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Master's thesis

Quantum quenches in Luttinger liquids

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Abstract

It is common in physics to treat $1D$ system as a toy model in order to test a physical theory because it is easier to work with low dimensional systems and find analytical solutions. One naively will say that one-dimensional models serve as toy models and they do not apply in the real world. But, one-dimensional systems, e.g quantum wires, exists in nature and have quite different physics from higher dimensional systems. It is remarkable that, in $1D$, an interacting fermionic problem can be turned to a free bosonic theory. An interesting question to ask is how the system evolves after a change in the systems internal parameters. This is called a quantum quench. In this thesis we investigate quantum quenches in the Luttinger liquid model, which is a $1D$ model and serves as the analog of the Fermi liquid theory in one dimension. More specifically, we study the kinetic energy, when we periodically drive the internal parameters of the system, and the two point correlation function, after a periodic quench of finite duration.

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Chapter 1

Introduction

In physics it is very common to start analyzing a problem in low dimensions because it is easier to solve it. One naively will say that one-dimensional models serve as toy models and they do not apply in the real world. As we will see in the next section one-dimensional systems exist and also have very interesting physics [1], which differs from higher dimensional systems.

1.1 Are there $1D$ systems?

The arrival of Quantum mechanics [2] changed the way that we see the world. New phenomena, that no one could imagine, was necessary in order to describe processes that happen at the microscopic level. Quantization of energy levels was crucial to describe the behaviour of the hydrogen atom and the black body radiation spectrum. Using quantum mechanics the realization of a $1D$ systems is possible. For simplicity we will consider a free particle inside a $2D$ box of length L and width d, with $L \gg d$.

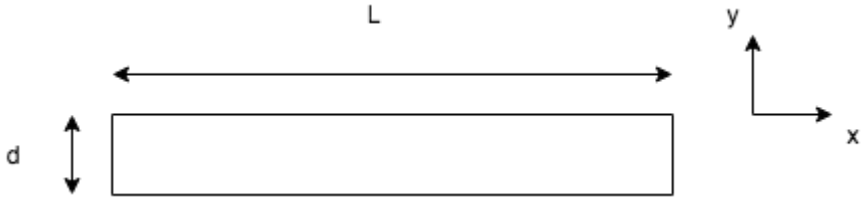

Figure 1.1: Box of length L and width d.

The energy of the particle depends on the two wavenumbers k_x, k_y

$$E(k_x, k_y) = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} \tag{1.1}$$

Because we confine the particle inside a box the wavefunction has to be zero in the walls, this yields the quantization of the two wavenumbers

$$k_x = \frac{\pi}{L} n_x \text{ and } k_y = \frac{\pi}{d} n_y \tag{1.2}$$

where n_x, $n_y \in \mathbb{Z}^+$ and the energy takes the form

$$E(n_x, n_y) = \frac{\hbar^2 \pi^2 n_x^2}{2mL^2} + \frac{\hbar^2 \pi^2 n_y^2}{2md^2} \tag{1.3}$$

Using $L \gg d$ we see that the states in the x-direction have a smaller energy difference than those in the y-direction and we can classify the energies in terms of the values n_y. Hence, $n_y = 1$ we have

$$E(n_x, 1) = \frac{\hbar^2 \pi^2 n_x^2}{2mL^2} + \frac{\hbar^2 \pi^2}{2md^2} \tag{1.4}$$

for $n_y = 2$

$$E(n_x, 2) = \frac{\hbar^2 \pi^2 n_x^2}{2mL^2} + \frac{\hbar^2 \pi^2 4}{2md^2} \tag{1.5}$$

and so on. Plotting the energy as a function of n_x

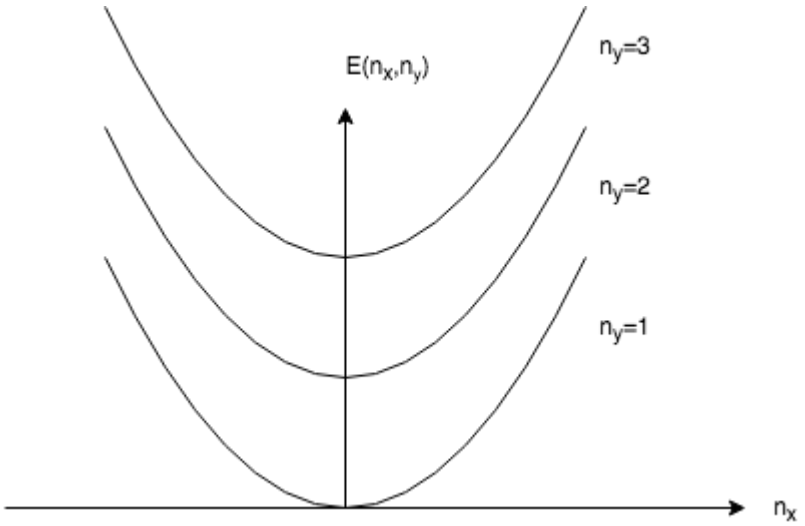

Figure 1.2: Energy eigenstates labeled from the quantum number n_y. Eigenstates are only for positive n_x.

where we have different eigenstates for every n_y (bands). Now if we include interactions or temprature we can see that if the distance between the minibands is larger than the temperature or the interaction energy then only one miniband can be excited ($n_y = 1$). The transverse degrees of freedom are thus frozen and only k_x matters. The system is a one-dimensional quantum system. One-dimensional system have different physics from higher dimensional systems. We discuss this briefly in the next section.

1.2 Why $1D$ is special?

To see why $1D$ is special we will consider interacting fermions. We start with free fermions in dimensions $d > 1$, because it is a very well known problem. From quantum statistical mechanics we know that non-interacting electrons obey the Fermi-Dirac distribution. At zero temperature the occupation number $n_{\mathbf{k}}$, for the individual momentum eigenstates is a step function. All states up to the Fermi energy ϵ_F are occupied. The occupation number $n_{\mathbf{k}}$ has a discontinuity of amplitude 1 at the Fermi surface.

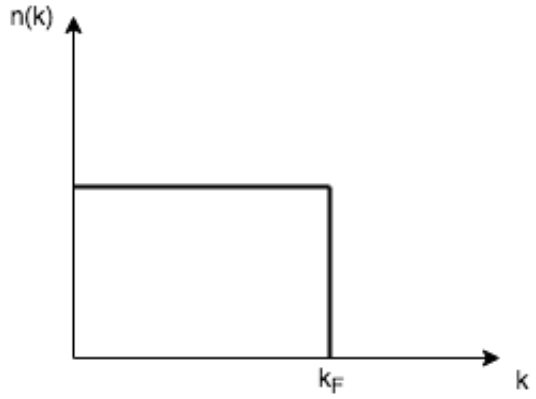

Figure 1.3: Fermi-Dirac distribution for free fermions and $T = 0$.

Excitations of the system consist of adding particles with a well defined momentum $\mathbf{k}$, having a well defined energy $\epsilon(\mathbf{k})$. The lifetime of the excited particles are infinite since they are eigenstates of the Hamiltonian. It is common to define the energy of a particle relative to the chemical potential μ, as $\xi(\mathbf{k}) = \epsilon(\mathbf{k}) - \mu$. The excitations are characterized by the spectral function $A(\mathbf{k}, \omega)$, which is the probability to find a state with frequency ω

and momentum $\mathbf{k}$, it is a delta function

$$A(\mathbf{k}, \omega) = \delta(\omega - \xi(\mathbf{k})) \tag{1.6}$$

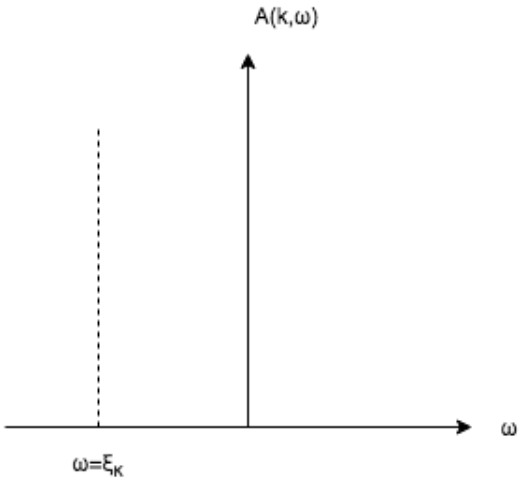

Figure 1.4: Spectral function $A(\mathbf{k}, \omega)$ for free fermions.

This is a consequence of the trivial time evolution $e^{i\xi(\mathbf{k})t}$ which does not decay in time since $\xi(\mathbf{k})$ is an eigen-energy of the system. We would like to know what happens when we turn on interactions between the fermions. This is described by Landau's Fermi liquid theory [3] which states that the properties of the systems remain similar to those of the free fermions. Instead of free fermions in Fermi liquid theory we have quasiparticles, which can be thought of as fermions dressed with the interactions. We mention here briefly two important differences with the free fermions. First, the occupation $n_{\mathbf{k}}$ still has a discontinuity at the Fermi surface $k = k_F$ but with amplitude smaller than 1.

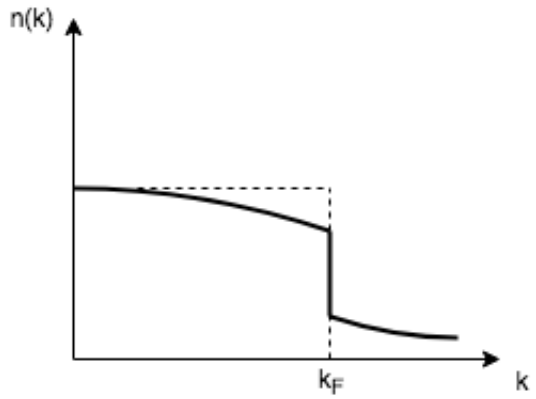

Figure 1.5: Fermi-Dirac distribution for Fermi liquid theory.

Second, the spectral function in this case is not a delta function but a Lorentzian of width $1/\tau_{\mathbf{k}}$, where the width becomes smaller for excitations near the Fermi surface. This means that quasiparticles have a finite lifetime and as we go closer to the Fermi surface they become well defined (the lifetime is very large).

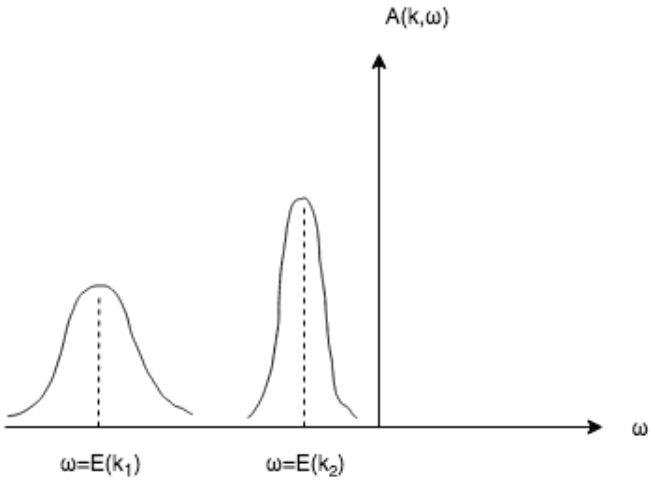

Figure 1.6: Spectral function $A(\mathbf{k}, \omega)$ for Fermi liquid theory. The closer to the Fermi surface the more narrow the Lorentzian becomes

Thus, singe particle excitations near the Fermi surface are still possible when the interactions are turned on. It can be shown that Fermi liquid theory

breaks down in $1D$ using perturbation theory [4], but we can also have a physical understanding why fails without any calculations. In one dimension a fermion that tries to propagate has to "push" its neighbors due to the interactions. Hence, no individual motion is possible and any individual excitation has to become a collective one in one dimension. This behaviour shows that the properties of one dimensional systems differ from those of higher dimensions. Excitations in one dimensional systems are bosonic in character and this can be used to map an interacting fermionic system to a free bosonic theory. The analog of the Fermi liquid theory in one dimension is the Luttinger liquid theory which is described in this thesis.

1.3 Thesis overview

Interaction in many-body quantum systems is one of the most important problems in physics. Nowdays, we can implement and simulate the dynamics of quantum many-body systems with a high degree of controllability on the system parameters even under nonequilibrium conditions. Utra-cold atomic systems can be used for experimental realizations of interacting many-body quantum system out of equilibrium, such as the Hubbard model [5]. There are a variety of non equilibrium driving protocols. In this thesis we will consider finite duration quantum quenches and periodic driving in the Luttinger-Tomonaga model. The term quantum quench was first introduced for the following procedure: Initially, we prepare a quantum system in some state $|\psi_0\rangle$, for example the ground state of a Hamiltonian H_0. Then, we suddenly change one of the systems parameters so that the time evolution of the system is determined by the new Hamiltonian H, and we study the dynamics. Sudden quenches have been studied in great detail [6],[7],[8]. The case of a periodic driving can be viewed as a finite duration quantum quench where the time of quenching goes to infinity. The outline of this thesis is as follows. In Chapter 2 we introduce the idea of a quantum quench in more detail giving two simple examples. In Chapter 3 we present the parametric driven oscillator and study the stability of such a system. In Chapter 4 we define the Tomonaga-Luttinger model and see how a fermionic problem can be mapped onto a bosonic one. In Chapter 5, we tackle the problem of Luttinger liquids out of equilibrium. First, we give the general method of how to describe a system under a finite duration quantum quench and then we apply it to calculate the kinetic energy under a periodic driving and the

fermionc Green's function under a finite duration periodic quench. Finally, in Chapter 6 conclude with some brief discussion of our results.

Chapter 2

Quantum quenches

In this chapter we introduce the idea of a quantum quench. We first describe what a quantum quench is and we present a simple example for the harmonic oscillator. After, we consider a quench in a many-particle quantum system in order to investigate the interesting phenomena that occur. Finally we briefly discuss various quench protocols.

2.1 What is a quantum quench?

Consider an isolated many-particle quantum system described by a Hamiltonian $H(g)$ where g some parameter of the system that can be tuned, such as an interaction strength in the Hubbard model or external magnetic-field strength in the transverse-field Ising model. We prepare the system initially in the ground state $|\Psi_0\rangle$ of the Hamiltonian $H(g_0)$. At $t = 0$ we suddenly change the system parameter to new a value g_1. The next figure show a schematic overview of the procedure described.

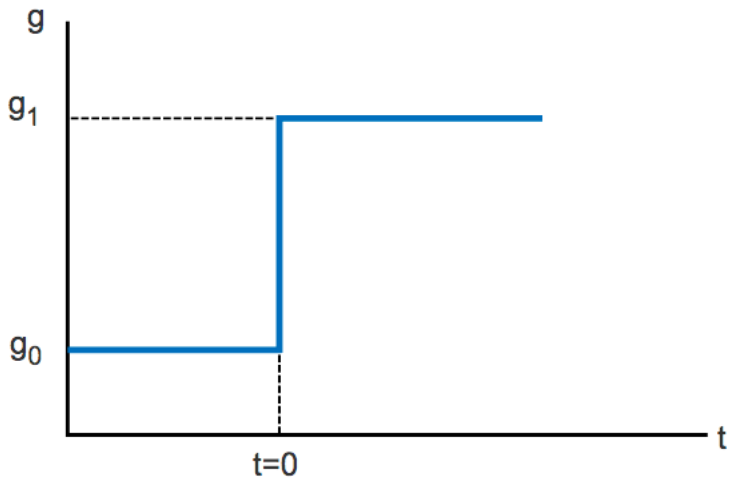

Figure 2.1: Sudden change of the parameter g at $t = 0$.

This procedure is called quantum quench. The ground state of the pre-quench Hamiltonian $|\Psi_0\rangle$ is in general not an eigenstate of the new Hamiltonian, and it will have a finite overlap with all the new eigenstates making the subsequent quantum evolution in general not trivial. After the quench the evolution of the system is governed by the new Hamiltonian $H(g_1)$ and the state of the system at times $t > 0$ is given by

$$|\Psi(t)\rangle = e^{-iH(g_1)t} |\Psi_0\rangle \tag{2.1}$$

where we used $\hbar = 1$. We can write the state at time t using the complete eigenstates of the new Hamiltonian $H(g_1)$

$$H(g_1) |n\rangle = E_n |n\rangle , \quad E_n \geq E_0 \tag{2.2}$$

as

$$|\Psi(t)\rangle = \sum_n e^{-iE_n t} \ \langle n|\Psi_0\rangle |n\rangle \tag{2.3}$$

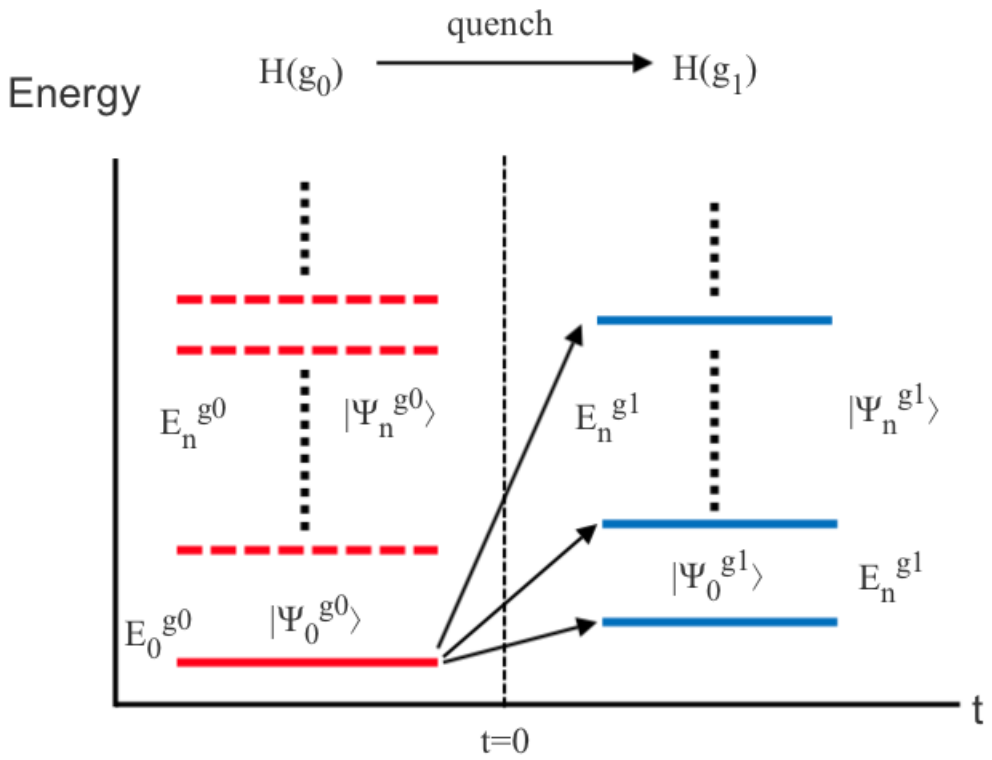

Figure 2.2: Hilbert space before and after the quantum quench. The initial state $|\Psi_0\rangle$ is not an eigenstate of the new Hamiltonian, $H(g_1)$. The system explores high-energy modes in the Hilbert space and not just states in the vicinity of the ground state

After the quench we would like to know how observables evovle with time. Suppose we want to know the value of a certain observable, say $\mathcal{O}$, at time t. This is given by the expected value

$$\langle\Psi(t)|\,\mathcal{O}\,|\Psi(t)\rangle \tag{2.4}$$

and using equation (2.3) we obtain

$$\langle\Psi(t)|\,\mathcal{O}\,|\Psi(t)\rangle = \sum_{n,m} \langle\Psi_0|n\rangle\,\langle n|\,\mathcal{O}\,|m\rangle\,\langle m|\Psi_0\rangle\,e^{-i(E_m-E_n)t} \tag{2.5}$$

From the above expectation value the exponential term signals quantum interference effects [9] in the dynamics of the system after the quench. Moreover, we can see that if the initial state $|\Psi_0\rangle$ is not an eigenstate of the new

Hamiltonian, $H(g_1)$, there is a finite probability for the system to occupy an excited state of the new Hamiltonian (figure 2.2). Hence, the system explores high-energy modes in the Hilbert space and not just states in the vicinity of the ground state [10]. In general, the quantum quench leads to violent dynamics at short time scales and usually to some kind of steady state at long times. The nature of the latter strongly depends on the nature of the Hamiltonian itself, and whether or not it is integrable. We will not go in detail about this, the interested reader can have a look at [10], [11], and [12]. Finally it should be mentioned that we can start from an initial thermal state [13], at temperature T, which is described from the density matrix

$$\rho_0 \equiv \rho(t=0) = \frac{e^{-H(g_0)/kT}}{Z_0} \tag{2.6}$$

where $Z_0 = Tr[e^{-H(g_0)/kT}]$ is the partition function, and then probe how the system evolves after the quench.

2.2 Quantum quench in Simple Harmonic oscillator

In this section we give a simple example in order to illustrate the idea of a quantum quench. We study the simple harmonic oscillator [2],[14] whose frequency is quenched from ω_0 to ω at $t = 0$. The hamiltonian of the oscillator before the quench is

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2 \tag{2.7}$$

whereas after performing the quench it is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 = \left(a^\dagger a + \frac{1}{2}\right) \tag{2.8}$$

Using the annihilation and creation operators a and $a^\dagger$ the energy eigenstates are denoted with $|n\rangle$ and the eigenvalues have the simple form $E_n = (n+\frac{1}{2})\omega$, where n is the number of quanta in the oscillator. We consider that the system initially is in the ground state $|\Psi_0\rangle$ of H_0 such that $a_0 |\Psi_0\rangle \equiv 0$, with a_0 the annihilation operator of the pre-quenched Hamiltonian. At time t the time evolved state is simply

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle \tag{2.9}$$

By expanding the state $|\Psi_0\rangle$ in energy eigenstate, $|n\rangle$, of the Hamiltonian H and using $e^{-iHt}\,|n\rangle = e^{-i(n+1/2)\omega t}\,|n\rangle$ we get

$$|\Psi(t)\rangle = \sum_n e^{-i(n+1/2)\omega t}\ \langle n|\Psi_0\rangle\,|n\rangle \tag{2.10}$$

From the above state in can be seen that the evolution of the system is periodic (quantum recurrence [15]) by doing the substitution $t \to t+T$, where $T = \frac{2\pi}{\omega}$

$$\begin{aligned} |\Psi(t+T)\rangle &= \sum_n e^{-i(n+1/2)\omega t} e^{-i(n+1/2)2\pi}\ \langle n|\Psi_0\rangle\,|n\rangle \\ &= \sum_n e^{-i(n+1/2)\omega t} e^{-i2\pi n} e^{-i\pi}\ \langle n|\Psi_0\rangle\,|n\rangle \\ &= -\,|\Psi(t)\rangle \end{aligned} \tag{2.11}$$

Now we would like to calculate various observables in the state $|\Psi(t)\rangle$. In quantum mechanics we are often interested in correlation functions. For those calculations it is useful to consider the Heisenberg picture of quantum mechanics, where the operators involve in time instead of states. We calculate the two point correlation function of the position operator x, which is given by

$$C_x(t_1,t_2) = \langle\Psi_0|\,\mathcal{T}\{x(t_1)x(t_2)\}\,|\Psi_0\rangle \tag{2.12}$$

and $\mathcal{T}$ denotes time ordering. It means that operators are ordered according to history with the later "times" to the left

$$\mathcal{T}\{x(t_1)x(t_2)\} = \Theta(t_1-t_2)x(t_1)x(t_2) + \Theta(t_2-t_1)x(t_2)x(t_1)$$

with $\Theta(t)$ being the Heaviside theta function.

The time evolution of x and p is given by the Heisenberg equation of motion

$$\frac{dx(t)}{dt} = i[H,x(t)],\ \ \frac{dp(t)}{dt} = i[H,p(t)] \tag{2.13}$$

because $H(t) = U(t)^\dagger H U(t) = H$. Computing the commutators using the canonical commutation relation $[x, p] = i$

$$[H, x(t)] = \frac{1}{2}[p(t)^2, x(t)] = \frac{1}{2}\Big(p(t)[p(t), x(t)] + [p(t), x(t)]p(t)\Big) = -ip(t)$$

$$\begin{aligned}[H, p(t)] &= \frac{\omega^2}{2}[x(t)^2, p(t)] = \frac{\omega}{2}\Big(x(t)[x(t), p(t)] + [x(t), p(t)]x(t)\Big) \\ &= i\omega^2 x p(t)\end{aligned}$$

we arrive at the two coupled differential equations for x and p

$$\frac{dx(t)}{dt} = p(t), \;\; \frac{dp(t)}{dt} = -\omega^2 x(t)$$

which yields the eom for x

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0 \tag{2.14}$$

with solution

$$x(t) = x(0)\cos\omega t + p(0)\frac{\sin\omega t}{\omega} \tag{2.15}$$

We therefore have for the correlation function

$$\begin{aligned}\langle\Psi_0|\, x(t_1)x(t_2)\,|\Psi_0\rangle &= \langle\Psi_0|\left(x(0)\cos\omega t_1 + p(0)\frac{\sin\omega t_1}{\omega}\right)\cdot \\ &\qquad \left(x(0)\cos\omega t_2 + p(0)\frac{\sin\omega t_2}{\omega}\right)|\Psi_0\rangle = \\ &= \langle\Psi_0|\, x(0)^2\,|\Psi_0\rangle \cos\omega t_1 \cos\omega t_2 \\ &+ \langle\Psi_0|\, p(0)^2\,|\Psi_0\rangle \frac{\sin\omega t_1 \sin\omega t_2}{\omega^2} \\ &+ \langle\Psi_0|\, x(0)p(0)\,|\Psi_0\rangle \cos\omega t_1 \frac{\sin\omega t_2}{\omega} \\ &+ \langle\Psi_0|\, p(0)x(0)\,|\Psi_0\rangle \cos\omega t_2 \frac{\sin\omega t_1}{\omega}\end{aligned}$$

In order to find an expression for the correlation function we have to calculate the four amplitudes. This we will do it via the creation and annihilation operators, $a_0^\dagger$ a_0, which are related with x and p via

$$x = \sqrt{\frac{1}{2\omega_0}}\left(a_0 + a_0^\dagger\right), \;\; p = i\sqrt{\frac{\omega_0}{2}}\left(a_0^\dagger - a_0\right)$$

So,

$$\begin{aligned}
\langle\Psi_0|\, x(0)^2\, |\Psi_0\rangle &= \frac{1}{2\omega_0} \langle\Psi_0|\, (a_0 + a_0^\dagger)^2\, |\Psi_0\rangle \\
&= \frac{1}{2\omega_0} \langle\Psi_0|\, (a_0 a_0 + a_0 a_0^\dagger + a_0^\dagger a_0 + a_0^\dagger a_0^\dagger)\, |\Psi_0\rangle \\
&= \frac{1}{2\omega_0}
\end{aligned}$$

$$\begin{aligned}
\langle\Psi_0|\, p(0)^2\, |\Psi_0\rangle &= -\frac{\omega_0}{2} \langle\Psi_0|\, (a_0^\dagger - a_0)^2\, |\Psi_0\rangle \\
&= -\frac{\omega_0}{2} \langle\Psi_0|\, (a_0^\dagger a_0^\dagger - a_0^\dagger a_0 - a_0 a_0^\dagger + a_0 a_0)\, |\Psi_0\rangle \\
&= \frac{\omega_0}{2}
\end{aligned}$$

$$\begin{aligned}
\langle\Psi_0|\, x(0)p(0)\, |\Psi_0\rangle &= i\frac{1}{2} \langle\Psi_0|\, (a_0 + a_0^\dagger)(a_0^\dagger - a_0)\, |\Psi_0\rangle \\
&= i\frac{1}{2} \langle\Psi_0|\, (a_0 a_0^\dagger - a_0 a_0 + a_0^\dagger a_0^\dagger - a_0^\dagger a_0)\, |\Psi_0\rangle \\
&= i\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\langle\Psi_0|\, p(0)x(0)\, |\Psi_0\rangle &= i\frac{1}{2} \langle\Psi_0|\, (a_0^\dagger - a_0)(a_0 + a_0^\dagger)\, |\Psi_0\rangle \\
&= i\frac{1}{2} \langle\Psi_0|\, (a_0^\dagger a_0 + a_0^\dagger a_0^\dagger - a_0 a_0 - a_0 a_0^\dagger)\, |\Psi_0\rangle \\
&= -i\frac{1}{2}
\end{aligned}$$

where we used that $a_0\,|\Psi_0\rangle \equiv 0$, $\langle\Psi_0|\, a_0^\dagger a_0^\dagger\, |\Psi_0\rangle = 0$ and $\langle\Psi_0|\, a_0 a_0^\dagger\, |\Psi_0\rangle = 1$. The correlation function is simplified to

$$\begin{aligned}
\langle\Psi_0|\, x(t_1)x(t_2)\, |\Psi_0\rangle &= \frac{1}{2\omega_0} \cos\omega t_1 \cos\omega t_2 + \frac{\omega_0}{2} \frac{\sin\omega t_1 \sin\omega t_2}{\omega^2} \\
&\quad + \frac{i}{2} \cos\omega t_1 \frac{\sin\omega t_2}{\omega} - \frac{i}{2} \cos\omega t_2 \frac{\sin\omega t_1}{\omega} \\
&= \frac{1}{2\omega_0} \cos\omega t_1 \cos\omega t_2 + \frac{\omega_0}{2} \frac{\sin\omega t_1 \sin\omega t_2}{\omega^2} \\
&\quad - \frac{i}{2} \frac{\sin(t_1 - t_2)}{\omega}
\end{aligned}$$

by algebraic manipulations the above relation can be cast to the following form

$$\langle\Psi_0|\, x(t_1)x(t_2)\,|\Psi_0\rangle = \frac{(\omega-\omega_0)^2}{4\omega^2\omega_0}\cos\omega(t_1-t_2) + \frac{\omega^2-\omega_0^2}{4\omega^2\omega_0}\cos\omega(t_1+t_2) + \frac{1}{2\omega}e^{-i\omega(t_1-t_2)}$$

For the calculation of $\langle\Psi_0|\, x(t_2)x(t_1)\,|\Psi_0\rangle$ we note that under the interchange $t_1 \leftrightarrow t_2$ the term of the exponential picks a minus sign. Hence the full correlation function will be

$$C_x(t_1,t_2) = \frac{(\omega-\omega_0)^2}{4\omega^2\omega_0}\cos\omega(t_1-t_2) + \frac{\omega^2-\omega_0^2}{4\omega^2\omega_0}\cos\omega(t_1+t_2) + \frac{1}{2\omega}e^{-i\omega|t_1-t_2|} \tag{2.16}$$

The last term is the known result of the two point correlation function of the harmonic oscillator without quench. For $\omega = \omega_0$ only the last term survives. The other two terms arise due to the quench. It can be seen that the existence of the term that depends on (t_1+t_2) breaks time translation invariance. This can be interpret as the system "remembers" that a quench happened in the past.

2.3 Quantum quench in linearly coupled oscillators

In order to have an idea about quantum quenches in many body quantum systems we present an example of a quantum quench in a chain of linearly coupled harmonic oscillators. This system can be described by a general Hamiltonian of the form

$$H = \frac{1}{2}\left[\sum_i \pi_i^2 + \sum_j K(i-j)(\phi_i-\phi_j)^2\right] \tag{2.17}$$

where π_i and ϕ_i are the momentum and position operators for the i-th oscillator. Using the Fourier transform of ϕ_i and π_i the above Hamiltonian can become diagonal in the momentum space having the general form

$$H = \sum_k \frac{1}{2}\pi_k\pi_{-k} + \frac{1}{2}\omega_k^2\phi_k\phi_{-k} \tag{2.18}$$

which is a sum of independent harmonic oscillators (one oscillator for each mode k). Moreover, we consider the continuum limit, where we replace sums with integrals, and also a relativistic dispersion relation

$$\omega_k = k^2 + m^2 \tag{2.19}$$

where we use ($c = 1$). As we done in the previous section we can perform a quantum quench in this system. We consider that the system is isolated and is at the ground state $|\Psi_0\rangle$. At $t = 0$ we perform a quantum quench, by suddenly changing the mass from m_0 to m. Again we want to study the two-point correlation function given by

$$C_\phi(t_1, t_2, r_1 - r_2) = \langle\Psi_0| \mathcal{T}\{\phi(r_1, t_1)\phi(r_2, t_2)\} |\Psi_0\rangle \tag{2.20}$$

where $\mathcal{T}$ denotes time ordering as in the previous section and because the theory is free, due to translational invariance the two point function depends on $r_1 - r_2$. The Hamiltonian (2.18) is a sum of independent harmonic oscillators, one for each mode k. Hence, we can use the result from the previous section and say that each momentum k oscillator has a two-point function of the form

$$\begin{aligned} C_\phi(t_1, t_2; k) = &\frac{(\omega_k - \omega_{0k})^2}{4\omega_k^2\omega_{0k}} \cos\omega_k(t_1 - t_2) + \frac{\omega_k^2 - \omega_{0k}^2}{4\omega_k^2\omega_{0k}} \cos\omega_k(t_1 + t_2) \\ &+ \frac{1}{2\omega_k} e^{-i\omega_k|t_1 - t_2|} \end{aligned} \tag{2.21}$$

with $\omega_k = \sqrt{k^2 + m^2}$ and $\omega_{0k} = \sqrt{k^2 + m_0^2}$. Thus the computation of the correlation function amounts to a Fourier transform with respect to k

$$C_\phi(t_1, t_2, r) = \int \frac{dk}{2\pi} e^{ikr} C_\phi(t_1, t_2; k) \tag{2.22}$$

To simplify our analysis we study the equal time correlation function and consider $m_0 \to \infty$. Then, equation (2.21) takes the form

$$C_\phi(t; k) = \frac{m_0}{4\omega_k^2}(1 - \cos 2\omega_k t) \tag{2.23}$$

and (2.22)

$$C_\phi(t, r) = \int \frac{dk}{2\pi} e^{ikr} \frac{m_0}{4\omega_k^2}(1 - \cos 2\omega_k t) \tag{2.24}$$

We solve the above integral numerically in order to see the behaviour of the correlation function. In the figure below is the solution of $C_\phi(t, r)$ at fixed r, for $m = r = 1$

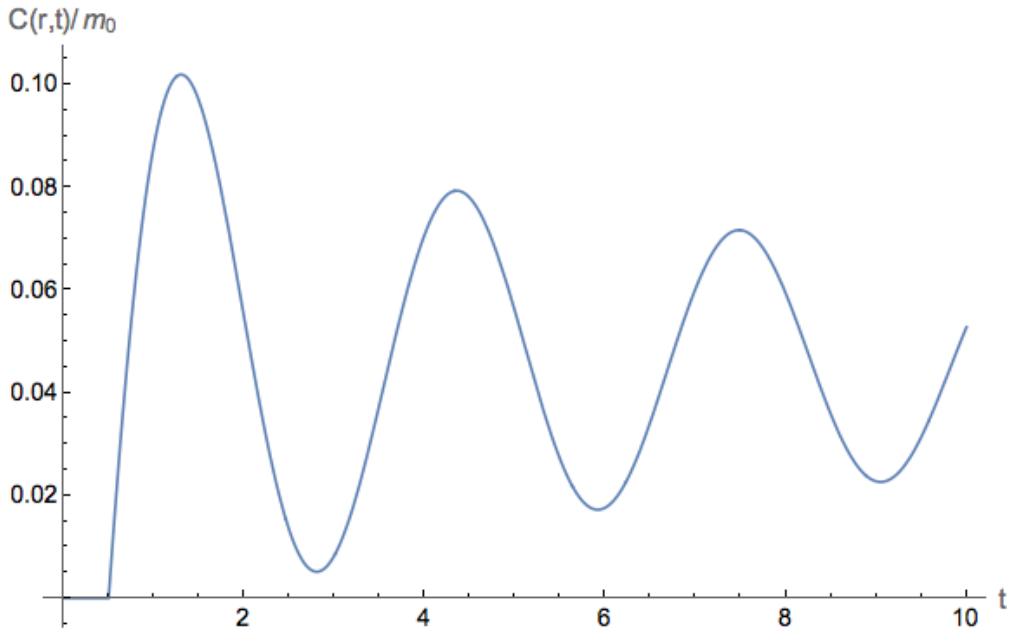

Figure 2.3: Solution of $C_\phi(t, r)$ at $r = 1$ for $m = 1$.

Moreover, we sketch the full solution of the correlation function for $0 \leq r \leq 10$ and $0 \leq t \leq 10$ as can be seen in the next figure.

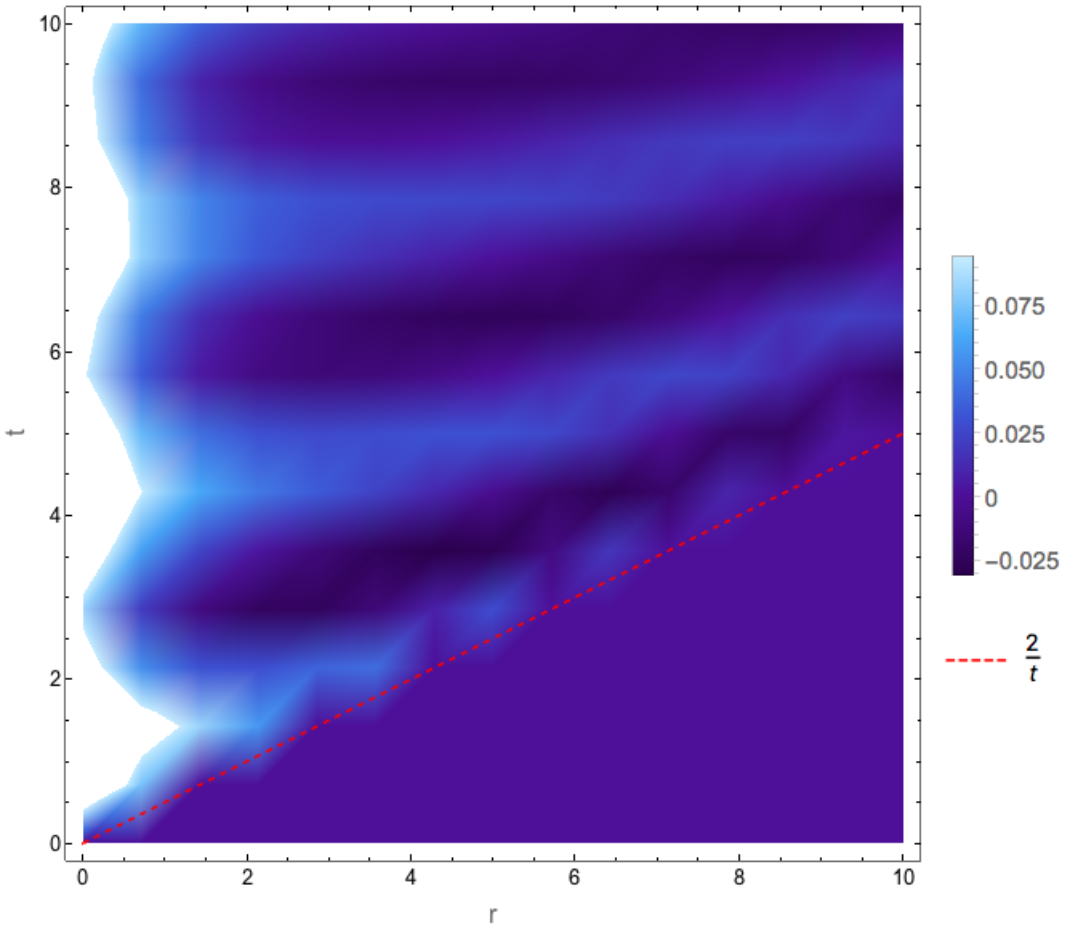

Figure 2.4: Contour plot of $C_\phi(t, r)$ for $m = 1$. The red line $r = 2t$ corresponds to the horizon. The colored values correspond to the value of the correlation between the origin and a given point at distance r at time t.

where the red line corresponds to $r = 2t$. From the solution we can see some interesting features. It is clear from figure 2.4 that there is a horizon at $r = 2t$. By taking a point at a distance r we have to wait a time $t = r/2$ in order to become correlated with the origin. Before this time there is no correlation between them. This feature, which is a direct consequence of the causality principle, is the light cone effect where first was introduced by Calabrese and Cardy [7], [6]. The physical interpretation goes as follows: at time $t = 0$, the initial state $|\Psi_0\rangle$ is a superposition of the energy eigenstates of the new Hamiltonian after the quench. Hence, the initial stete acts as

a source of quasi-particles. Those quasi-particles are entangled and they are generated at every point in space with equal and opposite momentum moving ballistically. If the quasi-particle dispersion relation is $E(p)$, they move with velocity $v(p) = dE/dp$. Thus, if a pair of quasi-particle produced in position x, then after a time t the right moving quasi-particle will be at position $x_1 = x + v(p)t$ and the other left moving at $x_2 = x - v(p)t$, and the total distance between them will be $|x_1 - x_2| = 2v(p)t$. Because those quasi-particles are entangled the two point x_1 and x_2 will become correlated at time t. The above procedure is represented in the following figure.

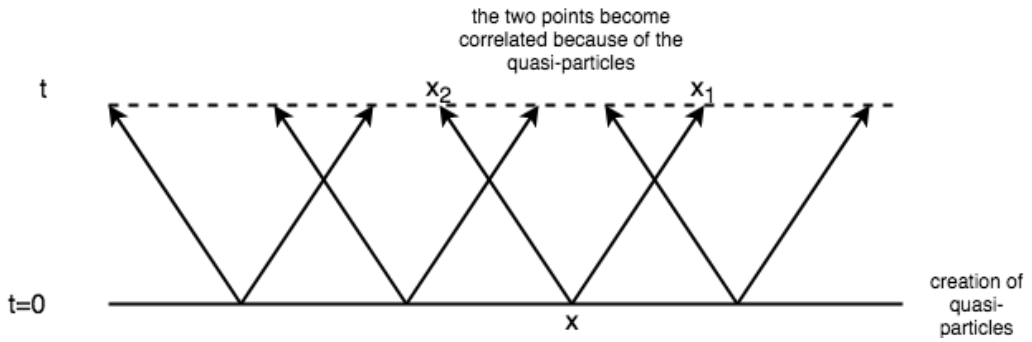

Figure 2.5: Schematic view of correlations induced by quasi-paricles. At $t = 0$ we have creation of quasi-particle at every point moving with equal and opposite momentum. After some time t, a pair of quasi-particles, created at a point x, arrive at the points x_1 and x_2 and the two points become correlated.

2.4 Quench protocols

So far we mentioned only a sudden change in the systems parameter, but in general we can have a quantum quench that has some finite duration time, say τ. Then, the evolution of the system in time breaks into two parts. First, we determine the evolution of the system during the quench, $t < \tau$ and then the evolution after the quench, $t > \tau$. This kind of problem we will consider in Chapter 5. Depending on how the systems parameter changes during the quench, the system will have different time evolution. This is called a quench protocol. There are various studied protocols in different models [16], the simplest one being the linear one [17]. In the figure below we can see different kinds of quench protocols of finite duration τ, over which the parameter g varies.

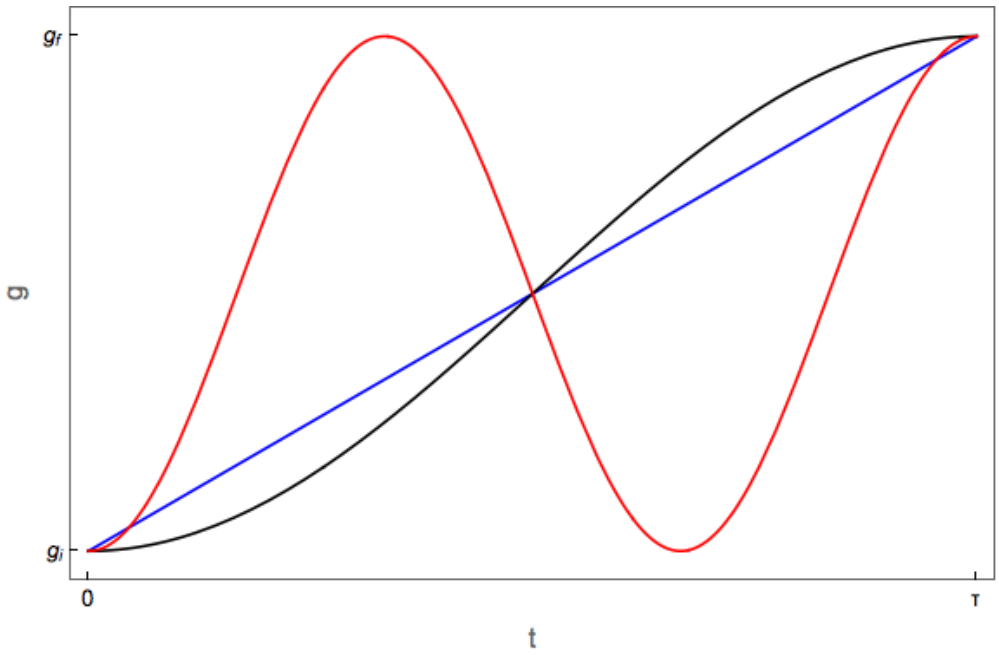

Figure 2.6: Different quench protocols for time duration τ. The solid blue line corresponds to a linear quench, the black line to cosine quench while the red to periodic quench.

Moreover, we can have a protocol of adiabatic quench, where the quench time $\tau \to \infty$. In this case the state of the system after the quench is again the ground state of the new Hamiltonian. In this thesis we will investigate periodic quenches of finite duration time τ and also we will examine the case where we periodically drive the system without stopping, meaning that there is no quench.

Chapter 3

Parametric oscillator

In this chapter we introduce the parametric oscillator [18]. Essentially a parametric oscillator is an oscillator where an intrinsic parameter is time-dependent. This is important because later we will consider a system where we periodically drive it and the same equation of motion will arise.

3.1 Mathieu equation

To illustrate the idea of parametric resonance we consider a one dimensional pendulum, without damping, of length l with point mass m in a gravitational field g. The equation of motion of the pendulum, for small oscillations around the origin ($\theta \ll 1$), is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{3.1}$$

Now imagine that we sinusoidally oscillate vertically the suspension point of the pendulum such that $y_0(t) = -A\cos(\Omega t)$, with A being the amplitude, Ω the driving frequency and y_0 denotes the position of the suspension point (figure 3.1).

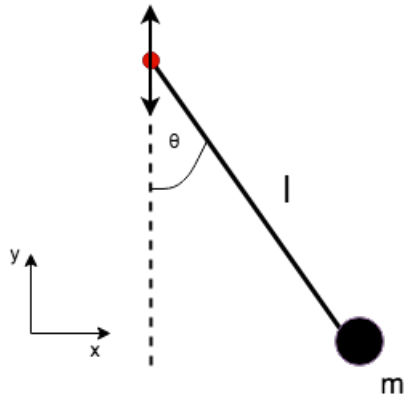

Figure 3.1: Pendulum with oscillating suspension point.

Using Langrangian formalism we can derive the equation of motion of this system. Performing our analysis in the $x-y$ plane, the position of the mass m, at time t, is given by:

$$x(t) = l\sin\theta(t),\ y(t) = y_0(t) - l\cos\theta(t)$$

and the corresponding velocities are:

$$\dot{x}(t) = l\dot{\theta}\cos\theta,\ \dot{y}(t) = \dot{y}_0(t) + l\dot{\theta}\sin\theta$$

The Lagrangian of the system is

$$\begin{aligned} L = T - V &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \\ &= \frac{1}{2}m\left(l^2\dot{\theta}^2\cos^2\theta + (\dot{y}_0 + l\dot{\theta}\sin\theta)^2\right) - mgy_0 + mlg\cos\theta \\ &= \frac{ml^2\dot{\theta}^2}{2} + ml\dot{\theta}\dot{y}_0\sin\theta + mlg\cos\theta + \frac{m\dot{y}_0^2}{2} - mgy_0 \end{aligned} \tag{3.2}$$

and for the equation of motion we use Euler-Lagrange equation

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \tag{3.3}$$

with

$$\frac{\partial L}{\partial\theta} = ml\dot{\theta}\dot{y}_0\cos\theta - mlg\sin\theta,\ \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + ml\dot{y}_0\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + ml\ddot{y}_0 \sin\theta + ml\dot{y}_0\dot{\theta}\cos\theta$$

From (3.3) we get

$$\ddot{\theta} + \left(\frac{\ddot{y}_0 + g}{l}\right)\sin\theta = 0 \tag{3.4}$$

By considering small oscillations near $\theta = 0$ we can linearize the above equation using $\sin\theta \approx \theta$, thus

$$\ddot{\theta} + \left(\frac{\ddot{y}_0 + g}{l}\right)\theta = 0 \tag{3.5}$$

substituting $\ddot{y}_0 = A\Omega^2\cos(\Omega t)$ we find

$$\ddot{\theta} + \omega_0^2[1 + h\cos(\Omega t)]\theta = 0 \tag{3.6}$$

with $\omega_0^2 = g/l$ and $h = \frac{A}{g}\Omega^2$. If we define $g(t) = \ddot{y}_0 + g$ we see that equation (3.5) is the same as (3.1) but with a time-dependent gravitational field. Hence, the problem that we have considered is equivalent to have the pendulum in a time-varying gravitational field. This is called a "parametric" pendulum because the motion depends on a time-dependent parameter. Equation (3.6) describes a driven harmonic oscillator but the driving force depends on the amplitude θ and is called Matheiu equation. Moreover, the driving force has a period given by $T = \frac{2\pi}{\Omega}$ whereas the natural period of the oscillator is $T_0 = \frac{2\pi}{\omega_0}$. An interesting question to ask is about the stability of this system, this will be investigated in the last section of this chapter.

Another useful form of the above equation can be obtained by changing the time variable to $\tau = \frac{\Omega t}{2}$ in order to make the equation dimensionless. With this transformation we have

$$\tau = \frac{\Omega t}{2} \rightarrow d\tau = \frac{\Omega}{2}dt$$

$$\frac{\Omega^2}{4}\ddot{\theta} + \frac{g + A\Omega^2\cos(2\tau)}{l}\theta = 0$$

and by defining two new parameters ϵ and h as

$$\epsilon = \left(\frac{2\omega_0}{\Omega}\right)^2,\ h = \frac{A\Omega^2}{g}$$

we finally obtain,

$$\ddot{\theta} + \epsilon[1 + h\cos(2\tau)]\theta = 0 \tag{3.7}$$

To investigate the stability of this system, we would like to know the form of the solutions it has. This is done in the next section where we consider Floquet theory [19].

3.2 Brief introduction to Floquet theory

We consider a first-order linear ordinary differential equation having periodic coefficient, which has the general form

$$\dot{\theta}(t) = A(t)\theta(t) \tag{3.8}$$

where A is periodic, $A(t+T) = A(t)$. We will make use of the following two observations:

- The coefficients are periodic, with period T, so if $\theta(t)$ is a solution of the differential equation so is $\theta(t+T)$
- The differential equation is linear and second order, so any solution $\theta(t)$ can be written as a liner combination of two linearly independent solutions $\theta_1(t)$ and $\theta_2(t)$:

  $$\theta(t) = A\theta_1(t) + B\theta_2(t) \tag{3.9}$$

  where A, B are determined from the initial conditions.

From periodicity, $\theta_1(t+T)$ and $\theta_2(t+T)$ are also solutions which we can write them in terms of $\theta_1(t)$ as and $\theta_2(t)$

$$\theta_1(t+T) = \alpha\theta_1(t) + \beta\theta_2(t),\ \theta_2(t+T) = \gamma\theta_1(t) + \delta\theta_2(t)$$

Hence, from (3.9) we have

$$\begin{aligned}
&\theta(t+T) = A\theta_1(t+T) + B\theta_2(t+T) \Rightarrow \\
&\theta(t+T) = A(\alpha\theta_1(t) + \beta\theta_2(t)) + B(\gamma\theta_1(t) + \delta\theta_2(t)) \Rightarrow \\
&\theta(t+T) = (A\alpha + B\gamma)\theta_1(t) + (A\beta + B\delta)\theta_2(t) \Rightarrow \\
&\theta(t+T) = A'\theta_1(t) + B'\theta_2(t)
\end{aligned} \tag{3.10}$$

where

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Choosing the vector $\begin{bmatrix} A \\ B \end{bmatrix}$ to be an eigenvector of the 2×2 matrix with eigenvalue λ, then $A' = \lambda A$ and $B' = \lambda B$ and

$$\theta(t+T) = \lambda\theta(t) \tag{3.11}$$

we observe that $\theta(t)$ is periodic with a scale factor λ. Depending on the magnitude of λ the solution may be stable or unstable. For stable solution we must have $|\lambda| < 1$, whereas for unstable $|\lambda| > 1$. Introducing

$$\mu = \frac{\ln|\lambda|}{T} \rightarrow \lambda = e^{\mu T}$$

then for all t, $\theta(t)$ takes the form

$$\theta(t) = e^{\mu T}\phi(t)$$

where $\phi(t)$ is a periodic function $\phi(t+T) = \phi(t)$ and stability depends on the sign of μ. Thus we found that the solutions of Mathieu equation have the form of an exponential factor multiplied with a periodic function.

3.3 Stability of the parametric pendulum

We now determine the conditions for parametric resonance to occur. The Mathieu equation is

$$\ddot{\theta} + \omega_0^2[1 + h\cos(\Omega t)]\theta = 0 \tag{3.12}$$

We take the driving frequency, Ω, to be close to the natural frequency of the oscillator $\Omega = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$, and we assume that $h \ll 1$. Then we write

$$\ddot{\theta} + \omega_0^2[1 + h\cos(2\omega_0 + \epsilon)t]\theta = 0 \tag{3.13}$$

and we seek a solution of the form

$$\theta(t) = a(t)\cos\left[\left(\omega_0 + \frac{1}{2}\epsilon\right)t\right] + b(t)\sin\left[\left(\omega_0 + \frac{1}{2}\epsilon\right)t\right] \tag{3.14}$$

where $a(t)$ and $b(t)$ are functions of time which they change slowly in comparison with the trigonometric functions. The stability of the solution will depend on whether the functions $a(t)$ and $b(t)$ grow or decay exponentially. We proceed by substituting equation (3.14) in (3.13) and we keep terms linear in h and ϵ.

$$\begin{aligned}\ddot{\theta} =&\ddot{a}(t)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]-2\dot{a}(t)\left(\omega_0+\frac{1}{2}\epsilon\right)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\\&-a(t)\left(\omega_0+\frac{1}{2}\epsilon\right)^2\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]+\ddot{b}(t)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\\&+2\dot{b}(t)\left(\omega_0+\frac{1}{2}\epsilon\right)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\\&-b(t)\left(\omega_0+\frac{1}{2}\epsilon\right)^2\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\end{aligned}$$

neglecting ϵ^2 terms and also dropping the $\ddot{a}(t)$ and $\ddot{b}(t)$ terms, because those functions are slowly varying, we arrive at

$$\begin{aligned}\ddot{\theta} =&-2\dot{a}(t)\left(\omega_0+\frac{1}{2}\epsilon\right)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\\&-a(t)\left(\omega_0^2+\omega_0\epsilon\right)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\\&+2\dot{b}(t)\left(\omega_0+\frac{1}{2}\epsilon\right)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\\&-b(t)\left(\omega_0^2+\omega_0\epsilon\right)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\end{aligned}$$

The above expression must be equal with

$$-\omega_0^2[1+h\cos\left[(2\omega_0+\epsilon)\,t\right]]\left[a(t)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]+b(t)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]\right]$$

using the trigonometric identities

$$\cos a\cos b=\frac{1}{2}[\cos\left(a+b\right)+\cos\left(a-b\right)],\ \sin a\sin b=\frac{1}{2}[\sin\left(a+b\right)+\sin\left(a-b\right)]$$

we write

$$\cos\left(\omega_0+\frac{1}{2}\epsilon\right)\cos\left(2\omega_0+\epsilon\right)=\frac{1}{2}\cos\left[3\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]+\frac{1}{2}\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]$$

and a similar expression fo the sin. The term with frequency $3\left(\omega_0+\frac{1}{2}\epsilon\right)t$ is higher order with respect to h so we drop it. Hence we have

$$\begin{aligned}
\ddot{\theta} &= -2a\dot{(t)}\left(\omega_0+\frac{1}{2}\epsilon\right)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] \\
&\quad - a(t)\left(\omega_0^2+\omega_0\epsilon\right)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] \\
&\quad + 2b\dot{(t)}\left(\omega_0+\frac{1}{2}\epsilon\right)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] \\
&\quad - b(t)\left(\omega_0^2+\omega_0\epsilon\right)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] \\
&= -\omega_0^2 a(t)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] - \omega_0^2 b(t)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] \\
&\quad - \omega_0^2\frac{1}{2}ha(t)\cos\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right] + \omega_0^2\frac{1}{2}hb(t)\sin\left[\left(\omega_0+\frac{1}{2}\epsilon\right)t\right]
\end{aligned}$$

canceling the zero order terms and collecting the terms $\sin\left(\omega_0+\frac{1}{2}\epsilon\right)t$ and $\cos\left(\omega_0+\frac{1}{2}\epsilon\right)t$ we get

$$\begin{aligned}
&\left(-2\dot{a}+b\epsilon+\frac{1}{2}h\omega_0 b\right)\omega_0\sin\left(\omega_0+\frac{1}{2}\epsilon\right) \\
&+\left(2\dot{b}-b\epsilon+\frac{1}{2}h\omega_0 b\right)\omega_0\cos\left(\omega_0+\frac{1}{2}\epsilon\right)=0
\end{aligned}$$

For this expression to be valid for all t, the coefficients of both cos and sin must equal zero, thus

$$\begin{aligned}
-2\dot{a}+b\epsilon+\frac{1}{2}h\omega_0 b &= 0 \\
2\dot{b}-b\epsilon+\frac{1}{2}h\omega_0 a &= 0
\end{aligned} \tag{3.15}$$

which a system of first order differential equations. We want solutions of the form $a(t)=ae^{\mu t}$ and $a(t)=be^{\mu t}$ and putting them in (3.15)

$$\begin{aligned}
\mu a+\frac{1}{2}b\left(\epsilon+\frac{1}{2}h\omega_0\right) &= 0 \\
\frac{1}{2}a\left(\epsilon-\frac{1}{2}h\omega_0\right)-\mu b &= 0
\end{aligned}$$

Which can be written in matrix form as

$$\mathbf{A}\mathbf{x} = 0 \tag{3.16}$$

with

$$\mathbf{A} = \begin{bmatrix} \mu & \frac{1}{2}\left(\epsilon + \frac{1}{2}h\omega_0\right) \\ \frac{1}{2}\left(\epsilon - \frac{1}{2}h\omega_0\right) & -\mu \end{bmatrix} \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

In order to have a solution the determinant of the matrix $\mathbf{A}$ must be zero, so

$$det(\mathbf{A}) = 0 \Rightarrow \mu^2 = \frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2 - \epsilon^2\right]$$

To have instability μ must be real and positive, because then the function $a(t)$ and $b(t)$ grow exponentially. Consequently this will occur when

$$-\frac{1}{2}h\omega_0 < \epsilon < \frac{1}{2}h\omega_0 \tag{3.17}$$

We deduce that a resonance occurs provided that the driving frequency is sufficiently close to twice the natural frequency with the range being proportional to the natural frequency itself. For larger amplitudes, the range of frequencies that we can have resonance grows. This features are summarized in the next phase diagram.

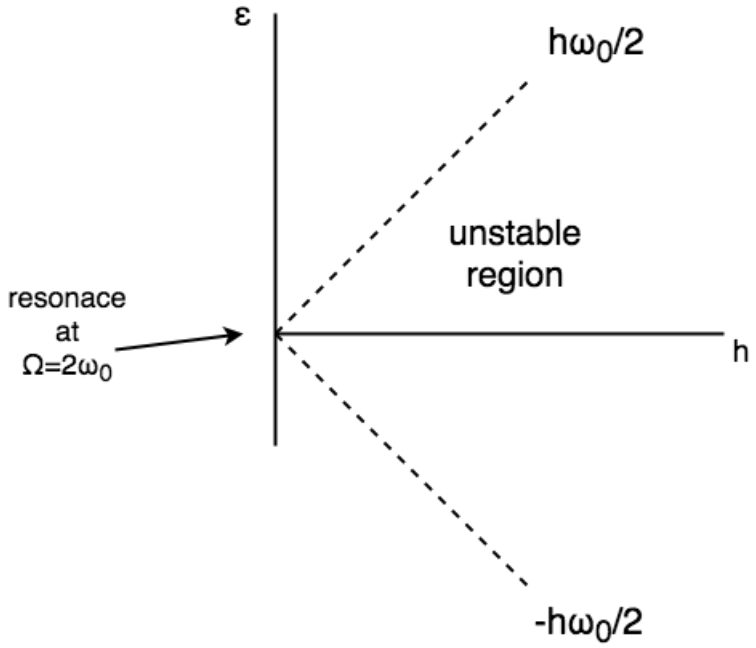

Figure 3.2: Phase diagram for parametric resonance. In the diagram are shown the regions where the oscillator becomes unstable.

Chapter 4

Luttinger liquids

In this chapter we discuss Luttinger liquids. First, we introduce the Tomonaga model for interacting fermions on a ring and then we move to Luttinger-Tomonaga model. The main result of this chapter is that a fermionic interacting problem can be mapped to a bosonic one. This is important because we can express fermionic field operators in terms of bosonic operators and calculate correlation functions. We present the main features of the model by following [20]. More on Luttinger liquids can be found on [1], [21]. In our analysis we use $\hbar = 1$.

4.1 Tomonaga model

Tomonaga [22] studied spinless fermions on a ring. He considered that the fermions are non-relativistic and worked in the high density limit, where the spatial range of the two-body interaction is much larger than the interparticle distance. The Hamiltonian of interacting fermions is given by

$$\hat{H} = \hat{T} + \hat{V} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} V(\hat{x}_i - \hat{x}_j) \tag{4.1}$$

and in second quantization [23], using position representation, takes the form

$$\begin{aligned} H = \int dx \hat{\psi}^\dagger(x) \left(-\frac{1}{2m} \frac{\partial^2}{\partial x^2} \right) \hat{\psi}(x) \\ + \frac{1}{2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') V(x - x') \hat{\psi}(x') \hat{\psi}(x) \end{aligned} \tag{4.2}$$

where $\hat{\psi}(x)$ and $\hat{\psi}^\dagger(x)$ are field operators, which destroy and create a particle at position x respectively and satisfy the canonical anticommutation relations

$$\{\hat{\psi}(x), \hat{\psi}(x')\} = 0,\ \{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x')\} = 0,\ \{\hat{\psi}(x), \hat{\psi}^\dagger(x')\} = \delta(x - x') \quad (4.3)$$

for $x, y \in [-L/2, L/2]$. Using the Fourier transform of the field $\hat{\psi}(x)$ in the momentum basis

$$\hat{\psi}(x) = \frac{1}{\sqrt{L}} \sum_n e^{ik_n x} \hat{c}_n \quad (4.4)$$

with k_n being quantized, due to the periodic boundary conditions, and equal to $k_n = 2\pi n/L$ with $n \in \mathbb{Z}$, $\hat{c}_n$ is the annihilation operator in momentum space and using a similar expression for $\hat{\psi}^\dagger(x)$ we can express the Hamiltonian in the momentum representation. The operators $\hat{c}_n$, $\hat{c}_n^\dagger$ obey the canonical anticommutation relations

$$\{\hat{c}_m, \hat{c}_n\} = 0,\ \{\hat{c^\dagger}_m, \hat{c^\dagger}_n\} = 0,\ \{\hat{c}_m, \hat{c}_n^\dagger\} = \delta_{m,n} \quad (4.5)$$

Next we expand the potential $V(x - x')$ in Fourier series

$$V(x - x') = \frac{1}{L} \sum_n e^{ik_n(x-x')} v(k_n) \quad (4.6)$$

with the inverse being

$$v(k_n) = \int_{-L/2}^{L/2} V(x) e^{-ik_n x} dx \quad (4.7)$$

If $V(0)$ exist, we can write $\hat{V}$ in terms of the particle density operator defined as

$$\hat{\rho}(x) = \sum_{i=1}^{N} \delta(x - \hat{x}_i) = \hat{\psi}^\dagger(x)\hat{\psi}(x) \quad (4.8)$$

where $\hat{x}_i$ is the position operator of the i-th particle. Then, from (4.3) we have

$$\{\hat{\psi}(x), \hat{\psi}(x')\} = 0 \Rightarrow \hat{\psi}(x)\hat{\psi}(x') = \hat{\psi}(x')\hat{\psi}(x)$$

$$\{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x')\} = 0 \Rightarrow \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x') = \hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x)$$

$$\{\hat{\psi}(x), \hat{\psi}^\dagger(x')\} = \delta(x - x') \Rightarrow \hat{\psi}^\dagger(x')\hat{\psi}(x) = \hat{\psi}(x)\hat{\psi}^\dagger(x') - \delta(x - x')$$

and $\hat{V}$ can be written as

$$\begin{aligned}
\hat{V} &= \frac{1}{2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x')V(x-x')\hat{\psi}(x')\hat{\psi}(x) \\
&= \frac{1}{2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' V(x-x')\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x')\hat{\psi}(x)\hat{\psi}(x') \\
&= \frac{1}{2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' V(x-x')\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}^\dagger(x')\hat{\psi}(x') \\
&\quad - \frac{1}{2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' V(x-x')\hat{\psi}^\dagger(x)\delta(x-x')\hat{\psi}(x')
\end{aligned}$$

Hence,

$$\hat{V} = \frac{1}{2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' V(x-x')\hat{\rho}(x)\hat{\rho}(x') - \frac{1}{2}V(0)\mathcal{N} \tag{4.9}$$

where $\mathcal{N} \equiv \int_{-L/2}^{L/2} dx\hat{\rho}(x)$ is the particle number operator. The Fourier decomposition of density operators is

$$\hat{\rho}(x) = \frac{1}{L}\sum_n \hat{\rho}_n e^{ik_n x} \tag{4.10}$$

with inverse

$$\hat{\rho}_n = \int_{-L/2}^{L/2} dx\hat{\rho}(x)e^{-ik_n x} \tag{4.11}$$

We can get an expression of $\hat{\rho}_n$ from $\hat{\rho}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$ using (4.4) as follows

$$\hat{\psi}^\dagger(x)\hat{\psi}(x) = \frac{1}{L}\sum_{n,m} e^{ix(k_m-k_n)}\hat{c}_n^\dagger\hat{c}_m$$

Hence,

$$\begin{aligned}
\hat{\rho}_l &= \frac{1}{L}\int_{-L/2}^{L/2} dx \sum_{n,m} e^{ix(k_m-k_n)}\hat{c}_n^\dagger\hat{c}_m e^{-ik_l x} \\
&= \frac{1}{L}\sum_{n,m}\int_{-L/2}^{L/2} dx e^{ix(k_m-k_n-k_l)}\hat{c}_n^\dagger\hat{c}_m \\
&= \sum_{n,m}\delta_{m,n+l}\hat{c}_n^\dagger\hat{c}_m = \sum_n \hat{c}_n^\dagger\hat{c}_{n+l}
\end{aligned}$$

After relabelling indices,

$$\hat{\rho}_n = \sum_m \hat{c}_m^\dagger \hat{c}_{m+n} \tag{4.12}$$

where we used

$$\int_{-L/2}^{L/2} dx e^{ix(k_m - k_n - k_l)} = L\delta_{m,n+l} \tag{4.13}$$

Moreover, it is easy to show that the following commutation relation holds

$$[\hat{\rho}_n, \hat{\rho}_m] = 0 \tag{4.14}$$

The operator (4.12) given an n, destroys a particle from the state $m+n$ and create it in the state m, meaning that creates particle-hole pairs (figure 4.1).

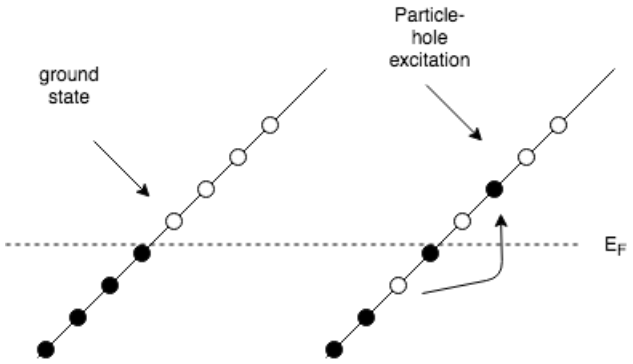

Figure 4.1: Schematic view of particle-hole excitations.

For $n = 0$ it counts the total number of particles in the system. Now we can rewrite the interaction $\hat{V}$ in terms of $\hat{\rho}_n$. By substitution of (4.10) in (4.9) and using (4.13) we obtain

$$\hat{V} = \frac{1}{2}\frac{1}{L}\sum_n v(k_n)\hat{\rho}_n\hat{\rho}_{-n} - \frac{1}{2}V(0)\mathcal{N} \tag{4.15}$$

$V(x)$ being real implies

$$V(x) = V(x)^* \Rightarrow v(k_n) = v(-k_n) \tag{4.16}$$

thus we have, by splitting the summations in (4.15)

$$\hat{V} = \frac{1}{2}\frac{1}{L}\sum_{n>0} v(k_n)\hat{\rho}_n\hat{\rho}_{-n} + \frac{1}{2}\frac{1}{L}\sum_{n<0} v(k_n)\hat{\rho}_n\hat{\rho}_{-n} + \frac{1}{2}\frac{1}{L}v(0)\hat{\rho}_0\hat{\rho}_0 - \frac{1}{2}V(0)\mathcal{N} \tag{4.17}$$

Relabelling the second term $n \to -n$ and with the help of (4.16) and (4.14) becomes equal with the first term

$$\frac{1}{2}\frac{1}{L}\sum_{n>0} v(-k_n)\hat{\rho}_{-n}\hat{\rho}_n = \frac{1}{2}\frac{1}{L}\sum_{n>0} v(k_n)\hat{\rho}_n\hat{\rho}_{-n}$$

so,

$$\hat{V} = \frac{1}{L}\sum_{n>0} v(k_n)\hat{\rho}_n\hat{\rho}_{-n} + \frac{1}{2L}v(0)\mathcal{N}^2 - \frac{1}{2}V(0)\mathcal{N} \tag{4.18}$$

We consider a momentum cut-off k_c, where above it the Fourier transform $v(k)$, of the two-body potential $V(x)$, decays exponentially. In figure 4.2 there are three examples of possible potentials $v(k)$

$$v_{\text{box}}(k) = \begin{cases} v & \text{for } |k| \le k_c \\ 0 & \text{for } |k| > k_c \end{cases} \quad , \quad v_{\text{gauss}}(k) = ve^{-(k/k_c)^2/2} \quad , \quad v_{\text{exp}}(k) = ve^{-|k|/k_c}$$

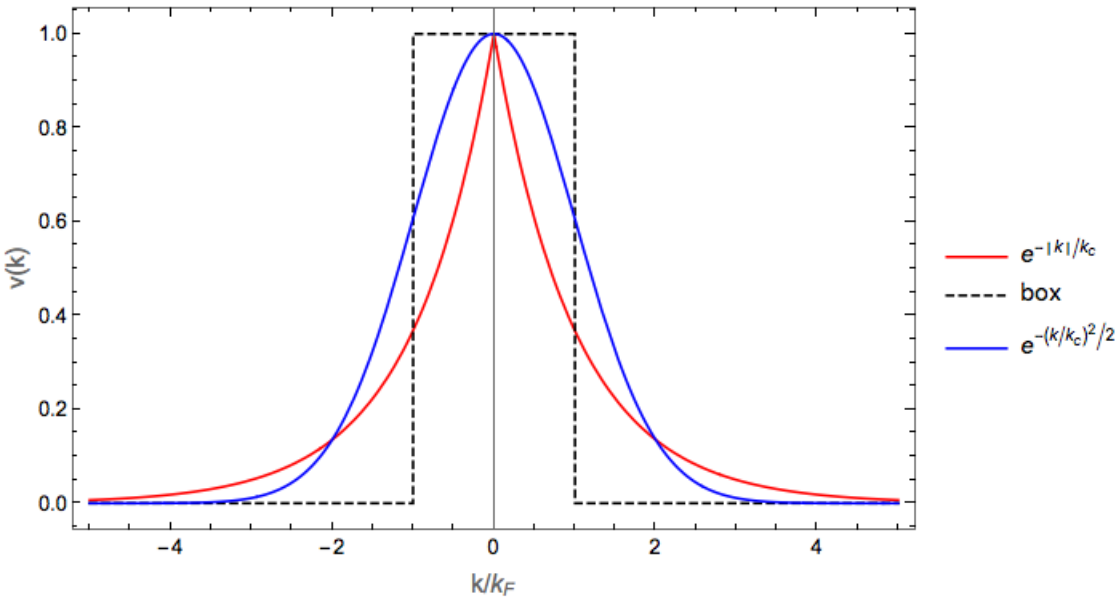

Figure 4.2: Example of potentials $v(k)$ with $v = 1$.

By doing this essentially we allow only low-energy interactions an thus we consider only the low-energy properties of the system. We can now use first order perturbation theory [2] in order to find an approximate expression for

the ground state of the interacting system. This will be helpful to understand the physics. The ground state of N (odd) non-interacting fermions on a ring of length L, is given by the Fermi state $|F\rangle$, where the electrons occupy momentum states from $-k_F$ to k_F, with $k_F = 2\pi n_F/L$ and $n_F = (N-1)/2$, where k_F is the Fermi momentum (fig with momentum states). Using first order perturbation theory in $\hat{V}$ we can write for the interacting ground state $|\Psi\rangle$

$$|\Psi\rangle = |F\rangle + \frac{\hat{Q}}{E^{(0)} - \hat{T}} \hat{V} |F\rangle + \dots \tag{4.19}$$

where $\hat{Q} = \hat{I} - |F\rangle \langle F|$ is a projector on the subspace of excited states of the system and $E^{(0)}$ is the total kinetic energy of the Fermi state. Because, when $\hat{V}$ acts to $|F\rangle$ gives a state which is orthogonal to $|F\rangle$ and two states proportional to $|F\rangle$ from

$$\hat{Q}\hat{V} |F\rangle = \hat{V} |F\rangle - \langle F| \hat{V} |F\rangle |F\rangle$$

remains only

$$\hat{Q}\hat{V} |F\rangle = \frac{1}{L} \sum_{n>0} v(k_n) \hat{\rho}_n \hat{\rho}_{-n} |F\rangle \tag{4.20}$$

The state $\hat{\rho}_n \hat{\rho}_{-n} |F\rangle$, using the physical interpretation of $\hat{\rho}_n$ as stated above, is an excited state of the system with two particles and two holes. From our assumption that $v(k_n)$ is zero for $k_n \gg k_c$, only states $\hat{\rho}_n \hat{\rho}_{-n} |F\rangle$ with n of the order $n_c = k_c L/2\pi$ or smaller contribute to the above expression. Hence, only low energy excitations are important. The density operator $\hat{\rho}_{-n}$, for $n_c > n > 0$ creates a particle-hole pair around the right Fermi point $k = k_F$ whereas $\hat{\rho}_n$ creates a particle-hole pair around the left Fermi point $k = -k_F$. We can rewrite (4.20) by considering low-energy excitations as

$$\hat{Q}\hat{V} |F\rangle = \frac{1}{L} \sum_{n>0} v(k_n) \sum_{m=-n_F-n}^{-n_F+1} \sum_{l=n_F+1}^{n_F+n} \hat{c^\dagger}_m \hat{c}_{m+n} \hat{c}^\dagger_l \hat{c}_{l-n} |F\rangle \tag{4.21}$$

To find the interacting ground state $|\Psi\rangle$ we need also the kinetic energy of the two-particle two-holes state $\hat{c^\dagger}_m \hat{c}_{m+n} \hat{c}^\dagger_l \hat{c}_{l-n} |F\rangle$, which equals $E^{(0)} + \epsilon_m - \epsilon_{m+n} + \epsilon_l - \epsilon_{l-n}$, where $\epsilon_n = (2\pi n/L)^2/2m$. The excitations of the system are in the neighbourhood of the two Fermi points and by linearizing the parabolic dispersion relation around those points we get two linear branches, one for left moving $(-)$ fermions and one for right moving $(+)$ (figure 4.3).

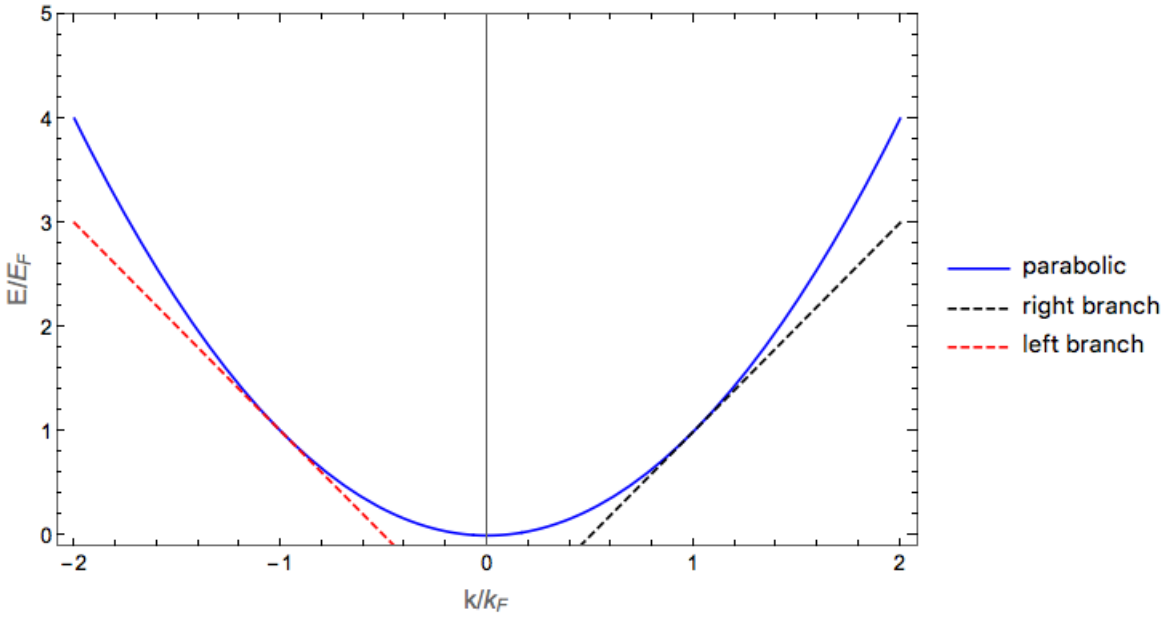

Figure 4.3: Linearization around the two Fermi points.

By doing this we have

$$\epsilon_{n,+} = \epsilon_F + v_F(n - n_F)2\pi/L,\ n > 0$$
$$\epsilon_{n,+} = \epsilon_F - v_F(n + n_F)2\pi/L,\ n < 0$$

where $\epsilon_F = k_F^2/2m$ is the Fermi energy and $v_F = k_F/m$ the Fermi velocity. Now the calculations of energies $E^{(0)}+\epsilon_m-\epsilon_{m+n}+\epsilon_l-\epsilon_{l-n}$ are straightforward and equal to $E^{(0)} + 2k_n v_F$ and the ground state $|\Psi\rangle$, up to first order, takes the form

$$|\Psi\rangle = |F\rangle - \frac{1}{L}\sum_{n>0}\frac{v(k_n)}{2k_n v_F}\sum_{m=-n_F-n}^{-n_F+1}\sum_{l=n_F+1}^{n_F+n}\hat{c^\dagger}_m\hat{c}_{m+n}\hat{c}_l^\dagger\hat{c}_{l-n}\,|F\rangle + \dots \qquad (4.22)$$

with dots denoting higher order terms which are multiple particle-hole pairs states. The analysis that we did shows that there are three types of electrons. Right moving electrons near the Fermi point $k \approx k_F$ with velocities approximately v_F, left moving electrons near the Fermi point $k \approx -k_F$ with velocities approximately $-v_F$ and the electrons deep in the Fermi sea, which do not contribute because we consider only low energy excitations. In the next section we will consider the Tomonaga-Luttinger model which is an extension of Tomonaga's model.

4.2 Tomonaga - Luttinger model (TLM)

Luttinger used the basic properties of the Tomonaga model to further modify it. He consider linear dispersion around the Fermi points but he extended the range of k-values for both the right and left movers to $2\pi n/L$ with n positive and negative integer. Essentially instead of the parabolic dispersion, in Luttinger model we have a Dirac cone (figure 4.4).

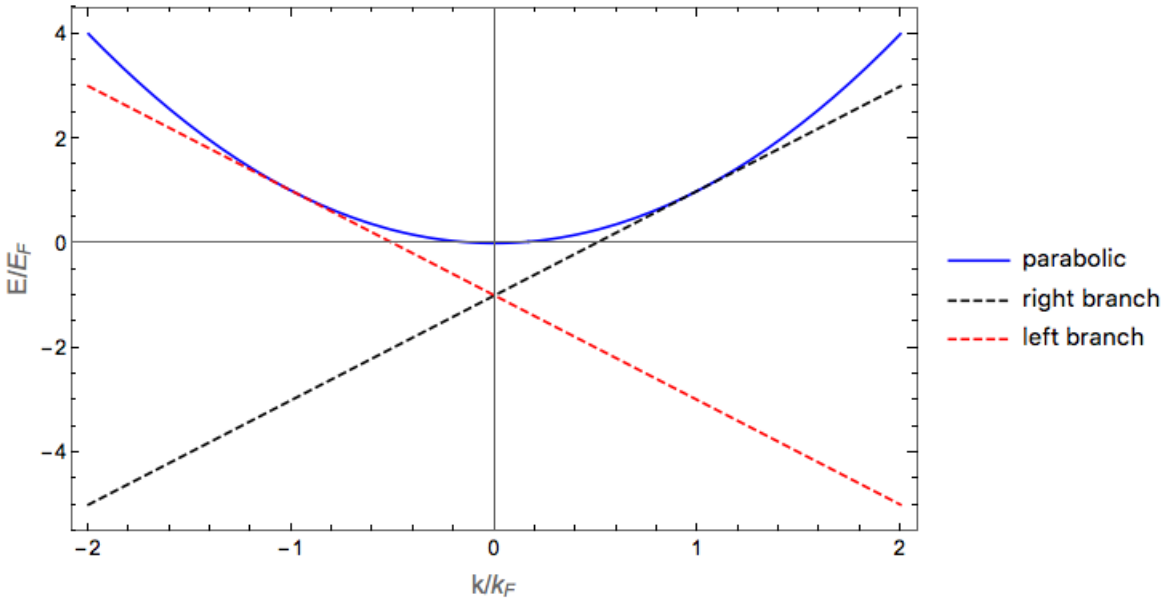

Figure 4.4: linear dispersion of Luttinger model.

This assumption simplifies the dispersion of the electrons, but introduces infinitely many negative energy electron states ("Dirac sea"), which have to be filled in the ground state. To proceed our analysis, we will consider finite instead of infinite negative energy electron states by introducing a band cut-off, $k_B = 2\pi m_0/L \geq 0$. Thus, we consider states with $k \geq -k_B$ for right movers and $k \leq k_B$ for left movers. This is done to avoid mathematical subtleties. Then Luttinger model corresponds to $k_B \to \infty$. The kinetic energy operator now takes the form

$$\hat{T}_{TL} = \sum_{n \in I_+} v_F k_n \hat{c}^\dagger_{n,+} \hat{c}_{n,+} + \sum_{n \in I_-} (-v_F k_n) \hat{c}^\dagger_{n,-} \hat{c}_{n,-} \tag{4.23}$$

where $I_+ = \{-m_0, -m_0+1, \ldots, \infty\}$ and $I_- = \{-\infty, \ldots, m_0-1, m_0\}$. The first term counts the kinetic energy in the right sector whereas the second in the left. We have two kinds of annihilation and creation operators, one for right movers (+) and one for left movers (−) which satisfy the following canonical anti-commutation relations

$$\{\hat{c}_{m,\alpha}, \hat{c}_{n,\beta}\} = 0, \ \{\hat{c}_{m,\alpha}, \hat{c}^\dagger_{n,\beta}\} = \delta_{n,m}\delta_{a,\beta} \tag{4.24}$$

where $\alpha, \beta = \pm$ denotes the right or left branch. We replace $\hat{\rho}_n\hat{\rho}_{-n}$ by $(\hat{\rho}_{n,+} + \hat{\rho}_{n,-})(\hat{\rho}_{-n,+} + \hat{\rho}_{-n,-})$ in the two-body interaction $\hat{V}$ (4.18), where

$$\hat{\rho}_{n,\alpha} \equiv \sum_m w^\alpha_{m,m+n}\hat{c}^\dagger_{m,\alpha}\hat{c}_{m+n,\alpha} \tag{4.25}$$

with

$$w^\alpha_{m,m+n} = \begin{cases} 1 & \text{for } m, m+n \in I_\alpha \\ 0 & \text{else} \end{cases} \tag{4.26}$$

are density operator for right and left moving electrons and $\hat{V}$ takes the form

$$\hat{V} = \frac{1}{L}\sum_{n>0} v(k_n)\left[(\hat{\rho}_{n,+}\hat{\rho}_{-n,+} + \hat{\rho}_{n,-}\hat{\rho}_{-n,-}) + (\hat{\rho}_{n,+}\hat{\rho}_{-n,-} + \hat{\rho}_{n,-}\hat{\rho}_{-n,+})\right] \tag{4.27}$$

where we dropped terms containing $\mathcal{N}$ because they are irrelevant to our considerations. Moreover, we have splitted the interaction potential in this way in order to distinguish the two kinds of interactions that we have. The first parenthesis corresponds to intra-branch scattering where the electrons stay on the same branch (figure 4.5), whereas the second term corresponds to inter-branch scattering where the electrons change branch (figure 4.6).

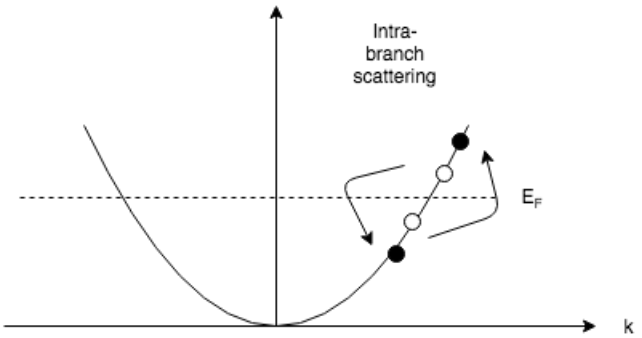

Figure 4.5: Schematic view of intra-branch interactions.

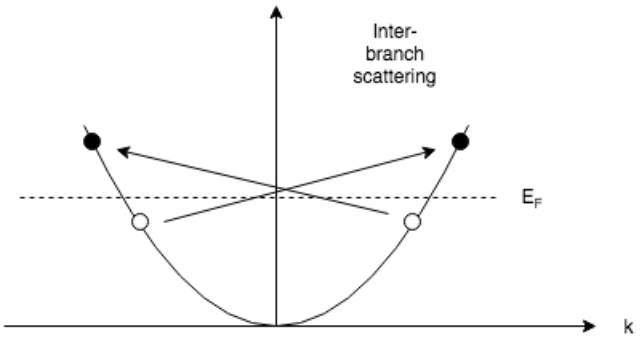

Figure 4.6: Schematic view of inter-branch interactions.

An important observation is that the commutation relations of $\hat{\rho}_{m,\alpha}$ are not all zero as was for $\hat{\rho}_m$. Using the anti-commutation relations (4.24), we have

$$[\hat{\rho}_{m,+}, \hat{\rho}_{n,-}] = 0 \tag{4.28}$$

and

$$\begin{aligned}
[\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\alpha}] &= \sum_{k,q} [\hat{c}^\dagger_{k,\alpha}\hat{c}_{k+m,\alpha}, \hat{c}^\dagger_{q,\alpha}\hat{c}_{q+n,\alpha}] w^\alpha_{k,k+m} w^\alpha_{q,q+n} \\
&= \sum_{k,q} \left(\delta_{q,k+m}\hat{c}^\dagger_{k,\alpha}\hat{c}_{q+n,\alpha} - \delta_{k,q+n}\hat{c}^\dagger_{q,\alpha}\hat{c}_{k+m,\alpha} \right) w^\alpha_{k,k+m} w^\alpha_{q,q+n} \\
&= \sum_{k} \left(w^\alpha_{k,k+m} - w^\alpha_{k,k+n} \right) w^\alpha_{k,k+m+n} \hat{c}^\dagger_{k,\alpha}\hat{c}_{k+m+n,\alpha}
\end{aligned} \tag{4.29}$$

From (4.29) we can see that the commutator is zero when m and n have the same sign. For example, by considering right movers ($\alpha = +$) and n, m positive, then $w^+_{k,k+m+n}$ is only non zero when $k \geq -m_0$ and $k+m+n \geq -m_0$. This implies also that $k + m \geq -m_0$ and $k + n \geq -m_0$, hence $w^+_{k,k+m} = 1$ and $w^+_{k,k+n} = 1$ and the commutator is zero. For $n = -m$ with $m > 0$ and $\alpha = +$ we have

$$\begin{aligned}
[\hat{\rho}_{m,+}, \hat{\rho}_{-m,+}] &= \sum_{k} \left(w^+_{k,k+m} - w^+_{k,k-m} \right) w^+_{k,k} \hat{c}^\dagger_{k,+}\hat{c}_{k,+} \\
&= \sum_{k=-m_0}^{-m_0+m-1} \hat{c}^\dagger_{k,+}\hat{c}_{k,+} \equiv \mathcal{N}_+(m)
\end{aligned} \tag{4.30}$$

where $\mathcal{N}_+(m)$ counts the number of right moving fermions on the lowest m states. Because the interaction $v(k)$ drops of rapidly for $|k| > k_c$, only

low energy eigenstates contribute to the interacting ground state, excitations of electrons deep in Fermi sea is not possible. In this subspace of the total Hilbert space it is an excellent approximation, which becomes asymptotically exact for $m_0 \to \infty$ to replace $\mathcal{N}_+(m)$, by $m\hat{I}$ on (4.30). For m, n having different sign we obtain

$$[\hat{\rho}_{m,+}, \hat{\rho}_{-n,+}] = \sum_{k=-m_0}^{-m_0-n-1} \hat{c}^\dagger_{k,+}\hat{c}_{k+m-n,+} = -\sum_{k=-m_0}^{-m_0-n-1} \hat{c}_{k+m-n,+}\hat{c}^\dagger_{k,+} \equiv 0 \quad (4.31)$$

where we use the anticommutation relation of fermions. The above commutator is zero because $\hat{c}^\dagger_{k,+}$ tries to create an electron in an occupied state deep in the Fermi sea. Same arguments apply to the left moving electrons and putting all together we arrive at

$$[\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\beta}] = \alpha m \delta_{\alpha,\beta}\delta_{m,-n} \quad (4.32)$$

The above commutation relations looks like bosonic commutation relations with a normalization factor. By defining new operators $\hat{b}_n$, $\hat{b}^\dagger_n$ as

$$\hat{b}_n \equiv \frac{1}{\sqrt{|n|}} \begin{cases} \hat{\rho}_{n,+} & \text{for } n > 0 \\ \hat{\rho}_{n,-} & \text{for } n < 0 \end{cases} \quad \text{and} \quad \hat{b}^\dagger_n \equiv \frac{1}{\sqrt{|n|}} \begin{cases} \hat{\rho}_{-n,+} & \text{for } n > 0 \\ \hat{\rho}_{-n,-} & \text{for } n < 0 \end{cases} \quad (4.33)$$

where we used $\hat{\rho}^\dagger_{n,\alpha} = \hat{\rho}_{-n,\alpha}$, then (4.32) implies the following commutation relations between b's

$$[\hat{b}_m, \hat{b}_n] = 0 \ , [\hat{b}^\dagger_m, \hat{b}^\dagger_n] = 0 \text{ and } [\hat{b}_m, \hat{b}^\dagger_n] = \delta_{m,n} \quad (4.34)$$

which are the familiar bosonic commutation realations of the creation and annihilation operators. Now the kinetic energy operator $\hat{T}_{TL}$ (4.23), using the Kronig identity [20] can be written in terms of the bosonic operators as

$$\hat{T}_{TL} = \sum_{n\neq 0} v_F|k_n|\hat{b}^\dagger_n\hat{b}_n = \sum_{n>0} v_F k_n \left(\hat{b}^\dagger_n\hat{b}_n + \hat{b}^\dagger_{-n}\hat{b}_{-n}\right) \quad (4.35)$$

and the two-body interaction (4.27) takes the form

$$\begin{aligned} \hat{V}_{TL} &= \sum_{n>0} \frac{v(k_n)n}{L} \left(\hat{b}^\dagger_n\hat{b}_n + \hat{b}^\dagger_{-n}\hat{b}_{-n} + \hat{b}^\dagger_n\hat{b}^\dagger_{-n} + \hat{b}_{-n}\hat{b}_n\right) \\ &= \sum_{n>0} \frac{v(k_n)k_n}{2\pi} \left(\hat{b}^\dagger_n\hat{b}_n + \hat{b}^\dagger_{-n}\hat{b}_{-n} + \hat{b}^\dagger_n\hat{b}^\dagger_{-n} + \hat{b}_{-n}\hat{b}_n\right) \end{aligned} \quad (4.36)$$

where we used $k_n = 2\pi n/L$ and we dropped terms proportional to particle number operators. Finally, the total Hamiltonian of the Tomonaga-Luttinger model is

$$\begin{aligned}\hat{H}_{TL} = \sum_{n>0} \Bigg\{ k_n \left(v_F + \frac{v(k_n)}{2\pi} \right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n} \right) \\ + \frac{k_n v(k_n)}{2\pi} \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n \right) \Bigg\}\end{aligned} \tag{4.37}$$

We can see that initially the Hamiltonian of the system was in terms of fermionic operators whereas now it is expressed in terms of bosonic operators. Thus, we have mapped fermions to bosons. This is called bosonization. The Hamiltonian (4.37) can be diagonalized by a Bogoliubov transformation. It is common to distinguish between intra and inter-branch scattering processes, this is done by replacing the potential $v(k_n)$ in the first term of the Hamiltonian (4.37) by a function $g_4(k_n)$ and that in the second term by $g_2(k_n)$. So the general form is

$$\begin{aligned}\hat{H}_{TL} = \sum_{n>0} \Bigg\{ k_n \left(v_F + \frac{g_4(k_n)}{2\pi} \right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n} \right) \\ + \frac{k_n g_2(k_n)}{2\pi} \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n \right) \Bigg\}\end{aligned} \tag{4.38}$$

4.3 Bosonization of field operators

In the previous section we saw that a fermionic problem was mapped to a bosonic one. In general bosons are more easy to handle and we would like also to express the fermionic field operators in terms of bosonic operators. Then we can calculate correlation functions in a straightforward manner. We will do it for right moving fermions, same reasoning applies also for left moving. The starting point is a simple relation which holds for the bosonic operators $\hat{b}^\dagger, \hat{b}$

$$[\hat{b}, e^{\lambda \hat{b}^\dagger}] = \lambda e^{\lambda \hat{b}^\dagger} \tag{4.39}$$

The proof is very simple and goes as follows. We define an operator as a function of a parameter λ, $\hat{b}(\lambda) \equiv e^{-\lambda \hat{b}^\dagger} \hat{b} e^{\lambda \hat{b}^\dagger}$ and we take the derivative with

respect to λ

$$\begin{aligned}\frac{d\hat{b}(\lambda)}{d\lambda} &= -e^{-\lambda\hat{b}^\dagger}\hat{b}^\dagger\hat{b}e^{\lambda\hat{b}^\dagger} + e^{-\lambda\hat{b}^\dagger}\hat{b}\hat{b}^\dagger e^{\lambda\hat{b}^\dagger} \\ &= e^{-\lambda\hat{b}^\dagger}\left(\hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\right)e^{\lambda\hat{b}^\dagger} = \hat{I}\end{aligned}$$

where we used $[\hat{b}, \hat{b}^\dagger] = \hat{I}$ and $[\hat{b}^\dagger, e^{\lambda\hat{b}^\dagger}] = 0$. So we have that

$$\hat{b}(\lambda) = \hat{b} + \lambda\hat{I}$$

and also

$$\hat{b} + \lambda\hat{I} = e^{-\lambda\hat{b}^\dagger}\hat{b}e^{\lambda\hat{b}^\dagger} \Rightarrow [\hat{b}, e^{\lambda\hat{b}^\dagger}] = \lambda e^{\lambda\hat{b}^\dagger}$$

where we multiplied from the left with $e^{\lambda\hat{b}^\dagger}$. Next we define the following operators which are linear combinations of the boson operators $\hat{b}_n, \hat{b}_n^\dagger$

$$\hat{A}_+ \equiv \sum_{n\neq 0}\lambda_n\hat{b}_n^\dagger\ ,\ \hat{B}_- \equiv \sum_{n\neq 0}\mu_n\hat{b}_n \tag{4.40}$$

with arbitrary constants λ_n and μ_n. Generalizing (4.39) the following commutation relations hold

$$[\hat{b}_m, e^{\hat{B}_-}e^{\hat{A}_+}] = \lambda_m e^{\hat{B}_-}e^{\hat{A}_+}\ \ ,\ \ [\hat{b}_m^\dagger, e^{\hat{B}_-}e^{\hat{A}_+}] = -\mu_m e^{\hat{B}_-}e^{\hat{A}_+} \tag{4.41}$$

Considering fermionic operators $\hat{S}$ which obey

$$[\hat{b}_m, \hat{S}] = -\lambda_m\hat{S}\ \ ,\ \ [\hat{b}_m^\dagger, \hat{S}] = \mu_m\hat{S}$$

the operator $\hat{O} \equiv \hat{S}e^{\hat{B}_-}e^{\hat{A}_+}$ commutes with all $\hat{b}_m$ and $\hat{b}_m^\dagger$,

$$\begin{aligned}[\hat{b}_m, \hat{O}] &= [\hat{b}_m, \hat{S}e^{\hat{B}_-}e^{\hat{A}_+}] = [\hat{b}_m, \hat{S}]e^{\hat{B}_-}e^{\hat{A}_+} + \hat{S}[\hat{b}_m, e^{\hat{B}_-}e^{\hat{A}_+}] \\ &= -\lambda_m\hat{S}e^{\hat{B}_-}e^{\hat{A}_+} + \lambda_m\hat{S}e^{\hat{B}_-}e^{\hat{A}_+} = 0\end{aligned}$$

and same for $\hat{b}_m^\dagger$. We therefore write

$$\hat{S} = \hat{O}e^{\hat{A}_+}e^{\hat{B}_-} \tag{4.42}$$

$\hat{O}$ is such that both sides of (4.42) yield identical matrix elements and preserves the anticommutation relations of the fermionic operator $\hat{S}$. Now we want to write the fermionic field operator in terms of bosonic operators. We

will do it for right moving fermions, the analysis of left moving being similar. We define an auxiliary fermionic field operator for right moving fermions as a linear combination of $\hat{c}_{n,+}$, which creates a right moving fermion in momentum state n

$$\tilde{\psi}_+(x) \equiv \sum_{n\in I_+} e^{inx}\hat{c}_{n,+} = \sum_{n=-\infty}^{\infty} e^{inx}\hat{c}_{n,+} \tag{4.43}$$

where we took the limit $m_0 \to \infty$. The $\hat{c}_{m,+}$ obey the following commutation relations [20], [24]

$$[\hat{b}_m, \hat{c}_{n,+}] = -\frac{1}{\sqrt{m}}\hat{c}_{n+m,+} \ , \ [\hat{b}_m^\dagger, \hat{c}_{n,+}] = -\frac{1}{\sqrt{m}}\hat{c}_{n-m,+} \tag{4.44}$$

Then the operator $\tilde{\psi}_+(x)$ obeys the same commutation relations as the operator $\hat{S}$ above

$$[\hat{b}_m, \tilde{\psi}_+(x)] = -\frac{1}{\sqrt{m}}e^{-imx}\tilde{\psi}_+(x) \ , \ [\hat{b}_m^\dagger, \tilde{\psi}_+(x)] = -\frac{1}{\sqrt{m}}e^{imx}\tilde{\psi}_+(x) \tag{4.45}$$

Hence we have

$$\tilde{\psi}_+(x) = \hat{O}_+(x)e^{i\hat{\phi}_+^\dagger(x)}e^{i\hat{\phi}_+(x)} \tag{4.46}$$

where $i\hat{\phi}_+(x)$ is given by

$$i\hat{\phi}_+(x) = \sum_{n>0}^{\infty} \frac{e^{inx}}{\sqrt{n}}\hat{b}_n \tag{4.47}$$

$\hat{O}(x)$ is called Klein factor and has the property to lower the fermion number by one when is acting in a state with N fermions and commutes with all boson operators. More details about the Klein can be found on [21] ,[20]. The physical field operators $\hat{\psi}_+(x)$ are related to the auxiliary field operators by

$$\hat{\psi}_+(x) = \frac{1}{\sqrt{L}}\tilde{\psi}_+\left(\frac{2\pi x}{L}\right) \tag{4.48}$$

Equations (4.46), (4.47) and (4.48) will be very useful for calculating a two-point correlation function in the next chapter. In our analysis we considered spinless fermions, including spin leads to a phenomenon called "spin-charge seperation", for details see [20].

Chapter 5

Non-equilibrium Luttinger liquids

In this chapter we will examine the Luttinger-Tomonaga model out of equilibrium. First, we will consider a finite duration quantum quench and we will derive the equations governing the dynamics during and after the quench [25]. Then, having the general formalism we will examine two cases. In the first case, we will periodically drive the system [26] and compute its kinetic energy as a function of time, which means that the duration of the quench, $\tau \to \infty$. In the second case we will consider a finite duration periodic quench and calculate the fernionic Green's function of the right moving fermions.

5.1 Time-dependent Tomonaga-Luttinger model

We consider the time-dependent Luttinger-Tomonaga model defined by the Hamiltonian

$$\begin{aligned}\hat{H}_{TL}(t) = \sum_{n>0} \Bigg\{ & k_n \left(v_F + \frac{g_4(k_n,t)}{2\pi} \right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n} \right) \\ & + \frac{k_n g_2(k_n,t)}{2\pi} \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n \right) \Bigg\}\end{aligned} \tag{5.1}$$

where $k_n = 2\pi n/L$, $n \in \mathbb{Z}$ is the momentum and L the length of the system, v_F is the Fermi velocity and the coupling functions g_2, g_4 depend also on time. The operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ destroy and create bosonic modes at momentum k_n

and satisfy the commutation relations

$$[\hat{b}_m, \hat{b}_n] = 0 \ , [\hat{b}_m^\dagger, \hat{b}_n^\dagger] = 0 \text{ and } [\hat{b}_m, \hat{b}_n^\dagger] = \delta_{m,n} \tag{5.2}$$

It is useful to define the following dimensionless coupling functions

$$\hat{g}_2(k_n, t) = \frac{g_2(k_n, t)}{2\pi v_F} \ , \ \hat{g}_4(k_n, t) = \frac{g_4(k_n, t)}{2\pi v_F} \tag{5.3}$$

and the Hamiltonian takes the form

$$\begin{aligned} \hat{H}_{TL}(t) = \sum_{n>0} \Big\{ & k_n v_F \left(1 + \hat{g}_4(k_n, t)\right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}\right) \\ & + k_n v_F \hat{g}_2(k_n, t) \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n\right) \Big\} \end{aligned} \tag{5.4}$$

Having a time dependent Hamiltonian we would like to know the time evolution of the system. We will consider the dynamics of the Tomonaga-Luttinger model after a quantum quench of finite duration τ. We suppose that at $t = 0$ the coupling functions have a certain value $\hat{g}_{2/4}(k_n, 0)$. Then for $t < \tau$ they depend on time $\hat{g}_{2/4}(k_n, t)$ and after the quench, $t > \tau$ they have another value $\hat{g}_{2/4}(k_n, \tau)$. In order to specify the dynamics we have to split the problem in two regimes, the during quench evolution and the post quench evolution. By doing this we will obtain a general scheme which can be used for different kinds of quench protocols.

Evolution during the quench, $t < \tau$

In order to find the time evolution of various observables we have to solve the Heisenberg equations of motion for the bosonic operators

$$i\frac{d}{dt}\hat{b}_n(t) = [\hat{b}_n(t), \hat{H}_{TL}(t)] \text{ and } i\frac{d}{dt}\hat{b}_n^\dagger(t) = [\hat{b}_n^\dagger(t), \hat{H}_{TL}(t)] \tag{5.5}$$

we will do it for $\hat{b}_n(t)$, the case of $\hat{b}_n^\dagger(t)$ being similar. For simplicity we will refer to $\hat{H}_{TL}$ as $\hat{H}$. Using $\hat{b}_n(t) = \hat{U}^\dagger(t)\hat{b}_n\hat{U}(t)$ we can write the commutator as

$$[\hat{b}_n(t), \hat{H}(t)] = \hat{U}^\dagger(t)[\hat{b}_n, \hat{H}(t)]\hat{U}(t) \tag{5.6}$$

Moreover we can write $\hat{H}$ as

$$\hat{H}(t) = \sum_{n>0} \hat{H}_n(t)$$

with,

$$\hat{H}_n(t) = k_n v_F \left(1 + \hat{g_4}(k_n, t)\right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}\right) + k_n v_F \hat{g_2}(k_n, t) \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n\right)$$

Thus,

$$[\hat{b}_m(t), \hat{H}(t)] = \sum_{n>0} \hat{U}^\dagger(t) [\hat{b}_m, \hat{H}_n(t)] \hat{U}(t)$$

where,

$$\begin{aligned} [\hat{b}_m, \hat{H}_n(t)] = {} & k_n v_F \left(1 + \hat{g_4}(k_n, t)\right) [\hat{b}_m, \hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}] \\ & + k_n v_F \hat{g_2}(k_n, t) [\hat{b}_m, \hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n] \end{aligned} \tag{5.7}$$

$\underline{[\hat{b}_m, \hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}]}$

$$\begin{aligned} [\hat{b}_m, \hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}] &= [\hat{b}_m, \hat{b}_n^\dagger \hat{b}_n] + [\hat{b}_m, \hat{b}_{-n}^\dagger \hat{b}_{-n}] \\ &= \hat{b}_n^\dagger [\hat{b}_m, \hat{b}_n] + [\hat{b}_m, \hat{b}_n^\dagger] \hat{b}_n + \hat{b}_{-n}^\dagger [\hat{b}_m, \hat{b}_{-n}] + [\hat{b}_m, \hat{b}_{-n}^\dagger] \hat{b}_{-n} \\ &= \delta_{m,n} \hat{b}_n + \delta_{m,-n} \hat{b}_{-n} \end{aligned}$$

$\underline{[\hat{b}_m, \hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n]}$

$$\begin{aligned} [\hat{b}_m, \hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n] &= [\hat{b}_m, \hat{b}_n^\dagger \hat{b}_{-n}^\dagger] + [\hat{b}_m, \hat{b}_{-n} \hat{b}_n] \\ &= \hat{b}_n^\dagger [\hat{b}_m, \hat{b}_{-n}^\dagger] + [\hat{b}_m, \hat{b}_n^\dagger] \hat{b}_{-n}^\dagger + \hat{b}_{-n} [\hat{b}_m, \hat{b}_n] + [\hat{b}_m, \hat{b}_{-n}] \hat{b}_n \\ &= \delta_{m,-n} \hat{b}_n^\dagger + \delta_{m,n} \hat{b}_{-n}^\dagger \end{aligned}$$

$$[\hat{b}_m, \hat{H}_n(t)] = k_n v_F \left(1 + \hat{g_4}(k_n, t)\right) \delta_{m,n} \hat{b}_n + k_n v_F \hat{g_2}(k_n, t) \delta_{m,n} \hat{b}_{-n}^\dagger \tag{5.8}$$

where the terms having $\delta_{m,-n}$ do not contribute. Hence,

$$i \frac{d}{dt} \hat{b}_n(t) = k_n v_F \left(1 + \hat{g_4}(k_n, t)\right) \hat{b}_n(t) + k_n v_F \hat{g_2}(k_n, t) \hat{b}_{-n}^\dagger(t) \tag{5.9}$$

We define

$$\omega_n(t) = |k_n| v_F \left(1 + \hat{g_4}(k_n, t)\right) \quad , \quad \lambda_n(t) = |k_n| v_F \hat{g_2}(k_n, t) \tag{5.10}$$

and equation (5.9) takes the form

$$i \frac{d}{dt} \hat{b}_n(t) = \omega_n(t) \hat{b}_n(t) + \lambda_n(t) \hat{b}_{-n}^\dagger(t) \tag{5.11}$$

For $\hat{b}^\dagger(t)$ we get

$$i\frac{d}{dt}\hat{b}_n^\dagger(t) = -\omega_n(t)\hat{b}_n^\dagger(t) - \lambda_n(t)\hat{b}_{-n}(t) \tag{5.12}$$

Now by making the ansatz

$$\hat{b}_n(t) = u_n(t)\hat{b}_n + v_n(t)^*\hat{b}_{-n}^\dagger \tag{5.13}$$

$$\hat{b}_n^\dagger(t) = u_n(t)^*\hat{b}_n^\dagger + v_n(t)\hat{b}_{-n} \tag{5.14}$$

where the operators on the right-hand side are time independent, and plugging those on (5.11), (5.12) we get differential equations for the unknown functions $u_n(t)$ and $v_n(t)$,

$$i\frac{d}{dt}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} \omega_n(t) & \lambda_n(t) \\ -\lambda_n(t) & -\omega_n(t) \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} \tag{5.15}$$

with initial conditions

$$u_n(t) = 1 \ , \ \ v_n(t) = 0 \tag{5.16}$$

because at $t = 0$ we want $\hat{b}_n(0) = \hat{b}_n$ and $\hat{b}_n^\dagger(0) = \hat{b}_n^\dagger$. In our analysis we will consider $\hat{g}_2(k_n, t) = \hat{g}_4(k_n, t)$, where exact solutions are possible for specific time dependences. Now we write $u_n(t)$ and $v_n(t)$ in terms of an auxiliary function $a_n(t)$ as

$$u_n(t) = \frac{1}{2}a_n(t) + \frac{i}{2v_F|k_n|}\frac{d}{dt}a_n(t) \tag{5.17}$$

$$v_n(t) = \frac{1}{2}a_n(t) - \frac{i}{2v_F|k_n|}\frac{d}{dt}a_n(t) \tag{5.18}$$

Substitution to (5.15) gives the following two equations

$$\begin{aligned} &i\frac{d}{dt}u_n(t) = \omega_n(t)u_n(t) + \lambda_n(t)v_n(t) \Rightarrow \\ &\Rightarrow \frac{i}{2}\frac{d}{dt}a_n(t) - \frac{1}{2v_F|k_n|}\frac{d^2}{dt^2}a_n(t) = \\ &\frac{1}{2}\omega_n(t)a_n(t) + \frac{\omega_n(t)i}{2v_F|k_n|}\frac{d}{dt}a_n(t) + \frac{1}{2}\lambda_n(t)a_n(t) - \frac{\lambda_n(t)i}{2v_F|k_n|}\frac{d}{dt}a_n(t) \end{aligned} \tag{5.19}$$

and

$$
\begin{aligned}
& i\frac{d}{dt}u_n(t) = \omega_n(t)u_n(t) + \lambda_n(t)v_n(t) \Rightarrow \\
& \Rightarrow \frac{i}{2}\frac{d}{dt}a_n(t) + \frac{1}{2v_F|k_n|}\frac{d^2}{dt^2}a_n(t) = \\
& -\frac{1}{2}\lambda_n(t)a_n(t) - \frac{\lambda_n(t)i}{2v_F|k_n|}\frac{d}{dt}a_n(t) - \frac{1}{2}\omega_n(t)a_n(t) + \frac{\omega_n(t)i}{2v_F|k_n|}\frac{d}{dt}a_n(t)
\end{aligned} \tag{5.20}
$$

Subtraction of (5.19), (5.20) gives

$$
-\frac{1}{v_F|k_n|}\frac{d^2}{dt^2}a_n(t) = \left[\omega_n(t) + \lambda_n(t)\right] a_n(t) \tag{5.21}
$$

For $\hat{g}_2(k_n,t) = \hat{g}_4(k_n,t)$

$$
\omega_n(t) + \lambda_n(t) = v_F|k_n|(1 + \hat{g}_2(k_n,t))
$$

and (5.21) takes the form

$$
\frac{d^2}{dt^2}a_n(t) + v_F^2k_n^2[1 + \hat{g}_2(k_n,t)]a_n(t) = 0 \tag{5.22}
$$

with initial conditions

$$
a_n(0) = 1 \ , \ \frac{d}{dt}a_n(t)\Big|_{t=0} = -iv_F|k_n| \tag{5.23}
$$

Evolution after the quench, $t > \tau$

After the quench ($t > \tau$) the system (5.15) has constant coefficients and is easy to solve. The two equations are

$$
i\frac{d}{dt}u_n(t) = \omega_n(\tau)u_n(t) + \lambda_n(\tau)v_n(t) \tag{5.24}
$$

and

$$
i\frac{d}{dt}v_n(t) = -\lambda_n(\tau)u_n(t) - \omega_n(\tau)v_n(t) \tag{5.25}
$$

Differentiation with respect to time of the first yiels

$$
i\frac{d^2}{dt^2}u_n(t) = \omega_n(\tau)\frac{d}{dt}u_n(t) + \lambda_n(\tau)\frac{d}{dt}v_n(t)
$$

and using (5.25) we get

$$i\frac{d^2}{dt^2}u_n(t) = \omega_n(\tau)\frac{d}{dt}u_n(t) + i\lambda_n^2(\tau)u_n(t) + i\lambda_n(\tau)\omega_n(\tau)v_n(t) \tag{5.26}$$

Plugging now (5.24) to the above equation gives

$$\frac{d^2}{dt^2}u_n(t) + (\omega_n^2(\tau) - \lambda_n^2(\tau))u_n(t) = 0 \tag{5.27}$$

with solution

$$u_n(t) = A_n \cos(\epsilon_n t) + B_n \sin(\epsilon_n t) \tag{5.28}$$

where

$$\epsilon_n = \sqrt{\omega_n^2(\tau) - \lambda_n^2(\tau)}$$

assuming $\omega_n(\tau) > \lambda_n(\tau)$ and $v_n(t)$ can be found from (5.24)

$$v_n(t) = \frac{i}{\lambda_n(\tau)}\frac{d}{dt}u_n(t) - \frac{\omega_n(\tau)}{\lambda_n(\tau)} \tag{5.29}$$

The two constants A_n and B_n are obtained from the initial conditions at $t = \tau$ and are given by

$$A_n = -i\frac{\lambda_n(\tau)}{\epsilon_n}\sin(\epsilon_n\tau)u_n(\tau) - \frac{i}{\epsilon_n}[i\epsilon_n\cos(\epsilon_n\tau) + \omega_n(\tau)\sin(\epsilon_n\tau)]v_n(\tau) \tag{5.30}$$

$$B_n = i\frac{\lambda_n(\tau)}{\epsilon_n}\cos(\epsilon_n\tau)u_n(\tau) - \frac{i}{\epsilon_n}[i\epsilon_n\sin(\epsilon_n\tau) - \omega_n(\tau)\cos(\epsilon_n\tau)]v_n(\tau) \tag{5.31}$$

Now we have the full machinery to calculate the dynamical evolution of observables for any quench scenario that we want. Moreover, we can study the dynamics of the system for quench times $\tau \to \infty$ by considering only the equations mentioned in the evolution during the quench. We will do this in the next section, where instead of a quantum quench we periodically drive the system without stopping and we calculate the kinetic energy of the system. Later, we will consider a periodic quench scenario to calculate the fermionic Green function.

5.2 Periodically driven Luttinger liquid

We consider the Hamiltonian (5.4) with $\hat{g}_2(k_n,t)=\hat{g}_4(k_n,t)$ where the interaction strength is varied periodically with driving frequency Ω. The coupling function has the form

$$\hat{g}_2(k_n,t)=\frac{g_2(k_n,t)}{2\pi v_F}=\frac{U(k_n,t)}{2\pi v_F}$$

where $U(k_n,t)$ is a time dependent potential and we denote g_2 as U and the Hamiltonian takes the form

$$\hat{H}(t)=\sum_{n>0}\left\{\omega_n(t)\left(\hat{b}_n^\dagger\hat{b}_n+\hat{b}_{-n}^\dagger\hat{b}_{-n}\right)+k_n\frac{U(k_n,t)}{2\pi v_F}\left(\hat{b}_n^\dagger\hat{b}_{-n}^\dagger+\hat{b}_{-n}\hat{b}_n\right)\right\} \quad (5.32)$$

with $\omega_n(t)=k_nv_F\left(1+\frac{U(k_n,t)}{2\pi v_F}\right)$. Assuming repulsive interaction potential, $U(k_n,t)>0$, and that the system is initially in the ground state $|\psi_0\rangle$. The state $|\psi_0\rangle$ is the ground state of a Hamiltonian with a constant interaction at times $t<0$ such that $U(k_n,t<0)=V(k_n)(1+\nu)$. Then at time $t=0$ the periodic driving starts and we assume the following form of the interaction potential

$$U(k_n,t)=V(k_n)(1+\nu\cos(\Omega t)) \quad (5.33)$$

The dimensionless coupling ν of the periodic driving is always $\nu<1$ to ensure that the potential is repulsive at all times. In our analysis we consider a momentum cut-off k_c above which the interaction decays exponentially. In the numerical simulations we use a Gaussian one

$$V(k)=V_0e^{-(k/k_c)^2} \quad (5.34)$$

In order to find the evolution under the periodic driving we use the analysis that we did, for the evolution during the quench, in the previous section. The equation that we have to solve is (5.22), where in our case is

$$\frac{d^2}{dt^2}a_n(t)+v_F^2k_n^2\left[1+\frac{U(k_n,t)}{2\pi v_F}\right]a_n(t)=0 \quad (5.35)$$

Using the transformation time $\tau=\Omega t/2$ in order to make the equation dimensionless and (5.33) we have

$$\tau=\frac{\Omega t}{2}\rightarrow d\tau=\frac{\Omega}{2}dt$$

$$\frac{\Omega^2}{4}\frac{d^2}{d\tau^2}a_n(\tau) + v_F^2 k_n^2 \left[1 + \frac{V(k_n)(1+\nu\cos(2\tau))}{2\pi v_F}\right] a_n(\tau) = 0 \Rightarrow$$
$$\frac{d^2}{d\tau^2}a_n(\tau) + \frac{4v_F^2 k_n^2}{\Omega^2}\left(1 + \frac{V(k_n)}{\pi v_F}\right)\left[1 + \frac{\nu V(k_n)}{\pi v_F + V(k_n)}\cos(2\tau)\right] a_n(\tau) = 0$$

Hence,

$$\frac{d^2}{d\tau^2}a_n(\tau) + \epsilon_n^2\left[1 + 2\gamma_n\cos(2\tau)\right] a_n(\tau) = 0 \tag{5.36}$$

where

$$\epsilon_n = \frac{2v_F k_n}{\Omega}\sqrt{\left(1 + \frac{V(k_n)}{\pi v_F}\right)} \tag{5.37}$$

is the "natural frequency" of the oscillator and

$$\gamma_n = \frac{\nu V(k_n)}{2\pi v_F + 2V(k_n)} \tag{5.38}$$

the coupling strength of the periodic perturbation. The initial conditions are given by (5.23). Equation (5.36) is the Mathieu equation as we found in Chapter 3. Hence, calculating the time evolution in the periodically driven Luttinger liquid is equivalent to a set of parametrically driven harmonic oscillators (one for each mode n). From the analysis that we did in Chapter 3 we have that equation (5.36) shows an instability with exponentially growing amplitudes in the case of parametric resonance which occurs for that particular mode n^* for which $\epsilon_{n^*} = 1$. So the driving frequency equals

$$\Omega = 2v_F k_n^* \sqrt{\left(1 + \frac{V(k_n^*)}{\pi v_F}\right)} \tag{5.39}$$

for parametric resonance. In the following we distinguish between slow and fast driving. The two regimes are determined from the energy scale Ω^*

$$\Omega^* = v_F k_c \tag{5.40}$$

Using the results from Chapter 3 we have the following. For slow driving $\Omega \ll \Omega^*$ the bosonic mode n^* where parametric resonance happens is determined by

$$\frac{k_n^*}{k_c} \to \frac{1}{\sqrt{1+2\alpha}}\frac{\Omega}{2\Omega^*} \tag{5.41}$$

where $\alpha = V_0/2\pi v_F$. And for the rate of the associated exponential growth in time we have

$$\Gamma = \frac{1}{4}\frac{\alpha\nu}{1+2\alpha}\Omega \tag{5.42}$$

The time scale t^* for the onset of the instability is

$$t^* = \Gamma^{-1} \tag{5.43}$$

For fast driving $\Omega \gg \Omega^*$ the resonant mode

$$\frac{k_n^*}{k_c} \to \frac{\Omega}{2\Omega^*} \tag{5.44}$$

is independent of the interaction potential up to corrections suppressed by the cutoff k_c and the rate is given by

$$\Gamma = \frac{1}{4}\Omega\nu V(k_n^*) \tag{5.45}$$

The behaviour for $\Omega \gg \Omega^*$ depends on the details of the large momentum behavior of the interaction potential. As we mentioned earlier we will use a Gaussian interaction potential.

5.2.1 Kinetic energy density

Now we calculate the dynamical evolution of the kinetic energy. The kinetic energy operator is defined from the Hamiltonian (5.4) of our model for $t < 0$ by considering the diagonal part of it. Thus we have :

$$\hat{H}_{kin} = \sum_{n>0} \omega_n(t<0)(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}) \tag{5.46}$$

where $\omega_n(t) = k_n v_F \left(1 + \frac{U(k_n,t)}{2\pi u_F}\right)$ and $U(k_n,t) = V(k_n)(1+\nu\cos(\Omega t))$ and for $t<0$ we have $U(k_n, t<0) = V(k_n)(1+\nu)$.

The kinetic energy of the periodically driven Luttinger liquid system at time t is given by

$$E_{kin}(t) = \langle\psi_0(t)|\, \hat{H}_{kin}\, |\psi_0(t)\rangle \tag{5.47}$$

with $|\psi_0(t)\rangle = \hat{U}(t)\,|\psi_0\rangle$ being the state at time t. In order to calculate this expectation value we first compute $\hat{U}(t)^\dagger \hat{H}_{kin}\hat{U}(t)$ and then we take the average value in the state $|\psi_0\rangle$ which is the ground state of the full Hamiltonian

at times $t < 0$.

Calculation of $\hat{U}(t)^\dagger \hat{H}_{kin} \hat{U}(t)$:

$$\begin{aligned}
\hat{U}(t)^\dagger \hat{H}_{kin} \hat{U}(t) =& \hat{U}(t)^\dagger \left(\sum_{n>0} \omega_n(t<0)(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n}) \right) \hat{U}(t) \\
&= \sum_{n>0} \omega_n(t<0) \left(\hat{U}(t)^\dagger \hat{b}_n^\dagger \hat{b}_n \hat{U}(t) + \hat{U}(t)^\dagger \hat{b}_{-n}^\dagger \hat{b}_{-n} \hat{U}(t) \right) \\
&= \sum_{n>0} \omega_n(t<0) \left(\hat{b}_n^\dagger(t) \hat{b}_n(t) + \hat{b}_{-n}^\dagger(t) \hat{b}_{-n}(t) \right)
\end{aligned}$$

Using $\hat{b}_n(t) = u_n(t)\hat{b}_n + v_n(t)^* \hat{b}_{-n}^\dagger$ and $\hat{b}_n^\dagger(t) = u_n(t)^* \hat{b}_n^\dagger + v_n(t)\hat{b}_{-n}$ we calculate the four terms in the previous equation.

$\underline{\hat{b}_n^\dagger(t)\hat{b}_n(t)}$

$$\begin{aligned}
b_n^\dagger(t)\hat{b}_n(t) =& (u_n(t)^* \hat{b}_n^\dagger + v_n(t)\hat{b}_{-n})(u_n(t)\hat{b}_n + v_n(t)^* \hat{b}_{-n}^\dagger) \\
&= |u_n(t)|^2 \hat{b}_n^\dagger \hat{b}_n + u_n(t)^* v_n(t)^* \hat{b}_n^\dagger \hat{b}_{-n}^\dagger + v_n(t) u_n(t) \hat{b}_{-n} \hat{b}_n \\
&\quad + |v_n(t)|^2 \hat{b}_{-n} \hat{b}_{-n}^\dagger
\end{aligned}$$

$\underline{\hat{b}_{-n}^\dagger(t)\hat{b}_{-n}(t)}$

$$\begin{aligned}
\hat{b}_{-n}^\dagger(t)\hat{b}_{-n}(t) =& (u_n(t)^* \hat{b}_{-n}^\dagger + v_n(t)\hat{b}_n)(u_n(t)\hat{b}_{-n} + v_n(t)^* \hat{b}_n^\dagger) \\
&= |u_n(t)|^2 \hat{b}_{-n}^\dagger \hat{b}_{-n} + u_n(t)^* v_n(t)^* \hat{b}_{-n}^\dagger \hat{b}_n^\dagger + v_n(t) u_n(t) \hat{b}_n \hat{b}_{-n} \\
&\quad + |v_n(t)|^2 \hat{b}_n \hat{b}_n^\dagger
\end{aligned}$$

Plugging in the above expressions in the kinetic energy and acting with the state $|\psi_0\rangle$ in the operator only the terms of the form $b^\dagger b$ and $bb^\dagger$ contribute. Thus we arrive at

$$E_{kin}(t) = 2 \sum_{n>0} k_n u_F \left(1 + \frac{V(k_n)(1+\nu)}{2\pi u_F} \right) \left(|u_n(t)|^2 + |v_n(t)|^2 \right) \tag{5.48}$$

We calculate the kinetic energy density numerically for different driving frequencies Ω for slow and fast driving. The energy scale Ω^* associated

with the crossover between the two limits is set by $\Omega^* = v_F k_c$. The kinetic energy is in units $\mathcal{E}^* = k_c \Omega^*$. For our results we use $\nu = 1/5$ and $V(k)/(2\pi v_F) = \alpha \exp\{-(k/k_c)^2\}$ with $\alpha = 1/2$. Furthermore we go to the continuum limit by converting the sum to integral with

$$\sum_{n>0} \to \frac{L}{2\pi} \int_0^\infty dk$$

and

$$\frac{E_{kin}(t)}{L} = \frac{1}{\pi} \int_0^\infty dk\, k u_F \left(1 + \frac{V(k)(1+\nu)}{2\pi u_F}\right) \left(|u(k,t)|^2 + |v(k,t)|^2\right) \quad (5.49)$$

Below we present the results for slow and fast driving. In the figures we show the kinetic energy density as a function of time.

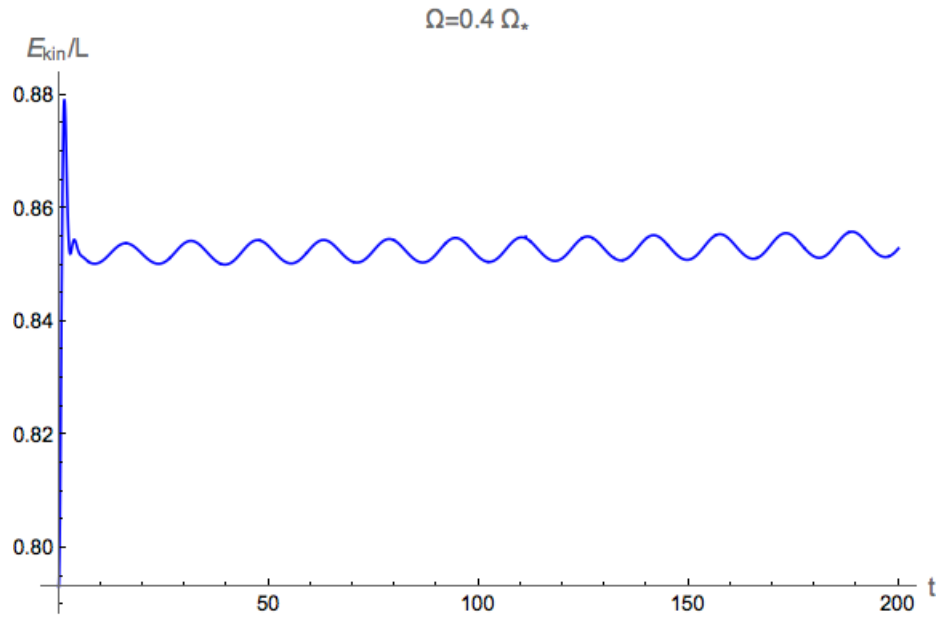

Figure 5.1: Evolution of kinetic energy density for $\Omega = 0.4\Omega^*$ with $\nu = 1/5$ and $\alpha = 1/2$.

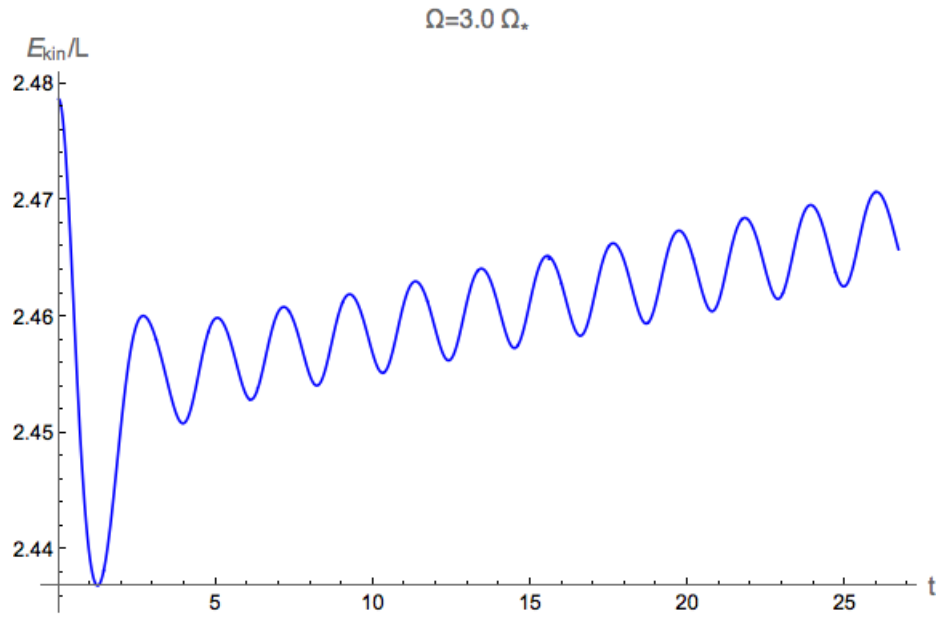

Figure 5.2: Evolution of kinetic energy density with $\nu = 1/5$ and $\alpha = 1/2$.

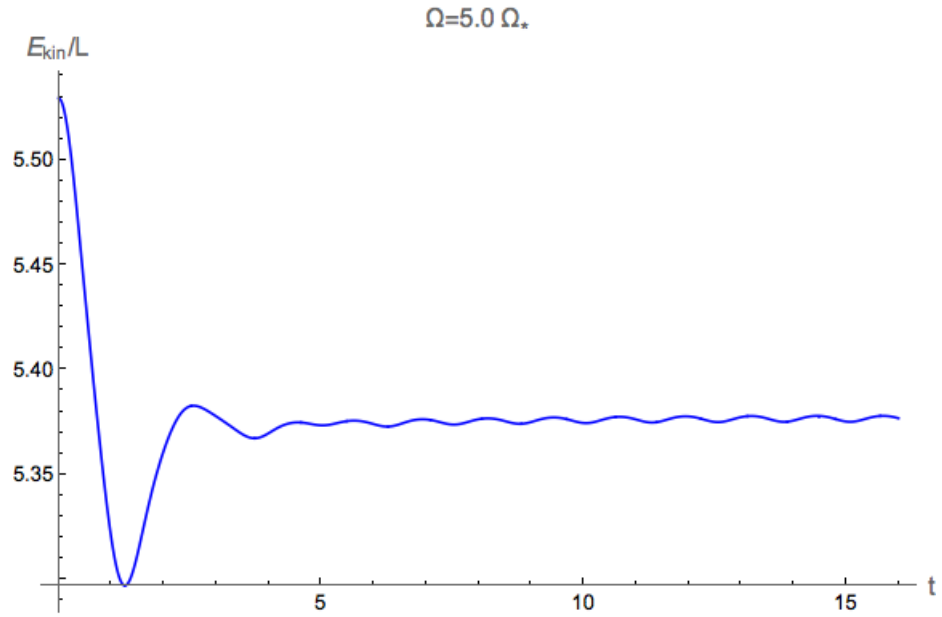

Figure 5.3: Evolution of kinetic energy density with $\nu = 1/5$ and $\alpha = 1/2$.

From the above figures we see that periodically driven systems can develop stationary states. For fast driving $\Omega \gg \Omega^*$ the unstable mode k_n^* of the parametric resonance becomes unstable at large times. After the initial transient dynamics the system settles to a stationary state. The rate Γ is small, which implies that the system will develop the instability at large times, $t > t^*$. Hence, this state is meta-stable. On the other hand, as we lower the driving frequency the rate of the instability Γ grows, so the instability starts at smaller times t^*. As Ω approaches Ω^*, the resonant mode becomes unstable for small times and there is no meta-stable state. Decreasing the driving frequency more, the unstable mode k_n^* of the parametric resonance becomes unstable at large times and a meta-stable state becomes again possible.

5.3 Correlation function of right moving fermions

We now turn to the calculation of the fermionic Green's function after a periodic quench of finite duration. We consider that the system is initially, at $t = 0$, in the ground state $|\psi_0\rangle$, where now $|\psi_0\rangle$ is the ground state of the boson operatos $\hat{b}_n$, such that $\hat{b}_n |\psi_0\rangle = 0$ for all n. The equal time Green's function of the right movers is defined as

$$G_F(x, y, t) = \langle \hat{\psi}_+^\dagger(x, t)\hat{\psi}_+(y, t)\rangle \tag{5.50}$$

In the previous chapter we mentioned that the bosonization of an auxiliary field operator is

$$\tilde{\psi}_+(x) = \hat{O}_+(x)e^{i\hat{\phi}_+(x)^\dagger}e^{i\hat{\phi}_+(x)} \tag{5.51}$$

with

$$i\hat{\phi}_+(x) = \sum_{n>0}^{\infty} \frac{e^{inx}}{\sqrt{n}}\hat{b}_n \tag{5.52}$$

and $\hat{O}(x)$ commutes with all boson operator. We will use those operators to calculate the Green function. It can be shown that the operators $\hat{O}(x)\hat{O}(y)$ lead to a factor $e^{-i\mathcal{N}_+(x-y)}$ [20], with $\mathcal{N}_+$ the number operator of right movers. Hence we have the following

$$\tilde{\psi}_+^\dagger(x, t)\tilde{\psi}_+(y, t) = e^{-i\hat{\phi}_+^\dagger(x,t)}e^{-i\hat{\phi}_+(x,t)}e^{i\hat{\phi}_+^\dagger(y,t)}e^{i\hat{\phi}_+(y,t)}e^{-i\mathcal{N}_+(x-y)} \tag{5.53}$$

Using the indentity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]} \tag{5.54}$$

we get

$$\tilde{\psi}_+^\dagger(x,t)\tilde{\psi}_+(y,t) = e^{-i(\hat{\phi}_+^\dagger(x,t)-\hat{\phi}_+^\dagger(y,t))}e^{-(i\hat{\phi}_+(x,t)-\hat{\phi}_+(y,t))}e^{[\hat{\phi}_+(x,t),\hat{\phi}_+^\dagger(y,t)]}e^{-i\mathcal{N}_+(x-y)} \tag{5.55}$$

Computation of the commutator gives

$$\begin{aligned}
[\hat{\phi}_+(x,t),\hat{\phi}_+^\dagger(y,t)] &= \sum_{n,m}\frac{e^{inx}}{\sqrt{n}}\frac{e^{imy}}{\sqrt{m}}[\hat{b}_n(t),\hat{b}_m^\dagger(t)] = \sum_{n,m}\frac{e^{inx}}{\sqrt{n}}\frac{e^{imy}}{\sqrt{m}}\delta_{n,m} \\
&= \sum_{n=1}^{\infty}\frac{e^{in(x-y)}}{n} = \lim_{\eta\to 0}\sum_{n=1}^{\infty}\frac{e^{in(x-y+i\eta)}}{n} \\
&= -\lim_{\eta\to 0}\left[\ln\left(1-e^{i(x-y+i\eta)}\right)\right]
\end{aligned} \tag{5.56}$$

where we added a convergence factor $e^{-k_n\eta}$ in order to make the series to converge and then take the limit $\eta \to 0$. Thus,

$$\tilde{\psi}_+^\dagger(x,t)\tilde{\psi}_+(y,t) = \frac{e^{-i\mathcal{N}_+(x-y)}}{1-e^{i(x-y+i0)}}e^{-i(\hat{\phi}_+^\dagger(x,t)-\hat{\phi}_+^\dagger(y,t))}e^{-i(\hat{\phi}_+(x,t)-\hat{\phi}_+(y,t))} \tag{5.57}$$

The next step is to take the expectation value of the above operator. First, we calculate the operators in the exponents

$$\begin{aligned}
i(\hat{\phi}_+(x,t)-\hat{\phi}_+(y,t)) &= \sum_{n>0}\frac{e^{inx}-e^{iny}}{\sqrt{n}}\hat{b}_n(t) \\
&= \sum_{n>0}\frac{e^{inx}-e^{iny}}{\sqrt{n}}(u_n(t)\hat{b}_n + v_n(t)^* b_{-n}^\dagger)
\end{aligned} \tag{5.58}$$

and acting with $|\psi_0\rangle$ gives

$$e^{-i(\hat{\phi}_+(x,t)-\hat{\phi}_+(y,t))}|\psi_0\rangle = e^{-\sum_{n>0}\frac{e^{inx}-e^{iny}}{\sqrt{n}}v_n(t)^*\hat{b}_{-n}^\dagger}|\psi_0\rangle \tag{5.59}$$

where we used that $\hat{b}_n|\psi_0\rangle = 0$. Similar for the other exponent we get

$$i(\hat{\phi}_+^\dagger(x,t)-\hat{\phi}_+^\dagger(y,t)) = -\sum_{n>0}\frac{e^{-inx}-e^{-iny}}{\sqrt{n}}(u_n(t)^*\hat{b}_n^\dagger + v_n(t)\hat{b}_{-n}) \tag{5.60}$$

and acting to $\langle\psi_0|$ gives

$$\langle\psi_0|\, e^{-i(\hat{\phi}_+^\dagger(x,t)-\hat{\phi}_+^\dagger(y,t))} = \langle\psi_0|\, e^{\sum_{n>0}\frac{e^{-inx}-e^{-iny}}{\sqrt{n}}v_n(t)\hat{b}_{-n}} \tag{5.61}$$

where we used that $\langle\psi_0|\,\hat{b}_n^\dagger = 0$. We define

$$\hat{A} = \sum_{n>0}\frac{e^{-ik_nx}-e^{-iny}}{\sqrt{n}}v_n(t)\hat{b}_{-n} \tag{5.62}$$

$$\hat{B} = -\sum_{n>0}\frac{e^{ik_nx}-e^{iny}}{\sqrt{n}}v_n(t)^*\hat{b}_{-n}^\dagger \tag{5.63}$$

and we write the correlation function as

$$\langle\tilde{\psi}_+^\dagger(x,t)\tilde{\psi}_+(y,t)\rangle = \frac{e^{-iN_+(x-y)}}{1-e^{i(x-y+i0)}}\,\langle\psi_0|\, e^{\hat{A}}e^{\hat{B}}\,|\psi_0\rangle \tag{5.64}$$

Using (5.54), only the exponent with the commutator contributes

$$\langle\tilde{\psi}_+^\dagger(x,t)\tilde{\psi}_+(y,t)\rangle = \frac{e^{-iN_+(x-y)}}{1-e^{i(x-y+i0)}}\,\langle\psi_0|\, e^{[\hat{A},\hat{B}]}\,|\psi_0\rangle \tag{5.65}$$

and calculation of $[\hat{A},\hat{B}]$ yields

$$\begin{aligned}
[\hat{A},\hat{B}] &= -\sum_{n,m}\frac{e^{-inx}-e^{-iny}}{\sqrt{n}}\frac{e^{imx}-e^{imy}}{\sqrt{m}}v_n(t)v_m(t)^*[\hat{b}_{-n},\hat{b}_{-m}^\dagger] \\
&= -\sum_{n,m}\frac{e^{-inx}-e^{-iny}}{\sqrt{n}}\frac{e^{imx}-e^{imy}}{\sqrt{m}}v_n(t)v_m(t)^*\delta_{n,m} \\
&= -\sum_{n}\frac{2-e^{-inx}e^{iny}-e^{-iny}e^{inx}}{n}|v_n(t)|^2 \\
&= -2\sum_{n>0}\frac{(1-\cos\left[n(x-y)\right])}{n}|v_n(t)|^2
\end{aligned} \tag{5.66}$$

equation (5.65) takes the form

$$\langle\tilde{\psi}_+^\dagger(x,t)\tilde{\psi}_+(y,t)\rangle = \frac{e^{-iN_+(x-y)}}{1-e^{i(x-y+i0)}}\exp\left(-2\sum_{n>0}\frac{(1-\cos\left[n(x-y)\right])}{n}|v_n(t)|^2\right) \tag{5.67}$$

Going to the physical fields using (4.48) and taking the continuum limit, for $y = 0$, yields

$$G_F(x,t) = \frac{i}{2\pi}\frac{e^{-ik_F x}}{x+i0}\exp\left(-\frac{1}{2}F_F(x,t)\right) \tag{5.68}$$

where

$$F_F(x,t) = 4\int_0^\infty \frac{dk}{k}[1-\cos(kx)]|v(k,t)|^2 \tag{5.69}$$

encodes the deviation of the Green function from the non-interacting result.

5.3.1 Correlation function after periodic quenches

We now consider a periodic quench and calculate (5.69). The two coupling functions have the form

$$g_2(k,t) = g_4(k,t) = \frac{g_2(k)}{2}\left[1-\cos\frac{\nu\pi t}{\tau}\right] \ , \ \nu\in\mathbb{Z} \tag{5.70}$$

where again we consider a Gaussian momentum cut-off of the coupling function

$$g_2(k) = g_0\exp\{-(k/k_c)^2/2\} \tag{5.71}$$

In our analysis we use $g_0 = 2\pi v_F$ and consider a quench time $\tau = 10/(v_F k_c)$. To calculate (5.69) we have to split the problem to two regimes, during the quench and after the quench, thus it has the form

- $$F_{F\,during}(x,t) = 4\int_0^\infty \frac{dk}{k}[1-\cos(kx)]|v(k,t)_{during}|^2 \tag{5.72}$$
- $$F_{F\,post}(x,t) = 4\int_0^\infty \frac{dk}{k}[1-\cos(kx)]|v(k,t)_{post}|^2 \tag{5.73}$$

The calculation of the function $v(k,t)$ in the two regimes is given by the analysis that we explained in previous section. Below we present how (5.69) evolves in time for $\nu = 9$.

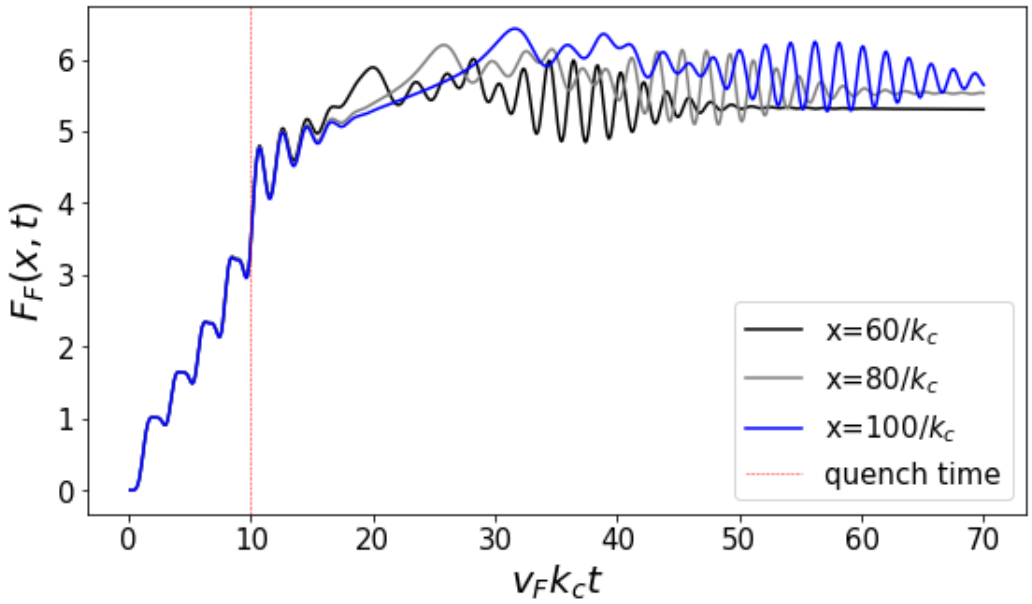

Figure 5.4: Function (5.69) for fixed separations x as a function of time t. We consider a periodic quench with quench time $\tau = 10/(v_F k_c)$ and final interaction strength $g_0 = 2\pi v_F$.

To analyse the light-cone effect in the fermionic Green function we consider the time dependence at fixed separations x. It is clear from the figure a propagation of a wave, which corresponds to the propagation of the correlations. Two main features can be distinguished from the propagating pulse. The first feature is the initial propagating "bump" while the second is the propagation of the wave-packet. We can extract the velocities of those features which for late times, follows a linear relation $v = x/(2t)$.

$features$	velocities
bump	$1.71563 v_F$
wave-packet	$0.91959 v_F$

Table 5.1: Velocities of the two features for periodic quench with quench time $\tau = 10/(v_F k_c)$ and final interaction strength $g_0 = 2\pi v_F$.

From the table we can see that the two features move with different velocities. The "bump" moves faster than the wave packet. We tried to find why this happens but the way we attacked the problem couldn't describe this behaviour, concluding that some other mechanism, more complicated, is responsible for this difference in the velocities.

Chapter 6

Conclusions

In this thesis we studied the dynamics of a Luttinger liquid under periodically driving of the internal parameters and also under a periodic quench, of finite duration. We saw that in the case of periodic driving the system develops an instability due to the parametric resonance. Calculating the kinetic energy density we found that meta-stable states can form for fast and slow driving until some characteristic time t^*, beyond which the system goes unstable again. Moreover, we calculated the equal time correlation function of the right moving fermions under a periodic quench. Clearly the light-cone effect is present, but further investigation of the correlation function is needed in order to explain why certain features move with different velocities.

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