Orbifold Compactifications of Type IIB and Supersymmetry Breaking

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Abstract

We study orbifold compactifications of type IIB string theory. In particular, we construct orbifolds of the form $(S^1 \times T^4)/Z_p$ preserving partial or no supersymmetry, giving rise to classical 5D Minkowski vacua. We demonstrate that these orbifold constructions are the lifts to string theory of type IIB supergravity reduced on a T^4 and furthermore reduced on a circle with a Scherk-Schwarz twist. We also study the vacua of the resulting theories and investigate if a one-loop cosmological constant can be generated, both from the supergravity and from the string-theoretic point of view.

Contents

Chapter 1

Introduction

Every process taking place in our universe is governed by four fundamental forces: gravity, electromagnetism, and the weak and strong nuclear forces. The first one is described by Einstein's theory of general relativity, which explains the large-scale physics, and the later three are included in the standard model of particle physics, which provides a quantum mechanical description of the subatomic world. Both theories are properly operating in their own domains and they have been experimentally confirmed. Nevertheless, a thorough knowledge of the laws of nature dictates the unification of gravity and quantum mechanics into one consistent theory of *quantum gravity*.

The most promising candidate for such a theory is string theory. The fundamental idea of string theory is that matter consists of one dimensional extended objects, namely strings. These strings oscillate, and different oscillation modes of strings correspond to different particles. Among all other particles, a vibrating string at low energies can produce the graviton, which is the particle mediating the gravitational force. Hence, string theory includes all the fundamental ingredients of our known universe. Furthermore, the low-energy limit of string theory is usually called *supergravity*.

As a theory of quantum gravity, string theory should also be able to explain the mystery around the cosmological constant Λ, which can be interpreted as a quantum vacuum energy density. The cosmological constant puzzle stems from the fact that quantum field theory predicts a very large value for Λ, while observations suggest that the cosmological constant is near vanishing [1, 2]. Although there have been many attempts to resolve this issue, it stills remains an open question. Therefore, it is very interesting to investigate this problem from the string theory point of view and understand how string theory handles the cosmological constant.

Another interesting aspect of string theory is that it lives consistently in ten dimensional spacetimes. Since our observed world is four dimensional, the extra spatial dimensions of string theory must be somehow invisible. This can be achieved by a procedure known as *compactification*, proposing that the extra dimensions are wrapped into each other in an internal compact space. There are many ways and manifolds on which string theories can be compactified, and the resulting theory depends on the geometry of the internal space.

In addition, string theory enjoys *supersymmetry*, which is a symmetry relating bosons and fermions and suggests that every boson has its fermionic superpartner and vice versa. However, supersymmetry has not yet been observed in our world and this indicates that it should be realized in a broken way. This can be reached by compactifying our theory on specific compact spaces such as an *orbifold*, which is the quotient of a manifold by a discrete group, and these kinds of compactifications were first examined by Dixon, Harvey, Vafa and Witten in the papers [3, 4]. A different way of supersymmetry breaking, known as generalized, twisted, or "Scherk-Schwarz" reduction, was proposed by Scherk and Schwarz in [5, 6]. Generalized reduction provides a technique for supersymmetry breaking by exploiting the extra dimensions of the theory and generating masses for some (not all) fields of the theory.

Scherk-Schwarz reductions have been considered in detail in the literature. In particular, we are motivated by a recent paper of Hull, Marcus, Stemerdink and Vandoren [7], where type IIB supergravity is reduced on a four-torus with the Kaluza-Klein method, and furthermore reduced on a circle with a Scherk-Schwarz twist. This model (type IIB on $T^4 \times S^1$) provides a set-up for studying black holes in five dimensional spacetimes, both microscopically and macroscopically. This was first done by Strominger and Vafa [8] and afterwards, similar models were constructed [9–11].

In this thesis we demonstrate that the Scherk-Schwarz reductions considered in [7] can be embedded into string theory, by compactification on *freely acting* orbifolds [12]. We compactify type IIB string theory on an orbifold $(S^1 \times T^4)/Z_p$, and we demonstrate that the low energy spectrum of our construction matches exactly the spectrum of type IIB supergravity, compactified on a T^4 with the Kaluza-Klein method, and furthermore compactified on a S^1 with a Scherk-Schwarz twist. Finally, we study the vacua of the theory and examine if a one-loop cosmological constant can be generated, both from the supergravity and from the string-theoretic point of view.

1.1 Outline

In chapter 2 we present some string theory preliminaries. We discuss type IIB string theory and compactification. In particular, we examine the massless spectrum of type IIB and we explain the Kaluza-Klein compactification on a circle and on a d-torus.

Subsequently, in chapter 3 we discuss the low-energy effective theory of type IIB string theory, namely type IIB supergravity. We start with the Kaluza-Klein reduction of type IIB supergravity on a four-torus and we further reduce it on a circle with a Scherk-Schwarz twist.

In chapter 4 we study orbifolds, which is the main topic of this thesis. We begin with toroidal orbifolds and then we discuss strings compactified on such spaces. We construct orbifolds preserving partial or no supersymmetry and we compare the resulting string spectra with the supergravity ones.

Finally, in chapter 5 we study the one-loop cosmological constant both from the supergravity and the string theory point of view, and we discuss how supersymmetry, or the absence of it, determines the one-loop vacuum energy density.

1.2 Conventions

We use the "mostly plus" spacetime metric $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,\ldots,+1)$. We choose natural units such that $c = k_B = \hbar = 1$, but we do not set Newton's constant G_N equal to one.

Chapter 2

String Theory

In this chapter we present some basic concepts of string theory. In principle, we are merely collecting results that will be necessary for the rest of this thesis, without proving or deriving them. We assume that readers are familiar with most of the discussed subjects. A thorough review on the topics of this chapter can be found in [13–20].

In section 2.1 we discuss superstrings. We focus on type IIB string theory and construct its massless spectrum. Afterwards, in section 2.2 we discuss compactification. We start with the Kaluza-Klein compactification of the bosonic string on a circle and subsequently on a d-torus.

2.1 Superstring Theory

Superstring theories are supersymmetric theories which live in ten dimensions and contain both bosons and fermions in their spectrum. There exist five different types of superstring theories, all emerging from one eleven dimensional theory, usually referred to as "M-theory". There are open and closed superstrings (type I and II), as well as heterotic strings. For our purposes, we consider type IIB string theory, but we also discuss some relevant results for type IIA.

Type IIB theory is based on oriented closed strings. It has $\mathscr{N} = 2$ supersymmetry, meaning that it contains two ten dimensional supersymmetry spinors. It is a chiral theory, hence these spinors are of the same chirality. This amount of supersymmetry gives rise to 32 conserved supercharges, which means that type IIB is maximally supersymmetric. On the other hand, type IIA is a non-chiral theory, suggesting that the supersymmetry spinors are of the opposite chirality.

2.1.1 Spectrum of Type IIB

When considering superstrings there is a subtlety regarding the boundary conditions of fermions. In the case of an open string, the equations of motion of the fermionic fields admit two possible boundary conditions. Periodic boundary conditions, which correspond to the Ramond (R) sector, or anti-periodic boundary conditions, corresponding to the Neveu-Schwarz (NS) sector. The mode expansion of the fermionic fields in the Ramond sector enforces integer moding, while in the Neveu-Schwarz sector half-integer modes arise.

We proceed with the analysis of the spectrum in both sectors. We denote the Neveu-Schwarz vacuum by $|0\rangle$ and the Ramond vacuum by $|\alpha\rangle$. They satisfy [14]

$$\begin{aligned} &a^\mu_m\,|0\rangle = b^\mu_r\,|0\rangle = 0, \qquad m = 1,2,\ldots \qquad && r = \frac{1}{2},\frac{3}{2},\ldots\,, \\ &a^\mu_m\,|\alpha\rangle = b^\mu_m\,|\alpha\rangle = 0, && m = 1,2,\ldots\,. \end{aligned} \tag{2.1.1}$$

The NS vacuum is a spacetime scalar with zero point energy $-\frac{1}{2}$. In general, the zero point energy of the NS vacuum due to a real boson is given by [15]

$$\frac{1}{48} - \frac{(2\theta-1)^2}{16}\,, \tag{2.1.2}$$

where $\theta = 0$ for integer modes, while $\theta = \frac{1}{2}$ for half-integer modes. For a (real) fermion we get the opposite of this quantity. We can construct the first excited state in the NS sector by acting with a fermionic oscillator on the vacuum, $b^i_{-1/2}\,|0\rangle$. This state is a massless spacetime vector of the Lorentz little group SO(8). Regarding the Ramond vacuum, it has a vanishing zero point energy and it turns out that it is degenerate. The Ramond vacuum is a spacetime spinor and it can be described by the 16 states $|s_1, s_2, s_3, s_4\rangle\,, s_i = \pm\frac{1}{2}$. In order to achieve spacetime supersymmetry we want to project out half of these states. This is done by the operation of "GSO" projection, which can be chosen such that

$$\sum_{i=1}^{4} s_i \in 2\mathbb{Z}\,, \qquad \text{or} \qquad \sum_{i=1}^{4} s_i \in 2\mathbb{Z}+1\,. \tag{2.1.3}$$

Subsequently, we can combine two open strings and make a closed one. This results in four different closed string sectors, giving rise to either spacetime bosons (NS-NS and R-R sectors), or spacetime fermions (NS-R and R-NS sectors). Taking the same projection on both sides gives type IIB, while taking different projection on each side yields type IIA. In order to obtain the spectrum of type IIB we take tensor products of the left- and right-moving open string spectra. In type IIB theory the left- and right-moving

Ramond vacua, which we denote by $|\alpha\rangle_{\mathscr{L}/\mathscr{R}}$, have the same chirality and represent eight component spinors.

Having said all this, we are ready to construct the massless spectrum of type IIB in ten dimensions. We list below the states that we find in each of the four different sectors[1] [13].

NS-NS sector:

$$\tilde{b}^i_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^j_{-1/2}\,|0\rangle_{\mathscr{R}} \to 1 + 28 + 35\,. \tag{2.1.4}$$

The spectrum contains the dilaton (1), a two-form gauge field (28) and the graviton (35), which is a spin-2 particle.

NS-R sector:

$$\tilde{b}^i_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha\rangle_{\mathscr{R}} \to 8 + 56\,. \tag{2.1.5}$$

Here, we find a gravitino (56), which is a spin-$\frac{3}{2}$ particle and a dilatino (8), which is a spin-$\frac{1}{2}$ particle.

R-NS sector:

$$|\alpha\rangle_{\mathscr{L}} \times b^i_{-1/2}\,|0\rangle_{\mathscr{R}} \to 8 + 56\,. \tag{2.1.6}$$

This spectrum is the same as in the NS-R sector.

R-R sector:

$$|\alpha\rangle_{\mathscr{L}} \times |\alpha\rangle_{\mathscr{R}} \to 1 + 28 + 35\,. \tag{2.1.7}$$

Finally, in the R-R sector we find a zero-form gauge field (1), a two-form gauge field (28) and a four-form gauge field (35).

Regarding type IIA, in the NS-NS sector the spectrum is exactly the same as in type IIB, while in the NS-R and R-NS sectors the only difference is that the two gravitini are of the opposite chirality. Finally, in the R-R sector one finds a one-form gauge field (8) and a three-form gauge field (56).

[1] A tilde above the oscillators means that they are left-moving and the absence of a tilde means that they are right-moving

2.2 Compactification

Superstring theories are consistent in ten dimensional spacetimes. However, our observed universe is four dimensional, thus we have to reduce the extra spatial dimensions. This can be done by a procedure known as *compactification.* This is an old concept in physics originating back in 1920's, when Gunnar Nordström and, a few years later, Kaluza and Klein worked on the unification of gravity and electromagnetism introducing a fifth dimension.

The fundamental idea of compactification is that the extra dimension(s) should be finite and as a consequence not observable. Therefore, concerning superstrings, the ten dimensional spacetime is split in a non-compact external spacetime $\mathscr{M}^{10-d}$ and in an internal compact manifold $\mathscr{M}^{d}$ as

$$\mathscr{M}^{10} = \mathscr{M}^{10-d} \times \mathscr{M}^{d}\,, \tag{2.2.1}$$

where d is the number of compactified dimensions. We always compactify on a compact manifold and additionally, we take the non-compact spacetime to be Minkowskian. In the following, we present a simple example of compactification of the bosonic string on a circle S^1 and subsequently on a d-torus T^d.

2.2.1 Compactification on S^1

Consider a closed bosonic string propagating in a background spacetime of the form $\mathbb{R}^{1,24} \times S^1$. We split the coordinates of the 26-dimensional spacetime, $X^M = (X^\mu, X^{25})$, $M = 0, \dots, 25$. We denote the coordinates on the external Minkowski spacetime by X^μ, $\mu = 0, \dots, 24$, and the coordinate on the circle direction by X^{25}. The radius of the circle is denoted by R. One consequence of this background is that the momentum in the circle direction is quantized [17]

$$p^{25} = \frac{k}{R}, \quad k \in \mathbb{Z}\,, \tag{2.2.2}$$

where the integer k is usually called the *momentum* number. The quantization of momentum is a necessary requirement for a singled-valued wave function $e^{ip^{25}X^{25}}$, since as we go once around the circle, $X^{25} \to X^{25} + 2\pi R$, we demand that the wave function returns to its initial value. Recall here that momentum states are defined as

$$e^{ip^{25}X^{25}}\,|0;0\rangle \equiv |0;k\rangle\ . \tag{2.2.3}$$

We now examine the effect of this background on a field. In general, all fields will depend both on the coordinates of the external spacetime and of the compact space. Suppose that we have one massless scalar field $\Phi(X^\mu, X^{25})$ obeying the wave equation

$$\partial_M \partial^M \Phi(X^\mu, X^{25}) = 0\,. \tag{2.2.4}$$

If we expand the X^{25} dependence of this field in Fourier modes on the circle

$$\Phi(X^\mu, X^{25}) = \sum_{k\in\mathbb{Z}} \Phi_k(X^\mu) e^{ikX^{25}/R}\,, \tag{2.2.5}$$

the equation of motion (2.2.4) becomes

$$\partial_\mu \partial^\mu \Phi_k(X^\mu) = \frac{k^2}{R^2}\Phi_k(X^\mu)\,. \tag{2.2.6}$$

This suggests that after compactification a single massless scalar field gives an infinite tower of lower-dimensional scalar fields with masses

$$M_k^2 = \frac{k^2}{R^2}\,. \tag{2.2.7}$$

These fields are usually called the Kaluza-Klein modes and all of them are massive, except for the $k = 0$ mode. We can see from the above equation that for small circle radius the masses of these modes become very large $M \sim 1/R$. Consequently, if we are probing energies $E \ll M$, we can safely ignore the massive Kaluza-Klein modes. We can repeat the above analysis for a massive scalar field $\Phi(X^\mu, X^{25})$ with mass m. We find that the whole Kaluza-Klein tower is shifted by m^2

$$M_k^2 = m^2 + \frac{k^2}{R^2}\,. \tag{2.2.8}$$

Another implication of the one compact direction is that the bosonic coordinate X^{25} obeys an altered boundary condition which reads [13]

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi w R, \;\; w \in \mathbb{Z}\,. \tag{2.2.9}$$

This is a purely stringy effect, as a string is an extended object and therefore can wind around the circle. The integer w tells us how many times a string winds around the S^1 and it is naturally called the *winding* number. A string wrapping around a circle stretches and its energy increases. Consequently, momentum and winding modes lead to a modified mass formula for the closed string, that is [15]

$$\begin{aligned} M^2 &= \frac{k^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(\tilde{N} + N - 2\right), \\ kw + \tilde{N} - N &= 0 \qquad k, w \in \mathbb{Z}\,. \end{aligned} \tag{2.2.10}$$

Here, we denote by $\tilde{N}$ (N), the level of the left- (right-) moving oscillators. Note that equation (2.2.10) is invariant under the simultaneous interchange

$$R \to \frac{\alpha'}{R}, \qquad k \leftrightarrow w\,. \tag{2.2.11}$$

This result implies that a string compactified on a circle of radius R has the same spectrum with a string compactified on a circle of radius $\frac{\alpha'}{R}$. This symmetry, which is actually an exact symmetry for the full interacting string theory, is known as *T-duality*.

Consider the mode expansion of $X^{25}_{\mathscr{L}/\mathscr{R}}$ (in the following we will omit the superscript 25 for convenience) [14]

$$\begin{aligned} X_{\mathscr{L}}(\tau+\sigma) &= x_{\mathscr{L}} + \frac{1}{2}\alpha' p_{\mathscr{L}}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{l\neq 0}\frac{1}{l}\tilde{a}_l e^{-il(\tau+\sigma)}\,, \\ X_{\mathscr{R}}(\tau-\sigma) &= x_{\mathscr{R}} + \frac{1}{2}\alpha' p_{\mathscr{R}}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{l\neq 0}\frac{1}{l}a_l e^{-il(\tau-\sigma)}\,, \end{aligned} \tag{2.2.12}$$

where $p_{\mathscr{L}/\mathscr{R}}$ are given by

$$\begin{aligned} p_{\mathscr{L}} &= \frac{k}{R} + \frac{wR}{\alpha'}\,, \\ p_{\mathscr{R}} &= \frac{k}{R} - \frac{wR}{\alpha'}\,. \end{aligned} \tag{2.2.13}$$

Under the T-duality action $p_{\mathscr{L}}$ is invariant, while $p_{\mathscr{R}}$ transforms to $-p_{\mathscr{R}}$. In addition, the left- and right-moving oscillators transform in a similar fashion [21]

$$\tilde{a}_l \to \tilde{a}_l\,, \qquad a_l \to -a_l\,. \tag{2.2.14}$$

Consequently, if we also require $x_{\mathscr{L}} \to x_{\mathscr{L}}$ and $x_{\mathscr{R}} \to -x_{\mathscr{R}}$, we conclude that T-duality acts as an asymmetric Z_2 reflection on the compact bosonic coordinate X^{25}

$$X^{25}_{\mathscr{L}} \to X^{25}_{\mathscr{L}}\,, \qquad X^{25}_{\mathscr{R}} \to -X^{25}_{\mathscr{R}}\,. \tag{2.2.15}$$

Regarding superstrings, the fermions are described exactly as in the non-compact spacetime, which is of course ten dimensional [22]. The mass formula is given by

$$
\begin{aligned}
&M^2 = \frac{k^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(\tilde{N}_B + \tilde{N}_F + N_B + N_F + \tilde{E}_0 + E_0\right) , \\
&kw + \tilde{N}_B + \tilde{N}_F - N_B - N_F + \tilde{E}_0 - E_0 = 0 \qquad k, w \in \mathbb{Z} .
\end{aligned} \tag{2.2.16}
$$

Here, we use the subscripts B, F to distinguish between the bosonic and fermionic level operators. In addition, $E_0 = -\frac{1}{2}$ for the NS sector, $E_0 = 0$ for the R sector, and similarly for $\tilde{E}_0$.

Concluding, we also examine how T-duality acts on superstrings. Due to supersymmetry, T-duality must act in the same way on the compact fermionic coordinate, flipping the sign of the right-moving coordinate, and leaving the left-moving one intact. The outcome of this transformation is that the chirality of the right-moving Ramond vacuum is reversed. Since the relative chirality between the left- and right-moving Ramond vacua is what distinguishes type II superstring theories, we conclude that T-duality exchanges type IIA with type IIB.

2.2.2 Compactification on T^d

In this section we wish to study a more general case of compactification, that is compactification of the bosonic string on a d-torus T^d, which can be thought as the product of d circles. The background spacetime is now of the form $\mathbb{R}^{1,25-d} \times T^d$. The coordinates split up as $X^M = (X^\mu, Y^i)$, where as before, we denote the coordinates on the external spacetime by X^μ, $\mu = 0, \ldots, 25-d$. The coordinates on the torus directions are denoted by Y^i, $i = 1, \ldots, d$. All the geometry of the torus is encoded in a non diagonal internal metric $\hat{G}_{ij}$, and an antisymmetric two-form background field $\hat{B}_{ij}$, where $i, j = 1, \ldots, d$. It is also useful to define the *dimensionless* fields $G_{ij} = (\alpha')^{-1}\,\hat{G}_{ij}$, $B_{ij} = (\alpha')^{-1}\,\hat{B}_{ij}$, and the background matrix $E_{ij} = G_{ij} + B_{ij}$. In addition, we rescale $Y^i \to Y^i R^{(i)}$, where we denote by $R^{(i)}$ the radius of the i'th circle. Consequently, the boundary conditions of the compact bosonic coordinates take the form

$$
Y^i(\tau, \sigma + 2\pi) = Y^i(\tau, \sigma) + 2\pi\, w^i, \quad w^i \in \mathbb{Z} , \tag{2.2.17}
$$

where w^i are the winding numbers which give the number of times that the string winds around each cycle of the torus. In addition, we define the quantized internal momenta as $p_i \in \mathbb{Z}$. Now, the mass formula reads

$$
\begin{aligned}
&M^2 = M_0^2 + \frac{2}{\alpha'}\left(\tilde{N} + N - 2\right) , \\
&p_i w^i + \tilde{N} - N = 0, \qquad p_i, w^i \in \mathbb{Z} ,
\end{aligned}
\tag{2.2.18}
$$

where M_0^2 is given by

$$
\alpha' M_0^2 = \begin{pmatrix} w & p \end{pmatrix}^I \mathscr{G}(E)_{IJ} \begin{pmatrix} w \\ p \end{pmatrix}^J \quad I, J = 1, \ldots, 2d , \tag{2.2.19}
$$

and the matrix $\mathscr{G}(E)$ is[2]

$$
\mathscr{G}(E) = \begin{pmatrix} G - B\,G^{-1}B & B\,G^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \tag{2.2.20}
$$

In addition, the mode expansions of the bosonic coordinates on the torus directions are

$$
\begin{aligned}
Y^i_{\mathscr{L}}(\tau + \sigma) &= y^i_{\mathscr{L}} + \tfrac{1}{2} P^i_{\mathscr{L}}(\tau + \sigma) + \text{oscillators} , \\
Y^i_{\mathscr{R}}(\tau - \sigma) &= y^i_{\mathscr{R}} + \tfrac{1}{2} P^i_{\mathscr{R}}(\tau - \sigma) + \text{oscillators} ,
\end{aligned}
\tag{2.2.21}
$$

where $P^i_{\mathscr{L}/\mathscr{R}}$ are given by

$$
\begin{aligned}
P^i_{\mathscr{L}}(E) &= w^i + G^{ij}\left(p_j - B_{jk} w^k\right) , \\
P^i_{\mathscr{R}}(E) &= -w^i + G^{ij}\left(p_j - B_{jk} w^k\right) .
\end{aligned}
\tag{2.2.22}
$$

In the case of toroidal compactification there is a larger symmetry compared to circle compactification, that is the T-duality group $O(d, d; \mathbb{Z})$. By definition, this group consists of matrices g satisfying

$$
g^t \, \tau \, g = \tau , \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad \tau = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} , \tag{2.2.23}
$$

where the superscript t denotes a transpose matrix, and 1_d denotes a $d \times d$ unit matrix. The T-duality group acts on the closed string background fields G_{ij}, B_{ij} and on momentum and winding modes, in such a way that it leaves the mass spectrum of the string unchanged. This symmetry can be written in terms of a matrix $g \in O(d, d; \mathbb{Z})$ as [23]

$$
\mathscr{G} \to g \, \mathscr{G} \, g^t , \qquad Z \to (g^{-1})^t \, Z , \tag{2.2.24}
$$

where we combined momentum (p_i) and winding (w^i) numbers in a $2d$-column vector Z, $Z^t = (w^i, p_i)$. At this point, we wish to examine how T-duality acts on $P^i_{\mathscr{L}/\mathscr{R}}$. Using

[2] Note that $\mathscr{G}$ is a $2d \times 2d$ matrix written in terms of $d \times d$ blocks.

the expression (2.2.22) and the action (2.2.24) we find (see appendix A.2.1)

$$\begin{aligned} P_{\mathscr{L}}(E) \to P'_{\mathscr{L}}(E') &= (d + c\,E)\, P_{\mathscr{L}}(E)\,, \\ P_{\mathscr{R}}(E) \to P'_{\mathscr{R}}(E') &= \left(d - c\,E^t\right) P_{\mathscr{R}}(E)\,, \end{aligned} \tag{2.2.25}$$

where c, d are integer $d\times d$ matrices. Now, we wish to furthermore examine the possibility that the T-duality action is *symmetric*, namely it acts in the same way both on $P_{\mathscr{L}}(E)$ and $P_{\mathscr{R}}(E)$. From (2.2.25) we can see that this is achieved only if $c = 0$, or $E = -E^t$. But the later is rejected because it would imply that $G = 0$. So, a *symmetric* action dictates that $c = 0$, and we get the following transformation rules

$$\begin{aligned} P'_{\mathscr{L}}(E') &= d\, P_{\mathscr{L}}(E)\,, \\ P'_{\mathscr{R}}(E') &= d\, P_{\mathscr{R}}(E)\,, \end{aligned} \tag{2.2.26}$$

where d is an integer $d \times d$ matrix.

In addition, as in the case of compactification on a circle, under the T-duality action the left- and right-moving oscillators transform in the same way as $P^i_{\mathscr{L}/\mathscr{R}}$ (cf. 2.2.14, appendix A.2.2). Consequently, we conclude that T-duality acts generally on the compact bosonic coordinates as

$$\begin{aligned} Y^i_{\mathscr{L}}(\tau+\sigma) &\to\ (d + c\,E)^i{}_j\, Y^j_{\mathscr{L}}(\tau+\sigma)\,, \\ Y^i_{\mathscr{R}}(\tau-\sigma) &\to\ \left(d - c\,E^t\right)^i{}_j\, Y^j_{\mathscr{R}}(\tau-\sigma)\,, \end{aligned} \tag{2.2.27}$$

and in the symmetric case as

$$\begin{aligned} Y^i_{\mathscr{L}}(\tau+\sigma) &\to\ d^i{}_j Y^j_{\mathscr{L}}(\tau+\sigma)\,, \\ Y^i_{\mathscr{R}}(\tau-\sigma) &\to\ d^i{}_j Y^j_{\mathscr{R}}(\tau-\sigma)\,. \end{aligned} \tag{2.2.28}$$

Finally, we note that the transformation (2.2.11) is now generalized to

$$\mathscr{G} \leftrightarrow \mathscr{G}^{-1}, \qquad p_i \leftrightarrow w^i\,. \tag{2.2.29}$$

The results (2.2.27) and (2.2.28) will play a prominent role in chapter 4 where we construct orbifolds and we demand that their action is an element of the T-duality group.

Chapter 3

Supergravity

Supergravity is a field theory combining the principles of supersymmetry and general relativity. Naturally, as a theory of gravity it contains the graviton. Moreover, supersymmetry requires the existence of gravitino, which is the superpartner of graviton. There is also another way to study supergravity, that is as the low-energy limit of string theory. In this thesis we examine supergravity from the later point of view.

In section 3.1 we write down the action and the field content of type IIB supergravity in ten dimensions. In section 3.2, using the Kaluza-Klein method, we reduce type IIB supergravity on a four-torus, and in section 3.3 we further reduce it on a circle with a Scherk-Schwarz twist.

3.1 Type IIB supergravity

We have already constructed in section 2.1.1 the massless spectrum of type IIB string theory, which consists of the graviton G_{MN}, an antisymmetric tensor B_{MN} usually called the Kalb-Ramond field, a scalar Φ known as the dilaton, the p-form gauge fields C_0, C_{MN}, C_{MNPS}, two gravitini ψ_μ and two dilatini χ. As a matter of fact, these fields compose the field content of type IIB supergravity. The bosonic part of the ten dimensional action of type IIB supergravity in string frame[1] reads [14]

[1]The action can also be written in the Einstein frame, by substituting the metric in the string frame G_{MN}, with the metric g_{MN} in the Einstein frame. These two are related by a rescaling $G_{MN} = e^{\Phi/2} g_{MN}$.

$$
\begin{aligned}
\mathrm{S} = & \frac{1}{2\kappa_{10}^2} \int d^{10}\mathrm{x}\, \sqrt{-G}\, e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{2}|H_3|^2 \right) \\
& - \frac{1}{4\kappa_{10}^2} \int d^{10}\mathrm{x}\, \sqrt{-G} \left(|F_1|^2 + |F_3|^2 + \frac{1}{2}|F_5|^2 \right) \\
& - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \,.
\end{aligned} \tag{3.1.1}
$$

The fermionic part of the action can be obtained from the bosonic part using supersymmetry transformations. Note that the last term in the action has a Chern-Simons structure and is independent of the metric. The various field strengths appearing in the action are defined as

$$
\begin{aligned}
& H_3 = \mathrm{d}B_2\,, \quad F_1 = \mathrm{d}C_0\,, \quad F_3 = \mathrm{d}C_2 - C_0\,\mathrm{d}B_2\,, \\
& F_5 = \mathrm{d}C_4 - \frac{1}{2}C_2 \wedge \mathrm{d}B_2 + \frac{1}{2}B_2 \wedge \mathrm{d}C_2\,.
\end{aligned} \tag{3.1.2}
$$

In addition, we denote by κ_{10}^2 the ten dimensional gravitational coupling, which is given by

$$
\kappa_{10}^2 = \frac{1}{4\pi}(4\pi^2\alpha')^4\,. \tag{3.1.3}
$$

As a last remark, we note that the field strength F_5 is self-dual in ten dimensions[2]. As a consequence, there is no covariant way to write down an action which takes it into account. Therefore, the self-duality constraint has to be imposed on the equations of motion[3].

3.2 Kaluza-Klein reduction

The ultimate goal of this section is the reduction of the ten dimensional fields of type IIB supergravity to six dimensions on a four-torus. At first, we explicitly present how fields are reduced from ten to nine dimensions. Afterwards, the generalization in more dimensions is straightforward, since a d-torus can be treated as the product of d circles and the same technique can be used step-by-step.

In principle, we follow the idea of Kaluza-Klein reduction [24]. In section 3.2.1 we analyze the reduction of the metric. Subsequently, in section 3.2.2 we examine the reduction of n-forms. Finally, in section 3.2.3 we present the field content of type IIB supergravity reduced on a T^4. For an instructive overview on the Kaluza-Klein reduction we refer the reader to [25].

[2] In d dimensions, a p-form field strength is dual to a $(d-p)$-form field strength.
[3] A nice discussion on this can be found on section B.4 of [18].

3.2.1 Reduction of the metric

Before all else, it is useful to present here the notation that is being used in the subsequent sections. We consider a background spacetime of the form $\mathbb{R}^{1,8} \times S^1$. The coordinates of the 10-dimensional spacetime split up as $x^M = (x^\mu, x^9) \equiv (x^\mu, z)$, with $M = 0, \ldots, 9$ and $\mu = 0, \ldots, 8$. The circle coordinate is denoted by z.

The 10-dimensional metric G_{MN} can be decomposed into $G_{\mu\nu}$, $G_{\mu 9}$ and G_{99}, and we can immediately identify these fields with a metric, a vector field and a scalar field in nine dimensions [26, 27]. Nevertheless, we also present here the Kaluza-Klein ansatz

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + e^{2\phi} A_\mu A_\nu & e^{2\phi} A_\mu \\ e^{2\phi} A_\nu & e^{2\phi} \end{pmatrix} . \tag{3.2.1}$$

From this ansatz we recognize the 9-dimensional fields, namely the metric $g_{\mu\nu}$, a U(1) gauge field A_μ which is usually referred to as the graviphoton, and the Kaluza-Klein scalar field ϕ. For the metric components the ansatz (3.2.1) implies

$$\begin{aligned} G_{\mu\nu} &= g_{\mu\nu} + e^{2\phi} A_\mu A_\nu \, , \\ G_{\mu 9} &= e^{2\phi} A_\mu \, , \\ G_{99} &= e^{2\phi} \, . \end{aligned} \tag{3.2.2}$$

Note that regardless of what parametrization we use, the 10-dimensional metric always gives rise to a metric, a vector field and a scalar field in nine dimensions. Now, suppose that we want to reduce the metric G_{MN} to six dimensions. It is decomposed as[4]

$$G_{MN} \quad \begin{cases} G_{\mu\nu} & (1) \text{ metric} \, , \\ G_{\mu i} & (4) \text{ vectors} \, , \\ G_{ij} & (10) \text{ scalars} \, . \end{cases} \tag{3.2.3}$$

In this case the indices take the values: $M = 0, \ldots, 9$, $\mu = 0, \ldots, 5$ and $i = 6, 7, 8, 9$. Also, in order to find the total number of scalars we used the fact that a totally symmetric two-tensor in d dimensions has $\frac{d(d+1)}{2}$ independent components.

[4]The generalization of the ansatz (3.2.1) is not so straightforward. A nice example can be found in section 2 of [7].

3.2.2 Reduction of n-forms

We consider a 10-dimensional n-form C_n defined as

$$C_n = \frac{1}{n!} C_{M_1 \dots M_n} \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_n} \,. \tag{3.2.4}$$

In terms of indices we observe that the reduction of $C_{M_1 \dots M_n}$ from ten to nine dimensions gives another n-form $C_{\mu_1 \dots \mu_n}$ with all indices in the x^μ-direction, as well as a $(n-1)$-form $C_{\mu_1 \dots \mu_{n-1} z}$ with one index[5] in the z-direction . This can be nicely expressed as [28]

$$C_n(x^\mu, z) = C_n(x^\mu) + C_{n-1}(x^\mu) \wedge \mathrm{d}z \,. \tag{3.2.5}$$

Now, we examine the reduction of the field strength $F_{n+1}(x^\mu, z) = \mathrm{d}C_n(x^\mu, z)$. From equation (3.2.5) it follows that

$$F_{n+1}(x^\mu, z) = \mathrm{d}C_n(x^\mu) + \mathrm{d}C_{n-1}(x^\mu) \wedge \mathrm{d}z \,. \tag{3.2.6}$$

At this point, we can define the reduced field strengths $F_{n+1}(x^\mu) = \mathrm{d}C_n(x^\mu)$ and $F_n(x^\mu) = \mathrm{d}C_{n-1}(x^\mu)$. Instead of $F_{n+1}(x^\mu)$, it is customary to define a slightly modified field strength $\widetilde{F}_{n+1}(x^\mu)$ in the following way [29]

$$\widetilde{F}_{n+1}(x^\mu) = F_{n+1}(x^\mu) - F_n(x^\mu) \wedge A_1, \tag{3.2.7}$$

where A_1 is the graviphoton which was discussed in the previous section. It turns out that this is a gauge invariant combination. Additionally, the last term in (3.2.7) can be understood as a Chern-Simons correction. Combining equations (3.2.6) and (3.2.7) we find

$$F_{n+1}(x^\mu, z) = \widetilde{F}_{n+1}(x^\mu) + F_n(x^\mu) \wedge (\mathrm{d}z + A_1) \,. \tag{3.2.8}$$

As an example, let us now study how a two-form C_2 is reduced from ten to six dimensions. It can be decomposed as follows

$$C_{MN} \quad \begin{cases} C_{\mu\nu} & (1)\ \text{2-form}\,, \\ C_{\mu i} & (4)\ \text{1-forms}\,, \\ C_{ij} & (6)\ \text{0-forms}\,. \end{cases} \tag{3.2.9}$$

As in the reduction of the metric to six dimensions, the indices take the values: $M = 0, \dots, 9$, $\mu = 0, \dots, 5$ and $i = 6, 7, 8, 9$. In addition, in order to find the total number of zero-forms we used that a totally antisymmetric rank n tensor in d dimensions has $\binom{d}{n}$ number of independent components. Finally, we note that a zero-form (scalar) is

[5] A second index in the circle direction would yield a wedge product $\mathrm{d}z \wedge \mathrm{d}z = 0$.

trivially reduced, producing one lower-dimensional zero-form.

As we saw in section 2.2.1, if we perform a *compactification* on S^1, all fields will pick up infinite Kaluza-Klein towers. At the limit where the radius of the circle goes to zero, we can neglect all massive modes and keep only the massless ones; we will be referring to this as *dimensional reduction*.

3.2.3 Reduction of type IIB supergravity on T^4

In this section we explicitly present the reduction of the bosonic field content of type IIB supergravity on a four-torus. In the two previous sections we performed the reduction of the metric G_{MN} which represents the graviton, and of the two-form C_2. The reduction of the Kalb-Ramond field B_{MN} follows the same way as of C_2. The dilaton Φ and the zero-form C_0 are trivially reduced. Therefore, we only need to work out the reduction of the four-form C_4. Following the same path as in section 3.2.2 we find

$$C_{MNPS} \begin{cases} C_{\mu\nu\rho\sigma} & (1)\ \text{4-form}\,, \\ C_{\mu\nu\rho i} & (4)\ \text{3-forms}\,, \\ C_{\mu\nu ij} & (6)\ \text{2-forms}\,, \\ C_{\mu ijk} & (4)\ \text{1-forms}\,, \\ C_{ijkl} & (1)\ \text{0-form}\,. \end{cases} \tag{3.2.10}$$

However, we should keep only half of the reduced fields because the four-form C_4 has a self-dual field strength $F_5 = \mathrm{d}C_4$ in ten dimensions and this eliminates half of its degrees of freedom. In addition, in six dimensions a four-form is dual to a zero-form, and a three-form is dual to a one-form. Consequently, from the reduction of the four-form we only keep 3 two-forms, 4 one-forms and 1 zero-form in six dimensions.

Concerning the reduction of fermions, in general one needs to be aware of what kind of spinors can be defined in various dimensions[6]. In our case, we know that all components of the ten dimensional gravitini will survive in six dimensions because toroidal reduction does not break supersymmetry. In ten dimensions we had maximal $\mathcal{N} = 2$ supergravity. This results in maximal $\mathcal{N} = 8$ supergravity in six dimensions. The fermionic spectrum consists of 8 gravitini ψ_μ (4 of each chirality) and 40 dilatini χ (20 of each chirality) [31].

Putting all these results together, we find the field content of type IIB supergravity

[6] See for example [30].

reduced on a T^4. It consists of the graviton, 5 tensors (two-forms), 16 vectors (one-forms), 25 scalars (zero-forms), 8 gravitini and 40 dilatini. These fields make up one single supergravity multiplet [32].

3.3 Scherk-Schwarz reduction

Using the Kaluza-Klein reduction and starting from ten dimensions, we analyzed how to obtain the spectrum of type IIB supergravity in six dimensions. We discussed that all massive modes are truncated away, and that all supersymmetry is preserved. In fact, the later is the main disadvantage of this method because for realistic models we have to break some (or all) supersymmetry. In 1979 Scherk and Schwarz proposed a mechanism of breaking supersymmetry [5, 6], which is known as generalized, twisted, or Scherk-Schwarz reduction.

Once again, we begin with the reduction of a $(d+1)$-dimensional supergravity theory[7] in d-dimensions, on a circle of radius R, with periodic coordinate $z \sim z + 2\pi R$. As we shall see in the following, the basic difference between Kaluza-Klein and Scherk-Schwarz reduction on a circle is that in the later we give the fields a specific dependence on the internal coordinate.

Suppose that our theory has a global symmetry G. An element $g \in G$ acts on a generic field ψ as $\psi \to g\,\psi$. The Scherk-Schwarz ansatz is [33]

$$\psi(x^\mu, z) = g(z)\,\psi(x^\mu)\,. \tag{3.3.1}$$

The group element $g(z)$ can be written in terms of a matrix M, which lies in the Lie algebra of G, as

$$g(z) = \exp\left(\frac{Mz}{2\pi R}\right)\,. \tag{3.3.2}$$

This ansatz guarantees that the d-dimensional theory is independent of the internal coordinate, which is a necessary condition for a proper dimensional reduction. Furthermore, from (3.3.2) we notice that if we go once around the circle, $z \to z + 2\pi R$,

$$g(z) \to e^M g(z)\,. \tag{3.3.3}$$

[7] We always have at the back of our minds type IIB supergravity. However, we keep the discussion general here.

The factor $e^M \equiv \mathbf{M}$ is called *monodromy* and the matrix M is usually called the mass matrix because it introduces mass parameters in the theory. Fields in non-trivial representations of G acquire masses in terms of M.

In addition, Scherk-Schwarz reduction generates a scalar (or Scherk-Schwarz) potential, which can be written in terms of the mass matrix M and naturally provides a mechanism for supersymmetry breaking. It can be shown that if the twist $g(z)$ is an element of the maximal compact subgroup of G, usually called the R-symmetry group, classically the potential is non-negative and has stable five dimensional Minkowksi vacua [12]. We will come back to this in chapter 5 where we discuss the one-loop scalar potential and the generating vacua.

3.3.1 Reduction of type IIB supergravity on $T^4 \times S^1$

In this section we want to furthermore reduce type IIB supergravity from six to five dimensions on a circle of radius R, using the Scherk-Schwarz reduction. We do not perform explicit calculations, since this has been done before in [7, 12, 34, 35].

Maximal $\mathscr{N} = 8$ supergravity in six dimensions has a global $G = \text{Spin}(5{,}5)$ symmetry. The maximal compact subgroup, or R-symmetry group, of this group is

$$\text{Spin}(5)_L \times \text{Spin}(5)_R = \text{USp}(4)_L \times \text{USp}(4)_R \,. \tag{3.3.4}$$

We decompose this group further as (using that $\text{USp}(2) \cong \text{SU}(2)$)

$$\text{USp}(4)_L \times \text{USp}(4)_R \to \text{SU}(2)_{L_1} \times \text{SU}(2)_{L_2} \times \text{SU}(2)_{R_1} \times \text{SU}(2)_{R_2} \,. \tag{3.3.5}$$

In this way, the 8 supersymmetries are realized in 4 groups. In addition, the various fields sit in the following representations of the R-symmetry group

$$\begin{aligned} \text{scalars}: &\quad (\mathbf{5{,}5})\,, \\ \text{vectors}: &\quad (\mathbf{4{,}4})\,, \\ \text{tensors}: &\ (\mathbf{5{,}1}) + (\mathbf{1{,}5})\,, \\ \text{gravitini}: &\ (\mathbf{4{,}1}) + (\mathbf{1{,}4})\,, \\ \text{dilatini}: &\ (\mathbf{5{,}4}) + (\mathbf{4{,}5})\,. \end{aligned} \tag{3.3.6}$$

Regarding the 5 tensors, we decompose them in self-dual B_2^+ and anti-self-dual B_2^- parts, transforming in the $(\mathbf{5{,}1})$ and $(\mathbf{1{,}5})$ representations of the R-symmetry group respectively.

From the representations of the fields under the R-symmetry group, we can also find the representations under the decomposition (3.3.5). This determines the charges of the fields under the subgroup $SU(2)^4$ and subsequently all masses. A field with charges (e_1, e_2, e_3, e_4) will be an eigenvector of the mass matrix with eigenvalue $i\mu$. The mass of the field will be then given by $|\mu|/2\pi R$, where

$$\mu = \sum_{i=1}^{4} e_i m_i \,, \tag{3.3.7}$$

and m_i are four real mass parameters, each representing one of the four SU(2)'s in (3.3.5). Note that the amount of supersymmetry that is broken from six to five dimensions is twice the number of SU(2)'s that we twist. In other words, for each mass parameter that we turn on, two gravitini become massive, thus we break two supersymmetries. The masses of the fields in five dimensions are presented in Table 3.1 [7]. The graviton is not charged under the subgroup $SU(2)^4$, hence it does not acquire mass.

Fields	Representation	$\lvert\mu(m_i)\rvert$
Scalars	**(5,5)**	$\lvert\pm m_1 \pm m_2 \pm m_3 \pm m_4\rvert$
		$\lvert\pm m_1 \pm m_2\rvert$
		$\lvert\pm m_3 \pm m_4\rvert$
		0
Vectors	**(4,4)**	$\lvert\pm m_{1,2} \pm m_{3,4}\rvert$
Tensors	**(5,1)**	$\lvert\pm m_1 \pm m_2\rvert, 0$
	(1,5)	$\lvert\pm m_3 \pm m_4\rvert, 0$
Gravitini	**(4,1)**	$\lvert\pm m_{1,2}\rvert$
	(1,4)	$\lvert\pm m_{3,4}\rvert$
Dilatini	**(5,4)**	$\lvert\pm m_1 \pm m_2 \pm m_{3,4}\rvert$
		$\lvert\pm m_{3,4}\rvert$
	(4,5)	$\lvert\pm m_{1,2} \pm m_3 \pm m_4\rvert$
		$\lvert\pm m_{1,2}\rvert$

TABLE 3.1: *The spectrum of type IIB supergravity on T^4 followed by a Scherk-Schwarz twist on S^1. The masses of the fields are equal to $|\mu(m_i)|/2\pi R$. The notation $m_{i,j}$ indicates that both indices i and j occur. Also, there is no correlation between the ij indices and the $\pm$ signs. For an expanded version of this table see appendix B.2.*

In order to construct a specific supergravity theory, we can turn on none, one, two, three or four mass parameters, resulting in $\mathcal{N} = 8, 6, 4, 2, 0$ theories respectively. Then, we can find the corresponding massive spectrum from table 3.1. Regarding the massless spectrum, the fields that do not acquire mass from the Scherk-Schwarz twist are reduced from six to five dimensions in the usual Kaluza-Klein way. In this way we find the supergravity spectrum from the Scherk-Schwarz *reduction* on S^1.

Let us now consider a *compactification* on the circle. Following section 2.2.1, we modify the Scherk-Schwarz ansatz (3.3.1) as [7]

$$\psi(x^\mu, z) = g(z) \sum_{n\in\mathbb{Z}} e^{inz/R} \psi_n(x^\mu) = \sum_{n\in\mathbb{Z}} \exp\left[i\left(\frac{\mu(m_i)}{2\pi} + n\right)\frac{z}{R}\right] \psi_n(x^\mu)\,. \tag{3.3.8}$$

The consequence of (3.3.8) is that all fields pick up infinite Kaluza-Klein towers. Furthermore, the masses of all fields (both of massless and massive) are shifted by n/R, and the mass of the n'th Kaluza-Klein mode is given by

$$\left|\frac{\mu(m_i)}{2\pi R} + \frac{n}{R}\right|, \quad n \in \mathbb{Z}\,. \tag{3.3.9}$$

Finally, note that if we shift $\frac{\mu(m_i)}{2\pi}$ by an integer k, which corresponds to changing $m_i \to m_i + 2\pi k$, we can compensate this by shifting $n \to n - k$. Thus, we will only consider values of mass parameters $m_i \in [0, 2\pi)$.

In the next chapter we will demonstrate that we can reproduce the type IIB supergravity spectrum following from the Scherk-Schwarz compactification on S^1, by compactifying type IIB string theory on orbifolds.

Chapter 4

Orbifolds

Toroidal compactifications are well understood and exactly solvable examples of superstring compactifications. Their main drawback is that they preserve all supersymmetry, since for a realistic model we need a manifold, or a more general space, that allows partially (or fully) broken supersymmetry. Orbifolds are examples of such spaces and string theories on orbifolds where first studied in [3, 4]. Of course, there are many other spaces that break supersymmetry, and the most well known examples are Calabi-Yau manifolds. However, in this thesis we focus only on orbifold compactifications.

Let $\mathscr{M}$ be a Riemannian manifold with a discrete isometry group G. An orbifold $\mathscr{O}$ is simply obtained by the quotient

$$\mathscr{O} = \frac{\mathscr{M}}{G} \,. \tag{4.0.1}$$

Suppose that we take a group element $g \in G$ acting on a point $x \in \mathscr{M}$ as $g\,x$. Then, these two points $(x, g\,x)$ are equivalent in the orbifold. If the action of G leaves invariant a point x (or more) in the manifold, this *fixed* point is identified as an orbifold singularity and the quotient space is *non-freely* acting. On the contrast, an orbifold with no fixed points is *freely* acting. In what follows, we take our manifold to be a d-dimensional torus T^d with isometry group, the cyclic group Z_p. The resulting quotient space is of the form T^d/Z_p, and we refer to p as the rank of the orbifold.

Even though orbifolds can be singular, string theory on such spaces is completely well-behaved. This is a result of the extended nature of strings, which gives rise to non-trivial *twisted* states, defined next.

Closed string states living in an orbifold $\mathscr{M}/G$ obey modified boundary conditions according to

$$X^\mu(\tau, \sigma + 2\pi) = g\, X^\mu(\tau, \sigma)\,, \tag{4.0.2}$$

for $g \in G$. These are defined as twisted states (or sectors), and depending on the group element g we can have multiple such states. For $g = 1$ we reproduce the usual boundary conditions for closed string states. Such states, which are invariant under the orbifold action, are called *untwisted*. In this thesis we study the latter states.

4.1 Toroidal orbifolds

One straightforward, yet non trivial, example of an orbifold, is the orbifold S^1/Z_2. A circle of radius R and coordinate z is obtained by identifying $z \sim z + 2\pi R$. The circle has a discrete Z_2 symmetry acting as a reflection with respect to the origin

$$z \to -z\,. \tag{4.1.1}$$

We mod out this symmetry and obtain the orbifold S^1/Z_2, which is a line segment. This is a *non-freely* acting orbifold, since there are two fixed points, at $z = 0$ and $z = \pi R$. However, we can view the Z_2 action in a different way, namely as a translation

$$z \to z + \pi R. \tag{4.1.2}$$

Now, the resulting orbifold has no fixed points, thus is *freely* acting. Actually, it is a smooth manifold, that is a circle of radius $R/2$. In one dimension we can generalize (4.1.2) to every discrete group Z_p acting as

$$z \to z + 2\pi R/p. \tag{4.1.3}$$

Let us now proceed and consider two dimensional toroidal orbifolds. It is convenient to use a complex coordinate w on the torus, with the identifications $w \sim w + 1 \sim w + \tau$, where τ is a complex number, usually called the complex structure modulus of the torus. In addition, the torus can be described by a lattice, and depending on the lattice that we want to construct we choose the appropriate value of τ. The discrete group Z_p acts on the torus complex coordinate as

$$w \to e^{2\pi i/p} w. \tag{4.1.4}$$

This rotation is a symmetry of the torus only if it leaves its lattice invariant. It can be shown that in two dimensions this can be realized only for $p = 2, 3, 4$ and 6 [36]. For $p = 2, 4$ we make the identifications $w \sim w+1 \sim w+i$ (square lattice), while for $p = 3, 6$ we identify $w \sim w+1 \sim w+e^{\pi i/3}$ (hexagonal lattice). For higher dimensional toroidal orbifolds, more values of p are allowed. We refer the reader to [37], for further discussion on such topics.

As an example, let us take a T^2/Z_2 orbifold. The two-torus is described by the square lattice, which is invariant under the Z_2 action, while the torus complex coordinate picks up a minus sign, $w \to -w$, under the orbifold action. This orbifold is non-freely acting because it has four fixed points, at $0, \frac{1}{2}, \frac{i}{2}$ and $\frac{1+i}{2}$ [36].

Consider now a T^2/Z_3 orbifold. The suitable lattice to describe the torus is the hexagonal, because it is unchanged under the action of the discrete group Z_3. This orbifold acts on the torus complex coordinate as a rotation by an angle $e^{2\pi i/3}$, or twice this amount. Again this is a non-freely acting orbifold, since it has three fixed points, at $0, \frac{e^{\pi i/6}}{\sqrt{3}}, \frac{i}{3}$ [36].

Finally, we generalize the above discussion and examine a d-torus, where we take d even. We denote by $R^{(m)}$ the radius of each circle and by $Y^m, m = 1, \ldots, d$, the real torus coordinates. We define the complex torus coordinates as

$$W^i = \frac{1}{\sqrt{2}}(Y^{2i-1} + iY^{2i}), \qquad i = 1, \ldots, d/2\,. \tag{4.1.5}$$

The action of Z_p on the complex torus coordinates is

$$W^i \to e^{2\pi i u_j/p} W^i, \tag{4.1.6}$$

for some integers $u_j, j = 1, \ldots, d/2$. As mentioned in 4.1.4, the rank of the orbifold can not be arbitrary. Thus, in order to construct the orbifold T^d/Z_p we have to ensure first that the torus lattice is invariant under the action of the discrete group Z_p 4.1.6. Afterwards, we can choose the desired values of u_j's.

4.2 Strings on orbifolds

In this section we consider strings propagating on orbifold background spacetimes. In particular, we wish to construct a T^5/Z_p orbifold which acts as a translation on one real torus coordinate, and as a rotation on the remaining four coordinates. In order to make this different action evident, we write our orbifold target space as $\mathbb{R}^{1,4} \times (S^1 \times T^4)/Z_p$[1]. We will restrict our analysis to orbifolds of rank $p = 2, 3, 4, 6$ and 12. In general, we can construct symmetric orbifolds, which act equally on the left and right movers, or asymmetric orbifolds, which act differently on the left and right movers.

We denote the ten world-sheet scalars by X^M, $M = 0, \ldots, 9$, as (X^μ, Z, Y^m), where $\mu = 0, \ldots, 4$ and $m = 1, \ldots, 4$. The coordinate on the circle is periodic. Thus, we identify $Z \sim Z + 2\pi r$, where r is the radius of the circle. The four torus coordinates can be combined as indicated in (4.1.5). In addition, we split them in left and right movers as

$$W^i(\tau,\sigma) = W^i_{\mathscr{L}}(\tau+\sigma) + W^i_{\mathscr{R}}(\tau-\sigma)\,. \tag{4.2.1}$$

As follows from (4.1.3) and (4.1.6), the orbifold acts naturally as a rotation on the complex torus coordinates

$$\begin{aligned}
W^1_{\mathscr{L}} &\to e^{2\pi i u_1/p} W^1_{\mathscr{L}} \\
W^1_{\mathscr{R}} &\to e^{2\pi i u_2/p} W^1_{\mathscr{R}} \\
W^2_{\mathscr{L}} &\to e^{2\pi i u_3/p} W^2_{\mathscr{L}} \\
W^2_{\mathscr{R}} &\to e^{2\pi i u_4/p} W^2_{\mathscr{R}},
\end{aligned} \tag{4.2.2}$$

and as a translation on the circle coordinate

$$Z \to Z + 2\pi r/p. \tag{4.2.3}$$

At this point we follow [38] and we make a particular choice for the integers u_i. We set

$$\begin{aligned}
\frac{2\pi u_1}{p} &= m_1 + m_3, & \frac{2\pi u_2}{p} &= m_2 + m_4 \\
\frac{2\pi u_3}{p} &= m_1 - m_3, & \frac{2\pi u_4}{p} &= m_2 - m_4.
\end{aligned} \tag{4.2.4}$$

We can see from (4.2.4) that the *mass parameters* m_i must satisfy

$$m_i = \frac{2\pi n_i}{p}, \;\; n_i \in \mathbb{Z}, \;\; i = 1, 2, 3, 4. \tag{4.2.5}$$

However, there is a subtlety regarding a rank 12 orbifold, in the sense that we do not construct it by setting directly the mass parameters equal to $\frac{2\pi n_i}{12}$. Instead, we construct it by combining appropriately different values of mass parameters, e.g. $\frac{\pi}{3}$ and $\frac{\pi}{2}$. So, for our purposes, in equation (4.2.5) we restrict to values of $p = 2, 3, 4$ and 6.

[1] We can also decompose T^4 into $T^2 \times T^2$ and work with different lattices on the two T^2, if it is necessary for the particular orbifold that we want to construct.

In terms of these four mass parameters the orbifold action is summarized as follows

$$\begin{aligned}
&W^1_{\mathscr{L}} \to e^{i(m_1+m_3)} W^1_{\mathscr{L}}\\
&W^1_{\mathscr{R}} \to e^{i(m_2+m_4)} W^1_{\mathscr{R}}\\
&W^2_{\mathscr{L}} \to e^{i(m_1-m_3)} W^2_{\mathscr{L}}\\
&W^2_{\mathscr{R}} \to e^{i(m_2-m_4)} W^2_{\mathscr{R}},\\
&Z \to Z + 2\pi r/p.
\end{aligned} \tag{4.2.6}$$

Note that momentum states will pick up a phase $e^{2\pi i n/p}$, where the momentum number is denoted by n, due to the shift on the circle (cf. 2.2.3). Also, the orbifold action on the fermionic coordinates is identical, provided that we choose the same complex basis, $\Psi^i = \frac{1}{\sqrt{2}}(\psi^{2i-1} + i\psi^{2i})$.

In general, the orbifold that we constructed has fixed points on the T^4. However, due to the shift on the circle, there are no points that are left invariant under the orbifold action (4.2.6), thus we constructed a freely acting orbifold. We would also like to highlight here that if the mass parameters (4.2.5) are integer multiples of $0, \frac{\pi}{3}, \frac{\pi}{2}$ and π, the orbifold action on the torus complex coordinates is an element of the T-duality group $O(4,4,\mathbb{Z})$[2] (see appendix A.3). These two conditions are of crucial importance because if they are satisfied, the supergravity theories that we discussed in section 3.3.1 have lifts to string theory.

As we discussed in the beginning of this chapter, closed strings living on orbifolds obey modified boundary conditions. For our orbifold construction the following boundary conditions have to be satisfied

$$\begin{aligned}
&X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma), && Z(\tau,\sigma+2\pi) = Z(\tau,\sigma) + 2\pi r(w + k/p),\\
&W^1_{\mathscr{L}}(\tau,\sigma+2\pi) = (e^{i(m_1+m_3)})^k\, W^1_{\mathscr{L}}(\tau,\sigma), && W^2_{\mathscr{L}}(\tau,\sigma+2\pi) = (e^{i(m_1-m_3)})^k\, W^2_{\mathscr{L}}(\tau,\sigma),\\
&W^1_{\mathscr{R}}(\tau,\sigma+2\pi) = (e^{i(m_2+m_4)})^k\, W^1_{\mathscr{R}}(\tau,\sigma), && W^2_{\mathscr{R}}(\tau,\sigma+2\pi) = (e^{i(m_2-m_4)})^k\, W^2_{\mathscr{R}}(\tau,\sigma).
\end{aligned} \tag{4.2.7}$$

where $k = 0, \ldots, p-1$ and $w \in \mathbb{Z}$. We have one untwisted sector ($k = 0$) and $p-1$ twisted sectors. Note that we omit windings on the torus, since we assume the torus to be much smaller than the circle.

We denote the fermionic oscillators by b^M_n, with $M = 0, \ldots, 9$. These oscillators split up in b^μ_n on $\mathbb{R}^{1,4}$ and b^z_n on the circle. On the torus, since we work in complex coordinates, we use complex oscillators b^i_n and their complex conjugates $\bar{b}^i_n$, with $i = 1, 2$. A tilde above all these oscillators means that they are left-moving and the absence of a tilde

[2] Note that this is also a reason why in equation (4.2.5) we do not allow the value $p = 12$.

means that they are right-moving. The bosonic oscillators are denoted by a_n^M and they split up in the same way as the fermionic oscillators.

We denote the Neveu-Schwarz vacua by $|0\rangle_{\mathscr{L}/\mathscr{R}}$ and the Ramond vacua by $|s_1, s_2, s_3, s_4\rangle_{\mathscr{L}/\mathscr{R}}$, with $s_i = \pm\frac{1}{2}$. The subscript $\mathscr{L}/\mathscr{R}$ is used to distinguish the left- and the right-moving vacua. We choose the GSO projection in such a way that both R vacua have to satisfy

$$\sum_{i=1}^{4} s_i \quad \in 2\mathbb{Z}. \tag{4.2.8}$$

The NS vacua are scalars and they are invariant under the orbifold action. The R vacua are ten dimensional spinors and we know how they transform under rotations. Hence, we also know how they transform under the orbifold action. Consider for example the left-moving R vacuum $|\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle_{\mathscr{L}}$ (even number of +signs). In light-cone gauge, the oscillators on the torus directions $\tilde{b}_0^i$ act as raising and lowering operators on the second two entries. The orbifold acts on these states as [39]

$$e^{2\pi i(v_3 S_3 + v_4 S_4)} \, |\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle_{\mathscr{L}}\,, \tag{4.2.9}$$

where $S_3 = J_{67}$ and $S_4 = J_{89}$ are the SO(4) Cartan generators acting as rotations on the (67) and (89) planes, with eigenvalues s_3 and s_4 respectively. Similarly, the orbifold action on the right-moving Ramond vacuum can be written as

$$e^{2\pi i(w_3 S_3 + w_4 S_4)} \, |\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle_{\mathscr{R}}\,. \tag{4.2.10}$$

We read of from (4.2.6) the values of $v_{3,4}$ and $w_{3,4}$

$$\begin{aligned} v_3 &= \frac{m_1 + m_3}{2\pi}, & w_3 &= \frac{m_2 + m_4}{2\pi}, \\ v_4 &= \frac{m_1 - m_3}{2\pi}, & w_4 &= \frac{m_2 - m_4}{2\pi}. \end{aligned} \tag{4.2.11}$$

We can see that the orbifold action on the Ramond vacua depends only on the values of s_3, s_4. In order to determine the orbifold action on the R vacua, we write down explicitly all possible values of these spins [38]

$$\begin{aligned} |\alpha_1\rangle_{\mathscr{L}/\mathscr{R}} &= |s_1, s_1, \tfrac{1}{2}, \tfrac{1}{2}\rangle_{\mathscr{L}/\mathscr{R}}\,, \\ |\alpha_2\rangle_{\mathscr{L}/\mathscr{R}} &= |s_1, s_1, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{\mathscr{L}/\mathscr{R}}\,, \\ |\alpha_3\rangle_{\mathscr{L}/\mathscr{R}} &= |s_1, -s_1, \tfrac{1}{2}, -\tfrac{1}{2}\rangle_{\mathscr{L}/\mathscr{R}}\,, \\ |\alpha_4\rangle_{\mathscr{L}/\mathscr{R}} &= |s_1, -s_1, -\tfrac{1}{2}, \tfrac{1}{2}\rangle_{\mathscr{L}/\mathscr{R}}\,. \end{aligned} \tag{4.2.12}$$

Here, the relative sign between s_1 and s_2 is fixed by the GSO projection. The orbifold action on each of the above states is

$$\begin{aligned}
|\alpha_1\rangle_{\mathscr{L}} &\to e^{im_1}\,|\alpha_1\rangle_{\mathscr{L}}\,, & |\alpha_1\rangle_{\mathscr{R}} &\to e^{im_2}\,|\alpha_1\rangle_{\mathscr{R}}\,,\\
|\alpha_2\rangle_{\mathscr{L}} &\to e^{-im_1}\,|\alpha_2\rangle_{\mathscr{L}}\,, & |\alpha_2\rangle_{\mathscr{R}} &\to e^{-im_2}\,|\alpha_2\rangle_{\mathscr{R}}\,,\\
|\alpha_3\rangle_{\mathscr{L}} &\to e^{im_3}\,|\alpha_3\rangle_{\mathscr{L}}\,, & |\alpha_3\rangle_{\mathscr{R}} &\to e^{im_4}\,|\alpha_3\rangle_{\mathscr{R}}\,,\\
|\alpha_4\rangle_{\mathscr{L}} &\to e^{-im_3}\,|\alpha_4\rangle_{\mathscr{L}}\,, & |\alpha_4\rangle_{\mathscr{R}} &\to e^{-im_4}\,|\alpha_4\rangle_{\mathscr{R}}\,.
\end{aligned} \tag{4.2.13}$$

In Table 4.1 [38] we write down all the massless states in the absence of momentum and/or winding modes in the NS and R sectors. These are general states that appear both left- and right-moving. Furthermore, we write down their charges under the orbifold action, as well as their representations under the massless little group SO(3)≅SU(2) and the massive little group SO(4)≅SU(2)×SU(2). The latter is important when momentum or windings are added and the state becomes massive.

Sector	State	$\mathscr{L}$ Orbifold charge	$\mathscr{R}$ Orbifold charge	SO(3) rep	SO(4) rep
NS	$b^{\mu}_{-1/2}\,|0\rangle$	1	1	3	(2,2)
	$b^{z}_{-1/2}\,|0\rangle$	1	1	1	
	$b^{1}_{-1/2}\,|0\rangle$	$e^{i(m_1+m_3)}$	$e^{i(m_2+m_4)}$	1	(1,1)
	$\bar{b}^{1}_{-1/2}\,|0\rangle$	$e^{-i(m_1+m_3)}$	$e^{-i(m_2+m_4)}$	1	(1,1)
	$b^{2}_{-1/2}\,|0\rangle$	$e^{i(m_1-m_3)}$	$e^{i(m_2-m_4)}$	1	(1,1)
	$\bar{b}^{2}_{-1/2}\,|0\rangle$	$e^{-i(m_1-m_3)}$	$e^{-i(m_2-m_4)}$	1	(1,1)
R	$|\alpha_1\rangle$	e^{im_1}	e^{im_2}	2	(2,1)
	$|\alpha_2\rangle$	e^{-im_1}	e^{-im_2}	2	(2,1)
	$|\alpha_3\rangle$	e^{im_3}	e^{im_4}	2	(1,2)
	$|\alpha_4\rangle$	e^{-im_3}	e^{-im_4}	2	(1,2)

TABLE 4.1: *In this table we present all massless states in the absence of momentum and/or winding. We write down general states that appear both left-moving and right-moving, omitting the tildes on the oscillators and the subscripts $\mathscr{L}/\mathscr{R}$. In addition, we present the charges of these states under the orbifold action, as well as their representations under the massless and massive little groups in five dimensions.*

At this point, we are ready to start discussing specific examples of orbifolds. Depending on the particular orbifold that we wish to construct, we shall choose the appropriate mass parameters as indicated in (4.2.5) and the suitable lattice, as we discussed in the previous section. Subsequently, we can take tensor products of the left and right movers in Table 4.1 and build the corresponding spectrum. In order to do so, we will use the following rules for tensoring SU(2) representations

$$3\times 3 = 5+3+1, \qquad 2\times 2 = 3+1, \qquad 3\times 2 = 4+2. \tag{4.2.14}$$

Recall that an irreducible representation of SU(2) is labelled by a non-negative (half)-integer $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and its dimension is $2l+1$. The tensor product of two representations labeled by l and m, with $l \geq m$ is given by $V_l \times V_m = V_{l+m} \times V_{l+m-1} \times \ldots \times V_{l-m}$ [40].

In addition, since we wish to compare the spectra arising from orbifold constructions with the spectra following from the Scherk-Schwarz reduction on the level of supergravity, we list in Table 4.2 the massless and massive representations corresponding to the various supergravity fields in five dimensions [38].

Massless field	SO(3) rep
$g_{\mu\nu}$	5
ψ_μ	4
A_μ	3
χ	2
ϕ	1

Massive field	SO(4) rep
$B^+_{\mu\nu}$ / $B^-_{\mu\nu}$	(3,1) / (1,3)
ψ^+_μ / ψ^-_μ	(2,3) / (3,2)
A_μ	(2,2)
χ^+ / χ^-	(2,1) / (1,2)
ϕ	(1,1)

TABLE 4.2: *In this table we list the various supergravity fields in five dimensions and their massless and massive representations under the appropriate little group.*

4.3 A symmetric $\mathscr{N} = 4$, Z_3 orbifold

Since we are interested in constructing a symmetric orbifold, we shall choose the mass parameters such that $m_1 = m_2$ and $m_3 = m_4$. Let us first set $m_1 = m_2 = \frac{2\pi}{3}$ and $m_3 = m_4 = 0$. This choice will result in a $\mathscr{N} = 4$ (0,2) theory in five dimensions. In addition, momentum states will pick up a phase $e^{2\pi i n/3}$. For the construction of the massless spectrum, we take combinations of states in Table 4.1 that are invariant under the orbifold action. We list below the states that we find in each sector.

NS-NS sector:

$$
\begin{aligned}
&\tilde{b}^\mu_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^\nu_{-1/2}\,|0\rangle_{\mathscr{R}} \to 5+3+1 \\
&\tilde{b}^\mu_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^z_{-1/2}\,|0\rangle_{\mathscr{R}} \to 3 \\
&\tilde{b}^z_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^\mu_{-1/2}\,|0\rangle_{\mathscr{R}} \to 3 \\
&\tilde{b}^z_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^z_{-1/2}\,|0\rangle_{\mathscr{R}} \to 1 \\
&\tilde{b}^i_{-1/2}\,|0\rangle_{\mathscr{L}} \times \bar{b}^j_{-1/2}\,|0\rangle_{\mathscr{R}} \to 4(1) \\
&\bar{\tilde{b}}^i_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^j_{-1/2}\,|0\rangle_{\mathscr{R}} \to 4(1)
\end{aligned}
\tag{4.3.1}
$$

R-R sector:

$$
\begin{aligned}
|\alpha_1\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to 3+1 \\
|\alpha_2\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to 3+1 \\
|\alpha_{3,4}\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 4(3)+4(1)
\end{aligned}
\tag{4.3.2}
$$

NS-R sector:

$$
\begin{aligned}
\tilde{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 2(4)+2(2) \\
\tilde{b}^{z}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 2(2) \\
\tilde{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to 2(2) \\
\bar{\tilde{b}}^{i}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to 2(2)
\end{aligned}
\tag{4.3.3}
$$

R-NS sector:

$$
\begin{aligned}
|\alpha_{3,4}\rangle_{\mathscr{L}} \times b^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(4)+2(2) \\
|\alpha_{3,4}\rangle_{\mathscr{L}} \times b^{z}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2) \\
|\alpha_2\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2) \\
|\alpha_1\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2)\,.
\end{aligned}
\tag{4.3.4}
$$

Note that the indices i and j denote the two complex torus coordinates and take the values $1,2$. Collecting together our results form the four sectors, for the massless states we find the graviton, 9 vectors, 4 gravitini, 16 dilatini and 16 scalars. They form one gravity multiplet, consisting of the graviton, 4 gravitini, 6 vectors, 4 dilatini and one scalar, coupled to three vector multiplets, each made up from one vector, 4 dilatini and 5 scalars [41].

For the massive spectrum, we take combinations of states in Table 4.1 that get a phase $e^{\pm 2\pi i/3}$ or $e^{\pm 4\pi i/3}$ and cancel it, by adding momentum modes appropriately. Whenever we add momentum and/or winding on a state, which is a tensor product of left and right movers, we denote it for convenience, only on the left movers, as $|\ldots; n, w\rangle_{\mathscr{L}}$, with n and w the momentum and winding numbers on the circle, respectively. In the following, we will add momentum only in the circle direction. In addition, we will use the notation $\hat{\tilde{b}}^{\mu}, \hat{b}^{\mu}$, for the fermionic oscillators in the μ- and z-directions. We list below the massive states.

NS-NS sector:

$$\begin{aligned}
&\bar{\hat{b}}^{\mu}_{-1/2}\,|0;+1,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(2,2)\\
&\bar{\tilde{b}}^{i}_{-1/2}\,|0;+1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(2,2)\\
&\bar{\tilde{b}}^{i}_{-1/2}\,|0;+2,0\rangle_{\mathscr{L}} \times \bar{b}^{j}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 4(1,1)\\
&\bar{\hat{b}}^{\mu}_{-1/2}\,|0;-1,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(2,2)\\
&\tilde{b}^{i}_{-1/2}\,|0;-1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(2,2)\\
&\tilde{b}^{i}_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times b^{j}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 4(1,1)
\end{aligned} \tag{4.3.5}$$

R-R sector:

$$\begin{aligned}
|\alpha_1;-2,0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to (3,1)+(1,1)\\
|\alpha_2;+2,0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to (3,1)+(1,1)\\
|\alpha_1;-1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 2(2,2)\\
|\alpha_2;+1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 2(2,2)\\
|\alpha_{3,4};-1,0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to 2(2,2)\\
|\alpha_{3,4};+1,0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to 2(2,2)
\end{aligned} \tag{4.3.6}$$

NS-R sector:

$$\begin{aligned}
\bar{\hat{b}}^{\mu}_{-1/2}\,|0;+1,0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to (3,2)+(1,2)\\
\bar{\hat{b}}^{\mu}_{-1/2}\,|0;-1,0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to (3,2)+(1,2)\\
\tilde{b}^{i}_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to 2(2,1)\\
\bar{\tilde{b}}^{i}_{-1/2}\,|0;+2,0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to 2(2,1)\\
\tilde{b}^{i}_{-1/2}\,|0;-1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 4(1,2)\\
\bar{\tilde{b}}^{i}_{-1/2}\,|0;+1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 4(1,2)
\end{aligned} \tag{4.3.7}$$

R-NS sector:

$$\begin{aligned}
|\alpha_2;+1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to (3,2)+(1,2)\\
|\alpha_1;-1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to (3,2)+(1,2)\\
|\alpha_1;-2,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2,1)\\
|\alpha_2;+2,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2,1)\\
|\alpha_{3,4};-1,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 4(1,2)\\
|a_{3,4};+1,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 4(1,2)\,.
\end{aligned} \tag{4.3.8}$$

We find that the massive spectrum consists of 4 gravitini of the same chirality (ψ^-_μ), 28 dilatini of either chiralities (20 χ^- and 8 χ^+), 16 vectors, 2 self-dual tensors and 10 scalars. They form four massive spin-$\frac{3}{2}$ multiplets and two massive tensor multiplets [42]. Each of the massive spin-$\frac{3}{2}$ multiplets consists of 1 gravitino, 4 vectors and 5 dilatini

$$(3,2)+4(2,2)+5(1,2)\,. \tag{4.3.9}$$

The tensor multiplet contains 1 self-dual tensor, 4 dilatini and 5 scalars, in the representations

$$(3,1)+4(2,1)+5(1,1)\,. \tag{4.3.10}$$

Note that these multiplets have different masses. The spin-$\frac{3}{2}$ multiplets have mass $|\frac{1}{r}|$, due to the contribution of the $n=\pm1$ momentum mode, while the tensor multiplets have mass $|\frac{2}{r}|$, as they are constructed using the $n=\pm2$ momentum mode. Furthermore, if we associate the orbifold radius r, with the Scherk-Schwarz radius R, by $r=\frac{R}{3}$, the masses of our multiplets can be written as $|\frac{1}{3R}|$ and $|\frac{2}{3R}|$ respectively.

All states, both massive and massless, are constructed such that the combination between left and right movers ensures zero phase. However, we can add to all these states a trivial phase $e^{(2\pi i/3)3k}$, where k is an integer number, or in other words, we can add momentum modes that are multiples of 3, i.e. $n=3k$. In this way we can construct Kaluza Klein towers, where the contribution of each momentum mode is $|\frac{3k}{r}|=|\frac{k}{R}|$. This entire spectrum arising from our orbifold construction matches exactly with the one found from the Scherk-Schwarz reduction on the level of supergravity.

We can also examine how the spectrum changes, if we pick other values for our mass parameters. The easiest case is to take $m_1=m_2=\frac{4\pi}{3}$ and $m_3=m_4=0$, which is almost identical to the previous one. We find the same states, with the only difference that the massive multiplets acquire twice the masses found before.

Let us now choose $m_1=m_2=0$ and $m_3=m_4=\frac{2\pi}{3}$. The resulting theory is a chiral $\mathscr{N}=4$ (2,0) theory in five dimensions. Regarding the massless spectrum, there is no difference comparing to our previous results. On the other hand, the massive spectrum consists now of 4 gravitini of the opposite chirality (ψ_μ^+), 28 dilatini (20 χ^+ and 8 χ^-), 16 vectors, 2 anti-self-dual tensors and 10 scalars. Now, each massive spin-$\frac{3}{2}$ multiplet consists of 1 gravitino, 4 vectors and 5 dilatini, which can be written in terms of their massive representations as

$$(2,3)+4(2,2)+5(2,1)\,. \tag{4.3.11}$$

The tensor multiplet contains 1 anti-self dual tensor, 4 dilatini and 5 scalars

$$(1,3)+4(1,2)+5(1,1)\,. \tag{4.3.12}$$

The masses of these multiplets are $|\frac{1}{r}|$ and $|\frac{2}{r}|$ respectively. Finally, if we take $m_1=m_2=0$ and $m_3=m_4=\frac{4\pi}{3}$, we find the same, $\mathscr{N}=4$ (2,0) theory, with the masses of the above multiplets changed accordingly to $|\frac{2}{r}|$ and $|\frac{4}{r}|$.

4.4 An asymmetric $\mathscr{N}=4$, Z_4 orbifold

In this section, we study an asymmetric Z_4 orbifold, which gives a non-chiral $\mathscr{N}=4$ (1,1) theory in five dimensions. An asymmetric orbifold should act differently on the left and right movers, hence we shall take $m_1 \neq m_2$ and/or $m_3 \neq m_4$. Let us set $m_1 = m_3 = \frac{\pi}{2}, m_2 = m_4 = 0$. In this case, momentum states get a phase of $e^{\pi i n/2}$.

As in the case of the symmetric Z_3 orbifold, for the construction of the massless spectrum we take combinations of states in Table 4.1 that are invariant under the orbifold action. We find the following states.

NS-NS sector:

$$
\begin{aligned}
&\tilde{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{\nu}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 5+3+1\\
&\tilde{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{z}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 3\\
&\tilde{b}^{z}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 3\\
&\tilde{b}^{z}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{z}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 1\\
&\tilde{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(3)\\
&\tilde{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(3)\\
&\tilde{b}^{z}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(1)\\
&\tilde{b}^{z}_{-1/2}\,|0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(1)\\
&\tilde{b}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 3\\
&\tilde{b}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{z}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 1\\
&\bar{\tilde{b}}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 3\\
&\bar{\tilde{b}}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{z}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 1\\
&\tilde{b}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(1)\\
&\tilde{b}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(1)\\
&\bar{\tilde{b}}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(1)\\
&\bar{\tilde{b}}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} \to 2(1)
\end{aligned}
\tag{4.4.1}
$$

NS-R sector:

$$
\begin{aligned}
&\tilde{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_m\rangle \to 4(4)+4(2)\\
&\tilde{b}^{z}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_m\rangle \to 4(2)\\
&\tilde{b}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_m\rangle \to 4(2)\\
&\bar{\tilde{b}}^{2}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |\alpha_m\rangle \to 4(2)\,,
\end{aligned}
\tag{4.4.2}
$$

where $m = 1,2,3,4$. In the R-NS and R-R sectors we find no states. In total, we find the graviton, 4 gravitini, 9 vectors, 16 dilatini and 16 scalars. These fields form one gravity multiplet coupled to three vector multiplets, which are exactly the same multiplets with those found in the symmetric Z_3 orbifold. That is reasonable, since the massless states should not be affected by the choice of the mass parameters, as long as we only take two of them non-zero.

We continue with the construction of massive states. Now, we take combinations of states in Table 4.1 that get a phase $e^{\pm\pi i}$ or $e^{\pm\pi i/2}$, and cancel it by adding momentum modes appropriately. We list below the states that we find in each sector.

NS-NS sector:

$$\begin{aligned}
\tilde{b}^1_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times \hat{b}^\mu\,|0\rangle_{\mathscr{R}} &\to (2,2)\\
\tilde{b}^1_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times b^i\,|0\rangle_{\mathscr{R}} &\to 2(1,1)\\
\tilde{b}^1_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times \bar{b}^i\,|0\rangle_{\mathscr{R}} &\to 2(1,1)\\
\bar{\tilde{b}}^1_{-1/2}\,|0;2,0\rangle_{\mathscr{L}} \times \hat{b}^\mu\,|0\rangle_{\mathscr{R}} &\to (2,2)\\
\bar{\tilde{b}}^1_{-1/2}\,|0;2,0\rangle_{\mathscr{L}} \times b^i\,|0\rangle_{\mathscr{R}} &\to 2(1,1)\\
\bar{\tilde{b}}^1_{-1/2}\,|0;2,0\rangle_{\mathscr{L}} \times \bar{b}^i\,|0\rangle_{\mathscr{R}} &\to 2(1,1)
\end{aligned} \tag{4.4.3}$$

R-R sector:

$$\begin{aligned}
|\alpha_1;-1,0\rangle_{\mathscr{L}} \times |\alpha_{1,2}\rangle &\to 2(3,1)+2(1,1)\\
|\alpha_1;-1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle &\to 2(2,2)\\
|\alpha_3;-1,0\rangle_{\mathscr{L}} \times |\alpha_{1,2}\rangle &\to 2(2,2)\\
|\alpha_3;-1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle &\to 2(1,3)+2(1,1)\\
|\alpha_2;1,0\rangle_{\mathscr{L}} \times |\alpha_{1,2}\rangle &\to 2(3,1)+2(1,1)\\
|\alpha_2;1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle &\to 2(2,2)\\
|\alpha_4;1,0\rangle_{\mathscr{L}} \times |\alpha_{1,2}\rangle &\to 2(2,2)\\
|\alpha_4;1,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle &\to 2(1,3)+2(1,1)
\end{aligned} \tag{4.4.4}$$

NS-R sector:

$$\begin{aligned}
\tilde{b}^1_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times |\alpha_{1,2}\rangle_{\mathscr{R}} &\to 2(2,1)\\
\tilde{b}^1_{-1/2}\,|0;-2,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 2(1,2)\\
\bar{\tilde{b}}^1_{-1/2}\,|0;2,0\rangle_{\mathscr{L}} \times |\alpha_{1,2}\rangle_{\mathscr{R}} &\to 2(2,1)\\
\bar{\tilde{b}}^1_{-1/2}\,|0;2,0\rangle_{\mathscr{L}} \times |\alpha_{3,4}\rangle_{\mathscr{R}} &\to 2(1,2)
\end{aligned} \tag{4.4.5}$$

R-NS sector:

$$
\begin{aligned}
|\alpha_1; -1, 0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}\, |0\rangle_{\mathscr{R}} &\to (3,2) + (1,2) \\
|\alpha_1; -1, 0\rangle_{\mathscr{L}} \times b^{i}\, |0\rangle_{\mathscr{R}} &\to 2(2,1) \\
|\alpha_1; -1, 0\rangle_{\mathscr{L}} \times \bar{b}^{i}\, |0\rangle_{\mathscr{R}} &\to 2(2,1) \\
|\alpha_3; -1, 0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}\, |0\rangle_{\mathscr{R}} &\to (2,3) + (2,1) \\
|\alpha_3; -1, 0\rangle_{\mathscr{L}} \times b^{i}\, |0\rangle_{\mathscr{R}} &\to 2(1,2) \\
|\alpha_3; -1, 0\rangle_{\mathscr{L}} \times \bar{b}^{i}\, |0\rangle_{\mathscr{R}} &\to 2(1,2) \\
|\alpha_2; 1, 0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}\, |0\rangle_{\mathscr{R}} &\to (3,2) + (1,2) \\
|\alpha_2; 1, 0\rangle_{\mathscr{L}} \times b^{i}\, |0\rangle_{\mathscr{R}} &\to 2(2,1) \\
|\alpha_2; 1, 0\rangle_{\mathscr{L}} \times \bar{b}^{i}\, |0\rangle_{\mathscr{R}} &\to 2(2,1) \\
|\alpha_4; 1, 0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}\, |0\rangle_{\mathscr{R}} &\to (2,3) + (2,1) \\
|\alpha_4; 1, 0\rangle_{\mathscr{L}} \times b^{i}\, |0\rangle_{\mathscr{R}} &\to 2(1,2) \\
|\alpha_4; 1, 0\rangle_{\mathscr{L}} \times \bar{b}^{i}\, |0\rangle_{\mathscr{R}} &\to 2(1,2)\,.
\end{aligned}
\tag{4.4.6}
$$

From the four different sectors we find 4 gravitini of either chiralities (2 ψ_{μ}^{+} and 2 ψ_{μ}^{-}), 28 dilatini (14 χ^{+} and 14 χ^{-}), 4 self-dual and 4 anti-self-dual tensors, 10 vectors and 16 scalars. We have 2 massive vector multiplets with mass $|\frac{2}{r}|$. Each of these multiplets consists of 1 vector, 4 dilatini and 4 scalars [42]

$$
(2,2) + 2(2,1) + 2(1,2) + 4(1,1)\,. \tag{4.4.7}
$$

In addition, we find 4 massive spin-$\frac{3}{2}$ multiplets of mass $|\frac{1}{r}|$, each consisting of 1 gravitino (ψ_{μ}^{+} or ψ_{μ}^{-}), 2 self-dual or anti-self-dual tensors, 2 vectors, 5 dilatini and 2 scalars

$$
(3,2) + 2(3,1) + 2(2,2) + (1,2) + 4(2,1) + 2(1,1)\,, \tag{4.4.8}
$$

$$
(2,3) + 2(1,3) + 2(2,2) + (2,1) + 4(1,2) + 2(1,1)\,. \tag{4.4.9}
$$

Now, we can relate the orbifold radius r with the Scherk-Schwarz radius R, by $r = \frac{R}{4}$. In addition, we can construct Kaluza Klein towers by adding momentum modes that are integer multiples of 4, as $e^{(\pi i/2)4k}$, and the contribution of each mode to the mass of a state is $|\frac{4k}{r}| = |\frac{k}{R}|$. Once again, the orbifold spectrum is identical with the one we obtain by the Scherk-Schwarz mechanism on the level of supergravity.

4.5 A symmetric $\mathscr{N} = 0$, Z_2 orbifold

In this section we wish to construct an orbifold that breaks all supersymmetry, resulting in a $\mathscr{N} = 0$ theory in five dimensions. Hence, we shall take all four mass parameters non-zero. The allowed values for the mass parameters in a Z_2 orbifold are

$$m_i = \pi\, n_i, \quad i = 1, 2, 3, 4\,. \tag{4.5.1}$$

Thus, we set $m_1 = m_2 = m_3 = m_4 = \pi$. States that carry momentum in the circle direction get a phase $e^{\pi i n}$, $n \in \mathbb{Z}$. This specific orbifold acts trivially on the left and right movers in the NS sector, while all states in the R sector obtain a phase $e^{\pm \pi i}$.

Concerning the massless states, we can see that all possible combinations between left and right movers in the NS sector are invariant under the orbifold action. The resulting spectrum in the NS-NS sector consists of the graviton, 11 vectors and 26 scalars. The same argument holds for the R-R sector where we find 16 vectors and 16 scalars.

$$|\alpha_m\rangle_{\mathscr{L}} \times |\alpha_l\rangle_{\mathscr{R}} \to 16(3) + 16(1) \qquad m, l = 1, \ldots, 4\,. \tag{4.5.2}$$

In the NS-R and R-NS sectors we find no states, thus there are no massless fermions.

We proceed with the construction of massive states. Once again, we take combinations in Table 4.1 that carry a phase $e^{\pm\pi i}$ and cancel it by adding appropriate momentum modes. We find the following states.

NS-R sector:

$$\begin{aligned}
\hat{\tilde{b}}^{\mu}_{-1/2}\, |0; -1, 0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to (3,2) + (1,2) \\
\hat{\tilde{b}}^{\mu}_{-1/2}\, |0; -1, 0\rangle_{\mathscr{L}} \times |\alpha_3\rangle_{\mathscr{R}} &\to (2,3) + (2,1) \\
\hat{\tilde{b}}^{\mu}_{-1/2}\, |0; 1, 0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to (3,2) + (1,2) \\
\hat{\tilde{b}}^{\mu}_{-1/2}\, |0; 1, 0\rangle_{\mathscr{L}} \times |\alpha_4\rangle_{\mathscr{R}} &\to (2,3) + (2,1) \\
\tilde{b}^{i}_{-1/2}\, |0; -1, 0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to 2(2,1) \\
\tilde{b}^{i}_{-1/2}\, |0; -1, 0\rangle_{\mathscr{L}} \times |\alpha_3\rangle_{\mathscr{R}} &\to 2(1,2) \\
\tilde{b}^{i}_{-1/2}\, |0; -1, 0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to 2(2,1) \\
\tilde{b}^{i}_{-1/2}\, |0; -1, 0\rangle_{\mathscr{L}} \times |\alpha_4\rangle_{\mathscr{R}} &\to 2(1,2) \\
\bar{\tilde{b}}^{i}_{-1/2}\, |0; 1, 0\rangle_{\mathscr{L}} \times |\alpha_1\rangle_{\mathscr{R}} &\to 2(2,1) \\
\bar{\tilde{b}}^{i}_{-1/2}\, |0; 1, 0\rangle_{\mathscr{L}} \times |\alpha_3\rangle_{\mathscr{R}} &\to 2(1,2) \\
\bar{\tilde{b}}^{i}_{-1/2}\, |0; 1, 0\rangle_{\mathscr{L}} \times |\alpha_2\rangle_{\mathscr{R}} &\to 2(2,1) \\
\bar{\tilde{b}}^{i}_{-1/2}\, |0; 1, 0\rangle_{\mathscr{L}} \times |\alpha_4\rangle_{\mathscr{R}} &\to 2(1,2)
\end{aligned} \tag{4.5.3}$$

R-NS sector:

$$\begin{aligned}
|\alpha_1;-1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to (3,2)+(1,2)\\
|\alpha_1;-1,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2,1)\\
|\alpha_1;1,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2,1)\\
|\alpha_2;1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to (3,2)+(1,2)\\
|\alpha_2;1,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2,1)\\
|\alpha_2;-1,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(2,1)\\
|\alpha_3;-1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to (2,3)+(2,1)\\
|\alpha_3;-1,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(1,2)\\
|\alpha_3;1,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(1,2)\\
|\alpha_4;1,0\rangle_{\mathscr{L}} \times \hat{b}^{\mu}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to (2,3)+(2,1)\\
|\alpha_4;1,0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(1,2)\\
|\alpha_4;-1,0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\,|0\rangle_{\mathscr{R}} &\to 2(1,2)\,.
\end{aligned} \tag{4.5.4}$$

In total, we find 8 massive gravitini (4 ψ^+_μ and 4 ψ^-_μ) and 40 massive dilatini (20 χ^+ and 20 χ^-). Note that in the NS-NS and R-R sectors we find no states, thus there are no massive bosons, while all fermions become massive and acquire the same mass $|\frac{1}{r}|$ from the contribution of one momentum mode $n=\pm 1$. Consequently, there are neither massive nor massless multiplets, which is expected in a non-supersymmetric theory. This will play a prominent role in the next chapter where we discuss the one-loop cosmological constant.

Finally, we can build Kaluza Klein towers by adding an even number of momentum modes to all states, both massless and massive. This will give an additional contribution of $|\frac{2k}{r}|$ to the mass of each state. If we associate the orbifold radius r, with the Scherk-Schwarz radius R, by $r=\frac{R}{2}$, the whole spectrum that we find here matches the supergravity one.

Chapter 5

Cosmological constant

In the previous chapter we studied orbifold compactifications of type IIB string theory, giving Minkowski vacua preserving partial, or no supersymmetry in five dimensions. We constructed the lightest states and we saw that the spectra of the various theories agreed precisely with the ones found from the Scherk-Schwarz reduction on the level of supergravity. In addition, we discussed how bosons and fermions fit in supermultiplets in supersymmetric theories, and we noticed that this was not the case for the $\mathscr{N} = 0$ theory. This bose-fermi degeneracy is a consequence of supersymmetry and it is not expected in a non-supersymmetric theory.

In this chapter, we wish to investigate how supersymmetry, or the absence of it, determines the one-loop vacuum energy density, namely the one-loop cosmological constant Λ. We examine this subject both from the supergravity, as well as form the full string theory point of view.

5.1 Supergravity calculation

In principle, the vacuum energy density can be determined by the minimum of the scalar potential. As we discussed in section 3.3, classically, the Scherk-Schwarz potential does not generate a cosmological constant. However, in one-loop approximation the (one-loop) cosmological constant, which is now determined by the minimum of the one-loop *effective* potential, may be non-vanishing. Furthermore, the effective potential can be expressed in terms of various *supertraces* over the masses of particles in the spectrum of the theory, which was first shown by Coleman and Weinberg in [43]. So, in order to determine the effective potential we have to compute supertraces.

Regarding supersymmetric theories, Zumino demonstrated that all supertraces vanish on Minkowski vacua [44]. For non-supersymmetric theories the situation is more complicated, due to the absence of bose-fermi degeneracy. However, for a non-supersymmetric theory it was shown in [45] that the second-order supertrace is vanishing. In what follows, we also want to compute supertraces in a non-supersymmetric supergravity theory, and we perform explicit supertrace calculations. In addition, we confirm that all supertraces vanish identically for theories preserving some (or all) supersymmetry.

Supertraces are defined as weighted sums over the masses of all particles in the spectrum of the theory

$$\mathrm{Str} M^{2\beta} = \sum_{\text{states } i} (-1)^{F_i} (M_i)^{2\beta}, \tag{5.1.1}$$

where F_i denotes the fermion number and $\beta \geq 0$ is an integer. It is important to emphasize here that each state has to be multiplied by the corresponding number of degrees of freedom. Therefore, the definition (5.1.1) can be written explicitly as

$$\mathrm{Str} M^{2\beta} = N_\phi (M_\phi)^{2\beta} - 2N_\chi (M_\chi)^{2\beta} + 3N_{B_{\mu\nu}} (M_{B_{\mu\nu}})^{2\beta} + 4N_{A_\mu} (M_{A_\mu})^{2\beta} - 6N_{\psi_\mu} (M_{\psi_\mu})^{2\beta}, \tag{5.1.2}$$

where by N_i we denote the number of particles with mass M_i and in (5.1.2) we considered the degrees of freedom for massive fields in five dimensions. Note that if we want to calculate $\mathrm{Str} M^0 \equiv \mathrm{Str}\mathbf{1}$, we also have to take into account the massless fields of the spectrum as follows

$$\begin{aligned} \mathrm{Str}\mathbf{1} = &\Big(N_\phi - 2N_\chi + 3N_{B_{\mu\nu}} + 4N_{A_\mu} - 6N_{\psi_\mu} \Big) \\ &+ \Big(N_\phi - 2N_\chi + 3N_{B_{\mu\nu}} + 3N_{A_\mu} - 4N_{\psi_\mu} + 5 \Big) . \end{aligned} \tag{5.1.3}$$

Here, the first line is derived from (5.1.2) by setting $\beta = 0$, and in the second line we have written down the massless fields multiplied by their degrees of freedom. The factor of 5 corresponds to the number of degrees of freedom of the graviton in five dimensions. Also, recall that a massless tensor is dual to a massless vector in five dimensions, and consequently, these fields have the same number of degrees of freedom. Finally, we observe that $\mathrm{Str}\mathbf{1}$ merely counts the difference between the bosonic and fermionic degrees of freedom.

Let us now present an example of supertrace calculation. Consider a $\mathscr{N} = 4$ (0,2) theory, with $m_1, m_2 \neq 0$ and $m_3 = m_4 = 0$ as follows from Table 3.1. Recall here that the fields that do not acquire mass from the Scherk-Schwarz twist on the circle will be reduced from six to five dimensions in the usual Kaluza-Klein fashion. We find that the massless spectrum of this theory consists of the graviton, 4 gravitini, 7 vectors, 8 dilatini and 6 scalars. These fields form one gravity multiplet and one vector multiplet. The massive fields are listed in Table 5.1.

Scalars	Vectors	Tensors	Gravitini	Dilatini
$10\,\lvert m_1+m_2\rvert$	$8\,\lvert m_1\rvert$	$2\,\lvert m_1+m_2\rvert$	$2\,\lvert m_1\rvert$	$8\,\lvert m_1+m_2\rvert$
$10\,\lvert m_1-m_2\rvert$	$8\,\lvert m_2\rvert$	$2\,\lvert m_1-m_2\rvert$	$2\,\lvert m_2\rvert$	$8\,\lvert m_1-m_2\rvert$
				$10\,\lvert m_1\rvert$
				$10\,\lvert m_2\rvert$

TABLE 5.1: *Massive field content for $\mathcal{N}=4$ (0,2) supergravity in five dimensions. We present the number of massive fields and the values $\mu(m_i)$ corresponding to these fields. The actual mass that a field acquires is given by $\mu(m_i)/2\pi R$. These fields make up four massive spin-$\frac{3}{2}$ multiplets and two massive tensor multiplets.*

Using (5.1.3) we immediately see that we have 128 bosonic and 128 fermionic degrees of freedom, which yields $\mathrm{Str}\mathbf{1}=0$. From (5.1.2) we get

$$\begin{aligned}\mathrm{Str}M^{2\beta} &= 16|m_1+m_2|^{2\beta}+16|m_1-m_2|^{2\beta}+32|m_1|^{2\beta}+32|m_2|^{2\beta}\\ &\quad -\left(16|m_1+m_2|^{2\beta}+16|m_1-m_2|^{2\beta}+32|m_1|^{2\beta}+32|m_2|^{2\beta}\right)=0,\quad \forall\beta>0.\end{aligned} \tag{5.1.4}$$

It is clear that that all supertraces vanish for every value of $\beta\geq 0$. This is a consequence of bose-fermi degeneracy and this result holds in every theory that preserves some (or all) supersymmetry, where all fields fit in supermultiplets. Indeed, if we repeat the same steps as above, we find that for $\mathcal{N}=8,6,4,2$ theories[1]

$$\mathrm{Str}M^{2\beta}=0,\qquad \forall\beta\geq 0. \tag{5.1.5}$$

Let us now study a theory preserving no supersymmetry in five dimensions. In order to obtain $\mathcal{N}=0$ supergravity, we take all mass parameters in Table 3.1 non-zero[2]. The massless and massive fields of this theory do not fit in multiplets. However, there is an equal number of bosonic and fermionic degrees of freedom (128), which means that $\mathrm{Str}\mathbf{1}$ vanishes. In addition, we compute higher order supertraces and we find

$$\mathrm{Str}M^2=\mathrm{Str}M^4=\mathrm{Str}M^6=0. \tag{5.1.6}$$

Note that this result is non-trivial, since in theories without supersymmetry there is no reason for these supertraces to vanish. We continue with our calculations and we find that the first non-vanishing supertrace is $\mathrm{Str}M^8$. It reads

$$\mathrm{Str}M^8=40320\,(m_1\,m_2\,m_3\,m_4)^2\,, \tag{5.1.7}$$

[1]The spectra of $\mathcal{N}=8,6,2$ theories are given in appendix B.1.
[2]For the complete spectrum of $\mathcal{N}=0$ theory see appendix B.2.

which is positive definite. The sign of $\mathrm{Str}M^8$ is of major importance because we expect it to determine the sign of the one-loop vacuum energy density. This is indeed the case for a four dimensional $\mathcal{N}=0$ supergravity theory examined in [46], where it is shown that for $\mathrm{Str}M^2=\mathrm{Str}M^4=\mathrm{Str}M^6=0$ and $\mathrm{Str}M^8>0$ the one-loop effective potential is negative definite, corresponding to a negative one-loop cosmological constant. In addition, we study supergravity as the low-energy limit of string theory and, as we shall see in the following, in superstring theory there is an explicit relation between the one-loop cosmological constant and $\mathrm{Str}M^8$; $\Lambda \propto -\mathrm{Str}M^8$ (cf. 5.2.2). So, we can see in these two cases that the one-loop cosmological constant and $\mathrm{Str}M^8$ have opposite signs. Based on the above, we also expect that the one-loop cosmological constant in our case, namely the five dimensional $\mathcal{N}=0$ supergravity theory, is negative.

5.1.1 Kaluza-Klein contributions

So far, we have studied supergravity spectra arising in five dimensions from Scherk-Schwarz reduction on S^1. Let us now consider a compactification on the circle. In this case the mass of the n'th Kaluza-Klein mode is given by (cf. 3.3.9)

$$\left|\frac{\mu(m_i)}{2\pi R}+\frac{n}{R}\right|, \quad n\in\mathbb{Z}\,. \tag{5.1.8}$$

We can see that our previous analysis is a truncation of (5.1.8) to the $n=0$ mode. In what follows, we want to generalize the discussion of section 5.1 by including the whole Kaluza-Klein towers in the supertrace calculations and examine how this alters our results.

Regarding $\mathcal{N}=2,4,6$ and 8 theories, Kaluza-Klein towers do not affect our previous supertrace results. As we have seen, both massive and massless fields in theories preserving some (or all) supersymmetry fit in supermultiplets. Hence, there is an exact multiplet by multiplet supertrace cancellation. On the other hand, for the $\mathcal{N}=0$ theory we show below that we find different results.

Practically we perform the same calculation as in section 5.1, but for an arbitrary $n\neq 0$ Kaluza-Klein mode, instead of the $n=0$ mode. In order to carry out the computation, we have to treat carefully some subtle details. Fist of all, we saw in section 3.3.1 that the graviton, 2 tensors and 1 scalar were not affected by the Scherk-Schwarz twist on the circle and acquired no mass. However, after compactification on S^1 these fields will pick up infinite Kaluza-Klein towers and they will acquire a mass $\frac{n}{R}$. In addition, the masses of all fields in Table 3.1 will be shifted by $\frac{n}{R}$ (see appendix B.3). Keeping all these in mind we compute $\mathrm{Str}M^{2\beta}$ for the n'th Kaluza-Klein mode. We find that the first non-vanishing supertrace is $\mathrm{Str}M^8$, yielding exactly the same answer as in (5.1.7), which is independent of n. Nevertheless, this is not the whole story, because each Kaluza-Klein mode will give the same constant result. Therefore we have to perform an infinite sum of the form

$$\sum_{n\in\mathbb{Z}} 40320\,(m_1 m_2 m_3 m_4)^2 \equiv \sum_{n\in\mathbb{Z}} \mathrm{A}. \tag{5.1.9}$$

We perform this sum using zeta function regularization. The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{5.1.10}$$

Using the definition (5.1.10) we find from (5.1.9)

$$\sum_{n\in\mathbb{Z}} \mathrm{A} = \mathrm{A}\left(\sum_{n>0} n^0 + \sum_{n<0} n^0 + 1\right) = \mathrm{A}\left(2\sum_{n>0} n^0 + 1\right) = \mathrm{A}\,[2\,\zeta(0) + 1] = 0, \tag{5.1.11}$$

where we used that $\zeta(0) = -\frac{1}{2}$. Therefore, we observe that Kaluza-Klein towers can make the final result vanishing. Of course, this answer is valid in the particular regularization scheme we used here, i.e. zeta function regularization, and we can not be sure if it holds in any other regularization scheme. Hence, we should be cautious with this answer, because an observable, such as the cosmological constant, must certainly not depend on the regularization procedure. In addition, for a more rigorous result, one could also consider a compactification on T^4 and take into account the Kaluza-Klein modes coming from the torus.

5.2 String theory perspective

In this section we wish to study the one-loop vacuum energy density from the full string theory point of view[3]. Once again we start our discussion with supertraces. Contrary to supergravity, in string theory there are *infinitely* many states. In order to handle this infinity, we introduce a regulator and we define the *regulated* string supertrace as [47, 48]

$$\mathrm{Str}M^{2\beta} \equiv \lim_{\gamma\to 0} \sum_{\text{states } i} (-1)^{F_i} (M_i)^{2\beta} e^{-\gamma M_i^2}, \tag{5.2.1}$$

where the regulator γ ensures that the sum over states will be convergent. At this point, we wish to highlight an exceptional expression relating explicitly the one-loop

[3] Although we are interested in studying type IIB string theory, we begin with a general discussion.

sting-theoretic cosmological constant Λ_{string} with supertraces [49]

$$\mathrm{Str} M^{D-2} = 6(-4\pi)^{D/2}(D/2-1)!\,\frac{\Lambda_{string}}{\alpha'}. \tag{5.2.2}$$

This relation is valid for any tachyon-free closed string living in an even number D of spacetime dimensions. In a string theory preserving partial or all supersymmetry there is an exact bose-fermi degeneracy in all mass levels. As a consequence, all supertraces vanish and we immediately conclude that $\Lambda_{string} = 0$. This is the same situation as with the supergravity theories $\mathscr{N} = 8, 6, 4, 2$ that we discussed in the previous section. Supersymmetry guarantees the vanishing of all supertraces and subsequently the vanishing of the one-loop cosmological constant.

Let us now consider a type IIB string theory preserving no supersymmetry. In section 4.5 we discussed such a model and we constructed the massless and the lightest massive states in the untwisted sector. Of course on top of these states we can also add momentum and/or winding modes. However, for the full string theory this is merely a small portion of the full spectrum[4]. Therefore, we have insufficient information to calculate supertraces, and unfortunately we can not draw a conclusion about the value of Λ_{string} from equation (5.2.2).

Nevertheless, there is another way of calculating the D-dimensional one-loop string cosmological constant, that is via the one-loop string partition function $Z(\tau)$

$$\Lambda^{(D)}_{string} \equiv -\frac{1}{2}\mathscr{M}^D \int_{\mathscr{F}} \frac{d^2\tau}{(\tau_2)^2}\, Z(\tau), \tag{5.2.3}$$

where $\mathscr{F} \equiv \{\tau : |\tau|^2 \geq 1, Im\tau > 0, |Re\tau| \leq 1/2\}$ is the fundamental domain of the modular group and $\tau = \tau_1 + i\tau_2$. The reduced string scale is $\mathscr{M} = M_{string}/2\pi$. Naturally, the full partition function $Z(\tau)$ for our orbifold constructions can be factorized as

$$Z(\tau) = Z(\tau)_{\mathbb{R}^{1,4}} \times Z(\tau)_{(S^1\times T^4)/Z_p} \tag{5.2.4}$$

For strings preserving some (or all) supersymmetry one can show that $Z(\tau) = 0$ [49], which implies that Λ_{string} vanishes[5]. However, the complete calculation of the partition function for strings preserving no supersymmetry is beyond of the scope of this thesis.

We should mention here that there have been many efforts to construct non-supersymmetric orbifolds with a vanishing one-loop cosmological constant [50–58]. The crucial feature

[4] For the full spectrum we have to take into account an infinite number of excited states, twisted sectors, as well as Kaluza-Klein and winding modes on the T^4.

[5] Note that this result is consistent with our supertrace analysis, where we also found $\Lambda_{string} = 0$.

of these models is bose-fermi degeneracy in the absence of supersymmetry, and this is achieved in asymmetric orbifolds. Finally, motivated from these attempts, we tried to construct a non-supersymmetric orbifold with an equal number of bosonic and fermionic degrees of freedom, but it turns out that in our set-up this is impossible, even in the massless level.

Chapter 6

Conclusion

In this thesis we studied orbifold compactifications of type IIB string theory. We demonstrated that the Scherk-Schwarz supergravity theories constructed in [7] can be embedded to string theories compactified on freely acting orbifolds of the form $(S^1 \times T^4)/Z_p$, only if the orbifold action is conjugate to an element of the T-duality group $SO(4,4,\mathbb{Z})$, which imposes a quantization condition on the four mass parameters $m_i, i = 1, \ldots, 4$.

In particular, we constructed a symmetric Z_3 orbifold and an asymmetric Z_4 orbifold preserving half of the supersymmetry, as well as a symmetric Z_2 orbifold breaking all supersymmetry. Also, we saw that the spectra of these theories (lightest states and Kaluza-Klein towers) agreed precisely with the spectra that we found for the Scherk-Schwarz reduction on the level of supergravity.

In addition, we performed explicit calculations of various supertraces, and we confirmed that all supertraces vanish identically in any supersymmetric supergravity theory due to an exact bose-fermi degeneracy, indicating that the one-loop cosmological constant is vanishing. The same holds true for supersymmetric string theories. For a non-supersymmetric $\mathcal{N} = 0$ supergravity theory we found $\mathrm{Str}M^2 = \mathrm{Str}M^4 = \mathrm{Str}M^6 = 0$ and $\mathrm{Str}M^8 > 0$, which we expect to give rise to a negative one-loop cosmological constant. Unfortunately, we were not able to draw a conclusion regarding the one-loop cosmological constant in a non-supersymmetric string theory.

Consequently, a future research is the calculation of the one-loop string-theoretic cosmological constant by directly computing the one-loop string partition function. In addition, the construction of a non-supersymmetric orbifold with equal number of bosonic and fermionic degrees of freedom in the massless level, which may lead to a vanishing one-loop cosmological constant, could be another challenging project.

Furthermore, from the supergravity point of view, we would like to understand better the relation between the one-loop effective potential and $\mathrm{Str}M^8$ in five dimensions, and determine if a negative one-loop cosmological constant is indeed generated in the 5D $\mathcal{N}=0$ theory. Another open question is to apply a more rigorous method of computing the Kaluza-Klein contributions (coming both from the circle and from the torus) in the supertrace calculations.

Regarding orbifolds, a follow-up project is to study the twisted sectors in both symmetric and asymmetric constructions because for the full string theory description both the untwisted and twisted sectors have to be taken into account. Moreover, string theory besides strings contains D-branes, hence we would also like to examine their behavior on orbifolds. One of the most interesting aspects of D-branes is that they can be used for the description of black holes in the context of string theory and, as we also stated in the beginning of this thesis, the main motivation for constructing the model type IIB in $T^4 \times S^1$ is exactly to study black holes in this set-up, which was also the main purpose of [7]. Inspired by this paper we would like to extend our work and study black holes in the context of compactifications of type IIB string theory on a $(S^1 \times T^4)/Z_p$ orbifold.

Appendix A

Group theory

In this appendix we present some basic properties of the group $O(d,d;\mathbb{R})$ and its subgroup $O(d;\mathbb{R})\times O(d;\mathbb{R}) \subset O(d,d;\mathbb{R})$. We also discuss how the T-duality group $O(d,d;\mathbb{Z})$ acts on the left- and right-moving momenta $P_{\mathscr{L}/\mathscr{R}}$ and on the left- and right-moving oscillators $\tilde{a}_n, a_n$. Finally, we demonstrate that the orbifold action that we discussed in section 4.2 is conjugate to an element of the T-duality group.

A.1 The groups $O(d,d;\mathbb{R})$ and $O(d;\mathbb{R}) \times O(d;\mathbb{R})$

Consider an element $g \in O(d,d;\mathbb{R})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad g^t = \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} , \tag{A.1.1}$$

where the superscript t denotes a transpose matrix. By definition, g is a $2d \times 2d$ matrix, hence a, b, c, d are $d \times d$ matrices. In addition, g satisfies

$$g^t \tau g = \tau \, , \qquad \tau = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} , \tag{A.1.2}$$

where 1_d denotes a $d \times d$ unit matrix. Since the group element g preserves the matrix τ, we say that it is written in the τ-frame. The definition (A.1.2) implies that

$$a^t\, c + c^t\, a = b^t\, d + d^t\, b = 0 \, , \; a^t\, d + c^t\, b = 1_d \, . \tag{A.1.3}$$

From the transpose matrix g^t, which also belongs in $O(d,d;\mathbb{R})$ and satisfies $g\,\tau\,g^t=\tau$, we find the relations

$$a\,b^t+b\,a^t=c\,d^t+d\,c^t=0\,,\ a\,d^t+b\,c^t=1_d\,. \tag{A.1.4}$$

From these expressions we can also find the inverse matrix g^{-1}, which is given by

$$g^{-1}=\begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix}. \tag{A.1.5}$$

There is a second basis in which we can write down the group $O(d,d;\mathbb{R})$, that is the η-frame. An element $\tilde{g}\in O(d,d;\mathbb{R})$ in the η-frame satisfies

$$\tilde{g}^t\,\eta\,\tilde{g}=\eta\,,\qquad \eta=\begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix}. \tag{A.1.6}$$

The matrix $\tilde{g}$ can be found by conjugation, as $\tilde{g}=X^{-1}\,g\,X$, with $X=X^t=X^{-1}$ given by

$$X=\frac{1}{\sqrt{2}}\begin{pmatrix} 1_d & 1_d \\ 1_d & -1_d \end{pmatrix}. \tag{A.1.7}$$

Thus, we find that $\tilde{g}$ reads

$$\tilde{g}=\frac{1}{2}\begin{pmatrix} a+b+c+d & a-b+c-d \\ a+b-c-d & a-b-c+d \end{pmatrix}. \tag{A.1.8}$$

Note that the matrix $\tilde{g}$ preserves the matrix η instead of the matrix τ.

There is a subgroup $O(d;\mathbb{R})\times O(d;\mathbb{R})\subset O(d,d;\mathbb{R})$ that, in the η-frame, is naturally embedded diagonally in elements of $O(d,d;\mathbb{R})$. For example, from (A.1.8) this can be achieved by taking $a=d$ and $b=c$. Consequently, an element $\tilde{h}\in O(d;\mathbb{R})\times O(d;\mathbb{R})$, in the η-frame, takes the form

$$\tilde{h}=\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}=\begin{pmatrix} d+c & 0 \\ 0 & d-c \end{pmatrix}. \tag{A.1.9}$$

Note that the matrices $(a\pm b)$ and $(d\pm c)$ are $d\times d$ matrices $\in O(d;\mathbb{R})$. Conjugating (A.1.9) with X, we find the form of an element $h\in O(d;\mathbb{R})\times O(d;\mathbb{R})$ in the τ-frame

$$h=\begin{pmatrix} a & b \\ b & a \end{pmatrix}=\begin{pmatrix} d & c \\ c & d \end{pmatrix}, \tag{A.1.10}$$

with $c^t\, d + d^t\, c = 0\,, d^t\, d + c^t\, c = 1_n$, (or $b^t\, a + a^t\, b = 0\,, a^t\, a + b^t\, b = 1_n$), as follows from (A.1.4). Finally, note that for element $\hat{g} \in SO(d,d;\mathbb{R})$ (or $SO(d;\mathbb{R}) \times SO(d;\mathbb{R})$) the above discussion is valid, with the additional constraint that $\det(\hat{g}) = 1$.

A.2 T-duality action

A.2.1 Transformation of momentum

In this section we wish to derive the transformation of the left- and right-moving momenta $P_{\mathscr{L}/\mathscr{R}}$ under the T-duality group $O(d,d;\mathbb{Z})$. First of all, we need to find a way of expressing the T-duality action. We define the matrix $\mathscr{G}(E)$

$$\mathscr{G}(E) = \begin{pmatrix} G - B\,G^{-1}B & B\,G^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \tag{A.2.1}$$

The matrix $\mathscr{G}(E)$ depends on the background matrix $E = G + B$, but for convenience we will not write down explicitly this dependence. We can also write $\mathscr{G}$ in terms of $d \times d$ matrices as

$$\mathscr{G} = \begin{pmatrix} \mathscr{G}_{11} & \mathscr{G}_{12} \\ \mathscr{G}_{21} & \mathscr{G}_{22} \end{pmatrix} , \qquad \begin{aligned} \mathscr{G}_{11} &= G - BG^{-1}B, & \mathscr{G}_{12} &= BG^{-1}, \\ \mathscr{G}_{21} &= -G^{-1}B, & \mathscr{G}_{22} &= G^{-1}. \end{aligned} \tag{A.2.2}$$

We combine momentum (p_i) and winding (w^i) numbers in a $2d$-column vector as

$$Z = \begin{pmatrix} w^i \\ p_i \end{pmatrix} . \tag{A.2.3}$$

T-duality acts on the background fields G_{ij}, B_{ij} and on momentum and winding modes, in such a way that it leaves the mass spectrum of the string unchanged. This action can be expressed as [23]

$$\mathscr{G} \to g \mathscr{G}\, g^t\,, \qquad Z \to (g^{-1})^t\, Z\,, \tag{A.2.4}$$

where g is an element of $O(d,d;\mathbb{Z})$ written in τ-frame. Having said all this, we can now find how $P_{\mathscr{L}/\mathscr{R}}$ transform under T-duality. The expressions for $P_{\mathscr{L}/\mathscr{R}}$ are given by

$$\begin{aligned} P^i_{\mathscr{L}}(E) &= w^i + G^{ij}\left(p_j - B_{jk} w^k\right) , \\ P^i_{\mathscr{R}}(E) &= -w^i + G^{ij}\left(p_j - B_{jk} w^k\right) . \end{aligned} \tag{A.2.5}$$

These expressions can be written in terms of the matrix $\mathscr{G}$ as

$$\begin{aligned} P^i_{\mathscr{L}}(E) &= w^i + (\mathscr{G}_{22})^{ij}\, p_j + (\mathscr{G}_{21})^i{}_j\, w^j \,, \\ P^i_{\mathscr{R}}(E) &= -w^i + (\mathscr{G}_{22})^{ij}\, p_j + (\mathscr{G}_{21})^i{}_j\, w^j \,. \end{aligned} \tag{A.2.6}$$

So, it suffices to find the transformation of p_i, w^i, $\mathscr{G}_{21}$ and $\mathscr{G}_{22}$. From (A.2.4) we get (we omit the matrix indices for convenience)

$$\begin{aligned} \mathscr{G}_{21} \to \mathscr{G}'_{21} &= c\,\mathscr{G}_{11}\, a^t + c\,\mathscr{G}_{12}\, b^t + d\,\mathscr{G}_{21}\, a^t + d\,\mathscr{G}_{22}\, b^t \,, \\ \mathscr{G}_{22} \to \mathscr{G}'_{22} &= c\,\mathscr{G}_{11}\, c^t + c\,\mathscr{G}_{12}\, d^t + d\,\mathscr{G}_{21}\, c^t + d\,\mathscr{G}_{22}\, d^t \,. \end{aligned} \tag{A.2.7}$$

For momentum and winding we have

$$\begin{aligned} w \to w' &= d\, w + c\, p \,, \\ p \to p' &= b\, w + a\, p \,. \end{aligned} \tag{A.2.8}$$

Another pair of useful relations is

$$\begin{aligned} & P^i_{\mathscr{L}}(E) - P^i_{\mathscr{R}}(E) = 2\, w^i \,, \\ & G_{ij}[P^j_{\mathscr{L}}(E) + P^j_{\mathscr{R}}(E)] + B_{ij}[P^j_{\mathscr{L}}(E) - P^j_{\mathscr{R}}(E)] = 2\, p_i \,. \end{aligned} \tag{A.2.9}$$

Combining all these together, we find that under T-duality $P_{\mathscr{L}/\mathscr{R}}(E)$ transform as

$$\left.\begin{aligned} P_{\mathscr{L}}(E) \to P'_{\mathscr{L}}(E') &= w' + \mathscr{G}'_{22}\, p' + \mathscr{G}'_{21} w' \,, \\ P_{\mathscr{R}}(E) \to P'_{\mathscr{R}}(E') &= -w' + \mathscr{G}'_{22}\, p' + \mathscr{G}'_{21} w' \end{aligned}\right\} \implies \tag{A.2.10}$$

$$\begin{aligned} P'_{\mathscr{L}}(E') &= (d + c\, E)\, P_{\mathscr{L}}(E) \,, \\ P'_{\mathscr{R}}(E') &= \left(d - c\, E^t\right) P_{\mathscr{R}}(E) \,. \end{aligned} \tag{A.2.11}$$

where c, d are integer $d \times d$ matrices.

A.2.2 Transformation of oscillators

In this section we want to confirm that the transformation of the left- and right-moving oscillators $\tilde{a}_n, a_n$ under the T-duality group is the same as the transformation of $P_{\mathscr{L}/\mathscr{R}}$. The derivation of the transformation of $\tilde{a}_n, a_n$ is adjusted from [23] (cf. section 4). Thus, it is helpful to start our discussion by presenting some useful formulas. First, we write down the expressions for the expansion of the bosonic coordinates $Y^i(\sigma)$ and of the conjugate momentum $P_i(\sigma)$ at $\tau = 0$ [1]. These read

$$\begin{aligned} Y^i(\sigma) &= y^i + w^i\sigma + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}[a_n^i(E)\,e^{in\sigma} + \tilde{a}_n^i(E)\,e^{-in\sigma}]\,, \\ 2\pi P_i(\sigma) &= p_i + \frac{1}{\sqrt{2}}\sum_{n\neq 0}[E_{ij}^t\,a_n^j(E)\,e^{in\sigma} + E_{ij}\,\tilde{a}_n^j(E)\,e^{-in\sigma}]\,. \end{aligned} \tag{A.2.12}$$

Note that the oscillators depend on the background matrix E, while $Y^i(\sigma)$ and $P_i(\sigma)$ are background independent. As stated in [23], this means that we can expand a string theory written around some background, using oscillators that correspond to a different background. Consider a background E and an element $g \in O(d,d;\mathbb{R})$ as given in (A.1.1). A new background E' can be obtained by acting with g on the background E as follows[2]

$$E' = g(E) = (a\,E + b)(c\,E + d)^{-1}\,. \tag{A.2.13}$$

The expression for $(E^t)'$ can be found by solving (A.2.13) for E, transposing, and taking the inverse. This yields

$$(E^t)' = (a\,E^t - b)(-c\,E^t + d)^{-1}, \tag{A.2.14}$$

Using the relations between the backgrounds E, E', and by writing $2\,G = E + E^t$, one can prove the validity of the following two expressions

$$\begin{aligned} (a^t + c^t\,E')^t\,G &= G'\,(a^t + c^t\,E')^{-1}\,, \\ (d + c\,E)^t\,G' &= G\,(d + c\,E)^{-1}\,. \end{aligned} \tag{A.2.15}$$

Now, we have all the necessary ingredients to actually start the derivation. First, we take a derivative of $Y^i(\sigma)$ with respect to σ, and rewrite (A.2.12) as

$$\begin{aligned} \partial_\sigma Y^i &= w^i + \frac{1}{\sqrt{2}}\sum_{n\neq 0}[\tilde{a}_{-n}^i(E) - a_n^i(E)]\,e^{in\sigma}\,, \\ 2\pi P_i &= p_i + \frac{1}{\sqrt{2}}\sum_{n\neq 0}[E_{ij}^t\,a_n^j(E) + E_{ij}\,\tilde{a}_{-n}^j(E)]\,e^{in\sigma}\,. \end{aligned} \tag{A.2.16}$$

Now, we make the following replacement (we omit the matrix indices for convenience)

$$\begin{pmatrix} \partial_\sigma Y \\ 2\pi P \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} \partial_\sigma Y \\ 2\pi P \end{pmatrix}. \tag{A.2.17}$$

[1] Note that the conjugate momentum P_i is completely different from the left- and right-moving momentum $P_{\mathscr{L}/\mathscr{R}}$.

[2] This definition is consistent with the group property $g(g'(E)) = g\,g'(E)$.

This map implies that w^i, p_i transform as

$$\begin{pmatrix} w \\ p \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix} , \tag{A.2.18}$$

which matches the action of T-duality on momentum (p_i) and winding (w^i) numbers only if the matrices a, b, c, d are integers (cf. A.2.4). So, in the following we will consider that these matrices take integer values. In order to find how the oscillators transform under the above replacement, we have to specify the backgrounds on the left- and right-hand side of (A.2.17) because the oscillators are background dependent. We choose a background E' for the LHS and a background E for the RHS of (A.2.17)

$$\begin{pmatrix} \partial_\sigma Y(E) \\ 2\pi P(E) \end{pmatrix} \to \begin{pmatrix} \partial_\sigma Y'(E') \\ 2\pi P'(E') \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} \partial_\sigma Y(E) \\ 2\pi P(E) \end{pmatrix} . \tag{A.2.19}$$

This map implies

$$\begin{aligned} &\tilde a_{-n}(E) - a_n(E) \to \tilde a'_{-n}(E') - a'_n(E') = d[\tilde a_{-n}(E) - a_n(E)] + c[E^t a_n(E) + E\tilde a_{-n}(E)] \\ &E^t a_n(E) + E\tilde a_{-n}(E) \to (E^t)' a'{}_n(E') - E'\tilde a'_{-n}(E') = b[\tilde a_{-n}(E) - a_n(E)] + a[E^t a_n(E) + E\tilde a_{-n}(E)] \end{aligned} \tag{A.2.20}$$

Let us now focus on the left movers. We multiply the first line in (A.2.20) with $(E^t)'$ and then we add the resulting expression with the second one. This gives

$$\begin{aligned} 2\,G'\,\tilde a'_{-n}(E') &= [(E^t)'(d + c\,E) + (b + a\,E)]\,\tilde a_{-n}(E) \\ &\quad + [(E^t)'(-d + c\,E^t) + (-b + a\,E^t)\,]a_n(E) \end{aligned} \tag{A.2.21}$$

Using that $(E^t)' = (a\,E^t - b)(-c\,E^t + d)^{-1}$ and $2\,G = E + E^t$, we find

$$\begin{aligned} G'\,\tilde a'_{-n}(E') &= [a + (E^t)'\,c]\,G\,\tilde a_{-n}(E) \\ G'\,\tilde a'_{-n}(E') &= (a^t + c^t\,E')^t\,G\,\tilde a_{-n}(E) \\ \tilde a'_{-n}(E') &= \,(a^t + c^t\,E')^{-1}\,\tilde a_{-n}(E) \end{aligned} \tag{A.2.22}$$

where in the last step we used that $(a^t + c^t\,E')^t\,G \;= G'\,(a^t + c^t\,E')^{-1}$. This transformation is actually the same as the transformation of $P_{\mathscr{L}}$ that we found in (A.2.11) because $(d + c\,E) = (a^t + c^t\,E')^{-1}$. This can be proven as follows

$$\begin{aligned} &(d + c\,E)(a^t + c^t\,E') = 1 \\ &(d + c\,E)[a^t + c^t\,(a\,E + b)(c\,E + d)^{-1}] = 1 \\ &(d + c\,E)[a^t\,c\,E + a^t\,d + c^t\,a\,E + c^t\,b] = (c\,E + d) \\ &(d + c\,E)[\underbrace{(a^t\,d + c^t\,b)}_{1} + \underbrace{(a^t\,c + c^t\,a)}_{0}\,E] = (c\,E + d)\,, \end{aligned} \tag{A.2.23}$$

where in the last line we used (A.1.4). The derivation of the transformation of the right-moving oscillators follows in a similar way and we find that

$$\begin{aligned} a_n(E) \to a'_n(E') &= [a^t - c^t\,(E^t)']^{-1}\,a_n(E) \\ a'_n(E') &= (d - c\,E^t)\,a_n(E)\,, \end{aligned} \tag{A.2.24}$$

which coincides with the transformation of $P_{\mathscr{R}}$.

A.3 Orbifold action

In this section we wish to demonstrate that the orbifold action discussed in section 4.2

$$\begin{aligned} W^1_{\mathscr{L}} &\to e^{i(m_1+m_3)} W^1_{\mathscr{L}} \\ W^1_{\mathscr{R}} &\to e^{i(m_2+m_4)} W^1_{\mathscr{R}} \\ W^2_{\mathscr{L}} &\to e^{i(m_1-m_3)} W^2_{\mathscr{L}} \\ W^2_{\mathscr{R}} &\to e^{i(m_2-m_4)} W^2_{\mathscr{R}}\,, \end{aligned} \tag{A.3.1}$$

is an element of the T-duality group $O(4,4;\mathbb{Z})$. Actually, it is sufficient to show that the orbifold action is *conjugate* to an element of $O(4,4;\mathbb{Z})$, since the conjugated element defines an equivalent theory. We start our discussion by considering that the mass parameters in (A.3.1) can take any real value. However, we will see that in order to conjugate the orbifold action to an element of the T-duality group, we have to impose a quantization condition on the mass parameters $m_i, i = 1, \ldots, 4$.

Using that $W^i = W^i_{\mathscr{L}} + W^i_{\mathscr{R}}$ and $W^i = \frac{1}{\sqrt{2}}(Y^{2i-1} + iY^{2i}), i = 1, 2$, we can rewrite (A.3.1) in terms of the real torus coordinates as

$$\begin{pmatrix} Y^1_{\mathscr{L}} \\ Y^2_{\mathscr{L}} \\ Y^3_{\mathscr{L}} \\ Y^4_{\mathscr{L}} \\ Y^1_{\mathscr{R}} \\ Y^2_{\mathscr{R}} \\ Y^3_{\mathscr{R}} \\ Y^4_{\mathscr{R}} \end{pmatrix} \longrightarrow \begin{pmatrix} \cos m & -\sin m & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin m & \cos m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\bar{m} & -\sin\bar{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\bar{m} & \cos\bar{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\mu & -\sin\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\mu & \cos\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\bar{\mu} & -\sin\bar{\mu} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\bar{\mu} & \cos\bar{\mu} \end{pmatrix} \begin{pmatrix} Y^1_{\mathscr{L}} \\ Y^2_{\mathscr{L}} \\ Y^3_{\mathscr{L}} \\ Y^4_{\mathscr{L}} \\ Y^1_{\mathscr{R}} \\ Y^2_{\mathscr{R}} \\ Y^3_{\mathscr{R}} \\ Y^4_{\mathscr{R}} \end{pmatrix}, \tag{A.3.2}$$

where $m = m_1 + m_3$, $\bar{m} = m_1 - m_3$, $\mu = m_2 + m_4$ and $\bar{\mu} = m_2 - m_4$. This action can be written in terms of an 8×8 matrix $R = \mathrm{diag}(U, V)$ as

$$\begin{pmatrix} Y_{\mathscr{L}} \\ Y_{\mathscr{R}} \end{pmatrix} \to R \begin{pmatrix} Y_{\mathscr{L}} \\ Y_{\mathscr{R}} \end{pmatrix} \equiv \begin{pmatrix} Y^i_{\mathscr{L}} \\ Y^i_{\mathscr{R}} \end{pmatrix} \to \begin{pmatrix} U^i{}_j & 0 \\ 0 & V^i{}_j \end{pmatrix} \begin{pmatrix} Y^j_{\mathscr{L}} \\ Y^j_{\mathscr{R}} \end{pmatrix} . \tag{A.3.3}$$

where $i, j = 1, \dots, 4$ and

$$U = \begin{pmatrix} \cos m & -\sin m & 0 & 0 \\ \sin m & \cos m & 0 & 0 \\ 0 & 0 & \cos \bar{m} & -\sin \bar{m} \\ 0 & 0 & \sin \bar{m} & \cos \bar{m} \end{pmatrix} , \quad V = \begin{pmatrix} \cos \mu & -\sin \mu & 0 & 0 \\ \sin \mu & \cos \mu & 0 & 0 \\ 0 & 0 & \cos \bar{\mu} & -\sin \bar{\mu} \\ 0 & 0 & \sin \bar{\mu} & \cos \bar{\mu} \end{pmatrix} . \tag{A.3.4}$$

Note that $U, V \in SO(4;\mathbb{R})$ and $R \in SO(4;\mathbb{R}) \times SO(4;\mathbb{R})$, preserving the matrix η. We can rewrite the matrix R in the τ-frame using (A.1.7)

$$R = \frac{1}{2} \begin{pmatrix} U+V & U-V \\ U-V & U+V \end{pmatrix} . \tag{A.3.5}$$

Let us now consider a symmetric orbifold, acting equally on the left and right movers, i.e. $U = V$. From the above expression we can see that a symmetric action implies that the matrix R takes the same form both in the τ- and η-frame. In addition, the elements of the matrix R are real numbers. However, if the mass parameters m_i are integer multiples of $\{0, \frac{\pi}{3}, \frac{\pi}{2}\}$, the matrix R is conjugate to an *integer* matrix $\tilde{R} = g\,R\,g^{-1}$, where $g \in SO(4,4;\mathbb{R})$ and $\tilde{R} \in SO(4,4;\mathbb{Z})$, implying that the orbifold action is conjugate to an element $\tilde{R}$ of the T-duality group.

If all mass parameters $m_i \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ the conjugation is trivial, since R is already integer, and the matrix g is the unit matrix. On the other hand, if at least one mass parameter $m_i \in \{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\}$, the matrix g (written in the τ-frame) is given by

$$g = \begin{pmatrix} h & 0 \\ 0 & (h^t)^{-1} \end{pmatrix} , \qquad g^{-1} = \begin{pmatrix} h^{-1} & 0 \\ 0 & h^t \end{pmatrix} , \tag{A.3.6}$$

where

$$h = \begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} , \qquad h^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} . \tag{A.3.7}$$

Now, we consider the case of the symmetric Z_3 orbifold that we discussed in section 4.3. In this case we set $m_1 = m_2 = \frac{2\pi}{3}, m_3 = m_4 = 0 \implies m = \bar{m} = \mu = \bar{\mu} = \frac{2\pi}{3}$, and the matrix R takes the form

$$R = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} . \tag{A.3.8}$$

We find that the matrix $\tilde{R} = g\,R\,g^{-1}$ is given by

$$\tilde{R} = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} . \tag{A.3.9}$$

We can see that the matrix $\tilde{R}$ is integer, preserves the matrix τ, and det$(\tilde{R})$=1. Thus, the orbifold action is indeed conjugate to an element of the T-duality group $SO(4,4;\mathbb{Z})$.

Let us now confirm that the conjugation is trivial for the other two orbifolds that we constructed, namely the Z_4, Z_2 orbifolds. First, consider the asymmetric Z_4 orbifold, discussed in section 4.4. In this case we set $m_1 = m_3 = \frac{\pi}{2}, m_2 = m_4 = 0 \implies m = \pi, \bar{m} = \mu = \bar{\mu} = 0$, and the matrix R takes the form

$$R = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.3.10}$$

We see that in this case we immediately get an integer matrix, written in the η-frame, with $\det(R)$=1. Hence, the conjugation is trivial, with the conjugation matrix g being the unit matrix, since $R \in SO(4,4;\mathbb{Z})$.

Finally, for the symmetric Z_2 orbifold that we constructed in section 4.5 where we set all mass parameters equal to π, the matrix R is simply the identity matrix.

Appendix B

Tables

In this appendix we present the massless and massive field content for the various supergravity theories in five dimensions as follow from Table 3.1.

B.1 $\mathscr{N} = 8, 6, 2$ supergravities

In order to obtain $\mathscr{N} = 8$ supergravity we keep all mass parameters zero. We find one graviton, 8 gravitini, 27 vectors, 48 dilatini and 42 scalars. These fields form one gravity multiplet. In total we have 128 bosonic and 128 fermionic degrees of freedom, which yields $\mathrm{Str}\mathbf{1} = 0$. Since all fields are massless $\mathrm{Str}M^{2\beta}$ with $\beta > 0$ vanish identically.

For $\mathscr{N} = 6$ supergravity we take only one mass parameter non-zero. Without loss of generality we choose $m_1 \neq 0$ and $m_2 = m_3 = m_4 = 0$. The massless spectrum of this theory consists of the graviton, 6 gravitini, 15 vectors, 20 dilatini and 14 scalars. These fields form one gravity multiplet. The massive field content consists of 2 gravitini, 4 self-dual tensors, 26 dilatini, 8 vectors and 20 scalars. All these fields have the same mass and fit in a (1,2) BPS supermultiplet. Again, we observe that we have the same (128) bosonic and fermionic degrees of freedom. From (5.1.2) for $\beta > 0$ we get

$$\mathrm{Str}M^{2\beta} = 64|m_1|^{2\beta} - 64|m_1|^{2\beta} = 0\,. \;\forall\beta > 0 \tag{B.1.1}$$

Finally, in order to get $\mathscr{N} = 2$ supergravity we have to take three mass parameters non-zero. Let us pick $m_1, m_2, m_3 \neq 0$ and $m_4 = 0$ (again there is no loss of generality). The massless spectrum of this theory consists of the graviton, 2 gravitini, 3 vectors, 4 dilatini and 2 scalars. These fields make up one gravity multiplet and two vector multiplets. The massive spectrum of $\mathscr{N} = 2$ theory is given in Table B.1

Scalars	Vectors	Tensors	Gravitini	Dilatini
$4\,\lvert m_1+m_2+m_3\rvert$	$2\,\lvert m_1+m_3\rvert$	$2\,\lvert m_1+m_2\rvert$	$2\,\lvert m_1\rvert$	$2\,\lvert m_1+m_2+m_3\rvert$
$4\,\lvert m_1+m_2-m_3\rvert$	$2\,\lvert m_1-m_3\rvert$	$2\,\lvert m_1-m_2\rvert$	$2\,\lvert m_2\rvert$	$2\,\lvert m_1+m_2-m_3\rvert$
$4\,\lvert m_1-m_2+m_3\rvert$	$2\,\lvert m_2+m_3\rvert$	$4\,\lvert m_3\rvert$	$2\,\lvert m_3\rvert$	$2\,\lvert m_1-m_2+m_3\rvert$
$4\,\lvert m_1-m_2-m_3\rvert$	$2\,\lvert m_2-m_3\rvert$			$2\,\lvert m_1-m_2-m_3\rvert$
$2\,\lvert m_1+m_2\rvert$	$4\,\lvert m_1\rvert$			$4\,\lvert m_1+m_2\rvert$
$2\,\lvert m_1-m_2\rvert$	$4\,\lvert m_2\rvert$			$4\,\lvert m_1-m_2\rvert$
$4\,\lvert m_3\rvert$				$4\,\lvert m_1+m_3\rvert$
				$4\,\lvert m_1-m_3\rvert$
				$4\,\lvert m_2+m_3\rvert$
				$4\,\lvert m_2-m_3\rvert$
				$2\,\lvert m_1\rvert$
				$2\,\lvert m_2\rvert$
				$2\,\lvert m_3\rvert$

TABLE B.1: *Massive field content for $\mathcal{N} = 2$ supergravity in five dimensions. We present the number of massive fields and the values $\mu(m_i)$ corresponding to these fields. The actual mass of a field is given by $\mu(m_i)/2\pi R$.*

We find 4 hypermultiplets with masses $|m_1 \pm m_2 \pm m_3|$ and one hypermultiplet with mass $|m_3|$, each consisting of 4 scalars and 2 gravitini. There are four vector multiplets, 2 with mass $|m_1 \pm m_3|$ and 2 with $|m_2 \pm m_3|$ made up from 2 vectors and 4 dilatini each. In addition, we have 2 tensor multiplets with masses $|m_1 \pm m_2|$, each consisting of 2 self-dual tensors, 4 dilatini and 2 scalars. In addition, there is one spin-$\frac{3}{2}$ multiplet with mass $|m_1|$ and one with $|m_2|$, formed by 2 gravitini, 4 vectors and 2 dilatini. Finally, we find one multiplet of mass m_3, consisting of 2 gravitini and 4 anti-self-dual tensors.

Once again, the total bosonic and fermionic degrees of freedom are equal and consequently we find $\mathrm{Str}\mathbf{1} = 0$. For $\mathrm{Str}M^{2\beta}$ with $\beta > 0$ there is an exact cancellation between bosons and fermions inside each multiplet.

B.2 $\mathcal{N} = 0$ supergravity

In order to obtain $\mathcal{N} = 0$ supergravity we take all mass parameters non-zero. The massless spectrum of this theory consists of the graviton, 3 vectors and 2 scalars. The massive field content of this theory is given in Table B.2.

Scalars	Vectors	Tensors	Gravitini	Dilatini
$2\,\lvert m_1+m_2+m_3+m_4\rvert$	$2\,\lvert m_1+m_3\rvert$	$2\,\lvert m_1+m_2\rvert$	$2\,\lvert m_1\rvert$	$2\,\lvert m_1+m_2+m_3\rvert$
$2\,\lvert m_1+m_2+m_3-m_4\rvert$	$2\,\lvert m_1-m_3\rvert$	$2\,\lvert m_1-m_2\rvert$	$2\,\lvert m_2\rvert$	$2\,\lvert m_1+m_2-m_3\rvert$
$2\,\lvert m_1+m_2-m_3+m_4\rvert$	$2\,\lvert m_1+m_4\rvert$	$2\,\lvert m_3+m_4\rvert$	$2\,\lvert m_3\rvert$	$2\,\lvert m_1-m_2+m_3\rvert$
$2\,\lvert m_1-m_2+m_3+m_4\rvert$	$2\,\lvert m_1-m_4\rvert$	$2\,\lvert m_3-m_4\rvert$	$2\,\lvert m_4\rvert$	$2\,\lvert m_1-m_2-m_3\rvert$
$2\,\lvert m_1+m_2-m_3-m_4\rvert$	$2\,\lvert m_2+m_3\rvert$			$2\,\lvert m_1+m_2+m_4\rvert$
$2\,\lvert m_1-m_2+m_3-m_4\rvert$	$2\,\lvert m_2-m_3\rvert$			$2\,\lvert m_1+m_2-m_4\rvert$
$2\,\lvert m_1-m_2-m_3+m_4\rvert$	$2\,\lvert m_2+m_4\rvert$			$2\,\lvert m_1-m_2+m_4\rvert$
$2\,\lvert m_1-m_2-m_3-m_4\rvert$	$2\,\lvert m_2-m_4\rvert$			$2\,\lvert m_1-m_2-m_4\rvert$
$2\,\lvert m_1+m_2\rvert$				$2\,\lvert m_1+m_3+m_4\rvert$
$2\,\lvert m_1-m_2\rvert$				$2\,\lvert m_1+m_3-m_4\rvert$
$2\,\lvert m_3+m_4\rvert$				$2\,\lvert m_1-m_3+m_4\rvert$
$2\,\lvert m_3-m_4\rvert$				$2\,\lvert m_1-m_3-m_4\rvert$
				$2\,\lvert m_2+m_3+m_4\rvert$
				$2\,\lvert m_2+m_3-m_4\rvert$
				$2\,\lvert m_2-m_3+m_4\rvert$
				$2\,\lvert m_2-m_3-m_4\rvert$
				$2\,\lvert m_1\rvert$
				$2\,\lvert m_2\rvert$
				$2\,\lvert m_3\rvert$
				$2\,\lvert m_4\rvert$

TABLE B.2: *Massive field content for $\mathcal{N} = 0$ supergravity in five dimensions. We present the number of massive fields and the values $\mu(m_i)$ corresponding to these fields. The actual mass of a field is given by $\mu(m_i)/2\pi R$.*

B.3 Kaluza-Klein towers

Scalars	Vectors	Tensors	Gravitini	Dilatini
$\pm(m_1+m_2+m_3+m_4)+n$	$\pm(m_1+m_3)+n$	$\pm(m_1+m_2)+n$	$\pm(m_1)+n$	$\pm(m_1+m_2+m_3)+n$
$\pm(m_1+m_2+m_3-m_4)+n$	$\pm(m_1-m_3)+n$	$\pm(m_1-m_2)+n$	$\pm(m_2)+n$	$\pm(m_1+m_2-m_3)+n$
$\pm(m_1+m_2-m_3+m_4)+n$	$\pm(m_1+m_4)+n$	$\pm(m_3+m_4)+n$	$\pm(m_3)+n$	$\pm(m_1-m_2+m_3)+n$
$\pm(m_1-m_2+m_3+m_4)+n$	$\pm(m_1-m_4)+n$	$\pm(m_3-m_4)+n$	$\pm(m_4)+n$	$\pm(m_1-m_2-m_3)+n$
$\pm(m_1+m_2-m_3-m_4)+n$	$\pm(m_2+m_3)+n$	n		$\pm(m_1+m_2+m_4)+n$
$\pm(m_1-m_2+m_3-m_4)+n$	$\pm(m_2-m_3)+n$	n		$\pm(m_1+m_2-m_4)+n$
$\pm(m_1-m_2-m_3+m_4)+n$	$\pm(m_2+m_4)+n$			$\pm(m_1-m_2+m_4)+n$
$\pm(m_1-m_2-m_3-m_4)+n$	$\pm(m_2-m_4)+n$			$\pm(m_1-m_2-m_4)+n$
$\pm(m_1+m_2)+n$				$\pm(m_1+m_3+m_4)+n$
$\pm(m_1-m_2)+n$				$\pm(m_1+m_3-m_4)+n$
$\pm(m_3+m_4)+n$				$\pm(m_1-m_3+m_4)+n$
$\pm(m_3-m_4)+n$				$\pm(m_1-m_3-m_4)+n$
n				$\pm(m_2+m_3+m_4)+n$
				$\pm(m_2+m_3-m_4)+n$
				$\pm(m_2-m_3+m_4)+n$
				$\pm(m_2-m_3-m_4)+n$
				$\pm(m_1)+n$
				$\pm(m_2)+n$
				$\pm(m_3)+n$
				$\pm(m_4)+n$

TABLE B.3: *Massive field content for $\mathscr{N}=0$ supergravity in five dimensions. We present the value $(\mu(m_i)+n)$ corresponding to the n'th Kaluza-Klein mode of each field. The actual mass of a field is given by $\mu(m_i)/2\pi R+n/R$. Note that besides the fields listed in this table, we also have to take into account the graviton, which also picks up an infinite Kaluza-Klein tower from the compactification on S^1.*

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