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Conformal quantum mechanics in holography and Hodge theory

Master Thesis

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Abstract

$\mathrm{AdS}_2/\mathrm{CFT}_1$ is a particularly interesting example of holography since it appears naturally as the near-horizon limit of extremal black holes. However it is also quite poorly understood. In this thesis aspects of a conformal quantum mechanics model are studied, using an infinite-dimensional representation which provides a natural candidate for the CFT_1 side of the correspondence. A similar model using a finite-dimensional representation, which has possible applications in Hodge theory, is also constructed. After reviewing some basic aspects of AdS/CFT and $\mathrm{AdS}_2/\mathrm{CFT}_1$ specifically, the conformal quantum mechanics model is introduced. This model captures some of the aspects of conformal field theories and forms a discrete series representation of $SL(2,\mathbb{R})$. We then propose a completeness relation inspired by the shadow transform in conformal field theories, which provides a method with which all higher correlation functions can be expressed as integrals. Afterwards we propose an interpretation of the conformal quantum mechanics as the boundary model of quantum mechanics on the Poincaré upper half plane, which is the canonical way the discrete series representation of $SL(2,\mathbb{R})$ is realized. Finally we propose a similar construction for a finite dimensional representation of $SL(2,\mathbb{R})$ which has possible applications in Hodge theory. For this representation we also calculate correlation functions and also use the shadow transform again to define a completeness relation. Finally we define an integral transformation in order to calculate this shadow transform explicitly.

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Contents

1 Introduction

Over the last decades the holographic principle has received much attention. Originally proposed by 't Hooft [1] and partially inspired by entropy calculations of black holes [2], it conjectures that quantum gravity in $d+1$ dimensions is equivalent to an ordinary quantum (field) theory in d dimensions. Since quantum gravity remains one of the biggest unsolved problems in theoretical physics, the possibility that quantum gravity theories can be rewritten in terms of relatively well-understood quantum field theories is quite alluring.[1] In 1997, Maldacena proposed that examples of holography could be found by considering superstring theory on asymptotic anti-de Sitter (AdS) space in $d+1$ dimensions and conformal field theories (CFTs) in d dimensions [3]. String theory very naturally incorporates gravity and therefore this duality provides strong hints towards some sort of holography. There are some caveats though. One of the strengths of AdS/CFT duality is that the strong coupling regime on one side corresponds to the weak coupling regime on the other. However, this also means that it is quite hard to prove the duality since strong coupling regimes are often quite difficult to work with. Furthermore, the AdS/CFT duality uses the fact that AdS has a conformal time-like boundary quite explicitly. Therefore it is somewhat unclear how the results generalize to different space-times, especially to de Sitter space-time which describes our universe, although there has been substantial work in this area [4–6]. Despite this studying the AdS/CFT duality has led to many insights in aspects of quantum gravity, such as the Hawking information paradox [7, 8] and led to an improvement of our understanding of quantum gravity in general. Furthermore it has found applications in studying quantum chromodynamics [9] and condensed matter systems [10]. Thus it seems like AdS/CFT has much more insights to give us and studying will lead to many more interesting results.

In this thesis we will mostly focus on aspects of the lowest dimensional version of AdS/CFT, namely $\mathrm{AdS}_2/\mathrm{CFT}_1$ where this notation means that the AdS theory is two-dimensional while the CFT is one-dimensional. Contrary to what one might expect, here lower dimensions do not result in a simpler theory or a better understood one. Many results that hold in higher dimensions do not immediately generalize and there are quite a few particularities [11] as we will also see later on. This is part of what makes $\mathrm{AdS}_2/\mathrm{CFT}_1$ so interesting since a better understanding here could lead to some fundamental insights into AdS/CFT in general. Especially since one would naively expect it $\mathrm{AdS}_2/\mathrm{CFT}_1$ to be the simplest. Furthermore, since (near) extremal black holes have been observed [12] and the AdS2 geometry naturally appears when studying these near their horizons [13, 14], this provides an exciting connection between AdS/CFT and physical experiments.

With these motivations in mind we turn towards possible models for $\mathrm{AdS}_2/\mathrm{CFT}_1$. Since the Einstein-Hilbert action is non-dynamical in two dimensions, it is necessary to modify it in order to obtain insights into quantum gravity. A common method is to introduce a scalar field called a dilaton which couples to the metric and possibly to matter. These kind of fields naturally appear when performing Kaluza-Klein compactifications in string theory [15, 16] as well as when taking the near boundary limits of extremal black holes discussed before

[1] We stress the word relative here.

[17], something we will discuss in more detail in section 3.1. One particularly interesting model of dilaton gravity is Jackiw-Teitelboim gravity [18, 19], mainly because it is quite simple but has AdS_2 as well as several black hole solutions. Some progress has been made in formulating dual theories with the most promising candidates being the Sachdev-Ye-Kitaev model [20] and conformal quantum mechanics [21, 22]. In this thesis we will mostly focus on this conformal quantum mechanics model, first introduced by de Alfaro, Fubini and Ferlan (dAFF) [23]. This model is quite interesting since it can be obtained as the boundary theory for the Jackiw-Teitelboim gravity described above [24], but also since it captures most of the properties of conformal field theories in a quantum mechanics setting.

There is also a different reason to study the dAFF model coming from string theory, or more explicitly the geometry of compactifactions. Famously string theory requires certain numbers of dimensions in order to be consistent. Since it is clear that in our day to day life the universe is 4 dimensional this makes one wonder where these other dimensions have gone. The answer provided by string theory is compactification, which intuitively is a means to make the extra dimensions extremely "small" and therefore only accessible at very high energies. One common way of compactifying is Kaluza-Klein compactification [25–27], which starts from string theory on a product manifold

$$\mathcal{M} \times K \tag{1.1}$$

where $\mathcal{M}$ and K are chosen such that the combined manifold allows for a consistent string theory. K is called the internal manifold and is compact, which allows us to speak of its size in a well-defined way. A string theory on this product manifold can now be rewritten as an effective string theory living on $\mathcal{M}$ and this effective theory will depend on the geometry and size of K. A natural question is now how the effective theory changes as we vary these. It turns out that much of this information about the size and geometry is captured in the cohomology groups $H^n(K)$ and specifically their Hodge decompositions [28]. Near certain singular geometries these cohomology groups decompose in terms of finite dimensional $SL(2,\mathbb{R})$ representations [29, 30]. Since the dAFF model is also obtained by considering an $SL(2,\mathbb{R})$ representation (albeit an infinite dimensional one) this invites us to ask if it is possible to apply similar constructions to these Hodge theoretical representations. This is one of the main goals of this thesis and we will present these constructions in chapter 5. An especially interesting aspect of these variations is that the resulting space of possible variations, the so-called moduli space, appears to have some holographic properties [31, 32]. Therefore a connection between the holographic aspects on the moduli space and those of AdS/CFT might lead to many interesting results on both sides.

In this thesis we will start to investigate this connection, mostly by providing a better understanding of the boundary $SL(2,\mathbb{R})$ representations. Firstly, we will review the basics of AdS/CFT duality in chapter 2. Thereafter we will take a closer look at the lowest dimensional case of $\mathrm{AdS}_2/\mathrm{CFT}_1$ in chapter 3 and discuss where it differs from the higher dimensional analogues. Then, in chapter 4, we will discuss the dAFF model. First reviewing the original construction and afterwards proposing a method of obtaining integral expressions for all n-point correlation functions, as well as suggest an interesting connection to a certain $SL(2,\mathbb{R})$ representation on the Poincaré upper half plane. From this we will then move on to the

Hodge theoretical setting in chapter 5, where we will first propose a similar model which is applicable in this setting and investigate its properties. To this model we then apply the same tools as used for the dAFF model and will see that it has much of the same structure. Finally we will review our results and provide an outlook in chapter 6.

2 An overview of anti-de Sitter/conformal field theory duality

Since its original conjecture by Maldacena [3] the anti-de Sitter/conformal field theory (AdS/CFT) duality has been a topic of active study. It is one of the important avenues for understanding quantum gravity and the hope is that it can be extended to a more general gauge/gravity duality [33, 34]. Unfortunately, it is at present still merely a conjecture although there are a number of examples where the conjecture passes some highly non-trivial tests, with the most well-known example being type-IIB string theory on $AdS_5 \times \mathbb{S}^5$ dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills [3, 35, 36]. One of the strengths of AdS/CFT is that it cross-links high coupling one one side to low coupling on the other side of the duality, this is however also why it is so difficult to prove since there is not yet a good definition of string theory at high coupling [37, 38]. Nevertheless AdS/CFT and the related gauge/gravity duality has already led to an increased understanding of quantum gravity, with notably new insights into the Hawking information paradox [7, 8].

In this chapter we will give an overview of the AdS/CFT correspondence in the most general setting, that is in d dimensions and with little reference to an explicit theory on either the CFT or the AdS side. Furthermore we will mostly focus on the case of a single scalar field for simplicity. If the reader is interested in more detail, one could consult references [33] and [39] although many other good references are available [34, 40, 41]. For discussions that focus more on bulk reconstruction from a boundary CFT, see references [42–45]. Taking this birds-eye view has the advantage of viewing the conjecture in a more isolated sense, separate from any explicit realization or even from string theory. In section 2.1 an overview of conformal field theories is given. We begin by defining the conformal group and afterwards take a look at its different representations. In section 2.3 we will introduce anti-de sitter space and motivate why it is the natural geometrization of the conformal group, afterwards we will introduce a scalar quantum field theory on this space and consider some of its general properties. Finally, in section 2.4 we will discuss the duality in full by explaining the holographic dictionary and performing some explicit tests.

2.1 Conformal field theories

Stated somewhat tautologically, the study of conformal field theories is the study of those field theories that posses a conformal symmetry. These conformal transformations can be defined on any (pseudo)-Riemannian space as those transformations that leave angles locally invariant. In slightly more precise terms, a conformal transformation is a coordinate transformation $x^\mu \mapsto x'^\mu$ such that the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \tag{2.1}$$

where $\Omega^2(x)$ is positive and smooth. Therefore, a conformal field theory that is invariant under these transformations has as one of its properties that it has no preferred scale. This property is manifest in various theories, for example $\mathcal{N} = 4$ super Yang-Mills and many statistical systems at their critical points [46]. The conformal transformations form a Lie group called the conformal group so in order to study conformal field theories it is first necessary to study this group some more.

2.2 The conformal group

In principle any pseudo-Riemannian manifold has a conformal group of transformations satisfying (2.1). However commonly the conformal group in d dimensions is defined as the conformal transformations of $(1, d-1)$-dimensional Minkowski space.[2] Thus it is the group of transformations $x^\mu \mapsto x'^\mu$ satisfying

$$\left(\frac{\partial x'^\alpha}{\partial x^\mu}\right)\left(\frac{\partial x'^\beta}{\partial x^\nu}\right)\eta_{\alpha,\beta} = \Omega^2(x)\eta_{\mu\nu}(x) \tag{2.2}$$

with η the Minkowski metric in $(1, d-1)$ dimensions and $\Omega^2(x)$ is again positive and smooth. Note that the special case $\Omega^2(x) = 1$ is the subgroup of transformations that leave the metric invariant, thus the conformal group is an extension of the Poincaré group. Equation (2.2) is a differential equation for the new coordinates x'^μ that can be solved and which for $d \geq 3$ has the following independent solutions[3]

$$\begin{array}{lr} x^\mu \mapsto x^\mu + a^\mu & \text{Translations} \\ x^\mu \mapsto \Lambda^\mu_\nu x^\nu & \text{Lorentz transformations} \\ x^\mu \mapsto \lambda x^\mu & \text{Dilations} \\ x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2} & \text{Special conformal transformations} \end{array} \tag{2.3}$$

and thus any conformal transformation in $d \geq 3$ dimensions is given by some combination of these. In $d \leq 2$ the conformal group is much larger, either generated by the infinite dimensional Virasoro algebra in the $d = 2$ case or full re-parametrization for $d = 1$ [47]. In this thesis however, when talking about the conformal group we will solely refer to transformations of the type (2.3), which can be defined in any dimension.

To study this group it is useful to consider the generators associated to (2.3). To obtain these, one can simply perform the coordinate transformations infinitesimally in order to obtain

$$\begin{array}{lllr} P_\mu & := -i\partial_\mu & & \text{Translations} \\ M_{\mu\nu} & := 2ix_{[\mu}\partial_{\nu]} & & \text{Lorentz transformations} \\ D & := -ix^\mu\partial_\mu & & \text{Dilations} \\ K_\mu & := -i(2x_\mu x\cdot\partial - x^2\partial_\mu) & & \text{Special conformal transformations} \end{array} \tag{2.4}$$

where the factor of $-i$ is due to convention. These generators then satisfy the following

[2] Sometimes Euclidian space is used instead.

[3] For some sources which go into the discussion below in some more detail, see for example references [34, 39, 46, 47].

commutation relations

$$\begin{aligned} [D, P_\mu] &= iP_\mu & [D, K_\mu] &= -iK_\mu \\ [P_\mu, K_\nu] &= -2i(M_{\mu\nu} + \eta_{\mu\nu} D) & [M_{\mu\nu}, P_\sigma] &= -i(\eta_{\mu\sigma} P_\nu - \eta_{\eta\sigma} P_\mu) \\ [M_{\mu\nu}, K_\sigma] &= -i(\eta_{\mu\sigma} K_\nu - \eta_{\nu\sigma} K_\mu) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}) \end{aligned} \tag{2.5}$$

and all other commutations resulting in zero. Note that this algebra is isomorphic to the algebra of $SO(2, d)$ which can be shown by defining

$$\begin{aligned} J_{-1,0} &= D & J_{-1,\alpha} &= \tfrac{1}{2}(P_{\alpha+1} - K_{\alpha+1}) \\ J_{\alpha\beta} &= M_{\alpha+1,\beta+1} & J_{0,\alpha} &= \tfrac{1}{2}(P_{\alpha+1} + K_{\alpha+1}) \end{aligned} \tag{2.6}$$

where α and β run from 1 to d. These generators then satisfy

$$[J_{\alpha,\beta}, J_{\gamma,\delta}] = i(\eta^{2,d-1}_{\alpha\beta} J_{\gamma\delta} + \eta^{2,d-1}_{\beta\gamma} J_{\alpha\delta} - \eta^{2,d-1}_{\alpha\gamma} J_{\beta\delta} - \eta^{2,d-1}_{\beta\delta} J_{\alpha\gamma}) \tag{2.7}$$

which are the commutation relations of the $\mathfrak{so}(2, d)$ algebra. Here $\eta^{2,d-1}$ is the Minkowski metric in $(2, d-1)$-dimensions. Note that this means that only the connected part of the conformal group and $SO(2, d)$ are isomorphic. This connection will be important in establishing the AdS/CFT duality. Furthermore when building representations of the conformal group it will be most convenient to build representations of $SO(2, d)$ instead and use that the two are locally isomorphic.

2.2.1 Representations of the conformal group

In this subsection we will briefly discuss how to obtain the unitary irreducible representations of the Lorentzian conformal group $SO(2, d)$.[4] In $3 + 1$ dimension the conformal group was first classified by Mack [48], for the Euclidian conformal group see references [49, 50] or [51] for analytical continuations thereof. Here however we will take a somewhat more heuristic approach by considering the representations of its maximal compact subgroup. For a more in-depth look into the structure and representation theory of Lie groups the reader can consult references [52–55]. Here we are briefly going to consider the $SO(2, d)$ group in its fundamental representation in order to study some of its properties, most importantly to find its maximal compact subgroup which we will use to label its representations. Afterwards we will connect this to the generators above.

The fundamental representation of the group $SO(2, d)$ is given by matrices $g \in SL(2 + d, \mathbb{R})$ that satisfy

$$g^T I g = I \tag{2.8}$$

where g^T denotes the transpose of a matrix and I is defined as

$$I = \begin{pmatrix} -I_{2\times 2} & 0_{2\times d} \\ 0_{d\times 2} & I_{d\times d} \end{pmatrix} \tag{2.9}$$

[4] Although we will only look at the connected component $SO_0(2, d)$, subtleties regarding the disconnected components will be ignored.

with $I_{n\times n}$ the identity matrix of size $n\times n$ and $0_{n\times m}$ the zero matrix of size $n\times m$. This then implies that following the same block notation as in the definition of I, an element g can be written as

$$g = \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix} \tag{2.10}$$

where $\alpha \in SL(2,\mathbb{R})$ and $\gamma \in SL(d,\mathbb{R})$ are anti-hermitian and β is an $2\times d$ matrix with real coefficients. The maximal compact subgroup $K = SO(2)\times SO(d) \subset SO(2,d)$ is then given by matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} . \tag{2.11}$$

Note that in terms of the infinitesimal generators these are spanned by $J_{-1,0} = D$ and $J_{\alpha\beta} = M_{\alpha+1,\beta+1}$. A unitary irreducible representation of $SO(2,d)$ must restrict to a unitary irreducible representation of $SO(2)\times SO(d)$. We can thus label the representation by $[\Delta, L]$ where Δ labels the $SO(2)$ representation and L the $SO(d)$ representation. Note that L denotes the familiar spin representation. From the commutation relation (2.5) one can see that K_μ and P_μ act as raising and lowering operators for D. Since we're interested in constructing a quantum field theory, we will consider (tensor) fields ϕ upon which the operators act by commutation. We now assume that there is a lowest weight state ϕ that satisfies

$$[D,\phi(0)] = i\Delta\phi(0)\ ; \quad [K_\mu,\phi(0)] = 0 \tag{2.12}$$

which we will call a primary field. This assumption is consistent with requiring positive energy states [48]. Now it is possible to generate its descendants by successively applying the commutator with P_μ, since then for $\phi_1(0) \coloneqq [P_\mu,\phi(0)]$ one has that

$$[D,\phi_1(0)] = i(\Delta+1)\phi_1(0) \tag{2.13}$$

and P_μ acts as a raising operator as expected. The action of the operators on $\phi(x)$ can now be obtained by using

$$\phi(x) = e^{ix^\mu P_\mu}\phi(0)e^{-ix^\mu P_\mu} \tag{2.14}$$

which results in the relations

$$\begin{aligned} [D,\phi_\alpha(x)] &= i(-\Delta + x\cdot\partial)\phi_\alpha(x) \\ [P_\mu,\phi_\alpha(x)] &= i\partial_\mu\phi_\alpha(x) \\ [K_\mu,\phi_\alpha(x)] &= \big(i(x^2\partial_\mu - 2x_\mu x\cdot\partial + 2\Delta x_\mu) - 2x^\nu\Sigma_{\mu\nu}\big)\phi_\alpha(x) \\ [M_{\mu\nu},\phi_\alpha(x)] &= \big(2ix_{[\mu}\partial_{\nu]} + \Sigma_{\mu\nu}\big)\phi_\alpha(x) \end{aligned} \tag{2.15}$$

after applying the commutation relations. Here $\Sigma_{\mu\nu}$ is the finite dimensional matrix associated to the representation L. Note that this is in agreement with (2.4). These fields are the main object of interest in conformal field theories.

Having discussed some basic properties of the conformal symmetry group and conformal field theories, we will take a look at anti-de Sitter space and field theories on it.

2.3 Anti-de Sitter space

The other side of the AdS/CFT duality is AdS space. In this section we will discuss its definition as well as look at some basic properties. The discussion below is mostly based on references [34], [39] and [56].

From the symmetry group of conformal field theories one is almost immediately led to introduce AdS space. This is because AdS is the unique (up to a radius) maximally symmetric manifold with the $SO(2,d)$ symmetry group, which is similar to the relation between Minkowski space and the Lorentzian group. To see this, note that by definition the $SO(2,d)$ symmetry group is the group of transformations in $\mathbb{R}^{2,d}$ that leaves

$$X_A X^A := -(X^{-1})^2 - (X^0)^2 + X_i X^i \tag{2.16}$$

invariant. From now on upper case Latin indices will denote the index of coordinates in $\mathbb{R}^{2,d}$ while lower case Latin letters will be reserved for the spatial coordinates. These transformations then define a family of $SO(2,d)$ invariant submanifolds $\mathcal{M}_L \subset \mathbb{R}^{2,d}$ which have as their defining equation

$$\mathcal{M}_L := \{ \ X \in \mathbb{R}^{2,d} \mid X_A X^A = L^2 \ \} \tag{2.17}$$

and these submanifolds are then precisely AdS_{d+1} space with radius L. Note that thus by construction the symmetry group of AdS_{d+1} is $SO(2,d)$.

While it is sometimes is convenient to work in the above embedding space there are also several other coordinate descriptions of AdS. One particularly useful set of global coordinates can be obtained by defining

$$X_{-1} = L\frac{\sin(t)}{\cos(\rho)} \tag{2.18}$$

$$X_0 = L\frac{\cos(t)}{\cos(\rho)} \tag{2.19}$$

$$X_i = L\tan(\rho)\Omega_i \tag{2.20}$$

with Ω_i satisfying $\sum_{i=1}^d (\Omega_i)^2 = 1$. In these coordinates the induced metric on AdS_{d+1} is given by

$$ds^2 = \frac{L^2}{\cos^2(\rho)}\big(- dt^2 + d\rho^2 + \sin^2(\rho) d\mathbf{\Omega}^2\big) \tag{2.21}$$

$$\rho \in [0, \tfrac{\pi}{2}) \ ; \quad t \in (-\infty, \infty) \ ; \quad \Omega_i \in [-1,1] \tag{2.22}$$

where $\mathbf{\Omega}$ denotes $\sum_{i=1}^d \Omega_i$. It should be noted that technically this is the universal cover of the space $\mathcal{M}_L$ defined above, although this subtlety will not matter for us.

Finally, a different set of useful coordinates is given by the Poincaré coordinates. These are defined by

$$X_{-1} = \frac{z}{2}\left(1 + \frac{L^2 + \mathbf{x}^2 - t^2}{z^2}\right) \tag{2.23}$$

$$X_0 = L\frac{t}{z} \tag{2.24}$$

$$X_i = L\frac{x_i}{z} \tag{2.25}$$

$$X_d = \frac{z}{2}\left(1 - \frac{L^2 - \mathbf{x}^2 + t^2}{z^2}\right) \tag{2.26}$$

where $\mathbf{x}$ denotes a vector of $d-1$ spacial coordinates and $1 \leq i \leq d-1$. In these the metric takes the following form

$$ds^2 = L^2\frac{dz^2 - dt^2 + d\mathbf{x}^2}{z^2} \tag{2.27}$$

$$z \in (0,\infty)\ ;\quad t,\ x_i \in (-\infty,\infty) \tag{2.28}$$

where $d\mathbf{x}^2 := \sum_{i=1}^{d-1}(dx^i)^2$. Because of its simplistic form these are the ones mostly used in actual calculations although one important limitation is that these don't cover the entire AdS space, only the Poincaré patch.

With these coordinates it is now possible to give explicit expressions for the AdS Killing vectors. Since by construction AdS inherits the $SO(2,d)$ symmetry group, its killing vectors are of the form

$$J_{(A,B)} = X_A\partial_B - X_B\partial_A \tag{2.29}$$

in the $\mathbb{R}^{2,d}$ coordinates used before. Note that here the subscript (A,B) is not an index but a label. These Killing vectors can be written in AdS coordinates by defining

$$(k_{(A,B)})_\mu := \frac{\partial X^C}{\partial x^\mu}(J_{(A,B)})_C \tag{2.30}$$

where x^μ denote the AdS coordinates. For the global coordinates defined in equation (2.18) these take the form

$$\begin{aligned}
k_{(-1,0)} &= \partial_t \\
k_{(-1,0)} &= \cos(\rho)\sin(t)\Omega_i\partial_\rho + \sin(\rho)\cos(t)\Omega_i\partial_t - \tfrac{\sin(t)}{\sin(\rho)}\partial_{\Omega_i} \\
k_{(0,i)} &= -\cos(\rho)\cos(t)\Omega_i\partial_\rho + \sin(\rho)\sin(t)\Omega_i\partial_t + \tfrac{\cos(t)}{\sin(\rho)}\partial_{\Omega_i} \\
k_{(i,j)} &= \Omega_i\partial_{\Omega_j} - \Omega_j\partial_{\Omega_i}
\end{aligned} \tag{2.31}$$

where again $1 \leq i \leq d$.

Another important property of AdS space is that it has a conformal time-like boundary, where a conformal boundary is defined as the boundary of the conformally compactified space. To see this, consider the metric of AdS_{d+1} in global coordinates, given by equation (2.21). Note that this has a singularity at $\rho = \frac{\pi}{2}$. However, it is possible to define a new metric as

$$d\tilde{s}^2 := f(\rho,t,\mathbf{x})ds^2 \tag{2.32}$$

where f is any smooth positive definite function that has a simple zero at $\rho = \frac{\pi}{2}$. The metric $d\tilde{s}^2$ defines a new manifold which is the conformal compactification of AdS_{d+1}. Note that f is non-unique and therefore it is possible to perform many different compactifications.[5] Choosing

$$f(\rho, t, \mathbf{x}) = \frac{\cos^2(\rho)}{L^2} \tag{2.33}$$

the new metric is given by

$$d\tilde{s}^2 = -dt^2 + d\rho^2 + \sin^2(\rho) d\mathbf{\Omega}^2 \ . \tag{2.34}$$

This metric now is actually defined on the boundary and thus it is possible to take the limit $\rho \to \frac{\pi}{2}$ and consider the resulting space. This has a metric given by

$$d\tilde{s}^2\Big|_{\rho=\frac{\pi}{2}} = -dt^2 + d\mathbf{\Omega}^2 \tag{2.35}$$

and thus using this conformal compactification the boundary is given by $\mathbb{R} \times \mathbb{S}^{d-1}$. Note that different choices of f would have given different boundary spaces.

2.3.1 Field theories on a fixed AdS background

We are now ready to study field theories on a fixed AdS background. For simplicity we will just consider a single scalar field that is not coupled to gravity. This will allow us to perform explicit calculations and even obtain an exact solution for the free field, with which we can then study its behaviour. Although this is only the simplest example, some key aspects such as its boundary behaviour also generalize to other fields [3, 35]. Most of the following analysis will be classical as we will only give an overview of the simplest arguments for AdS/CFT duality.

The scalar field on AdS is described by the action

$$S[\phi] = \int_{AdS_{d+1}} d^{d+1}x \, \sqrt{-g} \left(\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \frac{m^2}{2} \phi^2 + V(\phi) \right) \tag{2.36}$$

where g is the metric of AdS and V is a potential. This action implies that the classical equation of motion of ϕ is given by

$$(\Box - m^2)\phi = \frac{\partial V}{\partial \phi}(\phi) \tag{2.37}$$

with $\Box$ the Laplace-Belatrami operator of AdS space. Note that since AdS space has a time-like conformal boundary the boundary conditions need to be specified in order for the differential problem to be well-defined. For a free field of mass m (e.g $V(\phi) = 0$) this equation of motion reduces to

$$(\Box - m^2)\phi = 0 \tag{2.38}$$

[5] Note that the above procedure is equivalent to a Weyl rescaling of the metric, see [35] for a nice description of this procedure in Euclidian AdS.

which can be written explicitly in Poincaré coordinates (2.27) as

$$\left(z^2\partial_z^2 - (d-1)z\partial_z + z^2(-\partial_t^2 + \partial_i\partial^i) - m^2L^2\right)\phi = 0 \tag{2.39}$$

where the sum over i denotes a sum over spatial coordinates. Making the ansatz

$$\phi(z,t,\mathbf{x}) = \psi(z)e^{itp_0 + ix^ip_i} \tag{2.40}$$

the above differential can be rewritten as a modified Bessels equation with ψ having solutions of the form

$$\psi(z,t,x)_p = c_1 z^{d/2}K_{\Delta-d/2}(|p|z) + c_2 z^{d/2}I_{\Delta-d/2}(|p|z) \tag{2.41}$$

where K and I are modified Bessel functions and

$$\Delta \coloneqq \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2} \tag{2.42}$$

will play an important role in connecting to the CFT side. In the interior I diverges exponentially so to obtain normalizable solutions c_2 must be set to zero. Since near the boundary $z \to 0$ the functions K scales as

$$\lim_{z\to 0} z^{d/2}K_{\Delta-d/2}(|p|z) \sim z^{d-\Delta}f(p) \tag{2.43}$$

ϕ also scales as $z^{d-\Delta}$ near the boundary. Note that this diverges and thus some regularization will be necessary, this is usually done by defining a cutoff ϵ instead of taking the limit $z \to 0$. This process is called holographic renormalization and is often necessary when calculating correlation functions [57].

When one introduces interactions it is no longer possible to obtain an exact solution, however Witten introduced a general procedure for calculating correlation functions which can still be applied [35]. This procedure involves first solving the differential equation for a free field subject to the boundary conditions

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,t,\mathbf{x}) = \phi_0(t,\mathbf{x}) \tag{2.44}$$

where ϕ_0 is some to be specified function, and then iteratively solving for the full solution. In order to obtain the free solution in terms of the boundary conditions, one first defines a bulk-to-bulk propagator $G(z,t,\mathbf{x})$ satisfying

$$(\Box - m^2)G(z,t,\mathbf{x}) = \frac{\delta(z)\delta(t)\delta^{d-1}(\mathbf{x})}{\sqrt{-g}} \tag{2.45}$$

which is just the conventional Green's function. With this it is possible to define a bulk-to-boundary propagator K as[6]

$$K(z,t,x) \coloneqq \lim_{w\to 0} \frac{2\Delta - d}{w^{\Delta}} G(z-w,t,x) \tag{2.46}$$

[6]Note that this K is not the modified Bessel function used above, although this notation may be somewhat confusing we have used it here in order to be consistent with the notation from the literature. From here on out, K will always mean the bulk-to-boundary propagator unless stated otherwise.

chosen such that

$$\lim_{z\to 0} z^{\Delta-d}K(z,t,x) = \delta(t)\delta^{d-1}(\mathbf{x}) \tag{2.47}$$

which can be checked by explicitly calculating G and K, as has been done in reference [33]. An interesting property of K is that the free field solution $\widetilde{\phi}$ to

$$(\Box - m^2)\widetilde{\phi}(z,t,\mathbf{x}) = 0 \tag{2.48}$$

subject to

$$\lim_{z\to 0} z^{\Delta-d}\,\widetilde{\phi}(z,t,\mathbf{x}) = \phi_0(t,\mathbf{x}) \tag{2.49}$$

can be written as

$$\widetilde{\phi}(z,t,\mathbf{x}) = \int_{\partial AdS} dt' d^{d-1}\mathbf{x}'\; K(z,t-t',\mathbf{x}-\mathbf{x}')\phi_0(t',\mathbf{x}') \tag{2.50}$$

resulting in the free solution in terms of the boundary conditions. With this free solution it is now possible to expand ϕ iteratively as

$$\phi(z,t,x) = \widetilde{\phi}(z,t,\mathbf{x}) + c_1 \int_{AdS} d^{d+1}x'\; G(z-z',t-t',\mathbf{x}-\mathbf{x}')\widetilde{\phi}(z',t',\mathbf{x}') + \cdots \tag{2.51}$$

where the expansion continues with propagators acting on the free solution and the constants c_i are fixed by the potential V [38]. Inserting this expansion in to the action (2.36) gives an on-shell action which now depends on the boundary condition ϕ_0 instead of ϕ. Explicit expressions for K and G can be found [33, 38] which when inserted in the on-shell action and integrating by parts results in

$$S_{on-shell}[\phi_0] = \int_{\partial AdS}\int_{\partial AdS} dtd^{d-1}\mathbf{x}dt'd^{d-1}\mathbf{x}'\;\frac{\phi_0(t,\mathbf{x})\phi_0(t',\mathbf{x}')}{|t-t'|^{2\Delta}|\mathbf{x}-\mathbf{x}'|^{2\Delta}} + S_{interaction}[\phi_0]\,. \tag{2.52}$$

Note that the first term is identical to the generating functional $W_{CFT}[\phi_0]$ of a conformal field theory operator with conformal dimension Δ and source ϕ_0. This suggests that it might be possible to reinterpret the original field theory described by (2.36) also as a conformal field theory living on the boundary of AdS space, which leads us naturally to conjecture the AdS/CFT duality.

2.4 The AdS/CFT duality

The on-shell expansion (2.52) leads us to conjecture a connection between field theories on AdS space and conformal field theories living on its boundary. In its strongest form it posits a connection between the AdS partition functional $Z_{AdS}[\phi_0]$ where the ϕ_0 act as the boundary conditions of the theory and a CFT partition function $Z_{CFT}[\phi_0]$ where the ϕ_0 act as sources.[7]

[7]There is also the so-called extrapolate dictionary, which states that in a different limit actually the bulk theory field is actually equal to the CFT operator itself [34, 44] although they are conjectured to be equivalent for AdS [58].

Mathematically, the AdS/CFT duality can thus be summed up in the equation[8]

$$Z_{CFT}[\phi_0] = Z_{AdS}[\phi]\big|_{\lim_{z\to 0} z^{\Delta-d}\phi=\phi_0} \tag{2.53}$$

where the ϕ fields now include gravity and other non-scalar fields. The weaker forms of this conjecture state that this equality only holds in certain limits [33]. While the scalar field toy-model used above is interesting, the full extent of the AdS/CFT conjecture only comes in to play when considering different fields and especially gauge fields such as gravity. Equation (2.53) then implies a mapping where the boundary limit bulk gauge fields act as sources for conserved currents in the CFT side [35, 36]. An important special case for this is the mapping between the energy-momentum tensor on the CFT side and the metric on AdS, since any CFT has an energy-momentum tensor [47] this means that gravity appears on the AdS side quite naturally. All these combined lead to the so-called AdS/CFT dictionary which can be summed up as [59]

AdS		**CFT**
Fields	$\leftrightarrow$	Operators
Gauge fields	$\leftrightarrow$	Conserved currents
Boundary values of fields	$\leftrightarrow$	Sources for operators
Local isometries	$\leftrightarrow$	Global isometries

Besides this general idea we can quickly check some very basic requirements for this duality. Two somewhat trivial things to mention briefly are that as we saw in subsection 2.3 that AdS_{d+1} has an $\mathbb{R}\times\mathbb{S}^{d-1}$ boundary. Since $\mathbb{R}\times\mathbb{S}^{d-1}$ is conformally equivalent to $\mathbb{R}^{d-1,1}$ it at least seems possible that the boundary theory is described by a conformal field theory on this space. Furthermore taking the boundary limits of the Killing vectors given in equation (2.31) results in the $SO(2,d)$ Killing vectors defined in equation (2.6), signifying that the Killing vectors on AdS also get mapped to the conformal Killing vectors on the CFT side. There are quite a number of other checks on the AdS/CFT duality, such as calculations of correlation functions [60, 61] and the holographic anomaly [62, 63]. For brevity we will not go into those here but for an overview see references [38] and [33].

With this general overview of AdS/CFT finished we will now consider the lowest dimensional $\mathrm{AdS}_2/\mathrm{CFT}_1$ setting and as we will see it will turn out that even though initially this might seem like the simplest model it has quite a few peculiarities.

[8] This equation is also sometimes called the GKPW rule, in reference to Gubser, Klebanov, Polyakov [36] and Witten [35].

3 Lowest dimensional AdS/CFT

The lowest dimensional version of the Ads/CFT duality is quite interesting for a number of reasons. First of all it appears quite naturally when considering extremal black holes in higher dimensions, which makes it one of the few instances where the AdS/CFT duality might immediately provide testable predictions. Furthermore there are some very interesting differences in $\mathrm{AdS}_2/\mathrm{CFT}_1$ in comparison to the higher dimensions. For example, Lorentzian AdS_2 actually has two boundaries instead of one as is shown in figure 1. This makes it unclear how the ordinary AdS/CFT dictionary of relating bulk fields and boundary operators through simply taking the boundary limit should be applied, since now one has the choice of different limits to take. Furthermore calculating correlators by varying with respect to the boundary conditions as in equation (2.53) also becomes a bit unclear. It has been posited that therefore there are now two CFTs, one for each boundary, which have been used to calculate black hole entropies. [64]. Besides the double boundary there is also the fact that in $0+1$ dimension a quantum field theory reduces to ordinary quantum mechanics, thus one can not apply the general CFT formalisms without care. This might also turn out to be an advantage however, since usually quantum mechanical models are much easier to study than quantum field theories. A third difference between the higher dimensional analogues is that in two dimensions, Einstein-Hilbert gravity is non-dynamical. Therefore some modification is necessary in order to have interesting gravitational dynamics. Usually this is done by adding a dilaton field to the theory since as we will see these appear very naturally from higher-dimensional compactification. Finally, there is the fact that $\mathrm{AdS}_2 \times K$ with K compact admits no finite energy excitations [17, 65]. Therefore one has to settle for just studying the ground state, or one has to modify the space. We will discuss this more in section 3.2. In the end, all these differences are part of what makes $\mathrm{AdS}_2/\mathrm{CFT}_1$ so interesting, since overcoming them will surely lead to a better understanding of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ duality in general.

(a) (b)

Figure 1: (a) conformally compacted Euclidean AdS_2, originally from [66]. Note that the polygons are of equal volume. (b) conformally compactified Lorentzian AdS_2, note the two time-like boundaries in dark grey extending to $\pm\infty$.

In this chapter we will start by motivating the study of AdS_2 with dilaton gravity by considering the spherically reduced Reissner-Nordström black hole in section 3.1, afterwards in section 3.2 we will show why $\mathrm{AdS}_2 \times K$ admits no finite energy excitations for compact K. Finally in section 3.3 we will discuss motivate the particular conformal quantum mechanical model that we will study in the next chapter.

3.1 AdS from extremal black holes

One particular reason why AdS_2 is interesting it that it can be obtained as part of the near-horizon geometry of extremal black holes. In fact, this has been partially proven for general extremal black holes by Figueras et al. [14] and Kunduri et al. [13] using the $SO(2,1)$ symmetry near the boundary. As an example, here we will briefly show how the AdS_2 geometry arises from a Reisner-Nordström black hole. This discussion will mostly follow references [67] and [17].

Consider the Reissner Nordström metric in four dimensions given by

$$ds^2 = -\frac{(r-r^+)(r-r^-)}{r^2}dt^2 + \frac{r^2}{(r-r^+)(r-r^-)}dr^2 + r^2 d\mathbf{\Omega}^2 \tag{3.1}$$

$$r_\pm = QL_p + EL_p^2 \pm \sqrt{2QEL_p^3 + E^2L_p^4} \tag{3.2}$$

$$E = M - \frac{Q}{L_p} \tag{3.3}$$

here $d\mathbf{\Omega}^2$ denotes the S^2 metric, Q is the charge, E is the excitation energy above extremality and L_p is the Planck length. For an extremal black hole $E = 0$, simplifying the expression for $r_\pm$ to

$$r_+ = r_- = QL_p\ . \tag{3.4}$$

Taking the near-horizon limit means considering a coordinate near $r = r_\pm$ and taking the length scale to zero. Since L_p is the only remaining variable with dimension length, a natural choice is

$$z = \frac{Q^2L_p^2}{r - r_+} \tag{3.5}$$

and taking the limit $L_p \to 0$ while keeping z and Q constant. After performing this change of coordinates and taking these limits the new metric becomes

$$d\tilde{s}^2 = Q^2L_p^2\left(\frac{dz^2 - dt^2}{z^2} + d\mathbf{\Omega}^2\right) \tag{3.6}$$

which, since the first part is the metric of AdS_2 and the second part that of $\mathbb{S}^2$, describes the product space $\mathrm{AdS}_2 \times \mathbb{S}^2$. For different extremal black holes this procedure would result in the product space $\mathrm{AdS}_2 \times K$ where K is some compact manifold. It is also interesting to consider the near-extremal limits, this amounts to taking L_p to zero while simultaneously taking limits of combinations of E, Q and L_p or keeping them constant. This has been done by Maldacena, Michelson and Strominger [17] and one important result found there was that in the simplest limits the finite energy excitations on $\mathrm{AdS}_2 \times \mathbb{S}^2$ were supressed, something that we will discuss more in section 3.2.

Before we look at this in more detail we will take a look at the resulting theory of gravity in AdS_2 induced from extremal black holes, continuing with the Reisner-Nordström black hole example. As we will see this leads naturally to the introduction of a dilaton field. The metric (3.1) is a solution of the Einstein-Maxwell action

$$S = \frac{1}{L_p^2} \int d^4x \sqrt{-g} \left(R_g - \frac{L_p^2}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{3.7}$$

where g is the metric, R_g its curvature scalar and F the electro-magnetic field strength. Now consider the spherically symmetric anszats

$$\begin{aligned} ds^2 &= h_{ab}(r,t) dx^a dx^b + \Phi^2(r,t) d\mathbf{\Omega}^2 \\ F &= Q \sin\phi \, d\phi \wedge d\theta \end{aligned} \tag{3.8}$$

where $a, b \in \{t, r\}$ and ϕ, θ are the angular coordinates. Inserting this in the action above and integrating with respect to the angular coordinates applying Stokes theorem leads to[9]

$$S = \frac{4\pi}{L_p^2} \int dtdr \sqrt{-h} \left[\Phi^2 R_h + 2(\nabla\Phi)^2 + 2 - \frac{L_p^2 Q^2}{2\Phi^2} \right] \tag{3.9}$$

where R_h is the curvature scalar of h and ∇ is the covariant derivative with respect to h. This action is part of a family of $1+1$ dimensional models initially studied by Almheiri and Polchinski [65] with actions of the form

$$S = \frac{1}{16\pi G} \int drdt \sqrt{-h} \left[\Phi^2 R_h + \lambda(\nabla\Phi)^2 - U(\Phi) \right] \tag{3.10}$$

with $U(\Phi)$ some potential and G of dimension length squared. Many different theories give rise to an action of the form (3.10), for example spherically reduced gravity models such as above or certain string theories [69, 70]. Furthermore, some models of this form are exactly solvable even with matter making it possible to analyse them more in-depth [71].

With the motivation for studying these kind of models given, we will now discuss some of their properties which make $\mathrm{AdS}_2/\mathrm{CFT}_1$ especially difficult and interesting. Starting with the problem of finite energy excitations.

3.2 Backreaction and absence of finite matter excitations

Now that we have a general dilaton gravity action in two dimension we can start to couple it with matter. As we will see, requiring that the background-metric asymptotes to $\mathrm{AdS}_2 \times K$ for K a compact manifold implies that there are no finite energy matter excitations [17, 65]. In extremal black holes such as the one discussed above, this effect manifests by the thermodynamic description breaking down at extremality [72, 73]. Note that for non-compact K, finite matter excitations are allowed [74]. The general argument starts by coupling the action (3.10) to matter as

$$S = \int drdt \sqrt{-h} \left[\Phi^2 R_h + \lambda(\nabla\Phi)^2 - U(\Phi) \right] + S_{matter} \tag{3.11}$$

[9] See references [67] and [68] for more detail.

where the exact form of S_{matter} will not be important. In two dimensions, any Lorentzian metric can be rewritten as

$$ds^2 = -e^{2\omega(u,v)}dudv \tag{3.12}$$

by applying some coordinate transformation. The equation of motion for the metric component h^{uu} then implies that

$$-e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi^2) = T_{uu} \tag{3.13}$$

where T is the energy momentum tensor obtained by varying S_{matter} with respect to h. We impose that at the boundary the metric asymptotes to the AdS$_2$ metric, thus in accordance to AdS$_2$ we should find two boundaries which we will take to be at $u = v$ and $u = v + \pi$. Now integrating with the measure $e^{-2\omega}du$ along $v = 0$ results in the equation

$$e^{2\omega}\partial_u\Phi^2\big|_{u=0} - e^{2\omega}\partial_u\Phi^2\big|_{u=\pi} = \int_0^\pi du\, e^{-2\omega}T_{++}\ . \tag{3.14}$$

Importantly, the right hand side of this equation is greater than zero when matter is present. However imposing that the metric asymptotes to AdS$_2$ at the boundaries implies that $e^{2\omega}$ should scale as

$$\lim_{u\to v} e^{2\omega} \sim \frac{1}{(u-v)^2} \tag{3.15}$$

$$\lim_{u\to v+\pi} e^{2\omega} \sim \frac{1}{(u-v-\pi)^2} \tag{3.16}$$

and thus $e^{-2\omega}$ vanishes at the boundaries. But since the right hand side of equation (3.14) is non-zero we see that the $\partial_u\Phi^2$ term must diverge quadratically near at least one of the boundaries. This might not seem like an important problem since in higher dimensional AdS fields can also have divergences at the boundary,[10] however one must remember that in this context Φ^2 appears from an ansatz similar to equation (3.8). Thus Φ^2 should be interpreted as the volume of the compact space K, but since Φ^2 diverges this volume is infinite and K can no longer be compact. Therefore AdS$_2 \times K$ admits no finite energy excitations. A solution is available however by considering near-AdS (nAdS) geometries [75, 76] which we will briefly discuss now.

3.2.1 Near-AdS geometries and Schwarzian actions

Armed with the knowledge that some modification of the system above is necessary here we will briefly consider one possible solution proposed by Almheiri and Polchinski [65] which was developed more in references [75] and [76]. This solution involves a boundary parametrization resulting in a Schwarzian boundary action, which implies a possible relation between Jackiw-Teitelboim gravity and the Sachdev–Ye–Kitaev model [67]. The starting point is to consider small deformations

$$\Phi^2 = \phi_0 + \phi \tag{3.17}$$

[10] See for example equation (2.43).

of the dilaton around a constant value ϕ_0 that satisfies the equations motion. By expanding the action (3.10) around ϕ_0 and ignoring higher order and topological terms this action can be rewritten as [67, 75]

$$S = \frac{1}{16\pi G}\left[\int drdt\,\sqrt{-h}\phi(R_h + 2) + 2\int dt\;\phi_b K\right] \tag{3.18}$$

where the boundary integral is added in order to make the variational problem well-defined. Here K is the extrinsic curvature of the boundary and ϕ_b is the boundary limit of ϕ. Note that this action describes the well-known Jackiw-Teitelboim gravity [18, 19] and that the equation of motion for ϕ now assures that h has constant negative curvature. One important difference with before is that we will know work in Euclidean coordinates and consequentially that there is only one boundary. The equations of motion for h and ϕ can be solved resulting in

$$ds^2 = \frac{dr^2 + dt^2}{z^2} \tag{3.19}$$

$$\phi = \frac{a + bt + c(t^2 + r^2)}{r} \tag{3.20}$$

where a, b and c are integration constants. Since both the metric and ϕ diverge at the boundary $r \to 0$, we impose a cut-off by defining a boundary curve parametrized by a coordinate τ.[11] Furthermore, we impose that at along this cut-off, h and ϕ behave as

$$h|_{boundary} = \frac{1}{\epsilon^2} \tag{3.21}$$

$$\phi_b = \frac{\phi_\tau(\tau)}{\epsilon} \tag{3.22}$$

analogous to equation (3.19). This results in a set of equations for $t(\tau)$ and $r(\tau)$ along the path which can be solved resulting in

$$\epsilon\partial_\tau t(\tau) = z(\tau) \tag{3.23}$$

$$\frac{a + bt(\tau) + c(t(\tau))^2}{\partial_\tau t(\tau)} = \phi_\tau(\tau) \tag{3.24}$$

where it should be noted that ϕ_τ acts as the boundary condition imposed on ϕ. The second equation is the equation of motion for a boundary action given by

$$I = \frac{-1}{8\pi G}\int d\tau\phi_\tau(\tau)Sch(t,\tau) \tag{3.25}$$

where $Sch(t,\tau)$ is the Schwarzian defined as

$$Sch(t,\tau) \coloneqq \partial_\tau\left(\frac{\partial_\tau^2 t(\tau)}{\partial_\tau t(\tau)}\right) - \frac{1}{2}\left(\frac{\partial_\tau^2 t(\tau)}{\partial_\tau t(\tau)}\right)^2 . \tag{3.26}$$

[11] This and what follows is somewhat similar to holographic renormalization [57]. Although there is a substantial difference in that the cut-off is not uniform but along a path parametrized by τ.

The action (3.25) could also have been derived by imposing the equations of motion for ϕ on (3.18) and inserting the explicit formula for the extrinsic curvature [75].

Interestingly, a Schwarzian action also appears as the low temperature limit of the Sachdev–Ye–Kitaev model [20] leading to speculation that Jackiw-Teitelboim gravity might be the bulk dual of the Sachdev–Ye–Kitaev model [77, 78]. In this thesis however, we will mostly focus on an alternative CFT_1 system: conformal quantum mechanics.

3.3 Motivating conformal quantum mechanics

Although the model above is quite interesting, one may wonder if there are simpler CFT_1 systems one might construct. A natural starting point is to consider a conformal quantum mechanical system. Although it should be noted that the conformal group in one dimension consists of a full parametrization symmetry, while AdS_2 has a $SO(2,1)$ symmetry. Thus a natural starting point is to consider quantum mechanical systems with this same symmetry group. However, usually its double cover $SL(2,\mathbb{R})$ is used instead and therefore we will also focus on that. A canonical conformal quantum mechanics model with this symmetry group was introduced by de Alfaro et al. (dAFF) in reference [23]. In the following chapter we will study their construction more in-depth but here we will give some motivation for why it is a natural object to study in this context. First of all, as we mentioned it is a quantum mechanical system with the correct symmetry group and since many of its properties depend solely on the group properties in some sense the underlying system is interchangeable. The dAFF model can actually be obtained from the Jackiw-Teitelboim gravity described above [24] as well as from a symplectic reduction of hermitian matrix models [79]. This last one is particularly interesting since it has been argued that for pure AdS_2, the CFT_1 admits a $U(N)$ symmetry group [64].

Now that we have outlined the $\mathrm{AdS}_2/\mathrm{CFT}_1$ duality and given a motivation for the dAFF quantum mechanics model, we will turn our attention there. As we will see, it will turn out to have many properties similar to higher dimensional CFT's.

4 Conformal quantum mechanics

As we have seen, the dAFF quantum mechanics model provides a natural candidate for a CFT_1 theory. However as it turns out, many of their results are independent of the actual quantum mechanics and depend solely on the representation theory of $SL(2,\mathbb{R})$. Therefore in this chapter we will mostly take this representation theory view and not talk about the specific model. One of the main insights from dAFF was the introduction of a continuous state $|x\rangle$ that acts similar to a position basis in that it provides a way to move from discrete states in the Hilbert space to functions of x. It then turns out that, as was noted already by Chamon et al. [21] that there is an analogue to correlation functions that has very similar properties to those in ordinary CFT's.

We will start this chapter by giving an overview of the $SL(2,\mathbb{R})$ representation theory in section 4.1. Afterwards we will review the work of dAFF [23] and Chamon et al.[21] in the construction of the dAFF model in section 4.2. We will then expand on this in subsection 4.2.2 by introducing a completeness relation for the states $|x\rangle$ and find that it has some similar properties to the shadow operator from conformal field theories. This completeness relation then allows us to find formula's for all higher order correlation functions. Finally we propose an interpretation of the dAFF model as the boundary theory of a canonical discrete series representation of $SL(2,\mathbb{R})$ on the Poincaré upper half plane in section 4.3.

4.1 Representations of the 1-dimensional conformal group

Since conformal quantum mechanics consists of representations of the 1-dimensional conformal group, the first task at hand is to classify these and choose one to build our representation on. The irreducible representations of $SL(2,\mathbb{R})$ were first classified by Bargmann in [80] by first classifying the representations of the Lie algebra. Here we will take a similar approach and note that the representations of the Lie algebra can be lifted to a representation of the group in the usual way. Some of the general theory of Lie groups and algebras can be found in Lee [81], while a more in-depth look as well as an overview of the representation theory can be found in references [53] and [54] by Knapp, as well as [82] by Hall. Much of this theory is then applied to the $SL(2,\mathbb{R})$ group by Lang in [83].

The $SL(2,\mathbb{R})$ group is defined as the group of 2x2 matrices satisfying

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1 \tag{4.1}$$

where a, b, c and d are real. Or equivalently, with the function $f : GL(2,\mathbb{R}) \to \mathbb{R}$

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{4.2}$$

$SL(2,\mathbb{R})$ is defined by

$$f(A) = 1 \Leftrightarrow A \in SL(2,\mathbb{R}) \tag{4.3}$$

where $A \in GL(2,\mathbb{R})$. Since this group is a Lie group, it has a corresponding Lie algebra which is isomorphic to the tangent space of $SL(2,\mathbb{R})$ at the identity element. Using that $SL(2,\mathbb{R})$ has f as its defining function this Lie algebra this is then given by

$$\mathfrak{sl}(2,\mathbb{R}) \simeq T_e SL(2,\mathbb{R}) = \{v \in T_e GL(2,\mathbb{R}) \mid (df)_e(v) = 0\} \tag{4.4}$$

which results in the algebra of 2x2 traceless matrices.[12] A common basis of this Lie algebra is given by

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;;\quad X_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \;;\quad X_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{4.5}$$

and satisfy the commutation relations

$$[X_1, X_2] = -2X_3 \;;\quad [X_1, X_3] = -X_1 \;;\quad [X_2, X_3] = X_2 \tag{4.6}$$

where $[\cdot,\cdot]$ denotes the commutator for matrices. Now, for a Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot|\cdot\rangle$ we construct a representation $\pi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(\mathcal{H})$ as

$$\pi(X_1) = iK \;;\quad \pi(X_2) = iP \;;\quad \pi(X_3) = iD \tag{4.7}$$

which is taken as the definitions of D, P and K. Note that unitarity implies that these operators are self-adjoint with respect to the inner product of $\mathcal{H}$. The requirement that the representation is a Lie algebra homomorphism then fixes the commutation relations of P, K and D to be

$$\begin{aligned} [D,P] &= iP \\ [D,K] &= -iK \\ [P,K] &= -2iD \end{aligned} \tag{4.8}$$

and we see that D, P and K can be identified as the dilatation, translation and special conformal transformation operators of the conformal group. To see how these operators act on the Hilbert space $\mathcal{H}$, first note that

$$X_R \coloneqq \frac{1}{2}(X_1 + X_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.9}$$

generates the maximal compact subgroup $SO(2) \subset SL(2,\mathbb{R})$. The representation $\pi|_{X_R}$ restricts to a representation of $\mathfrak{so}(2)$ and therefore R defined by

$$\pi(X_R) = iR \tag{4.10}$$

is compact and there exists a complete orthonormal basis $|n\rangle$ of $\mathcal{H}$ such that

$$R|n\rangle = \lambda_n|n\rangle = (\lambda + n)|n\rangle \tag{4.11}$$

where λ and λ_n are integers or half-integers and n is an integer. Knowing this, it is useful to define the operators

$$L_\pm \coloneqq \frac{1}{2}(P - K) \pm iD \tag{4.12}$$

[12] Note that for matrix groups, their Lie algebra's themselves also consist of matrices.

Name of the representation	Value of Δ	Eigenvalues of R	Range of n
Principal series	$\Delta = \frac{1}{2} + i\nu$, $\nu \in \mathbb{R}$	$\lambda_n = n$ or $\lambda_n = \frac{1}{2} + n$	$n \in \mathbb{Z}$
Positive discrete series	$\Delta \in \mathbb{N} \setminus \{0,1\}$	$\lambda_n = \Delta + n$	$n \in \mathbb{N}$
Negative discrete series	$-\Delta \in \mathbb{N} \setminus \{0,1\}$	$\lambda_n = \Delta - n$	$n \in \mathbb{N}$
Complementary series	$\Delta \in (0,1)$	$\lambda_n = n$ or $\lambda_n = \frac{1}{2} + n$	$n \in \mathbb{Z}$

which satisfy the commutation relations

$$[R, L_\pm] = \pm L_\pm \; ; \quad [L_-, L_+] = 2R \tag{4.13}$$

as well as $(L^+)^\dagger = L^-$, which is a requirement for them to act as proper raising and lowering operators. The commutation relations then imply that the state $L_\pm|n\rangle$ is an eigenstate of R with eigenvalue $\lambda_n + 1$, thus we find that

$$L_\pm|n\rangle = C_{n,\pm}|n \pm 1\rangle \tag{4.14}$$

where $C_{n,\pm}$ is a constant. One final necessary ingredient is that the quadratic Casimir

$$C \coloneqq -D^2 + \frac{1}{2}(PK + KP) = R^2 - \frac{1}{2}(L_+L_- + L_-L_+) \tag{4.15}$$

commutes with all other generators. For a unitary irreducible representation any operator that commutes with all other operators must be proportional to the identity and thus for all $|\phi\rangle \in \mathcal{H}$ we should require that

$$C|\phi\rangle = \Delta(\Delta - 1)|\phi\rangle \tag{4.16}$$

where Δ is a constant which defines the representation. By applying equations (4.14) and (4.16) one obtains the relations

$$|C_{n,\pm}|^2 = \lambda_n(\lambda_n \pm 1) - \Delta(\Delta - 1) \tag{4.17}$$

which, if $\langle\cdot|\cdot\rangle$ is a well-defined inner product should be equal to or greater than zero. Thus one finds the inequality

$$\Delta(\Delta - 1) \leq \lambda_n(\lambda_n \pm 1) \tag{4.18}$$

which can be satisfied in the following ways which classify all unitary irreducible representations of $\mathfrak{sl}(2, \mathbb{R})$ [80, 83].

In this chapter we will mostly focus on the positive discrete series following the original constructions by dAFF [23], however interesting work has also been done to extend these constructions to the principal series [84]. Note that it is possible to obtain non-unitary finite dimensional representations by analytically continuing the principal series representations to $\Delta \in \mathbb{Z}$ [85]. These representations will be considered in the following chapter. Since we are working in the positive discrete series representation, there exists a lowest weight state $|0\rangle$ that satisfies

$$L_-|0\rangle = 0 \; ; \quad R|0\rangle = \Delta|0\rangle \tag{4.19}$$

and all other states in the representation can be generated by successively applying L_+ to this state. Thus noting that equation (4.17) implies that

$$L_\pm|n\rangle = \sqrt{\lambda_n(\lambda_n \pm 1) - \Delta(\Delta - 1)}|n \pm 1\rangle \tag{4.20}$$

assuming $C_{n,\pm}$ is real. Therefore for any state $|n\rangle$ it is possible to rewrite it as

$$|n\rangle = \sqrt{\frac{\Gamma(2\Delta)}{\Gamma(n+1)\Gamma(2\Delta+n)}}(L_+)^n|0\rangle \tag{4.21}$$

where Γ are the Gamma functions. Using this representation it is now possible to define a $|x\rangle$ state alongside a representation of D, K and P as differential operators. This will turn out to have some interesting properties similar to conformal field theories.

4.2 Continuous representations

While the above representations of $SL(2,\mathbb{R})$ are useful, it would be nice to construct states in terms of a continuous variable x in order to make it more similar to conventional CFT's. This was originally done by dAFF [23] who introduced a conformal quantum mechanical system alongside with a continuous state $|x\rangle$ was introduced which acts similar to a position basis in ordinary quantum mechanics. With this position basis dAFF calculated an analogue to a correlation function and found formulas very similar to those in ordinary CFT's, as was also noted by Chamon et al. [21] In this subsection we will briefly recall this construction, although slightly modified to fit in line with the conventional CFT operators described in subsection 2.1. With this we will then also calculate the two and three-point functions. In the following sections we will then expand on this by introducing some new calculational tools and try to interpret it as the boundary theory of a quantum field theory on AdS.

In [23] dAFF constructs the state $|x\rangle$ by defining a realization of the $\mathfrak{so}(2,1)$ algebra as differential operators on it. Following this, we define $|x\rangle$ such that[13]

$$\begin{aligned} \langle x|P = && -i\tfrac{d}{dx}\langle x| \\ \langle x|K = && -i(x^2\tfrac{d}{dx}+2\Delta x)\langle x| \\ \langle x|D = && -i(x\tfrac{d}{dx}+\Delta)\langle x| \end{aligned} \tag{4.22}$$

which satisfy the commutation relations (4.8) and give rise to the correct Casimir (4.16). Since the states $|n\rangle$ form a basis in $\mathcal{H}$ it is possible to expand

$$|x\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|x\rangle \tag{4.23}$$

and thus to obtain an explicit form for $|x\rangle$ it is enough to find an expression for

$$\beta_n(x) \coloneqq \langle x|n\rangle \ . \tag{4.24}$$

To obtain an expression for β_n, note that (4.22) together with the definition of R given in (4.10) imply that

$$\langle x|R|n\rangle = -\frac{i}{2}\left((1+x^2)\frac{d}{dx}+2\Delta x\right)\langle x|n\rangle \tag{4.25}$$

[13] Note again the difference in convention compared to [23] and [21].

while (4.11) implies that

$$\langle x|R|n\rangle = (\Delta + n)\langle x|n\rangle \ . \tag{4.26}$$

Combining the two gives the following differential equation for β_n

$$-\frac{i}{2}\left((1+x^2)\frac{d}{dx} + 2\Delta x\right)\beta_n(x) = (\Delta + n)\beta_n(x) \tag{4.27}$$

which has as its solution

$$\beta_n(x) = a_n\left(\frac{1+ix}{1-ix}\right)^{n+\Delta}(1+x^2)^{-\Delta} \tag{4.28}$$

with a_n a constant. This constant can then be fixed by calculating $\langle x|L_\pm|n\rangle$ and requiring that (4.20) holds. This results in

$$a_n = a_0\sqrt{\frac{\Gamma(2\Delta+n)}{\Gamma(n+1)}} \tag{4.29}$$

where a_0 is an arbitrary constant which will be set to 1 from now on. Formulas (4.28) and (4.21) can be inserted into the expansion (4.23) resulting in

$$|x\rangle = \sum_{n=0}^{\infty}\beta_n^*(x)|n\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1-ix}{1+ix}\right)^{n+\Delta}(1+x^2)^{-\Delta}(L_+)^n|0\rangle \tag{4.30}$$

which can be evaluated to

$$|x\rangle = \frac{e^{\frac{1-ix}{1+ix}L_+}}{(1+ix)^{2\Delta}}|0\rangle =: \mathcal{O}(x)|0\rangle \ . \tag{4.31}$$

Note that while schematically this looks similar to a conformal operator $\mathcal{O}$ acting on a vacuum state $|0\rangle$, one has to keep in mind that $|0\rangle$ is just the lowest weight state defined in equation (4.19). This is not annihilated by all operators and therefore not a true ground state as one would find in an ordinary quantum field theory. Similarly $\mathcal{O}$ does not satisfy the commutation relations with P, K and D that one would expect from a conformal primary. However both combined do act as a conformal primary, as we will see below.

Formulas (4.28) for $\beta_n(x)$ and (4.31) for $|x\rangle$ will now be used to calculate an analogue to correlation functions in a conventional conformal field theory and as we will see these will turn out to have the correct form.

4.2.1 Two and three-point correlation functions

In an ordinary conformal field theory the main objects of interests are the correlation functions. In fact, it is even possible to define a conformal field theory just using these and a small amount of other data [86]. Thus it is natural to consider the "correlation function"

$$G(x_1, x_2) \coloneqq \langle 0|\mathcal{O}^\dagger(x_1)\mathcal{O}(x_2)|0\rangle = \langle x_1|x_2\rangle \tag{4.32}$$

in this conformal quantum mechanics theory and see if it has similar behaviour, here $\mathcal{O}(t)$ is as defined in (4.31). There are two ways to evaluate equation (4.32), both originally performed in [23]. Firstly, since the eigenstates $|n\rangle$ form a complete orthonormal basis it is possible to insert these resulting in

$$G(x_1,x_2) = \langle x_1|x_2\rangle = \sum_n \langle x_1|n\rangle\langle n|x_2\rangle = \sum_n \beta_n(x_1)\beta_n^*(x_2) \tag{4.33}$$

which, when inserting 4.28 and performing the sum becomes

$$G(x_1,x_2) = \frac{\Gamma(2\Delta)}{\big(2i|x_2-x_1|\big)^{2\Delta}} \tag{4.34}$$

Note that this is precisely the form of the correlation function for a conformal operator with conformal dimension Δ in a conformal field theory. However, as [21] have noted this is despite the fact that $|0\rangle$ is not a fully invariant vacuum and $\mathcal{O}$ is not a covariantly transforming operator.

Interestingly, this correlation function could have also been derived using by inserting D, K and P in $\langle x_1|x_2\rangle$, and letting them act on both $\langle x_1|$ and $|x_2\rangle$ separately giving a differential equation for G. To do this, it is first necessary to calculate the action of the operators on $|x_1\rangle$, which can be done by taking equation (4.31) and acting on it with P, K and H. This results in

$$\begin{aligned} P|x_i\rangle &= i\tfrac{d}{dx_i}|x_i\rangle \\ K|x_i\rangle &= i(x_i^2\tfrac{d}{dx_i}+2\Delta x_i)\langle x_i| \\ D|x_i\rangle &= i(x_i\tfrac{d}{dx_i}|x_i\rangle \end{aligned} \tag{4.35}$$

Therefore, calculating $\langle x_1|P|x_2\rangle$ results in

$$\langle x_1|P|x_2\rangle = -i\frac{\partial}{\partial x_1}\langle x_1|x_2\rangle = i\frac{\partial}{\partial x_2}\langle x_1|x_2\rangle \tag{4.36}$$

giving a differential equation for $G(x_1,x_2)=\langle x_1|x_2\rangle$. Performing a similar procedure for D and K results in the set of differential equations

$$\begin{aligned} (\tfrac{\partial}{\partial x_1}+\tfrac{\partial}{\partial x_2})G &= 0 \\ (x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2})G &= 0 \\ \Big(x_1^2\tfrac{\partial}{\partial x_1}+x_2^2\tfrac{\partial}{\partial x_2}+2\Delta(x_1+x_2)\Big)G &= 0 \end{aligned} \tag{4.37}$$

which when solved also results in (4.34), as was originally shown by dAFF in reference [23]. Using this same method, dAFF also derived a 3-point correlator by inserting a conformal operator B satisfying

$$\begin{aligned} [P,B] &= i\tfrac{d}{dx}B \\ [D,B] &= i(x\tfrac{d}{dx}+\Delta_B)B \\ [K,B] &= i(x^2\tfrac{d}{dx}+2x\Delta_B)B \end{aligned} \tag{4.38}$$

where Δ_B is the conformal dimension of B. Defining

$$F(x_1,x_2,x_3) \coloneqq \langle x_1|B(x_2)|x_3\rangle \tag{4.39}$$

and calculating

$$\begin{aligned} &\langle x_1|\ [P, B(x)]\ |x_2\rangle \\ &\langle x_1|\ [K, B(x)]\ |x_2\rangle \\ &\langle x_1|\ [D, B(x)]\ |x_2\rangle \end{aligned} \tag{4.40}$$

this gives rise to a set of differential equations satisfied by F. With these dAFF then found that the unique (up to a constant) solution of this is given by

$$F(x_1, x_2, x_3) = ai^{2\Delta+\Delta_B} \frac{1}{|x_2 - x_1|^{\Delta_B}|x_3 - x_2|^{\Delta_B}|x_1 - x_3|^{\Delta-\Delta_B}} \tag{4.41}$$

where a is a real constant. Note again the similarity to the conformal three point functions found in conformal field theory.

While the above strategy of either performing the summation over β_n functions directly or solving a set of differential equations works for two- and three-point functions, it runs into trouble for higher order correlation functions. There is no clear way of performing the sum since the matrix element $\langle n|B|m\rangle$ is unknown while the resulting differential equations quickly become much too complicated. To solve this it is necessary to introduce an operator called the shadow operator [87] which makes it possible to obtain a modified completeness relation, greatly simplifying the calculations needed.

4.2.2 Completeness relations and the shadow operator

To calculate the higher point functions it would be useful to have a completeness relation for the states $|x\rangle$. Since then, any n-point correlation function could be split into a three-point function and an $n-1$ point function. For example, a four-point function $\langle 0|\mathcal{O}^\dagger(x_1)A(x_2)B(x_3)\mathcal{O}(x_4)|0\rangle$ could be split as

$$\langle 0|\mathcal{O}^\dagger(x_1)A(x_2)B(x_3)\mathcal{O}(x_4)|0\rangle = \langle 0|\mathcal{O}^\dagger(x_1)A(x_2)|X\rangle\langle X'|B(x_3)\mathcal{O}(x_4)|0\rangle \tag{4.42}$$

where $|X\rangle\langle X'|$ is schematic notation for some unknown completeness relation. In ordinary quantum mechanics for a position basis satisfying

$$\langle x|x'\rangle = \mu(x, x')\delta(x - x') \tag{4.43}$$

one would find a completeness relation of the form

$$\int_{\mathbb{R}} dx\ \frac{1}{\mu(x, x')}|x\rangle\langle x| = 1 \tag{4.44}$$

with μ an arbitrary non-zero function. This is a problem though since from equation (4.34) one can see that the states $|x\rangle$ do not satisfy the equation (4.43). Therefore a different approach is necessary. Interestingly in conformal field theories, a similar problem occurs in the conformal block decomposition used in the conformal bootstrap method [88]. There in order to calculate n-point correlators a conformal block expansion is introduced that splits it into two lower order correlation functions analogous to equation (4.42) [87]. This split is achieved by introducing a projection operator $P_{\mathcal{O}}$ that satisfies

$$\sum_{\mathcal{O}} P_{\mathcal{O}} = 1 \tag{4.45}$$

where the sum is over conformal primaries [86]. In other words, the sum over these projection operators is a completeness relation in conformal field theories.

A particularly convenient method of obtaining these projection operators is using the shadow operator formalism originally due to Ferrara, Grillo, Parisi and Gatto [89, 90] and generalized by Simmons-Duffin [87]. There for each scalar conformal operator $\mathcal{O}$ its shadow operator $\tilde{\mathcal{O}}$ is defined as[14]

$$\widetilde{\mathcal{O}}(x) := \int_{\mathbb{R}^d} d^d x' \langle \mathcal{O}(x)\mathcal{O}(x')\rangle\big|_{\Delta\to\widetilde{\Delta}}\mathcal{O}(x') \tag{4.46}$$

where $|_{\Delta\to\widetilde{\Delta}}$ denotes that in the correlation function Δ is replaced with the shadow dimension $\widetilde{\Delta} := d-\Delta$. This shadow dimension is defined such that the action of the conformal Casimir is invariant under $\Delta \to \widetilde{\Delta}$. Since for a scalar the 2-point correlation function in a conformal field theory is of the form $C|x-x'|^{-2\Delta}$ the shadow operator is given by

$$\widetilde{\mathcal{O}}(x) = \int_{\mathbb{R}^d} d^d x' \frac{C}{|x-x'|^{2(d-\Delta)}}\mathcal{O}(x') \ . \tag{4.47}$$

Note that this integral is formally divergent and therefore some regularization procedure will be required. The conformal projectors can now be written in terms of the shadow operator as

$$P_{\mathcal{O}} := \mathcal{N}_{\mathcal{O}} \int_{\mathbb{R}^d} d^d x |\mathcal{O}(x)\rangle\langle\widetilde{\mathcal{O}}(x)| \tag{4.48}$$

where $\mathcal{N}_{\mathcal{O}}$ is chosen such that (4.45) holds and $|\mathcal{O}(x)\rangle$ denotes $\mathcal{O}(x)$ acting on the CFT vacuum.

Motivated by this action of the shadow operator it would be tempting to try to immediately apply formulas (4.46) and (4.48) to the quantum mechanical system described above. There is however some subtlety that needs to be taken care of first. The operation of taking the shadow operator is a unitary transformation between two principal series representations with the same Casimir charges. Since the Casimir acts as $C = \Delta(\Delta-1)$ which is invariant under $\Delta \to \tilde{\Delta} = 1-\Delta$ [51]. However, the representation above is a discrete series representation and therefore one should expect that a different approach might be necessary. As we will see in the following chapter, for the Hodge theoretical setting the approach above will work. This is not surprising since there a finite dimensional representation is used, which is an analytical continuation of the principal series [85]. To give an alternative in this context, we will define a heuristic method of performing a similar operation.

Since the shadow transform is a map between representations the goal is to obtain a similar map for the discrete series. The first requirement is that the two representations have the

[14] This definition also extends naturally to operators with spin, however since our application will focus on scalar fields we will only consider those here.

same Casimir charge, which is achieved by the transformation $\Delta \to \tilde{\Delta} = 1 - \Delta$. However, since the states $|n\rangle$ in the quantum mechanics setting have eigenvalues $\Delta + n$ with respect to R, this action changes the eigenvalue to $\tilde{\Delta} + n = n + 1 - \Delta$. To correct for this it is then also necessary to replace $n \to \tilde{n} = 2\Delta + n - 1$. Combining these we obtain a heuristic "shadow transform" S. While this procedure might seem somewhat arbitrary, applying it to the β_n as given in (4.28) results in

$$S[\beta_n](x) = \beta_n(x)\big|_{(n,\Delta)\to(\tilde{n},\tilde{\Delta})} = \tilde{a}_n \left(\frac{1+ix}{1-ix}\right)^{n+\Delta} (1+x^2)^{\Delta-1} \tag{4.49}$$

where

$$\tilde{a}_n = a_n\big|_{(n,\Delta)\to(\tilde{n},\tilde{\Delta})} = \sqrt{\frac{\Gamma(n+1)}{\Gamma(2\Delta+n)}} = \frac{1}{a_n} \tag{4.50}$$

and a_n is as defined in equation (4.29) with $a_0 = 1$ as before.

With this expression it is possible to define a "shadow operator" such that

$$\tilde{\mathcal{O}}(x)|0\rangle \coloneqq \sum_{n=0}^{\infty} S[\beta_n](x)|n\rangle \tag{4.51}$$

analogous to the definition of $\mathcal{O}$ in (4.31).[15] With this operator it is now possible to define the projection operator (4.48)

$$P \coloneqq \mathcal{N} \int_{\mathbb{R}} dx\ \mathcal{O}(x)|0\rangle\langle 0|\tilde{\mathcal{O}}(x) \tag{4.52}$$

where $\mathcal{N}$ is a constant to be determined. Since there is only one "primary" operator in the conformal quantum mechanics setting the sum in equation (4.45) becomes trivial and P should be proportional to the identity. Since the states $|n\rangle$ form a complete orthonormal basis of $\mathcal{H}$ it is enough to check that

$$P_{n,m} \coloneqq \langle n|P|m\rangle = \delta_{n,m} \tag{4.53}$$

with δ the Kronecker delta. Inserting the explicit expressions for P, $\mathcal{O}$ and $\tilde{\mathcal{O}}$ as given in (4.52), (4.30) and (4.51) then results in

$$P_{n,m} = \mathcal{N} \int_{\mathbb{R}} dx\ \beta_n(x)\tilde{\beta}^*_n(x) = a_n\tilde{a}_m \int_{\mathbb{R}} dx\ \left(\frac{1+ix}{1-ix}\right)^{n-m} (1+x^2)^{-1} \tag{4.54}$$

which, when making the change of coordinates $x = \tan(\theta)$ evaluates to

$$P_{n,m} = a_n\tilde{a}_m\mathcal{N} \int_{-\pi/2}^{\pi/2} d\theta\ e^{2i\theta(n-m)} = a_n\tilde{a}_m\mathcal{N}\pi\delta_{n,m} \tag{4.55}$$

[15] Interestingly, since $|0\rangle$ is the actual state in the representation of $SL(2,\mathbb{R})$ one might wonder if this also needs to be transformed somehow. This however seems not to be the case as we will see below.

with δ again the Kronecker delta. Thus, setting $\mathcal{N} = \frac{1}{\pi}$ and noting that $a_n \tilde{a}_n = 1$ results in

$$P_{n,m} = \delta_{n,m} \tag{4.56}$$

as required. Thus P gives a pseudo-completeness relation for the states $|x\rangle$ which can be used to calculate arbitrarily high correlation functions by inserting P, inserting expressions for lower order correlation functions and performing the integrals. We will now briefly show how to find integral expressions for the higher order correlators.

4.2.3 Higher order correlators

In order to show the usefulness of this completion relation, lets consider the 4-point function

$$F_4(x_1, x_2, x_3, x_4) \coloneqq \langle x_1 | A(x_2) B(x_3) | x_4 \rangle \tag{4.57}$$

where A and B satisfy the commutation relations (4.38) with conformal dimensions Δ_A and Δ_B respectively. Now inserting the projection operator P from equation (4.52) and using that $P = 1$ results in

$$F_4(x_1, x_2, x_3, x_4) = \int_{\mathbb{R}} dx \; \langle x_1 | A(x_2) | x \rangle \langle 0 | \widetilde{\mathcal{O}}(x) B(x_3) | x_4 \rangle \tag{4.58}$$

where the first term in the integral is just the 3-point function from equation (4.41). The second term is similar to the 3-point function and can be obtained in a similar way by solving the differential equations it satisfies. To find these, first note that

$$\langle 0 | \widetilde{\mathcal{O}}(x) P = -i \frac{d}{dx} \langle 0 | \widetilde{\mathcal{O}}(x) \tag{4.59}$$

$$\langle 0 | \widetilde{\mathcal{O}}(x) D = -i (x \frac{d}{dx} + 1 - \Delta) \langle 0 | \widetilde{\mathcal{O}}(x) \tag{4.60}$$

$$\langle 0 | \widetilde{\mathcal{O}}(x) K = -i (x^2 \frac{d}{dx} + 2(1 - \Delta)) \langle 0 | \widetilde{\mathcal{O}}(x) \tag{4.61}$$

which are analogous to (4.22) but with $\Delta \to 1 - \Delta$. With these it is again possible to calculate the commutators

$$\begin{gathered} \langle 0 | \widetilde{\mathcal{O}}(x) \; [P, B(x_3)] \; | x_4 \rangle \\ \langle 0 | \widetilde{\mathcal{O}}(x) \; [K, B(x_3)] \; | x_4 \rangle \\ \langle 0 | \widetilde{\mathcal{O}}(x) \; [D, B(x_3)] \; | x_4 \rangle \end{gathered} \tag{4.62}$$

in order to find

$$\langle 0 | \widetilde{\mathcal{O}}(x) B(x_3) | x_4 \rangle \propto \frac{1}{|x - x_3|^{\Delta_B + 1 - 2\Delta} |x_3 - x_4|^{\Delta_B - 1 + 2\Delta} |x - x_4|^{1 - \Delta_B}} \tag{4.63}$$

analogous to a 3-point CFT correlator where the operators have weights $1 - \Delta$, Δ_B and Δ respectively. Inserting this and equation (4.41) into equation (4.58) then results in

$$\begin{aligned} F_4(x_1, x_2, x_3, x_4) \propto \int_{\mathbb{R}} dx \; & \frac{1}{|x_2 - x_1|^{\Delta_A} |x - x_2|^{\Delta_A} |x_1 - x|^{\Delta - \Delta_A}} \\ & \frac{1}{|x - x_3|^{\Delta_B + 1 - 2\Delta} |x_3 - x_4|^{\Delta_B - 1 + 2\Delta} |x - x_4|^{1 - \Delta_B}} \, . \end{aligned} \tag{4.64}$$

Unfortunately we were not able to solve this integral in complete generality, however these kind of integrals are very similar to those appearing in CFT 4-point functions and therefore it might be possible to apply tools used there such as Mellin-Barnes representations or the embedding formalism [87, 91]. One should note that the above procedure generalizes in an obvious way to higher order correlation functions, making it possible to find integral expressions for any n-point correlator. Even if the resulting integrals turn out to be unsolvable in full generality, having these integral expressions then still makes it possible to study their properties in more detail as well as calculate them in certain special cases or approximations.

We will now take a step back from the correlators and completeness relations discussed above by considering a certain discrete series representation of $SL(2,\mathbb{R})$ on the Poincaré upper half plane, and as we will see this turns out to have an interesting connection to the quantum mechanics model described above.

4.3 The discrete series representation on the Poincaré half-plane

To see the connection between the quantum mechanical system above and the discrete series on the Poincaré upper half-plane it is first necessary to construct the latter. We will mostly follow the excellent book by Lang on the $SL(2,\mathbb{R})$ group in doing this [83] and emphasize the connection to conformal quantum mechanics. To start, define the upper half plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \tag{4.65}$$

with its metric given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{4.66}$$

where we write $z = x + iy$. This metric is invariant under the action of $SL(2,\mathbb{R})$ on $\mathbb{H}^2$ given by

$$gz = \frac{az+b}{cz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \tag{4.67}$$

and in fact $SL(2,\mathbb{R})$ gives the full isometry group of $\mathbb{H}^2$. The Hilbert space our representation will act on is the space of holomorphic square integrable functions on $\mathbb{H}^2$ with the measure $d\mu_\Delta(z) := y^{2\Delta-2}dxdy$ which will be denoted by $\mathcal{H}_\Delta$. The discrete series representation $\pi_\Delta : SL(2,\mathbb{R}) \to GL(\mathcal{H}_\Delta)$ is then defined as

$$\pi_\Delta(g)f(z) := \frac{f(g^{-1}z)}{(cz+d)^\Delta} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \tag{4.68}$$

and $f \in \mathcal{H}_\Delta$. For the generators X_i defined in (4.5) this transformation rule implies that

$$\begin{aligned} \pi_\Delta(e^{tX_1})f(z) &= f(z+t) \\ \pi_\Delta(e^{tX_2})f(z) &= \tfrac{1}{(1+tz)^\Delta} f(\tfrac{z}{1+tz}) \\ \pi_\Delta(e^{tX_3})f(z) &= e^{t\Delta} f(e^t z) \end{aligned} \tag{4.69}$$

where $t \in \mathbb{R}$. Note that these are exactly the transformation rules of a primary operator under conformal transformations.

It is possible to associate an infinitesimal generator to each element $X \in \mathfrak{sl}(2, \mathbb{R})$ by calculating

$$\mathcal{L}_X(f) \coloneqq \frac{d}{dt}\pi_\Delta(e^{tX})f\big|_{t=0} \tag{4.70}$$

which, for the generators X_i defined in (4.5) results in

$$\begin{aligned} \mathcal{L}_1 &= \tfrac{\partial}{\partial x} \\ \mathcal{L}_2 &= (x^2 - y^2)\tfrac{\partial}{\partial x} + 2xy\tfrac{\partial}{\partial y} + 2\Delta(x + iy) \\ \mathcal{L}_3 &= x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + \Delta \end{aligned} \tag{4.71}$$

where $\mathcal{L}_i \coloneqq \mathcal{L}_{X_i}$. Lang then showed that this was a unitary irreducible representation of $SL(2, \mathbb{R})$ with lowest weight Δ. Furthermore, the functions

$$\phi_{\Delta,n}(z) \coloneqq a_n \left(\frac{1 + iz}{1 - iz}\right)^{\Delta+n} (1 + z^2)^{-\Delta} \tag{4.72}$$

form a complete orthonormal basis of $\mathcal{H}_\Delta$, where n is a positive integer and a_n is as in equation (4.29). Note the similarity with the β_n functions as given in equation (4.28). This is not coincidentally since as we will see the quantum mechanical system above can be thought of as the boundary theory associated to this representation.

To see this, first note that taking the limit $y \to 0$ of $\mathcal{L}_i$ as written in equation (4.71) results in

$$\begin{aligned} \lim_{y\to 0} \mathcal{L}_1 &= \tfrac{\partial}{\partial x} &&= iP \\ \lim_{y\to 0} \mathcal{L}_2 &= x^2\tfrac{\partial}{\partial x} + 2\Delta x &&= iK \\ \lim_{y\to 0} \mathcal{L}_3 &= x\tfrac{\partial}{\partial x} + \Delta &&= iD \end{aligned} \tag{4.73}$$

where P, K and D are as defined in equation (4.22). This gives a motivation for the original definition of the differential operators given in equation (4.22), which originally was just obtained by identifying P as the generator of translations and requiring that the commutators as well as the Casimir charge are correct. Furthermore, defining the operators

$$\begin{aligned} \mathcal{L}_R &\coloneqq \tfrac{1}{2}(\mathcal{L}_1 + \mathcal{L}_2) \\ \mathcal{L}_+ &\coloneqq \tfrac{1}{2}(\mathcal{L}_1 - \mathcal{L}_2) + i\mathcal{L}_3 \\ \mathcal{L}_- &\coloneqq \tfrac{1}{2}(\mathcal{L}_1 - \mathcal{L}_2) - i\mathcal{L}_3 \end{aligned} \tag{4.74}$$

and letting them act on $\phi_{\Delta,n}$ as defined in equation (4.72) results in

$$\begin{aligned} \mathcal{L}_R\phi_{\Delta,n} &= i(\Delta + n)\phi_{\Delta,n} \\ \mathcal{L}_+\phi_{\Delta,n} &= -i(2\Delta + n)\phi_{\Delta,n+1} \\ \mathcal{L}_-\phi_{\Delta,n} &= -in\phi_{\Delta,n-1} \end{aligned} \tag{4.75}$$

analogously to the action of R and $L_\pm$ on the β_n functions defined in equation (4.24). The $\phi_{\Delta,n}$ are in fact also eigenfunctions of the Casimir

$$\mathcal{L}_\mathcal{C} \coloneqq \mathcal{L}_3^2 - \frac{1}{2}(L_1L_2 + L_2L_1) = \Delta(\Delta - 1) - y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2y\Delta\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \tag{4.76}$$

with eigenvalue $\Delta(\Delta-1)$. Interestingly, the second term is the Laplacian on $\mathbb{H}^2$ and the third term is $\frac{\partial}{\partial \bar{z}}$ which is zero since we are only dealing with holomorphic functions. This combined with the fact that the functions $\phi_{\Delta,n}$ are annihilated by the Laplacian guarantees that they have eigenvalue $\Delta(\Delta-1)$. Note that this is different from ordinary free fields on AdS_2 which classically are eigenfunctions of the Laplacian, not annihilated by it.

Taking the limit of $\phi_{\Delta,n}$ to the boundary is trivial and results in a correspondence between the $\phi_{\Delta,n}$ of equation (4.72) and the β_n functions of equation (4.28). To see this, simply taking the limit

$$\lim_{y\to 0}\phi_{\Delta,n}(z)=\lim_{y\to 0}a_n\left(\frac{1+iz}{1-iz}\right)^{n+\Delta}(1+z^2)^{-\Delta}=\beta_n(x) \tag{4.77}$$

is enough. Note that this limit is slightly different than the limit usually taken in the AdS/CFT correspondence, since there for a field ψ with conformal dimension Δ the limit

$$\lim_{y\to 0}y^{\Delta-d}\psi(x,y) \tag{4.78}$$

is usually taken to be equivalent to the source of a conformal operator [33], while the limit

$$\lim_{y\to 0}y^{-\Delta}\psi(x,y) \tag{4.79}$$

is taken to be either the expectation value of the operator, or in the extended dictionary the operator itself [34]. The difference in limits however is simply due to the fact that the fields $\phi_{\Delta,n}$ are elements of $\mathcal{H}_\Delta=L(\mathbb{H}^2,\mu_\Delta)$ and thus appear with an extra factor of y^Δ in integrals. It is therefore possible to associate a function $f_{\mathbb{H}^2}\in L^2(\mathbb{H}^2)$ on $\mathbb{H}^2$ to any function $f_\Delta\in\mathcal{H}_\Delta$ by taking $f_{\mathbb{H}^2}(z):=y^\Delta f_\Delta$. This is then in $L^2(\mathbb{H}^2)$ since

$$\int_{\mathbb{H}^2}\frac{dxdy}{y^2}|f_{\mathbb{H}}(x,y)|^2=\int_{\mathbb{H}^2}\frac{dxdy}{y^2}y^{2\Delta}|f_\Delta(x,y)|^2=\int_{\mathbb{H}^2}d\mu_\Delta(z)|f_\Delta(x,y)|^2<\infty\,. \tag{4.80}$$

as required. Taking the limit to the boundary $y=0$ of f_Δ is then equivalent to taking the limit

$$\lim_{y\to 0}f_\Delta=\lim_{y\to 0}y^{-\Delta}f_{\mathcal{H}^2} \tag{4.81}$$

which the AdS/CFT limit relating bulk fields to boundary operators in the extrapolate dictionary [34, 44]. There seems to be no clear interpretation however for the other limit in equation (4.78) associated to sources in the conventional AdS/CFT dictionary.

The system above is interesting though because it gives some insights into the conformal quantum mechanics defined previously. The exact form of the differential operators defined in (4.22) may have seemed a bit arbitrary at first but now is realized as the boundary limits of differential operators acting on $\mathbb{H}^2$ in a canonical way. Furthermore the wave functions at the boundary are just directly the boundary limits of the wave functions in $\mathcal{H}_\Delta$. This means that a natural interpretation of the conformal quantum mechanics system is just simply as the boundary theory of quantum mechanics in $\mathcal{H}_\Delta$. While it might be tempting to imbue this with some sense of holography, this seems unlikely as the mapping seems much too simple to allow for the more interesting aspects of AdS/CFT. I.e. there is no theory of gravity in the bulk and while the analogue of correlation functions does match in the bulk and in the boundary, the calculation is much too trivial to obtain interesting information from since a lot of the complexity of field theories is missing. It would be interesting to promote both the boundary and the bulk theory to quantum field theories by second quantization and see if that yields interesting results. This could be done in further research.

Now that we have studied this conformal quantum mechanics model, we will construct a similar model for the Hodge theoretical setting. As we will see, much of the constructions can directly be carried over although there is the crucial difference that the resulting representation is not unitary.

5 Finite dimensional representations in Hodge theory

Having studied the conformal quantum mechanics setting in the previous chapter, we now turn towards the second main object of interest of this thesis: the finite dimensional $SL(2,\mathbb{R})$ representations from Hodge theory. In order to understand broadly what these are and why they are relevant, we need to take a brief look at string compactifications. Quite famously, string theory requires more than 4 dimensions in order to be consistent. Since at our every day energy-scales we see only 4 dimensions, string theory needs a method that hides these dimensions as these energy scales. This is what is is achieved by compactification. Usually, compactification starts by considering string theory on a product manifold[16]

$$\mathcal{M} \times K \tag{5.1}$$

where K is a compact manifold called the internal manifold. This decomposition then results in an effective theory over $\mathcal{M}$ which depends on the internal geometry of K. For example, if K is sufficiently small this effective theory will keep only the massless excitations on K. A natural question then is how this effective theory varies as this internal geometry changes.

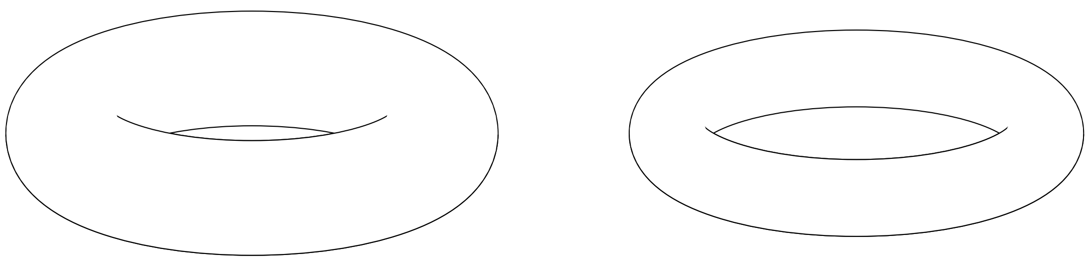

Figure 2: Two tori with different geometries, choosing either as the internal manifold K will lead to different effective theories on $\mathcal{M}$.

Much of this information about the geometry is captured by the de Rham cohomologies $H^n(K,\mathbb{C})$ and their Hodge decomposition, which have been used to study the infinite distance conjecture [92] and flux vacua [93]. These Hodge decompositions result in vector spaces that change over the space of all geometries, the moduli space. This moduli space has boundaries that denote singular geometries, for example for the torus this would be taking its width to zero. Interestingly near these boundaries these de Rham cohomologies split into a so-called limiting mixed Hodge structure. These cohomologies then decompose in to finite dimensional $SL(2,\mathbb{R})$ representations [29, 30]. Meaning that there are is a triple N^0, $N^\pm$ satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[N^0, N^\pm] = \pm 2N^\pm \; ; \quad [N^+, N^-] = N^0 \tag{5.2}$$

along with a basis $|n,\Delta\rangle$ of N^0 eigenstates that satisfy

$$N^0|n,\Delta\rangle = 2n|n,\Delta\rangle \tag{5.3}$$
$$N^\pm|n,\Delta\rangle \propto |n\pm 1,\Delta\rangle \tag{5.4}$$

[16]For a good reference see the excellent book by Blumenhagen, Lüst and Theisen [28], especially chapters 10, 14 and 17.

where Δ here is an integer that defines the highest weight of the representation, i.e. $-\Delta \leq n \leq \Delta$. Note however that these states $|n,\Delta\rangle$ do not directly form a basis of the F^p but of their so-called Deligne splitting, although for us this difference will not be important. A further property of these spaces is given by the nilpotent orbit theorem, which states that near the boundaries in moduli space these vector spaces F behave as

$$F(t) \approx e^{tN^0} F_0 + \mathcal{O}(e^{i2\pi t}) \tag{5.5}$$

where $t = x + iy$ is a coordinate on moduli space and the boundary is at $y \to \infty$. Finally the boundary vector spaces are endowed with an inner product such that the states $|n,\Delta\rangle$ are orthonormal and the $SL(2,\mathbb{R})$ generators satisfy

$$(N^0)^\dagger = N^0 \quad (N^+)^\dagger = N^- \ . \tag{5.6}$$

Since both these vector spaces and the conformal quantum mechanics from the previous chapter form $SL(2,\mathbb{R})$ representations, it is interesting to see if these same constructions from conformal quantum mechanics can also be applied to the Hodge theoretical setting. Applying these constructions was one of the main goals of this thesis and in this chapter we will discuss how this can be done. Before this, we will briefly discuss some properties of these finite dimensional representations.

5.1 The finite dimensional representation

Equations (5.2), (5.3) and (5.6) together completely determine the representation already. For example, using the commutation relations (5.2) and the adjoint relations (5.6) one can deduce that $N^\pm$ must act as

$$N^\pm|n,\Delta\rangle = \sqrt{(\Delta \mp n)(\Delta + 1 \pm n)}|n \pm 1,\Delta\rangle \tag{5.7}$$

which implies that any state $|n,\Delta\rangle$ can be written as

$$|n,\Delta\rangle = \sqrt{\frac{\Gamma(\Delta+1+n)}{\Gamma(\Delta+1-n)\Gamma(2\Delta+1)}} N^{\Delta-n}|\Delta,\Delta\rangle \tag{5.8}$$

where $|\Delta,\Delta\rangle$ is the highest weight state. Furthermore, the quadratic Casimir in this representation takes the form

$$C^2 := \Big((N^0)^2 + 2(N^+N^- + N^-N^+)\Big) \tag{5.9}$$

which acts as

$$C^2|n,\Delta\rangle = 4\Delta(\Delta+1)|n,\Delta\rangle \tag{5.10}$$

within this representation.

Finally since $SL(2,\mathbb{R})$ is non-compact it follows that its unitary irreducible representations are infinite dimensional. However, the representations considered in this chapter are finite dimensional and therefore not unitarizable. To see why this is the case, first notice that by comparing the $\mathfrak{sl}(2,\mathbb{R})$ commutators given in equation (4.6) to equation (5.2) above one can deduce that within this representation

$$X_1 \mapsto N^- \tag{5.11}$$

$$X_2 \mapsto -N^+ \tag{5.12}$$

$$X_3 \mapsto -2N^0 \tag{5.13}$$

where X_i are as defined in equation (4.5). From the Lie group-Lie algebra correspondence we know that it is possible to write any element $g \in SL(2,\mathbb{R})$ in the connected component to the identity as

$$g = e^{aX_1+bX_2+cX_3} \tag{5.14}$$

where a, b and c are real numbers. Thus g will be represented as

$$g \mapsto e^{aN^- - bN^+ - 2cN^0} \tag{5.15}$$

within our representation. Now applying the adjoint relations (5.6) to this equation results in

$$g^\dagger \mapsto e^{aN^- - bN^+ - 2cN^0} \tag{5.16}$$

while g^{-1} will be mapped to

$$g^{-1} = e^{-aX_1-bX_2-cX_3} \mapsto e^{-aN^- + bN^+ + 2cN^0} \tag{5.17}$$

and therefore $g^\dagger \neq g^{-1}$ for general $g \in SL(2,\mathbb{R})$. Which means that this representation is not unitary. One might wonder if by changing the inner product it would be possible to obtain a unitary representation. The problem here however is that as we mentioned $SL(2,\mathbb{R})$ is non-compact and therefore no finite dimensional unitary representations exist [83]. Therefore the representation considered in this chapter will not be unitary nor even unitarizable. This actually has some interesting consequences, first of all as we will see we will associated N^+ with translations in our continuous representation. In quantum mechanics the generator of time translations is usually interpreted as the Hamiltonian which is required to be self-adjoint. However, from the adjoint relations (5.6) it is immediately clear that $(iN^+) \neq iN^+$ and therefore an equation of the form

$$iN^+ f = i\frac{d}{dx} f \tag{5.18}$$

is not a proper Schrödinger equation.[17] Note that this is the notion of unitary usually meant in quantum mechanics while the notion of unitarity used above is the one more commonly used when talking about group representations. The representation from this chapter is neither.

[17] We have chosen to write the continuous coordinate as x here to connect more clearly with the Hodge theory setting where x also denotes the boundary coordinate. However it can also be interpreted as a time coordinate.

The equations (5.3) to (5.6) as well as equation (5.9) to (5.8) will be the most useful for further study, while the Hodge theory motivating them will play no further role from now on. Thus we will now solely consider the representation in isolation and construct a continuous representation analogous to that in chapter 4.

5.2 Continuous representations

To construct the continuous representations of the system described above we will take a similar approach to the one in chapter 4, namely we will first construct some differential operators acting on a state $\langle x|$ and then find an explicit expression for this state. Since there is a priori no natural definition of the differential operators we have a bit more freedom here. However since the goal is to end up with a state of the form (5.5) a natural requirement is

$$|x\rangle = e^{xN^-}|\phi\rangle \tag{5.19}$$

for some state $|\phi\rangle$. This in turn implies that

$$N^-|x\rangle = \frac{d}{dx}|x\rangle \tag{5.20}$$

and since $(N^-)^\dagger = N^+$ one should expect

$$\langle x|N^+ \quad = \tfrac{d}{dx}\langle x| \tag{5.21}$$

which we will therefore take as a starting point. Requiring the commutation relations in equation (5.2) as well as that the Casimir acts as in equation (5.9) then fixes the action of N^0 and N^- to be

$$\begin{aligned} \langle x|N^0 \quad &= (-2x\tfrac{d}{dx} + 2\Delta)\langle x| \\ \langle x|N^- \quad &= (-x^2\tfrac{d}{dx} + 2\Delta x)\langle x| \ . \end{aligned} \tag{5.22}$$

Now the next step is to find an expression for the state $|x\rangle$. In the conformal quantum mechanics setting it was necessary to define the operator R in order to obtain an orthonormal eigenbasis. Here however, there exists already an orthonormal eigenbasis of N^0 and therefore this step is unnecessary. Using this we can define

$$\beta_n(x) \coloneqq \langle x|n, \Delta\rangle \tag{5.23}$$

which makes it possible to express $|x\rangle$ as

$$|x\rangle = \sum_{n=-\Delta}^{\Delta} |n, \Delta\rangle\langle n, \Delta|x\rangle = \sum_{n=-\Delta}^{\Delta} \beta_n^*(x)|n, \Delta\rangle \tag{5.24}$$

where it was used that the states $|n, \Delta\rangle$ form an orthonormal basis. Note that in the conformal quantum mechanics setting the similar formula (4.23) involved an infinite sum while here the sum is finite.

An explicit expression for the β_n functions can be found by considering $\langle x|N^0|n,\Delta\rangle$ and using that $|n,\Delta\rangle$ is an eigenstate of N^0 as well as equation (5.22). This then results in the following differential equation

$$(-2t\frac{d}{dx}+2\Delta)\beta_n = 2n\beta_n \tag{5.25}$$

which has as its solution

$$\beta_n(x) = a_n x^{\Delta-n} \tag{5.26}$$

with a_n a constant. This constant can then be fixed by requiring that equation (5.7) holds resulting in

$$a_n \coloneqq a_\Delta \sqrt{\frac{\Gamma(2\Delta+1)}{\Gamma(\Delta+1+n)\Gamma(\Delta+1-n)}} \tag{5.27}$$

where a_Δ will be set to 1 from now on. With formulas (5.26) and (5.24) it is now possible to find an explicit expression for $|x\rangle$. This is done by simply inserting (4.24) and (5.8) in (5.24) and performing the sum explicitly, which results in

$$|x\rangle = \sum_{n=-\Delta}^{\Delta} \frac{(xN^-)^{\Delta-n}}{\Gamma(\Delta+1-n)}|\Delta,\Delta\rangle = e^{xN^-}|\Delta,\Delta\rangle \tag{5.28}$$

where it was used that $(L^-)^m|\Delta,\Delta\rangle = 0$ for $m > 2\Delta$. Note the relative simplicity in comparison with formula (4.21) from conformal quantum mechanics. This difference is due to that in the Hodge theory setting, we are working directly with an eigenbasis of N^0 while in conformal quantum mechanics it was necessary to work in an eigenbasis of R.[18] N^0 as a differential operator is much simpler than R was in the previous chapter, as one can see by comparing equations (4.25) and (5.22). The reason why it is possible now to work in an eigenbasis of N^0 instead of R is that we are no longer working in a unitary representation. In particular, while N^0 is self-adjoint, $N^\pm$ satisfy $(N^+)^\dagger = N^-$ which makes it possible for them to act properly as raising and lowering operators. Remember that in the conformal quantum mechanics setting the operators $L_\pm$ were defined that satisfied $(L_+)^\dagger = L_-$. In fact when working in an orthonormal eigenbasis the raising and lowering operators must satisfy these adjoint relations in order for the theory to be consistent.

With the expression for the β_n functions and the state $|x\rangle$ we will now calculate correlation functions, analogously to what was done in section 4.2.1.

5.2.1 Correlation functions

Similar to in the conformal quantum mechanics setting, it is possible to define an analogue to a correlation function. In order to make this more explicit, we first rewrite equation (5.24) as

$$|x\rangle = \mathcal{O}(x)|\Delta,\Delta\rangle \tag{5.29}$$

[18] The analogue of this in conformal quantum mechanics would be to work in an eigenbasis of D.

with

$$\mathcal{O}(x) := e^{xN^-} . \tag{5.30}$$

This then invites us to interpret

$$\langle x_1|x_2\rangle = \langle \Delta, \Delta|\mathcal{O}^\dagger(x_1)\mathcal{O}(x_2)|\Delta, \Delta\rangle \tag{5.31}$$

as a two point correlator of $\mathcal{O}$ acting on a "vacuum state" $|\Delta, \Delta\rangle$. Thus one can define the "correlator"

$$G(x_1, x_2) := \langle x_1|x_2\rangle \tag{5.32}$$

similar to equation (4.32). Inserting the full set of orthonormal states $|n, \Delta\rangle$ and performing the sum then results in

$$G(x_1, x_2) = \sum_{n=-\Delta}^{\Delta} \beta_n(x_1)\beta_n^*(x_2) = (1 + x_1x_2)^{2\Delta} . \tag{5.33}$$

Interestingly, this looks quite different from a normal correlation function. First of all it does not depend just on the distance $|x_1 - x_2|$, furthermore it diverges as either x_1 or x_2 go to infinity. While these properties make it quite unconventional as a correlation function, as we will see interpreting it this way does have its advantages when defining a shadow operator. Before that, we will show how these properties arise from the differential equations satisfied by G.

Remember that one way of obtaining the correlation function in the conformal quantum mechanics setting was by inserting an operator and letting it act on $\langle x_1|$ and $\langle x_2|$ separately, as described in section 4.2.1. One key property there was that the action of the operators on $\langle x_1|$ and $|x_2\rangle$ was obtained just by taking the complex conjugate. In the system considered in this chapter however, the action on $|x_i\rangle$ of N^0 and $N^\pm$ is given by

$$\begin{aligned} N^-|x_i\rangle &= \tfrac{d}{dx_i}|x_i\rangle \\ N^0|x_i\rangle &= (-x_i\tfrac{d}{dx_i} + 2\Delta)|x_i\rangle \\ N^+|x_i\rangle &= (-x_i^2\tfrac{d}{dx_i} + 2\Delta x_i)|x_i\rangle \end{aligned} \tag{5.34}$$

as can be obtained by explicit calculation. Heuristically, one can see that these equations can also be obtained by taking the adjoint of equations (5.21) and(5.22), although one has to be careful since taking the adjoint of the differential operators in this context is somewhat ill-defined.[19] This suggests the difference in the form of the correlation function is related to the fact that this is a non-unitary representation of $SL(2, \mathbb{R})$. With equations (5.21),(5.22) and (5.34) it is now possible to obtain the differential equations G must satisfy by calculating $\langle x_1|N^0|x_2\rangle$ and $\langle x_2|N^\pm|x_2\rangle$. This procedure then results in the following differential equations

$$\begin{aligned} (\tfrac{d}{dx_1} + x_2^2\tfrac{d}{dx_2} - 2\Delta x_2)G &= 0 \\ (x_1\tfrac{d}{dx_1} - x_2\tfrac{d}{dx_2})G &= 0 \\ (x_1^2\tfrac{d}{dx_1} + \tfrac{d}{dx_2} - 2\Delta x_1)G &= 0 \end{aligned} \tag{5.35}$$

[19]This is due to the fact that the states $|x\rangle$ behave somewhat analogous to a position basis, thus taking the differential of them is ill-defined until one actually considers wave functions of the form $\phi(x) := \langle x|\phi\rangle$. This is done in subsection 5.3.1.

which has the solution

$$G(x_1, x_2) = c(1 + x_1 x_2)^{2\Delta} \tag{5.36}$$

where c is an arbitrary constant. Setting $c = 1$ then results in equation (5.33).

Like in the conformal quantum mechanics setting, it is also possible to define a 3-point function

$$F(x_1, x_2, x_3) \coloneqq \langle x_1 | B(x_2) | x_2 \rangle \tag{5.37}$$

where B now satisfies

$$\begin{aligned} [N^-, B(x)] &= \tfrac{d}{dx} B(x) \\ [N^0, B(x)] &= (-2x\tfrac{d}{dx} + 2\Delta_B) B(x) \\ [N^+, B(x)] &= (-x^2 \tfrac{d}{dx} + 2\Delta_B x) B(x) \end{aligned} \tag{5.38}$$

similar to equation (4.38) in the conformal quantum mechanics setting. One may wonder whether the commutation relations between N^- and N^+ should be swapped or not, since their actions on $\langle x|$ and $|x\rangle$ are different. However, swapping their commutation relations with B above results in inconsistent differential equations. Furthermore, in the dAFF model the commutation relations were also chosen in accordance with how the operators acted on $|x\rangle$, although there the difference was simply a complex conjugation [**dede1976conformal**]. Inserting these commutators as $\langle x_1| [\cdot, B(x_2)] |x_3\rangle$ again results in a set of differential equations. These can be solved and result in

$$F(x_1, x_2, x_3) \propto (1 + x_1 x_2)^{\Delta_B} (1 + x_2 x_3)^{\Delta_B} |x_3 - x_2|^{2\Delta - \Delta_B} \tag{5.39}$$

which has some interesting properties. First of all, like before it is not dependant on just distances between the three points which is somewhat expected. More surprisingly though, it actually does depend on the distance $|x_3 - x_2|$ and therefore is not a product of the 2-point correlators either. This is a direct result of the differential equations (5.38) that B satisfies and therefore one may wonder if it is possible to find differential equations for B that would result in a more conventional product of 2-point correlators. Unfortunately we were not able to find any such differential equations. If these are possible then an immediate question arises which choice of differential equations is more natural, and also what interpretation the different choices have. These questions will hopefully be answered in further research.

Now we will continue following the conformal quantum mechanics constructions and try to find a completeness relation, which would make it possible to calculate higher order correlators. As we will see, the shadow transform will play an even more important role than in the conformal quantum mechanics setting.

5.3 Shadow operator

To calculate higher order correlators it is useful to obtain a completeness relation analogous to the one obtained in 4.2.2. It turns out that in the hodge theory setting the connection with the shadow transform is much more clear than in the conformal quantum mechanics setting. For the readers convenience, we will first briefly recall the shadow operator formalism. In conformal field theories, the shadow operator $\widetilde{\mathcal{O}}$ associated to an operator $\mathcal{O}$ with conformal dimension Δ is defined as [89, 90]

$$\widetilde{\mathcal{O}}(x) \coloneqq \int_{\mathbb{R}^d} d^d x' \langle \mathcal{O}(x)\mathcal{O}(x')\rangle\big|_{\Delta\to\widetilde{\Delta}}\mathcal{O}(x') \tag{5.40}$$

where $\widetilde{\Delta}$ is the shadow dimension and $|_{\Delta\to\widetilde{\Delta}}$ denotes that the in the correlation function of $\mathcal{O}$, Δ is replaced with $\widetilde{\Delta}$. This shadow dimension is defined such that the action of the Casimir is invariant under the transformation $\Delta \to \widetilde{\Delta}$. For example, in conformal field theories the Casimir acts as $[C, \mathcal{O}(x)] = \Delta(\Delta - d)\mathcal{O}(x)$ on a conformal scalar with weight Δ thus the shadow dimension is given by $d - \Delta$. The usefulness from this shadow operator comes when defining the projection operator $P_{\mathcal{O}}$ which can be interpreted as the projection operator on the subspace spanned by $|\mathcal{O}(x)\rangle$, here the state $|\mathcal{O}(x)\rangle$ denotes the state associated to $\mathcal{O}(x)$ in the state-operator correspondence. The projection operator is defined as

$$P_{\mathcal{O}} \coloneqq \int_{\mathbb{R}^d} d^d x' \; |\mathcal{O}(x)\rangle\langle\widetilde{\mathcal{O}}(x)| \tag{5.41}$$

and satisfies the completeness relation

$$\sum_{\mathcal{O}} P_{\mathcal{O}} = 1 \tag{5.42}$$

where the sum is over conformal primaries [87].

Motivated by these definition we now propose to define the shadow operator

$$\widetilde{\mathcal{O}}(x) \coloneqq \int_{\mathbb{R}} dx' \; \langle \mathcal{O}(x)\mathcal{O}(x')\rangle\big|_{\Delta\to\widetilde{\Delta}}\mathcal{O}(x') = \int_{\mathbb{R}} dx' \; \frac{e^{x'N^-}}{(1+xx')^{2(\Delta+1)}} \tag{5.43}$$

where $\mathcal{O}$ is as defined in (5.30) and the correlator found in equation (5.33). It was also used that since the Casimir from equation (5.9) is invariant under $\Delta \to -(\Delta+1)$ the shadow dimension is given by $\widetilde{\Delta} = -(\Delta+1)$. Two things should be noted from this definition, first of all it may seem that the integral diverges due to the exponential $e^{x'N^-}$, however this is not the case since N^- is nilpotent and therefore all terms of order $(N^-)^{2\Delta+1}$ and higher annihilate. There is an actual divergence in the integral at $x' = -(x)^{-1}$ though, which will need some regularization. We will ignore this for now and return to it in section 5.4 where we will find a consistent method of regularizing this integral and also find an explicit expression for $\widetilde{\mathcal{O}}$. For now, we can use this definition of $\widetilde{\mathcal{O}}$ to define a projection operator P as

$$P \coloneqq \mathcal{N} \int_{\mathbb{R}} dx \; |\mathcal{O}(x)\rangle\langle\widetilde{\mathcal{O}}(x)| \tag{5.44}$$

where

$$\begin{aligned} |\mathcal{O}(x)\rangle &\coloneqq \mathcal{O}(x)|\Delta, \Delta\rangle \\ |\widetilde{\mathcal{O}}(x)\rangle &\coloneqq \widetilde{\mathcal{O}}(x)|\Delta, \Delta\rangle \end{aligned} \tag{5.45}$$

are analogues to states obtained from the state-operator correspondence and $\mathcal{N}$ is a constant. Now to check that P satisfies a completeness relation, it is enough that

$$P_{n,m} \coloneqq \langle n, \Delta | P | m, \Delta \rangle = \delta_{n,m} \tag{5.46}$$

since the states $|n, \Delta\rangle$ form an orthonormal basis. Inserting the explicit expressions (5.43) and (5.44) then results in the integral

$$P_{n,m} = \mathcal{N} a_n \bar{a}_m \int_{\mathbb{R}^2} dx dx' \, \frac{x^{\Delta-n} x'^{\Delta-m}}{(1+xx')^{2(\Delta+1)}} \tag{5.47}$$

where a_n and a_m are given in equation (5.27). Making the change of coordinates

$$\begin{aligned} x &\to u+v \\ x' &\to u-v \end{aligned} \tag{5.48}$$

and performing a Wick rotation $v \to iv$ results in the integral[20]

$$P_{n,m} = i\mathcal{N} a_n \bar{a}_m \int_{\mathbb{C}} dz d\bar{z} \frac{z^{\Delta-n} \bar{z}^{\Delta-m}}{(1+|z|^2)^{2(\Delta+1)}} = 4\pi i(\Delta+1)\mathcal{N}\delta_{n,m} \tag{5.49}$$

with $z = u + v$ and the explicit formula (5.27) for a_n and $\bar{a}_m$ was inserted. Thus choosing

$$\mathcal{N} = \frac{1}{4\pi i(\Delta+1)} \tag{5.50}$$

results in equation (5.46) as required. Thus in the finite dimensional representation the shadow operator formalism works similar to how it does for conformal field theories.

A natural question is now why in this finite dimensional representation the above procedure works while in the conformal quantum mechanics setting of the previous chapter some modifications were necessary. One possible reason seems to be that the shadow transform of an operator is originally defined only for principal series representations of the conformal group, and that both the representations conventionally used in conformal field theories as well as the finite dimensional representations are given by analytical continuations of the principal series [51, 85]. Conversely, the representation described in the previous chapter is a discrete series representation. In any case both completeness relations make it possible to calculate higher order correlation functions explicitly (or at least find expressions for them), although these will have to be performed in further research. Now, we will take a quick aside and see how this completeness relation makes it possible to define a wave function quantum mechanics.

5.3.1 An aside: wave function mechanics

Interestingly, it is possible to define an analogue to wave function quantum mechanics using the projection operator above. To do this, note that in ordinary quantum mechanics it is

[20] This Wick rotation includes a Feynmann $i\epsilon$ prescription, regularizing the integral. An alternative derivation of this result will be presented in section 5.4.

possible to go from wave vectors in a Hilbert space to a wave function representation by using a position basis $|x\rangle$. This basis usually satisfies a completeness relation

$$\int_{\mathbb{R}} |x\rangle\langle x| = 1 \tag{5.51}$$

and therefore for states $|\phi\rangle$ and $|\psi\rangle$ in the Hilbert space this completeness relation can be inserted to obtain

$$\langle\phi|\psi\rangle = \int_{\mathbb{R}} dx\ \langle\phi|x\rangle\langle x|\psi\rangle = \int_{\mathbb{R}} dx\ \overline{\phi(x)}\psi(x) \tag{5.52}$$

where $\phi(x) \coloneqq \langle x|\phi\rangle$ and $\psi(x) \coloneqq \langle x|\psi\rangle$. Thus one obtains a natural embedding between the original Hilbert space and the space of square integrable functions, where the objects of interests are now wave functions instead of vectors.

To apply this to the representation from this chapter, note that the projection operator P defined in equation (5.44) can be rewritten as

$$P = \mathcal{N} \int_{\mathbb{R}^2} dxdx'\ \frac{|x\rangle\langle x'|}{(1+xx')^{2(\Delta+1)}} \tag{5.53}$$

and thus, for two general states $|\phi\rangle$ and $|\psi\rangle$

$$\langle\phi|\psi\rangle = \langle\phi|P|\psi\rangle = \int_{\mathbb{R}^2} dxdx'\ \frac{\overline{\phi(x)}\psi(x')}{(1+xx')^{2(\Delta+1)}} \tag{5.54}$$

where again $\phi(x) \coloneqq \langle x|\phi\rangle$ and $\psi(x) \coloneqq \langle x|\psi\rangle$. This inner product is slightly different than the one in equation (5.52) since there is now a double integral involved, inner products of this type also appear when constructing the complementary representation [80] or in quantum field theory [85], however this difference will not turn out to be important for us. This wave function representation now gives a natural interpretation of the the β_n functions defined in (5.26), which now span the wave function Hilbert space. Furthermore the actions of N^0 and $N^\pm$ on $\langle x|$ given in equation (5.34) now equates to a mapping of operators from the original representation to differential operators on the wave function representation. In that one can define differential operators on the wave functions as $n^\alpha\phi(x) \coloneqq \langle x|N^\alpha|\phi\rangle$ and use equation (5.34) to find an explicit expression. Finally the adjoints of the resulting differential operators

$$\tfrac{d}{dx}\ ;\quad -2x\tfrac{d}{dx} + 2\Delta\ ;\quad -x^2\tfrac{d}{dx} + 2\Delta x \tag{5.55}$$

with respect to the inner product

$$(\phi,\psi) \coloneqq \int_{\mathbb{R}^2} dxdx'\ \frac{\overline{\phi(x)}\psi(x')}{(1+xx')^{2(\Delta+1)}} \tag{5.56}$$

can now be calculated. Here $\phi(x)$ and $\psi(x)$ are elements of the wave function space. One can then easily check that

$$\left(\frac{d}{dx}\right)^\dagger = -x^2\frac{d}{dx} + 2\Delta x \tag{5.57}$$

$$\left(-2x\frac{d}{dx} + 2\Delta\right)^\dagger = -2x\frac{d}{dx} + 2\Delta \tag{5.58}$$

where the adjoint is now with respect to the inner product $(\cdot,\cdot)$ defined in equation (5.56). This is of course due to the fact that in the original representation $(N^+)^\dagger = N^-$ and $(N^0)^\dagger = N^0$ and that the differential operators from (5.55) are exactly those obtained from N^+, N^0 and N^- respectively.

We will now take a closer look at the shadow operator, and to perform some calculations we will define an integral transform related to the Mellin transform.

5.4 Integral transformations and analytical continuation in Mellin space

From the definition of the shadow operator given in equation (5.43) one may wonder if it is possible to find an explicit expression for $\widetilde{\mathcal{O}}$. One natural in-between step is defining

$$\widetilde{\mathcal{O}}_n(x) \coloneqq \int_{\mathbb{R}} dx' \, \frac{x^n}{n!} \frac{1}{(1+xx')^{2(\Delta+1)}} \tag{5.59}$$

such that

$$\widetilde{\mathcal{O}}(x) = \sum_{n=0}^{\infty} \widetilde{\mathcal{O}}_n(x)(N^-)^n \tag{5.60}$$

which is convenient when performing actual calculations. One problem however is that the integral (5.59) diverges due to the singularity at $xx' = -1$ thus some regularization is necessary.[21] With this in mind one might also notice that the expression for $P_{n,m}$ given in equation (5.47) involves a very similar integral of the form

$$\int_{\mathbb{R}^2} dxdx' \, \frac{x^n x'^m}{(1+xx')^{2(\Delta+1)}} \tag{5.61}$$

where here the integral was regularized by means of a Feynmann $i\epsilon$ prescription. Integrals of this kind will also appear when trying to calculate higher-order correlation functions by inserting P into

$$\langle \Delta, \Delta | \mathcal{O}^\dagger(x_1) A(x_2) B(x_3) \mathcal{O}(x_4) | \Delta, \Delta \rangle \tag{5.62}$$

where A and B have local expansions of the form

$$A(x) = \sum_{n\in\mathbb{Z}} A_n x^n \tag{5.63}$$

$$B(x) = \sum_{n\in\mathbb{Z}} B_n x^n \; . \tag{5.64}$$

From these examples it is clear that a systematic approach to working with these integrals would be convenient. Therefore we introduce the integral transform

$$\{\mathcal{T} f\}(n) \coloneqq \int_{\mathbb{R}} dx \; x^n f(x) \tag{5.65}$$

[21] The singularity at $x' \to \pm\infty$ is not a problem since $(N^-)^{2\Delta+1} = 0$ when acting within this representation, therefore the denominator always dominates in the integrand.

where we will usually be interested in the $n \in \mathbb{Z}$ case. With this we could write

$$\widetilde{\mathcal{O}}_n(x) = \frac{1}{n!}\{\mathcal{T}(1+xx')^{-2(\Delta+1)}\}(n) \tag{5.66}$$

as well as

$$P_{n,m} \propto \{\mathcal{T}_x\mathcal{T}_{x'}(1+xx')^{-2(\Delta+1)}\}(\Delta-n, \Delta-m) \tag{5.67}$$

where in this notation $\mathcal{T}_x$ denotes the $\mathcal{T}$ transform with respect to x and some regularization scheme is implicit. To actually calculate the $\mathcal{T}$ transform it is useful to note that

$$\{\mathcal{T}f\}(n) = \{\mathcal{M}f\}(n+1) + e^{i\pi n}\{\mathcal{M}f_-\}(n+1) \tag{5.68}$$

where $\{\mathcal{M}f\}$ is the Mellin transform defined as

$$\{\mathcal{M}f\}(s) \coloneqq \int_0^\infty dx\, x^{s-1}f(x) \tag{5.69}$$

and f_- is defined as

$$f_-(x) \coloneqq f(-x)\ . \tag{5.70}$$

This relation becomes useful since the Mellin transform is quite well-studied [94–96] and there are known results for analytical continuations of Mellin integrals [97]. Interestingly, the Mellin transform also appears when calculating AdS/CFT correlators [98, 99] although the context there is slightly different, we will discuss this more in subsection 5.4.3. Recently some work has also been done by Bianchi et al. [100] and the regularization procedures employed there and in reference [101] by Penedones et al. are quite reminiscent of the ones used here. Finally, the Mellin transform satisfies the following Mellin inversion formula

$$\phi(s) = \{\mathcal{M}f\}(s) \Leftrightarrow f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, x^{-s}\phi(s) \eqqcolon \{\mathcal{M}^{-1}\phi\}(x) \tag{5.71}$$

where c is a real number such that $\phi(s)$ is analytic on a strip $a < Re(s) = c < b$ and f and ϕ satisfy growth conditions such that both are well-defined [94, 95]. Note that the inverse Mellin transformation is independent of c because $\phi(s)$ is assumed to be analytic. Since the Mellin transform is unique this makes it possible to check possible analytical continuations by performing the inverse Mellin transform if possible.

In this section we will begin by studying this integral transform $\mathcal{T}$ and calculate the transform of some important functions. Afterwards we will apply this knowledge to calculate $P_{n,m}$ and $\widetilde{\mathcal{O}}_n(x)$ and we will conclude by giving some interpretations of the integral transform.

5.4.1 The general procedure and a useful integral

From equations (5.66) and (5.67) on may notice that they involve transformations of the function

$$f(x) \coloneqq (1+x)^{-2(\Delta+1)} \tag{5.72}$$

and therefore a natural starting point is to calculate $\{\mathcal{T}f\}(n)$, which we will do in this subsection. This will also showcase some of the general techniques used when calculating the $\mathcal{T}$ transform and the associated Mellin transforms. In order to do this we first split the integral into Mellin transforms as in equation (5.68), thus finding the Mellin transforms of f and $f_-(x) = f(-x)$ results in a formula for $\{\mathcal{T}f\}$. The Mellin transform of f can simply be obtained by evaluating the integral

$$\{\mathcal{M}f\}(s) = \int_0^\infty dx\, \frac{x^{s-1}}{(1+x)^{2(\Delta+1)}} = \frac{\Gamma(2\Delta+2-s)\Gamma(s)}{\Gamma(2\Delta+2)} \tag{5.73}$$

which is valid for a strip in the complex plane satisfying $0 < Re(s) < 2\Delta + 2$ and can be analytically continued to a meromorphic functions on the entire complex plane using the usual analytical continuation of the Gamma function. The Mellin transform of f_- however requires some more work due to the pole at $x = -1$. A trick one can use is to split the integral as

$$\{\mathcal{M}f_-\}(s) = \int_0^1 dx\, \frac{x^{s-1}}{(1-x)^{2(\Delta+1)}} + \int_1^\infty dx\, \frac{x^{s-1}}{(1-x)^{2(\Delta+1)}} \tag{5.74}$$

and note that these integrals separately do have regions $s \in \mathbb{C}$ for which they are well-defined, even though these regions do not overlap. This is equivalent to defining the functions

$$u_1(x) \coloneqq \begin{cases} f_-(x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases} \tag{5.75}$$

and

$$u_2(x) \coloneqq \begin{cases} 0 & \text{for } 0 \leq x \leq 1 \\ f_-(x) & \text{for } x > 1 \end{cases} \tag{5.76}$$

and to calculate their Mellin transforms separately. Then analytically continue them to meromorphic functions on the entire complex plane and finally define the Mellin transform of f_- as

$$\{\mathcal{M}f_-\}(s) = \{\mathcal{M}u_1\}(s) + \{\mathcal{M}u_2\}(s) \tag{5.77}$$

which is now also a meromorphic function on the entire complex plane, even though the original integral converges for no value of s. These kind of techniques are outlined by Bleistein and Handelsman in reference [97] and allow the calculation of Mellin transforms for very large classes of functions.

The Mellin transforms of u_1 and u_2 can be calculated explicitly resulting in

$$\{\mathcal{M}u_1\}(s) = \frac{\Gamma(s)\Gamma(-2\Delta-1)}{\Gamma(s-2\Delta-1)} \tag{5.78}$$

$$\{\mathcal{M}u_2\}(s) = \frac{\Gamma(-2\Delta-1)\Gamma(2\Delta+2-s)}{\Gamma(1-s)} \tag{5.79}$$

where it should be noted that since Δ is an integer, both are singular for non-integer s. However in the expressions (5.66) and (5.67) we are interested only in integer n and m, in this case it is possible to use the identity

$$\frac{\Gamma(n-z)}{\Gamma(-z)} = (-1)^n \frac{\Gamma(z+1)}{\Gamma(z+1-n)} \tag{5.80}$$

which is valid for integer n and can be derived by repeatedly applying $\Gamma(z+1) = z\Gamma(z)$ to the left hand side of this equation. Therefore for integer n the Mellin transforms of u_1 and u_2 are given by

$$\{\mathcal{M}u_1\}(n) = \{\mathcal{M}u_2\}(n) = (-1)^n \frac{\Gamma(s)\Gamma(2\Delta+2-s)}{\Gamma(2\Delta+2)} \,. \tag{5.81}$$

With this it is possible to use equation (5.77) to find the Mellin transform of f_-, inserting this with the Mellin transform of f into equation (5.68) results in the $\mathcal{T}$ transform of f which for integer n is given by

$$\{\mathcal{T}f\}(n) = \{\mathcal{T}(1+x)^{-2(\Delta+1)}\}(n) = -\frac{\Gamma(n+1)\Gamma(2\Delta+1-n)}{\Gamma(2\Delta+2)} \tag{5.82}$$

where it should be stressed that the original integral defining $\{\mathcal{T}f\}$ does not converge for any n and we are therefore including the analytical continuations above in the definition of $\{\mathcal{T}f\}$.

With this explicit formula and the knowledge of some of the general techniques, we are now ready to calculate the $\mathcal{T}$ transform of the monomial x^n.

5.4.2 An important special case, the integral transform of a monomial

Another integral that merits some special attention is the $\mathcal{T}$ transform of x^m. As an integral this would be written as

$$\{\mathcal{T}x^n\}(m) = \int_{\mathbb{R}} dx\, x^{n+m} \tag{5.83}$$

which clearly diverges for all n and m. Once again, it is possible to write this as a sum of Mellin transformations resulting in

$$\{\mathcal{T}x^m\}(n) = (1+e^{i\pi(n+m)})\{\mathcal{M}x^n\}(m+1) \tag{5.84}$$

and thus the problem is reduced to finding $\{\mathcal{M}x^n\}$. Using the same trick as before it is possible to split the Mellin transform integral by defining

$$u_1(x) := \begin{cases} x^n & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases} \tag{5.85}$$

and

$$u_2(x) := \begin{cases} 0 & \text{for } 0 \leq x \leq 1 \\ x^n & \text{for } x > 1 \end{cases} \tag{5.86}$$

such that $x^n = u_1(x) + u_2(x)$. The Mellin transform of u_1 and u_2 can easily be obtained by explicit calculation resulting in

$$\{\mathcal{M}u_1\}(s) = \frac{1}{n+s} \tag{5.87}$$

valid for $\mathrm{Re}(s) > -n$ and

$$\{\mathcal{M}u_2\}(s) = \frac{-1}{n+s} \tag{5.88}$$

valid for $\mathrm{Re}(s) < -n$. If one would now naively try to define

$$\{\mathcal{M}x^n\}(s) = \{\mathcal{M}u_1\}(s) + \{\mathcal{M}u_2\}(s) \tag{5.89}$$

one can clearly see that this would result in zero. However since the Mellin transform is unique this gives some problems, since clearly the zero function would also have $\{\mathcal{M}0\}(s) = 0$. Furthermore applying the inverse Mellin transformation from equation (5.71) to 0 would also result in zero thus something is going wrong. The key observation to make is that

$$\{\mathcal{M}^{-1}\mathcal{M}u_1\}(x) + \{\mathcal{M}^{-1}\mathcal{M}u_2\}(x) = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} ds\, \frac{x^{-s}}{n+s} - \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} ds\, \frac{x^{-s}}{n+s} \tag{5.90}$$

where $c_1 > -n$ and $c_2 < -n$. Since the integrands are meromorphic it is possible to choose c_1 and c_2 arbitrarily close to $-n$, this results in a closed contour around the pole at $s = -n$ and thus by Cauchy's integral formula we find

$$\{\mathcal{M}^{-1}\mathcal{M}u_1\}(x) + \{\mathcal{M}^{-1}\mathcal{M}u_2\}(x) = \frac{1}{2\pi i}\oint_\gamma \frac{x^{-s}}{s+n} = x^n \tag{5.91}$$

as one would expect. The solution thus is to define

$$\{\mathcal{M}x^n\}(s) = \{\mathcal{M}u_1\}(s+\epsilon) + \{\mathcal{M}u_2\}(s-\epsilon) \tag{5.92}$$

where the limit of $\epsilon \to 0$ is implicit and should be taken after the inverse Mellin transform.[22] This definition then ensures that there is a strip $-n-\epsilon < \mathrm{Re}(s) < -n+\epsilon$ where the two Mellin transforms are both defined and it is possible to take the inverse Mellin transform directly, furthermore it splits the coincident pole at $s = -n$ which was crucial in calculating the inverse Mellin transform above. Inserting the explicit formulas for u_1 and u_2 from equations results in

$$\{\mathcal{M}x^n\}(s) = \frac{1}{n+s+\epsilon} - \frac{1}{n+s-\epsilon}\ . \tag{5.93}$$

Two important observations are that firstly for $s \neq -n$ this still results in zero after the limit $\epsilon \to 0$ for $s \neq -n$, secondly this expression is actually a representation of $2\pi\delta(i(n+s))$ by the Sokhotsk-Plemelj theorem, with δ the Dirac delta distribution [103]. This relation could have also been derived by using the integral representation

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tag{5.94}$$

and applying the change of coordinates $x \to k := ln(x)$ to

$$\{\mathcal{M}x^n\}(s) = \int_0^\infty dx\, x^{n+s-1} = \int_{-\infty}^{\infty} dy e^{n+s} = 2\pi\delta(i(n+s)) \tag{5.95}$$

[22] Interestingly, this definition is very similar to the definition of a hyperfunction and the explicit formula obtained is identical to the representation of $2\pi i\delta(i(n+s))$ as a hyperfunction. See for example [102] for an overview that also discusses the Mellin transform quite nicely.

Note that all formulas above are only well defined for $\mathrm{Re}(s) = -n$ which is in accordance with $n - \epsilon < \mathrm{Re}(s) < n + \epsilon$ as $\epsilon \to 0$. This is also why the conventional approach to obtain the Mellin transform of x^n gave problems, as the resulting object is not actually an analytic function but a distribution (or a hyperfunction, depending on the interpretation). To check that (5.93) is the correct formula, lets first consider the inverse Mellin transform given by

$$\{\mathcal{M}^{-1}\mathcal{M}x^n i\}(x) = \frac{1}{2\pi i}\int_{-n-i\infty}^{-n+i\infty} ds\, x^{-s}\left[\frac{1}{n+s+\epsilon} - \frac{1}{n+s-\epsilon}\right] \tag{5.96}$$

$$= \frac{1}{2\pi i}\int_{-n-\epsilon-i\infty}^{-n-\epsilon+i\infty} ds\, x^{-s}\frac{1}{n+s} - \frac{1}{2\pi i}\int_{-n+\epsilon-i\infty}^{-n+\epsilon+i\infty} ds\, x^{-s}\frac{1}{n+s}\,. \tag{5.97}$$

Letting $\epsilon \to 0$ and closing the contour results in

$$\{\mathcal{M}^{-1}\mathcal{M}x^n\}(x) = x^n \tag{5.98}$$

as required.

For the above results for the Mellin transform of x^n it should be stressed that they only hold within the analytical continuations of the integrals, since clearly the integral representation of $\{\mathcal{M}x^n\}$ diverges. However, this result was also obtained in a similar way by Penedones et al. in reference [101] by a deformation of integration contours in the inverse Mellin transform. Furthermore this method was then applied by Bianchi et al. in reference [100] to calclate CFT_1 correlators.

With the Mellin transform of x^n obtained, the next step is to use this to find $\{\mathcal{T}x^n\}$. Inserting (5.93) into equation (5.84) results in

$$\{\mathcal{T}x^n\}(m) = (1 + e^{i\pi(n+m)})\left[\frac{1}{n+m+1+\epsilon} - \frac{1}{n+m+1-\epsilon}\right] \tag{5.99}$$

where again the $\epsilon \to 0$ limit is implicit and ensure that the above expression is equal to zero for $n \neq -m-1$. In the limit of $n \to -m-1$ however, we find that

$$\lim_{n\to -m-1}\{\mathcal{T}x^n\}(m) = 2\lim_{n\to -m-1}\frac{1+e^{i\pi(n+m)}}{\epsilon} \tag{5.100}$$

where since both the numerator and denominator go to zero this is undefined. Taking the limit simultaneously however results in $\{\mathcal{T}x^n\}(m) \to -2\pi i$ and would imply that

$$\{\mathcal{T}x^n\}(m) = -2\pi i\delta_{n,-m-1} \tag{5.101}$$

where $\delta_{n,-m-1}$ is the Kronicker delta. As we will see, this answer is consistent with the results from section 5.3 so it does seem correct although unfortunately we were unable to find a rigorous justification for it. Therefore one should be somewhat cautious in applying this formula.

With equations (5.82) and (5.101), we are now ready to demonstrate their usefulness by performing some actual calculations.

5.4.3 Applying the transformation

Using the result it becomes quite easy to calculate $\widetilde{\mathcal{O}}_n(x)$ and $P_{n,m}$, for example using a re-parametrization $x' \to x'/x$ it is possible to write

$$\widetilde{\mathcal{O}}_n(x) = \frac{1}{n!}\int_{\mathbb{R}} dx' \, \frac{x^n}{(1+xx')^{2(\Delta+1)}} = \begin{cases} \operatorname{sign}(x)\frac{x^{-(n+1)}}{n!}\{\mathcal{T}(1+x)^{-2(\Delta+1)}\}(n) & \text{for } x \neq 0 \\ \frac{1}{n!}\{\mathcal{T}x^n\}(0) & \text{for } x = 0 \end{cases} \tag{5.102}$$

which can be solved immediately by inserting formula's (5.82) and(5.101). This results in

$$\widetilde{\mathcal{O}}_n(x) = -\operatorname{sign}(x)x^{-(n+1)}\frac{\Gamma(2\Delta+1-n)}{\Gamma(2\Delta+2)} \quad \text{for } x \neq 0 \tag{5.103}$$

$$\widetilde{\mathcal{O}}_n(0) = \frac{-2\pi i}{n!}\delta_{n,-1} = 0 \tag{5.104}$$

where in the bottom equation it was used that $n \geq 0$. Similarly, note that rewriting equation (5.47) in terms of $\mathcal{T}$ transformations results in

$$P_{n,m} = \mathcal{N} a_n \bar{a}_m \{\mathcal{T}_x \mathcal{T}_{x'} (1+xx')^{-2(\Delta+1)} \tag{5.105}$$

while from the definition of $\mathcal{T}$ in equation (5.65) we see that for general g

$$\{\mathcal{T}_x \mathcal{T}_{x'} g(xy)\}(\Delta - n, \Delta - m) = \{\mathcal{T} g(x)\}(\Delta - n)\{\mathcal{T} x^n\}(-m-1) \tag{5.106}$$

which can be derived by performing a coordinate transformation $x \to xx'$ and $x' \to x'$. Combining these two results in

$$P_{n,m} = \mathcal{N} a_n \bar{a}_m \{\mathcal{T}(1+x)^{-2(\Delta+1)}\}\{\mathcal{T} x^n\}(-m-1) \tag{5.107}$$

which when inserting (5.82) and(5.101), as well as the equations (5.27) for a_n and (5.50) for $\mathcal{N}$ gives

$$P_{n,m} = \delta_{n,m} \tag{5.108}$$

as required.

As we have seen above the $\mathcal{T}$ transformation has made it possible to find an expression for $\widetilde{O}$, as well as perform another proof of the completeness relation for P. It should also help in performing calculations for the n-point correlators, although these will have to be carried out in future research. The integral space is also interesting because it seems to provide a different connection between the continuous variable x and the discrete variable n, which can appear either in a state $|n, \Delta\rangle$ or as the argument of a transformed function $\{\mathcal{T} f\}(n)$. Partly this is because in the transformed space the differential operator $-2x\frac{d}{dx} + 2\Delta$

$$\{\mathcal{T}(-2x\frac{d}{dx} + 2\Delta) f\}(n) = 2(\Delta + n + 1)\{T f\}(n) \tag{5.109}$$

in other words, in the transformed space the operator N^0 becomes diagonal. Since this is already clearly the case when acting on the states $|n, \Delta\rangle$ the spaces have a similar structure.

In this sense, there is some similarity to the Mellin transform for the dilaton in conformal field theories [94]. Interestingly there are some other similarities with the CFT Mellin amplitudes as well. In the CFT setting, the Mellin transform is often used to write correlation functions by applying the operator product expansion. Interestingly, $\mathcal{O}$ when acting on $|\Delta, \Delta\rangle$ plays a similar role since it generates all descendants of $|\Delta, \Delta\rangle$ by applying $(N^-)^n$. In the CFT setting the Mellin amplitudes are very tightly constrained by properties of the operator product expansion thus it would be interesting to see if similar constraints could be derived for the transformed functions here.

6 Conclusion and outlook

In this thesis we have studied representations of the 1-dimensional conformal group and specifically considered a mapping from discrete bases to continuous ones. With the goal of obtaining a better understanding of the CFT_1 side of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ duality. For this we constructed a conformal quantum mechanics model in chapter 4 originally due to dAFF [23]. Besides this we also introduced a similar finite dimensional representation in chapter 5 which has relevance in Hodge theory. For both of these theories an analogue to the shadow operator was introduced in order to obtain a completeness relation, which was used to obtain integral expressions for higher order correlation functions.

There is an interesting interplay between higher dimensional CFT's and this quantum mechanical model, where the latter has many properties simlar to conformal field theories but is fundamentally different simply due to not being a field theory. This difference makes it interesting that various tools from the conformal bootstrap program, such as the shadow operator, find their use naturally in conformal quantum mechanics. One is lead to wonder how this connection might develop further. For example one could ask what the analogue of the operator product expansion would be. Where we already have a complete basis of states, since all the states are known in the quantum mechanical model. Curiously, the continuous states $|x\rangle$ naturally have some of the same structure as an operator product expansion. Where for both there is an insertion of a primary operator (or the ground state) and its descendants, obtained by repeatedly acting with the raising or lowering operator. Hopefully a better understanding of the connection between conformal quantum mechanics and conformal field theories would lead to a better understanding of the individual theories too.

Besides this an interesting interpretation of the conformal quantum mechanics model was introduced in section 4.3, where it was used that the discrete series has a natural representation on the Poincaré upper half plane. The conformal quantum mechanics model studied before then naturally appears as the boundary theory of this representation. This seems very far from an AdS/CFT type duality since the models on both sides are much too simplistic and therefore it seems that there is no room for the more interesting aspects of AdS/CFT. Both models are just quantum mechanical systems on different spaces and there seems to be no clear way to introduce gravitational effects, gauge theories or other aspects that appear when dealing with quantum field theories. There does seem to be a natural way to promote both to field theories by a second quantization type procedure (or a many particle quantum mechanical model on the CFT_1 side) which would be interesting to investigate further. Especially since the two models do seem to share some of the basic properties of the AdS/CFT duality, such as the relation between the $SL(2, \mathbb{R})$ symmetry operators in the bulk and the boundary, as well as that the wave functions in the bulk and the boundary seem to be related with the correct limits. However as mentioned, one should not take the analogy too far.

A different direction that can be explored is the connection with Hodge theory. Even though the language and concepts in Hodge theory seem very different at first sight, there are some interesting similarities. First of all there is a similar problem of bulk reconstruction in Hodge theory [31, 32]. Where there is a boundary quantum mechanics model that, along with some different information, seems to be enough to describe the near-boundary behaviour of bulk fields. There are also other similarities, the operator product expansion blocks in CFT's can be related to geodesics on AdS [104] which seems at least superficially similar to Hodge theory where one of the main objects of interest is given by periods in the moduli space. This similarity deepens somewhat when one notices that this so-called kinematic space can be realized as coadjoint orbits [105], while in the Hodge theory setting much importance is placed on the nilpotent orbits. Furthermore, the kinematic space naturally carries symplectic and Kähler structure, both of which play an important role in the Hodge theory setting. Hopefully applying the model constructed in this thesis could shed some lights on these similarities, by providing a better understanding of the boundary theory.

Finally it turns out that there are some relations between the solution of certain differential operators and the periods of algebraic connections [106, 107]. Here a limited mixed Hodge structure also naturally appears. The differential operators studied in this thesis are of the same type and interestingly, the so-called Bloch-Vlasenko gamma functions defined there show significant similarity to the shadow transformed operators defined for the Hodge theory setting in chapter 5. In fact, both can be written in terms of Mellin transforms with coefficients of the type $e^{i\pi ns}$.[23] This provides a connection between Hodge theory and differential operators, so one may wonder if that connection can be used to relate the representations built in this thesis directly to the Hodge theoretical setting. If this turns out to be possible it might provide an interesting new perspective there. Furthermore, if this can be generalized to the conformal quantum mechanics case it might even provide interesting insights into the AdS/CFT setting, providing a new language into which one can view the duality.

At the moment much of what is written above remains speculative. However we think that the models studied in this thesis might provide interesting new viewpoints into both AdS/CFT and Hodge theory. Where in the best case scenario, it would provide connections between holography, Hodge theory, differential equations and group representation theory, hopefully leading to a better understanding of all of these. For now though, only the future will tell how much of that is actually possible.

[23] See for example page 5 in reference [106].

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