show that this approximation still satisfies the Maxwell equation (29) and the force-free condition (31). To prove that $dF_{BZ} = 0$, we note that $d\Omega = 0$. This is so because Ω only depends on a and a does not vary. Hence it follows:

$$\begin{aligned} dF_{BZ} &= d(-q_m \sin(\theta) d\theta \wedge d\phi + q_m \Omega \sin(\theta) d\theta \wedge d\phi) \\ &= -q_m \cos(\theta) d\theta \wedge d\theta \wedge d\phi + q_m \Omega \cos(\theta) d\theta \wedge d\theta \wedge d\phi \\ &= 0 \end{aligned}$$

since $d\theta \wedge d\theta = 0$.

Let us now control the force-free condition. We revert back to tensor notation. We want to compute the electric four-current j^μ. Since we have F_{BZ} only up to first order in a, we can only compute the electric four-current up to first order in a since it is defined as $4\pi j^\nu = \nabla_\mu F^{\mu\nu}$. We decompose this current as:

$$j^\mu = j^\mu_{(0)} + j^\mu_{(1)}$$

where $j_{(0)}$ is the zeroth order approximation and $j_{(1)}$ is the component which depends linearly on a. We note now that it is not true that $j^\nu_{(1)} = \nabla_\mu F^{\mu\nu}_{(1)}$ since the Kerr metric, which is implicitly encoded in the covariant derivative, has a component linear in a, namely the $g_{\phi t}$ component. This means that $\nabla_\mu F^{\mu\nu}_{(0)}$ also supplies a component to the linear part $j^{(1)}$. If $g_{\phi t}$ was not linear in a and none of the other components of the metric were linear in a, then the equation would hold true.

What we want is for this current to satisfy the force-free condition:

$$F^{BZ}_{\mu\nu} j^\nu = (F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu}(a))(j^\nu_{(0)} + j^\nu_{(1)}) = F^{(0)}_{\mu\nu} j^\nu_{(0)} + F^{(1)}_{\mu\nu}(a) j^\nu_{(0)} + F^{(0)}_{\mu\nu} j^\nu_{(1)} + F^{(1)}_{\mu\nu}(a) j^\nu_{(1)} = 0$$

We know that $F^{(0)}_{\mu\nu}$ is the non-rotating magnetic monopole and it serves as the zeroth-order approximation. We demand additionally that the current in this zeroth-order approximation satisfies the force-free equation i.e. $F^{(0)}_{\mu\nu} j^\nu_{(0)} = 0$. This is not unreasonable since what we want to achieve is a force-free solution and the easiest way to do that is to perturb another force-free solution. By equation (35), we know that the force-free solution of the magnetic monopole has no current flowing. Thus we get that $j_{(0)} = 0$.

Since our solution is the zeroth-order approximation, for any current to be force-free, it has to be force-free in the zeroth-order approximation. This implies that $F^{(0)}_{\mu\nu} j^\nu_{(1)} = 0$. From this follows:

$$F^{(0)}_{\theta\nu} j^\nu_{(1)} = F^{(0)}_{\theta\phi} j^\phi_{(1)} = 0$$

This is so because the only non-zero components of $F^{(0)}$ are $F^{(0)}_{\theta\phi}$ and $F^{(0)}_{\phi\theta}$. Thus we get that $j^\phi_{(1)} = 0$. Analogously, we can show that $j^\theta_{(1)} = 0$.

Differently put, if $J^{(1)}$ is the current three-form associated to $j^{(1)}$, then the force-free condition would be satisfied if:

$$d\theta \wedge J^{(1)} = d\phi \wedge J^{(1)} = 0$$

i.e the current three-form has $d\theta$ and $d\phi$ as components. To determine that this is indeed the case, we make use of the fact that by equation (30) we have $(d \star F)^{(1)}(a) = J^{(1)}$. Note a subtlety here: we first compute $d \star F$ and then throw out all the components which do not depend linearly on a. This gives us the current in first order perturbation. So let us compute:

$$\begin{aligned}\star F &= -q_m \sin(\theta) \star (d\theta \wedge d\phi) + q_m \Omega \sin(\theta) \star (d\theta \wedge du) \\ &= -q_m \sin(\theta) \star (d\theta \wedge d\phi) + q_m \Omega \sin(\theta) \star (d\theta \wedge dt) - \frac{(r^2 + a^2) q_m \Omega \sin(\theta)}{\Delta} \star (d\theta \wedge dr)\end{aligned} \tag{71}$$

where we plugged in the value for du in the second step. In chapter 5 we computed some Hodge duals, but we are not allowed to use those since they were computed using the Schwarzschild metric, but we work in Kerr spacetime. So we need to compute them again. We will just do it for one because the other equations follow trivially.

The formula for the components of the Hodge dual are given by:

$$\star(d\theta \wedge dt)_{\mu\nu} = \frac{1}{2} \tilde{\epsilon}^{\alpha\beta\gamma\delta} g_{\mu\gamma} g_{\delta\nu} (d\theta \wedge dt)_{\alpha\beta}$$

where we note again that $\tilde{\epsilon}^{\alpha\beta\gamma\delta}$ is the contravariant form of the volume element which is defined as $\tilde{\epsilon}^{\alpha\beta\gamma\delta} = |g|^{-1/2} \epsilon^{\alpha\beta\gamma\delta}$, where the $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor and g is the metric determinant. We also know that the components of the wedge product can be computed as:

$$(d\theta \wedge dt)_{\alpha\beta} = 2 d\theta_{[\alpha} dt_{\beta]} = d\theta_\alpha dt_\beta - d\theta_\beta dt_\alpha$$

So this gives us:

$$
\begin{aligned}
\star(d\theta \wedge dt)_{\mu\nu} &= \frac{1}{2}\tilde{\epsilon}^{\alpha\beta\gamma\delta} g_{\mu\gamma} g_{\delta\nu}(d\theta_\alpha dt_\beta - \theta_\beta dt_\alpha) \\
&= \frac{1}{2} g_{\mu\gamma} g_{\delta\nu}(\tilde{\epsilon}^{\theta t\gamma\delta} - \tilde{\epsilon}^{t\theta\gamma\delta}) \\
&= \tilde{\epsilon}^{\theta t\gamma\delta} g_{\mu\gamma} g_{\delta\nu} \\
&= \tilde{\epsilon}^{\theta tr\phi} g_{\mu r} g_{\phi\nu} + \tilde{\epsilon}^{\theta t\phi r} g_{\mu\phi} g_{\delta r} \\
&= 2\tilde{\epsilon}^{\theta tr\phi} g_{r[\mu} g_{\nu]\phi}
\end{aligned}
$$

where in the second step we have made use that the only non-zero components of $d\theta_\mu$ and dt_ν are $\mu = \theta$ and $\nu = t$ respectively. In the third step we made use of the antisymmetric property of the volume element. In the fourth step we made use of the fact that the volume element is non-zero only if all the indices are different.

Following the same steps we get:

$$
\begin{aligned}
\star(d\theta \wedge dr) &= 2\tilde{\epsilon}^{\theta rt\phi} g_{t[\mu} g_{\nu]\phi} \\
\star(d\theta \wedge d\phi) &= 2\tilde{\epsilon}^{\theta\phi tr} g_{t[\mu} g_{\nu]r}
\end{aligned}
$$

From this follows that:

$$
\begin{aligned}
\star(d\theta \wedge dt) &= 2\tilde{\epsilon}^{\theta tr\phi} g_{rr} g_{t\phi} dr \wedge dt + 2\tilde{\epsilon}^{\theta tr\phi} g_{rr} g_{\phi\phi} dr \wedge d\phi \\
\star(d\theta \wedge dr) &= 2\tilde{\epsilon}^{\theta rt\phi}(g_{tt} g_{\phi\phi} - g_{t\phi}^2) dt \wedge d\phi \\
\star(d\theta \wedge d\phi) &= 2\tilde{\epsilon}^{\theta\phi tr} g_{tt} g_{rr} dt \wedge dr + 2\tilde{\epsilon}^{\theta\phi tr} g_{t\phi} g_{rr} d\phi \wedge dr
\end{aligned}
\tag{72}
$$

So now we are going to make the following approximation: since we only regard terms which are linear in a, we can throw out all the terms in the Kerr metric which are not linear in a. If one looks at equation (4), one can see that the metric becomes:

$$
\begin{aligned}
g &= g_{Schw} + \frac{4Mar\sin^2(\theta)}{r^2} dt d\phi \\
&= -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) + \frac{4Ma\sin^2(\theta)}{r} dt d\phi
\end{aligned}
\tag{73}
$$

We reobtain the Schwarzschild metric with one off-diagonal component $g_{t\phi}$. We note that this $g_{t\phi}$ differs from the component in the Kerr metric (4) since in the Kerr metric the denominator is given by ρ^2. In this approximate case, since $\rho^2 = r^2 + a^2\cos^2(\theta)$ and we do not regard higher order components in a, we get r^2 in the denominator instead of ρ^2.

This allows us to write the equations (72) as follows:

$$
\Omega \star (d\theta \wedge dt) = \frac{4Ma\sin^2(\theta)}{r(1 - \frac{2M}{r})} \tilde{\epsilon}^{\theta tr\phi} \Omega dr \wedge dt + \frac{2r^2\sin^2(\theta)}{(1 - \frac{2M}{r})} \tilde{\epsilon}^{\theta tr\phi} \Omega dr \wedge d\phi \tag{74}
$$

$$
\Omega \star (d\theta \wedge dr) = -2\tilde{\epsilon}^{\theta rt\phi}((1 - \frac{2M}{r})r^2\sin^2(\theta) + (\frac{2Ma\sin^2(\theta)}{r})^2)\Omega dt \wedge d\phi \tag{75}
$$

$$
\star(d\theta \wedge d\phi) = -2\tilde{\epsilon}^{\theta\phi tr} dt \wedge dr + \frac{4Ma\sin^2(\theta)}{r(1 - \frac{2M}{r})} \tilde{\epsilon}^{\theta\phi tr} d\phi \wedge dr \tag{76}
$$

And the volume element is given by:

$$
\begin{aligned}
\tilde{\epsilon}^{\alpha\beta\gamma\delta} &= \frac{1}{\sqrt{g_{rr} g_{\theta\theta}(g_{tt} g_{\phi\phi} - g_{t\phi}^2)}} \epsilon^{\alpha\beta\gamma\delta} \\
&= \Big(- \frac{(1 - \frac{2M}{r})}{r^2((1 - \frac{2M}{r})r^2\sin^2(\theta) + (\frac{2Ma\sin^2(\theta)}{r})^2)} \Big)^{1/2} \epsilon^{\alpha\beta\gamma\delta}
\end{aligned}
$$

We have added Ω to equation (74) and equation (75) to make clear which components depend linearly on a. So as one can see, the first term of equation (74) has both a loose a as well as a Ω term. Since Ω also depends on a, this means that it does not depend linearly on a and hence we throw it out. We do this with all the terms which do not depend linearly on a. This gives us:

$$(\Omega \star (d\theta \wedge dt))^{(1)} = \frac{2r^2 \sin^2(\theta)}{(1 - \frac{2M}{r})} \tilde{\epsilon}^{\theta t r \phi} \Omega dr \wedge d\phi \tag{77}$$

$$(\Omega \star (d\theta \wedge dr))^{(1)} = -2\tilde{\epsilon}^{\theta r t \phi}(1 - \frac{2M}{r}) r^2 \sin^2(\theta) \Omega dt \wedge d\phi \tag{78}$$

$$(\star(d\theta \wedge d\phi))^{(1)} = \frac{4Ma \sin^2(\theta)}{r(1 - \frac{2M}{r})} \tilde{\epsilon}^{\theta \phi t r} d\phi \wedge dr \tag{79}$$

Also by throwing out the terms not linear in a, we can write:

$$\begin{aligned} \tilde{\epsilon}^{\alpha\beta\gamma\delta} &= \frac{1}{\sqrt{g_{rr} g_{\theta\theta} g_{tt} g_{\phi\phi}}} \epsilon^{\alpha\beta\gamma\delta} \\ &= \frac{1}{r^2 \sin(\theta)} \epsilon^{\alpha\beta\gamma\delta} \end{aligned}$$

Thus we get:

$$(\Omega \star (d\theta \wedge dt))^{(1)} = \frac{2 \sin(\theta)}{(1 - \frac{2M}{r})} \Omega dr \wedge d\phi \tag{80}$$

$$(\Omega \star (d\theta \wedge dr))^{(1)} = 2(1 - \frac{2M}{r}) \sin(\theta) \Omega dt \wedge d\phi \tag{81}$$

$$(\star(d\theta \wedge d\phi))^{(1)} = \frac{4Ma \sin(\theta)}{r^3(1 - \frac{2M}{r})} d\phi \wedge dr \tag{82}$$

If we plug these into equation (71), we get:

$$\star F = -\frac{4Maq_m \sin^2(\theta)}{r^3(1 - \frac{2M}{r})} d\phi \wedge dr + \frac{2q_m \sin^2(\theta)}{(1 - \frac{2M}{r})} \Omega dr \wedge d\phi - \frac{2(r^2 + a^2) q_m \Omega \sin^2(\theta)}{\Delta} (1 - \frac{2M}{r}) dt \wedge d\phi \tag{83}$$

Again we throw out the a^2 component. Then we note that we can simplify:

$$\frac{r^2}{\Delta} = (1 - \frac{2M}{r})^{-1}$$

Hence:

$$\star F = -\frac{4Maq_m \sin^2(\theta)}{r^3(1 - \frac{2M}{r})} d\phi \wedge dr + \frac{2q_m \sin^2(\theta)}{(1 - \frac{2M}{r})} \Omega dr \wedge d\phi - 2q_m \Omega \sin^2(\theta) dt \wedge d\phi \tag{84}$$

If we compute the current we get:

$$J^{(1)} = (d\star F)^{(1)} = -\frac{4Maq_m \sin(2\theta)}{r^3(1 - \frac{2M}{r})} d\theta \wedge d\phi \wedge dr + \frac{2q_m \sin(2\theta)}{(1 - \frac{2M}{r})} \Omega d\theta \wedge dr \wedge d\phi - 2q_m \Omega \sin(2\theta) d\theta \wedge dt \wedge d\phi \tag{85}$$

So now we see:

$$d\theta \wedge J^{(1)} = d\phi \wedge J^{(1)} = 0$$

As argued above, this means that our setup is force-free.

7.4.3 The Role of the Michel's Monopole

Up until now we have only assumed that Ω is a function of a, but in fact we can show it. The way to show it, is by assuming that the Faraday tensor needs to be regular on the event horizon. This is appears not to be the case if we continue to work with du since r_+ is a zero of Δ. There exists though a coordinate transformation in which $d\phi - \Omega du$ is not singular on the event horizon. Take:

$$d\tilde{\phi} = d\phi - \frac{a}{\Delta}dr$$
$$dv = du + 2\frac{r^2+a^2}{\Delta}dr$$

Plugging this into $d\phi - \Omega du$ gives us:

$$d\phi - \Omega du = d\phi - \frac{a}{\Delta}dr - \Omega dv + 2\Omega\frac{r^2+a^2}{\Delta}dr$$

We also note by definition of Ω_H, we have that $a = 2Mr_+\Omega_H$. By definition of r_+, we get that $2Mr_+ = r_+^2 + a^2$. This gives us:

$$\begin{aligned} d\phi - \Omega du &= d\phi - \Omega dv - \frac{\Omega_H(r_+^2+a^2)}{\Delta}dr + 2\Omega\frac{r^2+a^2}{\Delta}dr \\ &= d\phi - \Omega dv + \frac{-\Omega_H(r_+^2+a^2) + 2\Omega(r^2+a^2)}{\Delta}dr \end{aligned}$$

The singularity can be avoided if $\Omega = 2^{-1}\Omega_H$. Since Ω_H depends linearly on a, we get that it was correct to approach Ω as a term which is linear in a.

7.4.4 Energy and Angular Momentum Extracted

So now that we have argued that the electromagnetic fields around the black hole are in first order correctly described by the Michel's monopole, we can compute the energy extracted from the black hole. The problem is that we cannot use equations (69) and (7.3.3) to compute the energy extracted near the black hole. This has to do with the fact that we have derived these equations in the context of the Schwarzschild metric and not in the context of the Kerr metric.

The solution to this is that we can use them if we are far away enough from the black hole because note that in the $g_{t\phi}$ component in equation (73) vanishes if r becomes large enough. More general, the Kerr metric becomes flat far away from the black hole. So this allows us to compute the amount of energy extracted far away from the rotating black hole:

$$P_E(t) = \frac{8\pi}{3}q_m^2(\frac{1}{2}\Omega_H)^2 = \frac{2\pi}{3}q_m^2\Omega_H^2$$

8 Conclusion

As one can see, the Blandford-Znajek process is a huge programme and understanding every component of it requires a lot of background knowledge in general relativity, plasma physics and electrodynamics. In this thesis we have attempted to familiarise the reader with at least the general relativity and electrodynamics aspect of it. We have not been able to offer an in-depth introduction regarding the plasma outside the black hole. A systematic ordering of that aspect might offer a better understanding about the feasibility of the Blandford-Znajek process. As far as the writer is aware, no experimental data has been presented regarding the presence of a plasma around a rotating black hole which satisfies the force-free condition. Finding such a plasma, might offer the best evidence in support of the Blandford-Znajek process, but is a highly non-trivial feat to accomplish

Furthermore, questions regarding the efficiency and feasibility of the Blandford-Znajek process are still open. Following Blandford and Znajek, in this thesis we have offered a model of the energy extracted from a black hole computed through a perturbation of the Michel's rotating magnetic monopole. This approach has two problems: it only works for slowly rotating black holes as well as it presumes the existence of magnetic monopoles. Computing higher order terms in the perturbation will not offer us the analytic insight necessary to understand the feasibility of the Blandford-Znajek process. The other problem is in nature most black holes are fast spinning. I do not believe that the most efficient way of approaching these fast spinning black holes is by computing higher order terms certainly because for convergence one needs to compute very high order terms.

The problem regarding the magnetic monopole are clear: no magnetic monopole has ever been observed in nature. Brute force solutions which split the monopole have been proposed which are probably nearer to describing the reality. Sadly though, we have not been able to go into any detail regarding these split monopoles. Studying these might be a good starting point for a reader wishing to go into even more detail regarding the Blandford-Znajek process.

Of course, one should mention that the Blandford-Znajek process is classical and the amount of energy that can be extracted is bound by the irreducible mass of the black hole. For a more complete overview of this process, quantum electrodynamics has to be applied. As of the time of writing, the writer is not aware of any paper attempting to create a quantum equivalent of the Blandford-Znajek process. While understandable due to the fact that the physics is not even able to model a fast spinning black hole, attempting to link this might offer a new class of solutions which allow for easier generalisation to the fast spinning case. This is an extremely difficult feat and closely related to the Holy Grail of physics: the formulation of a quantum gravity.

9 Appendix A: Energy and Angular Momentum Conservation

In this appendix we will prove that the following formulas give us conserved quantities in Kerr spacetime:

$$E = -ds^2(\partial_t, u^\nu), \qquad L = ds^2(\partial_\phi, u^\nu) \tag{86}$$

To this end, let us look at the proper time between two events A and B which is defined as:

$$\tau_{AB} = \int_A^B (-g_{\mu\nu}(x)dx^\mu dx^\nu)^{1/2} \tag{87}$$

Suppose we have some parameter σ such that the four coordinates x^μ vary from $\sigma = 0$ at A and $\sigma = 1$ at B, then we can write the proper time as:

$$\tau_{AB} = \int_0^1 (-g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma})^{1/2} d\sigma \tag{88}$$

So now we can define the Lagrangian $\mathcal{L}$ as:

$$\mathcal{L}(x^\alpha, \frac{dx^\alpha}{d\sigma}) = (-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}) \tag{89}$$

Then the proper time is minimalised if the Lagrangian satisfies the Euler-Lagrange equations:

$$-\frac{d}{d\sigma}(\frac{d\mathcal{L}}{d\frac{dx^\alpha}{d\sigma}}) + \frac{d\mathcal{L}}{dx^\alpha} = 0 \tag{90}$$

So now we choose x^α equal to x^t. Since in Kerr spacetime the metric does not depend on time, we get that the second term on the left-hand side of the above equation equals 0. So now we have:

$$-\frac{d}{d\sigma}(\frac{d\mathcal{L}}{d\frac{dx^t}{d\sigma}}) = 0 \tag{91}$$

So an easy computation gives us:

$$\frac{d\mathcal{L}}{d\frac{dx^t}{d\sigma}} = -g_{t\nu}\frac{1}{\mathcal{L}}\frac{dx^\nu}{d\sigma} \tag{92}$$

By definition of the proper time, we also have that:

$$\mathcal{L} = \frac{d\tau_{AB}}{d\sigma} \tag{93}$$

If we plug this into equation (92) we get:

$$\frac{d\mathcal{L}}{d\frac{dx^t}{d\sigma}} = -g_{t\nu}\frac{dx^\nu}{d\tau_{AB}} \tag{94}$$

We note that $\frac{dx^\nu}{d\tau_{AB}}$ is a component of the four-velocity of a particle. So this now gives us:

$$\frac{d\mathcal{L}}{d\frac{dx^t}{d\sigma}} = -ds^2(\partial_t, u^\nu) \tag{95}$$

Now since the derivative with respect to σ is zero, we get that this is a conserved quantity. We define this conserved quantity to be the energy per unit rest mass of a particle. Analogously one could show the case for the angular momentum of a particle.

10 Appendix B: Some Mathematics

The goal of this appendix is to offer some background for some mathematical objects discussed in this thesis.

10.1 Semi-Riemannian Geometry

Semi-Riemannian geometry if the foundation for general relativity. Giving an overview of all the relevant aspects of it, goes outside the scope of this appendix and this thesis. So a lot is considered to be background knowledge, but we will refer the reader who wishes to extend his background on semi-Riemannian geometry to the following sources: [2] chapter 2 for a brief overview of all the important aspects, [12] for a more in depth mathematical treatment and [15] for a treatment of semi-Riemannian geometry which also touches on a lot of subjects in physics.

10.2 Killing Vectors

An object which is needed to analyse the symmetries of the metric are the Killing vectors.

Definition 10.1. *Suppose we are given a semi-Riemannian manifold (M, g) and a vector X_p in T_pM for some $p \in M$. X_p is called a* ***Killing vector*** *if the local flow $\phi^t_{X_p}$ defines a bijective isometry between T_pM and $T_{\phi^t_{X_p}(p)}M$. A vector field X is called a* ***Killing vector field*** *if at every point $p \in M$, X_p is a Killing vector.*

So what this definition tells us is that in an infinitesimal neighbourhood around $p \in M$, any tangent space which can be expressed as $T_{\phi^t_{X_p}(p)}M$, for some t in the domain of the local flow ϕ_{X_p}, are equivalent in the sense that the length of vectors is preserved. Or to put it in more physical terms: spacetime is invariant under translation along X.

This is a bit of an abstract definition, but it appears that there is an equivalent characterization of Killing vectors. This one tells us that if we move an infinitesimal distance along X_p, then the metric is unchanged. To put it formally:

Lemma 10.2. *X is a Killing vector field if and only if the Lie derivative of the metric g along X is zero i.e. $\mathcal{L}_X g = 0$.*

Proof. We note that the Lie derivative of the metric is defined as:

$$\mathcal{L}_X g(V, W) = \frac{d}{dt}|_{t=0} g(d\phi^t_X(V), d\phi^t_X(V))$$

where V and W are vector fields defined on M. From this it follows that:

$$\begin{aligned}\frac{d}{dt}|_{t=t_0} g(d\phi^t_X(V), d\phi^t_X(V)) =& \frac{d}{dt}|_{t=t_0} g(d(\phi^{t-t_0}_X \circ \phi^{t_0}_X)(V).d(\phi^{t-t_0}_X \circ \phi^{t_0}_X)(W)) \\ =& \frac{d}{dt}|_{t=t_0} g(d\phi^{t-t_0}_X d\phi^{t_0}_X(V).d\phi^{t-t_0}_X d\phi^{t_0}_X(W)) \\ =& \frac{d}{dt}|_{s=0} g(d\phi^s_X d\phi^{t_0}_X(V).d\phi^s_X d\phi^{t_0}_X(W)) \\ =& \mathcal{L}_X g(d\phi^{t_0}_X(V), d\phi^{t_0}_X(W))\end{aligned}$$

If we take $t_0 = 0$, then we note that the flow $\phi^{t_0}_X = \phi^0_X$ is just the identity. Now if $\mathcal{L}_X g = 0$ it follows that the metric is constant. This is precisely what an isometry requires. The bijective condition follows from the properties of the flow. Thus X is a Killing vector field.

Conversely we see that if X is a Killing vector field, the metric is constant and the left hand side of the equation is zero. Since this holds for any V and W it follows that $\mathcal{L}_X g = 0$. □

The following corollary to this lemma allows us to easily compute if a vector field is a Killing vector field.

Corollary 10.3. *Let (M, g) be a n-dimensional semi-Riemannian manifold. Let X be a vector field on M. X is a Killing vector field if and only if, for every local coordinate chart $(\partial_i)_{i=1}^n$, it satisfies:*

$$X^k \partial_k g_{ij} + g_{jk} \partial_i X^k + g_{ik} \partial_j X^k = 0 \tag{96}$$

where we have that $X = X^k \partial_k$ and $g = g_{ij} dx^i dx^j$, where $g_{ij} = g(\partial_i, \partial_j)$.

Proof. This lemma amounts to an application of corollary 12.33 from [11]. This corollary gives us that we can write the Lie derivative of g_{ij} as:

$$(\mathcal{L}_X g)(\partial_i, \partial_j) = X(g_{ij}) - g([X, \partial_i], \partial_j) - g(\partial_i, [X, \partial_j]) = X^k \partial_k g_{ij} - g([X^k \partial_k, \partial_i], \partial_j) - g(\partial_i, [X^k \partial_k, \partial_j])$$

Since $X^k \in C^\infty(M)$, it follows by property of the Lie brackets that $[X^k \partial_k, \partial_i] = -(\partial_i X^k)\partial_k - X^k[\partial_i, \partial_k]$. Furthermore, since we are working with coordinate charts we get that $[\partial_i, \partial_k] = 0$. Hence the above equation simplifies to:

$$(\mathcal{L}_X g)(\partial_i, \partial_j) = X^k \partial_k g_{ij} - g(-(\partial_i X^k)\partial_k, \partial_j) - g(\partial_i, -(\partial_j X^k)\partial_k)$$

Since the metric is linear over smooth functions, we get that:

$$(\mathcal{L}_X g)(\partial_i, \partial_j) = X^k \partial_k g_{ij} + g_{jk} \partial_i X^k + g_{ik} \partial_j X^k$$

Now it follows that, if X is a Killing vector field, then the left hand side of the equation is zero. If the right hand side of the equation is zero for all local coordinate charts, then we must have that $\mathcal{L}_X g = 0$ and hence a Killing vector field. □

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