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Generalized geometry as a model for heterotic string theory

Master Thesis

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Abstract
In this thesis, we study the relation between heterotic string theory and generalized geometry. First we construct consistent superstring theories with a focus on the heterotic versions. Then we define Courant algebroids and show their basic properties. Then we examine reduction of Courant algebroids and show that heterotic Courant algebroids can be obtained through the reduction of exact Courant algebroids. After that, we introduce geometric structures on Courant algebroids, including generalized metrics, generalized connections and generalized complex structures. Next we examine the low energy limit of heterotic string theory called heterotic supergravity. When compactified on a 6-dimensional internal space, we obtain the Strominger system. Using the geometric structures introduced earlier, we can formulate the Killing spinor equations on a heterotic Courant algebroid. We will then show that the Strominger system is equivalent to the Killing spinor equations, meaning that solutions to the Strominger system are preserved under Courant algebroid isomorphisms. Finally we explore the concept of T-duality. We show that two T-dual torus bundles have isomorphic Courant algebroids. We find the Buscher rules and show how they can be obtained from a canonical transformation.

Contents

1 Introduction

In the beginning of the twentieth century, two major theories of physics emerged. The first is general relativity, which describes gravity and the structure of spacetime on the largest scales. The second is quantum mechanics, which describes physics at the smallest scales. Both theories have shown remarkable results, but they are not compatible with each other. General relativity is a classical field theory like electromagnetism. However, unlike electromagnetism, there is no corresponding quantum field theory. This is because gravity is perturbatively nonrenormalizable.

To unite quantum mechanics and general relativity, we need a theory of quantum gravity. There are several candidates for this. The main candidates are string theory and loop quantum gravity, but there are more alternatives. Here we will focus on string theory.

The simplest version of string theory is bosonic string theory, which describes relativistic strings moving through an ambient space. Bosonic string theory has several shortcomings. The first problem is that it contains a tachyon in the spectrum, which is a state with a negative squared mass. This leads to inconsistencies in the theory. Another problem is that there are no fermions in the theory. Since we have observed many fermions like quarks and electrons in nature, a complete theory of physics should include fermions.

Starting from bosonic string theory, we add fermionic degrees of freedom to the action. After adding fermionic degrees of freedom the theory is reformulated in superspaces, which adds Grassmann variables to the coordinates. Like with bosonic string theory, the fermions need to satisfy certain constraints imposed by setting the conserved supercurrent and the energy momentum tensor to 0.

An analysis of the boundary conditions for the fermions leads to 2 possible options. The first option is to have periodic boundary conditions, which was first studied by Ramond in 1971 [18]. The second option is to have antiperiodic boundary conditions, which was first studied by Neveu and Schwarz in 1971 [16]. These different boundary conditions lead to different generators for the super Virasoro algebra. With the super Virasoro algebra, we quantize the theory. The super Virasoro algebra is also used to determine the critical dimension D and the normal ordering constant a. The theory so far still contains a tachyon. The tachyon is removed by imposing the GSO conditions, first proposed in 1977 by Gliozzi, Scherk and Olive[8].

Adding fermionic degrees of freedom doesn't lead to a unique theory. Instead, there are choices involved, leading to 5 different versions of superstring theory. These versions are called Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$.

The focus of this thesis is on heterotic string theories. The right and left moving fermions decouple in the action and heterotic string theories only impose supersymmetry on the right moving fermions. For the left moving part the heterotic theories introduce 32 fermions without supersymmetry. The internal symmetry group of the left moving part can be either $SO(32)$ or $E_8 \times E_8$.

The five superstring theories are not completely separated. There are dualities relating the different versions called S-duality and T-duality. T-duality relates both type II theories to each other and both heterotic theories to each other. The theories are compactified on a torus and conceptually the radius of the torus gets inverted. In the case of circle bundles we find the Buscher rules. The Buscher rules can also be obtained using canonical transformations, showing that both systems are physically equivalent.

Heterotic string theory has infinitely many states, most of which have very high masses. Even the lowest nonzero mass of a state is too high to measure experimentally. Thus we examine heterotic supergravity; the low energy limit of heterotic string theory. This theory has only massless fields. We formulate the action and compute the equations of motion and the supersymmetry transformations. Then we compactify this theory on a 6-dimensional internal space. In order to keep supersymmetry, the internal space needs to satisfy certain constraints. These constraints were formulated in 1986 by Strominger and are called the Strominger system [21].

The Strominger system is the following set of differential equations

$$
\begin{aligned}
\Lambda_\omega F_h = 0, \qquad & F_h^{0,2} = 0\,, \\
\Lambda_\omega R_\nabla = 0, \qquad & R_\nabla^{0,2} = 0\,, \\
d^*\omega - d^c log(||\Omega||_\omega) = 0\,, & \\
dd^c\omega - \alpha(Tr(R\wedge R) - Tr(F_h\wedge F_h)) = 0\,. &
\end{aligned}
$$

The first two lines describe Hermite-Yang-Mills connections. The third equation is called the dilatino equation and describes a conformally balanced metric. The last equation is called the Bianchi identity, and it couples the structure of the first 3 equations into one equation.

The mathematical language to describe this system is differential geometry. Differential geometry is the study of smooth manifolds and geometric structures on them. Examples of geometric structures include a Riemannian metric, a symplectic structure, a complex structure and a connection.

The Riemannian metric is the central object of study in Riemannian geometry, the natural language of studying general relativity. Symplectic geometry is useful for the study of Hamiltonian mechanics. As such, these structures naturally provide a language for many areas of theoretical physics. While these and other geometric structures play a role in the Strominger system, we will consider a different kind of geometric structure here.

In this thesis, we explore how a subfield of differential geometry known as generalized geometry can help us better understand the Strominger system. This area of study focuses on a mathematical object known as a Courant algebroid. The simplest example of a Courant algebroid over a manifold M is $TM \oplus T^*M$.

To better understand the bundle $TM \oplus T^*M$, we first consider the double of a vector space V which is given by $V \oplus V^*$. This vectorspace has a natural bilinear pairing

$$
\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\beta(X)+\alpha(Y))\,.
$$

Then we take this a step further and consider for a manifold M the double of the tangent bundle, given by $TM \oplus T^*M$. Pointwise this is the double of the tangent space and there is a natural projection ρ to the tangent bundle itself. Together with a Dorfman bracket given by

$$
[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X(\beta) - \iota_Y d\alpha\,,
$$

this is the first example of a Courant algebroid. The Dorfman bracket is analogous to the Lie bracket of vector fields. This is an example of an exact Courant algebroid. If a Lie group G acts on the manifold M, we can perform reduction to get a heterotic Courant algebroid. This is the type of Courant algebroid that is the most interesting for our purposes.

In generalized geometry we study geometric structures on these Courant algebroids. The first structure we introduce is the generalized metric, which is a bundle automorphism of the Courant algebroid. This bundle automorphism turns the pairing into a positive definite bilinear map. For both exact and heterotic Courant algebroids the generalized metric can be characterized in terms of more well-known objects. Next we consider generalized connections which behave similar to normal connections. We are particularly interested in generalized connections which are torsion free and compatible with the generalized metric. Unlike in Riemannian geometry, such a generalized connection is no longer unique. The third structure we study is a generalized complex structure. This is a simultaneous generalization of complex en symplectic strutures. We then combine multiple structures to get generalized Kähler and generalized SKT structures.

On a heterotic Courant algebroid we construct a family of torsion free generalized connections which are compatible with a given generalized metric. This generalized connection gives rise to a Dirac operator on spinors. With these Dirac operators, we formulate the Killing spinor equations:

$$
\begin{aligned}
D_+^\varphi \eta &= 0\,, \\
\not{D}_-^\varphi \eta &= 0\,.
\end{aligned}
$$

Solutions of these equations are preserved under Courant algebroid isomorphisms. We will show that these Killing spinor equations are equivalent to the Strominger system, meaning that solutions of one system translate to solutions of the other system. This means that the Strominger system is a natural system of equations on a heterotic Courant algebroid.
This thesis is organised as follows:
Section 2 gives an introduction to superstring theory and heterotic superstring theory. Section 3 introduces Courant algebroids. In section 4 we study several geometric structures on Courant algebroids. In section 5 we formulate the Strominger system and the Killing spinor equations and prove their equivalence. In section 6 we study T-duality in relation to generalized geometry and as a canonical transformation.

2 Superstring Theory

2.1 Introduction

This section follows the book *Superstring Theory: Volume 1, Introduction*, by Green, Schwarz and Witten [12]. Recall the Polyakov action for bosonic string theory:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\,.$$

In two dimensions every (pseudo-)metric is conformally flat. This means that locally we can write $h_{\alpha\beta} = e^{2f}\eta_{\alpha\beta}$ for a particular choice of coordinates and a particular smooth function f. This choice of coordinates is called conformal gauge. In conformal gauge this action simplifies to:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\partial_\alpha X_\mu\partial^\alpha X^\mu\,.$$

We modify this action by including D fermions $\psi^\mu_A(\sigma,\tau)$ transforming in the vector representation of the Lorentz group $SO(D-1,1)$. Thus the new action becomes

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma(\partial_\alpha X_\mu\partial^\alpha X^\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu)\,,$$

where ρ^α is a two dimensional Dirac matrix given by

$$\rho^0 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\rho^1 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

These Dirac matrices satisfy the Clifford algebra $\{\rho^\alpha,\rho^\beta\} = -2\eta^{\alpha\beta}$. The components of ψ^μ are given by

$$\psi^\mu = \begin{pmatrix}\psi^\mu_-\\ \psi^\mu_+\end{pmatrix}.$$

Since the ρ^α are imaginary, the operator $i\rho^\alpha\partial_\alpha$ is real. Hence it makes sense to demand that the components of ψ^μ are real as well. In this basis of Dirac matrices charge conjugation corresponds to complex conjugation. This means that the requirement for the spinors to be real is equivalent to requiring the spinors to be invariant under charge conjugation. These two-component real spinors are called Majorana spinors.

The equal τ commutation relations for the bosonic coordinates are given by $[X^\mu(\sigma,\tau),\dot{X}^\nu(\sigma',\tau)] = i\pi\eta^{\mu\nu}\delta(\sigma-\sigma')$. Similarly, the equal τ commutation relation for the fermions is given by $[\psi^\mu_A(\sigma,\tau),\psi^\nu_B(\sigma',\tau)] = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma')$. Since the Lorentz pseudometric $\eta_{\mu\nu}$ is not positive definite, the oscillators of X^0 will have negative norm squared. These states are called ghosts. In the critical dimension for bosonic string theory, $D = 26$, the Virasoro algebra can be used to decouple the ghosts from the theory. For the fermions the same problem occurs with ψ^0. In order to get a theory without ghosts the fermions also admit a certain symmetry. This symmetry is superconformal symmetry.

Let ε be a constant anticommuting spinor. The action S is invariant under the infinitesimal transformations

$$\delta X^\mu = \bar{\varepsilon}\psi^\mu\,,$$
$$\delta\psi^\mu = -i\rho^\alpha\partial_\alpha X^\mu\varepsilon\,.$$

These transformations are known as the supersymmetry transformations.

Lemma 2.1.1. *Let X^μ and ψ^μ be the bosonic and fermionic fields and assume that ψ^μ is on-shell. Then the commutator of two supersymmetry transformations with different values of ε will generate a spacetime translation in the fields X^μ and ψ^μ.*

Proof. The parameter for the translation is given by

$$a^\alpha = 2i\bar{\varepsilon}_1\rho^\alpha\varepsilon_2 \,.$$

First we consider the bosonic field X^μ:

$$\begin{aligned}
[\delta_1, \delta_2]X^\mu &= \delta_1\delta_2 X^\mu - \delta_2\delta_1 X^\mu \\
&= \delta_1(\bar{\varepsilon}_2\psi^\mu) - \delta_2(\bar{\varepsilon}_1\psi^\mu) \\
&= -i\bar{\varepsilon}_2\rho^\alpha\partial_\alpha X^\mu\varepsilon_1 + i\bar{\varepsilon}_1\rho^\alpha\partial_\alpha X^\mu\varepsilon_2 \\
&= (-i\bar{\varepsilon}_2\rho^\alpha\varepsilon_1 + i\bar{\varepsilon}_1\rho^\alpha\varepsilon_2)\partial_\alpha X^\mu \\
&= (2i\bar{\varepsilon}_1\rho^\alpha\varepsilon_2)\partial_\alpha X^\mu \\
&= a^\alpha\partial_\alpha X^\mu \,.
\end{aligned}$$

Now consider the fermionic field ψ^μ. Since ψ^μ is on-shell it satisfies the Dirac equation $\rho^\alpha\partial_\alpha\psi^\mu = 0$. In components this equation is given by:

$$\begin{aligned}
\partial_0\psi_+^\mu &= \partial_1\psi_+^\mu \\
\partial_0\psi_-^\mu &= -\partial_1\psi_-^\mu \,.
\end{aligned}$$

This allows us to work out an expression for $[\delta_1, \delta_2]\psi^\mu$:

$$\begin{aligned}
[\delta_1, \delta_2]\psi^\mu &= \delta_1\delta_2\psi^\mu - \delta_2\delta_1\psi^\mu \\
&= -i\delta_1\rho^\alpha\partial_\alpha X^\mu\varepsilon_2 + i\delta_2\rho^\alpha\partial_\alpha X^\mu\varepsilon_1 \\
&= i\partial_\alpha(\bar{\varepsilon}_2\psi^\mu\rho^\alpha\varepsilon_1 - \bar{\varepsilon}_1\psi^\mu\rho^\alpha\varepsilon_2) \\
&= (\varepsilon_{2-}\partial_0\psi_+^\mu - \varepsilon_{2+}\partial_0\psi_-^\mu)\begin{pmatrix} -i\varepsilon_{1+} \\ i\varepsilon_{1-} \end{pmatrix} + (\varepsilon_{1+}\partial_0\psi_-^\mu - \varepsilon_{1-}\partial_0\psi_+^\mu)\begin{pmatrix} -i\varepsilon_{2+} \\ i\varepsilon_{2-} \end{pmatrix} \\
&\quad + (\varepsilon_{2-}\partial_1\psi_+^\mu - \varepsilon_{2+}\partial_1\psi_-^\mu)\begin{pmatrix} i\varepsilon_{1+} \\ i\varepsilon_{1-} \end{pmatrix} + (\varepsilon_{1+}\partial_1\psi_-^\mu - \varepsilon_{1-}\partial_1\psi_+^\mu)\begin{pmatrix} i\varepsilon_{2+} \\ i\varepsilon_{2-} \end{pmatrix} \\
&= 2i\begin{pmatrix} \varepsilon_{1+}\varepsilon_{2+}\partial_0\psi_-^\mu - \varepsilon_{1+}\varepsilon_{2+}\partial_1\psi_-^\mu \\ \varepsilon_{1-}\varepsilon_{2-}\partial_0\psi_+^\mu + \varepsilon_{1-}\varepsilon_{2-}\partial_1\psi_+^\mu \end{pmatrix} \\
&= 2i\begin{pmatrix} (\varepsilon_{1+}\varepsilon_{2+} + \varepsilon_{1-}\varepsilon_{2-})\partial_0\psi_-^\mu + (\varepsilon_{1-}\varepsilon_{2-} - \varepsilon_{1+}\varepsilon_{2+})\partial_1\psi_-^\mu \\ (\varepsilon_{1+}\varepsilon_{2+} + \varepsilon_{1-}\varepsilon_{2-})\partial_0\psi_+^\mu + (\varepsilon_{1-}\varepsilon_{2-} - \varepsilon_{1+}\varepsilon_{2+})\partial_1\psi_+^\mu \end{pmatrix} \\
&= a^\alpha\partial_\alpha\psi^\mu \,.
\end{aligned}$$

□

Using the Noether method we can find the conserved supercurrent and the energy momentum tensor. In the supersymmetry transformations ε is a constant. If ε is not a constant, the action is not invariant. The variation of the action will then be of the form

$$\delta S = \frac{2}{\pi}\int d^2\sigma(\partial_\alpha\bar{\varepsilon})J^\alpha \,.$$

This means that J^α is the conserved supercurrent satisfying $\partial_\alpha J^\alpha = 0$. This method will give

$$J_\alpha = \frac{1}{2}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu \,.$$

Using this method for the translations where $\delta\sigma^\alpha$ is a constant gives the energy momentum tensor

$$T_{\alpha\beta} = \partial_\alpha X_\mu\partial_\beta X^\mu + \frac{i}{4}\bar{\psi}^\mu(\rho_\alpha\partial_\beta + \rho_\beta\partial_\alpha)\psi_\mu \,.$$

Since the energy momentum tensor is traceless, the off-diagonal entries in lightcone coordinates vanish. Because of the identity $\rho^\alpha\rho^\beta\rho_\alpha = 0$ for Dirac matrices in two dimensions, the supercurrent also satisfies the condition $\rho^\alpha J_\alpha = 0$.

2.2 Superspace

Bosonic string theory lives on a worldsheet Σ. Superstring theory introduces a new superspace called $\bar{\Sigma}$. Besides the wordsheet coordinates, the superspace also contains two Grassmann variables θ^A forming a two-component Majorana spinor. A general function Y^μ in superspace is of the form

$$Y^\mu(\sigma^\alpha, \theta^A) = X^\mu(\sigma^\alpha) + \bar{\theta}\psi^\mu(\sigma^\alpha) + \bar{\psi}^\mu(\sigma^\alpha)\theta + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma^\alpha)\,.$$

This is the most general power series expansion in powers of θ^A. Higher orders in θ^A vanish since it is an anticommuting variable. Such a function is called a superfield. In superspace, supersymmetry is generated by the generator

$$Q_A = \frac{\partial}{\partial\bar{\theta}^A} + i(\rho^\alpha\theta)_A\partial_\alpha\,.$$

Instead of this generator, it is often convenient to introduce an anticommutating ε_A and work with $\bar{\varepsilon}Q$ instead. This will then generate the following transformations of the superspace coordinates:

$$\begin{aligned}
\delta\theta^A &= [\bar{\varepsilon}Q, \theta^A] = \varepsilon^A\,,\\
\delta\sigma^\alpha &= [\bar{\varepsilon}Q, \sigma^\alpha] = i\bar{\varepsilon}\rho^\alpha\theta\,.
\end{aligned}$$

On the coordinates of superspace the commutator of these transformations is given by $[\delta_1, \delta_2]Y^\mu = -(2i\bar{\varepsilon}_1\rho^\alpha\varepsilon_2)\partial_\alpha Y^\mu = -a^\alpha\partial_\alpha Y^\mu$. Using the two dimensional Fierz identity

$$\theta_A\bar{\theta}_B = -\frac{1}{2}\delta_{AB}\bar{\theta}_C\theta_C\,,$$

we find that the transformation $\delta Y^\mu = \bar{\varepsilon}QY^\mu$ leads to

$$\begin{aligned}
\delta X^\mu &= \bar{\varepsilon}\psi^\mu\,,\\
\delta\psi^\mu &= -i\rho^\alpha\varepsilon\partial_\alpha X^\mu + B^\mu\varepsilon\,,\\
\delta B^\mu &= i\bar{\varepsilon}\rho^\alpha\partial_\alpha\psi^\mu\,.
\end{aligned}$$

In the case of $B^\mu = \rho^\alpha\partial_\alpha\psi^\mu = 0$, this reduces to the previous supersymmetry transformations. In order to write interesting Lagrangians, we need a derivative operator which is invariant under supersymmetry. The operator which satisfies this is given by

$$D_A = \frac{\partial}{\partial\bar{\theta}^A} - i(\rho^\alpha\theta)_A\partial_\alpha\,.$$

This operator is called the superspace covariant derivative. It anticommutes with Q so $\{D_A, Q_B\} = 0$. This means that covariant derivatives of superfields transform in the same way as the superfields themselves. The superspace covariant derivative has self anti-commutation relation

$$\{D_A, \bar{D}_B\} = 2i(\rho^\alpha)_{AB}\partial_\alpha\,.$$

Like the derivative we also need an integral to write down a theory. In superspace this integral is given by

$$\int d^2\sigma d^2\theta\,.$$

where $d^2\theta$ is the Berezin integral for Grassmann variables. An important property of this integral is that it is invariant under supersymmetry transformations, meaning that for any superfield Y with action

$$S = \int d^2\sigma d^2\theta Y\,.$$

we get

$$\delta S = \int d^2\sigma d^2\theta \bar{\varepsilon} Q Y = 0\,.$$

Note that the B^μ field is the only contributor to the Berezin integral. Since this field transforms as a total derivative under the supersymmetry transformation, the integral over a closed manifold will be 0.
Using superspace formalism we can build a new action for a string in D dimensions. It is given by

$$S = \frac{i}{4\pi} \int d^2\sigma d^2\theta \bar{D} Y^\mu D Y_\mu\,.$$

The covariant derivatives are

$$DY^\mu = \psi^\mu + \theta B^\mu - i\rho^\alpha \theta \partial_\alpha X^\mu + \frac{i}{2}\bar{\theta}\theta\rho^\alpha\partial_\alpha\psi^\mu\,,$$
$$\bar{D}Y^\mu = \psi^\mu + B^\mu\bar{\theta} + i\partial_\alpha X^\mu\bar{\theta}\rho^\alpha - \frac{i}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha\,.$$

Since the Berezin integral only keeps terms proportional to $\bar{\theta}\theta$, we need to compute the coefficient in front of $\bar{\theta}\theta$ in the expansion of $\bar{D}Y^\mu DY_\mu$. This coefficient is given by

$$(-\partial_\alpha X_\mu \partial^\alpha X^\mu + i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu + B^\mu B_\mu)\bar{\theta}\theta\,.$$

Since $\bar{\theta}\theta = -2i\theta^1\theta^2$, the Berezin integral picks up another factor of $-2i$. Hence after the Berezin integral is performed, the action reduces to

$$S = \frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - B_\mu B^\mu)\,.$$

The equation of motion for B^μ is $B^\mu = 0$, meaning that the new field drops out of the theory and we recover the previous action. This shows how the previous theory can be obtained from the most general superfield, despite the lack of B^μ-term in the action.

2.3 Constraints

The commutator of two supersymmetry transformations is a translation. In bosonic string theory the translations are generated by L_0 and $\tilde{L}_0$. Like L_0 and $\tilde{L}_0$ we need to extend Q_A to an infinite component supersymmetry. The equations of motion for the fermion fields are $\rho^\alpha\partial_\alpha\psi^\mu = 0$. In components, this is given by

$$\left(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\right)\psi^\mu_- = 0\,,$$
$$\left(\frac{\partial}{\partial\sigma} - \frac{\partial}{\partial\tau}\right)\psi^\mu_+ = 0\,.$$

Writing $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, the fermionic part of the action can be written as

$$S_F = \frac{i}{\pi}\int d^2\sigma(\psi^\mu_-\partial_+\psi_{\mu-} + \psi^\mu_+\partial_-\psi_{\mu+})\,.$$

This means that the positive and negative chirality components of the spinor are decoupled. The equations of motion for the fermionic components are similar to the equations of motion of the bosons. We have $\partial_+\psi^\mu_- = \partial_+(\partial_- X^\mu) = 0$ and the same for $+$ and $-$ reversed. Supersymmetry is the symmetry between ψ^μ_- and $\partial_- X^\mu$ or between ψ^μ_+ and $\partial_+ X^\mu$. This decoupling can be used to simplify expressions for the supersymmetry current and the energy momentum tensor. The conserved supercurrent has

4 component. In lightcone coordinates these are written as J_{++}, J_{+-}, J_{-+} and J_{--}. Here the first index denotes the lightcone coordinate and the second index denotes the chirality. The components J_{+-} and J_{-+} vanish, so we will write J_+ and J_- for the nonvanishing components. These are given by

$$J_+ = \psi_+^\mu \partial_+ X_\mu \,,$$
$$J_- = \psi_-^\mu \partial_- X_\mu \,.$$

These currents satisfy $\partial_- J_+ = \partial_+ J_- = 0$. We want to know what algebra these J_+ and J_- generate. Recall that the nonvanishing equal τ (anti)commutation relations of $\psi_\pm^\mu$ and $\partial_\pm X^\mu$ are given by

$$\{\psi_\pm^\mu(\sigma), \psi_\pm^\nu(\sigma')\} = \pi\delta(\sigma - \sigma')\eta^{\mu\nu} \,,$$
$$[\partial_\pm X^\mu(\sigma), \partial_\pm X^\nu(\sigma')] = \pm\frac{i\pi}{2}\delta'(\sigma - \sigma')\eta^{\mu\nu} \,.$$

With this, we find the equal τ anticommutation relations for J_+ and J_-:

$$\{J_+(\sigma), J_+(\sigma')\} = \pi\delta(\sigma - \sigma')T_{++}(\sigma) \,,$$
$$\{J_-(\sigma), J_-(\sigma')\} = \pi\delta(\sigma - \sigma')T_{--}(\sigma) \,,$$
$$\{J_+(\sigma), J_-(\sigma')\} = 0 \,.$$

In lightcone coordinates, the energy momentum tensor is given by

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{\mu+} \,,$$
$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{\mu-} \,.$$

In bosonic string theory the constraints $T_{++} = T_{--} = 0$ solved the problem of the timelike component X^0. We may try a similar approach and demand that $T_{++} = T_{--} = 0$ for string theory. Since the energy momentum tensor appears in the anticommutation relations of J_+ and J_-, it is natural to demand $J_+ = J_- = 0$ as well. The derivation of the constraints is a lot more rigorous than this simple guess for superstring theory. A more rigorous approach for superstring theory confirms this guess.

2.4 Boundary conditions

We start by analyzing the possible boundary conditions in the unconstrained theory. For the bosonic coordinates X^μ, the boundary conditions from bosonic string theory still apply. The diffferent boundary conditions give rise to open and closed strings. For the fermionic degrees of freedom a surface term arises in the variation of the Lagrangian. This surface term vanishes when $\psi_+\delta\psi_- - \psi_-\delta\psi_+ = 0$ at the boundary. Here the index is supressed, but this equation should hold for each μ independently. This constraint is satisfied by setting $\psi_+^\mu = \pm\psi_-^\mu$ at each end of the string. Since the overall signs of ψ_- and ψ_+ are not physical, we can set $\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau)$ without loss of generality. The sign at the other end now becomes physically meaningful. The Ramond boundary conditions impose $\psi_+^\mu(\pi,\tau) = \psi_-^\mu(\pi,\tau)$. These conditions lead to the following mode expansion:

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \,,$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \,.$$

The Neveu-Schwarz boundary conditions impose $\psi_+^\mu(\pi,\tau) = -\psi_-^\mu(\pi,\tau)$. They lead to the following mode expansions:

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau-\sigma)} \,,$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau+\sigma)} .$$

The strings with Ramond boundary conditions turn out to describe spacetime fermions whereas the strings with Neveu-Schwarz boundary conditions describe spacetime bosons. For closed strings the boundary conditions are periodicity and antiperiodicity for the separate components. The mode expansions for this are given by

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} ,$$

or

$$\psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir(\tau-\sigma)} ,$$

and

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} ,$$

or

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} .$$

In super string theory the Virasoro algebra gets expanded to the super-Virasoro algebra. Like in the bosonic case, it contains the L_m modes which are given by

$$L_m = \frac{1}{\pi}\int_0^\pi d\sigma(e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}) = \frac{1}{\pi}\int_{-\pi}^\pi d\sigma e^{im\sigma}T_{++} .$$

Depending on the boundary conditions for the fermionic degrees of freedom we define a different set of generators. In the case of Ramond boundary conditions, we define for each integer $m \in \mathbb{Z}$ the generator

$$F_m = \frac{\sqrt{2}}{\pi}\int_0^\pi (e^{im\sigma}J_+ + e^{-im\sigma}J_-) = \frac{\sqrt{2}}{\pi}\int_{-\pi}^\pi d\sigma e^{im\sigma}J_+ .$$

In the case of Neveu-Schwarz boundary conditions, we define for each half integer $r \in \mathbb{Z} + \frac{1}{2}$ the generator

$$G_r = \frac{\sqrt{2}}{\pi}\int_0^\pi (e^{ir\sigma}J_+ + e^{-ir\sigma}J_-) = \frac{\sqrt{2}}{\pi}\int_{-\pi}^\pi d\sigma e^{ir\sigma}J_+ .$$

For closed strings there are two sets of generators of the super-Virasoro algebra. One set contains the mode of T_{++} and J_+ and the other contains the modes of T_{--} and J_-. Classically all modes should vanish, but in the quantum theory this is more subtle. This will become clear in the quantization of the super string.

2.5 Covariant quantization

The equal τ commutation relations for the bosonic coordinates is given by

$$[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\pi\delta(\sigma,\sigma')\eta^{\mu\nu} .$$

From this, we easily calculate the commutation relations for the coefficients of the mode expansion:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} .$$

These relations remain the same in super string theory. For the closed string the same relations hold for the $\tilde{\alpha}_m^\mu$. For the fermionic degrees of freedom we have the equal τ anticommutation relation

$$\{\psi_A^\mu(\sigma,\tau), \psi_B^\nu(\sigma',\tau)\} = \pi\delta(\sigma,\sigma')\delta_{AB}\eta^{\mu\nu} .$$

From this we deduce that the coefficients b_r^μ and d_n^μ of the mode expansion satisfy

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}\,,$$

and

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n}\,.$$

The mass shell condition for the Virasoro constraints states that

$$\alpha' M^2 = N + constant\,.$$

The constant comes from normal ordering. The number operator N decomposes as $N = N^\alpha + N^d$ or $N = N^\alpha + N^b$, where N^α, N^d and N^b are given by

$$N^\alpha = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n\,,$$

$$N^d = \sum_{n=1}^{\infty} n d_{-n}\cdot d_n\,,$$

$$N^b = \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}\cdot b_r\,.$$

The ground state is the state which is annihilated by all α_n^μ and d_n^μ with positive n or by all α_n^μ and b_r^μ with positive n and r. Acting with α_{-n}^μ or d_{-n}^μ on a state raises the value of $\alpha' M^2$ by n units, while acting with b_{-r}^μ will raise the value by r units. The generalized Virasoro algebra generators decompose similarly. For the Ramond boundary conditions we have $L_m = L_m^{(\alpha)} + L_m^{(d)}$ and for the Neveu Schwarz boundary conditions we have $L_m = L_m^{(\alpha)} + L_m^{(b)}$. Here

$$L_m^{(\alpha)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_{m+n}:\,,$$

like in bosonic string theory, and

$$L_m^{(d)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} (n+\frac{1}{2}m) :d_{-n}\cdot d_{m+n}:\,,$$
$$L_m^{(b)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} (r+\frac{1}{2}m) :b_{-r}\cdot b_{m+r}:\,.$$

The normal ordering is only needed in the case $n = 0$. The fermionic generators are given by

$$F_m = \sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot d_{m+n}:\,,$$

$$G_r = \sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot b_{r+n}:\,.$$

The super-Virasoro algebra in the bosonic (NS) sector is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{8}D(m^3-m)\delta_{m+n}\,,$$
$$[L_m, G_r] = (\frac{1}{2}m - r)G_{m+r}\,,$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2}D(r^2-\frac{1}{4})\delta_{r+s}\,.$$

In the fermionic (R) sector, it is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{8}Dm^3\delta_{m+n}\,,$$
$$[L_m, F_n] = (\frac{1}{2}m - n)F_{m+n}\,,$$
$$\{F_m, F_n\} = 2L_{m+n} + \frac{1}{2}Dm^2\delta_{m+n}\,.$$

In the R sector, the entire algebra is generated by F_0, L_{-1} and L_1. In the NS sector, the linear span of L_{-1}, L_0, L_1, $G_{-\frac{1}{2}}$ and $G_{\frac{1}{2}}$ forms a subalgebra known as $OS_P(1|2)$. Now we impose the constraints on the quantum theory by requiring the physical states to be annihilated by the positive-frequency components of the super-Virasoro algebra. In particular, a physical bosonic state $|\phi\rangle$ should satisfy

$$L_m|\phi\rangle = 0\,,$$
$$G_r|\phi\rangle = 0\,,$$
$$(L_0 - a)|\phi\rangle = 0\,.$$

for all $n, r > 0$ and some constant a which will be determined later. The conditions for all $n, r > 0$ all follow from the cases $r = \frac{1}{2}$ and $r = \frac{3}{2}$. This means that it is sufficient for a state $|\psi\rangle$ to satisfy $(L_0 - a)|\phi\rangle = G_{\frac{1}{2}}|\phi\rangle = G_{\frac{3}{2}}|\phi\rangle = 0$. Now we want to find the critical values for a and D. In bosonic string theory, these parameters represented the boundary between ghosts and no ghosts, which resulted in a variety of extra zero-norm states. We can study this by looking at the ground state $|0;k\rangle$. This state is on-shell for $\frac{k^2}{2} = a$. The excited state $|\phi\rangle = G_{-\frac{1}{2}}|0;k\rangle$ is on-shell for $\frac{k^2}{2} = a - \frac{1}{2}$. For $a = \frac{1}{2}$ this state satisfies $G_{\frac{1}{2}}|\phi\rangle = 0$ and the state has zero norm. For higher values of a the state will have a negative norm. Because $a = \frac{1}{2}$ sits at this boundary, it is the critical value.
With $a = \frac{1}{2}$ there is a family of zero norm states given by $G_{-\frac{1}{2}}|\tilde{\phi}\rangle$, where $|\tilde{\phi}\rangle$ in annihilated by $G_{\frac{1}{2}}$, $G_{\frac{3}{2}}$ and L_0. For the critical dimension we follow a similar approach. We construct a family of zero norm states of the form

$$|\phi\rangle = (G_{-\frac{3}{2}} + \lambda G_{-\frac{1}{2}}L_{-1})|\tilde{\phi}\rangle\,,$$

where $|\tilde{\phi}\rangle$ satisfies

$$G_{\frac{1}{2}}|\tilde{\phi}\rangle = G_{\frac{3}{2}}|\tilde{\phi}\rangle = (L_0+1)|\tilde{\phi}\rangle = 0\,.$$

Using the commutation and anticommutation relations we find

$$G_{\frac{1}{2}}|\phi\rangle = (2-\lambda)L_{-1}|\tilde{\phi}\rangle\,,$$
$$G_{\frac{3}{2}}|\phi\rangle = (D - 2 - 4\lambda)|\tilde{\phi}\rangle\,.$$

This means that $|\phi\rangle$ is physical for $\lambda = 2$ and $D = 10$, since these are the only values for which $|\psi\rangle$ is annihilated by $G_{\frac{1}{2}}$ and $G_{\frac{3}{2}}$ simultaneously. Thus we conclude that $D = 10$ is the critical dimension.
In the fermionic sector any physical state has to satisfy $F_n|\psi\rangle = L_n|\psi\rangle = 0$ for any $n > 0$. The zero mode also gives the condition $(F_0 - \mu)|\psi\rangle = 0$. From this we find that

$$(F_0+\mu)(F_0-\mu)|\psi\rangle = (F_0^2 - \mu^2)|\psi\rangle = (L_0 - \mu^2)|\psi\rangle = 0$$

The arbitrary constant μ does not arise from a normal ordering ambiguity when passing from the classical to the quantum theory. Since F_0 is anticommuting, a non-zero value of μ would cause the operator $(F_0 - \mu)$ to be neither commuting nor anticommuting. The positive frequency super-Virasoro algebra is generated by L_0 and F_1. States of the form $|\psi\rangle = F_0|\tilde{\psi}\rangle$ with $L_0|\tilde{\psi}\rangle = F_1|\tilde{\psi}\rangle = 0$ have zero norm, but they are only on-shell for $\mu = 0$. This is the first family of zero-norm states. The second family is given by $|\psi\rangle = F_0F_{-1}|\tilde{\psi}\rangle$ where $F_1|\tilde{\psi}\rangle = (L_0+1)|\tilde{\psi}\rangle = 0$. These states satisfy $F_0|\psi\rangle = 0$. In order for this to be a physical state, it also has to be annihilated by L_1. Using the (anti-)commutation relations, we find $L_1|\psi\rangle = (\frac{1}{4}D - \frac{5}{2})|\tilde{\psi}\rangle$, from which we conclude that $D = 10$ is the critical dimension for the fermionic sector as well.

2.6 Vertex Operators

We now turn our attention to vertex operators. First we consider the boson emission from a bosonic state. A physical vertex operator needs to have conformal dimension $J = 1$ in order to map physical states to physical states. For super strings this is still a necessary but no longer a sufficient condition. The vertex operator must have the right commutation relations with the G_r's as well.
Let $V = V(\tau = 0)$ be a candidate for a physical vertex operator, and assume that there is another vertex operator W such that for each $r \in \mathbb{Z} + \frac{1}{2}$ we get

$$V(0) = [G_r, W(0)] \,.$$

This works if W is a bosonic vertex operator. In the case of a fermionic vertex operator, the commutator should be replaced by an anticommutator.
Since $G_r^2 = L_{2r}$, we find that $\{G_r, V(0)\} = [L_{2r}, W(0)]$. Now $V(\tau)$ is given by

$$V(\tau) = e^{iL_0\tau} V(0) e^{-iL_0\tau} \,.$$

And by definition of conformal dimension J we get

$$[L_m, V(\tau)] = e^{im\tau}(-i\frac{d}{d\tau} + mJ)V(\tau) \,.$$

From this we deduce that V has conformal dimension $J = 1$ if and only if W has conformal dimension $J = \frac{1}{2}$. As a first example, we consider the operator

$$W(0) =: e^{ik\cdot X(0)} : ,$$

which has conformal dimension $J = \frac{1}{2}$ for $k^2 = 1$. This is the mass-shell condition for the bosonic tachyon. The corresponding operator is given by

$$V(0) = [G_r, W(0)] = k \cdot \psi(0) : e^{ik\cdot X(0)} : ,$$

at $\tau = 0$ and

$$V(\tau) = k \cdot \psi(\tau) : e^{ik\cdot X(\tau)} : ,$$

for general τ. Here $\psi^\mu(\tau)$ is given by

$$\psi^\mu(\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\tau} \,.$$

This vertex operator has conformal dimension $J = 1$ since both factors have $J = \frac{1}{2}$ and they commute. Thus V is the vertex operator for the emission of the tachyon. The first excited state is the massless vector of polarization ζ^μ and momentum k^μ and is given by $\zeta \cdot b_{-\frac{1}{2}}|0; k\rangle$. The $G_{\frac{1}{2}}$ condition requires $\zeta \cdot k = 0$ for this state to be physical. For the construction of the vertex operator corresponding to this state we consider the operator

$$W_1(0) = \zeta \cdot \psi(0) e^{ik\cdot X(0)} \,,$$

which has conformal dimension $J = \frac{1}{2}$ for $k^2 = 0$. The corresponding vertex operator is given by

$$V_1(0) = \{G_r, W_1(0)\} = (\zeta \cdot \dot{X}(0) - \zeta \cdot \psi(0) k \cdot \psi(0)) e^{ik\cdot X(0)} \,.$$

The vertex operators are seperated into two groups. One group has an even number of ψ excitations and the other group has an odd number of ψ excitations. If W is a bosonic operator then the commutator $[G_r, W]$ gives a fermionic vertex operator V while if W is a fermionic operator then the anti-commutator $\{G_r, W\}$ gives a bosonic vertex operator. String states corresponding to a fermionic vertex operator V correspond to Fock states with an even number of b excitations and are called states of odd G-parity. These states lead to anomalies at the one-loop level, so they will be removed from the spectrum. One of the removed states is the tachyon.

2.7 The GSO conditions

The model we discussed so far is called the RNS-model after Ramond, Neveu and Schwarz. Even for $D = 10$ and $a = \frac{1}{2}$ (NS sector) or $a = 0$ (R sector), this is an inconsistent quantum theory. Additional conditions need to be imposed on the model in order to get a consistent quantum theory. These conditions were proposed by Gliozzi, Scherk and Olive and are known as the GSO conditions.
One reason why the current theory needs further restrictions on the spectrum is that it contains a tachyon. The theory can only have the potential to describe nature if it doesn't predict tachyons.
The second reason is that the theory has anti-commuting operators ψ^μ which map bosons to bosons. While this is not contradictory, it is still unnatural. It is preferred to have only commuting operators which map bosons to bosons. The proposal is to simply discard all states which are obtained by acting with an odd number of ψ^μ-operators on the good states. To do this, we introduce the operator $(-1)^F$, which acts as $(-1)^F X^\mu = X^\mu$, $(-1)^F \psi^\mu = -\psi^\mu$. We give the massless vector positive sign. The GSO projection is obtained by removing the -1-eigenspace of the operator $(-1)^F$.
The third reason for introducing this GSO projection is that the theory in the 10-dimensional ambient space becomes supersymmetric. This means that both the 2-dimensional wordsheet theory and the 10-dimensional spacetime theory become supersymmetric, which results in a more elegant theory.
The massless part of the spectrum contains a vector and a spinor. The massless vector is given by $b_{-\frac{1}{2}}|0;k\rangle$. The massless spinor is given by $|a;k\rangle u^a(k)$, where $u^a(k)$ satisfies the massless Dirac equation and a is a spinor index. Unbroken supersymmetry requires that each masslevel contains a supersymmetric pair. The massless vector has 8 independent transverse degrees of freedom. For even dimensions D, spinors have $2^{\frac{D}{2}}$ complex components. In 10 dimensions, this means 32 complex components. By simultaneously imposing Weyl and Majorana constraints, the number of degrees of freedom can be reduced to a quarter of the original number, which is 16 real components in this case. These degrees of freedom need to satisfy the Dirac equation, which relates half of the degrees of freedom to the other half. Thus both the massless vector and the massless spinor have 8 degrees of freedom which means they can form a supersymmetry pair.
The Majorana condition states that the fermion fields are real. This is the result of a choice of basis in which the intertwiner is real. In general a fermion ψ is a Majorana fermion if it satisfies $\psi = \psi^c$, where ψ^c is the charge conjugate of ψ. In order for this to make sense, the Dirac matrices all have to be real or completely imaginary. A representation where all Dirac matrices are either real or imaginary is called a Majorana representation. We will work with a representation in which the Dirac matrices are purely imaginary. We write Γ^μ for these 32×32 imaginary Dirac matrices.
For even dimensions D there is a matrix analogous to γ_5 in 4 dimensions. In $D = 10$ we introduce $\Gamma_{11} = \Gamma^0 \dots \Gamma^9$. Like γ_5, this matrix satisfies $\{\Gamma_{11}, \Gamma^\mu\} = 0$, $(\Gamma_{11})^2 = \mathrm{Id}_{32}$. Spinors with eigenvalue 1 are called spinors of positive chirality and spinors with eigenvalue -1 are called spinors with negative chirality. The operators $\frac{1}{2}(\mathrm{Id}_{32} \pm \Gamma_{11})$ project spinors of definite chirality, called Weyl spinors. Restricting to spinors of positive chirality or to spinors of negative chirality is called a Weyl condition.
In a Majorana representation the matrix Γ_{11} is real. This means that the positive chirality and the negative chirality parts of real fermions are also real. Thus the Majorana condition and the Weyl condition are compatible and we can impose both conditions simultaneously. These conditions are not compatible in 4 dimensions, since γ_5 is imaginary. It is only possible to impose both conditions at the same time in $D \equiv 2(\mathrm{mod} 8)$ and spinors satisfying both conditions are called Majorana-Weyl spinors.
We can generalize this to states which have mass by the operator

$$\bar{\Gamma} = \Gamma_{11}(-1)^{N_\Gamma} ,$$

where N_Γ is given by

$$N_\Gamma = \sum_{n=1}^{\infty} d_{-n} \cdot d_n .$$

This operator satisfies $\{\bar{\Gamma}, d_n^\mu\} = 0$. Since $\psi^\mu(\sigma,\tau)$ is linear in the coefficients d_n^μ, we also get $\{\bar{\Gamma}, \psi^\mu(\sigma,\tau)\} = 0$. The operator $\bar{\Gamma}$ represents $(-1)^F$ in the R sector. The corresponding notion in the NS sector is given by

$$G = -(-1)^{N_G} \, ,$$

where N_G is given by

$$N_G = \sum_{r=\frac{1}{2}}^{\infty} b_{-r} \cdot b_r \, .$$

2.8 Heterotic string theory

With the Chan-Patton method it is possible to place charges at the ends of an open string. This is possible due to the existence of special points on the strings; the ends. For closed strings there are no such special points. Hence if we wish to place charges on them, the only way to do so is by distributing them over the entire string. So far we have added fermionic fields with a spacetime index μ to the string. We will now consider adding fermionic fields with more internal quantum numbers, which will introduce internal symmetry groups. As we have seen before, the components of the Majorana fermions in the action decouple. Therefore it is possible to include left moving modes of one type and right moving modes of another type. We will take the right moving modes to be the supersymmetric fermions we have discussed so far. The left moving modes will be non-supersymmetric Majorana-Weyl fermions λ^A, with $1 \leq A \leq n$ for some n. These carry internal symmetry groups and will be discussed later. The action for this theory in conformal gauge is given by

$$S = \frac{-1}{2\pi} \int d^2\sigma \left(\sum_{\mu=0}^{9} (\partial_\alpha X_\mu \partial^\alpha X^\mu - 2i\psi_-^\mu \partial_+ \psi_{\mu-}) - 2i \sum_{A=1}^{n} \lambda_+^A \partial_- \lambda_+^A \right) .$$

The right moving part of the theory is supersymmetric and leads to a critical dimension of $D = 10$. Since the left-moving part is not supersymmetric, it will have the same ghost fields as the bosonic string theory. These ghost fields have a central charge of -26, which is cancelled by 26 bosonic fields. We only have 10 bosonic fields, so the rest of the cancelation has to come from the left-moving fermions λ^A. Since Majorana-Weyl fermions contribute $\frac{1}{2}$ to the central charge, we need 32 Majorana fermions for the cancelation of the Weyl anomaly.

2.8.1 Heterotic $SO(32)$

If all λ^A satisfy the same boundary conditions, their symmetry group is $SO(32)$. The first option is to assign periodic boundary conditions to all fermions. This is known as the periodic sector (P sector) and is analogous to the Ramond sector. The fermions have a mode expansion given by

$$\lambda^A(\sigma^+) = \sum_{n\in\mathbb{Z}} \lambda_n^A e^{-2in\sigma^+} \, .$$

The second option is to assign antiperiodic boundary conditions to all fermions, which is known as the antiperiodic sector (A sector) and is analogous to the Neveu-Schwarz sector. The mode expansion is given by

$$\lambda^A(\sigma^+) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \lambda_r^A e^{-2ir\sigma^+} \, .$$

The canonical anticommutation relations for these are given by $\{\lambda_n^A, \lambda_m^B\} = \delta^{AB}\delta_{n+m}$ and $\{\lambda_r^A, \lambda_s^B\} = \delta^{AB}\delta_{r+s}$.

The left-moving modes give rise to a different Virasoro algebra. The Virasoro constraints for the right-moving modes is given by $L_m|\psi\rangle = (L_0 - a)|\psi\rangle = 0$ for all $m > 0$ and for some normal ordering constant a. These operators are the same as before. For the left-moving modes we also get the Virasoro constraints $\tilde{L}_m|\psi\rangle = (\tilde{L}_0 - \tilde{a})|\psi\rangle = 0$ for all $m > 0$, but for different operators $\tilde{L}_m$ and a different normal ordering constant $\tilde{a}$. The operator $\tilde{L}_0$ is given by $\tilde{L}_0 = \frac{p^2}{8} + \tilde{N}$, where $\tilde{N}$ is given by

$$\tilde{N} = \sum_{n=1}^{\infty}(\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + n\lambda^A_{-n}\lambda^A_n)$$

in the P sector and

$$\tilde{N} = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \sum_{r=\frac{1}{2}}^{\infty} r\lambda^A_{-r}\lambda^A_r$$

in the A sector. For the right-moving modes $a = 0$. The contribution of a bosonic coordinate to the normal ordering constant is $\frac{1}{24}$. We have 8 transverse bosonic degrees of freedom, giving a contribution of $\frac{8}{24}$ or $\frac{1}{3}$. Fermionic degrees of freedom with integer modes give a contribution of $-\frac{1}{24}$, while fermionic degrees of freedom with half-integer modes give a contribution of $\frac{1}{48}$. Hence for the P sector and for the A sector we find different values for the normal ordering constant:

$$\tilde{a}_P = \frac{8}{24} - \frac{32}{24} = -1\,,$$
$$\tilde{a}_A = \frac{8}{24} + \frac{32}{48} = 1\,.$$

With this we find that in the P sector, the mass is given by $\frac{1}{4}M^2 = N + \tilde{N} + 1$, whereas in the A sector it is given by $\frac{1}{4}M^2 = N + \tilde{N} - 1$. The mass-shell condition from the right-moving modes implies $N = \tilde{N} + 1$ in the P sector and $N = \tilde{N} - 1$ in the A sector. Since N and $\tilde{N}$ are non-negative, the P sector has no massless states. In the A sector, the massless states are given by $N = 0$, $\tilde{N} = 1$.
For the right-moving modes, the operator N is always an integer. The condition $N = \tilde{N} - 1$ in the A sector means that $\tilde{N}$ also has to be an integer. This implies that we need to remove all states with an odd number of λ^A excitations. This is analogous to the GSO conditions for the right-moving modes. A natural thing to do is to also apply a GSO-like condition on the P sector. This turns out to be necessary for unitarity at one-loop level.
The operator $(-1)^F$ in the P sector is given by

$$(-1)^F = \bar{\lambda}_0(-1)^{N_\lambda}\,,$$

where

$$N_\lambda = \sum_{n=1}^{\infty}\lambda^A_{-n}\lambda^A_n\,,$$

and

$$\bar{\lambda}_0 = \lambda^1_0\lambda^2_0\dots\lambda^{32}_0\,.$$

The states with $\tilde{N} = 0$ in the P sector are given by $Spin(32)$ spinors $|a\rangle$ satisfying the condition $\bar{\lambda}_0|a\rangle = |a\rangle$. All excited states can be obtained by acting with an even number of λ^A excitations on such states to ensure they satisfy the GSO-like condition.

2.8.2 Heterotic $E_8 \times E_8$

So far we assigned the same boundary conditions to all left-moving Majorana-Weyl fermions. We will now explore what happens when we take mixed boundary conditions into account. As a first impression this gives a lot of new possibilities at the cost of losing symmetry.
We will consider the case where some of the fermions satisfy periodic boundary conditions and the rest of the fermions satisfy antiperiodic boundary conditions. If we take n fermions with periodic boundary conditions, then the normal ordering constant will be given by

$$\tilde{a} = 1 - \frac{n}{16}\,.$$

Recall that N has integer eigenvalues. In the P sector $\tilde{N}$ also has integer eigenvalues. In the A sector $\tilde{N}$ has integer and half integer eigenvalues, but by level matching we discarded the states with half integer eigenvalues. It turns out that half integer values for $\tilde{a}$ lead to one-loop anomalies, so we won't consider those cases. This leaves us with the natural choice of $n = 16$, with $\tilde{a} = 0$ and symmetry group $Spin(16) \times Spin(16)$.
Since $\tilde{a} = 1$ in the A sector, massless states are obtained by acting with two $\lambda^A_{-\frac{1}{2}}$ operators on the ground state. These states are given by

$$\lambda^A_{-\frac{1}{2}}\lambda^B_{-\frac{1}{2}}|0\rangle_L \,.$$

Under the action of the group $Spin(16) \times Spin(16)$, these states transform as:

$$\begin{aligned}
&(\mathbf{120}, \mathbf{1}) \text{ for } 1 \leq A, B \leq 16\,, \\
&(\mathbf{1}, \mathbf{120}) \text{ for } 17 \leq A, B \leq 32\,, \\
&(\mathbf{16}, \mathbf{16}) \text{ for } 1 \leq A \leq 16, 17 \leq B \leq 32\,.
\end{aligned}$$

Here **16** and **120** denote the vector representation and the adjoint representation of $SO(16)$. Together they form the adjoint representation of $SO(32)$ given by $\mathbf{496} = \mathbf{120} \oplus \mathbf{120} \oplus (\mathbf{16} \otimes \mathbf{16})$.
In the case of 16 fermions with periodic boundary conditions and 16 antiperiodic boundary conditions we get $\tilde{a} = 0$. Because of this, there are new massless states. If we denote the two spinor representations of $Spin(16)$ as **128** and **128**', then the massless states transfrom as either $(\mathbf{128}, \mathbf{1}) \oplus (\mathbf{128'}, \mathbf{1})$ or $(\mathbf{1}, \mathbf{128}) \oplus (\mathbf{1}, \mathbf{128'})$, depending on whether the first 16 fermions satisfy periodic or antiperiodic boundary conditions.
Again we will need to impose GSO conditions on these states. Now there are two candidates for $(-1)^F$. The first candidate, $(-1)^{F_1}$, anticommutes with the first 16 fermions and commutes with the last 16 fermions. The second candidate, $(-1)^{F_2}$, commutes with the first 16 fermions and anticommutes with the last 16 fermions. We will keep those states which have eigenvalue 1 for both $(-1)^{F_1}$ and $(-1)^{F_2}$. We impose that the groundstate of the A sector $|0\rangle_L$ is even under both operators. The allowed states can be constructed by acting on the ground state with two fermions that are either both from the first group of 16 or both from the second group of 16. The allowed states transform as:

$$(\mathbf{120}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{120})\,.$$

The two spinor representations have opposite signs under $(-1)^F$, so we only keep one representation from each pair. This leaves us with two copies of $\mathbf{120} \oplus \mathbf{128}$. These form the exceptional Lie algebra $\mathfrak{e}_8$, the Lie algebra of the exceptional Lie group E_8. Thus the gauge group becomes $E_8 \times E_8$.

2.8.3 The bosonized approach to the heterotic string

Instead of 32 fermions we will now consider 16 additional bosons in the left moving part of the theory. The 16 new dimensions introduced in this way will be compactified on a suitable torus. The suitable torus will be the one which correctly describes the bosonized fermions with boundary conditions.
For the 16 newly introduced bosons we write the mode expansion as

$$X^I_L(\sigma^+) = x^I_L + p^I_L(\tau + \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}a^I_n e^{-2in(\tau+\sigma)}\,.$$

For bosons with both a left moving part and a right moving part, we have a center of mass coordinate x^I and momentum p^I. The canonical commutation relations for x^I and p^I are given by

$$[x^I, p^J] = i\delta^{IJ}\,.$$

The left moving parts and right moving parts both contribute half of this relation, such that for x^I_L and p^I_L the commutation relations are

$$[x^I_L, p^J_L] = \frac{i}{2}\delta^{IJ}\,.$$

Upon quantization, the left moving momentum operator becomes $p_L^I = -\frac{i}{2}\frac{\partial}{\partial x_L^I}$.
We construct the torus by taking 16 linearly independent vectors e_i^I, where the lower index denotes different vectors and the upper index denotes the coordinates of the vector. We define Γ to be the lattice consisting of all points of the form $\sum_{n_i} n_i e_i^I$. We consider the torus $T^{16} = \mathbb{R}^{16}/\pi\Gamma$. We introduce the metric

$$g_{ij} = \sum_{I=1}^{16} e_i^I e_j^I \,.$$

We consider the case where this matrix has integer entries and 2 on all diagonal entries. The momenta K^I are required to give a well defined value for

$$e^{2iK\cdot x}\,,$$

since this quantity is observable. This is satisfied when $K^I e_i^I$ is an integer for all $1 \leq i \leq 16$. This means that K lies in the dual lattice $\bar{\Gamma}$, consisting of integer multiples of e_i^{*I} satisfying

$$\sum_{I=1}^{16} e_i^{*I} e_j^I = \delta_{ij}$$

In order to ensure that X_L only has left-moving modes, K also needs to lie in Γ. The requirement that the entries of g_{ij} are integers ensures that $\Gamma \subset \bar{\Gamma}$. If $\Gamma = \bar{\Gamma}$ then the lattice is said to be self-dual. if the entries of g_{ij} are integers, the lattice is called an integral lattice. If on top of that the diagonal entries are even, the lattice is said to be an even lattice. All $v \in \Gamma$ have even norm squared: $\langle v, v\rangle \in 2\mathbb{Z}$. Lattice sites with $K^2 = 2$ correspond to massless states.
Even self-dual lattices only exist in dimensions $d \equiv 0(mod 8)$. In $d = 8$, the only even self-dual lattice is Γ_8, the root lattice of E_8. In $d = 16$, the only even self-dual lattices are $\Gamma_8 \times \Gamma_8$ and Γ_{16}, which contains the root lattice of $SO(32)$ as a sublattice. Choices for either $\Gamma_8 \times \Gamma_8$ or Γ_{16} correspond to the gauge groups $E_8 \times E_8$ and $SO(32)$.

3 Courant Algebroids

We will now turn our attention to a new topic. This section will be the first introduction to Courant algebroids. We will use these results in section 4 to introduce geometrical structures on Courant algebroids. In section 5 we will show the interplay between Courant algebroids and superstring theory through the Strominger system.

3.1 Introduction

Let V be a real n-dimensional vector space. The double of V is defined as $V \oplus V^*$. It is equipped with a canonical non-degenerate bilinear form of signature (n, n) given by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)) \,,$$

for $X, Y \in V$ and $\alpha, \beta \in V^*$. In addition, the double comes with a canonical orientation. For the choice of orientation, first note that $\Lambda^{2n}(V \oplus V^*) \simeq \Lambda^n(V) \otimes \Lambda^n(V^*)$. With this identification we define the map $\Lambda^{2n}(V \oplus V^*) \to \mathbb{R}$, $(u_1 \wedge \cdots \wedge u_n) \otimes (u^1 \wedge \cdots \wedge u^n) \mapsto det(u^i(u_j))$. The orientation is given by the bases which map to $\mathbb{R}_+$.
The symmetry group of the double is $SO(V \oplus V^*) \simeq SO(n, n)$, the indefinite special orthogonal group. This group is given by

$$SO(n, n) = \{A \in End(V \oplus V^*) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \; \forall x, y \in V \oplus V^*, det(A) = 1\}$$

This is a subgroup of the split orthogonal group $O(n, n)$. The Lie algebra of $SO(n, n)$ is given by

$$\mathfrak{so}(n, n) = \{T \in End(V \oplus V^*) \mid \langle Tx, y \rangle + \langle x, Ty \rangle = 0\} \,.$$

The extended double is given by $\mathcal{H} = V \oplus \mathfrak{g} \oplus V^*$, where $\mathfrak{g} \simeq \mathbb{R}^m$ is equipped with a bilinear form c of arbitrary signature. The extended double has a canonical bilinear form given by

$$\langle X + s + \alpha, Y + t + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)) + c(s, t)$$

Similar to the double we can define a symmetry group and its Lie algebra. The extended double can be oriented once an orientation on $\mathfrak{g}$ is chosen.

3.2 Clifford algebras and spinors

Let V be a real n-dimensional vector space with a non-degenerate bilinear form of arbitrary signature. Its Clifford algebra is given by

$$\mathcal{C}\ell(V) = \otimes^\bullet V / (u^2 - \langle u, u \rangle)$$

Here $\otimes^\bullet V$ denotes the free tensor algebra. The Clifford algebra contains the spin group as a subgroup. The spin group is given by

$$Spin(V) = \{v_1 \cdot \dots \cdot v_{2k} | v_i \in V, \langle v_i, v_j \rangle = \pm\delta_{ij}\} \,.$$

The spin group is the double cover of $SO(V)$ through the map $\rho : Spin(V) \to SO(V)$, $\rho(x)(v) = x \cdot v \cdot x^{-1}$, where $v \in V$, $x \in Spin(V)$.

3.3 Courant Algebroids

This section follows the approach of [1]. Let M be an n-dimensional manifold. Consider the double of the tangent bundle, i.e., $TM \oplus T^*M$. Since this is pointwise the double of a vector space, it comes with a non-degenerate bilinear form of signature (n, n). It also has a projection $\pi : TM \oplus T^*M \to$

$TM, X+\alpha \mapsto X$ called the anchor. We will equip this with the structure of a bracket similar to the Lie bracket for vector fields. Vector fields act on differential forms through the interior product:

$$X(\varphi) = \iota_X(\varphi)\,,$$

where $X \in \mathfrak{X}(M)$, $\varphi \in \Omega^\bullet(M)$. This is a map of degree -1. There is also a canonical map of degree 1, namely the exterior derivative. The Lie bracket of two vector fields is the unique vector field satisfying

$$\iota_{[X,Y]}\varphi = [\mathcal{L}_X, \iota_Y]\varphi = [[\iota_X, d], \iota_Y]\varphi\,,$$

where $X, Y \in \mathfrak{X}(M)$, $\varphi \in \Omega^\bullet(M)$. The brackets denote the supercommutator of graded algebras, which is given by

$$[A, B] = A \circ B - (-1)^{|A|\cdot|B|} B \circ A\,.$$

This approach to the Lie bracket will be used in defining the Dorfman bracket. The double bundle $TM \oplus T^*M$ acts on differential forms through the Clifford action:

$$(X+\alpha)(\varphi) = \iota_X\varphi + \alpha \wedge \varphi\,,$$

where $X \in \mathfrak{X}(M)$, $\alpha \in \Omega^1(M)$ and $\varphi \in \Omega^\bullet(M)$. This action has mixed degree, but both degrees are odd. With this, we can define the Dorfman bracket of two sections $e_1, e_2 \in \Gamma(TM \oplus T^*M)$ as the unique section satisfying

$$[e_1, e_2](\varphi) = [e_1, [d, e_2]](\varphi)\,,$$

where $\varphi \in \Omega^\bullet(M)$ and $[\cdot,\cdot]$ denotes the supercommutator. Explicitly, this bracket is given by

$$[X+\alpha, Y+\beta] = [X, Y] + \mathcal{L}_X(\beta) - \iota_Y d\alpha\,.$$

This bracket is no longer skew-symmetric, but it does satisfy a Jacobi-like identity

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]\,.$$

The structure $(TM \oplus T^*M, [\cdot,\cdot], \pi, \langle\cdot,\cdot\rangle)$ is the first example of a Courant algebroid.

Definition 3.3.1. A **Courant algebroid** is a 4-tuple $(E, [\cdot,\cdot], \rho, \langle\cdot,\cdot\rangle)$ consisting of a vector bundle $E \to M$, a bilinear bracket $[\cdot,\cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$, a bundle map $\rho : E \to TM$ and a symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on E, such that the following conditions hold for all $e_1, e_2, e_3 \in \Gamma(E)$:

- $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$
- $\langle [e_1, e_2] + [e_2, e_1], e_3\rangle = d\langle e_1, e_2\rangle(\rho(e_3))$
- $\rho(e_1)(\langle e_2, e_3\rangle) = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$

We call $[\cdot,\cdot]$ the Dorfman bracket, ρ the anchor and $\langle\cdot,\cdot\rangle$ the pairing of E.

Lemma 3.3.2. *Let E be a Courant algebroid. Then we have the following identities:*

- $[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2$
- $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$*, where the right hand side denotes the Lie bracket for vectorfields.*

Proof. For the first identity consider the expression $\rho(e_1)\langle fe_2, e_3\rangle$. Using the axioms of Courant algebroids, we can express this as

$$\rho(e_1)(\langle fe_2, e_3\rangle) = \langle [e_1, fe_2], e_3\rangle + \langle fe_2, [e_1, e_3]\rangle\,.$$

By linearity of the pairing, we can take the function f in front of the pairing and use the Leibniz rule to get

$$\rho(e_1)(f\langle e_2, e_3\rangle) = \rho(e_1)(f)\langle e_2, e_3\rangle + f\rho(e_1)\langle e_2, e_3\rangle\,.$$

Using the axioms of Courant algebroid, this becomes

$$\rho(e_1)(f\langle e_2, e_3\rangle) = \rho(e_1)(f)\langle e_2, e_3\rangle + f\langle [e_1, e_2], e_3\rangle + f\langle e_2, [e_1, e_3]\rangle\,.$$

Comparing these expressions, we find

$$[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2\,.$$

For the second identity we consider how both sides act on a function $\langle e_3, e_4\rangle$. First we consider $\rho([e_1, e_2])\langle e_3, e_4\rangle$:

$$\begin{aligned}\rho([e_1, e_2])\langle e_3, e_4\rangle &= \langle [[e_1, e_2], e_3], e_4\rangle + \langle e_3, [[e_1, e_2], e_4]\rangle\,,\\ &= \langle [e_1, [e_2, e_3]], e_4\rangle - \langle [e_2, [e_1, e_3]], e_4\rangle + \langle e_3, [e_1, [e_2, e_4]]\rangle - \langle e_3, [e_2, [e_1, e_4]]\rangle\,.\end{aligned}$$

Now we consider $[\rho(e_1), \rho(e_2)]\langle e_3, e_4\rangle$:

$$\begin{aligned}[\rho(e_1), \rho(e_2)]\langle e_3, e_4\rangle &= \rho(e_1)\rho(e_2)\langle e_3, e_4\rangle - \rho(e_2)\rho(e_1)\langle e_3, e_4\rangle\,,\\ &= \rho(e_1)\langle [e_2, e_3]e_4\rangle + \rho(e_1)\langle e_3, [e_2, e_4]\rangle - \rho(e_2)\langle [e_1, e_3], e_4\rangle - \rho(e_2)\langle e_3, [e_1, e_4]\rangle\,,\\ &= \langle [e_1, [e_2, e_3]], e_4\rangle + \langle [e_2, e_3], [e_1, e_4]\rangle + \langle [e_1, e_3], [e_2, e_4]\rangle + \langle e_3, [e_1, [e_2, e_4]]\rangle\\ &\quad - \langle [e_2, [e_1, e_3]], e_4\rangle - \langle [e_1, e_3], [e_2, e_4]\rangle - \langle [e_2, e_3], [e_1, e_4]\rangle - \langle e_3, [e_2, [e_1, e_4]]\rangle\,,\\ &= \langle [e_1, [e_2, e_3]], e_4\rangle - \langle [e_2, [e_1, e_3]], e_4\rangle + \langle e_3, [e_1, [e_2, e_4]]\rangle - \langle e_3, [e_2, [e_1, e_4]]\rangle\,.\end{aligned}$$

Thus we conclude that $\rho([e_1, e_2])\langle e_3, e_4\rangle = [\rho(e_1), \rho(e_2)]\langle e_3, e_4\rangle$. By non-degeneracy of the pairing it is always possible to find e_3, e_4 such that $\langle e_3, e_4\rangle$ doesn't vanish locally. By linearity in the first argument of the pairing we can multiply e_3 by a function such that $\langle e_3, e_4\rangle$ can be an arbitrary function locally. Since both vectorfields have the same action on any arbitrary function, we conclude that $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$. □

Using the pairing and the anchor, we can produce a map $\Gamma(T^*M) \to \Gamma(E)$. A 1-form $\alpha \in \Omega^1(M)$ is mapped to $\alpha' = \frac{1}{2}\rho^*(\alpha) \in \Gamma(E)$ such that for all $e \in \Gamma(E)$ the equation $\langle \alpha', e\rangle = \frac{1}{2}\alpha(\rho(e))$ holds.
A Courant algebroid is called transitive if the anchor is surjective. It is called exact if the following sequence is exact:

$$0 \longrightarrow T^*M \xrightarrow{\frac{1}{2}\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0\,.$$

For transitive Courant algebroids the map $\frac{1}{2}\rho^*$ is injective. Since we will be concerned with transitive Courant algebroids only, we will identify 1-forms with their image under the map $\frac{1}{2}\rho^*$. An isotropic splitting for E is a section $s : TM \to E$ of ρ such that the image $s(TM)$ is isotropic with respect to the pairing. Such an isotropic splitting gives an identification $E \simeq TM \oplus T^*M$, where the anchor is the projection and the pairing is given by $\langle X + \alpha, Y + \beta\rangle = \frac{1}{2}(\beta(X) + \alpha(Y))$. There exists a closed 3-form H such that the Dorfman bracket can be written as [20]:

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X(\beta) - \iota_Y d\alpha + \iota_Y \iota_X H\,.$$

This 3-form is explicitly given by the equation

$$H(X, Y, Z) = \langle [s(X), s(Y)], s(Z)\rangle\,,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. A different choice of the splitting results in a different 3-form H', but the difference between H and H' is exact. Hence the cohomology class $[H] \in H^3(M)$ is independent of the choice of the splitting and is called the Ševera class. Conversely, any closed 3-form H defines an exact Courant Algebroid structure on $E = TM \oplus T^*M$. The bracket is called the H-twisted Dorfman bracket. This gives a bijection between the set of equivalence classes of exact Courant algebroids on M and $H^3(M)$.
After the introduction of the objects of study, the next question is always what the maps between those objects are. This is where the notion of a Courant algebroid isomorphism comes in.

Definition 3.3.3. Let $(E, [\cdot,\cdot], \rho, \langle\cdot,\cdot\rangle)$ and $(E', [\cdot,\cdot]', \rho', \langle\cdot,\cdot\rangle')$ be two Courant algebroid structures on M. A **Courant algebroid isomorphism** $\varphi : E \to E'$ consists of a diffeomorphism $f : M \to M$ and a bundle isomorphism $\varphi : E \to E'$ covering f, such that the induced map of sections $\varphi_* : \Gamma(E) \to \Gamma(E')$ given by $\varphi_*(e) = \varphi \circ e \circ f^{-1}$ interchanges the Courant algebroid structures. This means that the following equations hold for all $e_1, e_2 \in \Gamma(E)$:

- $\varphi_*([e_1, e_2]) = [\varphi_* e_1, \varphi_* e_2]$
- $\rho'(\varphi_* e_1) = f_* \rho(e_1)$
- $\langle \varphi_* e_1, \varphi_* e_2 \rangle = (f^{-1})^* \langle e_1, e_2 \rangle$

The corresponding infinitesimal notion is given by derivations.

Definition 3.3.4. A **derivation** $D : E \to E$ consists of a bundle endomorphism $D : E \to E$ and a vector field X such that:

- $D[e_1, e_2] = [De_1, e_2] + [e_1, De_2]$
- $X(\langle e_1, e_2 \rangle) = \langle De_1, e_2 \rangle + \langle e_1, De_2 \rangle$

holds for all $e_1, e_2 \in \Gamma(E)$.

From the second condition it becomes clear that X is uniquely determined by D. For this reason we write $\rho(D) = X$. The space $Der(E)$ of derivations forms a Lie algebra.
Derivations satisfy two important identities. For all $D \in Der(E)$, $e \in \Gamma(E)$ and $F \in C^\infty(M)$ the following relations hold:

- $D(fe) = fD(e) + \rho(D)(f)e$
- $\rho(De) = [\rho(D), \rho(e)]$

The first identity motivates why these endomorphisms are called derivations. The map $\rho : Der(E) \to \mathfrak{X}(M)$ is a homomorphism of Lie algebras. The adjoint action defined by $\mathrm{ad}_{e_1}(e_2) = [e_1, e_2]$ is a derivation for all $e_1 \in \Gamma(E)$. Derivations of this form are called inner derivations.

3.4 Courant algebroid reductions

Let M be a manifold and G a Lie group which acts on M on the right by diffeomorphisms. Let $\mathfrak{g}$ be the Lie algebra of G defined using left invariant vector fields on G. By differentiation the right action of G determines a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{X}(M)$.
A lifted action of G on E is a right action of G on E by automorphisms covering the right action of G on M. A lifted infinitesimal action is a Lie algebra homomorphism $\tilde{\psi} : \mathfrak{g} \to Der(E)$ covering the homomorphism $\psi : \mathfrak{g} \to \mathfrak{X}(M)$. This means that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\tilde{\psi}"] \arrow[dr, "\psi"'] & Der(E) \arrow[d, "\rho"] \\ & \mathfrak{X}(M)\,. \end{tikzcd}$$

Consider a lifted action of G on E and assume that M is a principal G-bundle. This turns the quotient space E/G into a vector bundle over M/E. There is a canonical identification between $\Gamma(E/G)$ and $\Gamma(E)^G$, the set of G-invariant sections of E. If $e_1, e_2 \in \Gamma(E)$ are G-invariant, then both $[e_1, e_2]$ and $\langle e_1, e_2 \rangle$ are also G-invariant. We define the anchor ρ^G as the composition

$$\Gamma(E/G) \simeq \Gamma(E)^G \xrightarrow{\ \rho\ } \Gamma(TM)^G \longrightarrow \Gamma(T(M/G))\,.$$

Definition 3.4.1. The Courant algebroid $(E/G, [\cdot,\cdot], \rho^G, \langle\cdot,\cdot\rangle)$ constructed above is called the **simple reduction** of E by the lifted action of G.

Another notion of an action is the extended action introduced in [4]. Let G be a compact, connected, semisimple Lie group with an action on M and a lifted action $G \to Aut(E)$. Differentiating this lifted action gives the lifted infinitesimal action $\mu : \mathfrak{g} \to Der(E)$. Let $\mathrm{ad} : \Gamma(E) \to Der(E)$ denote the adjoint action of the sections of E. If the image of μ lies in the set of inner derivations, we can lift the action of μ to an action $\mathfrak{g} \to \Gamma(E)$. However, this doesn't necessarily respect the different bracket structures of $\mathfrak{g}$ and $\Gamma(E)$. In the case of interest this is not the case, so we will look at trivially extended actions only.

Definition 3.4.2. Let G be a Lie group acting on the manifold M with induced infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and E a transitive Courant algebroid over a manifold M. A **trivially extended action** $\alpha : \mathfrak{g} \to \Gamma(E)$ is an action α satisfying $\rho \circ \alpha = \psi$ which integrates to an action of G on E.

The case of compact Lie groups G and exact Courant algebroids E can be characterized in terms of the Cartan complex of equivariant cohomology. Since G is compact there exists a G-invariant splitting $s : TM \to E$. Let H be the associated 3-form of this splitting. We can compute the action of $\mathfrak{g}$ on E in two different ways. Firsly, for every $e \in \mathfrak{g}$ we get $e(X+\theta) = \mathcal{L}_{\psi(e)}(X+\theta)$. Secondly, writing $\alpha(e) = \psi(e) + \xi(e)$, we get

$$e(X+\theta) = [\alpha(e), X+\theta] = \mathcal{L}_{\psi(e)}(X+\theta) + \iota_Y(\iota_{\psi(e)}H - d\xi(e))\,.$$

By comparing these two expressions we conclude that

$$d\xi(e) = \iota_{\psi(e)}H\,.$$

Furthermore, the condition that α is a Lie algebra homomorphism is given by

$$\begin{aligned}
\alpha([e_1,e_2]) &= \psi([e_1,e_2]) + \xi([e_1,e_2]) \\
&= [\psi(e_1)+\xi(e_1), \psi(e_2)+\xi(e_2)] \\
&= [\psi(e_1),\psi(e_2)] + \mathcal{L}_{\psi(e_1)}\xi(e_2) - \iota_{\psi(e_2)}(\iota_{\psi(e_1)}H - d\xi(e_1)) \\
&= \psi([e_1,e_2]) + \mathcal{L}_{\psi(e_1)}\xi(e_2)\,,
\end{aligned}$$

from which we conclude that $\mathcal{L}_{\psi(e_1)}\xi(e_2) = \xi([e_1,e_2])$. If we view ξ as a $\mathfrak{g}^*$-valued 1-form then this means that it is equivariant.

Definition 3.4.3. The **Cartan complex of equivariant forms** is the differential graded algebra of equivariant polynomials $\mathfrak{g} \to \Omega^\bullet(M)$. In degree k, it is given by

$$\Omega^k_G(M) = \bigoplus_{2p+q=k} (S^p\mathfrak{g}^* \otimes \Omega^q(M))^G\,,$$

where S^p denotes the symmetric algebra of degree p. The equivariant derivative d_G is given by

$$d_G(\varphi)(e) = d(\varphi(e)) - \iota_{\psi(e)}\varphi\,,$$

with $\varphi \in \Omega^k_G(M)$ and $e \in \mathfrak{g}$.

We consider the element φ given by $\varphi(e) = H + \xi(e)$. Its equivariant derivative is given by $d_G\varphi(e) = -\langle\alpha(e),\alpha(e)\rangle$. We introduce the bilinear form $c(e) = -\langle\alpha(e),\alpha(e)\rangle$ and write this equation as $d_G\varphi = c$. An extended action of this form acts on $E = TM \oplus T^*M$ by Lie derivative, so there is no obstruction to integrating this to a G-action on E.

Definition 3.4.4. Two extended actions are **equivalent** if they are related by an automorphism.

If two extended actions are equivalenttheir corresponding equivariant 3-forms differ by a d_G-exact term. A different invariant splitting leads to the addition of an exact term $\varphi \to \varphi + d_G\beta$, with $\beta \in \Omega^2(M)^G$. This leads to theorem 2.13 of [4]:

Theorem 3.4.5. *Let G be a compact Lie group. Then trivially extended G-actions on a fixed exact Courant algebroid with prescribed quadratic form $c(e) = -\langle \alpha(e), \alpha(e)\rangle$ are up to equivalence in bijection with solutions to the equation $d_G\varphi = c$, with $\varphi(e) = H + \xi(e)$ and H a representative of the Ševera class of E.*

3.4.1 Reduction by extended actions

An extended action $\alpha : \mathfrak{g} \to \Gamma(E)$ is called non-degenerate if for every $x \in M$ the bilinear form $c_x(a, b) = -\langle \alpha(a)_x, \alpha(b)_x\rangle$ is non-degenerate. We denote the image of α by K and its annihilator by $K^\perp$. Both K and $K^\perp$ are G-invariant. The action of G on P is free, so the rank of K is equal to the dimension of G. Both $K^\perp$ and $K \cap K^\perp$ also have constant rank. We consider the vector bundle

$$E_{red} = \frac{K^\perp}{K \cap K^\perp}/G\,.$$

Taking the quotient ensures that the induced pairing is non-degenerate. If we take $e_1, e_2 \in \Gamma(K^\perp)$, then for every $x \in \mathfrak{g}$ we get

$$\langle \alpha(x), [e_1, e_2]\rangle = \rho(e_1)\langle \alpha(x), e_2\rangle - \langle e_2, [e_1, \alpha(x)]\rangle\,.$$

The first term vanishes because $\alpha(x) \in K$ and $e_2 \in K^\perp$. The second term can be rewritten as

$$-\langle e_2, [e_1, \alpha(x)]\rangle = -\rho(e_2)\langle e_1, \alpha(x)\rangle + \langle e_2, [\alpha(x), e_1]\rangle\,.$$

The first term vanishes again for the same reason. The second term contains $[\alpha(x), e_1]$, which is the action of x on e_1. By G-invariance this term also vanishes. We conclude that for $e_1, e_2 \in \Gamma(K^\perp)$ we get $[e_1, e_2] \in \Gamma(K^\perp)$. This means that E_{red} with the induced structures is a Courant algebroid.
Let $e_1, \ldots, e_n$ be a basis for $\mathfrak{g}$. Take $X \in \mathcal{X}(P)$, then $X \in \Gamma(\rho(K^\perp))$ if there exists an $\beta \in \Omega^1(P)$ such that for all $1 \leq i \leq n$ we get

$$\langle X + \beta, \psi(e_i) + \xi(e_i)\rangle = \frac{1}{2}(\xi(e_i)(X) + \beta(\psi(e_i)) = 0\,.$$

Since $\psi(e_i)$ are nowhere vanishing and $dim(P) > dim(\mathfrak{g})$, we can always find solutions for β. This means that ρ, when restricted to $K^\perp$, is surjective. Thus we conclude that the Courant algebroid obtained through this reduction procedure is transitive.
For non-degenerate extended actions we get $K \cap K^\perp = \{0\}$, so we get

$$E_{red} = K^\perp/G\,.$$

3.5 Lie Algebroids

Definition 3.5.1. A **Lie algebroid** L over M is a vector bundle L together with a skew-symmetric bilinear form $[\cdot,\cdot] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ and a bundle map $\pi : L \to TM$ such that for all $e_1, e_2, e_3 \in \Gamma(L)$, $f \in C^\infty(M)$ we get:

- $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$
- $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$

- $[e_1, fe_2] = \pi(e_1)(f)e_2 + f[e_1, e_2]$

A Lie algebroid is called regular if π has constant rank and transitive if π is surjective. A transitive Lie algebroid A is called quadratic if $V = Ker(\pi)$ is equipped with a non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle$ which is preserved in the sense that for all $a \in \Gamma(A)$, $b, c \in \Gamma(V)$ we get:

$$\pi(a)(\langle b, c \rangle) = \langle [a, b], c \rangle + \langle b, [a, c] \rangle \,.$$

Definition 3.5.2. Let G be a Lie group and $P \to M$ a principal G-bundle. The quotient of TP by the action of G defines a vector bundle $\mathcal{A} = TP/G$, such that sections of $\mathcal{A}$ can be identified with G-invariant vectorfields on P. The Lie bracket of two G-invariant vectorfields is again G-invariant, so this turns $\mathcal{A}$ into a Lie algebroid called the **Atiyah algebroid** of P.

We will assume that G is compact, connected and semisimple. Let $\mathfrak{g}$ be the Lie algebra of G and equip it with a G-invariant, non-degenerate bilinear form $c(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. For any principal bundle $P \to M$ with Atiyah algebroid $\mathcal{A}$ the kernel V of the anchor π can be identified with the adjoint bundle $\mathfrak{g}_P = P \times_G \mathfrak{g}$. The pairing $\langle \cdot, \cdot \rangle$ is G-invariant, so it induces a pairing on $\mathfrak{g}_P$ which we will also call $\langle \cdot, \cdot \rangle$. With this pairing the Atiyah algebroid is a quadratic Lie algebroid.

3.6 Heterotic Courant Algebroids

Let $\mathcal{H}$ be a transitive Courant algebroid. The anchor ρ dualises to an injective map $\frac{1}{2}\rho^* : T^*M \to \mathcal{H}$.

Lemma 3.6.1. *The quotient $\mathcal{A} = \mathcal{H}/T^*M$ carries the structure of a quadratic Lie algebroid.*

Proof. The three relations between the bracket and the anchor are automatically satisfied since they already hold for any Courant Algebroid. Hence we only need to show that the Dorfman bracket is skew-symmetric on sections of $\mathcal{A}$. For any 3 sections e_1, e_2, e_3 of $\mathcal{H}$ we have the following identity:

$$\langle [e_1, e_2] + [e_2, e_1], e_3 \rangle = d\langle e_1, e_2 \rangle(\rho(e_3)) \,.$$

Setting $e_2 = e_1$, we get

$$\langle [e_1, e_1], e_3 \rangle = \frac{1}{2} d\langle e_1, e_1 \rangle(\rho(e_3)) \,.$$

This means that

$$[e_1, e_1] = \frac{1}{2}\rho^*(d\langle e_1, e_1 \rangle) \,.$$

Hence in the quotient $\mathcal{A} = \mathcal{H}/T^*M$ we find $[e_1, e_1] = 0$ and the bracket is skew-symmetric. Thus $\mathcal{A}$ carries the structure of a Lie algebroid. The condition for the pairing is automatically satisfied for any Courant algebroid. $\square$

Definition 3.6.2. A **heterotic Courant algebroid** is a transitive Courant algebroid $\mathcal{H}$ such that there exists a principal G-bundle P for which $\mathcal{A} = \mathcal{H}/T^*M$ is isomorphic to the Atiyah algebroid of P as quadratic Lie algebroids.

Let $\mathcal{H}$ be a heterotic Courant algebroid associated to the Atiyah algebroid $\mathcal{A} \simeq \mathcal{H}/T^*M$. Let $\mathcal{K}$ be the kernel of ρ and $\mathfrak{g}_P$ be the adjoint bundle. We get the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{H} \xrightarrow{\rho} TM \longrightarrow 0 \quad .$$

An isotropic splitting s splits the sequence above. The subspace $s(TM) \cap \mathcal{K}$ gives a lift of $\mathfrak{g}_P$ to $\mathcal{K}$ orthogonal to T^*M. As such, it also splits the following sequence:

$$0 \longrightarrow T^*M \longrightarrow \mathcal{K} \longrightarrow \mathfrak{g}_P \longrightarrow 0 \quad .$$

This determines a decomposition $\mathcal{H} = TM \oplus \mathfrak{g}_P \oplus T^*M$ where the anchor is the projection onto TM and the paring is given by

$$\langle X + s + \alpha, Y + t + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)) + \langle s, t \rangle \,.$$

Not all quadratic Lie algebroids $\mathcal{A}$ associated to a principal G-bundle $P \to M$ come from a quotient of transitive Courant algebroids $\mathcal{H}$ by the cotangent bundle. Let A be a connection with curvature F. The closed 4-form $\langle F, F \rangle$ represents the first Pontryagin class of P. The quadratic Lie algebroid $\mathcal{A}$ comes from a transitive Courant algebroid if and only if the first Pontryagin class vanishes [4].

Theorem 3.6.3. *Let $P \to M$ be a principal G-bundle with Atiyah algebroid $\mathcal{A}$ and a connection ∇ with curvature F. Let H^0 be a 3-form such that $dH^0 = \langle F, F \rangle$. This pair determines a heterotic Courant algebroid $\mathcal{H}$ such that $\mathcal{H}/T^*M \simeq \mathcal{A}$. The bundle decomposes as $\mathcal{H} = TM \oplus \mathfrak{g}_P \oplus T^*M$, the anchor is given by the projection and the pairing is given by:*

$$\langle X + s + \alpha, Y + t + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)) + \langle s, t \rangle \,.$$

The bracket is given by

$$\begin{aligned} [X + s + \alpha, Y + t + \beta] = {} & [X, Y] + \nabla_X t - \nabla_Y s - [s, t] - F(X, Y) \\ & + \mathcal{L}_X \beta - \iota_Y d\alpha + \iota_Y \iota_X H^0 + 2\langle t, \iota_X F \rangle \\ & - 2\langle s, \iota_Y F \rangle + 2\langle \nabla s, t \rangle \,. \end{aligned} \tag{1}$$

Conversely, for a heterotic Courant algebroid with splitting $s : TM \to \mathcal{H}$ there is a pair (∇, H^0) with $dH^0 = \langle F, F \rangle$ with the bracket given above.

Proof. The proof for this follows from the classification of transitive Courant algebroids [22]. □

Let $\mathcal{H} = TM \oplus \mathfrak{g}_P \oplus T^*M$ be a heterotic Courant algebroid. For a 2-form B we define the B-shift as

$$e^B(X + s + \alpha) = X + s + \alpha + \iota_X B \,.$$

And for the $\mathfrak{g}_P$-valued 1-form A we define the A-transform

$$e^A(X + s + \alpha) = X + s - AX + \alpha + \langle 2s - AX, A \rangle \,.$$

Both of these transforms preserve the anchor and the pairing. If we write the dependence of the bracket on ∇ and H^0 explicitly, i.e., $[\cdot, \cdot] = [\cdot, \cdot]_{\nabla, H^0}$, then the A-shift and the B-shift transform the bracket as follows:

$$\begin{aligned} [e^B u, e^B v]_{\nabla, H^0} &= e^B [u, v]_{\nabla, H^0 + dB} \,, \\ [e^A u, e^A v]_{\nabla, H^0} &= e^A [u, v]_{\nabla + A, H^0 + 2\langle A, F_\nabla \rangle + \langle A, d_\nabla A \rangle + \frac{1}{3}\langle A, [A, A] \rangle} \,. \end{aligned}$$

Heterotic Courant algebroids can be obtained by reduction of exact Courant algebroids. Let $\sigma : P \to M$ be a principal G-bundle. Every exact Courant algebroid is associated to a closed G-invariant 3-form H. The Courant algebroid is then given by $E = TP \oplus T^*P$ with the projection as anchor, the usual pairing and the H-twisted Dorfman bracket. The action of G on P extends to an action of G on E since H is G-invariant. As mentioned earlier, trivially extended actions $\alpha : \mathfrak{g} \to \Gamma(E)$ correspond, up to equivalence, to solutions to the equation $d_G \varphi = c$. Here $\varphi(e) = H + \xi(e)$ and $c(e) = -\langle \alpha(e), \alpha(e) \rangle$. We will consider the case where the pairing is given by c, i.e., $\langle e_1, e_2 \rangle = c(e_1, e_2) = -\langle \alpha(e_1), \alpha(e_2) \rangle$. This leads to proposition 3.3 of [1]:

Theorem 3.6.4. *Equivalence classes of solutions to the equation $d_G(H+\xi) = c = \langle\cdot,\cdot\rangle$ correspond to pairs (H^0, A), where H^0 is a 3-form on M and A a connection on P with curvature F satisfying $dH^0 = \langle F, F\rangle$. The corresponding solution has H, ξ given by*

$$H = \sigma^*(H^0) - CS_3(A)\,,$$
$$\xi = -cA\,,$$

where we view c as a map $\mathfrak{g} \to \mathfrak{g}^*$. Here $CS_3(A)$ is the Chern-Simons 3-form of A given by

$$CS_3(A) = c(A, F) - \frac{1}{3!}c(A, [A, A])\,.$$

Proof. We write the extended action $\alpha : \mathfrak{g} \to \Gamma(E)$ as $\alpha(x) = \psi(x)+\xi(x)$. Since c is non-degenerate, we can write $\xi = -cA'$ for some g-valued 1-form A'. From the G-invariance of ξ and c we get that A' has to be G-invariant as well. With $\alpha(x) = \psi(x) - c(A', x)$ the condition $c(x, y) = -\langle\alpha(x), \alpha(y)\rangle$ becomes $2c(x, y) = c(A'(\psi(x)), y) + c(A'(\psi(y)), x)$. We choose a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ with corresponding dual basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. Let $A_0 = A_0^i e_i$ be a connection 1-form on P, where $A_0^i(\psi(e_j)) = \delta^i_j$. We can write A' as

$$A' = a^i_j A^j e_i + B^i e_i\,,$$

for some G-invariant functions a^i_j and some 1-forms $B^i \in \Omega^1(M)$ satisfying $B^i(\psi(e_j)) = 0$. Note that we left out pull-back notation. In this basis, the constraint $2c(x, y) = c(A'(\psi(x)), y) + c(A'(\psi(y)), x)$ reads

$$2c_{ij} = a^k_i c_{jk} + a^k_j c_{ik}\,.$$

Hence $a^k_i c_{jk} = c_{ij} + \beta_{ij}$, where $\beta_{ji} = -\beta_{ij}$. This means that $a^j_i = \delta^j_i + c^{jk}\beta_{ki}$. Substituting this gives

$$A' = A_0 + B^i e_i + \beta_{ik}c^{kj}A_0^i e_j\,.$$

We define the connection A by $A = A_0 + B^i e_i$. With this notation, ξ is given by $\xi = -cA - \beta_{ij}A_0^i e^j$. If $\varphi = H+\xi$ is a solution to $d_G\varphi = c$, then we can construct another solution $\varphi' = \varphi - d_G(\frac{1}{2}\beta_{ij}A_0^i A_0^j)$ $= H'+\xi'$, with $H' = H - d(\beta_{ij}A_0^i A_0^j)$ and $\xi' = \xi+\beta_{ij}A_0^i e^j = -cA$. This shows that up to equivalence every solution of $d_G(H+\xi) = c$ is given by an extended action $\xi = -cA$ for some connection A.
We now need to find the closed invariant 3-form H for which the condition $d_G(H-cA) = c$ is satisfied. We write the connection A as $A = A^i e_i$. We can now decompose H as

$$H = H^0 + H^1_i A^i + \frac{1}{2}H^2_{ij}A^i \wedge A^j + \frac{1}{6}H^3_{ijk}A^i \wedge A^j \wedge A^k\,.$$

Note that pull-back notation is again suppressed. We write the structure constants with respect to the basis as c^k_{ij}, meaning that $[e_i, e_j] = c^k_{ij}e_k$. We also write $c_{ijk} = c^l_{ij}c_{lk}$. The curvature of A is given by $F^i e_i$ with $F^k = dA^k + \frac{1}{2}c^k_{ij}A^i \wedge A^j$. With this notation, the condition $d_G(H - cA) = c$ becomes $d_G H - c_{ij}F^i e^j + \frac{1}{2}c_{ijk}A^i \wedge A^j e^k = 0$. By comparing coefficients we find $H^3_{ijk} = c_{ijk}$, $H^2_{ij} = 0$ and $H^1_i = -c_{ij}F^j$. We recognize this as $H = H^0 - CS_3(A)$. Since H has to be closed and $dCS_3(A) = c(f, F)$, we get $dH^0 = c(F, F)$. Conversely any pair (H^0, A) satisfying $dH^0 = c(F, F)$ gives a solution to $d_G(H - cA) = c$, where $H = H^0 - CS_3(A)$. □

4 Generalized geometry

4.1 Generalized metrics

Definition 4.1.1. A **generalized metric** on a Courant algebroid E is a self-adjoint orthogonal bundle automorphism $\mathcal{G} : E \to E$ which satisfies

$$\langle \mathcal{G} e_x, e_x \rangle > 0$$

for all non-zero vectors $e_x \in E_x$.

Since $\mathcal{G}$ is self-adjoint and orthogonal it squares to the identity. It has eigenvalues 1 and -1 and we denote the eigenspaces by V_+ and V_-. This means that $\mathcal{G}$ restricted to $V_\pm$ gives $\pm\mathrm{Id}$. Therefore a generalized metric is completely determined by its eigenspaces V_+ and V_-. For this reason the choice of an eigenspace V_+ is equivalent to the generalized metric $\mathcal{G}$.

Lemma 4.1.2. *The restriction $\rho_\pm$ of ρ to $V_\pm$ is surjective.*

Proof. Assume that there is an $X \in \mathfrak{X}(M)$ which does not lie in the image of ρ_+. Consider $\alpha \in \Omega^1(M)$ such that $\alpha(X) \neq 0$. As an element of E, α is orthogonal to V_+, so it has to be a section of V_-. However, it has zero norm, contradicting the fact that 0 is the only element of zero norm in V_-. Therefore we conclude that ρ_+ is surjective. The same proof applies to ρ_-. $\square$

Theorem 4.1.3. *Let E be an exact Courant algebroid over a manifold M. A generalized metric $\mathcal{G}$ is equivalent to a metric g on M and an isotropic splitting $s : TM \to E$.*

Proof. First we assume there is a generalized metric $\mathcal{G}$ with eigenspaces V_+ and V_-. The signature of the pairing of E is (n, n) and ρ_- is surjective by lemma 4.1. By dimension counting ρ_- is an isomorphism. Denote the inverse of ρ_- by λ such that $\lambda : TM \to E$, $\lambda = (\rho_-)^{-1}$. This induces a Riemannian metric g on M given by

$$g(X, Y) = -\langle \lambda(X), \lambda(Y) \rangle .$$

We define the splitting $s : TM \to E$ by $s(X) = \lambda(X) + g(X)$. For vectorfields $X, Y \in \mathfrak{X}(M)$ we get

$$\begin{aligned} \langle s(X), s(Y) \rangle &= \langle \lambda(X) + g(X), \lambda(Y) + g(Y) \rangle \\ &= \langle \lambda(X), \lambda(Y) \rangle + \frac{1}{2}(\iota_{\rho(\lambda(X))} g(Y) + \iota_{\rho(\lambda(Y))} g(X)) \\ &= \langle \lambda(X), \lambda(Y) \rangle + \frac{1}{2}(g(Y, X) + g(X, Y)) \\ &= 0 . \end{aligned}$$

Hence this splitting is indeed isotropic. Now assume we have a metric g on M and an isotropic splitting $s : TM \to E$. We use the splitting to identify $E \simeq TM \oplus T^*M$. With this identification, the eigenspaces are given by

$$\begin{aligned} V_+ &= \{X + g(X) | X \in TM\} , \\ V_- &= \{X - g(X) | X \in TM\} . \end{aligned}$$

These eigenspaces uniquely determine the generalized metric $\mathcal{G}$. Since the eigenspaces are orthogonal complements either V_+ or V_- would already give enough information to reconstruct $\mathcal{G}$. $\square$

Note that a choice of splitting s corresponds to a choice of a 3-form H representing the Ševera class of E. This means that a generalized metric corresponds to a pair (g, H), where g is a metric on M and H is a representative of the Ševera class of E.
A different choice of the splitting leads to a B transform. Therefore, in a general splitting the eigenspaces have the form

$$V_+ = \{X + g(X) + \iota_X B | X \in TM\} ,$$
$$V_- = \{X - g(X) + \iota_X B | X \in TM\} .$$

For heterotic Courant algebroids the situation is very similar. We can identify $\mathcal{H} \simeq TM \oplus \mathfrak{g}_P \oplus T^*M$ using a splitting $s : TM \to \mathcal{H}$. The pairing is the canonical pairing and the bracket is given by Equation 1. Hence this identification corresponds to a choice of connection and a choice of representative of the Ševera class. With the identification, the eigenspaces are given by

$$\begin{aligned} V_+ &= \{X + t + g(X) | X \in TM, t \in \mathfrak{g}_P\} , \\ V_- &= \{X - g(X) | X \in TM\} . \end{aligned}$$

This shows that a generalized metric on $\mathcal{H}$ is equivalent to a triple (g, ∇, H), where g is a metric on M, ∇ a connection on $\mathfrak{g}_P$ and H a representative of the Ševera class of $\mathcal{H}$. If we denote the projections to $\mathfrak{g}_P$ and T^*M by $\pi_{\mathfrak{g}_P}$ and π_{T^*M}, the connection ∇ and the 3-form H are given by

$$\begin{aligned} \nabla_X a &= \pi_{\mathfrak{g}_P}([X, a]_{\mathcal{H}}) , \\ \iota_Y \iota_X H &= \pi_{T^*M}([X, Y]_{\mathcal{H}}) . \end{aligned}$$

A different choice of splitting now leads to a (B, A)-transform. Therefore, in a general splitting the eigenspaces have the form

$$\begin{aligned} V_+ &= \{X + t - AX + g(X) + \iota_X B + \langle 2t - AX, A\rangle | X \in TM, t \in \mathfrak{g}_P\} , \\ V_- &= \{X - AX - g(X) + \iota_X B - \langle AX, A\rangle | X \in TM\} . \end{aligned}$$

For any Courant algebroid E with generalized metric $\mathcal{G}$ we define the projections $\Pi_\pm : E \to V_\pm$, $\Pi_\pm = \frac{1}{2}(Id \mp \mathcal{G})$. We will use the notations $e^+ = \Pi_+ e$ and $e^- = \Pi_- e$.

4.2 Generalized connections

Definition 4.2.1. A **generalized connection** D on a Courant algebroid E is a bundle map

$$D : \Gamma(E) \to \Gamma(E^* \otimes E) ,$$

satisfying the Leibniz rule

$$D_{e_1} f e_2 = \rho(e_1) f e_2 + f D_{e_1} e_2 ,$$

and which is compatible with the pairing:

$$\rho(e_1)\langle e_2, e_3\rangle = \langle D_{e_1} e_2, e_3\rangle + \langle e_2, D_{e_1} e_3\rangle ,$$

for all $e_1, e_2, e_3 \in \Gamma(E)$, $f \in C^\infty(M)$.

Generalized connections always exist. We can construct one by taking any normal connection ∇^E on E compatible with the pairing and define

$$D_{e_1} e_2 = \nabla^E_{\rho(e_1)} e_2 .$$

The torsion $T_D \in \Lambda^3(E^*)$ is defined as

$$T_D(e_1, e_2, e_3) = \langle D_{e_1} e_2 - D_{e_2} e_1 - [[e_1, e_2]], e_3\rangle + \frac{1}{2}(\langle D_{e_3} e_1, e_2\rangle - \langle D_{e_3} e_2, e_1\rangle) ,$$

where $[[e_1, e_2]] = \frac{1}{2}([e_1, e_2] - [e_2, e_1])$ is the skew-symmetrization of the Dorfman bracket. The divergence of a section $e \in \Gamma(E)$ with respect to a generalized connection D is defined as

$$div_D(e) = Tr(De) = \sum_{i=1}^{n} \langle D_{e_i} e, e^i \rangle \,,$$

where $e_1, \ldots, e_n$ is a basis of E and $e^1, \ldots, e^n$ is the dual basis. For normal metrics there is a unique connection which is both torsion free and compatible with the metric, known as the Levi-Civita connection. However, in the generalized setting uniqueness is lost. The divergence satisfies the Leibniz rule

$$div_D(fe) = \rho(e)(f) + f div_D(e) \,,$$

for all $e \in \Gamma(E)$, $f \in C^\infty(M)$. A generalized connection D is compatible with a generalized metric $\mathcal{G}$ if it preserves the eigenspaces, meaning

$$D(\Gamma(V_\pm)) \subset \Gamma(E^* \otimes V_\pm) \,.$$

For sections $e_1^+ \in \Gamma(V_+)$, $e_2^- \in \Gamma(V_-)$ we define the generalized curvature as

$$GR(e_1^+, e_2^-) = D_{e_1^+} D_{e_2^-} - D_{e_2^-} D_{e_1^+} - D_{[[e_1^+, e_2^-]]} \,.$$

Explicit calculations of the generalized curvature can be found in [10].

4.3 Generalized complex geometry

Generalized complex geometry was first introduced by Nigel Hitchin [15] and later developed by Marco Gualtieri [14].

Definition 4.3.1. Let V be a real vector space. A **generalized complex structure** on V is an endomorphism $\mathcal{J}$ of $V \oplus V^*$ satisfying $\mathcal{J}^2 = -\mathrm{Id}$ and $\mathcal{J}^* = -\mathcal{J}$.

Here $V \oplus V^*$ is identified with its dual by the canonical pairing.
Any generalized complex structure is orthogonal, meaning that $\mathcal{J}^* \mathcal{J} = \mathrm{Id}$. Let J be a linear complex structure on V and define

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} .$$

The endomorphism $\mathcal{J}_J$ satisfies $\mathcal{J}_J^2 = -\mathrm{Id}$ and $\mathcal{J}_J^* = -\mathcal{J}_J$, meaning that it is a generalized complex structure.
Let ω be a linear symplectic structure, viewed as a map $V \to V^*$. Then we define

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} .$$

The endomorphism $\mathcal{J}_\omega$ satisfies $\mathcal{J}_\omega^2 = -\mathrm{Id}$ and $\mathcal{J}_\omega^* = -\mathcal{J}_\omega$, meaning that it is a generalized complex structure. Thus a linear generalized complex structure is a generalization of both linear complex and linear symplectic structures.
A generalized linear complex structure on V is equivalent to a maximal isotropic linear subspace $L \subset (V \oplus V^*) \otimes \mathbb{C}$ of real index 0, meaning that $L \cap \bar{L} = \{0\}$. The subspace L is given by the i-eigenspace. For $x, y \in L$, we find $\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle x, y \rangle$ by orthogonality of $\mathcal{J}$, but since $x, y \in L$ we also get $\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle ix, iy \rangle = -\langle x, y \rangle$. Hence we get $\langle x, y \rangle = 0$, meaning that L is indeed isotropic. Now $\bar{L}$ is the $-i$-eigenspace, so $L \oplus \bar{L} = (V \oplus V^*) \otimes \mathbb{C}$ and $L \cap \bar{L} = \{0\}$, which means that L is a maximally isotropic subspace of real index 0.
For a given subspace L, we can define $\mathcal{J}$ to be multiplication by i on L and multiplication by $-i$ on

$\bar{L}$. The real transformation then defines a linear generalized complex structure $\mathcal{J}$ on V.
We can define complex and symplectic structures on manifolds. These structures correspond to smoothly varying linear complex and symplectic structures together with an integrability condition. For complex structures the integrability condition is the vanishing of the Nijenhuis tensor. For symplectic structures the integrability condition is closedness of the symplectic form. Generalized complex structures follow the same approach.

Definition 4.3.2. A **generalized almost complex structure** $\mathcal{J}$ on a manifold M is a smoothly varying linear generalized complex structure $\mathcal{J} \in \Gamma(End(TM \oplus T^*M))$. A generalized almost complex structure is integrable if the i-eigenspace $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ in involutive under the Dorfman bracket. An integrable generalized almost complex structure is called a generalized complex structure.

We decompose $(TM \oplus T^*M) \otimes \mathbb{C} = L \oplus \bar{L}$ and define the projection $E = \rho(L) \subset TM \otimes \mathbb{C}$, satisfying $E \oplus \bar{E} = TM \otimes \mathbb{C}$. The type of a generalized complex manifold at a point $x \in M$ is the real codimension of E_x in $T_xM \otimes \mathbb{C}$. This type may or may not be constant over the manifold M. For generalized complex structures of constant type, the two extremal cases of type 0 and type $n = dim(M)$ correspond to symplectic and complex structures.
The generalized complex structure $\mathcal{J}_\omega$ induced by the symplectic form ω is of constant type 0 and given by

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} .$$

The maximal isotropic subbundle L is given by

$$L = \{X - i\omega(X) | X \in TM \otimes \mathbb{C}\} .$$

This subbundle is involutive for a non-twisted Dorfman bracket if and only if $d\omega = 0$.
The generalized complex structure $\mathcal{J}_J$ induced by the complex structure J is of constant type n and given by

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} .$$

The maximal isotropic subbundle L is given by

$$L = T^{0,1}M \oplus T^{1,0*}M .$$

This subbundle is involutive for a non-twisted Dorfman bracket if and only if J is integrable as an almost complex structure.
Each maximal isotropic subbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ corresponds to a line bundle $K \subset \Lambda^\bullet T^*M$ [5]. The bundle L is the annihilator of K with respect to Clifford multiplication:

$$L = \{X + \alpha \in (TM \oplus T^*M) \otimes \mathbb{C} | (X + \alpha) \cdot K = 0\} .$$

The bundle K is called the canonical bundle. We can explicitly construct it as follows:
First, for any vector space V, subspace $E \subset V$ and $\varepsilon \in \Lambda^2 E^*$ we define

$$L(E, \varepsilon) = \{X + \alpha \in E \otimes V^* | \alpha_{|_E} = \iota_X \varepsilon\} .$$

Claim 1. This subspace is maximally isotropic. Moreover, any maximally isotropic subspace L is of this form.

To show this, consider the projection $\pi : V \otimes V^* \to V$ and define $E = \pi(L)$. Since L is isotropic we have $L \cap V^* \subset Ann(E)$, and since L is maximally isotropic we get an equality. For any $X + \alpha \in L$ we define $\varepsilon(X) = [\alpha] \in V^*/Ann(E) \simeq E^*$. Now every element of L can be written as $X + \iota_X \varepsilon + \beta$,

where $\beta \in Ann(E)$. Since $X + \iota_X\varepsilon + \beta \in L(E,\varepsilon)$, we conclude that $L \subset L(E,\varepsilon)$. By the maximality of L, we conclude that $L = L(E,\varepsilon)$.
For any nonzero spinor $\varphi \in \Lambda^\bullet V^*$ the null space is defined as

$$L_\varphi = \{e \in V \oplus V^* | e \cdot \varphi = 0\}\,.$$

For any nonzero spinor this null space is isotropic. If in addition it is maximal, the spinor is called a pure spinor. This means that pure spinors are elements of canonical bundles of maximal isotropic subspaces.

Lemma 4.3.3. *The canonical bundle of $L(E,\varepsilon)$ is given by $K = exp(\varepsilon) \cdot det(Ann(E))$, where $det(Ann(E))$ is the determinant bundle of $Ann(E)$ and $exp(\varepsilon) \cdot \varphi = \varphi - \varepsilon \wedge \varphi$ for any $\varphi \in U$.*

Proof. First we note that $L(E,\varepsilon) = exp(\varepsilon)L(E,0)$, where $exp(\varepsilon)$ denotes the ε-transform. Now if $X+\alpha$ annihilates φ, then $exp(\varepsilon)(X+\alpha) = X+\alpha+\iota_X\varepsilon$ annihilates $exp(\varepsilon)\cdot\varphi$. Hence we can restrict ourselves to the case $\varepsilon = 0$. Take any nonzero $\varphi \in K = det(Ann(E))$, then $(X+\alpha)\cdot\varphi = \iota_X\varphi + \alpha\wedge\varphi = 0$ holds if and only if $\iota_X\varphi = 0$ and $\alpha \wedge \varphi = 0$. This means that $X \in E$ and $\alpha \in Ann(E)$, showing that $L(E,0) = L_\varphi$. □

With this, we can define an alternative grading on the complex of differential forms. We define $U_0 = K$ and $U_k = \Lambda^k \bar{L} \cdot U_0$ for $1 \leq k \leq 2n$. Thus we decompose $\Lambda^\bullet T^*M \otimes \mathbb{C}$ as

$$\Lambda^\bullet T^*M \otimes \mathbb{C} = U_0 \oplus \cdots \oplus U_{2n}\,.$$

Clifford multiplication by sections of L has degree -1 and Clifford multiplication by sections of $\bar{L}$ has degree 1. Furthermore, we have the symmetry $\bar{U_k} = U_{2n-k}$.
In complex geometry the exterior derivative splits in the Dolbeault operators, $d = \partial = \bar\partial$. In generalized complex geometry it is also possible to define operators ∂, $\bar\partial$ analogous to the Dolbeault operators. Explicitly, they are given by:

$$\begin{aligned}\partial &= \pi_{k-1} \circ d : \Gamma(U_k) \to \Gamma(U_{k-1})\,,\\ \bar\partial &= \pi_{k+1} \circ d : \Gamma(U_k) \to \Gamma(U_{k+1})\,.\end{aligned}$$

Here π_k is the projection onto U_k. The generalized almost complex structure $\mathcal{J}$ is integrable if and only if $d = \partial + \bar\partial$. The proof can be found in [14]. Just as in complex geometry, we find $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial = -\bar\partial\partial$ as a consequence of $d^2 = 0$. This is because ∂^2, $\bar\partial^2$ and $\partial\bar\partial + \bar\partial\partial$ all map into a different U_n.
For a generalized complex structure $\mathcal{J}_J$ associated to a complex structure J, the new grading is well known. Here $U_0 = \Lambda^{n,0}T^*M$ and

$$U_k = \bigoplus_p \Lambda^{n-p,k-p}T^*M\,,$$

for $k \geq 1$. The operators ∂ and $\bar\partial$ are the Dolbeault operators and $d^{\mathcal{J}_J} = d^c$.

4.4 Generalized Kähler structure

We want to combine the notion of generalized complex structures with that of a generalized metric. For this, we impose the condition that $\mathcal{J}$ leaves the eigenspaces V_+ and V_- invariant. This implies that $\mathcal{G}$ and $\mathcal{J}$ commute. From this we find that $(\mathcal{G}\mathcal{J})^2 = -\mathrm{Id}$ and $(\mathcal{G}\mathcal{J})^* = \mathcal{J}^*\mathcal{G}^* = -\mathcal{J}\mathcal{G} = -\mathcal{G}\mathcal{J}$, meaning that $\mathcal{G}\mathcal{J}$ defines a new generalized almost complex structure. Imposing that this new generalized complex structure is integrable leads to the notion of generalized Kähler structures.

Definition 4.4.1. A **generalized Kähler structure** is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures for which $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric.

Any Kähler structure (g, ω, J) defines a generalized Kähler structure by taking $\mathcal{J}_1 = \mathcal{J}_J$ and $\mathcal{J}_2 = \mathcal{J}_\omega$. The corresponding generalized metric is given by

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure. Write the generalized metric $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ as follows:

$$\mathcal{G} = \begin{pmatrix} A & g^{-1} \\ \sigma & A^* \end{pmatrix}.$$

If we square this, we get

$$\mathcal{G}^2 = \begin{pmatrix} A^2 + g^{-1}\sigma & Ag^{-1} + g^{-1}A^* \\ \sigma A + A^*\sigma & \sigma g^{-1} + (A^*)^2 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{TM} & 0 \\ 0 & \mathrm{Id}_{T^*M} \end{pmatrix}.$$

Now we define the 2-form $b = -gA$. Using the equation above, we can write

$$\mathcal{G} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g + bg^{-1}b & bg^{-1} \end{pmatrix}.$$

From this we see that a generalized Kähler metric is completely determined by a Riemanian metric g and a 2-form b. In terms of these maps, the eigenspaces $V_\pm$ are given as the graph of $b \pm g$. The 2-form b is not necessarily closed. The torsion of a generalized Kähler structure is the 3-form $h = db$.

4.4.1 Relation to Bi-Hermitian geometry

The restrictions of the anchor $\rho_\pm = \rho_{|V_\pm}$ are isomorphisms $V_\pm \simeq TM$. We can use these isomorphisms to transform geometric structures on $V_\pm$ to geometric structures on TM. By compatibility with the generalized metric we find $\mathcal{J}_{1|V_+} = \mathcal{J}_{2|V_+}$ and $\mathcal{J}_{1|V_-} = -\mathcal{J}_{2|V_-}$. Because of this we only need to transform $\mathcal{J}_1$. This results in two almost complex structures $J_\pm$ on TM which are compatible with the induced generalized metric g.
The generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is equivalent to the quadruple (g, b, J_+, J_-) of a Riemannian metric g, a 2-form b and two almost complex structures $J_\pm$ compatible with the metric g. This quadruple is subject to integrability conditions.
We first note that $\rho[e_1, e_2] = [\rho(e_1), \rho(e_2)]$, since ρ is the anchor. This means that subbundles closed under the Dorfmann bracket get projected onto subbundles closed under the Lie bracket. This implies that the almost complex structures $J_\pm$ are integrable turning (g, J_-, J_+) into a bi-Hermitian structure.
We define the 2-forms $\omega_\pm$ by

$$\omega_\pm(X, Y) = g(J(X), Y).$$

Let $T^{1,0}_\pm$ be the i-eigenbundle of $J_\pm$ and $L_1^\pm = (\rho_{|V_\pm})^{-1}T^{1,0}_\pm$ be the preimage under the restriction of the anchor. These are explicitly given as

$$\begin{aligned} L_1^+ &= \{X + (b+g)(X) | X \in T^{1,0}_+\} \\ &= \{X + (b - i\omega_+)(X) | X \in T^{1,0}_+\}, \end{aligned}$$

$$\begin{aligned} L_1^- &= \{X + (b-g)(X) | X \in T^{1,0}_-\} \\ &= \{X + (b + i\omega_-)(X) | X \in T^{1,0}_-\}. \end{aligned}$$

Now we can find the integrability conditions for these bundles from Proposition 6.16 from [13]: A subbundle $F \subset (TM \oplus T^*M) \otimes \mathbb{C}$ given by

$$F = \{X + c(X) | X \in E\} .$$

For some complex 2-form c and a subbundle E is Courant integrable if and only if E is Lie integrable and c satisfies

$$\iota_X \iota_Y dc = 0 ,$$

for all $X, Y \in E$.
This means that $J_\pm$ are integrable almost complex structures and

$$db = d^c_- \omega_- = -d^c_+ \omega_+ ,$$

where $d^c_\pm$ is the d^c-operator associated with $J_\pm$.

4.5 SKT structures

Definition 4.5.1. A **strong Kähler structure with torsion**, or SKT structure, is a Hermitian structure (g, J) on an manifold M for which the corresponding Hermitian 2-form ω satisfies $dd^c\omega = 0$.

In the context of Courant algebroids, this definition generalizes to the following [6]:

Definition 4.5.2. A **SKT structure** on an exact Courant algebroid E is a generalized metric $\mathcal{G}$ with a complex structure $\mathcal{J}$ on the positive eigenspace V_+ which is orthogonal with respect to the pairing and for which the i-eigenspace is involutive.

Since $\rho_+ : V_+ \to TM$ is an isomorphism interchanging the brackets, we can use $\mathcal{J}$ to define a complex structure J on M by setting $J = \mathcal{J} \circ \rho_+^{-1}$. Integrability of $\mathcal{J}$ implies the integrability of J. We take the metric g to be the one induced by ρ_+, meaning that $g(X, Y) = \langle \rho_+^{-1}(X), \rho_+^{-1}(Y) \rangle$. In fact, this metric is the same as the metric induced by ρ_-. Now (g, J) defines an SKT structure on M [6]. Furthermore, $H = d^c\omega$ represents the Ševera class of E.

Definition 4.5.3. A **hyper KT structure** on a Riemannian manifold (M, g) is a triple I, J, K of complex structures which are Hermitian with respect to g satisfying $IJ = K$ and

$$d^c_I \omega_I = d^c_J \omega_J = d^c_K \omega_K = H .$$

for a closed 3-form H.

This also has an analogous definition for exact Courant algebroids:

Definition 4.5.4. A **hyper KT structure** on an exact Courant algebroid E is a generalized metric $\mathcal{G}$ together with 3 generalized complex structures $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ on V_+ satisfying $\mathcal{I}\mathcal{J} = \mathcal{K}$.

5 The Strominger system and Killing spinors

5.1 Supergravity

Heterotic string theory leads to a spectrum of massless states and higher excited massive states. The massless states consist of the spacetime metric g, the field strength H, the dilaton φ and the gauge connection A. The higher excited states have masses associated to the string scale α'. This scale has to be close to the Planck scale in order to give Newtonian gravity in the low energy limit. This energy scale is so high that there is currently no hope to approach it in experiments. Because of this, we will consider the low energy limit of the theory. The resulting effective field theory is called heterotic supergravity. This theory only contains the massless fields having truncated out all the massive fields. The action of the supergravity theory with these fields can be constructed using the Noether method [24]. The bosonic part of the action of heterotic supergravity is given by [17]:

$$S = \int_{M_{10}} e^{-2\varphi}\left(\mathcal{R} - 4|d\varphi|^2 + \frac{1}{12}|H|^2 + \frac{\alpha'}{4}(Tr(|F|^2 - Tr|R|^2)\right) dvol\,.$$

Here $\mathcal{R}$ is the Ricci scalar of g, F the curvature of the gauge connection of the $SO(32)$ or $E_8 \times E_8$ gauge bundle and R is the curvature with respect to g. The field strength H is given by

$$H = dB + \frac{\alpha'}{4}(CS_3(A) - CS_3(\nabla))\,,$$

with ∇ a connection constructed as follows. Let ∇^{LC} be the Levi-Civita connection of g. Then we define a new connection ∇ by

$$\nabla_X Y = \nabla^{LC}_X Y + \frac{1}{2} g^{-1}(\iota_Y \iota_X H)\,.$$

The norm of a differential form α is given by $|\alpha|^2 dvol = \alpha \wedge *\alpha$, where $*$ is the Hodge star operator. By varying this action with respect to the inverse metric $g^{\mu\nu}$, the Kalb-Ramond field B, the gauge connection A and the dilaton φ, we get the equations of motion:

$$\begin{aligned} R_{\mu\nu} - \nabla_\mu \nabla_\nu \varphi - \frac{1}{4} H_{\mu\alpha\beta} H^{\alpha\beta}_{\nu} + \alpha'(Tr(F_{\mu\alpha}F^{\alpha}_{\nu}) - Tr(R_{\mu\alpha}R^{\alpha}_{\nu})) &= 0\,, \\ d^*(e^{-2\varphi}H) &= 0\,, \\ d^*_A(e^{-2\varphi}F) + \frac{e^{-2\varphi}}{2} * (F \wedge *H) &= 0\,, \\ \mathcal{R} - 4\Delta\varphi - 4|d\varphi|^2 - \frac{1}{2}|H|^2 - \alpha'(Tr(|F|^2 - Tr|R|^2) &= 0\,. \end{aligned}$$

The supersymmetric partners of the bosonic fields are the gravitino ψ_M, the dilatino λ and the gaugino χ. We denote the 10-dimensional gamma matrices by Γ_M for $0 \leq M \leq 9$. The supersymmetry transformations of these fields are given by

$$\begin{aligned} \delta\psi_M &= (\nabla^{LC}_M + \frac{1}{8} H_{MNP}\Gamma^N \wedge \Gamma^P)\varepsilon\,, \\ \delta\lambda &= (\nabla\!\!\!/\varphi + \frac{1}{12} H_{MNP}\Gamma^M \wedge \Gamma^N \wedge \Gamma^P)\varepsilon\,, \\ \delta\chi &= -\frac{1}{2} F_{MN}\Gamma^M \wedge \Gamma^N \varepsilon\,. \end{aligned}$$

Once we have a solution which extremizes the action, we need to impose that the transformations vanish, $\delta\psi_M = \delta\lambda = \delta\chi = 0$, in order to satisfy supersymmetry.
We consider the theory on a compactified space $M \times X$, where M is 4-dimensional Minkowski space

and X is a compact 6 dimensional manifold known as the internal space. These constraints impose certain restrictions on the geometry of the internal space which were first formulated by Strominger [21]. The conditions $\delta\psi_M = \delta\lambda = 0$ imply the existence of spinors $\eta_\pm$ satisfying $\nabla^+\eta_\pm = 0$ which we will assume to be normalized. With these spinors we can define

$$J_M^N = i\eta_+^\dagger g^{NP}\Gamma_M \wedge \Gamma_P \eta_+ \,. \tag{2}$$

This tensor J satisfies $J^2 = -\mathrm{Id}$, which means that it is an almost complex structure. In addition the Nijenhuis tensor vanishes, meaning that the almost complex structure is integrable [21]. If the 2-form ω defined by $\omega(X, Y) = g(J(X), Y)$ is closed, the manifold is Kähler. The field strength H is given in terms of ω as $H = d^c\omega = i(\bar{\partial} - \partial)\omega$, where ∂ and $\bar{\partial}$ are the Dolbeault operators. The restrictions on the geometry of the space are combined in the form of a system of differential equations called the Strominger system.

5.2 The Strominger system

For this section we will use the formulation of the Strominger system given by Mario Garcia Fernandez in his lectures on the Stominger system [9].

Definition 5.2.1. A **topological Calabi Yau manifold** is a pair (X, Ω) consisting of a complex manifold X of dimension n and a nowhere vanishing section Ω of the bundle $\Lambda^n T^*X$.

A hermitian metric on X is a metric g which is compatible with the almost complex structure J, meaning that $g(J(X), J(Y)) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(X)$. We will write $g(J(X), Y) = \omega(X, Y)$ for all $X, Y \in \mathfrak{X}(X)$. We denote by $\Lambda_\omega : \Omega^k(X) \to \Omega^{k-2}(X)$ the operator

$$\psi \mapsto \iota_{\omega^\#}\psi = *(\omega \wedge *\psi) \,,$$

which in coordinates is given by $\omega^{i_1 i_2}\psi_{i_1 \ldots i_n}$, where $\omega^{ij} = (\omega^{-1})_{ij}$. The norm $||\Omega||_\omega^2$ is given by

$$||\Omega||_\omega^2 \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \bar{\Omega} \,.$$

The dilatino equation for a hermitian metric g on (X, Ω) is given by

$$d^*\omega = d^c log(||\Omega||_\omega) \,.$$

This equation is equivalent to the conformally balanced equation

$$d(||\Omega||_\omega \omega^{n-1}) = 0 \,.$$

Let g be a balanced metric on X, meaning that $d\omega^{n-1} = 0$. Let E be a holomorphic vector bundle over X of rank r and write $\Omega^{p,q}(E)$ for the E-valued (p, q)-forms. The holomorphic structure on E is equivalent to a Dolbeault operator

$$\bar{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E) \,,$$

satisfying $\bar{\partial}_E^2 = 0$. For a hermitian metric h on E, there is an associated unitary Chern connection A. This is the unique connection satisfying

$$d_A^{0,1} = \bar{\partial}_E \,,$$
$$d(s, t)_h = (d_A s, t) + (s, d_A t) \,,$$

where $s, t \in \Gamma(E)$ and d_A is the covariant derivative associated to A. The curvature of the unitary Chern connection is the $End(E)$-valued $(1, 1)$-form

$$F_h = d_A^2 \,.$$

For a hermitian metric h on E, the Hermite-Einstein equation is given by

$$\Lambda_\omega F_h = i\lambda \mathrm{Id}\,,$$

for a real constant λ. A unitary connection is called a Hermite-Yang-Mills connection if

$$\begin{aligned} \Lambda_\omega F_A &= i\lambda \mathrm{Id}\,, \\ F_A^{0,2} &= 0\,. \end{aligned}$$

We now have all the ingredients for the Strominger system.

Definition 5.2.2. Let (X, Ω) be a topological Calabi-Yau manifold with $E \to X$ a holomorphic vector bundle and g, h hermitian metrics on TX and E respectively. Let A be a unitary connection on (E, h) and ∇ a unitary connection on (TX, g, J). Then the **Strominger system** is given by

$$\begin{aligned} \Lambda_\omega F_h = 0, &\qquad F_h^{0,2} = 0\,, \\ \Lambda_\omega R_\nabla = 0, &\qquad R_\nabla^{0,2} = 0\,, \\ d^*\omega - d^c log(||\Omega||_\omega) = 0\,, & \\ dd^c\omega - \alpha(Tr(R\wedge R) - Tr(F_h \wedge F_h)) = 0\,, & \end{aligned}$$

where α is a real constant.

The first two equations are the condition that both A and ∇ are Hermite-Yang-Mills connections. The third equation is the dilatino equation. The last equation is called the Bianchi identity and couples two Hermite-Yang-Mills connections A and ∇ to a conformally balanced metric g with induced 2-form ω.

5.3 Killing spinors

For a given generalized metric we can define the Gualtieri-Bismut connection [13]. Let C^+ denote the orthogonal complement of $\mathfrak{g}_P$ in V^+, meaning

$$C^+ = \mathfrak{g}_P^\perp \subset V^+\,.$$

If we restrict ρ to C^+, we get an isomorphism

$$\rho|_{C^+} : C^+ \xrightarrow{\sim} TM\,.$$

With this, we define the map

$$C = \rho|_{V_-}^{-1} \circ \rho \circ \Pi_+ + \rho|_{C^+}^{-1} \circ \rho \circ \Pi_-\,.$$

This map sends V_- to C^+ and V_+ to V_-. In the splitting $E \simeq TM \oplus \mathfrak{g}_P \oplus T^*M$, this map is given by

$$\begin{aligned} C(X + g(X) + t) &= X - g(X)\,, \\ C(X - g(X)) &= X + g(X)\,, \end{aligned}$$

for $X \in \mathfrak{X}(M)$, $t \in \mathfrak{g}_P$ and induced metric g seen as a map $TM \to T^*M$.

Definition 5.3.1. Let E be a transitive Courant algebroid with generalized metric $\mathcal{G}$. The **Gualtieri-Bismut connection** D^B associated to the generalized metric $\mathcal{G}$ is given by

$$D^B_{e_1} e_2 = [e_{1-}, e_{2+}]_+ + [e_{1+}, e_{2-}]_- + [C(e_{1+}), e_{2+}]_+ + [C(e_{1-}), e_{2-}]_-\,.$$

The torsion of D^B is denoted by T_{D^B}. From this, we construct the canonical generalized Levi-Civita connection D^{LC}. For $e_1, e_2, e_3 \in \Gamma(E)$, the generalized connection D^{LC} is given by

$$\langle D^{LC}_{e_1} e_2, e_3 \rangle = \langle D^B_{e_1} e_2, e_3 \rangle - \frac{1}{3} T_{D^B}(e_1, e_2, e_3)\,.$$

In Riemannian geometry the Levi-Civita connection is the unique torsion free connection which is compatible with the metric. In the generalized case, uniqueness is lost. We can deform this generalized connection by a Weyl term, which keeps the generalized connection torsion free and compatible with the generalized metric. For $\varphi \in E^*$, the Weyl term χ^φ is given by

$$\langle \chi^\varphi_{e_1} e_2, e_3 \rangle = \varphi(e_2)\langle e_1, e_3 \rangle - \varphi(e_3)\langle e_1, e_2 \rangle\,.$$

All Weyl terms are of this form. With respect to the eigenspaces of the generalized metric, we write

$$\chi^{\varphi\pm\pm\pm}_{e_1} e_2 = \Pi_\pm \chi^\varphi_{e_1^\pm} e_2^\pm\,.$$

Using this, we define a new torsion free generalized connection which is compatible with the generalized metric as follows:

$$D^\varphi = D^{LC} + \frac{1}{3(\mathrm{rank}(V_+) - 1)}\chi^{\varphi+++} + \frac{1}{3(\mathrm{rank}(V_-) - 1)}\chi^{\varphi---}\,.$$

Explicit calculations for these connections come from [11].
We will now assume that M is a spin manifold of dimension 6. Since $\mathrm{rank}(V_-) = 6$, we can decompose the spin bundle in positive and negative half spinor bundles

$$Spin(V_-) = Spin_-(V_-) \oplus Spin_+(V_-) \subset \mathcal{C}\ell(V_-)\,.$$

For $\psi \in C^\infty(M)$, we take the generalized connection D^φ associated to V_+ and the 1-form

$$\varphi = 6d\psi\,.$$

We will write

$$D^\varphi_\pm : \Gamma(V_-) \to \Gamma((V_-) \otimes V^*_\pm)$$

for the induced differential operators. These operators work on spinors as

$$D^\varphi_\pm : \Gamma(Spin_+(V_-)) \to \Gamma(Spin_+(V_-)) \otimes V^*_\pm)\,.$$

This gives rise to an associated Dirac operator

$$\not{D}^\varphi_- : \Gamma(Spin_+(V_-)) \to \Gamma(Spin_-(V_-))\,.$$

Definition 5.3.2. Let M be a spin manifold of dimension 6, E a heterotic Courant algebroid over M, $\mathcal{G}$ a generalized metric and $\eta \in Spin_+(V_-)$ a positive chirality spinor. The **Killing spinor equations** are given by

$$D^\varphi_+ \eta = 0\,,$$
$$\not{D}^\varphi_- \eta = 0\,.$$

Solutions to the Killing spinor equations are respected by Courant algebroid isomorphisms. This means that if η is a solution to the Killing spinor equations obtained from V_+ and ψ and (φ, f) is an isomorphism, then $\varphi_*\eta$ satisfies the Killing spinor equations for $\varphi(V_+)$ and $f_*\psi$.

Recall that a generalized metric on a heterotic Courant algebroid is equivalent to a triple (g, ∇^θ, H). We define the connections $\nabla^\pm$ in terms of the Levi-Civita connection ∇^{LC} and H by

$$g(\nabla^\pm_X Y, Z) = g(\nabla^{LC}_X, Y, Z) \pm \frac{1}{2} H(X, Y, Z)\,.$$

Consider the following sections of E:

$$\begin{aligned} a_+ &= X + s + g(X)\,, \\ b_- &= Y - g(Y)\,, \\ c_+ &= Z + t + g(Z)\,, \\ d_- &= W - g(W)\,. \end{aligned}$$

The Bismut-Gualtieri connection for these sections is given by

$$\begin{aligned} D^B_{a_+} c_+ &= 2\Pi_+ \left(\nabla^+_X Z + g^{-1} c(\iota_X F, t)\right) + \nabla^\theta_X t - F(X, Z)\,, \\ D^B_{b_-} c_+ &= 2\Pi_+ \left(\nabla^+_Y Z + g^{-1} c(\iota_Y F, t)\right) + \nabla^\theta_Y t - F(Y, Z)\,, \\ D^B_{a_+} d_- &= 2\Pi_- \left(\nabla^-_X W + g^{-1} c(\iota_W F, s)\right)\,, \\ D^B_{b_-} d_- &= 2\Pi_- \left(\nabla^-_Y W\right)\,. \end{aligned}$$

Now we consider the generalized connection $D' = \nabla^g \oplus \nabla^\theta \oplus \nabla^{g^{-1}}$ on $E = TM \oplus \mathfrak{g}_P \oplus T^*M$, where ∇^g and $\nabla^{g^{-1}}$ denote the Levi-Civita connection and the induced Levi-Civita connection respectively. On $e_1 = X + s + \alpha$ and $e_2 = Y + t + \beta$, the generalized connection D' acts as

$$D'_{e_1} e_2 = \nabla^g_X Y + \nabla^\theta_X t + \nabla^{g^{-1}}_X \beta\,.$$

When comparing this expression to the expressions for D^B, we see that a couple of terms are missing. For $e = X + s + \alpha$, we construct the following endomorphisms to make up for this difference:

$$\chi'_e = \begin{pmatrix} 0 & 0 & 0 \\ -\iota_X F & -c^{-1}(c(s, [\cdot,\cdot])) & 0 \\ \iota_X H - 2c(F, s) & 2c(\iota_X F, \cdot) & 0 \end{pmatrix}\,,$$

and

$$\chi'_{Ce} = \begin{pmatrix} 0 & 0 & 0 \\ -\iota_X F & 0 & 0 \\ \iota_X H & 2c(\iota_X F, \cdot) & 0 \end{pmatrix}\,.$$

These endomorphisms are precicely constructed such that

$$D^B = D' + (\chi'_C)^{+++} + (\chi')^{---} + (\chi')^{-+-} + (\chi')^{+-+}\,. \tag{3}$$

Using Lemma 3.5 of [10] we can now easily compute the torsion of D^B. The torsion of D^B is given by

$$T_{D^B}(e_1, e_2, e_3) = \langle \chi^B_{e_{1+}} e_{2+}, e_{3+} \rangle + \langle \chi^B_{e_{1-}} e_{2-}, e_{3-} \rangle\,.$$

Here χ^B is given by

$$\chi^B_e = \begin{pmatrix} 0 & 0 & 0 \\ -\iota_X F & c^{-1}(c(s, [\cdot,\cdot])) & 0 \\ 2\iota_X H - 2c(F, s) & 2c(\iota_X F, \cdot) & 0 \end{pmatrix}\,,$$

with $e = X + s + \alpha$. From this we get an explicit expression for D^{LC}, namely:

$$D^{LC} = D^B - \frac{1}{3}(\chi^B)^{+++} - \frac{1}{3}(\chi^B)^{---}\,.$$

Now we introduce new connections $\nabla^{\pm\frac{1}{3}}$ on TM given by

$$g(\nabla^{\pm\frac{1}{3}}_X Y, Z) = g(\nabla^{LC}_X Y, Z) \pm \frac{1}{6} H(X, Y, Z)\,.$$

The generalized Levi-Civita connection now acts on a_+, b_-, c_+ and d_- as

$$\begin{aligned}
D^{LC}_{a_+} c_+ &= 2\Pi_+ \left(\nabla^{+\frac{1}{3}}_X Z + \frac{2}{3} g^{-1} c(\iota_X F, t) + \frac{1}{3} g^{-1} c(\iota_Z F, s) \right) \\
&\quad + \nabla^\theta_X t - \frac{2}{3} F(X, Z) - \frac{1}{3} c^{-1}(c(s, [t, \cdot]))\,, \\
D^{LC}_{b_-} c_+ &= 2\Pi_+ \left(\nabla^+_Y Z + g^{-1} c(\iota_Y F, t) \right) + \nabla^\theta_Y t - F(Y, Z)\,, \\
D^{LC}_{a_+} d_- &= 2\Pi_- \left(\nabla^-_X W + g^{-1} c(\iota_W F, s) \right)\,, \\
D^{LC}_{b_-} d_- &= 2\Pi_- \left(\nabla^{-\frac{1}{3}}_Y W \right)\,.
\end{aligned}$$

Lastly, we are able to explicitly compute D^φ for 1-forms φ. This generalized connection is given by

$$\begin{aligned}
D^\varphi_{a_+} c_+ &= D^{LC}_{a_+} c_+ + \frac{1}{3(\mathrm{rank}(V_+) - 1)} \Pi_+ \left(\varphi(Z) a_+ - 2(g(X, Z) + c(s, t))\varphi \right)\,, \\
D^\varphi_{b_-} c_+ &= D^{LC}_{b_-} c_+\,, \\
D^\varphi_{a_+} d_- &= D^{LC}_{a_+} d_-\,, \\
D^\varphi_{b_-} d_- &= D^{LC}_{b_-} d_- + \frac{1}{3(\mathrm{rank}(V_-) - .1)} \Pi_- \left(\varphi(W) b_- - 2g(Y, W)\varphi \right)
\end{aligned} \tag{4}$$

Now we can formulate the Killing spinor equations more explicitly.

Lemma 5.3.3. *The Killing spinor equations are equivalent to the following set of equations:*

$$\begin{aligned}
F \cdot \eta &= 0\,, \\
\nabla^- \eta &= 0\,, \\
(H + 2d\psi) \cdot \eta &= 0\,, \\
dH - c(F \wedge F) &= 0\,.
\end{aligned}$$

Note that the anchor provides an isomorphism $\rho_{|V_-}$ between V_- and TM, which we use to identify Spin(V_-) with Spin(TM). The differential forms F, H and $d\varphi$ act on Spin(TM) as sections of $\Lambda^\bullet(T^*M) \subset \mathcal{C}\ell(TM) = End(Spin(TM))$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame with respect to g and let $\{e^1, \ldots, e^n\}$ be the dual frame. We can write any $A \in End(TM)$ as

$$A = g(Ae_i, e_j) e^i \otimes e_j\,.$$

Since $e^i \otimes e_j - e^j \otimes e_i$ embeds as $\frac{1}{2} e^j e^i$ in $\mathcal{C}\ell(TM)$, any skew symmetric $A \in End(TM)$ acts on $\mathcal{C}\ell(TM)$ as

$$A = \frac{1}{2} \sum_{i<j} g(Ae_i, e_j) e^j e^i\,.$$

Now we can compute $D^{\varphi}_{+}\eta$ as follows:

$$\begin{aligned}
D^{\varphi}_{+}\eta &= D^{LC}_{+}\eta \\
&= D'_{+}\eta + \chi'^{-+-}\eta \\
&= \nabla^g \eta + \frac{1}{2}\sum_{i<j} g(\chi'^{-+-}e_i, e_j)e^j e^i \cdot \eta \\
&= \nabla^g \eta - \frac{1}{2}\sum_{i<j} H(e_i, e_j, \cdot)e^j e^i \cdot \eta + \sum_{i<j} c(F(e_i, e_j), \cdot)e^j e^i \cdot \eta \\
&= \nabla^{-}\eta - c(F\cdot\eta, \cdot)\,.
\end{aligned}$$

Here the first line follows from equation 4 and the second line follows from equation 3. This shows that the equation $D^{\varphi}_{+}\eta = 0$ is equivalent to the first two equations. Now we compute $D^{\varphi}_{-}\eta$:

$$\begin{aligned}
D^{\varphi}_{-}\eta &= D^{LC}_{-}\eta + \frac{1}{15}(\chi^{\varphi})^{---}\eta \\
&= D'_{-}\eta + \chi'^{---}\eta - \frac{1}{3}(\chi^{B})^{---}\eta + \frac{1}{15}(\chi^{\varphi})^{---}\eta \\
&= \nabla^g \eta + \frac{1}{2}\sum_{i<j} g((\chi'^{-+-} - \frac{1}{3}(\chi^{B})^{---})e_i, e_j)e^j e^i \cdot \eta + \frac{1}{30} g((\chi^{\varphi})^{---}e_i, e_j)e^j e^i \cdot \eta \\
&= \nabla^g \eta - \frac{1}{6}\sum_{i<j} H(e_i, e_j, \cdot)e^j e^i \cdot \eta \\
&+ \frac{1}{30}\sum_{i<j}(6d\psi(e_i)e^j e^i \cdot \eta \otimes e^j - 6d\psi(e_j)e^j e^i \cdot \eta \otimes e^i) \\
&= \nabla^g \eta - \frac{1}{2}\sum_{i<j} H(e_i, e_j, \cdot)e^j e^i \cdot \eta + \frac{1}{3}\sum_{i<j} H(e_i, e_j, \cdot)e^j e^i \cdot \eta \\
&+ \frac{1}{5}\sum_{i<j}(d\psi(e_i)e^j e^i \cdot \eta \otimes e^j - d\psi(e_j)e^j e^i \cdot \eta \otimes e^i) \\
&= \nabla^{-}\eta + \frac{1}{3}\sum_{i<j} H(e_i, e_j, \cdot)e^j e^i \cdot \eta + \frac{1}{5}\sum_{i<j}(d\psi(e_i)e^j e^i \cdot \eta \otimes e^j - d\psi(e_j)e^j e^i \cdot \eta \otimes e^i)\,.
\end{aligned}$$

Since $\nabla^{-}\eta = 0$, we get

$$\begin{aligned}
\not{D}^{\varphi}_{-}\eta &= \frac{1}{3}\sum_{i<j} H(e_i, e_j, e_k)e^k e^j e^i \cdot \eta + \frac{1}{5}\sum_{i<j}(d\psi(e_i)e^k e^j e^i \cdot \eta \otimes e^j(e_k) - d\psi(e_j)e^k e^j e^i \cdot \eta \otimes e^i(e_k)) \\
&= -\frac{1}{6}\sum H(e_k, e_j, e_i)e^k e^j e^i \cdot \eta + \frac{1}{5}\sum_{i<j}(d\psi(e_i)e^k e^j e^i \cdot \eta\delta^j_k + d\psi(e_j)e^k e^i e^j \cdot \eta\delta^i_k + 2d\psi(e_j)e^k \delta^{ij}\eta\delta^i_k) \\
&= -(H + 2d\psi)\cdot\eta\,.
\end{aligned}$$

Hence this gives us the third equation. The last equation holds for any heterotic Courant algebroid. Conversely, the last equation can be used to reconstruct the generalized metric and hence the generalized connections. □

Lemma 5.3.4. *Let $H \in \Omega^3(M)$ and $\varphi \in C^\infty(M)$, then a solution (g, η) with nonvanishing $\eta \in Spin_+(V_-)$ to the system*

$$\nabla^{-}\eta = 0\,,$$
$$(H + 2d\varphi)\cdot\eta = 0\,,$$

is equivalent to a Calabi-Yau structure (ω, Ω)*, consisting of a 2-form* ω*, an integrable almost complex structure* J *and a holomorphic volume form* Ω *satisfying*

$$\begin{aligned} H &= d^c\omega\,, \\ \varphi &= -\frac{1}{2}log||\Omega||_\omega - \kappa\,, \\ d^*\omega &= d^c log||\Omega||_\omega\,, \end{aligned}$$

for some constant κ *[21].*

Proof. Here we follow the proof of [11]. Let (g, η) be a solution to the equations. Define J as in equation 2 and define ψ by

$$\psi_{NMP} = \eta_-^\dagger \Gamma_N \wedge \Gamma_M \wedge \Gamma_P \eta_-\,.$$

Then ψ is a parallel completely holomorphic 3-form and the Nijenhuis tensor of J vanishes [21]. The antiholomorphic vectorfields can be characterized as follows:

$$T^{0,1}M = \{X \in TM \otimes \mathbb{C} | \iota_X \psi = 0\}\,.$$

The 2-form ω is the corresponding parallel Kähler form defined by $\omega(X,Y) = g(JX,Y)$ for all $X, Y \in \mathfrak{X}(M)$. Using the $SU(3)$ structure, we find a isomorphic bundle for the half spinor bundle $Spin_+(V_-)$, namely the Clifford module $Spin_+(V_-) \simeq \Omega^{0,even}(M)$. We also get the isomorphism $Spin_-(V_-) \simeq \Omega^{0,odd}$. The Clifford action is given by

$$\alpha \cdot \sigma = \sqrt{2}(\iota_{g^{-1}\alpha^{1,0}} + \alpha^{0,1} \wedge \sigma)\,.$$

This Clifford action sends $Spin_\pm(V_-)$ to $Spin_\mp(V_-)$. For $SU(3)$, the space of even parallel spinors is 1-dimensional and η is identified with a non-vanishing function which we take to be the constant function 1 for simplicity [23]. We choose a basis $\{dz_j, d\bar{z}_j\}$ of 1-forms such that locally

$$g = \delta^{ij} dz_i \cdot d\bar{z}_j\,.$$

In this basis, we get

$$\begin{aligned} dz_j \cdot 1 &= 0\,, \\ d\bar{z}_j \cdot 1 &= \sqrt{2} d\bar{z}_j\,, \\ dz_i \wedge dz_j \wedge d\bar{z}_k \cdot 1 &= 0\,, \\ d\bar{z}_i \wedge d\bar{z}_j \wedge dz_k \cdot 1 &= \sqrt{2}(\delta_{ik} d\bar{z}_j - \delta_{jk} d\bar{z}_i)\,. \end{aligned}$$

Now the second equation becomes

$$(H + 2d\varphi) \cdot 1 = 2\sqrt{2}\left(H^{0,3} + \sum_{i<j}(H^{1,2}_{\bar{i}\bar{j}i} d\bar{z}_j - H^{1,2}_{\bar{i}\bar{j}j} d\bar{z}_i) + \bar{\partial}\varphi\right)\,.$$

From this we find that $H^{0,3} = 0$ and $i\Lambda_\omega H^{1,2} = -2\bar{\partial}\varphi$, since the sum here is exactly $i\Lambda_\omega H^{1,2}$. From the equation $\nabla^- J = 0$ we get $H = d^c\omega$ [21], which implies

$$\Lambda_\omega d\omega = 2d\varphi\,.$$

Now we define $\Omega = e^{-2\varphi}\psi$. Then

$$\varphi = -\frac{1}{2}(||\Omega||_\omega - ||\psi||_\omega)\,,$$

which gives $\kappa = -\frac{1}{2}||\psi||_\omega$. Using that $\Lambda_\omega d\omega = Jd^*\omega$ ([9]), we get

$$d^*\omega = d^c log(||\Omega||_\omega)\,.$$

We get the opposite direction by defining $\eta = 1$ in the $Spin_+(V_-)$ model induced by the $SU(3)$ structure. □

This brings us to theorem 1.2 of [11]:

Theorem 5.3.5. *The Strominger system is equivalent to the Killing spinor equations on a heterotic Courant algebroid.*

Proof. Assume that we have solutions to the Killing spinor equations. This gives rise to a topological Calabi-Yau structure on M with conformaly balanced Kähler form ω and $H = d^c\omega$. Additionally, we have $F \cdot \eta = 0$, which implies that both A and ∇ are Hermite-Yang-Mills [23]. On top of that, the equation $dH - c(F \wedge F) = 0$ combined with $H = d^c\omega$ gives the Bianchi identity.
Conversely, for a solution to the Strominger system we define $\theta = \nabla \oplus A$, $H = d^c\omega$ and φ as before. Now the half spinor determined by ω and the complex structure satisfies the Killing spinor equations. $\square$

As a consequence, the Strominger system is a natural system of equations in generalized geometry, meaning, solutions are exchanged under isomorphisms of Courant algebroids.

6 T-duality

T-duality, short for target space duality, is a duality between two string theories described on different target spaces. It relates a theory on a torus bundle $M \to B$ to a theory on a torus bundle $\tilde{M} \to B$, both over the same base manifold B. Both torus bundles are equipped with a 3-form H and $\tilde{H}$. Here we follow the approach of [7].

Definition 6.0.1. Let M and $\tilde{M}$ be two T^k bundles over a base manifold B and $H \in \Omega^3_{T^k}(M)$, $\tilde{H} \in \Omega^3_{T^k}(\tilde{M})$ be two invariant closed 3-forms. Consider the fiber product $M \times_B \tilde{M}$ with projections $p : M \times_B \tilde{M} \to M$ and $\tilde{p} : M \times_B \tilde{M} \to \tilde{M}$. Now M and $\tilde{M}$ are **T-dual** if there exists a T^{2k}-invariant 2-form $F \in \Omega^2_{T^{2k}}(M \times_B \tilde{M})$ such that

- $dF = p^*H - \tilde{p}^*\tilde{H}$
- $F : \mathfrak{t}^k_M \times \mathfrak{t}^k_{\tilde{M}} \to \mathbb{R}$ is non-degenerate

where $\mathfrak{t}^k_M (\mathfrak{t}^k_{\tilde{M}})$ is the tangent space to the fiber of $\tilde{p}(p)$.

In the literature one often imposes that H and $\tilde{H}$ represent integral cohomology classes and that

$$(F(\partial_{\theta_i}, \partial_{\tilde{\theta}_j})) \in GL(k, \mathbb{Z})\,,$$

where $\{\partial_{\theta_i}\}$ $(\{\partial_{\tilde{\theta}_j}\})$ is a basis of invariant period 1 elements of $\mathfrak{t}^k_M$ $(\mathfrak{t}^k_{\tilde{M}})$. This last conditions means that F is unimodular. Consider the filtration

$$\Omega^k(B) = \mathcal{F}^0 \subset \cdots \subset \mathcal{F}^k = \Omega^k_{T^k}(M)\,,$$

with $\mathcal{F}^i = Ann(\Lambda^{i+1}\mathfrak{t}^k_M)$. For T-dual bundles M and $\tilde{M}$ we find that $H \in \mathcal{F}^1$, which follows from the equation $dF = p^*H - \tilde{p}^*\tilde{H}$.

Theorem 6.0.2. *Given a principal T^k-bundle M over B and an invariant 3-form $H \in \mathcal{F}^1 \subset \Omega^3_{T^k}(M)$ representing an integral cohomology class, there is another principal T^k-bundle $\tilde{M}$ over B with 3-form $\tilde{H}$ dual to M [3].*

Proof. The 3-form H is given by

$$H = \langle \tilde{c}, \theta \rangle + h\,,$$

where $h \in \mathcal{F}^0$, $\tilde{c} \in \Omega^2(B, \mathfrak{t}^{k*}_M)$, $\theta \in \Omega^1(M, \mathfrak{t}^k_M)$ a connection and $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathfrak{t}^k_M$ and its dual $\mathfrak{t}^{k*}_M$. We take $\tilde{M}$ to be a principal T^k-bundle over B with Chern class $[\tilde{c}]$. Closedness of $\tilde{c}$ follows from the closedness of H. Let $\tilde{\theta} \in \Omega^1(\tilde{M}, \mathfrak{t}^{k*}_M)$ be a connection such that $d\tilde{\theta} = \tilde{c}$. Then define $c = d\theta$ and

$$\tilde{H} = \langle c, \tilde{\theta} \rangle + h\,.$$

Now we find that

$$p^*H - \tilde{p}^*\tilde{H} = \langle \tilde{c}, \theta \rangle - \langle c, \tilde{\theta} \rangle = \langle d\tilde{\theta}, \theta \rangle - \langle d\theta, \tilde{\theta} \rangle = -d\langle \theta, \tilde{\theta} \rangle\,.$$

Hence $F = -\langle \theta, \tilde{\theta} \rangle$ is unimodular and M and $\tilde{M}$ are T-dual. □

This brings us to a theorem relating the twisted cohomology of the T-dual bundles.

Theorem 6.0.3. *Let (M,H) and $\tilde{M},\tilde{H})$ be two T-dual bundles over B with F such that $dF = p^*H - \tilde{p}^*H$. Then the map $\tau : \Omega^\bullet_{T^k}(M) \to \Omega^\bullet_{T^k}(\tilde{M})$ given by*

$$\tau(\varphi) = \int_{T^k} e^F \wedge \varphi$$

gives an isomorphism of twisted differential complexes $(\Omega^\bullet_{T^k}(M), d_H)$ and $(\Omega^\bullet_{T^k}(\tilde{M}), d_{\tilde{H}})$. Here the domain of the integral consists of the fibers of $\tilde{p}$ and $d_H\varphi = d\varphi + H \wedge \varphi$ for differential forms φ. [2]

Proof. Invertibility of τ follows from the non-degeneracy of F. The fact that τ intertwines the differentials follows from the following calculation:

$$\begin{aligned}
d_{\tilde{H}}\tau\varphi &= \int_{T^k} d_{\tilde{H}}(e^F \wedge \varphi) \\
&= \int_{T^k} de^F \wedge \varphi + e^F \wedge d\varphi + \tilde{H} \wedge e^F \wedge \varphi \\
&= \int_{T^k} (H - \tilde{H}) \wedge e^F \wedge \varphi + e^F \wedge d\varphi + \tilde{H} \wedge e^F \wedge \varphi \\
&= \int_{T^k} e^F \wedge (d\varphi + H \wedge \varphi) \\
&= \tau d_H \varphi\,.
\end{aligned}$$

□

The isomorphism τ can be seen as a composition of a pull-back, a B transform and a push-forward, which are all operations on the Clifford module of T^k-invariant forms. In order to turn τ into an isomorphism of Clifford modules, we need to specify how it acts on T^k-invariant sections. In other words, we need to find a map $\varphi : (TM \oplus T^*M)/T^k \to (T\tilde{M} \oplus T^*\tilde{M})/T^k$ such that

$$\tau(v \cdot \rho) = \varphi(v) \cdot \tau\rho$$

holds for all $v \in (TM \oplus T^*M)/T^k$ and $\rho \in \Omega^\bullet_{T^k}(M)$. We want to apply the same approach to φ as for τ, namely to define it as a composition of a pull-back, a B-transform and a push-forward. However, this is not directly well-defined, so we have to make some choices. Consider $X + \alpha \in TM \oplus T^*M/T^k$. Now $p^*\alpha$ is well-defined, but this doesn't hold for X. For X we get a lift $\tilde{X} \in T(M \times_B \tilde{M}$, which is unique up to vectors in the fiber of p. Applying a B-transform with filed $-F$ now yields $\tilde{X} + p^*\alpha - \iota_{\tilde{X}}F$. We want to take the push-forward of this section to get a section of $(T\tilde{M}^*\tilde{M})/T^k$. This is dependent on the choice of lift for $\tilde{X}$. Also, the push-forward of $p^*\alpha - \iota_{\tilde{X}}F$ is only defined if this form is basic, meaning that

$$p^*\alpha(Y) - F(\tilde{X}, Y) = 0$$

must hold for all $Y \in \mathfrak{t}^k_M$. Fortunately, by the non-degeneracy of F, there is a unique lift $\tilde{X}$ of X for which this holds. For this choice of $\tilde{X}$, we define

$$\varphi(X + \alpha) = \tilde{p}_*\tilde{X} + p^*\alpha - \iota_{\tilde{X}}F$$

Now φ satisfies $\tau(v \cdot \rho) = \varphi(v) \cdot \tau(\rho)$. Using this equation, we can prove the following theorem:

Theorem 6.0.4. *Let (M,H) and $(\tilde{M},\tilde{H})$ be two T-dual bundles over a common base B. Then φ is an isomorphism of the Courant algebroids M and $\tilde{M}$ with the H-twisted and $\tilde{H}$-twisted Dorfman brackets [7].*

Proof. Let $v_1, v_2 \in TM \oplus T^*M$, $\rho \in \Omega^{\bullet}_{T^k}(M)$, then

$$\langle v_1, v_1 \rangle \tau(\rho) = \tau(\langle v_1, v_1 \rangle \rho) = \tau(v_1 \cdot (v_1 \cdot \rho)) = \varphi(v_1) \cdot (\varphi(v_1) \cdot \tau(\rho)) = \langle \varphi(v_1), \varphi(v_1) \rangle \tau(\rho) \, .$$

Hence $\langle v_1, v_1 \rangle = \langle \varphi(v_1), \varphi(v_1) \rangle$. For the bracket, we get

$$\begin{aligned} \varphi([v_1, v_2]_H) \cdot \tau(\rho) &= \tau([v_1, v_2]_H \cdot \rho) \\ &= \tau([[d_H, v_1], v_2] \cdot \rho) \\ &= [[d_{\tilde{H}}, \varphi(v_1)], \varphi(v_2)] \cdot \tau(\rho) \\ &= [\varphi(v_1), \varphi(v_2)]_{\tilde{H}} \cdot \tau(\rho) \, . \end{aligned}$$

Hence $\varphi([v_1, v_2]_H) = [\varphi(v_1), \varphi(v_2)]_{\tilde{H}}$. □

We can write the T^k-invariant double bundles as

$$\begin{aligned} (TM \oplus T^*M)/T^k &\simeq TB \oplus T^*B \oplus \mathfrak{t}^k_M \oplus \mathfrak{t}^{k*}_M \, , \\ (T\tilde{M} \oplus T^*\tilde{M})/T^k &\simeq TB \oplus T^*B \oplus \mathfrak{t}^{k*}_M \oplus \mathfrak{t}^k_M \, . \end{aligned}$$

In this splitting the map φ simply interchanges the $\mathfrak{t}^{k*}_M$ and $\mathfrak{t}^k_M$ coordinate [7]. This means that any T^k-invariant geometric structure can be transported to the T-dual space using φ and τ. For a pair of T-dual spaces (M, H) and $(\tilde{M}, \tilde{H})$, and a generalized complex structure $\mathcal{J}_M$ on M, we get a corresponding generalized complex structure $\mathcal{J}_{\tilde{M}}$ on $\tilde{M}$. This means that for the grading of differential forms U^k_M and $U^k_{\tilde{M}}$ and generalized Dolbeault operators ∂_M and $\partial_{\tilde{M}}$ we get

$$\begin{aligned} \tau(U^k_M) &= U^k_{\tilde{M}} \, , \\ \tau(\partial_M \psi) &= \partial_{\tilde{M}} \tau(\psi) \, , \\ \tau(\bar{\partial}_M \psi) &= \bar{\partial}_{\tilde{M}} \tau(\psi) \, . \end{aligned}$$

Since φ interchanges the tangent and cotangent direction, the properties of T-dual geometric structures may differ. See [7] for how different structures change under T-duality.
The relation between T-duality and exact Courant algebroids gives different perspectives on T-duality. The first perspective is that of a different reduction of the same base space. This is described by Theorem 6.4 of [7]:

Theorem 6.0.5. *Let $(\mathcal{M}, \mathcal{H})$ be the total space of a principal $T^k \times \tilde{T}^k$-bundle and let $\Psi : \mathfrak{t}^k \times \tilde{\mathfrak{t}}^k \to \Gamma(T\mathcal{M} \oplus T^*\mathcal{M})$ be a lift of of the $T^k \times \tilde{T}^k$ action for which the pairing on $\Psi(\mathfrak{t}^k \times \tilde{\mathfrak{t}}^k)$ is non-degenerate, of split signature, and such that $\Psi(\mathfrak{t}^k)$ and $\Psi(\tilde{\mathfrak{t}}^k)$ are isotropic. Then the spaces M and $\tilde{M}$ obtained by reducing $\mathcal{M}$ by the action of T^k and $\tilde{T}^k$ are T-dual and all T-dual pairs arise from such a reduction.*

As a consequence, the Courant algebroid of T^k-invariant sections of $TM \oplus T^*M$ and the Courant algebroid of $\tilde{T}^k$-invariant sections of $T\tilde{M} \oplus T^*\tilde{M}$ are isomorphic, since both are isomorphic to the reduction of $\mathcal{M}$ by the action of $T^k \times \tilde{T}^k$.
A second perspective on T-duality is as a generalized submanifold.

Definition 6.0.6. For a pair $(\mathcal{N}, \mathcal{H})$, with $\mathcal{N}$ a manifold and $\mathcal{H}$ a 3-form, a **generalized submanifold** is a pair $(\mathcal{M}, F)$ of a submanifold $\mathcal{M}$ and a 2-form F, such that $dF = \iota^* H$.

This gives rise to the notion of a generalized tangent bundle:

$$\tau_F = \{X + \alpha \in T\mathcal{M} \oplus T^*\mathcal{N} | \alpha_{|_{\mathcal{M}}} = \iota_X F\} \, .$$

Theorem 6.5 of [7] provides a description of T-dual spaces in terms of generalized submanifolds:

Theorem 6.0.7. *Let (M, H) and $(\tilde{M}, \tilde{H})$ be two principal T^k bundles over a base manifold B. Define $(\mathcal{N}, \mathcal{H}) = (M \times \tilde{M}, H - \tilde{H})$ and $\mathcal{M} = M \times_B \tilde{M}$. Then M and $\tilde{M}$ are T-dual if and only if there exists a 2-form F on $\mathcal{M}$ such that $(\mathcal{M}, F)$ is a generalized submanifold of $(\mathcal{N}, \mathcal{H})$ and such that τ_F is everywhere transversal to $TM \oplus T^*M$ and to $T\tilde{M} \oplus T^*\tilde{M}$.*

6.0.1 Heterotic T-duality

Heterotic Courant algebroids come from a reduction of an exact Courant algebroid by a principal G-bundle. Recall that G is a compact, connected, semisimple Lie group. With an exact Courant algebroid on the total space we get a heterotic Courant algebroid on the reduced space. This raises the question of what happens with T-dual Courant algebroids after reduction to a heterotic Courant algebroids. For this, consider a principal $G \times T^k$-bundle P over a base manifold M. We define projections $\sigma : P \to P/G = X$, $\pi : P \to P/T^k = P_0$, $\pi_0 : X \to M$ and $\sigma_0 : P_0 \to M$, resulting in a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\sigma"] & P_0 \arrow[d, "\sigma_0"] \\ X \arrow[r, "\pi_0"] & M \end{tikzcd}$$

Let $H \in \mathcal{F}^1(P)$ be a 3-form representing an integral cohomology class. This means that there is a T-dual space $(\tilde{P}, \tilde{H})$ over the base P_0. We need to lift the G-action on P_0 to a G-action on $\tilde{P}$ which commutes with the T^k-action. This is possible whenever $\tilde{P}$ is the pullback of a $\tilde{T}^k$-bundle $\tilde{\sigma} : \tilde{X} \to M$ by σ_0. This turns out to always be case. Thus we have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\sigma"] & P_0 \arrow[d, "\sigma_0"] & \tilde{P} \arrow[l, "\tilde{\pi}"'] \arrow[d, "\tilde{\sigma}"] \\ X \arrow[r, "\pi_0"] & M & \tilde{X} \arrow[l, "\tilde{\pi}_0"'] \end{tikzcd}$$

Let θ be a principal connection for the T^k-bundle $X \to M$, then $\sigma^*\theta$ is a G-invariant connection on P. Take H to be $G \times T^k$-invariant such that it decomposes as

$$H = \langle \tilde{c}, \theta \rangle + h \,,$$

with $h \in \mathcal{F}^0$ and $\tilde{c} \in \Omega^2(P, \mathfrak{t}^k_P)$. Let ψ and $\tilde{\psi}$ be the actions of $\mathfrak{g}$ on P and $\tilde{P}$ coming from the G-action. This gives a commuting diagram

$$\begin{tikzcd} \Gamma(P) \arrow[r, "\pi"] & \Gamma(P_0) & \Gamma(\tilde{P}) \arrow[l, "\tilde{\pi}"'] \\ & \mathfrak{g} \arrow[ul, "\psi"] \arrow[u, "\psi_0"] \arrow[ur, "\tilde{\psi}"'] & \end{tikzcd}$$

Theorem 6.0.8. *The heterotic Courant algebroids X/T^k and $\tilde{X}/T^k$ described above are isomorphic.*

Proof. Since the map φ corresponding to the T-duality is independent of splitting we can show the desired property in a certain splitting corresponding to specic extended actions. This is easiest done by using pullback connections.

Let $\theta \in \Omega^1(X, \mathfrak{t}^k)$ and $\tilde{\theta} \in \Omega^1(\tilde{X}, \tilde{\mathfrak{t}}^k)$ be torus connections on X and $\tilde{X}$ and pull them back to P and $\tilde{P}$ and let $A \in \Omega^1(P_0, \mathfrak{g})$ be G connection which we pull back to P and $\tilde{P}$. We can find $G \times T^k$ and $G \times \tilde{T}^k$ invariant representatives of the Ševera classes written as

$$H = \sigma^* h - CS_3(A) \,,$$
$$\tilde{H} = \tilde{\sigma}^* \tilde{h} - CS_3(A) \,.$$

Since H and $\tilde{H}$ are T-dual we can find a $G \times T^k \times \tilde{T}^k$-invariant 2-form F on $P \times_{P_0} \tilde{P}$ such that

$$H - \tilde{H} = h - \tilde{h} = dF \,,$$

where we omit pullback notation. As before, we can further rewrite this to

$$\begin{aligned} H &= H^0 + \langle \tilde{c}, \theta \rangle - CS_3(A)\,, \\ \tilde{H} &= H^0 + \langle c, \tilde{\theta} \rangle - CS_3(A)\,. \end{aligned}$$

The extended actions are given by $\xi = -cA$, so the corresponding G-equivariant forms are given by

$$\begin{aligned} \Phi &= H + \xi = H^0 + \langle \tilde{c}, \theta \rangle - CS_3(A) - cA\,, \\ \tilde{\Phi} &= \tilde{H} + \xi = H^0 + \langle c, \tilde{\theta} \rangle - CS_3(A) - cA\,. \end{aligned}$$

On the correspondence space we get $\Phi - \tilde{\Phi} = dF = d_G F$, since the torus connections are basic. We use the splitting

$$\begin{aligned} (TP \oplus T^*P)/T^k &\simeq TP_0 \oplus T^*P_0 \oplus \mathfrak{t}^k \oplus \mathfrak{t}^{k*}\,, \\ (T\tilde{P} \oplus T^*\tilde{P})/T^k &\simeq TP_0 \oplus T^*P_0 \oplus \mathfrak{t}^{k*} \oplus \mathfrak{t}^k\,. \end{aligned}$$

In this splitting, $\psi : \mathfrak{g} \to TP_{T^k} \simeq TP_0 \oplus \mathfrak{t}^k$ is given by

$$\psi(x) = (\psi_0(x), \iota_{\psi(x)}\theta) = (\psi_0(x), 0)\,,$$

and similar for $\tilde{\psi}$. Hence the extended actions are given by

$$\begin{aligned} \alpha(x) &= (\psi_0(x), -c(A, x), 0, 0)\,, \\ \tilde{\alpha}(x) &= (\psi_0(x), -c(A, x), 0, 0)\,. \end{aligned}$$

Since φ simply interchanges the tangent and cotangent directions, we get $\varphi \circ \alpha = \tilde{\alpha}$, hence X/T^k and $\tilde{X}/\tilde{T}^k$ are isomorphic. □

6.0.2 Buscher rules

Now we consider the case of a circle bundle. Let (M, H) and $(\tilde{M}, \tilde{H})$ be T-dual circle bundles over the base manifold B. M and $\tilde{M}$ are endowed with connections θ and $\tilde{\theta}$ with $F = -\theta \wedge \tilde{\theta}$. These connections give splittings $TM/S^1 \simeq TB \oplus \langle \partial_\theta \rangle$, $t^*M/S^1 \simeq T^*B \oplus \langle \theta \rangle$ and similar for $\tilde{M}$. Here TB is the space of invariant horizontal vector fields and ∂_θ is an invariant period 1 generator of the circle action. A general section of $(TM \oplus T^*M)/S^1$ can be written as

$$X + f\partial_\theta + \alpha + g\theta\,,$$

with X horizontal and α basic. A pullback of this section is given by

$$X + f\partial_\theta + h\partial_{\tilde{\theta}} + \alpha + g\theta\,,$$

where we will need to find h later. The B-field transform by F is given by

$$X + f\partial_\theta + h\partial_{\tilde{\theta}} + \alpha + g\theta + f\tilde{\theta} - h\theta\,.$$

Now we require $\alpha + g\theta + f\tilde{\theta} - h\theta$ to be basic for $M \times \tilde{M} \to_B \tilde{M}$, which means that $h = g$. Now we take the push forward of this section to $\tilde{M}$ to get

$$\varphi(X + f\partial_\theta + \alpha + g\theta) = X + g\partial_{\tilde{\theta}} + \alpha + f\tilde{\theta}\,. \tag{5}$$

With this concrete formula for φ we can compute how a generalized metric transforms under T-duality. Let $\mathcal{G}$ be an invariant generalized metric with positive eigenspace V_+. We can write V_+ as the graph of $g + b$, with g a Riemannian metric and b a 2-form. Since both g and b are S^1-invariant, we can decompose them as

$$\begin{aligned} g &= g_0\theta \cdot \theta + g_1 \cdot \theta + g_2\,, \\ b &= b_1 \wedge \theta + b_2\,, \end{aligned}$$

with g_i and b_i basic tensors of degree i. A general section of V_+ can be written as

$$X + f\partial_\theta + (\iota_X g_2 + fg_1 + \iota_x b_2 - fb_1) + (g_1(X) + fg_0 + b_1(X))\theta\,.$$

Using equation 5 we compute the image of this section under φ:

$$X + (g_1(X) + fg_0 + b_1(X))\partial_{\tilde{\theta}} + (\iota_X g_2 + fg_1 + \iota_x b_2 - fb_1) + f\tilde{\theta}\,.$$

This is the graph of $\tilde{g} + \tilde{b}$ with

$$\begin{aligned}\tilde{g} &= \tilde{g}_0\tilde{\theta}\cdot\tilde{\theta} + \tilde{g}_1\cdot\tilde{\theta} + \tilde{g}_2\,,\\ \tilde{b} &= \tilde{b}_1\wedge\tilde{\theta} + \tilde{b}_2\,,\end{aligned}$$

where $\tilde{g}_i$ and $\tilde{b}_i$ are given by

$$\begin{aligned}\tilde{g}_0 &= \frac{1}{g_0}\,,\\ \tilde{g}_1 &= -\frac{b_1}{g_0}\,,\\ \tilde{g}_2 &= g_2 + \frac{b_1\cdot b_1 - g_1\cdot g_1}{g_0}\,,\\ \tilde{b}_1 &= -\frac{g_1}{g_0}\,,\\ \tilde{b}_2 &= b_2 + \frac{g_1\wedge b_1}{g_0}\,.\end{aligned}$$

These transformations are known as the Buscher rules. There is also a dilaton shift, given by

$$\tilde{\varphi} = \varphi - \frac{1}{4}log\left(\frac{G_{00}}{\tilde{G}_{00}}\right)\,.$$

6.1 T-duality as a canonical transformation

A T-duality transformation can be seen as a canonical transformation [19]. We consider a bosonic string in flat space with 1 compactified dimension of radius R. The Lagrangian density for the scalar associated with the compactified dimension is given by

$$\mathcal{L} = \frac{R^2}{2\alpha'}(\dot{\theta}^2 - \theta'^2)\,,$$

where θ is the scalar and $\dot{\theta}$ and θ' denote the partial derivatives with respect to τ and σ respectively. With the canonical momentum

$$P_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{R^2}{\alpha'}\dot{\theta}\,,$$

we get the Hamiltonian density

$$\mathcal{H} = \frac{\alpha'}{2R^2}P_\theta^2 + \frac{R^2}{2\alpha'}\theta'^2\,.$$

Now we apply the transformation to the dual variables

$$\begin{aligned}\tilde{\theta}' &= -P_\theta\,,\\ P_{\tilde{\theta}} &= -\theta'\,.\end{aligned} \tag{6}$$

This transformation preserves the Poisson bracket, so it is a canonical transformation. The Hamiltonian density in terms of these dual variables is given by

$$\mathcal{H} = \frac{R^2}{2\alpha'} P_{\tilde{\theta}}^2 + \frac{\alpha'}{2R^2} \tilde{\theta}'^2 \,.$$

And similarly, the Lagrangian density is given by

$$\mathcal{L} = \frac{\alpha'}{R^2} (\dot{\theta}^2 - \tilde{\theta}'^2) \,.$$

This Lagrangian density is the same as before the transformation but with a new radius $\tilde{R} = \frac{\alpha'}{R}$. Since the transformation was a canonical transformation, both Lagrangian densities describe the same physical system, which shows that the T duality transformation $\tilde{R} = \frac{\alpha'}{R}$ holds.
Next consider the bosonic σ-model with Lagrangian density

$$\mathcal{L} = \frac{1}{2} G_{MN} (\dot{x^M} \dot{x^N} - x'^M x'^N) + \frac{1}{2} B_{MN} (\dot{x}^M x'^N - \dot{x}^N x'^M) \,.$$

By applying a Legendre transform to the compactified coordinate $x^0 = \theta$ we get a Routhian. A Routhian is a mixed version of the Lagrangian and the Hamiltonian, which is a function of coordinates and both velocities and momenta. We apply the transformation of equation 6 again. This results in a similar Lagrangian, but with new metric and 2-form. The new tensors are again given by the Buscher rules:

$$\begin{aligned}
\tilde{G}_{00} &= \frac{1}{G_{00}} \,, \\
\tilde{G}_{0i} &= \frac{B_{0i}}{G_{00}} \,, \\
\tilde{G}_{ij} &= G_{ij} - \frac{G_{0i} G_{0j} - B_{0i} B_{0j}}{G_{00}} \,, \\
\tilde{B}_{0i} &= \frac{G_{0i}}{G_{00}} \,, \\
\tilde{B}_{ij} &= B_{ij} - \frac{G_{0i} B_{0j} - B_{0i} G_{0j}}{G_{00}} \,.
\end{aligned}$$

Additionally there is a shift for the dilaton. This shift is given by

$$\tilde{\varphi} = \varphi - \frac{1}{4} log \left(\frac{G_{00}}{\tilde{G}_{00}} \right) \,.$$

This approach of the bosonic σ-model can be extended to the heterotic string. The transformations for G_{MN} and B_{MN} remain the same. Additionally, the gauge fields A_0 and A_i transform as: [19]

$$\begin{aligned}
\tilde{A}_0 &= -\frac{A_0}{G_{00}} \,, \\
\tilde{A}_i &= -A_i + A_0 \frac{G_{0i} - B_{0i}}{G_{00}} \,.
\end{aligned}$$

However, this leads to wrong results due to the possible appearance of anomalies at the one-loop level. Now we consider a compact scalar θ coupled to a $U(1)$ Wilson line. The degrees of freedom are θ and a scalar y associated to a $U(1)$ gauge symmetry satisfying the constraint equation $\dot{y} = y'$. The Lagrangian density for this system is

$$\mathcal{L} = \frac{1}{2} G(\dot{\theta}^2 - \theta'^2) + \frac{1}{2} (\dot{y}^2 - y'^2) + A(\dot{y}\theta' - y'\dot{\theta}) \,.$$

Here $G = \frac{R^2}{\alpha'}$ and A represents a Wilson line along S^1. The constraint $\dot{y} = y'$ will be required for the equations of motion. The canonical momenta are given by

$$
\begin{aligned}
P_\theta &= G\dot{\theta} - Ay' \,, \\
P_y &= \dot{y} + A\theta' \,.
\end{aligned}
$$

The corresponding Hamiltonian is given by

$$
\mathcal{H} = \frac{1}{2}G^{-1}(P_\theta + Ay')^2 + \frac{1}{2}(P_y - A\theta')^2 + \frac{1}{2}G\theta'^2 + \frac{1}{2}y'^2 \,.
$$

In these coordinates the constraint is given by

$$
P_y - A\theta' - y' = 0 \,.
$$

Now we need to find a canonical transformation. This transformation needs to leave the Dirac bracket invariant[19]. We consider the following tranformation:

$$
\begin{aligned}
\tilde{\theta}' &= -P_\theta + AP_y \,, \\
\tilde{y}' &= -(1 + A\tilde{A})P_y + \tilde{A}P_\theta \,, \\
P_{\tilde{\theta}} &= -(1 + A\tilde{A})\theta' - \tilde{A}y' \,, \\
P_{\tilde{y}} &= -y' - A\theta' \,.
\end{aligned}
\tag{7}
$$

Here $\tilde{A}$ is the dual Wilson line. For any $\tilde{A}$ this transformation has period 2, meaning that applying the transformation twice result in the original coordinates. However, the transformation is not a canonical transformation for arbitrary $\tilde{A}$. Hence we determine $\tilde{A}$ by requiring equation 7 to be a canonical transformation. This leads to the following relations:

$$
\begin{aligned}
\tilde{G} &= \frac{G}{(G + A^2)^2} \,, \\
\tilde{A} &= -\frac{\alpha' A}{G + A^2} \,.
\end{aligned}
$$

And since $G = \frac{R^2}{\alpha'}$ ($\tilde{G} = \frac{\tilde{R}^2}{\alpha'}$), these relations can also be expressed as

$$
\begin{aligned}
\tilde{R} &= \frac{\alpha'}{R(1 + \frac{\alpha' A^2}{R^2})} \,, \\
\tilde{A} &= -\frac{A}{R^2(1 + \frac{\alpha' A^2}{R^2})} \,.
\end{aligned}
$$

This approach can be generalized. Instead of taking 1 compact dimension and 1 gauge field, we can consider n toroidal dimensions and several gauge fields. For multiple dimensions with several gauge fields, inverting a single radius leads to the following transformations: [19]

$$\begin{aligned}
\tilde{G}_{00} &= \frac{G_{00}}{(G_{00}+A_0^2)^2}\,, \\
\tilde{G}_{0i} &= \frac{G_{00}B_{0i}+A_0^2G_{0i}-A_0\cdot A_iG_{00}}{(G_{00}+A_0^2)^2}\,, \\
\tilde{G}_{ij} &= G_{ij}+\frac{G_{0i}G_{0j}-B_{0i}B_{0j}}{G_{00}+A_0^2}\,, \\
&+\frac{1}{(G_{00}+A_0^2)^2}(G_{00}(B_{0i}A_0\cdot A_j+B_{0j}A_0\cdot A_i-(A_0\cdot A_i)(A_0\cdot A_j))\,, \\
&+A_0^2((G_{0i}-B_{0i})(G_{0j}-B_{0j})+G_{0i}A_0\cdot A_j+G_{0j}A_0\cdot A_i))\,, \\
\tilde{A}_0^a &= -\frac{A_0^a}{G_{00}+A_0^2}\,, \\
\tilde{A}_i^a &= -A_i^a+A_0^a\frac{G_{0i}-B_{0i}+A_0\cdot A_i}{G_{00}+A_0^2}\,, \\
\tilde{B}_{0i} &= \frac{G_{0i}+A_0\cdot A_i}{G_{00}+A_0^2}\,, \\
\tilde{B}_{ij} &= B_{ij}-\frac{(G_{0i}A_0\cdot A_i)B_{0j}-(G_{0j}A_0\cdot A_j)B_{0i}}{G_{00}+A_0^2}\,, \\
\tilde{\varphi} &= \varphi+\frac{1}{4}log\left(\frac{det(\tilde{G})}{det(G)}\right)\,.
\end{aligned}$$

For inverting all radii you need to introduce canonical momenta for all coordinates θ_M and self dual gauge scalars y^a. This way you end up with the correct T duality transformations for general toroidal compactifications and arbitrary Wilson lines and a constant background for the Kalb-Ramond B-field. We can choose a gauge in which the couplings between the scalars θ_M and y^a vanish. In this gauge, we find

$$\begin{aligned}
\tilde{G}_{MN} &= G_{MN}+A_M\cdot A_N\,, \\
\tilde{B}_{MN} &= B_{MN}\,.
\end{aligned}$$

The transformation of A_M is determined by the constraint

$$D_\tau y^a=\dot{y}^a+A_M\dot{\theta}^M=y'^a+A_M\theta'^M=D_\sigma y^a\,.$$

6.2 Comparison of T-duality between string theory and Courant algebroids

We have discussed two notions of T-duality so far. The first notion was T-duality in Courant algebroids. The first step here is to understand T-duality for exact Courant algebroids. Two torus bundles over the same base manifold with closed invariant 3-forms are T dual if the fiber product of the bundles satisfies some conditions. By theorem 6.5 of [7] we found that this condition is equivalent to the statement that the fiber product of the torus bundles is a generalized submanifold of the Cartesian product of the torus bundles. There is an isomorphism of Courant algebroids between two T-dual torus bundles. Heterotic Courant algebroids arise from reducing exact Courant algebroids by the action of a principal G bundle. We have shown that T-dual exact Courant algebroids give an isomorphism of heterotic Courant algebroids after reduction. We found this by interchanging the tangent and cotangent coordinates in a particular splitting, up to a sign difference.

The second notion of T-duality is the T-duality of string theory. First we considered a string theory with 1 compactified dimension of radius R. This system is equivalent to a system where the radius is inverted. We showed this by showing that this corresponds to a canonical transformation. In this transformation we interchanged the momenta with the spatial derivatives, up to a sign difference. For the heterotic case we considered compact scalars coupled to Wilson lines. This approach was needed in order to get the right α' corrections, since the simpler model would lead to anomalies at the one-loop level.

In both cases of T-duality the spaces are equivalent. In mathematics this is called an isomorphism of Courant algebroids, in physics we say the systems are related by a canonical transformation. For Courant algebroids there is no unique T-dual torus bundle. However, with a particular construction we arrive at the Buscher rules. In string theory the T-dual is unique. We can find α' corrections for string theory, but there is no analogous construction on the Courant algebroid side for the α' corrections. In a sense, heterotic Courant algebroids only describe the low energy limit of the heterotic string.

7 Conclusion and outlook

This thesis has covered two seemingly unrelated areas of study. On one side there is heterotic superstring theory. String theory combines quantum mechanics and general relativity, which leads to extra dimensions and higher excited energy states. The higher excited states have masses comparable to the Planck scale, so we truncated the theory to end up with only massless states. The resulting theory is called heterotic supergravity. We then compactified the theory to 4 non-compact dimensions and 6 compact dimensions. This lead to a system of equations for the internal space called the Strominger system.

On the other side there is generalized geometry. The central object of study of generalized geometry is the Courant algebroid. The simplest example is that of an exact Courant algebroid. If a Lie group acts on the manifold, we can use a reduction procedure to obtain a heterotic Courant algebroid. We then introduced several structures, such as generalized metrics and generalized connections. With these structures we formulated the Killing spinor equations. These equations are equivalent to the Strominger system, showing that the Strominger system is a natural system of equations in generalized geometry.

Then we considered T-duality. T-duality is a relation between different torus bundles over the same base manifold. We showed that two T-dual bundles over the same base manifold have isomorphic exact Courant algebroids. Then we showed that reducing T-dual bundles by the action of a Lie Group G results in two isomorphic heterotic Courant algebroids. For two T-dual bundles we can transform generalized geometric structures from one bundle to the other. When transforming a generalized metric over a T-dual circle bundle we obtained a new generalized metric. This transformation is described by the Buscher rules.

There is another way to obtain the same Buscher rules. We showed that the same transformations arise from a coordinate transformation. Since this particular coordinate transformation preserves the Poisson brackets, it is called a canonical transformation. Canonical transformations describe the same physical system, which means that T-dual bundles describe equivalent physical systems.

We have found a connection between two different areas of study. At first sight there seems to be no relation between heterotic string theory and generalized geometry, but when we looked deeper we found multiple interactions between these areas. However, we first need to remove the higher energy states from the theory in order for this connection to emerge. Thus generalized geometry provides a useful description of heterotic string theory at low energy, but it is unable to describe the full theory. In this way, it gives an infinitesimal or low energy approximation to heterotic string theory.

In mathematics there are many structures with an infinitesimal counterpart. Lie groups have Lie algebras, Lie groupoids have Lie algebroids and so on. We say that the infinitesimal structure integrates to the global counterpart. We could wonder if Courant algebroid should also be seen as infinitesimal objects. Perhaps integrating Courant algebroids to their (currently unknown) global counterparts could help understand the higher energy behaviour of heterotic string theory. This could be the subject of further study.

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