Utrecht University

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Twisted Cohomology and Feynman Integrals

Master Thesis

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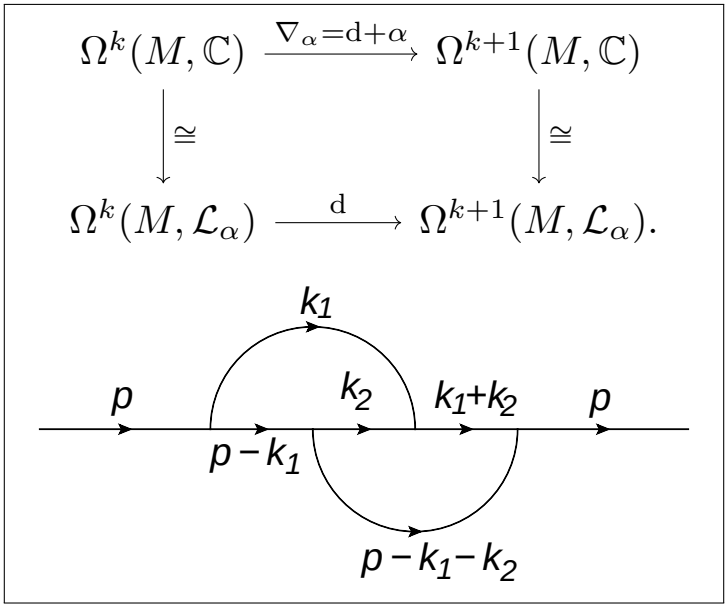
TWIN Master: Mathematical Sciences and Theoretical Physics



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Abstract

This thesis has two aims. The primary aim is to introduce the intersection product method of evaluating families of Feynman integrals, and to put it into physical and mathematical context. In order to achieve this, we consider Feynman integrals and IBP relations between such integrals. We then define and consider a local system and the twisted cohomology of said system, proving multiply properties of either. This includes the existence of a Morse function which can be used to calculate the cohomology. We then define the intersection product on the twisted cohomology and consider its properties.

The secondary aim is to judge how well the method does in practice. We compute a few examples and argue that the method is suitable for Feynman diagrams with a low number of loops, while being less suitable for higher loop counts.

Contents

1 Introduction

Before we start with the content, let us first consider the context of this thesis.

1.1 Motivation

At the moment of writing, the best way to test a quantum field theory is by comparing the predictions of the theory to the experimental data from CERN. Specifically, CERN has the Large Hadron Collider where they repeatedly measure the output particles of a collision of a known set of input particles (with known momenta). This allows them to measure the probability of each possible output given the input.

To then test a theory, one has to use the theory to similarly predict the probability of each possible output given the input. This is done using Feynman diagrams. For any input and output one can determine infinitely many Feynman diagrams corresponding to said input and output. Calculating the value of each Feynman diagram gives the scattering amplitude for the output, and squaring the scattering amplitude gives a likelihood. Normalising this likelihood with the likelihoods of all other outputs gives a probability. This means we have to calculate the value of each Feynman diagram.

Of course, there are generally infinitely many diagrams. The number of diagrams with N edges is finite for any N, but the number generally increases as N increases. We therefore may generally only make predictions in the perturbative regime, i.e. when the value of the diagrams with N edges goes to 0 sufficiently quickly for $N \to \infty$. Even in the perturbative regime, we can only calculate the value of finitely many independent Feynman diagrams. We therefore have to estimate the value of all complicated Feynman diagrams which gives an error to the theoretical prediction.

This error might make two different theories indistinguishable if the difference between the predictions is within the error range. It is therefore of utmost importance that we try to get this error as low as possible.

The value of a Feynman diagram is calculated using the corresponding Feynman integral, which is derived from the diagram by using the Feynman rules of the quantum field theory. This means that the size of the error strongly depends on the time it takes to evaluate the Feynman integrals.

It is in this context that we look at a recent technique for evaluating families of Feynman integrals.

1.2 Aims

The technique we consider is the intersection product as described in [Miz20]. The primary aim is to give the necessary background to understand the technique. This involves studying current methods, investigating the mathematical background and explaining the technique in detail. The secondary aim is to see how this new technique performs, by calculating a few examples and comparing it to current techniques.

This document has six sections. The first section is this introduction. The second section describes the preliminaries which are required to respectively understand the physics and mathematics of this thesis. The third section describes the current methods used to evaluate families of Feynman integrals, both the main technique and supplementary techniques. The fourth section sketches the mathematical background for the intersection product, and the fifth section defines the intersection product itself and shows a few examples. The final section attempts to answer the secondary aim and discusses possible future research.

1.3 Acknowledgement

I would like to thank my supervisors Eric Laenen and Gil Cavalcanti for guiding me through this project from start to finish. I also want to thank Sebastian Mizera[1] for helpful answers and suggestions.

On a personal level, I would like to thank my parents and friends who offered emotional support during this lockdown. I could not be at Utrecht University or at Nikhef while working on the thesis, but physical contact with my parents and online contact with my friends made the year bearable.

Finally I thank the LaTeX project, MiKTeX, TeXStudio and JaxoDraw [BT04] for their open source software which were used in the writing of this thesis.

[1] See [Miz20; Miz19; Fre+19a; Fre+19b; MM19; Man+19].

2 Preliminaries

This thesis assumes a certain level of knowledge in high-energy theoretical physics, differential geometry and algebraic geometry. In this section we briefly discuss what knowledge is required for the physical and mathematical parts of this thesis respectively, and where the knowledge can be obtained.

2.1 Physical preliminaries

A field theory is described by its Lagrangian, which is the integral of the Lagrangian density over space (generally $\mathbb{R}^3$). Equivalent descriptions are in terms of the action, which is the integral of the Lagrangian over time hence the integral of the Lagrangian density over space-time, or the Hamiltonian, which is the Legendre transform of the Lagrangian. The Lagrangian determines the equation of motion of the physical system via the Euler-Lagrange equation. See a standard textbook on field theory such as [Sop08] or [Sre07] for the Lagrangian, Lagrangian density, action, Euler-Lagrange equation and Hamilton's principle. The Legendre transform is explained and proven in [MS17].

A quantum field theory is a specific kind of field theory which incorporates canonical quantisation, as explained in [Sre07]. The Lagrangian density contains a kinetic term for each particle field and interaction terms. (We see the quadratic self-interaction of any field as part of the kinetic term.) For any quantum field theory and for each particle field in that theory, one can extract the propagator of that field from the kinetic part. The form of the propagator depends on the spin of the particle. For spin 0 particle fields the propagator in momentum representation has the value

$$\frac{1}{q^2 - m^2},$$

where $q = (q_0, q_1, q_2, q_3)$ is the four-vector momentum of the particle and m^2 is the mass term. These propagators give the possible edges in the Feynman diagram. The other interaction terms can then be translated into vertices of the Feynman diagram. For instance, a term $\lambda\phi^2 H$ for λ constant and particle fields ϕ and H gives a vertex with value λ connecting two ϕ propagators and one H propagator. In this way, the Lagrangian determines which propagators and vertices might appear, hence which Feynman diagrams appear in the theory.

The Feynman diagram is then turned into a Feynman integral as follows. First we choose a momentum for each edge, such that the incoming and outgoing momenta at each vertex agree. The choice of external momenta is fixed, but there is generally some choice for internal momenta. We then replace all edges and vertices by their values (up to some symmetry factors and similar details) to get an integrand, and integrate over all possible choices of internal momenta. See [Sre07] for the full procedure and all details.

This then gives us an integral, but this integral typically has issues such as divergences. This is because the procedure of taking the value of a Feynman diagram implicitly assumes analytical continuations, so we similarly have to see our integral as the analytic continuation of a well-defined integral rather than the actual integral we wrote down. This analytic continuation is done using regularisation. We use dimensional regularisation; this involves formally changing the dimension of our space-time from $4 = 3 + 1$ to $d = 4 - 2\varepsilon$ for some small $\varepsilon > 0$. The integral over a space of dimension d is not generally mathematically well-defined, so one typically rewrites the integral before applying this substitution. Dimensional regularisation and other kinds of regularisation are explained in [Sre07].

Another issue is that $1/(q^2 - m^2)$ diverges for $q^2 = m^2$, which often occurs in Feynman integrals as well. In order to solve this we instead use the Feynman propagator $1/(q^2 - m^2 + i\varepsilon)$ for a small $\varepsilon > 0$. We implicitly see the time direction $\mathbb{R}$ as a real contour inside $\mathbb{C}$. This is not the same ε as in the dimensional regularisation; some literature uses the notation $i0$ to avoid confusion. The Feynman propagators can also be found in [Sre07].

Finally, Feynman integrals generally correspond to Feynman diagrams with external particles with non-zero spin. In that case, the result of the Feynman integral is generally a tensor rather than a scalar. However any such Feynman integral can be reduced to a sum of scalar Feynman integrals by repeated inner products with external momenta [Smi06]. We therefore only consider scalar Feynman integrals.

In this thesis we will not evaluate any single given loop diagram such as the bubble, triangle or box diagram. Instead, we focus on using the outcomes of a few known Feynman integrals in a family of Feynman integral in order to evaluate the entire family.

2.2 Mathematical preliminaries

This thesis uses a lot of concepts and results from differential geometry and algebraic topology, as well as some concepts from algebraic geometry.

As for differential geometry, we assume the reader is familiar with differential forms and related notions. For physicists, a differential k-form is locally equivalent to a fully anti-symmetric k-tensor as in general relativity. A differential k-form ϕ is closed if the derivative $\mathrm{d}\phi$ is 0.[2] A differential form is exact if it is the derivative of a form. Note that $\mathrm{d}^2 = 0$ so each exact form is closed. Details of all these concepts can be found in [Lee12], while more physics minded explanations are in [Nak03].

For algebraic topology we assume that the reader is familiar with CW-complexes[3] and cellular cohomology. The main idea of CW-complexes is that we can iteratively build a topological space by starting with points (0-discs), gluing in lines (1-discs), gluing in surfaces (2-discs), etc. The space one obtains after gluing all discs up to k-discs is known as the k-skeleton. One can then use these glued-in discs (known as cells) to define the cellular (co)homology. Both CW-complexes and their cellular (co)homology can be found in [Hat05]. We also assume that the reader knows about the fundamental group of loops up to continuous deformations (also known as the first homotopy group) and the fundamental groupoid of paths up to continuous deformations (which is the generalisation of the fundamental group to any homotopy class of paths, not just loops). The former can be found in [Hat05; FF16] while the latter is in more advanced textbooks such as [May99]. Some results assume the readers knows of the Serre spectral sequence as discussed in [Hat04] or [FF16] and of Morse theory as discussed in [Mil16].

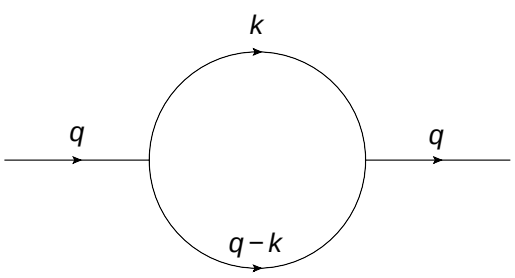
Algebraic geometry is used implicitly in the statement of some results and explicitly in the proofs. We only sketch said proofs for this reason, as going into algebraic geometry was not the aim of this thesis. A full understanding of the proofs requires knowledge of sheaf theory and algebraic varieties. A geometric introduction to presheaves can be found in [BT+82]. Details on sheaf theory can be found in [Bre12].

Finally, we assume the reader knows of the Residue Theorem from complex analysis, which can be found in any modern complex analysis textbook.

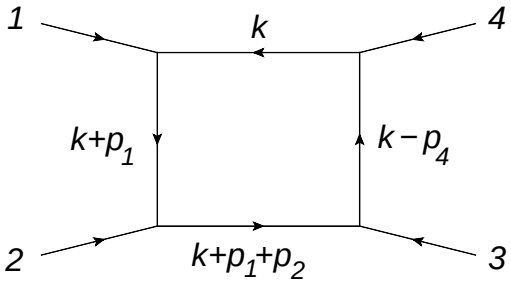
Many results in this thesis require that we work on a compact manifold. However, the manifolds we find are generally not compact. This can be solved by using a physical assumption; since the integrals we consider come from physics, they have certain properties. The relevant property here is the existence of an ultraviolet (UV) and infrared (IR) cut-off. We may always cut off the parts where the energy becomes too large or too small, although they typically have to be restored at the end of a computation by taking the limit of the lower bound to 0 and of the upper bound to ∞. Mathematically, this simply means that we may assume there exists a compact manifold with boundary $M_{c.o.}$ which is a deformation retract of M.

[2] For $k = 2$ this can physically be stated as $(\mathrm{d}\phi)_{\mu\nu\rho} = \partial_\mu\phi_{\nu\rho} + \partial_\nu\phi_{\rho\mu} + \partial_\rho\phi_{\mu\nu} = 0$.

[3] The 'CW' stands for 'closure-finite weak' complexes, as it is required that the closure of any cell is contained in a finite union of other cells and the complex is given the weak topology.



(a) The bubble diagram.



(b) The box diagram.

Figure 1: The diagrams of Subsection 3.1 and Example 3.2.1.

3 Evaluating and representing Feynman Integrals

In the Preliminaries we have seen the basic ideas used to evaluate a single Feynman integral. In practice it is rarely sufficient to calculate a single integral, and we instead need to evaluate entire families. Fortunately the Feynman integrals are not independent, so there are techniques to evaluate large number of Feynman integrals at once. The intersection product which we will discuss is one such technique, but there are also other and complementary techniques.

We discuss a few relevant techniques. We focus on the iterative reduction to master integrals, but we also discuss compatible techniques such as evaluation using differential equations and different representations of the Feynman integral.

3.1 Recursively one-loop diagrams

We consider a useful technique which is applicable to diagrams containing a bubble sub-diagram. Consider the bubble diagram with external momentum q as in Figure 1a. The family of Feynman integrals corresponding to the diagram is given by

$$F(\lambda_1, \lambda_2; d) = \int \mathrm{d}^d k \, \frac{1}{[k^2]^{\lambda_1} \, [(q-k)^2]^{\lambda_2}}.$$

Using Feynman parameters this integral can be evaluated, and the result is [Smi06:A.7]

$$F(\lambda_1, \lambda_2; d) = \frac{i\pi^{d/2}}{[q^2]^{\lambda_1+\lambda_2-d/2}} \, \frac{\Gamma(d/2-\lambda_1)\Gamma(d/2-\lambda_2)\Gamma(\lambda_1+\lambda_2-d/2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(d-\lambda_1-\lambda_2)} =: \frac{i\pi^{d/2}}{[q^2]^{\lambda_1+\lambda_2-d/2}} \, G(\lambda_1, \lambda_2; d).$$

The idea is to apply this result to bubble sub-diagrams of a larger Feynman diagram. Rather than requiring q to be an external momentum, we allow it to be any linear combination of external momenta and internal momenta other than k. As such, anytime there is some internal momentum k for which the only k-dependent terms have the form $[k^2]^{-\lambda_1}[(k-q)^2]^{-\lambda_2}$, we can replace the integral with a propagator $1/[q^2]^{\lambda_1+\lambda_2-d/2}$ up to some constant factor which can be pulled out of the integral. This diagrammatically corresponds to replacing the bubble diagram by a single line with weight $\lambda_1 + \lambda_2 - d/2$.

Any diagram which can be iteratively reduced to a one-loop diagram using the above procedure is called *recursive one-loop* and the corresponding integral is relatively easy to evaluate. We will see how to use these recursively one-loop integrals in Example 3.2.2.

3.2 IBP reduction to Basis Integrals

In this subsection we will discuss a way of relating Feynman integrals called the IBP reduction, and use it to calculate a family as an example. This entire subsection is based on Chapter 5 of [Smi06].

The goal of IBP reduction is to write any element of a family of Feynman integrals in terms of a basis[4] of a few chosen integrals, known as *master* integrals. The strategy is to use relations which resemble Integration

[4]Unlike the strict mathematical definition, it is common in physics to be contend with a non-minimal spanning set. For the intersection product we will have to use minimal bases, but in this section it is sufficient to find any spanning set.

By Parts (IBP) for general integrals, hence use the product rule of differentiation and Stokes' theorem. In particular, we know for an arbitrary orientable manifold with boundary O and differentiable functions f, g on O that

$$\int_O \frac{\partial f}{\partial x}\, g(x)\, \mathrm{d}x + \int_O f(x)\, \frac{\partial g}{\partial x}\, \mathrm{d}x = \int_O \frac{\partial}{\partial x}(fg)\, \mathrm{d}x = \int_{\partial O} f(x)g(x)\, \mathrm{d}x.$$

If $\partial O = \emptyset$, we find that the final term is 0. This gives us a straightforward relation between the two integrals in the first term. A particular choice is $g(x) = x$, which then tells us that

$$\int_O f(x)\, \mathrm{d}x = -\int_O \frac{\partial f}{\partial x}\, x\, \mathrm{d}x. \tag{1}$$

We apply the same idea to Feynman integrals. Remark that this is non-trivial in the case of Feynman diagrams, since it is unclear what effect dimensional regularisation formally has on the manifold and its boundary. Fortunately we may still use Equation 1 for dimensionally regularised Feynman integral, which is shown in Appendix E of [Smi06]. Alternatively, we can instead use the same principle on alternative representations of the Feynman integral where the dimensional regularisation has no effect on the domain of integration. Such representations are described in Subsection 3.4.

To explain the method, we first apply it to generic one-loop integrals. Consider a family of integrals

$$F(a_\ell; p_i) = \int \mathrm{d}^d k\, \frac{1}{E_1^{a_1} \dots E_N^{a_N}},$$

where $\ell \in \{1, \dots, N\}$ for N the number of separate terms, and where every E_ℓ is of the form

$$E_\ell = (k - p_\ell)^2 - m_\ell^2$$

where m_ℓ is the mass of the corresponding propagator and p_ℓ is a linear combination of external momenta p_i. Applying the d-dimensional vector equivalent of Equation 1 we find

$$\begin{aligned}
d \cdot F(a_\ell; p_i) &= \int \mathrm{d}^d k\, \frac{d}{E_1^{a_1} \dots E_N^{a_N}} \\
&= -\int \mathrm{d}^d k\, k \cdot \frac{\partial}{\partial k} \left(\frac{1}{E_1^{a_1} \dots E_N^{a_N}} \right) \\
&= -\sum_{\ell=1}^{N} \int \mathrm{d}^d k\, \frac{k}{E_1^{a_1} \dots E_{\ell-1}^{a_{\ell-1}} E_{\ell+1}^{a_{\ell+1}} \dots E_N^{a_N}} \cdot -a_\ell \frac{2k - 2p_\ell}{E_\ell^{a_\ell+1}} \\
&= \sum_{\ell=1}^{N} \int \mathrm{d}^d k\, 2a_\ell\, \frac{k^2 - p_\ell \cdot k}{E_1^{a_1} \dots E_\ell^{a_{\ell+1}} \dots E_N^{a_N}}
\end{aligned}$$

We can now rewrite $k^2 - p_\ell \cdot k$ as a linear combination of $E_\ell - p_\ell^2 + m_\ell^2 = k^2 - 2p_\ell \cdot k$ and k^2. Assume without loss of generality that $p_1 = 0$ such that $E_1 = k^2 - m_1^2$. We then get:

$$\begin{aligned}
d\, F(a_\ell; p_i) &= \sum_{\ell=1}^{N} \int \mathrm{d}^d k\, 2a_\ell\, \frac{\frac{1}{2}(E_\ell - p_\ell^2 + m_\ell^2) + \frac{1}{2}(E_1 + m_1^2)}{E_1^{a_1} \dots E_\ell^{a_{\ell+1}} \dots E_N^{a_N}} \\
&= \sum_{\ell=1}^{N} \int \mathrm{d}^d k\, a_\ell \left(\frac{1}{E_1^{a_1} \dots E_\ell^{a_\ell} \dots E_N^{a_N}} + \frac{m_\ell^2 - p_\ell^2 + m_1^2}{E_1^{a_1} \dots E_\ell^{a_{\ell+1}} \dots E_N^{a_N}} + \frac{1}{E_1^{a_1-1} \dots E_\ell^{a_{\ell+1}} \dots E_N^{a_N}} \right) \\
&= N \sum_{\ell=1}^{N} a_\ell\, F(a_\ell; p_i) + \sum_{\ell=1}^{N} a_\ell\, F(a_1, a_2, \dots, a_\ell + 1, \dots, a_N; p_i) \\
&\quad + \sum_{\ell=1}^{N} a_\ell\, F(a_1 - 1, a_2, \dots, a_\ell + 1, \dots, a_N; p_i) \\
&= N \sum_{\ell=1}^{N} a_\ell\, F(a_\ell; p_i) + \sum_{\ell=1}^{N} a_\ell\, \boldsymbol{\ell}^+ F(a_\ell; p_i) + \sum_{\ell=1}^{N} a_\ell\, \boldsymbol{\ell}^+ \mathbf{1}^- F(a_\ell; p_i).
\end{aligned}$$

Here we defined $\boldsymbol{\ell}^+$ to be the operator that raises the value of a_ℓ by one. This gives us an IBP relation between $F(a_\ell; p_i)$, $\boldsymbol{\ell}^+ F(a_\ell; p_i)$ and $\boldsymbol{\ell}^+ \mathbf{1}^- F(a_\ell; p_i)$. Note that $\boldsymbol{\ell}^+ F(a_\ell; p_i)$ has a higher order than the other two terms. This allows us to iteratively express higher order Feynman integrals in terms of lower order Feynman integral. In particular, if we know a basis of integrals $F(a_\ell; p_i)$ then we can use them to evaluate any higher-order diagram. It is however clear that we find no useful relations for

$$\sum_{\ell=1}^{N} a_\ell = 0.$$

This gives a boundary for how far the order of the integrals can be reduced. In order to give intuition and explicitly describe a basis we look at an example.

Example 3.2.1 (Box diagram)**.** Consider the box diagram with external (incoming) momenta p_1, p_2, p_3 and p_4 such that

$$\sum_{i\in\{1,2,3,4\}} p_i = 0, \qquad p_i^2 = 0,$$

which shows up in massless ϕ^3-theory. See Figure 1b. The corresponding family of Feynman integrals is (up to some normalising factors depending on convention) of the form

$$F(a_1,a_2,a_3,a_4) := F(a_1,a_2,a_3,a_4;p_1,p_2,p_3,p_4;d) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_4)^2]^{a_4}}.$$

We now produce an IBP relation as follows. Our integration domain has no boundaries, hence we have

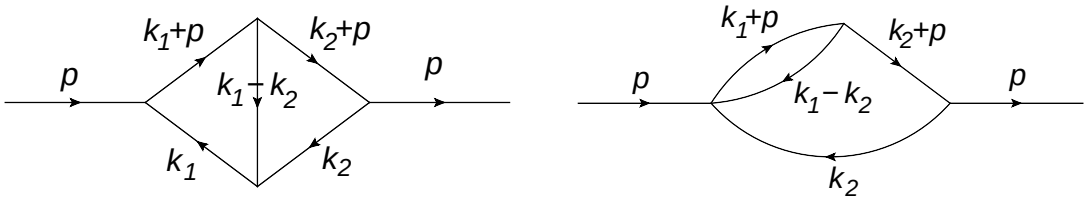
$$\begin{aligned}
0 &= \int \mathrm{d}^d k \, \frac{\partial}{\partial k} \cdot \left(\frac{k}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_4)^2]^{a_4}} \right) \\
&= d \cdot F(a_1,a_2,a_3,a_4) + \sum_{i\in\{1,2,3,4\}} \int \mathrm{d}^d k \, \frac{-2a_i \; k \cdot \left(k + \sum_{j<i} p_j\right)}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_4)^2]^{a_4}[(k+\sum_{j<i} p_j)^2]} \\
&= d \cdot F(a_1,a_2,a_3,a_4) + \sum_{i\in\{1,2,3,4\}} \int \mathrm{d}^d k \, \frac{-a_i k^2 - a_i \left(k + \sum_{j<i} p_j\right)^2 + a_i \left(\sum_{j<i} p_j\right)^2}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_4)^2]^{a_4}[(k+\sum_{j<i} p_j)^2]} \\
&= d \cdot F(a_1,a_2,a_3,a_4) + \sum_{i\in\{1,2,3,4\}} \left[-a_i + a_i \left(\sum_{j<i} p_j\right)^2 \mathbf{i}^+ - a_i \mathbf{1}^- \mathbf{i}^+ \right] F(a_1,a_2,a_3,a_4) \\
0 &= \big((d - 2a_1 - a_2 - a_3 - a_4) + 2a_3 \; p_1 \cdot p_2 \; \mathbf{3}^+ - (a_2 \mathbf{1}^+\mathbf{2}^- + a_3 \mathbf{1}^+\mathbf{3}^- + a_4 \mathbf{1}^+\mathbf{4}^-)\big) F(a_1,a_2,a_3,a_4). \qquad (2)
\end{aligned}$$

Note that we used $\mathbf{1}^+\mathbf{1}^- F = F$. The IBP relation allows us to reduce $a_1 + a_2 + a_3 + a_4$ as long as $a_3 \neq 0$ by using the $\mathbf{3}^+$ term. This iteratively allows us to reduce a_3 to 1 when starting with $a_3 > 1$. However, we then still have a_1, a_2 and a_4 which are not reduced. In order to find more relations, we use a trick: instead of starting with k, we start with $k + \sum_{j<i} p_j$ corresponding to a propagator, e.g. $k + p_1 + p_2$. Since p_1 and p_2 are constant in k this does not affect the integration by parts. Since we already did k we still have $k + p_1$, $k + p_1 + p_2$ and $k - p_4 = k + p_1 + p_2 + p_3$. Going through a similar calculation and similarly completing the squares as e.g.

$$-2a_3(k+p_1)\cdot(k+p_1+p_2+p_3) = -a_3(k+p_1)^2 - a_3(k+p_1+p_2+p_3)^2 + a_3(p_2+p_3)^2,$$

we find:

$$\begin{aligned}
0 &= \big((d - a_1 - 2a_2 - a_3 - a_4) + 2a_4 \; p_2 \cdot p_3 \; \mathbf{4}^+ - (a_1 \mathbf{2}^+\mathbf{1}^- + a_3 \mathbf{2}^+\mathbf{3}^- + a_4 \mathbf{2}^+\mathbf{4}^-)\big) F(a_1,a_2,a_3,a_4) \qquad (3) \\
&= \big((d - a_1 - a_2 - 2a_3 - a_4) + 2a_1 \; p_1 \cdot p_2 \; \mathbf{1}^+ - (a_1 \mathbf{3}^+\mathbf{1}^- + a_2 \mathbf{3}^+\mathbf{2}^- + a_4 \mathbf{3}^+\mathbf{4}^-)\big) F(a_1,a_2,a_3,a_4) \qquad (4) \\
&= \big((d - a_1 - a_2 - a_3 - 2a_4) + 2a_2 \; p_2 \cdot p_3 \; \mathbf{2}^+ - (a_1 \mathbf{4}^+\mathbf{1}^- + a_2 \mathbf{4}^+\mathbf{2}^- + a_3 \mathbf{4}^+\mathbf{3}^-)\big) F(a_1,a_2,a_3,a_4). \qquad (5)
\end{aligned}$$



(a) The double triangle diagram.

(b) The double triangle with edge 1 collapsed.

Figure 2: The diagrams of Example 3.2.2.

We see that we can reduce any $a_i > 1$ to 1. Therefore, assuming we started with all $a_i \geq 0$, we find that any Feynman integral of this family can iteratively be written as a sum of Feynman integrals $F(e_1, e_2, e_3, e_4)$ where $e_i \in \{0, 1\}$. Hence, these integrals are our master integrals.

The iterative reduction can be done systematically and algebraically by a computer using the above relations. This means we only have to calculate $2^4 = 16$ different Feynman integrals to know all massless ϕ^3 box diagram Feynman integrals with $a_i \geq 0$. These integrals therefore form a basis for the family. In fact, a smaller subset is sufficient; for instance, using Equation 4 on $F(0, 1, 0, 1)$ gives

$$(d-2)F(0,1,0,1) = F(0,0,1,1) + F(0,1,1,0).$$

This shows that many of the 16 different Feynman integrals are still related, although via less systemically applicable equations.

Remark that the relations depend on the external momenta via $p_1 \cdot p_2$ and $p_2 \cdot p_3$. This means that some of the relations may break down in the limit $p_1 \cdot p_2 = 0$ or $p_2 \cdot p_3 = 0$, no longer giving relations between different values of $a_1+a_2+a_3+a_4$. However, these product of external momenta precisely are the Mandelstam variables corresponding to the s-channel respectively the t-channel of $2 \to 2$ scattering [Smi06:pg. 41], so this limit corresponds to leaving a physical sector. As such, we only lose all our relations in unphysical cases. △

Of course, any linear combination of the above relations is itself a relation. While this fact is not immediately useful in the above Example, it is at times needed to rewrite the relations in a form where e.g. one $\mathbf{i}^+$ is isolated, or to invert the relations for negative a_i. We considered the massless case $p_i^2 = 0$, but when $p_i^2 \neq 0$ more terms appear which can be removed via linear combinations. See [Smi06:pg. 123] for an example of this using a triangle diagram with two massive legs.

Note that setting $a_i = 0$ for any i is equivalent to contracting the corresponding edge (representing a propagator) to a point. In particular, Setting $a_1 = 0$ in the box will give a triangle diagram, while also setting $a_3 = 0$ gives a bubble diagram and finally setting $a_2 = 0$ gives a tadpole diagram. This shows that the basis we obtained consists of relatively easy integrals.

For a higher number of loops, we correspondingly have more relations. The goal in these cases is to express all Feynman integrals in terms of a subset for which the diagram simplifies, such that known one-loop results can be used. This is also the strategy for one-loop diagrams for which the IBP relations do not relate different values of $\sum_i a_i$. We again look at an example to clarify the method.

Example 3.2.2 (Double triangle)**.** Let us consider the easiest example of a two-loop diagram which does not decompose into two one-loop diagrams, the double triangle in ϕ^3–theory with external momentum p. See Figure 2a. Denoting the loop momenta by k_1 and k_2, the family of Feynman integrals is given by

$$F(a_1,a_2,a_3,a_4,a_5) = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \, \frac{1}{[k_1^2]^{a_1}[(p+k_1)^2]^{a_2}[(k_1-k_2)^2]^{a_3}[(p+k_2)^2]^{a_4}[k_2^2]^{a_5}},$$

where we assume each $a_i \geq 0$. Note that for $a_3 = 0$ the loops decompose into two bubbles and the integral into a product of integrals, for which we assume know the answer[5]. As such, our goal is to write any

[5]When evaluating higher loop diagrams we assume that the integral of certain one-loop diagrams like bubble and triangle diagrams are known, see e.g. Equation A.7 from [Smi06] for the bubble diagram solution. These integrals are expressed in terms of gamma functions or polylogarithms.

$F(a_1, a_2, a_3, a_4, a_5)$ in terms of $F(e_1, e_2, 0, e_4, e_5)$ for some integers $e_i \geq 0$; the $F(e_1, e_2, 0, e_4, e_5)$ form our basis.

The IBP are determined as in the one-loop case. For k_1 we find three relations corresponding to k_1, $k_1 + p$ and $k_1 - k_2$ respectively:

$$\begin{aligned} 0 &= \big((d - 2a_1 - a_2 - a_3) + a_2\, p^2 \mathbf{2}^+ - (a_2 \mathbf{1}^- \mathbf{2}^+ + a_3 \mathbf{1}^- \mathbf{3}^+ - a_3 \mathbf{5}^- \mathbf{3}^+)\big) F(a_1, a_2, a_3, a_4, a_5) && (6)\\ &= \big((d - a_1 - 2a_2 - a_3) + a_1\, p^2 \mathbf{1}^+ - (a_1 \mathbf{2}^- \mathbf{1}^+ + a_3 \mathbf{2}^- \mathbf{3}^+ - a_3 \mathbf{4}^- \mathbf{3}^+)\big) F(a_1, a_2, a_3, a_4, a_5) && (7)\\ &= \big((d - a_1 - a_2 - 2a_3) - (a_1 \mathbf{3}^- \mathbf{1}^+ - a_1 \mathbf{5}^- \mathbf{1}^+ + a_2 \mathbf{3}^- \mathbf{2}^+ - a_2 \mathbf{4}^- \mathbf{2}^+)\big) F(a_1, a_2, a_3, a_4, a_5), && (8) \end{aligned}$$

where the surprising $\mathbf{5}^-$ shows up because of the factor $k_2^2 = 2k_1 \cdot (k_1 - k_2) - (k_1 - k_2)^2 - k_1^2$ and similarly for the $\mathbf{4}^-$. For k_2 we similarly have relations corresponding to $k_2 - k_1$, $k_2 + p$ and k_2 respectively:

$$\begin{aligned} 0 &= \big((d - 2a_3 - a_4 - a_5) - (a_4 \mathbf{3}^- \mathbf{4}^+ - a_4 \mathbf{2}^- \mathbf{4}^+ + a_5 \mathbf{3}^- \mathbf{5}^+ - a_5 \mathbf{1}^- \mathbf{5}^+)\big) F(a_1, a_2, a_3, a_4, a_5) && (9)\\ &= \big((d - a_3 - 2a_4 - a_5) + a_5\, p^2 \mathbf{5}^+ - (a_3 \mathbf{4}^- \mathbf{3}^+ + a_5 \mathbf{4}^- \mathbf{5}^+ - a_3 \mathbf{2}^- \mathbf{3}^+)\big) F(a_1, a_2, a_3, a_4, a_5) && (10)\\ &= \big((d - a_3 - a_4 - 2a_5) + a_4\, p^2 \mathbf{4}^+ - (a_3 \mathbf{5}^- \mathbf{3}^+ + a_4 \mathbf{5}^- \mathbf{4}^+ - a_3 \mathbf{1}^- \mathbf{3}^+)\big) F(a_1, a_2, a_3, a_4, a_5). && (11) \end{aligned}$$

Clearly, the above six relations allow us to reduce a_1, a_2, a_4 or a_5 without much effort from $a_i \geq 1$ to a sum of integrals with value $a_i = 1$. However, this does not work for a_3, since we do not have any expression with an isolated $\mathbf{3}^+$. We therefore need to take linear combinations of the above relations to find an expression for $\mathbf{3}^+$. We try $a_1 \mathbf{1}^+$ (6) $+ a_2 \mathbf{2}^+$ (7) $- a_3 \mathbf{3}^+$ (8), where we note that any $\boldsymbol{\ell}^+$ does not commute with the weight a_ℓ in front of a $F(a_1, a_2, a_3, a_4, a_5)$ and instead shifts the value of the corresponding a_ℓ by one when pulled through. We leave out the F for notational convenience, but remember that this is an equality of operators. We then find that

$$\begin{aligned} 0 =& a_1\big((d - 2a_1 - 2 - a_2 - a_3)\mathbf{1}^+ + a_2\, p^2\, \mathbf{1}^+\mathbf{2}^+ - (a_2 \mathbf{2}^+ + a_3 \mathbf{3}^+ - a_3 \mathbf{1}^+ \mathbf{5}^- \mathbf{3}^+)\big)\\ &+ a_2\big((d - a_1 - 2a_2 - 2 - a_3)\mathbf{2}^+ + a_1\, p^2\, \mathbf{2}^+\mathbf{1}^+ - (a_1 \mathbf{1}^+ + a_3 \mathbf{3}^+ - a_3 \mathbf{2}^+ \mathbf{4}^- \mathbf{3}^+)\big)\\ &- a_3\big((d - a_1 - a_2 - 2a_3 - 2)\mathbf{3}^+ - (a_1 \mathbf{1}^+ - a_1 \mathbf{3}^+ \mathbf{5}^- \mathbf{1}^+ + a_2 \mathbf{2}^+ - a_2 \mathbf{3}^+ \mathbf{4}^- \mathbf{2}^+)\big)\\ =& a_1(d - 2a_1 - 2a_2 - 2)\mathbf{1}^+ + a_2(d - 2a_1 - 2a_2 - 2)\mathbf{2}^+ - a_3(d - 2a_3 - 2)\mathbf{3}^+ + 2a_1a_2\, p^2\, \mathbf{1}^+\mathbf{2}^+. \end{aligned}$$

Rewriting gives

$$a_3(d - 2a_3 - 2)\mathbf{3}^+ = a_1(d - 2a_1 - 2a_2 - 2)\mathbf{1}^+ + a_2(d - 2a_1 - 2a_2 - 2)\mathbf{2}^+ + 2a_1a_2\, p^2\, \mathbf{1}^+\mathbf{2}^+,$$

acting on $F(a_1, a_2, a_3, a_4, a_5)$.

We see a slight problem, at some point we reach the boundary $a_3(d - 2a_3 - 2) = 0$ and can not reduce a_3 further. In dimensional regularisation d is non-integer so we only have to worry about $a_3 = 0$; but that still implies we can only reduce $a_3 \geq 1$ to 1, not to 0. Of course, this is not the only relation we can use; we can also try to use Equations 8 or 9. Unfortunately those Equations don't solve our fundamental problem: we aim to write everything in terms of integrals with $a_3 = 0$, but that simply is too small a subset of the family and does not work as a basis. We will need to include at least some master integral with $a_3 = 1$. For this we use the recursively one-loop diagrams introduced in Subsection 3.1.

Note that for $a_1 = 0$ the left triangle collapses to some bubble diagram, see Figure 2b. We can use the evaluation of the bubble diagram as a substitution to get a triangle diagram, for which we know the general evaluation. As such, we know the value of any integral with $a_1 = 0$. Similarly we know the value for any integral with $a_2 = 0$. It is therefore sufficient to reduce either a_1, a_2 or a_3 to 0. Moreover, our situation is symmetric in $(1, 2, 3) \leftrightarrow (5, 4, 3)$ so the same holds for a_4 and a_5. We then can evaluate any integral of the given family by iteratively using Equations 8, 9, since for any $a_4 \geq 1$ or $a_5 \geq 1$ the latter equation reduces one of a_1, a_2, a_3 to 0, while for $a_4 = 0$ or $a_5 = 0$ we also know the value of the integral. This concludes the evaluation; we see that in the end we only need prior knowledge of a lower dimension (the triangle diagram and bubble diagram values) and two IBP relations in order to determine all integrals of this family. △

In the example above we used two standard techniques: the recursively one-loop diagrams and the triangle sub-diagrams. The former is very commonly used to a find a manageable basis, while the latter gives comparatively easy IBP relations. These techniques are rarely sufficient but are nevertheless very helpful when using IBP relations.

Remark 3.2.3. In the above examples we always worked with the $a_i \geq 0$ integers. However, the bubble diagram evaluation as seen in Subsection 3.1 shows how we may find integrals with a_i non-integer. In that case one can not work with a basis of $a_i = 0$, but instead a basis with $a_i \leq 0$ is used. Fortunately integrals with $a_i < 0$ can often still be evaluated using known integrals, see for instance Equation (A.2) in [Smi06:Appendix A]. Nevertheless, non-integer a_i complicate the choice of a basis. One positive side effect of a_i non-integer is that $a_i - k \neq 0$ for any $k \in \mathbb{Z}$, so there are typically less boundaries on the IBP relations. •

This concludes our discussion of the IBP relations. We shortly consider two similar relations which are not sufficient by themselves but can be used along with the IBP relations.

3.2.1 Other relations

We consider two extra relations. The first is called the Lorentz Invariance identity and follows from the fact that scalar Feynman integrals are invariant under infinitesimal Lorentz transformations of the external momenta. [Smi06] This tells us that

$$F(a_\ell; p_i) = F(a_\ell; p_i^\mu + \varepsilon_\nu^\mu p_i^\nu),$$

for ε an infinitesimal transformation. The precise form of this identity differs depending on the shape of $F(a_\ell; p_i)$, but generally this provides a relation between derivatives of $F(a_\ell; p_i)$ with respect to the p_i. There is some discussion whether or not the LI relations are just linear combinations of IBP relations, see for instance [Smi06; Gro11]. However, there is little discussion that the LI relations can be useful.

The second relation uses the dimension of the Feynman integral. Consider the operator $\mathbf{d}^{2-}$ which reduces the dimension of the Feynman integral by two:

$$\mathbf{d}^{2-} F(a_\ell; p_i; d) = F(a_\ell; p_i; d-2).$$

This operator can be written explicitly using the Symanzik polynomial which we will see in Subsection 3.4:

$$\mathbf{d}^{2-} = \frac{i^L}{\pi}\, \mathcal{U}(z_1, \ldots, z_L)\Big|_{z_\ell \mapsto i\, a_\ell\, \boldsymbol{\ell}^+}. \quad \text{[Smi06]}$$

This means that if we can calculate $F(a_\ell; p_i; d+2)$ for all a_ℓ then we can derive $F(a_\ell; p_i; d)$ as well. This is useful as it allows us to move boundaries such as the boundary $d - 2a_3 - 2$ in Example 3.2.2.

We have seen multiple identities which linearly relate Feynman integrals in a family. We now look at an accompanying technique which is typically used after reduction to a basis of integrals to evaluate said integrals.

3.3 Evaluation using Differential Equations

In this subsection we look into a way of evaluation individual Feynman integrals using the result of the IBP relations derived above in order to make a system of differential equations (DE). We assume that we have a family of Feynman integrals and a basis $I_1, \ldots, I_k$ of master integrals such any element of the family can be expressed in terms of the basis. Any derivative of a basis integral I_j with respect to external parameters $x = (x_1, \ldots, x_\ell)$ (e.g. mass or other Mandelstam invariants of the external momenta) will then again be a member of the family, hence will give an expression in terms of the I_j. This means that for each external parameter x_i we can find some matrix A_{x_i} depending on the external parameters and on the dimension d such that:

$$\frac{\partial}{\partial x_i} I_j(x; d) = \big(A_{x_i}(x; d)\big)_{jj'} \cdot I_{j'}(x; d).$$

By construction, this matrix A_{x_i} can be calculated by taking the derivatives with respect to the external parameters and then using the algorithms described in Subsection 3.2. We can do this for any i, meaning we know the derivative of $I_j(x; d)$ in each direction. We can then integrate over the derivative to find $I_j(x; d)$. The steps one needs to take in order to evaluate the I_j are to choose a base point x', to determine the boundary conditions to fix $J_{j'}$, and to integrate the exponent of the matrix over a path γ from x' to x:

$$I_j(x; d) = \left[\prod_{t \in [0,1]} \exp\left(\sum_{i=1}^{\ell} \gamma_i'(t)\; A_{x_i}(\gamma(t); d)\; \mathrm{d}t \right) \right]_{jj'} \cdot J_{j'}. \tag{12}$$

Here $\boldsymbol{\pi}$ denotes a path-ordered integral or product integral[6], i.e.

$$\boldsymbol{\pi}_{t\in[0,1]} \exp(X(\gamma(t))) := \lim_{N\to\infty} \prod_{i=1}^{N} \exp\left(X(\gamma(t_i))\,\Delta t\right) \neq \exp\left(\lim_{N\to\infty} \sum_{i=1}^{N} X(\gamma(t_i))\,\Delta t\right) = \exp\left(\int_{\gamma:x'\rightsquigarrow x} X(\gamma)\right),$$

where $t_i = i/N \in [0,1]$ for $i \in \{0,1,\ldots,N\}$ and $\Delta t = 1/N$. The product is sorted such that t_N is leftmost and t_0 is rightmost. We need to use such an integral since for matrices X, Y we generally have $\exp(X+Y) \neq \exp(X)\exp(Y)$.

The above expression for $I_j(x,d)$ is not uniquely defined, as it depends on the choice of a path γ. This behaviour is expected, as Feynman integrals generally need branch cuts to be single-valued and we haven't chosen branch cuts. After choosing branch cuts in x, the expression no longer depends on the path γ as long as we demand that the path never crosses a branch cut.

3.3.1 The canonical form

The above path-ordered integral is typically very hard to evaluate, but is very simple for e.g. A_{x_i} diagonal or more generally if the matrix $A_{x_i}(\gamma(t))$ commutes with $A_{x_i}(\gamma(t'))$ for all $t,t' \in [0,1]$. As such, the difficulty of the integral strongly depends on the choice of a basis I_j. Indeed the main complication in evaluating using DE is to find an alternate basis for which the matrix A_{x_i} simplifies to an easier form. We will denote one such easier form as the *canonical form* [Hen15], that is the form

$$\mathrm{d}\vec{I}(x;d) = \varepsilon(\mathrm{d}\tilde{A})\ \vec{I}(x;d), \qquad \tilde{A} = \sum_k A_k\ \log(\alpha_k(x)),$$

where $d = 4-2\varepsilon$, A_k is a constant matrix and $\alpha_k(x)$ is an algebraic expression in x, e.g. $x - p_k$ where p_k denotes the k-th singularity of the canonical form. Remark that in the above formula the derivative d only derives with respect to the parameters x, not the dimension d. The set $\{\alpha_k(x)\}$ is called the *alphabet*.

In the canonical form Equation 12 becomes:

$$\vec{I}(x;d) = \left[\boldsymbol{\pi}_{\gamma:x'\rightsquigarrow x} \exp(\varepsilon\ \mathrm{d}\tilde{A})\right] \cdot \vec{J}. \tag{13}$$

Using the work of Carl Miller on exponential iterated integrals [Mil05:Thm.3.2] we can write this path-ordered integral as a sum of Chen iterated integrals:

$$\vec{I}(x;d) = \left[\sum_{n\in\mathbb{N}} \int_{\gamma:x'\rightsquigarrow x} (\varepsilon\ \mathrm{d}\tilde{A})^n\right] \cdot \vec{J} = \left[\sum_{n\in\mathbb{N}} \varepsilon^n \int_{\gamma:x'\rightsquigarrow x} (\mathrm{d}\tilde{A})^n\right] \cdot \vec{J}, \tag{14}$$

where

$$\int_{\gamma:x'\rightsquigarrow x} \omega^n := \int_{0\leq t_1<\cdots<t_n\leq 1} \gamma^*\omega(t_1)\cdot\ldots\cdot\gamma^*\omega(t_n).$$

Note that the exponential has no factor $1/n!$ since the volume of the integration domain $0 \leq t_1 < \cdots < t_n \leq 1$ already is $1/n!$; equivalently we could allow the $t_i \in [0,1]$ to be independent which would require a factor $1/n!$ and an ordering convention.

Let us stress that Equation 14 is very useful. The equation describes the value of the Feynman integral as a sum of powers of ε and integrals. In particular, the integrals have no further ε-dependence hence do not have to be simplified further and can easily be integrated at the end of the calculation (although numerical integration may require a choice of a branch cut). This means that the canonical form provides a very explicit value for $\vec{I}(x;d)$.

Remark that Miller's exponential iterated integral is not the only way to expand the product integral in Equation 13. Another useful method is the Magnus expansion [IMN99], which writes the product integral as an exponential of a sum of commutators:

$$\vec{I}(x;d) = \exp(\Omega)\cdot\vec{J}, \qquad \Omega = \sum_{k\in\mathbb{N}} \Omega_k,$$

[6]Other notations for the path-ordered integral $\boldsymbol{\pi}\exp(A)$ include $\mathcal{P}\exp\left(\int A\right)$, $\mathbb{P}\exp\left[\int A\right]$ and $\prod\exp(A)$.

where

$$\Omega_1 = \varepsilon^1 \int_{\gamma:x'\rightsquigarrow x} \mathrm{d}\tilde{A} = \varepsilon^1 \int_{t\in[0,1]} \gamma^* \mathrm{d}\tilde{A}(t), \qquad \Omega_2 = \frac{1}{2}\varepsilon^2 \int_{0\leq t_1<t_2\leq 1} [\gamma^*\mathrm{d}\tilde{A}(t_1), \gamma^*\mathrm{d}\tilde{A}(t_2)],$$

and generally the term Ω_k has power ε^k and contains $k-1$ nested commutators. Because of the exponential $\exp(\Omega)$ this expansion mixes powers of ε and is hence less useful as an quantitative expansion. The expansions however has the property that truncations still qualitatively resemble the full expansion. For instance, any singularity of $\int_{\gamma:x'\rightsquigarrow x} \mathrm{d}\tilde{A}$ might give rise to an essential singularity in the product integral. Any truncation of the expansion in Equation 14 will only give a singularity of finite order, while the first order term of the Magnus expansion indeed gives an essential singularity.

The result of the repeated integrals is not always expressible in terms of known functions, but will often be made of logs and iterative integrals over logs such as polylogarithms [Hen15]

$$\mathrm{Li}_n(x) = \sum_{k>1} \frac{x^k}{k^n}, \qquad x\,\partial_x \mathrm{Li}_n(x) = \mathrm{Li}_{n-1}(x), \qquad \mathrm{Li}_1(x) = -\log(1-x),$$

or Goncharov polylogarithms [Hen15]

$$\begin{aligned} G(a_1,\ldots,a_n;z) &:= \int_0^z \frac{dt}{t-a_1}\quad G(a_2,\ldots,a_n;t) && n>1,\\ G(a_1;z) &:= \int_0^z \frac{dt}{t-a_1} && a_1\neq 0,\\ G(\vec{0}_n;z) &= \frac{1}{n!}\log^n(z). \end{aligned}$$

The questions are now how to determine the $\vec{J}$ from the boundary conditions and how to generally write the DE in the canonical form. We will investigate these questions next.

3.3.2 Boundary conditions

We shortly consider the boundary conditions. The only boundaries are given by the points $x = p_j$ where the matrices $A_{x_i}(x;d)$ become singular or zero. As such, the boundary conditions are generally specified by the expansion of matrices around the singularities. This is often enough to specify the boundary condition at each singularity.

There are unfortunately cases where the matrix of the DE does not fully fix the boundary conditions. In this case there is a trick, which works as follows. Introduce a new parameter y and replace the occurrences of the singularity p_j by yp_j in the master integrals $\vec{I}$. For $y = 1$ the old Feynman integrals $\vec{I}(y=1) = \vec{I}$ are recovered, while for $y = 0$ one has a completely different set of Feynman integrals $\vec{I}(0)$. Now the derivative with respect to y gives a DE $\partial_y \vec{I}(y)$, which provides extra restrictions on the form of $I(y=1)$ and helps determine the boundary condition. A version of the above trick is often used to compute single-scale integrals. [Hen15]

One nuance for the limit $x \to p_j$ is that most generally the limits $x \to p_j$ and $\varepsilon \to 0$ do not commute. This gives another reason to use a simplified form like the canonical form above, as the separated x and ε–dependence in the canonical form guarantees that the limits commute. In practice the boundary conditions are often calculated through first fixing the limit $x \to 0$, then the limit $x \to \infty$ or to other singular points if necessary, and using the above trick only if the boundary conditions are still not clear. These steps are generally comparatively easy to do, but are hard to implement in a computer program and are typically done by hand. See e.g. [PTW16] for more details on calculating boundary conditions by hand.

3.3.3 Transformation into the canonical form

We now look at writing the DE in canonical form. As one might expect from how powerful the form is, it is hard to rewrite a generic DE into that form. We will only give some generic steps for a DE in a single external parameter x; see dedicated literature for more details [Hen15; Lee15]. We start with a general expression

$$\partial_x \vec{I}(x;d) = A \cdot \vec{I}(x;d).$$

The first step is to makes the poles of A explicit. Since the singularities of A must correspond to singularities of the original family of Feynman integrals, we know that the 'physical' poles of A are all regular. Here we emphasize the 'physical' to separate them from the 'spurious' singularities which may appear due to the nature of matrix transformations. Moreover, we also know that the eigenvalues of A can only have at most simple poles, since for any higher order pole at x_0 and for any eigenvector $\vec{f}$ we have [Hen15: 3.23]:

$$\partial_x \vec{f} = \frac{a}{(x-x_0)^{m+1}}\ \vec{f}(x) \qquad \text{gives} \qquad \vec{f}(x) = \exp\left(-\frac{a}{m\ (x-x_0)^m}\right)\ \vec{f}_0,$$

which has an essential singularity at $x = x_0$. This can not appear in a Feynman integral as the integrand is the product of a rational function in x.[7] We find that any non-simple poles must be spurious. There exist logarithms which use this fact to bring the DE in the following form:

$$\partial_x \vec{I}(x;d) = \left[\sum_{1\leq j\leq k} \frac{A_j(\varepsilon)}{x-p_j}\right] \cdot \vec{I}(x;d),$$

where the $A_k(\varepsilon)$ are matrices[8], the $\vec{I}$ have been changed using a basis transformation, and the p_k denote the singularities. [Lee15] In general this form is not possible without introducing a new singular point p_{k+1} and matrix $A_{k+1}(\varepsilon)$ to balance out the limit to ∞ [Hen15]; this is to be expected, since we have

$$\frac{A_\infty(\varepsilon)}{x-\infty} = 0,$$

so we would lose the information contained in A_∞ if we did not 'back it up' in another A_{k+1}.

It should be noted that the algorithm mentioned above can be done with only rational transformations; meaning the resulting $\vec{I}$ is still a rational function of x. This is relevant for some of the mathematical properties of $\vec{I}$. In particular, $\vec{I}$ will still only have regular poles and knowing the explicit form of the DE above should help when deriving the boundary conditions of the DE.

Note that the above equation already has the right shape in terms of x, since

$$\frac{1}{x-p_j}\,\mathrm{d}x = \mathrm{d}\log(x-p_j) = \mathrm{d}\log(\alpha_j(x)).$$

However, the A_j still depend on ε. The next step is to simplify this ε dependence. This is hard [Hen15], although it seems that large steps have been made in slightly more recent papers [Lee15]. The main idea here is that we expect the eigenvalues of the $A_j(\varepsilon)$ to be linear in ε for the following reason. Near a singular point p_j the DE has a solution

$$\vec{I} = P(x,\varepsilon)\ \exp(A_j(\varepsilon)\ x)\ \vec{I}_{p_j}(\varepsilon),$$

where $\vec{I}_{p_j}(\varepsilon)$ is a boundary vector at $x = p_j$ and $P(x,\varepsilon)$ is a matrix polynomial. We see that the scaling of $\vec{I}$ near $x = p_j$ is determined by $\exp(A_j(\varepsilon)\ x)$, hence by the eigenvalues of $A_j(\varepsilon)$. But from the Feynman parameter representation of the Feynman integral (which we will see in Subsection 3.4, Equation 18) we know that the scaling of $\vec{I}$ near the poles is given by terms linear in ε, i.e. $(x-p_j)^{a\,\varepsilon}$ for some $a \in \mathbb{R}$. This tells us the eigenvalues have to be linear in ε after normalisation.

However, the matrix itself can contain non-linear terms because of the nature of matrix transformations. As such, we need to transform the matrix to make the ε-dependence explicit and indeed write the differential equation in canonical form. There are algorithms for that relying on linear algebra [Lee15], but they generally take an extremely long time to compute as the size of the matrix increases and hence one has to use block-matrix simplifications. Moreover, it is not clear how to extend this algorithm to multi-variable cases with more than one x. Henn [Hen15] has proposed other methods of simplifying the ε-dependence by integrating out the ε^0 term, but they are not always usable and therefore hard to turn into an algorithm.

[7] In the representations in Subsection 3.4 the integrand is no longer purely rational, but it is still the quotient of finite powers of polynomials hence the integrand still does not have essential singularities.

[8] The $A_k(\varepsilon)$ are not generally linear in ε hence this is not the canonical form; but we will see that the eigenvalues are linear in ε.

We see that algebraically reducing the DE is hard, and there are no full generality algorithms. It is for this reason that Henn [Hen15] introduces a more geometrical manner of looking at the DE using cuts. We will shortly look at cuts after considering alternative representations of Feynman integrals such as the Feynman parameter representation we mentioned above.

3.4 Representing Feynman Integrals

Like any integral, the Feynman integral can be put into different forms using substitutions and tricks. Since the standard loop momentum representation has no straightforward way to implement dimensional regularisation mathematically (as it requires integrating over a space of dimension d for d generally non-integer) we consider a few alternative representations of the integral with a continuous dependence on the dimension. These representations also write the Feynman integral in such a manner that intersection theory can easily be applied to the integrands, as we will see in later chapters.

Below, we consider three representations. For each representation we consider the (up to constant factors) generic Feynman integral

$$F(\alpha_\ell; p_i) = \int \cdots \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \quad \frac{1}{D_1^{\alpha_1} \ \dots \ D_N^{\alpha_N}}.$$

Here the D_ℓ are quadratic expression in the external momenta p_i and internal momenta k_j. The number L denotes the number of loop momenta, N denotes the number of lines in the Feynman diagram and the number of external momenta is E (such that there are $E+1$ external lines). Note that in this subsection we use α for the weights rather than a for (subjective) notational clarity.

3.4.1 Schwinger Parameter Representation

One well-known representation is the Schwinger parameter representation, also known as the alpha parameter representation. [Smi06] This representation is easy to derive and is often used in the derivation of other representations such as the Feynman parameter representation in the next subsection. We follow [Smi06; IZ80] in the derivation of this representation; other sources such as [LP13] use a different convention which results in different factors i and different signs.

The core idea of the Schwinger parameter representation is to use the identity

$$\frac{1}{A} = -i \int_0^\infty \mathrm{d}\alpha \ \exp(i\,A\,\alpha) = -i \left[\frac{\exp(i\,A\,\alpha)}{i\,A}\right]_{\alpha=0}^{\alpha\to\infty}$$

on the propagators D_ℓ. Remark that this identity only holds when the imaginary part of A satisfies $\Im A > 0$; this is no issue since we are working with Feynman propagators which means we add $i\,\varepsilon$ to any of the propagators for a small $\varepsilon > 0$ in order to avoid poles, and we take the $\varepsilon \to 0$ limit at the end.[9] More generally, one can take the partial derivative of the above expression with respect to A to derive

$$\frac{1}{A^n} = \frac{-i^n}{\Gamma(n)} \int_0^\infty \mathrm{d}\alpha \ \alpha^{n-1} \ \exp(i\,A\,\alpha).$$

While partial derivatives give the above identity for integer n, it is possible to use the definition of the Gamma matrix to derive the identity for any $n \in \mathbb{R}$, $n > 0$, where i^n is replaced with $\exp(i\,\pi\,n/2)$.

We will be using the parameters z rather than α, as we α to denote the weights. This gives:

$$\frac{1}{D_1^{\alpha_1} \ \dots \ D_N^{\alpha_N}} = \frac{\exp(i\,\pi\,\alpha/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_N)} \int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \cdots \int_0^\infty \mathrm{d}z_N \ z_1^{\alpha_1-1} \dots z_N^{\alpha_N-1} \ e^{i\left(D_1 z_1 + D_2 z_2 \ + D_N z_N\right)},$$

where

$$\alpha = \sum_{\ell=1}^{L} \alpha_\ell.$$

[9] As we remarked in the Preliminaries, this ε is different from the ε in dimensional regularisation. Unfortunately, it seems that the common convention is to use ε for this value as well.

All terms which do not depend on D_ℓ commute with the integral over $\mathrm{d}^d k_j$, so inserting the above identity into the Feynman integral we find that the integral over $\mathrm{d}^d k_j$ simplifies to

$$\int \cdots \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_L \; e^{iD_1 z_1 + iD_2 z_2 \; \ldots \; + iD_N z_N}.$$

This is simply a product of (complex) Gaussian integrals, and hence something we can make sense of mathematically. Indeed we know the usual Gaussian integral over $\mathbb{R}^l$ with signature $(-+^{l-1})$ for any $l \in \mathbb{N}$ in terms of exponents and powers:

$$\int \mathrm{d}^{(1,l-1)} k_1 \ldots \mathrm{d}^{(1,l-1)} k_L \; \exp\left[i \left(\sum_{i,j} A_{ij} \; k_i \cdot k_j + 2 \sum_i b_i \cdot k_i + c \right) \right] = \frac{\pi^{lL/2} \; e^{-i\pi \; L(l-2)/4}}{\det(A)^{l/2}} \; \exp\left[i\, c - i \sum_{i,j} A^{-1}_{ij} \; b_i \cdot b_j \right].$$

Remark that the signature here is $(-+^{l-1})$ hence if edge ℓ has momentum q_ℓ and mass m_ℓ then $D_\ell = q_\ell^2 - m_\ell^2 = -(q_\ell^0)^2 + (\vec{q}_\ell)^2 - m_\ell^2$. This means that the q_ℓ^0 had to be integrated with a different sign from the $\vec{q}_\ell$ part, hence the factor $e^{i\pi L(l-2)/4}$ uses $(l-1)-1 = l-2$ instead of l.

The right hand side is well-defined for arbitrary $l > 1$, which solves the problem of dimensional regularisation; we can set $l = d$ instead of $l = 4$. We apply this to

$$i \left(\sum_{i,j} A_{ij} \; k_i \cdot k_j + 2 \sum_i b_i \cdot k_i + c \right) = i\, D_1 z_1 + i\, D_2 z_2 \; \ldots \; + i\, D_N z_N.$$

Note that this A_{ij} depends linearly on the z_ℓ and consists of product of the form $k_i \cdot k_j$, while b_j depends linearly on the z_ℓ and consists of product of the form $p_i \cdot k_j$. The term c contains all the masses and products of external parameters $p_i \cdot p_j$, and is also linear in the z_ℓ. It follows that $\det A$ is a homogeneous polynomial in the z_ℓ of degree L (as A is an $L \times L$ matrix) while $\sum_{ij} A^{-1}_{ij} b_i \cdot b_j$ is a homogeneous rational function (that is, the quotient of a homogeneous polynomial by another homogeneous polynomial) of total degree 1. We claim [LP13] that the product of this homogeneous rational function with $\det A$ gives a homogeneous polynomial of degree $L+1$; in other words, the denominator of the rational function is a divisor of $\det A$. We define

$$\mathcal{U} := \det A, \qquad \mathcal{V} := \det A \sum_{i,j} A^{-1}_{ij} b_i \cdot b_j.$$

This allows us to rewrite the Feynman integral as

$$F(\alpha_\ell; p_i) = \frac{\pi^{dL/2} \; e^{i\pi(\alpha/2 - L(d-2)/4)}}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_N)} \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_N \; z_1^{\alpha_1 - 1} \ldots z_N^{\alpha_N - 1} \; \mathcal{U}^{-d/2} \; \exp(ic - i\mathcal{V}/\mathcal{U}). \tag{15}$$

This is known as the Schwinger Representation. In this derivation we assumed that every propagator is of the form $\sim 1/(q^2 - m^2)$; the Schwinger representation can also be extended to more general forms where the numerator is a polynomial in the loop momenta. [Smi06]

Invariance under $k_i \mapsto k_i + q_i$

Many authors instead use the combination

$$\mathcal{F} = \mathcal{V} - c\,\mathcal{U}.$$

This is because $\mathcal{V}$ is not invariant under constant transformations such as $k_i \mapsto k_i + q_i$ for some constant vector q_i. The combination $\mathcal{F}$ is invariant under this transformation, which can be shown as follows. We

have

$$\sum_{i,j} A_{ij}\; k_i \cdot k_j + 2\sum_i b_i \cdot k_i + c \mapsto \sum_{i,j} A_{ij}\; (k_i + q_i) \cdot (k_j + q_j) + 2\sum_i b_i \cdot (k_i + q_j) + c$$
$$= \sum_{i,j} A_{ij}\; k_i \cdot k_j + 2\sum_i \left(b_i + \sum_j A_{ij} q_j \right) \cdot k_i$$
$$+ \left[c + \sum_{i,j} A_{ij} q_i \cdot q_j + 2\sum_i b_i \cdot q_i \right].$$

This shows how b_i and c transform under $k_i \mapsto k_i + q_j$; the matrix A_{ij} does not change. Then the combination $\sum_{ij} A^{-1}_{ij} b_i \cdot b_j$ transforms as

$$\sum_{ij} A^{-1}_{ij} b_i \cdot b_j \mapsto \sum_{i,j} A^{-1}_{ij} \left(b_i + \sum_k A_{ik} q_k \right) \cdot \left(b_j + \sum_l A_{jl} q_l \right) = \sum_{i,j} \left[A^{-1}_{ij} b_i \cdot b_j + A_{ij} q_i \cdot q_j \right] + 2\sum_i b_i \cdot q_i.$$

Comparing this to the change in c and recalling that A_{ij} is invariant, it follows directly that $\mathcal{F} = \mathcal{V} - c\mathcal{U}$ is invariant under $k_i \mapsto k_i + q_i$ as well.

One can show [IZ80] (up to convention) that the two polynomials can be written using the structure of the Feynman diagram as follows,

$$\mathcal{U} = \sum_{T \in T^1} \prod_{\ell \notin T} z_\ell \tag{16}$$

and

$$\mathcal{F} = \mathcal{V} - c\mathcal{U} = \sum_{T \in T^2} \left[-(p^T)^2 \prod_{\ell \notin T} z_\ell \right] - \mathcal{U} \sum_\ell c'_\ell\; z_\ell. \tag{17}$$

Here T^1 is the set of subgraphs of the Feynman diagrams that are (maximal) trees (e.g. have no loops and one connected component), while T^2 is the set of subgraphs consisting of a (maximal) pair of trees that don't touch. The complement of a $T \in T^1$ consists of L edges, while the complement of $T \in T^2$ consists of $L + 1$ edges. The vector p^T is the sum of all external momenta that flow into one of the connected components of the tree $T \in T^2$. This expression does not depend on the choice of component, since the other component has precisely $-p^T$ and we take the square. Finally c'_ℓ is the invariant part of the constant term in the propagator of the edge ℓ. Roughly speaking, c'_ℓ contains internal mass term. In particular, if the propagator has a denominator of the form $ak^2 + b \cdot k + c$ where k is a sum of loop momenta then c'_ℓ is given by

$$c'_\ell = c - \frac{b \cdot b}{4a}.$$

The reason we need this invariant part rather than the entire constant term c is again because c is not $k_i \mapsto k_i + q_i$ invariant, but the product over trees in Equation 17 is.

3.4.2 Feynman Parameter Representation

The Feynman parameter representation strongly resembles the Schwinger parameter representation, but is slightly different and can be written in a nicer form. The main idea [IZ80] is to reduce the integrals

$$\int_0^\infty \mathrm{d}z_\ell \quad \text{to integrals} \quad \int_0^1 \mathrm{d}z_\ell.$$

This is achieved by introducing a factor

$$\int_0^\infty \mathrm{d}s\; \delta\left(s - \sum_\ell z_\ell \right) = 1,$$

scaling each z_ℓ by s, and then integrating over s. This is sensible because $\mathcal{U}$ and $\mathcal{V}$ are homogeneous, such that we can extract the obtained factors s explicitly. The derivation is as follows.

$$\begin{aligned}F(\alpha_\ell;p_i) =& \frac{\pi^{dL/2}\, e^{i\pi(\alpha/2-L(d-2)/4)}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_N)} \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_N\; z_1^{\alpha_1-1}\ldots z_N^{\alpha_N-1}\, \mathcal{U}^{-d/2}\; \exp(ic - i\mathcal{V}/\mathcal{U}) \\ =& \frac{\pi^{dL/2}\, e^{i\pi(\alpha/2-L(d-2)/4)}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_N)} \cdot \\ & \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_N\; \delta\left(s - \sum_\ell z_\ell\right) z_1^{\alpha_1-1}\ldots z_N^{\alpha_N-1}\, \mathcal{U}^{-d/2}\; \exp(ic - i\mathcal{V}/\mathcal{U}).\end{aligned}$$

We now scale $z \mapsto sz$ which gives $z_\ell^{\alpha_\ell-1}\mathrm{d}z_\ell \mapsto s^{\alpha_\ell} z_\ell^{\alpha_\ell-1}\mathrm{d}z_\ell$ and note that $\delta\left(s - s\sum_\ell z_\ell\right) = \frac{1}{s}\delta\left(1 - \sum_\ell z_\ell\right)$. This gives us:

$$\begin{aligned}F(\alpha_\ell;p_i) =& \frac{\pi^{dL/2}\, e^{i\pi(\alpha/2-L(d-2)/4)}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_N)} \cdot \\ & \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_N\; \delta\left(1 - \sum_\ell z_\ell\right) s^{\alpha-dL/2-1}\; z_1^{\alpha_1-1}\ldots z_N^{\alpha_N-1}\, \mathcal{U}^{-d/2}\; \exp[is(c - \mathcal{V}/\mathcal{U})].\end{aligned}$$

We then evaluate the integral over s. This integral has integrand $s^{\alpha-dL/2-1}e^{is(c-\mathcal{V}/\mathcal{U})}$, for which it is not a priori clear that the integral converges. Remark that the c contains the constant parts of the propagators such as the mass terms, and hence also inherits the $i\varepsilon$ from the Feynman propagator procedure. This means that the integrand overall scales as $e^{-\varepsilon s}$ in the limit $s \to \infty$, hence the integral converges at infinity. On the other hand, the integral converges at $s = 0$ if and only if $\alpha \geq dL/2$. This is not a significant restraint, considering that for $\alpha < dL/2$ we already typically have UV divergences.[10] Since we are using dimensional regularisation, we may ignore this divergence at this point. We as such can evaluate the integral (using the definition of the Gamma function, partial derivatives and smooth continuations) to obtain:

$$\int_0^\infty \mathrm{d}s\; s^{\alpha-dL/2-1}\; \exp[is(c - \mathcal{V}/\mathcal{U})] = \Gamma(\alpha - dL/2)\; \left[e^{\pi\, i/2}\; \left(-c + \frac{\mathcal{V}}{\mathcal{U}}\right)\right]^{-\alpha+dL/2}.$$

Substituting this in, we arrive at the Feynman parameter representation:

$$F(\alpha_\ell;p_i) = \pi^{dL/2}\, e^{i\pi L/2} \frac{\Gamma(\alpha - dL/2)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_N)} \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_N\; \delta\left(1 - \sum_\ell z_\ell\right) z_1^{\alpha_1-1}\ldots z_N^{\alpha_N-1}\; \frac{\mathcal{U}^{\alpha-d(L+1)/2}}{\mathcal{F}^{\alpha-dL/2}}, \tag{18}$$

where we used $\mathcal{F} = \mathcal{V} - c\,\mathcal{U}$ as mentioned before. Once again, note that other sources such as [LP13] use different conventions or definitions resulting in different factors π or $e^{i\pi/2}$.

There is another way to write this integral which was popularised by [LP13], which is also called the Feynman parameter representation:

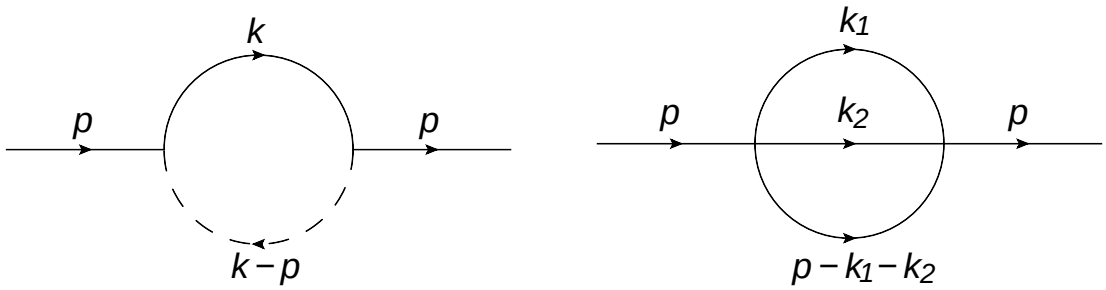
$$F(\alpha_\ell;p_i) = \frac{\pi^{dL/2}\, e^{i\pi L/2}\, \Gamma(d/2)}{\Gamma((L+1)d/2 - \alpha)\Gamma(\alpha_1)\cdots\Gamma(\alpha_N)} \int_0^\infty \mathrm{d}z_1 \ldots \int_0^\infty \mathrm{d}z_N\; z_1^{\alpha_1-1}\ldots z_N^{\alpha_N-1}\; (\mathcal{F} + \mathcal{U})^{-d/2}. \tag{19}$$

This representation has a very simple integrand for any values of α_ℓ and is therefore very suitable for intersection product calculations. In the rest of this thesis "Feynman parameter representation" refers to this latter form rather than the earlier form in Equation 18.

The former representation is obtained from the latter by inserting $\int_0^\infty \mathrm{d}s\; \delta\left(s - \sum_\ell z_\ell\right) = 1$ as before, scaling $z \mapsto sz$ and integrating over s. Note that the different scaling of $\mathcal{F}$ and $\mathcal{U}$ with respect to s creates a term of the form

$$\frac{1}{\mathcal{U} + s\mathcal{F}},$$

[10] When $\alpha < dL/2$ you generally integrate a function which scales as $k^{-2\alpha}$ in the limit $k \to 0$, over L spaces of dimension d. That integral will diverge at $k = 0$, which is known as the UV divergence.



(a) The (massive) sunrise diagram. This is non-standard terminology. The dashed line represents the particle without mass.

(b) The massless sunset diagram.

Figure 3: The sunrise and sunset diagrams. Both are specific cases of what are generally referred to as sunset diagrams.

which upon integration gives rise to quotient of powers of the two terms.

Let us consider two sunset diagrams as an example.

Example 3.4.1 (Sunrise diagram)**.** We consider the on-shell sunrise diagram with an incoming and outgoing massive particle with momentum p and mass $p^2 = m^2$, see Figure 3a. This sunset diagram has a Feynman integral of the form:

$$F(\alpha_1, \alpha_2; p) = \int \mathrm{d}^d k \quad \frac{1}{[k^2 - m^2]^{\alpha_1} [(k-p)^2]^{\alpha_2}}.$$

In order to determine the Feynman parameter representation, we only need to determine $\mathcal{U}$ and $\mathcal{F}$. We use Equations 16 and 17 to determine $\mathcal{U}$ and $\mathcal{V}$; there are two 1-trees $T \in T^1$ corresponding to the upper path and lower path, and one only 2-tree in T^2 consisting of only the two external edges. The only 2-tree has $p^T = p$. We therefore find

$$\mathcal{U} = z_1 + z_2, \qquad \mathcal{F} = -p^2\, z_1 z_2 - (-m^2\, z_1 + 0\, z_2)(z_1 + z_2) = m^2 z_1^2.$$

We can then immediately write down:

$$F(\alpha_1, \alpha_2; m^2) = \frac{\pi^{d/2}\, i\, \Gamma(d/2)}{\Gamma(d - \alpha_1 - \alpha_2)\, \Gamma(\alpha_1)\, \Gamma(\alpha_2)} \int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2\, \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{z_1 z_2\, (z_1 + z_2 + m^2 z_1^2)^{d/2}}. \qquad \triangle$$

Example 3.4.2 (Sunset diagram)**.** As a two-loop example, we consider the massless sunset diagram of Figure 3b. The Feynman integral for this diagram in the loop momentum representation is

$$F(\alpha_1, \alpha_2, \alpha_3; p) = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \quad \frac{1}{[k_1^2]^{\alpha_1} [k_2^2]^{\alpha_1} [(p - k_1 - k_2)^2]^{\alpha_3}}.$$

We again have to find $\mathcal{U}$ and $\mathcal{F}$, and we us Equations 16 and 17. There are three 1-trees corresponding to the upper, middle and lower path, and there is one 2-tree consisting of only the external edges. The only 2-tree has $p^T = p$. There are no invariant constant contributions. This gives us:

$$\mathcal{U} = z_1 z_2 + z_2 z_3 + z_3 z_1, \qquad \mathcal{F} = -p^2\, z_1 z_2 z_3.$$

We thus obtain

$$F(\alpha_1, \alpha_2, \alpha_3; p) = \frac{-\pi^d\, \Gamma(d/2)}{\Gamma(d - \alpha)\, \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \int_0^\infty \mathrm{d}z_3\, \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{z_1 z_2 z_3\, (z_1 z_2 + z_2 z_3 + z_3 z_1 - p^2\, z_1 z_2 z_3)^{d/2}}. \qquad \triangle$$

3.4.3 Baikov Representation

We finally consider and derive the Baikov representation, following [Gro11; FP17]. Also relevant is the loop-by-loop approach as described in Appendix A of [Fre+21].

We still consider the Feynman integral

$$F(\alpha_\ell; p_1, \dots, p_E) = \int \cdots \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \quad \frac{1}{D_1^{\alpha_1} \dots D_N^{\alpha_N}}.$$

with D_ℓ, p_i, L, E and N as before.

We will now assume without loss of generality that $N = \frac{1}{2}L(L+1) + LE$; if $N < \frac{1}{2}L(L+1) + LE$ we can add some D_ℓ and set the corresponding α_ℓ to 0. Remark that this is the maximal amount of different propagators assuming the graph has no vertices connecting only one or two edges and assuming each equal momentum corresponds to the same mass.[11] We can always rewrite the Feynman integral such that the internal momentum parts of the D_ℓ are linearly independent, so we assume this as well. Finally, we assume the Feynman integral is Euclidean so that we have no complications with signs or powers of i; this can be achieved by means of a Wick rotation.

The core idea of the Baikov Representation is that the value of the Feynman integral does not directly depend on the external momenta p_i and the interal momenta k_j, since the value is a scalar rather than a vector. Moreover, the value can not depend on specific components of the vectors due to Special Relativity (this is a physical assumption rather than mathematical). Instead, the value only depends on the invariant products $p_i \cdot k_j$, $p_i \cdot p_j$ and $k_i \cdot k_j$. Moreover, the D_ℓ precisely consist of sums such expressions and are therefore invariant. Since the momentum parts of the D_ℓ are linearly independent, any product $p_i \cdot k_j$ or $k_i \cdot k_j$ can be written as a sum of some D_ℓ and some constant mass term. This means it is possible to write the integral as an integral over the D_ℓ, where the $D_\ell = x_\ell$ are now parameters. Of course this gives us some Jacobian which we have to determine. This Jacobian will reflect the symmetries between the D_ℓ which are lost when we treat them as independent variables.

To make this explicit, we write each D_ℓ as follows:

$$D_\ell = \sum_{1 \le i \le j \le L} A_\ell^{ij}\ k_i \cdot k_j + \sum_{i=1}^{E} \sum_{j=1}^{L} B_\ell^{ij}\ p_i \cdot k_j + f_\ell.$$

The term f_ℓ contains terms not depending on internal momenta, such as internal masses and products of external momenta. Note that A_ℓ^{ij} is an $\frac{1}{2}L(L+1) \times N$–matrix and B_ℓ^{ij} is an $EL \times N$–matrix, such that they can be added to form a $N \times N$–matrix (AB).

As mentioned before, the D_ℓ are chosen such that the internal momentum parts of all D_ℓ are linearly independent. In other words, $\{D_1 - f_1, D_2 - f_2, \dots, D_N - f_N\}$ is a basis for the space

$$\langle D_1 - f_1, D_2 - f_2, \dots, D_N - f_N \rangle = \langle p_i \cdot k_j, k_i \cdot k_j \rangle.$$

The matrix (AB) then simply is a transformation between bases. With this in mind, the Baikov representation is achieved in two steps.

Parameter transformation from k_j to $p_i \cdot k_j, k_i \cdot k_j$ This transformation is achieved by repeatedly using the Gram determinant of K vectors $q_i \in \mathbb{R}^K$:

$$G(q_1, \dots, q_K) = \det\left(q_i \cdot q_j\right)_{1 < i,j < k} = \|q_1 \wedge q_2 \wedge \cdots \wedge q_K\|^2.$$

Equivalently, we may write

$$G(q_1, \dots, q_K) = \det\left[Q^T(q_1, \dots, q_K) \cdot Q(q_1, \dots, q_K)\right] = \det\left(Q(q_1, \dots, q_K)\right)^2$$

[11] In other words, this holds if we assume that if any two edges of the Feynman diagram both have momentum q, then their masses are the same. This condition does not generally hold in QCD.

where Q is the square matrix

$$Q(q_1,\dots,q_K) = \begin{pmatrix} q_1 & q_2 & \dots & q_K \end{pmatrix}, \quad |\det Q(q_1,\dots,q_K)| = \|q_1 \wedge \dots \wedge q_K\|.$$

This determinant is, by construction, the square of the volume of the parallelotope in $\mathbb{R}^K$ formed by the vectors $q_1,\dots,q_K$. As such, we can use it to translate between measures.

We split each internal momentum k_j into a perpendicular part $k_{j\perp}$ and parallel part $k_{j\parallel}$, where the parallel part lies in the span $\langle p_i, k_{l>j}\rangle$. In this case, the parallel part is fully determined by the products $p_i \cdot k_j$ and $k_i \cdot k_j$. In particular, we have the following.

Lemma 3.4.3. *For any vector $v = (v_1, v_2, \dots, v_K) \in \mathbb{R}^K$ and basis $\{q_1,\dots,q_K\}$ of $\mathbb{R}^K$ we have*

$$\mathrm{d}v = \frac{\mathrm{d}(v\cdot q_1)\,\mathrm{d}(v\cdot q_2)\,\dots\,\mathrm{d}(v\cdot q_K)}{\sqrt{G(q_1,\dots,q_K)}}. \tag{20}$$

Proof. We use the fact that $\mathrm{d}v = \mathrm{d}v_1\,\mathrm{d}v_2\,\dots\,\mathrm{d}v_K$ together with $v_j = v\cdot e_j$ for $\{e_j\}$ the standard basis of $\mathbb{R}^K$. We then apply the transformation matrix $Q(q_1,\dots,q_K)$, which gives a Jacobian.

$$\begin{aligned}
\mathrm{d}v &= \mathrm{d}v_1\,\mathrm{d}v_2\,\dots\,\mathrm{d}v_K \\
&= \mathrm{d}(v\cdot e_1)\,\mathrm{d}(v\cdot e_2)\,\dots\,\mathrm{d}(v\cdot e_K) \\
&= \mathrm{d}(v\cdot q_1)\,\mathrm{d}(v\cdot q_2)\,\dots\,\mathrm{d}(v\cdot q_K)\cdot\frac{1}{|\det Q(q_1,\dots,q_K)|} \\
&= \frac{\mathrm{d}(v\cdot q_1)\,\mathrm{d}(v\cdot q_2)\,\dots\,\mathrm{d}(v\cdot q_K)}{\sqrt{G(q_1,\dots,q_K)}}.
\end{aligned}$$

This is the required expression. □

Applying Equation 20 to $k_{j\parallel}$ gives us

$$\mathrm{d}k_{j\parallel} = \frac{\mathrm{d}(k_{j+1}\cdot k_j)\,\mathrm{d}(k_{j+2}\cdot k_j)\,\dots\,\mathrm{d}(k_L\cdot k_j)\,\mathrm{d}(p_1\cdot k_j)\,\dots\,\mathrm{d}(p_E\cdot k_j)}{\sqrt{G(k_{j+1},\dots,k_L,p_1,\dots,p_E)}}.$$

One remark here is that we have to be slightly careful with the interpretation of vectors and products. The term $\mathrm{d}(k_{j+1}\cdot k_j)$ is not a derivative of $k_{j+1}\cdot k_j$, but is $\mathrm{d}s$ from some parameter s which happens to coincide with $k_{j+1}\cdot k_j$. In particular, the two terms are treated as elements of different spaces. Similarly $\mathrm{d}k_{j\parallel}$ is treated as an $E+L-j$ form, although is possible for $E+L-j$ to be greater than d, the number of spacial dimensions.[12] Moreover, we treat $k_{j+1},\dots,p_E$ as $E+L-j$–vectors.[13]

We now consider the perpendicular part, $k_{j\perp}$. We may ignore the product of k_j with any k_l for $l<j$, since those products are already covered in $\mathrm{d}k_{l\parallel}$. (Recall that $(\mathrm{d}y + a\mathrm{d}x)\mathrm{d}x = \mathrm{d}y\mathrm{d}x$ for any x, y and a, since $\mathrm{d}x\wedge\mathrm{d}x = 0$.) On the other hand, the product of $k_{j\perp}$ with any p_i or k_l with $l>j$ is 0 by definition. Therefore, the only relevant product of $k_{j\perp}$ is the product $k_{j\perp}\cdot k_{j\perp}$. In particular, this product corresponds to the norm of $k_{j\perp}$, hence to a radius. This means we can write $\mathrm{d}k_{j\perp}$ in terms of $\mathrm{d}\|k_{j\perp}\|$ by switching to radial coordinates and integrating over the sphere S^{k-1}. This last integral is well-known:

$$\mathrm{Vol}(S^n_r) = \frac{2\,\pi^{n/2}}{\Gamma(n/2)}\,r^{n-1},$$

where r denotes the radius of the sphere, n denotes the dimension and Γ is the Euler gamma function. This formula work for any n, even when n is non-integer, and is therefore usable in our case. We find that

$$\begin{aligned}
\mathrm{d}k_{j\perp} &\sim \frac{2\,\pi^{(d+j-E-L)/2}}{\Gamma((d+j-E-L)/2)}\,\|k_{j\perp}\|^{d+j-E-L-1}\,\mathrm{d}\|k_{j\perp}\| \\
&\sim \frac{\pi^{(d+j-E-L)/2}}{\Gamma((d+j-E-L)/2)}\,\|k_{j\perp}\|^{d+j-E-L-2}\,\mathrm{d}(k_{j\perp}\cdot k_{j\perp})
\end{aligned}$$

[12] We assumed that the Feynman integral is Euclidean, such that all space-time dimensions are spacial dimensions.

[13] Remark that this is possible even when $E+L-j$ is greater than d, the number of spacial dimensions, since the vectors generally depend on external variables and as such should be viewed as abstract objects rather than concrete vectors. In other words, the span is an abstract vector space rather than a physical one. Compare this to how the vector space of polynomials of degree ≤ 2 has basis $\{1, x, x^2\}$, but for any concrete value of $x\in\mathbb{R}$ this set is degenerate and the subset $\{1\}$ is sufficient to span the space.

where we used that $\mathrm{d}(r^2) = 2r\,\mathrm{d}r$ and use $\sim$ to emphasize we ignore any terms such as $\mathrm{d}(k_j \cdot k_1)$.

Finally we want to relate $\mathrm{d}(k_{j\perp} \cdot k_{j\perp})$ to $\mathrm{d}(k_j \cdot k_j)$ and rewrite $\|k_{j\perp}$ in terms of k_j, k_l for $l > j$ and p_i. To do the former, we in principle have to determine $\mathrm{d}(k_{j\|} \cdot k_{j\|})$. However, $\mathrm{d}(k_{j\|} \cdot k_{j\|})$ only contributes via terms $\mathrm{d}(k_j \cdot k_l)$ for $l > j$ or $\mathrm{d}(k_j \cdot p_i)$, per construction of $k_{j\|}$. We therefore find that

$$\mathrm{d}(k_j \cdot k_j) \sim \mathrm{d}(k_{j\perp} \cdot k_{j\perp}).$$

In order to rewrite $\|k_{j\perp}\|$ note that

$$\begin{aligned}\|k_{j\perp}\|\,\|k_{j+1} \wedge \cdots \wedge k_L \wedge p_1 \wedge \cdots \wedge p_E\| &= \|k_{j\perp} \wedge k_{j+1} \wedge \cdots \wedge k_L \wedge p_1 \wedge \cdots \wedge p_E\| \\ &= \|k_j \wedge k_{j+1} \wedge \cdots \wedge k_L \wedge p_1 \wedge \cdots \wedge p_E\|,\end{aligned}$$

where we used that $k_{j\perp}$ is perpendicular to include it in the norm. We therefore see

$$\|k_{j\perp}\| = \frac{\|k_j \wedge k_{j+1} \wedge \cdots \wedge k_L \wedge p_1 \wedge \cdots \wedge p_E\|}{\|k_{j+1} \wedge \cdots \wedge k_L \wedge p_1 \wedge \cdots \wedge p_E\|} = \sqrt{\frac{G(k_j, \ldots, k_L, p_1, \ldots, p_E)}{G(k_{j+1}, \ldots, k_L, p_1, \ldots, p_E)}}.$$

Substituting these two equations in the earlier expression for $\mathrm{d}k_{j\perp}$, we find

$$\mathrm{d}k_{j\perp} \sim \frac{\pi^{(d+j-E-L)/2}}{\Gamma((d+j-E-L)/2)} \left(\frac{G(k_j, \ldots, k_L, p_1, \ldots, p_E)}{G(k_{j+1}, \ldots, k_L, p_1, \ldots, p_E)}\right)^{(d+j-E-L-2)/2} \mathrm{d}(k_j \cdot k_j).$$

We finally combine this with the equation for $\mathrm{d}k_{j\|}$ to obtain

$$\mathrm{d}k_j \sim C_j \frac{G(k_j, \ldots, k_L, p_1, \ldots, p_E)^{(d+j-E-L-2)/2}}{G(k_{j+1}, \ldots, k_L, p_1, \ldots, p_E)^{(d+j-E-L-1)/2}}\, \mathrm{d}(k_j \cdot k_j)\, \ldots\, \mathrm{d}(k_L \cdot k_j)\, \mathrm{d}(p_1 \cdot k_j)\, \ldots\, \mathrm{d}(p_E \cdot k_j),$$

where

$$C_j = \frac{\pi^{(d+j-E-L)/2}}{\Gamma((d+j-E-L)/2)}.$$

Remark that the Gram determinant in the denominator and numerator differ by 1, such that the denominator for j cancels with the numerator for $j+1$. This means that the product only contains the numerator for $j = 1$ and the denominator for $j = L$. Using the short-hand notation $s_{ij} = k_i \cdot k_j$ and $t_{ij} = p_i \cdot k_j$, we find that

$$\mathrm{d}k_1 \mathrm{d}k_2 \ldots \mathrm{d}k_L = C'\, G(k_1, \ldots, k_L, p_1, \ldots, p_E)^{(d-E-L-1)/2}\, \mathrm{d}s_{11}\, \mathrm{d}s_{12}\, \ldots\, \mathrm{d}s_{L-1,L}\, \mathrm{d}t_{11}\, \ldots\, t_{LE},$$

where

$$C' = G(p_1, \ldots, p_E)^{(-d+E+1)/2} \prod_{j=1}^{L} \frac{\pi^{(d+j-E-L)/2}}{\Gamma((d+j-E-L)/2)}.$$

Recall that

$$G(k_1, \ldots, k_L, p_1, \ldots, p_L) = \det \begin{pmatrix} k_i \cdot k_j & p_i \cdot k_j \\ k_i \cdot p_j & p_i \cdot p_j \end{pmatrix} = \det \begin{pmatrix} s_{ij} & t_{ij} \\ t_{ij} & p_i \cdot p_j \end{pmatrix},$$

so $G(k_1, \ldots, k_L, p_1, \ldots, p_L) = G(s_{ij}, t_{ij})$ is a polynomial in s_{ij}, t_{ij} and the product of external variables $p_i \cdot p_j$.

The integration domain corresponding to this measure is the space of all s_{ij}, t_{ij} where the Gram determinant is non-zero. (This is a consequence of us treating the vectors p_i, k_j as independent vectors in $\mathbb{R}^{E+L}$, which implies the Gram determinant is non-zero.) This was the first and largest step.

Transformation from $p_i \cdot k_j$, $k_i \cdot k_j$ to $D_\ell - f_\ell$ We now have the second step, to transform from the $\mathrm{d}s_{ij}$ and $\mathrm{d}t_{ij}$ to the $D_\ell - f_\ell$. As mentioned before, the s_{ij}, t_{ij} span the same (abstract) vector space as the $D_\ell - f_\ell$, and the transition matrix from the latter basis to the former is precisely given by (AB). We can then easily rewrite the measure using a Jacobian:

$$\begin{aligned}\mathrm{d}k_1 \mathrm{d}k_2 \ldots \mathrm{d}k_L &= C'\, G(k_1, \ldots, k_L, p_1, \ldots, p_E)^{(d-E-L-1)/2}\, \mathrm{d}s_{11}\, \mathrm{d}s_{12}\, \ldots\, \mathrm{d}s_{L-1,L}\, \mathrm{d}t_{11}\, \ldots\, t_{LE} \\ &= C\, P(D_\ell - f_\ell)^{(d-E-L-1)/2}\, \mathrm{d}D_1\, \mathrm{d}D_2\, \ldots\, \mathrm{d}D_N,\end{aligned}$$

where

$$P(D_\ell - f_\ell) = G(s_{ij}, t_{ij})$$

is the Gram determinant rewritten in terms of of the $D_\ell - f_\ell$ (and hence again a polynomial) and where

$$C = C' \ \det(AB)^{-1} = G(p_1, \ldots, p_E)^{(-d+E+1)/2} \ \det(AB)^{-1} \ \frac{\pi^{Ld/2 - EL/2 - L(L-1)/4}}{\prod_{j=1}^{L} \Gamma((d+j-E-L)/2)}.$$

The usual notation is to write x_ℓ for the parameter corresponding to D_ℓ. Substituting this into the initial integral, we find:

$$F(\alpha_\ell; p_i) = C \int \cdots \int \frac{\mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_N}{x_1^{\alpha_1} \, x_2^{\alpha_2} \ldots x_N^{\alpha_N}} \ P(x_\ell - f_\ell)^{(d-E-L-1)/2}. \tag{21}$$

This is known as the Baikov representation. The integration domain is all $(x_1, \ldots, x_N)$ such that the polynomial $P(x_\ell - f_\ell)$ is non-zero.

Loop-by-loop approach In the above, we assumed that each $k_j \cdot k_l$ for $l > j$ and $k_j \cdot p_i$ was relevant. However, there are situations where $k_i \cdot k_j$ is not relevant as the loop momenta belong to different loops which do not touch or intersect, and there are similar situations where $p_i \cdot k_j$ can be ignored. In that case, there is one D_ℓ whose exponential α_ℓ is always zero. It is then unnecessary and not sensible to integrate over the corresponding parameter x_ℓ.

The loop-by-loop approach is a variant of the Baikov representation which keeps this in mind when going through the reduction of $\mathrm{d}k_j = \mathrm{d}k_{j\perp} \, \mathrm{d}k_{j\parallel}$. In practice, this means that the Baikov representation is applied to each k_j iteratively rather than to all k_j at once, in each step treating all other k_i as external parameters. The result is still of the form

$$C \int \cdots \int \frac{dx_1 \ldots dx_N}{x_1^{\alpha_1} \ldots x_N^{\alpha_N}} \ F(x_1, \ldots, x_N),$$

where $F(x_1, \ldots, x_N)$ is the product of (non-integer) powers of polynomials. However, unlike in Equation 21, the power is not the same for all polynomials and generally $N \neq LE + L(L+1)/2$.

Remark that the loop-by-loop approach works especially well on families of recursively one-loop Feynman diagrams as in Subsection 3.1.

Example 3.4.4 (Sunrise)**.** Recall the sunrise integral (see Figure 3a) from Example 3.4.1:

$$F(\alpha_1, \alpha_2; p) = \int \mathrm{d}^d k \quad \frac{1}{[k^2 - m^2]^{\alpha_1} [(k-p)^2]^{\alpha_2}}.$$

We will write this Feynman integral using the Baikov representation. We have $E = 1$ and $L = 1$, so we want $N = 2$. The sunrise diagram indeed has two internal edges, so this is satisfied. We then have to determine C and $P(x_\ell - f_\ell)$, which in turn means we first have to determine A_ℓ^{ij}, B_ℓ^{ij} and f_ℓ. We have

$$\begin{aligned} D_1 &= k^2 - m^2 \\ D_2 &= k^2 - 2p \cdot k + p^2, \end{aligned}$$

so we find

$$A_\ell^{ij} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_\ell^{ij} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad f_1 = -m^2, \ \ f_2 = p^2.$$

In other words,

$$\begin{pmatrix} D_1 + m^2 \\ D_2 - p^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} k^2 \\ p \cdot k \end{pmatrix}, \qquad \begin{pmatrix} k^2 \\ p \cdot k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} D_1 + m^2 \\ D_2 - p^2 \end{pmatrix}.$$

The inverted matrix gives us $P(D_1 + m^2, D_2 - p^2)$:

$$P(D_1 + m^2, D_2 - p^2) = p^2 \, (D_1 + m^2) - \left(\frac{1}{2}(D_1 + m^2) - \frac{1}{2}(D_2 - p^2) \right)^2 = p^2 \, k^2 - (k \cdot p)^2 = G(p, k).$$

The calculation for C is more direct:

$$\begin{aligned} C &= G(p)^{-d/2+1}\ \det(AB)^{-1}\ \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \\ &= (p^2)^{-d/2+1}\ \cdot -\frac{1}{2}\ \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2)}. \end{aligned}$$

Putting things together we find

$$F(\alpha_1,\alpha_2;p) = \frac{-\pi^{(d-1)/2}\ (p^2)^{-d/2+1}}{2\,\Gamma((d-1)/2)} \int\int \frac{\mathrm{d}x_1\ \mathrm{d}x_2}{x_1^{\alpha_1}\ x_2^{\alpha_2}} \left(p^2\,(x_1+m^2) - \frac{1}{4}\left(x_1 - x_2 + 2m^2\right)^2\right)^{(d-3)/2},$$

where we slightly simplified the polynomial using $p^2 = m^2$.

We can also explicitly write the measures in the way we described above. We know that $\mathrm{d}^d k$ splits into the part $\mathrm{d}k_\perp$ orthogonal to p and the part $\mathrm{d}k_\parallel$ parallel to p, hence $\mathrm{d}k_\perp$ can be written in terms of $\mathrm{d}(k\cdot k)$ and $\mathrm{d}k_\parallel$ in terms of $\mathrm{d}(k\cdot p)$. According to the formula found above, we have:

$$\begin{aligned} \mathrm{d}k_\parallel &= \frac{\mathrm{d}(k\cdot p)}{\sqrt{G(p)}} = \frac{\mathrm{d}(k\cdot p)}{\sqrt{p\cdot p}}, \\ \mathrm{d}k_\perp &\sim \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \left(\frac{(k\cdot k)(p\cdot p) - (p\cdot k)^2}{(p\cdot p)}\right)^{(d-3)/2} \mathrm{d}(k\cdot k). \end{aligned}$$

We can indeed verify the former directly: we can write $k_\parallel$ in terms of the basis p as $c\,p$, in which case

$$\mathrm{d}k_\parallel = \mathrm{d}(c\,p) = \mathrm{d}(c\,\|p\|) = \frac{\mathrm{d}(c\,\|p\|^2)}{\|p\|} = \frac{\mathrm{d}(c\,p\cdot p)}{\sqrt{p\cdot p}} = \frac{\mathrm{d}(k\cdot p)}{\sqrt{p\cdot p}}.$$

Since we have found above (using the matrix (AB)) that

$$\mathrm{d}(k\cdot k) = \mathrm{d}(D_1 + m^2) = \mathrm{d}D_1, \qquad \mathrm{d}(p\cdot k) = \frac{1}{2}\mathrm{d}(D_1+m^2) - \frac{1}{2}\mathrm{d}(D_2 - p^2) = \frac{1}{2}\mathrm{d}D_1 - \frac{1}{2}\mathrm{d}D_2,$$

and since

$$\mathrm{d}(k\cdot k)\ \mathrm{d}(k\cdot p) = \mathrm{d}D_1\ \left[\frac{1}{2}\mathrm{d}D_1 - \frac{1}{2}\mathrm{d}D_2\right] = -\frac{1}{2}\,\mathrm{d}D_1\ \mathrm{d}D_2,$$

we find

$$\begin{aligned} \mathrm{d}^d k &= \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2)}[(k\cdot k)(p\cdot p) - (p\cdot k)^2]^{(d-3)/2}\ (p\cdot p)^{-d/2+1}\ \mathrm{d}(k\cdot k)\ \mathrm{d}(k\cdot p) \\ &= \frac{-\pi^{(d-1)/2}}{2\,\Gamma((d-1)/2)}[(k\cdot k)(p\cdot p) - (p\cdot k)^2]^{(d-3)/2}\ (p\cdot p)^{-d/2+1}\ \mathrm{d}D_1\ \mathrm{d}D_2. \end{aligned}$$

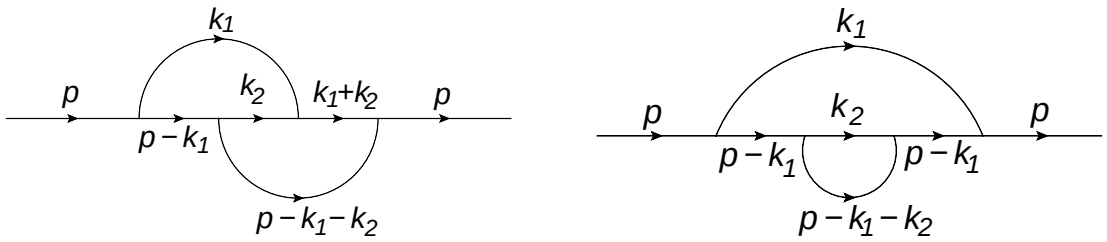
Is it clear that this agrees with the $F(\alpha_1,\alpha_2;p)$ found above after substituting $k\cdot k$, $k\cdot p$ and D_i with expressions in x_i. △

Example 3.4.5 (Sunset)**.** We again consider the sunset integral (see Figure 3b) from Example 3.4.2:

$$F(\alpha_1,\alpha_2,\alpha_3;p) = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \quad \frac{1}{[k_1^2]^{\alpha_1}[k_2^2]^{\alpha_1}[(p-k_1-k_2)^2]^{\alpha_3}}.$$

This is a two-loop diagram so we can use the regular Baikov method or the loop-by-loop Baikov approach. We will first consider the regular Baikov and describe the loop-by-loop Baikov at the end. Note that we have $E=1$ and $L=2$, so we want $N = 2+3 = 5$. We only have $N=3$, so we have to add another two edges. We do this by stretching the two 4–vertices into pairs of 3–vertices connected with an edge, see Figure 4a. This adds a fourth and fifth propagator for the momentum $p-k_1$ respectively k_1+k_2. We therefore can write our Feynman integral as

$$F(\alpha_1,\alpha_2,\alpha_3,\alpha_4=0,\alpha_5=0;p) = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \quad \frac{1}{[k_1^2]^{\alpha_1}[k_2^2]^{\alpha_1}[(p-k_1-k_2)^2]^{\alpha_3}[(p-k_1)^2]^{\alpha_4}[(k_1+k_2)^2]^{\alpha_5}},$$



(a) The sunset diagram of Figure 3b with 4–vertices stretched into pairs of 3–vertices.

(b) The sunset diagram with 4–vertices stretched in a different way for the loop-by-loop Baikov approach. Remark that this is a recursively one-loop diagram.

Figure 4: The stretched sunset diagrams of Example 3.4.5.

with new weights α_4 and α_5. The matrices A_ℓ^{ij} and B_ℓ^{ij} are now (we order (ij) as $\{(11),(12),(22)\}$):

$$A_\ell^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad B_\ell^{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & -2 \\ -2 & 0 \\ 0 & 0 \end{pmatrix},$$

and the constants f_ℓ are

$$f_1 = 0, \quad f_2 = 0, \quad f_3 = p^2, \quad f_4 = p^2, \quad f_5 = 0.$$

In other words,

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 - p^2 \\ D_4 - p^2 \\ D_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & -2 & -2 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1^2 \\ k_1 \cdot k_2 \\ k_2^2 \\ p \cdot k_1 \\ p \cdot k_2 \end{pmatrix}, \quad \begin{pmatrix} k_1^2 \\ k_1 \cdot k_2 \\ k_2^2 \\ p \cdot k_1 \\ p \cdot k_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 - p^2 \\ D_4 - p^2 \\ D_5 \end{pmatrix}.$$

We can now find $P(x_1, x_2, x_3 - p^2, x_4 - p^2, x_5)$ by calculating the determinant of the 6×6–matrix $G(k_1, k_2, p)$ and replacing all $k_i \cdot k_j$ and $p_i \cdot k_j$ using the above inverted matrix. This is trivial for a computer but the result will not be written as it is not enlightening for humans.

As before we can directly calculate C:

$$\begin{aligned} C &= G(p)^{-d/2+1} \det(AB) \frac{\pi^{d-3/2}}{\Gamma((d-2)/2)\Gamma((d-1)/2)} \\ &= (p^2)^{-d/2+1} \, 8 \cdot \frac{\pi^{d-3/2}}{\Gamma((d-2)/2)\Gamma((d-1)/2)}. \end{aligned}$$

This gives us the Baikov representation:

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0, \alpha_5 = 0; p) = C \int \cdots \int \frac{\mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}x_4 \, \mathrm{d}x_5}{x_1^{\alpha_1} \, x_2^{\alpha_2} \, x_3^{\alpha_3}} \; P(x_\ell - f_\ell)^{(d-4)/2}.$$

A clear disadvantage of the Baikov representation for this Feynman diagram is the integral over x_4 and x_5 which are unnecessary. One can integrate over x_4 and x_5 to obtain an easier representation, [FP17] but these integrals are generally complicated. The loop-by-loop method attempts to minimize this disadvantage by changing the construction such that less extra x_ℓ are necessary.

In the loop-by-loop Baikov method we first choose one internal momentum to be the internal momentum and see the other internal momenta as external. This means that L is always 1 in the loop-by-loop Baikov method. We choose to first represent the integral over k_1 using the Baikov method. We turn the 4-vertices into pairs of 3-vertices connected by edges as in Figure 4b. Then there is a single external momentum on

the inner loop, given by $p - k_2$. We as such have $E = 1$ hence $N = 1 + 1 = 2$, meaning the two propagators with momenta k_1 and $p - k_2 - k_1$ which form the loop are sufficient. This means we can apply Baikov to determine the polynomial $P_1(x_1, x_2 - (p - k_2)^2)$ (which involves taking the determinant of a 2×2–matrix) and the constant C_1. The result will be an integral over some x_1 and x_2 which linearly span the same space as k_1^2 and $k_1 \cdot (p - k_2)$. Remark that the polynomial P_1 generally depends on product with k_2, which means we have to remember to replace those products in the next step.

We then integrate over k_2. We again have $L = 1$ and the only external momentum is p so $E = 1$. This means we again have $N = 2$ and require two propagators, which will have denominators k_2^2 and $(p - k_2)^2$; the former was already in the Feynman integral while the latter appeared when we stretched the 4-vertices into pairs of 3-vertices. The two corresponding products are k_2^2 and $p \cdot k_2$. The translation between the products and propagators is again a 2×2–matrix. The result is of the form:

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0; p) = C_1\, C_2 \int \cdots \int \frac{\mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x_4}{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}\, P_1(x_\ell - f_\ell)^{(d-4)/2}\, P_2(x_\ell - f_\ell)^{(d-3)/2}.$$

The result is slightly more complicated than before, with multiple polynomials, but it also has an integral less (the integral over x_5) and it is easier to calculate as it only contains determinants and inverses of small matrices rather than the inverse of a 5×5–matrix. The loop-by-loop method is not clearly better for this particular Feynman integral, but the above should give some intuition on how the loop-by-loop method can be superior in higher loop order calculations. △

3.5 Cuts

We will shortly consider cuts, as they are very useful in combination with master integrals. The idea of a cut is to put a chosen edge in the Feynman diagram on-shell. This creates a simpler Feyman integral which can often be calculated easily. The cut Feynman integral then gives information on the uncut Feynman integral, which can typically be recovered using another integral. [Smi06:Appendix F] There are also similar methods which reduce the Feynman integral to an easier form but have no clear diagrammatic interpretation; these are also referred to as 'cuts'. [FP17] For our use, however, the most relevant property is the fact that cuts preserve relations between Feynman integrals. In particular, if F is a Feynman integral, $\{I_i\}$ is a basis of master integrals and $C[A]$ denotes a cut version of a Feynman integral A, we find that

$$F = \sum_i c_i\, I_i \qquad \text{implies} \qquad C[F] = \sum_i c_i\, C[I_i].$$

This is extremely useful when there is some cut C such that $C[I_i] = 0$ for most i, as it allows us to more easily determine the other c_i. [Fre+19b] Similarly, a cut version of a Feynman integral satisfies the same DE as the uncut version, albeit with different boundary conditions. [Hen15]

The most well known example of a cut is the phase space cut, which is the application of Cutkosky's rule in the loop momentum representation. Cutkosky's rule allows one to calculate the discontinuity of a Feynman integral with respect to a pole. For example, if the Feynman integral has a propagator $1/(p^2 - m^2)$ then the integrand has a pole at $p^2 = m^2$ i.e. $p_0^2 = \vec{p}^{\,2} + m^2$. We normally solve this by changing the propagator to a Feynman propagator $1/(p^2 - m^2 + i\varepsilon)$, which guarantees that the integral over p_0 evades the pole in the complex plane. However, this is not always possible in degenerate cases and even if it is we may be interested how much the outcome of the integral would differ if we would pass by the pole on the other side. This can be calculated using the rule:

$$\frac{1}{p^2 - m^2 + i\varepsilon} \mapsto (-2\pi i)\, \delta(p^2 - m^2)\, \theta(p_0) =: (-2\pi i)\, \delta_+(p^2 - m^2),$$

where θ_0 is the Heaviside step function. [Ell+12:Appendix C]

It is possible to cut more than one edge in a diagram, a collection of multiple cuts is still called a cut. When the cut diagram has no more edges which can be cut, the cut is a *maximal cut*. The number cuts in a maximal cut is generally dL corresponding to the d components of each internal momentum; for $L = 1$ in dimensional regularisation typically at most 5 component cuts are used, 4 for the space-time components and the final for the decoupled extra components [Ell+12]. Most physical cuts cut multiple components at once, hence giving a lower number of cuts.

For us, cuts in the Baikov representation are more useful. In the Baikov representation, a possible cut is given by the replacement of the integral $\int_0^\infty \mathrm{d}x_\ell$ by a loop integral around 0:

$$\int_0^\infty \mathrm{d}x_\ell \quad \mapsto \quad \oint_{x_\ell=0} \mathrm{d}x_\ell.$$

This cut is not identical to the phase space cut and hence is not directly related to the discontinuity of the Feynman integral, but nevertheless satisfies the above properties of preserving relations between Feynman integrals and preserving the differential equation. [FP17] Note that the loop integrals can be calculated using the residue theorem and require no actual integration, making this cut very easy to generally compute.

We have now seen the most important techniques that are currently used in evaluating families of Feynman integrals. This gives us sufficient context to look at the intersection product from a physics perspective. We can now consider the mathematical background of the intersection product, which we will do in the next section.

4 The Twisted Cohomology

We will now consider the twisted cohomology, which gives the mathematical background for the intersection product. We will define the twisted cohomology and related notions, give a few examples, and consider a few theorems which are relevant and important. Finally we will shortly discuss higher-dimensional twisted cohomologies and their relevance and issues.

The main definitions and most results are adapted from the book [Aom+11] with some results from [FF16]. However, [Aom+11] takes a very algebraic point of view and proves the results for algebraic varieties rather than differential manifolds in general. We therefore try to give a more geometrical view whenever possible.

4.1 Definition using Local Systems

Before we can (geometrically) define the twisted cohomology on a connected complex manifold, we first need to define a specific local system using monodromy. We do this below.

Let M be a connected complex manifold and let $\widetilde{M}$ be its universal cover. We use monodromy on M to construct a local system, then define a connection on the local system in a natural manner and use the connection to define a cohomology on M. We follow [Miz19] for the general idea and [Aom+11] for the algebraic details.

Consider a closed 1-form $\alpha \in \Omega^1(M)$ and a loop γ on M. Then the integral $\int_\gamma \alpha$ is a well-defined complex number, we call this number the *monodromy* of α on γ. By fixing α, we can make a map

$$\gamma \mapsto \int_\gamma \alpha,$$

defined on the set of loops on M. This map has two main properties. Firstly, we have that two path-homotopic loops $\gamma_1 \sim \gamma_2$ give the same value. This can be seen as follows: use the homotopy to choose a surface $S \subseteq M$ with as boundary the union of the images of γ_1 and γ_2^{-1}, where we follow γ_2 in the opposite direction. We see that

$$\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \int_{\gamma_1} \alpha + \int_{\gamma_2^{-1}} \alpha = \int_{\partial S} \alpha = \int_S \mathrm{d}\alpha = 0.$$

This shows that the integrals indeed agree. Note that we used that α is closed. Secondly, we have that the concatenation of loops is mapped to the sum of their complex numbers,

$$\int_{\gamma_1 \cdot \gamma_2} \alpha = \int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha.$$

These properties imply that the function descends to the fundamental group. Indeed, we have proven the following proposition.

Proposition 4.1.1. *For any 1-form α, the map $\pi_1(M) \to \mathbb{C}$ given by*

$$[\gamma] \mapsto \int_\gamma \alpha$$

is a well-defined homomorphism of groups (where $\mathbb{C}$ is a group under addition). □

We call this homomorphism the *monodromy homomorphism* of α. We want to use it to define a local system, and to do that we first want to turn the homomorphism into a representation. Therefore we compose the integration with the homomorphism $\exp : \mathbb{C} \to \mathbb{C}^* = \mathrm{GL}(1;\mathbb{C})$ to get a representation

$$\pi_1(M) \to \mathbb{C}^* = \mathrm{GL}(1,\mathbb{C}) : \qquad [\gamma] \mapsto \exp \int_\gamma \alpha. \tag{22}$$

We use the above representation to define a complex line bundle $\mathcal{L}_\alpha$ (known as a local system) on M.

Construction 4.1.2 (Local system)**.** Let $\widetilde{M}$ be the universal cover of M. By the definition of a universal cover, the fundamental group $\pi_1(M)$ has a free action on $\widetilde{M}$ such that the quotient of the action is precisely M. In particular, this means that $\widetilde{M}$ is a principal $\pi_1(M)$–bundle over M. Since we have a principal $\pi_1(M)$–bundle and a representation $\pi_1(M) \to \mathbb{C}^*$ we can take the associated vector bundle $\big(\widetilde{M}\times\mathbb{C}\big)/\pi_1(M) \to M$; see [Nak03] for details of such a construction. Because $\mathbb{C}^* = \mathrm{GL}(1;\mathbb{C})$ is a group of 1×1 matrices, the associated vector bundle is a line bundle. We will call this line bundle $\mathcal{L}_\alpha \to M$, where the subscript emphasises that the construction depends on the choice of α. •

This line bundle is not interesting from a topological standpoint and is in fact trivial. This can be thought of as follows. Consider the cylinder and Möbius ring as real line bundles over the circle S^1. The former is trivial, while the latter has no global trivialisation since looping around once gives a factor -1 and there is no smooth way to go from 1 to -1 without passing through 0. The main difference between our case and the Möbius ring is that we are working with a *complex* line bundle. In the complex plane there is a smooth path from 1 to -1 evading 0, e.g. $t \mapsto e^{i\pi\cdot t}$ for $t \in [0,1]$. This means we can untwist the complex Möbius ring by multiplying the entire ring by a position-dependent phase, hence the complex Möbius ring is trivial and isomorphic to the complex cylinder. We use the same idea below.

Proposition 4.1.3. *The line bundle $\mathcal{L}_\alpha \to M$ constructed above is isomorphic to the trivial line bundle $M\times\mathbb{C} \to M$.*

Proof. The line bundle $\widetilde{M}\times\mathbb{C} \to \widetilde{M}$ is trivial by construction, so we can choose a global section $s : \widetilde{M} \to \widetilde{M}\times\mathbb{C}$ which is nowhere 0. We choose $s(x) = (x,1)$ as our section. This section does not project down to $\mathcal{L}_\alpha$ since it is not covariant under the action of $\pi_1(M)$. We therefore have to twist it using a phase. However, unlike the action—which is discrete—the twist must be smooth, so we need to generalise the action in some sense.

We lift the one-form $\alpha \in \Omega^1(M)$ to a one-form $\tilde\alpha \in \Omega^1\big(\widetilde{M}\big)$. Then for any path $\tilde\gamma$ on $\widetilde{M}$ the expression

$$\exp\int_{\tilde\gamma}\tilde\alpha \in \mathbb{C}^*$$

is a well-defined non-zero complex number. We fix a point $x_0 \in \widetilde{M}$ and we define a new section s' as follows:

$$s'(x) = \exp\int_{\tilde\gamma^{-1}}\tilde\alpha\cdot s(x), \qquad \tilde\gamma : x_0 \rightsquigarrow x,$$

where $\tilde\gamma$ is an arbitrary path from x_0 to x and the multiplication is understood to act on the $\mathbb{C}$ part of $s(x)$. The universal cover $\widetilde{M}$ is simply connected so the above definition does not depend on the choice of a path $\tilde\gamma$ and the section s' is well-defined. We also see that s' is smooth, since we can at each point x strategically choose local paths to show $\exp\int_{\tilde\gamma}\tilde\alpha$ is smooth at said point x, and we can attach smooth paths after reparametrising.[14]

We finally have to show that s' is indeed covariant under the action of $\pi_1(M)$. The action of some loop γ on some $(x_1,\lambda) \in \widetilde{M}\times\mathbb{C}$ is given by

$$(x_1,\lambda) \mapsto \left(x_2, \exp\int_{\gamma^{-1}}\alpha\cdot\lambda\right)$$

for x_1 a pre-image of $\gamma(0)$ and x_2 the endpoint of the unique lift of γ starting in x_1. Letting this act on $s'(x_1)$, we find for any path $\tilde\gamma_{01} : x_0 \rightsquigarrow x_1$ that

$$\gamma\cdot s'(x_1) = \exp\int_{\gamma^{-1}}\alpha\cdot\exp\int_{\tilde\gamma_{01}^{-1}}\tilde\alpha\cdot s(x_2).$$

Since $\widetilde{M}$ is the universal cover of M, we can use $\widetilde{M}$ as a parametrisation of M hence the integral of α over γ^{-1} is the same as the integral over the lift $\tilde\alpha$ over the lifted path $\tilde\gamma : x_1 \rightsquigarrow x_2$. We therefore have:

$$\gamma\cdot s'(x_1) = \exp\int_{\tilde\gamma^{-1}}\tilde\alpha\cdot\exp\int_{\tilde\gamma_{01}^{-1}}\tilde\alpha\cdot s(x_2) = \exp\int_{(\tilde\gamma_{01}\cdot\tilde\gamma)^{-1}}\tilde\alpha\cdot s(x_2) = s'(x_2).$$

[14] Here we use the fact that a function is smooth if and only if all its directional derivatives are smooth, and each directional derivative can be shown to be smooth by choosing $\tilde\gamma$ along the direction.

We used that $\tilde{\gamma}_{01} \cdot \tilde{\gamma}$ is a path from x_0 to x_2 which is piecewise smooth hence can be made smooth via reparametrisation.

We conclude that s' is $\pi_1(M)$–covariant, hence s' projects down to a well-defined section $\bar{s}' \in \Gamma(\mathcal{L}_\alpha)$. Since this section is by construction non-zero, the linear map of bundles sending $\bar{s}'$ to the constant section with value 1 on $M \times \mathbb{C} \to M$ gives a bundle isomorphism between $\mathcal{L}_\alpha$ and $M \times \mathbb{C}$. We find that $\mathcal{L}_\alpha$ is trivial. □

As such, the interest in this line bundle comes from the additional (smooth) structures which are natural to define on the bundle, like the connection. We equip this line bundle with a connection as follows. We start by choosing the null connection $\tilde{\nabla}$ on the trivial line bundle $\widetilde{M} \times \mathbb{C}$ (hence $\tilde{\nabla} = \mathrm{d}$ locally everywhere), and we choose the unique flat connection ∇_α on $\mathcal{L}_\alpha = \big(\widetilde{M} \times \mathbb{C}\big)/\pi_1(M)$ which pulls back to $\tilde{\nabla}$. This unique flat connection is best described using parallel transport.

Let γ be a loop from $q \in M$ to itself, and let $(q, \lambda) \in \mathcal{L}_\alpha$. In order to parallel transport (q, λ) along γ, we lift both to $\widetilde{M}$. The loop γ is lifted to a path $\tilde{\gamma} : \tilde{q}_1 \rightsquigarrow \tilde{q}_2$ and (q, λ) is lifted to $(\tilde{q}_1, \lambda)$. Since we have the null connection on $\widetilde{M} \times \mathbb{C}$ the parallel transport is simply

$$(\tilde{q}_1, \lambda) \xrightarrow{\tilde{\gamma}:\tilde{q}_1 \rightsquigarrow \tilde{q}_2} (\tilde{q}_2, \lambda).$$

We now want to project this down again to $\mathcal{L}_\alpha$; however, we are now at $\tilde{q}_2$ instead of $\tilde{q}_1$. In order to return to $\tilde{q}_1$, we apply the action of $\pi_1(M)$ on $\widetilde{M}$:

$$[(\tilde{q}_2, \lambda)] = [\gamma^{-1} \cdot (\tilde{q}_2, \lambda)] = \left[\left(\tilde{q}_1, \exp \int_\gamma \alpha \cdot \lambda\right)\right].$$

Projecting this down to $\mathcal{L}_\alpha$, we find that the parallel transport has result

$$(q, \lambda) \xrightarrow{\gamma} \left(q, \exp \int_\gamma \alpha \cdot \lambda\right).$$

Remark 4.1.4. We similarly assume that the parallel transport along any path $\gamma : q \rightsquigarrow q'$, i.e. for any $\lambda \in \mathbb{C}$, is given by:

$$(q, \lambda) \mapsto \left(q', \exp \int_\gamma \alpha \cdot \lambda\right).$$

In particular, the infinitesimal limit shows that $\nabla_\alpha = \mathrm{d} + \alpha$. This is true in the specific global trivialisation described in Proposition 4.1.3. However, this is not generally true in any other local trivialisation $\mathcal{L}_\alpha|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}$. Since the formula does hold in the mentioned global trivialisation, we will write $\nabla_\alpha = \mathrm{d} + \alpha$ for the connection constructed above. In particular, we get the commutative diagram:

$$\begin{array}{ccc} \Omega^k(M, \mathbb{C}) & \xrightarrow{\nabla_\alpha = \mathrm{d}+\alpha} & \Omega^{k+1}(M, \mathbb{C}) \\ \downarrow\cong & & \downarrow\cong \\ \Omega^k(M, \mathcal{L}_\alpha) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(M, \mathcal{L}_\alpha). \end{array} \quad \bullet$$

We assumed that α is closed. This allows us to easily check that this connection is flat (i.e. has no curvature) using the above notation, as follows. For any complex-valued section s we have

$$\nabla_\alpha^2 s = (\mathrm{d} + \alpha)^2 s = \mathrm{d}^2 s + \mathrm{d}(\alpha s) + \alpha \wedge \mathrm{d}s + \alpha \wedge \alpha s = 0 + (\mathrm{d}\alpha)s + 0 + 0 = 0,$$

where we used the anti-commutativity of d and α in $\mathrm{d}(\alpha s) = (\mathrm{d}\alpha)s - \alpha \wedge \mathrm{d}s$. We see that the connection is flat.

Since we have $\nabla_\alpha^2 = 0$ we also find that $\mathrm{d}^2 = 0$ on $\mathcal{L}_\alpha$-valued k-forms

$$\Omega^k_\alpha := \Omega^k(M, \mathcal{L}_\alpha) := \Gamma\left(\bigwedge^k TM^* \otimes \mathcal{L}_\alpha\right).$$

We therefore find the following (geometrical) definition.

Definition 4.1.5. Let $k \geq 0$ an integer, let $\alpha \in \Omega^1(M)$ be a closed one-form. The vector space

$$H^k_\alpha := H^k(M, \mathcal{L}_\alpha) := \frac{\ker \mathrm{d} : \Omega^k_\alpha \to \Omega^{k+1}_\alpha}{\mathrm{im}\, \mathrm{d} : \Omega^{k-1}_\alpha \to \Omega^k_\alpha} \tag{23}$$

is called the *k-th α-twisted cohomology.* •

This definition is hard to work with, so we rewrite it using the global trivialisation. In this trivialisation, any section of $\mathcal{L}_\alpha$ corresponds to a section $M \to M \times \mathbb{C}$ hence corresponds to a smooth function $M \to \mathbb{C}$. As mentioned before, the connection has the form $\nabla_\alpha = \mathrm{d} + \alpha$ in the trivialisation. We therefore find the following alternate (algebraic) description:

$$H^k_\alpha = \frac{\ker(\mathrm{d} + \alpha) : \Omega^k \to \Omega^{k+1}}{\mathrm{im}(\mathrm{d} + \alpha) : \Omega^{k-1} \to \Omega^k}, \tag{24}$$

using complex-valued k-forms instead of $\mathcal{L}_\alpha$-valued k-forms.

We consider an example, both to investigate how the twisted cohomology can be calculated and to give some intuition for the results in later subsections.

Example 4.1.6 ($\mathrm{d}\log(z)$ on the punctured plane $\mathbb{C}^*$)**.** Let the punctured plane $\mathbb{C}^*$ be our manifold and choose $\alpha = \lambda\, \mathrm{d}\log(z) = \frac{\lambda}{z}\mathrm{d}z$ with $\lambda \in \mathbb{C}^*$ constant. This α is closed and non-exact, since $\log(z)$ is not a (single-valued) function on $\mathbb{C}^*$.

The fundamental group $\pi_1(\mathbb{C}^*)$ is generated by a single element, the class of the loop $\circlearrowleft_0$ which circles once around the puncture $z = 0$ in the counter-clockwise direction. We can represent this class by the loop

$$\gamma : [0, 1] \to \mathbb{C}^*, \qquad t \mapsto e^{2\pi i\, t}.$$

We then have

$$\exp \int_\gamma \frac{\lambda}{z}\, \mathrm{d}z = \exp \int_0^1 \frac{\lambda}{e^{2\pi i\, t}} \cdot 2\pi i\, e^{2\pi i\, t}\, \mathrm{d}t = \exp \int_0^1 2\pi i\lambda\, \mathrm{d}t = \exp\big(2\pi i\lambda\big).$$

We see that there is a non-trivial monodromy for non-integer λ. In the following we will assume that λ is indeed non-integer.

To calculate the zeroth twisted cohomology, we need to characterise all non-zero sections $s \in \Omega^0_\alpha = \Gamma(\mathcal{L}_\alpha)$ which satisfy

$$\mathrm{d}s = 0.$$

This is hard to work with. We see that the solutions can not be a 'constant non-zero function' because the monodromy shows that the line bundle is non-trivial, and a constant non-zero function would be a global section. However, it is more practical to use the (algebraic) description in terms of complex-valued forms.

In that case, we are looking for any non-zero function $f : \mathbb{C}^* \to \mathbb{C}$ satisfying

$$\mathrm{d}f + \alpha f = \mathrm{d}f + \frac{\lambda}{z} \cdot f\, \mathrm{d}z = 0.$$

We can naively solve this differential equation and find $f(z) = z^{-\lambda}$ or $f(z) = 0$. Since λ is not an integer, the former is not a well-defined (single-valued) function. We find that there are no non-zero $f \in \Omega^0$ such that $(\mathrm{d} + \alpha)f = 0$, hence

$$H^0_\alpha = \frac{\{0\}}{\{0\}} \cong 0.$$

We next want to calculate the first twisted cohomology; we immediately use the description in terms of differential forms. The image of Ω^0_α in Ω^1_α is given in terms of forms by $\{\mathrm{d}f + \frac{\lambda}{z} \cdot f\, \mathrm{d}z | f \in \Omega^0\} \subseteq \Omega^1$, and the kernel is given by $\{\phi \in \Omega^1 \mid \mathrm{d}\phi + \alpha \wedge \phi = 0\}$. This second set can be simplified by using $\phi = \phi_1 \mathrm{d}z + \phi_2 \mathrm{d}\bar{z}$. Since α purely has a $\mathrm{d}z$ component, we find that

$$\mathrm{d}\phi + \alpha \wedge \phi = 0$$

implies

$$\mathrm{d}\phi_1 \wedge \mathrm{d}z + \mathrm{d}\phi_2 \wedge \mathrm{d}\bar{z} = -\frac{\lambda}{z} \cdot \phi_2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

hence

$$-\frac{\partial \phi_1}{\partial \bar{z}} + \frac{\partial \phi_2}{\partial z} + \frac{\lambda}{z} \, \phi_2 = 0.$$

Unfortunately, the solutions of this differential equation are difficult to determine. We use a formal trick which will be explained later: we multiply the entire differential equation with z^λ. Note that this can only be done formally and algebraically, as z^λ is not a (geometrically) well-defined function on $\mathbb{C}^*$. This trick gives us

$$0 = -z^\lambda \frac{\partial \phi_1}{\partial \bar{z}} + z^\lambda \frac{\partial \phi_2}{\partial z} + z^\lambda \frac{\lambda}{z} \phi_2 = -\frac{\partial (z^\lambda \phi_1)}{\partial \bar{z}} + \frac{\partial (z^\lambda \phi_2)}{\partial z}.$$

This is a much easier condition. We change the image of the 0-forms in the same manner:

$$\left\{ \mathrm{d}f + \frac{\lambda}{z} \cdot f \, \mathrm{d}z \mid f \in \Omega^0 \right\} = \{ z^{-\lambda} \mathrm{d}(z^\lambda f) \mid f \in \Omega^0 \}.$$

We now have the equations in a form similar to that of the untwisted (complex) de Rham cohomology, and can as such take the same steps. For any closed $\phi = \phi_1 \, \mathrm{d}z + \phi_2 \, \mathrm{d}\bar{z}$ we can integrate $z^\lambda \phi_1$ with respect to z to find $z^\lambda f$ for some well-defined f such that

$$z^\lambda \phi - \mathrm{d}(z^\lambda f) = z^\lambda \left(\phi_2 - \frac{\partial (z^\lambda f)}{\partial \bar{z}} \right) \, \mathrm{d}\bar{z}$$

has no $\mathrm{d}z$ component. In the usual (untwisted complex) de Rham cohomology this is not possible for all ϕ_1 but the factor z^λ lets us avoid cases like $\phi_1(z) = \frac{1}{z}$. This means that any class $[\phi]$ of the twisted cohomology group contains an element $\phi' = \phi'_2 \, \mathrm{d}\bar{z}$ with zero $\mathrm{d}z$ component. Looking back at the differential equation, we see that $\partial(z^\lambda \phi'_2)/\partial z = 0$, so $z^\lambda \phi'_2$ purely depends on $\bar{z}$.

We do the same in the $\mathrm{d}\bar{z}$ component. Note that $z^\lambda \phi'_2$ can not contain a term of the form $1/\bar{z}$ since $z^{-\lambda}/\bar{z}$ is not a single-valued function. Therefore, we can integrate $z^\lambda \phi'_2$ over $\bar{z}$ to find $z^\lambda g$ for some well-defined g such that

$$z^\lambda \phi' = z^\lambda \, \phi'_2 \, \mathrm{d}\bar{z} = d(z^\lambda g).$$

This shows that ϕ' is $\mathrm{d} + \alpha$–exact, hence ϕ was exact. We find that every class in H^1_α is of the form $[0]$, hence $H^1_\alpha = 0$.

Finally, we consider H^2_α. Since $\mathbb{C}^*$ retracts to the circle S^1 we know we will find $H^2_\alpha = 0$, but we still compute it as practice. Since all of Ω^2 is $\mathrm{d} + \alpha$–closed, we consider some $f(z, \bar{z}) \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$ and want to find out when it is exact. For this, we use the radial decomposition of the punctured plane as follows.

We use coordinates $(\mathrm{d}r, \mathrm{d}\theta)$ instead of $(\mathrm{d}z, \mathrm{d}\bar{z})$. Remark that θ is a multi-valued function, but $\mathrm{d}\theta$ is well-defined. On the other hand, r is well-defined and positive on all of $\mathbb{C}^*$. Any closed 2-form ψ will now be of the form

$$\psi = f(r, \theta) \, \mathrm{d}r \wedge \mathrm{d}\theta.$$

This form can be directly be written as

$$\psi = \mathrm{d} \left(\left[\int_1^r f(r', \theta) \mathrm{d}r' \right] \mathrm{d}\theta \right)$$

which tells us ψ is d–exact. However, we need it to be $\mathrm{d} + \alpha$–exact, or equivalently we need $z^\lambda \psi$ to be d–exact over one-forms multiplied by z^λ. We have $z^\lambda = r^\lambda e^{\lambda i \theta}$, so this gives

$$z^\lambda \psi = \mathrm{d} \left(\left[\int_1^r f(r', \theta) \cdot r'^\lambda e^{\lambda i \theta} \mathrm{d}r' \right] \mathrm{d}\theta \right) = \mathrm{d} \left(\left[\int_1^r f(r', \theta) \cdot r'^\lambda \mathrm{d}r' \right] \, e^{\lambda i \theta} \, \mathrm{d}\theta \right).$$

Since r is a strictly positive function on $\mathbb{C}^*$, this integral over r' is well-defined (algebraically, not geometrically) and we find that ψ is $\mathrm{d}+\alpha$–exact. Since this holds for any closed 2-form ψ, we find that $H^2_\alpha = 0$.

We conclude that

$$H^0_\alpha = 0, \quad H^1_\alpha = 0, \quad H^2_\alpha = 0.$$

Therefore, this twisted cohomology agrees with the (untwisted) cohomology of a topologically contractible space. This is intuitively somewhat expected because the cohomology ring of $\mathbb{C}^*$ is generated by $\alpha = \lambda\,\mathrm{d}\log(z)$ and contains no other elements.

As a final remark, for integer $\lambda = k \in \mathbb{Z}$ we find that $f(z) = z^{-k}$ is well-defined on $\mathbb{C}^*$, hence $H^0_\alpha = \{[z^{-k}]\}$ is non-empty and of the same dimension as H^0. In this case it similarly turns out that $H^1_\alpha \cong H^1$ and $H^2_\alpha \cong H^2 = 0$, with the isomorphisms given by $[\phi] \mapsto [\phi z^{-k}]$. This is part of a more general fact which we explore in Proposition 4.2.1. △

We see that the twisted cohomology of a manifold can be 0, even when the (untwisted) cohomology is non-zero. We generally expect the ℓ–th twisted cohomology to have a smaller dimension than the ℓ–th untwisted cohomology, as we will see in Theorem 4.2.5. Also note that the fundamental group of $\mathbb{C}^*$ is generated by a single loop around the origin, and the monodromy of α on this loop is non-zero. This means that the subgroup of trivial monodromy $\{[\gamma] \in \pi_1(M) \mid \exp\int_\gamma \alpha = 1\}$ is trivial for this M and α. While we do not prove an equivalent of the Hurewicz theorem (see [Hat05]) for twisted cohomology (and there exists no such theorem as far as the author is aware), this still suggests that $H^1_\alpha = 0$ is the expected outcome.

4.2 Relevant Results

We will now discuss some relevant and important results relating to the twisted cohomology. Our first result involves closed differential forms $\beta \in \Omega^1$ with the property that $\int_\gamma \beta \in \mathbb{Z}$ is an integer for any loop γ on M. Such a differential form is called *integral* and its cohomology class is called an *integral class*. Note that any exact differential 1-form $\mathrm{d}f$ is integral since $\int_\gamma \mathrm{d}f = 0$ for any loop γ.

Proposition 4.2.1. *Let M be a connected complex manifold, let $\alpha \in \Omega^1(M)$ be closed, let $\beta \in \Omega^1(M)$ be integral, and let $\alpha' = \alpha + 2\pi i\,\beta$. Then for any integer $k \geq 0$ we have*

$$H^k_\alpha \cong H^k_{\alpha'}. \tag{25}$$

Proof. We first show that the statement holds for β exact, i.e. $\alpha' = \alpha + \mathrm{d}f$ for a smooth function $f \in C^\infty(M)$. We will define a map $h_{\mathrm{d}f} : \Omega^k \to \Omega^k$ and show it induces a well-defined map $h : H^k_\alpha \to H^k_{\alpha'}$ on the twisted cohomologies, then we show the map gives an isomorphism. We define h to be the linear map

$$h_{\mathrm{d}f} : \Omega^k \to \Omega^k, \qquad \phi \mapsto e^{-f}\phi.$$

Note that this map is a linear bijection with inverse $\phi \mapsto e^f\phi$.

We first need to show that any $\phi \in \Omega^k$ is $\mathrm{d}+\alpha$–closed if and only if $h(\phi)$ is $\mathrm{d}+\alpha'$–closed, then show a similar statement for $\mathrm{d}+\alpha$–exact respectively $\mathrm{d}+\alpha'$–exact. For the first statement, note that

$$\begin{aligned}(\mathrm{d}+\alpha'\wedge)(e^{-f}\phi) &= e^{-f}(\mathrm{d}+\alpha\wedge+\mathrm{d}f\wedge)\phi + (\mathrm{d}e^{-f})\wedge\phi \\ &= e^{-f}(\mathrm{d}+\alpha\wedge)\phi + e^{-f}(\mathrm{d}f - \mathrm{d}f)\wedge\phi = e^{-f}(\mathrm{d}+\alpha\wedge)\phi.\end{aligned}$$

Since e^{-f} is invertible we find that ϕ is $\mathrm{d}+\alpha$–closed if and only if $h_{\mathrm{d}f}(\phi)$ is $\mathrm{d}+\alpha'$–closed. For the second statement, let $\phi = (\mathrm{d}+\alpha)\chi$ for some $(k-1)$-form χ. Then we have

$$e^{-f}\phi = e^{-f}(\mathrm{d}+\alpha\wedge)\chi = e^{-f}(\mathrm{d}+\alpha\wedge+\mathrm{d}f\wedge)\chi - e^{-f}\mathrm{d}f\wedge\chi = (\mathrm{d}+\alpha\wedge+\mathrm{d}f\wedge)(e^{-f}\chi).$$

We see that $h_{\mathrm{d}f}(\phi) = e^{-f}\phi$ is $\mathrm{d}+\alpha'$–exact. Following the same equation in the opposite direction we see that ϕ is $(\mathrm{d}+\alpha)$–exact if and only if $h_{\mathrm{d}f}(\phi)$ is $(\mathrm{d}+\alpha')$–exact. Therefore, the map $h_{\mathrm{d}f}$ is well-defined on the level of cohomologies:

$$h_{\mathrm{d}f} : H^k_\alpha \to H^k_{\alpha'}, \qquad [\phi] \mapsto [e^{-f}\phi].$$

The same holds for the inverse map $\phi \mapsto e^f\phi$, hence $h_{\mathrm{d}f}$ is a bijection on the cohomology groups. Since $h_{\mathrm{d}f}$ is also linear, we conclude it is an isomorphism.

We secondly show that the statement holds for $\alpha' = \alpha + 2\pi i\ \beta$. While we can not generally write $\beta = \mathrm{d}g$, we can still define an equivalent of e^{-g} by using the integral property. By the definition of the integral property we know that for any loop γ on M, the monodromy of β on γ is an integer. Thus we have for any loop γ on M that

$$\exp\left(\int_\gamma 2\pi i\ \beta\right) = \exp\left(2\pi i\ k_\gamma\right) = 1,$$

for some $k_\gamma \in \mathbb{Z}$. As a consequence we have that for any $p, q \in M$ the complex number

$$\exp\left(\int_{p\rightsquigarrow q} 2\pi i\ \beta\right) \in \mathbb{C}^*$$

is well-defined as it does not depend on the choice of a path $p \rightsquigarrow q$. This means we define an equivalent of e^g as follows. We choose a basepoint $z_0 \in M$ and set

$$G : M \to \mathbb{C}, \qquad z \mapsto \exp\left(\int_{z_0\rightsquigarrow z} 2\pi i\ \beta\right).$$

This is again smooth, following the same argument as we used in Proposition 4.1.3 for $s'(x)$. Note that G takes values in $\mathbb{C}^*$, so it has a well-defined smooth inverse G^{-1}. This inverse will be our equivalent of e^{-g}.

We define h_G by

$$h_\beta : \Omega^k \to \Omega^k, \qquad \phi \mapsto G^{-1}\phi.$$

Using the fact that

$$\mathrm{d}G = 2\pi i\ \beta\ G \qquad \text{hence} \qquad \mathrm{d}G^{-1} = -2\pi i\ \beta\ G^{-1},$$

we similarly see that the induced map $h_\beta : H^k_\alpha \to H^k_{\alpha'}$ is a well-defined isomorphism.

We find that $H^k_\alpha \cong H^k_{\alpha'}$ for any $\alpha' = \alpha + 2\pi i\ \beta$. □

Remark 4.2.2. The converse of the above theorem does *not* hold true: the cohomologies of α and α' can agree even if their difference is non-exact and non-integral. We saw this in Example 4.1.6 on $\mathbb{C}^*$, as $\frac{1}{2}\,\mathrm{d}\log(z)$ and $\frac{1}{3}\,\mathrm{d}\log(z)$ give the same cohomology but their difference $\frac{1}{6}\,\mathrm{d}\log(z)$ is not exact or integral. •

Remark 4.2.3. One might wonder whether the above maps $h_{\mathrm{d}f}, h_\beta : \Omega^k \to \Omega^k$ also induces a isomorphism on the cohomology ring. However, the twisted cohomology does not actually form a ring under multiplication, as the wedge of two $\mathrm{d}+\alpha$–closed differential forms is generally not $\mathrm{d}+\alpha$–closed:

$$(\mathrm{d}+\alpha)\phi\wedge\theta = \mathrm{d}\phi\wedge\theta + (-1)^k\phi\wedge\mathrm{d}\theta + \alpha\wedge\phi\wedge\theta = [(\mathrm{d}+\alpha)\phi]\wedge\theta + (-1)^k\phi\wedge\mathrm{d}\theta = (-1)^k\phi\wedge\mathrm{d}\theta \neq 0,$$

for $\theta \in \Omega^k$ and $\phi \in \Omega^l$ for some k, l both $\mathrm{d}+\alpha$–closed.

Instead, the wedge product of a $\mathrm{d}+\alpha$–closed ϕ and d–closed θ will be $\mathrm{d}+\alpha$–closed, which we also see from the above computation. This gives an action of the untwisted cohomology H^* on the twisted cohomology H^*_α, making the latter into a H^*–module. The map $h_{\mathrm{d}f}$ does give an isomorphism of H^*–modules because the map intertwines the action of H^*:

$$[\theta\wedge(e^{-f}\phi)] = [e^{-f}\theta\wedge\phi]$$

for representatives of classes $[\theta] \in H^*$, $[\phi] \in H^*_\alpha$. •

The map $h_\beta : \Omega^k \to \Omega^k$ is not well-defined for non-integral $\beta \in \Omega^1$. Nevertheless, we can try to define it and see what happens to form intuition. Following Example 4.1.6, we take $\alpha = \lambda\cdot\mathrm{d}\log(z)$ on $\mathbb{C}^*$ with λ non-integer. We then have "$f = \lambda\log(z)$" hence

$$h_{\lambda\cdot\mathrm{d}\log(z)}(\phi) = e^{-\lambda\log(z)}\phi = z^{-\lambda}\phi.$$

Indeed $z^{-\lambda}$ is not a single-valued well-defined function for non-integer λ, so the map $h_{\lambda\cdot\mathrm{d}\log(z)}$ is not well-defined. However, this does tell us that we can think of the cohomology classes of $H^k_{\lambda\cdot\mathrm{d}\log(z)}$ as classes $[z^{-\lambda}\phi]$

for some d-closed $\phi \in \Omega^k$. For example, instead of seeing $H^0_{\lambda\cdot \mathrm{d}\log(z)}$ as $\{f \in \Omega^0(\mathbb{C}^*) | \mathrm{d}f + \lambda f \,\mathrm{d}\log(z) = 0\}$ we can describe the same cohomology as

$$H^0_{\lambda\cdot \mathrm{d}\log(z)} = \{z^{-\lambda} f \mid \mathrm{d}f = 0, \quad z^{-\lambda} f \in \Omega^0(\mathbb{C}^*)\} = \{g \mid \mathrm{d}(z^\lambda g) = 0, \quad g \in \Omega^0(\mathbb{C}^*)\}.$$

From the former description it follows that any f must be locally constant and $fz^{-\lambda}$ must be smooth and single-valued; the only f satisfying both conditions is $f = 0$ hence $H^0_{\lambda\cdot \mathrm{d}\log(z)} = 0$ like we found above. Similarly for higher k, we have the (formal algebraic) expression

$$H^k_{\lambda\cdot \mathrm{d}\log(z)} \cong \{z^{-\lambda}\phi \mid z^{-\lambda}\phi \in \Omega^k, \quad \mathrm{d}\phi = 0\}/\{z^{-\lambda}\mathrm{d}\psi \mid z^{-\lambda}\psi \in \Omega^{k-1}\}. \tag{26}$$

This generally simplifies the differential equation for the kernel, but can make solutions more complicated. The formula is equally true for $\mathcal{M}_{0,n}$ as the steps in the derivation do not depend on the used manifold.

In the physics context we will typically be looking at 1-forms such as

$$\alpha = \sum_i \lambda_i \,\mathrm{d}\log(P_i),$$

where λ_i is constant (or only depends on external parameters) and each P_i is a polynomial defined on $\mathbb{C}^n$. Mirroring the steps in Example 4.1.6 it is clear that we will be assuming that there is some i such that $\lambda_i \notin \mathbb{Z}$, since any integer term $k \,\mathrm{d}\log(P_i)$ can be scaled away using a smooth, well-defined factor P_i^{-k}. One might wonder what happens if e.g. λ_1 is non-integer but λ_2 is integer or vice versa (integer respectively non-integer) or both non-integer; are the twisted cohomologies in each case isomorphic? To illustrate these cases, we look at another example.

Example 4.2.4 (Twice punctured plane)**.** We consider the twice punctured plane $\mathbb{C}^{**} := \mathbb{C} \setminus \{0, 1\}$. Note that $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$, so we can see manifold as $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ as well. We look at the twisted cohomology with respect to a one-form

$$\alpha = \lambda_1 \,\mathrm{d}\log(z) + \lambda_2 \,\mathrm{d}\log(z - 1),$$

in three different cases: $\lambda_1 \in \mathbb{Z}, \lambda_2 \notin \mathbb{Z}$ or $\lambda_1 \notin \mathbb{Z}, \lambda_2 \in \mathbb{Z}$ or finally $\lambda_1, \lambda_2 \notin \mathbb{Z}$.

Before we calculate the twisted cohomologies, recall the untwisted cohomology: the zeroth cohomology is $H^0(\mathbb{C}^{**}) \cong \mathbb{C}$ which is the space of constant functions, while the first cohomology $H^1(\mathbb{C}^{**}) \cong \mathbb{C}^2$ is spanned by $\mathrm{d}\log(z)$ and $\mathrm{d}\log(z - 1)$. All higher cohomology groups are 0.

Integer λ_1, non-integer λ_2. We first assume $\lambda_1 \in \mathbb{Z}$. By Proposition 4.2.1 we may set $\lambda_1 = 0$, giving us the one form $\alpha = \lambda_2 \,\mathrm{d}\log(z - 1)$. Similarly to Example 4.1.6 we look at the space of differential forms multiplied with $\exp(\lambda_2 \,\log(z - 1)) = (z - 1)^{\lambda_2}$.

The zeroth twisted cohomology consists of all well-defined and smooth functions f such that $(z - 1)^{\lambda_2} f$ is constant. Since $(z - 1)^{\lambda_2}$ is not well-defined and constant functions are, we have a contradiction unless $f = 0$. We find

$$H^0_{\lambda_2 \,\mathrm{d}\log(z-1)} = 0.$$

We move on to the first cohomology group. We use the same idea as we did in Example 4.1.6. Let $\phi = \phi_1 \,\mathrm{d}z + \phi_2 \,\mathrm{d}\bar{z}$ a closed 1-form. We try to find a primitive of $(z - 1)^{\lambda_2}\phi_1$ with respect to z. If we can find such a primitive f, then adding $\mathrm{d}f$ to ϕ gives a new 1-form $\phi' = \phi'_2 \,\mathrm{d}\bar{z}$. It then follows that $(z - 1)^{\lambda_2} \,\phi'_2$ purely depends on $\bar{z}$, but that means that ϕ'_2 is of the form $(z - 1)^{-\lambda_2} \,g(\bar{z})$ for some function g, and demanding that this is single-valued means that $g(\bar{z})$ contains no terms $1/\bar{z}$ hence can be integrated. This means that ϕ' is exact.

We therefore have to find when a primitive of $(z - 1)^{\lambda_2}\phi_1$ with respect to z exists. While in Example 4.1.6 the only problem was $1/z$, we now have $1/z$ and $1/(z - 1)$ which may give a multi-valued contribution, as they have a non-zero residue around $z = 0$ and $z = 1$ correspondingly. The latter is solved by the factor $(z - 1)^{\lambda}$, which means we can find a primitive in that case. However, the former persist. This means that the only 1-forms which we can not reduce to exact terms can be written in the form

$$\phi = \frac{a}{z} \,\mathrm{d}z.$$

As such, we find that

$$H^1_{\lambda_2 \,\mathrm{d}\log(z-1)} \cong \mathbb{C} \cdot \left[\frac{1}{z}\,\mathrm{d}z\right] = \mathbb{C}[\mathrm{d}\log z]$$

is one dimensional.

Finally we look at the second cohomology and higher. Since $\mathbb{C}^{**}$ deformation retracts to the wedge of two circles $S^1 \wedge S^1$, we know that the higher cohomology groups are all 0. The $H^2_{\lambda_2 \,\mathrm{d}\log(z-1)} = 0$ case can also be explicitly calculated analogously to Example 4.1.6.

Non-integer λ_1, integer λ_2. This case is very similar to the previous case, except the roles of z and $z-1$ have been swapped. We find:

$$H^0_{\lambda_1 \,\mathrm{d}\log(z)} = 0, \quad H^1_{\lambda_1 \,\mathrm{d}\log(z)} \cong \mathbb{C}[\mathrm{d}\log(z-1)], \quad H^k_{\lambda_1 \,\mathrm{d}\log(z)} = 0 \quad k \geq 2.$$

Non-integer λ_1 and λ_2. This appears to be a very different case, since both $\mathrm{d}\log(z)$ and $\mathrm{d}\log(z-1)$ contribute. However as we will see, in fact little has changed compared to the first case.

We remark that the algebraic scale factor corresponding to the trick in Equation 26 is $z^{\lambda_1}(z-1)^{\lambda_2}$. Since the product of this factor with a smooth function will never be constant, we find

$$H^0_{\lambda_1 \,\mathrm{d}\log(z)+\lambda_2 \,\mathrm{d}\log(z-1)} = 0.$$

We can also again remark that the second cohomology group should be 0, so as in the previous cases the main goal is to calculate the first cohomology group. Since $z^{\lambda_1}(z-1)^{\lambda_2}$ does not depend on $\bar{z}$, the same argument as in the first case tells us that the $\alpha_2 \mathrm{d}\bar{z}$ part of any 1-form α can be removed using exact terms, giving a part $\mathrm{d}z$ in return. As such, we may restrict our search for closed forms to 1-forms $\alpha = \alpha_1 \mathrm{d}z$. In that case, we have to solve

$$0 = \mathrm{d}\big(z^{\lambda_1}(z-1)^{\lambda_2}\alpha_1 \,\mathrm{d}z\big) = z^{\lambda_1}(z-1)^{\lambda_2}\,\frac{\partial \alpha_1}{\partial \bar{z}}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z.$$

Again, this is solved by $\partial\alpha_1/\partial\bar{z} = 0$. This means the closed forms are given by

$$\alpha = \alpha_1(z)\,\mathrm{d}z.$$

We have to find out for which α_1 this is non-exact. We again do this by integrating $z^{\lambda_1}(z-1)^{\lambda_2}\alpha_1$. We see that the integration is not well-defined for $\alpha_1 = 1/z$ but also not for $\alpha_1 = 1/(z-1)$. However, we do know that the integration is well-defined for at least one linear combination of the two:

$$z^{\lambda_1}(z-1)^{\lambda_2}\left[\frac{c\lambda_1}{z} + \frac{c\lambda_2}{z}\right]\mathrm{d}z = \mathrm{d}\big(z^{\lambda_1}(z-1)^{\lambda_2}c\big)$$

for c a constant. For any other linear combination of $1/z$ and $1/(z-1)$ the integration fails. Since $\lambda_1 \neq 0, \lambda_2 \neq 0$ we find that

$$[\mathrm{d}\log(z)] = -\frac{\lambda_2}{\lambda_1}\,[\mathrm{d}\log(z-1)]\,,$$

such that we can take either of the two and either linear combination which is not of the form above as the representative. We conclude that

$$H^1_{\lambda_1 \,\mathrm{d}\log(z)+\lambda_2 \,\mathrm{d}\log(z-1)} \cong \mathbb{C}[\mathrm{d}\log(z-1)] = \mathbb{C}[\mathrm{d}\log(z)].$$

We have found that the cohomology groups are actually isomorphic for different values of λ_1 and λ_2, assuming at least one is non-integer. △

We see that $\mathbb{C}^{**}$ only has a single twisted cohomology. Later theorems, the Middle Dimension Theorem 4.2.10 and Euler characteristic 4.3.9, will tell us that this is typically what we expect. Before we look at those theorems, we first attempt to compare the twisted cohomology to the untwisted cohomology.

Theorem 4.2.5. *Let M be a connected complex manifold and let α be a closed 1-form on M. Assume that*

$$\operatorname{im}\left[\mathrm{d}:\Omega^k(M)\to\Omega^{k+1}(M)\right]\cap\ker\left[\alpha\wedge:\Omega^k(M)\to\Omega^{k+1}(M)\right]=\operatorname{im}\left[\alpha\wedge\mathrm{d}:\Omega^{k-1}(M)\to\Omega^k(M)\right]$$

for all k. Then we have an isomorphism of graded H^*–modules,

$$\frac{\ker\left[[\alpha]\wedge:H^*\to H^*\right]}{\operatorname{im}\left[[\alpha]\wedge:H^*\to H^*\right]}\cong H^*_\alpha, \tag{27}$$

where H^* is the cohomology ring, $[\alpha]\wedge:H^*\to H^*$ is the ring product with $[\alpha]\in H^1$, and H^*_α is the twisted cohomology module.

Proof. We will prove this isomorphism by sending a map from the kernel $\ker\left[[\alpha]\wedge:H^*\to H^*\right]$ to H^*_α, then proving it is surjective and has kernel $\operatorname{im}\left[[\alpha]\wedge:H^*\to H^*\right]$. In order to construct the map we need the following lemma.

Lemma 4.2.6. *Let*

$$I^*_\alpha=\{\phi\in\Omega^*(M)\mid\alpha\wedge\phi=0\}=\{\alpha\wedge\phi\mid\phi\in\Omega^*(M)\}$$

be the differential ideal generated by α and let ψ be a k–form such that $\mathrm{d}\psi\in I^{k+1}_\alpha$. Then there exist $\phi\in\Omega^{k-1}(M)$ and $\chi\in\Omega^k(M)$ such that

$$\psi=\alpha\wedge\phi+\chi,\qquad\mathrm{d}\chi=0.$$

Proof. This follows from the assumption on $\operatorname{im}\mathrm{d}\cap\ker(\alpha\wedge)=\operatorname{im}(\alpha\wedge\mathrm{d})$. The form $\mathrm{d}\psi$ is per definition in $\operatorname{im}\mathrm{d}$ and per assumption in $\ker(\alpha\wedge)$, hence there is some $\beta\in\Omega^{k-1}$ such that

$$\mathrm{d}\psi=\alpha\wedge\mathrm{d}\beta=-\mathrm{d}(\alpha\wedge\beta).$$

This shows that $\psi=-\alpha\wedge\beta+\chi$ for some closed χ, hence choosing $\phi=-\beta$ gives

$$\psi=\alpha\wedge\phi+\chi,$$

as required. □

Construction We use the lemma to construct a map from $\ker([\alpha]\wedge)$ to H^*_α. Let $[\psi]\in\ker([\alpha]\wedge)$ be represented by $\psi\in\Omega^k(M)$, then $\mathrm{d}\psi=0$ and $[\alpha\wedge\psi]=[0]$ hence $\alpha\wedge\psi=\mathrm{d}\chi$ for some $\chi\in\Omega^k(M)$. Now $\mathrm{d}\chi\in I^{k+1}_\alpha$, so using the lemma we can write $\chi=\alpha\wedge\chi'+\xi$ with $\mathrm{d}\xi=0$. This gives us

$$(\mathrm{d}+\alpha\wedge)\psi=\mathrm{d}\chi=-\alpha\wedge\mathrm{d}\chi',$$

so

$$(\mathrm{d}+\alpha\wedge)(\psi+\mathrm{d}\chi')=-\alpha\wedge\mathrm{d}\chi'+\alpha\wedge\mathrm{d}\chi'=0.$$

We see that $\psi+\mathrm{d}\chi'$ is $(\mathrm{d}+\alpha)$–closed, hence defines a class in H^k_α. This gives a map from $\ker([\alpha]\wedge)$ to H^*_α.

Well-definedness We first have to check this map is well-defined. Remark that we did not use ξ in the construction and ξ can absorb any exact terms, so the map does not depend on the choice of ψ and we only need to show the map does not depend on the choice of χ. To show this, we choose another χ_2 such that $\alpha\wedge\psi=\mathrm{d}\chi_2$. We can again use the lemma to split $\chi_2=\alpha\wedge\chi_2'+\xi_2$ with $\mathrm{d}\xi_2=0$. We then have

$$-\alpha\wedge\mathrm{d}\chi'=\mathrm{d}\chi=\alpha\wedge\psi=\mathrm{d}\chi_2=-\alpha\wedge\mathrm{d}\chi_2'.$$

This shows that $\alpha\wedge(\mathrm{d}\chi'-\mathrm{d}\chi_2')=0$, hence there exists some $\zeta\in\Omega^{k-1}(M)$ such that

$$\mathrm{d}\chi'=\mathrm{d}\chi_2'+\alpha\wedge\zeta.$$

Note that $\mathrm{d}\chi'$ and $\mathrm{d}\chi_2'$ are both closed, hence $\alpha\wedge\mathrm{d}\zeta=0$. Again using the lemma we find ζ',β such that

$$\zeta=\alpha\wedge\zeta'+\beta,\quad\mathrm{d}\beta=0.$$

This finally tells us

$$\mathrm{d}\chi'=\mathrm{d}\chi_2'+\alpha\wedge\beta=\mathrm{d}\chi_2'+(\mathrm{d}+\alpha\wedge)\beta,$$

hence

$$[\psi+\mathrm{d}\chi']=[\psi+\mathrm{d}\chi_2']\in H^k_\alpha.$$

We see that the map does not depend on the choice of χ, hence is well-defined.

Surjectivity We next have to show surjectivity. Let $\psi \in H^k_\alpha$. Then $\mathrm{d}\psi = -\alpha \wedge \psi \in I^{k+1}_\alpha$, so using the lemma we can find $\phi \in \Omega^{k-1}(M)$ and $\chi \in \Omega^k(M)$ such that

$$\psi = \alpha \wedge \phi + \chi, \qquad \mathrm{d}\chi = 0.$$

Then $[\chi] \in H^k$ and we have

$$[\alpha] \wedge [\chi] = [\alpha \wedge \chi] = [\alpha \wedge \psi] = [-\mathrm{d}\psi] = [0].$$

Therefore, $[\chi] \in \ker([\alpha]\wedge)$. We show that this class is mapped to $[\psi] \in H^k_\alpha$.

We have

$$\mathrm{d}\chi = 0, \qquad \alpha \wedge \chi = \alpha \wedge \psi = -\mathrm{d}\psi$$

where we recall that

$$\psi = \alpha \wedge \phi + \chi.$$

The map then sends $[\chi] \in \ker([\alpha]\wedge)$ to $[\chi - \mathrm{d}\phi] \in H^k_\alpha$. This class is precisely $[\psi]$:

$$[\chi - \mathrm{d}\phi] = [\chi - \mathrm{d}\phi + (\mathrm{d} + \alpha\wedge)\phi] = [\chi + \alpha \wedge \phi] = [\psi].$$

We see that the map is surjective.

Kernel We now consider the kernel of the map; we want to show this kernel is precisely $\mathrm{im}([\alpha]\wedge)$ using mutual inclusion. Let $[\alpha \wedge \psi] \in \mathrm{im}([\alpha]\wedge)$ be represented by $\alpha \wedge \psi$ for $\psi \in \Omega^{k-1}(M)$. Then $\alpha \wedge (\alpha \wedge \psi) = 0$, so $[\alpha \wedge \psi]$ is simply mapped to $[\alpha \wedge \psi] \in H^*_\alpha$. However, we know $\mathrm{d}(\alpha \wedge \psi) = 0$ hence $\alpha \wedge \mathrm{d}\psi = 0$, hence using the lemma we can find ψ' and χ such that

$$\psi = \alpha \wedge \psi' + \chi, \qquad \mathrm{d}\chi = 0.$$

Now

$$\alpha \wedge \psi = \alpha \wedge \chi = (\mathrm{d} + \alpha\wedge)\chi,$$

so we find that

$$[\alpha \wedge \psi] = [(\mathrm{d} + \alpha\wedge)\chi] = [0] \in H^k_\alpha.$$

We find that $\mathrm{im}([\alpha]\wedge)$ lies in the kernel.

Now assume we have $[\psi] \in \ker([\alpha]\wedge)$, $\psi \in \Omega^k(M)$ such that $[\psi]$ is mapped to $[0] \in H^k_\alpha$. Then after choosing χ, χ' and ξ as in the construction we find that $[\psi + \mathrm{d}\chi'] = [0]$. This implies there is some $\phi \in \Omega^{k-1}(M)$ such that

$$\psi + \mathrm{d}\chi' = \mathrm{d}\phi + \alpha \wedge \phi.$$

But then in particular

$$\psi = \alpha \wedge \phi + \mathrm{d}\phi - \mathrm{d}\chi', \qquad [\psi] = [\alpha \wedge \phi].$$

Now note that

$$\alpha \wedge \mathrm{d}\phi = \alpha \wedge (\psi + \mathrm{d}\chi' - \alpha \wedge \phi) = 0,$$

so we can use the lemma to find ϕ', β such that

$$\phi = \alpha \wedge \phi' + \beta, \qquad \mathrm{d}\beta = 0.$$

Then finally

$$[\psi] = [\alpha \wedge \phi] = [\alpha \wedge \beta] = [\alpha] \wedge [\beta],$$

hence $[\psi] \in \mathrm{im}([\alpha]\wedge)$. We see that the kernel is precisely $\mathrm{im}([\alpha]\wedge)$.

Intertwines ring action To show that the two spaces are isomorphic as H^*–modules, we also have to show the map intertwines the action of H^*. We show this directly: let $[\psi] \in H^k$ be represented by $\psi \in \Omega^k(M)$, let $[\phi] \in H^\ell$ be represented by $\phi \in \Omega^\ell(M)$ and let χ, χ', ξ such that

$$\alpha \wedge \psi = \mathrm{d}\chi, \qquad \chi = \alpha \wedge \chi' + \xi, \qquad \mathrm{d}\xi = 0.$$

The map sends

$$[\psi] \mapsto [\psi + \mathrm{d}\chi'].$$

We then also have

$$\alpha \wedge (\phi \wedge \psi) = \mathrm{d}(\phi \wedge \chi), \qquad \phi \wedge \chi = (-1)^\ell \alpha \wedge (\phi \wedge \chi') + \phi \wedge \xi, \qquad \mathrm{d}(\phi \wedge \xi) = 0.$$

Therefore, the map sends

$$[\phi] \wedge [\psi] = [\phi \wedge \psi] \mapsto [\phi \wedge \psi + (-1)^\ell \mathrm{d}(\phi \wedge \chi')] = [\phi \wedge (\psi + \mathrm{d}\chi')].$$

We see that the map intertwines the action of H^* on both modules.

Conclusion We have constructed a surjective intertwining map from $\ker([\alpha]\wedge)$ to H^*_α with kernel $\mathrm{im}([\alpha]\wedge)$. We conclude that

$$\frac{\ker\left[[\alpha]\wedge : H^* \to H^*\right]}{\mathrm{im}\left[[\alpha]\wedge : H^* \to H^*\right]} \cong H^*_\alpha$$

as graded H^*–modules. □

This theorem is very useful, as it gives us a concrete way of calculating the twisted cohomology if we already know the cohomology ring. The cohomology ring can be calculated using the de Rham cohomology but also with other cohomology theories and results such as the Künneth formula and the Mayer-Vietoris Sequence can be used; the same is not true for the twisted cohomology.

We first see two important corollaries before we shortly discuss the strange condition $\mathrm{im}\,\mathrm{d} \cap \ker(\alpha\wedge) = \mathrm{im}(\alpha \wedge \mathrm{d})$.

Corollary 4.2.7 (Euler Characteristic)**.** *Let M and α be as in the Theorem and again assume that*

$$\mathrm{im}\left[\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)\right] \cap \ker\left[\alpha\wedge : \Omega^k(M) \to \Omega^{k+1}(M)\right] = \mathrm{im}\left[\alpha \wedge \mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M)\right]$$

for all k. Then we have

$$\sum_{i=0}^{\infty} (-1)^i \ \dim H^i_\alpha = \sum_{i=0}^{\infty} (-1)^i \ \dim H^i = \chi(M),$$

where $\chi(M)$ is the Euler characteristic.

Proof. We have the following:

$$\begin{aligned}
\sum_{i=0}^{\infty} (-1)^i \ \dim H^i_\alpha &= \sum_{i=0}^{\infty} (-1)^i \ \left(\dim\ker([\alpha]\wedge : H^i \to H^{i+1}) - \dim\mathrm{im}([\alpha]\wedge : H^{i-1} \to H^i)\right) \\
&= \sum_{i=0}^{\infty} (-1)^i \ \left(\dim\ker([\alpha]\wedge : H^i \to H^{i+1}) + \dim\mathrm{im}([\alpha]\wedge : H^i \to H^{i+1})\right) \\
&= \sum_{i=0}^{\infty} (-1)^i \ \dim H^i.
\end{aligned}$$

□

Corollary 4.2.8. *Let M and α be as in the Theorem and again assume that*

$$\mathrm{im}\left[\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)\right] \cap \ker\left[\alpha\wedge : \Omega^k(M) \to \Omega^{k+1}(M)\right] = \mathrm{im}\left[\alpha \wedge \mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M)\right]$$

for all k. Then we have

$$\dim H^k_\alpha \leq \dim H^k,$$

for any $k \geq 0$. □

The above two corollaries tell us a lot about the structure of the twisted cohomology, which can be useful when combined with other results such as the Middle Dimension Theorem 4.2.10 or the Morse Theory of Subsection 4.4.

Discussion 4.2.9. The above Theorem 4.2.5 and its corollaries have a non-standard assumption:

$$\operatorname{im}\left[\mathrm{d}:\Omega^k(M)\to\Omega^{k+1}(M)\right]\cap\ker\left[\alpha\wedge:\Omega^k(M)\to\Omega^{k+1}(M)\right]=\operatorname{im}\left[\alpha\wedge\mathrm{d}:\Omega^{k-1}(M)\to\Omega^k(M)\right].$$

This condition states that if there are $\psi,\phi\in\Omega^k(M)$ such that

$$\mathrm{d}\psi=\alpha\wedge\phi,$$

then there is some $\chi\in\Omega^{k-1}(M)$ such that

$$\mathrm{d}\psi=\alpha\wedge\mathrm{d}\chi=\alpha\wedge\phi.$$

We needed to assume this since we found no way to generally prove this and since the condition is not true for all M and α, although we do know that it is true in certain cases. One can for instance easily verify that it is true for the twice punctured plane in Example 4.2.4.

If M is closed (compact without boundary) and α is exact and nowhere vanishing, then it is most likely possible to prove the condition using the Atiyah-Singer index theorem, but when M is only compact it already becomes more difficult since the Atiyah-Patodi-Singer index theorem might not behave well on the boundary. This means we can not prove the condition for bounded cut-off of M either, so there is no clear way to argue that this condition should be true due to physical assumption. •

4.2.1 Middle Dimension Theorem

We have another Theorem which is useful and often true, but not generally. This is the Middle Dimension Theorem. It states:

Theorem 4.2.10 (Middle Dimension Theorem)**.** *Let M be a connected complex manifold and α a closed, non-exact 1-form on M such that α is not $2\pi i$ times an integral closed form (i.e. U is multi-valued). Assume that one of the following conditions is satisfied:*

- *The dimension of M is $\dim_{\mathbb{C}} M=1$.*
- *The 1-form satisfies $\alpha=\sum_i\alpha_i\,\mathrm{d}\log P_i$ for a finite number of polynomials $\{P_i\}$ such that the highest homogeneous components $\overline{P}_j$ satisfy Assumption 2 of [Aom+11:pg.84], and such that $\sum_i\alpha_i\deg\overline{P}_i\notin\mathbb{Z}$. Moreover, M is the algebraic variety defined as the complement of the zero set of the P_j in $\mathbb{C}^n$.*
- *There are finitely many zeroes of α and all zeroes are non-degenerate, and there exists a cut-off of M such that the boundary has zero twisted cohomology.*

Then we have

$$H^k_\alpha=0\qquad\text{for}\quad k\neq n=\dim_{\mathbb{C}}(M).$$

Proof sketch. At this point we will not prove the theorem, but we will give the main idea of each scenario.

The theorem can be proven directly under the assumption $\dim_{\mathbb{C}} M=1$ by using that

$$H^0_\alpha=\{f:M\to\mathbb{C}\mid\mathrm{d}(Uf)=0\}=0,$$

since a function f can never be single-valued on M if it is of the form cU^{-1} for c constant. One then finds $H^2_\alpha=0$ using a duality theorem (see Subsection 4.3) together with the (physical) fact that M has a compact cut-off $M_{c.o.}$ such that $H^2_\alpha(M)=H^2(M_{c.o.},\mathcal{L}_\alpha|_{M_{c.o.}})$. This proves directly that only H^1_α is generally non-zero.

When the second possible condition is true, the proof from [Aom+11] can be used; see pages 60 to 84 of the book for all details. The main idea is as follows. Let D be the union of the sets of zeroes of the P_j. Instead of all smooth differential forms (which one considers in de Rham theory) we can limit the twisted cohomology to only forms which are rational functions with poles in D, or even only forms which diverge at

most logarithmic in D (hence which are a product of polynomials and logarithms). Denoting the rational k–forms by $\Omega^k(*D)$ and the logarithmic k–forms by $\Omega^k(\log D)$, we clearly have

$$\Omega^k(\log D) \subset \Omega^k(*D) \subset \Omega^k.$$

These subcomplexes respect $\mathrm{d}+\alpha$, in the sense that the image of $\Omega^k(\log D)$ under $\mathrm{d}+\alpha$ lies entirely inside $\Omega^{k+1}(\log D)$, and similarly for $\Omega^k(*D)$. This means it is sensible to take the cohomology over a subcomplex rather than the whole complex. It turns out that the cohomology over $\Omega^\bullet(*D)$ is isomorphic to the regular twisted cohomology over $\Omega^\bullet$, while the cohomology over $\Omega^\bullet(\log D)$ is generally different. However we also have

$$\Omega^\bullet(*D) = \sum_{k\geq 0} P^{-k}\Omega^\bullet(\log D),$$

so using the rescaling $\alpha \mapsto \alpha - k\cdot \mathrm{d}\log P$ and using Proposition 4.2.1 we see that the cohomology of $\Omega^\bullet(*D)$ can be computed using the cohomology over $\Omega^\bullet(\log D)$.

To now show that the cohomology is zero for $k \neq n$ we use the degree of each logarithmic form. Since a logarithmic form is rational it has a well-defined degree. It turns out[15] that each logarithmic k–form (for $k \leq n-1$) of a certain degree can be written as the sum of an exact form plus a form of strictly lower degree. Inductively reducing the forms then shows that all logarithmic k–forms are exact, hence the cohomology is 0.

Finally let us consider the third condition, the finitely many zeroes of α are non-degenerate and M has a compact cut-off whose boundary has zero twisted cohomology. In this case, the theorem is true due to Morse Theory which we will discuss in Subsection 4.4. The Morse function allows us to construct a CW-structure on M relative to the cut-off boundary. If the cut-off boundary has zero twisted cohomology, this implies the twisted cohomology of M is precisely the twisted cohomology of the CW-complex. But general Morse theory on complex manifolds [Mil16] tells us that the CW-complex only has cells of dimension $\dim_{\mathbb{C}} M$, so only the middle dimension twisted cohomology can be non-zero. □

We see that the Theorem holds for several different and seeming unrelated conditions. This is not very satisfactory, but we do now know of a general proof. At least conditions 1 and 2 guarantee that the Theorem will hold for any physical system where all poles have been properly regularised, as we will see in Section 5.

4.3 The Twisted Homology and Duality Theorems

In this subsection we will consider two duality theorems: the de Rham duality which relates the twisted cohomology H^k_α to the twisted homology $H^{\otimes U}_k$, and the Poincaré duality which relates the twisted cohomology H^k_α to the inversely twisted cohomology $H^{2n-k}_{-\alpha}$. We first discuss the de Rham duality. To do this, we first need to define the twisted homology. Throughout this section we will use U to denote the multi-valued function $\tilde{M} \to \mathbb{C}$ such that $\alpha = \mathrm{d}\log U$.

There are multiple different but isomorphic definitions of the homology, such as the singular homology of a topological space, simplicial homology of a topological space or the cellular homology of a CW complex. [Hat05] Since every manifold has the homotopy type of a CW complex [Mil16] we can use either definition. We therefore choose to use the cellular homology since it is the easiest to compute. Remark that [Aom+11] instead uses the simplicial homology.

Nevertheless, our definition is based on the definition of a twisted homology by Aomoto and Kita [Aom+11]. The main idea in either definition is that simplexes and cells are contractible to a point, so one can choose a branch of U on the simplex/cell. This turns U into a single-valued function on the simplex/cell. The main difference between our cellular definition and the simplicial definition by Aomoto and Kita is then that they consider all possible branch cuts for each simplex, while we fix a single possible branch cut for each cell. The latter approach uses complex multiplication, so we work over $\mathbb{C}$ while Aomoto and Kita work over $\mathbb{Z}$.

This section assumes familiarity with the (untwisted) cellular homology, see for instance [Hat05]. We define the k-th chain complex as the abstract complex span of the k-cells rather than as the relative homology of the k and $k-1$-skeletons for simplicity. This is equivalent. [Hat05:Lemma 2.34]

[15] This fact requires a long and complicated proof, which among other things uses the Euler vector field, Lie derivatives, and the graded complex of a fibration.

The definition of the twisted cellular cohomology we use is also shortly discussed in Section 22.2 of [FF16].

4.3.1 Definition

As mentioned above, the main idea of the twisted CW-complex is to choose a branch for each cell.

Let M be a manifold endowed with a CW-structure and let $\{\sigma_k^1 : D^k \to e_k^1, \dots, \sigma_k^\ell : D^k \to e_k^\ell\}$ denote the k-cells of M. Then the multi-valued function $U : \widetilde{M} \to \mathbb{C}$ is not generally single-valued on a k-cell e_k^i. However, the domain of the cell is a k-disc hence simply connected, so we can choose a branch of U on the cell. We call this choice $U_k^i : D^k \to \mathbb{C}$, and we denote the pair by

$$e_k^i \otimes U_k^i := \big(\sigma_k^i : D^k \to e_k^i,\ U_k^i : D^k \to \mathbb{C}\big).$$

Definition 4.3.1 (Twisted CW-structure)**.** Given a manifold M and a multi-valued map $U : \widetilde{M} \to \mathbb{C}$. A twisted CW-structure is a CW-structure together with a choice of a branch U_k^i for every cell e_k^i. •

To define the twisted cellular homology from the twisted CW-structure, we do two things. We start with the usual cellular chain complex. First of all we change the chain complex to not only be the integer multiples of cells, but any complex multiple;

$$\langle e_k^1, \dots, e_k^\ell\rangle_{\mathbb{Z}} \mapsto \langle e_k^1, \dots, e_k^\ell\rangle_{\mathbb{C}}.$$

Secondly we replace the cells e_k^i by pairs $e_k^i \otimes U_k^i$. This gives us the following chain groups.

$$C_k^{\otimes U}(M) := \langle e_k^1 \otimes U_k^1, \dots, e_k^\ell \otimes U_k^\ell\rangle_{\mathbb{C}}.$$

We then have to change the boundary map. Recall the untwisted boundary map,

$$\partial_k : C_k(M) \to C_{k-1}(M), \qquad e_k^i \mapsto \sum_j d_k^{ij}\ e_{k-1}^j,$$

where the degree d_k^{ij} is the number of how often the cell e_{k-1}^j appears in the boundary of the cell e_k^i weighted with orientation.[16] The twisted boundary map is mostly similar, but with one remark. The cells are replaced by $e_k^i \otimes U_k^i$ and $e_{k-1}^j \otimes U_{k-1}^j$, but generally U_k^i and U_{k-1}^j will not belong to the same branch. Let us make this statement more precise.

Let $e_{k-1}^j \subseteq \partial e_k^i$ and consider the pre-image $(\sigma_k^i)^{-1}(e_{k-1}^j) \subseteq \partial D^k$. The pre-image consists of finitely many connected components, each isomorphic to D^{k-1}. Let D refer to one such connected component and let $\phi : D \to D^{k-1}$ be the isomorphism $\phi = (\sigma_{k-1}^j)^{-1} \circ \sigma_k^i|_D$. Then we have U_{k-1}^j on D^{k-1} and U_k^i on D, so we can compare the two functions via the isomorphism ϕ. Both functions are branches of U on D^{k-1}, hence they differ by some constant factor u in the monodromy. In that case our boundary map has a contribution $u \deg\phi$ rather than a contribution $\deg\phi$, where $\deg\phi = 1$ if ϕ preserves the orientation and $\deg\phi = -1$ if ϕ changes the orientation.

We can repeat the above for each of the finitely many connected components of $(\sigma_k^i)^{-1}(e_{k-1}^j)$. We use this to define the twisted boundary map.

Definition 4.3.2 (Twisted boundary map and twisted homology)**.** Given a twisted CW-structure. Define

$$d_k^{ij} = \sum_{D \subseteq (\sigma_k^i)^{-1} e_{k-1}^j} \deg(\phi)\ \frac{U_k^i\big|_D}{U_{k-1}^j \circ \phi} \in \mathbb{C},$$

where the sum is over connected components of $(\sigma_k^i)^{-1} e_{k-1}^j$ and for each D we have the isomorphism $\phi : D \to D^{k-1}$. Then the twisted boundary map is the map

$$\partial_k^{\otimes U} : C_k^{\otimes U}(M) \to C_{k-1}^{\otimes U}(M), \qquad e_k^i \otimes U_k^i \mapsto \sum_j d_k^{ij} \cdot e_{k-1}^j \otimes U_{k-1}^j.$$

[16] This d_k^{ij} is often defined abstractly via a map $\chi_k^{ij} : S^{k-1} \to S^{k-1}$, but that does not generalise well for our use so we have to use a concrete counting definition. This is no problem mathematically since each cell is compact hence the sum only consist of finitely many addends.

The twisted cellular homology is the homology of the twisted chain complexes with the twisted boundary map,

$$H_k^{\otimes U}(M) := \frac{\ker \partial_k^{\otimes U} : C_k^{\otimes U}(M) \to C_{k-1}^{\otimes U}(M)}{\operatorname{im} \partial_{k+1}^{\otimes U} : C_{k+1}^{\otimes U}(M) \to C_k^{\otimes U}(M)}.$$

•

Remark 4.3.3. The above definition is well-defined. The boundary map satisfies $\partial_k^{\otimes U} \circ \partial_{k+1}^{\otimes U} = 0$ by the usual proof [Hat05]; the complex numbers don't affect the proof since $u^{ij} \cdot u^{jl} = u^{ij'} \cdot u^{j'l}$ for any $(k+1)$-cell e_{k+1}^i, k-cells e_k^j and $e_k^{j'}$ and $(k-1)$-cell e_{k-1}^l. Moreover, the choice of a branch for each cell does not influence the outcome since the boundary map commutes with complex multiplication by a scalar, and change the choice of a branch corresponds to multiplying with an element of the monodromy. •

To fix a branch U_k^i over a cell it is sufficient to fix U_k^i at a chosen point p_0 of the cell e_k^i with a chosen pre-image $x_0 \in D^k$, $p_0 = \sigma_k^i(x_0)$. The value of U_k^i at some other point y can then be calculated by choosing a path $\gamma : x_0 \rightsquigarrow y$, mapping the path to e_k^i using σ_k^i and following the path to find the value of U at the endpoint.

4.3.2 Examples

While the above definition strongly resembles the definition of the untwisted cellular cohomology, we look at 2 examples to further clarify the details. We look at the circle S^1 and at the wedge of circles $S^1 \vee S^1$.

Example 4.3.4 (S^1)**.** Consider the manifold S^1 with the (generally) multi-valued function $z \mapsto z^\lambda$ for some $\lambda \in \mathbb{R}$. Note that for this U we have $\alpha = \lambda \,\mathrm{d}\log(z)$.

We endow S^1 with the simplest CW-structure: one 0-cell p and one 1-cell a, hence both endpoints of the 1-cell are at p. We assume without loss of generality that p corresponds to the point $1 \in S^1$. To make this CW-structure into a twisted CW-complex we need to choose a branch of z^λ for each of the cells.

- For $p = 1$, the possible values of z^λ are $\exp(k \cdot \lambda\, 2\pi i)$ for any integer k. For the sake of simplicity we choose the branch with $k = 0$, so the value 1.
- The branch of a can be defined in a few ways, for instance by fixing the value of U at $-1 \in a$. The possible values are $\exp((2k+1) \cdot \lambda\, \pi i)$ for integer k. We choose $k = 0$ to get $\exp(\lambda\, \pi i)$. Note that in this branch the clockwise value of p is 1, while the counter-clockwise value is $\exp(\lambda\, 2\pi i)$.

We then calculate the only boundary map, $\partial_1^{\otimes U} a$. Note that p once appears in the boundary map with same orientation and once with opposite orientation, so in the untwisted case we would have $d_1^{ij} = 1 - 1 = 0$. We chose 1 as the value of p, but the boundary of a has value 1 on one side and $\exp(\lambda\, 2\pi i)$ on the other. For the former we find $u^{ap} = 1$, for the latter $u^{ap} = \exp(\lambda\, 2\pi i)$. This gives us (up to an overall minus depending on the choice of the orientations)

$$\partial_1^{\otimes U}\big(a \otimes U_a\big) = (1 - \exp(\lambda\, 2\pi i)) \cdot \big(p \otimes U_p\big) = \big(1 - e^{\lambda\, 2\pi i}\big)\big(p \otimes U_p\big).$$

There are now two significantly different options, depending on whether λ is integer or not.

λ is integer In that case, we find that $1 - e^{\lambda\, 2\pi i} = 0$. Therefore $\partial_1^{\otimes U}(a \otimes U_a) = 0$ and we find that a is in the kernel of the boundary map. Therefore the 1-st homology is non-zero:

$$H_1^{\otimes U}(M) = \langle a \otimes U_a \rangle_{\mathbb{C}} \cong \mathbb{C}.$$

Furthermore the image of $\partial_1^{\otimes U}$ is 0, so we similarly find

$$H_0^{\otimes U}(M) = \langle p \otimes U_p \rangle_{\mathbb{C}} \cong \mathbb{C}.$$

This is identical to the untwisted case.

λ is non-integer In that case, $1 - e^{\lambda\, 2\pi i} \neq 0$. That implies that a is not in the kernel of the boundary map, hence

$$H_1^{\otimes U}(M) = 0.$$

Moreover, the image of the boundary map is spanned by $(1 - e^{\lambda\, 2\pi i})p \otimes U_p$. Since the pre-factor is non-zero, this has the same span as $p \otimes U_p$. We therefore find

$$H_0^{\otimes U}(M) = \frac{\langle p \otimes U_p \rangle_{\mathbb{C}}}{\langle (1 - e^{\lambda\, 2\pi i})\; p \otimes U_p \rangle_{\mathbb{C}}} = 0.$$

We see that in this case all homology groups are 0. △

Example 4.3.5 ($S^1 \vee S^1$)**.** The wedge of two circles $S^1 \vee S^1$ is two circles attached at one point. This is not a manifold, but it is a deformation retract of the twice punctured plane $\mathbb{C}^{**}$ so we will use it to calculate the twisted homology of $\mathbb{C}^{**}$.

We recall $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ and choose the multi-valued function $z \mapsto z^{\lambda_1}(z-1)^{\lambda_2}$. Note that this gives $\alpha = \lambda_1\, \mathrm{d}\log(z) + \lambda_2\, \mathrm{d}\log(z-1)$, as in Example 4.2.4. We embed $S^1 \vee S^1$ into $\mathbb{C}^{**}$ as

$$S^1 \vee S^1 = \left\{ z \in \mathbb{C}^{**} \mid |z - 0| = \frac{1}{2} \;\text{ or }\; |z - 1| = \frac{1}{2} \right\}.$$

We make it into a CW-complex with one 0-cell p at $z = 1/2$ and two 1-cells a and b, both starting and ending at p. We choose that a passes through $z = -1/2$ while b passes through $z = 3/2$. We choose the following branches.

- For p we can choose any value

$$\left|\frac{1}{2}\right|^{\lambda_1+\lambda_2} \cdot \exp(k \cdot \lambda_1\, 2\pi i + (2\ell+1) \cdot \lambda_2 \pi i).$$

  We choose $k = \ell = 0$, giving us $|1/2|^{\lambda_1+\lambda_2} \cdot \exp(\lambda_2\, \pi i)$. This fixes U_p.

- The branch of a is fixed by choosing the value at $z = -1/2$. We can choose any

$$\left|\frac{1}{2}\right|^{\lambda_1} \cdot \left|\frac{3}{2}\right|^{\lambda_2} \cdot \exp((2k+1) \cdot \lambda_1 \pi i + (2\ell+1) \cdot \lambda_2 \pi i).$$

  We choose $k = \ell = 0$, giving us $|1/2|^{\lambda_1}|3/2|^{\lambda_2} \cdot \exp((\lambda_1 + \lambda_2)\, \pi i)$.
  The clockwise boundary value of p for this branch is $|1/2|^{\lambda_1+\lambda_2} \cdot \exp(\lambda_2\, \pi i)$, while the counter-clockwise value is $|1/2|^{\lambda_1+\lambda_2} \cdot \exp((2\lambda_1 + \lambda_2)\, \pi i)$.

- The branch of b is fixed by choosing the value at $z = 3/2$. We can choose any

$$\left|\frac{3}{2}\right|^{\lambda_1} \cdot \left|\frac{1}{2}\right|^{\lambda_2} \cdot \exp(k \cdot \lambda_1\, 2\pi i + \ell \cdot \lambda_2\, 2\pi i).$$

  We choose $k = \ell = 0$ to get $|3/2|^{\lambda_1}|1/2|^{\lambda_2} \cdot 1$.
  The clockwise value of p for this branch is $|1/2|^{\lambda_1+\lambda_2} \cdot \exp(-\lambda_2\, \pi i)$, while the counter-clockwise value is $|1/2|^{\lambda_1+\lambda_2} \cdot \exp(\lambda_2\, \pi i)$.

We can directly write down the boundary map using the values:

$$\partial_1^{\otimes U}\big(a \otimes U_a\big) = \big(1 - \exp(\lambda_1\, 2\pi i)\big)\; \big(p \otimes U_p\big),$$
$$\partial_1^{\otimes U}\big(b \otimes U_b\big) = \big(\exp(-\lambda_2\, 2\pi i) - 1\big)\; \big(p \otimes U_p\big).$$

We see that the coefficients are $1 - e^{\lambda_1\, 2\pi i}$ and $e^{-\lambda_1\, 2\pi i} - 1$. This gives us in total 4 cases, as λ_1 can be an integer or not and λ_2 can be an integer or not. However, there are only 2 true cases.

Both λ_1, λ_2 are integer In this case, we find that both a and b are in the kernel of the boundary map, and that the image of the boundary map is 0. In that case we find the usual homology

$$H_1^{\otimes U} \cong \mathbb{C}^2, \qquad H_0^{\otimes U} \cong \mathbb{C}.$$

At least one is non-integer In this case, we find that either a or b is not in the kernel of the boundary map, hence the image of the boundary map is non-zero. Since there is only one 0-cell, this image has dimension 1. Since the space spanned by a and b has dimension 2, we find that the first cohomology has dimension $2-1=1$ and the first cohomology has dimension $1-1=0$:

$$H_1^{\otimes U} \cong \mathbb{C}, \qquad H_0^{\otimes U} = 0. \qquad \triangle$$

Remark that the twisted homology found in Example 4.3.4 corresponds to the twisted cohomology found in Example 4.1.6 while the same holds for Example 4.3.5 and Example 4.2.4. This is not a coincidence, and is a consequence of the de Rham duality. We will now discuss this duality.

4.3.3 De Rham duality

We want to show that the twisted cellular homology defined above is dual to the twisted de Rham cohomology in the same way that the (untwisted) cellular homology is dual the the (untwisted) de Rham cohomology, i.e. using de Rham's theorem. Recall de Rham's theorem: [Lee12]

$$H^k(M;\mathbb{R}) \cong \mathrm{Hom}(H_k(M),\mathbb{R}), \qquad [\omega] \mapsto \left(\Delta \mapsto \int_\Delta \omega\right),$$

where $H^k(M;\mathbb{R})$ is the k-th de Rham cohomology with real coefficients, $H_k(M)$ is the k-th cellular homology with coefficients in $\mathbb{Z}$, and Δ is a cell of the CW-structure seen as a subset of M. Using the linearity of homomorphisms, this isomorphism implies an isomorphism

$$H^k(M;\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(H_k(M;\mathbb{C}),\mathbb{C}),$$

where all coefficients are now complex. Our goal is to adapt this to an isomorphism

$$H^k_\alpha(M) \overset{?}{\cong} \mathrm{Hom}_{\mathbb{C}}(H_k^{\otimes U}(M),\mathbb{C}),$$

of the twisted cohomology and twisted homology.

To obtain an isomorphism, we need to give a map from $H^k_\alpha(M)$ to $\mathrm{Hom}_{\mathbb{C}}(H_k^{\otimes U}(M),\mathbb{C})$ and then show it is surjective and injective.

We first need to define our equivalent of the integral on the chain groups. For a twisted cell $e^i_k \otimes U^i_k$ and a twisted k-form ψ we define

$$\int_{e^i_k \otimes U^i_k} U\ \psi := \int_{D^k} U^i_n\ (\sigma^i_k)^*\psi.$$

Note that both functions on the right-hand-side are single-valued, so this is a well-defined integral. We linearly extend this definition to any linear combinations of twisted cells,

$$\int_{\sigma\otimes U_\sigma} U\ \psi = \sum_i c_i \int_{e^i_k\otimes U^i_k} U\ \psi, \qquad \text{for} \quad \sigma\otimes U_\sigma = \sum_i c_i\ e^i_k \otimes U^i_k.$$

We next have to check that this definition satisfies Stokes' theorem with respect to the boundary $\partial^{\otimes U}$ and the differential ∇_α, so it descends to the (co)homology. Remark that $\alpha = \mathrm{d}\log U$ and U^i_k is the pull-back of a branch of U, such that

$$U^i_k\ (\sigma^i_k)^*\nabla_\alpha\big(\dots\big) = \mathrm{d}\big(U^i_k\ \dots\big),$$

where $(\sigma^i_k)^*\nabla_\alpha = \mathrm{d} + (\sigma^i_k)^*\alpha$ is the pull-back of ∇_α from M to D^k. We then have:

$$\int_{e^i_k\otimes U^i_k} U\ \nabla_\alpha\psi = \int_{D^k} U^i_k\ (\sigma^i_k)^*\big(\nabla_\alpha\psi\big)$$

$$
\begin{aligned}
&= \int_{D^k} U_k^i \, (\sigma_k^i)^* \nabla_\alpha \big((\sigma_k^i)^* \psi \big) \\
&= \int_{D^k} \mathrm{d}\big(U_k^i \, (\sigma_k^i)^* \psi \big) \\
&= \int_{\partial D^k} U_k^i \, (\sigma_k^i)^* \psi \\
&= \sum_j \sum_{D \subseteq (\sigma_k^i)^{-1} e_{k-1}^j} \int_D U_k^i \, (\sigma_k^i)^* \psi \\
&= \sum_j \sum_{D \subseteq (\sigma_k^i)^{-1} e_{k-1}^j} \frac{U_k^i|_D}{U_{k-1}^j \circ \phi} \int_D \big(U_{k-1}^j \circ \phi \big) \, (\sigma_k^i)^* \psi \\
&= \sum_j \sum_{D \subseteq (\sigma_k^i)^{-1} e_{k-1}^j} \deg(\phi) \, \frac{U_k^i|_D}{U_{k-1}^j \circ \phi} \int_{D^{k-1}} U_{k-1}^j \, (\sigma_{k-1}^j)^* \psi \\
&= \sum_j \sum_{D \subseteq (\sigma_k^i)^{-1} e_{k-1}^j} \deg(\phi) \, \frac{U_k^i|_D}{U_{k-1}^j \circ \phi} \int_{e_{k-1}^j \otimes U_{k-1}^j} U \, \psi \\
&= \sum_j d_k^{ij} \int_{e_{k-1}^j \otimes U_{k-1}^j} U \, \psi \\
\int_{e_k^i \otimes U_k^i} U \, \nabla_\alpha \psi &= \int_{\partial_k^{\otimes U} (e_k^i \otimes U_k^i)} U \, \psi .
\end{aligned}
$$

We see that an equivalent of Stokes' theorem holds for the twisted (co)homology as well, which tells us that indeed the defined integral map descends to a map

$$H_\alpha^k(M) \to \mathrm{Hom}_{\mathbb{C}}(H_k^{\otimes U}(M), \mathbb{C}).$$

Finally, we need to show that the map is injective and surjective. This is very hard to do directly, so we use another space as an intermediate step. We will define $H_{\otimes U}^k(M)$, show that $H_{\otimes U}^k \cong \mathrm{Hom}_{\mathbb{C}}(H_k^{\otimes U}(M), \mathbb{C})$ and finally show that $H_\alpha^k(M) \cong H_{\otimes U}^k(M)$. The space we need to define is the *twisted cellular cohomology*.

Definition 4.3.6 (Twisted Cellular Cohomology)**.** Given a twisted CW-structure, let $(C_\bullet^{\otimes U}, \partial_\bullet^{\otimes U})$ be the chain complex of the twisted cellular homology. Then we can take the dual complex by applying the covariant exact functor $\mathrm{Hom}_{\mathbb{C}}(\bullet, \mathbb{C})$. This gives a cochain complex,

$$(C_{\otimes U}^\bullet, \ (\partial_{\otimes U}^*)^\bullet).$$

The cohomology with respect to this cochain complex is the twisted cellular cohomology,

$$H_{\otimes U}^k := \frac{\ker(\partial_{\otimes U}^*)^k : C_{\otimes U}^k \to C_{\otimes U}^{k+1}}{\mathrm{im}(\partial_{\otimes U}^*)^{k-1} : C_{\otimes U}^{k-1} \to C_{\otimes U}^k}.$$

•

The above definition uses the dual cochain complex. This is a standard construction, see [Hat05] for more details.

We now first need to show that $H_{\otimes U}^k \cong \mathrm{Hom}_{\mathbb{C}}(H_k^{\otimes U}(M), \mathbb{C})$. This follows directly from Lemma 2 of [Aom+11] upon applying $C_\bullet = C_\bullet^{\otimes U}$. We show this Lemma below.

Lemma 4.3.7. *Let $(C_\bullet, \partial_\bullet)$ a chain complex of complex vector spaces. Then there exists a natural isomorphism,*

$$H^k(\mathrm{Hom}_{\mathbb{C}}(C_\bullet, \mathbb{C})) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(H_k(C_\bullet), \mathbb{C}).$$

Proof. By the definition of a chain complex, we have for each C_k an exact sequence:

$$0 \to \ker \partial_k \hookrightarrow C_k \overset{\partial_k}{\twoheadrightarrow} \operatorname{im} \partial_k \to 0.$$

Here the first arrow is given by inclusion. Remark that $\ker \partial_k \subseteq C_k$ and $\operatorname{im} \partial_k \subseteq \ker \partial_{k-1} \subseteq C_{k-1}$. We define $Z_k = \ker \partial_k$ and $B_k = \operatorname{im} \partial_k$. Since this definition works for any k and since $\partial_k(Z_k) = 0 \subseteq Z_{k-1}$ and $\partial_{k-1} B_k = 0 \subseteq B_{k-1}$, we can define new chain complexes $(Z_\bullet, \partial_\bullet)$ and $(B_\bullet, \partial_{\bullet+1})$. Remark, however, that these chain complexes have $\partial_\bullet = 0$.

We now have a short exact sequence of chain complexes,

$$0 \to Z_\bullet \to C_\bullet \xrightarrow{\partial_\bullet} B_\bullet \to 0.$$

Applying the contravariant exact functor $\operatorname{Hom}_{\mathbb{C}}(\bullet, \mathbb{C})$ to each chain complex, we get another short exact sequence:

$$0 \to B^\bullet \xrightarrow{(\partial^*)^\bullet} C^\bullet \to Z^\bullet \to 0.$$

This short exact sequence of cochain complexes implies a long exact sequence of their cohomologies. However, we know that $\partial_\bullet = 0$ on $Z_\bullet$ and on $B_{\bullet+1}$. Therefore, we similarly have that ∂^* is zero on $Z^\bullet$ and $B^{\bullet+1}$. We therefore simply have

$$H^k(Z^\bullet) = Z^k = \operatorname{Hom}_{\mathbb{C}}(Z_k, \mathbb{C}),$$
$$H^k(B^\bullet) = B^k = \operatorname{Hom}_{\mathbb{C}}(B_k, \mathbb{C}).$$

Moreover, the connecting homomorphism

$$\delta_k^* : \quad Z^k = H^k(Z^\bullet) \;\to\; H^{k+1}(B^\bullet) = B^{k+1}$$

in the long exact sequence precisely coincides with the dual of the inclusion map of $B_{k+1} \subseteq Z_k$. Since the inclusion map is injective and B_k, Z_k are both vector spaces over $\mathbb{C}$, the dual is surjective. This implies the long exact sequence is cut into multiple shorter exact sequences,

$$0 \to H^k(C^\bullet) \to H^k(Z^\bullet) \to H^{k+1}(B^\bullet) \to 0,$$

for each integer $k \geq 0$. Using the cohomologies of $Z^\bullet$ and $B^\bullet$, this simply gives

$$0 \to H^k(C^\bullet) \to Z^k \to B^{k+1} \to 0.$$

Finally, note that the homology $H^k(C_\bullet)$ is defined as quotient of the kernel of ∂_k by the image of ∂_{k+1}, hence it satisfies the following short exact sequence:

$$0 \to B_{k+1} \to Z_k \to H_k(C_\bullet) \to 0.$$

Taking the dual of this short exact sequence by applying $\operatorname{Hom}_{\mathbb{C}}(\bullet, \mathbb{C})$ gives

$$0 \to \operatorname{Hom}_{\mathbb{C}}(H_k(C_\bullet), \mathbb{C}) \to Z^k \to B^{k-1} \to 0.$$

Comparing this short exact sequence to the sequence we obtained above, one finds that there exists a natural isomorphism

$$H^k(C^\bullet) \cong \operatorname{Hom}_{\mathbb{C}}(H_k(C_\bullet), \mathbb{C}). \qquad \square$$

We still have to show that $H^k_{\otimes U}(M) \cong H^k_\alpha(M)$, and write the isomorphism explicitly. For this part, we will assume M is compact or use a compact cut-off of M if it is not. We will later shortly discuss what happens when M is not compact.

Proposition 4.3.8 (De Rham Duality)**.** *Let M be a connected complex compact manifold (possibly with boundary) and let α be a closed 1-form on M. Let U be a complex map on the universal cover $\widetilde{M}$ of M such that $\alpha = \mathrm{d} \log U$. Consider the twisted (de Rham) cohomology H^k_α and twisted cellular cohomology $H^k_{\otimes U}$. Then the map*

$$H^k_\alpha \ni [\phi] \mapsto \left[\int_\bullet U\ \phi : \quad \sigma \mapsto \int_\sigma U\ \phi \right] \in H^k_{\otimes U}$$

is an isomorphism for each k.

Proof sketch. We do not give all the details, see Section 2.2 of [Aom+11] for the precise details. Remark that the usual proof with the Mayer-Vietoris sequence will not work since we do not generally have an equivalent of the Mayer-Vietoris sequence for twisted (co)homology. We therefore need a significantly different proof.

This proof unfortunately requires the twisted singular cohomology rather than the twisted cellular cohomology, although they are equivalent. [Hat05] The difference is that cellular (co)homology considers a few fixed cells (which makes computations much easier) while singular (co)homology simultaneously considers all possible 'cells' one could have chosen (which makes proofs much easier).

The idea behind the proof is that rather than only considering the singular cochain complex with complex coefficients, $C^\bullet = \mathrm{Hom}_{\mathbb{C}}(C_\bullet, \mathbb{C})$, one can also consider cochain complexes such as $C^\bullet(\Omega^k_M) = \mathrm{Hom}_{\mathbb{C}}(C_\bullet, \Omega^k_M)$, where Ω^k_M is the collection of all smooth k–forms up to local equivalence classes in the sense that two elements of Ω^k are equivalent at a point p if their expansions at the point p are the same.[17] These Ω^k_M have the property that there is an exact sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \dots$$

This sequence is to be understood as that each closed k–form is everywhere locally equivalent to an exact k–form for $k > 0$, and each closed function is everywhere locally constant. This sequence respects $\mathcal{L}_\alpha$ and we have the commutative diagram

$$\begin{array}{ccc} \Omega^k_M & \xrightarrow{\nabla_\alpha} & \Omega^{k+1}_M \\ \downarrow\cong & & \downarrow\cong \\ \Omega^k_M \otimes \mathcal{L}_\alpha & \xrightarrow{\mathrm{d}} & \Omega^{k+1}_M \otimes \mathcal{L}_\alpha, \end{array}$$

hence we find a new exact sequence

$$0 \longrightarrow \mathcal{L}_\alpha \longrightarrow \Omega^0_M \xrightarrow{\nabla_\alpha} \Omega^1_M \xrightarrow{\nabla_\alpha} \Omega^2_M \xrightarrow{\nabla_\alpha} \dots$$

Since this sequence is exact, it splits into many short exact sequences of the form

$$0 \longrightarrow \nabla_\alpha \Omega^k_M \xrightarrow{\subseteq} \Omega^{k+1}_M \xrightarrow{\nabla_\alpha} \nabla_\alpha \Omega^{k+1}_M \longrightarrow 0.$$

This gives corresponding short exact sequences on the cochain complexes of the form $\mathrm{Hom}_{\mathbb{C}}(C_\bullet, \nabla_\alpha \Omega^k_M)$ and of the form $\mathrm{Hom}_{\mathbb{C}}(C_\bullet, \Omega^k_M)$, and hence long exact sequences on their cohomologies.

The big trick is now that the cohomologies of the chain with coefficients in $\mathcal{L}_\alpha$ are the usual twisted singular cohomologies (as choosing map $e_\ell \otimes U_\ell \mapsto a \in \mathbb{C}$ for each ℓ is equivalent to choosing a map $e_\ell \mapsto U_\ell\ a \in \mathcal{L}_\alpha$ for each ℓ), while the 0-th cohomologies with coefficients in Ω^k_M correspond to closed or exact k-forms, hence a quotient of these cohomologies gives the twisted (de Rham) cohomology. Using that certain cohomologies are 0 and working with multiple long exact sequence, one can finally proof

$$H^k_{\otimes U} \cong H^k(\mathrm{Hom}_{\mathbb{C}}(C_\bullet, \mathcal{L}_\alpha)) \cong H^k_\alpha. \qquad \square$$

This is a very useful result, as it shows the duality between H^k_α and $H^{\otimes U}_k$. In particular, this Proposition implies that the bilinear product

$$H^k_\alpha \times H^{\otimes U}_k \to \mathbb{C}, \qquad [\phi],\ [\sigma] \to \int_\sigma U\ \phi$$

is non-degenerate, hence can be used as an inner product. This is however not yet the inner product we want to use, as it requires the explicit computation of $H^{\otimes U}_k$. We will instead use this to show yet another duality which we may use to induce an inner product on H^k_α, but we first consider a Corollary.

Corollary 4.3.9. *Let M and α be as in the Proposition, then*

$$\sum_i (-1)^i \dim H^i_\alpha = \chi(M).$$

[17] This is a commonly used equivalence relation in algebraic geometry, and the set of all such equivalence classes is called the *germ*.

Proof. This follows from the fact that $H^k_\alpha \cong H^k_{\otimes U} \cong \mathrm{Hom}_{\mathbb{C}}(H^{\otimes U}_k, \mathbb{C})$, since this shows that $\dim H^k_\alpha = \dim H^{\otimes U}_k$. We then note that

$$\begin{aligned}
\sum_i (-1)^i \dim H^{\otimes U}_k &= \sum_i (-1)^i \left[\dim\ker \partial^{\otimes U}_k : C^{\otimes U}_k(M) \to C^{\otimes U}_{k-1}(M) - \dim \mathrm{im}\, \partial^{\otimes U}_{k+1} : C^{\otimes U}_{k+1}(M) \to C^{\otimes U}_k(M)\right] \\
&= \sum_i (-1)^i \left[\dim\ker \partial^{\otimes U}_k : C^{\otimes U}_k(M) \to C^{\otimes U}_{k-1}(M) + \dim \mathrm{im}\, \partial^{\otimes U}_k : C^{\otimes U}_k(M) \to C^{\otimes U}_{k-1}(M)\right] \\
&= \sum_i (-1)^i \dim C^{\otimes U}_k \\
&= \sum_i (-1)^i \dim C_k.
\end{aligned}$$

This is valid since M is compact hence each C_k is of finite dimension. This last expression is precisely the Euler characteristic $\chi(M)$, which can be shown by using the duality of (untwisted) cohomologies. □

This gives us another and more general way to show Corollary 4.2.7. Remark that this only works for M indeed compact, since $\dim C_k$ may be infinite for a general (complex connected) manifold. We now consider another duality.

4.3.4 Poincaré Duality

The above duality was written as $H^k_\alpha \cong \mathrm{Hom}_{\mathbb{C}}(H^{\otimes U}_k, \mathbb{C})$, but since we are working with finite dimensional (complex) vector spaces we know that the dual of a dual is isomorphic to the original space. We therefore similarly have

$$\mathrm{Hom}_{\mathbb{C}}(H^k_\alpha) \cong H^{\otimes U}_k,$$

where the isomorphism sends $[\sigma] \in H^{\otimes U}_k$ to $\int_\sigma U \bullet : H^k_\alpha \to \mathbb{C}$.

Given a class $[\psi] \in H^{2n-k}_{-\alpha}$, we know that for any $[\phi] \in H^k_\alpha$ we have $[\psi \wedge \phi] \in H^{2n}$. In particular, $\psi \wedge \phi$ is a top form hence can be integrated over M. This integral gives a complex number since M is compact. Therefore, we find that for a fixed $[\psi] \in H^{2n-k}_{-\alpha}$,

$$H^k_\alpha \to \mathbb{C}, \qquad [\phi] \mapsto \int_M \psi \wedge \phi$$

gives a homomorphism on H^k_α. Then by the above, this must correspond to some element of $H^{\otimes U}_k$. This fact is the core of the Poincaré duality. It turns out that this correspondence is an isomorphism.

Fact 4.3.10 (Poincaré Duality)**.** *Let M be a compact manifold, let U be a complex multi-valued function on M, and let $\alpha = \mathrm{d}\log U$ be well-defined. The map*

$$H^{2n-k}_{-\alpha} \to H^{\otimes U}_k$$

is an isomorphism.

This fact is much more difficult to prove than the de Rham duality, as the trick with germs does not work. A proof assuming that M is an "n-dimensional weak homology manifold over $\mathcal{L}_\alpha$" can found in [Bre12:Ch. V. 9]. The conditions that M is an "n-dimensional weak homology manifold over $\mathcal{L}_\alpha$" is satisfied whenever the Middle Dimension Theorem 4.2.10 applies, which makes it sufficiently general for us to assume it as a fact.

Remark that the Poincaré duality tells us the bilinear product

$$H^n_\alpha \times H^n_{-\alpha} \to \mathbb{C}, \qquad [\phi],\ [\psi] \mapsto \int_M \phi \wedge \psi$$

is non-degenerate. This product is known as the *intersection product* in physics, and is what we will use to define an inner product on H^n_α. Before we can move to apply this result to physics, we first have a short look at Morse theory and at the generalisation of the twisted cohomology to vector bundles rather than the line bundle $\mathcal{L}_\alpha$.

Remark 4.3.11. In the above, we have used the cut-off to make sure we always integrate over a compact space. However, our manifold M is not generally compact. In that case, the Poincaré duality does not relate generic forms to generic forms, but demands that one of the two spaces contains compactly supported forms instead. A form ϕ is compactly supported if there is some compact $K \subseteq M$ such that $\phi = 0$ on $M \setminus K$.

The space of compactly supported k-forms is usually denoted by a subscript c, hence the cohomology is also denoted

$$H^k_{c,\alpha}.$$

The de Rham duality relates compactly supported k-forms to locally finite elements of $H_k^{\otimes U}$. The elements of $H_k^{\otimes U}$ are generally infinite linear combinations of cells in the CW complex, since a non-compact manifold can have an infinite CW complex. Such an infinite linear combination of cells is locally finite if for each compact subset $K \subseteq M$, the number of cells with intersection with K is finite. •

4.4 Morse Theory

By the previous sections it has become clear that the Euler characteristic is highly relevant in the discussion of the twisted cohomology. Fortunately, the twisted cohomology itself often gives us a Morse function which we can use to determine the Euler Characteristic.

Consider the following multi-valued function. We choose a base-point $z_0 \in \widetilde{M}$ and define

$$G : \widetilde{M} \to \mathbb{C}, \qquad z \mapsto \int_{z_0 \rightsquigarrow z} \tilde{\alpha},$$

where $z_0 \rightsquigarrow z$ is any smooth path from z_0 to z and we recall $\tilde{\alpha}$ is a lift of α. Following similar arguments as in the definition of the representation (Equation 22), we see that G is well-defined and smooth. We want to investigate the real part $\Re(G)$ of G, and show that it becomes a Morse function on M for certain α.

We first have to demand that $\Re(G)$ descends to M.

Lemma 4.4.1. *Let M be a connected complex manifold and let α be a closed one-form on M. Let the map $G : \widetilde{M} \to \mathbb{C}$ be defined as above. Then the following are equivalent:*

1. *$\Re(G) : \widetilde{M} \to \mathbb{R}$ descends to a well-defined function $\Re(G) : M \to \mathbb{R}$.*
2. *For any $x_1, x_2 \in \widetilde{M}$ projecting to the same $x \in M$, we have*
$$\exp(G(x_1))/\exp(G(x_2)) \in S^1.$$
3. *The monodromy representation $\pi_1(M) \to \mathbb{C}^*$ induced by α takes values in S^1.*
4. *The absolute value $|U| : \widetilde{M} \to \mathbb{R}$ of the map*
$$U : \widetilde{M} \to \mathbb{C}, \qquad U(z) = \exp(G(z)) = \exp \int_{z_0 \rightsquigarrow z} \tilde{\alpha}$$
descends to a map $|U| : M \to \mathbb{R}$.

Proof. We first prove 1. and 2. are equivalent. Per definition, $\Re(G)$ descends to M if and only if for any two $x_1, x_2 \in \widetilde{M}$ with the same projection $x \in M$ we have $G(x_1) - G(x_2) \in i\mathbb{R}$. Taking the exponent of both sides gives us $\exp(G(x_1))/\exp(G(x_2)) \in \exp(i\mathbb{R}) = S^1$. Note that exp and log are both well-defined single valued on $\widetilde{M}$ since the latter is simply connected, so the function exp is invertible and we conclude that 1. holds if and only if 2. is true.

We secondly prove 3. is also equivalent with the previous two. Assume 3. is true and let $x_1, x_2 \in \widetilde{M}$ project to the same $x \in M$. Then

$$\exp(G(x_1))/\exp(G(x_2)) = \exp(G(x_1) - G(x_2)) = \exp \int_{x_1 \rightsquigarrow x_2} \tilde{\alpha}.$$

Since $\tilde{\alpha}$ is defined as the lift of α, the latter integral is the same as the integral over α of the projected path. However x_1 and x_2 both project to x, so the resulting path is a loop. We find that the integral must be equal

to the integral of α over a loop, hence the right-hand-side is precisely the representation of that loop. We conclude that

$$\exp(G(x_1))/\exp(G(x_2)) = \exp\int_{x_1\rightsquigarrow x_2}\tilde{\alpha} \in S^1.$$

Conversely let γ be a loop on M. Then we can choose a base-point and lift the loop to a path in $\widetilde{M}$, starting at a chosen x_1 and ending at some x_2. Then we have

$$\exp\int_\gamma \alpha = \exp\int_{x_1\rightsquigarrow x_2}\tilde{\alpha} = \exp(G(x_1))/\exp(G(x_2)) \in S^1.$$

The loop γ was arbitrary, so we see that the representation $\pi_1(M)\to\mathbb{C}^*$ takes values in S^1.

Finally we prove 4. is equivalent to 2. The map $|U|$ descends to M if and only if for any two $x_1, x_2 \in \widetilde{M}$ projecting to $x\in M$ we have $U(x_1)/U(x_2)\in S^1$. That is precisely the statement of 2. □

Generally the above conditions are not satisfied. However in physics we are interested in α which have the form $\alpha = \lambda\,\mathrm{d}\log(P)$ for some $\lambda\in\mathbb{R}$ and some polynomial P, or a finite sum of such expressions. We will therefore consider that form for the rest of this subsection.

In that case we find

$$G = \lambda\log(P), \qquad U = \exp(G) = P^\lambda, \qquad |U| = |P|^\lambda,$$

from which it is immediately clear that condition 4. is satisfied. We see that for $\alpha = \lambda\,\mathrm{d}\log(P)$, the function

$$\Re(G) = \Re(\lambda\log(P)) : M\to\mathbb{R} \tag{28}$$

is well-defined.

We then have to determine whether $\Re(G): M\to\mathbb{R}$ is Morse. Recall that a Morse function is a smooth real-valued function on a compact manifold which has no degenerate critical points, i.e. every critical point of the function has a non-singular Hessian matrix. [Mil16] In particular, the compactness of M together with the non-degeneracy of each critical point implies there are only finitely many critical points.

Because of the Cauchy-Riemann equations, the critical points of $\Re(G)$ on M precisely lift to the critical points of G on $\widetilde{M}$, hence are given by $dG = 0$. By the construction of G, that is simply equal to $\alpha = 0$. This gives us an easy classification of the critical points of $\Re(G)$. However, we have to check the critical points are non-degenerate and that the manifold is compact.

The latter is problematic, as the manifold is very clearly non-compact in even simple cases (such as the punctured plane, see Example 4.1.6). We will instead have to use a compact subset of M which has the same topology. We could use the existence of a cut-off to find some $M_{c.o.}$ which is compact with boundary, but we don't want to deal with a generic boundary. Instead, we will define a specific cut-off which respects the function $\Re(G)$.

Let ℓ be a real number and consider the equation $\Re(G) = \ell$, or equivalently $|U| = e^\ell$. Notate the set of $z\in M$ satisfying this equation by V_ℓ. Choose ℓ such that for every critical point p, we have $\Re(G(p)) > \ell$; this is possible as there are only finitely many critical values. Then by the implicit function theorem we find that V_ℓ is an $(n-1)$-dimensional submanifold of M. Remark that generally $\ell \ll 0$, so the cut-off $|U| = e^\ell$ corresponds to an UV cut-off in physics.

Choose similarly Λ such that $\Lambda > \Re(G(p))$ for each critical point p, and define V_Λ as solutions to the equation $\Re(G) = \Lambda$. Then V_Λ is also an $(n-1)$-dimensional submanifold of M. Generally $\Lambda \gg 0$, so the cut-off $|U| = e^\Lambda$ corresponds to a IR cut-off in physics.

Definition 4.4.2 (Cut-off)**.** Let M be a connected complex manifold and let α a closed one-form on M. Assume M and α satisfy the condition of Lemma 4.4.1. The *cut-off* of M with is the manifold triad $(M_{c.o.}, V_\ell, V_\Lambda)$ where

$$M_{c.o.} := \Re(G)^{-1}([\ell,\Lambda]).$$

The spaces V_ℓ and V_Λ are referred to as the *cut-off boundaries*. •

The cut-off of M is a manifold triad, which means it is a manifold with boundary. The choice of ℓ and Λ ensures that all critical points of $\Re(G)$ still lie inside $M_{c.o.}$. This means we have not lost any of the topology, and indeed the (co)homology of $M_{c.o.}$ must agree with the (co)homology of M.

We can always do this construction for a given function $\Re(G)$. However, this does not guarantee that $M_{c.o.}$ is compact. For a manifold $M \subseteq \mathbb{C}^N$ for some N, we know that M is compact if and only if M is bounded and closed as a subset of $\mathbb{C}^N$, i.e. there are no paths to 'infinity' and there are no holes such as punctures. In order for $M_{c.o.}$ to be bounded, the polynomial should contain no factors such as $x - y^2$, whose zero set is clearly not bounded. If such a factor does occur we can for instance move from $\mathbb{C}^N$ to $\mathbb{CP}^N$ and extend the polynomial by making it homogeneous, which gives us a compact $M_{c.o.} \subseteq \mathbb{CP}^N$. In that case, however, we have to be more careful as the extension from $\mathbb{C}^N$ to $\mathbb{CP}^N$ may change the Euler characteristic.

In order to guarantee that $M_{c.o.}$ has no holes we have to similarly demand that the UV cut-off excludes any closed holes in M. This is not generally true, as it depends on the choice of M. In particular, if $p \in M$ then $M \setminus \{p\}$ is again a manifold but the new manifold clearly has a problematic hole at p. We will therefore always choose M (and occasionally tweak U) such that there are no remaining unbounded holes, see Section 5.

After making sure M is compact, we still have to make sure that each critical point is non-degenerate and in particular whether there are only finitely many points. If the polynomial P is known then it is highly trivial to determine this; solve $\alpha = 0$ and for each critical point calculate all second partial derivatives and determine the eigenvalues of the matrix of said second partial derivatives. We can not make any general statements about this, but the computation is trivial for computers hence can be determined for any α. In particular, for $\alpha = \lambda \mathrm{d}\log(P)$ we find that

$$\alpha = \lambda \sum_{i=1}^{N} \frac{1}{P} \frac{\partial P}{\partial z_i} \, \mathrm{d}z_i,$$

such that $\alpha = 0$ at some $z = p$ only when each derivative is zero at p. The equation

$$\frac{\partial P}{\partial z_i} = 0$$

will generally be satisfied for a codimension 1 subspace of M and there are $\dim_{\mathbb{C}} M$ such equations, hence we typically expect that the solution to $\alpha = 0$ is a dimension 0 subspace. This suggests the condition that the critical points are non-degenerate is not impossible, at the very least.

Assuming the compactness and non-degeneracy are satisfied, we apply Morse theory to the function $\Re(G)$ on $M_{c.o.}$. According to Morse theory, we can write M as V_0 with various cells attached. [Mil16] We can describe the homology of M assuming we known the homology of V_0 and the glue maps of the various cells as follows:

- Consider a k-cell. The glue map sends the boundary of the k-cell to the union of V_0 with other cells. If the image of the boundary is contractible in this union, then we can contract it to a single point in V_λ, hence we would wedge the k-cell to the earlier union. This means we simply add one dimension to the k-th homology.

- If the image of the boundary is not contractible, then $[\partial e_k]$ must be a non-zero class in the $(k-1)$-th homology of the union. Most generally that means $[\partial e_k]$ is the sum of some class in $H_{k-1}(V_\ell)$ with a linear combination of $(k-1)$-cells which have been previously attached via gluing. This means we can easily calculate the change in cohomology when we glue the k-cell; either $\dim H_{k-1}$ changes by -1 or $\dim H_k$ changes by 1 depending on $[\partial e_k] \stackrel{?}{=} [0]$ before gluing.

- We can use the above two options to inductively compute the homology of M, and in particular the Euler characteristic of M is equal to the characteristic of V_λ plus the alternating sum of the number of cells.

Because M is a complex manifold, it is a standard result that the index of any critical point of $\Re(G)$ is equal to the (complex) dimension of M. [Mil16] This means we can always up to homotopy assume the boundary ∂e_n^i lies entirely within V_λ, and in particular the boundary is a linear combination of classes in $H_{k-1}(V_\ell)$.

We conclude that in our case, $H_k(M) = H_k(V_\ell)$ for $k \neq n-1, n$; while $H_n(M)$ and $H_{n-1}(M)$ depend on $H_{n-1}(V_\ell)$ and on the boundary maps of the n-cells.

In particular, we see that $\chi(M) = \chi(V_\ell) + (-1)^n K_n$ where K_n is the number of critical points of the map $\Re(\log U)$. A final question is then what $\chi(V_\ell)$ is. That will strongly depend on M and U, but we state some expectations. For M a submanifold of $\mathbb{C}$ we know that V_ℓ will be a finite union of circles S^1. The Euler characteristic of a circle is $\chi(S^1) = 0$ and the Euler characteristic of a finite union is the sum of the Euler characteristics of the parts, so in that case we find that

$$\chi(M) = (-1)^n K_n.$$

More generally, if M is of the form $\mathbb{C}^n \setminus S$ with S a discrete set of finitely many points, then we expect that the boundary will be a union of spheres S^{2n-1}, which still have Euler characteristic $\chi(S^{2n-1}) = 0$. On the other hand, if M is $\mathbb{C}^n \setminus \ell$ with a single real line ℓ removed, then the boundary around the line will be a cylinder $S^{2n-2} \times \mathbb{R}$, which instead has Euler characteristic $\chi(S^{2n-2} \times \mathbb{R}) = 2$.

Assuming the critical points are non-degenerate and the 'holes' $M_{c.o.}$ are taken care of (both of which we will need to determine case-by-case), we have found that $\Re(G) : M \to \mathbb{R}$ is Morse. As such, the Morse inequalities immediately tell us that the Euler characteristic of M is equal to the Euler characteristic of V_ℓ plus or minus the number of critical points of $\Re(G)$, i.e. the dimension of the twisted cohomology is equal to the number of zeroes of α. We have proven the following.

Theorem 4.4.3. *Let M be a connected complex manifold and let α be a closed one-form on M such that Lemma 4.4.1 is satisfied. Let G be defined by*

$$G : \widetilde{M} \to \mathbb{C}, \qquad z \mapsto \int_{z_0 \rightsquigarrow z} \tilde{\alpha}$$

and let $\Re(G) : \widetilde{M} \to \mathbb{R}$ be the real part of G. Then

- *if the representation $\pi_1(M) \to \mathbb{C}^*$ only takes values in $S^1 \subseteq \mathbb{C}^*$, then $\Re(G)$ descends to $M \to \mathbb{C}$;*
- *if additionally the critical points of G are non-degenerate and $M_{c.o.}$ is compact, then the Euler characteristic is equal to the Euler characteristic of V_ℓ plus or minus the number of zeroes of α.*

The first condition is satisfied for $\alpha = \lambda \, \mathrm{d}\log(P)$ for $\lambda \in \mathbb{R}$ and P a polynomial on $M \subseteq \mathbb{C}^n$, the second condition is to be checked for each individual case.

This Theorem is particularly potent when V_ℓ is a union of spheres and the manifold M and function U are well-behaved, as in that case we find the following Corollary.

Corollary 4.4.4. *Let M, U and α be such that the Middle Dimension Theorem 4.2.10, De Rham Duality Theorem 4.3.8 and Morse Theorem 4.4.3 are satisfied. Assume moreover that V_ℓ is a union of spheres. Then:*

$$\dim H^n_\alpha = \#\{\alpha(z) = 0\}, \qquad H^k_\alpha = 0 \quad \text{for } k \neq n. \tag{29}$$

□

We above corollary tells us everything we need to calculate the twisted cohomology. However, we still need slightly more information on how to calculate the intersection product without having to integrate over $M_{c.o.}$. The calculation can be simplified slightly using vector bundle twisted cohomology and spectral sequences. We will discuss this next.

4.5 Twisted Cohomology of a Fibration

It is well known [Hat04; BT+82] that for a Serre fibration $M \xrightarrow{\pi} B$ with fibres $F_x = \pi^{-1}(x)$, the cohomology of M can be calculated from the cohomologies of B and $F_\bullet$ via the Serre-Leray spectral sequence

$$E_2^{p,q} = H^p(B, H^q(F_\bullet)) \Rightarrow H^{p+q}(M).$$

We would like to find a similar spectral sequence for the twisted cohomology. It turns out that such a spectral sequence indeed exists. [FF16] In order to describe this spectral sequence, we first have to generalise the notion of a local system from line bundles to vector bundles.

4.5.1 Generalisation to Vector Bundles

We defined the local system as the associated vector bundle of $\widetilde{M} \to M$ where the action of $\pi_1(M)$ on $\mathbb{C}$ was given by the representation $\pi_1(M) \to \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$, and we further endowed this associated vector bundle with parallel transports. This definition can be generalised as follows; we include the parallel transports into the definition immediately for clarity.

Definition 4.5.1. Let M be a smooth manifold with universal cover $\widetilde{M}$. A *local system* of dimension m is a pair

$$\big(E, \rho : \Pi_1(M) \to \mathrm{GL}(m, \mathbb{C})\big)$$

of a vector bundle and a representation, where E is an associated vector bundle

$$E = (\widetilde{M} \times \mathbb{C}^m)/\pi_1(M) \to M,$$

and where the action of $\pi_1(M)$ on $\mathbb{C}^m$ is given by the representation $\rho : \Pi_1(M) \to \mathrm{GL}(m, \mathbb{C})$. •

We see that E is trivial as a topological vector bundle, since we can use ρ to untwist it. The interesting structure is again the smooth structure: parallel transport and the connection.

The representation of the fundamental groupoid described the parallel transport; for any path $\gamma : x_0 \rightsquigarrow x_1$ the equivalence class $[\gamma] \in \Pi_1(M)$ is send to some $\rho([\gamma]) \in \mathrm{GL}(m, \mathbb{C})$ which describes that the parallel transports sends $v \in \mathbb{C}^m$ at x_0 to

$$\gamma \cdot (x_0, v) = \big(x_1, \rho([\gamma]) \cdot v\big).$$

In our earlier definition, the representation $\rho : \Pi_1(M) \to \mathbb{C}^*$ was given by

$$[\gamma] \mapsto \exp \int_\gamma \alpha.$$

This indeed describes both the parallel transport and the representation $\pi_1(M) \to \mathbb{C}^*$.

Remark 4.5.2. An equivalent definition is possible where one does not take the associated vector bundle but rather the associated principal bundle $(\widetilde{M} \times \mathrm{GL}(m, \mathbb{C}))/\pi_1(M)$, such that the result is a $\mathrm{GL}(m, \mathbb{C})$–bundle over M. This definition is used in [FF16:Section 22.2]. Taking the associated vector bundle of that definition gives our definition, while taking the fibre bundle of our definition returns their definition.

Furthermore remark that our definition is not the most general; instead of taking a representation ρ it is also possible to take a function which sends any homotopy class of paths $a \rightsquigarrow b$ to a homomorphism from the fibre over a to the fibre over b. •

We would like to again have some $A \in \Omega^1(M, \mathrm{Mat}_m(\mathbb{C}))$ (where $\mathrm{Mat}_m(\mathbb{C})$ denotes the space of complex $m \times m$–matrices) such that ρ can be written as the exponential of an integral over A. However, we run into the same problem as in Subsection 3.3: the exponent of an integral is not the same as a path integral, and for some multi-valued function $U : \widetilde{M} \to \mathrm{GL}_n(\mathbb{C})$ we generally have

$$U^{-1}\,\mathrm{d}U \neq \mathrm{d}U\,U^{-1} \neq \mathrm{d}\log U.$$

Given A, we could define

$$U = \prod_\gamma \exp(A),$$

such that

$$\mathrm{d}U = A \cdot U, \qquad A = \mathrm{d}U \cdot U^{-1}.$$

Unfortunately, this means that for any k-form ϕ we have

$$\mathrm{d}(U \cdot \phi) = A \cdot U \cdot \phi + U \cdot \mathrm{d}\phi \neq U \cdot (A \cdot \phi + \mathrm{d}\phi).$$

This is problematic, so we instead define U with the opposite ordering. Since U^{-1} has the opposite ordering compared to U, the easiest way to define U is as follows.

Definition 4.5.3. Let M be a smooth manifold and let $A \in \Omega^1(M, \mathrm{Mat}_m(\mathbb{C}))$. Then the (generally multi-valued) source of A is defined as $U : \widetilde{M} \to \mathrm{GL}_n(\mathbb{C})$ such that

$$U^{-1}(z) = \prod_{\gamma : z_0 \rightsquigarrow z} \exp(-A),$$

for some fixed point $z_0 \in M$. •

We then see that the source of A satisfies:

$$-U^{-1} \cdot \mathrm{d}U \cdot U^{-1} = \mathrm{d}U^{-1} = -A \cdot U^{-1}, \qquad \mathrm{d}U = U \cdot A.$$

In particular, we have

$$\mathrm{d}(U \cdot \phi) = U\,(A \cdot \phi + \mathrm{d}\phi).$$

The question is whether for any representation ρ we can find some A such that parallel transport using ρ is locally given by $\mathrm{d} + A$, hence such that ρ corresponds to U. Since M is manifold, we know that any homotopy class $[\gamma] \in \Pi_1(M)$ contains a smooth path $\gamma' \in [\gamma]$. Now for any $t \in (0, 1]$ the restriction of γ' to $[0, t]$ gives a new path up to rescaling by a factor t. Each of these paths has a corresponding value $\rho([\gamma'_{[0,t]}])$. Then A exists only if $t \mapsto \rho([\gamma'_{[0,t]}])$ is smooth in t. In particular, A exists only if the same holds for any class $[\gamma]$ and any smooth path $\gamma' \in [\gamma]$. This is of course not generally true.

We will therefore put this as a specific assumption: we assume that A exists such that

$$\rho([\gamma]) = \prod_{\gamma}^{T} \exp(A),$$

where we use the T to denote that we take the opposite ordering. We then notate the local system by $\mathcal{L}_A$ and we indeed have the connection $\nabla_A = \mathrm{d} + A$ on $\Omega^k(M, \mathbb{C}^m)$ satisfying

$$\begin{array}{ccc} \Omega^k(M, \mathbb{C}^m) & \xrightarrow{\nabla_A} & \Omega^{k+1}(M, \mathbb{C}^m) \\ \downarrow \cong & & \downarrow \cong \\ \Omega^k(M, \mathcal{L}_A) & \xrightarrow{\mathrm{d}} & \Omega^{k+1}(M, \mathcal{L}_A). \end{array}$$

We can then define the twisted cohomology over M with coefficients in $\mathcal{L}_A$. Let $\Omega^k_A = \Omega^k(M, \mathcal{L}_A)$ and $\Omega^k_m = \Omega^k(M, \mathbb{C}^m)$. Then we have.

Definition 4.5.4. Let M be smooth manifold and let $\mathcal{L}_A$ be a local system in the sense of Definition 4.5.1. The twisted cohomology over M with coefficients in $\mathcal{L}_A$ is

$$H^k_A := H^k(M, \mathcal{L}_A) := \frac{\ker \mathrm{d} : \Omega^k_A \to \Omega^{k+1}_A}{\mathrm{im}\, \mathrm{d} : \Omega^{k-1}_A \to \Omega^k_A} = \frac{\ker(\mathrm{d} + A) : \Omega^k_m \to \Omega^{k+1}_m}{\mathrm{im}(\mathrm{d} + A) : \Omega^{k-1}_m \to \Omega^k_m}. \tag{30}$$

•

If M has a CW structure, we can also again define the cellular homology $H^{\otimes U}_k$ as in Definition 4.3.2 and cellular cohomology $H^k_{\otimes U}$ as in Definition 4.3.6, except that we have to use m–tuples of cells as elements e^k_i of the chain complex and that the quotient of two matrices is not well-defined so that we instead have to use

$$d^{ij}_k = \sum_{D \subseteq (\sigma^i_k)^{-1} e^j_{k-1}} \deg(\phi)\ \left(U^j_{k-1} \circ \phi\right)^{-1} \cdot U^i_k\big|_D \in \mathrm{Mat}_m(\mathbb{C}).$$

We can again use Lemma 4.3.7 to see that

$$H^k_{\otimes U} \cong \mathrm{Hom}_{\mathbb{C}}(H^{\otimes U}_k, \mathbb{C}).$$

Remark 4.5.5. The proofs of the de Rham Duality Theorem 4.3.8 [Aom+11] and of Poincaré Duality 4.3.10 [Bre12] both use complicated sheaf theory. However, this has the advantage that the proofs generalise to

cohomologies with coefficients in any sheaf. In particular, the former proof can directly be adapted to higher dimensions by instead using the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C}^m \longrightarrow \Omega^0_M \otimes \mathbb{C}^m \xrightarrow{\mathrm{d}} \Omega^1_M \otimes \mathbb{C}^m \xrightarrow{\mathrm{d}} \Omega^2_M \otimes \mathbb{C}^m \xrightarrow{\mathrm{d}} \dots$$

as a starting point and noting that $\Omega^1_M \otimes \mathbb{C}^m = \Omega^1(M, \mathbb{C}^m)$, while the latter is completely agnostic of the dimension of the local system. This means that both dualities apply to the generalised twisted cohomology as well. •

Corollary 4.5.6. *Let M be a connected complex manifold and let A be a matrix of closed one-forms corresponding to a local system $\mathcal{L}_A$. If $\dim_{\mathbb{C}} M = 1$ and the source U is multi-valued, then $H^0_A = 0$ and $H^2_A = 0$.*

Proof. Similarly to the $m = 1$ case, we know that H^0_A is the vector space all $\mathrm{d} + A$–closed m–tuples of functions. In particular, that implies that H^0_A consists of functions of the form $U \cdot c$, where c is a constant m–tuple. But that can never be a single-valued function for $c \neq 0$, hence we find that

$$H^0_A = 0.$$

Now Poincaré duality implies that $H^2_A = 0$ as well. □

This gives us an effective Middle Dimension Theorem for $\dim_{\mathbb{C}} M = 1$.

With the definitions and properties defined, we can finally look at the Serre-Leray spectral sequence.

4.5.2 Serre-Leray Spectral Sequence

We will discuss the idea here from a mathematical point of view, see the next section for the physical details.

Let $M \xrightarrow{\pi} B$ be a Serre fibration with fibres $F_\bullet$. Let moreover $U : \widetilde{M} \to \mathbb{C}^*$ a multi-valued function on M such that $\alpha = \mathrm{d} \log U \in \Omega^1$ is well-defined. Then we have a local system $\mathcal{L}_\alpha \to M$.

We first have to find the twisted equivalent of the Serre-Leray spectral sequence before we can prove it. Let us shortly consider the original. We recall: [Hat04]

Theorem 4.5.7 (Serre-Leray Spectral Sequence)**.** *Let $F_\bullet \to M \to B$ be a Serre fibration with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F_\bullet)$. Then there is a spectral sequence $\{E^{p,q}_r, d_r\}$ such that:*

- *$d_r : E^{p,q}_r \to E^{p+r,q-r+1}_{r+1}$ and*
$$E^{p,q}_{r+1} = \frac{\ker d_r : E^{p,q}_r \to E^{p+r,q-r+1}_{r+1}}{\operatorname{im} d_r : E^{p-r,q+r-1}_{r+1} \to E^{p,q}_r}.$$
- *The stable terms $E^{p,k-p}_\infty$ are isomorphic to successive terms F^k_p / F^k_{p+1} of a filtration*
$$0 \subseteq F^k_k \subseteq F^k_{k-1} \subseteq \dots \subseteq F^k_0 = H^k(M).$$
- *$E^{p,q}_2 \cong H^p(B, H^q(F_\bullet))$.*

This stylised as

$$E^{p,q}_2 \cong H^p(B, H^q(F_\bullet)) \Rightarrow H^{p+q}(M).$$

The main obstacle in generalising this to twisted cohomologies is the action of $\pi_1(B)$ on $H^*(F)$. This would generalise to the action of $\pi_1(B)$ on the twisted cohomology of F. This action has to be trivial. However, we know that the action of $\pi_1(M)$ on $\mathbb{C}$ is not trivial, as it is given by the representation

$$[\gamma] \mapsto \exp \int_\gamma \alpha.$$

This means we cannot expect the action of $\pi_1(B)$ to be trivial on the twisted cohomology of F if we see the latter as $\mathbb{C}^m$ for some m. Instead, we have to treat the twisted cohomology similarly to $\mathcal{L}_\alpha$, since we do know that the action of $\pi_1(M)$ is trivial on $\mathcal{L}_\alpha$. In other words, we have to treat the twisted cohomology of F into

a (generalised) local system using the action of $\pi_1(B)$. With this in mind, we can replace each term with a twisted version to create the twisted Serre-Leray spectral sequence.

The right-hand-side is simple, we replace

$$H^{p+q}(M) \mapsto H^{p+q}(M, \mathcal{L}_\alpha) = H^{p+q}_\alpha.$$

The left-hand-side is non-trivial. We first need to find a local system on $F_\bullet$ for the twisted cohomology on F, and then we have to interpret the twisted cohomology over $F_\bullet$ as local coefficients of the cohomology over B.

Construction 4.5.8. The local system $\mathcal{L}_\alpha$ restricts to a local system on each fibre $F_\bullet$. In particular, let F one such fibre. Then $U|_F$ is a (multi-valued or single-valued) function on F and $\mathcal{L}_\alpha|_F \to F \subset M$ is a local system. This local system is then given by $\mathcal{L}_{\alpha'}$, where $\alpha' \in \Omega^1(F)$ is the restriction of $\alpha \in \Omega^1(M)$ to F, hence the pull-back of α under the inclusion map $i : F \subset M$. This gives us a unique local system $\mathcal{L}_{\alpha'} \to F$ of which we can determine the cohomology in the usual sense. We find the twisted cohomology $H^k(F, \mathcal{L}_{\alpha'})$. We can repeat this for any fibre F.

Now consider the base B. Since $H^k(F_\bullet)$ can be replaced with $H^k(F_\bullet, i^*_\bullet \mathcal{L}_\alpha)$ we have to find out what a cohomology over B with coefficients in $H^k(F_\bullet, i^*_\bullet \mathcal{L}_\alpha)$ means. It turns out that

$$\{H^k(F_\bullet, i^*_\bullet \mathcal{L}_\alpha)\}_\bullet \to B$$

is indeed a generalised local system over each connected component of B.[18] For any fibre F the space $H^k(F, i^*\mathcal{L}_\alpha)$ is a finite dimensional vector space over $\mathbb{C}$, and the action $\rho : \Pi_1(M) \to \mathrm{GL}(m, \mathbb{C})$ for $m = \dim H^k(F, i^*\mathcal{L}_\alpha)$ is given as follows.

Let $\gamma : x_0 \rightsquigarrow x_1$ for $x_0, x_1 \in B$. Then the fibres F_{x_1} and F_{x_2} are weakly homotopy equivalent, with the weak homotopy equivalence given by conjugation with γ. [FF16:pg. 114] Since the fibres are the pre-image of a singleton under a projection, the fibres are submanifolds of M, hence they have the homotopy type of a CW-complex [Mil16] so by Whitehead's Theorem [Hat05] the fibres are in fact homotopy equivalent. This tells us that γ induces a linear isomorphism

$$H^k(F_{x_0}, i^*_{x_0}\mathcal{L}_\alpha) \to H^k(F_{x_1}, i^*_{x_1}\mathcal{L}_\alpha).$$

The precise isomorphism is induced by the parallel transport of a k–form from F_{x_0} to F_{x_1} along the path γ with respect to the connection $\mathrm{d} + \alpha$.

However, both sides are isomorphic to $\mathbb{C}^m$. This means that the linear isomorphism corresponds to a linear isomorphism $\mathbb{C}^m \to \mathbb{C}^m$, hence it corresponds to an element of $\mathrm{GL}(m, \mathbb{C})$. The map ρ sends $[\gamma]$ to the corresponding element of $\mathrm{GL}(m, \mathbb{C})$. •

With these definitions, we can now write down the Serre-Leray spectral sequence for twisted cohomologies.

Theorem 4.5.9. *Let $F_\bullet \to M \to B$ be a Serre fibration with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F_\bullet)$, let α be a closed one-form on M defining a local system. Then there exists a spectral sequence with second page*

$$E_2^{p,q} = H^p\big(B, H^q(F_\bullet, i^*_\bullet \mathcal{L}_\alpha)\big)$$

which converges to $H^{p+q}(M, \mathcal{L}_\alpha)$. [FF16; Hat04]

Proof sketch. We adapt the proof of the usual Serre Spectral sequence to the twisted case, see [Hat04:Ch. 5.1] for details. The proof uses several claims which are generalisations of results from usual (co)homologies but which we haven't shown in this Thesis, such as the long exact sequence of a CW-complex and a subcomplex, and excision of relative twisted (co)homologies. The proof works with cellular (co)homologies, so we implicitly use the de Rham duality to proof the result for de Rham cohomology as well.

Choose a CW-complex structure on B and pull-back the skeleta B_p for a filtration $M_p = \pi^{-1}(B_p)$ on M, then for any successive pair (M_p, M_{p+1}) there is a long exact sequence with coefficients in $\mathcal{L}_\alpha$. This long

[18] This means that the cohomology forms a local system over each connected component of B, but the local systems for different connected components may have different dimension. The cohomology is a generalised local system over all of B whenever B is connected.

exact sequence can be used to make a staircase diagram. Upon defining exact couples and derived couples in the usual way, one finds a spectral sequence which converges to a filtration of $H^{p+q}_{\otimes U}(M)$, and the first page of this spectral sequence is given by the relative twisted cohomology

$$E_1^{p,q} = H^p_{\otimes U}(M_q, M_{q-1}).$$

One then finally has to show that the second page of of the spectral sequence is indeed

$$E_2^{p,q} \overset{?}{\cong} H^p\big(B, H^q(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big).$$

One finds an isomorphism of chain complexes

$$\begin{array}{ccccccc}
\dots & \xrightarrow{d_1} & H^{p+q}_{\otimes U}(M_p, M_{p-1}) & \xrightarrow{d_1} & H^{p+q+1}_{\otimes U}(M_{p+1}, M_p) & \xrightarrow{d_1} & \dots \\
 & & \Psi\uparrow\cong & & \Psi\uparrow\cong & & \\
\dots & \xrightarrow{\partial^*} & \mathrm{Hom}_{\mathbb{C}}(H_p(B_p, B_{p-1}; \mathbb{C}), H^q(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)) & \xrightarrow{\partial^*} & \mathrm{Hom}_{\mathbb{C}}(H_{p+1}(B_{p+1}, B_p; \mathbb{C}), H^q(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)) & \xrightarrow{\partial^*} & \dots
\end{array}$$

where the map Ψ is constructed and shown to be an isomorphism. This involves a lot of diagram chasing, using the CW-structure and excision to write relative (twisted) cohomologies as finite sums of easier (twisted) cohomologies, and using the homotopy lifting property. Using Lemma 4.3.7 on the bottom row gives the required twisted cohomologies

$$\mathrm{Hom}_{\mathbb{C}}(H_{p+1}(B_{p+1}, B_p; \mathbb{C}), H^q(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)) \cong H^p\big(B, H^q(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big).$$

We see that the second page is indeed given by $H^p\big(B, H^q(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big)$. □

Corollary 4.5.10. *Let $F_\bullet \to M \to B$ be a Serre fibration with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F_\bullet)$, let α be a closed one-form on M defining a local system. If $\dim_{\mathbb{C}} B = 1$, then*

$$H^{k+1}(M, \mathcal{L}_\alpha) \cong H^1\big(B, H^k(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big).$$

Proof. Since the dimension of B is 1, we can use the Middle Dimension Theorem to note that

$$H^0\big(B, H^k(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big) = 0, \qquad H^2\big(B, H^k(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big) = 0.$$

This means that the page $E_2^{p,q}$ is only non-zero for $p = 1$. But the map d_l sends

$$d_l : E_l^{p,q} \to E_l^{p+l, q-l+1},$$

so for $l \geq 2$ the map d_l always maps to or from 0. This means that the kernel at $(p, q) = (1, k)$ is everything while the image is 0, so we have

$$E_{l+1}^{1,k} = E_l^{1,k}$$

and all other elements of E_{l+1} are 0. This inductively implies that

$$E_\infty^{1,k} \cong E_2^{1,k} \cong H^1\big(B, H^k(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big)$$

are the only non-zero elements of E_∞. We therefore find that

$$H^{k+1}(M, \mathcal{L}_\alpha) = \bigoplus_{j=0}^{k} E_\infty^{j,k+1-j} = E_\infty^{1,k} \cong H^1\big(B, H^k(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)\big).$$ □

This is a useful Corollary if we can find a Serre fibration to a space of dimension 1. A similar results holds when $\dim_{\mathbb{C}} F = 1$ instead. The Corollary implies that knowing the Euler characteristic of B (or F) and knowing the twisted cohomology of the other is sufficient to know the twisted cohomology of M. Of course, if we just want to know the dimension of the twisted cohomology of M it is sufficient to use Corollary 4.4.4. But Corollary 4.5.10 can be used to find a basis, assuming we know a basis on the fibre.

We will see in the next section that the above Theorem can be very useful when doing the integration in the intersection product.

5 The Intersection Product

We can now define the intersection product. The intersection product is about applying the dualities of Subsection 4.3 to families of Feynman integrals written in a suitable representation. In Subsection 3.4 we have seen the Schwinger Parameter Representation (Equation 15), Feynman Parameter Representation (Equation 19) and Baikov Representation (Equation 21). These representations have in common that they all write the Feynman integral in the form

$$F = c \int_C U\ \phi,$$

where c is some constant, C is some real domain of integration, U is a function which is generally multi-valued due to dimensional regularisation, and ϕ is a rational function which depends on the Feynman diagram. In particular, we have:

$$\begin{aligned}
\text{Schwinger Parameter:}\quad & U = z_1^{-1}\dots z_N^{-1}\,\mathcal{U}^{-d/2}\,\exp(ic - i\mathcal{V}/\mathcal{U}), && \phi = z_1^{\alpha_1}\dots z_N^{\alpha_N}\,\mathrm{d}z_1\dots\mathrm{d}z_N,\\
\text{Feynman Parameter:}\quad & U = z_1^{-1}\dots z_N^{-1}\,(\mathcal{F}+\mathcal{U})^{-d/2}, && \phi = z_1^{\alpha_1}\dots z_N^{\alpha_N}\,\mathrm{d}z_1\dots\mathrm{d}z_N,\\
\text{Baikov:}\quad & U = P(x_\ell - f_\ell)^{(d-E-L-1)/2}, && \phi = \frac{\mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_N}{x_1^{\alpha_1}\,x_2^{\alpha_2}\dots x_N^{\alpha_N}}.
\end{aligned}$$

Of course, we still have to determine what the corresponding manifold is on which this U is defined. This is not clear for the Schwinger Parameter Representation, but for the other two Representations we can define the manifold as follows. We know that $\mathcal{F}+\mathcal{U}$ is an polynomial, and similarly for $P(x_\ell - f_\ell)$. When using the loop-by-loop Baikov Method we instead find that U is a product of powers of polynomials $P_1^{\lambda_1}\cdots P_k^{\lambda_k}$, in that case we consider the polynomial $P_1\cdots P_k$. For either Representation, we obtain a polynomial in N variables. We can therefore determine the zero set of this polynomial in $\mathbb{C}^N$, we call this zero set D. Then the manifold is defined as:

$$M = \mathbb{C}^N \setminus D.$$

Remark that this manifold M is an algebraic variety, since if we denote the polynomial by P then

$$M = \{(z_1,\dots,z_N)\in\mathbb{C}^N \mid P(z)\neq 0\} \cong \{(z_1,\dots,z_N,t)\in\mathbb{C}^{N+1} \mid P(z)\cdot t - 1 = 0,\}$$

where the isomorphism sends $(z_1,\dots z_N)$ to $(z_1,\dots,z_N,1/P(z))$. This manifold no longer contains any zeroes of the polynomial nor does it contain infinities (since a polynomial can not become infinite on $\mathbb{C}^N$), hence the function U is everywhere locally well-defined on the manifold and indeed U is a well-defined function on the universal cover $\widetilde{M}$.

This definition is however not completely satisfactory, since any ϕ in Baikov representation has a pole at $x_i = 0$, but generally $x_i = 0$ is not contained in D. This means that our ϕ are not well-defined on M. We solve this by slightly changing P to also include the factor $x_1\,x_2\,\dots\,x_N$, which corresponds to picking some $\delta_1,\delta_2,\dots,\delta_N > 0$ and changing U to

$$U = P(x_\ell - f_\ell)^{(d-E-L-1)/2}\,x_1^{\delta_1}\,x_1^{\delta_2}\,\dots\,x_N^{\delta_N}.$$

This process is essentially another regularisation on top of dimensional regularisation; [Miz20] refers to this process as regularising relative boundaries. A similar step is required in Feynman Parameter Representation when one allows $\alpha_i < 0$ as well.

After this step, we see that U is locally well-defined and each ϕ is globally well-defined. This means that we have now reached a point that we can apply mathematical theorems to the physical situation. First of all, we define $\alpha = \mathrm{d}\log U$ which (generally) gives us an exact 1-form. We now have M and α, hence there exists the twisted cohomology H^k_α. The above definition of M makes it plausible that condition 2 of the Middle Dimension Theorem 4.2.10 is satisfied, indeed we will generally assume this is true. This tells us that only the N-th cohomology H^N_α is relevant.

Now consider some physical N–form ϕ. While the function U is multi-valued on M, we can make is single-valued on the real integration domain C. We therefore find that

$$\int_C U\ \phi$$

is well-defined. Moreover, we can use Stoke's law and the fact that C has no boundary in M, $\partial C = \emptyset$, to find

$$0 = \int_{\partial C} U\ \psi = \int_C \mathrm{d}(U\ \psi) = \int_C U\ (\mathrm{d}+\alpha)\psi,$$

where ψ is any $N-1$–form on M. Therefore, the integral over ϕ only depends on the value of ϕ modulo terms of the form $(\mathrm{d}+\alpha)\psi$ for a $N-1$–form ψ. Moreover, ϕ and α only depend on $z_1,\dots,z_N$ and not on the complex conjugates $\bar{z}_1,\dots,\bar{z}_N$. Therefore, $\mathrm{d}\phi$ and $\alpha\wedge\phi$ are both 0 for any physical ϕ. We therefore find that ϕ is $\mathrm{d}+\alpha$–closed and that the integral over ϕ only depends on ϕ up to exact terms; this means that the integral only depends on the class

$$[\phi] \in H^N_\alpha.$$

In other words, if we can evaluate

$$\int_C U\ [\phi]$$

for any class

$$[\phi] \in H^N_\alpha,$$

we can evaluate any Feynman integral in the family.

The twisted cohomology H^N_α is finite whenever Theorem 4.2.5 or Theorem 4.3.8 is satisfied, since H^N_α is $\pm\chi(M)$. This means that one only needs to know how to evaluate the integral over finitely many classes in H^k_α in order to determine all Feynman integrals in the family. This is not too surprising, since that idea was also the idea behind calculating Feynman integrals using the IBP relations and master integrals.

What is new, however, is that we can use the duality theorems from Subsection 4.3 in order to directly write any Feynman integral in terms of the master integrals. The Poincaré duality gives a non-degenerate bilinear product between H^N_α and $H^N_{c,-\alpha}$, which we can use to define an inner product on H^k_α. This means we no longer have to inductively reduce any Feynman integral in terms of the master integrals, but we can project its integrand onto the integrands of the master integrals.

We will introduce this inner product on H^k_α in the following Subsection. The inner product is known as the intersection product. Unfortunately, the intersection product is not trivial to compute. We will therefore use rest of the Subsections to discuss the computation of the intersection product and consider a few examples.

5.1 The Intersection Product

As we have seen in Remark 4.2.3, the cohomology classes with coefficients in $\mathcal{L}_\alpha$ do not form a ring, but we instead have that the wedge product gives a map

$$H^n_\alpha \times H^m_\beta \to H^{n+m}_{\alpha+\beta}, \quad (\phi,\psi)\mapsto \phi\wedge\psi.$$

In particular, if the manifold M has complex dimension n then there is a product

$$H^n_\alpha \times H^n_{-\alpha} \to H^{2n}.$$

This product is not useful to us, since our manifold M is generally non-compact. We therefore limit $H^n_{-\alpha}$ to compactly-supported forms, which we denote by a subscript c. That then gives a product

$$H^n_\alpha \times H^n_{c,-\alpha} \to H^{2n}_c.$$

The latter space contains compactly-supported top-dimensional forms which can be integrated over M to give a complex number. It is the composition of the wedge product and integration

$$H^n_\alpha \times H^n_{c,-\alpha} \to \mathbb{C}, \quad (\phi,\psi)\mapsto \langle\phi,\psi\rangle := \int_M \phi\wedge\psi$$

which we refer to as the *intersection product*. This bilinear form is non-degenerate according to Fact 4.3.10.

Since it is non-degenerate it can be used as an inner product, and indeed we will use this inner product to expand an arbitrary element of H^n_α in terms of a given basis.

This bilinear product is between two different spaces which slightly complicates expansions. For any basis $\{h_1, h_2, \ldots, h_k\}$ of $H^n_{c,-\alpha}$ and for any basis $\{e_1, e_2, \ldots, e_k\}$ of H^n_α the expansion of an element ϕ in terms of the e_i is given by

$$\phi = \sum_{i=1}^{k} \phi_i \, e_i, \qquad \phi_i = \sum_{j=1}^{k} [\langle e_i, h_j \rangle]^{-1}_{ij} \, \langle \phi, h_j \rangle,$$

where $[\langle e_i, h_j \rangle]^{-1}_{ij}$ denotes the inverse of the square matrix $[\langle e_i, h_j \rangle]_{ij}$.

5.2 Explicit Evaluation

We now see how to actually calculate the intersection product for 1-forms and more general n-forms. Instead of integrating over M, which is generally difficult, we use a version of Stokes' Theorem and the Residue Theorem, and only calculate expansions and residues. We follow [Fre+19b] for the explicit definitions and [Aom+11] for the mathematical background.

Throughout this chapter we assume that the manifold M and the 1-form α satisfy the conditions of the Middle Dimension Theorem, see Subsection 4.2.1.

5.2.1 One-forms

We start with 1-forms, corresponding to M having complex dimension 1. Using Stoke's Theorem and the Residue theorem the above form is equivalent to

$$\langle \phi, \psi \rangle := \sum_{p \in P} \operatorname*{Res}_{z=p} [\phi \, \Psi], \tag{31}$$

where $\phi \in H^1_\alpha$, $\psi \in H^1_{c,-\alpha}$, P is the collection of poles of α and Ψ is a local primitive of ψ:

$$(\mathrm{d} - \alpha)\Psi = \psi.$$

Proposition 5.2.1 (Cho, Matsumoto 1995)**.** *For $\phi \in H^1_\alpha$ and $\psi \in H^1_{c,-\alpha}$ on a complex manifold M with $\dim_{\mathbb{C}} M = 1$ we have*

$$\int_M \phi \wedge \psi \sim c \sum_{p \in P} \operatorname*{Res}_{z=p} [\phi \, \Psi],$$

where c is a fixed constant, P denotes the poles of α together with α, and Ψ is a local primitive of ψ at each point p. [MM19]

Proof. We first use the Morse function (or a similar construction) to find a compact cut-off $M_{c.o.} \subseteq M$ which is a manifold with boundary and has the same (co)homology as M. Note that the intersection product over $M_{c.o.}$ is then equivalent to that over M, so we may use that instead.[19]

Since M is of complex dimension 1, the boundary of $M_{c.o.}$ must be of real dimension 1. Since $M_{c.o.}$ is also compact, we find that the boundary must be a finite union of circles S^1.

We consider the (generally multi-valued) primitive Ψ of ψ on each of the circles in the boundary. Remark that

$$\mathrm{d}(\phi \, \Psi) = (\mathrm{d}\phi) \, \Psi - \phi \wedge \mathrm{d}\Psi = -\alpha \wedge \phi \, \Psi - \phi \wedge \mathrm{d}\Psi = -\phi \wedge (\mathrm{d}\Psi - \alpha \, \Psi) = -\phi \wedge \psi.$$

We therefore find that

$$\int_{M_{c.o.}} \phi \wedge \psi = -\int_{M_{c.o.}} \mathrm{d}(\phi \, \Psi) = -\int_{\partial M_{c.o.}} \phi \, \Psi.$$

Since $\partial M_{c.o.}$ is a union of circles, the integral over each of these circles can be expressed as a residue over the interior (here the 'interior' refers to the subset of M bounded by the circle and in the complement of $M_{c.o.}$).

[19] Remark that we can use the intersection product over M or over $M_{c.o.}$ to get equivalent results, but we cannot mix intersection products from both. In practice, we will always use $M_{c.o.}$ as it is easier to integrate over compact sets.

The Morse function has isolated critical points hence the poles of α are isolated as well. This implies that $M_{c.o.}$ can be chosen such that the interior of each circle contains exactly one pole of α. Moreover, ϕ and ψ are smooth on M, so the only poles of $\phi\,\Psi$ can be poles of α. Therefore, the residues are precisely residues over the poles of α. We conclude that

$$\int_{M_{c.o.}} \phi \wedge \psi = -\int_{\partial M_{c.o.}} \phi\,\Psi = -2\pi i \sum_{p\in P} \operatorname*{Res}_{z=p}[\phi\,\Psi].$$

Remark that we have to add ∞ to P since one circle in $\partial M_{c.o.}$ will be the upper cut-off, which can be interpreted in $\mathbb{CP}^1$ as a circle around ∞. □

Note that the poles of α all lie outside the interior of M per definition of α, so to make full sense of the above expression we may need to have a suitable embedding of M into the compact manifold $\mathbb{CP}^1$. Since P can be explicitly determined and since the expansion Ψ around each $z = p$ can be calculated, we see that the above definition can be programmed in Python or Mathematica to largely automate the calculation. We consider a few examples calculated this way.

Remark 5.2.2. In practice we integrate over $M_{c.o.}$, which is compact. We therefore can use any differential form $\psi \in H^1_{-\alpha}$ and do not have to worry about making the differential form compactly supported. •

Example 5.2.3 (Euler Beta)**.** The Euler Beta integrals are defined by [AS64:Sec. 6.2]

$$\beta(a,b) := \int_{(0,1)} z^{a-1}\,(1-z)^{b-1}\,\mathrm{d}z.$$

We consider $a = n + \gamma + 1$ and $b = 1 + \gamma$ to get the integrals

$$I_n := \int_{(0,1)} B^\gamma\, z^n\,\mathrm{d}z,$$

where $B = z(1-z)$ is a polynomial, γ is a non-integer real number and n is an integer. We want to use the intersection product to relate the integrals to each other. For this, we first have to determine the manifold and the relevant twisted cohomology.

Since B is a polynomial, we can use B^γ to construct the one-form; we will choose the one-form

$$\alpha = \mathrm{d}\log(B^\gamma) = \gamma\,\mathrm{d}\log(z(1-z)) = \gamma\left(\frac{1}{z} + \frac{1}{z-1}\right)\mathrm{d}z.$$

From the log, it is clear the one-form α is singular in the limits to 0, 1 and ∞. We find the manifold $\mathbb{C}\setminus\{0,1\} = \mathbb{CP}^1 \setminus \{0,1,\infty\}$.

Our next step is to determine the twisted cohomology of $\mathbb{C}\setminus\{0,1\}$ with respect to α. In this case, we can use the result from Example 4.2.4 to immediately conclude that

$$H^0_\alpha = 0, \quad H^1_\alpha \cong \mathbb{C}, \quad H^2_\alpha = 0.$$

We therefore have to choose one master integral. We only have to be careful we choose a non-zero integral, otherwise we can choose anything. We choose the integral I_0 as the master integral. This gives us $z^0\,\mathrm{d}z = \mathrm{d}z$ as the relevant integrand, as we remove the integral and the multi-valued B^γ. Remark that I_0 is generally non-zero because $\mathrm{d}z$ is non-exact.

Since the cohomology has dimension 1 and integration is linear, there should be some c_n such that $I_n = c_n \cdot I_0$ for any $n \in \mathbb{N}$. We will determine c_1, the other coefficients are calculated similarly. To calculate c_1, we have to intersect $\mathrm{d}z$ and $z^1 \cdot \mathrm{d}z = z\,\mathrm{d}z$ with some (compact) one-form in $H^1_{c,-\alpha}$. The form $\mathrm{d}z$ is in $H^1_{-\alpha}$ and is simple, and is compactly supported on any cut-off submanifold (with boundary) of $\mathbb{C}\setminus\{0,1\}$.

We can then start to calculate the intersection $\langle \mathrm{d}z, \mathrm{d}z\rangle$ first. Using Equation 31 we know:

$$\langle \mathrm{d}z, \mathrm{d}z\rangle = \sum_{p=0,1,\infty} \operatorname*{Res}_{z=p}[1\,\mathrm{d}z \cdot \Psi]$$

where

$$\left[\mathrm{d} - \gamma \left(\frac{1}{z} + \frac{1}{z-1}\right) \mathrm{d}z\right] \Psi = \mathrm{d}z.$$

We have to solve for Ψ near the poles $0, 1$ and ∞, so we solve Ψ as a power series in z, $z-1$ and $w = 1/z$ respectively.

We start by calculating the residue near $p = 0$. The power series for Ψ is given by

$$\Psi = \sum_{k=-\infty}^{\infty} c_k \, z^k$$

for some $c_k \in \mathbb{C}$. Note that

$$\frac{1}{z-1} = -1 - z - z^2 - z^3 - \dots$$

for $|z| < 1$. Substituting the power series into the differential equation, we locally find:

$$\sum_{k=-\infty}^{\infty} \left[c_k \, k \, z^{k-1} - \gamma \, c_k \, z^{k-1} + \gamma \sum_{l \geq 0} c_{k-l} \, z^k \right] = 1 \cdot z^0.$$

Shifting the k of the rightmost term by one, we find

$$c_k \, k - \gamma \, c_k + \gamma \sum_{l \geq 0} c_{k-l-1} = \delta_{k1},$$

for any k. We can iteratively solve this as follows. The lowest contribution on the right-hand-side is at $k = 1$, so we can set $c_k = 0$ for any $k \leq 0$. We then start with $k = 1$, which gives us

$$c_1 - \gamma \, c_1 = 1, \qquad \text{hence} \qquad c_1 = \frac{1}{1-\gamma}.$$

We then calculate c_2, we similarly have

$$2 \, c_2 - \gamma \, c_2 + \gamma \, c_1 = 0, \qquad \text{hence} \qquad c_2 = \frac{-\gamma \, c_1}{2-\gamma} = \frac{-\gamma}{(1-\gamma)(2-\gamma)}.$$

The higher orders are calculated in the same way, as are the expansion around $z = 1$ and $z = \infty$. For the latter we used $w = 1/z$ with $\mathrm{d}z = -1/w^2 \, \mathrm{d}w$ and instead expand around $w = 0$ for simplicity.

We find respectively: (remark that Mastrolia used a different convention in his MathAmp2019 talk, which is equal to replacing γ with $-\gamma$)

$$\begin{aligned}
\Psi|_0 &= 0 \, z^{-1} + 0 \, z^0 + \frac{1}{1-\gamma} \, z^1 - \frac{\gamma}{(1-\gamma)(2-\gamma)} \, z^2 - \frac{2\gamma}{(2-\gamma)(3-\gamma)} \, z^3 + \dots \\
\Psi|_1 &= 0 \, (z-1)^{-1} + 0 \, (z-1)^0 + \frac{1}{1-\gamma} \, (z-1)^1 + \frac{\gamma}{(1-\gamma)(2-\gamma)} \, (z-1)^2 - \frac{2\gamma}{(2-\gamma)(3-\gamma)} \, (z-1)^3 + \dots \\
\Psi|_\infty &= \frac{1}{1-2\gamma} \, w^{-1} - \frac{1}{2(1-2\gamma)} \, w^0 + \frac{\gamma}{2(-1-2\gamma)(1-2\gamma)} \, w^1 + \frac{\gamma}{4(-1-2\gamma)(1-2\gamma)} \, w^2 \\
&\quad - \frac{2\gamma+\gamma^2}{4(-3-2\gamma)(-2-2\gamma)(-1-2\gamma)} \, w^3 + \dots
\end{aligned}$$

We can now directly determine the residues of $\Psi|_p \, \mathrm{d}z$ at $p = 0$, 1 and ∞. For the first two we clearly have

$$\operatorname*{Res}_{z=0}[\Psi|_0 \, \mathrm{d}z] = c_{-1} = 0, \qquad \operatorname*{Res}_{z=1}[\Psi|_1 \, \mathrm{d}z] = \operatorname*{Res}_{z=1}[\Psi|_1 \, \mathrm{d}(z-1)] = c_{-1} = 0.$$

For $z = \infty$, we have to be more careful since $\mathrm{d}z \neq \mathrm{d}w$. We instead have to transform the $\mathrm{d}z$ to $-1/w^2 \, \mathrm{d}w$, which gives:

$$\operatorname*{Res}_{z=\infty}[\Psi|_\infty \, \mathrm{d}z] = \operatorname*{Res}_{w=0}\left[\Psi|_\infty \cdot -\frac{1}{w^2} \, \mathrm{d}w\right] = -c_1 = \frac{-\gamma}{2(-1-2\gamma)(1-2\gamma)}.$$

We conclude that

$$\langle \mathrm{d}z, \mathrm{d}z\rangle = \frac{-\gamma}{2(-1-2\gamma)(1-2\gamma)}.$$

We similarly determine $\langle z\,\mathrm{d}z, \mathrm{d}z\rangle$. We again write Ψ as a power series near 0, 1 and ∞; we can actually use the same Ψ as in the above since we didn't change the $\mathrm{d}z$. We calculate

$$\langle z\,\mathrm{d}z, \mathrm{d}z\rangle = \sum_{p=0,1,\infty} \operatorname*{Res}_{z=p}[z\,\mathrm{d}z\cdot\Psi|_p]$$

for the same $\Psi|_0, \Psi|_1$ and $\Psi|_\infty$. In particular, the residues of $\Psi|_p \cdot z\,\mathrm{d}z$ are respectively:

$$\begin{aligned}
\operatorname*{Res}_{z=0}[\Psi|_0 \cdot z\,\mathrm{d}z] &= c_{-2} = 0,\\
\operatorname*{Res}_{z=1}[\Psi|_1 \cdot z\,\mathrm{d}z] &= \operatorname*{Res}_{z=1}[\Psi|_1 \cdot (z-1)\,\mathrm{d}(z-1) + \Psi|_0 \cdot 1\,\mathrm{d}(z-1)] = c_{-2} + c_{-1} = 0,\\
\operatorname*{Res}_{z=\infty}[\Psi|_\infty \cdot z\,\mathrm{d}z] &= \operatorname*{Res}_{w=0}\left[\Psi|_0 \cdot -\frac{1}{w^3}\,\mathrm{d}w\right] = -c_2 = \frac{-\gamma}{4(-1-2\gamma)(1-2\gamma)}.
\end{aligned}$$

We conclude that

$$\langle z\,\mathrm{d}z, \mathrm{d}z\rangle = \frac{-\gamma}{4(-1-2\gamma)(1-2\gamma)}.$$

By the linearity of multiplication and integration and the linearity of the intersection product, we now know that

$$\frac{I_1}{I_0} = \frac{\langle z\,\mathrm{d}z, \mathrm{d}z\rangle}{\langle \mathrm{d}z, \mathrm{d}z\rangle} = \frac{-\gamma}{4(-1-2\gamma)(1-2\gamma)} \Big/ \frac{-\gamma}{2(-1-2\gamma)(1-2\gamma)} = \frac{1}{2}.$$

We found $I_1 = \frac{1}{2} I_0$. We can similarly compute I_{-1} in terms of I_0. We first take the new residues

$$\begin{aligned}
\operatorname*{Res}_{z=0}\left[\Psi|_0 \cdot z^{-1}\,\mathrm{d}z\right] &= c_0 = 0,\\
\operatorname*{Res}_{z=1}\left[\Psi|_1 \cdot z^{-1}\,\mathrm{d}z\right] &= \operatorname*{Res}_{z=1}\left[\Psi|_1 \cdot \sum_{l\geq 0}(-1)^l\,(z-1)^l\,\mathrm{d}(z-1)\right] = \sum_{l\geq 0}(-1)^l\,c_{-1-l} = 0,\\
\operatorname*{Res}_{z=\infty}\left[\Psi|_\infty \cdot z^{-1}\,\mathrm{d}z\right] &= \operatorname*{Res}_{w=0}\left[\Psi|_0 \cdot -\frac{1}{w}\,\mathrm{d}w\right] = -c_0 = \frac{1}{2(1-2\gamma)}.
\end{aligned}$$

We then find $\langle z^{-1}\,\mathrm{d}z, \mathrm{d}z\rangle = 1/(2(1-2\gamma))$ hence

$$\frac{I_{-1}}{I_0} = \frac{\langle z^{-1}\,\mathrm{d}z, \mathrm{d}z\rangle}{\langle \mathrm{d}z, \mathrm{d}z\rangle} = \frac{1}{2(1-2\gamma)} \Big/ \frac{-\gamma}{2(-1-2\gamma)(1-2\gamma)} = \frac{1+2\gamma}{\gamma}, \qquad I_{-1} = \frac{1+2\gamma}{\gamma}\, I_0.$$

To confirm that these results are correct, we compare them to know results. We know [AS64] that

$$I_n = \frac{\Gamma(1+\gamma)\,\Gamma(1+\gamma+n)}{\Gamma(2+2\gamma+n)}.$$

Here Γ is the Euler gamma function. Putting the values for $n = -1, 0, 1$ into the formula and using $\Gamma(x+1) = x\,\Gamma(x)$, we directly confirm that our above calculations are correct. △

The above example had no external parameters, making it easy to determine the manifold. Moreover, the twisted cohomology had dimension 1 such that is was easy to choose a basis. These things are not true for the next example.

Example 5.2.4 (Hypergeometric ${}_2F_1$)**.** We consider an Euler integral representation of a product of the Euler Beta function and the Gauss ${}_2F_1$ hypergeometric function:

$$f(a,b,c;x) := \beta(b, c-b)\ {}_2F_1(a,b,c;x) = \int_0^1 z^{b-1}\,(1-z)^{c-b-1}\,(1-x\,z)^{-a}\,\mathrm{d}z.$$

For any fixed a, b, c the integrand has two variables, x and z. Here z is an internal variable and x is an external variable. We assume that a, b and $c-b$ are all fixed non-integers and determine a relation between $f(a,b,c;x)$ and $f(a+n,b+m,c+k;x)$ for some integers n,m,k. Note that the latter can be rewritten as

$$f(a+n,b+m,c+k;x) = \int_0^1 u\cdot z^m(1-z)^{k-m}(1-x\,z)^{-n}\,\mathrm{d}z, \qquad u(z) = z^{b-1}\,(1-z)^{c-b-1}\,(1-x\,z)^{-a}.$$

We therefore choose the one-form

$$\alpha = \mathrm{d}\log u = \left(\frac{b-1}{z} + \frac{c-b-1}{1-z} + \frac{-a}{1-x\,z}\right)\mathrm{d}z.$$

Since we also allow the integers n,m,k to be negative, we can assume without loss of generality that $a<0$, $b>1$ and $c>b+1$.

We use this one-form to determine the manifold. It is clear that α is singular at $z=0$, $z=1$, $z=\infty$ and at $x\,z=1$. We have two variables, so our manifold becomes

$$M = \mathbb{C}^2 \setminus \{(z,x) \mid z=0 \text{ or } z=1 \text{ or } z\,x=1\}.$$

Note that M can be embedded into $\mathbb{CP}^2$ by adding a new variable t into the polynomial u such that the result is homogeneous,

$$u_{hom}(z,x,t) = z^{b-1}(t-z)^{c-b-1}(t^2-x\,z)^{-a}.$$

Then the singularities at $z=\infty$ become singularities at $t=0$ instead as $[x:\infty:1]=[0:1:0]$ for $x\in\mathbb{C}$. The singularities of α are no longer isolated poles, but instead form lines in $\mathbb{CP}^2$.

We are not interested in the entire manifold, as our integral does not integrate over x. We therefore want to look at the submanifolds obtained by choosing a fixed x. For most values of x this gives 4 isolated poles. However for $x=0,1,\infty$ we find only 3 distinct poles, with one of the poles being on the intersection of two of the lines described above. We therefore expect that the coefficient relating $f(a+n,b+m,c+k;x)$ to $f(a,b,c;x)$ will have singular behaviour at $x=0$, $x=1$ and $x=\infty$.

With these considerations in mind, we assume $x\neq 0,1,\infty$. In that case the submanifold for z is

$$M = \mathbb{C}\setminus\{0,1,1/x\} = \mathbb{CP}^1\setminus\{0,1,1/x,\infty\}.$$

We haven't already calculated the twisted cohomology of this (sub)manifold, so we use the Morse theory from Subsection 4.4.

We first determine the cut-off boundaries. We have

$$\Re\log u(z) = (b-1)\log|z| + (c-b-1)\log|1-z| - a\log|1-xz|,$$

so the precise boundaries depend on the values of a,b,c. We assumed before that $a<0$, $b>1$ and $c>b+1$. In particular, that means that the coefficient in front of each log is strictly positive. We therefore get compact cut-off boundaries. For $\ell\ll 0$ sufficiently small the space $\Re\log u(z)=\ell$ will consist of three circles around $z=0$, $z=1$ and $z=1/x$. This can be seen by the fact that the points on a sufficiently small circle around $z=0$ will always have approximately the same distance to $z=1$ and $z=1/x$, at distances 1 and $1/x$ respectively. Therefore, the solutions of $\Re\log u(z)=\ell$ can be approximated by

$$(b-1)\log|z| + (c-b-1)\log|1| - a\log\left|x\cdot\frac{1}{x}\right| = \ell.$$

Since $\log|1|=0$, we find that the equation is approximately solved by $|z|=e^{\ell/(b-1)}$ which gives a circle around 0 with radius $e^{\ell/(b-1)}\ll 1$. Similarly, small circles around $z=1$ and $z=1/x$ also provide solutions. Therefore, the lower cut-off boundary consists of three circles.

Note that the upper cut-off boundary is approximately a large circle around $z=0$ since for $\Lambda\gg 0$ and and for $z\gg 0$ we can approximate $|1-z|\sim|z|$ and $|1-xz|\sim|x|\,|z|$. That gives us

$$(b-1)\log|z| + (c-b-1)\log|z| - a\log(|x|\,|z|) = \Lambda,$$

hence solutions $|z| = \exp\big(\Lambda/(c - a\,|x| - 2)\big)$. This gives us a circle with a large radius.

We then need to determine the cohomology of the lower cut-off boundary M_ℓ. Since the circles are disjoint, their twisted cohomology is simply the sum of the twisted cohomologies of each circle. Each of the circles is a special case of Example 4.1.6, so we simply find that $H^0_\alpha(M_\ell) = 0$ and $H^1_\alpha(M_\ell) = 0$ for a, b and $c - b$ non-integer. This is indeed what we expected, since the each circle corresponds to a loop and the monodromy of α along any loop around $z = 0$, $z = 1$ or $z = 1/x$ is non-zero.

In that case, our manifold has $H^0_\alpha = 0$, $H^2_\alpha = 0$, and the first cohomology class is given by the number of zeroes of α. The equation

$$0 = z(1-z)(1-xz)\cdot\alpha = \big[(b-1)(1-z)(1-xz) + (c-b-1)\,z\,(1-xz) - a\,z\,(1-z)\big]\mathrm{d}z$$

is a quadratic polynomial in z, and has two distinct roots. We find that $H^1_\alpha \cong \mathbb{C}^2$ and we have to find two linearly independent 1-forms to form a basis.

There are multiple possible bases. We first try the basis $\{\mathrm{d}z, z\,\mathrm{d}z\}$ in analogy to Example 5.2.3. Remark that $\mathrm{d}z$ corresponds to $f(a,b,c;x)$ and $z\,\mathrm{d}z$ to $f(a,b+1,c+1;x)$. We similarly have to choose 2 elements of $H^1_{c,-\alpha}$ to intersect with; we again choose $\mathrm{d}z$ and $z\,\mathrm{d}z$. We then get the following:

$$\begin{aligned}
\langle \mathrm{d}z \mid \mathrm{d}z\rangle &= \sum_{p=0,1,1/x,\infty} \operatorname*{Res}_{z=p}[\mathrm{d}z\cdot\Psi|_p]\\
\langle z\,\mathrm{d}z \mid \mathrm{d}z\rangle &= \sum_{p=0,1,1/x,\infty} \operatorname*{Res}_{z=p}[z\,\mathrm{d}z\cdot\Psi|_p]\\
\langle \mathrm{d}z \mid z\,\mathrm{d}z\rangle &= \sum_{p=0,1,1/x,\infty} \operatorname*{Res}_{z=p}[\mathrm{d}z\cdot\Psi'|_p]\\
\langle z\,\mathrm{d}z \mid z\,\mathrm{d}z\rangle &= \sum_{p=0,1,1/x,\infty} \operatorname*{Res}_{z=p}[z\,\mathrm{d}z\cdot\Psi'|_p].
\end{aligned}$$

Here $\Psi|_p$ is the local solution at p of the equation $(\mathrm{d}-\alpha)\Psi = \mathrm{d}z$, while $\Psi'|_p$ similarly solves $(\mathrm{d}-\alpha)\Psi' = z\,\mathrm{d}z$.

To see whether $\{\mathrm{d}z, z\,\mathrm{d}z\}$ for H^1_α and $\{\mathrm{d}z, z\,\mathrm{d}z\}$ for $H^1_{c,-\alpha}$ are indeed bases we calculate the above four intersections; the two sets are bases if and only if the matrix formed by the four intersections has non-zero determinant. The calculations themselves are identical to the calculation in Example 5.2.3. We calculate a power series expansion for Ψ and Ψ' at each point before taking the residues. This gives the following.

$$\begin{aligned}
\langle \mathrm{d}z, \mathrm{d}z\rangle &= \frac{a\left((b+1)x^2 - 2(b+1)x - c(x-1)^2 + 2\right) - (b-1)x^2(b-c+1)}{x^2(a-c+1)(a-c+2)(a-c+3)}\\
\langle \mathrm{d}z, z\,\mathrm{d}z\rangle &= \frac{\left[\begin{array}{l} a^2(x-1)^2x(b-c+1) + a\big(\left(-2b^2+b+3\right)x^3 + \left(3b^2+b-2\right)x^2 + c(x-1)^2((2b-3)x-5)\\ \quad -5(b+1)x + c^2(x-1)^2 + 6\big) + \left(b^2-3b+2\right)x^3(b-c+1)\end{array}\right]}{x^3(a-c+1)(a-c+2)(a-c+3)(a-c+4)}\\
\langle z\,\mathrm{d}z, \mathrm{d}z\rangle &= \frac{\left[\begin{array}{l} a^2(x-1)^2x(b-c+1) + a\Big(-\left(2b^2+b-1\right)x^3 + c(x-1)^2((2b-1)x-3)\\ \quad +3b(b+1)x^2 - 3(b+1)x + c^2(x-1)^2 + 2\Big) + (b-1)bx^3(b-c+1)\end{array}\right]}{x^3(a-c)(a-c+1)(a-c+2)(a-c+3)}\\
\langle z\,\mathrm{d}z, z\,\mathrm{d}z\rangle &= \frac{\left[\begin{array}{l} -a^3(x-1)^2x^2(b-c+1) + a^2(x-1)^2x(b-c+1)(3(b-1)x+2c-4)\\ \quad +a\big(c(x-1)^2\left(\left(3b^2-6b+2\right)x^2 + (6-8b)x + 11\right) + 4b\left(b^2-1\right)x^3\\ \quad +\left(-3b^3+3b^2+4b-2\right)x^4 + 2c^2(x-1)^2((b-1)x-3) - 6b(b+1)x^2\\ \quad +8(b+1)x + c^3(x-1)^2 - 6\big) + b\left(b^2-3b+2\right)x^4(b-c+1)\end{array}\right]}{x^4(a-c)(a-c+1)(a-c+2)(a-c+3)(a-c+4)}
\end{aligned}$$

Writing the above values into a matrix and calculating the determinant gives

$$\langle \mathrm{d}z, \mathrm{d}z\rangle\langle z\,\mathrm{d}z, z\,\mathrm{d}z\rangle - \langle \mathrm{d}z, z\,\mathrm{d}z\rangle\langle z\,\mathrm{d}z, \mathrm{d}z\rangle = -\frac{a(b-1)(1+b-c)(x-1)^2}{(a-c)(1+a-c)(2+a-c)(3+a-c)(4+a-c)x^4},$$

which is indeed generally non-zero. In particular, the expression is 0 only if $x = 1$ or $x = \infty$, and diverges for $x = 0$. We already predicted singular behaviour at $x = 0$, $x = 1$ and $x = \infty$ so this is no surprise.

We see that the $\{\mathrm{d}z, z\,\mathrm{d}z\}$ can form a basis for H^1_α and for $H^1_{c,-\alpha}$. That means that we can now express any element of H^1_α in terms of $\mathrm{d}z$ and $z\,\mathrm{d}z$ via the intersection product. We will do this later, we first consider another basis.

The above basis works, but clearly gives rather large intersection products. This is because α has poles which are not present in $\mathrm{d}z$ and $z\,\mathrm{d}z$, such that the poles instead appear in the intersection product. To avoid this we can choose a basis whose form is more reminiscent of α, such as

$$\left\{\frac{1}{z}\mathrm{d}z - \frac{1}{z-1}\mathrm{d}z, \frac{1}{z-1}\mathrm{d}z - \frac{x}{xz-1}\mathrm{d}z\right\} = \{\mathrm{d}\log(z) - \mathrm{d}\log(z-1), \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1)\}.$$

This is an example of a dlog basis. Note that this basis is degenerate at $x = 1$ and at $x = \infty$. We will show the basis is indeed a proper basis for any other value of x by calculating the intersection product. After solving a local power series equation and summing the residues one finds the following.

$$\begin{aligned}
\langle \mathrm{d}\log(z) - \mathrm{d}\log(z-1), \mathrm{d}\log(z) - \mathrm{d}\log(z-1)\rangle &= \frac{1}{1-b} - \frac{1}{c-b-1} \\
\langle \mathrm{d}\log(z) - \mathrm{d}\log(z-1), \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1)\rangle &= \frac{1}{c-b-1} \\
\langle \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1), \mathrm{d}\log(z) - \mathrm{d}\log(z-1)\rangle &= \frac{1}{c-b-1} \\
\langle \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1), \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1)\rangle &= \frac{1}{a} - \frac{1}{c-b-1}.
\end{aligned}$$

These intersection numbers are clearly easier to work with, although they are not easier to compute. The determinant is now

$$\left(\frac{1}{1-b} - \frac{1}{c-b-1}\right)\left(\frac{1}{a} - \frac{1}{c-b-1}\right) - \left(\frac{1}{c-b-1}\right)\left(\frac{1}{c-b-1}\right) = \frac{c-a-2}{-a(b-1)(c-b-1)}.$$

We have assumed $a < 0$ and $c > b + 1 > 2$, so this is non-zero. The dlog basis is therefore indeed a good basis.

Before we calculate the intersection of another element with respect to these bases, we first note that so far we have used the same basis for H^1_α and $H^1_{c,-\alpha}$. This is not necessary, we could for instance take $\{\mathrm{d}z, z\,\mathrm{d}z\}$ for H^1_α and $\{\mathrm{d}\log(z) - \mathrm{d}\log(z-1), \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1)\}$ for $H^1_{c,-\alpha}$ as that gives:

$$\begin{aligned}
\langle \mathrm{d}z, \mathrm{d}\log(z) - \mathrm{d}\log(z-1)\rangle &= \frac{-1}{c-a-1} \\
\langle \mathrm{d}z, \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1)\rangle &= \frac{x-1}{(c-a-1)x} \\
\langle z\,\mathrm{d}z, \mathrm{d}\log(z) - \mathrm{d}\log(z-1)\rangle &= \frac{a-ax+bx}{(a-c)(c-a-1)x} \\
\langle z\,\mathrm{d}z, \mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1)\rangle &= \frac{(x-1)(c-1+bx-ax)}{(c-a)(c-a-1)x^2}.
\end{aligned}$$

These intersection products are still quite simple and indeed the determinant is

$$\left(\frac{-1}{c-a-1}\right)\left(\frac{(x-1)(c-1+bx-ax)}{(c-a)(c-a-1)x^2}\right) - \left(\frac{x-1}{(c-a-1)x}\right)\left(\frac{a-ax+bx}{(a-c)(c-a-1)x}\right) = \frac{x-1}{(c-a)(c-a-1)^2x^2},$$

which is zero only for $x = 1$ or $x = \infty$ and singular only for $x = 0$.

We now have seen three possible pairs of bases, and we will use to find relations. To do so, we first have to determine to which integrals the chosen forms correspond. We find

$$\begin{aligned}
\mathrm{d}z \Rightarrow \quad & f(a,b,c;x) = \beta(b,c-b)\ {}_2F_1(a,b,c;x) \\
z\,\mathrm{d}z \Rightarrow \quad & f(a,b+1,c+1;x) = \beta(b+1,c-b)\ {}_2F_1(a,b+1,c+1;x) \\
\mathrm{d}\log(z) - \mathrm{d}\log(z-1) \Rightarrow \quad & -f(a,b-1,c-2;x) = -\beta(b-1,c-b-1)\ {}_2F_1(a,b-1,c-2;x) \\
\mathrm{d}\log(z-1) - \mathrm{d}\log(xz-1) \Rightarrow \quad & f(a+1,b+1,c;x) = \beta(b+1,c-b-1)\ {}_2F_1(a+1,b+1,c;x),
\end{aligned}$$

where we used

$$\frac{1}{z}-\frac{1}{z-1}=\frac{-1}{z(z-1)},\qquad \frac{1}{z-1}-\frac{1}{xz-1}=\frac{(x-1)z}{(z-1)(xz-1)}.$$

We will express $f(a,b+2,c+2;x)=\beta(b+2,c-b)\ {}_2F_1(a,b+2,c+2;x)$ in terms of $f(a,b,c;x)$ and $f(a,b+1,c+1;x)$, and in terms of $-f(a,b-1,c-2;x)$ and $f(a+1,b+1,c;x)$. Note that $f(a,b+2,c+2;x)$ corresponds to the integrand $z^2\,\mathrm{d}z$. We write

$$f(a,b+2,c+2;x)=c_1\cdot f(a,b,c;x)+c_2\cdot f(a,b+1,c+1;x),$$

where the coefficients c_1,c_2 are calculated using the intersection product,

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}\langle \mathrm{d}z\mid\phi_1\rangle & \langle \mathrm{d}z\mid\phi_2\rangle\\ \langle z\,\mathrm{d}z\mid\phi_1\rangle & \langle z\,\mathrm{d}z\mid\phi_2\rangle\end{pmatrix}^{-1}\cdot\begin{pmatrix}\langle z^2\,\mathrm{d}z\mid\phi_1\rangle\\ \langle z^2\,\mathrm{d}z\mid\phi_2\rangle\end{pmatrix}.$$

Here $\{\phi_1,\phi_2\}$ is a basis for $H^1_{c,-\alpha}$; the choice of a basis should not matter. We will use $\{\mathrm{d}z, z\,\mathrm{d}z\}$ and $\{\mathrm{d}\log(z)-\mathrm{d}\log(z-1),\mathrm{d}\log(z-1)-\mathrm{d}\log(xz-1)\}$ to confirm this.

For $\{\mathrm{d}z, z\,\mathrm{d}z\}$ the rightmost vector becomes too long to write down , but generally has the form

$$\begin{aligned}\langle z^2\,\mathrm{d}z,\mathrm{d}z\rangle &= \frac{\text{polynomial in } a,\ b,\ c \text{ and } x}{(c-a+1)(c-a)(c-a-1)(c-a-2)(c-a-3)x^4}\\ \langle z^2\,\mathrm{d}z,z\,\mathrm{d}z\rangle &= \frac{\text{another polynomial in } a,\ b,\ c \text{ and } x}{(c-a+1)(c-a)(c-a-1)(c-a-2)(c-a-3)(c-a-4)x^5}.\end{aligned}$$

Inverting the intersection matrix and doing a matrix multiplication, we find

$$\begin{pmatrix}c_1 & c_2\end{pmatrix}^T=\begin{pmatrix}\dfrac{-b}{(c-a+1)x} & \dfrac{c+(b-a+1)x}{(c-a+1)x}\end{pmatrix}^T.$$

We conclude that

$$f(a,b+2,c+2;x)=\frac{-b}{(c-a+1)x}\cdot f(a,b,c;x)+\frac{c+(b-a+1)x}{(c-a+1)x}\cdot f(a,b+1,c+1;x).$$

This gives a useful contiguity relation for $f(a,b,c;x)$.

We repeat the procedure for the $\{\mathrm{d}\log(z)-\mathrm{d}\log(z-1),\mathrm{d}\log(z-1)-\mathrm{d}\log(xz-1)\}$ basis, which should give the same answer. We have

$$\begin{aligned}\langle z^2\,\mathrm{d}z,\mathrm{d}\log(z)-\mathrm{d}\log(z-1)\rangle &= \frac{a^2\,x(1-x)-b(b+1)x^2-a(1-x)(c+x+2bx)}{(c-a+1)(c-a)(c-a-1)x^2}\\ \langle z^2\,\mathrm{d}z,z\,\mathrm{d}\log(z-1)-\mathrm{d}\log(xz-1)\rangle &= \frac{\left[\begin{array}{l}(1-x)(c^2+c((1-2a+b)x-1)-x(a^2x+a(1+b-x-2bx)\\ \quad+(1+b)(bx-1)))\end{array}\right]}{(c-a+1)(c-a)(c-a-1)x^3}.\end{aligned}$$

Again inverting the intersection matrix and doing matrix multiplication, we find

$$\begin{pmatrix}c_1 & c_2\end{pmatrix}^T=\begin{pmatrix}\dfrac{-b}{(c-a+1)x} & \dfrac{c+(b-a+1)x}{(c-a+1)x}\end{pmatrix}^T.$$

Indeed, we find the same coefficients. We can similarly write $f(a,b+2,c+2;x)$ in terms of $-f(a,b-1,c-2;x)$ and $f(a+1,b+1,c;x)$. The only difference is in the intersection matrix, not in the intersection itself. We have already calculated the matrix elements and the intersections above. We write

$$f(a,b+2,c+2;x)=d_1\cdot -f(a,b-1,c-2;x)+d_2\cdot f(a+1,b+1,c;x),$$

then we find (using either basis for $H^1_{c,-\alpha}$)

$$d_1=\frac{\left[\begin{array}{l}(b-1)(-a^2(c-b-1)(x-1)^2x-b(1+b)(c-b-1)x^3\\ \quad-a(c^2(x-1)^2+c(x-1)^2(2bx+x-1)\\ \quad-(1+b)x(1-(2+3b)x+(1+2b)x^2)))\end{array}\right]}{(c-a+1)(c-a)(c-a-1)(c-a-2)x^3}$$

and

$$d_2 = \frac{\left[\begin{array}{l} a(c^3(x-1) - c^2(x-1)(3-(1+b-2a)x) + c(x-1)(2-3(1+b)x \\ \quad + 2a(2+b)x + (-a+a^2+b-2ab+b^2)x^2) - (1+b)x(2-2(1+b)x \\ \quad + a^2(x-1)x + b(1+b)x^2 - a(x-1)(2bx+x-2))) \end{array}\right]}{(c-a+1)(c-a)(c-a-1)(c-a-2)x^3}.$$

This again gives us a contiguity relation.

Both contiguity relations we found were for $f(a, b+2, c+2; x)$ since they were calculated using $z^2\,\mathrm{d}z$. We can similarly express any $f(a+n, b+m, c+k; x)$ using the more complicated integrand $z^m(1-z)^{k-m}(1-x\,z)^{-n}\,\mathrm{d}z$ by taking the intersection product. This means that we do not have to inductively use contiguity relations, but we can express $f(a+n, b+m, c+k; x)$ more directly. △

5.2.2 Bases

In the above Examples we have seen bases of the form

$$\{\mathrm{d}z, z\,\mathrm{d}z, z^2\,\mathrm{d}z, \dots\}$$

and the basis of the form

$$\{\mathrm{d}\log(a_0(z)) - \mathrm{d}\log(a_1(z)), \mathrm{d}\log(a_1(z)) - \mathrm{d}\log(a_2(z)), \mathrm{d}\log(a_2(z)) - \mathrm{d}\log(a_3(z)), \dots\}$$

where the a_k are linear functions in z. The basis we have seen are examples of two general classes of bases which are often used.

- The monomial basis.

  A monomial basis is of the form $\{\mathrm{d}z, z\,\mathrm{d}z, z^2\,\mathrm{d}z, \dots\}$. These 1-forms have no poles and are therefore typically easy to take the primitive or the residue of. However, this also means that any relevant pole must instead appear in the intersection product, such that the expressions for the intersection products are typically large.

- The dlog basis. [MM19; Hen15]

  As the name implies, the dlog basis has elements of the form $\mathrm{d}\log(a_\ell(z))$.Here a_ℓ is a linear or rational function of z chosen such that $a_\ell(p) = 0$ or ∞ at some critical point p of α. More generally dlog can refer to elements of the form $\mathrm{d}\log(q(z))$ where q is a rational function of polynomials in z.

  The dlog basis can be seen as the natural counterpart of the twisted cycle description of the twisted homology, and as such most of the information of α (such as poles) and be reconstructed from the basis. The basis also has applications in the theory of differential equations for Feynman integrals, where it is related to the canonical form discussed in Subsection 3.3.

In the above Example we have seen that the basis for H^1_α and the basis of $H^1_{c,-\alpha}$ do not necessarily have to be induced by the same 1-forms. This is known as 'mixed bases'. [Man+19] Mixing bases can be useful as the roles of the bases differ. The basis $\{\phi_1, \phi_2, \dots, \phi_k\}$ of H^1_α should be chosen such that the outcome of the corresponding Feynman integrals F_i are usable. On the other hand, the basis $\{\psi_1, \dots, \psi_k\}$ of $H^1_{c,-\alpha}$ should be chosen such that the computation of the intersection with ψ_i takes the least time. In particular, we are not interested in the corresponding Feynman integrals but only in the corresponding local primitives Ψ_i.

However, there are also advantages to using the dlog basis for both spaces. For instance, when both ϕ and ψ are dlog, the intersection product can be further simplified; see [Fre+19a:Section 3.5]. According to [Miz20] the general question of when which basis is optimal has yet to be answered.

5.2.3 Higher forms

For higher forms, the calculation stays similar in principle but becomes inductively more difficult. Since we want to use residues, we can only integrate over one dimension at a time. This means we have to split an integral over n variables $z_1, \dots, z_n$ into n integrals over one variable z_i each. We do this inductively by

choosing an 'internal dimension' and integrating over that dimension. Formally speaking this means we turn our manifold into the total space of a fibration and integrate over the fibre, using Corollary 4.5.10. We start with $n = 2$ for simplicity and clarity. This section is loosely based on [Fre+19a].

Since we have $n = 2$, we know that $M \subseteq \mathbb{C}^2$. We split the space into fibres such that each value of z_2 denotes a fibre and the coordinate along the fibre is z_1, so the projection is the restriction of M to

$$\mathbb{C}^2 \to \mathbb{C}, \qquad (z_1, z_2) \to z_2.$$

We denote B for the image of M in this projection, such that the fibration is $M \to B$. We denote the fibres by $F_\bullet$. For clarity, $B \subseteq \mathbb{C}$ corresponds to z_2 and $F_\bullet$ to elements $(z_1, \bullet)$ for any $\bullet \in B$.

Then $\alpha = \alpha_1 \,\mathrm{d}z_1 + \alpha_2 \,\mathrm{d}z_2$ induces $\alpha_1 \mathrm{d}z_1$ on the fibre, giving rise to a twisted cohomology $H^k(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1})$. The fibre is a manifold of dimension 1, so we can calculate the intersection along the fibre using the theory described above.

Let $\{e_1, \ldots, e_k\}$ be a basis of $H^1(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1})$ on the fibre, and $\{h_1, \ldots, h_k\}$ a basis of $H^1_c(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})$ also on the fibre. We want to calculate the inner product ${}_{(2)}\langle \phi, \psi \rangle$ for $\phi \in H^2_\alpha$ and $\psi \in H^2_{c,-\alpha}$ on the entire manifold. The main idea will be much like the intersection of 1-forms: we want to use residues. However, the product $\phi \wedge \psi$ is now a 4-form, so even if we take a local primitive of either we still have a 3-form. We need a 1-form in order to use the residue.

The main trick is to turn every 2-form into a product of a 1-form of the form $\mathrm{d}z_1$ and a 1-form of the form $\mathrm{d}z_2$ by using Corollary 4.5.10. Then the two $\mathrm{d}z_1$ terms can be intersected and the two $\mathrm{d}z_2$ terms can be intersected, giving a single function as a result. This means that our calculation has the following steps.

0. Choose a basis $\{e_1, \ldots, e_k\}$ for $H^1(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1})$ and $\{h_1, \ldots, h_k\}$ for $H^1_c(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})$ and determine their intersection product.

1. Start with $\phi \in H^2_\alpha$ and $\psi \in H^2_{c,-\alpha}$ of which you want to determine the intersection product.

2. Compute a local primitive of the 4-form $\phi \wedge \psi$ by determining an 1-form Ψ such that

$$(\mathrm{d} - \alpha)\Psi = \psi, \qquad \mathrm{d}(\phi \wedge \Psi) = (-1)^2 \,\phi \wedge \psi.$$

   To find this Ψ, use that

$$H^2_c(M, \mathcal{L}_{-\alpha}) \cong H^1_c\big(B, H^1_c(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})\big)$$

   to see ψ as an element of the right-hand-side, then locally integrate over B (hence over z_2) using a series expansion. The result is a $H^1_c(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})$–valued function Ψ over B, hence indeed a 1-form over M

3. Write Ψ in terms of the basis h_j, so $\Psi = \sum_j \Psi_j h_j$. Since Ψ was a function with values in $H^1_c(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})$, the coefficients Ψ_j are (generally multi-valued) functions over B. This gives

$$\mathrm{d}\left[\sum_j \Psi_j \,\phi \wedge h_j\right] = (-1)^2 \,\phi \wedge \psi.$$

4. Calculate the intersection product of ϕ with the h_j over the fibre (hence over z_1). During this calculation we temporarily ignore that ϕ has a $\mathrm{d}z_2$ part, and simply take the intersection over the $\mathrm{d}z_1$ parts (along the fibre). In particular, we calculate a local primitive H_j of the 1-form h_j and take the residue over z_1:

$$\int_{F_\bullet} (-1)^2 \phi \wedge \psi = 2\pi i \sum_j \Psi_j \operatorname*{Res}_{z_1}[\phi \, H_j].$$

   Here we use the residue over z_1 to denote the sum of residues over poles in z_1, given by $\mathbb{C} \setminus F_\bullet$ and ∞. The result of the intersection is a 1-form of the form $\mathrm{d}z_2$, not a function.

5. We find that

$$\int_M \phi \wedge \psi = \int_B \int_{F_\bullet} \phi \wedge \psi = \int_B (-1)^2 (2\pi i) \sum_j \Psi_j \operatorname*{Res}_{z_1}[\phi \, H_j]$$

is a 1-form of the form $\mathrm{d}z_2$. We take the z_2–residue of that expression and find a function.

$$\int_M \phi \wedge \psi = \int_B (-1)^2(2\pi i) \sum_j \Psi_j \operatorname*{Res}_{z_1}[\phi\, H_j] = (-1)^2(2\pi i)^2 \operatorname*{Res}_{z_2}\Big[\sum_j \Psi_j \operatorname*{Res}_{z_1}[\phi\, H_j]\Big].$$

Again, the residue over z_2 refers to the residue over the poles of z_2 hence the points of $\mathbb{C} \setminus B$ and ∞.

6. Collecting the above expressions, we find

$$\int_M \phi \wedge \psi = (-2\pi i)^2 \operatorname*{Res}_{z_2}\left[\sum_j \Psi_j \operatorname*{Res}_{z_1}[\phi\, H_j]\right] = (-2\pi i)^2 \sum_j \operatorname*{Res}_{z_2}\left[\Psi_j \operatorname*{Res}_{z_1}[\phi\, H_j]\right].$$

7. Finally, the functions Ψ_j were functions over B and do not depend on $F_\bullet$. Therefore, we can move the Ψ_j inside the residue for the slightly easier expression

$$\int_M \phi \wedge \psi = (-2\pi i)^2 \sum_j \operatorname*{Res}_{z_2}\left[\operatorname*{Res}_{z_1}[\phi\, H_j\, \Psi_j]\right].$$

Based on the above, we define the intersection product on M as follows:

$$_{(2)}\langle \phi, \psi\rangle := \sum_{p_2 \in P_2} \sum_{p_1 \in P_1} \sum_{i=1}^{k} \operatorname*{Res}_{z_2=p_2} \operatorname*{Res}_{z_1=p_1} [\phi\, H_i\, \Psi_i]\,, \tag{32}$$

where $k = \dim H^1_{\alpha_1\,\mathrm{d}z_1}$,

$$\begin{aligned} \mathrm{d}\Psi_i - \widehat{A}^{(2)}_{ij}\mathrm{d}z_2\ \Psi_j &= \sum_{l=1}^{k} [\langle e_i, h_j\rangle]^{-1}_{il} \langle e_l, \psi\rangle, \\ \mathrm{d}H_i - \alpha_1 \mathrm{d}z_1\ H_i &= h_i, \end{aligned}$$

and

$$\widehat{A}^{(2)}_{ij} = -\sum_{l=1}^{k} [\langle e_i, h_j\rangle]^{-1}_{il}\ \langle e_l, (\partial_{z_2} - \alpha_2) h_j\rangle.$$

We see that we do not need to know Ψ for the intersection product, as long as we know Ψ_j. This means we do not have to explicitly use the isomorphism $H^2(M, \mathcal{L}_\alpha) \cong H^1\big(B, H^1(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1})\big)$, we only use it implicitly.

In this formula we have made the poles of the residues explicit, using $P_2 = \mathbb{C} \setminus B$ and $P_1 = \mathbb{C} \setminus F_\bullet$. However, we still have to find these sets, and P_1 generally depends on the corresponding base point z_2. The set P_1 is the set of poles in z_1 on the fibre, hence P_1 is the set of poles of $\alpha_1\,\mathrm{d}z_1$ for the fixed value of z_2. On the other hand, P_2 is a bit more complicated. It turns out that $\widehat{A}^{(2)}_{ij}$ (a matrix of 1-forms) takes over the role of α on the base space, and the poles P_2 are precisely the poles of $\widehat{A}^{(2)}_{ij}$. Note that the above formula for Ψ_i treats $\widehat{A}^{(2)}_{ij}$ as a replacement for α.

Indeed, it turns out that the local system $H^1_c(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})$ over B is the local system $\mathcal{L}_A$ corresponding to $A = \widehat{A}^{(2)}$ as in Subsubsection 4.5.1.

In order to explain the formula for Ψ_i and give more context for Ψ and $\widehat{A}^{(2)}_{ij}$, we consider the following. We found that taking the primitive of $\phi\ in H^2_{c,-\alpha}$ gives some

$$\Psi = \sum_{j=1}^{k} \Psi_j\, h_j.$$

Here Ψ_j is a (generally multi-valued) function on B, and was given by locally taking the primitive of ψ with respect to z_2. We therefore have the equation

$$\mathrm{d}\Psi - \alpha_2\, \Psi = \partial_{z_2}\Psi\, \mathrm{d}z_2 - \alpha_2 \Psi = \psi.$$

We want to express this in terms of the Ψ_j. We find

$$\sum_{j=1}^{k}\Big[(\partial_{z_2}\Psi_j)h_j+\Psi_j\partial_{z_2}h_j-\alpha_2\Psi_j\,h_j\Big]=\partial_{z_2}\sum_{j=1}^{k}\Psi_j\,h_j-\alpha_2\sum_{j=1}^{k}\Psi_j\,h_j=\psi.$$

We finally have to remove the h_j from the equation. To do this, we take the intersection product of both sides with e_l and then multiply both sides with $[\langle e_i,h_j\rangle]^{-1}_{il}$ and sum over l. Note that Ψ_j does not depend on z_1, so we can move it out of the intersection product. This gives:

$$\partial_{z_2}\Psi_j+\sum_{l=1}^{k}[\langle e_i,h_j\rangle]^{-1}_{il}\langle e_l,\partial_{z_2}h_j\rangle\Psi_j-\sum_{l=1}^{k}[\langle e_i,h_j\rangle]^{-1}_{il}\langle e_l,\alpha_2\,h_j\rangle\Psi_j=\sum_{l=1}^{k}[\langle e_i,h_j\rangle]^{-1}_{il}\langle e_l,\psi\rangle.$$

Gathering the second and third terms, we indeed find the earlier expression for $\widehat{A}^{(2)}_{ij}$:

$$\widehat{A}^{(2)}_{ij}=-\sum_{l=1}^{k}[\langle e_i,h_j\rangle]^{-1}_{il}\ \langle e_l,(\partial_{z_2}-\alpha_2)h_j\rangle.$$

This shows that the local system $H^k_c(F_\bullet,\mathcal{L}_{-\alpha_1\mathrm{d}z_1})$ is indeed the local system generated by $\widehat{A}^{(2)}$.

We can now in theory calculate 2-forms. We first generalise the above the n-forms before looking at an example.

Higher forms

For higher n-forms we inductively repeat the above process of splitting off one dimension. To be precise, given the n–intersection we choose bases $\{e^{(n)}_1,\dots,e^{(n)}_\ell\}$ and $\{h^{(n)}_1,\dots,h^{(n)}_\ell\}$ for the n-th intersection product and define $\widehat{A}^{(n+1)}$ as

$$\widehat{A}^{(n+1)}_{ij}=-\sum_{l=1}^{\ell}\big[{}_{(n)}\langle e^{(n)}_i,h^{(n)}_j\rangle\big]^{-1}_{il}\ \ {}_{(n)}\langle e^{(n)}_l,(\partial_{z_{n+1}}-\alpha_{n+1})h^{(n)}_j\rangle.$$

We then define P^{n+1} as the set of poles of $\widehat{A}^{(n+1)}$, use it to determine the local primitive and calculate the residue at every pole to find the $(n+1)$–intersection:

$${}_{(n+1)}\langle\phi^{(n+1)},\psi^{(n+1)}\rangle:=\sum_{p_n\in P_n}\operatorname*{Res}_{z_n=p_n}\sum_{i=1}^{\ell}\Big[{}_{(n)}\langle\phi^{(n+1)},h^{(n)}_i\rangle\ \Psi^{(n+1)}_i\Big],$$

where

$$\mathrm{d}\Psi^{(n+1)}_i-\widehat{A}^{(n+1)}_{ij}\ \Psi^{(n+1)}_j\,\mathrm{d}z_{n+1}=\sum_{l=1}^{\ell}\Big[{}_{(n)}\langle e^{(n)}_a,h^{(n)}_b\rangle\Big]^{-1}_{il}\langle e^{(n)}_l,\psi^{(n+1)}\rangle.$$

In words, $\Psi^{(n+1)}_i$ is the $\big(\mathrm{d}-\widehat{A}^{(n+1)}_{ij}\big)$–primitive of the projection of $\psi^{(n+1)}$ in terms of the $h^{(n)}_i$. Inductively expanding the $(n+1)$–intersection one finds (we use a summation convention for clarity)

$${}_{(n+1)}\langle\phi^{(n+1)},\psi^{(n+1)}\rangle:=\sum_{p_n\in P_n}\cdots\sum_{p_2\in P_2}\sum_{p_1\in P_1}\operatorname*{Res}_{z_n=p_n}\dots\operatorname*{Res}_{z_2=p_2}\operatorname*{Res}_{z_1=p_1}\Big[\phi^{(n+1)}\Psi^{(n+1)}_{i_n}H^{(n)}_{i_ni_{n-1}}\dots H^{(2)}_{i_2i_1}H^{(1)}_{i_1}\Big],$$

where the $H^{(k)}_{i_ki_{k-1}}$ are the primitives of the $h^{(k)}_{i_k}$, where $\{h^{(k)}_1,\dots,h^{(k)}_{\nu_k}\}$ is a basis at the k-th step of the induction. Note that the above makes mathematical sense due to iterated applications of Theorem 4.5.9. This gives

$$\begin{aligned}H^n(M,\mathcal{L}_{-\alpha})&\cong H^1(B_{(n)},\{H^{n-1}(F_{(n-1)},\mathcal{L}_{-\alpha_1\,\mathrm{d}z_1-\cdots-\alpha_{n-1}\,\mathrm{d}z_{n-1}})\}_{z_n})\\&\cong H^1(B_{(n)},\{H^1(B_{(n-1)},\{H^{n-2}(F_{(n-2)},\mathcal{L}_{-\alpha_1\,\mathrm{d}z_1-\cdots-\alpha_{n-2}\,\mathrm{d}z_{n-1}})\}_{z_{n-1}})\}_{z_n})\\&\cong H^1(B_{(n)},\{H^1(B_{(n-1)},\{\dots,\{H^1(F_{(1)},\mathcal{L}_{-\alpha_1\,\mathrm{d}z_1})\}_{z_2}\dots\}_{z_{n-1}})\}_{z_n}),\end{aligned}$$

where the Serre fibration $M \to B_{(n)}$ has fibre $F_{(n-1)}$ of dimension $n-1$, the fibration $F_{(n-1)} \to B_{(n-1)}$ has fibre $F_{(n-2)}$ of dimension $n-2$, etc. Note that one has to make sure all these projections are indeed Serre fibrations.

The above formula is very useful, but it perhaps not the most intuitive. We will now look at an example to build intuition.

5.2.4 Higher form example

It is clear that the calculation becomes more complicated quickly when $n > 2$, so we will look at an example where $n = 2$. We do the same example as in Section 3.3 of [Fre+19a] but with a slightly different method; we only use the spectral sequence to write the 2-form in $H^2_{c,-\alpha}$ as a '(1-form)-valued 1-form', while Frellesvig et al. also write the 2-form in H^2_α in that form.

Example 5.2.5. We start with

$$U(z_1, z_2) := \big(z_1\ z_2\ (1 - z_1 - z_2)\big)^\gamma,$$

for non-integer γ. This is not directly related to any common integral such as the Euler Beta function or the Gamma function, but is a generalisation of the Euler Beta function from Example 5.2.3. Setting $\alpha = \mathrm{d}\log U$ as usual gives

$$\alpha = \gamma\left(\frac{1}{z_1} - \frac{1}{1 - z_1 - z_2}\right)\mathrm{d}z_1 + \gamma\left(\frac{1}{z_2} - \frac{1}{1 - z_1 - z_2}\right)\mathrm{d}z_2.$$

Note that our manifold will be

$$M = \mathbb{C}^2 \setminus \{z \in \mathbb{C}^2 \mid z_1 = 0 \text{ or } z_2 = 0 \text{ or } z_1 + z_2 = 1\}.$$

Therefore we have basis

$$B = \mathbb{C} \setminus \{z_2 \in \mathbb{C} \mid z_2 = 0\},$$

and for each $z_2 \in B$ the fibre over z_2 is

$$F_{z_2} = \mathbb{C} \setminus \{z_1 \in \mathbb{C} \mid z_1 = 0 \text{ or } z_1 = 1 - z_2\}.$$

Note that the projection $M \to B$ is ramified at $z_2 = 1$; the fibre F_1 only has one puncture $z_1 = 0 = 1 - z_2$ while all other fibres have two. This means $M \to B$ is not a Serre fibration. We therefore have to remove the line $z_2 = 1$ from the manifold, and remove the point $z_2 = 1$ from B. We then get a new M and B:

$$M = \mathbb{C}^2 \setminus \{z \in \mathbb{C}^2 \mid z_1 = 0 \text{ or } z_2 = 0 \text{ or } z_2 = 1 \text{ or } z_1 + z_2 = 1\},$$

$$B = \mathbb{C} \setminus \{z_2 \in \mathbb{C} \mid z_2 = 0 \text{ or } z_2 = 1\}.$$

We have changed M, so we generally have to change U as well such that M is again the complement of the zero set of U. This can be achieved by multiplying U with $(1 - z_2)^\delta$ for some non-integer $\delta > 0$, and correspondingly adding $-\delta/(1 - z_2)\ \mathrm{d}z_2$ to α.

However, this change is not required in this specific case. This is because the cut-off boundary around the line $z_2 = 1$ is a cylinder with zero twisted cohomology, which can be seen in the same way as Example 4.1.6. This means that in this particular example we can add this cut-off manifold by hand without changing U. We will do this to improve the clarity of the equations. Finally remark that the intersection products we find will differ slightly from the ones we would have found with the factor $(1 - z_2)^\delta$, since our intersection products generally do not depend on δ. The values will agree in the limit $\delta \to 0$.

The new projection $M \to B$ is a Serre fibration, and we have $P_1 = \{0, 1 - z_2, \infty\}$ and $P_2 = \{0, 1, \infty\}$. Note that $\alpha = 0$ has only one solution, $z_1 = z_2 = 1/3$, and the solution is non-degenerate so we can use the Morse theory to find that

$$\dim H^2_\alpha = 1.$$

The other twisted cohomologies are 0 by the Middle Dimension Theorem 4.2.10, using the third case.

In this example, we will calculate

$${}_{(2)}\langle \mathrm{d}z_1 \wedge \mathrm{d}z_2, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \rangle.$$

This is one of the easiest intersection products to calculate, but any other intersection product can be calculated similarly. We first have to calculate the intersection product on $H^1(F_\bullet, i_\bullet^*\mathcal{L}_\alpha)$. For any fixed z_2 we simply have

$$i_{z_2}^*\mathcal{L}_\alpha = \mathcal{L}_{\alpha_1 \mathrm{d}z_1}, \qquad \alpha_1 = \gamma\left(\frac{1}{z_1} - \frac{1}{1-z_1-z_2}\right).$$

This $\alpha_1 \mathrm{d}z_1$ has a single zero at $(1-z_2)/2$, and this zero is non-degenerate. Another Morse function therefore implies that $\dim H^1(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1}) = 1$, hence in particular also $\dim H_c^1(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1}) = 1$.

We can directly see that the 1-form $z_1\,\mathrm{d}z_1$ is $(\mathrm{d} - \alpha_1\,\mathrm{d}z_1)$–closed, and we claim it is non-exact. Then $\{z_1\,\mathrm{d}z_1\}$ is a basis for $H^1(F_\bullet, \mathcal{L}_{-\alpha_1 \mathrm{d}z_1})$. We here drop the condition that the 1-form has to be compactly supported by implicitly using the compact cut-off $M_{c.o.}$; in the end our intersection is defined using residues, such that it is no longer problematic if the integral over M diverges. We similarly take $\{z_1\,\mathrm{d}z_1\}$ as a basis for $H^1(F_\bullet, \mathcal{L}_{\alpha_1 \mathrm{d}z_1})$.

To then calculate ${}_{(2)}\langle \mathrm{d}z_1 \wedge \mathrm{d}z_2, \mathrm{d}z_1 \wedge \mathrm{d}z_2\rangle$ we need to know H_i and Ψ_i (see Equation 32), and for the former we need to know $\widehat{A}_{ij}^{(2)}$. Let us calculate these in turn. We have $e_1 = z_1\,\mathrm{d}z_1$ and $h_1 = z_1 \mathrm{d}z_2$, so we have

$$\widehat{A}_{11}^{(2)} = -\langle z_1\,\mathrm{d}z_1, z_1\,\mathrm{d}z_1\rangle^{-1}\langle z_1\,\mathrm{d}z_1, \left(\partial_{z_2} - \gamma\left(\frac{1}{z_2} - \frac{1}{1-z_1-z_2}\right)\right) z_1\,\mathrm{d}z_1\rangle.$$

Here the 1-form intersection on $F_\bullet$ can be calculated using Equation 31, so we need to find a local $(\mathrm{d}-\alpha_1 \mathrm{d}z_1)$–primitive H_1 of $h_1 = z_1\,\mathrm{d}z_1$ on $F_\bullet$ as well as the local primitive of

$$\left(\partial_{z_2} - \gamma\left(\frac{1}{z_2} - \frac{1}{1-z_1-z_2}\right)\right) z_1\,\mathrm{d}z_1 = -\gamma\left(\frac{z_1}{z_2} - \frac{z_1}{1-z_1-z_2}\right)\mathrm{d}z_1.$$

Moreover, we do not need just one primitive but a primitive at each point of $P_1 = \{0, 1-z_2, \infty\}$. Using Mathematica to calculate local series expansions iteratively, we find for H_1:

$$\begin{aligned}
H_1|_{z_1=0} &= \frac{z_1^2}{2-\gamma} + \frac{z_1^3\,\gamma}{(z_2-1)(3-\gamma)(2-\gamma)} + \frac{z_1^4\,(3\gamma-2\gamma^2)}{(z_2-1)^2(4-\gamma)(3-\gamma)(2-\gamma)} + \mathcal{O}(z_1^5),\\
H_1|_{z_1=1-z_2} &= -\frac{(z_1-1+z_2)\,(z_2-1)}{1-\gamma} + \frac{(z_1-1+z_2)^2}{(2-\gamma)(1-\gamma)} + \frac{(z_1-1+z_2)^3\,\gamma}{(z_2-1)(3-\gamma)(2-\gamma)} + \mathcal{O}((z_1-1+z_2)^5),\\
H_1|_{z_1=\infty} &= \frac{z_1^2}{2(1-\gamma)} + \frac{z_1\,(1-z_2)\gamma}{2(1-\gamma)(1-2\gamma)} - \frac{(1-z_2)^2}{4(1-2\gamma)} - \frac{\gamma(1-z_2)^3}{z_1\,4(1-2\gamma)(1+2\gamma)}\\
&\quad - \frac{\gamma(1-z_2)^4}{z_1^2\,8(2\gamma-1)(2\gamma+1)} + \mathcal{O}\left(\frac{1}{z_1^3}\right).
\end{aligned}$$

We get a similar set of expansions for the primitive of $-\alpha_1\,z_1 \mathrm{d}z_1$, which we denote by G:

$$\begin{aligned}
G|_{z_1=0} &= \frac{z_1^2\,(2z_2-1)\gamma}{(z_2-1)z_2(2-\gamma)} + \mathcal{O}(z_1^3),\\
G|_{z_1=1-z_2} &= (1-z_2) + \frac{(z_1-1+z_2)\,(z_2-1)\gamma}{z_2(1-\gamma)} + \mathcal{O}((z_1-1+z_2)^2),\\
G|_{z_1=\infty} &= -\frac{z_1^2\,\gamma}{2z_2(1-\gamma)} + \frac{z_1\,(3\gamma^2 z_2 - 2\gamma z_2 - \gamma^2)}{2z_2(1-\gamma)(1-2\gamma)} + \frac{2z_2 - 2z_2^2 + \gamma - 4z_2\gamma + 3z_2^2\gamma}{4z_2(1-2\gamma)}\\
&\quad + \frac{(z_2-1)^2\gamma(2z_2+\gamma-3z_2\gamma)}{z_1\,4z_2(1-2\gamma)(1+2\gamma)} - \frac{(z_2-1)^3\gamma(2z_2+\gamma-3z_2\gamma)}{z_1^2\,8z_2(1-2\gamma)(1+2\gamma)} + \mathcal{O}\left(\frac{1}{z_1^3}\right).
\end{aligned}$$

We now have:

$$\begin{aligned}
\langle z_1\,\mathrm{d}z_1, z_1\,\mathrm{d}z_1\rangle &= \operatorname*{Res}_{z_1=0}[H_1|_{z_1=0}\,z_1\,\mathrm{d}z_1] + \operatorname*{Res}_{z_1=1-z_2}[H_1|_{z_1=1-z_2}\,z_1\,\mathrm{d}z_1] + \operatorname*{Res}_{z_1=\infty}[H_1|_{z_1=\infty}\,z_1\,\mathrm{d}z_1]\\
&= 0 + 0 + \frac{\gamma(1-z_2)^4}{8(2\gamma-1)(2\gamma+1)}\\
&= \frac{\gamma(1-z_2)^4}{8(2\gamma-1)(2\gamma+1)},
\end{aligned}$$

where we note that in the calculation of the residue at $z_1 = 1 - z_2$ we expanded $z_1 = (z_1 - 1 + z_2) + (1 - z_2)$ and that the residue at ∞ is given by minus the $1/z_1$ term.[20] Similarly, we have

$$\begin{aligned}\langle z_1\,\mathrm{d}z_1, -\alpha_1\ z_1\,\mathrm{d}z_1\rangle &= \operatorname*{Res}_{z_1=0}[G|_{z_1=0}\ z_1\,\mathrm{d}z_1] + \operatorname*{Res}_{z_1=1-z_2}[G|_{z_1=1-z_2}\ z_1\,\mathrm{d}z_1] + \operatorname*{Res}_{z_1=\infty}[G|_{z_1=\infty}\ z_1\,\mathrm{d}z_1]\\ &= 0 + 0 + \frac{-(z_2-1)^3\gamma(2z_2+\gamma-3z_2\gamma)}{8z_2(1-2\gamma)(1+2\gamma)}\\ &= \frac{-(z_2-1)^3\gamma(2z_2+\gamma-3z_2\gamma)}{8z_2(1-2\gamma)(1+2\gamma)}.\end{aligned}$$

We therefore find that

$$\widehat{A}^{(2)}_{11} = -\frac{8(2\gamma-1)(2\gamma+1)}{\gamma(1-z_2)^4}\cdot\frac{-(z_2-1)^3\gamma(2z_2+\gamma-3z_2\gamma)}{8z_2(1-2\gamma)(1+2\gamma)} = -\frac{2z_2+\gamma-3z_2\gamma}{(z_2-1)z_2}.$$

We see that A is singular at $z_2 = 0$ and at $z_2 = 1$, which agrees with $P_2 = \{0, 1, \infty\}$.

The calculations so far have been about the cohomology of $F_\bullet$, and are the same no matter which 2-form we started with. We now recall that we are computing the intersection product of $[\mathrm{d}z_1 \wedge \mathrm{d}z_2] \in H^2_\alpha$ with $[\mathrm{d}z_1 \wedge \mathrm{d}z_2] \in H^2_{-\alpha}$. We start by calculating the Ψ_i.

We know that

$$\mathrm{d}\Psi_i - \widehat{A}^{(2)}_{ij}\mathrm{d}z_2\ \Psi_j = \sum_{l=1}^{k}[\langle e_i, h_j\rangle]^{-1}_{il}\langle e_l, \psi\rangle.$$

Since the only e_l is $e_1 = z_1\,\mathrm{d}z_1$ and the only h_j is $h_1 = z_1\,\mathrm{d}z_1$, we find that there is only one Ψ_1, which is given by

$$\mathrm{d}\Psi_1 - \widehat{A}^{(2)}_{11}\mathrm{d}z_2\ \Psi_1 = \langle z_1\,\mathrm{d}z_1, z_1\,\mathrm{d}z_1\rangle^{-1}\langle z_1\,\mathrm{d}z_1, \mathrm{d}z_1\wedge\mathrm{d}z_2\rangle.$$

We therefore first have to compute $\langle z_1\,\mathrm{d}z_1, \mathrm{d}z_1\wedge\mathrm{d}z_2\rangle$. Computing $\langle z_1\,\mathrm{d}z_1, \mathrm{d}z_1\rangle$ as above, we find

$$\langle z_1\mathrm{d}z_1, \mathrm{d}z_1\wedge\mathrm{d}z_2\rangle = \langle z_1\,\mathrm{d}z_1, \mathrm{d}z_1\rangle\ \mathrm{d}z_2 = \frac{(z_2-1)^3\gamma}{16\gamma^2-4}\mathrm{d}z_2.$$

We already knew $\langle z_1\,\mathrm{d}z_1, z_1\,\mathrm{d}z_1\rangle$ hence we find

$$\langle z_1\,\mathrm{d}z_1, z_1\,\mathrm{d}z_1\rangle^{-1}\langle z_1\,\mathrm{d}z_1, \mathrm{d}z_1\wedge\mathrm{d}z_2\rangle = \frac{8(2\gamma-1)(2\gamma+1)}{\gamma(1-z_2)^4}\cdot\frac{(z_2-1)^3\gamma}{16\gamma^2-4}\,\mathrm{d}z_2 = \frac{-2}{z_2-1}\,\mathrm{d}z_2.$$

We then have to find the local $[\mathrm{d} - \widehat{A}^{(2)}_{11}]$–primitive of the right-hand-side. We are fortunate here since $\widehat{A}^{(2)}_{11}$ is just 1 by 1 matrix; generally it can be any $k \times k$–matrix which means it is much harder to iteratively find the local primitive. Using the same algorithm as for the 1-form intersection product, we find

$$\begin{aligned}\Psi_1|_{z_2=0} &= \frac{z_2\cdot 2}{1-\gamma} + \mathcal{O}(z_2^2),\\ \Psi_1|_{z_2=1} &= \frac{-1}{1-\gamma} - \frac{(z_2-1)\,\gamma}{(1-\gamma)(3-2\gamma)} + \mathcal{O}((z_2-1)^2),\\ \Psi_1|_{z_2=\infty} &= \frac{-2}{2-3\gamma} + \frac{2\gamma}{z_2\,(2-3\gamma)(1-3\gamma)} + \frac{2(1+\gamma)}{z_2^2\,3(2-3\gamma)(1-3\gamma)}\\ &\quad + \frac{2(2+3\gamma+\gamma^2)}{z_2^3\,3(2-3\gamma)(1-3\gamma)(1+3\gamma)} + \frac{2(6+11\gamma+6\gamma^2+\gamma^3)}{z_2^4\,3(2-3\gamma)(1-3\gamma)(1+3\gamma)(2+3\gamma)} + \mathcal{O}\left(\frac{1}{z_2^5}\right).\end{aligned}$$

This tells us that

$${}_{(2)}\langle \mathrm{d}z_1\wedge\mathrm{d}z_2, \mathrm{d}z_1\wedge\mathrm{d}z_2\rangle = \operatorname*{Res}_{z_2=0,1,\infty}\ \operatorname*{Res}_{z_1=0,1-z_2,\infty}\mathrm{d}z_1\wedge\mathrm{d}z_2\cdot\Psi_1\cdot H_1.$$

[20] This is because for $w = 1/z_1$ we have $\mathrm{d}z_1 = -1/w^2\ \mathrm{d}w$, and the residue at $z_1 = \infty$ is the residue at $w = 0$, hence the $1/w$ term.

This gives us a total of 9 double residues. However, the only non-zero residue in z_1 is at $z_1 = \infty$, which we can directly find since Ψ_1 has no z_1 dependence. We therefore find:

$$ {}_{(2)}\langle \mathrm{d}z_1 \wedge \mathrm{d}z_2, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \rangle = \operatorname*{Res}_{z_2=0,1,\infty} \mathrm{d}z_2 \cdot \Psi_1 \cdot \frac{\gamma(1-z_2)^3}{4(1-2\gamma)(1+2\gamma)}. $$

We now only have a power series in z_2 left. The factor $(1-z_2)^3$ only raises the powers at $z_2 = 0$ and $z_2 = 1$, hence those have no residue. However there is a residue at $z_2 = \infty$. The factor $(1-z_2)^3$ means that terms of Ψ_1 of order -1 to -4 all get mixed. Calculating the product and taking the -1st coefficient, we find

$$ {}_{(2)}\langle \mathrm{d}z_1 \wedge \mathrm{d}z_2, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \rangle = \frac{\gamma^2}{3(2-3\gamma)(1-3\gamma)(1+3\gamma)(2+3\gamma)}. $$

Remark that [Fre+19a] arrives at the same outcome via a slightly different computation. Also remark that if we had multiplied U by $(1-z_2)^\delta$ we would instead have found

$$ {}_{(2)}\langle \mathrm{d}z_1 \wedge \mathrm{d}z_2, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \rangle = \frac{\gamma^2\ (-1+2\gamma+\delta)(2\gamma+\delta)(1+2\gamma+\delta)}{2(2-3\gamma+\delta)(1-3\gamma+\delta)(3\gamma+\delta)(1+3\gamma+\delta)(2+3\gamma+\delta)\ (-1+2\gamma)(1+2\gamma)}, $$

which indeed gives the same value in the limit $\delta \to 0$.

Let us also consider how the intersection product ${}_{(2)}\langle z_1^k\ z_2^l\ \mathrm{d}z_1 \wedge \mathrm{d}z_2, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \rangle$ can be computed. In principle, we already have the primitives H_1 and Ψ_1, so this calculation merely requires us to calculate the residues. However, we have only calculated H_1 and Ψ_1 up to a certain order at each point; we will now have to calculate more orders depending on the value of k and l. When $k > 1$ we will need to calculate $H_1|_{z_1=\infty}$ to a higher order, and similarly for $l > 1$ and $\Psi_1|_{z_2=\infty}$. On the other hand, if $k < 0$ we will need to calculate both $H_1|_{z_1=0}$ and $H_1|_{z_1=1-z_2}$ to a higher order, and similarly for $\Psi|_{z_2=0}$ and $\Psi|_{z_2=1}$.

This shows that we generally have to compute H_1 and Ψ_1 to rather high orders if we want to compute the other intersection product, and the order increases linearly with the degree of the integrand. This unfortunately means that we can not easily compute very high orders without computing the primitive to very high orders, which requires a large amount of time. △

The above Example can be generalised to higher forms and cases where $\widehat{A}^{(2)}$ truly is a matrix rather than just a single number. This generalisation should be rather easy, but the author of this thesis ran into computer algebra issues in the latter cases, where Mathematica would not find any local solutions of Ψ_i such that

$$ \mathrm{d}\Psi_i - \widehat{A}^{(2)}_{ij} \mathrm{d}z_2\ \Psi_j = \sum_{l=1}^{k} [\langle e_i, h_j \rangle]^{-1}_{il} \langle e_l, \psi \rangle. $$

This problem should be solvable if given some more time.

To finally see how the intersection product can be applied to calculate actual Feynman diagrams, we borrow an example from [Fre+19b].

Example 5.2.6 (Appendix B of [Fre+19b])**.** Consider a massless box diagram with a self-energy insertion, see Figure 5. This diagram has 7 internal edges, but edges 6 and 7 have the same momentum hence the same propagator. This means we effectively only have 6 internal edges, while we have $E = 3$ and $L = 2$ hence $LE + \frac{1}{2}L(L+1) = 9$. This means we can't use the Baikov representation without introducing several new edges. Frellesvig et al. therefore instead use the loop-by-loop Baikov approach.

The loop-by-loop Baikov representation gives

$$U(z_1, z_2, z_3, z_4, z_5, z_5) = \mathcal{B}_1^{1-d/2}\, \mathcal{B}_2^{d/2-3/2}\, \mathcal{B}_3^{d/2-5/2},$$

where (for $t = (p_2+p_3)^2 = (p_1+p_4)^2$ and $s = (p_1+p_2)^2 = (p_3+p_4)^2$)

$$\begin{aligned}
&\mathcal{B}_1 = z_6,\\
&\mathcal{B}_2 = 2(z_5+z_6)z_4 - z_4^2 - (z_5-z_6)^2,\\
&\mathcal{B}_3 = t^2 z_1^2 + s^2 z_2^2 = 2tz_1\big((2s+t)z_3 = s(t-z_2-z_6)\big) - 2sz_2(st - tz_3 + (s+2t)z_6) + (tz_3 + s(z_6-t)).
\end{aligned}$$

Here $\mathcal{B}_2$ is the Baikov polynomial corresponding to the loop consisting of edges 4 and 5 with 6 as the external edge, and $\mathcal{B}_3$ is the Baikov polynomial corresponding to the leftover loop over edges 1, 2, 3 and the new edge which replaced the first loop. The factor $\mathcal{B}_1$ is not a Baikov polynomial, but appears because we replaced the self-energy insertion with a propagator; this is comparable to the power $\lambda_1 + \lambda_2 - d_2$ in the formula for recursively one-loop diagrams as seen in Subsection 3.1.

We now have U hence can calculate α and calculate the intersection product using the iterative method. However, the Feynman diagrams are 6 forms, which means the calculation requires six expansions around poles per tuple of poles. This calculation is rather inefficient. Instead, Frellesvig et al. used the cuts of [FP17] to reduce the problem to easier cases.

They first determine the master integrals. They assume each master integral only has weights 0 and 1, with at least one weight 1. This gives $2^6 - 1 = 63$ possible master integrals. For each potential master integral they determine the corresponding cut. Taking the cut, they get another U; then using Morse theory tells them which cuts correspond to a non-zero sector. They find that the non-zero cases are $F(1,1,1,1,1,0)$, $F(1,0,1,1,1,0)$ and $F(0,1,0,1,1,0)$. These will therefore be their master integrals. They then use two cuts: the cut $\{1,3,4,5\}$ and the cut $\{2,4,5\}$. Note that the former sends $F(0,1,0,1,1,0)$ to 0 as it has no edges 1 or 3, while the latter sends $F(1,0,1,1,1,0)$ to 0 as it has no edge 2.

Now either cut gives a twisted cohomology for which we can compute the intersection product. The $\{1,3,4,5\}$ cut leaves edges 2 and 6 hence gives 2-forms, the $\{2,4,5\}$ cut leaves edges 1, 3 and 6 hence gives 3-forms. Frellesvig et al. choose bases for each and can finally compute the intersection product.

They then consider a specific element of the family to calculate the intersection products of. They consider $F(1,2,1,1,1,1)$ and write in terms of the master integrals using coefficients c_1, c_2 and c_3 as in Figure 6. Since the cuts do not change the coefficients, the coefficients can be determined on the cuts. See Figures 7a and 7b for an illustration of the cuts.

Frellesvig et al. then use their bases they chose before to calculate the coefficients, and find

$$\begin{aligned}
c_1 &= \frac{-3(3d-16)(3d-14)(2s+t)}{2(d-6)\,st^3},\\
c_2 &= \frac{-3(3d-16)(3d-14)(3d-10)(2ds-10s-t)}{4(d-6)(d-5)(d-4)\,s^2t^3},\\
c_3 &= \frac{3(3d-16)(3d-14)(3d-10)(3d-8)}{2(d-6)^2(d-4)\,st^4}.
\end{aligned}$$

They claim the results agree with IBP decomposition. △

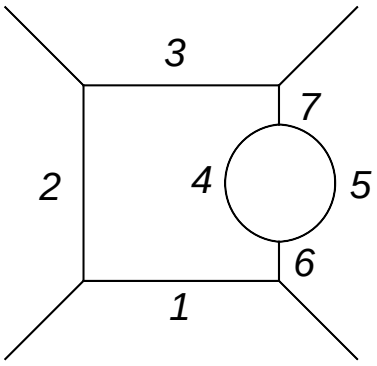


Figure 5: The massless box with self-energy considered in Example 5.2.6.

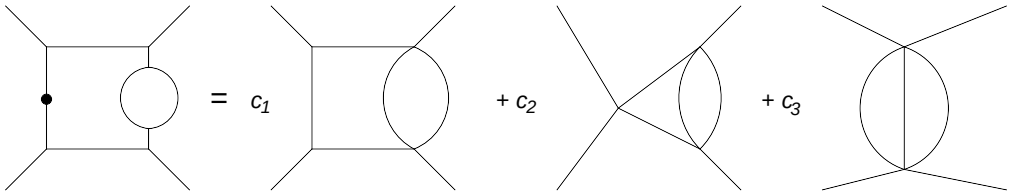
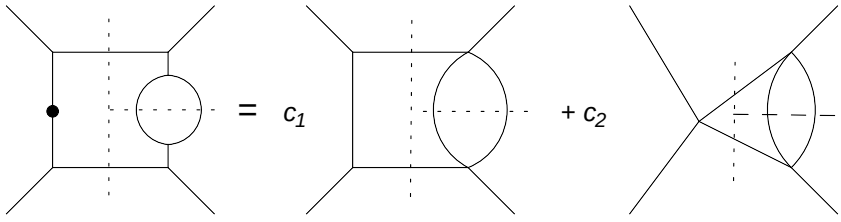
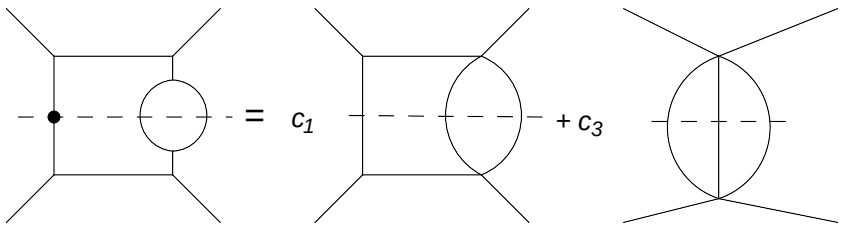


Figure 6: The specific Feynman diagram with weights $(1, 2, 1, 1, 1, 1)$ considered in Example 5.2.6.



(a) The $\{1, 3, 4, 5\}$ cut.



(b) The $\{2, 4, 5\}$ cut.

Figure 7: The cuts used in Example 5.2.6.

6 Discussion

We summarise observations that have been made during this thesis project and add the personal view of the author.

6.1 Observations and Thoughts

We have seen the twisted cohomology and the intersection product. The twisted cohomology is nearly trivial to compute because of Corollary 4.4.4, so that gives us no trouble. However, the intersection product is quite hard to evaluate. For $\dim_{\mathbb{C}} M = 1$ the calculation merely requires that one calculates M and α, then solves $\alpha(z) = 0$, chooses a basis of the cohomology and the Poincaré dual, determines the local primitives of the dual basis to sufficiently high order, and calculates the relevant residue. Important here is that as the integrand increases in complexity, the more orders of the local primitive are required. In particular, in Example 5.2.3 we see that calculating the intersection product

$$\langle z^k \, \mathrm{d}z, \mathrm{d}z \rangle$$

requires the local primitive of $\mathrm{d}z$ at ∞ to be calculated to order $1/z^{k+1}$. This shows that we can generally expect the calculation for the intersection product to increase linearly with the complexity of the integrand of the relevant Feynman integral. In the same Example, calculating I_2 or I_3 in terms of I_0 is simple, but calculating I_{30} is already very slow.

However, the different intersection products use the same dual basis, hence the local primitives can be reused. In particular, if the local primitive of $\mathrm{d}z$ is calculated to order $1/z^{k+1}$ then we can use that primitive to calculate not only $\langle z^k \, \mathrm{d}z, \mathrm{d}z \rangle$ but any $\langle z^\ell \, \mathrm{d}z, \mathrm{d}z \rangle$ for $0 \leq \ell \leq k$. The latter calculation now only requires taking the residue of a product over each pole and adding the residues; this should be a very quick process.

It is for this reason that the author of this thesis thinks that the intersection product might become a useful technique in the actual calculation of Feynman integrals.

However, the situation quickly becomes worse as $\dim_{\mathbb{C}} M$ increases. Not only is there more work needed to write M as a fibration and determine (dual) bases and their primitives, but even if we know all relevant primitives we still have to calculate $\dim_{\mathbb{C}} M$ residues for each intersection product. Each residue has to be taken over a number of poles (generally more than 1), hence the total amount of combinations of residues can grow exponentially (in the worst case). This may cause memory issues for larger Feynman integrals. Since the number $\dim_{\mathbb{C}} M$ corresponds to the number of edges in Feynman Parameter or Baikov representation, this number quickly become larger for higher loop orders.

Moreover, we have to calculate the primitives before we can use them. There might be very efficient algorithms to find the local primitives, but the author of this thesis could only find iterative methods. This means that the setup to use the intersection product takes rather long.

The author therefore thinks that the intersection product in its current form is not (yet) suitable for higher loop calculations. The intersection product will have to be combined with the theory of cuts or a similar supplementary technique to become viable.

6.2 Further Research

There are then multiple things which could be considered in further research.

- The influence of the choice of representation: Is the intersection product generally easier to compute in the Feynman Parameter representation than in the (loop-by-loop) Baikov representation? Is there another representation which is even better?
- The influence of the choice of a fibration: For $\dim_{\mathbb{C}} M > 1$ there are multiple possible choices of a fibration. Changing the fibration might give an easier to calculate intersection product. If so, is there any way to determine the best fibration effectively?
- Differential Equations on the fibration: It is well known that the 1-form α is related to the matrices A_x from Subsection 3.3, as for any external parameter x, x-independent ϕ and Feynman integral $F = \int_C U\phi$ we have

$$\partial_x F = \partial_x \int_C U\phi = \int_C \partial_x(U\phi) = \int_C U(\partial_x\phi + \alpha(\partial_x)\ \phi) = \int_C U\ \alpha(\partial_x)\ \phi,$$

hence A_x is just (plus or minus) the contraction $\alpha(\partial_x)$ in terms of the master integrals. The same can then be done at any step in the calculation of the intersection product for higher forms; the Serre fibration $M \to B$ gives a local system $H^k(F_\bullet, i_\bullet^* \mathcal{L}_\alpha)$ over B, which has some generating $\widehat{A}$. This $\widehat{A}$ should similarly give a differential equation. Perhaps this could be used to derive properties of the local system, or maybe the differential equation could even help calculate the intersection product itself.

- Cuts: Intersection theory can be used in combination with cuts. Cuts reduce the twisted cohomology to a smaller twisted cohomology. How do cuts affect the twisted cohomology precisely? Can we mathematically predict which cuts give the best result and can we construct cuts dual to the basis of H^N_α? Are cuts related to the twisted homology?

There are also a few questions which are not directly related to the intersection product itself, but rather to the twisted cohomology. These questions are most likely less useful physically, but can still be studied out of mathematical interest:

- Homotopy groups and the Hurewicz theorem: Assume $\dim_{\mathbb{C}} M = 1$. Is there an equivalent of the Hurewicz theorem for twisted cohomologies where the fundamental group is replaced with the trivial monodromy subgroup

$$\left\{ [\gamma] \in \pi_1(M) \mid \exp \int_\gamma \alpha = 1 \right\} \leqslant \pi_1(M)?$$

  If there is, can we similarly find a theorem for $\dim_{\mathbb{C}} M > 1$ using higher homotopy groups? The latter might require the generalisation of monodromy to spheres; can this be achieved?

  When $\dim_{\mathbb{C}} M = 2$ and $\alpha = \alpha_1 \, \mathrm{d}z_1 + \alpha_2 \, \mathrm{d}z_2$ it might be tempting to define the monodromy using the integral of $\alpha_1 \, \alpha_2 \, \mathrm{d}z_1 \wedge \mathrm{d}z_2$ over elements of $\pi_2(M)$. Is this well-defined, or is it at the very least possible to use this to define the subgroup of $\pi_2(M)$ of spheres with trivial monodromy?

  Would this definition respect the long exact sequence of homotopy groups in the case of a Serre fibration, such that the the sequence descends to a sequence of trivial monodromy subgroups?

  Remark that works by M. Yoshinaga and R. Randell have already partially answered a related question in the context of arrangements of hyperplanes and $(n-1)$–connected topological pairs respectively. However, they use a map

$$\pi_\ell(M) \otimes_{\mathbb{Z}} \mathcal{L}_\alpha \to H_\ell(M, \mathcal{L}_\alpha)$$

  instead. This map does not seem to generalise to e.g. Example 4.1.6 hence is not generally applicable to the twisted cohomology of smooth manifolds. A question is whether a Hurewicz theorem can be created which is applicable even when the Middle Dimension Theorem 4.2.10 does not hold.

- The condition in Theorem 4.2.5: Can we say more about this condition? It is generally true, or are there easy counterexamples? The condition strongly resembles the dd^c–lemma from complex geometry, but it should be much easier to classify the condition because we know more about the map $\alpha\wedge$.

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