Equivariant K-Theory

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Contents

1 Introduction

In [3] it is shown that for a compact space X, the groups $K^{-n}(X)$ can be described using Clifford-algebras. In this thesis, we will show that it is possible to do something similar for equivariant K-theory. Moreover, we will show how Clifford algebras can be used to show that the groups $K_G^{-n}(X)$ are periodic.
We will now indicate per section what it contains, what references we used and what material is new. Note that since we of course wrote everything in our own way, combined multiple sources and applied the theory to different cases, all material differs from the sources. In this section, we will only mention the most important new results/proofs.

The material in this thesis is ordered as follows:

- In Section 2, the notion of a G-space and a G-vector bundle for a group G will be introduced. We will for example show how G-vector bundles can be constructed using clutching functions, show that under certain conditions (equivariant)-sections of a G-vector bundle can be extended and show that G-vector bundles can be embedded in 'trivial' G-vector bundles. This section will mostly be based on [8] and [1] where we changed some of the proofs to hold for G-vector bundles.
  The most notable material that is new with regard to [8] and [1] is:
  (i) We give a different proof of Theorem 2.31 because we want to use a different version of the Peter-Weyl theorem.
  (ii) We also introduce quaternionic G-vector bundles and show how Theorem 2.31 can be proven for real and quaternionic G-vector bundles
- In Section 3, we will give the definition of the group $K_G^{-n}(X)$. We will also construct the long exact sequence for these groups and prove some of the basic properties of the groups $K_G^{-n}(X)$. This section is based on [8]. The most notable material that is new with regard to [8] is:
  (i) We prove Theorem 3.20 differently.
  (ii) We work out an example for Proposition 3.14. (Example 3.15)
- In Section 4, we will introduce Banach categories and pseudo abelian categories to view equivariant K-theory from a more general perspective. We will show that the category of G-vector bundles is a Banach category. This section is based on [3] and the definitions found on [5], [7] and [6]. Unless otherwise stated, the proofs in these sections are our own.
- In this section we will define the group $K(C)$ and $K^{-1}(C)$ of a Banach category C. The group $K(\varphi)$ of a quasi surjective Banach functor φ will also be introduced. We will prove some basic results for these groups and show how these groups give an 'alternative' definition of $K_G(X)$ and $K_G^{-1}(X)$ we defined in section 3 This section will be based on chapter II. 1, II.2 and III.3 of [3] and section 8 and 9 of [4]. The most notable material that is new in this section is:
  (i) We define the group $K^{-1}(C)$ differently and consequently we need to prove Lemma 5.10 because it no longer holds by definition.

(ii) We show how the theory of this section can be applied to G-vector bundles.

(iii) We give our own proof of Theorem 5.18.

(iv) We use [4] to show that the group $K_G(X,Y)$ in section 3 coincides with the group $K(i^*)$ in Section 4.

- In Section 6 we introduce Clifford algebras. The main result of this section is that the Clifford algebras are in some sense periodic. In the next three sections, we will use this to show show that the groups $K_G^{-n}(X)$ are periodic. This section will be based on chapter III.3 of [3] and [4].

- In Section 7 the groups $K_G^{l,m}(X)$ are introduced. We will use the results from the previous section to show that they are periodic. We will then explain how these groups can be used to show that

$$(K_G^{\mathbb{R}})^{-n}(X,Y) \cong (K_G^{\mathbb{R}})^{-n-8}(X,Y) \qquad \text{and} \qquad (K_G^{\mathbb{C}})^{-n}(X,Y) \cong (K_G^{\mathbb{C}})^{-n-2}(X,Y),$$

which is the main result of this section. This section is based on section III. 4 of [3]. The most notable material that is new in this section is:

(i) We apply the theory to G-vector bundles and relate it to the groups $K_G^{-n}(X,Y)$.

(ii) Lemma 7.9 and Lemma 7.10.

(iii) We give a different proof of Proposition 7.2.

- In Section 8, we will introduce gradations and will use them to define the groups $K_G^{l,m}(X,A)$ from the previous section. The main results of this section will be that the groups $K_G^{l,m}(X,A)$ are periodic (Proposition **??**), that

$$K_G^{l,m}(X,\emptyset) \cong K_G^{l,m}(X)$$

(Lemma 8.9) and the proof of Equation 7.1 (Theorem 8.16) which says that

$$K_G^{0,0}(X,A) \cong K_G(X,A).$$

This section is based on chapter III.4 and III.5 of [3] The most notable material that is new in this section is:

(i) We apply gradations as defined in [3] to G-vector bundles.

(ii) We give our own proof of Proposition 8.11 and Lemma 8.15.

- In Section 9 we will finish the proof of Theorem 7.16. We will prove that

$$K_G^{l,m+1}(X,A) \cong K_G^{l,m}(X \times [0,1], X \times \{0,1\} \cup A \times I) \tag{1.1}$$

and describe the isomorphism explicitly. We will base the proof on the proof in chapter III.5 and III.6 of [3] of Theorem 5.10, but change it such that it applies to the equivariant case. The most notable material that is new in this section is:

(i) We change the proof in [3] to prove Equation 1.1 if G is finite.

(ii) We give another (new) proof based on the case where G is finite to prove Equation 1.1 for the case that G is a compact Lie group.

- Section A is the appendix. In A.1 we define and construct the Haar measure and show it can be used to integrate vector valued functions. This appendix is based on chapter 19 of [9]. But the proofs in Section A.1.2 are our own. We also have an appendix about representations of groups. The aim of this appendix is to develop enough theory to state the Peter-Weyl theorem. This appendix is based on chapter 20, 21 and 23 of [9]. But the proofs in Section A.2.1 and A.2.2 are our own.

2 Vector bundles

In this section the notion of a G-space and a G-vector bundle for a group G will be introduced. We will for example show how G-vector bundles can be constructed using clutching functions, show that under certain conditions (equivariant)-sections of a G-vector bundle can be extended and show that G-vector bundles can be embedded in 'trivial' G-vector bundles. This section will mostly be based on [8] and [1] where we changed some of the proofs to hold for G-vector bundles.
In this section a compact space X is a Hausdorff space, with the property that any open cover $\{U_i\}_{i\in I}$ of X has a finite subcover. We start with some definitions:

Definition 2.1. *Let G be a topological group and let X be a topological space. Let $\alpha : G \times X \to X$ be a left action on X. We then call the pair (X, α) a G-space. We will usually just say that X is a G-space and use the notation $gx := \alpha(g, x)$ or $l_g(x) := \alpha(g, x)$.*

Definition 2.2. *Let X be a G-space and $Y \subset X$. We call Y a G-invariant subspace, if $gy \in Y$ for all $y \in Y$ and $g \in G$.*

Since a G-space has more structure than a topological space, a morphisms between G-spaces also must 'preserve' this structure. This motivates the following definition:

Definition 2.3. *Let G be a group and let X, Y be G-spaces. A continuous function $f : X \to Y$ is called a G-map if*

$$f(g \cdot x) = g \cdot f(x),$$

for all $g \in G$ and $x \in X$.

Example 2.4. Let G be a finite group. The pair (G, α), where we endow G with the discrete topology and define α by $\alpha(g, h) = l_g(h) = g \cdot h$ is an example of a G-space. The map $r_h : G \to G$, defined by $r_h(g) = gh$ is an example of a G-map

With the terminology introduced above, we can give the following definition:

Definition 2.5. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a G-space. A G-vector bundle over X is a G-space E together with a G-map $p : E \to X$ and, for all $x \in X$, the fiber $E_x := p^{-1}(x)$ of x is endowed with the structure of a $\mathbb{K}$-vector. It must satisfy the following conditions:*

(i) The map $l_g : E_x \to E_{gx}$ is a linear map.

(ii) For all $x \in X$, there exists an open neighbourhood U_x of X and a homeomorphism $\Psi : p^{-1}(U_x) \to U_x \times \mathbb{K}^n$, such that $P \circ \Psi = p|_{U_x}$, where $P : U_x \times \mathbb{K}^n \to U_x$ is the projection $P(y, v) = y$, and the map $\Psi|_{p^{-1}(y)} : p^{-1}(y) \to \{y\} \times \mathbb{K}^n \cong \mathbb{K}^n$ is a linear isomorphism for all $y \in U_x$. We will call the isomorphism Ψ a local trivialisation.

If $\mathbb{K} = \mathbb{R}$ we call $p : E \to X$ a real G-vector bundle and if $\mathbb{K} = \mathbb{C}$, we call $p : E \to X$ a complex G-vector bundle.

There is also a notion of a morphism between G- vector bundles:

Definition 2.6. *Let $p : E \to X$ and $q : E' \to X$ be G-vector bundles. A G-map $F : E \to E'$ is a G-vector bundle morphism if*

(i) We have $p = q \circ F$.

(ii) For all $x \in X$, the map $F|_{E_x} : E_x \to E'_x$ is linear.

We call F a (G-vector bundle) isomorphism if there exists a G-vector bundle morphism $F' : E' \to E$, such that $F \circ F' = id$ and $F : \circ F = id$.

Remark 2.7. Notice that a G-vector bundle morphism $F : E \to E'$ is already an isomorphism if $F|_x : E_x \to F_x$ is an isomorphism for all $x \in X$. The map $F' : E' \to E$ can then be defined by $F'|_x = (F|_x)^{-1}$ and is continuous.

Remark 2.8. There is also a more general notion of a G-vector bundle morphism. If $p : E \to X$ and $q : E' \to X'$ are G-vector bundles. Then a G-map $F : E \to F$ is a morphism of vector bundles if:

(i) There exists a continuous G-map $f : X \to X'$ such that $f \circ p = q \circ F$.

(ii) For all $x \in X$, the map $F|_{E_x} : E_x \to E'_{f(x)}$ is linear.

However, unless otherwise stated, by a morphism of G-vector bundles, we will mean the morphism defined in Definition 2.6.

Example 2.9. Let G be a group X a G-space and $\alpha : G \to \mathbb{C}^n$ a continuous representation of G. Then the bundle $p : X \times \mathbb{C}^n \to X$, where $p(x, g) = x$ and $h \cdot (x, v) = (hx, \alpha(g)v)$ is a complex G-vector bundle. Notice that the vector space $\mathbb{C}^n$ together with the multiplication $g \cdot v = \alpha(g)v$ is a G-module. We will often denote the representation above by $p : X \times M \to X$, where M is a G-module.

Example 2.10. Let G be a Lie group viewed as a G space as in example 2.4. Let TG denote its tangent bundle. Then the bundle $p : TG \to G$, where the G-action on TG is given by $h \cdot v_g = Tl_h(g)v_g$ for $v_g \in TG_g$, is a real G-vector bundle. Notice that the map $Tr_g : TG \to TG$, defined by $Tr_h(v_g) = Tr_h(g)v_g$ for $v_g \in TG_g$, is an example of a morphism of vector bundles in the sense of Remark 2.8.

Now that we have some intuition about what a G-vector bundle is, we will show how known G-vector bundles can be used to construct new G-vector bundles.

Definition 2.11. *Let G be a group and $p : E \to X$ and $p' : E' \to X$ be G-vector bundles, then the following bundles are G-vector bundles:*

1. *The bundle $q : E \oplus E' \to X$, where $(E \oplus E')_x = E_x \oplus E'_x$ for all $x \in X$ and the G action is given by $g(v, w) = (gv, gw)$, for all $(v, w) \in E_x \oplus E'_x$*
2. *The bundle $q : E \otimes E' \to X$, where $(E \otimes E')_x = E_x \otimes E'_x$ for all $x \in X$ and the G-action is given by the map $g\cdot : E_x \otimes E'_x \to E_{gx} \otimes E'_{gx}$ induced by the map $g\cdot : E_x \times E'_x \to E_{gx} \otimes E_{gy}$ defined by $g(v, w) = (gv) \otimes (gw)$.*
3. *The bundle $q : \mathrm{Hom}(E, E') \to X$, where $\mathrm{Hom}(E, E')_x = \mathrm{Hom}(E_x, E'_x)$, which is the set of linear maps from E_x to E'_x and the action of g is defined by $g(A : E_x \to E'_x) = l_g \circ A \circ l_{g^{-1}} : E_{gx} \to E'_{gx}$.*

In the following lemma, we will define the topology show that they are indeed G-vector bundles.

Lemma 2.12. *The bundles defined above are indeed G-vector bundles.*

Proof. We will prove the lemma for (i), the proof of the other cases is similar. Let $\{(U_i, \Psi_i, \Psi'_i)\}_{i\in I}$ be a cover of X, such that $\Psi : p^{-1}(U_i) \to U \times \mathbb{K}^n$ and $\Psi' : (p')^{-1}(U_i) \to U_i \times \mathbb{K}^{n'_i}$ are local trivialisations. Notice that for all $g \in G$, the map

$$\psi_i \circ l_g : l_g^{-1}((p^{-1}(U))) \to U_i \times \mathbb{K}^n$$

is also a local trivialisation. Therefore, we can add local trivializations to the cover I, such that if $(U_i, \psi_i, \psi'_i) \in \{(U_i, \Psi_i, \Psi'_i)\}_{i\in I}$, then $(l_g^{-1}(U_i), \psi_i \circ l_g, \psi'_i \circ l_g) \in \{(U_i, \Psi_i, \Psi'_i)\}_{i\in I}$. Let $F = \coprod_{i\in I} U_i \times \mathbb{K}^{n_i} \times \mathbb{K}^{n'_i}$ and let $\pi : F \to E \oplus E'$ be the map defined by $\pi(x, v, v') = (\Psi_i^{-1}(v), \Psi_i^{-1}(v'))$ for $i \in I$. We now endow $E \oplus E'$ with the quotient topology ($U \subset E \oplus E'$ is open if and only if $\pi^{-1}(U)$ is open). With respect to this topology the map $p : E \oplus E' \to X$ is continuous and the map $(\Psi_i, \Psi'_i) : p^{-1}(U_i) \to U_i \times \mathbb{K}^{n_i} \times \mathbb{K}^{n'_i}$ are local trivializations. Since

$$\pi \circ l_g : \coprod_{i\in I} U_i \times \mathbb{K}^{n_i} \times \mathbb{K}^{n'_i} \to E \oplus E'$$

which maps a $x \in l_g^{-1}((U_i) \times \mathbb{K}^n \times \mathbb{K}^{n_i}$ to $gx \in U_i \times \mathbb{K}^n \times \mathbb{K}^{n_i}$ is continuous and constant on the fibers of π, it induces a continuous map l_g on $E \oplus E'$, which is precisely the multiplication we already defined. Therefore, $E \oplus E'$ is a G-vector bundle. $\square$

Another construction which is often useful is the pull-back of a G-vector bundle.

Definition 2.13. *Let $p : E \to X$ be a G-vector bundle and let $f : Y \to X$ be a G-map, then we can define the pull-back bundle $q : f^*E \to Y$ by $f^*E_y = E_{f(y)}$ and endow it with the G-action which maps a $v \in E_y$ to $gv \in E_{gf(y)} = E_{f(gy)} = E_{gy}$.*

Lemma 2.14. *This bundle is indeed a G-vector bundle.*

Proof. We endow f^*E with the topology such that the following diagram is a pull-back square:

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "q"] & E \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd} .$$

Thus $f^*E = \{(y, v) \in Y \times E \mid f(y) = p(v)\}$, with the subspace topology, where $F(y, v) = v$ and $q(y, v) = y$. With this definition, the action of G on f^*E becomes $g \cdot (y, v) = (gy, gv)$, which is continuous and well defined. We now show that the bundle has local trivialisations. Let $y \in Y$. Then there exists a local trivialisation (U, Ψ), such that $f(y) \in U$. We now define the map $H : q^{-1}(f^{-1}(U)) \to f^{-1}(U) \times \mathbb{R}^n$ by $H(y, v) = (y, \mathrm{pr}_2 \circ \Psi(v))$, where pr_2 is the projection on the second coordinate. Notice that H has a continuous inverse which is given by $H^{-1}(y, v) = (y, \Psi^{-1}(f(y), v))$. Therefore, Ψ is a local trivialisation. $\square$

Example 2.15. Let $p : E \to X$ be a vector bundle, $A \subset X$ a G-invariant subset and $i : A \to X$ the inclusion. The bundle bundle $q : i^*E \to A$ is an example of a pull-back bundle. Notice that this bundle is isomorphic to the bundle $p|_{p^{-1}(A)} : p^{-1}(A) \to A$. We will often denote this bundle by $E|_A$.

Example 2.16. If $p : X \times M \to X$ is a G vector bundle and $f : Y \to X$ is a G-map, then

$$f^*(X \times M) := \{Y \times (X \times M) \mid x = f(y)\} \cong Y \times M,$$

where the isomorphism is given by $(y, f(y), v) \to (y, v)$.

The following lemma shows how the pull-back 'commutes' with the sum and product of G-vector bundles we defined before:

Lemma 2.17. *Let $p : E \to X$ and $q : F \to X$ be G-vector bundles and $f : Y \to X$ a G-map, then*

$$f^*(E \oplus F) \cong f^*E \oplus f^*F$$

and

$$f^*(E \otimes F) \cong f^*(E) \otimes f^*(F).$$

Proof. The isomorphisms are given by the map $\Psi : f^*(E \oplus F) \to f^*E \oplus f^*F$, defined by

$$\Psi(y, (v_E, V_F)) = ((y, v_E), (y, V_F))$$

and $\Phi : f^*(E \otimes F) \to f^*(E) \otimes f^*(F)$, defined by

$$\Phi(y, V_E \otimes V_F) = (y, V_E) \otimes (y, V_F).$$

□

The pull-back the composition of functions and pullback are related in the following sense:

Lemma 2.18. *Let X, Y and Z be G-spaces, $f : X \to Y$ and $h : Y \to Z$ be G-maps and $p : E \to Z$ a G-vector bundle. Then*

$$(h \circ f)^*E \cong f^*h^*E.$$

Proof. By definition, we have

$$(h \circ f)^*E = \{(x, v) \in X \times E \mid h \circ f(x) = p(v)\}$$

and

$$f^*h^*E = \{(x, (y, v) \in X \times Y \times E \mid f(x) = y \text{ and } h(y) = p(v)\}.$$

The map $\Psi : f^*h^*E \to (h \circ f)^*E$ defined by

$$\Psi(x, y, v) = (x, v)$$

gives the required G-vector bundle isomorphism. □

We will now prove some basic results about sections of G-vector bundles. We first give the definition:

Definition 2.19. *Let $p : E \to X$ be a G-vector bundle. We call a map $s : X \to E$ a section if $p(s(x)) = x$ for all $x \in X$. We will denote the space of sections over X by $\Gamma(E)$. We call a section $s : X \to E$ an equivariant section if s is a G-map and will denote the set of equivariant sections by $\Gamma^G(E)$.*

Example 2.20. If $X \times M$ is a G-vector bundle, then the map $s : X \to X \times M$, defined by $s(x) = (x, v)$ for a $v \in M$, is a section of $X \times M$. If $v = 0$, then

$$s(gx) = g(x, 0) = (gx, 0) = g(x, 0) = g(sx)$$

and the section is an equivariant section.

The following lemma and theorem motivates why it can be useful to look at sections of a G-vector bundle

Lemma 2.21. *Let $p : E \to X$ and $q : F \to X$ be G-vector bundles, let $\mathrm{Hom}_{vb}(E, F)$ denote the space of morphisms of vector bundles between E and F. Then,*

$$\Gamma^G(\mathrm{Hom}(E, F)) \cong \mathrm{Hom}_{vb}(E, F).$$

Proof. Let $f \in \mathrm{Hom}_{\mathrm{vb}}(E, F)$, then for all $x \in X$, the map $f|_{E_x} : E_x \to F_x$ is a linear map and hence and element of $\mathrm{Hom}(E_x, F_x)$. Let $H : \mathrm{Hom}_{\mathrm{vb}}(E, F) \to \Gamma^G(\mathrm{Hom}(E, F))$ be the map defined by $H(f)(x) = f|_{E_x}$. We first show that $H(f)$ is continuous. Let (U, Ψ_E) and (U, Ψ_F) be local trivialisations of E and F, then f induce a vector bundle homomorphism $f : U \times \mathbb{K}^n \to U \times \mathbb{K}^m$ over U. The map $\tilde{f} : U \to M_{m,n}(\mathbb{K})$, with $\tilde{f}(x) = f(x, \cdot)$ is continuous. Because $\tilde{f} = H(f)|U$, it follows that $H(f)$ is continuous. Also notice that $f \circ l_g = l_g \circ f$, which implies that

$$g \cdot (H(f)(x)) = l_g \circ f|_{E_x} \circ l_{g^{-1}} = f_{E_{gx}} \circ l_g \circ l_{g^{-1}} = f_{E_{gx}} = G(f)(gx).$$

Hence H is well defined.
We now show that H has an inverse. Let $s \in \Gamma^G(\mathrm{Hom}(E, F))$. Then s induces an element $s' \in \mathrm{Hom}_{\mathrm{vb}}(E, F)$, which is defined by $s'(v) = s(p(v))(v)$. As before, it can be shown that s is continuous by restricting to a local trivialisation. We show that s' is a G-map. Since s is an equivariant section, we have $s(gx) = gs(x) = l_g \circ s(x) \circ l_{g^{-1}}$. Therefore,

$$\begin{aligned}
s'(gv) &= s(p(gv))(gv) \\
&= s(gp(v))(gv) \\
&= l_g(l_{g^{-1}} s(gp(v)) \circ l_g)(v) \\
&= l_g s(g^{-1} gp(v))(v) \\
&= l_g \circ s(pv)(v) \\
&= gs'(v).
\end{aligned}$$

and s is a G-map. We claim that H^{-1} is the map which maps s to s'. Notice that

$$H \circ H^{-1}(s)(x) = H(s')(x) = s'|_{E_x} = s(x)$$

and

$$H^{-1} \circ H(f)(v) = H(f)'(v) = H(f)(p(v))v = f|_{p(v)}(v) = f(v).$$

Thus, H^{-1} is indeed the inverse. $\square$

To state the next theorem, we need the following definition:

Definition 2.22. *Let X and Y be G-spaces and let $f, g : Y \to X$ be G-maps. We call f and g G-homotopic if there exists a homotopy $H : Y \times [0, 1] \to X$, such that:*

(i) The map $H_t := H(\cdot, t)$ is a G-map for all $t \in [0,1]$.

(ii) $H_0 = f$.

(iii) $H_1 = g$.

Theorem 2.23. *Let $p : E \to X$ be a G-vector bundle with X compact, Y a compact G-space and $f, g : Y \to X$ be G-homotopic. Then*

$$f^*E \cong g^*E.$$

To prove this theorem we will need a view lemmas:

Lemma 2.24. *(Tietze extension theorem) Let X be a compact (Hausdorff) space, let $A \subset X$ be a closed subset and let $f : A \to X$ be a continuous map, then there exists a map $\tilde{f} : X \to \mathbb{R}$, such that $\tilde{f}|_A = f$.*

Lemma 2.25. *Let $p : E \to X$ be a G-vector bundle, with X compact and let $Y \subset X$ be a closed subset. Let $s : Y \to E|_Y$ be a section, then there exists a section $\tilde{s} : X \to E$, such that $\tilde{s}|_Y = s$.*

Proof. For each $y \in Y$, let (U_y, Ψ_y) be a local trivialisation of E. Since X is compact Hausdorff, X is normal, which implies that there exists an open $V_y \subset U_y$, such that $\{y\} \subset V_y \subset \overline{V_y} \subset U$.

Notice that $\{(V_y|_Y)\}_{y \in Y}$ is a cover of Y. Because Y is a closed subset of a compact space X, it follows that Y is compact. This implies that the cover $\{(V_y|_Y)\}_{y \in Y}$ has a finite subcover $\{V_i \cap Y\}_{i \in I}$. Let $\{\chi_i\}_{i \in I} \cup \{\chi_{Y-X}\}$ be a partition of unity subordinated to the cover $\{V_i\}_{i \in I} \cup \{X - Y\}$. Notice that $\mathrm{pr}_2 \circ \Psi_i \circ s : U_i \to \mathbb{R}^n$ is a vector valued function. Since for all $i \in I$, the set $Y \cap \overline{V_i}$ is a closed subset of $\overline{V_i}$, we can use Lemma 2.24 to extend the coordinate functions of $\mathrm{pr}_2 \circ \Psi_i \circ s$ and obtain a map $s_i : \overline{V_i} \to \mathbb{R}^n$ such that $s_i|_{\overline{V} \cap Y} = \mathrm{pr}_2 \circ \Phi_i \circ s|_{U_i}$ and a section $s_i' := \Psi_i^{-1} \circ (id \times s_i) : \overline{V_i} \to E|_{V_i}$ such that $s_i'|_{\overline{V_i}} = s|_{\overline{V_i}}$. Let $\tilde{s} : X \to E$ be the section defined by

$$\tilde{s}(x) = \sum_{i \in I} \chi_i(x) s'(x),$$

where $s_i'(x) := 0$ if $x \notin \overline{V_i}$. For $y \in Y$, we have

$$\tilde{s} = \sum_{i \in I} \chi(y) s'(y) = \sum_{i \in I} \chi_i(y) s(y) = s(y).$$

Therefore, $\tilde{s}$ is the required section. □

If we assume that Y is G-invariant, then the following holds:

Lemma 2.26. *Let $p : E \to X$ and $s : Y \to E|_Y$ as above. Assume in addition that Y is G-invariant and that s is an equivariant section, then there exists an equivariant section $\tilde{s} : X \to E$ such that $\tilde{s}|_Y = s$.*

Proof. Lemma 2.25 implies that there is a section $s' : X \to E$, such that $s'|_Y = s$. Let $\tilde{s} : X \to E$ be the section defined by

$$\tilde{s}(x) = \int_G g s'(g^{-1}x) dg.$$

We first show that for all $x \in X$, the integral is well defined. Let $f : X \times G \to E$ denote the function $f(x, g) = gs'(g^{-1}x)$. Notice that $f(x, g) = \alpha_E(g, s'(\alpha_X(g^{-1}, x)))$, which implies that f is continuous. Therefore, $f_x := f(x, \cdot)$ is continuous. We have $s(g^{-1}x) \in E_{g^{-1}x}$, which implies that $gs'(g^{-1}(x)) \in E_x \cong \mathbb{K}^n$ for all $g \in G$. Therefore, the function $f : X \times G \to E$ is continuous and f_x can thus be integrated as explained in definition A.17 in the appendix.

We now show that $\tilde{s}$ is continuous. Let (U, Ψ) be a local trivialisation and let $x \in U$. Let $V \subset U \times \mathbb{R}^n$ open. We will show that there is an open $W \subset U$, such that $(x, s'(x)) \in W$ and $W \subset \tilde{s}^{-1}(U)$. Since V is open, there exists an open neighbourhood U_0 of x and an $\epsilon > 0$, such that $U_0 \times B_{2\epsilon}(s'(x)) \subset V$, where $B_{2\epsilon}(s'(x)) := \{y \in \mathbb{R}^n, \|s'(x) - y\| < 2\epsilon\}$. For $g \in G$, we let $U_g = (\mathrm{pr}_2 \circ f)^{-1}(B_\epsilon(f(x, g))) \subset G \times X$. Notice that $\{\mathrm{pr}_G(U_g)\}_{g\in G}$ is an open cover of G. Since G is compact, the cover $\{\mathrm{pr}_G(U_g)\}_{g\in G}$ has a finite subcover $\{\mathrm{pr}_G(U_{g_i})\}_{i\in I}$. Let $W := \cap_{i\in I}\mathrm{pr}_X(U_{g_i})$. Notice that W is open and $x \in W$. For $(y, g) \in W \times G$, we have $(y, g) \in U_{g_i}$ for a $i \in I$, by construction we also have $(x, g) \in U_{g_i}$ and thus

$$\|f(x, g) - f(y, g)\| < 2\epsilon.$$

Lemma A.18 now implies that for all $y \in W$, we have

$$\begin{aligned}\|\tilde{s}(x) - \tilde{s}(y)\| &= \| \int_G f(x, g)dg - \int_G f(y, g)dg\| = \| \int_G f(x, g) - f(y, g)dg\| \\ &\leq \int_G \|f(x, g) - f(y, g)\|dg < \int_G 2\epsilon dg < 2\epsilon.\end{aligned}$$

Therefore,

$$W \subset \tilde{s}^{-1}(W \times B_{2\epsilon}(s(x))) \subset \tilde{s}^{-1}(V)$$

and $\tilde{s}$ is continuous.

We now show that $\tilde{s}|_Y = s$. Let $y \in Y$. Since Y is G-invariant and s is equivariant, we have $gy \in Y$ and

$$gs'(g^{-1}y) = gs(g^{-1}y) = gg^{-1}s(y) = s(y)$$

for all $(g, y) \in G \times Y$. Therefore,

$$\tilde{s}(y) = \int_G gs'(g^{-1}y)dg = \int_G s(y)dg = s(y).$$

Lastly, we prove that $\tilde{s}$ is equivariant. Let $x \in X$ and $h \in G$. Then using lemma A.18, it follows that

$$\tilde{s}(hx) = \int_G gs(g^{-1}hx)dg = \int_G hh^{-1}gs'((h^{-1}g)^{-1}x)dg = h\int_G l^*_{h^{-1}}f_x(g)dg = h\int_G f_x(g)dg = h\tilde{s}(x).$$

The section $\tilde{s}$ is thus the required section. □

Before we can prove the theorem, we need one more lemma:

Lemma 2.27. *Let X a compact G-space , $Y \subset X$ be a G-invariant subset and let $p : E \to X$ and $q : F \to Y$ be G-vector bundles. Assume that there exists an isomorphism $f : E|_Y \to F|_Y$ over Y. Then there exists an open $Y \subset U \subset X$ and an isomorphism $\tilde{f} : E|_U \to F|_U$, such that $\tilde{f}|_Y = f$.*

Proof. In Lemma 2.21, we showed that the map $f : E|_Y \to F|_Y$ can be viewed as an equivariant section $f' \in \Gamma^G(\mathrm{Hom}(E,F)|_Y)$. Lemma 2.26 implies that this section can be extended to a section $\tilde{f} \in \Gamma^G(\mathrm{Hom}(E,F))$. Notice that the function $\det \circ \tilde{f}$ is continuous and let $U = (\det \circ \tilde{f})^{-1}(\mathbb{C} - \{0\})$. Since for all $u \in U$, the map $\tilde{f}(u)$ is an isomorphism, Remark 2.7 implies that $\tilde{f} : E|_U \to F|_U$ is an isomorphism. It remains to show that $Y \subset U$ and that U is G-invariant. We first show that $Y \subset U$. Since $f : E|_Y \to F|_Y$ is an isomorphism, it follows that for all $y \in Y$ the map $f|_y : E_y \to F_y$ is an isomorphism. Therefore, $\det(\tilde{f}(y)) = \det(f(y)) \neq 0$ and $y \in U$.
We now show that U is G-invariant. Let $g \in G$ and $u \in U$. Then

$$\det(\tilde{f}(gu)) = \det(g\tilde{f}) = \det(l_g \circ \tilde{f}) = \det(l_g)\det(\tilde{f}) \neq 0,$$

where we view l_g as the linear map $l_g : F_u \to F_{gu}$. Therefore, $gu \in U$. □

We are now ready to prove Theorem 2.23

Proof of Theorem 2.23. Let $H : Y \times [0,1] \to X$ be a G-homotopy such that $H_0 = f$ and $H_1 = g$. The space $Y \times [0,1]$, with the action $g(y,t) = (gy,t)$ is a G-space and with this action, the homotopy $H : Y \times [0,1] \to X$ is a G-map. Let $\mathrm{pr}_Y : Y \times [0,1] \to Y$ denote the projection onto Y. Notice that for all $t \in [0,1]$, the map $H_t \circ \mathrm{pr}_Y$ is a G map. By definition, we have

$$(f_t \circ \mathrm{pr}_Y)^* Ev = \{((y,t),v) \in (Y \times [0,1]) \times E \mid H_t \circ \mathrm{pr}_Y(x,t) = p(v)\}$$

and

$$H^*E = \{((x,t),v) \in (Y \times [0,1]) \times E =| H(x,t) = p(v))\}.$$

Since $Y \times \{t\}$ is a closed G-subspace and $H|_{Y,t} = H_t \circ \mathrm{pr}_Y$, we have

$$(H_t \circ \mathrm{pr}_Y)^* Ev|_{Y \times \{t\}} \cong H^*E|_{Y \times \{t\}},$$

where the isomorphisms is given by $\Phi((y,t),v) = ((y,t),v)$. Lemma 2.27 now implies that there exists a G-invariant open neighbourhood $Y \times \{t\} \subset U$, such that Φ extends to an isomorphism $\Phi' : (H_t \circ \mathrm{pr}_Y)^*E|_U \to H^*E|_U$. Since Y is compact, there exists an open $t \in W_t \subset [0,1]$ such that $Y \times W_t \subset U$. For all $s \in W$, we have that Φ' restricts to an isomorphism

$$\Phi' : (H_t \circ \mathrm{pr}_Y)^*E|_{Y \times s} \to H^*E|_{Y \times s}.$$

Since

$$(H_t \circ \mathrm{pr}_Y)^*E|_{Y \times s} = i_s^*(H_t \circ \mathrm{pr}_Y)^*E = (H_t \circ \mathrm{pr}_Y \circ i_s)^*E = H_t^*E$$

and

$$H^*E|_{Y \times s} = i_s^* H^*E| = (H \circ i_s)^*E = H_s^*E,$$

where $i_s(y) = (y,s)$, it follows that $H_t^*E \cong H_s^*E$ for all $s \in W_t$. We now show that this implies that $H_0^*E \cong H_1^*E$. Let $\sim$ be an equivalence relation on $[0,1]$ defined by $s \sim t$ if $H_s^*E \cong H_t^*E$. The argument above implies that for all $t \in T$, the class $[t]_\sim$ an open set. However, because $[t]_\sim = [0,1] - \cup_{s \notin [t]_\sim}[s]_\sim$, the set $[t]_\sim$ is also closed. Because $[0,1]$ is connected, it follows that $[0]_\sim = [0,1]$ and $H_0^*E \cong H_1^*$. □

This theorem is will turn out to be very useful. We now give some first applications of this theorem to illustrated its usefulness.

Theorem 2.28. *Let X be compact and G-contractible (G-homotopy equivalent to a point) and let $p : E \to X$ be a G-vector bundle. Then there exists a G-module M, such that $E \cong X \times M$.*

Proof. Since X is G-contractible, there exists a $x_0 \in X$ and a G-homotopy $H : X \times I \to X$ such that $H_0 = id$, $H_1(x) = x_0$ for all $x \in X$. Theorem 2.23 implies that

$$E \cong id^* E \cong H_1^*(E) \cong H_1^*(E|_{\{x_0\}}).$$

Notice that $E|_{x_0} \cong \{x_0\} \times M$ for some G-module M. Example 2.16 implies that

$$H_1^*(E_{x_0}) \cong H_1^*(\{x_0\} \times M) \cong X \times M.$$

Thus, $E \cong X \times M$. □

Let X be a compact G-space and let X_1, X_2 be closed G-invariant subsets, such that $X_1 \cup X_2 = X$. Let $p_1 : E_1 \to X_1$ and $p_2 : E_2 \to X_2$ be G-vector bundles Let $Y = X_1 \cap X_2$ and let $f : E_1|_Y \to E_2|_Y$ be an isomorphism. Notice that Y is indeed a G-invariant subset of X, which implies that the bundles $E_1|_Y$ and $E_2|_Y$ are indeed G-vector bundles We can then define the bundle

$$p : E_1 \cup_f E_2 \to X,$$

whith $E_1 \cup_f E_2 := E_1 \coprod E_2 / \sim$ where identify a $v \in E_1$, with its image $f(v)$.

Theorem 2.29. *The bundle $E_1 \cup_f E_2$ is indeed a G-vector bundle and if g is homotopic to f through G-vector bundle isomorphisms, then $E_1 \cup_f E_2 \cong E_1 \cup_g E_2$.*

Proof. We first show that $E_1 \cup_f E_2$ is indeed a G-vector bundle. We will only show that $E_1 \cup_f E_2$ has local trivialisation, the other requirements follow directly from the construction. First assume that $x \notin Y$. We can assume without loss of generality that $x \in X_1$. Let (U, Ψ) be a local trivialisation of E_1, then $(U \cap (X_1 - Y), \Psi|_{U \cap (X_1 - Y)})$ is a local trivialisation of $E_1 \cup_f E_2$.
Now assume that $x \in Y$. Let (U_1, Ψ) be a local trivialisation of E_1, such that $x \in U_1$. Since Y is compact, Y is normal and there exists an open neighbourhood V of x, such that $\{x\} \subset V \subset \overline{V} \subset X$. Notice that the map

$$\Psi \circ f^{-1} : E_2|_{\overline{V} \cap Y} \to (\overline{V} \cap Y) \times \mathbb{K}^n \cong (X_2 \times \mathbb{K}^n)|_{\overline{V} \cap Y},$$

is an isomorphisms and $\overline{V} \cap Y$ is a closed subset of X_2. Therefore, if we apply lemma 2.27 (where we endow E_2 and $X_2 \times \mathbb{K}^n$ with the trivial action), it follows that the map $\Psi \circ f^{-1}$ extends and we obtain a local trivialisation (V_2, Ψ_2) of E_2, such that $\Psi_2|_{Y \cap \overline{V}} = \Psi \circ f^{-1}$. We now define the map $\Psi \coprod \Psi_2 : E_1|_{V \cup V_2} \coprod E_2|_{V \cup V_2} \to (V \cup V_2) \times \mathbb{K}^n$, by

$$\Psi \coprod \Psi_2 = \begin{cases} \Psi(x) & \text{if} \quad x \in V \\ \Psi_2(x) & \quad\quad x \in V_2 \end{cases}$$

This map is constant on the equivalence classes of $\sim$ and hence induces a map $\Psi \cup_f \Psi_2 : E_1 \cup_f E_2|_{V \cup V_2} \to (V \cup V_2) \times \mathbb{K}^n$, which is the desired local trivialisation.

We now show that if g is homotopic to f through G-vector bundle isomorphism, then $E_1 \cup_f E_2 \cong E_1 \cup_g E_2$.

Let $H : E_1|_Y \times [0,1] \to E_2|_Y$ denote the homotopy, with $H_0 = f$ and $H_1 = g$. Let $\mathrm{pr}_Y : Y \times [0,1] \to Y$ be the map defined by $\mathrm{pr}_Y(y,t) = y$ and let $i_t : Y \to Y \times I$ be the map given by $i_t(y) = (y,t)$. The map H induces a map $\tilde{H} : \mathrm{pr}_Y^* E_1|_{Y\times[0,1]} \to \mathrm{pr}_Y^* E_2|_{Y\times[0,1]}$ defined by $\tilde{H}((y,t),v) = H(v,t)$. The idea of the prove is that we show that

$$E_1 \cup_{H_t} E_2 \cong i_t^*(\mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2).$$

Since $F(y,t) = (y,t)$ is a G-homotopy from i_0 to i_1, Theorem 2.23 then implies that

$$E_1 \cup_{H_0} E_2 \cong i_0^*(\mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2) \cong i_1^*(\mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2) \cong E_1 \cup_{H_1} E_2.$$

It thus remains to show that $E_1 \cup_{H_t} E_2 \cong i_t^*(\mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2)$. By definition, we have

$$i_t^*(\mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2) = \{(y,v) \in X \times \mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2 \mid (y,t) = q(v)\}.$$

Define the map $\Phi : E_1 \coprod E_2 \to \mathrm{pr}_Y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_Y^* E_Y$ by

$$\Phi(v) = \begin{cases} (p_{E_1}(v), t, v) & \text{if} \quad v \in E_1 \\ (p_{E_2}(v), t, v) & \quad\quad v \in E_2 \end{cases}$$

Notice that this map is continuous and that if $v_2 = H_t v_1$, then $\Phi(v_1) = \Phi(v_2)$. Therefore, this map induces a map $\tilde{\Phi} : E_1 \cup_{H_t} E_2 \to \mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2$. Since $p(\tilde{\Phi}(E_1 \cup_{H_t} E_2)) \subset Y \times \{t\}$, this map is in fact a map $\tilde{\Phi} : E_1 \cup_{H_t} E_2 \to i_t^*(\mathrm{pr}_y^* E_1 \cup_{\tilde{H}} \mathrm{pr}_y^* E_2)$, which is the required isomorphism. □

This construction has the following nice properties, which follow directly from the construction:

Lemma 2.30. *(i) Let $p_i : E_i \to X_i$ and $q_i : F_i \to X_i$ be vectorbundles for $i \in \{1,2\}$ and let $f : E_1 \to E_2$ and $g : F_1 \to F_2$ be isomorphism, then*

$$(E_1 \oplus F_1) \cup_{(f,g)} (E_2 \oplus F_2) \cong (E_1 \cup_f F_1) \oplus (E_2 \cup_g F_2).$$

(ii) Let $p : E \to X$ be a vectorbundle, $E_1 = E|_{X_1}$, $E_2 = E|_{X_2}$ and $f : E_1|_Y \to E_2|_Y$ te cannonical isomorphism, then

$$E \cong E_1 \cup_f E_2.$$

We will end this section by proving a theorem that will need in the next section:

Theorem 2.31. *Let X be a compact G-sapce and let $p : E \to X$ be a G-vectorbundle, then there exists a G-module M and a vectorbundle $E^\perp$, such that $X \times M \cong E \oplus E^\perp$.*

To prove this theorem, we will need to introduce the notion of a metric on a G-vector bundle:

Definition 2.32. *Let $p : E \to X$ be a G-vector bundle. A map $\mu : E \oplus E \to \mathbb{K}$ is a metric on E if $\mu|_{E_x \times E_x}$ is an inner product. We wil call μ invariant if $\mu(gv, gw) = \mu(v,w)$.*

Lemma 2.33. *Let $p : E \to X$ be a G-vector bundle, then E has an invariant metric.*

Proof. We first construct a metric and then use the metric to construct an invariant metric. Let $\{(U_i\}_{i\in I}$ be a cover of X of local trivialisations. Since X is compact, we can assume that it is a finite cover. Let $\mu_i : (U_i \times \mathbb{K}^n) \times (U_i \times \mathbb{K}^n) \to \mathbb{K}$ be the metric defined by

$\mu_i(v, w) = \langle v, w\rangle$, where $\langle \cdot, \cdot\rangle$ is the Euclidean inner product. Since $U_i \times \mathbb{K}^n \cong E_{U_i}$, this defines a metric on $E|_{U_i}$. Let χ_i be a partition of unity subordinated to $\{U_i\}$. The function $\mu' : E \oplus E \to \mathbb{K}$, defined by

$$\mu'(v, w) = \sum_{i\in I} \chi_i(p(v, w))\mu_i(v, w),$$

is a metric on E. The function $\mu(v, w) : E \oplus E \to \mathbb{K}$, with

$$\mu(v, w) := \int_G \mu'(gv, gw)dg,$$

also defines a metric on G. Notice that

$$\mu(hv, hw) = \int_G \mu'(ghv, ghw)dg = \int_G l_h^*\mu'(\cdot v, \cdot w)dg = \int_G \mu(gv, gw)dg = \mu(v, w).$$

which implies that the metric is invariant. □

The following proposition shows why metrics can be useful

Proposition 2.34. *Let $p : E \to X$ be a sub bundle of $p : F \to X$, then there exists a G-vector bundle $p : E^\perp \to X$, such that $E \oplus E^\perp = F$.*

Proof. Let μ be an invariant metric on F. Then we define the bundle $E^\perp$ by

$$E_x^\perp := \{v \in F_x \mid \mu(v, w) = 0 \text{ for all } w \in E_x\}.$$

The action on $E^\perp$ is the action on F restricted to $E^\perp$. This is well defined because

$$\mu(gv, w) = \mu(gvg(g^{-1}w)) = \mu(v, g^{-1}w) = 0,$$

for all $v \in E_x^\perp$ and $w \in E_{gx}$. We now show that the bundle is locally trivial. Let $x_0 \in X$ and $s_1, \ldots, s_n$ be sections such that $s_1(y), \ldots, s_k(y)$ is a basis of E_y for all y in some neighbourhood of x_0 and $s_{k+1}(x_0), \ldots, s_n(x_0)$ is a basis of $E_{x_0}^\perp$. Let U be an open neighbourhood of x, such that $s_i(x) \neq 0$ for all $x \in U$ and $1 \leq i \leq n$. By setting

$$s_1'(x) := \frac{s_1(x)}{\mu(s_1(x), s_1(x))}$$

and

$$s_{i+1}'(x) = s_{i+1} - (\frac{\mu(s_{i+1}(x), s_1'(x))}{\mu(s_1'(x), s_1'(x))}s_1'(x)) - \ldots - \frac{\mu(s_{i+1}(x), s_i'(x))}{\mu(s_i'(x), s_i'(x))}s_i'(x),$$

it follows that the local sections $\{s_{k+1}'(y), \ldots s_n'(y)\}$ form a basis of $E_y^\perp$ for all y is some open neighbourhood W of x_0, which implies that $E^\perp$ is locally trivial. □

We will use this result to prove the following lemma, which will enable us to prove Theorem 2.31:

Lemma 2.35. *Let M be a G-module, $p : E \to X$ a G-vector bundle and $\pi : X \times M \to E$ a surjective map, then there exists a bundle $Ker(E)$, such that $Ker(E) \oplus E \cong X \times M$.*

Proof. Let $q : \mathrm{Ker}(E) \to X$ be the G-vector bundle, such that $\mathrm{Ker}(E)_x = \ker(\pi|_{\{x\}\times M})$. We leave it to the reader to check that $q : \ker(E) \to X$ is well defined. Notice that locally π is a surjective map $U \times \mathbb{K}^n \to U \times \mathbb{K}^m$, which implies that the kernel has a locally constant rank. The following sequence of maps is pointwise an exact sequence

$$0 \longrightarrow \mathrm{Ker}(E) \xrightarrow{\ i\ } X \times M \xrightarrow{\ \pi\ } E \longrightarrow 0\ .$$

Notice that $\mathrm{Ker}(E)$ is a G-vector bundle, which is also a subset of of $X \times M$. If we let μ be an G-invariant metric on $X \times M$, then Proposition 2.34 implies that we have $\mathrm{Ker}(E) \oplus \mathrm{Ker}(E)^{\perp} \cong X \times M$. Let $j : \mathrm{Ker}(E) \oplus \mathrm{Ker}(E)^{\perp} \to \mathrm{Ker}(E)$ denote the projection on the first coordinate. Then since $j \circ i = id$, the sequence is pointwise a split exact seqence and $X \times M \cong \ker(E) \oplus E$, where the isomorphism is given by the map $\Psi : X \times M \to \ker(E) \oplus E$, with $\Psi(x, m) = (j(x, m), p(x, m))$. $\square$

Remark 2.36. Notice that the proof this lemma, also holds if we replace the bundle $X \times M$ by an arbitrary vector bundle. The statement then becomes:
Let $\pi : F \to E$ be a surjective G-vector bundle morphism, then there exists a bundle Ker(E), such that Ker$(E) \oplus E \cong F$.

We are now ready to prove Theorem 2.31 for a complex G-vector bundle.

Proof of Theorem 2.31 for a complex G-vector bundle. We first assume that E is a complex vector bundle. Lemma 2.35 implies that it is sufficient to construct a surjective G-vector bundle morphism $p : X \times M \to E$.

Let $x_0 \in X$. Let (U_{x_0}, Ψ_i) be a local trivialisation, such that U_{x_0} is a compact neighbourhood of x_0. Let $s_i : U_{x_0} \to X \times \mathbb{C}^n$ be the section defined by $s_i(x) = (x, e_i)$ for $1 \leq i \leq n$. Let $\beta : G \times U_{x_0} \to U_i \times \mathbb{C}^n$ be the map $\beta(g, x) = (x, g s_i(g^{-1}x))$ and let $V_x^i := \beta^{-1}(U_{x_0} \times B(e_i, \epsilon_i))$, with $\epsilon_i > 0$. Since $\{e\} \times U_{x_0} \subset V_i$ and U_{x_0} is compact, there exists an open set $W_i \subset G$, such that $W_i \times U_x \subset V_i$. Let $\{\chi_{W_i}, \chi_{G-\{e\}}\}$ be a partition of unity with respect to the cover $\{W_i, G - \{e\}\}$ of G. Notice that $\chi_{W_i} \in L^2(G)$. Theorem A.36 implies that there exists a $f_i \in \bigoplus_{\alpha \in [G]} C_\alpha(G)$, such that $\|f - \chi_W\|_{L^2} < \epsilon_i'$. Notice that there is a finite subset $J_i \subset [G]$, such that $f_i \in \bigoplus_{\alpha \in J_i} C_\alpha(G)$, which is a finite dimensional vector space. Let $\Phi : \bigoplus_{\alpha \in J_i} C_\alpha(G) \to \Gamma_G(E)$ be the map such that

$$\Phi(f)(x) \frac{1}{\int_G \chi_{W_i} dx} \int_G f(g) g s(g^{-1}x) dg.$$

Notice that Φ is linear. We endow $\mathrm{Im}(\Phi)$ with the G-action $g \cdot s = l_g \circ g \circ l_{g^{-1}}$. We first show that this action maps the image of Φ to itself:

$$\begin{aligned} h \cdot \Phi(f)(x) &= l_h \frac{1}{\int_G \chi_{W_i} dx} \int_G f(g) g s(g^{-1}h^{-1}x) dg \\ &= \frac{1}{\int_G \chi_{W_i} dx} \int_G (L_h f)(hg) hg s((hg)^{-1}x) dg \\ &= \frac{1}{\int_G \chi_{W_i} dx} \int_G L_h f(g) g s(g^{-1}x) dg \\ &= \Phi(L_h f)(x). \end{aligned}$$

Since by Lemma A.35, we know that $\bigoplus_{\alpha\in J_i} C_\alpha(G)$ is L_h invariant, it follows that $L_h f \in \bigoplus_{\alpha\in J_i} C_\alpha(G)$, which implies that $h\cdot\Phi(f)\in \mathrm{im}(\Phi)$. This computation shows that

$$id\times\Phi : X\times\bigoplus_{\alpha\in J_i} C_\alpha(G)\to X\times \mathrm{im}(\Phi),$$

is a morphism of G-vector bundles, where the action on $X\times\bigoplus_{\alpha\in J_i} C_\alpha(G)$ is $g\cdot f = L_g(f)$. Let $J_x = \cup_{i=1}^n J_i$. We now show that there is an open set U'_x and a surjective morphism $\tau : X\times\bigoplus_{\alpha\in J} C_\alpha(G)|_{U'_x}\to E|_{U'_x}$.

Notice that

$$\begin{aligned}
\|\Phi(f_i)(x_0)-s_i(x_0)\| &= \|\frac{1}{\int_G \chi_{W_i} dg}\int_G f_i(g) g s(g^{-1}x_0)dg - s_i(x_0)\| \\
&= \frac{1}{\int_G \chi_{W_i} dg}\int_G \|\chi_{W_i}(g)s_i(x_0) - f_i(g) g s(g^{-1}x_0)dg\| \\
&\leq \frac{1}{\int_G \chi_{W_i} dg}(\int_G \|\chi_{W_i}(g)s_i(x_0)-f_i(g)s_i(x_0)\|dg + \int_G \|f_i(g)(s_i(x_0)-gs(g^{-1}x_0))\|dg) \\
&\leq \frac{1}{\int_G \chi_{W_i} dg}(\epsilon_i'\|s_i(x_0)\| + \int_G |f_i(g)|\epsilon_i dg) \\
&\leq \frac{\epsilon'\|s_i(x_0)\|}{\int_G \chi_{W_i} dg} + \epsilon_i\frac{\int_G (|f_i(g)-\chi_i(g)| + |\chi_i(g)|dg)}{\int_G \chi_{W_i} dg} \\
&\leq \frac{\epsilon'\|s_i(x_0)\|}{\int_G \chi_{W_i} dg} + \epsilon(\frac{\int_G \chi_{W_i} dg + \epsilon'}{\int_G \chi_{W_i} dg}).
\end{aligned}$$

Therefore, we can approximate $s_i(x_0)$ as close as we want, which implies that we can assume that $\{\Phi(f_1)(x_0),\ldots,\Phi(f_2)(x_0)\}$ is a basis of E_{x_0}. This implies that $\{\phi(f_1)(x),\ldots,\phi(f_n)(x)\}$ is a basis of E_x for all x in a neighbourhood U'_{x_0} of x_0. It follows that the map $\tau : X\times\bigoplus_{\alpha\in J_x} C_\alpha(G)|_{U'_x}\to E|_{U'_x}$ defined by $\tau(x,f)=\Phi(f)(x)$ is a surjective G-vector bundle morphism.

We now use this map to obtain the surjection $F : X\times M\to E$.

Notice that $\{U'_x\}_{x\in X}$ is a cover of X. Since X is compact, this cover has a finite subcover $\{U_{x_0},\ldots,U_{x_n}\}$. Let $J=\cup_{i=1}^n J_{x_i}$. Let $F : X\times\bigoplus_{\alpha\in J} C_\alpha(G)\to E$ be the map defined by $F(x,f)=\Phi(f)(x)$. The argument above implies that F is a surjective G-vector bundle morphism, which proves the theorem for the complex case.

□

We now prove the real case:

Proof of Theorem 2.31 for a real G-vector bundle. We again construct a surjective map $F : X\times M\to E$. If $p : E\to X$ is a real G-vector bundle, we can construct the complex G-vector bundle bundle $q : E\otimes\mathbb{C}\to X$. This bundle is defined by $(E\otimes\mathbb{C})_x = E_x\otimes_{\mathbb{R}}\mathbb{C}$ and the G-action is defined by $g(V\otimes z)=(gv)\otimes z$. Since $E\otimes\mathbb{C}$ is a complex G-vector bundle, there exists a surjective map $F : X\times M\to E\otimes\mathbb{C}$, where M is a complex G-module. Notice that we can also view $X\times M$ and E as real G-vector bundles, by forgetting the complex structure. Since there is a surjective real G-vector bundle morphism $\tau : E\otimes\mathbb{C}\to E$, defined by $\tau(v \otimes z) = \mathrm{Re}(z)v$, the map $\tau \circ F : X \times M \to E$ is a surjective G-vector bundle morphism. $\square$

We will also need a variation of Theorem 2.31 for a quaternionic G-vector bundle. We first give the definition of such a bundle:

Definition 2.37. *Let X be a compact G-space. A quaternionic vector bundle over X is a pair (E, σ), where $p : E \to X$ is a real G-vector bundle and $\sigma : \mathbb{H} \to End(E)$ is a unital $\mathbb{R}$-algebra homomorphism. A map $f : (E, \sigma) \to (F, \rho)$ is a quaternionic G-vector bundle morphism if it is a G-vector bundle morphisms and $\rho(h) \circ f = f \circ \sigma(h)$ for all $h \in \mathbb{H}$.*

Example 2.38. If X is a G-space, then the bundle $q : X \times \mathbb{H} \to X$, where the $\mathbb{H}$-action is left multiplication and the G-action is given by $g(x, h) = (gx, h)$ is an example of a quaternionic G-vector bundle. Moreover, if $p : E \to X$ is a real G-vector bundle, then $E \otimes_{\mathbb{R}} (X \times \mathbb{H})$ is a quaternionic G-vector bundle, where the $\mathbb{H}$-action is given by $x \cdot (v \otimes h) = (v \otimes xh)$.

Remark 2.39. If (E, σ) is a quaternionic G-vector bundle, then for each $x \in X$, there exists a neighbourhood U of X and a $n \in \mathbb{N}$ such that there is a $\mathbb{H}$-module isomorphism

$$E|_U \to U \times \mathbb{H}^n.$$

One can construct this isomorphism as follows: Let $v \in E_x$ and $s_1 : X \to E$ a section such that $s_1(x) = v$. Then the vectors $s_1(x), \sigma(i)s_1(x), \sigma(j)s_1(X), \sigma(k)s_1(x)$ are linearly independent. If the span of these vectors is not yet E_x, let

$$w \in E_x - \mathrm{Span}(s_1(x), \sigma(i)s_1(x), \sigma(j)s_1(X), \sigma(k)s_1(x))$$

and $s_2 : X \to E_x$ a section such that $s_2(x) = w$. Since

$$w \notin \mathrm{Span}(s_1(x), \sigma(i)s_1(x), \sigma(j)s_1(X), \sigma(k)s_1(x)),$$

the vectors

$$s_1(x), \sigma(i)s_1(x), \sigma(j)s_1(X), \sigma(k)s_1(x), s_2(x), \sigma(i)s_2(x), \sigma(j)s_2(X), \sigma(k)s_2(x)$$

are linearly independent. We can repeat this process to eventually obtain a basis

$$s_1(x), \sigma(i)s_1(x), \sigma(j)s_1(X), \sigma(k)s_1(x), \ldots, s_n(x), \sigma(i)s_n(x), \sigma(j)s_n(X), \sigma(k)s_n(x)$$

of E_x. We can restrict to a neighbourhood U of x, such that these sections form a basis of E_y for all $y \in Y$. The isomorphism is now given by the map $F : U \times \mathbb{H}^n \to E|_U$, with

$$F(u, h_1, \ldots, h_n) = \sigma(h_1)s_1(u) + \ldots + \sigma(h_n)s_n(u).$$

We also need the following definitions:

Definition 2.40. *Let V be a finite dimensional $\mathbb{H}$-module. We call a map $g : V \times V \to \mathbb{H}$ a quaternionic inner product on V, iff*

1. $g(u, v) = \overline{g(v, u)}$ *for all* $u, v \in V$.

2. $g(v, v) \geq 0$ *for all* $v \in V$ *and* $g(v, v) = 0$ *if and only if* $v = 0$.

3. $g(u_1 + u_2, v) = g(u_1, v) + g(u_2, v)$ *for all* $u_1, v_1, v_2 \in V$

4. $g(\lambda u, v) = \lambda g(u, v)$ *for all* $\lambda \in \mathbb{H}$ *and* $u, v \in V$.

Remark 2.41. Notice that part 1 of the definition implies that

$$g(v, v) = \overline{g(v, v)}.$$

Therefore, we have $g(v, v) \in \mathbb{R}$, which is needed for part 2 of the definition.

Example 2.42. The map $\langle \cdot, \cdot \rangle : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$, defined by

$$\langle u, v \rangle = \sum_{i=1}^{n} u\overline{v}.$$

is an example of an quaternionic inner product.

Definition 2.43. *Let* (E, σ) *be a quaternionic* G*-vector bundle. We call a map* $\mu : (E, \sigma) \oplus (E, \sigma) \to \mathbb{H}$ *a quaternionic metric if* $\mu|_{E_x \times E_x}$ *is a quaternionic inner product. It is invariant if* $\mu(gv, gw) = \mu(v, w)$ *for all* $g \in G$.

Remark 2.44. Notice that Lemma 2.33, Proposition 2.34, Lemma 2.35 and Remark 2.36 also holds for quaternionic bundles. (The direct sum is defined by $(E, \sigma) \oplus (F, \rho) = (E \oplus F, \sigma \oplus \rho)$). The proof of these statements for quaternionic G-vector bundles are similar to the proofs we already gave, where we now use quaternionic metrics instead of metrics, use the quaternionic inner product from the Example 2.42 instead of the Euclidean inner product and replace the G-module M in Lemma 2.35 by the G-module $M \otimes \mathbb{H}$.

With this remark, we are ready to state and prove the theorem:

Theorem 2.45. *Let* (E, σ) *be a quaternionic* G*-vector bundle. Then there exists a real* G*-module* M *and a quaternionic* G*-vector bundle* $(E^\perp, \sigma^\perp)$*, such that*

$$X \times (M \otimes \mathbb{H}) \cong (E \oplus E^\perp, \sigma \oplus \sigma^\perp).$$

Proof. Notice that since E is a real vector bundle, there exists a real G-vector bundle $E^\perp$ and a real G-module M, such that $E \oplus E^\perp \cong X \times M$ as real G-vector bundles. Let $\Phi : X \times M \to E \oplus E^\perp$ denote this isomorphism and let $P : E \oplus E^\perp \to E$ denote the projection. The map $\pi : X \times (M \otimes \mathbb{H}) \to E$ defined by

$$\pi(m, h) = \sigma(h)P(\Phi(m)),$$

is a surjective quaternionic G-vector bundle morphism. Remark 2.44 together with Lemma 2.35 implies that there exists a quaternionic G-vector bundle (F, ρ), such that

$$(E \oplus F, \sigma \oplus \rho) \cong X \times (M \otimes \mathbb{H}).$$

□

3 Equivariant K-theory

In this section, we will give the definition of the group $K_G^{-n}(X)$. We will also construct the long exact sequence for these groups and prove some of the basic properties of the groups $K_G^{-n}(X)$. This section is based on [8].
In this section, all topological spaces are compact and G will always be a compact Lie group. Let $p : E \to X$ be a complex (or real) G-vector bundle. We let

$$[E] := \{q : F \to X \mid q : F \to X \text{ is isomorphic to } p : E \to X\}$$

denote its isomorphism class. We will denote the set of complex (or real) G-vector bundles over X by $\mathcal{E}_G^{\mathbb{K}}(X)$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We are now ready to give the following definition:

Definition 3.1. *Let X be a compact G-space. Then, for $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, we define the group $K_G^{\mathbb{K}}(X)$ by*

$$K_G^{\mathbb{K}}(X) := \{([E],[F]) \mid E, F \in \mathcal{E}_G^{\mathbb{K}}(X)\}/\sim,$$

Where $([E_+],[E_-]) \sim ([F_+],[F_-])$ if there exists a G-vector bundle $L \in \mathcal{E}_G^{\mathbb{K}}(X)$, such that

$$E_+ \oplus F_- \oplus L \cong F_+ \oplus E_- \oplus L.$$

We will often denote $([E_+],[E_-])$ by $[E_+] - [E_-]$ or simply $E_+ - E_-$.

Remark 3.2. We will often denote $K_G^{\mathbb{K}}(X)$ by $K_G(X)$.

It turns out that $K_G(X)$ 'naturally' has the structure of an abelian group:

Lemma 3.3. *The direct sum $(E_+ - E_-) \oplus (F_+ - F_-) = (E_+ \oplus F_+) - (E_- \oplus F_-)$ gives $K_G(X)$ the structure of an abelian group.*

Proof. First, notice that $\oplus$ is associative. Let E be a G-vector bundle on X. We show that $E - E$ is the identity. For $F_+ - F_- \in K_G(X)$, we have $(F_+ - F_-) \oplus (E - E) = (F_+ \oplus E) - (F_- \oplus E)$. Since

$$F_+ \oplus (F_- \oplus E) \cong (F_+ \oplus E) \oplus F_-$$

via the isomorphism $\Phi(f_+, f_-, e) = (f_+, e, f_-)$, it follows that $(F_+ \oplus E) - (F_- \oplus E) = F_+ - F_-$ in $K_G(X)$. We now prove that each element in $K_G(X)$ has an inverse. Let $E_+ - E_- \in K_G(X)$. Then $E_- - E_+ \in K_G(X)$ and

$$(E_+ - E_-) \oplus (E_- - E_+) = (E_+ \oplus E_-) - (E_- \oplus E_+).$$

Since $E_- \oplus E_+ \cong E_+ \oplus E_-$, via the isomorphism $\Phi(v_+, v_-) = (v_-, v_+)$, it follows that

$$(E_+ \oplus E_-) - (E_- \oplus E_+) = (E_+ \oplus E_-) - (E_+ \oplus E_-) = e$$

in $K_G(X)$.
Lastly, we show that $K_G(X)$ is indeed abelian. Let $E_+ - E_-, F_+ - F_- \in K_G(X)$. We have that

$$(E_+ - E_-) \oplus (F_+ - F_-) = (E_+ \oplus F_+) - (E_- \oplus F_-)$$

and

$$(F_+ - F_-) \oplus (E_+ - E_-) = (F_+ \oplus E_+) - (F_- \oplus E_-).$$

Since $E_\pm \oplus F_\pm \cong F_\pm \oplus E_\pm$ via the isomorphism $\Phi(v_e, v_f) = (v_f, v_e)$, it follows that

$$(F_+ \oplus E_+) - (F_- \oplus E_-) = (E_+ \oplus F_+) - (E_- \oplus F_-).$$

□

It is sometimes useful to describe $K_G(X)$ in the following way:

Proposition 3.4. *Let $(E_+, E_-), (F_+, F_-) \in \mathcal{E}_G(X) \times \mathcal{E}_G(X)$ and $\sim$ be the relation from Definition 3.1. Then $(E_+, E_-) \sim (F_+, F_-)$ iff There is a G-module M, such that*

$$E_+ \oplus F_- \oplus M_X \cong F_+ \oplus E_- \oplus M_X,$$

Where M_X denotes the trivial bundle $X \times M$.

Proof. Since $M_X \in \mathcal{E}_G(X)$, it follows that if $E_+ \oplus F_- \oplus M_X \cong F_+ \oplus E_- \oplus M_X$, then $(E_+, E_-) \sim (F_+, F_-)$. Now assume that $(E_+, E_-) \sim (F_+, F_-)$. Then, there is a $F \in \mathcal{E}_G^{\mathbb{K}}(X)$ and an isomorphism $\Phi : E_+ \oplus F_- \oplus F \to F_+ \oplus E_- \oplus F$ Theorem 2.31 implies that there exists a bundle $F^\perp$, such that $F \oplus F^\perp \cong M_X$, for some G-module M. It follows that

$$E_+ \oplus F_- \oplus M_X \cong E_+ \oplus F_- \oplus F \oplus F^\perp \cong F_+ \oplus E_- \oplus F \oplus F^\perp \cong F_+ \oplus E_- \oplus M_X,$$

which proves the proposition. □

Remark 3.5. Let $\{e\}$ be the trivial group. Notice that the only finite dimensional complex (or real) $\{e\}$-modules are the spaces $\mathbb{C}^n$ (or $\mathbb{R}^n$) for $n \in \mathbb{N}_0$, with the trivial action. The proposition above thus implies that in $\mathcal{E}_{\{e\}}(X) \times \mathcal{E}_{\{e\}}(X)$, we have $(E_+, E_-) \simeq (F_+, F_-)$ if and only if

$$E_+ \oplus F_- \oplus (X \times \mathbb{C}^n) \cong F_+ \oplus E_- \oplus (X \times \mathbb{C}^n),$$

or $(X \times \mathbb{R}^n)$ in the real case. We will often denote $K_{\{e\}}^{\mathbb{K}}(X)$ by $K^{\mathbb{K}}(X)$.

This remark enable us to compute some examples:

Example 3.6. Let $X = \{\mathrm{pt}\}$. We will compute $K_{\{e\}}^{\mathbb{K}}(X)$. Since every $\{e\}$-vector bundle over pt is trivial, the previous remark implies that

$$X \times \mathbb{K}^{n_+} - X \times \mathbb{K}^{n_-} = X \times \mathbb{K}^{k_+} - X \times \mathbb{K}^{k_-},$$

if and only if there exists a $l \in \mathbb{N}_0$ such that

$$X \times \mathbb{K}^{n_+ + k_- + l} \cong X \times \mathbb{K}^{k_+ + n_- + l}.$$

This is true if and only if $n_+ - n_- = k_+ - k_-$. Therefore, the function $\Phi : K_{\{e\}}^{\mathbb{K}}(X) \to \mathbb{Z}$ defined by

$$\Phi(X \times \mathbb{K}^{n_+} - X \times \mathbb{K}^{n_-}) = n_+ - n_-$$

is an isomorphism.

Example 3.7. We will now compute $K_G(G)$, where the G-action is given by left multiplication. We will show that

$$K_G(G) \cong K_{\{e\}}(\mathrm{pt}) \cong \mathbb{Z}.$$

Let $i_e : \mathrm{pt} \to G$ be the G-map defined by $i_e(\mathrm{pt}) = e$, where e denotes the identity in G. The map $i_e^* : K_G(G) \to K_{\{e\}}(\mathrm{pt})$ defined by

$$i_e^*(E_+ - E_-) = i_e^* E_+ - i_e^* E_-,$$

where we restrict the G-action to a $\{e\}$ action, is a well defined group homomorphism. We will show that i_e^* is an isomorphism. Let $r : K_{\{e\}}(\mathrm{pt}) \to K_G(G)$ be the map defined by

$$r(\mathrm{pt} \times \mathbb{K}^{n_+} - \mathrm{pt} \times \mathbb{K}^{n_-}) = G \times \mathbb{K}^{n_+} - G \times \mathbb{K}^{n_-},$$

where the G action on $G \times \mathbb{K}^{n_\pm}$ is defined by

$$g(h, v) = (gh, v).$$

Notice that r is a well defined group homomorphism and $i_e^* \circ r = id$. If $E \in \mathcal{E}_G(G)$, then the map $f : G \times E_e \to E$, defined by

$$f : (g, v) = (g, gv)$$

is a G-vector bundle isomorphism (the G-action on $G \times E_e$ is defined by $g(h, v) = (gh, v)$). This implies that

$$r \circ i_e^* = id.$$

The elements of $K_G(X)$ can all be written in the following way:

Lemma 3.8. *Let $E_+ - E_- \in K_G(X)$. Then there exists a G-module M and a G-vector bundle F, such that*

$$E_+ - E_- = F - M_X.$$

Proof. In Theorem 2.31, we showed that there is a G-vector bundle $E_-^\perp$ and a G-module M such that $E_- \oplus E_-^\perp \cong M_X$. Therefore,

$$E_+ - E_- = (E_+ - E_-) \oplus (E_-^\perp - E_-^\perp) = (E_+ \oplus E_-^\perp) - (E_- \oplus E_-^\perp) = (E_+ \oplus E_-^\perp) - M_X.$$

□

We can endow $K_G(X)$ with the structure of a ring, by giving it the following multiplication:

$$(E_+ - E_-) \otimes (F_+ - F_-) = (E_+ \otimes E_-) \oplus (E_- \oplus E_+) - (E_- \otimes F_+ \oplus E_+ \otimes F_-).$$

Lemma 3.9. *This multiplication indeed gives $K_G(X)$ the structure of a ring.*

Proof. We will leave most of the verifications to the reader. We will only show what the unit of the ring is. The unit is given by the bundle $\mathbb{K}_X := X \times \mathbb{K}$, where the G-action is defined by $g(x, v) = (gx, v)$. The isomorphism $\Phi : E \times \mathbb{K}_X \to E$ is given by

$$\Phi(v \otimes c) = cv.$$

□

The rings $K_G(X)$ have even more structure:

Lemma 3.10. *The assignment $X \to K_G(X)$ is a functor $K_G : Top_G^{op} \to Ring$, where Top_G is the category of compact G-spaces.*

Proof. To show that it is a functor, we first must define how K_G acts on a G-map $f : X \to Y$. The map $f^* : K_G(Y) \to K_G(X)$ is defined by $f^*(E_+ - E_-) = f^*E_+ - f^*E_-$. Notice that if $(E_+, E_-) \sim (F_+, F_-)$, then there exists an $F \in \mathcal{E}_G^{\mathbb{K}}(Y)$ such that

$$E_+ \oplus F_- \oplus F \cong F_+ \oplus E_- \oplus F.$$

Lemma 2.17 implies that

$$f^*E_+ \oplus f^*F_- \oplus f^*F \cong f^*(E_+ \oplus F_- \oplus F) \cong f^*(F_+ \oplus E_- \oplus F) \cong f^*F_+ \oplus f^*E_- \oplus f^*F.$$

Therefore, the map is well defined. We now show that f^* is a ring homomorphism. Notice that

$$f^*(0) = f^*(E - E) = f^*E - f^*E = 0$$

and

$$f^*(Y \otimes \mathbb{K}) \cong X \otimes \mathbb{K}.$$

In Lemma 2.17, we showed that $f^*(E \oplus F) = f^*(E) \oplus f^*(F)$ and $f^*(E \otimes F) = f^*(E) \otimes f^*(F)$, which implies that $f^* : K_G(Y) \to K_G(X)$ is indeed a ring homomorphism.
In Lemma 2.17 we also showed that for all $E \in \mathcal{E}_G^{\mathbb{K}}(Y)$ and G-maps $f : Z \to X$ and $g : X \to Y$, we have

$$f^*g^*E \cong (g \circ f)^*E,$$

as G-vector bundles. Because we also have $id^*E \cong E$, for all G-vector bundles $E \in \mathcal{E}_G^{\mathbb{K}}(Y)$, the assignment $X \to K_G(X)$ is a functor. □

Before we study $K_G(X)$ further, we will calculate $K_G^{\mathbb{C}}(X)$ for some G-spaces X. We will need the following lemma:

Lemma 3.11. *Let $p : E \to X$ be a G-vector bundle and let $P : E \to E$ be a morphism of G-vector bundles, such that $P^2 = P$. Then $ImP := \{P(v) \mid v \in E\}$ and $Im(id - P)$ are G-vector bundles, such that*

$$ImP \oplus Im(id - P) \cong E.$$

Proof. Notice that

$$(id - P)(id - P) = id - 2P + P^2 = id - 2P + P = id - P$$

and

$$p(id - p) = (id - p)p = p - p = 0.$$

We first show that for all $x \in X$, we have $E_x \cong \mathrm{Im}(P)_x \oplus \mathrm{Im}(id - p)_x$. Let $v \in \mathrm{Im}(P) \cap \mathrm{Im}(id - P)$. Then $v = (id - P)y$ and $v = P(z)$, for some $y, z \in E_x$. It follows that

$$0 = P(id - P)y = P((id - P)y) = P(P(z)) = P(z) = v.$$

Notice that for all $v \in E_x$, we have $v = Pv + (id - P)v$. Therefore,

$$E_x \cong \mathrm{Im}(P)_x \oplus \mathrm{Im}(id - p)_x.$$

We now show that $\mathrm{Im}(P)$ is indeed a G-vector bundle. We only show that $\mathrm{Im}(P)$ has local trivialisations, the other properties follow directly from the definition of $\mathrm{Im}(P)$.

Let $x \in X$. Let $s_1, \dots, s_n$ be sections, such that $s_1(x), \dots, s_k(x)$ form a basis of $\mathrm{Im}(P)(x)$ and $s_{k+1}(x), \dots, s_n(x)$ form a basis of $\mathrm{Im}(id - P)(x)$. The sections

$$Ps_1, \dots, Ps_k, (id - P)s_{k+1}, \dots (id - P)s_n,$$

Now have the property that $\det(Ps_1(x), \dots, Ps_k(x), (id - P)s_{k+1}(x), \dots (id - P)s_n(x)) \neq 0$. Therefore, there exists an open neighbourhood U of x, such that for all $y \in U$, we have

$$\det(Ps_1(y), \dots, Ps_k(y), (id - P)s_{k+1}(y), \dots (id - P)s_n(y)) \neq 0.$$

It follows that the map $\Psi : \mathrm{im}P|_U \to U \times \mathbb{K}^k$, defined by

$$\Psi^{-1}(u, (x_1, \dots, x_k)) = (x_1 Ps_1(u), \dots x_k Ps_k(u)),$$

is a local trivialisation. $\square$

Remark 3.12. We will often denote the bundle $\mathrm{Im}(P)$ by PE.

Remark 3.13. If $Q : E \to E$ is also a G-vector bundle morphism such that $Q^2 = Q$ and $H : E \times [0, 1] \to E$ is a G-homotopy between P and Q such that $H_t : E \to E$ is a G-vector bundle morphism which satisfies $H_t^2 = H_t$, then $PE \cong QE$. This can be shown as follows: Let $\pi_X : I \times X \to X$ denote the projection and $i_t : X \to I \times X$ be the inclusion $i_t(x) = (t, x)$. The map H induces a G-vector bundle morphism $\tilde{H} : \pi_X^* E \to \pi_X^* E$ defined by $\tilde{H}((x, t), v) = H(t, v)$. Notice that $\tilde{H}^2 = \tilde{H}$, which implies that we can construct the bundle $\tilde{H}(\pi^* E)$. We have $i_0^* \tilde{H}(\pi^* E) \cong PE$ and $i_1^* \tilde{H}(\pi^* E) \cong QE$. Since there is a G-homotopy between i_0 and i_1, Theorem 2.23 implies that $PE \cong QE$.

Proposition 3.14. *Let X be a G-space, such that $gx = x$, for all $(g, x) \in G \times X$. Then, there is a ring isomorphism:*

$$\Phi : K^{\mathbb{C}}(X) \otimes R(G) \to K_G(X),$$

where $R(G) := K_G^{\mathbb{C}}(\{pt\})$.

Proof. Let $p : X \to \{\mathrm{pt}\}$ denote the map defined by $p(x) = \mathrm{pt}$. We define the map $\tilde{\Phi} : K_G^{\mathbb{C}}(X) \times R(G) \to K^{\mathbb{C}}(X)$, by

$$\tilde{\Phi}((M_{\mathrm{pt}} - N_{\mathrm{pt}}) \otimes (E_+ - E_-)) := (p^* M_{\mathrm{pt}} - p^* N_{\mathrm{pt}}) \otimes (E_+ - E_-) = (M_X \otimes E_+ \oplus N_X \otimes E_-) - (N_X \otimes E_+ \oplus M_X \otimes E_-).$$

The group action on $M_X \otimes E$ is defined by $g(m \otimes v) = (gm) \otimes v$ and is defined similarly on $N_X \otimes E_-$, $N_X \otimes E_+$ and $M_X \otimes E_-$. Notice that $\tilde{\Phi}$ is a bilinear map, which implies that $\tilde{\Phi}$ induces a map $\Phi : K_G^{\mathbb{C}}(X) \otimes R(G) \to K^{\mathbb{C}}(X)$. We show that this map is an isomorphism by constructing an inverse. To construct this inverse, we first need to define some other maps.

Let $p : E \to X$ be a G-vector bundle over X. we can define the map $P : E \to E$, by

$$P(v) = \int_G (gv) dg.$$

Notice that P is continuous and $P|_{E_x}$ is linear for all $x \in X$. We now show that P is a G-map. Let $h \in G$, then

$$hP(v) = h \int_G gvdg = \int_G (hg)\ vdg = \int_G gvdg = \int_G (gh)vdg = \int_G g(hv)dg = P(hv). \quad (3.1)$$

Since for all $h \in G$, we have $hP(v) = P(v)$, it follows that

$$P(P(v)) = \int_G gP(v)dg = \int_G P(v)dg = P(v).$$

Lemma 3.11 now implies that $PE := \mathrm{Im}P$ is a G-vector bundle. In Equation 3.1 we showed that $gP(v) = P(v)$. Therefore, the G-action on PE is trivial. Since the action of G on X is also trivial, we can view PE as an element of $K^{\mathbb{C}}(X)$. Let $\Psi_M : K_G^{\mathbb{C}}(X) \to K^{\mathbb{C}}(X)$ be the ring homomorphism defined by $\Psi_M(E_+ - E_-) = P\mathrm{Hom}(M_X, E_+) - P\mathrm{Hom}(M_X, E_-)$, where M is a G-module.

With this notation, we are finally ready to define the inverse $\varphi : K_G^{\mathbb{C}} \to K^{\mathbb{C}}(X) \otimes R(G)$ by

$$\varphi(E_+ - E_-) = \sum_{M \in [G]} M_{\mathrm{pt}} \otimes \Psi_M(E_+ - E_-).$$

Notice that φ is a ring homomorphism. We now show that $\Phi \circ \varphi = id$. Notice that

$$\Phi \circ \varphi(E_+ - E_-) = \bigoplus_{M \in [G]} M_X \otimes P\mathrm{Hom}(M_X, E_+) - \bigoplus_{M \in [G]} M_X \otimes P\mathrm{Hom}(M_X, E_-).$$

We will show that

$$E_+ \cong \bigoplus_{M \in [G]} M_X \otimes P\mathrm{Hom}(M_X, E_+) =: F.$$

Let $x \in X$. Then

$$F_x = \bigoplus_{M \in [G]} M \otimes P\mathrm{Hom}(M, (E_+))_x.$$

Notice that $f \in P\mathrm{Hom}(M_X, (E_+))_x$ implies that $f : M \to E_x$ is a map such that $l_g \circ f \circ l_{g^{-1}} = f$ and thus that f is an equivariant map between representations of G. Since $(E_+)_x$ is a finite dimensional representation of G, Proposition A.28 implies that there exists irreducible representations $(\alpha_1, M_1), \ldots, (\alpha_n, M_n)$, such that $(E_+)_X \cong \bigoplus_{i=1}^n M_i$. Lemma A.31 now implies that

$$\begin{aligned}
F_X &= \bigoplus_{M \in [G]} M \otimes \mathrm{Hom}(M, \bigoplus_{i=1}^n M_i) \\
&\cong \bigoplus_{M \in [G]} \bigoplus_{i=1}^n M \otimes \mathrm{Hom}(M, M_i) \\
&\cong \bigoplus_{i=1}^n M_i \otimes \mathrm{Hom}(M_i, M_i) \\
&\cong \bigoplus_{i=1}^n M_i \otimes_{\mathbb{C}} \mathbb{C} \\
&\cong \bigoplus_{i=1}^n M_i \\
&= E_x.
\end{aligned}$$

Chasing through all the identifications, we see that the isomorphism $F_x \to E_x$ is given by $(m \otimes f) \to f(m)$. Notice however that this isomorphism is the restriction of the morphism of G-vector bundles $h : F \to E$, defined by $h(m \otimes f) = f(m)$, to F_x. Therefore, the map h is a morphism of vector bundles, which is fiberwise an isomorphism and thus an isomorphism.

We now show that $\varphi \circ \Phi = id$. Let $(M'_{\mathrm{pt}} - N_{\mathrm{pt}}) \otimes (E_+, E_-) \in R(G) \otimes K^{\mathbb{C}}(X)$. Then

$$\begin{aligned}\varphi \circ \Phi((Q_{\mathrm{pt}} - N_{\mathrm{pt}}) \otimes (E_+, E_-)) &= \sum_{M \in [G]} M_{\mathrm{pt}} \otimes P\,\mathrm{Hom}(M_X, Q_X \otimes E_+ \oplus N_X \otimes E_-)) \\ &\quad - \sum_{M \in [G]} M_{\mathrm{pt}} \otimes P\,\mathrm{Hom}(M_X, (N_X \otimes E_+ \oplus Q_X \otimes E_-)).\end{aligned}$$

Since

$$P\,\mathrm{Hom}(M_X, Q_X \otimes E_+ \oplus N_X \otimes E_-) \cong P\,\mathrm{Hom}(M_X, Q_X \otimes E_+) \oplus P\,\mathrm{Hom}(M_X, N_X \otimes E_-),$$

to show that $\varphi \circ \Phi$ is an isomorphism, it is sufficient to show that

$$\sum_{M \in [G]} M_{pt} \otimes P\,\mathrm{Hom}(M_X, Q_X \otimes E_+) = Q_{pt} \otimes E,$$

the proof for the other terms is then similar and we obtain

$$\begin{aligned}\varphi \circ \Phi((Q_{\mathrm{pt}} - N_{\mathrm{pt}}) \otimes (E_+, E_-)) &= (Q_{\mathrm{pt}} \otimes E_+ + N_{\mathrm{pt}} \otimes E_-) - (N_{\mathrm{pt}} \otimes E_+ + Q_{\mathrm{pt}} \otimes E_-) \\ &= (Q_{pt} - N_{pt}) \otimes (E_+ - E_-).\end{aligned}$$

We first show that the map $\mu : P\,\mathrm{Hom}(M_X, Q_X) \otimes E_+ \to P\,\mathrm{Hom}(M_X, Q_X \otimes E_+)$ given by

$$\mu(A \otimes v) = A_v : m \to Am \otimes v$$

is an isomorphism. Notice that μ is a G-vector bundle morphism and μ is injective. We now show that μ is surjective. Let $B \in P\,\mathrm{Hom}(M_X, Q_X \otimes E_+)_x$ and let $m \in M$. We have

$$B(m) = \sum_{i=1}^{n} n_i \otimes v_i,$$

where $v_1, \ldots, v_n$ is a basis of E_x. Because $B \in P\,\mathrm{Hom}(M_X, Q \otimes E_+)_x$, we have

$$B(gm) = gB(m) = g\sum_{i=1}^{n} n_i \otimes v_i = \sum_{i=1}^{n} (gn_i) \otimes v_i,$$

for all $g \in G$. Let for all $i \in I$, let

$$B_i := id \otimes \mathrm{pr}_{v_i} \circ B : M \to Q \otimes \mathrm{Span}(v_i) \cong Q.$$

Then, by construction:

$$B = \mu(\sum_{i=1}^{n} B_i \otimes v_i).$$

Therefore,

$$P\,\mathrm{Hom}(M_X, Q_X \otimes E_+) \cong P\,\mathrm{Hom}(M_X, Q_X) \otimes E_+.$$

In Proposition A.28, we showed that $M \cong \bigoplus_{i=1}^k M_k$, where M_k are irreducible representations. Lemma A.31 now implies that

$$\begin{aligned}
\sum_{M\in[G]} M_{\mathrm{pt}} \otimes P\operatorname{Hom}(M_X, Q_X \otimes E_+) &= \sum_{M\in[G]} M_{pt} \otimes P\operatorname{Hom}(M_X, Q_X) \otimes E_+ \\
&= \sum_{M\in[G]} M_{\mathrm{pt}} \otimes (\bigoplus_{i=1}^k P\operatorname{Hom}(M_X, (M_i)_X) \otimes E_+) \\
&= \sum_{i=1}^k (M_i)_{\mathrm{pt}} \otimes (P\operatorname{Hom}((M_i)_X, (M_i)_X)) \otimes E_+) \\
&= \sum_{i=1}^k (M_i)_{\mathrm{pt}} \otimes (\mathbb{C}_X \otimes E_+) \\
&= \sum_{i=1}^k (M_i)_{pt} \otimes E_+ \\
&= (\bigoplus_{i=1}^k (M_i)_{\mathrm{pt}}) \otimes E_+ \\
&= Q_{\mathrm{pt}} \otimes E_+,
\end{aligned}$$

which proves the theorem. $\square$

We now give an example to show how this proposition can be used to compute the group $K_G(X)$ for a G-space X.

Example 3.15. We will show how to compute $K^{\mathbb{C}}_{S^1}(S^2)$, where the action on S^2 is such that $gx = x$ for all $(g, x) \in S^1 \times X$. In section 7, we will show that $K^{\mathbb{C}}(S^2) \cong \mathbb{Z}$. We now need to determine $R(S^1)$. The irreducible representations of S^1 are given by the maps $\gamma_n : S^1 \to \mathbb{C} - \{0\} = \operatorname{Aut}(\mathbb{C})$, defined by $\gamma_n(e^{i\theta}) = (e^{i\theta})^n$ for $n \in \mathbb{Z}$. Let $M_n := \{\oplus_{i=1}^k \gamma_n \mid k \in \mathbb{N}\} \cup \{0\}$ and notice that the direct sum gives M_n the structure of a monoid isomorphic to $\mathbb{N}_0$. Since every representation is isomorphic to the direct sum of irreducible representations, every isomorphism class of $\mathcal{E}^{\mathbb{C}}_G(\{\mathrm{pt}\})$ corresponds to an element of

$$M := \oplus_{k\in\mathbb{Z}} M_k \cong \oplus_{k\in\mathbb{Z}} \mathbb{N}_0.$$

Therefore, the group $K^{\mathbb{C}}_G(\{\mathrm{pt}\})$ isomorphic to the group

$$K^{\mathbb{C}}_G(\{\mathrm{pt}\}) \cong \oplus_{k\in\mathbb{Z}} \mathbb{Z}. \tag{3.2}$$

We now determine the ringstructure of $K^{\mathbb{C}}_G(\{\mathrm{pt}\})$. If $k, l \in \mathbb{Z}$, then the representation $\gamma_k \otimes \gamma_l$ on $\mathbb{C} \otimes \mathbb{C}$ is isomorphic to the representation γ_{k+l} on $\mathbb{C}$. It follows that $\gamma_k \cong (\gamma_1)^k$. Since in Equation 3.2, the representations γ_k, γ_l and γ_l correspond to the elements $(k, 1)$, $(l, 1)$ and $(1, 1)$, we must have $(k, j) \otimes (l, i) = (k + l, i \cdot j)$ on $\oplus_{k\in\mathbb{Z}} \mathbb{Z}$. Notice that this implies that the elements $(1, 1)$ and $(-1, 1)$ generate the ring $\oplus_{k\in\mathbb{Z}} \mathbb{Z}$. Therefore, we have

$$K^{\mathbb{C}}_G(\{\mathrm{pt}\}) \cong \mathbb{Z}[X, \frac{1}{X}],$$

with the relation $X \cdot \frac{1}{X} = 1$. The element X corresponds to the element $(1, 1)$ of $\oplus_{i\in\mathbb{Z}} \mathbb{Z}$ and thus to the representation γ_1.

To further study the groups/rings $K_G(X)$, we will define the reduced groups $\tilde{K}_G(X)$.

Definition 3.16. *Let X be a compact G-space. Then, for $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ we define*

$$\tilde{K}_G^{\mathbb{K}}(X) = \{[E] \mid E \in \mathcal{E}_G^{\mathbb{K}}\}/\sim',$$

Where $[E] \sim' [F]$ if there exists G-modules M and N, such that

$$E \oplus M_X \cong F \oplus N_X.$$

We will often denote an element of $\tilde{K}_G$ by $[E]_{\sim'}$.

Lemma 3.17. *The sum $[E] \oplus [F] = [E \oplus F]$ give $\tilde{K}_G(X)$ the structure of an abelian group.*

Proof. We will only prove that that there is an identity element and that every element has an inverse, the other properties follow almost directly from the definition of the sum. The identity element of the group 0, is the vector bundle $X \times \{0\}$, where $\{0\}$ is viewed as a zero dimensional vector space. Notice that for all $E \in \mathcal{E}_G^{\mathbb{K}}(X)$, we have

$$0 + E \cong E \cong E + 0,$$

which implies that 0 is the identity.
Let $E \in \mathcal{E}_G^{\mathbb{K}}(X)$. We will construct the inverse of E. In Theorem 2.31, it is shown that there exists a G-vector bundle $E^\perp$ and a G-module M, such that $E \oplus E^\perp \cong M_X \cong M_X$. This implies that

$$E \oplus E^\perp \oplus 0 \cong 0 \oplus M_X,$$

and thus that $[E \oplus E^\perp]_{\sim'} = [0]_{\sim'}$. ☐

As before, the map $X \to \tilde{K}_G(X)$ can be viewed as functor $\mathrm{Top}_G \to \mathrm{Ab}$, where $\tilde{K}_G(f)(E) = f^*E$, for a G-map $f : Y \to X$.
The functors $\tilde{K}_G$ and K_G have the following useful property:

Proposition 3.18. *Let X and Y be G-spaces and let $f, g : Y \to X$ be G-maps. Then,*

(i) If f and g are G-homotopic, then $f^* = g^*$, viewed as maps $K_G(X) \to K_G(Y)$ or $\tilde{K}_G(X) \to \tilde{K}_G(Y)$.

(ii) If f is a G-homotopy equivalence, then $K_G(X) \cong K_G(Y)$ and $\tilde{K}_G(X) \cong \tilde{K}_G(Y)$

Proof. We will prove the statement for K_G, the proof for $\tilde{K}_G$ is similar. We first prove (i). Let $E_+ - E_- \in K_G(X)$. Since f and g are G-homotopic, Theorem 2.23 implies that $f^*E_\pm \cong g^*E_\pm$. Therefore,

$$f^*(E_+ - E_-) = f^*E_+ - f^*E_- = g^*E_+ - g^*E_- = g^*(E_+ - E_-).$$

We now show (ii). Since f is a G-homotopy equivalence, there exists a map $h : X \to Y$, such that $f \circ h$ and $h \circ f$ are G-homotopic to the identity. Part (i) of our lemma now implies that

$$f^* \circ h^* = (h \circ f)^* = (id)^* = id,$$

viewed as a map from $K_G(Y)$ to $K_G(Y)$, and

$$h^* \circ f^* = (f \circ h)^* = (id)^* = id,$$

viewed as a map from $K_G(X)$ to $K_G(X)$. Thus, the map $f^* : K_G(X) \to K_G(Y)$ is an isomorphism, which proves the proposition. ☐

As the notation might suggest, the group $\tilde{K}_G(X)$ is closely related to the group $K_G(X)$.

Proposition 3.19. *Let X be a compact G-space and $x_0 \in X$ a point such that $gx_0 = x_0$ for all $g \in G$. The following sequence is a split short exact sequence of abelian groups:*

$$0 \longrightarrow K_G(\mathrm{pt}) \underset{i^*}{\overset{p^*}{\rightleftarrows}} K_G(X) \xrightarrow{\ \pi\ } \tilde{K}_G(X) \longrightarrow 0 \ ,$$

where $p : X \to \{pt\}$ is the projection, $i : \{pt\} \to X$ defined by $i(x) = x_0$ and $\pi(E - M_X) = E$.

Proof. We first show that π is well defined. Let $E_+ - E_- \in K_G(X)$. In Lemma 3.8 we showed that there exists a G-vector bundle F and an G-module M, such that $E_+ - E_- = F - M_X$. Assume that we also have that $E_+ - E_- = F' - M'_X$. Then, we have $F - M_X = F' - M'_X$ and Lemma 3.4 implies that there exists a G-module N, such that

$$F \oplus (M'_X \oplus N_X) \cong F' \oplus (M_X \oplus N_X).$$

Thus, $[F]_{\sim'} = [F']_{\sim'}$ and π is well defined.
We now show that the sequence is exact. Notice that since $i^*p^* = (p \circ i)^* = (id)^* = id$, it follows that p^* is injective. Also notice that since $\pi(E) = [E]_{\sim'}$, the map π is surjective. We now show that $\mathrm{Im}(P) = \mathrm{Ker}(\pi)$. We have

$$\pi(p^*(M_{\mathrm{pt}} - N_{\mathrm{pt}}) = \pi(M_X - N_X) = [M_X]_{\sim'}.$$

Because $M_X \oplus 0 = 0 \oplus M_X$, it follows that $[M_X]_{\sim'} = 0$ and $\mathrm{Im}(P) \subset \mathrm{Ker}(\pi)$. We now prove that $\mathrm{Ker}(\pi) \subset \mathrm{Im}(\pi)$. Let $E_+ - M_X \in \mathrm{Ker}(\pi)$. Then there exists G-modules N and N', such that $E_+ \oplus N = N'$. This implies that $E_+ = N'_X - N_X$ and

$$E_+ - M_X = (N'_X - N_X) - M_X = N'_X - (N \oplus M_X) = p^*(N'_{\mathrm{pt}} - (N_{\mathrm{pt}} \oplus M_{\mathrm{pt}})).$$

Thus, we have $E_+ - M_X \in \mathrm{Im}(p^*)$. Since $i^* \circ p^* = id$, the sequence is a split short exact sequence. ∎

Let X be a compact G-space. We will denote the one-point compactification of X by X^+ and we will denote the point that is added to X by pt. Notice that if we endow X^+ with the G-action $gx = gx$ if $x \in X$ and $g\mathrm{pt} = \mathrm{pt}$, the space X^+ is a G-space. Also notice that since X is compact, we have $X^+ \cong X \coprod \{\mathrm{pt}\}$. With this notation, we can state the following important consequence of this proposition:

Theorem 3.20. *Let X be a compact G-space. Then,*

$$\tilde{K}_G(X^+) \cong K_G(X),$$

as abelian groups.

Proof. First, notice that $K^G(X) \oplus K^G(\mathrm{pt}) \cong K^G(X^+)$, where the isomorphism is given by

$$\Phi(E_+ - E_-, M_{\mathrm{pt}} - N_{\mathrm{pt}}) = (E_+ \coprod M_{\mathrm{pt}} - E_- \coprod N_{\mathrm{pt}}),$$

with $E_+ \coprod M_{\mathrm{pt}}|_X := E_+$ and $E_+ \coprod M_{\mathrm{pt}}|_{\mathrm{pt}} := M_{\mathrm{pt}}$. Let $i_{pt} : K_G(\mathrm{pt}) \to K_G(\mathrm{pt}) \oplus K_G(X)$ be the map defined by

$$i_{\mathrm{pt}}(M_{\mathrm{pt}} - N_{\mathrm{pt}}) := (M_{\mathrm{pt}} - N_{\mathrm{pt}}, 0)$$

and let $\pi : K_G(\mathrm{pt}) \oplus K_G(X) \to K_G(X)$ be the map defined by

$$\pi_X((M_{\mathrm{pt}} - N_{\mathrm{pt}}), (E_+ - E_-)) := E_+ \oplus N_X - E_{\oplus} M_X.$$

This yields following the exact sequence:

$$0 \longrightarrow K_G(\mathrm{pt}) \xrightarrow{\Phi \circ p_{\mathrm{pt}}} K_G(X^+) \xrightarrow{\pi_X \circ \Phi^{-1}} K_G(X) \longrightarrow 0 \ ,$$

Let $i : \{pt\} \to X$ denote the inclusion. We have

$$i^*(\phi \circ i_{pt}(M_{\mathrm{pt}} - N)\mathrm{pt}) = i^*(0 \coprod M_{\mathrm{pt}} - 0 \coprod N_{\mathrm{pt}}) = M_{\mathrm{pt}} - N_{\mathrm{pt}}.$$

Therefore, the sequence above is a split short exact sequence exact sequence. It follows that $K_G(X) \cong \mathrm{Ker}(i^*)$. Applying Proposition 3.19 to X^+, with $x_0 = \mathrm{pt}$, we see that the sequence

$$0 \longrightarrow K_G(\mathrm{pt}) \underset{i^*}{\overset{p^*}{\rightleftarrows}} K_G(X^+) \xrightarrow{\ \pi\ } \tilde{K}_G(X^+) \longrightarrow 0 \ ,$$

is also split short exact sequence. Thus, we also have $\tilde{K}_G(X^+) \cong \mathrm{Ker}(i^*)$ and

$$\tilde{K}_G(X^+) \cong K_G(X).$$

$\square$

Remark 3.21. The following notation will be useful to describe G-vector bundles over X^+. If X and Y are compact G-spaces and $f : Y \to X$ is a G-map, then f induces a G-map $\tilde{f} : X^+ \to Y^+$, which is given by $\tilde{f}(x) = f(x)$ if $x \in X$ and $\tilde{f}(\mathrm{pt}) = \mathrm{pt}$. We will usually denote this map by f. If E is a G-vector bundle over X, we will also use the notation E for the bundle $E \coprod 0_{\mathrm{pt}}$ over X^+

It might not be clear why it would be useful to look at the groups $\tilde{K}(X^+)$. It will turn out that the groups $\tilde{K}_G(X)$ can be used to construct an exact sequence which will be of vital importance for the development of the theory.
To construct this exact sequence, we must first introduce some notation. Let X be a G-space, and $x_0 \in X$, such that $gx_0 = x_0$, for all $g \in G$. The reduced cone over X is defined by

$$CX := X \times I/(\{x_0\} \times [0,1] \cup \{1\} \times X).$$

The reduced suspension of X is denoted by

$$\Sigma X := X \times I/(\{x_0\} \times [0,1] \cup \{1,0\} \times X).$$

Notice that CX and ΣX are G-spaces, where the action on an element $[(x,t)]$ is given by $g[(x,t)] = [(gx,t)]$.
Lastly, if X, Z_1, Z_2 are G-spaces and $i_1 : X \to Z_1$ and $i_2 : X \to Z_2$ is an inclusion which is also a G-map, then

$$Z_1 \coprod_X Z_2 := Z_1 \coprod Z_2/\sim,$$

where $i_1(x) \sim i_2(x)$ for all $x \in X$. Notice that $\Sigma X \cong CX \coprod_X CX$, where we identify $(x,0)$ with$(x,0)$ for all $x \in X$. The isomorphism is given by the map $\Phi : CX \coprod_X CX \to \Sigma X$

defined by

$$\Phi[(x,t)] = \begin{cases} (x, \frac{1}{2}+\frac{t}{2}) & \text{if} \quad (x,t) \text{ is an element of the first cone} \\ (x, \frac{1}{2}-\frac{t}{2}) & \quad (x,t) \text{ is an element of the second cone} \end{cases} .$$

With this notation, we are ready to state the following Lemma:

Lemma 3.22. *Let X be a G-space, A a closed G-invariant subspace and $x_0 \in A \subset X$, such that $gx_0 = x_0$ for all $g \in G$. Let $X \coprod_A CA$ be defined by $a \sim (a,0)$, for all $a \in A$ and let $i_A : A \to X$ and $i_X : X \to X \coprod_A CA$ denote the inclusion. Then, the following sequence is exact:*

$$\tilde{K}_G(X \textstyle\coprod_A CA) \xrightarrow{i_X^*} \tilde{K}_G(X) \xrightarrow{i_A^*} \tilde{K}_G(A)\ . \tag{3.3}$$

Proof. We first show that $\mathrm{Im}(i_X^*) \subset \mathrm{Ker}(i_A^*)$. Notice that $i_A^* \circ i_X^* = (i_X \circ i_A)^*$, where $i_X \circ i_A$ is the inclusion of A into $X \coprod_A CA$. Let $H : A \times [0,1] \to X \coprod_A CA$ be the map defined by $H(a,t) = (a,t)$. Then H is a G-homotopy, such that $H_0 = i_X \circ i_A$ and $H_1(a) = [(x_0, 1)]$ for all $a \in A$. Therefore, Theorem 2.23 and example 2.16 imply that

$$(i_X \circ i_A)^* E = (H_1)^* E \cong M_A$$

and

$$[(i_X \circ i_A)^* E]_{\sim'} = [M_A]_{\sim'} = 0.$$

We now show that $\mathrm{Ker}(i_A^*) \subset \mathrm{Im}(i_X^*)$. Assume that $[E]_{\sim'} \in \mathrm{Ker}(i_A^*)$. Then $[E|_A]_{\sim'} = 0$ and there exists G-modules M and N such that $E|_A \oplus N_A \cong M_A$. Let $\Phi : E|_A \oplus N_A \to M_A$ be an isomorphism. Because $CA \cup X = X \coprod_A CA$ and $X \cap CA = A$, which is a closed G-subspace of A. Theorem 2.29 implies that

$$(E \oplus N_X) \cup_\Phi M_{CA}$$

is a G-vector bundle over $X \coprod_A CA$. Since $i_X^*((E \oplus M_X) \cup_\Phi N_{CA}) \cong E \oplus M_X$, it follows that

$$[i_X^*((E \oplus M_X) \cup_\Phi N_{CA}]_{\sim'} = [E \oplus M_X]_{\sim'} = [E]_{\sim'}.$$

□

Remark 3.23. If X_1 and Y_1 are compact G-spaces, $X_2 \subset X_1$ and $Y_2 \subset Y_1$ and $f : Y_1 \to X_1$ is a map such that $f(Y_2) \subset X_2$, then f induces a G-map $\tilde{f} : Y_1 \coprod_{Y_2} CY_2 \to X_1 \coprod_{X_2} CX_2$, defined by $\tilde{f}(y) = f(y)$ if $x \in Y_1$ and $\tilde{f}(y,t) = (f(y), t)$ if $(y,t) \in CY_2$. Notice that this map makes the following diagram commutes:

$$\begin{tikzcd}
\tilde{K}_G(Y_1 \coprod_{Y_2} CY_2) \arrow[r, "i_{Y_1}^*"] & \tilde{K}_G(Y_1) \arrow[r, "i_{Y_2}^*"] & \tilde{K}_G(Y_2) \\
\tilde{K}_G(X_1 \coprod_{X_2} CX_2) \arrow[r, "i_{X_1}^*"] \arrow[u, "\tilde{f}^*"] & \tilde{K}_G(X_1) \arrow[r, "i_{X_2}^*"] \arrow[u, "f^*"] & \tilde{K}_G(X_2) \arrow[u, "f^*"]
\end{tikzcd} .$$

We will often denote the map $\tilde{f}$ by f and write f as $f : (Y_1, Y_2) \to (X_1, X_2)$ to emphasize that $f(Y_2) \subset X_2$.

We can iterate this process from the lemma to extend the exact sequence. If in the lemma above, we replace A, X and $X\coprod_A CA$ by X, $X\coprod_A CA$ and $(X\coprod_A C_A)\coprod_X C_X \cong CX\coprod_A C_A$, we obtain the exact sequence

$$\tilde{K}_G(CX\textstyle\coprod_A CA) \xrightarrow{i*} \tilde{K}_G(X\textstyle\coprod_A C_A) \xrightarrow{i_X^*} \tilde{K}_G(X) \ . \tag{3.4}$$

If we repeat this process and now attach on $CX\coprod_A CA$ a cone along $X\coprod_A CA$, we get

$$(CX\coprod_A CA)\coprod_{X\coprod_A CA}(C(X\coprod_A CA)\cong(CX\coprod_A CA)\coprod_{X\coprod_A CA}(CX\coprod_{CA} CCA)\cong CX\coprod_X(CX\coprod_{CA} CCA)$$

and the following exact sequence:

$$\tilde{K}_G(CX\textstyle\coprod_X(CX\coprod_{CA} CCA)) \longrightarrow \tilde{K}_G(CX\textstyle\coprod_A CA) \longrightarrow \tilde{K}_G(X\textstyle\coprod_A CA) \ . \tag{3.5}$$

The following lemma shows why this sequence makes sense:

Lemma 3.24. *Let X and A be as above. We have*

$$\tilde{K}_G(\Sigma A)\cong\tilde{K}_G(CX\coprod_A CA),$$

and

$$\tilde{K}_G(\Sigma X)\cong\tilde{K}_G(CX\coprod_X(CX\coprod_{CA} CCA).$$

Proof. We will prove that $\tilde{K}_G(\Sigma A)\cong\tilde{K}_G(CX\coprod_A CA)$, the proof of the other statement is similar. Notice that it, by Proposition 3.18, is sufficient to show that ΣA is G-homotopy equivalent to $CX\coprod_X(CX\coprod_A CA)$. We claim that the inclusion $i:\Sigma A\cong CA\coprod_A CA\to CX\coprod_A CA$ is a G-homotopy equivalence. Let $p: CX\coprod_A CA\to\Sigma A$ be the map defined by

$$p([(x,t)])=\begin{cases}[(x,0)] & \text{if} \quad [(x,t)]\in CX\\ [(a,t)] & \text{if} \quad [(x,t)]\in CA\end{cases}$$

We have

$$p\circ i([x,t])=\begin{cases}[(x,0)] & \text{if} \quad t\in[0,\frac{1}{2}]\\ [x,2t-1] & \text{if} \quad t\in[\frac{1}{2},1]\end{cases}.$$

The homotopy $H:\Sigma A\times I\to\Sigma A$, defined by

$$H((x,t),s)=([(x,\frac{1}{1-\frac{1}{2}s}(t-\frac{1}{2}s))]),$$

where we have the convention that $([x,t])=[(x,0)]$ if $t\le 0$, gives the G-homotopy between id and $p\circ i$.
Notice that the G-homotopy

$$F: CX\coprod_A CA\times I\to CX\coprod_A CA,$$

defined by

$$F((x,t),s)=\begin{cases}[(x,t+s] & \text{if} \quad [(x,t)]\in CX\\ [x,(1+s)(t)-s] & \text{if} \quad [(x,t)]\in CA\end{cases}$$

where we have the convention that $[(x,t)]=([x,1])$ if $t>1$ and $([a,-t])=([a,t])\in CX$, is a G-homotopy between $i\circ p$ and id. Therefore, the inclusion i is a G-homotopy equivalence. $\square$

We are now ready to construct the exact sequence:

Theorem 3.25. *Let X be a G-space, $A \subset X$ a G-invariant subspace and and $x_0 \in A$, such that $gx_0 = x_0$ for all $g \in G$. The following sequence is exact:*

$$\tilde{K}_G(\Sigma X) \xrightarrow{i^*_{\Sigma A}} \tilde{K}_G(\Sigma A) \longrightarrow \tilde{K}_G(X \textstyle\coprod_A CA) \xrightarrow{i_X} \tilde{K}_G(X) \xrightarrow{i_A} \tilde{K}_G(X) \ . \tag{3.6}$$

Consider the diagram:

$$\begin{array}{ccc} CA \coprod_A CA & \longrightarrow & CX \coprod_A CA \\ \downarrow & & \downarrow \\ CX \coprod_X CX & \longrightarrow & (CX \coprod_X (CX \coprod_{CA} CCA) \end{array} \ , \tag{3.7}$$

where every map is the inclusion. Let $i : CA \coprod_A CA \to CX \coprod_X (CX \coprod_{CA} CCA$ denote the composition of maps in the upper half of the diagram and j the composition of maps of the lower half of the diagram. We will show that i and j are G-homotopic.
Notice that the maps i and j coincide on the first copy of CA. On the second copy of CA, the map i maps into CCA via the map

$$i(x,t) = ((x,t),0)$$

and j maps into CCA via the map

$$j(x,t) = ((x,0),t).$$

These map are homotopic relative to X via the homotopy

$$H(((x,s),t),\phi) = (x, \sqrt{1+\tan^2(f(\phi))}\begin{pmatrix} \cos(\frac{\pi}{2}\phi) & -\sin(\frac{\pi}{2}\phi) \\ \sin(\frac{\pi}{2}\phi) & \cos(\frac{\pi}{2}\phi) \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}).$$

where $f(\phi) := \frac{\pi}{2}\phi$ if $t \leq 0.5$ and $f(\phi) := \frac{\pi}{2}(1-\phi)$ if $t \geq 0.5$. Therefore, the maps i and j are G-homotopic. Proposition 3.18 now implies that the following diagram, induced by equation 3.7, commutes:

$$\begin{array}{ccc} \tilde{K}_G(CX \coprod_X (CX \coprod_{CA} CCA) & \longrightarrow & \tilde{K}_G(CX \coprod_A CX) \\ \downarrow & & \downarrow \\ \tilde{K}_G(CX \coprod_A CA) & \longrightarrow & \tilde{K}_G(CA \coprod_A CA). \end{array} \tag{3.8}$$

Also notice that, by Lemma 3.24 the horizontal arrows are isomorphisms. Combining this with Equation 3.3 , Equation 3.4 and Equation 3.5, we obtain the desired exact sequence:

$$\tilde{K}_G(\Sigma X) \xrightarrow{i^*_{\Sigma A}} \tilde{K}_G(\Sigma A) \longrightarrow \tilde{K}_G(X \textstyle\coprod_A CA) \xrightarrow{i_X} \tilde{K}_G(X) \xrightarrow{i_A} \tilde{K}_G(X) \ .$$

This theorem motivates the following definitions:

Definition 3.26. *Let X, A and x_0 be as above, then for all $n \in \mathbb{N}_0$ we define:*

$$\tilde{K}_G^{-n}(X) := \tilde{K}(\Sigma^n X)$$

and

$$\tilde{K}_G^{-n}(X, A) := \tilde{K}_G(\Sigma^n X \coprod_{\Sigma^n A} C\Sigma^n A).$$

Definition 3.27. *Let X be a compact G-space and $A \subset X$ a closed G-invariant subset. For all $n \in \mathbb{N}_0$, we define:*

$$K_G^{-n}(X) := \tilde{K}_G^{-n}(X^+)$$

and

$$K_G^{-n}(X, A) := \tilde{K}_G^{-n}(X^+, A^+).$$

Remark 3.28. Notice that Theorem 3.20 implies that $K_G^0(X) \cong K_G(X)$ as a group.

Remark 3.29. Using Equation 3.6, we obtain the exact sequence

$$\ldots \longrightarrow \tilde{K}_G^{-(k+1)}(X, A) \longrightarrow \tilde{K}_G^{-(k+1)}(X) \longrightarrow \tilde{K}_G^{-(k+1)}(A) \longrightarrow \tilde{K}_G^{-k}(X, A) \longrightarrow$$

$$\ldots \longrightarrow \tilde{K}_G^{0}(X, A) \longrightarrow \tilde{K}_G^{0}(X) \longrightarrow \tilde{K}_G^{0}(A).$$

and a similar one for the K^{-n} groups

We will end this section by giving another interpretation of the group $K_G(X, A)$ and $\tilde{K}_G(X, A)$. Let X and A be defined as before. Notice that the space $X/A := X/\sim$, where $x \sim y$ if $x = y$ or $x, y \in A$ is a G-space. Let $\pi : X \to X/A$ denote the projection. Then, we have

Proposition 3.30. *Let X be a G-space, A a closed G-invariant subset. If A is G-contractible, then the map $\pi^* : \tilde{K}_G(X/A) \to \tilde{K}_G(X)$ is an isomorphism.*

Proof. We will prove the proposition by constructing an inverse i. Let $E \in \mathcal{E}_G(X)$. Because A is G-contractible, it follows that there exists a G-module M and an isomorphism $f : E|_A \to A \times M$. In Lemma 2.27, we showed that there exists an open G-invariant set $A \subset U$, such that f extends to an isomorphism $\tilde{f} : E|_U \to U \times M$. We now define the G-vector bundle $i(E)$ as follows: $i(E) := E/\sim$, where $v \sim w$ if $v = w$ or $v, w \in E|A$ and $\mathrm{pr}_M(f(v)) = \mathrm{pr}_M(f(w))$. Let $\pi' : E \to i(E)$ denote the projection. Notice that the G-action on E induces a G-action on $i(E)$. Let $q : i(E) \to X/A$ denote the projection defined by $q([v]) = [p(v)]$. To show that $q : i(E) \to X/A$ is a G-vector bundle, we will prove that the G-vector bundle is locally trivial and leave the verification of the other properties to the reader. First assume that $x \in X/A - \pi(A)$. Let (U, Ψ) be a local trivialisation of E, with $x \in U$, then $(U \cap (X - A), \Psi|_{U \cap (X-a)})$ is a local trivialisation of $i(E)$, with $x \in U \cap (X - A)$.
Now assume that $x \in A$. Let $\tilde{f} : E|_U \to U \times M$ the isomorphism we defined before. Notice that $\pi' \circ \tilde{f}$ is constant on the fibers of π'. This implies that there exists a unique continuous map $g : i(E)|_{\pi(U)} \to \pi(U) \times M$ such that $g \circ \pi' = \pi' \circ \tilde{f}$. The pair $(\pi(U), g)$ is the desired local trivialisation around x.
The map i induces a map $i : \tilde{K}_G(X) \to \tilde{K}_G(X/A)$ by $i(E - F) = i(E) - i(F)$. Notice that i is well defined. By construction, we have $i\pi^* E \cong E$ and $\pi^* i(E) \cong E$, which implies that i is the inverse of π^* and $\pi^* : \tilde{K}_G(X/A) \to \tilde{K}_G(X)$ is an isomorphism. □

Remark 3.31. A direct consequence of this theorem is that, because CA and $C(A^+)$ are G-contractible, we have

$$\tilde{K}_G(X, A) \cong \tilde{K}_G(X/A)$$

and

$$K_G(X, A) \cong \tilde{K}_G(X^+/A^+) \cong \tilde{K}_G(X/A).$$

In the next sections, we will develop the theory further, to eventually show that the groups $K_G^{-n}(X)$ are periodic:

$$(K_G^{\mathbb{C}})^{-n}(X) \cong (K_G^{\mathbb{C}})^{-n-2}(X)$$

and

$$(K_G^{\mathbb{R}})^{-n}(X) \cong (K_G^{\mathbb{C}})^{-n-8}(X).$$

To show this, we will first look at the group $K(X)$ from the perspective of Banach categories, then introduce Clifford algebras and show that they are in a certain sense periodic. After his, we will link these algebras to the groups $K^{-n}(X)$ to prove that the groups $K_G^{-n}(X)$ are periodic.

4 Banach categories

To view equivariant K-theory from a more general perspective, we will introduce Banach categories and pseudo abelian categories. We will show that the category of G-vector bundles is a Banach category. This section is based on [3] and the definitions found on [5], [7] and [6].

We will start with a definition:

Definition 4.1. *Let $\mathcal{C}$ be a locally small category. We call $\mathcal{C}$ an additive category if*

(i) For each object $C, C' \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ has the structure of an abelian group.

(ii) For each object $C, C', C'' \in \mathcal{C}$, the composition $\circ : \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{C}}(C', C'') \to \mathrm{Hom}_{\mathcal{C}}(C, C'')$ is bilinear.

(iii) The category $\mathcal{C}$ has finite products and co-products.

Remark 4.2. Since $\mathrm{Hom}_{\mathbb{C}}(C, C')$ is an abelian group, it contains a morphism $0_{C,C'}$, which is the identity of the group. Since composition is bilinear, we have

$$\begin{aligned} 0_{C',C''} \circ f &= 0_{C',C''} \circ f + 0_{C',C''} \circ f + (-0_{C',C''} \circ f) \\ &= (0_{C',C''} + 0_{C',C''}) \circ f + (-0_{C',C''} \circ f) \\ &= 0_{C',C''} \circ f + (-0_{C',C''} \circ f) = 0_{C,C''} \end{aligned}$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ and similarly we have $g \circ 0_{C,C'} = 0_{C,C''}$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(C', C'')$.

Example 4.3. The category of vector spaces over $\mathbb{C}$ is an example of an additive category.

We start with proving some basic properties of additive categories:

Lemma 4.4. *Let $\mathcal{C}$ be an additive category and let $A, C_1, \ldots, C_n$ be objects of $\mathcal{C}$. Let $f_i, g_i : A \to C_i$ be morphisms for $1 \leq i \leq n$. Let $(f_1, \ldots, f_n) : A \to \prod_{i=1}^n C_i$ denote the unique map with the property that $p_i \circ (f_1, \ldots, f_n) = f_i$, where $p_i : \prod_{i=1}^n C^i \to C^i$ denotes the projection. Then,*

$$(f_1, \ldots, f_n) + (g_1, \ldots, g_n) = (f_1 + g_1, \ldots, f_n + g_n).$$

Proof. Since composition of functions is bilinear, we have

$$p_i \circ ((f_1, \ldots, f_n) + (g_1, \ldots, g_n)) = p_i \circ (f_1, \ldots, f_n) + p_i \circ (g_1, \ldots, g_n) = f_i + g_i.$$

for all $1 \leq i \leq n$, which proves the lemma. □

Lemma 4.5. *Let $\prod_{i=1}^n C_i$ be the product from the previous lemma. Let*

$$i_{C_i} := (0_{C_i,C_1}, \ldots, 0_{C_i,C_{i-1}}, id_{C_i}, 0_{C_i,C_{i+1}}, \ldots, 0_{C_i,C_n})$$

We have

$$id_{\prod_{i=1}^n C_i} = \sum_{i=1}^n i_{C_i} \circ p_i.$$

Proof. Notice that by definition, we have

$$p_j \circ i_{C_i} := \begin{cases} 0_{C_i,C_j} & \text{if} \quad i \neq j \\ id_{C_i} & \text{if} \quad i = j \end{cases}. \tag{4.1}$$

This implies that

$$p_j(\sum_{i=1}^{n} i_{C_i} \circ p_i) = \sum_{i=1}^{n} (p_j \circ i_{C_i}) \circ p_i = id_{C_j} \circ p_j + \sum_{i \neq j} 0_{C_i,C_j} \circ p_i = p_j + 0 = p_j,$$

for all $1 \leq j \leq n$. The fundamental property of the product now implies that

$$\sum_{i=1}^{n} i_{C_i} \circ p_i = (p_1, \dots, p_n).$$

Since $p_j \circ id = p_j$ for all $1 \leq j \leq n$, we also have $id = (p_1, \dots, p_n)$, which proves the lemma. □

We can use these lemmas to show that abelian categories have the following useful popery. The proof is based on the proof proposition 2.1 from [5].

Proposition 4.6. *Let $\mathcal{C}$ be an additive category and let $C_1, \dots, C_n$ be objects of $\mathcal{C}$. Then*

$$\coprod_{i=1}^{n} C_i \cong \prod_{i=1}^{n} C_i.$$

Proof. To show that the product and co-product are isomorphic, it is sufficient to show that the product has the fundamental property of the co-product. We first consider the case $n = 0$. In this case our diagram is empty and we have to show that a terminal object also is an initial object. Let C denote an terminal object. Then by definition, there is a unique morphism $f : C \to C$. Since $0_{C,C}, id_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$, it follows that $f = 0_{C,C} = id_C$. Let C' be an object of C and $g \in \mathrm{Hom}_{\mathcal{C}}(C, C')$. Remark 4.2 implies that

$$g = g \circ id_C = g \circ 0_{C,C} = 0_{C,C'}.$$

Therefore, we have $\mathrm{Hom}_{\mathcal{C}}(C, C') = \{0_{C,C'}\}$ and C is an initial object.
We now consider the case where the diagram is not empty. Let $p_i : \prod_{i=1}^{n} C^i \to C_i$ denote the projection and let i_{C_j} be defined as in the lemma above. We show that $\prod_{i=1}^{n} C_i$, where the inclusions are given by the maps $i_{C_j} : C_j \to \prod_{i=1}^{n} C_i$ for $1 \leq j \leq n$ is a co-product. Let $A \in \mathcal{C}$ and let $f_i : C_i \to A$ be morphisms in $\mathcal{C}$. We claim that the map $g := \sum_{i=1}^{n} f_i \circ p_i$ is the unique map with the property that $g \circ i_{C_j} = f_j$ for all $1 \leq j \leq n$. We first show that g indeed has this property. Equation 4.1 and Remark 4.2 imply that

$$g \circ i_{C_j} = \sum_{i=1}^{n} f_i \circ (p_i \circ i_{C_j}) = f_j \circ id_j + \sum_{i \neq j} f_i \circ 0_{C_j,C_i} = f_j.$$

We now show that g is unique. Assume that $h : \prod_{i=1}^{n} C_i \to A$ also satisfies $h \circ i_{C_j} = f_j$. Lemma 4.1 implies that

$$h = h \circ id = h \circ (\sum_{i=1}^{n} i_{C_i} \circ p_i) = \sum_{i=1}^{n} (h \circ i_{C_i}) \circ p_i = \sum_{i=1}^{n} f_i \circ p_i = g.$$

Therefore,

$$\coprod_{i=1}^{n} C^i \cong \prod_{i=1}^{n} C^i.$$

□

Remark 4.7. In the proof of Proposition 4.6, we did not use that $\mathcal{C}$ has co-products. The proposition shows that a category $\mathcal{C}$ is already a additive category if it satisfies (i) and (ii) of Definition 4.1 and has finite products.

Remark 4.8. If $i^{C_i} : C_i \to \coprod_{i=1}^{n} C_i$ is the inclusion. Then the isomorphism of the previous proposition can be described explicitly by the map $\Phi : \coprod_{i=1}^{n} C_i \to \prod_{i=1}^{n} C_i$, defined by

$$i^{C_i} \circ \Phi = i_{C_i}.$$

This Lemma shows that every finite product in an additive category $\mathcal{C}$ also has the structure of a co-product and every finite co-product also has the structure of a product. We will therefore no longer distinguish between them and will write the product/co-product of object $C_1, \ldots C_n$ of $\mathcal{C}$ by $\bigoplus_{i=1}^{n} C_i$.

Remark 4.9. Let $\mathcal{C}$ be an additive category and C_i, D_j be objects in $\mathcal{C}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Let $f : \oplus_{i=1}^{n} C_i \to \oplus_{i=1}^{k} D_j$ be morphisms in $\mathcal{C}$. Lemma 4.5 implies that

$$f = id \circ f \circ id = \sum_{i=1}^{k} \sum_{j=1}^{n} i_{D_i} p_{D_i} f i_{C_j} p_{C_j}.$$

If we let $f_{i,j} := p_{D_i} f i_{C_j} \in \mathrm{Hom}_{\mathcal{C}}(C_j, D_i)$, then we can write f as the matrix:

$$\begin{pmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & & \vdots \\ f_{k,1} & \cdots & f_{k,n} \end{pmatrix}.$$

We can recover f from this matrix via the formula

$$f = \sum_{i=1}^{k} \sum_{j=1}^{n} i_{D_i} f_{i,j} p_{C_j}.$$

Notice that this matrix is unique, because if the is a matrix $(f'_{i,j})_{1 \leq i \leq k, 1 \leq j \leq l})$ such that

$$f = \sum_{i=1}^{k} \sum_{j=1}^{n} i_{D_i} f'_{i,j} p_{C_j},$$

then

$$f_{p,q} = p_{D_p} \circ (\sum_{i=1}^{k} \sum_{j=1}^{n} i_{D_i} f_{i,j} p_{C_j}) \circ i_q = p_{D_p} \circ (\sum_{i=1}^{k} \sum_{j=1}^{n} i_{D_i} f'_{i,j} p_{C_j}) \circ i_q = f'_{i,j}.$$

If $g : \oplus_{i=1}^{l} B_i \to \oplus_{i=1}^{n} C_i$ is also a morphism, then these matrices have the property that

$$f \circ g = \begin{pmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & & \vdots \\ f_{k,1} & & f_{k,n} \end{pmatrix} \begin{pmatrix} g_{1,1} & \cdots & g_{1,l} \\ \vdots & & \vdots \\ g_{n,1} & & g_{n,l} \end{pmatrix}.$$

This is true because Equation 4.1 implies that

$$\begin{aligned}(f\circ g)_{p,q} &= p_{D_p} f\circ id\circ id\circ g i_{B_q}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} p_{D_p} f i_{C_i}\circ (p_{C_i} i_{C_j})\circ p_{C_j} g i_{C_q}\\ &= \sum_{i=1}^{n} (p_{D_p} f i_{C_i} p_{C_i})\circ (i_{C_i} p_{C_i} g i_{C_q})\\ &= p_{D_p}(\sum_{i=1}^{n} f_{p,i} g_{i,q}) i_{C_q}.\end{aligned}$$

If h_1 and h_2 are morphisms in $\mathcal{C}$, we wil often use the notation $h_1 \oplus h_2$ to denote the matrix

$$\begin{pmatrix} h_1 & 0\\ 0 & h_2\end{pmatrix}.$$

As the $\oplus$ notation might suggest, the sum $\oplus$ of G-vector bundles can be viewed as the product of those bundles in a certain additive category.

Proposition 4.10. *Let $\mathcal{E}_G^{\mathbb{K}}(X)$ be the set of G-vector bundles over a compact G-space X. The category $\mathcal{E}_G^{\mathbb{K}}(X)$, where the objects are G-vector bundles and the morphism are G-vector bundle morphism (as defined in Definition 2.6), is an additive category.*

Proof. Let $p : E \to X$ and $q : F \to X$ be G-vector bundles. If $f, g \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F)$, then the sum $(f+g)(x) := f(x)+g(x)$ gives the set $\mathrm{Hom}_G(E, F)$ the structure of an abelian group such that the composition of maps is bilinear. We now show that $E_G^{\mathbb{K}}(X)$ has finite products. First, notice that the G-vector bundle $X \times 0$ is a terminal object. Now let $p_i : E_i \to X$ be G-vector bundles for $1 \le i \le n$. We claim that the bundle $q : E_1 \oplus \ldots \oplus E_n \to X$, with the projection $\pi_i : E_1 \oplus \ldots \oplus E_n \to E_i$, defined by $p_i(v_1, v_2, \ldots, v_n) = v_i$, is the product of these vector bundles. To prove the claim, we check the fundamental property of the product. Let $q' : A \to X$ be a G-vector bundle and let $f_i : A \to E_i$ be G-vector bundle morphism for $1 \le i \le n$. The map $f : A \to E_1 \oplus \ldots \oplus E_n$, defined by $f(v) = (f_1(v), \ldots, f_n(v))$ is the unique G-vector bundle morphism with the property that $p_i \circ f = f_i$. Hence $E_1 \oplus \ldots \oplus E_n$ is indeed the product of the G-vector bundles. Therefore, the categoy $E_G^{\mathbb{K}}(X)$ is an additive category. □

In the proposition above, the set $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E.F)$ has more structure then an abelian group; it is a $\mathbb{K}$ vector space. This motivates the definition of a Banach category (Definition 2.1 on page 59 of [3]):

Definition 4.11. *Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let $\mathcal{C}$ be a category. We call $\mathcal{C}$ a Banach category if $\mathcal{C}$ is an additive category, such that for all object C, C' of C, the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ has the structure of a Banach space over $\mathbb{K}$, such that the composition map $\circ$ is bilinear and continuous.*

Proposition 4.12. *The category $E_G^{\mathbb{K}}(X)$ is a Banach category.*

Proof. Let $p : E \to X$ and $q : F \to X$ be G-vector bundles. We want to give $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F)$ the structure of a Banach space. Let μ_E and μ_F be invariant metrics on E and F. For each

$x \in X$ and $v_E \in E_x$ and $v_f \in F_x$, we let $\|v_E\|_E := \sqrt{\mu_E(x_E, x_E)}$ and $\|v_F\|_F := \sqrt{\mu_F(x_F, x_F)}$. We now define the norm $\|\cdot\|$ on $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(X, Y)$ by

$$\|A\| := \sup_{\substack{v \in E \\ \|v\|_E = 1}} (\|Av\|_F).$$

It is clear from the definition that if $\|A\| < \infty$ for all $A \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F)$, then $\|\cdot\|$ is a norm. We show that $\|A\| < \infty$. Since $\|Av\|_F = (\sqrt{\cdot} \circ \mu_f \circ \Delta_F \circ A)(v)$, where $\Delta_F(v_f) := (v_f, v_f)$, it follows that

$$\|A\| = \sup(\sqrt{\cdot} \circ g_f \circ \Delta_F \circ A|_{(\mu_E \circ \Delta_E)^{-1}(1)}).$$

Since $(g_E \circ \Delta_E)^{-1}(1)$ is a closed subset of a compact space, it is compact. Therefore, the map $\sqrt{\cdot} \circ \mu_f \circ \Delta_F \circ A|_{(\mu_E \circ \Delta_E)^{-1}(1)}$ attains its maximum and $\|A\| < \infty$.
We now show that $(\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F), \|\cdot\|)$ is complete.
Let $\{A_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence. Let $x \in X$ and

$$\|A_i\|_x := \sup_{v \in E_x, \|v\|_E = 1} (\|Av\|_F).$$

We have $\|A_i - A_j\|_x \leq \|A_i - A_j\|$. Therefore, The sequence $\{A_i|_{E_x}\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathrm{Hom}_{\mathrm{Lin}}(E_x, F_x)$ (the set of linear maps from $E_x \to F_x$) with the operator norm. Since this is a Banach space, there is a unique $A_x \in \mathrm{Hom}_{\mathrm{Lin}}(E_x, F_x)$, such that $\lim_{n \to \infty} A_i|_x = A_x$. We define $A : E \to F$ by $A|_{E_x} = A_x$. We show that A is a G-map. Notice that for all $x \in X$, we have

$$l_g \circ A_i|_{E_x} \circ l_{g^{-1}} = A_i|_{E_x}.$$

Therefore,

$$\lim_{i \to \infty} l_g \circ A_i|_{E_x} \circ l_{g^{-1}} - A_i|_{E_x} = \lim_{i \to \infty} 0 = 0$$

and

$$A_x = \lim_{i \to \infty} A_i|_{E_x} = \lim_{i \to \infty} l_g \circ A_i|_{E_x} \circ l_{g^{-1}} = l_g A_x l_{g^{-1}}.$$

Thus, we have $l_g A = A l_g$. We leave it to the reader to check that A is indeed continuous. Therefore, The map A is a morphism of G-vector bundles. We now show that we indeed have $\lim_{n \to \infty} A_i = A$. Let $\epsilon > 0$. Since A_i is a Cauchy sequence, there is a $N \in \mathbb{N}$, such that for all $i, j > N$, we have $\|A_i - A_j\| < \epsilon$. Let $i > n$. For each $x \in X$, there exists a $i_x > i$, such that $\|A_{i_x} - A\|_x < \epsilon$. Therefore,

$$\|A - A_i\| = \sup_{x \in X} \|A - A_i\|_x = \leq \sup_{x \in X}(\|A - A_{i_x}\| + \|A_{i_x} - A_i\|) < \sup_{x \in X} 2\epsilon = 2\epsilon$$

and $\lim_{n \to \infty} A_i = A$. Thus $(\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F), \|\cdot\|)$ is a Banach space.
We now show that the composition map

$$\circ : \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F) \times \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(F, F') \to \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F')$$

is continuous. Let $A \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E, F)$ and $B \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(F, F')$. Notice that for all $v \in F$, with $g_F(v, v) \neq 0$, we have

$$\|Bv\| = \|B \frac{v}{\|v\|_F}\| \|v\|_F \leq \|B\| \|v\|_F.$$

Therefore, $\|Bv\| \leq \|B\|\|v\|$. This implies that

$$\|BA\| = \sup_{v\in E,\ \|v\|_E=1}(\|BAv\|_{F'}) \leq \sup_{v\in E,\ \|v\|_E=1}\|B\|\|Av\| = \|B\|\sup_{v\in E,\ \|v\|_E=1}\|Av\| = \|B\|\|A\|,$$

which implies that the composition is continuous. Thus, the category $\mathcal{E}_G^{\mathbb{K}}(X)$ is a Banach category. $\square$

Remark 4.13. In our definition of the Banach structure on $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,F)$, we defined the Banach structure using the metric μ_E and μ_F. It is at first not clear what happens to the Banach structure if we choose other metrics μ_E' and μ_F'. Let $\|.\|'$ denote the norm on $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,F)$ induced by μ_E' and μ_F'. We show that the map

$$id\circ\cdot : (\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,F), \|\cdot\|) \to (\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,F), \|\cdot\|')$$

is an isomorphism. First notice that $id\circ\cdot$ is bijective and linear. We show that $id\circ\cdot$ is continuous. In the proof of the proposition above, we showed that $\|id\circ A\| \leq \|id\|\|A\|$, which implies that $id\circ\cdot$ is bounded and thus continuous. Notice that

$$\cdot\circ id : (\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,F), \|\cdot\|') \to (\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,F), \|\cdot\|)$$

is the inverse of $id\circ\cdot$ and $\|A\circ id\| \leq \|A\|\|id\|$, which implies that $id\circ\cdot$ is continuous. Therefore, $id\circ\cdot$ is an isomorphism.
Thus the choice of metric on the G-vector bundles does not matter in the sense that a different choice of metrics will give an isomorphic Banach space.

The Banach structure on $\mathrm{Hom}_{\mathcal{E}_G(X)}(E,F)$ has the following useful property:

Proposition 4.14. *There is a bijection between paths in* $\mathrm{Hom}_{\mathcal{E}_G(X)}(E,F)$ *and homotopies between G-vector bundle morphism.*

Proof. First, Let $H : [0,1] \to \mathrm{Hom}_{\mathcal{E}_G(X)}$ be a path in $\mathrm{Hom}_{\mathcal{E}_G(X)}(E,F)$. We claim that the map $\tilde{H} : E\times[0,1] \to F$ defined by $\tilde{H}(x,t) = H(t)(x)$ is a homotopy between H_0 and H_1 though G-vector bundle morphism. This is the case if $\tilde{H}$ is continuous. Let $x \in X$ and let (U,Ψ_E) and (U,Ψ_F) be local trivialisations of E and F, such that U is a compact neighbourhood of x and let $v \in E_x$. Let μ_E and μ_f be the metrics which induce the norm on $\mathrm{Hom}_{\mathcal{E}_G(X)}$. In the local trivialisation $U\times\mathbb{R}^n$, we have

$$\begin{aligned}\|\mathrm{pr}_{\mathbb{R}^n}H(t)(v) - \mathrm{pr}_{\mathbb{R}^n}H(s)(w)\| &\leq \|\mathrm{pr}_{\mathbb{R}^n}H(t)(v) - \mathrm{pr}_{\mathbb{R}^n}H(t)(w)\| \\ &\quad + \|\mathrm{pr}_{\mathbb{R}^n}H(t)(w) - \mathrm{pr}_{\mathbb{R}^n}H(s)(w)\|.\end{aligned}$$

Since $H(t)$ is a G-vector bundle morphism, there exists an open neighbourhood of $U_1 \subset U\times\mathbb{R}^n$ of v such that if $w \in U_1$, then $\|\mathrm{pr}_{\mathbb{R}^n}H(t)(v) - \mathrm{pr}_{\mathbb{R}^n}H(t)(w)\| < \epsilon$. We may assume that there exists an $N \in \mathbb{N}$, such that $U_1 \subset U\times B(0,N)$. Remark 4.13 implies that we may assume that $\mu_E|_{U\times\mathbb{R}^n}$ and $\mu_F|_{U\times\mathbb{R}^m}$ are the Euclidean metric. Therefore, we have

$$\|\mathrm{pr}_{\mathbb{R}^n}H(t)(w) - \mathrm{pr}_{\mathbb{R}^n}H(s)(w)\| \leq \|H(t)-H(s)\|\|w\|.$$

Since H is continuous, there exists an open neighbourhood I_0 of t such that if $s \in I$, then $\|H(t)-H(s)\| < \frac{\epsilon}{N}$. Therefore, if $(w,s) \in U_1\times I_0$, then

$$\|\mathrm{pr}_{\mathbb{R}^n}H(t)(v) - \mathrm{pr}_{\mathbb{R}^n}H(s)(w)\| \leq 2\epsilon.$$

Therefore, the map $\mathrm{pr}_U \circ \tilde{H}|_{E|_U \times [0,1]}$ and $\mathrm{pr}_{\mathbb{R}^n} \circ \tilde{H}|_{E|_U \times [0,1]}$ are continuous, which implies that $\tilde{H}|_{E|_U \times [0,1]}$ is continuous and thus that $\tilde{H}$ is continuous.
Now let $H : E \times [0,1] \to F$ be a homotopy through G-vector bundle morphism. We define the path $H' : [0,1] \to \mathrm{Hom}_{\mathcal{E}_G(X)}(E,F)$ by $H'(t) = H(\cdot, t)$. We show that H' is continuous. Let $t \in [0,1]$, $x \in X$ and let $v \in (g_E \circ \Delta_E)^{-1}(1)$ and $t \in [0,1]$. Since the map $h : [0,1] \times (\mu_E \circ \Delta_E)^{-1}(1) \to \mathbb{R}$ defined by

$$h(s,v) = \sqrt{\mu_F(\Delta_F(H(v,t) - H(v,s)))},$$

is continuous, there exists open neighbourhoods U_v of v in $(g_E \circ \Delta_E)^{-1}(1)$ and I_v of t in $[0,1]$ such that $U_v \times I_v \subset h^{-1}((-\epsilon, \epsilon))$. Since $\{U_v\}_{v \in (g_E \circ \Delta_E)^{-1}(1)}$ is an open cover of a compact space, it has a finite sub cover $U_{v_1}, \dots U_{v_n}$. Let $I = \cap_{i=1}^n I_i$. Then, by construction, we have that $\|H'(t) - H'(s)\| < 2\epsilon$ if $s \in I$, which implies that H' is continuous. The operation $'$ and $\tilde{}$ are by construction each others inverse, which proves the proposition. □

The set $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,E)$ has even more structure then just being a Banach space, it has the structure of a Banach algebra, where the multiplication is given by composition.

Definition 4.15. *A Banach algebra is a $\mathbb{K}$-Banach space $(V, \|\cdot\|)$, together with a multiplication $\cdot : V \times V \to V$ such that*

(i) The triple $(V, +, \cdot)$ is an associative $\mathbb{K}$-algebra.

(ii) For all $U, v \in V$, we have $\|uv\| \leq \|u\|\|v\|$

Remark 4.16. The element $id : E \to E$ has the property that $v \cdot id = id \cdot v = v$ for all $v \in V$. The Banach algebra $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,E)$ is thus a Banach algebra with unit.

Example 4.17. Because the set is $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}}(E,E)$ a Banach algebra, we can define the exponential and logarithm on $\mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}}(E,E)$. The exponential for a $v \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}}(E,E)$ is defined by

$$\exp(v) := \sum_{k=0}^{\infty} \frac{v^k}{k!},$$

where $v^0 = id$. For $v \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}}(E,E)$, with $\|id - v\| < 1$, we can define the logarithm by

$$\log(v) := \sum_{k=1}^{\infty} \frac{(-1)^{n-1}(v - id)}{n}.$$

We leave it to the reader to check that both sums are indeed convergent. The exponential function still has some of the properties from the exponential function on $\mathbb{K}$. One can check by writing out the power series that

$$\exp(\lambda_1 v)\exp(\lambda_2 v) = \exp((\lambda_1 + \lambda_2)v),$$

for all $v \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E,E)$ and $\lambda_1, \lambda_2 \in \mathbb{K}$. Notice that this implies that $\exp(v)\exp(-v) = \exp(0) = id$ and hence that $\exp(v) \in \mathrm{Aut}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E)$. Another property that caries over is

$$\exp(\log(v)) = v,$$

for all $v \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}}(E,E)$, with $\|id - v\| < 1$. Again, this can be shown by writing out the power series.

We have now seen examples of additive categories and Banach categories. These categories have more structure than a category and therefore the functors between them should 'preserve' this extra structure. This motivates the following definition, which is based on Definition 1.1 of [6].

Definition 4.18. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an additive functor if:*

(i) the functor maps initial objects to initial objects and terminal objects to terminal objects.

(ii) The functor F preserves finite products and co-products.

Remark 4.19. A functor F preserves finite products if for a product $\prod_{i=1}^n X_i$ with projections $p_j : \prod_{i=1}^n X_i \to X_j$, we have $F(\prod_{i=1}^n X_i) \cong \prod_{i=1}^n F(X_i)$ and the projections on $F(\prod_{i=1}^n X_i))$ are given by $F(p_j) : F(\prod_{i=1}^n X_i)) \to F(X_j)$. The definition for co-products is similar.

This notion of morphism might seem the wrong notion of morphism, because we do not require $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ to be a group homomorphism. However, the following proposition shows that an additive functor already has this property.

Proposition 4.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories and let C, C' be objects of $\mathcal{C}$. Then for each $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, we have*

$$F(f_1 + f_2) = F(f_1) + F(f_2).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
& C \arrow[dl, "id"'] \arrow[d, "g"] \arrow[dr, "id"] & \\
C \arrow[r, shift left, "i_1"] \arrow[dr, "f_1"'] & C \oplus C \arrow[l, shift left, "p_1"] \arrow[r, shift left, "p_2"] \arrow[d, "h"] & C \arrow[l, shift left, "i_2"] \arrow[dl, "f_2"] \\
& C' &
\end{tikzcd},$$

Where p_j and i_j are defined as before, the function g is defined as the unique function such that $p_j \circ g = id$ and h is defied as the unique function such that $h \circ i_j = f_j$. Notice that $g = i_1 + i_2$ and $h = f_1 \circ p_1 + f_2 \circ p_2$. This implies that

$$h \circ g = (f_1 \circ p_1 + f_2 \circ p_2) \circ (i_1 + i_2) = f_1 + f_2.$$

If we apply F to all objects and morphisms in this diagram, we obtain the following diagram:

$$
\begin{tikzcd}
& F(C) \arrow[dl, "id"'] \arrow[d, "F(g)"] \arrow[dr, "id"] & \\
F(C) \arrow[r, shift left, "F(i_1)"] \arrow[dr, "F(f_1)"'] & F(C \oplus C) \arrow[l, shift left, "F(p_1)"] \arrow[r, shift left, "F(p_2)"] \arrow[d, "F(h)"] & F(C) \arrow[l, shift left, "F(i_2)"] \arrow[dl, "F(f_2)"] \\
& F(C') &
\end{tikzcd},
$$

Since F preserves limits, $F(g)$ is the unique map such that $F(p_j) \circ F(g) = id$. Therefore, we have $F(g) = F(i_1) + F(i_2)$. Since F preserves co-products, the map $F(h)$ is the unique map such that $F(h) \circ F(i_j) = F(f_j)$. This implies that $F(h) = F(f_1) \circ F(p_1) + F(f_2) \circ F(p_2)$. Thus,

$$F(f_1+f_2) = F(h\circ g) = F(h)\circ F(g) = (F(f_1)\circ F(p_1)+F(f_2)\circ F(p_2))\circ(F(i_1)+F(i_2)) = F(f_1)+F(f_2),$$

which proves the proposition. $\square$

There is also a notion of a morphism between Banach categories (The definition is based on Definition 2.6 on page 60 of [3]):

Definition 4.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be Banach categories. We call a functor $F : \mathcal{C} \to \mathcal{D}$ a Banach functor if:*

(i) The functor F is an additive functor.

(ii) For each $C, C' \in \mathcal{C}$, the map $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ is linear and continuous.

Example 4.22. Let X be a compact G-space and $A \subset X$ an closed G-invariant subset. The inclusion $i : A \to X$ induces a functor $i^* : \mathcal{E}_G^{\mathbb{K}}(X) \to \mathcal{E}_G^{\mathbb{K}}(A)$ which maps an object E to i^*E and acts on morphism by $i^*f = f|_A$. Since $i^*(X \times 0) = A \times 0$, the functor i^* preserves the initial and terminal object. Also notice that $i^*(E_1 \oplus E_2) \cong i^*(E) \oplus i^*(E)$ and that i^* maps the inclusion and projections of $E_1 \oplus E_2$ to those of $i^*(E_1 \oplus E_2)$. Therefore, the functor i^* is an example of an additive functor. Because we also have that

$$i^*(f + \lambda g) = (f + \lambda g)|_A = f|_A + \lambda g|_A = i^*(f) + \lambda i^*(g),$$

for $f, g \in \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E_1, E_2)$ and

$$\|i^* f\| = \|f \circ i\| \leq \|f\|,$$

the map $i^* : \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(X)}(E_1, E_2) \to \mathrm{Hom}_{\mathcal{E}_G^{\mathbb{K}}(A)}(i^*E_1, i^*E_2)$ is linear and continuous. Therefore, the functor i^* is a Banach functor.

In some cases, a Banach functor preserves even more structure (The definition is based on Definition 2.6 on page 60 of [3]):

Definition 4.23. *Let $\mathcal{C}$ and $\mathcal{D}$ be Banach functors. We call a Banach functor $F : \mathcal{C} \to \mathcal{D}$ quasi surjective, if for every object D of $\mathcal{D}$ there exists an object C of $\mathcal{C}$ and an object $D^\perp$ of $\mathcal{D}$ such that*

$$D \oplus D^\perp \cong F(C).$$

Example 4.24. We claim that the functor i^* from the previous example is also an example a quasi surjective functor. Let $D \in \mathcal{E}_G^{\mathbb{K}}(X)$. Theorem 2.31 implies that there is a G-module M and a $D^\perp \in \mathcal{E}_G^{\mathbb{K}}(X)$, such that $D \oplus D^\perp \cong A \times M$. In Example 2.16, we showed that $i^*(X \times M) \cong A \times M$. Therefore, we have

$$i^*(X \times M) \cong D \oplus D^\perp,$$

which implies that i^* is quasi surjective.

We end this section by looking at another special type of additive categories. The definition we give is based on the Definition in [7].

Definition 4.25. *Let $\mathcal{C}$ be an additive category. We call $\mathcal{C}$ a pseudo-abelian category if for each object C_0 of C and each morphism $P : C_0 \to C_0$, with $P^2 = P$, the functor $Im_P : \mathcal{C}^{op} \to Set$ defined on objects by*

$$Im_P(C) := \{f \in \mathrm{Hom}_{\mathcal{C}}(C, C_0) \mid p \circ f = f\}.$$

and a morphisms by

$$F(g)(f) = f \circ g$$

is representable.

Remark 4.26. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable if there exists an object C of $\mathcal{C}$ such that the functor $y_C : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is naturally isomorphic to F, where $y_C(X) := \mathrm{Hom}_{\mathcal{C}}(X, C)$ and $y_C(g)(f) = f \circ g$

Example 4.27. We claim that the category $\mathcal{E}_G^{\mathbb{K}}(X)$ from the previous examples is an example of a pseudo-abelian category. Let $E \in \mathcal{E}_G^{\mathbb{K}}(X)$ and let $P : E \to E$ be a G-vector bundle morphism such that $P^2 = P$. In Lemma 3.11 we showed that $\mathrm{Im}P$ is a G-vector bundle. We claim that $y_{\mathrm{Im}P} \cong \mathrm{Im}_P$. We define the natural isomorphism $\sigma : y_{\mathrm{Im}P} \to \mathrm{Im}_P$ by

$$\sigma_F(f) = P \circ f.$$

First, notice that σ is indeed a natural transformation. We now show that σ_F is an isomorphism. Let $\tau : \mathrm{Im}_P \to y_{\mathrm{Im}P}$ be the natural transformation defined by $\tau_F(f) = i \circ f$. Notice that τ is a natural transformation. We claim that τ is the inverse of σ. Let $F \in \mathcal{E}_G^{\mathbb{K}}(X)$. We have $\tau_F \circ \sigma_F(f) = i \circ P \circ f$. Since $f = P \circ f$, it follows that $\mathrm{Im}(f) \subset \mathrm{Im}(P)$. Since $P|_{\mathrm{Im}P} = id$, it follows that $i \circ P \circ f = f$. With a similar argument, it follows that

$$\sigma_F \circ \tau_F g = P \circ i \circ g = g.$$

Therefore, the functor Im_P is representable and $\mathcal{E}_G^{\mathbb{K}}(X)$ is a pseudo-abelian category.

The following lemma about pseudo-abelian categories will often be usefull:

Lemma 4.28. *Let $\mathcal{C}$ be a pseudo-abelian category, C an object of $\mathcal{C}$ and $P_i : C \to C$ be morphisms such that $P_i^2 = P_i$ for all $1 \leq i \leq n$, $P_i \circ P_j = P_j \circ P_i = 0$ and $\sum_{i=1}^n P_i = id$. Then, we have*

$$C \cong \oplus_{i=1}^n C_i,$$

where C_i is an object such that $ImP_i \cong y_{C_i}$.

Proof. First, notice that since $\mathcal{C}$ is pseudo-abelian, there exists objects C_i and a natural isomorphisms $\sigma_i : y_{C_i} \to \mathrm{Im}_{P_i}$. We claim that the map $L := (\sigma_1^{-1}(P_1), \ldots, \sigma_n^{-1}(P_n))^T : C \to \oplus_{i=1}^n C_i$ is an isomorphisms. Consider the map $Q := (\sigma_1(id_{C_1}), \ldots, \sigma_n(id_{C_n})) : \oplus_{i=1}^n C_i \to C$. We have

$$QL = \sum_{i=1}^n \sigma_i(id_{C_i})\sigma_i^{-1}(P_i) = \sum_{i=1}^n \sigma_i(id_{C_i}\sigma_i^{-1}(P_i)) = \sum_{i=1}^n P_i = id.$$

If $i \neq j$, then

$$\begin{aligned} LQ_{i,j} &= \sigma_i^{-1}(P_i)\sigma_j(id_{C_j}) = \sigma_i^{-1}(P_i\sigma_j(id_{C_j})) = \sigma_i^{-1}(P_iP_j\sigma_j(id_{C_j})) \\ &= \sigma_i^{-1}(0_{C_j,C}) = \sigma_i^{-1}(P_i \circ 0_{C_j,C}) = \sigma_i^{-1}(P_i) \circ 0_{C_j,C} = 0_{C_j,C_i} \end{aligned}$$

and if $i = j$, then

$$LQ_{i,i} = \sigma_i^{-1}(P_i)\sigma_i(id_{C_i}) = \sigma_i^{-1}(P_i\sigma_i(id_{C_i})) = \sigma_i^{-1}(\sigma_i(id_{C_i})) = id_{C_i}.$$

Therefore, $LQ = id_{\oplus_{i=1}^n C_i}$.

$\square$

5 K-theory for Banach categories

In this section we will define the group $K(C)$ and $K^{-1}(C)$ of a Banach category C. The group $K(\varphi)$ of a quasi surjective Banach functor φ will also be introduced. We will prove some basic results for these groups and show how these groups give an 'alternative' definition of $K_G(X)$ and $K_G^{-1}(X)$ we defined in section 3 This section will be based on chapter II. 1, II.2 and III.3 of [3] and section 8 and 9 of [4].

To define the group $K(C)$ for a Banach category C, we will need the following definition:

Definition 5.1. *Let M be a commutative monoid. An abelian group group M_G together with a monoid monomorphism $i : M \to M_G$ is called the Grothendieck group of M, if for each monoid homomorphism $f : M \to A$, with A an abelian group, there exists a unique group homomorphism $f' : M_G \to A$, such that*

$$f' \circ i = f.$$

Lemma 5.2. *Let M be a commutative monoid. Then the group M_G exists and is unique up to isomorphism.*

Proof. We first show uniqueness. Assume that an abelian group N, with the map $j : M \to N$ is also a Grothendieck group of M. Since $j : M \to N$ is a monoid homomorphism and M_G is a Grothendieck group of M, Definition 5.1 implies that there exists a unique group homomorphism $j' : M_G \to N$ such that $j' \circ i = j$. Similarly, there exists a unique group homomorphism $i' : N \to M_G$, such that $i' \circ j = i$. Notice that

$$i' \circ j' \circ i = i' \circ j = i.$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd} M_G \arrow[dr, shift right, "id"'] \arrow[dr, shift left, "i'\circ j'"] & \\ M \arrow[u, "i"] \arrow[r, "i"] & M_G \end{tikzcd} .$$

Notice that id and $i' \circ j'$ both have $id \circ i = i$ and $(i' \circ j') \circ i = i$. Definition 5.1 says that there is a unique map with this property, hence $i' \circ j' = id$. With a similar argument it can be shown that $j' \circ i' = id$. Thus, the map $j' : M_G \to N$ is an isomorphism.
We now show existence of the group. We define $M_G := M \times M / \sim$, where $(m, n) \sim (a, b)$ if there exists a $l \in M$ such that $m + b + k = a + n + k$ and the sum of two elements is given by $(m_+, m_-) + (n_+, n_-) = (m_+ + n_+, m_- + m_-)$. We leave it to the reader to verify that $\sim$ is an equivalence relation and M_G is a group. The proof is somewhat similar to the proof of Lemma 3.3. We now show that M_G, with the monoid homomorphism $i : M \to M_G$ defined by $i(m) = (m, 0)$ satisfies definition 5.1. Let $h : M \to A$ be a monoid homomorphism, where A is an abelian group. Assume that there exists a $h' : M_G \to A$ such that $h' \circ i = h$. Then, since $i(m) = (m, 0)$, we must have

$$h'(m, 0) = h(m).$$

Because $(m,0)+(0,m)=0$, we must have

$$0 = h'(0) = h'((m,0)+(0,m)) = h'(m,0)+h'(0,m) = h(m)+h'(0,m)$$

and thus that $h'(0,m) = -h(m)$ and

$$h'(m,n) = h'(m,0)+h'(0,n) = h(m)-h(n).$$

Thus if h' exists, it is defined as above. Since the map $h'(m,n) = h(m)-h(n)$ is a well defined group homomorphism, the group M_G is indeed the Grothendieck group of M. $\square$

The construction of the Grothendieck group of a monoid looks very similar to our construction of the group $K_G^{\mathbb{K}}(X)$. In fact, if we let $E_G^{\mathbb{K}}(X) := ([E] \mid E \in \mathcal{E}_G^{\mathbb{K}}(X))$ and define the operation $\oplus : E_G^{\mathbb{K}}(X) \times E_G^{\mathbb{K}}(X) \to E_G^{\mathbb{K}}(X)$ by $[E]\oplus[F] = [E\oplus F]$. Then $(E_G^{\mathbb{K}}(X), \oplus)$ is a commutative monoid and the construction in Lemma 5.2 shows that $K_G^{\mathbb{K}}(X)$ is the Grothendieck group of $E_G^{\mathbb{K}}(X)$. This motivates the following definition:

Definition 5.3. *Let $\mathcal{C}$ be an additive category. Let $C_M := ob(\mathcal{C})/\sim$, where $C \sim D$ if C and D are isomorphic, denote the commutative module where the addition is given by $[C]\oplus[D] = [C\oplus D]$. Then we define $K(C)$ as the Grothendieck group of C_M.*

We can also define a group similar to the group $K(X,Y)$ for a Banach functor $\varphi : C \to D$. We will define this groups using triples (C_+, C_-, α), where $C_+, C_- \in \mathcal{C}$ and $\alpha : \varphi(C_+) \to \varphi(C_-)$ is an isomorphism. We will call the triple elementary if $C_+ = C_-$ and α is homotopic to the identity trough automorphism (In this context, a homotopy between maps f_1 and f_2 is thus a map $H : [0,1] \to \mathrm{Aut}(C_+)$, such that $H(0) = f_0$ and $H(1) = f(1)$. We will call the triples (C_+, C_-, α) and (C'_+, C'_-, α') isomorphic if there exists isomorphism $f : C_+ \to C'_+$ and $g : C_- \to C'_-$ such that $\varphi(g)\circ\alpha = \alpha'\circ\varphi(f)$. We will define the sum of two triples (C_+, C_-, α) and (C'_+, C'_-, α') by

$$(C_+, C_-, \alpha)\oplus(C'_+, C'_-, \alpha') = (C_+\oplus C'_+, C_-\oplus C'_-, \alpha\oplus\alpha').$$

With this terminology, we are ready to give the definition:

Definition 5.4. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a Banach functor. We define*

$$K(\varphi) := \{(C_+, C_-, \alpha) \mid C_+, C_- \in \mathcal{C} \text{ and } \alpha : \varphi(C_+) \to \varphi(C_-) \text{ is an isomorphism.}\}/\sim .$$

Where $(C_+, C_-, \alpha) \sim (C'_+, C'_-, \alpha')$ if there exist elementary triples (D, D, β) and (D', D', β') such that

$$(C_+, C_-, \alpha)\oplus(D, D, \beta) \cong (C'_+, C'_-, \alpha')\oplus(D', D', \beta').$$

We will denote the equivalence class of (C_+, C_-, α) by $[C_+, C_-, \alpha]$.

Lemma 5.5. *The set $K(\varphi)$, with the addition $[C_+, C_-, \alpha] = [C'_+, C'_-, \alpha'] = [C_+\oplus C'_+, C'_-\oplus C'_-, \alpha\oplus\alpha']$ is a group.*

Proof. First notice that $\oplus$ is well defined and $(K(\varphi), \varphi)$ is a monoid, where a triple $[C_+, C_-, \beta]$ is 0 if and only if there exist elementary triples (D, D, γ) and (D', D', γ') such that

$$(C_+, C_-, \beta)\oplus(D, D, \gamma) \cong (D', D', \gamma').$$

We now show that $\oplus$ is commutative. Let $C_1, C_2 \in C$ and $\beta : C_1 \oplus C_2 \to C_2 \oplus C_1$ be the map defined by

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since φ is a Banach functor, we have

$$\varphi(\beta) = \varphi(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} \varphi(0) & \varphi(1) \\ \varphi(1) & \varphi(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, the following diagram commutes:

$$\begin{tikzcd}
\varphi(C_1) \oplus \varphi(C_2) \arrow[r, "\alpha \oplus \alpha'"] \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"'] & \varphi(C_1) \oplus \varphi(C_2) \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"] \\
\varphi(C_2) \oplus \varphi(C_1) \arrow[r, "\alpha' \oplus \alpha"] & \varphi(C_2) \oplus \varphi(C_1)
\end{tikzcd}.$$

and the addition is commutative. We now show that every element has an inverse. Let $[C_+, C_-, \alpha] \in K(\varphi)$. We show that $[C_-, C_+, \alpha^{-1}]$ is the inverse. Notice that

$$[C_+, C_-, \alpha] \oplus [C_-, C_+, \alpha^{-1}] = [C_+ \oplus C_-, C_- \oplus C_+, \alpha \oplus \alpha^{-1}] \tag{5.1}$$

and that the following diagram commutes:

$$\begin{tikzcd}
\varphi(C_+) \oplus \varphi(C_-) \arrow[r, "\alpha \oplus \alpha^{-1}"] \arrow[d, "\varphi(id)"'] & \varphi(C_-) \oplus \varphi(C_+) \arrow[d, "\varphi(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})"] \\
\varphi(C_+) \oplus \varphi(C_-) \arrow[r, "\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix}"'] & \varphi(C_+) \oplus \varphi(C_-)
\end{tikzcd}.$$

This implies that $[C_+ \oplus C_-, C_- \oplus C_+, \alpha \oplus \alpha^{-1}] = [C_+ \oplus C_-, C_+ \oplus C_-, a]$, where a is the matrix on the lower horizontal arrow. If we view α a a number, we can use Gaussian elimination to write the matrix a as a product of elementary matrices and we obtain:

$$a = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}. \tag{5.2}$$

Notice however that if we view α again as a morphism, all the matrices are well defined automorphism of $\varphi(C_+) \oplus \varphi(C_-)$ and the equality also holds in this case. The homotopy $H : [0, 1] \to \mathrm{Aut}(\varphi(C_+) \oplus \varphi(C_-))$ given by

$$H(t) = \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \tag{5.3}$$

shows that $[C_+ \oplus C_-, C_+ \oplus C_-, a]$ is an elementary triple, which implies that $[C_+ \oplus C_-, C_+ \oplus C_-, a] = 0$. $\square$

Remark 5.6. Notice that if α is homotopic to a morphism β, then

$$\begin{pmatrix} 0 & -\alpha^{-1} \\ \beta & 0 \end{pmatrix} \text{ is homotopic to } \begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix}.$$

If we replace in $[C_+, C_-, \alpha]$by $[C_+, C_-, \beta]$ in Equation 5.1 in the lemma above, we see that this implies that

$$[C_+, C_-, \beta] \oplus [C_-, C_+, \alpha^{-1}] = 0$$

and thus that

$$[C_+, C_-, \beta] = [C_+, C_-, \alpha].$$

Before we proceed, we give an example which motivates why this group could be useful:

Example 5.7. Let X be a compact G-space and $Y \subset X$ be a closed G-invariant subset. In Example 4.22 we showed that the functor $i^* : \mathcal{E}_G^{\mathbb{K}}(X) \to \mathcal{E}_G^{\mathbb{K}}(Y)$ is a Banach functor. The group $K(i^*)$ consists of triples $[E, F, \alpha]$, with $E, F \in \mathcal{E}_G^{\mathbb{K}}(X)$ and $\alpha : E|_Y \to F|_Y$ a G-vector bundle isomorphism. We will later give a geometric interpretation of these groups and will show that $K(X, Y) \cong K(i^*)$.

To show that $K(i^*) \cong K_G(X, Y)$, we must first understand the group $K(i^*)$ better. To do this, we need to introduce the group $K^{-1}(\mathcal{C})$ of a Banach category $\mathcal{C}$.

Definition 5.8. *Let $\mathcal{C}$ be a Banach category. We let*

$$K^{-1}(C) := \{(C, \alpha) \mid C \in ob(\mathcal{C}) \text{ and } \alpha : C \to C \text{ is an isomorphism}\}/\sim.$$

Where $(C, \alpha) \sim (C', \alpha')$ if there exists an object D of C, such that $\alpha \oplus id_{C'} \oplus id_D$ is homotopic through isomorphism to $id_C \oplus \alpha' \oplus id_D$ (which we will denote by $\alpha \oplus id_{C'} \oplus id_D \simeq id_C \oplus \alpha' \oplus id_D$) .

We will denote the equivalence class of an element by $[C, \alpha]$. The sum of two elements of $K^{-1}(C)$ is defined by

$$[C, \alpha] \oplus [C', \alpha'] = [C \oplus C', \alpha \oplus \alpha'].$$

It will turn out that $(K^{-1}(\mathcal{C}), \oplus)$ is a group. To show this, we first need the following lemmas:

Lemma 5.9. *The set $K^{-1}(\mathcal{C})$ together with the map $\oplus$ is a monoid.*

Proof. First, notice that $\oplus$ is well defined and associative. We now show that $K^{-1}(\mathcal{C})$ has a zero element and determine all triples which are zero. We claim that that $[C, \alpha] = 0$ if and only if there exists a $D \in \mathcal{C}$, such that $\alpha \oplus id_D$ is homotopic to $id_C \oplus id_D$ through isomorphisms. If $[C, \alpha] = 0$, then $[*, id] \oplus [C, \alpha] = [*, id]$, where $*$ is the initial/terminal object in $\mathcal{C}$. This implies that there exists a $D \in \mathcal{C}$, such that

$$id_* \oplus \alpha \oplus id_D \simeq id_* \oplus id_C \oplus id_D.$$

Since $\mathrm{End}(*) = \{0\}$, this implies that $\alpha \oplus id_D \simeq id_C \oplus id_D$. If there exists a $D \in \mathcal{C}$, such that $\alpha \oplus id_D$ is homotopic to $id_C \oplus id_D$ through isomorphisms, then for all $[E, \gamma] \in K^{-1}(\mathcal{C})$ we have $[E, \gamma] \oplus [C, \alpha] = [E \oplus C, \gamma \oplus \alpha]$ and

$$\begin{aligned} \gamma \oplus \alpha \oplus id_E \oplus (id_D \oplus id_E) &\simeq \gamma \oplus id_C \oplus id_E \oplus id_D \oplus id_E \\ &\simeq id_E \oplus id_C \oplus id_E \oplus id_D \oplus \gamma \\ &\simeq id_E \oplus id_C \oplus \gamma \oplus id_E \oplus id_D \end{aligned}$$

where in the last two lines, we used that $\gamma \oplus id_E \simeq id_E \oplus \gamma$, where the homotopy is given by

$$H(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & id_E \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

This implies that $[E \oplus C, \gamma \oplus \alpha] = [E, \gamma]$. The proof for the case $[C, \alpha] \oplus [E, \gamma]$ is similar. $\square$

Lemma 5.10. *Let $[C_1, \alpha_1], [C_2, \alpha_2] \in K(C)$ and let $h : C_1 \to C_2$ be an isomorphism in $\mathcal{C}$ such that $h \circ \alpha_1 = \alpha_2 \circ h$. Then*

$$[C_1, \alpha_1] = [C_2, \alpha_2].$$

Proof. We first show that $[C, \alpha_1] \oplus [C_2, \alpha_2^{-1}] = 0$. Notice that

$$[C_1, \alpha_1] \oplus [C, \alpha_2^{-1}] = [C_1 \oplus C_2, \alpha_1 \oplus \alpha_2^{-1}] = [C_1 \oplus C_2, \alpha_1 \oplus (h\alpha_1^{-1}h^{-1})].$$

We have

$$\alpha_1 \oplus h\alpha_1^{-1}h^{-1} = \begin{pmatrix} 0 & -(h \circ \alpha_1^{-1})^{-1} \\ h \circ \alpha_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & h^{-1} \\ -h & 0 \end{pmatrix}. \tag{5.4}$$

It follows from Equation 5.2 and Equation 5.3 that both matrices in Equation 5.4 are homotopic to the identity. Remark 5.9 now implies that

$$[C_1 \oplus C_2, \alpha \oplus \alpha^{-1}] = 0.$$

If we let $h = id$ and $\alpha_1 = \alpha_2$, then this computation shows that

$$[C, \alpha_2] \oplus [C, \alpha_2^{-1}] = [C, \alpha_2^{-1}] \oplus [C, \alpha_2] = 0.$$

For a general h, we now have

$$[C_1, \alpha_1] = [C_1, \alpha_1] \oplus [C_2, \alpha_2^{-1}] \oplus [C_2, \alpha_2] = [C_2, \alpha_2].$$

$\square$

Proposition 5.11. *The set $K^{-1}(\mathcal{C})$, with the operation $[C, \alpha] \oplus [D, \beta] = [C \oplus D, \alpha \oplus \beta]$ is indeed a group.*

Proof. Notice that $\oplus$ is well defined and $K^{-1}(\mathcal{C})$ is a monoid. We now show that it is commutative. Let $[C, \alpha], [D, \beta] \in K^{-1}(\mathcal{C})$. Then, we have

$$[C, \alpha] \oplus [D, \beta] = [C \oplus D, \alpha \oplus \beta] \text{ and } [D, \beta] \oplus [C, \alpha] = [D \oplus C, \beta \oplus \alpha].$$

Notice that $h \circ (\alpha \oplus \beta) = (\beta \oplus \alpha) \circ h$, where

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 5.10 now implies that $[C \oplus D, \alpha \oplus \beta] = [D \oplus C, \beta \oplus \alpha]$.
Lastly, notice that in the proof Lemma 5.10 we showed that $[C, \alpha] \oplus [C, \alpha^{-1}] = 0$. Therefore, $K^{-1}(C)$ is a group. $\square$

The following lemma will sometimes be useful:

Lemma 5.12. *Let $[C, \alpha_1], [C, \alpha_2] \in K(\mathcal{C})$. Then*

$$[C, \alpha_1] \oplus [C, \alpha_2] = [C, \alpha_1\alpha_2].$$

Proof. Notice that $[C,\alpha_1]\oplus[C,\alpha_2]=[C\oplus C,\alpha_1\oplus\alpha_2]$ and

$$[C,\alpha_1\alpha_2]=0\oplus[C,\alpha_1\alpha_2]=[C,id]\oplus[C,\alpha_1\alpha_2]=[C,id\oplus\alpha_1\alpha_2].$$

Because

$$(id\oplus\alpha_1\alpha_2)=(\alpha_1^{-1}\oplus\alpha_1)\circ(\alpha_1\oplus\alpha_2),$$

Equation 5.4, with $h=id$ and $\alpha=\alpha_1^{-1}$ implies that $\alpha_1^{-1}\oplus\alpha_1$ is homotopic to the identity. Therefore,

$$[C,\alpha_1\alpha_2]=[C,\alpha_1\oplus\alpha_2].$$

□

We will now calculate the group $K^{-1}(\mathcal{C})$ for some Banach categories $\mathcal{C}$.

Example 5.13. We will calculate $K^{-1}(\mathcal{E}_e^{\mathbb{R}}(\mathrm{pt}))$. Notice that if $[E,A]\in K^{-1}(\mathcal{E}_e^{\mathbb{R}}(\mathrm{pt}))$, then $E\cong\mathbb{R}^n$ for some $n\in\mathbb{N}$ and $A\in\mathrm{GL}_n(\mathbb{R})$. Notice that $[E,A_1]=[E,A_2]$ if and only if there exists $n\in\mathbb{N}$ and $[E_3,id]\in K^{-1}(\mathcal{E}_e^{\mathbb{R}}(\mathrm{pt}))$ such that

$$A_1\oplus id_{E_2}\oplus id_{E_3}\simeq id_{E_1}\oplus A_2\oplus id_{E_3}. \tag{5.5}$$

These two morphism are elements of $GL_k(\mathbb{R})$ for some $k\in\mathbb{N}$. Since elements of $GL_k(\mathbb{R})$ are homotopic if and only if their determinants have the same sign, it follows that Equation 5.5 holds if and only if

$$\mathrm{sgn}(det(A_1\oplus id_{E_2}\oplus id_{E_3}))=\mathrm{sgn}(\det(id_{E_1}\oplus A_2\oplus id_{E_3})).$$

Since $\det(A_1\oplus id_{E_2}\oplus id_{E_3})=\det(A_1)$ and $\det(id_{E_1}\oplus A_2\oplus id_{E_3})=\det(A_2)$, we have $[E,A_1]=[E,A_2]$ if and only if $\mathrm{sgn}(\det(A_1))=\mathrm{sgn}(\det(A_2))$. Therefore,

$$K^{-1}(\mathcal{E}_e^{\mathbb{R}}(\mathrm{pt}))=\mathbb{Z}/2\mathbb{Z}.$$

Example 5.14. We now determine $K^{-1}(\mathcal{E}_G^{\mathbb{C}}(\mathrm{pt}))$. Let $[M,\alpha]\in K^{-1}(\mathcal{E}_G^{\mathbb{C}}(\mathrm{pt}))$. Since M is a finite dimensional representation and $\alpha\in\mathrm{Aut}_G(M)$, Theorem A.38 implies that $\alpha\simeq id$. Therefore,

$$[M,\alpha]=[M,id]=0$$

and

$$K^{-1}(\mathcal{E}_G^{\mathbb{C}}(\mathrm{pt}))=\{0\}.$$

It is not obvious how the group $K^{-1}(\mathcal{C})$ is related to the group $K(\mathcal{C})$. The following theorem shows how they are connected:

Theorem 5.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be Banach categories and let $\varphi:\mathcal{C}\to\mathcal{D}$ be a quasi-surjective Banach functor. Then there exists a map $\partial:K^{-1}(\mathcal{D})\to K(\varphi)$ such that the following sequence is exact:*

$$K^{-1}(\mathcal{C})\xrightarrow{i_{-1}}K^{-1}(\mathcal{D})\xrightarrow{\partial}K(\varphi)\xrightarrow{j}K(\mathcal{C})\xrightarrow{i}K(\mathcal{D}),$$

Where $i_{-1}([C,\alpha])=[\varphi(C),\varphi(\alpha)]$, $j([C_+,C_-,\alpha])=(C_+,C_-)$ and $i(C_+-C_-)=(\varphi(C_+),\varphi(C_-))$.

Proof. We start by defining ∂. Let $[D, \alpha] \in K^{-1}(D)$. Since φ is quasi surjective, there exists a $D' \in \mathrm{ob}(D)$ and $C \in \mathrm{ob}(C)$ such that there exists an isomorphism $f : D \oplus D' \to \varphi(C)$. We define

$$\partial(D, \alpha) := [C, C, f(\alpha \oplus id)f^{-1}].$$

Notice that this map is well defined, because if we have an isomorphism $g : D \oplus D'' \to \varphi(C')$, then we have

$$[C, C, f(\alpha \oplus id)f^{-1}] + [C', C', g(\alpha^{-1} \oplus id)g^{-1}] = [C \oplus C', C \oplus C', f(\alpha \oplus id)f^{-1} \oplus g(\alpha^{-1} \oplus id)g^{-1}].$$

Since the following diagram commutes:

$$\begin{tikzcd}
(D \oplus D') \oplus (D \oplus D')' \arrow[d, "\alpha \oplus id \oplus \alpha^{-1} \oplus id"] & \varphi(C) \oplus \varphi(D) \arrow[l, "f^{-1} \oplus g^{-1}"'] \arrow[d, "f(\alpha \oplus id)f^{-1} \oplus g(\alpha^{-1} \oplus id)g^{-1}"] \\
(D \oplus D') \oplus (D \oplus D') \arrow[r, "f \oplus g"] & \varphi(C) \oplus \varphi(C')
\end{tikzcd},$$

and Equation 5.2 and Equation 5.3 imply that $\alpha \oplus id \oplus \alpha^{-1} \oplus id$ is homotopic to the identity, it follows that $f(\alpha \oplus id)f^{-1} \oplus g(\alpha^{-1} \oplus id)g^{-1}$ is homotopic to the identity. Therefore, we have

$$[C, C, f(\alpha \oplus id)f^{-1}] + [g(\alpha^{-1} \oplus id)g^{-1}] = 0,$$

which implies that ∂ is well defined. We leave it to the reader to check that i_{-1}, j and i are well defined.
We now check exactness of the sequence. We first show that $\mathrm{Im}(j) \subset \mathrm{Ker}(i)$. We have

$$i \circ j([C_+, C_-, \alpha]) = (\varphi(C_+), \varphi(C_-)).$$

Since $\alpha : \varphi(C_+) \to \varphi(C_-)$ is an isomorphism, it follows that $(\varphi(C_+), \varphi(C_-)) = 0$.

We now show that $\mathrm{Ker}(i) \subset \mathrm{Im}(j)$. Assume that $i(C_+, C_-) = 0$. Then we have $(\varphi(C_+), \varphi(C_-)) = 0$, which implies that there exists an object D of $\mathcal{D}$ such that $\varphi(C_+) \oplus D \cong \varphi(C_-) \oplus D$. Since φ is quasi surjective, there is an object C_0 of C and an object $D^\perp$ of $\mathcal{D}$ such that $D \oplus D^\perp \cong \varphi(C_0)$. Therefore,

$$\varphi(C_+) \oplus \varphi(C_0) \cong \varphi(C_+) \oplus D \oplus D^\perp \cong \varphi(C_-) \oplus D \oplus D^\perp \cong \varphi(C_-) \oplus \varphi(C_0).$$

Let $\beta : \varphi(C_+) \oplus \varphi(C_0) \to \varphi(C_-) \oplus \varphi(C_0)$ be the isomorphism from the equation above. Then, by construction, we have

$$j(C_+ \oplus C_0, C_- \oplus C_0, \beta) = (C_+ \oplus C_0, C_- \oplus C_0) = (C_+, C_-).$$

We now prove that $\mathrm{Im}\partial \subset \mathrm{Ker}(j)$. Notice that

$$j \circ \partial([D, \alpha]) = j([C, C, f(\alpha \oplus id)f^{-1}]) = (C, C) = 0.$$

We now show that $\mathrm{Ker}(j) \subset \mathrm{Im}\partial$. Let $[C_+, C_-, \alpha] \in \mathrm{Ker}(j)$, then $j([C_+, C_-, \alpha]) = (C_+, C_-) = 0$, which implies that there exists a $C_0 \in \mathrm{ob}(C)$ and an isomorphism $f : C_+ \oplus C_0 \to C_- \oplus C_0$.

Because the following diagram commutes

$$\begin{tikzcd}
\varphi(C_+) \oplus \varphi(C_0) \arrow[r, "\alpha\oplus id"] \arrow[d, "\varphi(g)"] & \varphi(C_-) \oplus \varphi(C_0) \arrow[d, "\varphi(id)"] \\
\varphi(C_-) \oplus \varphi(C_0) \arrow[r, "\alpha\oplus id\circ\varphi(g^{-1})"] & \varphi(C_-) \oplus \varphi(C_0)
\end{tikzcd},$$

it follows that

$$[C_+, C_-, \alpha] = [C_+ \oplus C_0, C_- \oplus C_0, \alpha \oplus id] = [C_- \oplus C_0, C_- \oplus C_0, \alpha \oplus id \circ \varphi(g^{-1})].$$

Notice that

$$\partial([\varphi(C_-) \oplus \varphi(C_+), \alpha \oplus id \circ \varphi(g^{-1})]) = [C_- \oplus C_0, C_- \oplus C_0, \alpha \oplus id \circ \varphi(g^{-1})],$$

which proves the claim.

We prove that $\mathrm{Im}(i_{-1}) \subset \mathrm{Ker}(\partial)$. Notice that

$$\partial \circ i_{-1}([C, \alpha]) = \partial([\varphi(C), \varphi(\alpha)]) = [C, C, \varphi(\alpha)].$$

The commutative diagram:

$$\begin{tikzcd}
\varphi(C) \arrow[r, "\varphi(\alpha)"] \arrow[d, "\varphi(\alpha)"] & \varphi(C) \arrow[d, "id"] \\
\varphi(C) \arrow[r, "id"] & \varphi(C)
\end{tikzcd}$$

implies that

$$[C, C, \varphi(\alpha)] = [C, C, id] = 0.$$

Lastly, we show that $\mathrm{Ker}(\partial) \subset \mathrm{Im}(i_{-1})$. If $\partial([D, \alpha]) = 0$, then there exists an object D' of $\mathcal{D}$, an object C of $\mathcal{C}$ and an isomorphism $f : D \oplus D' \to \varphi(C)$, such that

$$\partial([D, \alpha]) = [C, C, f \circ \alpha \oplus id \circ f^{-1}] = 0.$$

By definition, this implies that there exists elementary triple $[C_E, C_E, \beta]$ and $[C_0, C_0, \gamma]$, such that

$$(C, C, f \circ \alpha \oplus id \circ f^{-1}) \oplus (C_E, C_E, \beta) \cong (C_0, C_0, \gamma).$$

Therefore, there are isomorphisms $h : C \oplus C_E \to C_0$ and $g : C \oplus C_E \to C_0$ such that the following diagram commutes:

$$\begin{tikzcd}
D \oplus D' \oplus \varphi(C_E) \arrow[r, "f\oplus id"] \arrow[d, "\alpha\oplus id\oplus\beta"] & \varphi(C) \oplus \varphi(C_E) \arrow[r, "\varphi(h)"] \arrow[d, "f\circ\alpha\oplus id\circ f^{-1}\oplus\beta"] & \varphi(C_0) \arrow[d, "\gamma"] \\
D \oplus D' \oplus \varphi(C_E) \arrow[r, "f\oplus id"] & \varphi(C) \oplus \varphi(C_E) \arrow[r, "\varphi(g)"] & \varphi(C_0)
\end{tikzcd},$$

This implies that the diagram

$$\begin{tikzcd}
D \oplus D' \oplus \varphi(C_E) \arrow[r, "f\oplus id"] \arrow[d, "\alpha\oplus id\oplus\beta"] & \varphi(C)\oplus\varphi(C_E) \arrow[r, "\varphi(h)"] \arrow[d, "f\circ\alpha\oplus id\circ f^{-1}\oplus\beta"] & \varphi(C_0) \arrow[d, "\varphi(h)\varphi(g^{-1})\gamma"] \\
D \oplus D' \oplus \varphi(C_E) \arrow[r, "f\oplus id"] & \varphi(C)\oplus\varphi(C_E) \arrow[r, "\varphi(h)"] & \varphi(C_0)
\end{tikzcd},$$

commutes. Since γ is homotopic to the identity, Lemma 5.10 implies that

$$\begin{aligned}
[D, \alpha] &= [D \oplus D' \oplus \varphi(C_E), \alpha \oplus id \oplus \beta] \\
&= [\varphi(C_0), \varphi(C_0), \varphi(h)\varphi(g^{-1})\gamma] \\
&= [\varphi(C_0), \varphi(C_0), \varphi(hg^{-1})] \\
&= i_{-1}([C_0, C_0, hg^{-1}]).
\end{aligned}$$

□

Example 5.16. Let X be a G-space and $x_0 \in X$ such that $gx_0 = x_0$ for all $g \in G$. In Example 4.22 and Example 4.24, we showed that the functor $i^* : \mathcal{E}_G^{\mathbb{K}}(X) \to \mathcal{E}_G^{\mathbb{K}}(\{x_0\})$ is a quasi-surjective Banach functor. Theorem 5.15 now implies that the following sequence is exact:

$$K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X)) \xrightarrow{i_{-1}} K^{-1}(\mathcal{E}_G^{\mathbb{K}}(\{x_0\})) \xrightarrow{\partial} K(i^*) \xrightarrow{j} K(X) \xrightarrow{i} K(\{x_0\}) \ .$$

Notice that if $[M, \alpha] \in K^{-1}(E_G^{\mathbb{K}}(\{x_0\})$, then $[X \times M, \tilde{\alpha}] \in K^{-1}(X)$, where $\tilde{\alpha}(x, m) = (x, \alpha(m))$ and $i_{-1}([X \times M, \tilde{\alpha}]) = [M, \alpha]$. Thus, the map i_{-1} is surjective, which implies that j is is injective.

We will now give geometric interpretations of the groups $K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X))$ and $K(i^*)$. These interpretations will also show how these groups are related to the groups we introduced in section 3. We will start with the K^{-1} groups.

Lemma 5.17. *Let X be a G-space and let $[E, \alpha] \in K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X))$. Then, there exists a G-module M and a bijective G-vector bundle morphism $\beta : XM_X \to M_X$ such that*

$$[E, \alpha] = [M_X, \beta].$$

Proof. Let $[E, \alpha] \in K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X))$. In Theorem 2.31, we showed that there exists a G-vector bundle $E^\perp$ and a G-module M, such that $E \oplus E^\perp \cong M_X$. Let $f : E \oplus E^\perp \cong M_X$ denote the isomorphism. Notice that $[E^\perp, id] = 0$. Lemma 5.10 now implies that

$$[E, \alpha] = [E \oplus E^\perp, \alpha \oplus id] = [M_X, f \circ \alpha \oplus id \circ f^{-1}].$$

□

Theorem 5.18. *There exists an isomorphism*

$$\sigma : K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X)) \to \tilde{K}_G^{\mathbb{K}}(\Sigma X^+).$$

Proof. We first give the definition of σ. Notice that $\Sigma X^+ \cong CX^+ \coprod_{X^+} CX^+$. We will denote the first cone by C_1X^+ and the second cone by C_2X^+. We have $C_1X^+ \cap C_2X^+ \cong X^+$. Let $[E,\alpha] \in K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X))$. Lemma 5.17 implies that

$$[E,\alpha] = [M_X, f \circ \alpha \oplus id \circ f^{-1}].$$

We define

$$\sigma([E,\alpha]) := (M_{C_1X^+} \cup_{f\circ\alpha\oplus id_{E^\perp}\circ f^{-1} \coprod id_{E_{\mathrm{pt}}}} M_{C_2X^+}).$$

We will often drop the $\coprod id$ from the notation, to shorten the notation. We first show that σ is well defined. Assume that have that $[E',\beta] = [E,\alpha]$ Then, by definition, there exists a G-vector bundle F, such that

$$\alpha \oplus id_{E'} \oplus id_F \simeq id_E \oplus \beta \oplus id_F.$$

Theorem 2.31 implies that we may assume that $F = Q_X$ for some G-module Q. By definition, we have $\sigma([E',\beta]) = (N_{C_1X} \cup_{g\circ\beta\oplus id_{(E')^\perp}\circ g^{-1}} N_{C_2X})$. Theorem 2.29 and Lemma 2.30 now imply that

$$\begin{aligned}
\sigma([E,\alpha]) &= [M_{C_1X^+} \cup_{f\circ\alpha\oplus id_{E^\perp}\circ f^{-1}} M_{C_2X^+})]_{\sim'}\\
&= [M_{C_1X^+} \cup_{f\circ\alpha\oplus id_{E^\perp}\circ f^{-1}} M_{C_2X^+})]_{\sim'}\\
&+ [(N\oplus Q)_{C_1X^+} \cup_{(g\oplus id)\circ((id\oplus id)\oplus id)\circ(g^{-1}\oplus id)} (N\oplus Q)_{C_2X^+}]_{\sim'}\\
&= (M\oplus N\oplus Q)_{C_1X^+} \cup_{(f\oplus g\oplus id)\circ(\alpha\oplus id)\oplus(id\oplus id)\oplus id)\circ(f^{-1}\oplus g^{-1}\oplus id)} (M\oplus N\oplus Q)_{C_1X^+}\\
&= [(M\oplus N\oplus Q)_{C_1X^+} \cup_{(f\oplus g\oplus id)\circ(id\oplus id)\oplus(\beta\oplus id)\oplus id)\circ(f^{-1}\oplus g^{-1}\oplus id)} (M\oplus N\oplus Q)_{C_2X^+}]_{\sim'}\\
&= [M_{C_1X^+} \cup_{id} M_{C_2X^+}]_{\sim'} \oplus [N_{C_1X^+} \cup_{g\circ\beta\oplus id\circ g^{-1}} N_{C_2X^+}] \oplus [Q_{C_1X^+} \cup_{id} Q_{C_2X^+}]_{\sim'}\\
&= [N_{C_1X^+} \cup_{g\circ\beta\oplus id\circ g^{-1}} N_{C_2X^+}]_{\sim'}\\
&= \sigma([E',\beta]).
\end{aligned}$$

We now show that σ is a group homomorphism. Let $[E,\alpha],[F,\beta] \in K^{-1}(\mathcal{E}_G^{\mathbb{K}}(X))$. Lemma 5.17 implies that $[E,\alpha] = [M_X, f\circ\alpha\oplus id_{E^\perp}\circ f^{-1}]$ and $[F,\beta] = [N_X, g\circ\beta\oplus id_{E^\perp}\circ g^{-1}]$. It follows that

$$\begin{aligned}
[E,\alpha]\oplus[F,\beta] &= [M_X, f\circ\alpha\oplus id\circ f^{-1}] \oplus [N_X, g\circ\beta\oplus id\circ g^{-1}]\\
&= [(M\oplus N)_X, f\oplus g\circ(\alpha\oplus id)\oplus(\beta\oplus id)\circ g^{-1}\oplus f^{-1}].
\end{aligned}$$

Lemma 2.30 implies that,

$$\begin{aligned}
\sigma([E,\alpha]\oplus[F,\beta] &= [(M\oplus N)_{C_1X} \cup_{f\oplus g\circ(\alpha\oplus id)\oplus(\beta\oplus id)\circ g^{-1}\oplus f^{-1}} (M\oplus N)_{C_2X}]_{\sim'}\\
&= [(M_{C_1X} \cup_{f\circ\alpha\oplus id\circ f^{-1}} M_{C_2X}) \oplus (N)_{C_1X} \cup_{g\circ\beta\oplus id\circ g^{-1}} N_{C_2X}]_{\sim'}\\
&= \sigma([E,\alpha]) \oplus \sigma([F,\beta]).
\end{aligned}$$

It remains to show that σ is bijective. We first show that σ is surjective. Let $[E]_{\sim'} \in \tilde{K}_G(X)$. In Lemma 2.30, we showed that

$$E \cong E|_{C_1X^+} \cup_{id} E|_{C_2X^+}. \tag{5.6}$$

Since CX^+ is G-contractible, there exists a G-module M and isomorphism $\alpha : M_{C_1X^+} \to E|_{C_1X^+}$ and $\beta : M_{C_2X^+} \to E|_{C_2X^+}$. If we combine this with Equation 5.6, we obtain:

$$E \cong M_{C_1X^+} \cup_{\beta^{-1}\alpha} M_{C_2X^+}.$$

Notice however that we need that $\beta^{-1}\alpha|_{M_{\mathrm{pt}}} = id$. Theorem A.38 implies that (after a change of basis) the map $\beta^{-1}\alpha|_{M_{\mathrm{pt}}}$ is homotopic to a direct sum $\oplus_{i=1}^n \oplus_{\pm_i} : \oplus_{i=1}^n(\oplus_{j=1}^{k_i} M_i) \to \oplus_{i=1}^n \oplus_{j=1}^{k_i} M_i M_i$, where $M \cong \oplus_{i=1}^n M_i$ and M_i are irreducible G-modules. Let $I := \{1 \leq i \leq n \mid \pm_i = -\}$ and $N := \oplus_{I \in i} \oplus_{j=1}^{k_i} M_i$. Notice that

$$\sigma X^+ \times N \cong N_{CX^+} \cup_{-id \coprod -id} N_{CX^+}.$$

By construction, the map $\beta^{-1}\alpha|_{M_{\mathrm{pt}}} \oplus -id : M \oplus N \to M \oplus N$ is homotopic to the identity. Theorem 2.29 now implies that

$$\begin{aligned}
[E]_{\sim'} &= [M_{C_1X^+} \cup_{\beta^{-1}\alpha} M_{C_2X^+} \oplus (N_{CX^+} \cup_{-id \coprod -id} N_{CX^+})]_{\sim'} \\
&= [M \oplus N_{C_1X^+} \cup_{\beta^{-1}\alpha \oplus -id} M \oplus N_{C_2X^+}]_{\sim'} \\
&= [M \oplus N_{C_1X^+} \cup_{\beta^{-1}\alpha \oplus -id|_{M \oplus N_X} \coprod id_{M \oplus N_{\mathrm{pt}}}} M \oplus N_{C_2X^+}]
\end{aligned}$$

Therefore,

$$[E]_{\sim'} = \sigma([M \oplus N_X, \beta^{-1}\alpha \oplus (-id)]).$$

Now assume that $\sigma[E, \alpha] = 0$. As before, we have

$$\sigma[E, \alpha] = M_{C_1X^+} \cup_{f \circ \alpha \oplus id \circ f^{-1}} M_{C_2X^+}.$$

Since $\sigma([E, \alpha]) = 0$, it follows that there exists G-modules N and Q, such that

$$(C_1X^+ \times M \cup_{f \circ \alpha \oplus id \circ f^{-1}} C_2X^+ \times M) \oplus N_{\Sigma X^+} \cong Q_{\Sigma X^+}.$$

Notice that

$$(M_{C_1X^+} \cup_{f \circ \alpha \oplus id \circ f^{-1}} M_{C_2X^+}) \oplus N_{\Sigma X^+} \cong (M \oplus N)_{C_1X^+} \cup_{(f \circ \alpha \oplus id \circ f^{-1}) \oplus id} (M \oplus N)_{C_2X^+}.$$

and

$$Q_{\Sigma X^+} \cong Q_{C_1X^+} \cup_{id} Q_{C_2X^+}.$$

Let

$$h : (M \oplus N)_{C_1X^+} \cup_{(f \circ \alpha \oplus id \circ f^{-1}) \oplus id} (M \oplus N)_{C_2X^+} \to Q_{C_1X^+} \cup_{id} Q_{C_2X^+}$$

be the composition of all these isomorphism and let $i_j : (M \oplus N)_{C_jX^+} \to (M \oplus N)_{C_1X^+} \cup_{(f \circ \alpha \oplus id \circ f^{-1}) \oplus id} (M \oplus N)_{C_2X^+}$ denote the inclusion. Notice that $h \circ i_j : (M \oplus N)_{CX^+} \to Q_{X^+}$ is an isomorphism. Chasing trough al these identifications, we see that the following diagram commutes

$$\begin{tikzcd}
(M \oplus N)_{X^+} \arrow[r, "h \circ i_1"] \arrow[d, "(f \circ \alpha \oplus id \circ f^{-1}) \oplus id \coprod id"'] & Q_{X^+} \arrow[d, "id \coprod id"] \\
(M \oplus N)_{X^+} \arrow[r, "h \circ i_2"] & Q_{X^+}
\end{tikzcd}.$$

It follows that

$$(f \circ \alpha \oplus id \circ f^{-1}) \oplus id = (h \circ i_2)^{-1} \circ (h \circ i_1)|_{X^+}.$$

Let $H : (M \oplus N)_{X^+} \times I \to (M \oplus N)_{X^+}$ be defined by

$$H(x, v, t) = (x, \mathrm{pr}_{M \oplus N} \circ (i_2)^{-1} \circ (i_1)([x, t], v)).$$

The map H is a G-homotopy from $(f \circ \alpha \oplus id \circ f^{-1}) \oplus id$ to $H_1 = H(\cdot, 1)$. Since

$$H(x, v, 1) = (x, \mathrm{pr}_{M\oplus N} \circ (i_2)^{-1} \circ (i_1)([x, 1], v)) = (x, \mathrm{pr}_{M\oplus N} \circ (i_2)^{-1} \circ i_1([\mathrm{pt}, 0], v)) = (x, v).$$

the function $(f \circ \alpha \oplus id \circ f^{-1}) \oplus id_{N_X}$ is homotopic to the identity. This implies that

$$\begin{aligned}
[E, \alpha] &= [E \oplus E^{\perp}, \alpha \oplus id] \\
&= [M_{X^+}, f \circ \alpha \oplus id \circ f^{-1}] \\
&= [M_{X^+} \oplus N_{X^+}, f \circ \alpha \oplus id \circ f^{-1}] \\
&= [M_{X^+}, H_1] = [M_{X^+}, id] = 0.
\end{aligned}$$

□

Example 5.19. In Example 5.13, we showed that $K^{-1}(\mathcal{E}^{\mathbb{R}}_{\{e\}}(\{\mathrm{pt}\})) \cong \mathbb{Z}$. The Theorem above now implies that

$$\mathbb{Z}/2\mathbb{Z} \cong K^{-1}(\mathcal{E}^{\mathbb{R}}_{\{e\}}(\{\mathrm{pt}\}) \cong \tilde{K}_{\{e\}}(\Sigma\{\mathrm{pt}\}^+) \cong \tilde{K}_{\{e\}}(S^1).$$

We will now give a geometric interpretation of the group $K(i^*)$ for the inclusion $i^* : \mathcal{E}_G(X) \to \mathcal{E}_G(Y)$. If $Y = \emptyset$, there is an easy interpretation:

Proposition 5.20. *Let X be a compact G-space and $i : \emptyset \to X$ be the inclusion. We have*

$$K(i^*) \cong K_G(X).$$

Proof. The elements of $K(i^*)$ are triples $[E_+, E_-, \alpha]$. Notice however that $\alpha : \emptyset \to \emptyset$ is the empty function. By looking closely at the definition of $K(i^*)$, we thus have $[E_+, E_-, \alpha] = [F_+, F_-, \beta]$ if and only if there exists G-vector bundles Q_1 and Q_2 such that $E_+ \oplus Q_1 \cong F_+ \oplus Q_2$ and $E_- \oplus Q_1 \cong F_- \oplus Q_2$. We claim that $j : (E^{\mathbb{K}}_G(X), \oplus) \to K(i^*)$ defined by $j(E) = ([E, 0_X, \alpha])$ is the Grothendieck group of $(E^{\mathbb{K}}_G(X), \oplus)$. We check the fundamental property. Let A be an abelian group and $h : E_G(X) \to A$ a monoid homomorphism. If $\tilde{h} : K(i^*) \to A$ is a group homomorphism such that $\tilde{h} \circ j = h$, then Since $[E, 0, \alpha] \oplus [0, E, \alpha] = [E, E, \alpha] = 0$, we must have $\tilde{h}([0, E, \alpha]) = -\tilde{h}([E, 0, \alpha]) = -h(E)$ and $\tilde{h}([E, F, \alpha]) = h(E) - h(F)$. We leave it to the reader to check that $\tilde{h}$ is well defined. Thus, $K(i^*)$ is the Grothendieck group of $E_G(X)$ and hence isomorphic to $K(X)$. One can check that, in fact, the isomorphism is given by $\Phi([E_+, F_-, \Phi]) = (E_+, F_-)$. □

For the case where $Y \neq \emptyset$, we will need the following lemma:

Lemma 5.21. *Let X be a compact G-space and $Y \subset X$ a closed G-invariant subset. Let $i : Y \to X$ denote the inclusion. If there exists a map $P : X \to Y$, such that $P \circ i = id$, then the sequence*

$$0 \longrightarrow \tilde{K}_G(X \textstyle\coprod_Y CY) \xrightarrow{j^*} \tilde{K}_G(X) \xrightarrow{i^*} \tilde{K}_G(Y) \longrightarrow 0\ .$$

from Lemma 3.22 is a split short exact sequence.

Proof. In Lemma 3.22 we already showed that $\mathrm{Im}(j^*) = \mathrm{Ker}(i^*)$. We first show that i^* is surjective. Let $[E_+]_{\sim'} \in \tilde{K}_G(Y)$. We have $[P^*E]_{\sim'} \in \tilde{K}_G(X)$ and

$$i^*([P^*E]_{\sim'}) = [i^*P^*E]_{\sim'} = [(P \circ i)^*E]_{\sim'} = [(id)^*E]_{\sim'} = [E]_{\sim'}.$$

Second, we show that j^* is injective. Let $[E]_{\sim'} \in \tilde{K}_G(X \coprod_Y CY)$ and assume that $[j^*E]_{\sim'} = 0$. By definition, there exists G-modules M and N such that $j^*E \oplus N_X \cong M_X$. Since CY is G-contractible and $E|_Y \oplus N_Y \cong M_Y$, we have $E \oplus N_{X \coprod_Y CY} \cong M_{CY}$. Notice that

$$E \oplus N_{X \coprod_Y CY} \cong (E|_X \oplus N_X) \cup_{id} (E_{CY} \oplus N_{CY}) \cong M_X \cup_\alpha M_{CY},$$

for some isomorphism $\alpha : M_Y \to M_Y$. We define the map $f : M_X \cup_\alpha M_{CY} \to M_{X \coprod_Y CY}$ by

$$f(x,m) = \begin{cases} (x,m) & \text{if} \quad (x,m) \in CY \\ (x, \mathrm{pr}_M(\alpha(P(x),m))) & \quad (x,m) \in X \end{cases}$$

Notice that f is well defined and a G-vector bundle isomorphism, which implies that

$$[E]_{\sim'} = [E \oplus N_{X \coprod_Y CY}]_{\sim'} = [M_{X \coprod_Y CY}]_{\sim'} = 0.$$

□

Let X and Y be compact G-spaces, and $X_0 \subset X$ and $Y_0 \subset Y$ be closed invariant subsets and $f : (Y, Y_0) \to (X, X_0)$ a G-map. The map $f : Y \to X$ induces a group homomorphism $f^* : K(i^*_{X_0}) \to K(i^*_{Y_0})$ defined by

$$f^*([E_+, E_-, \alpha]) = [f^*E_+, f^*E_-, \tilde{\alpha}],$$

where $\tilde{\alpha}(v) = \alpha(v) \in (E_-)_{p_+(v)}$. With this notation, we are ready to state the following proposition:

Proposition 5.22. *Let X be a compact G-space and $Y \subset X$ a closed G-invariant subset. Let $i : Y \to X$ be the inclusion. There exists a natural group morphism*

$$\Phi : K(i^*) \to K(X, Y),$$

such that Φ is an isomorphism if $Y = \emptyset$

Proof. Let $X_0, X_1 = X$. Let $Z := X_0 \coprod_Y X_1$. Let $P_j : Z \to X_j$ be the projection defined by $P(i,x) = x$ and let $i_j : X_j \to Z$ be the inclusion. Notice that $i_j P_j = id$. Lemma 5.21 implies that the following sequence is a split short exact sequence:

$$0 \longrightarrow K(Z, X_i) \xrightarrow{\pi_j} K(Z) \xrightarrow{i_j^*} K(X_j) \longrightarrow 0 \ . \tag{5.7}$$

Since $X_i^+/Y^+ \cong Z^+/X_{1-i}^+$, we have $K(X_i, Y) \cong K(Z, X_{i-1})$. Moreover, this isomorphism is induced by the inclusion $f_i : (X_i, Y) \to (Z, X_{i-1})$, (which induces an inclusion $f_i : X_i^+ \coprod_{Y^+} CY^+ \to Z^+ \coprod_{X_{i-1}^+} CX_{i-1}^+$). If E is a G-vector bundle over a compact G-space X, we will also use the notation E to denote the G-vector bundle $E \coprod 0_{\mathrm{pt}}$ over X^+. With this notation in place, we are ready to define Φ. Let $[E_+, E_-, \alpha] \in K(i^*)$. Then $F := (E_+ \cup_\alpha E_-)$ defines a G-vector bundle over Z^+.Theorem 2.31 implies that there exists a bundle $E_-^\perp \in \mathcal{E}_G^{\mathbb{K}}(X^+)$ and a G-module M such that $E_- \oplus E_-^\perp \cong M_{X^+}$. Notice that

$$[i_1^*(F \oplus P_1^*(E_-^\perp))]_{\sim'} = [F|_{X_1^+} \oplus (P_1 \circ i_1)^*(E_-^\perp)]_{\sim'} = [E_- \oplus (E_-^\perp)]_{\sim'} = 0.$$

Equation 5.7 implies that there exists a unique $F' \in K(Z, X_1)$ such that

$$\pi_1^*(F') = [F \oplus P_1^*(E_-^\perp)]_{\sim'}.$$

We now define $\Phi([E_+, E_-, \alpha])$ as the unique $x \in K(X, Y)$ such that

$$\pi_1^*(f_0^*)^{-1}x = [F \oplus P_1^* E_-^\perp]_{\sim'}.$$

We first check that Φ is well defined. First, notice that $\Phi((E_+ \oplus F_+, E_- \oplus F_-, \alpha \oplus \beta)) = \Phi(E_+, E_-, \alpha) \oplus \Phi(F_+, F_-, \beta)$. This implies that if Φ is constant on the equivalence classes, then Φ is a group homomorphism. We first show that all the elementary triples are in the kernel of Φ. Let (E, E, α) be an elementary triple. Since α is homotopic to the identity, we have $F := E \cup_\alpha E \cong E \cup_{id} E \cong P_1^*(E)$. Therefore,

$$[F \oplus P_1^*(E^\perp)]_{\sim'} = [P_1^*(E \oplus E^\perp)]_{\sim'} = P_1^*[E \oplus E^\perp] = 0$$

and $\Phi(E, E, \alpha) = 0$. We now show that if two triples are isomorphic, they are mapped to the same element. Let (E_+, E_-, α) and (F_+, F_-, β) be isomorphic triples. Then there exists isomorphism $g : E_+ \to F_+$ and $h : E_- \to f_-$ such that $h|_{E_-|_Y} \circ \alpha = \beta \circ g|_{E_+|_Y}$. Notice that the map $g \cup h : E_+ \cup_\alpha E_1 \to F_+ \cup_\beta F_-$ defined by

$$g \cup h(v) := \begin{cases} g(v) & \text{if} \quad v \in E_+ \\ h(v) & \quad\;\; v \in E_- \end{cases}.$$

is a well defined G-vector bundle isomorphism and the map $\tilde{h} : P_1^* E_- \to P_1^* F_-$, defined by

$$\tilde{h}(x, v) = (x, h(v)),$$

is also a G-vector bundle isomorphism. Therefore,

$$[E_+ \cup_\alpha E_1 \oplus P_1^* E_-^\perp = [F_+ \cup_\beta F_- \oplus P_1^* F_1^\perp]_{\sim'}$$

and $\Phi(E_+, E_-, \alpha) = (F_+, F_-, \beta)$.
Thus, the map Φ is a well defined group homomorphism.
We now prove that Φ is an isomorphism if $Y = \emptyset$. Let $[E_+, E_-, \alpha] \in K(i^*)$. In this case, $X_0 \coprod_\alpha X_1 = X_0 \coprod X_1$ and $F := E_+ \cup_\alpha E_- = E_+ \coprod E_-$. Notice that

$$E_+ \coprod E_- \oplus P_1^*(E_-^\perp) = (E_+ \oplus (E_-^\perp)|_{X_0} \coprod (E_- \oplus E_-^\perp)|_{X_1^+} = \pi_1^*((E_+ \oplus E_-^\perp)|_{X_0} \coprod M_{C(X_1^+)}).$$

Since

$$f_0^*(E_+ \oplus E_-^\perp \coprod M_{CX_1}) = (E_+ \oplus E_-^\perp)|_X \coprod M_{\text{pt}}.$$

, Using the isomorphism from Theorem 3.20, we see that this element corresponds to the element

$$E_+ \oplus E_-^\perp - M_X = E_+ - E_-,$$

of $K(X) = K(X, \emptyset)$. It follows that $\Phi([E_+, E_-, \alpha]) = E_+ - E_-$. Proposition 5.20 implies that Φ is an isomorphism.

Lastly, we show that Φ is natural. Let $h : (X, Y) \to (X', Y')$ be a continuous G-map. As we noted in remark 3.21, the map h induces a map $h : X \coprod_Y CY \to X' \coprod_{Y'} CY'$. The map h also induces a G-map $h : X \cup_Y X \to X' \cup_{Y'} X'$ which we will all denote by h to keep the notation simple. This map than induces a map $h : Z^+ \coprod_{X^+} X^+ \to (Z')^+ \coprod_{(X')^+} C(X')^+$.

With this notation, the following diagram commutes:

$$\begin{tikzcd}
Z^+ \arrow[r, "\pi_1"] \arrow[d, "h"] & Z^+ \cup_{X^+} CX^+ \arrow[d, "h"] & X^+ \cup_{Y^+} CY^+ \arrow[l, "f_0"'] \arrow[d, "h"] \\
(Z')^+ \arrow[r, "\pi_1"] & (Z')^+ \cup_{(X')^+} C(X')^+ & (X')^+ \cup_{(Y')^+} C(Y')^+ \arrow[l, "f_0"']
\end{tikzcd}.$$

Let $[E_+, F_-, \Phi] \in K(i^*_{X'})$, with $i : X' \to Y'$ and assume that $\Phi([E_+, E_-, Y]) = E$. We show that $\Phi(f^*([E_+, E_-, \alpha])) = f^*E$. The diagram inplies that if $\pi_1^*(f_0^*)^{-1}E = x$, then

$$h^*x = \pi_1^*(f_0^*)^{-1}(h^*E).$$

In our case, we have $x = E_+ \cup_\alpha E_- \oplus P_1^* E_-^\perp$. Notice that

$$h^*x = h^*E_+ \cup_{\tilde{\alpha}} h^*E_- \oplus P_1^*((h^*E_-)^\perp),$$

where $\tilde{\alpha}$ is defined as in the remark before the proposition . Therefore,

$$h^*E = \Phi[h^*E_+, h^*E_-, \tilde{\alpha}].$$

□

The map Φ is actually always an isomorphism.

Theorem 5.23. *The map Φ is a natural isomorphism.*

Proof. Let X be a compact G-space and $\{x_0\} \in X$ a G-invariant subset. We will first show that $\Phi : K(i^*) \to K_G(X, \{x_0\})$ is an isomorphism. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K(i^*) \arrow[r] \arrow[d, "\Phi"] & K(\mathcal{E}_G(X)) \arrow[r] \arrow[d, "\Phi"] & K(\mathcal{E}_G(\{x_0\})) \arrow[d, "\Phi"] \\
0 \arrow[r] & K_G(X, Y) \arrow[r] & K_G(X) \arrow[r] & K_G(\{x_0\})
\end{tikzcd}.$$

Lemma 3.22 implies that the lower horizontal sequence is exact and Example 5.16 shows that the upper horizontal sequence is exact. Since Φ is natural and all the maps are induced by inclusions (we view $K_G(\cdot)$ as $K_G(\cdot, \emptyset)$ and $K(j^*)$, with $j : \emptyset \to X$ as $K(\mathcal{E}_G(X))$), the diagram commutes. Since the last two vertical arrows are isomorphism, the first vertical arrow is also an isomorphism.

Now assume that X is a compact G-space and $Y \subset X$ is a closed G-invariant subset. Let $\pi : (X, Y) \to (X/Y, Y/Y)$ be the projection. Using Remark 3.31, we see that

$$K_G(X, Y) = \tilde{K}_G(X^+, Y^+) \cong \tilde{K}_G(X^+/Y^+) \cong \tilde{K}_G(X^+/Y^+, Y/Y^+) = K_G(X/Y, Y/Y).$$

One can check that this isomorphism is induces by $\pi^* : K_G(X, Y) \to K_G(X/Y, Y/Y)$. Let $i : Y \to X$ and $j : Y/Y \to X/X$ be inclusions. Since Φ is natural, the following diagram commutes:

$$\begin{array}{ccc} K_G(i^*) & \xrightarrow{\Phi} & K_G(X/Y, Y/Y) \\ \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} \\ K_G(j^*) & \xrightarrow{\Phi} & K_G(X, Y) \end{array} .$$

Since the upper half of the diagram is a composition of isomorphism, it follows that $\Phi \circ \pi^*$ is an isomorphism. This implies that π^* is injective. To show that $\Phi : K_G(j^*) \to K_G(X, Y)$ is an isomorphism, it is sufficient to show that $\pi^* : K(i^*) \to K(j^*)$ is surjective and hence an isomorphism. Let $[E_+, E_-, \alpha] \in K(j^*)$. In Theorem 2.31, we showed that there exists a G-vector bundle $E_-^{\perp}$ and a G-module M such that $E_- \oplus E_-^{\perp} \cong M_X$. Notice that $[E_+, E_-, \alpha] = [E_+ \oplus E_-^{\perp}, M_X, \beta]$, for some G-vector bundle isomorphism $\beta : E_+ \oplus E_-^{\perp}|_Y \to M_Y$. Since $E_+ \oplus E_-^{\perp}|_Y \cong M_Y$, we can use the same construction as the construction in Proposition 3.30 to obtain the G-vector bundle $q : E_+ \oplus E_-^{\perp}/\sim \to X/Y$ on X/Y, where $v \sim w$ if $\mathrm{pr}_M \circ \beta(v) = \mathrm{pr}_M \circ \beta(w)$. Notice that β induces a map $\tilde{\beta} : E_+ \oplus E_-^{\perp}/\sim \to M_{X/Y}$ defined by $\tilde{\beta}(v) = (q(v), \mathrm{Pr}_M(\beta(v)))$. The triple $[E_+ \oplus E_-^{\perp}/\sim .M_{X/Y}, \tilde{\beta}]$ now defines an element of $K(i^*)$, such that

$$\pi^*[E_+ \oplus E_-^{\perp}/\sim, M_{X/Y}, \tilde{\beta}] = [E_+ \oplus E_-^{\perp}, M_X, \beta] = [E_+, E_-, \alpha].$$

$\square$

6 Clifford algebras

In this section we introduce Clifford algebras. The main result of this section is that the Clifford algebras are in some sense periodic. In the next to chapters, we will use this to show show that the groups $K_G^{-n}(X)$ are periodic. This section will be based on chapter III.3 of [3] and [4].

We start with a definition:

Definition 6.1. *A quadratic form on $\mathbb{K}^n$ is a function $Q : \mathbb{K}^n \to \mathbb{K}$, defined by*

$$Q(x) := \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} x_i x_j,$$

with $\lambda_{i,j} \in \mathbb{K}$.

Example 6.2. The map $x \to \langle x, x \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$, is an example of a quadratic form.

Let $T^k\mathbb{K}^n := \mathbb{K}^n \otimes \ldots \otimes \mathbb{K}^n$ be the tensor product of k copies of $\mathbb{K}^n$. We can then construct the space

$$T\mathbb{R}^n := \mathbb{R} \oplus \bigoplus_{i=1}^{\infty} T^i(\mathbb{R}^n).$$

Notice that $T\mathbb{R}^n$ has the structure of a $\mathbb{R}$-algebra, where the multiplication is given by $v \cdot w = v \otimes w$. With this notation, we can give the following definition:

Definition 6.3. *Let $Q : \mathbb{K}^n \to \mathbb{K}$ be a quadratic form. We define the Clifford algebra of Q by the quotient algebra*

$$C(Q) := T\mathbb{K}^n / I,$$

where I is the two sided ideal generated by the set $\{x \otimes x - 1 \cdot Q(x) \mid x \in \mathbb{K}^n\}$.

Remark 6.4. Notice that the map $i : \mathbb{K}^n \to C(Q)$ defined by $i(x) = x$ is injective, which implies that $\mathbb{K}^n \subset C(Q)$.

The algebras have the following useful property:

Proposition 6.5. *Let M be a $\mathbb{K}$-algebra with unit, the map Q a quadratic form on $\mathbb{K}^n$ and let $L : \mathbb{K}^n \to M$ be a linear map such that $(Lx)^2 = 1 \cdot Q(x)$. Then, there exists a unique (unital) algebra morphism $\tilde{L} : C(Q) \to M$ such that $\tilde{L} \circ i = L$.*

Proof. Assume that $\tilde{L}$ exists. Notice that by definition we have $\tilde{L}(1) = 1$. Let $\sum_{i=0}^{n} \lambda_i e_{k_{i,1}} \otimes \ldots \otimes e_{k_{i,n_i}} \in C(Q)$, with $\lambda_i \in \mathbb{K}^n$, the vectors e_j the standard basis and $1 \leq k_{i,n_i} \leq n$. Because $\tilde{L}$ is an $\mathbb{K}$-algebra morphism, we must have

$$\tilde{L}(\sum_{i=0}^{n} \lambda_i e_{k_{i,1}} \otimes \ldots \otimes e_{k_{i,n_i}}) = \sum_{i=0}^{n} \lambda_i \tilde{L}(e_{k_{i,1}}) \cdot \ldots \cdot \tilde{L}(e_{k_{i,n_i}}) = \sum_{i=0}^{n} \lambda_i (L e_{k_{i,1}}) \cdot \ldots \cdot (L e_{k_{i,n_i}}).$$

Therefore, if $\tilde{L}$ exists, it must be defined as above. Notice that $\tilde{L}$ is well defined if it map the elements of the ideal I to 0. This is the case if for all $x \in \mathbb{K}^n$, we have $\tilde{L}(x \otimes x - Q(x)1) = 0$. Let $x \in \mathbb{K}^n$. We have

$$\tilde{L}(x \otimes x - 1 \cdot Q(x)) = L(x) \cdot L(x) - 1 \cdot Q(x) = 1 \cdot Q(x) - 1 \cdot Q(x) = 0.$$

□

Example 6.6. Before we proceed we will give another example of a Clifford algebra. Let $Q : \mathbb{R}^{k+l} \to \mathbb{R}$ be the quadratic form defined by

$$Q_{k,l}(x) = -(\sum_{i=1}^{k} x_i^2) + \sum_{i=k+1}^{k+l} x_i^2.$$

We will denote the algebra $C(Q_{k,l})$ by $C^{k,l}$. Notice that

$$e_i^2 = Q(e_i) = \begin{cases} -1 & \text{if} \quad 1 \leq i \leq k \\ 1 & \quad k+1 \leq i \leq k+l \end{cases}. \tag{6.1}$$

We claim that

$$e_i e_j + e_j e_i = 0 \tag{6.2}$$

if $i \neq j$. This holds because

$$Q(e_i + e_j) - Q(e_i) - Q(e_j) = 0$$

and in $C^{k,l}$ we have $Q(v) = v \otimes v$ for al $v \in V$, which implies that

$$(e_i + e_j) \otimes (e_i + e_j) - e_i \otimes e_i - e_j \otimes e_j = 0$$

and

$$e_i e_j + e_j e_i = 0.$$

Using these relations, it follows that the set $B = \cup_{i=0}^{l+k} B_i$, with

$$B_i := \{e_{j_1} \otimes \ldots \otimes e_{j_i} \mid 1 \leq j_1 < \ldots < j_i \leq k+l\}$$

is a basis of C^{k+1} and $\dim C^{k+l} = 2^{k+l}$. One can check that this, together with equation 6.1 and 6.2, implies that

$$C^{0,0} \cong \mathbb{R}, \qquad C^{1,0} \cong \mathbb{C} \qquad \text{and} \qquad C^{2,0} \cong \mathbb{H}.$$

The remainder of this section, we will compute the algebras $C^{k,l}$ for all $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$. The following lemmas will enable us to do this:

Lemma 6.7. *Let $(\mathbb{R}^{p+q}, C^{p,q})$ and $(\mathbb{R}^{l+m}, C^{l,m})$ be Clifford algebras. If $p+q$ is even and $(e_1 e_2 \ldots e_{p+q})^2 = 1$ (in $C^{p,q}$), then*

$$C^{p+l,q+m} \cong C^{p,q} \otimes_{\mathbb{R}} C^{l,m}.$$

Proof. First, notice $C^{p+l,q+m} \cong C(Q_{p,q} \oplus Q_{l,m})$, where $Q_{p,q} \oplus Q_{l,m}$ is a quadratic form on $\mathbb{R}^{p+q} \oplus \mathbb{R}^{l+m}$ defined by

$$Q_{p,q} \oplus Q_{l,m}(v_1, v_2) = Q_{p,q}(v_1) + q_{l,m}(v_2).$$

Let $c := e_1 e_2 \ldots e_{p+q}$ and $L : \mathbb{R}^{p+q} \oplus \mathbb{R}^{l+m} \to C(Q_{p,q} \oplus Q_{l,m})$ be the map defined by

$$L(v_1, v_2) = v_1 \otimes 1 + c \otimes v_2.$$

Since $ce_i = (-1)^{p+q-1} e_i c$ for $1 \le i \le p+q$, we have $ce_i = -e_i c$ and $cv = -vc$ for all $v \in \mathbb{R}^{p+q}$. Therefore,

$$\begin{aligned}(L(v_1, v_2))^2 &= (v_1 \otimes 1 + c \otimes v_2)^2 = v_1^2 \otimes 1 + v_1 c \otimes v_2 - v_1 c \otimes v_2 + c^2 \otimes v_2^2 \\ &= Q_{p,q}(v_1) \otimes 1 + 1 \otimes Q_{l,m}(v_1) = Q_{p,q}(v_1) + Q_{l,m}(v_2) \\ &= (Q_{p,q} \oplus Q_{l,m})(v_1, v_2).\end{aligned}$$

Proposition 6.5 implies that this map induces a unique map $\tilde{L} : C(Q_{p,q} \oplus Q_{l,m}) \to C^{p,q} \otimes_{\mathbb{R}} C^{l,m}$, such that $\tilde{L} \circ i = L$. Since both algebras have dimension $2^{p+q+l+m}$ it is sufficient to show that $\tilde{L}$ is surjective. To show this, it suffices to show that the image of $\tilde{L}$ contains a multiplicative basis of $C^{p,q} \otimes_{\mathbb{R}} C^{l,m}$. We must thus show that the image of $\tilde{L}$ contains the elements $e_i \otimes 1$ and $1 \otimes e_j$, for $1 \le i \le p+q$ and $1 \le j \le l+m$. Notice that

$$\tilde{L}(e_i, 0) = e_i \otimes 1$$

and

$$\tilde{L}(c \otimes e_j) = (c \otimes 1) \cdot c \otimes e_j = c^2 \otimes e_j = 1 \otimes e_j.$$

□

Lemma 6.8. *We have*

$$C^{0,2} \otimes C^{k,l} \cong C^{l,k+2}$$

and

$$C^{2,0} \otimes C^{k,l} \cong C^{l+2,k}.$$

Proof. We will prove the first statement, the proof of the second statement is similar. As in the previous lemma, we have $C^{l,k+2} \cong C(Q_{0,2} \oplus Q_{l,0} \oplus Q_{0,k})$. We now define the map

$$L : \mathbb{R}^2 \oplus \mathbb{R}^l \oplus \mathbb{R}^k \to C^{0,2} \otimes C^{k,l}$$

by

$$L(u, v, w) = u \otimes 1 + e_1 e_2 \otimes (w, v).$$

Notice that

$$\begin{aligned}L(u, v, w)^2 &= u^2 \otimes 1 - 1 \otimes (w, v)^2 = Q_{0,2}(u) - Q_{k,l}(w, v) \\ &= Q_{0,2}(u) + Q_{l,k}(v, w) = (Q_{0,2} \oplus Q_{l,0} \oplus Q_{0,k})(u, v, w).\end{aligned}$$

Proposition 6.5 implies that this map induces am map $\tilde{L} : C(Q_{0,2} \oplus Q_{l,0} \oplus Q_{0,k}) \to C^{0,2} \otimes C^{k,l}$. As before, these algebras have the same dimension and we have to show that the image of $\tilde{L}$ contains a multiplicative basis. We have $\tilde{L}((e_i, 0, 0)) = e_i \otimes 1$ and $\tilde{L}(-c \otimes (0, e_i, 0)) = 1 \otimes (0, e_i)$ and $\tilde{L}(-c \otimes (0, 0, e_i)) = 1 \otimes (0, e_i)$. □

Remark 6.9. This Lemma implies that

$$C^{0,4} \cong C^{0,2} \otimes_{\mathbb{R}} C^{2,0}$$

and

$$C^{4,0} \cong C^{2,0} \otimes_{\mathbb{R}} C^{0,2}.$$

Since $C^{0,2} \otimes_{\mathbb{R}} C^{2,0} \cong C^{2,0} \otimes_{\mathbb{R}} C^{0,2}$, it follows that

$$C^{0,4} \cong C^{4,0}.$$

Lastly, we need the following statements about matrices. In this lemma, if B is a $\mathbb{K}$-algebra, we will use the notation $M_n(B)$ for the algebra of $n \times n$ matrices with coefficients in B.

Lemma 6.10. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let B be a $\mathbb{K}$-algebra. Then we have the following isomorphisms:*

(i) $M_n(M_m(B)) \cong M_{nm}(B)$.

(ii) $B \otimes_{\mathbb{K}} M_n(\mathbb{K}) \cong M_n(B)$.

(iii) $M_n(B) \otimes_{\mathbb{K}} M_m(\mathbb{K}) \cong M_{nm}(B)$

Proof. We leave the verification of these statements to the reader. □

Example 6.11. Before we show how to determine all the Clifford algebras $C^{p,q}$, we compute some more examples. We will determine $C^{0,1}$, $C^{0,2}$ and $C^{1,1}$. We start with $C^{1,1}$. Notice that the map

$$L : C^{0,1} \to \mathrm{Span}(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}).$$

Which maps 1 to the first matrix and e_1 to the second matrix is an isomorphism. Since

$$L(\frac{1+e_1}{2}) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \text{ and } L(\frac{1-e_1}{2}) \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

The map $\tilde{L} : C^{0,1} \to \mathbb{R} \oplus \mathbb{R}$ defined by $\tilde{L}(a + be_1) = (a + b, a - b)$ is an isomorphism, where $\mathbb{R} \oplus \mathbb{R}$ is the $\mathbb{R}$ algebra with addition $(a, b) + (c, d) = (a + c, b + d)$, multiplication $(a, b)(c, d) = (ac, bd)$ and scalar multiplication $\lambda(a, b) = (\lambda a, \lambda b)$.
We have $C^{0,2} \cong M_2(\mathbb{R})$, where the isomorphism is given by

$$1 \to \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad e_1 \to \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad e_2 \to \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

We also have $C^{1,1} \cong M_2(\mathbb{R})$. in this case the isomorphism is given by

$$1 \to \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad e_1 \to \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad e_2 \to \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

We will now use the algebras we have already computed to compute the other Clifford algebras.

Proposition 6.12. *The following identities hold:*

(i) If $l \geq m$, then $C^{l,m} \cong M_{2^m}(C^{l-m,0})$.

(ii) If $m \leq l$, then $C^{l,m} \cong M_{2^m}(C^{0,m-l})$

(iii) $C^{l+8,k} \cong C^{l,k+8} \cong M_{16}(C^{l,m})$.

(iv) $C^{0,k+2} \cong M_2(C^{k,0})$

(v) $C^{k+2,0} \cong C^{0,k} \otimes_{\mathbb{R}} \mathbb{H}$

Proof. We first prove (i). Notice that in $C^{1,1}$, we have $(e_1e_2)^2 = 1$. Lemma 6.7 now implies that

$$C^{l,m} \cong C^{l-1,m-1} \otimes_{\mathbb{R}} C^{1,1}.$$

If we now use lemma 6.7 repeatedly, then, because $l \geq m$, we obtain

$$C^{l,m} \cong C^{l-m,0} \otimes (C_1^{1,1} \otimes \ldots \otimes C_m^{1,1}). \tag{6.3}$$

Since $C^{1,1} \cong M_2(\mathbb{R})$. Lemma 6.10 (iii) implies that

$$C_1^{1,1} \otimes \ldots \otimes C_m^{1,1} \cong M_2(\mathbb{R}) \otimes \ldots M_2(\mathbb{R}) \cong M_{2^m}(\mathbb{R}).$$

We can now apply Lemma 6.10 (ii) to Equation 6.3 to obtain

$$C^{l,m} \cong C^{l-m,0} \otimes_{\mathbb{R}} M_{2^m} \cong M_{2^m}(C^{l-m}).$$

The proof of (ii) is similar to the proof of (i). We now show (iii). We will prove the theorem for $C^{l+8,k}$, the proof for the other case is similar. Lemma 6.7 together with Remark 6.9 implies that

$$C^{0,8} \cong C^{0,4} \otimes C^{0,4} \cong C^{0,4} \otimes C^{4,0} \cong C^{4,4}.$$

Using Part (i) of this proposition, we get $C^{4,4} \cong M_{2^4}(C^{0,0}) = M_{16}(\mathbb{R})$. We can now apply Lemma 6.7 to $C^{l+8,k}$ to obtain

$$C^{l+8,k} \cong C^{l,k} \otimes_{\mathbb{R}} C^{8,0} \cong C^{l,k} \otimes_{\mathbb{R}} M_{16}(\mathbb{R}) \cong M_{16}(C^{l,k}),$$

where the last isomorphism follows from Lemma 6.10 (ii).

Lastly, we prove (iv) and (v). We will only prove (iv), the proof of (v) is similar. In Lemma 6.8, we showed that

$$C^{0,k+2} \cong C^{k,0} \otimes C^{0,2}$$

and in Example 6.11 we showed that $C^{0,2} \cong M_2(\mathbb{R})$. Therefore, Lemma 6.10 (ii) implies that

$$C^{0,k+2} \cong C^{k,0} \otimes M_2(\mathbb{R}) \cong M_2(C^{k,0}).$$

□

Remark 6.13. A direct consequence of part $(i) - (iii)$ of this proposition is that every Clifford algebra $C^{l,m}$ is isomorphic to $M_{2^{n_{l,m}}}(C^{p_{l,m},q_{l,m}})$ for a $(p_{l,m}, q_{l,m}) \in \{(0,k) \mid 0 \leq k \leq 7\} \cup \{(k,0) \mid 0 \leq k \leq 7\}$ and $n_{l,m} \in \mathbb{N}_0$.

Remark 6.14. Notice that part (iii) of the proposition above implies that the algebras are periodic with period 8 in the sense that increasing one of the indices of $C^{l,m}$ by 8 will give an algebra isomorphism to $M_{16}(C^{l,m})$.

Remark 6.13 implies that to be able to determine all Clifford algebras, we must only determine $C^{0,k}$ and $C^{k,0}$ for all $0 \leq k \leq 7$. Using that we already determined the algebras for $k \leq 2$ and Proposition 6.12 (iv) and (v) we can determine the remaining algebras: The first few calculations are as follows:

$$C^{0,3} \cong M_2(C^{1,0}) \cong M_2(\mathbb{C}),$$

$$C^{4,0} \cong C^{0,4} \cong M_2(C^{2,0}) \cong M_2(\mathbb{H}),$$

k	$C^{k,0}$	$C^{0,k}$
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R}\oplus\mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$

Table 1: The algebras $C^{k,0}$ and $C^{0,k}$.

$$C^{3,0}\cong C^{0,1}\otimes_{\mathbb{R}}\mathbb{H}\cong(\mathbb{R}\oplus\mathbb{R})\otimes\mathbb{H}\cong\mathbb{H}\oplus\mathbb{H}.$$

If we continue this process, we eventually obtain Table 1, which is table 1 of [4]. We will end this chapter by examining what happens in the complex case. Let $Q^{\mathbb{C}}_{k,l}:\mathbb{C}^{k+l}\to\mathbb{C}$ denote the quadratic form defined by '

$$Q^{\mathbb{C}}_{k,l}(x)=-(\sum_{i=1}^{k}x_i^2)+\sum_{i=k+1}^{k+l}x_i^2$$

and let

$$C^{k,l}_{\mathbb{C}}:=C(Q^{\mathbb{C}}_{k,l}).$$

Before we show how to compute these algebras, we will first give some examples:

Example 6.15. Notice that $C^{0,0}_{\mathbb{C}}\cong\mathbb{C}$. Using the same isomorphism as in Example 6.11, where the matrices now represents objects in $M_2(\mathbb{C})$, we obtain $C^{0,1}_{\mathbb{C}}\cong\mathbb{C}\oplus\mathbb{C}$, $C^{1,1}_{\mathbb{C}}\cong M_2(\mathbb{C})$ and $C^{0,2}_{\mathbb{C}}\cong M_2(\mathbb{C})$. We also have $C^{1,0}_{\mathbb{C}}\cong\mathbb{C}\oplus\mathbb{C}$, where the isomorphism is given by

$$L(z+e_1w)=(z+iw,z-iw).$$

Lastly, there is an isomorphism $C^{2,0}_{\mathbb{C}}\cong M_2(\mathbb{C})$, which is defined by

$$1\to\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad e_1\to\begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad\text{and}\quad e_2\to\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

We will again use these algebras to determine the other algebras. To do this, we will use the following Lemma:

Lemma 6.16. *Let $(\mathbb{C}^{p+q},C^{p,q}_{\mathbb{C}})$ and $(\mathbb{C}^{l+m},C^{l,m}_{\mathbb{C}})$ be Clifford algebras. Then,*

(i) If $p+q$ is even and $(e_1e_2\dots e_{p+q})^2=1$ (in $C^{p,q}_{\mathbb{C}}$), then

$$C^{p+l,q+m}_{\mathbb{C}}\cong C^{p,q}_{\mathbb{C}}\otimes_{\mathbb{C}}C^{l,m}_{\mathbb{C}}.$$

(ii) we have

$$C^{0,2}_{\mathbb{C}}\otimes_{\mathbb{C}}C^{p,q}_{\mathbb{C}}\cong C^{q,p+2}_{\mathbb{C}}$$

and

$$C^{2,0}\otimes_{\mathbb{C}}C^{p,q}_{\mathbb{C}}\cong C^{q+2,p}_{\mathbb{C}}.$$

Proof. The proof of the statements is almost identical to the proof of Lemma 6.7 and Lemma 6.8. □

Since $C_{\mathbb{C}}^{0,2} \cong C_{\mathbb{C}}^{2,0} \cong M_2(\mathbb{C})$, part (ii) of this lemma implies that

$$C_{\mathbb{C}}^{k,l+2} \cong C_{\mathbb{C}}^{k+2,l} \cong M_2(C_{l,k}). \tag{6.4}$$

If $k = 2m + i$ and $l = 2n + j$, with $i, j \in \{0, 1\}$ and we apply Equation 6.4 repeatedly, then we obtain

$$C_{\mathbb{C}}^{k,l} \cong M_{2^{m+n}}(C_{\mathbb{C}}^{i,j}),$$

where, since $C_{\mathbb{C}}^{0,1} \cong C_{\mathbb{C}}^{1,0}$, the order of the indices i, j in $C_{\mathbb{C}}^{i,j}$ does not matter. Therefore, we have

$$C_{\mathbb{C}}^{k,l} \cong \begin{cases} M_{2^{m+n}}(\mathbb{C}) & (i,j) = 0 \\ M_{2^{m+n}}(\mathbb{C} \oplus \mathbb{C}) \cong M_{2^{m+n}}(\mathbb{C}) \oplus M_{2^{m+n}}(\mathbb{C}) \quad \text{if} & (i,j) = (0,1) \text{ or } (1,0) \\ M_{2^{m+n+1}}(C_{\mathbb{C}}^{i,j}) & (i,j) = (1,1) \end{cases}.$$

This equation implies that

$$C_{\mathbb{C}}^{k+2,l} \cong C_{\mathbb{C}}^{k,l+2} \cong M_2(C_{\mathbb{C}}^{k,l}).$$

Thus, in the complex case, the Clifford algebras are periodic with period 2.

7 The groups $K^{l,m}$

In this section the groups $K_G^{l,m}(X)$ are introduced. We will use the results from the previous section to show that they are periodic. We will then explain how these groups can be used to show that

$$(K_G^{\mathbb{R}})^{-n}(X,Y) \cong (K_G^{\mathbb{R}})^{-n-8}(X,Y) \qquad \text{and} \qquad (K_G^{\mathbb{C}})^{-n}(X,Y) \cong (K_G^{\mathbb{C}})^{-n-2}(X,Y),$$

which is the main result of this section.

This section is based on section III. 4 of [3].
We start with a definition:

Definition 7.1. *Let $\mathcal{C}$ be a Banach category and let B be a finite dimensional $\mathbb{K}$-algebra with unit. We let $\mathcal{C}^B$ be the category whose objects are pairs (C, f), where $C \in ob(\mathcal{C})$ and $f : B \to End(C)$ is a (unital) algebra homomorphism. A morphism $\sigma : (C, f) \to (D, g)$ is a map $\sigma \in Hom_{\mathcal{C}}(C, D)$ such that $\sigma \circ f(b) = g(b) \circ \sigma$, for all $b \in B$.*

The construction we defined above 'preserves' some of the structure of $\mathcal{C}$.

Proposition 7.2. *The category $\mathcal{C}^B$ defined above is still a Banach category. Moreover, if $\mathcal{C}$ is pseudo-abelian, then $\mathcal{C}^B$ is also pseudo-abelian.*

Proof. Since $\mathrm{Hom}_{\mathcal{C}^B}((C, f), (D, g))$ is a linear subspace of $\mathrm{Hom}_{\mathcal{C}}(C, D)$, restricting the norm on $\mathrm{Hom}_{\mathcal{C}}(C, D)$ to $\mathrm{Hom}_{\mathcal{C}^B}((C, f), (D, g))$ endows it has the structure of a normed vector space, such that composition is bilinear and continuous. We now show that the norm is complete. Let $\epsilon > 0$ and $(A_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Hom}_{\mathcal{C}^B}((C, f), (D, g))$. Since $\mathrm{Hom}_{\mathcal{C}}(C, D)$ is complete, this sequence has a limit A in $\mathrm{Hom}_{\mathcal{C}}(C, D)$. Notice that

$$\begin{aligned}\|Af(b) - g(b)A\| = \|Af(b) - A_if(b) + A_if(b) - g(b)A\| &= \|Af(b) - A_if(b) + g(b)A_if - g(b)A\| \\ &\leq \|(A - A_i)f(b)\| + \|g(b)(A - A_i)\|.\end{aligned}$$

Since composition is continuous, we have $\|(A - A_i)f(b)\| + \|g(b)(A - A_i)\| < \epsilon$ for i sufficiently large, which implies that $\|Af(b) - g(b)A\| < \epsilon$. Therefore, $Af(b) = g(b)A$ and $A \in \mathrm{Hom}_{\mathcal{C}^B}((C, f), (D, g))$. It remains to show that $\mathcal{C}^B$ has finite products and co-products. If (C_i, f_i) are objects of $\mathcal{C}$ for $1 \leq i \leq n$, then the object

$$(C_1 \oplus \ldots \oplus C_n, f_1 \oplus \ldots \oplus f_n),$$

with same projections and inclusions as in $\mathcal{C}$ has the structure of a product and co-product.

Now assume that $\mathcal{C}$ is a pseudo-abelian category. Let (C, f) be an object of $\mathcal{C}^B$ and let $P \in \mathrm{Hom}_{\mathcal{C}^B}((C, f), (C, f))$ be a morphism such that $P^2 = P$. Since C is an object of $\mathcal{C}$ and $\mathcal{C}$ is a pseudo-abelian category, there exists an object $C_0 \in \mathcal{C}$, such that the functor $\mathrm{Im}_P^{\mathcal{C}} : \mathcal{C}^{op} \to \mathrm{Set}$, which acts on objects by

$$\mathrm{Im}_P^{\mathcal{C}}(X) := \{h \in \mathrm{Hom}_{\mathcal{C}}(X, C) \mid p \circ f = f\}.$$

and a morphisms by

$$F(g)(h) = h \circ g$$

is naturally isomorphic to $y_{C_0} : \mathcal{C}^{op} \to \mathrm{Set}$ (which acts on objects by $y_{C_0}(X) = \mathrm{Hom}_{\mathcal{C}}(X, C_0)$ and on morphisms by $y_{C_0}(g)(h) = h \circ g$). Let $\sigma : y_{C_0} \to \mathrm{Im}_P^{\mathcal{C}}$ denote this isomorphism and let σ^{-1} denote its inverse. Now consider the functors $\mathrm{Im}_P : \mathcal{C}^B \to \mathrm{Set}$ and $y_{(C_0,f_0)}$, where $f_0 : B \to \mathrm{End}(C_0)$ is defined by

$$f_0(b) := \sigma^{-1}(P) \circ f(b) \circ \sigma(id_{C_0}).$$

Notice that because

$$\sigma^{-1}(P) \circ \sigma(id) = \sigma^{-1}(P \circ \sigma(id)) = \sigma^{-1}(\sigma(id)) = id,$$

we have $f_0(ab) = f_0(a)f_0(b)$ and $f(1) = id$, which implies that f_0 is indeed a (unital) algebra homomorphism.

We claim that $\sigma|y_{(C_0,f_0)} : y_{(C_0,f_0)} \to \mathrm{Im}_P$ induces an isomorphism. We first show that σ is well defined. Let $h \in y_{(C_0,f_0)}(X, g)$, then,

$$\begin{aligned}
\sigma(h) \circ g(b) &= \sigma(h \circ g(b)) \\
&= \sigma(f_0(b) \circ h) \\
&= \sigma(\sigma^{-1}(P) \circ f(b) \circ \sigma(id_{C_0}) \circ h) \\
&= \sigma(\sigma^{-1}P) \circ (f(b) \circ \sigma(id_{C_0}) \circ h) \\
&= P \circ f(b) \circ \sigma(h) \\
&= f(b) \circ P \circ \sigma(h) \\
&= f(b) \circ \sigma(h).
\end{aligned}$$

Therefore, the map $\sigma|y_{(C_0,f_0)}$ is well defied. We now show that $\sigma^{-1}|_{\mathrm{Im}_P}$ is also well defined, which implies that σ is a natural isomorphism. Let $h \in \mathrm{Im}P((X, g))$. Then

$$\begin{aligned}
\sigma^{-1}(h) \circ g(b) &= \sigma^{-1}(h \circ g(b)) \\
&= \sigma^{-1}(P \circ h \circ g(b)) \\
&= \sigma^{-1}(P \circ f(b) \circ h) \\
&= \sigma^{-1}(P) \circ f(b) \circ h \\
&= \sigma^{-1}(P) \circ f(b) \circ \sigma(id_{C_0}\sigma^{-1}(h)) \\
&= (\sigma^{-1}(P) \circ f(b) \circ \sigma(id_{C_0})) \circ \sigma^{-1}(h). \\
&= f_0(b) \circ \sigma^{-1}(h)
\end{aligned}$$

□

Remark 7.3. Let $\mathcal{C} = \mathcal{E}^{\mathbb{K}}_G(X)$ and $(E, f) \in \mathcal{E}^{\mathbb{K}}_G(X)^B$. In section 4, we showed that $\mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)}$ is a Banach algebra, where the multiplication is given by composition. The Lemma above implies that

$$\mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^B}(E, f) \subset \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)}(E)$$

is a Banach space. Notice that if $\sigma_1, \sigma_2 \in \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^B}(E, f)$, then

$$\sigma_1\sigma_2 f = \sigma_1 f \sigma_2 = f\sigma_1\sigma_2,$$

which implies that $\sigma_1\sigma_2 \in \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^B}(E, f)$. Therefore, the set $\mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^B}(E, f)$ is also a Banach algebra.

Although it is in general difficult to determine all the morphisms of $\mathcal{C}$ in $\mathrm{Hom}_{\mathcal{C}^B}(C, D)$, in some cases we can at least find some of the morphisms:

Example 7.4. Let C be a Banach category and B be a finite dimensional $\mathbb{K}$-algebra. Let $(C,f) \in \mathcal{C}^B$. Then $\bigoplus_{i=1}^n (C,f) = (\bigoplus_{i=1}^n C, f\oplus,\dots,f) \in \mathcal{C}^B$. As before, we can view a matrix $A \in M_n(\mathbb{K})$ as an element of $\mathrm{End}_{\mathbb{C}}(\bigoplus_{i=1}^n C)$ by viewing the matrix elements $A_{i,j}$ as the map $A_{i,j} id_C$. Since $\lambda id_C f = f \lambda id_C$ for all $\lambda \in \mathbb{K}$, it follows that $Af = fA$ and thus that

$$A \in \mathrm{End}_{C^B}(\bigoplus_{i=1}^n C, f\oplus,\dots,f)$$

for all $A \in M_n(\mathbb{K})$.

We will be interested in the categories $\mathcal{C}_{\mathbb{R}}^{l,m} := \mathcal{C}^{C^{p,q}}$, where $\mathcal{C}$ is a real Banach category and $\mathcal{C}_{\mathbb{C}}^{l,m} := \mathcal{C}^{C_{\mathbb{C}}^{p,q}}$ if $\mathcal{C}$ is a complex Banach category.

We will now translate the propositions about the structure of the algebras $C^{p,q}$ to statements about the categories $\mathcal{C}^{p,q}$. To do this, we need the following Lemmas:

Lemma 7.5. *Let $\mathcal{C}$ be a pseudo-abelian Banach category and let A and B be $\mathbb{K}$-algebras. Then the following categories are equivalent:*

1. $(\mathcal{C}^A)^B \cong \mathcal{C}^{A\otimes_{\mathbb{K}} B}$.
2. $(\mathcal{C}^{A\oplus B}) \cong \mathcal{C}^A \times \mathcal{C}^B$, *where $\mathcal{C}^A \times \mathcal{C}^B$ denotes the product of the categories.*

Proof. We first show (i). Let $F : (\mathcal{C}^A)^B \to \mathcal{C}^{A\otimes_{\mathbb{K}} B}$ be the functor that maps an object $((C,f_A),f_B)$ of $(\mathcal{C}^A)^B$ to the object $(C, f_A \otimes f_B)$ of $\mathcal{C}^{A\otimes_{\mathbb{K}} B}$, where $f_A \otimes f_B$ is defined by $f_A \otimes f_B(a \otimes b) = f_A(a) \circ f_B(b)$, and maps a map $\sigma : ((C,f_A),f_B) \to ((D,g_A),g_B)$ to the map $\sigma : (C, f_A \otimes f_B) \to (D, g_A \otimes g_b)$. We first show that this functor is well defined. We show that $f_A \otimes f_B : A \otimes_{\mathbb{K}} B \to \mathrm{End}_{\mathbb{C}}(C)$ is indeed an algebra homomorphism. Notice that $f_A \otimes f_B$ is a $\mathbb{K}$-module homomorphism. Since for all $b \in B$, we have $f_B(b) \in \mathrm{End}_{\mathcal{C}^A}(C)$, it follows that for all $b \in B$ and $a \in A$ the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{f_A(a)} & C \\ \Big\downarrow{\scriptstyle f_B(b)} & & \Big\downarrow{\scriptstyle f_B(b)} \\ C & \xrightarrow{f_A(a)} & C \end{array} .$$

Therefore, we have

$$\begin{aligned} f_A \otimes f_B(ac \otimes bd) &= f_A(ac)f_B(bd) = f_A(a)f_A(c)f_B(b)f_B(d) \\ &= f_A(a)f_B(b)f_A(c)f_B(d) = f_A \otimes f_B(a,b) f_A \otimes f_B(c,d). \end{aligned}$$

We now show that σ is well defined. By definition, we have $\sigma \in \mathrm{Hom}_{\mathcal{C}^A}((C,f_A),(D,g_A))$ and $\sigma \circ f_B(b) = g_B(b) \circ \sigma$ for all $b \in B$. This implies that

$$\sigma \circ f_A(a) \circ f_B(b) = g_A(a) \circ \sigma \circ f_B(b) = g_A(a) \circ g_B(b) \circ \sigma$$

and $\sigma \in \mathrm{Hom}_{\mathbb{C}^{A\otimes_{\mathbb{K}} B}}((C, f_A \otimes f_B),(D, g_A \otimes g_B))$. Moreover, if $\sigma \in \mathrm{Hom}_{\mathcal{C}^{A\otimes_K B}})((C, f_A \otimes f_B),(D, g_A \otimes g_B))$, then, by definition we have

$$\sigma \circ f_A(a) = \sigma \circ f_A \otimes f_B(a \otimes 1) = g_A \otimes g_B(a \otimes 1) \circ \sigma = g_A(a) \circ \sigma$$

and similarly $\sigma \circ f_B(b) = g_B(b) \circ \sigma$. This implies that $\sigma \in \mathrm{Hom}_{(\mathcal{C}^A)^B}(((C, f_A), f_B), ((D, g_A), g_B))$. Therefore, the map $\sigma \to \sigma$ gives a bijection

$$\mathrm{Hom}_{(\mathcal{C}^A)^B}(((C, f_A), f_B), ((D, g_A), g_B)) \cong \mathrm{Hom}_{\mathbb{C}^{A \otimes_{\mathbb{K}} B}}((C, f_A \otimes f_B), (D, g_A \otimes g_B))$$

and F is full and faithful. We now show that F is essentially surjective. Let $(C, h) \in \mathcal{C}^{A \otimes_{\mathbb{K}} B}$. Then, we have

$$(C, h) = F((c, h(\cdot \otimes 1), h(1 \otimes \cdot))).$$

We now show (ii). Let $F : \mathcal{C}^A \times \mathcal{C}^B \to \mathcal{C}^{A \oplus B}$ be the functor, which is on objects defined by $F((C_1, f_1), (C_2, f_2)) = (C_1 \oplus C_2, f_1 \oplus f_2)$, where $f_1 \oplus f_2 : A \oplus B \to \mathrm{End}_{\mathcal{C}}(C_1 \oplus C_2)$ is defined by $(f_1 \oplus f_2)(a, b) = (f_1(a), f_2(b))$, and a morphism $(\sigma, \rho) : ((C_1, f_1), (C_2, f_2)) \to (D_1, g_1), (D_2, g_2))$ is mapped to the morphism $\sigma \oplus \rho : (C_1 \oplus C_2, f_1 \oplus f_2) \to (D_1 \oplus D_2, g_1 \oplus g_2)$. Notice that the functor is well defined. We now prove that F is an equivalence. We first show that F is full. Let $\sigma \in \mathrm{Hom}_{\mathcal{C}^{A \oplus B}}(C_1 \oplus C_2, f_1 \oplus f_2, D_1 \oplus D_2, g_1 \oplus g_2)$. By definition, we have

$$\sigma \circ f_A(a) \oplus f_B(b) = g_A(a) \oplus g_B(b) \circ \sigma.$$

for all $(a, b) \in A \oplus B$.Therefore,

$$0 = p_{D_2} \circ (g_A(1) \oplus g_B(0) \circ \sigma) = p_{D_2} \circ (\sigma \circ g_A(1) \oplus g_B(0)) = p_{D_2} \circ \sigma \circ (i_{C_1} \circ p_{C_1}).$$

Therefore, $0 = p_{D_2} \sigma i_{C_1}$. With a similar computations, we can show that $0 = p_{D_1} \sigma i_{C_2}$. This implies that there exists morphisms $\sigma_1 \in \mathrm{Hom}_{\mathcal{C}}(C_1, D_1)$ and $\sigma_2 \in \mathrm{hom}_{\mathcal{C}}(C_2, D_2)$ such that $\sigma = \sigma_1 \oplus \sigma_2$. Notice that

$$\begin{aligned} g_A(a)\sigma_1 &= p_{D_1} \circ (g_A(a) \oplus g_B(b) \circ \sigma_1 \oplus \sigma_2) \circ i_{C_1} = p_{D_1}(\sigma_1 \oplus \sigma_2 \circ f_A(a) \oplus f_B(b)) \circ i_{C_1} \\ &= p_{D_1}(\sigma_1 f_A(a) \oplus \sigma_2 f_B(b) \circ) \circ i_{C_1} = \sigma_1 \circ f_A(a). \end{aligned}$$

It follows that $\sigma_1 \in \mathrm{Hom}_{\mathcal{C}^A}((C_1, f_A), (D_1, g_A))$ and similarly that $\sigma_2 \in \mathrm{Hom}_{\mathcal{C}^A}((C_2, f_B), (D_2, g_B))$. Therefore, we have $\sigma = F((\sigma_1, \sigma_2))$.
We now show that F is faithful. Assume that $\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2 \in \mathrm{Hom}_{\mathcal{C}^{A \oplus B}}(C_1 \oplus C_2, f_1 \oplus f_2, D_1 \oplus D_2, g_1 \oplus g_2)$ and $\mu_1 \oplus \mu_2 = \sigma_1 \oplus \sigma_2$. By definition, this implies that $\mu_1 = \sigma_1$, $\mu_2 = \sigma_2$ and thus that $(\mu_1, \mu_2) = (\sigma_1, \sigma_2)$. Lastly, we show that F is essentially surjective. Let $(C, h) \in \mathcal{C}^{A \oplus B}$. Notice that $P_1 := h(1, 0)$ and $P_2 := h(0, 1)$ are maps in $\mathcal{C}^{A \oplus B}$ such that $P_1^2 = P_1$, $P_2^2 = P_2$ and $P_1 \circ P_2 = P_2 \circ P_1 = 0$. Since $\mathcal{C}^{A \oplus B}$ is pseudo-abelian by Proposition7.2, Lemma 4.28 implies that $C \cong C_1 \oplus C_2$, where C_i is an object such that $\mathrm{Im} P_i \cong y_{C_i}$. Let $\sigma_i : y_{C_i} \to \mathrm{Im} P_i$ denote this isomorphism. By chasing through the isomorphisms in the proof of Lemma 4.28 we obtain that the $A \oplus B$ acts on C_1 via the map

$$h_1(a, b) = \sigma_1^{-1}(P_1) \circ h(a, b) \circ \sigma_1(id_{C_1}) = \sigma_1^{-1}(P_1 \circ h(a, b) \circ \sigma_1(id_{C_1})) = \sigma_1^{-1}(h(a, 0)\sigma_1(id_{C_1})).$$

Therefore, the map h_1 only depends on a we can view (C_1, h_1) as an elemt of $\mathcal{C}^A$. Similarly, we can view (C_2, h_2) as an elemnt of $\mathcal{C}^B$ and we obtain

$$(C, h) \cong F((C_1, h_1), (C_2, h_2)).$$

$\square$

Example 7.6. A consequence of this lemma is that the category $(\mathcal{E}_G^{\mathbb{K}}(X))^{\mathbb{K} \oplus \mathbb{K}}$ is equivalent to the category $\mathcal{E}_G^{\mathbb{K}} \times \mathcal{E}_G^{\mathbb{K}}$.

Lemma 7.7. *Let $\mathcal{C}$ be a pseudo abelian Banach category. Then, the categorys $\mathcal{C}$ and $\mathcal{C}^{M_n(\mathbb{K})}$ are equivalent.*

Proof. Let C be an object of $\mathcal{C}$. We define the functor $F : \mathcal{C} \to \mathcal{C}^{M_n(\mathbb{K})}$ by $F(C) = (\oplus_{i=1}^n C, h)$, where the action h maps a matrix $A \in M_n(\mathbb{K})$ to the matrix $A : (\oplus_{i=1}^n C, h) \to (\oplus_{i=1}^n C, h)$, where we interpret the indices of the matrix a a multiple of the identity. The functor F maps a morphism $f : C \to D$ to the morphism $f \oplus \ldots \oplus f$. Notice that F is well defined. It is clear from the definition that F is faithful. We now show that F is full. Let $f := (f_{i,j})_{1\leq i,j\leq n} : F(C) = (\oplus_{i=1}^n C, h) \to (\oplus_{i=1}^n D, g) = F(D)$ be a morphism in $\mathcal{C}^{M_n(\mathbb{K})}$. By definition, we must have $g(A) \circ f = f \circ h(A)$ for all $A \in M_n(\mathbb{K})$. Let A denote a matrix which is everywhere zero except at $A_{p,q}$. Then, Because $Af = fA$, it follows that $f_{q,i} = 0$ if $i \neq q$, $f_{j,p} = 0$ if $j \neq p$ and $f_{p,p} = f_{q,q}$. Since p and q where arbitrary, the matrix $f_{1\leq i,j\leq n}$ must be a diagonal matrix and

$$f = f_{1,1} \oplus \ldots \oplus f_{1,1} = F(f_{1,1}).$$

Lastly, we show that F is essentially surjective. Let $(C, h) \in \mathcal{C}^{M_n(\mathbb{K})}$. Let $P_i \in M_n(\mathbb{K})$ denote the projection to the i-th coordinate and let $h_i := h(P_i)$. Notice that $h_i^2 = h_i$ and $h_i \circ h_j = h_j \circ h_i = 0$. Since $\mathcal{C}$ is pseudo-abelian, Lemma 4.28 implies that

$$C \cong \oplus_{i=0}^n C_i,$$

where $\mathrm{Im}h_i \cong y_{C_i}$ in $\mathcal{C}$. We first show that the C_i are isomorphic. Let $L_{i,j} \in M_n(\mathbb{K})$, for $i \neq j$, denote the linear map, such that $L_{i,j}(e_i) = e_j$, $L_{i,j}(e_j) = e_i$ and $L_{i,j}(e_k) = e_k$ if $k \neq i, j$. Notice that $P_i = L_{i,j} P_j L_{i,j}$. If $f \in Im(h_i)$, then

$$h_j h(L_{i,j}) f = h(L_{i,j}) h(L_{i,j} P_j L_{i,j}) f = h(L_{i,j}) h_i f = h(L_{i,j}) f.$$

This implies that the map $f \to h(L_{i,j}) f$ is a well defined natural transformation from $\mathrm{Im}(h_i)$ to $\mathrm{Im}(h_j)$. Since $f \to h(L_{i,j}) f$ is also a natural transformation from $\mathrm{Im}(h_j)$ to $\mathrm{Im}(h_i)$ and $h(L_{i,j})^2 = id$ the map is a natural isomorphism. Therefore,

$$y_{C_i} \cong \mathrm{Im}h_i \cong \mathrm{Im}h_j \cong y_{C_j}$$

and $C_i \cong C_j$.
If $\sigma_i : y_{C_i} \to \mathrm{Im}h_i$ denotes the natural isomorphism from the definition of C_i, then the isomorphism above is given by the map $I_{j,i} := \sigma_j^{-1}(P_j) \circ h(L_{i,j}) \sigma_i(id_{C_i}) : C_i \to C_j$. We claim that the map

$$f := id_{C_1} \oplus \sigma_1^{-1}(P_1) \circ h(L_{2,1}) \sigma_2(id_2)) \oplus \ldots \oplus \sigma_1^{-1}(h_1) \circ h(L_{n,1}) \sigma_n(id_n)) : (\oplus_{i=0}^n C_i, h) \to F(C_1),$$

is an isomorphism in $\mathcal{C}^{M_n(\mathcal{C})}$.

We first determine the induced $M_n(\mathbb{K})$ action $\tilde{h}$ on $\oplus_{i=1}^n C_i$ induced by h. The action of a matrix A on $\oplus_{i=1}^n C_j$ is given by

$$(\sigma_1^{-1}(h_1), \ldots, \sigma_n^{-1}(h_n))^T \circ h(A) \circ (\sigma_1(id_{C_1}), \ldots, \sigma_n(id_{C_n})).$$

This implies that

$$h(P_i)_{p,q} = \begin{cases} 0 & \text{if} \quad (p,q) \neq (i,i) \\ id_{C_i} & \qquad \text{else} \end{cases}$$

and

$$\tilde{h}(L_{i,j})_{l,k} = \sigma_l^{-1}(h(P_l \circ L_{i,j}) \circ \sigma_k(id_k)) = \begin{cases} id_{C_i} & \text{if} \quad l = k \text{ and } l \neq i,j \\ \sigma_l^{-1}(h_l)h(L_{i,j})\sigma_k(id_k)) & (l,k) = (i,j) \text{ or } (j,i) \\ 0 & \text{else} \end{cases}$$

Because $I_{1,j} = I_{1,i}I_{i,j}$ and $I_{i,j} = I_{j,i}$, the action of the matrices P_i and $L_{i,j}$ is compatible with f. Since the matrices P_i and $L_{i,j}$ generate $M_n(\mathbb{K})$ and these matrices are compatible with f, the morphism f is indeed an isomorphism in $\mathcal{C}^{M_n(\mathcal{C})}$. □

Let $\mathcal{C}$ denote a pseudo-abelian Banach category. The isomorphisms $C^{l,m+8} \cong C^{l+8,m} \cong M_{16}(C^{l,m})$ and $C^{l+1,m+1} \cong M_2(C^{l,m})$, together with the lemmas above now imply that $\mathcal{C}_{\mathbb{R}}^{l,m} \simeq \mathcal{C}_{\mathbb{R}}^{p,q}$, if $l - m = p - q$ mod 8. Similarly, we obtain $\mathcal{C}_{\mathbb{C}}^{l,m} \simeq \mathcal{C}_{\mathbb{C}}^{p,q}$ if $l - m = p - q$ mod 2. This is the periodicity we hinted at earlier. If we specialize to the case where $\mathcal{C} = \mathcal{E}_G^{\mathbb{K}}(X)$, we can use Lemma 7.5 and 7.7 to obtain Table 2, where $\mathcal{E}_G^{\mathbb{H}}(X)$ denotes the category of quaternionic

$l - m$ mod 8	$(\mathcal{E}_G^{\mathbb{R}}(X))^{l,m}$	$(\mathcal{E}_G^{\mathbb{C}}(X))^{l,m}$
0	$\mathcal{E}_G^{\mathbb{R}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X)$
1	$\mathcal{E}_G^{\mathbb{C}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X) \times \mathcal{E}_G^{\mathbb{C}}(X)$
2	$\mathcal{E}_G^{\mathbb{H}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X)$
3	$\mathcal{E}_G^{\mathbb{H}}(X) \times \mathcal{E}_G^{\mathbb{H}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X) \times \mathcal{E}_G^{\mathbb{C}}(X)$
4	$\mathcal{E}_G^{\mathbb{H}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X)$
5	$\mathcal{E}_G^{\mathbb{C}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X) \times \mathcal{E}_G^{\mathbb{C}}(X)$
6	$\mathcal{E}_G^{\mathbb{R}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X)$
7	$\mathcal{E}_G^{\mathbb{R}}(X) \times \mathcal{E}_G^{\mathbb{R}}(X)$	$\mathcal{E}_G^{\mathbb{C}}(X) \times \mathcal{E}_G^{\mathbb{C}}(X)$

Table 2: The categories $(\mathcal{E}_G^{\mathbb{R}}(X))^{l,m}$ and $(\mathcal{E}_G^{\mathbb{C}}(X))^{l,m}$

G-vector bundles, where the objects and morphism are defined as in Definition 2.37.
We will now explore some of the consequences of these equivalences in the case $\mathcal{C} = \mathcal{E}_G^{\mathbb{K}}(X)$.

Remark 7.8. Let A and B be $\mathbb{K}$ algebras. The equivalence preserves surjective G-vector bundle morphisms in the following way:

(i) A surjective morphism $f \in (\mathcal{E}_G^{\mathbb{K}}(X))^{A \otimes_{\mathbb{K}} B}$ corresponds to a surjective morphism $f \in ((\mathcal{E}_G^{\mathbb{K}}(X))^A)^B$ and vice versa.

(ii) A surjective morphism $f \in (E_G^{\mathbb{K}}(X))^{A \oplus B}$ corresponds to a pair of surjective morphisms $(f_1, f_2) \in (\mathcal{E}_G^{\mathbb{K}}(X))^A \times (\mathcal{E}_G^{\mathbb{K}}(X))^B$ and vice versa.

(iii) A surjective morphism $f \in (\mathcal{E}_G^{\mathbb{K}}(X))$ corresponds to the morphism $\oplus_{i=1}^n f \in (\mathcal{E}_G^{\mathbb{K}}(X))^{M_n(\mathbb{K})}$, which is surjective. Similarly, a morphism $f \in (\mathcal{E}_G^{\mathbb{K}}(X))^{M_n(\mathbb{K})}$ can, after possibly pre and post composing with an isomorphism, be viewed as a morphism $\oplus_{i=1}^n \tilde{f}$ and hence corresponds to a surjective morphism $\tilde{f}$ in $\mathcal{E}_G^{\mathbb{K}}(X)$.

Lemma 7.9. *Let $(E, \sigma) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$. Then, there exists a G-module M and a $(E^\perp, \sigma^\perp) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ such that*

$$(E \oplus E^\perp, \sigma \oplus \sigma^\perp \cong X \times ((M \otimes C^{l,m}), \rho),$$

where $\rho(x)(m \otimes c) = (m \otimes xc)$ and $g(m \otimes c) = (gm \otimes c)$, for all $x \in C^{l,m}$ and $g \in G$.

Proof. First, notice that since $E \in \mathcal{E}_G^{\mathbb{K}}(X)$, there exists a G-vector bundle $F \in \mathcal{E}_G^{\mathbb{K}}(X)$ and a G-module M, such that

$$E \oplus F \cong X \times M.$$

Let $\Phi : E \oplus F \to X \times M$ denote the isomorphism and let $p : E \oplus F \to E$ denote the projection. The map $\pi : (X \times (M \otimes C^{l,m}), \rho) \to (E, \sigma)$, defined by

$$\pi(m \otimes c) = \sigma(c)p(\Phi(x, m))$$

is a surjective morphism in $(\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$. Remark 7.8 implies that the equivalences from Lemma 7.5 and 7.7 preserve surjective vector bundle morphisms. Since $(\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ is equivalent to one of the categories in Table 2, via these equivalences, the morphism π corresponds to a surjective morphism or a pair of surjective morphisms in $\mathcal{E}_G^{\mathbb{R}}(X)$, $\mathcal{E}_G^{\mathbb{C}}(X)$ or $\mathcal{E}_G^{\mathbb{H}}(X)$. To keep the exposition simple, we assume that the morphism $\pi : (X \times (M \otimes C^{l,m}), \rho) \to E$ corresponds to a pair surjective of morphisms $(\pi_1, \pi_2) : (F_1, F_2) \to (L_1, L_2)$ in $\mathcal{E}_G^{\mathbb{H}}(X)$. The proof for the other cases is similar. Remark 2.44 together with Remark 2.36 implies that there exists $L_1^\perp, L_2^\perp$ in $\mathcal{E}_G^{\mathbb{H}}(X)$ such that

$$(L_1, L_2) \oplus (L_1^\perp, L_2^\perp) = (L_1 \oplus L_1^\perp, L_2 \oplus L_2^\perp) \cong (F_1, F_2).$$

Since an equivalence of categories preserves and reflects limits and co-limits. This implies that there exists a $L \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ such that $(E \oplus L, \sigma \oplus \sigma_L) \cong ((M \otimes C^{l,m}), \rho)$. $\square$

If G is a finite group, we can slightly improve this lemma.

Lemma 7.10. *Let G be a finite group. Let $(E, \sigma) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$. Then, there exists a $(E^\perp, \sigma^\perp) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ such that*

$$(E \oplus E^\perp, \sigma \oplus \sigma^\perp \cong X \times (\bigoplus_{i=1}^{n} \mathbb{K}[G]) \otimes C^{l,m}).$$

Proof. Let $\pi : (X \times (M \otimes C^{l,m}), \rho) \to (E, \sigma)$ be the morphism in $(\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ from the previous lemma. Lemma A.43 says that there exists a surjective G-module morphism $\Phi : \bigoplus_{i=1}^n G \to M$. Notice that the map $P : X \times (\bigoplus_{i=1}^n G) \otimes C^{l,m}) \to X \times (M \otimes C^{l,m}$, defined by

$$P(x, (g \otimes c)) = (x, \Phi(g) \otimes c),$$

is a surjective morphism in $(\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$. We can now proceed analogously to the previous lemma to obtain the desired result. $\square$

We are now ready to define the group $K^{l,m}(X)$.

Definition 7.11. *Let $\mathcal{C}$ be a Banach category and Let $i_{l,m} : \mathcal{C}_{\mathbb{K}}^{l,m+1} \to \mathcal{C}_{\mathbb{K}}^{l,m}$ be the functor that restricts the $C_{\mathbb{K}}^{l,m+1}$ actions on the objects of $\mathcal{C}$ to a $C^{l,m}$ action and maps a morphism to itself. We then define*

$$K^{l,m}(\mathcal{C}) := K(i_{l,m}).$$

If $\mathcal{C} = \mathcal{E}^{\mathbb{K}}_G(X)$, we will use the notation

$$K^{l,m}_G(X)_{\mathbb{K}} := K(i_{l,m} : (\mathcal{E}^{\mathbb{K}}_G(X))^{l,m+1} \to (\mathcal{E}^{\mathbb{K}}_G(X))^{l,m})$$

and will often omit the $\mathbb{K}$ from the notation.

Remark 7.12. Because for objects $C, D \in \mathcal{C}$ the Banach structure on $\mathrm{Hom}_{\mathcal{C}^{l,m+1}}((C,f),(D,g))$ and $\mathrm{Hom}_{\mathcal{C}^{l,m}}((C, i_{l,m}(f)),(D, i_{l,m}(g)))$ is obtained by restricting the Banach structure on $\mathrm{Hom}_{\mathcal{C}}(C,D)$, the functor $i_{l,m}$ is a Banach functor, which implies that the group $K(i_{l,m})$ is well defined.

Remark 7.13. If $\mathcal{C} = \mathcal{E}^{\mathbb{K}}_G(X)$, then we claim that $i_{p,q}$ is quasi surjective. The claim holds, because if $E \in (\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$, then there is a $E^{\perp} \in (\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$ such that $E \oplus E^{\perp} \cong X \times ((M \otimes C^{l,m})$. Notice that the projection $\pi : X \times (M \otimes C^{l,m+1}) \to X \times (M \otimes C^{l,m})$, where we view both bundles as an elements of $\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$, is a surjective morphism in $\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$. As before, this implies that there is a surjective morphism $\pi' : X \times (M \otimes C^{l,m+1}) \to E$ in $\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$ and hence that there exists a $F \in \mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$ such that

$$E \oplus F \cong X \times (M \otimes C^{l,m+1}).$$

Thus, the functor $i_{l,m}$ is quasi-surjective.

The periodicity of the categories extends to the groups $K^{l,m}$:

Proposition 7.14. *Let $\mathcal{C}$ be a pseuso-abelian category, then we have*

$$K^{l,m}(\mathcal{C}) \cong K^{p,q}(\mathcal{C}),$$

if $l - m = p - q$ mod 8 (or mod2) in the complex case.

Proof. We will prove the proposition for the real case. The proof for the complex case is similar. First, notice that it is sufficient to show that $K^{l+8,m}(\mathcal{C}) \cong K^{l,m}(\mathcal{C})$, $K^{l,m+8}(\mathcal{C}) \cong K^{l,m}(\mathcal{C})$ and $K^{l+1,m+1}(\mathcal{C}) \cong K^{l,m}(\mathcal{C})$. We will show that $K^{l+8,m}(\mathcal{C}) \cong K^{l,m}(\mathcal{C})$. Let $F_{l,m} : \mathcal{C}^{l,m} \to \mathcal{C}^{l+8,m}$ denote the equivalence from Lemma 7.7 combined with Lemma 7.5 (i) . We claim the map $F : K^{l,m}(\mathcal{C}) \to K^{l+8,m}(\mathcal{C})$ defined by

$$F([C_+, C_-, \alpha]) = [F_{l,m+1}(C_+), F_{l,m+1}(C_-), F_{l,m}(\alpha)].$$

Is an isomorphism. We first show that F is well defined. Notice that $F_{l,m}(\alpha)$ is a map from $F_{l,m}(i_{l,m}(C_+))$ to $F_{l,m}(i_{l,m}(C_-)$. We need that it is a map from $i_{l+8,m}(F_{l,m+1}(C_+))$ to $i_{l+8,m}(F_{l,m+1}(C_+))$. This holds if $i_{l+8} \circ F_{l,m+1} = F_{l,m} \circ i_{l,m}$. We will now show that this is the case. The functor $F_{l,m+1}$ sends an object (C, σ) to the object

$$(\oplus_{i=1}^{16} C, \sigma \otimes \rho),$$

where $\rho : M_{16}(\mathbb{R}) \to \mathrm{End}_{\mathcal{C}}(\oplus_{i=1}^{16} C)$ is the map that sends a matrix to the endomorphism associated to the matrix and $\sigma \otimes \rho : C^{l,m+1} \otimes M_{16}(\mathbb{R}) \to \mathrm{End}_{\mathcal{C}}(\oplus_{i=1}^{16} C,)$ is defined by $\sigma \otimes \rho(a \otimes b) = \sigma(a) \oplus \ldots \oplus \sigma(a) \circ \rho(b)$. The element $e_{l+8+m+1}$ of $C^{l,m+1+8}$ corresponds to the element $e_1 \ldots e_8 \otimes e_{m+1}$ of $M_8(\mathbb{R}) \otimes C^{l,m+1}$. This implies that

$$i_{l+8,m}(\bigoplus_{i=1}^{n} C, \sigma \otimes \rho) = (\bigoplus_{i=1}^{16} C, \sigma|_{C^{l,m}} \otimes \rho = F_{l,m}(i_{l,m})(C, \sigma).$$

It is clear that the functors also coincide on morphism. Since $F_{l,m}$ is a Banach functor and an equivalence, it F preserves morphisms and maps elementary triples to elementary triples. Therefore, F is a well defined group homomorphism.
We now show that it is injective and surjective. We first proof infectivity. Assume that $F([C_+, C_-, \alpha]) = [F_{l,m+1}(C_+), F_{l,m+1}(C_-), F_{l,m}(\alpha)] = 0$. Then there exists elementary triples $[D, D, \beta]$ and $[D', D', \beta']$ such that $(C_+, C_-, \alpha) \oplus (D, D, \beta) \cong (D', D', \beta')$. Because $F_{l,m}$ is an equivalence, there are objects $E, E' \in \mathcal{C}^{l,m+1}$ and isomorphisms $f : D \to F_{l,m+1}(E)$ and $f' : D' \to F_{l,m+1}E'$ in $\mathcal{C}^{l+8,m+1}$. Therefore,

$$F(C_+, C_-, \alpha) \oplus (F_{l,m+1}(E), F_{l,m+1}(E), f^{-1}\beta f) \cong (F_{l,m+1}(E'), F_{l,m+1}(E'), f^{-1}\beta' f).$$

Since the morphisms between these objects in $(\mathcal{C}^{l,m})^{M_{16}(\mathbb{K})}$ are diagonal matrices with the same morphism on each entry on the diagonal, we can project to first coordinate to obtain

$$(C_+, C_-, \alpha) \oplus (E, E, (f^{-1}\beta f)_{1,1}) \cong (E', E', (f^{-1}\beta' f)_{1,1}),$$

where the last to triples are elementary triples.
We now prove surjectivity. If $[C_+, C_-, \alpha]$ is a triple in $K^{l+8,m+1}(\mathcal{C})$, then since $F_{l,m}$ is essentially surjective, there exists an object $D_+, D_- \in \mathcal{C}^{l,m+1}$ and an isomorphisms $f_\pm : C_\pm \to F_{l,m+1}(D_\pm)$. Then, we have

$$[C_+, C_-, \alpha] = [F_{l,m+1}(D_+), F_{l,m+1}(D_-), f_-\alpha \circ f_+^{-1}].$$

Since the morphism in $(\mathcal{C}^{l,m})^{M_n(\mathcal{C})}$ are diagonal matrices with the same morphism on each entry on the diagonal, we can project to the first coordinate to obtain

$$[F_{l,m+1}(D_+), F_{l,m+1}(D_-), f_- \circ \alpha \circ f_+^{-1}] = F([D_+, D_-, (f_-\alpha \circ f_+^{-1})_{1,1}].$$

$\square$

Before we study the groups $K^{l,m}(X)$ further, it might be good to explain why we are interested in these groups. Now that we have defined the groups $K_G^{p,q}(X)$, we will slightly generalise the definition to obtain a group $K_G^{p,q}(X, A)$, where A is a closed G-invariant subspace of X. These groups again have the property that $K_G^{p,q}(X, A) \cong K_G^{l,m}(X, A)$ if $p - q = l - m \bmod 8$ (or $\bmod\ 2$ in the complex case). We will then show that

$$K_G^{0,0}(X, A) \cong K_G(X, A) \tag{7.1}$$

and

$$K_G^{l,m+1}(X, A) \cong K_G^{l,m}(X \times [0,1], X \times \{0,1\} \cup A \times I). \tag{7.2}$$

Applying this equation repeatedly, we obtain the equation:

$$K_G^{l,m}(X, A) \cong K_G^{l,0}(X \times [0,1]^m, X \times \partial[0,1]^{m-1} \cup A \times [0,1]^m) \tag{7.3}$$

These equation and the following lemma will then enable us to prove the periodicity of the groups $K_G^{-n}(X, A)$.

Lemma 7.15. *Let X be a compact G-space and Y a closed G-invariant subset. We have:*

$$K_G(X \times [0,1]^n, X \times \partial[0,1]^{n-1} \cup A \times [0,1]^n)) \cong K_G^{-n}(X, A).$$

Proof. Remark 3.31 implies that it is sufficient to show that $\Sigma^n(X^+)/\Sigma^n(A^+)$ is G-homotopy equivalent to

$$Y := X^+ \times [0,1]^n/(X^+ \times \partial[0,1]^{m-1} \cup A^+ \times [0,1]^n).$$

Notice that the map

$$f : Y \to \Sigma^n(X^+)/\Sigma^n(A^+),$$

defined by $f([x,v]) = [x,v]$ is a G-isomorphisms, which proves the Lemma. □

Theorem 7.16. *Let X be a compact G-space and $A \subset X$ a closed G-invariant subset. We have the following group isomorphism*

$$(K_G^{\mathbb{R}})^{-n}(X,Y) \cong K_G^{0,n}(X,Y)_{\mathbb{R}} \qquad \textit{and} \qquad (K_G^{\mathbb{C}})^{-n}(X,Y) \cong K_G^{0,n}(X,Y)_{\mathbb{C}}.$$

Proof. Equation 7.3 together with Lemma 7.15 implies that

$$\begin{aligned} K_G^{0,n}(X,A) &\cong K^{0,0}(X \times [0,1]^m, X \times \partial[0,1]^{m-1} \cup A \times [0,1]^m) \\ &\cong K_G((X \times [0,1]^m, X \times \partial[0,1]^{m-1} \cup A \times [0,1]^m) \\ &\cong K_G^{-n}(X,Y). \end{aligned}$$

□

Remark 7.17. A direct consequence of this theorem and the periodicity of the groups $K_G^{l,m}(X,Y)$ is that

$$K^{l,m}(X,Y) \cong K_G^{l-m \bmod 8}(X,Y)$$

(or mod 2 in the complex case) and that

$$(K_G^{\mathbb{R}})^{-n}(X,Y) \cong (K_G^{\mathbb{R}})^{-n-8}(X,Y) \qquad \text{and} \qquad (K_G^{\mathbb{C}})^{-n}(X,Y) \cong (K_G^{\mathbb{C}})^{-n-2}(X,Y).$$

Example 7.18. Let $X = \{\mathrm{pt}\}$. We have

$$(K_G^{\mathbb{C}})^{-n}(X,\emptyset) = (\tilde{K}_G^{\mathbb{C}})^{-n}(X^+) = \tilde{K}_G^{\mathbb{C}}(\Sigma^n(X^+)) \cong \tilde{K}_G^{\mathbb{C}}(S^n).$$

where $S^n := \{x \in \mathbb{R}^{n+1} \mid \|X\| = 1\}$ is viewed as a G-space with a trivial action. We already showed that

$$\tilde{K}_G^{\mathbb{C}}(X^+) \cong K_G^{\mathbb{C}}(\mathrm{pt}) \cong R(G)$$

and

$$\tilde{K}_G^{\mathbb{C}}(\Sigma\,\mathrm{pt}^+) \cong K^{-1}(\mathcal{E}_G^{\mathbb{C}}(\mathrm{pt})) \cong 0.$$

Remark 7.17 now implies that

$$\tilde{K}_G^{\mathbb{C}}(S^n) \cong \begin{cases} R(G) & \text{if} \quad n = 2k \\ 0 & \quad\;\; n = 2k+1 \end{cases}$$

8 Gradations

In this section, we will introduce gradations and will use them to define the groups $K_G^{l,m}(X,A)$ from the previous section.

The main results of this section will be that the groups$K_G^{l,m}(X,A)$ are periodic (Proposition 8.11, that

$$K_G^{l,m}(X,\emptyset) \cong K_G^{l,m}(X)$$

(Lemma 8.9) and the proof of Equation 7.1 (Theorem 8.16) which says that

$$K_G^{0,0}(X,A) \cong K_G(X,A).$$

This section is based on chapter III.4 and III.5 of [3] We start with a definition:

Definition 8.1. *Let $(E,f) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$. We call a morphism $h \in End_{(\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}}(E,f)$ a gradation on (E,f) if:*

(i) $f(e_i)h = -hf(e_i)$ for all $1 \leq i \leq l+m$

(ii) $h^2 = 1$

We will denote the set of gradations on (E,f) by $Gr^{l,m}(E,f)$ and endow it with the subspace topology.

Remark 8.2. Notice that a gradation is an extension of the $C^{l,m}$ structure on E into an $C^{l,m+1}$ structure.

We will be interested in triples $((E,f),\alpha_+,\alpha_-)$, where $(E,f) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ and $\alpha_1,\alpha_2 \in \mathrm{Gr}^{l,m}(E,f)$. We will define the sum of gradations by

$$((E_1,f_1),\alpha_+,\alpha_2) \oplus ((E_2,f_2),\beta_+,\beta_-) = ((E_1 \oplus E_2, f_1 \oplus f_2), \alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2).$$

If $\alpha_+,\alpha_- \in \mathrm{Grad}^{l,m}((E,f)$, then the notation $\alpha_+ \simeq \alpha_-$ will mean that α_+ is G-homotopic to α_- through gradations. We are now ready to define the relative group $K^{l,m}(X,A)$:

Definition 8.3. *Let X be a G-space and $A \subset X$ a closed G-invariant subset. We define:*

$$K^{l,m}(X,A) := \{((E,f),\alpha_+,\alpha_-) \mid (E,f) \in ob((\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}),\ \alpha_+,\alpha_- \in Gr^{l,m}(E,f) \text{ and } \alpha_+|_A = \alpha_-|_A\}/\sim.$$

where $((E,f),\alpha_+,\alpha_-) \sim ((E',f'),\beta_+,\beta_-)$ if there exists a (F,γ,γ), such that

$$\alpha_+ \oplus \beta_- \oplus \gamma \simeq \alpha_- \oplus \beta_+ \oplus \gamma,$$

relative to A (thus the G-homotopy H between the gradations has the property that $H_t|_{E\oplus E'\oplus F|_Y} = \alpha_+ \oplus \beta_- \oplus \gamma|_Y$ for all $t \in [0,1]$). We will denote the equivalence class of a triple $((E,f),\alpha_+,\alpha_-)$ by $[(E,f),\alpha_+,\alpha_-]$. The operation

$$[(E,f),\alpha_+,\alpha_-] \oplus [(E',f'),\beta_+,\beta_-] = [(E_1 \oplus E_2, f \oplus f'), \alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2]$$

gives $K_G^{l,m}(X,A)$ the structure of an abelian group.

Lemma 8.4. *The set $K^{l,m}(X,A)$, together with the operation $\oplus$ is indeed a group.*

Proof. First, notice that a triple $[E, \alpha_+, \alpha_-]$ is 0 if and only if there exists a triple (F, γ, γ) such that

$$\alpha_+ \oplus \gamma \simeq \alpha_- \oplus \gamma,$$

relative to A. Also notice that $\oplus$ is well defined and associative. We now show that every element has an inverse. Let $[E, \alpha_1, \alpha_2]$ be a triple. Then, we have

$$[E, \alpha_+, \alpha_-] \oplus [E, \alpha_+, \alpha_-] = [E \oplus E, \alpha_+ \oplus \alpha_-, \alpha_- \oplus \alpha_+].$$

Notice that the homotopy $H : [0, 1] \to \mathrm{Gr}^{l,m}(E \oplus E)$

$$H(\theta) := \begin{pmatrix} \cos(\frac{\pi}{2}\theta) & -\sin(\frac{\pi}{2}\theta) \\ \sin(\frac{\pi}{2}\theta & \cos(\frac{\pi}{2}) \end{pmatrix} (\alpha_1 \oplus \alpha_2) \begin{pmatrix} \cos(\frac{\pi}{2}\theta) & \sin(\frac{\pi}{2}\theta) \\ -\sin(\frac{\pi}{2}\theta & \cos(\frac{\pi}{2}) \end{pmatrix}. \tag{8.1}$$

is a homotopy through gradations and since $\alpha_+|_{E|_A} = \alpha_-|_{E|_A}$, it follows that $H(t)|_{E|_A} = \alpha_+ \oplus \alpha_-|_A = \alpha_- \oplus \alpha_+|_A$ and thus that it is a homotopy relative to A. Therefore,

$$[E \oplus E, \alpha_+ \oplus \alpha_-, \alpha_- \oplus \alpha_+] = 0.$$

Lastly, we show that $\oplus$ is commutative. Let $x := [E, \alpha_+, \alpha_-], y := [F, \beta_+, \beta_-] \in K^{l,m}(X, Y)$. Then, we have

$$(x \oplus y) \oplus (y + x)^{-1} = [E \oplus F \oplus F \oplus E, \alpha_+ \oplus \beta_+ \oplus \beta_- \oplus \alpha_-, \alpha_- \oplus \beta_- \oplus \beta_+ \oplus \alpha_+].$$

Using the homotopy from Equation 8.1 twice, we obtain:

$$\alpha_+ \oplus \beta_+ \oplus \beta_- \oplus \alpha_- \simeq \alpha_- \oplus \beta_+ \oplus \beta_- \oplus \alpha_+ \simeq \alpha_- \oplus \beta_- \oplus \beta_+ \oplus \alpha_+.$$

Therefore, we have $(x \oplus y) \oplus (y \oplus x)^{-1} = 0$ and $\oplus$ is commutative. $\square$

Remark 8.5. If $f : (Y, B) \to (X, A)$ is a G-map, then f induces a map $f^* : K_G^{l,m}(X, A) \to K_G^{l,m}(Y, B)$, defined by

$$f^*([E, \alpha_+, \alpha_-]) = (f^*E, f^*\alpha_+, f^*\alpha_-),$$

where $f^*\alpha_\pm : f^*E \to f^*E$ is defined by $f^*\alpha|_{f*E|_y} = \alpha|_{f(y)}$. Therefore, the assignment $(X, Y) \to K_G^{l,m}(X, Y)$ is a contravariant functor.

There is also a notion of an isomorphism between triples:

Definition 8.6. *Let $x := ((E, f), \alpha_+, \alpha_-)$ and $y := ((F, h), \beta_+, \beta_-)$ be triples, with $(E, f), (F, h)) \in (\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}$ an isomorphism $\phi \in \mathrm{Hom}_{(\mathcal{E}_G^{\mathbb{K}}(X))^{l,m}}((E, f), (F, h))$ is an isomorphism between x and y if $\phi \circ \alpha_\pm = \beta_\pm \circ \phi$.*

Lemma 8.7. *Let $x = ((E, f), \alpha_+, \alpha_-)$,$y = ((F, h), \beta_+, \beta_-)$ and ϕ be defined as above, and assume that $[x], [y] \in K^{l,m}(X, A)$. then*

$$[(E, f), \alpha_+, \alpha_-] = [(F, h), \beta_+, \beta_-]$$

Proof. The proof of the Lemma is somewhat similar to the proof Lemma 5.10. Notice that

$$[E, \alpha_+, \alpha_-] \oplus [F, \beta_-, \beta_+] = [E \oplus F, \alpha_+ \oplus \beta_-, \alpha_- \oplus \beta_+].$$

We have $\beta_\pm = \phi \circ \alpha_\pm \circ \phi^{-1}$. Equation 5.4 implies that

$$\alpha_+ \oplus \phi \circ \alpha_- \circ \phi^{-1} = \begin{pmatrix} 0 & -\alpha_+\phi^{-1} \\ \phi\alpha_- & 0 \end{pmatrix} \begin{pmatrix} 0 & \phi^{-1} \\ -\phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\phi^{-1} \\ \phi & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_- & 0 \\ 0 & \phi \circ \alpha_+ \circ \phi \end{pmatrix} \begin{pmatrix} 0 & \phi^{-1} \\ -\phi & 0 \end{pmatrix}.$$

The leftmost matrix in the final equation is homotopic to the identity via the homotopy $H(t)$ from equation 5.3, with α replaced by ϕ. One can check that the map

$$\tilde{H}(t) := H(t) \circ (\alpha_- \oplus \beta_+) \circ (H(t))^{-1},$$

is a homotopy from $\alpha_- \oplus \beta_+$ to $\alpha_+ \oplus \beta_-$ relative to A and through gradations. $\square$

We will now show that $K_G^{l,m}(X, \emptyset) \cong K_G^{l,m}(X)$. To show this, we need the following lemma:

Lemma 8.8. *Let $(E, f) \in (\mathcal{E}_G(X))^{l,m}$ and $\alpha_+, \alpha_- \in Gr^{l,m}(E, f)$. If $\alpha_+ \simeq \alpha_-$ via a homotopy $H : I \to Gr^{l,m}(E, f)$, then there exists a map $\tilde{H} : I \to Aut_{(\mathcal{E}_G(X))^{l,m}}(E, f)$ such that $\tilde{H}(0) = 1$ and $\tilde{H}(1)\alpha_+\tilde{H}(1)^{-1} = \alpha_-$.*

Proof. Let $h : [0, 1]^2 \to \mathrm{End}^{l,m}_{(\mathcal{E}_G(X))^{l,m}}(E, F)$ be the map defined by

$$h(t, s) = \frac{1 + H(t)H(s)}{2}.$$

Notice that $h(t, t) = id$ and is hence invertible for all $t \in [0, 1]$. Since h is continuous and the subset of automorphism is open, there exists an open set $\{(t, t) \mid t \in I\} \subset U \subset I^2$, such that $h(s, t)$ is invertible if $(s, t) \in U$. Since I^2 is compact, there exists an $\epsilon > 0$, such that $B((t, t), \epsilon) \subset U$ for all $t \in I$. Let $n = \lfloor \frac{2}{\epsilon} \rfloor$ and $x_i = i\frac{\epsilon}{2}$ for $1 \leq i \leq n$ and $x_{n+1} = 1$. Then, this implies that the map $\tilde{H} : I \to \mathrm{Aut}_{(\mathcal{E}_G(X))^{l,m}}(E, f)$ defined on $[x_i, x_{i+1}]$ by

$$\tilde{H}(t) = h(t, x_i)h(x_i, x_{i-1}) \ldots h(x_2, x_1)h(x_1, 0)$$

is a well defined automorphism. Since $h(t, s)H(s) = \frac{H(s)+H(t)}{2} = H(t)h(t, s)$ it follows that

$$\tilde{H}(1)\alpha_+\tilde{H}(1)^{-1} = \tilde{H}(1)H(0)\tilde{H}(1)^{-1} = H(1)\tilde{H}(1)\tilde{H}(1)^{-1} = \alpha_-.$$

$\square$

Proposition 8.9. *Let X be a compact G-space, Then,*

$$K_G^{l,m}(X, \emptyset) \cong K_G^{l,m}(X).$$

Proof. Let $\Phi : K_G^{l,m}(X) \to K_G^{l,m}(X, \emptyset)$ be the map defined by

$$\Phi([(E_+, \sigma_+), (E_-, \sigma_-), \alpha]) = [(E_+, \sigma|_{C^{l,m}}), \alpha^{-1}\sigma_+(e_{l+m+1}), \sigma_-(e_{l+m+1})\alpha].$$

We first check that Φ is well defined. Assume that $y := ((F\rho), (F, \rho), \beta)$ is an elementary triple, then and let $x := ((E_+, \sigma_+), (E_-, \sigma_-), \alpha)$. Then,

$$\Phi([x+y]) = [(E_+, \sigma|_{C^{l,m}}), \sigma_+(e_{l+m+1}), \alpha^{-1}\sigma_-(e_{l+m+1})\alpha] \oplus [(F, \rho|_{C^{l,m}}), \rho(e_{l+m+1}), \beta^{-1}\rho(e_{l+m+1})\beta].$$

Since β is homotopic to the identity through automorphisms in $\mathcal{E}_G(X)^{l,m}$, it follows that $\rho(e_{l+m+1}) \simeq \beta^{-1}\rho(e_{l+m+1})\beta$ and $[(F, \rho|_{C^{l,m}}), \rho(e_{l+m+1}), \beta^{-1}\rho(e_{l+m+1})\beta] = 0$. Now let x as before and assume that $y := ((F_+, \rho_+), (F_-, \rho_-), \alpha)$ is isomorphic to x. Then, there exist morphisms $f \in \mathrm{Hom}((E_+,\sigma_+),(F_+,\rho_+))$ and $g \in \mathrm{Hom}((E_-,\sigma_-),(F_-,\rho_-))$ such that $g \circ \alpha = \beta \circ f$. This implies that

$$\begin{aligned}
\Phi([y]) &= [(F_+, \rho_+|_{C^{l,m}}), \rho_+(e_{l+m+1}), \beta^{-1}\rho_+(e_{l,m+1})\beta] \\
&= [(F_+, \rho_+|_{C^{l,m}}), \rho_+(e_{l+m+1}), \beta^{-1}g\sigma_-(e_{l+m+1})g^{-1}\beta] \\
&= [(F_+, \rho_+|_{C^{l,m}}), f\sigma_+(e_{l+m+1})f^{-1}, f\alpha^{-1}\sigma_+(e_{l+m+1})\alpha f^{-1}] \\
&= [(F_+, \rho_+|_{C^{l,m}}), f\sigma_+(e_{l+m+1})f^{-1}, f\alpha^{-1}\sigma_+(e_{l+m+1})\alpha f^{-1}]
\end{aligned}$$

Therefore, Lemma 8.7 now implies that $\Phi([y]) = \Phi([x])$. The map Φ is therefore well defined. Also notice that Φ is indeed a group homomorphism. We now show that Φ is an isomorphism. We first show that Φ is surjective. Let $[(E,\sigma),\alpha_+,\alpha_-] \in K^{l,m}(X,\emptyset)$. Then

$$\Phi((E,\sigma_+),(E,\sigma_-),id) = [(E,\sigma),\alpha_+,\alpha_-],$$

where $\sigma_\pm$ is the $C^{l,m+1}$ structure induced by σ and $\alpha_\pm$.
We now show that Φ is injective. Assume that

$$\Phi([(E_+,\sigma_+),(E_-,\sigma_-),\alpha]) = [(E,\sigma_+|_{C^{l,m}}),\sigma_+(e_{l+m+1}),\alpha^{-1}\sigma_-(e_{l+m+1})\alpha] = 0.$$

Let $\tilde{\sigma}_+$ be the $C^{l,m+1}$ action on E_+ such that $\tilde{\sigma}_+|_{C^{l,m}} = \sigma_+|_{C^{l,m}}$ and $\tilde{\sigma}_+(e_{l+m+1}) = \alpha^{-1}\sigma_-(e_{l+m+1})\alpha$. Notice that $\alpha : (E_+,\tilde{\sigma_+}) \to (E_-,\sigma_-)$ is a morphism in $\mathcal{E}_G(X)^{l,m+1}$. Therefore, the diagram

$$\begin{array}{ccc}
(E_+,\sigma_+|_{C^{l,m}}) & \xrightarrow{id} & (E_+,\tilde{\sigma}_+|_{C^{l,m}}) \\
\Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \alpha} \\
(E_+,\sigma_+|_{C^{l,m}}) & \xrightarrow{\alpha} & (E_-,\sigma_-|_{C^{l,m}})
\end{array} .$$

commutes and

$$[(E_+,\sigma_+),(E_-,\sigma_-),\alpha] = [(E_+,\sigma_+),(E_+,\tilde{\sigma_+}),id].$$

We may thus assume that $\alpha = id$. Since

$$\Phi([(E_+,\sigma_+),(E_-,\sigma_-),\alpha]) = [(E,\sigma_+|_{C^{l,m}}),\sigma_+(e_{l+m+1}),\alpha^{-1}\sigma_-(e_{l+m+1})\alpha] = 0,$$

There exist a triple $((F,\rho),\beta,\beta)$ such that

$$\sigma_+(e_{l+m+1}) \oplus \beta \simeq (\alpha^{-1}\sigma_-(e_{l+m+1})\alpha) \oplus \beta.$$

Lemma 8.8 implies that there exists a map $h : I \to \mathrm{Aut}_{(\mathcal{E}_G(X))^{l,m}}(E_+ \oplus F, \sigma|_{C^{l,m}\oplus\rho})$ such that $h(0) = id$ and

$$h(1) \circ \sigma_+(e_{l+m+1}) \oplus \beta \circ h(1)^{-1} = (\alpha^{-1}\sigma_-(e_{l+m+1})\alpha) \oplus \beta.$$

This implies that $h(1)^{-1} : (E_- \oplus F, \sigma_- \oplus \beta) \to ((E_+ \oplus F), \sigma_+ \oplus \beta)$ is a morphism in $\mathcal{E}_G(X)^{l,m+1}$. Now notice that the diagram

$$\begin{array}{ccc}
(E_+ \oplus F, \sigma_+ \oplus \rho_\beta|_{C^{l,m}}) & \xrightarrow{\alpha} & (E_- \oplus F, \tilde{\sigma}_- \oplus \rho_\beta|_{C^{l,m}}) \\
\Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle h(1)^{-1}} \\
(E_+ \oplus F, \sigma_+ \oplus \rho_\beta|_{C^{l,m}}) & \xrightarrow{h(1)^{-1}\alpha} & (E_+ \oplus F, \sigma_+ \oplus \rho_\beta|_{C^{l,m}})
\end{array} .$$

commutes,where ρ_β denotes the $C^{l,m+1}$ action induce by ρ and β, which implies that

$$\begin{aligned}[(E_+,\sigma_+),(E_-,\sigma_-),\alpha] &= [(E_+,\sigma_+),(E_-,\sigma_-),\alpha]) \oplus [(F,\rho_\beta),(F,\rho_\beta),id] \\ &= [(E_+\oplus F,\sigma_+\oplus\rho_\beta\beta),(E_+,\sigma_+\oplus\rho_\beta),h(1)^{-1}] \\ &= 0.\end{aligned}$$

It is zero because $h(1)$ is homotopic to the identity through automorphisms. □

Remark 8.10. This proposition implies that the groups $K_G^{l,m}(X,\emptyset)^{\mathbb{K}} \cong K_G^{p,q}(X,\emptyset)$ if $l-m = p-q \mod 8$ if $\mathbb{K}=\mathbb{R}$ or $\mod 2$ if $\mathbb{K}=\mathbb{C}$.

The remark above also holds if we replace $\emptyset$ by a closed G-invariant subset $A\subset X$.

Proposition 8.11. *The groups $K_G^{l,m}(X,A)_{\mathbb{K}}$ and $K_G^{p,q}(X,A)$ are isomorphic if $l-m = p-q$ mod 8 if $\mathbb{K}=\mathbb{R}$ or mod 2 if $\mathbb{K}=\mathbb{C}$.*

Proof. As before, we will prove this theorem for $\mathbb{K}=\mathbb{R}$. It is sufficient to show that $K_G^{l,m}(X,A)\cong K_G^{l+8,m}(X,A)$, $K_G^{l,m}(X,A)\cong K_G^{l,m+8}(X,A)$ and $K_G^{l,m}(X,A)\cong K_G^{l+1,m+1}(X,A)$. We will show that $K_G^{l,m}(X,A)\cong K^{l+8,m}(X,A)$. The proof for the other cases is similar. We will define the isomorphism using the map of the previous proposition. Let $x := [(E,\sigma),\alpha_+,\alpha_-]\in K_G^{l,m}(X,A)$. By forgetting the 'extra' structure, the triple $[(E,\sigma),\alpha_+,\alpha_-]$ defines an element of $K_G^{l,m}(X,\emptyset)$. Using the isomorphism from Proposition 8.9, this element corresponds to the triple $[(E,\sigma_+),(E,\sigma_-),id]\in K_G(X)^{l,m}$, where $\sigma_\pm$ is the $C^{l,m+1}$ action induced by σ and $\alpha_\pm$. Using the isomorphism from Proposition 7.14 we see that this triple corresponds to the triple

$$[((\oplus_{i=1}^{16}E,\sigma_+\otimes\rho),(\oplus_{i=1}^{16}E,\sigma_+\otimes\rho),id]\in K_G(X)^{l,m},$$

where ρ sends a matrix $A\in M_{16}(\mathbb{R})$ to the corresponding isomorphism in $\oplus_{i=1}^{16}E$ and $\sigma_\pm\otimes\rho : C^{l+8,m+1}\otimes M_{16}(\mathbb{R})\to \mathrm{End}(\oplus_{i=1}^{16}E)$ is defined by

$$\sigma_\pm\otimes\rho(a,b)=\sigma_\pm(a)\oplus\ldots\oplus\sigma_\pm(a)\circ\rho(b).$$

Using again the isomorphism from Proposition 8.9, we see that this element corresponds to the triple

$$[(\oplus_{i=1}^{16}E,\sigma\otimes\rho),\alpha_+\oplus\ldots\oplus\alpha_+\circ\rho(e_1\ldots e_8),\alpha_-\oplus\ldots\oplus\alpha_-\circ\rho(e_1\ldots e_8)].$$

Notice that the gradations coincide on $\oplus_{i=1}^{16}E|_Y$ and this triple can thus be viewed as an element

$$[(\oplus_{i=1}^{16}E,\sigma\otimes\rho),\alpha_+\oplus\ldots\oplus\alpha_+\circ\rho(e_1\ldots e_8),\alpha_-\oplus\ldots\oplus\alpha_-\circ\rho(e_1\ldots e_8)]\in K^{l+8,m}(X,A).$$

Notice that the composition of all these identification is a well defined group homomorphism $\Phi : K^{l,m}(X,A)\to K^{l+8,m}(X,A)$. We now show that Φ is an isomorphism by defining an inverse. Let $[(F,\sigma),\beta_+,\beta_-]\in K^{l+8,m}$. The equivalence of Lemma 7.7 implies that $(F,\sigma)\cong(\oplus_{i=1}^{16}E,\tilde\sigma\otimes\rho)$ for some $(E,\tilde\sigma)\in E_G(X)^{l,m}$. Let $f:F\to\oplus_{i=1}^{16}E$ denote this isomorphism. Then, we have

$$[(F,\sigma),\beta_+,\beta_-]=[(\oplus_{i=1}^{16}E,\tilde\sigma\otimes\rho),f\beta_+f^{-1},f\beta_-f^{-1}].$$

As before, we can we view this triple as an element of $K_G^{l+8,m}(X, \emptyset)$ and hence as an element of $K_G^{l+8,m}(X)$ and use the isomorphisms from Lemma 7.14 and Proposition 8.9 to obtain the triple

$$[(E, \tilde{\sigma}), (\rho(e_1 \dots e_8) f \beta_+ f^{-1})_{1,1}, (\rho(e_1, \dots e_8)(f \beta_+ f^{-1})_{1,1}]$$

in $K_G^{l,m}(X, A)$. where the subscript refers to the entry of the matrix that represents the morphism $\rho(e_1 \dots e_8) f \beta_\pm f^{-1}$. Notice that this is a well defined group homomorphism. It is clear from the definition that this morphism is the inverse of Φ. □

We will end this section by showing that $K_G^{0,0}(X, Y) \cong K_G(X, Y)$. To show this, we will need to prove some lemmas:

Lemma 8.12. *Let $p : E \to X$ be a G-vector bundle and $Y \subset X$ a closed G-invariant subset. If $H : E|_Y \times [0,1] \to E|_Y$ is a homotopy between G-vector bundle isomorphisms and through G-vector bundle isomorphism such that $H_0 = id$, then there exists a homotopy $\tilde{H} : E \times [0,1] \to E$, such that $\tilde{H}_t$ is an isomorphisms for $t \in [0,1]$, $\tilde{H}_0 = id$ and $\tilde{H}_{Y \times [0,1]} = H$.*

Proof. Let $\pi_X : X \times [0,1] \to X$ denote the projection and let $h : \pi_X^* E|_{Y \times [0,1]} \times [0,1] \to \pi_X^* E|_{Y \times [0,1]}$ be the map defined by $h((v,s),t) = H(v, st)$. Notice that we can view h as a continuous map $h : [0,1] \to \mathrm{Aut}_{\mathcal{E}_G^{\mathbb{K}}(Y \times [0,1])}(\pi_X^* E|_{Y \times [0,1]})$. Let $f : [0,1]^2 \to \mathrm{Aut}_{\mathcal{E}_G^{\mathbb{K}}(Y \times [0,1])}(\pi_X^* E|_{Y \times [0,1]})$ be the map defined by $f(s,t) = h(s)^{-1} h(t)$. Notice that f is continuous and $f(t,t) = id$ for all $t \in [0,1]$. Let $U = \{(s,t) \in [0,1]^2 \mid \|id - \beta(s,t)\| < 1\}$. Notice that U is open. With a similar argument as in the proof of Lemma 8.8, we obtain $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1 \in [0,1]$ such that for all $0 \le i \le n$ and $t \in [t_i, t_{i+1}]$, we have $f(t_i, t) \in U$. Since $\|id - f(t_i, t_{i+1})\| \in U$ and $\mathrm{End}(\pi_X^* E|_{Y \times [0,1]}))$ is a Banach algebra, Example 4.17 implies that the function $\log(\cdot)$ is defined on $f(t_i, t_{i+1})$. Therefore, we have $\log(f(t_i, t_{i+1})) \in \mathrm{End}(\pi_X^* E|_{Y \times [0,1]})$. Using Lemma 2.26 and 2.21, we can extend this map to obtain a map $\alpha_i \in \mathrm{End}(\pi_X^* E)$. As mentioned in Example 4.17, the exponential function is also defined on $\mathrm{End}(\pi_X^* E)$. We now define:

$$\tilde{H} := \exp(\alpha_0) \dots \exp(\alpha_1).$$

Notice that

$$\begin{aligned}
\tilde{H}|_Y &:= \exp(\alpha_0)|_Y \dots \exp(\alpha_1)|_Y = \exp(\alpha_0|_Y) \dots \exp(\alpha_1|_Y) \\
&= \exp(\log(f(t_0, t_1)) \dots \exp(\log(f(t_n, t_{n+1})) = f(t_0, t_1) \dots f(t_n, t_{n+1}) \\
&= h(0)^{-1} h(1) = id \circ H \\
&= H
\end{aligned}$$

Therefore, the map $\pi_X^* \tilde{H}_0^{-1} \tilde{H}$ is the required map. □

Remark 8.13. Since $\mathrm{End}_{\mathcal{E}_G^{\mathbb{K}}(A)^{l,m}}((E, \sigma))$ is a Banach space and a sub algebra of $\mathrm{End}_{\mathcal{E}_G^{\mathbb{K}}(A)}(E)$ for any G-space A, it follows that if H is a homotopy through isomorphisms in $\mathcal{E}_G^{\mathbb{K}}(Y)^{l,m}$, the extension $\tilde{H}$ is a homotopy through isomorphisms in $\mathcal{E}_G^{\mathbb{K}}(X)^{l,m}$.

This lemma will enable us to prove that the group $K_G^{l,m}(X, A)$ fits into a short exact sequence:

Lemma 8.14. *The following sequence is an exact sequence:*

$$K_G^{l,m}(X, A) \xrightarrow{j} K_G^{l,m}(X) \xrightarrow{i_*} K_G^{l,m}(A) \ ,$$

where $j([E,\alpha_+,\alpha_-]) = [E,\alpha_+,\alpha_-]$*. Moreover, if* A *is a* G*-retract of* X*, then we obtain the short exact sequence:*

$$0 \longrightarrow K_G^{l,m}(X,A) \xrightarrow{j} K_G^{l,m}(X) \xrightarrow{i_*} K_G^{l,m}(A) \longrightarrow 0 \ .$$

Proof. The proof of this Lemma is quite long, so we will postpone it to the end of the section. □

Notice that this Lemma is very similar to Lemma 5.21. The following lemma is a reformulation of Proposition 3.30, but now for the groups $K^{p,q+1}(X,A)$.

Lemma 8.15. *Let* X *be a compact* G*-space,* $A \subset X$ *a closed* G*-invariant subspace and* $\pi : (X,A) \to (X/A, A/A)$ *the projection. The map*

$$\pi^* : K_G^{l,m}(X/A, A/A) \to K_G^{l,m}(X,A)$$

is an isomorphism.

Proof. We will prove the theorem by constructing an inverse. Let $([(E,\sigma),\alpha_+,\alpha_-] \in K_G^{l,m}(X,A)$. As discussed before, Lemma 7.9 implies that there exists a G-module M and a triple $[(E^\perp,\sigma^\perp),\gamma,\gamma]$ such that

$$[(E,\sigma),\alpha_+,\alpha_-] \oplus [(E^\perp,\sigma^\perp),\gamma,\gamma] = [(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}}), \rho_X(e_{l+m+1}), \beta)],$$

For some $\beta \in \mathrm{Gr}^{l,m}(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}})$. Notice that $i^*\beta = i^*\rho(e_{l+m+1})$, where $i : A \to X$ is the inclusion. We now define the inverse $j : K_G^{l,m}(X,A) \to K_G^{l,m}(X/A, A/A)$ by

$$j([(E,\sigma),\alpha_+,\alpha_-]) = [(X/A \times (M \otimes C^{l,m+1}), \rho_{X/A}|_{C^{l,m}}), \rho_{X/A}(e_{l,m+1}), \tilde{\beta}],$$

Where $\tilde{\beta}([x],v) = \beta(x,v)$. We leave it to the reader to verify that j is well-defined. It is clear from the definition that $\pi^* \circ j = id$. We now show that $j \circ \pi^* = id$. Let $[(E,\sigma),\alpha_+,\alpha_-] \in K_G^{l,m}(X/A,A)$. As before, we have

$$[(E,\sigma),\alpha_+,\alpha_-] = [(X/A \times (M \otimes C^{l,m+1}), \rho_{X/Y}|_{C^{l,m}}), \rho_{X/Y}(e_{l,m+1}), \beta],$$

Now notice that

$$\begin{aligned} j \circ \pi^*[(E,\sigma),\alpha_+,\alpha_-] &= j \circ \pi^*[(X/A \times (M \otimes C^{l,m+1}), \rho_{X/Y}|_{C^{l,m}}), \rho_{X/Y}(e_{l,m+1}), \beta] \\ &= j[[(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}}), \rho_X(e_{l,m+1}), \pi^*\beta] \\ &= [(X/A \times (M \otimes C^{l,m+1}), \rho_{X/Y}|_{C^{l,m}}), \rho_{X/Y}(e_{l,m+1}), \beta] \end{aligned}$$

□

With these lemmas in place, we are ready to prove the theorem:

Theorem 8.16. *There is a natural isomorphism*

$$\Phi : K_G^{0,0}(X,A) \to K_G(X,A).$$

Proof. Our approach will be the same as in the proof of Proposition 5.22 and Theorem 5.23. Thus, we will define a natural homomorphism $\Phi : K^{0,0}(X, A) \to K(X, A)$ and proof that it is an isomorphism if $A = \emptyset$. We can then proceed analogously to the proof of Theorem 5.23 to show that Φ is also an isomorphism if $A \neq \emptyset$.
We will first define Φ and show that Φ is natural. Let $[(E, \sigma), \alpha_+, \alpha_-] \in K^{0,0}(X, A)$. Let

$$P_\pm := \frac{1 + \alpha\pm}{2}.$$

Notice that $P_\pm^2 = P_\pm$ and $P_+|_{E|_A} = P_-|_{E|_A}$. We now define

$$\Phi([(E, \sigma), \alpha_+, \alpha_-]) = [P_+E, P_-E, \beta],$$

where $\beta : P_+E|_Y \to P_-E|_Y$ is the identity. Remark 3.13 implies that this map is well defined. Also notice that Φ is indeed a group homomorphism.

We now show that Φ is natural. Let $f : (Y, B) \to (X, A)$ be a G-map. By definition, we have

$$f^*([(E, \sigma), \alpha_+, \alpha_-]) = [(f^*E, f^*\sigma), f^*\alpha_+, f^*\alpha_-].$$

Notice that $\frac{1+f^*\alpha_\pm}{2} = f^*P_\pm$. Therefore, we have

$$\Phi([(f^*E, f^*\sigma), f^*\alpha_+, f^*\alpha_-]) = [(f^*P_+)f^*E, (f^*P_-)f^*E, f^*\beta].$$

The bundle $(f^*P_\pm)f^*E$ is canonically isomorphic to $f^*(PE_\pm)$. Therefore, we have

$$[(f^*P_+)f^*E, (f^*P_-)f^*E, \beta] = f^*[P_+E, P_-E, \beta].$$

We now complete the proof by showing that the map is an isomorphism if $A = \emptyset$. We will construct an inverse and show that this inverse is an isomorphism.

Let $h : K_G(X) \to K_G^{0,0}(X, \emptyset)$ be the map defined by

$$h(E_+ - E_-) = [E_+ \oplus E_-, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}],$$

We leave it to the reader to verify that h is well defined. We first show that $\Phi \circ h = id$. If we let $h(E_+ - E_-)$ as above, then the corresponding projections P_+ and P_- are

$$P_+ = \frac{1}{2}(id + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_- = \frac{1}{2}(id + \begin{pmatrix} -1 & 0 \\ 0 & -0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $P_+(E_+ \oplus E_-) = E_+$ and $P_-(E_+ \oplus E_-) = E_-$ and

$$\Phi \circ h(E_+ - E_-) = E_+ - E_-.$$

We now show that $h \circ \Phi([E, \alpha_+, \alpha_-]) = [E, \alpha_+, \alpha_-]$. By definition, we have

$$h \circ \Phi([E, \alpha_+, \alpha_-]) = [P_+E \oplus P_-E, id \oplus -id, -id \oplus id].$$

If we add the triple $[(1 - P_+)E \oplus (1 - P_-)E, -id, -id] = 0$ and then rearrange the terms, we obtain

$$\begin{aligned} h \circ \Phi([E, \alpha_+, \alpha_-]) &= [P_+E \oplus P_-E \oplus (1 - P_+)E \oplus (1 - P_-)E, id \oplus -id \oplus -id \oplus -id, -id \oplus id \oplus -id \oplus -id] \\ &= [P_+E \oplus (1 - P_+)E \oplus P_-E \oplus (1 - P_-)E, id \oplus -id \oplus -id \oplus -id, -id \oplus -id \oplus id \oplus -id] \\ &= [E \oplus E, \alpha_+ \oplus (-id), (-id) \oplus \alpha_-]. \end{aligned}$$

Where the last equality holds because $\alpha_\pm|_{P_\pm E} = id$ and $\alpha_\pm|_{1-P_\pm} = -id$. We claim that

$$[E \oplus E, \alpha_+ \oplus (-id), (-id) \oplus \alpha_-] = [E, \alpha_+, \alpha_-].$$

The claim holds because

$$\alpha_+ \oplus (-id) \oplus \alpha_- = A^{-1}(-id) \oplus \alpha_- \oplus \alpha_+ A,$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $\det(B) = 1$ the matrix B is homotopic to id through invertible matrices in $M_3(\mathbb{K})$, which implies that

$$\alpha_+ \oplus (-id) \oplus \alpha_- \simeq (-id) \oplus \alpha_- \oplus \alpha_+.$$

□

As promised, we end with the proof of Lemma 8.14.

Proof of Lemma 8.14. First, notice that since $\alpha_+|_{E|A} = \alpha_-|_{E|A}$, it follows that $i^*\alpha_+ = i^*\alpha_-$ and hence that

$$i^* \circ j([E, \alpha_+, \alpha_-]) = [i^*E, i^*\alpha_+, i^*\alpha_-] = 0.$$

We now show that $\mathrm{Ker}(i^*) \subset \mathrm{Im}(j)$. Let $[(E, \sigma), \alpha_+, \alpha_-] \in \mathrm{Ker}(i^*)$. By definition, this implies that there exists a $(F, \tau) \in \mathcal{E}_G^{l,m}(A)$ and a gradation γ on F such that

$$i^*(\alpha_+) \oplus \gamma \simeq i^*(\alpha_-) \oplus \gamma. \tag{8.2}$$

Proposition 8.9 implies that (F, τ) together with γ corresponds to an element of $(F, \tau_0) \in \mathcal{E}_G(A)^{l,m+1}$. Using Lemma 7.9 we see that there exists a $(F^\perp, \tau_0^\perp) \in \mathcal{E}_G(A)^{l,m+1}$ and a G-module M such that

$$(F \oplus F^\perp, \tau_0 \oplus \tau_0^\perp) \cong (A \times (M \otimes C^{l,m+1}), \rho_A).$$

Notice that

$$i^*(X \times (M \otimes C^{l,m+1}), \rho_X) = (A \times (M \otimes C^{l,m+1}), \rho_A).$$

Also notice that

$$\tau_0^\perp(e_{l+m+1}) \in \mathrm{Gr}_{l,m}(E^\perp, \tau^\perp|_{C^{l,m}})$$

and

$$\rho_X(e_{p+q+1}) \in \mathrm{Gr}_{l,m}(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}})).$$

If we define $\beta_\pm := \alpha_\pm \oplus \rho_X(e_{p+q+1})$, then we obtain

$$[(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}}), \beta_+, \beta_-] = [(E, \sigma), \alpha_+, \alpha_-]$$

and using Equation 8.2 we see that

$$i^*\beta_+ \simeq i^*\beta_-.$$

Lemma 8.7 implies that there exists a map $h : [0,1] \to \mathrm{Aut}_{\mathcal{E}_G^{\mathbb{K}}(A)^{l,m}}(A \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}})$, such that $h(0) = id$ and $h(1) i^*\beta_+ h(1)^{-1} = i^*\beta_-$ and Lemma 8.12 implies that this map extends to a map $\tilde{h} : [0,1] \to \mathrm{Aut}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}})$, with $\tilde{h}(0) = id$. Therefore, we have

$$[(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}}), \beta_+, \beta_-] = [(X \times (M \otimes C^{l,m+1}), \rho_X|_{C^{l,m}}), \tilde{h}(1)\beta_+\tilde{h}(1)^{-1}, \beta_-].$$

Since $i^*(\tilde{h}(1)\beta_+\tilde{h}(1)^{-1}) = h(1)i^*\beta_+h(1)^{-1} = \beta_-$, the last triple lies in the image of j.
We now show that the sequence is short exact if A is a G-retract of X.
We first show that i^* is surjective. Let $[E, \alpha_+, \alpha_-] \in K_G^{l,m}(X, A)$. Then, we have

$$i^*[r^*E, r^*\alpha_+, r^*\alpha_-] = [E, \alpha_+, \alpha_-].$$

We now show that j is injective. Assume that $j([(E,\sigma), \alpha_+, \alpha_-]) = 0$. Then, there exists a gradation γ over a $(F, \tau) \in \mathcal{E}_G(X)^{l,m}$ such that $\alpha_+ \oplus \gamma \simeq \alpha_- \oplus \gamma$. Lemma 7.9 implies that we can add a gradation $0 = [((E \oplus F)^\perp, \mu), \delta, \delta]$ to $[(E \oplus F, \sigma \oplus \tau), \alpha_+, \alpha_-]$, to obtain

$$[E, \alpha_+, \alpha_-] = [(X \times (M \otimes C^{p,q+1})), \rho|_{C^{p,q}}, \beta_+, \beta_-],$$

with $\beta_+ \simeq \beta_-$ and $\beta_+ = \rho(e_{p+q+1})$. We show that we also have that $\beta_+ \simeq \beta_-$ relative to A. Since $\beta_+ \simeq \beta_-$, there exists a homotopy $H : (X \times (M \otimes C^{p,q+1})) \times [0,1] \to X \times (M \otimes C^{p,q+1})$ such that

(i) $H_t \in \mathrm{Hom}_{\mathcal{E}_G(X)^{l,m}}((X \times (M \otimes C^{p,q+1}), (X \times (M \otimes C^{p,q+1}))$,

(ii) H_t is a G-vector bundle isomorphism for all $t \in [0,1]$,

(iii) $H_0 = id$,

(iv) $H_1\beta_+H_1^{-1} = \beta_-$.

Let $\tilde{H} : (X \times (M \otimes C^{p,q+1}) \times [0,1] \to (X \times (M \otimes C^{p,q+1})$ be the map defined by

$$\tilde{H}((x,v),t) = H(x,t)H((r(x),v),t)^{-1}.$$

The homotopy

$$t \to \tilde{H}_t\beta_+\tilde{H}^{-1}$$

is a homotopy between β_- and

$$H_1H((r(\cdot),\cdot),1)^{-1}\beta_+H((r(\cdot),\cdot),1)H_1^{-1} = H_1\beta_-H_1^{-1} = \beta_+.$$

Where

$$H((r(\cdot),\cdot),1)^{-1}\beta_+H((r(\cdot),\cdot),1) = \beta_+,$$

because $\beta_+ : X \times (M \otimes C^{p,q+1}) \to X \times (M \otimes C^{p,q+1})$ does not depend on X and the equality holds on the vector bundle restricted to Y. Notice that by construction, it is a homotopy relative to Y. □

9 Periodicity theorem

In this section we will finish the proof of Theorem 7.16. We will prove that

$$K_G^{l,m+1}(X,A) \cong K_G^{l,m}(X\times[0,1], X\times\{0,1\}\cup A\times I) \tag{9.1}$$

and describe the isomorphism explicitly. We will base the proof on the proof in chapter III.5 and III.6 of [3] of Theorem 5.10, but change it such that it applies to the equivariant case.

9.1 The group G is finite and $A=\emptyset$

We will start with the case that G is finite and $A=\emptyset$ We will prove that Equation 9.1 holds by showing that it follows from a more general statement about Banach algebras (Theorem 9.12). To do this, we must first obtain more convenient descriptions of $K_G^{l,m+1}(X,A)$ and $K_G^{l,m}(X\times[0,1], X\times\{0,1\}\cup A\times I)$. We will first provide another description of $K_G^{l,m+1}(X,A)$. To do this it will be convenient to have the following definition:

Definition 9.1. *Let X be a G-space. We call a sequence $((F_n,\sigma_n))_{n\in\mathbb{N}}$ in $\mathcal{E}_G^{\mathbb{K}}(X,A)^{l,m}$ a cofinal sequence if:*

(i) For all $n,k\in\mathbb{N}$, we have $(F_n,\sigma_n)\oplus(F_k,\sigma_k)=(F_{n+k},\sigma_n\oplus\sigma_k)$.

(ii) For all $(E,\tau)\in\mathcal{E}_G^{\mathbb{K}}(X,A)^{l,m}$, there exists a $k\in\mathbb{N}$ and a bundle $(E^\perp,\tau)$ such that

$$(E\oplus E^\perp,\tau\oplus\tau^\perp)\cong(F_k,\sigma_k).$$

Example 9.2. If G is a finite group, then Lemma 7.10 implies that the sequence $(F_n)_{n\in\mathbb{N}}$ defined by

$$F_n := \bigoplus_{i=1}^{n}\mathbb{K}[G]\otimes C^{l,m},$$

is a cofinal sequence.

Remark 9.3. If $((F_n,\sigma_n))_{n\in\mathbb{N}}$ is a cofinal sequence in $\mathcal{E}_G^{\mathbb{K}}(X)^{l,m+1}$, then the sequence

$$n\to((F_n,\sigma_n|_{C^{l,m}}),\sigma_n(e_{m+l+1}))$$

is a sequence of gradations with the property that

(i) For all $n,k\in\mathbb{N}$, we have $\sigma_n(e_{m+l+1})\oplus\sigma_k(e_{m+l+1})=\sigma_{n+k}(e_{m+l+1})$.

(ii) For each gradation $((E,\tau),\alpha)$, with $(E,\tau)\in\mathcal{E}_G^{\mathbb{K}}(X)$ and $\alpha\in\mathrm{Gr}^{l,m}(E,\tau)$, then there exists a $n\in\mathbb{N}$ and a gradation $\alpha^\perp$ on a $(E^\perp,\tau^\perp)\in\mathcal{E}_G^{\mathbb{K}}(X)^{l,m}$ such that

$$((E\oplus E^\perp,\tau\oplus\tau^\perp),\alpha\oplus\alpha^\perp)\cong((F_n,\sigma_n|_{C^{l,m}}),\sigma_n(e_{m+l+1})).$$

We will often denote $((F_n,\sigma_n|_{C^{l,m}}),\sigma_n(e_{m+l+1}))$ by $(F_n,\sigma_n(e_{l+m+1}))$.

The following lemma shows why a cofinal sequence is a useful notion to have:

Lemma 9.4. *Let* $[E, \alpha_+, \alpha_-] \in K_G^{l,m}(X, A)$ *and* $(F_n, \sigma_n)_{n \in N}$ *be a cofinal sequence in* $\mathcal{E}_G^{\mathbb{K}}(X)^{l,m+1}$. *Then, there exists an* $n \in \mathbb{N}$ *such that*

$$[E, \alpha_+, \alpha_-] = [F_n, \sigma_n(e_{l+m+1}), \beta],$$

for some $\beta \in Gr^{l,m}(F_n)$. *Moreover, we have*

$$[F_n, \sigma_n(e_{l+m+1}), \beta] = [F_k, \sigma_k(e_{l+m+1}), \gamma]$$

iff there exists an $p \in \mathbb{N}$ *such that*

$$\sigma_n(e_{l+m+1}) \oplus \gamma \oplus \sigma_p(e_{l+m+1}) \simeq \beta \oplus \sigma_{k+p}(e_{l+m+1}).$$

relative to A.

Proof. Remark 9.3 implies that there exists a gradation $(E^\perp, \alpha_+^\perp)$ and an $n \in \mathbb{N}$ such that

$$(E \oplus E^\perp, \alpha \oplus \alpha_+^\perp) \cong (F_n, \sigma_n(e_{l+m+1})).$$

It follows that

$$[E, \alpha_+, \alpha_-] = [E, \alpha_+, \alpha_-] + [E^\perp, \alpha_+^\perp, \alpha_+^\perp] = [F_n, \sigma_n(e_{l+m+1}), \alpha_- \oplus \alpha_+^\perp].$$

We now prove the second statement. Assume that

$$[F_n, \sigma_n(e_{l+m+1}), \beta] = [F_k, \sigma_k(e_{l+m+1}), \gamma].$$

By definition, there exists a gradation (L, δ) such that

$$\sigma_n(e_{l+m+1}) \oplus \gamma \oplus \delta \simeq \beta \oplus \sigma_k(e_{l+m+1}) \oplus \delta. \tag{9.2}$$

As before, there exists a gradation $(L^\perp, \delta^\perp)$ and a $p \in \mathbb{N}$ such that

$$(L \oplus L^\perp, \delta \oplus \delta^\perp) \cong (F_p, \sigma_p(e_{l+m+1})).$$

If we add $\delta^\perp$ to equation 9.2, we obtain

$$\sigma_n(e_{l+m+1}) \oplus \gamma \oplus \sigma_p(e_{l+m+1}) \simeq \beta \oplus \sigma_{k+p}(e_{l+m+1}).$$

□

Let $(F_n)_{n \in \mathbb{N}}$ denote a cofinal row in $\mathcal{E}_G^{l,m+1}(X)$, then we define the topological space

$$\mathrm{Gr}^{l,m}((F_n)_{n \in \mathbb{N}}) := \mathrm{colim}_{n \in \mathbb{N}} \mathrm{Gr}^{l,m}(F_n, \sigma_n(e_{l,m+1})),$$

where the inclusion $\mathrm{Gr}^{l,m}(F_n) \to \mathrm{Gr}^{l,m}(F_{n+1})$ is given by $\alpha \to \alpha \oplus \sigma_1(e_{l+m+1})$. We choose $\sigma(e_{l+m+1}) := \sigma_1(e_{l+m+1}) \in \mathrm{Gr}^{l,m}((F_n)_{n \in \mathbb{N}})$ as a base point for $\pi_0(\mathrm{Gr}^{l,m}((F_n)_{n \in \mathbb{N}}))$ and give $\pi_0(\mathrm{Gr}^{l,m})((F_n)_{n \in \mathbb{N}})$ the structure of a commutative monoid by endowing it with the addition $([f] \oplus [g]) = [f \oplus g]$. More precisely, the addition is defined by choosing a representatives $f \in \mathrm{Gr}^{l,m}(F_k)$ and $g \in \mathrm{Gr}^{l,m}(F_n)$ for some $k, n \in \mathbb{N}$ and map it to the class of the direct sum $f \oplus g \in \mathrm{Gr}^{l,m}(F_{k+n}) \subset \mathrm{Gr}^{l,m}((F_n)_{n \in \mathbb{N}})$. This is well defined, because we have

$$(A \oplus \pm id_{F_1}) f \oplus \sigma_1(e_{l+m+1}) \oplus g \oplus \sigma_1(e_{l+m+1})(A^{-1} \oplus \pm id_{F_1}) = f \oplus g \oplus \sigma_1(e_{l+m+1}) \oplus \sigma_1(e_{l+m+1}).$$

for some invertible matrix $A \in M_{n+k+1}(\mathbb{K})$. Since either $\det(A \oplus id)$ or $\det(A \oplus -id)$ is positive, we may assume that

$$B(f \oplus \sigma_1(e_{l+m+1}) \oplus g \oplus \sigma_1(e_{l+m+1}))B^{-1} = f \oplus g \oplus \sigma_1(e_{l+m+1}) \oplus \sigma_1(e_{l+m+1}),$$

for an invertible matrix $B \in M_{k+n+2}(\mathbb{K})$ with $\det(B) = 1$. Since $\det(B) = 1$. It follows that B is homotopic the identity through invertible matrices, which implies that

$$f \oplus \sigma_1(e_{l+m+1}) \oplus g \oplus \sigma_1(e_{l+m+1}) \simeq f \oplus g \oplus \sigma_1(e_{l+m+1}) \oplus \sigma_1(e_{l+m+1}).$$

Theorem 9.5. *Let $(F_n)_{n\in\mathbb{N}}$ be a cofinal sequence in $\mathcal{E}_G^{l,m+1}(X)$. The map $\Phi : \pi_0(Gr^{l,m}((F_n)_{n\in\mathbb{N}}) \to K_G^{l,m}(X,)$ defined by*

$$\Phi([f]) = [F_k, \sigma_k(e_{l,m+1}), f_k],$$

for some $k \in \mathbb{N}$, where $f_k \in Gr^{l,m}(F_k) \subset Gr^{l,m}((F_n)_{n\in\mathbb{N}})$ is a representative of f, is a monoid isomorphism and $(\pi_0(Gr^{l,m}((F_n)_{n\in\mathbb{N}}), \oplus)$ is an abelian group.

Proof. Since $\mathrm{Gr}^{l,m}((F_n)_{n\in\mathbb{N}})$ is Hausdorff, the image of a map $h : [0,1] \to \mathrm{Gr}^{l,m}((F_n)_{n\in\mathbb{N}})$ is contained in $\mathrm{Gr}^{l,m}(F_k)$ for some k depending on h. Lemma 9.4 now implies that Φ is well defined. Also notice that Φ is indeed a monoid homomorphism. We now show that Φ is surjective. Let $[E, \alpha_+, \alpha_-] \in K^{l,m}(X,)$. By Lemma 9.4, there exists a $k \in \mathbb{N}$ and a $\beta \in \mathrm{Gr}^{l,m}(F_k)$ such that

$$[E, \alpha_+, \alpha_-] = [F_k, \sigma_k(e_{l+m+1}), \beta].$$

Therefore, we have

$$\Phi(\beta) = [E, \alpha_+, \alpha_-].$$

Lastly, we show that Φ is injective. Assume that

$$\Phi(\beta) = [F_k, \sigma_k(e_{l+m+1}), \beta].$$

Lemma 9.4 implies that there exists a $n \in \mathbb{N}$, such that

$$\beta \oplus \sigma_n(e_{l+m+1}) \simeq \sigma_{n+k}(e_{l+m+1}).$$

This homotopy is a homotopy between β and $\sigma_{n+k}(e_{l+m+1})$ in $\mathrm{Gr}^{l,m}((F_n)_{n\in\mathbb{N}})$. Thus,

$$[\beta] = [\sigma_{n+k}(e_{l+m+1})] = 0.$$

□

This theorem gives a nice description of the group $K_G^{l,m}(X,)$ provided that we can find a cofinal sequence. We have shown in Example 9.2 that this is possible if G is finite. We will now choose a cofinal sequence $(F_n)_{n\in\mathbb{N}}$ to get a better description of $\mathrm{Gr}^{l,m+1}((F_n)_{n\in\mathbb{N}})$. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}_G^{l,m+2}(X)$, where $F_n := (X \times (\oplus_{i=1}^n \mathbb{K}[G] \otimes C^{l,m+2}, \rho_n)$, where ρ_n is the $C^{l,m+2}$ action as defined in Lemma 7.9. Remark 7.13 implies that (F_n, ρ_n) is a cofinal row. It will be useful to also consider the cofinal row $(\tilde{F}_n, \tilde{\rho}_n)|_{n\in\mathbb{N}}$ defined by $F_1 = (F_1 \oplus F_1, \rho_1 \oplus \rho_1')$, where $\rho_1'|_{C^{l,m+1}} = \rho_1|_{C^{l,m+1}}$ and $\rho_1'(e_{l+m+2}) = -\rho|_1(e_{l+m+2})$.

With this choice of cofinal sequence, we will define the set $\mathrm{GL}(F_1, \rho_1|_{C^{l,m}})$ and show how this set relates to the set of gradations. First, notice that Remark 7.3 says that

$\mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}((F_1,\rho|_{C^{l,m}}))$ is a Banach algebra, where the multiplication is given by composition. As explained in Remark 4.9, we can view each morphism

$$A \in \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}((F_n,\rho_n|_{C^{l,m}})) = \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(\bigoplus_{i=1}^{n} F_1, \rho_1|_{C^{l,m}} \oplus \ldots \oplus \rho_1|_{C^{l,m}})$$

as a matrix $(A_{i,j})_{1\leq i,j\leq n}$, with $A_{i,j} \in \mathrm{End}_{E^{\mathbb{K}}_G(X)^{l,m}}(F_1,\rho_1|_{C^{l,m}})$. We set $\mathrm{GL}_n(F_1)$ to be the set of matrices with coefficients in $\mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(F_1,\rho_1|_{C^{l,m}})$ which are invertible. Notice that there is a canonical bijection

$$\mathrm{GL}_n(F_1) \cong \mathrm{Aut}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}((F_n,\rho_n|_{C^{l,m}})),$$

which sends a matrix to the corresponding morphism. This isomorphism induces a topology on $\mathrm{GL}_k(F_1,\rho_1|_{C^{l,m}})$. We now define the space

$$\mathrm{GL}(F_1,\rho_1|_{C^{l,m}}) := \mathrm{colim}_{k\to\infty}\, \mathrm{GL}_{2k}(F_1,\rho_1|_{C^{l,m}}),$$

where the inclusion $\mathrm{GL}_{2k}(F_1,\rho_1|_{C^{l,m}}) \to \mathrm{GL}_{2k+2}(F_1,\rho_1|_{C^{l,m}})$ is given by $f \to f \oplus \epsilon \oplus -\epsilon$, where $\epsilon := \rho(e_{p+q+2})\rho(e_{p+q+1})$.

We now define the map $(\overline{}) : \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(F_1,\rho_1|_{C^{l,m}}) \to \mathrm{End}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(F_1,\rho_1|_{C^{l,m}})$ by

$$\overline{f} = \rho_1(e_{l+m+1}) f \rho_1(e_{l+m+1}).$$

Notice that $\overline{\overline{f}} = f$. This map induces a map $\overline{} : \mathrm{GL}_{2k}(F_1,\rho_1|_{C^{l,m}}) \to \mathrm{GL}_{2k}(F_1,\rho_1|_{C^{l,m}})$, by letting it act on the matrix elements. With this map, we can define the subsets

$$\mathrm{GL}^-(F_1,\rho_1|_{C^{l,m}}) := \{f \in \mathrm{GL}(F_1,\rho_1|_{C^{l,m}}) \mid \overline{f} = -f\}$$

and

$$I(F_1,\rho_1|_{C^{l,m}}) := \{f \in \mathrm{GL}^-(F_1,\rho_1|_{C^{l,m}}) \mid f^2 = -1\}.$$

Notice that since $\overline{\epsilon} = -\epsilon$ and $\epsilon^2 = -1$, these definition make sense. The following propositions show how these definitions relate to the group $K^{l,m+1}_G(X,\emptyset)$.

Proposition 9.6. *There is an isomorphism*

$$K^{l,m+1}_G(X,\emptyset) \cong \pi_0(I(F_1,\rho_1|_{C^{l,m}})),$$

where we choose $\epsilon \in I(F_1,\rho_1|_{C^{l,m}})$ as our base point and the group structure on $\pi_0(I(F_1,\rho_1|_{C^{l,m}}))$ is defined by $[f]\oplus[g] = [f\oplus g]$.

Proof. Since the sequence $(\tilde{F}_n,\tilde{\rho}_n)_{n\in\mathbb{N}}$ is a cofinal sequence in $\mathcal{E}^{\mathbb{K}}_G(X)^{l,m+2}$, Theorem 9.5 implies that $K^{l,m+1}(X,\emptyset) \cong \pi_0(\mathrm{Gr}^{l,m+1}(\tilde{F}_{n\in\mathbb{N}}))$, where the group operation is endowed by the direct sum. It is therefore sufficient to construct a homeomorphism between $\Phi : I(F_1,\rho_1|_{C^{l,m}}) \to \mathrm{Gr}^{l,m+1}(\tilde{F}_{n\in\mathbb{N}})$ that preserves the base point and show that the group structure on $\pi_0(I(F_1,\rho_1|_{C^{l,m}}))$ induced by Φ and the group structure on $\mathrm{Gr}^{l,m+1}(\tilde{F}_{n\in\mathbb{N}})$ is the direct sum.

We first define Φ. Let

$$I_k(F_1,\rho_1|_{C^{l,m}}) := \{f \in \mathrm{GL}_{2k}(F_1,\rho_1|_{C^{l,m}}) \mid \overline{f} = -f \text{ and } f^2 = -1\}.$$

Notice that $I(F_1, \rho_1|_{C^{l,m}}) \cong \text{colim}_{k\to\infty} I_k(F_1, \rho_1|_{C^{l,m}})$, where the inclusion $I_k(F_1, \rho_1|_{C^{l,m}}) \subset I_{k+1}F_1, \rho_1|_{C^{l,m}})$ is given by $f \to f \oplus \epsilon \oplus -\epsilon$. We now define Φ on $I_k(F_1, \rho_1|_{C^{l,m}}$ by $\Phi_k(\alpha) = \alpha\rho_{2k}(e_{l+m+1})$ for all $k \in \mathbb{N}$. We first check that $\alpha\rho_{2K}(e_{l+m+1})$ is indeed a gradation. Notice that

$$\Phi_k(\alpha)^2 = \alpha\rho_{2k}(e_{l+m+1})\alpha\rho_{2k}(e_{l+m+1}) = \alpha\overline{\alpha} = \alpha(-\alpha) = id,$$

$$\alpha\rho_{2k}(e_{l+m+1})\tilde{\rho}(e_p) = -\alpha\tilde{\rho}(e_p)\rho_{2k}(e_{l+m+1}) = \tilde{\rho}(e_p)\alpha\rho(e_{l+m+1})$$

for $1 \leq p \leq l+m$ and

$$\tilde{\rho}_k(e_{l+m+1})\alpha = \overline{\alpha}\tilde{\rho}_k(e_{l+m+1}) = -\alpha\tilde{\rho}_k(e_{l+m+1}).$$

This implies that $\Phi_k(\alpha)$ is a gradation. Also notice that Φ_k is continuous. Since

$$\Phi_{k+1}(\alpha \oplus \epsilon \oplus -\epsilon) = \alpha \oplus \rho(e_{l+m+2}) \oplus -\rho(e_{l+m+2}) = \alpha \oplus \tilde{\rho}_k(e_{l+m+2}),$$

it follows that the maps Φ_k for $k \in \mathbb{N}$ induce a unique map $\Phi : I(F_1, \rho_1|_{C^{l,m}}) \to \text{Gr}^{l,m+1}(\tilde{F}_{n\in\mathbb{N}})$. This map is a homeomorphism, because it has an inverse, which is on $(\tilde{F}_k, \tilde{\rho}_k|_{C^{l,m+1}}$ defined by $\Phi^{-1}(\alpha) = \alpha\tilde{\rho}_k(e_{l+m+1})$. We leave to the reader to check that this map is well defined and indeed the inverse.
Notice that $\Phi(\epsilon) = \tilde{\rho}(e_{l+m+2})$, which implies that the Φ maps the base point to the base point. Also notice that $\Phi_*[\alpha \oplus \beta] = \Phi_*[\alpha] \oplus \Phi_*[\beta]$ and $\Phi_*^{-1}[\alpha \oplus \beta] = \Phi_*^{-1}[\alpha] \oplus \Phi_*^{-1}[\beta]$. Therefore, the direct sum give $\pi_0(I(F_1, \rho_1|_{C^{l,m}}))$ the structure of an abelian group isomorphic to $K^{l,m+1}(X, \emptyset)$. □

We will now describe the group

$$K^{p,q}(X \times [0,1], \{0,1\} \times X)$$

in terms of $\text{GL}(F_1, \rho_1|_{C^{l,m}})$. Let $(\tilde{F}_n, \tilde{\rho}_n)$ be defined as before. Notice that $(F_n, \rho_n|_{C^{l,m+1}})_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}_G^{l,m+1}(X)$. Remark 7.13 implies that $(\tilde{F}_n, \tilde{\rho}_n)_{n\in\mathbb{N}}$ is a cofinal sequence in $\mathcal{E}_G^{l,m+1}(X)$. Let $p : X \times I \to X$ denote the projection. Then, since

$$(p^*F_n, p^*\rho_n|_{C^{l,m+1}}) = ((X \times [0,1]) \times (\oplus_{i=1}^{2n}\mathbb{K}[G] \otimes C^{l,m+2}), \rho_{2n}|_{C^{l,m+1}}),$$

the sequence $((p^*\tilde{F}_n, p^*\tilde{\rho}_n))_{n\in\mathbb{N}}$ is a cofinal sequence in $\mathcal{E}_G^{l,m+1}(X\times[0,1])$. We will often denote $p^*\tilde{\rho}_n$ by $\tilde{\rho}_n$. With this notation in place, we can show the following:

Lemma 9.7. *Let $(E, \sigma) \in \mathcal{E}_G^{\mathbb{K}}(X \times [0,1])^{l,m}$. Then there exists a bundle $(E^\perp, \sigma^\perp) \in \mathcal{E}_G^{\mathbb{K}}(X \times [0,1])^{l,m}$ such that*

$$(E \oplus E^\perp, \sigma \oplus \sigma^\perp) \cong (p_X^* F_n, \tau),$$

where $\tau_n|_{C^{l,m}} = \tilde{\rho}|_{C^{l,m}}$ and

$$\tau_n(e_{l+m+1})((x,t), v) := \beta_n((x,t), v) := (\rho_n(e_{p+q+1})\cos(\pi t) + \rho_n(e_{l+m+2})\sin(\pi t))v$$

for some $n \in \mathbb{N}$.

Proof. Since $(p^*\tilde{F}_n, \tilde{\rho}|_{C^{l,m+1}})_{n\in\mathbb{N}}$ is a cofinal sequence, there exists a bundle $(E^\perp, \sigma^\perp)$ such that

$$(E \oplus E^\perp, \sigma \oplus \sigma^\perp) \cong (p^*\tilde{F}_n, \tilde{\rho}_n|_{C^{l,m+1}}).$$

We now define the morphism $f : p^*\tilde{F}_n \to p^*\tilde{F}_n)$ by

$$f((x,t),v) = (\cos(\frac{\pi}{2}t) - \rho_n(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t))v.$$

Notice that $\rho_n(e_k)\circ f = f\circ\rho_n(e_k)$, for $1 \le k \le l+m$, which implies that $f \in \mathrm{End}_{\mathcal{E}_G^{\mathbb{K}}(X\times I)^{l,m}}(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}})$. Also notice that h is invertible, with

$$f^{-1}((x,t),v) = (\cos(\frac{\pi}{2}t) + \rho_n(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t)).$$

Therefore, the map h is an isomorphism from $(p^*\tilde{F}_n, \rho|_{C^{l,m+1}})$ to $(p^*\tilde{F}_n, \tau) \in \mathcal{E}_G^{\mathbb{K}}(X\times I)^{l,m+1})$, where $\tau|_{C^{l,m}} = \rho_n|_{C^{l,m}}$ and $\tau(e_{l+m+1}) = f\circ\rho(e_{l+m+1})f^{-1}$. Writing out the definition we obtain

$$\begin{aligned}
\tilde{\rho}(e_{l+m+1})((x,t),v) &= (\cos(\frac{\pi}{2}t) - \rho_n(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t))\rho(e_{l+m+1})(\cos(\frac{\pi}{2}t) + \rho_n(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t))v \\
&= (\cos(\frac{\pi}{2}t) - \rho_n(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t))(\rho_n(e_{l+m+1})\cos(\frac{\pi}{2}t) + \rho_n(e_{l+m+2})\sin(\frac{\pi}{2}t))v \\
&= ((\cos(\frac{\pi}{2})^2 - \sin(\frac{\pi}{2})^2)\rho(e_{l+m+1}) + 2\sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2})\rho(e_{l+m+1}))v \\
&= \beta_n((x,t),v).
\end{aligned}$$

□

Notice that since $(p^*F_n, \tau_n) \oplus (p^*F_k, \tau_k) = (p^*(F_{n+k}, \tau_{n+k})$, the bundles $(p^*F_n, \tilde{\rho}_n)$ are a cofinal sequence in $\mathcal{E}_G^{\mathbb{K}}(X \times I)$. This yields the following result:

Lemma 9.8. *Let $[E, \alpha_+, \alpha_-] \in K_G^{l,m}(X \times [0,1], X \times \{0,1\})$, then we have*

$$[E, \alpha_+, \alpha_-] = [p^*\tilde{F}_n, \beta_n, f] \tag{9.3}$$

where, $f \in Gr^{l,m}(X \times [0,1])$. Moreover, we have

$$[p^*\tilde{F}_n, \beta_n, f] = [p^*\tilde{F}_k, \beta_k, h],$$

iff there exists a $p \in \mathbb{N}$ such that

$$\beta_n \oplus h \oplus \beta_p \simeq f \oplus \beta_{k+p}$$

relative to $X \times \{0,1\}$.

Proof. This is a direct consequence of Lemma 9.4. □

We now want to relate the map f from the previous Lemma to a map

$$\tilde{f} : I \to \mathrm{Aut}_{\mathcal{E}_G^{\mathbb{K}}(X)^{l,m}}(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}}).$$

Lemma 9.9. *We may assume that the map f in equation 9.3 of the previous Lemma is given by*

$$f|_{\tilde{F}_n\times\{t\}} = \tilde{f}(t)\tilde{\rho}_n(e_{l+m+1})\tilde{f}^{-1}(t),$$

Where $\tilde{f} : [0,1] \to Aut_{\mathcal{E}_G^{\mathbb{K}}(X)^{l,m}}(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}})$ is a map such that $\tilde{f}(0) = id$ and

$$\tilde{\rho}_n(e_{p+q+1})f(1) = -f(1)\tilde{\rho}(e_{p+q+1}).$$

Proof. We can view the gradation $f \in \mathrm{Gr}^{l,m}(p^*\tilde{F}_n, \tilde{\rho}|_{C^{l,m}})$ as a continuous map $f : [0,1] \to \mathrm{Gr}^{l,m}(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}})$ defined by $f(t) = f|_{p^*E|_{X\times\{t\}}}$. Lemma 8.8 implies that there exists a map $\tilde{f} : I \to \mathrm{Aut}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}})$ such that $\tilde{f}(0) = id$ and $\tilde{f}(1)f(0)\tilde{f}(1)^{-1} = f(1)$. Notice that the construction in Lemma 8.8 is such that we even have $\tilde{f}(t)f(0)\tilde{f}(t)^{-1} = f(t)$ for all $t \in [0,1]$. Since $f|_{p^*E|_{X\times\{0,1\}}} = \beta_n|_{p^*E|_{X\times\{0,1\}}}$, it follows that $f(0) = \rho(e_{l+m+1})$, $f(1) = -\rho(e_{l+m+1})$ and thus

$$\tilde{f}(1)\rho(e_{l+m+1})\tilde{f}(1)^{-1} = -\rho(e_{l+m+1}).$$

Therefore, the map $\tilde{f}$ has the required properties. □

These map have the following useful property:

Lemma 9.10. *We have $[p^*\tilde{F}_n, \beta_n, f\tilde{\rho}_n(e_{l+m+1})f^{-1}] = [p^*\tilde{F}_k, \beta_k, h\tilde{\rho}_k(e_{l+m+1})h^{-1}]$, for maps $f : [0,1] \to Aut_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}})$ and $h : [0,1] \to Aut_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(\tilde{F}_k, \tilde{\rho}|_{C^{l,m}})$ as in Lemma 9.9 if and only if there exists a homotopy*

$$H : [0,1] \times [0,1] \to Aut_{\mathcal{E}^{\mathbb{K}}_G(X)}(\tilde{F}_{n+k+p}, \tilde{\rho}_{n+k+p}|_{C^{l,m}}),$$

such that $H_0 = f$, $H_1 = h$ and H_t is a map such that $H_t(0) = id$ and $H_t(1)\rho(e_{l+m+1}) = -\rho(e_{l+m+1})H_t(1)$.

Proof. It is clear that if such a homotopy $H : [0,1] \times [0,1] \to \mathrm{Aut}_{\mathcal{E}^{\mathbb{K}}_G(X)}(\tilde{F}_{n+k+p}, \tilde{\rho}_{n+k+p}|_{C^{l,m}})$ exists, then

$$[p^*\tilde{F}_n, \beta_n, f\tilde{\rho}_n(e_{l+m+1})f^{-1}] = [p^*\tilde{F}_k, \beta_k, h\tilde{\rho}_k(e_{l+m+1})h^{-1}].$$

Now assume that $[F_n, \beta_n, f\tilde{\rho}_n(e_{l+m+1})f^{-1}] = [F_k, \beta_k, h\tilde{\rho}_k(e_{l+m+1})h^{-1}]$ Lemma 9.8 implies that there exists a $p \in \mathbb{N}$ such that

$$\beta_n \oplus h\tilde{\rho}_k(e_{l+m+1})h^{-1} \oplus \beta_p \simeq f\tilde{\rho}_n(e_{l+m+1})f^{-1} \oplus \beta_{k+p}$$

relative to $X \times \{0,1\}$. We can view the homotopy $L : I \to \mathrm{Gr}^{l,m}(p^*\tilde{F}_{n+k+p}, \tilde{\rho}_{n+l+p}|_{C^{l,m}})$ as a map $L' : [0,1] \times [0,1] \to \mathrm{Gr}^{l,m}(\tilde{F}_{n+k+p}, \tilde{\rho}_{n+l+p}|_{C^{l,m}})$, where $L'(t,s) = L(s)|_{p^*F_{n+k+p}|_{X\times\{t\}}}$. We have $L_0(t) := L(\cdot, 0) = \beta_{n+p}(t) \oplus h(t)$ and $L_1 = \beta_{n+p}(t) \oplus h(t)$. Since the homotopy is relative to $X \times \{0,1\}$, we have $L(0,s) = \tilde{\rho}_{n+k+p}(e_{l+m+1})$ and $L(1,t) = -\tilde{\rho}_{n+k+p}(e_{l+m+1})$. We now have the following commutative diagram:

$$\begin{array}{ccc}
[0,1] \times \{0,1\} & \xrightarrow{\tilde{L}'} & \mathrm{Aut}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(\tilde{F}_{n+k+p}, \tilde{\rho}_{n+k+p}|_{C^{l,m}}) \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle x \to x\tilde{\rho}_{n+k+p}(e_{p+q+1})x^{-1}} \\
[0,1] \times [0,1] & \xrightarrow{L'} & \mathrm{Gr}^{l,m}(\tilde{F}_{n+k+p}, \tilde{\rho}_{n+k+p}|_{C^{l,m}})
\end{array},$$

where i is the inclusion an $\tilde{L}'$ is defined by

$$\tilde{L}'(t,0) = \beta_n(t) \oplus h(t) \oplus \beta_p(t) \text{ and } \tilde{L}'(t,1) = f(t) \oplus \beta_{k+p}(t),\ \tilde{L}'(0,t) = id,$$

for $t \in [0,1]$. Since $([0,1] \times [0,1], [0,1] \times \{0,1\} \cup \{0\} \times [0,1])$ is homeomorphic to $([0,1] \times [0,1], \{0\} \times [0,1])$. We can use Lemma 8.8 to obtain a map

$$H : [0,1] \times [0,1] \to \mathrm{Aut}_{\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}}(\tilde{F}_{n+k+p}, \tilde{\rho}_{n+k+p}|_{C^{l,m}}),$$

such that

$$H|_{[0,1]\times\{0,1\}\cup\{0\}\times[0,1]} = \tilde{L}'$$

and

$$He_{p+q+1}H^{-1} = L'.$$

The map H now is the required homotopy. $\square$

We will now rephrase this Lemma to make it into a statement about $\mathrm{GL}(F_1, \rho|_{C^{l,m}})$. Let $\Omega(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}}))$ be the space of maps $f : [0,1] \to \mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}})$ with the property that $f(0) = id$ and $f(1) \in \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}}))$ endowed with the compact open topology. We now define

$$\pi_1(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}})) := \mathrm{colim}_{k\to\infty}\pi_0(\Omega(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}}))),$$

where the inclusion

$$\pi_0(\Omega(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}}))) \to \pi_0(\Omega(\mathrm{GL}_{2k+2}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k+2}(F_1, \rho|_{C^{l,m}})))$$

is given by

$$f \to f \oplus \gamma_1,$$

where

$$\gamma_1(t) := \cos(\frac{\pi}{2}t) + (\epsilon \oplus -\epsilon)\sin(\frac{\pi}{2}t).$$

We choose $[\gamma_1]$ as our base point. On $\pi_1(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}}))$, we can define the addition $[f]\oplus[g] = [f\oplus g]$ (We need to choose representatives $f_k \in \pi_0(\Omega(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}})))$ of $[f]$ for some $k \in \mathbb{N}$ and
$g_n \in \pi_0(\Omega(\mathrm{GL}_{2n}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2n}(F_1, \rho|_{C^{l,m}})))$ of $[g]$ for a $n \in \mathbb{N}$ to take the direct sum and the addition is thus defined by

$$[f] \oplus [g] = [f_k \oplus g_k])$$

We leave to the reader to verify that this addition is well defined and gives

$$\pi_1(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}})),$$

the structure of a commutative monoid. With these definitions, we can rephrase Lemma 9.9 and Lemma 9.10 to obtain the following theorem:

Theorem 9.11. *The map*

$$\Psi : \pi_1(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}})) \to K^{l,m}_G(X \times [0,1], X \times \{0,1\}),$$

which is defined on $\pi_0(\Omega(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}})))$ *by*

$$\Psi_{2k}(f) = [(p_X^*\tilde{F}_k, \tilde{\rho}|_{C^{l,m}}), \beta_k, f\tilde{\rho}|_k(e_{l+m+1})f^{-1}].$$

is a monoid isomorphism and hence an isomorphism of groups.

Proof. Notice that

$$\begin{aligned}
\gamma_1(t)\tilde{\rho}_1(e_{l+m+1})\gamma_1(t)^{-1} &= (\cos(\frac{\pi}{2}t) + \tilde{\rho}_1(e_{l+m+2}e_{l+m+1})\sin(\frac{\pi}{2}t))e_{l+m+1}(\cos(\frac{\pi}{2}t) - \tilde{\rho}_1(e_{l+m+2}e_{l+m+1})\sin(\frac{\pi}{2}t)) \\
&= (\tilde{\rho}_1(e_{l+m+1})\cos(\frac{\pi}{2}t) + \tilde{\rho}_1(e_{l+m+2})\sin(\frac{\pi}{2}t))(\cos(\frac{\pi}{2}t) - \tilde{\rho}_1(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t)) \\
&= \cos(\pi t)\tilde{\rho}_1(e_{l+m+1}) + \sin(\pi t)\tilde{\rho}_1(e_{l+m+2}) \\
&= \beta_1
\end{aligned}$$

Therefore, the map Ψ is well defined. It is clear from the definition that Φ is a monoid homomorphism and Lemma 9.9 and Lemma 9.10 imply that Ψ is an isomorphism. $\square$

We have now expressed the groups $K_G^{l,m+1}(X,\emptyset)$ and $K_G^{l,m+1}(X\times[0,1],X\times\{0,1\})$ in terms of the Banach algebra $(F_1,\rho_1|_{C^{l,m}})$. The following general theorem about Banach algebras shows why we have reduced to these definitions:

Theorem 9.12. *Let A be a Banach algebra with an algebra homomorphism $(\overline{\ }) : A \to A$ such that $\overline{\overline{a}} = a$ for all $a \in A$. If A contains an element ϵ such that $\epsilon^2 = -1$ and $\overline{\epsilon} = -\epsilon$, then the map*

$$j : \pi_0(I(A)) \to \pi(\mathrm{GL}(A), \mathrm{GL}^-(A)),$$

defined by

$$j(f)(t) = \cos(\frac{\pi}{2}t) + \sin(\frac{\pi}{2}t)f$$

is a bijection.

Remark 9.13. For a general Banach algebra A, the definition of the spaces $\mathrm{GL}(A)$, $\mathrm{GL}^-(A)$, $I(A)$, $\pi_0(I(A))$ and $\pi(\mathrm{GL}(A),\mathrm{GL}^-(A))$ are the same as the definitions we gave earlier for the Banach algebra $(F_1,\rho|_{C^{l,m}})$.

Proof. The proof of this theorem is a bit beyond the focus of our thesis. The proof can, for example, be found in [3] as Theorem 6.12 on page 167.
The original proof can be found in [10]. □

With this theorem and all the work we have done, we can at last prove the theorem:

Theorem 9.14. *Let G be a finite group, then there exists an isomorphism*

$$K_G^{l,m+1}(X,\emptyset) \cong K^{l,m}(X\times[0,1], X\times\{0,1\}).$$

Proof. In Proposition 9.6 we showed that

$$K^{l,m+1}(X,\emptyset) \cong \pi_0(I(F_1,\rho_1|_{C^{l,m}})).$$

Theorem 9.12 implies that

$$\pi_0(I(F_1,\rho_1|_{C^{l,m}})) \cong \pi_1(\mathrm{GL}_{2k}(F_1,\rho|_{C^{l,m}}), \mathrm{GL}_{2k}^-(F_1,\rho|_{C^{l,m}})).$$

Lastly, in Theorem 9.11 we showed that

$$\pi_1(\mathrm{GL}_{2k}(F_1,\rho|_{C^{l,m}}), \mathrm{GL}_{2k}^-(F_1,\rho|_{C^{l,m}})) \cong K^{l,m}(X\times[0,1], X\times\{0,1\}).$$

Which proves the theorem. □

Remark 9.15. We can also describe this isomorphism more explicitly. Let $[(E,\sigma),\alpha_+,\alpha_-] \in K^{l,m+1}(X,\emptyset)$. Lemma 9.4 implies that there exists a $n\in\mathbb{N}$ and a $\beta \in \mathrm{Gr}^{l,m+1}(\tilde{F}_n,\tilde{\rho}|_{C^{l,m+1}})$ such that

$$[(E,\sigma),\alpha_+,\alpha_-] = [(\tilde{F}_n,\tilde{\rho}|_{C^{l,m+1}}),\tilde{\rho}_n(e_{l+m+2}),\beta].$$

This corresponds to the class

$$\beta \in \pi_0(\mathrm{Gr}^{l,m}((\tilde{F}_n,\tilde{\rho}_n|_{C^{l,m1}})_{n\in\mathbb{N}})$$

and hence to the class

$$[\beta\rho_{2n}(e_{l+m+1})] \in \pi_0(I(F_1,\rho_1|_{C^{l,m}}).$$

This class corresponds to the path

$$j(t) = \cos(\frac{\pi}{2}t) + \sin(\frac{\pi}{2}t)\beta\rho_{2n}(e_{l+m+1})$$

in $\pi_1(\mathrm{GL}_{2k}(F_1, \rho|_{C^{l,m}}), \mathrm{GL}^-_{2k}(F_1, \rho|_{C^{l,m}}))$. We have

$$\begin{aligned} j(t)\rho_{2n}(e_{l+m+1})j(t)^{-1} &= (\cos(\frac{\pi}{2}t) + \sin(\frac{\pi}{2}t)\beta\rho_{2n}(e_{l+m+1}))(\rho_{2n}(e_{l+m+1})\cos(\frac{\pi}{2}t) + \sin(\frac{\pi}{2}t)\beta) \\ &= \cos(\pi t)\rho_{2n}(e_{l+m+1}) + \sin(\pi t)\beta. \end{aligned}$$

Using the isomorphism from Theorem 9.11 on $[\beta\rho_{2n}(e_{l+m+1})]$, we thus obtain the element

$$[(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}}), \beta_n, \cos(\pi t)\rho_{2n}(e_{l+m+1}) + \sin(\pi t)\beta].$$

Let $\mu : K^{l,m+1}(X, \emptyset) \to K^{l,m}(X \times [0,1], X \times \{0,1\})$ be the map defined by

$$\mu((E,\sigma), \alpha_+, \alpha_-) = ((p^*E, p^*\sigma|_{C^{l,m}}), \cos(t)\sigma(e_{l+m+1}) + \sin(t)\alpha_+, \cos(t)\sigma(e_{l+m+1}) + \sin(t)\alpha_-).$$

Notice that μ is a well defined group homomorphism and

$$\mu([(\tilde{F}_n, \tilde{\rho}|_{C^{l,m+1}}), \tilde{\rho}_n(e_{l+m+2}), \beta]) = [(\tilde{F}_n, \tilde{\rho}|_{C^{l,m}}), \beta_n, \cos(\pi t)\rho_{2n}(e_{l+m+1}) + \sin(\pi t)\beta],$$

which implies that the isomorphism from Theorem 9.14 is induced by μ.

9.2 The group G is a Lie group and $A = \emptyset$.

We will now prove that if G is a Lie-group, the group homomorphism

$$\mu : K^{l,m+1}(X, \emptyset) \to K^{l,m}(X \times [0,1], X \times \{0,1\})$$

defined by

$$\mu([(E,\sigma), \alpha_+, \alpha_-]) = [(p^*E, p^*\sigma|_{C^{l,m}}), \cos(t)\sigma(e_{l+m+1}) + \sin(t)\alpha_+, \cos(t)\sigma(e_{l+m+1}) + \sin(t)\alpha_-].$$

is an isomorphism. As mentioned before, since a Lie group G can have an infinite number of irreducible representations, we can no longer use a cofinal sequence in $\mathcal{E}^{\mathbb{K}}_G(X)^{l,m}$ to show this. However, we can still do something similar.

Let $[G]_{\mathbb{K}}$ denote the set of isomorphisms classes of irreducible (real or complex) finite dimensional representations of G. For each $[N] \in [G]$, we choose a representative N and a G-invariant inner product $\langle \cdot, \cdot \rangle_N$ on N. This inner product induces an inner product $\langle \cdot, \cdot \rangle_N$ on $N \otimes C^{l,m+2}$ which is defined by

$$\langle n \otimes (e_{i_1} \dots e_{i_k}), m \otimes (e_{j_1}, \dots, e_{j_n}) \rangle = \begin{cases} \langle n, m \rangle & \text{if} \quad e_{i_1} \dots e_{i_k} = e_{j_1}, \dots, e_{j_n} \\ 0 & \text{else} \end{cases}.$$

For $n \in \mathbb{N}$. We let $E_n(X) := X \times \prod_{N \in [G]} \oplus_{i=1}^n (N \otimes C^{l,m+2})$ denote the product of these representations/ G-modules in Top, let $\pi_n : E_n(X) \to X$ denote the projection and let $p_N : \prod_{N \in I} N \to N$ denote the projection. We endow the space with a G-action defined by

$$g(x, v) = (gx, gv),$$

where $g \cdot v$ is the unique map with the property that $p_N \circ (gv) = gp_N(v)$. Similarly, we can endow $E_n(X)$ with a $C^{l,m+2}$-action ρ defined by

$$\rho(u)(x,v) = (x, \rho(u)v).$$

Notice that the space $E_n(X)$ almost has the structure of a G-vector bundle, with a $C^{l,m+2}$-action, expect for the fact that the fibers of the projection $\pi : X \times \prod_{N\in[G]} \oplus_{i=1}^{n} (N \otimes C^{l,m+2})$ can be infinite dimensional vector spaces. This motivates the following definition:

Definition 9.16. *We define $(E_n(X))_{n\in\mathbb{N}}$ as the category, whose objects are the spaces $E_n(X)$ for $n \in \mathbb{N}$ and the G-vector bundle $X \times \{0\}$ an whose morphisms are continuous G-maps $f : E_k \to E_n$ such that :*

1. $\pi_k = \pi_n \circ f$.
2. *For all $x \in X$, the map $f|_{\{x\}\times\prod_{N\in[G]}\oplus_{i=1}^{k}(N\otimes C^{l,m+2})} : (E_k(X))_x \to (E_n(X))_x$ is linear.*
3. *We have*
$$\|f\| := \sup_{n\in\mathbb{N}} \|f|_{X\times\oplus_{i=1}^{k}N\otimes C^{l,m+2}}\|_N < \infty.$$
Where $\|f|_{X\times\oplus_{i=1}^{k}N\otimes C^{l,m+2}}\|$ is the norm of the G-vector bundle morphism
$$f|_{X\times\oplus_{i=1}^{k}N\otimes C^{l,m+2}} : X \times \oplus_{i=1}^{k} N \otimes C^{l,m+2} \to X \times \oplus_{i=1}^{n} N \otimes C^{l,m+2}.$$

Remark 9.17. Notice that with the norm on the morphisms we defined above, the category $(E_n(X))_{n\in\mathbb{N}}$ is a Banach category. Therefore, the categories $(E_n(X))_{n\in\mathbb{N}}^{l,m}$ are also Banach categories and the space $\mathrm{End}_{(E_n(X))_{n\in\mathbb{N}}^{l,m}}((E_n), \rho)$ is a Banach algebra.

For the category $(E_n(X))_{n\in\mathbb{N}}$, we can define gradations and the group $K_G^{l,m}((E_n(X))_{n\in\mathbb{N}},)$ as follows:

Definition 9.18. *Let $(E, f) \in (E_n)_{n\in\mathbb{N}}^{l,m}$. We call a morphism $h \in End_{(E_n(X))_{n\in\mathbb{N}}^{l,m}}(E, f)$ a gradation on (E, f) if:*

(i) $f(e_i)h = -hf(e_i)$ *for all* $1 \leq i \leq l + m$

(ii) $h^2 = 1$

We will denote the set of gradations on (E, f) by $Gr^{l,m}(E, f)$ and endow it with the subspace topology.

and

Definition 9.19. *Let X be a G-space. We define:*

$$K^{l,m}((E_n(X))_{n\in\mathbb{N}}) := \{((E, f), \alpha_+, \alpha_-) \mid (E, f) \in ob((E_n)_{n\in\mathbb{N}}^{l,m}),\ \alpha_+, \alpha_- \in Gr^{l,m}(E, f)\}/\sim.$$

where $((E, f), \alpha_+, \alpha_-) \sim ((E', f'), \beta_+, \beta_-)$ if there exists a triple (F, γ, γ), such that

$$\alpha_+ \oplus \beta_- \oplus \gamma \simeq \alpha_- \oplus \beta_+ \oplus \gamma$$

through gradations.

Remark 9.20. The proof that this is indeed a group is similar to the proof of Lemma 8.4. Moreover, Lemma 8.7 and Lemma 8.8 still hold for the group $K^{l,m}((E_n(X))_{n\in\mathbb{N}})$.

Remark 9.21. If $A \subset X$ is a closed G-invariant subspace, we can also define the group $K^{l,m}((E_n(X))_{n\in\mathbb{N}}, (E_n(A)_{n\in\mathbb{N}}))$ as the group of triples

$$((E_n(X), f), \alpha_+, \alpha_-),$$

where $\alpha_+|_{E_n(A)} = \alpha_-|_{E_n(A)}$ and we identify $((E_n(X), f), \alpha_+, \alpha_-)$ with $((E_k(X), f'), \beta_+, \beta_-)$ if there exists a triple $(E_p(X), \gamma, \gamma)$, such that

$$\alpha_+ \oplus \beta_- \oplus \gamma \simeq \alpha_- \oplus \beta_+ \oplus \gamma$$

through gradations relative to $E_{n+k+p}(A)$. As before, the direct sum gives $K^{l,m}((E_n(X))_{n\in\mathbb{N}}, (E_n(A)_{n\in\mathbb{N}}))$ the structure of a group.

The idea is that we use the group $K^{l,m}((E_n(X))_{n\in\mathbb{N}})$ to define the cofinal sequences. To do this, we define the sequence $(\tilde{E}_n, \tilde{\rho}|_{C^{l,m+1}})_{n\in\mathbb{N}})^{l,m+1}$ in $(E_n(X \times I))_{n\in\mathbb{N}}$ by

$$\tilde{E}_1 := E_1(X \times [0,1]) \oplus E_1(X \times [0,1])$$

and $\tilde{\rho}_1 = \rho \oplus \rho'$, where $\rho'|_{C^{l,m+1}} = \rho$ and $\rho'(e_{l+m+2}) = -\rho(e_{l+m+2})$ and

$$\tilde{E}_{n+1} = \tilde{E}_n \oplus \tilde{E}_1.$$

We will now shows how we can relate this sequence to the group $K^{l,m}_G(X \times [0,1], X \times \{0,1\})$.

Definition 9.22. *Let $I \subset [G]$ be a finite subset and Y be a compact G-space. We define the map*

$$P_I^* : End_{(E_n(Y))_{n\in\mathbb{N}}}(E_n(Y)) \to End_{\mathcal{E}_G(X)}(Y \times \bigoplus_{N\in I} \bigoplus_{i=1}^{n} N).$$

by

$$P_I^*(\sigma) = \sigma|_{Y \times \bigoplus_{N\in I} \bigoplus_{i=1}^{n} N}.$$

Remark 9.23. Notice that if $M, N \in [G]$ and $M \neq N$, then for the set of equivariant maps between the representations, we have $\mathrm{Hom}_G(M, N) = 0$. This implies that in the definition above we have

$$\mathrm{Im}(P_I^*\sigma) \subset Y \times \bigoplus_{N\in I} \bigoplus_{i=1}^{n} N.$$

Remark 9.24. The map P_I^* induces a homomorphism

$$P_I^* : K^{l,m}((E_n(Y \times [0,1])_{n\in\mathbb{N}}, (E_n(Y \times \{0,1\})_{n\in\mathbb{N}}) \to K^{l,m}(Y \times [0,1], Y \times \{0,1\}),$$

defined by

$$p_I^*([(E_n, \sigma), \alpha_+, \alpha_-]) = ([(Y \times \bigoplus_{N\in I} \bigoplus_{i=1}^{n} N, P_I^*\sigma), P_I^*\alpha_+, P_I^*\alpha_-]).$$

We can define the gradation β_r on $(\tilde{E}_n, \tilde{\rho}|_{C^{l,m}})$ by

$$\beta_r|_{\tilde{E}|_{X\times\{t\}}} = \tilde{\rho}(e_{l+m+1})\cos(\pi t) + \tilde{\rho}(e_{l+m+2})\sin(\pi t).$$

and proceed analogously to the case where G was finite to obtain the following results:

Lemma 9.25. *Let* $[(E,\sigma),\alpha_+,\alpha_-] \in K_G^{l,m}(X\times[0,1], X\times\{0,1\})$*. Then, there exists a* $n\in\mathbb{N}$*, a* $\delta \in Gr^{l,m+1}((\tilde{E}_n, \tilde{\rho}|_{C^{l,m+1}})$ *and finite set* $I \subset [G]$ *such that*

$$[(E,\sigma),\alpha_+,\alpha_-] = P_I^*[(\tilde{E}_n,\tilde{\rho}_n),\beta_n,\delta]$$

and

$$P_N^*(\delta) = P_N^*(\beta_n) \tag{9.4}$$

for all $N \in [G]-I$*.*

Proof. Lemma 7.9 implies that there exists a $(E^\perp,\sigma^\perp)) \in \mathcal{E}_G^{l,m+1}(X\times[0,1])$, a $\alpha_+^\perp \in \mathrm{Gr}^{l,m+1}(E^\perp,\sigma^\perp)$ and a G-module M such that

$$(E\oplus E^\perp, \sigma\oplus\sigma^\perp) = ((X\times[0,1])\times M\otimes C^{l,m+2}, \rho|_{C^{l,m+1}})$$

and

$$\alpha\oplus\alpha^\perp = \rho(e_{l+m+2}).$$

Therefore, we have

$$[(E,\sigma),\alpha_+,\alpha_-] = [((X\times[0,1])\times M\otimes C^{l,m+2}, \rho|_{C^{l,m}}), \rho(e_{l,m+1}), \gamma],$$

fore some $\gamma \in \mathrm{Gr}^{l,m}(((X\times[0,1])\times M\otimes C^{l,m+2}, \rho|_{C^{l,m+2}})$. Notice that

$$M \cong \oplus_{i=1}^n \oplus_{j=1}^{k_i} N_i,$$

as a representation, where N_i are distinct irreducible representations. We can assume that $k_1 = k_2 = \ldots = k_n$. Let $I = \cup_{i=1}^n\{N_i\}$ and let $\delta' \in \mathrm{Gr}^{l,m}(\tilde{E}_n)$ be the gradation defined by

$$P_I^*\delta'(v) = \gamma \oplus \rho_n(e_{l+m+1}),$$

and

$$P_N^*\delta'(v) = \tilde{\rho}_n(e_{l+m+1}), \tag{9.5}$$

for $N\in[G]-I$. Notice that

$$P_I^*[(\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}}), \tilde{\rho}_n(e_{l+m+1}), \delta'] = [(E,\sigma),\alpha_+,\alpha_-].$$

As in Lemma 9.8, we let $f : (\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}}) \to (\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}})$ be the automorphisms in $\mathrm{End}_{(E_n(X\times[0,1]))^{l,m}_{n\in\mathbb{N}}}((\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}}))$ defined by

$$f(t) = \cos(\frac{\pi}{2}t) - \tilde{\rho}_n(e_{l+m+1}e_{l+m+2})\sin(\frac{\pi}{2}t).$$

Remark 9.20 implies that

$$\begin{aligned}[(\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}}), \tilde{\rho}_n(e_{l+m+1}), \delta'] &= [(\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}}), f\tilde{\rho}_n(e_{l+m+1})f^{-1}, f\delta' f^{-1}] \\ &= [(\tilde{E}_n, \tilde{\rho}_n|_{C^{l,m}}), \beta_n, \delta]\end{aligned}$$

with $\delta := f\delta' f^{-1}$. Equation 9.5 implies that we also have that

$$P_N^*(\delta) = P_I^*\beta_n,$$

for $N\in[G]-I$. ∎

This Lemma allows us to prove the following proposition

Proposition 9.26. *The map*

$$\Phi : K_G^{l,m}(X \times [0,1], X \times \{0,1\}) \to K^{l,m}((E_n(X \times [0,1]))_{n\in\mathbb{N}}, (E_n(X \times \{0,1\}))_{n\in\mathbb{N}})$$

defined by

$$\Phi([(E,\sigma),\alpha_+,\alpha_-]) = [(\tilde{E}_n,\tilde{\rho}_n),\beta_n,\delta],$$

where δ is defined as in the previous Lemma, is an injective group homomorphism. Moreover, we have that if

$$x := [(\tilde{E}_n,\tilde{\rho}_n),\beta_n,\delta], y := [(\tilde{E}_k,\tilde{\rho}_k),\beta_k,\epsilon] \in Im(\Phi),$$

were δ and ϵ are as in the previous Lemma, then $x = y$ if there exist a $p \in \mathbb{N}$ such that

$$\beta_n \oplus \epsilon \oplus \beta_p \simeq \delta \oplus \beta_{k+p},$$

relative to $X \times \{0,1\}$.

Proof. The proof of this statement is similar to the proof of Lemma 9.4. We will only proof that the map injective and leave the verifications of the other properties to the reader. Assume that $\Phi([(E,\sigma),\alpha_+,\alpha_-]) = x = 0$. By definition, there exists a finite subset $I \subset [G]$ such that

$$P_I^* x = [(E,\sigma),\alpha_+,\alpha_-].$$

Since $x = 0$ and P_I^* is a group homomorphism, we also have

$$P_I^* x = P_I^* 0 = 0$$

and thus $x = 0$. □

Lemma 9.27. *We may assume that the map δ in Proposition 9.26 is given by*

$$\delta(t) = f(t)\tilde{\rho}(e_{l+m+1})f^{-1}(t)$$

where

$$f : [0,1] \to Aut_{(E_n(X\times[0,1]))^{l,m}_{n\in\mathbb{N}}}(E_{2n}(X \times [0,1]), \rho_{2n}|_{C^{l,m}})$$

is such that $f(0) = id$ and $\tilde{\rho}(e_{l+m+1})f(1) = -f(1)\tilde{\rho}(e_{l+m+1})$. Moreover, if $\epsilon = h\tilde{\rho}(e_{l+m+1})h^{-1}$ is as in Proposition 9.26, then

$$[(\tilde{E}_n,\tilde{\rho}_n),\beta_n,\delta_0\tilde{\rho}(e_{l+m+1})\delta_0^{-1}] = [(\tilde{E}_n,\tilde{\rho}_n),\beta_n,\delta_1\tilde{\rho}(e_{l+m+1})\delta_1^{-1}]$$

iff there exists a homotopy $H : [0,1]^2 \to Aut_{(E_n(X\times[0,1]))^{l,m}_{n\in\mathbb{N}}}(E_{2n},\rho_{2n}|_{C^{l,m}}$ is a homotopy such that $H_0 = f$, $H_1 = h$, $H_t(0) = id$ and $\rho_{2n}(e_{l+m+1})H_t(1) = -\rho_{2n}(e_{l+m+1})H_t(1)$ for all $t \in [0,1]$.

Remark 9.28. The map $\delta(t)$ denotes the map $\delta|_{E|_{X\times\{t\}}}$

Proof. The proof of the statement is similar to the proof of Lemma 9.9 and Lemma 9.10. □

We can do something similar for the group group $K_G^{l,m+1}(X,\emptyset)$.
For $n \in \mathbb{N}$, Let $(F_n,\tilde{\rho}_n)$ be defined by

$$F_1 = (E_1(X) \oplus E_1(X), \rho \oplus \rho')$$

where $\rho'|_{C^{l,m+1}} = \rho|_{C^{l,m+1}}$ and $\rho'(e_{l+m+1}) = -e_{l+m+1}$ and

$$(F_{n+1},\tilde{\rho}_{n+1}) = (F_n \oplus F_1, \tilde{\rho}_n \oplus \tilde{\rho}_1).$$

Lemma 9.29. *Let $[(E,\sigma),\alpha_+,\alpha_-] \in K_G^{l,m+1}(X,\emptyset)$. Then, there exists a $k \in \mathbb{N}$, a finite subset $I \subset G$ and a $\beta \in Gr^{l,m+1}(F_n)$ such that*

$$P_I^*[(F_n,\tilde{\rho}_n), r\tilde{h}o(e_{l+m+2}),\beta] = [(E,\sigma),\alpha_+,\alpha_-].$$

Moreover, if $N \in [G]-I$, then

$$P_N^*\beta = \tilde{\rho}(e_{l+m+2}).$$

Proof. The proof is similar to the proof of Lemma 9.25 □

Proposition 9.30. *The map $\chi : K_G^{l,m+1}(X,\emptyset) \to K^{l,m+1}((E_n(X))_{n\in\mathbb{N}})$ defined by*

$$\chi([(E,\sigma),\alpha_+,\alpha_-]) = [(F_n,\tilde{\rho}_n),\tilde{\rho}(e_{l+m+2}),\beta],$$

where β is defined as in the previous Lemma, is a well defined injective group homomorphism. Moreover, if $x := [(F_n,\tilde{\rho}_n),\tilde{\rho}(e_{l+m+2}),\beta]$ and $y := [(F_k,\tilde{\rho}_k),\tilde{\rho}(e_{l+m+2}),\gamma]$ are in the image of χ, then $x = y$ if and only if there exists a $p \in \mathbb{N}$ such that

$$\tilde{\rho}_n(e_{l+m+2}) \oplus \gamma \oplus \tilde{\rho}_p(e_{l+m+2}) \simeq \beta \oplus \tilde{\rho_{k+p}}.$$

As before, we define

$$\mathrm{Gr}^{l,m+1}((E_n)_{n\in\mathbb{N}}) := \mathrm{colim}_{n\to\infty}\mathrm{Gr}^{l,m+1}((F_n,\tilde{\rho}|_{C^{l,m+1}})),$$

where the inclusion is given by

$$f \to f \oplus \tilde{\rho}_1(e_{l+m+2}).$$

and obtain:

Theorem 9.31. *The morphism*

$$\chi_0 : K_G^{l,m+1}(X,\emptyset) \to \pi_0(Gr^{l,m+1}((E_n)_{n\in\mathbb{N}}))$$

defined by

$$[(E,\sigma),\alpha_+,\alpha_-] \to [\beta],$$

where β is defined as in Lemma 9.29, is a well defined injective monoid homomorphism.

Proof. The proof is similar to the proof of Theorem 9.5. □

We have now shown how we can use the category $(E_n)_{n\in\mathbb{N}}$ to describe the group $K^{l,m+1}(X,\emptyset)$ and $K^{l,m}(Y\times[0,1], Y\times\{0,1\})$. As in the case where G was a finite group, we will use these result to reduce the bijectivity of μ to Theorem 9.12.

As mentioned before, the space $B := (E_1(X),\rho|_{C^{l,m}})$ has the structure of a Banach algebra, where the multiplication is given by composition. The map $(\overline{}) : B \to B$ defined by $\overline{\alpha} = \rho(e_{l+m+1})\alpha\rho(e_{l+m+1})$ is an algebra homomorphism such that $\overline{\overline{\alpha}} = \alpha$ for all $\alpha \in B$. We choose the element $\epsilon := \rho(e_{l+m+2}e_{l+m+1})$ as our base point. Notice that $\epsilon^2 = -1$ and $\overline{\epsilon} = -\epsilon$. We now define the spaces $\mathrm{GL}((E_1(X),\rho_1|_{C^{l,m}})$, $\mathrm{GL}^-((E_1(X),\rho_1|_{C^{l,m}})$, $I(E_1(X),\rho_1|_{C^{l,m}})$, $\pi_0(I(E_1(X),\rho_1|_{C^{l,m}}))$ and

$$\pi_1(\mathrm{GL}((E_1(X),\rho_1|_{C^{l,m}})),\mathrm{GL}^-((E_1(X),\rho_1|_{C^{l,m}})))$$

as before. We have

Lemma 9.32. *The map* $h : Gr^{l,m+1}((E_n(X))_{n\in\mathbb{N}}) \to I((E_1(X), \rho|_{C^{l,m}}))$ *induced by the maps* $h_n : Gr^{l,m+1}(F_n, \tilde{\rho}|_{C^{l,m+1}}) \to I((E_1(X), \rho|_{C^{l,m}}))$ *with*

$$h_k(f) = f\rho_{2k}(e_{l+m+1})$$

is a homeomorphism and hence induces a bijection

$$h_* : \pi_0(Gr^{l,m+1}((E_n(X))_{n\in\mathbb{N}})) \to \pi_0(I((E_1(X), \rho|_{C^{l,m}}))).$$

Proof. The proof goes analogous to the proof of Proposition 9.6. □

Lemma 9.33. *The map*

$$\Psi : \pi_1(\mathrm{GL}((E_1, \rho_1|_{C^{l,m}})), \mathrm{GL}^-((E_1, \rho_1|_{C^{l,m}}))) \to K_G^{l,m}((E_n(X\times[0,1]))_{n\in\mathbb{N}}, (E_n(X\times\{0,1\}))_{n\in\mathbb{N}}) \tag{9.6}$$

defined on $\pi_0(\Omega(\mathrm{GL}_{2k}(E_1, \rho_1|_{C^{l,m}}), \mathrm{GL}^-_{2k}(E_1, \rho_1|_{C^{l,m}}))$ *by*

$$\Psi_{2k}(f) = [(\tilde{E}_k, \tilde{\rho}_k|_{C^{l,m}}), \beta_k, f\tilde{\rho}(e_{l+m+1})f^{-1}].$$

is a well defined monoid homomorphism. Moreover, we have

$$Im(\Phi) \subset Im(\Psi)$$

and $\Psi|_{\Psi^{-1}(Im(\Phi))}$ *is injective. (Where* Φ *is the map from Proposition 9.26)*

Proof. The proof that Ψ is a well defined monoid homomorphism is the same as in the proof of Theorem 9.11. Lemma 9.27 implies that

$$\mathrm{Im}(\Phi) \subset \mathrm{Im}(\Psi)$$

and that $\Psi|_{\Psi^{-1}(\mathrm{Im}(\Phi))}$ is injective

□

Theorem 9.12 now says the following:

Theorem 9.34. *The map* $j : \pi_0(I(E_1(X), \rho_1|_{C^{l,m}})) \to \pi_1(\mathrm{GL}((E_1(X), \rho_1|_{C^{l,m}})), \mathrm{GL}^-((E_1(X), \rho_1|_{C^{l,m}})))$ *defined by*

$$j(f)(t) = \cos(\frac{\pi}{2}t) + \sin(\frac{\pi}{2}t)f.$$

is a bijection.

With this theorem, we can finish the proof:

Theorem 9.35. *The map* $\mu : K_G^{l,m+1}(X, \emptyset) \to K_G^{l,m}(X \times [0,1], X \times \{0,1\})$. *is surjective.*

Proof. We first show that the map

$$\Psi \circ j \circ h : \mathrm{Im}(\chi_0) \to \mathrm{Im}(\Phi),$$

is a monoid isomorphism. Since h and j are isomorphism and Ψ is injective on $\Psi^{-1}(\mathrm{Im}(\Phi))$, it is sufficient to show that $j \circ h$ is well defined and surjective. Let $\gamma \in \mathrm{Im}(\chi_0)$. By definition, there exists a finite set $I \subset [G]$ such that $P_N(\beta) = \tilde{\rho}(e_{l+m+2})$ for all $N \in [G] - I$. We have $h(\gamma) = \gamma\tilde{\rho}_k(e_{l+m+1})$ and

$$j(h(\gamma))(t) = \cos(\frac{\pi}{2}t) + \sin(\frac{\pi}{2}t)\gamma\tilde{\rho}_k(e_{l+m+1}).$$

Therefore, we have

$$\Psi \circ j \circ h(\gamma) = [(\tilde{E}_k, \tilde{\rho}_k|_{C^{l,m}}), \beta_k, j(h(\gamma))\tilde{\rho}_k(e_{l+m+1})j(h(\gamma))^{-1}].$$

Notice that for $N \in [G] - I$, we have

$$P_N^*(j(h(\gamma))\tilde{\rho}_k(e_{l+m+1})j(h(\gamma))^{-1}) = P_N^*\beta_k.$$

Since Φ is well defined, Lemma 9.25 implies that

$$\Phi(P_I^*(\Psi \circ j \circ h(\gamma))) = \Psi \circ j \circ h(\gamma).$$

We now show that $\Psi \circ j \circ h$ is surjective. Let $x := [(E, \sigma), \alpha_+, \alpha_-] \in K_G^{l,m}(X \times [0,1], X \times \{0,1\})$ and

$$\Phi(x) := [(\tilde{E}_k, \tilde{\rho}_k), \beta_k, \gamma].$$

Notice that there exists a finite set $I \subset [G]$, such that

$$P_N(\gamma) = P_N(\beta_k)$$

for $N \in [G] - I$ and

$$P_I^*[(\tilde{E}_k, \tilde{\rho}_k), \beta_k, \gamma] = [(E, \sigma), \alpha_+, \alpha_-].$$

Lemma 9.33 implies that $\Phi(x) \in \mathrm{Im}(\Psi)$. Therefore, there exists a $f \in \pi_1(\mathrm{GL}((E_1(X), \rho_1|_{C^{l,m}})), \mathrm{GL}^-((E_1(X)$ such that

$$\Phi(x) = [(\tilde{E}_k, \tilde{\rho}|_{C^{l,m}}), \beta|_r, f\tilde{\rho}(e_{l+m+1})f^{-1}].$$

Theorem 9.34 says that we may assume that $f = j(\delta)$, for some $\delta \in \pi_0(I(E_1(X), \rho_1|_{C^{l,m}}))$. Let $\delta' := \delta\tilde{\rho}_k(e_{l+m+1})$. Notice that $\delta' \in \mathrm{Gr}^{l,m+1}((E_n)_{n\in\mathbb{N}})$. Let $\delta_0 \in \mathrm{Gr}^{l,m+1}((E_n)_{n\in\mathbb{N}})$ be defined by

$$P_I^*\delta_0 := \delta'$$

and

$$P_N^*\delta := \tilde{\rho}(e_{l+m+1}).$$

By construction, we have

$$P_I^*(\Psi \circ j \circ h(\delta_0)) = P_I^*(\Psi \circ j \circ h(\delta')) = x,$$

Moreover, we have

$$P_N^*(\Psi \circ j \circ h(\delta_0)) = \tilde{\rho}(e_{l+m+1}),$$

which, because Φ is well defined, implies that

$$\Phi(x) = \Psi \circ j \circ h(\delta_0)).$$

Therefore, the map $\Psi \circ j \circ h$ is an isomorphism. Since χ_0 and Φ are injective, the maps $\chi_0 : K_G^{l,m+1}(X, \emptyset) \to \mathrm{Im}(\chi_0)$ and $\Phi : K_G^{l,m}(X \times [0,1], X \times \{0,1\}) \to \mathrm{Im}(\Phi)$ are isomorphism and the map $\Psi \circ j \circ h$ induces an isomorphism

$$K_G^{l,m+1}(X, \emptyset) \to K^{l,m}(X \times I, X \times \{0,1\}).$$

By chasing through all the identifications as in Remark 9.15 , we see that this isomorphism is given by μ. Therefore, the group homomorphism μ is an isomorphism. $\square$

9.3 The group G is a Lie group and $A \neq \emptyset$

We are now almost done. We will show that the map

$$\mu : K_G^{l,m+1}(X,A) \to K_G^{l,m}(X \times [0,1], (X \times \{0,1\}) \cup (A \times [0,1])) \tag{9.7}$$

defined as before is also an isomorphism if $A \neq \emptyset$. We will proceed in three steps. We will first explain that μ is natural, then prove that μ is an isomorphism if Y is a point and will then prove the general case.

If $f : (X,Y) \to (Z,W)$ is a G-map, then f induces a map

$$\tilde{f} : (X \times [0,1], (X \times \{0,1\}) \cup (Y \times [0,1])) \to (Z \times [0,1], (Z \times \{0,1\}) \cup (W \times [0,1])).$$

A computations shows that we have

$$\tilde{f}^* \mu = \mu f^* \tag{9.8}$$

which implies that μ is natural. We will often denote $\tilde{f}^*$ as f^*. With this observation, we are ready for the case $A = \{\text{pt}\}$.

Proposition 9.36. *Let $x_0 \in X$, such that $\{x_0\}$ is a G-invariant subset. The map μ gives an isomorphism*

$$\mu : K_G^{l,m+1}(X, \{x_0\}) \to K_G^{l,m}(X \times [0,1], X \times \{0,1\} \cup \{x_0\} \times [0,1]).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_G^{l,m+1}(X,\{x_0\}) \arrow[r, "i_X^*"] \arrow[d, "\mu"] & K_G^{l,m}(X,\emptyset) \arrow[r, "i_{x_0}^*"] \arrow[d, "\mu"] & K_G^{l,m+1}(x_0,\emptyset) \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & K_G^{l,m+1}(X \times [0,1], X \times \{0,1\} \cup \{x_0\} \times [0,1]) \arrow[r, "i_X^*"] & K_G^{l,m}(X \times [0,1], X \times \{0,1\}) \arrow[r, "i_{x_0}^*"] & K_G^{l,m+1}(\{x_0\} \times [0,1], \{x_0\} \times \{0,1\}) \arrow[r] & 0
\end{tikzcd}$$

where $i_X : (X, \emptyset) \to (X, \{x_0\})$ and $i_{x_0}(\{x_0\}, \emptyset) \to (X, \emptyset)$ are the inclusions. Lemma 8.14 implies that the upper row is exact.

The map $r : (X \times [0,1], X \times \{0,1\}) \to (\{x_0\} \times [0,1], \{x_0\} \times \{0,1\})$ defined by $r(x,t) = (x_0,t)$ has the property that $r \circ i_{x_0} = id$. We can show with a proof similar to the proof of Lemma 8.14 that the lower row is also exact.

Since the map μ in the middle and rightmost column are isomorphism, the map μ in the first row is an isomorphism. $\square$

We can now finally prove the Theorem:

Theorem 9.37. *Let X be a compact G-space and let $A \subset X$ be a closed G-invariant subset. The map*

$$\mu : K^{l,m+1}(X,A) \to K^{l,m}(X \times [0,1], X \times \{0,1\} \cup A \times [0,1]).$$

defined by

$$\mu((E,\sigma),\alpha_+,\alpha_-) = ((p^*E, p^*\sigma|_{C^{l,m}}), \cos(t)\sigma(e_{l+m+1}) + \sin(t)\alpha_+, \cos(t)\sigma(e_{l+m+1}) + \sin(t)\alpha_-).$$

is a natural isomorphism

Proof. The map μ is natural in the sense of equation 9.8. We now show that μ is an isomorphism. In Lemma 8.15 we showed that the projection $\pi : (X, A) \to (X/A, A/A)$ defines an isomorphism

$$\pi^* : K_G^{l,m}(X/A, A/A) \to K_G^{l,m}(X, A).$$

Now consider the diagram:

$$\begin{tikzcd}
K_G^{l,m+1}(X/A, A/A) \arrow[r, "\pi^*"] \arrow[d, "\mu"] & K_G^{l,m}(X, A) \arrow[d, "\mu"] \\
K^{l,m}(X/A \times [0,1], A/A \times [0,1] \cup X/A \times \{0,1\}) \arrow[r, "\tilde{\pi}^*"] & K^{l,m}(X \times [0,1], X \times \{0,1\} \cup A \times [0,1])
\end{tikzcd}$$

Where the map $\tilde{\pi} : (X \times [0,1], X \times \{0,1\} \cup A \times [0,1]) \to (X/A \times [0,1], A/A \times [0,1] \cup X/A \times \{0,1\})$ is the projection. Since μ is natural, the diagram commutes. With a reasoning similar to that of Lemma 8.15, we can also show that $\tilde{\pi}^*$ is an isomorphism. Proposition 9.36 implies that the map μ in the left column is an isomorphism. Therefore, the map

$$\mu : K^{l,m+1}(X, A) \to K^{l,m}(X \times [0,1], X \times \{0,1\} \cup A \times [0,1])$$

is an isomorphism. $\square$

A Appendix

A.1 Haar measure/integration of vector valued functions

A.1.1 Haar Measure

In this section, we will show how a continuous function $f : G \to \mathbb{C}$, where G is a Lie group, can be integrated over the Lie group G. To accomplish this we will define the Haar measure, show how it can be constructed and prove some basic properties of the Haar measure. We will follow the approach from section 19 of [9].
We will start with the definition of a density on a vector space.

Definition A.1. *Let V be a n-dimensional real vector space. We call a function $\mu : V^n \to \mathbb{C}$ a density if for every linear map $L : V \to V$ the pullback $T * \mu := \mu \circ (T, \ldots, T)$ satisfies*

$$T^*\mu = |\det(T)|\mu.$$

Example A.2. If $V = \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, then the map $\mu(v_1, \ldots, v_n) = \lambda|\det((v_1, \ldots, v_n))|$, where we view v_i as a column of a matrix, is a density.

Let DV denote the complex vector space of densities on V.The example above actually already describes DV.

Lemma A.3. *The vector space DV is a one dimensional complex vector space.*

Proof. Let $e_1, \ldots e_n$ be a basis of V and let λ be densities on V. Let $(v_1, \ldots, v_n) \in V^n$. Then

$$\begin{aligned} \lambda(v_1, \ldots v_n) &= \lambda((v_1, \ldots, v_n)(e_1), \ldots, (v_1, \ldots, v_n)(e_n)) \\ &= |\det(v_1, \ldots, v_n)|\lambda(e_1, \ldots, e_n). \end{aligned}$$

Thus the density is uniquely determined by its value on $(e_1, \ldots, e_n)$ and the dimension of $D(V)$ is at most 1. The example above shows that the dimension of $D(V)$ is at leat 1. Therefore, $D(V) = 1$. □

If $A : V \to W$ is a linear map and $\mu : W^n \to \mathbb{C}$ is a density on W, then we can use μ to define a density $A^* : V^n \to \mathbb{C}$ by

$$A^*\mu := \mu \circ (A, \ldots, A).$$

Now let M be a smooth manifold and let TM denote its tangent bundle. Then we can define the density bundle DM, by $DM_m := D(TM_m)$.

Lemma A.4. *The density bundle is a complex line bundle.*

Proof. Let $\chi_U : U \to \mathbb{R}^n$ be a chart . Notice that $TU \cong U \times \mathbb{R}^n$. Let $\mu : \mathbb{R}^n \to \mathbb{C}$ denote the density defined by $\mu(e_1, \ldots, e_n) = 1$, where $(e_1, \ldots, e_n)$ denotes the standard basis. We can now define the map $s_x : U \to DU$, defined by $s(x) = \mu$. The map s_U is a local frame of DM. If $\chi_V : V \to \mathbb{R}^n$ is a chart such that $U \cap V \neq \emptyset$, then we also obtain a section $s_V : U \to DU$. Notice that

$$\begin{aligned} s_V(x) &= \mu(e_1^V, \ldots, e_n^V) \\ &= \mu(T(\chi_U\chi_V^{-1})_x)(e_1^U, \ldots, e_n^U) \\ &= |\det(T(\chi_U\chi_V^{-1})_x)|\mu(e_1^U, \ldots, e_n^U) \\ &= |\det(T(\chi_U\chi_V^{-1})_x)|s_U(x). \end{aligned}$$

This implies that the frames we defined are smoothly compatible. One can check that the charts $\Phi_U DM \to \mathbb{C}$, defined by $\Phi_U^{-1}(x, v) = (\chi_U^{-1}(x), vs_U)$ give DU a smooth structure such that $DM \to M$ is a line bundle. $\square$

We denote the space of continuous sections of the density bundle of a manifold M by $\Gamma(DM)$. We will call a $s \in \Gamma(DM)$ a density on M. We call a density $\lambda \in \gamma(M)$ positive if $\text{Image}(\lambda_m) \subset [0, \infty[$ and $\lambda_m \neq 0$ for all $m \in M$.
In example A.2 we saw that a density assigns to a vector space a multiple of (the absolute value of) the determinant. Since the determinant 'measures' the n-dimensional volume of vectors, a sections $s \in \gamma(DM)$ assigns to each $m \in M$ a way to 'measure' the volume enclosed by the tangent vectors. We need one more definition before we can use this intuition to define how to integrate over a density.

Definition A.5. *Let M, N be manifolds, $F : M \to N$ be a diffeomorphism and $\lambda \in \Gamma(DN)$, then we define the pull-back of λ allong F by*

$$F^*(\lambda)(m) := TF(x)^*\lambda(F(x)).$$

Notice that $F^*(\lambda) \in \Gamma(DM)$.

We are now ready to define how to integrate over a density. Let $U \subset \mathbb{R}^n$, $f \in C_0(U, \mathbb{C})$ and Let $\mu \in \Gamma(DU)$ denote the section of the density bundle such that $\mu_x(e_1, \dots e_n) = 1$. Then we define

$$\int_U f\mu := \int_{\mathbb{R}^n} f(x)dx,$$

where the second integral is the Lebesgue integral.
Now let M be a smooth manifold, $(U', \chi_{U'} : U' \to U)$ a chart and $\lambda \in \Gamma(DU')$ a compactly supported density. Notice that $(\chi_{U'}^{-1})^*\lambda = g\mu$ for a $g \in C_0(U, \mathbb{C})$. Therefore, we can define

$$\int_{U'} \lambda := \int_U (\chi_U^{-1})^*\lambda = \int_U g dx.$$

It can be shown with the change of variable formula that this definition does not depend on the choice of chart. We can now define the integral of a density over a compact manifold:

Definition A.6. *Let M be a compact manifold, λ be a density on M and $\{(U_i, \chi_{U_i})\}_{i \in I}$ be a finite cover of local trivialisations. Let ψ_i be a partition of unity subordinated to this cover, then we have*

$$\int_M \lambda := \sum_{i \in I} \int_{U_i} \psi_i \lambda.$$

This integral has the following useful properties:

Lemma A.7. *Let M, N be compact smooth manifolds and let $F : N \to M$ be a diffeomorphism, then*

(i) The integral $\int_M \lambda$ does not depend on the choice of cover or partition of unity.

(ii) $\int_N F^*\lambda = \int_M \lambda$.

(iii) If $\lambda(v_1, \dots, v_n) \geq 0$ for all $v_1, \dots, v_n \in TM_m$ and all $m \in M$, then $\int_G \lambda \geq 0$ and $\int_G \lambda = 0$ implies that $\lambda = 0$.

Proof. We leave the verification of part (i) to the reader. Part (ii) is a corollary of the change of variable theorem. We now prove part (iii). Notice that

$$\int_M \lambda = \sum_{i\in I} \int_{U_i} \psi_i \lambda.$$

the integral $\int_{U_i} \psi_i \lambda = \int_{\mathbb{R}^n} \psi_i g_i dx$, where g_i is a continuous non-negative function. Hence, for all $i \in I$, $\int_{U_i} \psi_i \lambda \geq 0$ and $\int_{U_i} \psi_i g_i = 0$ if and only if $\psi_i g_i = 0$, which implies that $\int_M \lambda \geq 0$ and $\int_M \lambda = 0$, if and only if $\lambda = 0$. □

We can also use densities to integrate a function over a compact manifold:

Definition A.8. *Let $f : G \to \mathbb{C}$ be continuous and $\lambda \in \Gamma(DM)$. Then, we have $f \cdot \lambda \in \Gamma(DM)$ and we define*

$$\int_G f\gamma := \int_G (f\gamma).$$

We will now switch our attention to invariant densities on a compact Lie group G.

Definition A.9. *Let G be a compact Lie group and $\omega \in \Gamma(DG)$. We cal ω left invariant if $l_g^* \omega = \omega$, where $l_g(h) = gh$. We will denote the set of invariant sections by $\Gamma_G(DG)$.*

It turns out that there is a simple characterisation of the left invariant densities on a Lie group.

Lemma A.10. *The map $pr_e : \Gamma_G(DG) \to D(TG_e)$ defined by $pr_e(\lambda) = \lambda(e)$, is an isomorphism.*

Proof. First, notice that pr_e is linear. Let $\lambda \in \Gamma_G(DG)$. Then

$$\lambda(g) = (l_{g^{-1}}^* \lambda)(g) = Tl_{g^{-1}}(g)^* \lambda(e).$$

It follows that if $\lambda(e) = 0$, then $\lambda = 0$. Thus pr_e is injective. We now show that pr_e is surjective. Let $\gamma \in D(TG_e)$, we define $\tilde{\lambda} : G \to DTG$, by

$$\tilde{\lambda}(g) := Tl_{g^{-1}}(g)^* \lambda = \lambda \circ (Tl_{g^{-1}}(g), \ldots, Tl_{g^{-1}}(g)).$$

Notice that $\tilde{\lambda}$ is continuous and that for $h, g \in G$, we have

$$\begin{aligned}
l_h^* \tilde{\lambda}(g) &= (Tl_h)(g)^* \tilde{\lambda}(hg) \\
&= (Tl_h)(g)^* ((Tl_{(hg)^{-1}}(hg))^* \lambda(hg)) \\
&= (Tl_h)(g)^* (\lambda(Tl_{g^{-1}}(g) Tl_{h^{-1}}(hg), \ldots, Tl_{g^{-1}}(g) Tl_{h^{-1}}(hg)) \\
&= \lambda \circ (Tl_{g^{-1}}(g) Tl_{h^{-1}} Tl_h(g), \ldots, Tl_{g^{-1}}(g) Tl_{h^{-1}} Tl_h(g)) \\
&= \lambda \circ (Tl_{g^{-1}}(g), \ldots, Tl_{g^{-1}}(g)) \\
&= \tilde{\lambda}(g).
\end{aligned}$$

Therefore, $\tilde{\lambda} \in \Gamma_G(DG)$ and pr_e is surjective. □

Remark A.11. Notice that if $\lambda \in D(TG_e)$ is positive, then $\tilde{\lambda}$ is also positive.

We are now ready to define the Haar measure.

Theorem A.12. *Let G be a Lie group, then there is a unique positive density $dx \in \Gamma_G(DTG)$, such that $\int_G dx = 1$. We call dx the Haar measure.*

Proof. Let $\lambda \in DTG_e$, such that $\lambda > 0$. Then $\tilde{\lambda} \in \Gamma_G(DTG)$ is positive and Lemma A.7 implies that $\int_G \tilde{\lambda} = c \neq 0$. Therefore, $dx := \frac{\tilde{\lambda}}{c}$ is the required density.
We now show uniqueness. Assume that dx' is also a Haar measure. Lemma A.10 and the fact that DTG_e is one dimensional implies that $dx = adx'$, for a $a \in \mathbb{C}$. Since $a = a\int_G dx = \int_G adx = \int_G dx' = 1$, it follows that $dx = dx'$. □

Remark A.13. We will usually denote the Haar measure by dx or dg, depending on the variable we use.

We now state some useful properties of the Haar measure.

Lemma A.14. *Let G be a compact Lie group and let dx denote the Haar measure. Let $f, g : G \to \mathbb{C}$ be continuous and let $\lambda \in \mathbb{C}$. then the following statements hold:*

(i) $\int_G \lambda f + gdx = \lambda \int_G fdx + \int_G gdx$.

(ii) $\int_G f \circ l_g dx = \int_G fdx$

(iii) $r_g^* dx = dx$.

Proof. Part (i) follows directly from the definition. We now show part (ii). Notice that

$$\int_G f \circ l_g dx = \int_G (f \circ l_g) l_g^*(dx) = \int_G l_g^*(f \cdot dx).$$

Because l_g is a diffeomorphism, it follows that $\int_G l_g^*(f \cdot dx) = \int_G fdx$, which proves part (ii). We now show that (iii) holds. We first show that $r_g^*(dx) = |\det(TC_{g^{-1}}(e))|dx$, where $C_g := l_g \circ r_g^{-1} = r_g^{-1} \circ l_g$. Since dx is left invariant, we have for all $g, h \in G$ that

$$\int_G l_h^* r_g^* dx = \int_G (r_g \circ l_h)^* dx = \int_G (l_h \circ r_g)^* dx = \int_G (r_g)^*(l_h^* dx) = \int_G (r_g)^* dx.$$

Therefore, $r_g^* dx \in \Gamma_G(DG)$ and $r_g^* dx = cdx$. It follows that $C_{g^{-1}}^* dx = l_{g^{-1}}^*(r_g^* dx) = cdx$ Applying pr_e to the equation, we obtain

$$C_{g^{-1}}^* dx(e) = TC_{g^{-1}}(e)^* dx(e) = |\det(TC_{g^{-1}}(e))| dx(e),$$

which implies that $c = |\det(TC_{g^{-1}}(e))|$ We now show that $|\det(TC_{g^{-1}}(e))| = 1$. Since $TC_{gh} = TC_g(e)TC_h(e)$, it follows that

$$|\det(TC_{(gh)^{-1}}(e))| = |\det(TC_{h^{-1}}(e)TC_{g^{-1}}(e)| = |\det(TC_{h^{-1}}(e))||\det(TC_{g^{-1}}(e))|.$$

Therefore, the map $h : G \to (0, \infty)$ defined by $h(g) = |\det(TC_{g^{-1}}(e))|$ gives a group homomorphism between G and $(0, \infty)$, where the group structure on $(0, \infty)$ is given by multiplication.. Since G is compact, it follows that $h(G)$ is a compact subgroup of $(0, \infty)$. Since the only compact subgroup of $(0, \infty)$ is 1, we have $h(g) = 1$ for all $g \in G$, which proves (iii). □

Remark A.15. If G is a zero dimensional compact Lie group (a finite group), we can not use densities to define the Haar measure. However, we can define our Haar-measure as

$$\int_G fdx := \frac{1}{|G|} \sum_{g \in G} f(g),$$

where $|G|$ denotes the number of elements of G.

Remark A.16. In this section, we have defined the Haar measure using densities, which requires our group to be a Lie group. It is possible to define the Haar measure for any locally compact (Hausdorff) group. The construction for the more general case can for example be found in section 9 of [2]

A.1.2 Integration of vector valued functions

In the previous section, we showed how the integral $\int_G f dx$ is defined for a continuous function $f : G \to \mathbb{C}$. In this section, we show how a function $f : G \to \mathbb{C}^n$ can be integrated using the Haar measure.

Definition A.17. *Let G be a Lie-group, let dx denote the Haar measure and let $f = (f_1, \dots, f_n) : G \to \mathbb{K}^n$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be a continuous function. Then we define:*

$$\int_G f dx := (\int_G f_1 dx, \dots, \int_G f_n dx).$$

This integral has the following useful properties:

Lemma A.18. *Let $f, g : G \to \mathbb{K}^n$ be continuous functions. Then the following statements hold:*

(i) $\int_G \lambda f + g dx = \lambda \int_G f dx + \int_G g dx$

(ii) If $A : \mathbb{K}^n \to \mathbb{K}^m$ is a linear map, then

$$\int_G Af dx = A \int_G f dx.$$

(iii) If $g : \mathbb{K}^n \to \mathbb{K}^n \to \mathbb{K}$ is an inner product and $\| \cdot \|$ is the norm associated to the inner product, then

$$\| \int_G f dx \| \leq \int_G \|f\| dx.$$

(iv) We have $\int_G f \circ l_g dx = \int_G f dx$.

Remark A.19. Part (ii) of this lemma implies that the integral does not depend on the choice of basis and is hence defined for functions $f : G \to V$, where V is an n-dimensional $\mathbb{K}$ vector space. This holds because if $f = (f_1, \dots, f_n)$ with respect to a basis $(e_1, \dots, e_n)$ and $f = (f_1', \dots, f_n')$ with respect to a basis $(f_1, \dots, f_n)$ and $A : V \to V$ the linear map such that $A(f_i) = f_i'$, then

$$\int_G (f_1', \dots, f_n') dx = \int_G A(f_1, \dots, f_n) dx = A \int_G (f_1, \dots, f_n).$$

Proof. Part (i) and (ii) follow directly from the fact that for a functions $f, g : G \to \mathbb{K}$ and $\mu \in \mathbb{K}$, we have

$$\int_G \lambda f + g dx = \lambda \int_G f dx + \int_G g dx.$$

Part (iv) follows directly from part (ii) of Lemma A.14, because

$$\int_G f \circ l_g dx = (\int_G f_1 \circ l_g dx, \dots, \int_G f_n \circ l_g dx) = (\int_G f_1 dx, \dots, \int_G f_n dx) = \int_G f dx.$$

We now prove statement (iii). Part (ii) implies that we can assume that the standard basis is an orthonormal basis of $\mathbb{K}^n$. Notice that

$$\| \int_G f dx \| = \| \sum_{i \in I} \int_{U_i} \psi_i f dx \| \leq \sum_{i \in I} \| \int_{U_i} \psi_i f dx \|,$$

and

$$\int_G \|f\| dx = \int_G \sum_{i \in I} \psi_i \|f\| dx = \sum_{i \in I} \int_{U_i} \|\psi_i f\| dx,$$

it is sufficient to show that for all $i \in I$,

$$\| \int_{U_i} \psi_i f dx \| \leq \int_{U_i} \|\psi_i f\| dx.$$

Therefore, it is sufficient to show that

$$\| \int_{\mathbb{K}^n} f dx \| \leq \int_{\mathbb{K}^n} \|f\| dx,$$

where $f = (f_1, \ldots, f_n)$ is a continuous function with compact support and dx denotes the Lebesgue integral. Because for all $1 \leq i \leq n$, the function f_i is integrable, there exists a sequence of simple functions (functions with a finite image finite) $\{\phi_i^k\}_{k \in \mathbb{N}}$ such that $\mathrm{Re}|\phi_i^k(v)| \leq |\mathrm{Re} f(v)|$ and $|\mathrm{Im}\phi_i^k(v)| \leq |\mathrm{Im}(v)|$ for all $v \in \mathbb{K}^n$ and

$$\lim_{k \to \infty} \int_{\mathbb{K}^n} \phi_i^k dx = \int_G f_i dx.$$

For all $k \in \mathbb{N}$, we can assume that $\phi_i^k = \sum_{j=1}^m a_i^j \chi_{A_j}$, where for all $1 \leq i \leq k$ and $1 \leq j \leq m$, $a_i^j \in \mathbb{K}$, $A_i \subset \mathbb{K}^n$ are disjoint and $\chi_{A_i} : \mathbb{K}^n \to \mathbb{K}$ is defined by $\chi(x) = 1$ if $x \in A_i$ and 0 else. Let $\varphi^k = (\phi_1^k, \ldots, \phi_n^k)$, then

$$\| \int_G \varphi^k dx \| = \| (\sum_{j=1}^m a_1^j dx(A_i), \ldots, \sum_{j=1}^m a_n^j dx(A_j)) \| \leq \sum_{j=1}^m \|(a_1^j, \ldots, a_n^j)\| dx(A_i),$$

where $dx(A_i)$ is the measure/volume of A_i. Because $|\mathrm{Re}\phi_i^k(g)| \leq |\mathrm{Re} f(g)|$ and $|\mathrm{Im}\phi_i^k(g)| \leq |\mathrm{Im} f(g)|$ for all $g \in G$ and $1 \leq i \leq k$, we have $|a_i^j| \leq |f_i(g)|$ for all $g \in G$. Since our basis is orthonormal, we have $\|(a_1^j, \ldots, a_n^j)\| \leq \|f(g)\|$ for all $g \in A_j$. Since the A_i are disjoint, the function $\|\phi^k\|$ is a simple function such that $\|\phi^k\| \leq \|f\|$. Therefore,

$$\| \int_G \phi^k dx \| \leq \sum_{j=1}^m \|(a_1^j, \ldots, a_n^j)\| dx(A_i) = \int_G \|\phi^k\| dx \leq \int_G \|f\| dx$$

and

$$\| \int_G f dx \| = \lim_{k \to \infty} \| \int_G \varphi^k dx \| \leq \lim_{k \to \infty} \int_G \|\phi^k\| dx \leq \int_G \|f\| dx.$$

$\square$

A.2 Representation of groups

In this section, we will introduce some notions of representation theory of Lie groups. The aim of this section is to develop enough theory to state the Peter-Weyl Theorem. This section is based on chapter 20, 21 and 23 of [9].
In this section, G will always be a compact Lie group. We start with the definition of a representation:

Definition A.20. *Let V be a Banach space over $\mathbb{C}$ (or $\mathbb{R}$). A complex (or real) representation (α, V) of G in V is a continuous left action $\alpha : G \times V \to V$, such that $\alpha(g) := \alpha|_{\{g\}\times V} : V \to V$ is a linear map. We will call the representation finite dimensional if V is finite dimensional.*

Example A.21. If we set $V = \mathbb{C}$ and $G = S^1$, then the map $\alpha : S^1 \times \mathbb{C} \to \mathbb{C}$, defined by

$$\alpha(e^{i\phi}, z) = e^{i\phi} \cdot z,$$

is an example of a complex representation.

Example A.22. If we set $V = \mathbb{R}^2$ and $G = S^1$. The map $\alpha : S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\alpha((\cos(\phi), \sin(\phi))) := \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

is an example of a real representation of S^1. Notice that this representation is obtained from the example above, by forgetting the complex structure.

Example A.23. Let G be a Lie group and let $V = L^2(G) := \{f : \mathbb{C} \to G : \int_G |f^2| dx < \infty\}/\sim$, where we identify functions if $\|f - g\|_{L^2(G)} = \int_G |f - g|^2 dx = 0$. Notice that $(L^2(G), \|\cdot\|_G)$ is a Banach space. The maps $L, R : G \times L^2(G) \to G$, defined by

$$L_g f = f \circ l_{g^{-1}}$$

and

$$R_g = g \circ r_g$$

where $r_g(x) = xg$ and $l_g(x) = gx$, are examples of infinite dimensional representations.

We have now seen some examples of representations. Just like G-vector bundles, some representations can be constructed using other representations. This motivates the following definition:

Definition A.24. *Let (α, V) be a representation of G and let $V' \subset V$ be a linear subspace. We call V' an invariant subspace if $\alpha(g)V' \subset V'$ for all $g \in G$. We call V an irreducible representation if 0 and V are the only closed invariant subspaces and $V \neq 0$.*

Remark A.25. The representation of Example A.22 is an example of an irreducible representation.

To write a representation (α, V) as a sum of irreducible representation our representation must be unitary (V has an inner product $\langle \cdot, \cdot \rangle$, such that $\langle \alpha(g)v, \alpha(g)w \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in V$). The following lemma says that this is possible for finite dimensional representations:

Lemma A.26. *Let (α, V) be a representation of G. Then we can endow V with an inner product, such that the representation is unitary.*

Proof. This is a special case of Lemma 2.33 where we view our representation as a G-vector bundle over a point. $\square$

This lemma has the following useful consequences:

Lemma A.27. *Let (α, V) be a finite dimensional representation of G and let U be an invariant subspace, then there exists an invariant subspace $H^{\perp}$, such that $V = H \oplus H^{\perp}$.*

Proof. This is a special case of Proposition 2.34, where we again view our representation as a G-vector bundle over a point. $\square$

Proposition A.28. *Let (α, V) be a finite dimensional representation. Then there exists finite dimensional irreducible representations (α_i, V_i) for $1 \leq i \leq n$, such that $V = \oplus_{i=1}^{n} V_i$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$*

Proof. We proof the theorem with induction on the dimension of V. Notice that the proposition holds if $\dim(V) = 1$. Now assume that the proposition holds for representations of dimension n and that $\dim(V) = n + 1$. If V is irreducible, then the proposition holds. If V is not irreducible, then V has an invariant subspace $V \neq W \neq 0$. Lemma A.27 implies that $V = W \oplus W^{\perp}$, with $\dim(W), \dim(W^{\perp}) < \dim(V)$. Therefore, the proposition holds for W and $W^{\perp}$ and thus for $V = W \oplus W^{\perp}$. $\square$

For the rest of this section, we will assume that our representations are complex representations.
We will now define the notion of a morphism between representations.

Definition A.29. *Let (V, α) and (W, β) be representations. A linear map $L : V \to W$ is an equivariant map if for all $g \in G$, we have $L \circ \alpha(g) = \beta(g) \circ L$.*

Remark A.30. If we view V and W as G-vector bundles over a point, then the notion of an equivariant map between these G-vector bundles coincides with the definition above.

It turns out that there are not many morphisms between finite dimensional irreducible representations:

Lemma A.31. *If (V, α) and (W, β) are finite dimensional irreducible representations, the following statements hold:*

(i) If $A : V \to V'$ is equivariant and not an isomorphism, then $A = 0$.

(ii) We have $End_G(V) := \{A : V \to V \mid A \text{ is an equivariant map}\} = \mathbb{C}id$.

Proof. We first prove (i). Since A is equivariant, it follows that $\beta(g)(Av) = A(\alpha(g)v)$. Therefore, $\mathrm{im}(A)$ is an invariant subspace of W. With a similar argument, it can be shown that $\ker(A)$ is an invariant subspace of V. Since V is irreducible, we have $\ker(A) = V$ or 0. If $\ker(A) = V$, the proposition holds. Now assume that $\ker(A) = 0$. Since W is irreducible, we have $\mathrm{im}(A) = 0$ or W. If $\mathrm{im}(A) = 0$, it follows that $W = 0$, which leads to a contradiction. If $\mathrm{im}(A) = W$, then A is an isomorphism, which also leads to a contradiction. Therefore, $\ker(A) = V$ and $A = 0$.
We now show (ii). Let $A \in \mathrm{End}(V)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Since we are

working with complex vector spaces, there exists a $0 \neq v \in V$, such that $Av = \lambda v$. Let $w \in V$. Since (α, V) is irreducible, there exists $g_1, \ldots, g_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, such that $w = \sum_{i=1}^n \lambda_i \alpha(g_i)v$. We can now compute

$$Aw = A(\sum_{i=1}^n \lambda_i \alpha(g_i)v) = \sum_{i=1}^n \lambda_i A\alpha(g_i)v = \sum_{i=1}^n \lambda_i \alpha(g_i)Av = \sum_{i=1}^n \lambda_i \alpha(g_i)\lambda v = \lambda \sum_{i=1}^n \lambda_i \alpha(g_i)v = \lambda w,$$

which implies that $A = \lambda id$. Because $(\mu \cdot id)\alpha(g) = \alpha(g)(\mu \cdot id)$ for all $\mu \in \mathbb{C}$ and $g \in G$, it follows that

$$\mathrm{End}_G(V) = \mathbb{C}id.$$

□

We will now assign a space of functions to a representation.

Definition A.32. *Let (α, V) be a finite dimensional unitary representation. A matrix element of (α, V) is a function $m_{v,w} : G \to \mathbb{C}$, defined by*

$$m_{v,w}(x) = \langle \pi(x)v, w \rangle,$$

where $v, w \in V$. We denote the space of matrix elements of α by $C_\alpha(G)$.

Remark A.33. Notice that because V is finite dimensional, the space $C_\alpha(G)$ is also finite dimensional.

The space $C_\alpha(G)$ can also be described in the following way:

Lemma A.34. *Let $f \in C_\alpha(G)$, then there exists a linear map $A : V \to V$, such that $f(x) = tr(\pi(x)A)$. Also, if $A : V \to V$ is a linear map, then $tr(\pi(\cdot)A) \in C_\alpha(G)$.*

Proof. For $u, v \in V$, we define the map $T_{u,v} : V \to V$ by $T_{u,v}(x) = \langle x, u \rangle v$. Notice that $\mathrm{tr}(T_{w,v}) = v$. This holds, because if $(v, e_1, \ldots, e_{n-1})$ is a basis of V, then with respect to this basis we have

$$T_{w,v} = \begin{pmatrix} \langle v, w \rangle & \langle e_1, w \rangle & \ldots & \langle e_n, w \rangle \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

which implies that $\mathrm{tr}(T_{w,v}) = \langle v, w \rangle$.
Therefore, $m_{v,w}(x) = \mathrm{tr}(\alpha(x)T_{w,v})$ and

$$\sum_{i=0}^k m_{v_i,w_i}(x) = \mathrm{tr}(\alpha(x) \sum_{i=0}^k T_{w_i,v_i}),$$

which proves the first claim. Now let $A : V \to V$ be a lineair map. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then

$$A = \sum_{1 \leq i,j \leq n} \langle Av_j, v_i \rangle T_{v_j,v_i}.$$

This holds, because we have

$$(\sum_{1 \leq i,j \leq n} \langle Av_j, v_i \rangle T_{v_j,v_i})v_k = \sum_{1 \leq i \leq n} \langle Av_k, v_i \rangle T_{v_k,v_i})v_k = \sum_{1 \leq i \leq n} \langle Av_k, v_i \rangle v_i = Av_k.$$

for all $1 \leq k \leq n$. Therefore,

$$\begin{aligned}
\operatorname{tr}(\alpha(x)A) &= \operatorname{tr}(\alpha(x) \sum_{1\leq i,j\leq n} \langle Av_j, v_i\rangle T_{v_j,v_i}) \\
&= \sum_{1\leq i,j\leq n} \langle Av_j, v_i\rangle \operatorname{tr}(\alpha(x)T_{v_j,v_i}) \\
&= \sum_{1\leq i,j\leq n} \langle Av_j, v_i\rangle m_{v_i,v_j}(x) \in C_\alpha(G).
\end{aligned}$$

□

On $L^2(G)$, we define the representation $(R, L) : (G \times G) \times C_\alpha(G) \to C_\alpha(G)$, by $(R_g, L_h)(f) = R_x \circ L_h(f)$. The matrix elements now have the following properties:

Lemma A.35. *Let (V, α) and (W, β) be finite dimensional irreducible unitary representations. Then the following holds:*

(i) $C_\alpha(G)$ is an (R_g, L_h) invariant subspace of $L^2(G)$.

(ii) If there exists an equivariant isomorphism $L : V \to W$, then $C_\alpha(G) = C_\beta(G)$.

(iii) If (V, α) and (W, β) are not isomorpic, then $C_\alpha(G) \perp C_\beta(G)$ in $L^2(G)$.

Proof. We first show (i). Notice that

$$(R_g, L_h)(\operatorname{tr}(\alpha(\cdot)A))(x) = \operatorname{tr}(\alpha(h^{-1}xg)A) = \operatorname{tr}(\alpha(h^{-1})\alpha(x)\alpha(g)A) = \operatorname{tr}(\alpha(x)(\alpha(g)A\alpha(h^{-1}))).$$

Since $\alpha(g)A\alpha(h^{-1})$ is lineair, this implies that $\operatorname{tr}(\alpha(x)(\alpha(g)A\alpha(h^{-1}))) \in C_\alpha(G)$. Second, we show (ii). Notice that for all $g \in G$, we have

$$\operatorname{tr}(\alpha(g)A) = \operatorname{tr}(\alpha(g)L^{-1}LA) = \operatorname{tr}(L^{-1}\beta(g)LA) = \operatorname{tr}(\beta(g)LAL^{-1}).$$

Since $A \to LAL^{-1}$ is a bijection from $\operatorname{End}(V)$ onto $\operatorname{End}(V')$, It follows with lemma A.34 that $C_\alpha(G) = C_\beta(G)$.
Lastly, we prove (iii). Let $L_{w',w} : V \to V'$ be the map defined by

$$L_{w,u}v = \int_G \langle \alpha(g)v, u\rangle \beta(g^{-1})wdg,$$

where $u \in V$ and $w \in W$. Notice that

$$\begin{aligned}
L_{w,u}\alpha(h)v &= \int_G \langle \alpha(gh)v, u\rangle \beta(h)\beta(gh)^{-1})wdg \\
&= \beta(h) \int_G \langle \alpha(gh)v, u\rangle \beta(gh)^{-1})wdg \\
&= \beta(h) \int_G \langle \alpha(g)v, u\rangle \beta(g^{-1})wdg \\
&= \beta(h)L_{w,u}v.
\end{aligned}$$

Thus $L_{w,u}$ is equivariant and lemma A.31 implies that $L_{w,u} = 0$. Let $w' \in W$. Notice that the map $\langle \cdot, w' \rangle : W \to \mathbb{C}$ is linear. Lemma A.18 implies that

$$\begin{aligned} 0 = \langle L_{w,u} v, w' \rangle &= \langle \int_G \langle \alpha(g) v, u \rangle \beta(g^{-1}) w dg, w' \rangle dg \\ &= \int_G \langle \alpha(g) v, u \rangle \langle \beta(g^{-1}) w, w' \rangle dg \\ &= \int_G m^\alpha_{v,u}(g) \overline{m^\beta_{w',w}(g)} dg \\ &= \langle m^\alpha_{v,u}, m^\beta_{w',w} \rangle_{L^2(G)}. \end{aligned}$$

Thus $C_\alpha(G) \perp C_\beta(G)$ in $L^2(G)$. □

We are now ready to state the Peter-Weyl theorem. Let $[G]$ denote the set of isomorphism classes of finite dimensional irreducible representations of G. Then the following statement holds:

Theorem A.36 (Peter-Weyl theorem)**.** *The subspace*

$$\bigoplus_{\alpha \in [G]} C_\alpha(G)$$

is dense in $L^2(G)$*.*

Proof. The proof of this theorem can, for example, be found in chapter 25 of [9] □

A.2.1 Equivariant maps between representations

In section we will elaborate a bit on Lemma A.31 and will describe the set $\mathrm{Hom}(V, W)$ of equivarinat maps between two finite dimensional representations (V, α) and (W, β) of G.

Lemma A.37. *Let* (V, α) *be an irreducible real representation of* G*. We have*

$$End_G(V) \cong \mathbb{R},\ \mathbb{C} \text{ or } \mathbb{H}.$$

Proof. We show that End_G is a finite dimensional associative division algebra over $\mathbb{R}$. First, notice that in the proof of part (i) of A.31 we did not use that the representation was complex. Hence, if $A \in \mathrm{End}_G(V)$ and A is not an isomorphism, then $A = 0$. Also notice that $\mathrm{End}_G(V) \subset \mathrm{End}(V)$ is a sub vector space and hence finite dimensional. If $A, B \in \mathrm{End}_G(V)$, then $AB \in \mathrm{End}_G(V)$. Therefore, $\mathrm{End}_G(V)$ is a finite dimensional associative algebra over $\mathbb{R}$. Notice that $id_V \in \mathrm{End}_G(V)$. If $A \in \mathrm{End}_V(G)$ and $A \neq 0$, then A is an isomorphism and A^{-1} exists . Since $l_g \circ A = A \circ l_g$ it follows that

$$A^{-1} \circ l_g = A^{-1} \circ l_g \circ A \circ A^{-1} = A^{-1} \circ A \circ l_g \circ A^{-1} = l_g \circ A^{-1}.$$

Therefore, we have $A^{-1} \in \mathrm{End}_G(V)$ and $\mathrm{End}_G(V)$ is a finite dimensional associative division algebra over $\mathbb{R}$. This implies that $\mathrm{End}_G(V) \cong \mathbb{R},\ \mathbb{C}$ or $\mathbb{H}$ □

Theorem A.38. *Let* (V, α) *be a complex (or real) representation and let* $A \in Aut_G(V)$*. Then* A *is homotopic through equivariant automorphism of* G *to id (or* $\pm_1 id \oplus \ldots \oplus \pm_n id$*)*

Proof. We showed that there exists (non-isomorphic) irreducible representations (V_i, α_i) for $1 \leq i \leq n$, such that

$$(V, \alpha) \cong (\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} V_j, \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} \alpha_j).$$

Since $\mathrm{Hom}_G(V_i, V_j) = 0$ if $i \neq j$, it follows that

$$\mathrm{Hom}_G(V, V) \cong \mathrm{Hom}_G(\bigoplus_{i=1}^{n} \bigoplus_{i=1}^{k_i} V_i, \bigoplus_{i=1}^{n} \bigoplus_{i=1}^{k_i} V_i) \cong \mathrm{End}_G(\bigoplus_{i=1}^{k_1} V_1) \oplus \ldots \oplus \mathrm{End}_G(\bigoplus_{i=1}^{k_j} V_n).$$

If the representation is complex, then $\mathrm{End}(V_i) = \mathbb{C}id$, which implies that

$$\mathrm{End}_G(\bigoplus_{i=1}^{k_j} V_j) \cong \mathrm{GL}_{k_j}(\mathbb{C}).$$

and

$$\mathrm{Hom}_G(V, V) \cong \mathrm{GL}_{k_1}(\mathbb{C}) \oplus \ldots \oplus \mathrm{GL}_{k_n}(\mathbb{C}).$$

Since $\mathrm{GL}_n(\mathbb{C})$ is connected, it follows that $\mathrm{Hom}_G(V, V)$ is connected and hence that A is homotopic to the identity. If the representation is real, then we have $\mathrm{End}_G(V_j) \cong \mathrm{GL}_{k_j}(\mathbb{K}_j)$, with $\mathbb{K}_j = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Notice that this implies that

$$\mathrm{Hom}_G(V, V) \cong \mathrm{GL}_{k_1}(\mathbb{K}_1) \oplus \ldots \oplus \mathrm{GL}_{k_1}(\mathbb{K}_n).$$

Since $\mathrm{GL}_{k_j}(\mathbb{K}_j)$ has at most two connected components and each component contains id and/or $-id$, the claim follows. $\square$

A.2.2 Group ring

In this section, we will introduce the group ring and prove some basic results about it. We start with the definition:

Definition A.39. *Let G be a group and $\mathbb{K}$ a field. The group ring is $\mathbb{K}[G]$ is a $\mathbb{K}$-algebra which is defined as follows: It is the free $\mathbb{K}$-vector space over G, with a multiplication which is defined on its basis elements by*

$$(\lambda_1 g_1) \cdot (\lambda_2 g_2) = (\lambda_1 \lambda_2)(g_1 g_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{K}$, $g_1, g_2 \in G$, $\lambda_1\lambda_2$ is the product in $\mathbb{K}$ and $g_1 g_2$ is the product in G.

Remark A.40. Notice that $\mathbb{K}[G]$ also has the structure of a representation of G, where the G-action is given by

$$g \cdot (\sum_{i=1}^{n} \lambda_i g_i) = \sum_{i=1}^{n} \lambda_i (g g_i).$$

The group ring has the following useful property:

Lemma A.41. *Let (V, α) be an irreducible representation of G, $v \in V - \{0\}$ and let $\pi : \mathbb{K}[G] \to M$ be the linear map defined by*

$$\pi(\sum_{i=0}^{n} \lambda_i g_i) = \sum_{i=0}^{n} \lambda_i \alpha(g_i) v,$$

Then π is surjective.

Proof. Notice that by construction $\mathrm{Im}(\pi)$ is an invariant subspace of M. Since $v \in \mathrm{Im}(\pi)$, we have $\mathrm{Im}(\pi) \neq 0$. Since M is irreducible, this implies that $\mathrm{Im}(\pi) = M$, $\square$

Remark A.42. Notice that if G is finite, then $\mathbb{K}[G]$ is a finite dimensional representation of G and π is a surjective equivariant map between these representations.

Lemma A.43. *Let (V, α) be a finite dimensional representation of a finite group G. Then, there exists a finite dimensional representation $(V^{\perp}, \alpha^{\perp})$ such that*

$$(V \oplus V^{\perp}, \alpha \oplus \alpha^{\perp}) \cong \bigoplus_{i=1}^{n} \mathbb{K}[G].$$

Proof. Proposition A.28 implies that there are representations (V_i, α_i), such that

$$(V, \alpha) \cong (\bigoplus_{i=1}^{n} V_i, \bigoplus_{i=1}^{n} \alpha_i).$$

Lemma A.41 implies that there exists a surjective equivariant map

$$\pi : \bigoplus_{i=1}^{n} \mathbb{K}[G] \to (\bigoplus_{i=1}^{n} V_i, \oplus_{i=1}^{n} \alpha_i).$$

If we view these representations as G-vector bundles over a point, then Lemma 2.35 says that there exists a representation $(V^{\perp}, \alpha^{\perp})$, such that

$$V \oplus V^{\perp} \cong (\bigoplus_{i=1}^{n} V_i) \oplus V^{\perp} \cong \bigoplus_{i=1}^{n} \mathbb{K}[G].$$

$\square$

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