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MASTER'S THESIS

Spin Current in Hydrodynamics

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"After sleeping through a hundred million centuries we have finally opened our eyes on a sumptuous planet, sparkling with color, bountiful with life. Within decades we must close our eyes again. Isn't it a noble, an enlightened way of spending our brief time in the sun, to work at understanding the universe and how we have come to wake up in it?"

Richard Dawkins

Abstract

Hydrodynamic description of the spin degree of freedom has been drawing an increasing amount of attention in recent years. The reason for that attention lies in its diverse range of applications from condensed matter physics to astrophysics and high energy physics. After a brief discussion of the recent developments in hydrodynamics, we investigate relativistic hydrodynamics in the presence of a spin current. We generalize the results in the literature to describe a 2+1D fluid that is constrained only by Lorentz symmetry. We build the hydrodynamic equations for the energy-momentum tensor and the spin current. Then, we carry out an exhaustive analysis to identify constitutive relations, transport coefficients, and entropy. This setup enables us to solve hydrodynamic equations up to second order in gradients, i.e., in a Navier-Stokes level of approximation. Our framework reproduces well-known phenomena in the literature such as the spin Seebeck effect, thermal vorticity, Benett effect, and spin hydrodynamic generation. We further characterize more than 20 novel transport coefficients.

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I have carried out this research in one of the most unfortunate times of my lifetime, namely, the COVID-19 pandemic. That is the reason why I would like to acknowledge the people who helped me overcome every obstacle that I have dealt with as comprehensively as possible. Please accept my apologies if I left some of you great people unacknowledged.

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List of Abbreviations

w/	**w**ith
w/o	**w**ith**o**ut
GR	**G**eneral **R**elativity
SM	**S**tatistical **M**echanics
QM	**Q**uantum **M**echanics
SQM	**S**tatistical **Q**uantum **M**echanics
QFT	**Q**uantum **F**ield **T**heory
SFT	**S**tatistical Quantum **F**ield **T**heory
FTFT	**F**inite-**T**emperature Quantum **F**ield **T**heory
NS	**N**avier-**S**tokes
RHIC	**R**elativistic **H**eavy **I**on **C**ollider
QGP	**Q**uark-**G**luon **P**lasma

List of Symbols

$U(N)$	unitary group of degree N
$SU(N)$	special unitary group of degree N
$SO(1,N)$	N+1 dimensional proper, orthochronous Lorentz group
M_{ab}	generators of Lorentz algebra
$\left(M_{ab}^{\text{adjoint}}\right)^{gh}{}_{cd}$	adjoint representation of the generators of Lorentz algebra
$f_{abcd}{}^{gh}$	structure constants of Lorentz algebra
ϕ	arbitrary matter field
ψ	arbitrary Dirac spinor
γ_μ	Dirac matrices
$\mathcal{S}$	arbitrary scalar
$\mathcal{V}^\mu$	arbitrary vector
$\mathcal{T}^{\mu\nu}$	arbitrary rank-2 tensor
$g_{\mu\nu}$	arbitrary metric tensor (mostly positive signature)
$\eta_{\mu\nu}$	Minkowski metric tensor (mostly positive signature)
δ_μ^ν	Kronecker delta symbol
$\epsilon_{\mu\nu\rho}$	Levi-Civita *tensor*
$\Lambda^\mu{}_\nu$	arbitrary Lorentz transformation
$\Gamma^\rho_{\mu\nu}$	affine connection
$\mathring{\Gamma}^\rho_{\mu\nu}$	Levi-Civita connection
$T^\lambda{}_{\mu\nu}$	torsion tensor
$K_\mu{}^{\alpha\beta}$	contorsion tensor
$e_\mu{}^a$	vielbein
$\omega_\mu{}^{ab}$	spin connection (w/ torsion)
$\mathring{\omega}_\mu{}^{ab}$	spin connection (w/o torsion)
Ω_{abc}	objects of anholonomity
$G_{\mu\nu}{}^{ab}$	field strength tensor of spin connection
D_μ	gauge covariant derivative of Lorentz group (w/ torsion)
$\mathring{D}_\mu$	gauge covariant derivative of Lorentz group (w/o torsion)
∇_μ	covariant derivative (w/ torsion)
$\mathring{\nabla}_\mu$	covariant derivative (w/o torsion)
$R^\lambda{}_{\sigma\mu\nu}$	Riemann curvature tensor (w/ torsion)
$\mathring{R}^\lambda{}_{\sigma\mu\nu}$	Riemann curvature tensor (w/o torsion)

$R_{\mu\nu}$	Ricci tensor (w/ torsion)
$\mathring{R}_{\mu\nu}$	Ricci tensor (w/o torsion)
R	Ricci scalar (w/ torsion)
$\mathring{R}$	Ricci scalar (w/o torsion)
$Z[\cdot]$	generating functional
$W[\cdot]$	effective action
$T^{\mu\nu}$	energy-momentum tensor
$S^{\lambda}{}_{\mu\nu}$	spin current
T	temperature
μ^{ab}	spin chemical potential
u^{μ}	fluid velocity
a^{μ}	fluid acceleration
Θ	fluid expansion
$\sigma^{\mu\nu}$	fluid stress tensor
Ω	fluid vorticity

Dedicated to Cemal Ulugöl

Chapter 1

Introduction

One way to think about hydrodynamics is that it is a low-energy effective field theoretic description of an underlying classical or quantum many-body system at finite temperature [29]. In particular, hydrodynamics is concerned with the characterization of the collective motion of particles in relatively long distances. This thesis builds on the relativistic description of hydrodynamics, that is to say, we investigate the fluids whose microscopic descriptions exhibit Lorentz symmetry, for instance, quantum field theories. This symmetry constraint does not mean that relativistic hydrodynamics describe the matter only when it travels close to the speed of light, c. We solely focus on the symmetry itself, and one can always take the non-relativistic limit by making a $1/c$ expansion.

In addition to the Lorentz symmetry, we are particularly interested in the hydrodynamic characterization of spin currents. Hydrodynamic description of the spin degree of freedom is drawing an increasing amount of attention in recent years. The reason for that attention lies in the diverse range of applications from condensed matter physics to astrophysics and high energy physics [4, 12, 16, 19, 21].

This hydrodynamic description is especially relevant for heavy-ion collisions in the ultra-relativistic limit. Relativistic heavy-ion collisions produce quark-gluon plasmas (QGPs) along with strong magnetic fields. Consequently, those magnetic fields magnetize the plasma and cause the macroscopic flow of spin degrees of freedom [9, 15, 25, 26]. Specifically, the recent observation of global spin polarization of the Λ and $\overline{\Lambda}$ particles in heavy-ion collisions at RHIC [1, 2] serves as an experimental realization of the phenomenon.

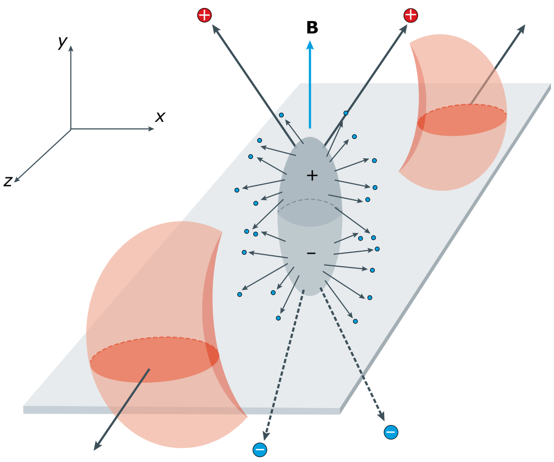

FIGURE 1.1: Schematic of the aftermath of a heavy ion collision. Spectator ions are depicted in orange and the resulting QGP is given in gray. Figure is taken from [24]

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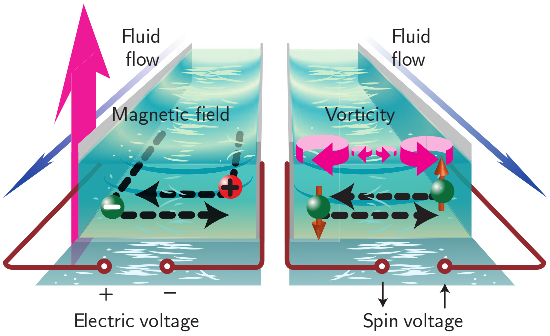


FIGURE 1.2: Schematic of electron hall effect (left) and analogous effect in spintronics due to non-vanishing vorticity (right). Figure is taken from [45]
.

Collective description of the spin degree of freedom is equally essential in spintronics [5, 10] and quantum spin liquids [43]. In (nearly) defect-free crystals, hydrodynamic description of spin can elucidate the spin transport, energy dissipation, and efficiency of the system [13]. On the other hand, the hydrodynamic framework is crucial by definition for quantum spin liquids. The recent observation of spin currents induced by vorticity in liquid metals [45] is merely an example of that.

Aside from its wide range of applications on experimental observations, hydrodynamics is appealing from a purely theoretical perspective as well. We can model gravitational fluctuations of a black hole as hydrodynamic fluctuations, and vice versa [39, 40]. As a matter of fact, this connection goes beyond the linear response. It turns out, Einstein's field equations of general relativity embeds the hydrodynamic equations surpassing the Navier-Stokes level of approximation. This link between hydrodynamic equations and Einstein's field equations is often called "fluid-gravity correspondence" [20].

This correspondence is more than just an academic exercise. One can use holography [3, 38] on top of fluid-gravity correspondence to map the hydrodynamics of a black hole to the hydrodynamics of a quantum system whose Hamiltonian is explicitly known. Therefore, we can compute transport coefficients of the quantum system using black hole physics. This mapping is particularly beneficial because the quantum systems that appear in holography are strongly interacting. Furthermore, the usual tools we use to compute transport coefficients such as loop corrections and Monte-Carlo simulations break down in those strongly correlated systems since they live in the non-perturbative regime. That is why holography – consequently hydrodynamics – plays an increasingly prominent role in the understanding of strongly interacting systems like QGPs, strongly correlated graphenes, and transition metal dichalcogenide monolayers [33].

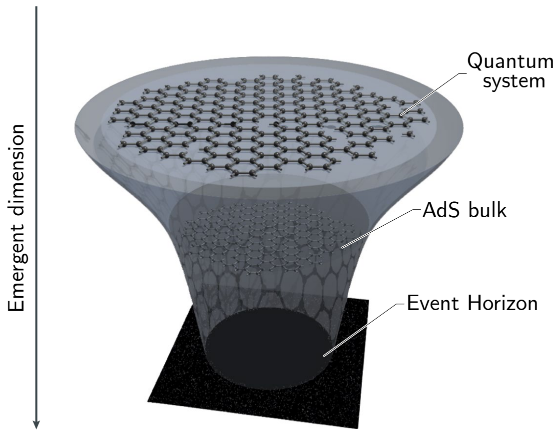


FIGURE 1.3: AdS/CFT correspondance relates a d dimensional many-body system to a black hole in $d+1$ dimensional AdS space. Figure is adapted from [14]
.

All those reasons above stimulated the growing interest in the hydrodynamic description of spin currents. Until this year, that branch of hydrodynamics stood as a gap in the literature. The first attempt to fill this gap was proposed by [18]. Their framework builds the hydrodynamic description of spin currents that appear in Lorentz and parity invariant 3+1D systems. Moreover, they presented the implementation of holography in their framework for the spin liquids that are dual to Lovelock Chern-Simons gravity [17].

In this thesis, we close the gap even further and generalize this framework to 2+1D systems that are only Lorentz invariant. Likewise, our framework is ready to be used for the calculations of hydrodynamic degrees of freedom both self-consistently and holographically. In the following chapters, we give the preliminary knowledge that is required to build the framework. Then, we derive the hydrodynamics equations from the corresponding general hydrodynamic action. We proceed with the introduction of hydrodynamic decompositions and show the constitutive relations of the energy-momentum tensor and the spin current. Using those results, we solve the hydrodynamic equations for an ideal spin fluid and investigate its entropy. Moreover, we introduce first- and second-order corrections to the ideal fluid to characterize a real fluid. Consequently, we derive the corrections to the constitutive relations. Finally, we present the transport coefficients of the theory and investigate the entropy of the real fluid in first-order.

Chapter 2

Preliminaries

2.1 Non-relativistic Hydrodynamics

Hydrodynamics is the branch of physics that focuses on the description and prediction of the flow profile of a fluid [30]. We could have called this branch “fluid dynamics," but we chose not to do so for historical reasons. When we are dealing with hydrodynamics, we describe the fluid as a continuum. Continuum description means that we do not have particles even in infinitesimal volume elements.

We have to identify which physical quantities affect the fluid flow to construct the mathematical description. The first quantity is, obviously, the velocity field, $\mathbf{v}(t, \mathbf{x})$ since this field describes the mass flow rate and direction of the infinitesimal fluid element at position $\mathbf{x}$ in space. At first sight, one might think velocity field should be sufficient to describe fluid flow, but this is not the case. The velocity field knows nothing about the microscopic properties of the fluid to carry their effects to the macroscopic world. Thus, we need other quantities that describe how the particles, forming the fluid, interact with each other and its surroundings effectively. Luckily, we have a whole branch of physics to do this job for us, namely, thermodynamics. In particular, we need two thermodynamic degrees of freedom to give a complete description of a fluid. At this point, one might ask what is so special about the number two, and it has a fathomable explanation. If we know two thermodynamic degrees of freedom of the system, we can determine all thermodynamic quantities using the equation of state! Consequently, if we know all thermodynamic quantities, then, we know all the macroscopic effects of the inner structure of the fluid. Now, the question is which quantities we should choose. Although we are free to choose any two, it is customary to pick pressure, $p(t, \mathbf{x})$, and mass density, $\rho(t, \mathbf{x})$. This discussion finalizes our choice of relevant physical quantities to describe a fluid mathematically.

We can start deriving the fundamental equations of hydrodynamics right away since we know which physical quantities to consider. First of all, we can neither generate mass out of thin air nor destroy it in non-relativistic physics. Therefore, we need to conserve it. Let us consider some finite volume Γ in space. The total mass, M_Γ in this volume is given by

$$M_\Gamma(t) = \int_\Gamma \mathrm{d}V \rho(t, \mathbf{x}), \tag{2.1}$$

where $\mathrm{d}V$ is the infinitesimal volume element and the integration is taken within the volume Γ. It is apparent in the expression that total mass within the volume is time dependent, i.e., the volume is not isolated and it can exchange mass with its surroundings. We can write mass flux through an infinitesimal surface element $\mathrm{d}\mathbf{S}$ as $\rho(t,\mathbf{x})\mathbf{v}(t,\mathbf{x})\cdot\mathrm{d}\mathbf{S}$. Thus the total mass flux, $\Phi_{M,\Gamma}$ is given by

$$\Phi_{M,\Gamma}(t) = \oint_{\partial\Gamma} \mathrm{d}\mathbf{S}\cdot\mathbf{v}(t,\mathbf{x})\rho(t,\mathbf{x}), \tag{2.2}$$

where the integration is taken over the surface enclosing the volume Γ. We note that the infinitesimal surface element $\mathrm{d}\mathbf{S}$ is pointing away from the volume Γ. On the other hand, we can express the mass flux as the negative time derivative of the total mass. Consequently, we can present the equality

$$\frac{\partial}{\partial t}\int_{\Gamma}\mathrm{d}V\rho(t,\mathbf{x}) = -\oint_{\partial\Gamma}\mathrm{d}\mathbf{S}\cdot\mathbf{v}(t,\mathbf{x})\rho(t,\mathbf{x}), \tag{2.3}$$

and we can simplify the expression by making use of Stoke's theorem,

$$\int_{\Gamma}\mathrm{d}V\left(\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\mathbf{v})\right)=0, \tag{2.4}$$

where we dropped space and time dependence of mass density and velocity field for notational clarity. Moreover, we put a vector arrow on gradient operator although we employed boldface vector notation. We chose this notation in order not to confuse space gradient with the covariant derivatives, which we will introduce in the upcoming chapters. Equation (2.4) gives us the mass conservation in volume Γ. However, the choice of Γ is arbitrary as we have stated at the beginning. This implies a stronger conservation law,

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\mathbf{v})=0. \tag{2.5}$$

This equation is called **mass continuity equation** and the vector $\mathbf{j}=\rho\mathbf{v}$ is called **mass flux density**.When we expand the divergence term, we obtain,

$$\frac{\partial\rho}{\partial t}+\mathbf{v}\cdot\vec{\nabla}\rho+\rho\vec{\nabla}\cdot\mathbf{v}=0, \tag{2.6}$$

and we notice that the first two terms are nothing but a total time derivative to give the final form of the equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\vec{\nabla}\cdot\mathbf{v}=0. \tag{2.7}$$

Even though, mass continuity equation is a crucial part of hydrodynamics, it does not describe how the velocity field evolves in time. We have to consider Newton's second law to describe the dynamics of flow velocity. We will divide the total force applied on the fluid into two parts, namely, an external force and the force caused by the pressure.

The total force emerging from the pressure in an arbitrary volume Γ of the fluid can be expressed as

$$-\oint_{\partial\Gamma} \mathrm{d}\mathbf{S} p = -\int_{\Gamma} \mathrm{d}V \vec{\nabla} p, \tag{2.8}$$

and we can read the emergent force density caused by pressure from the volume integral. Therefore, velocity dynamics is given by

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\vec{\nabla} p + \mathbf{f}, \tag{2.9}$$

where we introduced $\mathbf{f}$ as the external force density. If we expand the total time derivative and rearrange the equation, we find

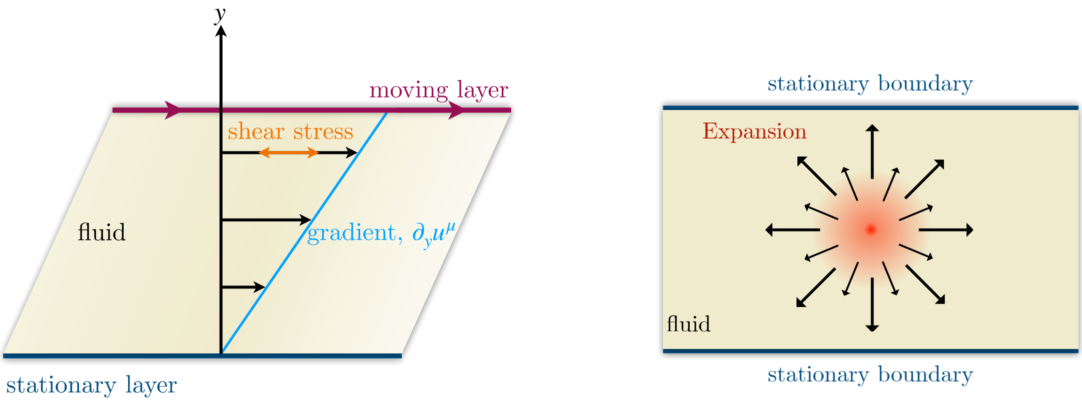
$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \vec{\nabla}\right) \mathbf{v} = -\frac{1}{\rho} \vec{\nabla} p + \frac{1}{\rho} \mathbf{f}. \tag{2.10}$$

This equation is called **Euler's equation** and it describes velocity dynamics as we desired. We should bear in mind that we did not include any dissipative process, like friction, in the derivation. Therefore, the motion is adiabatic and entropy density is constant in time. The family of fluids that undergo adiabatic motion is called **ideal fluids**. If we want to take the internal friction of the fluid into account, we need to modify the equation such that it reads

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \vec{\nabla}\right) \mathbf{v} = -\vec{\nabla} p + \eta \vec{\nabla}^2 \mathbf{v} + \left(\zeta + \frac{1}{3}\eta\right) \vec{\nabla} \left(\vec{\nabla} \cdot \mathbf{v}\right) + \mathbf{f} \tag{2.11}$$

where η is shear viscosity (Fig.2.1a) and ζ is bulk viscosity (Fig.2.1b). This equation is called **Navier-Stokes momentum equation** and full derivation can be found in [30].

Navier-Stokes momentum equation coupled to mass continuity equation is a monumental tool to describe non-relativistic fluids [7, 8, 32, 31, 35, 37, 44]. It enabled humankind to make excellent developments in technology such as climate modeling, vehicle design, liquid cooling, blood flow modeling. On the other hand, our ultimate goal is to describe the hydrodynamics of spin currents. Spin, itself, is a relativistic notion. It arises from the irreducible representations of the Lorentz group. Furthermore, it is inherently a quantum phenomenon. However, Navier-Stokes equations are neither relativistic nor quantum. Therefore, we need to develop a piece of machinery to describe quantum hydrodynamics in the relativistic regime. To reach our goal, we need to understand how a quantum field theory couples to a heat bath. Then, we need to make the necessary approximation to wind up at the hydrodynamic regime. Essentially, we have already mentioned this approximation at the beginning of this section. We assumed that particles within the liquid are so strongly correlated that we can ignore their individual identity and describe them as a density field. The approximation will be the same for relativistic quantum hydrodynamics. Now, we need to introduce a finite temperature field theory (FTFT) to describe how a quantum field theory couples to a heat bath.



(A) Depiction of shear stress caused by the spatial gradient of velocity. The energy price of non-trivial shear stress is related to shear viscosity, η.

(B) Depiction of expansion. The energy price of non-trivial expansion is related to bulk viscosity, ζ.

FIGURE 2.1: 2D descriptions of shear stress and expansion. Here, we use the term "expansion" interchangeably with compression since they are the same thing up to a minus sign.

2.2 Quantum Field Theory in Finite Temperature

2.2.1 Statistical field theory

Statistical (quantum) field theory (SFT) gives us a description of the physical properties of a quantum system in the thermal equilibrium state. In this description, we trade in time to introduce temperature to the system. Precisely, we make a Wick rotation to Euclidean time, and we compactify this coordinate to a circle of circumference L. Then, we identify temperature, T, with $1/L$. From now on, we will call this compactified time coordinate as "time circle" and circumference L as the invariant length of the time circle. We can express SFT partition function as

$$Z_{SFT} = \int \mathscr{D}[\phi] e^{-S_E}, \tag{2.12}$$

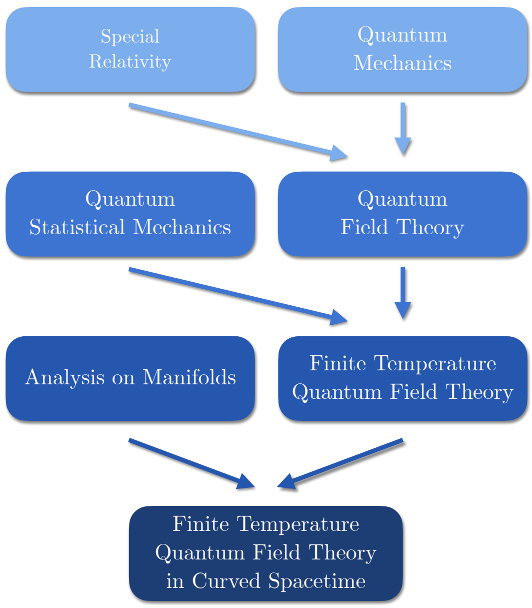


FIGURE 2.2: The chart of how different branches of physics relate to each other.

where

$$S_E[\phi] = \int_0^{1/T_0} \mathrm{d}\tau \int \mathrm{d}x^{d-1} \sqrt{g_E} \mathcal{L}[\phi] \tag{2.13}$$

is the Euclidean action, g_E is the determinant of the Euclidean metric, $1/T_0$ is the coordinate periodicity of the time circle, ϕ is a matter field, and $\mathcal{L}[\phi]$ is the Lagrangian of the underlying quantum field theory. Furthermore, we introduce the time-like vector V^μ, which can be expressed as $V^\mu = (1, 0, 0, \dots, 0)^T$ in suitable coordinates. This vector has to be a Killing vector since SFT describes an equilibrium state. Hence, spacetime is stationary. Therefore, we can calculate the invariant length of the time circle by starting in the aforementioned suitable coordinates and plug V^μ back in the expression to regain coordinate independence. Thus, the invariant length of the time circle is

$$\begin{aligned} L &= \int_0^{1/T_0} \mathrm{d}\tau \sqrt{g_{\tau\tau}} \\ &= i \int_0^{-i/T_0} \mathrm{d}t \sqrt{-g_{tt}} \\ &= \frac{\sqrt{-g_{tt}}}{T_0} \\ &= \frac{\sqrt{-V^2}}{T_0}, \end{aligned} \tag{2.14}$$

where we rotated back to real time in second line and used the fact that spacetime is stationary in the third line. Then, the temperature is given by

$$T = \frac{T_0}{\sqrt{-V^2}}. \tag{2.15}$$

Now, we can calculate all thermodynamic quantities of the underlying system by introducing source terms $J(x)$ to the action that couple to matter fields $\phi(x)$ linearly, then, taking functional derivatives of the partition function with respect to the sources. This description is exquisite for an equilibrium state. However, hydrodynamics describes a system in "near-equilibrium," thus, we cannot afford to trade in time to get the temperature. That is why we need to introduce a real-time finite temperature field theory. In this section, we chose to introduce close-time-path FTFT for pedagogical reasons, an interested reader may choose another FTFT to show what we will do in the upcoming part is choice-invariant.

2.2.2 Closed-Time-Path Finite Temperature Field Theory

For simplicity, we introduce close-time-path FTFT in Schrödinger picture following [11]. We start with first-quantized description and we will go back to path integral quantization at the end of our discussion. Quantum mechanical description of a mixed state embedded in an external surrounding is described by the density matrix $\rho(t)$ of the system which is given by

$$\rho(t) = \sum_n p_n \left|\psi_n(t)\right\rangle \left\langle\psi_n(t)\right|, \tag{2.16}$$

where p_n is the probability of finding the system in the state $|\psi_n(t)\rangle$, which abides to the orthonormality condition $\langle\psi_n(t)|\psi_m(t)\rangle = \delta_{n,m}$. Here, we assumed the system exhibits discrete states, still, it can be generalized to a continuum of states trivially. Moreover, we can find the expectation value of an operator $\mathcal{O}$ using the density matrix by

$$\langle\mathcal{O}\rangle(t) = \mathrm{Tr}\left[\rho(t)\mathcal{O}\right] = \sum_n p_n \langle\psi_n(t)|\mathcal{O}|\psi_n(t)\rangle\,. \tag{2.17}$$

Following this, we can define the quantum version of the entropy, $S = -\ln\Omega$, as an expectation value,

$$S = -\langle\ln p\rangle = -\sum_n p_n \ln p_n, \tag{2.18}$$

which coincides to Shannon entropy. Assuming the states satisfy the Schrödinger equation, we find that the time evolution of the density matrix is given by quantum Liouville equation,

$$i\frac{\partial\rho(t)}{\partial t} = \left[H,\rho(t)\right], \tag{2.19}$$

where H is the Hamiltonian of the system. We should bear in mind that we assumed probabilities, p_n, do not change in time to derive this equation. This coincides to the assumption of an adiabatic evolution, thus, entropy is constant. We need to remember this when using closed-time-path FTFT to show relativistic quantum hydrodynamics is a low energy effective FTFT. Now, we define the time evolution operator as

$$U(t,t') = T\left[\exp\left(-i\int_{t'}^{t}\mathrm{d}t''H(t'')\right)\right], \tag{2.20}$$

where T is time ordering. At that point, we can write $\rho(t)$ as

$$\rho(t) = U(t,0)\rho(0)U(0,t). \tag{2.21}$$

Moreover, we know that density matrix is a positive definite matrix with unit trace. Therefore, we can write it in terms of a Hermitian operator, H_i, whose meaning will become obvious shortly after,

$$\rho(0) = \frac{e^{-\beta H_{\mathrm{i}}}}{\mathrm{Tr}\left[e^{-\beta H_{\mathrm{i}}}\right]} \tag{2.22}$$

where β is inverse temperature. Now, we suppose that the Hamiltonian of the system has the form

$$H(t) = H_{\mathrm{i}}\Theta_H\left(-\mathfrak{Re}(t)\right) + \mathscr{H}(t)\Theta_H\left(\mathfrak{Re}(t)\right), \tag{2.23}$$

where $\Theta_H(x)$ is the Heaviside step function. Let us elucidate this supposition. We prepare an equilibrium state in negative times, then, we turn on perturbations adiabatically after $\mathfrak{Re}(t) = 0$. Obviously, system evolves in equilibrium if the Hamiltonian is time independent. With this supposition, we can re-express the density matrix as

$$\rho(0) = \frac{U(t' - i\beta, t')}{\mathrm{Tr}\left[U(t' - i\beta, t')\right]}, \tag{2.24}$$

and its time evolution is

$$\rho(t) = \frac{U(t,0)U(t' - i\beta, t')U(0,t)}{\mathrm{Tr}\left[U(t' - i\beta, t')\right]}, \tag{2.25}$$

where t' is a large negative time. Using this expression and some algebraic manipulations, one can show straightforwardly that expectation value of an operator, $\mathcal{O}$ for a large positive time t'' is

$$\langle \mathcal{O} \rangle (t) = \frac{\mathrm{Tr}\left[U(t' - i\beta, t')U(t',t)\mathcal{O}U(t,t')\right]}{\mathrm{Tr}\left[U(t' - i\beta, t')\right]}. \tag{2.26}$$

Let us again illuminate the physical meaning of this expression. Initially, the system sits on large negative time t'. Then it evolves to some time t and at that point, we insert an operator $\mathcal{O}$. Then, the system continues evolving to a large positive time t''. After that, system evolves back in time to the initial time t' and thermalizes to inverse temperature β. Therefore, it follows the time contour, $C = C_+ + C_- + C_\beta$ given in fig.(2.3).

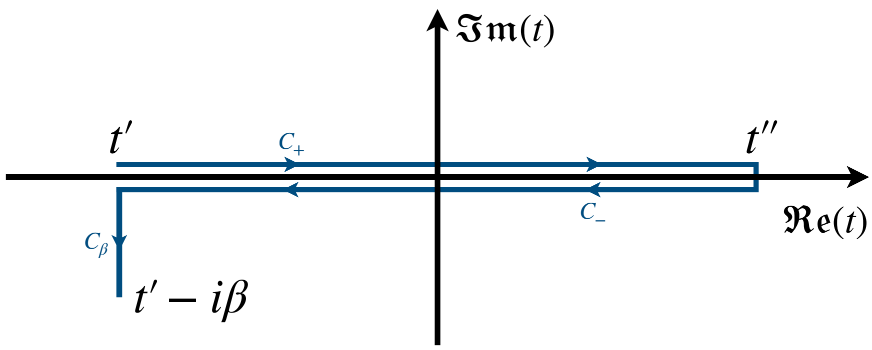


FIGURE 2.3: The time contour the system follows in closed-time-path FTFT..

Now, we can introduce the path integral description of the generating functional as

$$Z_{FT}[J_c] = \int \mathscr{D}[\phi] e^{iS_{FT}[\phi, J_c]} \tag{2.27}$$

where

$$S_{FT}[\phi, J_c] = \int_C \mathrm{d}t \int \mathrm{d}x^{d-1} \left(\mathcal{L}[\phi] + \mathcal{L}_J[\phi, J_c]\right), \tag{2.28}$$

and we already introduced the source term $\mathcal{L}_J[\phi, J_c]$ that is linear in fields ϕ. Still, this expression is in flat spacetime.

After this point, we will not introduce any new derivations in closed-time-path FTFT but we will give a sketch of what it should be done to obtain the presented result. The derivations are out of the scope of this thesis, an interested reader can consult [11] for the complete derivations. Now, let us proceed. When we calculate the propagator of the theory, we see that propagating in imaginary time is prohibited. Therefore, the contribution of the contour piece c_β is non-dynamic. Then, we can integrate out this contribution and absorb it into the integral measure of the generating functional. To eliminate the contour integration on time coordinate and introduce the usual volume integration, we can decompose that integration into,

$$\int_C \mathrm{d}t = \int \mathrm{d}t_+ - \int \mathrm{d}t_-, \tag{2.29}$$

which coincides to the introduction of another timelike coordinate. To eliminate this further, we need to promote fields (Φ) and sources ($\mathbf{J}$) into doublets. Moreover, Green's function gains a matrix structure. Nevertheless, FTFT reduces to the underlying quantum field theory when we take the zero temperature limit. Finally, the generating functional in curved spacetime becomes,

$$Z_{FT}[g_{\mu\nu}, \mathbf{J}] = \int \mathscr{D}[\phi] e^{iS_{FT}[g_{\mu\nu}, \Phi, \mathbf{J}]} \tag{2.30}$$

where

$$S_{FT}[g_{\mu\nu}, \Phi, \mathbf{J}] = \int \mathrm{d}x^d \sqrt{-g} \left(\Phi^\dagger \mathscr{G}^{-1} \Phi + \mathcal{L}_J[\Phi, \mathbf{J}] \right), \tag{2.31}$$

and we define the effective action to satisfy

$$Z_{FT}[g_{\mu\nu}, \mathbf{J}] = e^{iW_{FT}[g_{\mu\nu}, \mathbf{J}]}. \tag{2.32}$$

Now, we can compare our result from FTFT with SFT. We have traded in time coordinate to get temperature in SFT. This was not a feasible option to introduce relativistic quantum hydrodynamics since we needed a time coordinate to describe out-of-equilibrium behavior. When we introduced FTFT, we saw that we can keep the time coordinate in expense of introducing doublet fields, i.e., doubling the degrees of freedom of the underlying system. This completes all the machinery we need to introduce relativistic quantum hydrodynamics. From now on, we will focus on that description.

2.3 Relativistic Hydrodynamics

Despite its success in describing many physical phenomena, Navier-Stokes equations are not applicable for every fluid description. The obvious example is the description of relativistic fluids. The reason for this inability is quite fundamental. Navier-Stokes equations are not Lorentz covariant. Thus, we need to find a relativistic description, which reduces to Navier-Stokes equations in the non-relativistic limit.

Just like our non-relativistic discussion, let us start by thinking about which physical degrees of freedom we need to characterize a relativistic fluid. The first choice is, obviously, promoting the velocity field ($\mathbf{v}$) into 4-velocity (u^μ), which is normalized to $u^\mu u_\mu = -1$. Furthermore, we can promote non-relativistic mass density to an energy density since they are equivalent. Finally, we keep the pressure and the equation of state in the relativistic portrayal (with relativistic corrections of course.) This completes the relativistic promotions, but there is one more modification left. We can write energy density and pressure covariantly using the energy-momentum tensor, $T^{\mu\nu}$, whose elements we depict in fig.(2.4).
Therefore, it is enough to describe the dynamics of the energy-momentum tensor to describe the fluid. In addition, the dynamics of the energy-momentum tensor should reduce to the mass continuity equation and Navier-Stokes momentum equation in the non-relativistic limit. Now, the question is how we describe the dynamics of the energy-momentum tensor. Luckily, it is well-known that the energy-momentum tensor is the conserved current of diffeomorphisms. Thus, we can characterize its dynamics by requiring diffeomorphism invariance on the underlying theory. On top of that, nothing is restricting us from coupling the underlying theory to a conserved gauge current, $\mathcal{J}^\mu$, which is conjugate to a gauge field, $\mathcal{B}_\mu$. We will derive the hydrodynamic equations of motion following method presented in [22]. Suppose that we know the effective action, $W[g_{\mu\nu}, \mathcal{B}_\mu]$, of the underlying theory. Then, we can classify the energy-momentum tensor and the gauge current by the response under the variations of the metric and the gauge field in respective order. Therefore, we define the energy-momentum tensor and the gauge current by

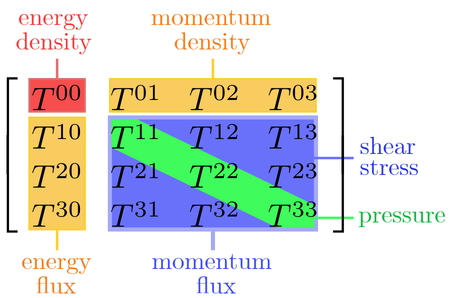


FIGURE 2.4: Individual elements of energy-momentum tensor. Figure is adapted from public domain.

$$
\begin{aligned}
T^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}}, \\
\mathcal{J}^\mu &= \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta \mathcal{B}_\mu}.
\end{aligned}
\tag{2.33}
$$

Equivalently, we can write the variation of the effective action as

$$
\begin{aligned}
\delta W &= \int \mathrm{d}^d x \left(\frac{\delta W}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta W}{\delta \mathcal{B}_\mu} \delta \mathcal{B}_\mu \right) \\
&= \int \mathrm{d}^d x \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + \mathcal{J}^\mu \delta \mathcal{B}_\mu \right),
\end{aligned}
\tag{2.34}
$$

Notice that we have not assumed any particular gauge group for the gauge field yet. We just suppressed the group indices in our notation and nothing more. We, now, suppose the metric is invariant under gauge transformations to derive the hydrodynamic equations of motion. At that point, giving a hint about the future is in order. When we consider hydrodynamics of spin current, the aforementioned supposition will turn out to be incorrect. However, let us not get bothered by that now. We will discuss every intricacy of the hydrodynamics of spin current in the following chapter. Using the supposition, we can express the gauge transformations of the background fields in the presence of an infinitesimal gauge transformation λ as

$$\begin{aligned} \delta_\lambda g_{\mu\nu} &= 0, \\ \delta_\lambda \mathcal{B}_\mu &= \mathring{\nabla}_\mu \lambda \end{aligned} \tag{2.35}$$

where δ_λ denotes the variation with respect to λ, and the covariant derivative defined as $\mathring{\nabla}_\mu \mathcal{V}^\nu := \mathcal{D}_\mu \mathcal{V}^\nu + \mathring{\Gamma}^\nu_{\mu\rho}\mathcal{V}^\rho$, $\mathcal{D}_\mu$ is the gauge covariant derivative of $\mathcal{B}_\mu$, and $\mathring{\Gamma}^\nu_{\mu\rho}$ is Levi-Civita connection. Then, the variation of the effective action is given by

$$\begin{aligned} \delta_\lambda W &= \int \mathrm{d}^d x \sqrt{-g} \left(\frac{1}{2} T^{\mu\nu} \delta_\lambda g_{\mu\nu} + \mathcal{J}^\mu \delta_\lambda \mathcal{B}_\mu \right) \\ &= \int \mathrm{d}^d x \sqrt{-g}\, \mathcal{J}^\mu \mathring{\nabla}_\mu \lambda \\ &= - \int \mathrm{d}^d x \sqrt{-g}\, \lambda \mathring{\nabla}_\mu \mathcal{J}^\mu. \end{aligned} \tag{2.36}$$

Gauge invariance of the theory dictates $\delta_\lambda W = 0$, which implies $\mathring{\nabla}_\mu \mathcal{J}^\mu = 0$. This equation is our first equation of motion, namely, conservation of gauge current.

Now, we turn to the diffeomorphism invariance. When we introduce an infinitesimal diffeomorphism ξ to the system, the change in background fields is given by a Lie derivative. We can express their variation as

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu} \\ &= \mathring{\nabla}_\mu \xi_\nu + \mathring{\nabla}_\nu \xi_\mu, \\ \delta_\xi \mathcal{B}_\mu &= \mathcal{L}_\xi \mathcal{B}_\mu \\ &= \xi^\nu \mathring{\nabla}_\nu \mathcal{B}_\mu + \mathcal{B}_\nu \mathring{\nabla}_\mu \xi^\nu \end{aligned} \tag{2.37}$$

where δ_ξ denotes the variation with respect to diffeomorphism ξ. Then, the variation of the effective action is given by

$$\begin{aligned}
\delta_\xi W &= \int \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta_\xi g_{\mu\nu} + \mathcal{J}^\nu \delta_\xi \mathcal{B}_\nu \right] \\
&= \int \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \left(\mathring{\nabla}_\mu \xi_\nu + \mathring{\nabla}_\nu \xi_\mu \right) + \mathcal{J}^\nu \left(\xi^\mu \mathring{\nabla}_\mu \mathcal{B}_\nu + \mathcal{B}_\mu \mathring{\nabla}_\nu \xi^\mu \right) \right] \\
&= \int \mathrm{d}^d x \sqrt{-g} \left[T^{\mu\nu} \mathring{\nabla}_\nu \xi_\mu + \mathcal{J}^\nu \left(\xi^\mu \mathring{\nabla}_\mu \mathcal{B}_\nu + \mathcal{B}_\mu \mathring{\nabla}_\nu \xi^\mu \right) \right] \\
&= \int \mathrm{d}^d x \sqrt{-g} \xi_\mu \left[-\mathring{\nabla}_\nu T^{\mu\nu} + \mathcal{F}^{\mu\nu} \mathcal{J}_\nu \right]
\end{aligned} \tag{2.38}$$

where we used conservation of gauge current in the third line. Just like gauge invariance, diffeomorphism invariance of the theory dictates $\delta_\xi W = 0$, which implies $\mathring{\nabla}_\nu T^{\mu\nu} = \mathcal{F}^{\mu\nu} \mathcal{J}_\nu$. This equation is our second (and the more fundamental) equation of motion, namely, conservation of energy-momentum. To summarize our discussion so far, we re-present the hydrodynamic equations of motion as

$$\boxed{\begin{aligned} &\mathring{\nabla}_\nu T^{\mu\nu} = \mathcal{F}^{\mu\nu} \mathcal{J}_\nu, \\ &\mathring{\nabla}_\mu \mathcal{J}^\mu = 0. \end{aligned}} \tag{2.39}$$

To understand the physical meaning of energy-momentum tensor and the gauge current better, we can decompose energy-momentum tensor and gauge current into scalars, vectors and tensors of $SO(d-1) \subset SO(1, d-1)$, and Lorentz symmetry is preserved by u^μ. The decomposition is given by

$$\begin{aligned}
T^{\mu\nu} &= \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + u^{(\mu} q^{\nu)} + \tau^{\mu\nu}, \\
\mathcal{J}^\mu &= \mathcal{N} u^\mu + j^\mu,
\end{aligned} \tag{2.40}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor, $u_\mu q^\mu = u_\mu j^\mu = g_{\mu\nu} \tau^{\mu\nu} = 0$, $u_\mu \tau^{\mu\nu} = 0$, and $\tau^{[\mu\nu]} = 0$. We identify $\mathcal{E}$ with energy density, $\mathcal{P}$ with pressure, $\mathcal{N}$ with charge density, q^μ with heat current, $\tau^{\mu\nu}$ with shear stress, and j^μ with transverse charge flow.

At the end of the previous section, We mentioned that relativistic hydrodynamics would unfold as a low-energy effective field theory. It is time to fulfill this promise. We start by expanding the effective action in gradients. Since the effective Lagrangian transforms as a scalar, we can decompose it into independent scalars within the background fields and their derivatives. We denote i^{th} scalar of order n in gradients as $\mathcal{S}_{\mathrm{i}}^{(n)}$ and we define the set of all n^{th} order scalars as

$$\mathcal{S}^{(m)} = \left\{ \mathcal{S}_{\mathrm{i}}^{(m)} \middle| 0 < i \leq N_m \right\} \tag{2.41}$$

Moreover, we suppose there are N_n many scalars that are n^{th} order in gradients so that we can decompose m^{th} order corrected effective action, $W^{(m)}$ as

$$W^{(m)} = \int \mathrm{d}^d x \sqrt{-g} \left[P\left(\mathcal{S}^{(0)}\right) + \sum_{n=1}^{m} \sum_{i=1}^{N_n} \chi_{\mathrm{i}}^{(n)} \left(\mathcal{S}^{(0)}\right) \mathcal{S}_{\mathrm{i}}^{(n)} \right] \tag{2.42}$$

where P and $\chi_{\mathrm{i}}^{(n)}$ are functions of zeroth-order scalars. By extending our discussion in subsection (2.2.1), we define the equilibrium state by making use of the timelike vector V^μ. We characterize hydrostatic equilibrium by demanding the Lie derivatives of the background fields with respect to V^μ to vanish. This is still a valid condition since we depict hydrostatic equilibrium as a stationary (up to a Lorentz boost) state.

Furthermore, we can characterize the zeroth-order scalars of the theory in equilibrium by using the equilibrium condition. First, notice that these quantities have to be gauge-invariant, local in space but non-local in imaginary time. The non-locality emerges from the fact that we integrated out the imaginary time in both SFT and FTFT. The first zeroth-order scalar is the temperature, and we already derived its equilibrium definition in (2.15). However, the temperature is not the only one. We need to build scalar(s) using the gauge field as well to describe the effects of the gauge field. To introduce the gauge field, we need a way to delocalize it in imaginary time. At this point, Polyakov loops, $P_{\mathcal{B}}$, come to the rescue. They are defined to be Wilson loops around the time circle [34] and they are gauge-invariant by definition. In addition, the loop integral around the time circle guarantees non-locality in imaginary time. Thus, Polyakov loops are cut out for our needs. In equilibrium, Polyakov loops of the gauge field are given by

$$\begin{aligned} \ln P_{\mathcal{B}} &:= i\mathcal{P} \oint \mathrm{d}x^\mu \mathcal{B}_\mu \\ &= i \int_0^{-i/T_0} \mathrm{d}t V^\mu \mathcal{B}_\mu \\ &= \frac{1}{T_0} V^\mu \mathcal{B}_\mu \end{aligned} \tag{2.43}$$

where we used the equilibrium condition after the second line. Therefore, we identify the chemical potential of the gauge field as $\mu_{\mathcal{B}} = \ln(P_{\mathcal{B}})/L$. Finally, we obtain the velocity profile simply by normalizing V^μ. Consequently, degrees of freedom of the hydrodynamic system are given by

$$T = \frac{T_0}{\sqrt{-V^2}}, \qquad \mu_{\mathcal{B}} = \frac{\mathcal{B}_\mu V^\mu}{\sqrt{-V^2}}, \qquad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}. \tag{2.44}$$

This finalizes the construction of relativistic hydrodynamics. Notice that we did not restrict ourselves to a specific order in gradients. Thus, we can describe hydrodynamics in any order. In the non-relativistic limit, zeroth-order hydrodynamics would coincide with ideal hydrodynamics, i.e., Euler's equation and mass continuity equation. First-order hydrodynamics would coincide with Navier-Stokes equations. Now, let us demonstrate this correspondence for the ideal case. Demonstration of the non-relativistic equivalence of first-order relativistic hydrodynamics and Navier-Stokes equations is out of the scope of this thesis. However, a full discussion can be found in [41].

2.3.1 Ideal Relativistic Hydrodynamics: a special case

Suppose we have an ideal relativistic fluid in absence of any gauge field in 3+1D spacetime. In this case, we can write the effective action as

$$W_{(0)} = \int \mathrm{d}^4x \sqrt{-g} P\left(T\right). \tag{2.45}$$

Then, the variation of the action is simply given by

$$\delta W_{(0)} = \int \mathrm{d}^4x \sqrt{-g}\frac{1}{2}\left(Pg^{\mu\nu} + u^\mu u^\nu \frac{\partial P}{\partial T}T\right)\delta g_{\mu\nu}. \tag{2.46}$$

Therefore, the energy-momentum tensor has the simple form

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} \tag{2.47}$$

where

$$\epsilon = -P + \frac{\partial P}{\partial T}T \tag{2.48}$$

is the energy density. Notice that the scalar function P turned out to be the pressure of the ideal fluid. This identification is not a special case, we can identify the scalar function P with the ideal pressure of the underlying theory. In addition, this identification leads to the definition of the ideal entropy density $s = \frac{\partial P}{\partial T}$. Finally, we present the projections of the hydrodynamic equation of motion as

$$\begin{aligned} u_\mu \mathring{\nabla}_\nu T^{\mu\nu} &= u^\mu \mathring{\nabla}_\nu \epsilon + (\epsilon + P)\, \mathring{\nabla}_\mu u^\mu = 0, \\ \Delta^\lambda{}_\mu \mathring{\nabla}_\nu T^{\mu\nu} &= (\epsilon + P)\, u^\mu \mathring{\nabla}_\mu u^\lambda + \Delta^{\lambda\mu}\mathring{\nabla}_\mu P = 0. \end{aligned} \tag{2.49}$$

These equations give the complete description of a free ideal relativistic fluid. Now, let us introduce the non-relativistic limit.

2.3.2 Non-relativistic Limit of Ideal Relativistic Hydrodynamics

Here, we present a step-by-step guide on how to take the non-relativistic limit of relativistic hydrodynamics. Even though we are using ideal hydrodynamics as an example, our arguments are general and can be used in any order in gradients. First of all, we take the special relativity limit of the fundamental equations by introducing the flat background metric $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\mathring{\nabla}_\mu \to \partial_\mu$. Then, the equations reduce to

$$\begin{aligned} u^\mu \partial_\nu \epsilon + (\epsilon + P)\, \partial_\mu u^\mu &= 0, \\ (\epsilon + P)\, u^\mu \partial_\mu u^\lambda + \Delta^{\lambda\mu} \partial_\mu P &= 0. \end{aligned} \tag{2.50}$$

Now, we introduce the small spatial velocity limit $u^i \ll u^0$ so that we can express the velocity as $u^\mu = (1, \mathbf{v})^T$ for $|\mathbf{v}| \ll 1$. This limit reduces the projections of the gradient to

$$u^\mu \partial_\mu \to \frac{\partial}{\partial t} + \mathbf{v} \cdot \vec{\nabla}, \qquad \Delta^{i\mu} \partial_\mu \to \partial^i \tag{2.51}$$

and $\Delta^{00} = 0$. Consequently, the equations further reduce to

$$\begin{aligned} \frac{\partial \epsilon}{\partial t} + \mathbf{v} \cdot \vec{\nabla} \epsilon + (\epsilon + P)\, \vec{\nabla} \cdot \mathbf{v} &= 0, \\ (\epsilon + P)\, \frac{\partial \mathbf{v}}{\partial t} + (\epsilon + P) \left(\mathbf{v} \cdot \vec{\nabla} \right) \mathbf{v} + \vec{\nabla} P &= 0. \end{aligned} \tag{2.52}$$

Finally, we impose a non-relativistic equation of state where $P \ll \epsilon$, and we realize the energy density is dominated by mass density. This limit gives the final non-relativistic form of hydrodynamic equations:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \mathbf{v} &= 0, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \vec{\nabla} \right) \mathbf{v} + \vec{\nabla} P &= 0. \end{aligned} \tag{2.53}$$

The equations are precisely mass continuity equation and Euler's equation as we expected! Therefore, we conclude that both relativistic and non-relativistic hydrodynamics are nothing but the conservation of energy momentum tensor.

Chapter 3

Spin Hydrodynamics Formalism

In the previous chapter, we have discussed non-relativistic hydrodynamics and extended it to a relativistic quantum case. Moreover, we have shown the two descriptions are equivalent in the proper limit. Now, it is time to build the theory of spin currents in hydrodynamics.

We have stated that every current has a conjugate gauge field in the discussion of relativistic hydrodynamics. Then, the gauge invariance of the theory produced the equation of motion of the current we are interested in. Here, we can give some examples. Electromagnetism has $U(1)$ symmetry, and that gives us electric current. Weak interaction has $SU(2)$ symmetry, which corresponds to weak current. Strong interaction has $SU(3)$ symmetry, which produces color current [48]. Now, the question is, what gives the spin current? We need to look at a very fundamental symmetry to answer this question, i.e., Lorentz symmetry. Lorentz symmetry is the symmetry to build a relativistic theory, and we have a canonical way to relate it to spin. When we look at Wigner's construction of unitary irreducible representations of the Poincaré group, we see that the corresponding charge of the Lorentz group, $SO(1,N)$, is spin [47]!. Thus, we need to use the Lorentz group as a gauge group to obtain spin current and its equation of motion. Therefore, we need to understand how the Lorentz group behaves as a gauge group, and that is what we will do in the following section.

3.1 Gauge Theory of $SO(1,N)$

In this section, we construct the gauge theory for the Lorentz group, $SO(1,N)$, as promised. Since the Lorentz group is a non-abelian group, we can build the gauge theory like any non-abelian gauge theory. Let us begin with the matter fields. Under a group transformation, fields rotate as:

$$\phi(x) \to \phi'(x) = U\phi(x), \tag{3.1}$$

where ϕ denotes the fields written as a column vector. When an object transform as above, we will call that object transforms *covariantly* under gauge transformations. For compact Lie groups, the matrices U can be written in exponential form. This is not true for non-compact Lie groups but still valid for infinitesimal rotations. Thus, $SO(1,N)$ being a non-compact Lie group, its group action on fields can be written in exponential form for the set of infinitesimal parameters λ^{ab} as follows:

$$U(\lambda) = \exp\left(-\frac{1}{2}\lambda^{ab}M_{ab}\right), \tag{3.2}$$

where M_{ab}s are the *generators* of the Lie algebra, $\mathfrak{so}(1,N)$, and we can decompose the generators into rotations and boosts. In N+1D spacetime, we obviously have N boosts. On the other hand, number of rotations is not as straightforward, yet still not very complicated. We need to count the maximum number of mutually orthogonal planes in ND space since we can write every rotation as a combination of rotations in those planes. Moreover, we can count the maximum number of mutually orthogonal planes by choosing two axes from an orthogonal coordinate system. Thus, we have $\binom{N}{2} = \frac{N(N-1)}{2}$ rotations in ND space. This gives us a total number of $N + \frac{N(N-1)}{2} = \frac{(N+1)N}{2}$ generators. Therefore, the representation of the generators, M_{ab} has to be anti-symmetric in its indices. Furthermore, the Lie algebra, $\mathfrak{so}(1,N)$, is given by:

$$[M_{ab}, M_{cd}] = f_{abcd}{}^{gh}M_{gh}, \tag{3.3}$$

where $f_{abcd}{}^{gh}$ are called *structure constants* and they form the adjoint representation of the generators. Their closed form is given by:

$$f_{abcd}{}^{gh} = \left(M_{ab}^{\text{adjoint}}\right)^{gh}{}_{cd} = 2\eta_{a[c}\delta^{g}_{d]}\delta^{h}_{b} - 2\eta_{b[c}\delta^{g}_{d]}\delta^{h}_{a}, \tag{3.4}$$

where the Minkowski metric has the signature $(-,+,+,\ldots,+)$ and $[\cdot,\cdot]$ in the indices stands for anti-symmetrization. Furthermore, the group action of the generators on vectors, $\mathcal{V}^c$, and spinors, ψ is given by:

$$\begin{aligned} M_{ab}\mathcal{V}^c &= 2\delta^c_{[a}\mathcal{V}_{b]}, \\ M_{ab}\psi &= \frac{1}{4}\left[\gamma_a, \gamma_b\right]\psi. \end{aligned} \tag{3.5}$$

In order to couple the theory to matter fields, we need a gauge symmetry preserving derivative. One can see easily that the usual derivative $\partial_\mu\phi$ does not transform covariantly under gauge transformations, to overcome this problem, we construct the *gauge covariant derivative* as

$$D_\mu := \partial_\mu + \frac{1}{2}\omega_\mu{}^{ab}M_{ab}, \tag{3.6}$$

where $\omega_\mu{}^{ab}$ are a set of gauge fields, which we will call as *spin connection* from now on. Moreover, we demand $D_\mu\phi$ to transform covariantly to find the transformation rule for spin connection which is given by:

$$\omega_\mu{}^{ab} \to \omega'_\mu{}^{ab} = \omega_\mu{}^{ab} + \partial_\mu\lambda^{ab} + \frac{1}{2}f_{cdgh}{}^{ab}\omega_\mu{}^{cd}\lambda^{gh} = \omega_\mu{}^{ab} + D_\mu\lambda^{ab}. \tag{3.7}$$

Therefore, spin connection transforms in the adjoint representation of the Lorentz group. For completeness, the field strength $G_{\mu\nu}{}^{ab}$ is given by:

$$G_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \frac{1}{2} f_{cdgh}{}^{ab} \omega_\mu{}^{cd} \omega_\nu^{gh}. \tag{3.8}$$

Here, we choose a pragmatic approach and end the construction of this gauge theory since this is all we need to know to proceed building up our ultimate goal. In addition, we built the theory for the charge $-1/2$, again for the same pragmatic reasons.

Previously, we have claimed that spin connection would couple to the spin of the matter fields. To convince ourselves that the claim is true, we can test the claim in minimal coupling. For a spin-0 field, ϕ, it is obvious that there is no coupling since $D_\mu \phi = \partial_\mu \phi$. In addition, we can use Dirac spinors ψ as a non-trivial example. In the minimal coupling case, Dirac Lagrangian reads:

$$\begin{aligned} \mathcal{L}_{\text{Dirac}} &= i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi \\ &= i\overline{\psi}\gamma^\mu \left(\partial_\mu + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} \right) \psi - m\overline{\psi}\psi \\ &= i\overline{\psi}\gamma^\mu \partial_\mu \psi + \frac{i}{8}\omega_\mu{}^{ab}\overline{\psi}\gamma^\mu \left[\gamma_a, \gamma_b\right] \psi - m\overline{\psi}\psi \\ &= i\overline{\psi}\gamma^\mu \partial_\mu \psi + \omega_\mu{}^{ab}\overline{\psi}\Sigma^\mu{}_{ab}\psi - m\overline{\psi}\psi, \end{aligned} \tag{3.9}$$

where we identified the second term with the Dirac spin tensor $\Sigma^\mu{}_{\alpha\beta}$. We see that spin connection directly couples to the spin of the matter field as we expected. At this point, a careful reader might notice that the spin tensor has one spacetime index and two group indices in the Lagrangian. However, the spin tensor has three spacetime indices by definition. Then, the question raises canonically. Is there a way to relate group indices to spacetime indices? The answer is yes, there is but to understand that relation, we need to introduce a new formalism, i.e, the vielbein formalism.

3.2 Vielbein Formalism

Throughout the remaining part of this thesis, we work on constructing a quantum effective field theory in most general curved spacetime. The generality includes non-vanishing torsion. Although having a general framework is crucial for describing as many physical phenomena as possible, it has its downsides in a calculational perspective. Objects like Dirac matrices, γ^μ, are spacetime independent in flat background. However, this is not the case for a curved background. That is why our calculations would get quite cumbersome rapidly if we tried to work directly in a curved background. The workaround of this problem is to use so-called vielbein (meaning many-legs in German) formalism. In this formalism, we consider a map between the spacetime coordinates and a set of locally inertial coordinates. From now on, we call these “Lorentz” coordinates. We can use this mapping to push-forward the tensorial objects in our theory to the flat tangent manifold, do the necessary calculations, then, pull them back using the inverse map. This approach will ease our work significantly.

Now, let us denote the locally inertial coordinate frame at x_0 as $y^a(x_0; x)$. Then the mapping, vielbein, between spacetime frame and the locally inertial coordinate frame is given by:

$$e_\mu{}^a(x_0) := \left.\frac{\partial y^a(x_0; x)}{\partial x^\mu}\right|_{x=x_0}, \tag{3.10}$$

where Greek indices are for spacetime coordinates and Latin ones are for Lorentz coordinates. Under a general coordinate transformation, vielbein transforms as a 1-form, i.e.

$$e'_\mu{}^a(x') = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu{}^a(x) \tag{3.11}$$

and under Lorentz transformations, it transforms as a vector, i.e.,

$$e'_\mu{}^a(x) = e_\mu{}^b(x)\Lambda_b{}^a. \tag{3.12}$$

As we have stated before, we can use the vielbein to change between spacetime and Lorentz frame of coordinates. The mapping for vectors and one-forms are given by

$$\begin{aligned} \mathcal{V}^a &= e_\mu{}^a \mathcal{V}^\mu, \qquad & \mathcal{V}_\mu &= e_\mu{}^a \mathcal{V}_a, \\ \mathcal{V}^\mu &= e^\mu{}_a \mathcal{V}^a, \qquad & \mathcal{V}_a &= e^\mu{}_a \mathcal{V}_\mu, \end{aligned} \tag{3.13}$$

where we dropped the explicit spacetime dependence for brevity, and we will keep on doing so in the remaining part of the thesis. Using these mappings, we can express the spacetime metric as

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}. \tag{3.14}$$

As a consequence of this expression, we find two additional identities given by

$$e_\mu{}^a e^\nu{}_a = \delta^\nu_\mu, \qquad e_\mu{}^a e^\mu{}_b = \delta^a_b. \tag{3.15}$$

To couple matter fields in Lorentz coordinates to our theory, we need a Lorentz covariant derivative and that is precisely what we have built in the previous section. The Lorentz covariant derivative is given by (3.6). Therefore, we define the new *covariant derivative* as $\nabla_\mu V^\nu := D_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$ where Γ is the vielbein compatible connection, which is determined by:

$$0 = \nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \Gamma^\rho_{\mu\nu} e_\rho{}^a. \tag{3.16}$$

This expression is also called the vielbein postulate which is equivalent to $D_{[\mu} e_{\nu]}{}^a = \frac{1}{2} T^a{}_{\mu\nu}$ where $T^\alpha{}_{\mu\nu} := 2\Gamma^\alpha_{[\mu\nu]}$ is the torsion tensor and the connection is given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + K_\mu{}^\lambda{}_\nu, \tag{3.17}$$

where $K_{\mu}{}^{\lambda}{}_{\nu} := \frac{1}{2}\left(T_{\nu}{}^{\lambda}{}_{\mu} + T_{\mu}{}^{\lambda}{}_{\nu} + T^{\lambda}{}_{\mu\nu}\right)$ is the contorsion tensor and we denote torsionless connection, which is the Levi-Civita connection, by putting a ring on its symbol, i.e. $\mathring{\Gamma}$. Then, we can write the connection equivalently in the form $\Gamma^{\lambda}_{\mu\nu} = \mathring{\Gamma}^{\lambda}_{\mu\nu} + K_{\mu}{}^{\lambda}{}_{\nu}$. Moreover, we can solve vielbein postulate for the spin connection in the absence of torsion, i.e,

$$\mathring{\nabla}_{\mu} e_{\nu}{}^{a} = 0 \quad \Rightarrow \quad \mathring{\omega}_{\mu}{}^{ab} = \frac{1}{2} e_{\mu c}\left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right), \tag{3.18}$$

where $\Omega_{abc} := 2e^{\mu}{}_{a}e^{\nu}{}_{b}\partial_{[\mu}e_{\nu]c}$ are the objects of anholonomity. Subsequently, we can use this result to find the spin connection in non-vanishing torsion:

$$\begin{aligned}
0 = \nabla_{\mu} e_{\nu}{}^{a} &= \partial_{\mu} e_{\nu}{}^{a} + \omega_{\mu}{}^{a}{}_{b} e_{\nu}{}^{b} - \Gamma^{\rho}_{\mu\nu} e_{\rho}{}^{a} \\
&= \partial_{\mu} e_{\nu}{}^{a} + \mathring{\omega}_{\mu}{}^{a}{}_{b} e_{\nu}{}^{b} + \left(\omega_{\mu}{}^{a}{}_{b} - \mathring{\omega}_{\mu}{}^{a}{}_{b}\right) e_{\nu}{}^{b} - \mathring{\Gamma}^{\rho}_{\mu\nu} e_{\rho}{}^{a} - K_{\mu}{}^{a}{}_{\nu} \\
&= \mathring{\nabla}_{\mu} e_{\nu}{}^{a} + \left(\omega_{\mu}{}^{a}{}_{b} - \mathring{\omega}_{\mu}{}^{a}{}_{b}\right) e_{\nu}{}^{b} - K_{\mu}{}^{a}{}_{\nu} \\
&= \left(\omega_{\mu}{}^{a}{}_{b} - \mathring{\omega}_{\mu}{}^{a}{}_{b}\right) e_{\nu}{}^{b} - K_{\mu}{}^{a}{}_{\nu} \\
&\Rightarrow \omega_{\mu}{}^{ab} = \mathring{\omega}_{\mu}{}^{ab} + K_{\mu}{}^{ab}.
\end{aligned} \tag{3.19}$$

Furthermore, Riemann tensor is given by the well-known identity

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{3.20}$$

and we can use the vielbein postulate once more to plug in $\Gamma^{\rho}_{\mu\nu} = e^{\rho}{}_{a}\partial_{\mu}e_{\nu}{}^{a} + e^{\rho}{}_{a}e_{\nu}{}^{b}\omega_{\mu}{}^{a}{}_{b}$ to arrive at the expression

$$R^{ab}{}_{\mu\nu} = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}, \tag{3.21}$$

which gives the Riemann tensor purely in terms of the spin connection. This concludes our discussion on the vielbein formalism. Now, we can use this formalism to construct spin hydrodynamics.

Chapter 4

Effective Theory for Spin Hydrodynamics

In this section, we build an effective theory for the hydrodynamics of the spin currents using our discussion in section 2.3. Suppose that the generating functional Z of the underlying theory is given by:

$$Z[e_\mu{}^a, \omega_\mu{}^{ab}] = \exp\left(iW[e_\mu{}^a, \omega_\mu{}^{ab}]\right), \tag{4.1}$$

where W is the effective action, $e_\mu{}^a$ is the veilbein, and $\omega_\mu{}^{ab}$ is the spin connection. We define energy-momentum tensor and spin current as:

$$T^{\mu\nu} = \frac{1}{|e|}\frac{\delta W}{\delta e_\mu{}^a} e^{\nu a}, \qquad S^\mu{}_{ab} = \frac{2}{|e|}\frac{\delta W}{\delta \omega_\mu{}^{ab}}. \tag{4.2}$$

Consequently, the variation of the effective action is given by:

$$\begin{aligned}\delta W &= \int \mathrm{d}^d x \left(\frac{\delta W}{\delta e_\mu{}^a}\delta e_\mu{}^a + \frac{\delta W}{\delta \omega_\mu{}^{ab}}\delta \omega_\mu{}^{ab}\right) \\ &= \int \mathrm{d}^d x |e| \left(T^{\mu\nu} e_{\nu a}\delta e_\mu{}^a + \frac{1}{2} S^\mu{}_{ab}\delta\omega_\mu{}^{ab}\right).\end{aligned} \tag{4.3}$$

These definitions are nontrivial and worth investigating. First of all, we defined energy-momentum tensor as the variation of effective action with respect to vielbein rather than the metric. This definition breaks the symmetry of the energy-momentum tensor and allows it to have an anti-symmetric part. This anti-symmetric part corresponds to an intrinsic torque in the fluid, which can couple to spin degrees of freedom. Furthermore, we have defined spin current as the variation of effective action with respect to the spin connection. This definition does not deviate from our previous discussion. However, there is a subtlety that one can be miss easily. In the absence of torsion, we have shown that we could write the spin connection in terms of the vielbein by making use of the vielbein postulate. As a consequence, variation of the spin connection becomes dependent on the variation of vielbein. This situation makes the definition of a spin current ambiguous. Let us investigate this ambiguity and outline how to lift it.

4.1 Spin Current Ambiguity and Belinfante-Rosenfeld Transformation

To illustrate the spin current ambiguity, suppose spacetime has no torsion. Thus, any term we add to the effective action that is proportional to contorsion is essentially zero. Now, we introduce a "zero" to the effective action as

$$W'[e_\mu{}^a, \omega_\mu{}^{ab}] = W[e_\mu{}^a, \omega_\mu{}^{ab}] + W_{BR}[e_\mu{}^a, \omega_\mu{}^{ab}], \tag{4.4}$$

where

$$W_{BR}[e_\mu{}^a, \omega_\mu{}^{ab}] = \int \mathrm{d}^d x |e| B^\mu{}_{ab} K_\mu{}^{ab}, \tag{4.5}$$

and $B^\mu{}_{ab}$ is an arbitrary tensor (anti-symmetric in the last two indices), which we assume to be independent of the vielbein and the spin connection for brevity. After the introduction of the new term, we express the modifications to the currents as

$$T'^{\mu\nu} \to T^{\mu\nu} + T'^{\mu\nu}_{BR}, \qquad S'^\mu{}_{ab} \to S^\mu{}_{ab} + S^\mu_{BRab} \tag{4.6}$$

where $T^{\mu\nu}$ and $S^\mu{}_{ab}$ are the original currents while $T^{\mu\nu}_{BR}$ and S^μ_{BRab} are the modifications. To find the closed form of the modifications, we calculate the contorsion variation using Eqs. (3.18, 3.19), then, the variation is given by

$$\begin{aligned} \delta K_\mu{}^{ab} &= \delta\omega_\mu{}^{ab} - \delta\mathring{\omega}_\mu{}^{ab} \\ &= \delta\omega_\mu{}^{ab} - \left[\mathring{D}_\mu\left(e^{\sigma[a}\delta e_\sigma{}^{b]}\right) - \mathring{D}_\sigma\left(e^{\sigma[a}\delta e_\mu{}^{b]}\right) - \mathring{D}_\sigma\left(e^{\sigma[a|}e^{\rho|b]}e_{\mu c}\delta e_\rho{}^c\right)\right]. \end{aligned} \tag{4.7}$$

Now, we can calculate the modifications to the currents since we assumed $B^\mu{}_{ab}$ to be independent of the sources. The variation of the modification term becomes,

$$\begin{aligned} \delta W_{BR} &= \int \mathrm{d}^d x |e| B^\mu{}_{ab} \delta K_\mu{}^{ab} \\ &= \int \mathrm{d}^d x |e| B^\mu{}_{ab} \Bigg\{ \delta\omega_\mu{}^{ab} - \bigg[\mathring{D}_\mu\left(e^{\sigma[a}\delta e_\sigma{}^{b]}\right) \\ &\quad - \mathring{D}_\sigma\left(e^{\sigma[a}\delta e_\mu{}^{b]}\right) - \mathring{D}_\sigma\left(e^{\sigma[a|}e^{\rho|b]}\delta e_\rho{}^c\right)\bigg]\Bigg\} \\ &= \int \mathrm{d}^d x |e| \bigg[\mathring{\nabla}_\lambda\left(B^{\lambda\mu\nu} - B^{\mu\lambda\nu} - B^{\nu\lambda\mu}\right) e_{\nu a}\delta e_\mu{}^a \\ &\quad + B^\mu{}_{ab}\delta\omega_\mu{}^{ab}\bigg], \end{aligned} \tag{4.8}$$

where we used integration by part to obtain third line from the second. Then, the currents have the form

$$T^{\mu\nu}_{BR} = \mathring{\nabla}_\lambda \left(B^{\lambda\mu\nu} - B^{\mu\lambda\nu} - B^{\nu\lambda\mu} \right), \qquad S^{\mu}_{BRab} = 2B^{\mu}{}_{ab}. \tag{4.9}$$

These modifications terms cannot describe any physics since they are zero contributions. This implies that they are reminiscents of a residual degree of freedom, and one can choose any $B^{\mu}{}_{ab}$ to redefine the energy-momentum tensor and the spin current without changing the physics. This redefinition is called **Belinfante-Rosenfeld Transformation** [6, 42]. This degree of freedom has an interesting consequence. One is free to choose $B^{\mu}{}_{ab} = -\frac{1}{2}S^{\mu}{}_{ab}$ to set $S'^{\mu}{}_{ab} = 0$ and claim the spin current is nothing but a residual degree of freedom which does not contribute to the physics of the fluid. This claim makes the spin current definition ambiguous since the spin degree of freedom is a fundamental physical degree of freedom.

To lift this ambiguity, we need to relax the vielbein postulate to include a non-vanishing torsion. When spacetime has non-trivial torsion, it fixes Belinfante-Rosenfeld degree of freedom automatically because the underlying contribution to the effective action is not zero anymore. Therefore, we need to introduce torsion to spacetime to have a well-defined spin current.

At this point, we would like to emphasize that we are not claiming there has to be torsion in spacetime fundamentally. We are using torsion merely as a mathematical tool to resolve an ambiguity. We can always take the vanishing torsion limit at the end of our calculations but not at the beginning. How we are using torsion here is analogous to how we use curvature to define the energy-momentum tensor. Even if we are working with a flat metric, we define the energy-momentum tensor as the variation of the action with respect to the metric, i.e., we introduce an infinitesimal curvature. This definition of the energy-momentum tensor makes it symmetric automatically and that form is the one that couples to gravity without additional improvement terms.

Now, we proceed to the symmetries of the effective theory to find the equations of motion of the energy-momentum tensor and the spin current. Following our previous discussion, we demand the effective action to be gauge and diffeomorphism invariant.

4.2 Gauge Symmetry

The spin connection, $\omega_\mu{}^{ab}$ of the effective theory is the gauge field of $SO(1, d-1)$ by construction. Thus, we demand gauge invariance of the theory. For an infinitesimal Lorentz transformation, parametrized by λ, the change in fields can be read from Eqs.(3.7, 3.12), which are:

$$\delta e_\mu{}^a = -\lambda^a{}_b e_\mu{}^b, \qquad \delta\omega_\mu{}^{ab} = D_\mu \lambda^{ab}. \tag{4.10}$$

Therefore, we find the implication of local Lorentz symmetry by setting the variation of the action to zero, i.e.,

$$\begin{aligned} 0 &= \int \mathrm{d}^d x |e| \left(-\lambda^{ab} e_{\mu b} T^{\mu}{}_{a} + \frac{1}{2} (D_\mu \lambda^{ab}) S^{\mu}{}_{ab} \right) \\ &= \frac{1}{2} \int \mathrm{d}^d x |e| \left(2\lambda^{ab} T_{[ab]} + \mathring{\nabla}_\mu \left(\lambda^{ab} S^{\mu}{}_{ab} \right) - \mathring{\Gamma}^{\mu}_{\mu\rho} \lambda^{ab} S^{\rho}{}_{ab} - \lambda^{ab} D_\mu \left(S^{\mu}{}_{ab} \right) \right) \\ &= \frac{1}{2} \int \mathrm{d}^d x |e| \lambda^{ab} \left(2T_{[ab]} + K^{\mu}{}_{\mu\rho} S^{\rho}{}_{ab} - \nabla_\mu \left(S^{\mu}{}_{ab} \right), \right) \end{aligned} \tag{4.11}$$

and we obtain $\nabla_\mu S^{\mu}{}_{\rho\sigma} = 2T_{[\rho\sigma]} + K^{\mu}{}_{\mu\lambda} S^{\lambda}{}_{\rho\sigma}$, which is the equation of motion for the spin current.

4.3 Diffeomorphism Invariance

To obtain the dynamics of energy-momentum tensor, we demand diffeomorphism invariance of the effective theory. Transformation rules for an infinitesimal diffeomorphism is given by a Lie derivative along the vector ξ:

$$\begin{aligned} \delta e_\mu{}^{a} &= \mathcal{L}_\xi e_\mu{}^{a} = \xi^\nu \partial_\nu e_\mu{}^{a} + \left(\partial_\mu \xi^\nu \right) e_\nu{}^{a} \\ &= \xi^\nu \partial_\nu e_\mu{}^{a} + \partial_\mu \left(\xi^\nu e_\nu{}^{a} \right) - \xi^\nu \partial_\mu e_\nu{}^{a} \\ &= 2\xi^\nu \left(K_{[\nu}{}^{a}{}_{\mu]} - \omega_{[\nu|}{}^{a}{}_{b} e_{|\mu]}{}^{b} \right) + \partial_\mu \left(\xi^\nu e_\nu{}^{a} \right) \\ &= D_\mu \left(\xi^\nu e_\nu{}^{a} \right) + \xi^\nu \left(2K_{[\nu}{}^{a}{}_{\mu]} - \omega_\nu{}^{a}{}_{b} e_\mu{}^{b} \right), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \delta \omega_\mu{}^{ab} &= \mathcal{L}_\xi \omega_\mu{}^{ab} = \xi^\nu \partial_\nu \omega_\mu{}^{ab} + \left(\partial_\mu \xi^\nu \right) \omega_\nu{}^{ab} \\ &= \xi^\nu \partial_\nu \omega_\mu{}^{ab} - \xi^\nu \partial_\mu \omega_\nu{}^{ab} + \partial_\mu \left(\xi^\nu \omega_\nu{}^{ab} \right) \\ &= \xi^\nu \left(R^{ab}{}_{\nu\mu} - 2\omega_{[\nu}{}^{ac} \omega^{b}_{\mu]c} \right) + \partial_\mu \left(\xi^\nu \omega_\nu{}^{ab} \right) \\ &= \xi^\nu R^{ab}{}_{\nu\mu} + D_\mu \left(\xi^\nu \omega_\nu{}^{ab} \right). \end{aligned} \tag{4.13}$$

When we demand diffeomorphism invariance, we find

$$
\begin{aligned}
0 =& \int \mathrm{d}^d x |e| \left(T^{\mu}{}_{a} \delta e_{\mu}{}^{a} + \frac{1}{2} S^{\mu}{}_{ab} \delta \omega_{\mu}{}^{ab} \right) \\
=& \int \mathrm{d}^d x |e| \Big[T^{\mu}{}_{a} \left(D_{\mu} \left(\xi^{\nu} e_{\nu}{}^{a} \right) + \xi^{\nu} \left(2K_{[\nu}{}^{a}{}_{\mu]} - \omega_{\nu}{}^{a}{}_{b} e_{\mu}{}^{b} \right) \right) \\
&+ \frac{1}{2} S^{\mu}{}_{ab} \left(\xi^{\nu} R^{ab}{}_{\nu\mu} + D_{\mu} \left(\xi^{\nu} \omega_{\nu}{}^{ab} \right) \right) \Big] \\
=& \int \mathrm{d}^d x |e| \xi^{\nu} \Big[- e_{\nu}{}^{a} D_{\mu} T^{\mu}{}_{a} - \mathring{\Gamma}^{\mu}_{\mu\rho} T^{\rho}{}_{\nu} + T^{\mu}{}_{a} \left(2K_{[\nu}{}^{a}{}_{\mu]} - \omega_{\nu}{}^{a}{}_{b} e_{\mu}{}^{b} \right) \\
&+ \frac{1}{2} S^{\mu}{}_{ab} R^{ab}_{\nu\mu} - \omega_{\nu}{}^{ab} T_{[ab]} \Big] \\
=& \int \mathrm{d}^d x |e| \xi^{\nu} \left[-\nabla_{\mu} T^{\mu}{}_{\nu} + K_{\mu}{}^{\mu}{}_{\rho} T^{\rho}{}_{\nu} + 2K_{[\nu}{}^{\rho}{}_{\mu]} T^{\mu}{}_{\rho} + \frac{1}{2} S^{\mu}{}_{ab} R^{ab}{}_{\nu\mu} \right],
\end{aligned}
\tag{4.14}
$$

where we used the Spin current equation of motion in the third line. Finally, we obtain:

$$
\nabla_{\mu} T^{\mu}{}_{\nu} = K_{\mu}{}^{\mu}{}_{\rho} T^{\rho}{}_{\nu} + 2K_{[\nu}{}^{\rho}{}_{\mu]} T^{\mu}{}_{\rho} + \frac{1}{2} S^{\mu}{}_{ab} R^{ab}{}_{\nu\mu}, \tag{4.15}
$$

which is the equation of motion of the energy-momentum tensor.

Furthermore, we can rewrite the equations of motion in a more convinient way using Levi-Civita connection:

$$
\boxed{
\begin{aligned}
\mathring{\nabla}_{\mu} T^{\mu\nu} &= \frac{1}{2} R^{\rho\sigma\nu\lambda} S_{\lambda\rho\sigma} - T_{\rho\sigma} K^{\nu ab} e^{\rho}{}_{a} e^{\sigma}{}_{b}, \\
\mathring{\nabla}_{\lambda} S^{\lambda}{}_{\mu\nu} &= 2T_{[\mu\nu]} + 2S^{\lambda}{}_{\rho[\mu} e_{\nu]}{}^{a} e^{\rho b} K_{\lambda ab}\,.
\end{aligned}
}
\tag{4.16}
$$

To solve the equations of motion, we can decompose every tensorial object into scalars, vectors and tensors of $SO(d-1) \subset SO(1, d-1)$, where the Lorentz symmetry is preserved by the velocity profile u^{μ}. We call this decomposition as *hydrodynamic decomposition*. From now on, we set $d = 3$ and work in 2+1D spacetime.

Chapter 5

Field Decompositions in 2+1D

In this section, we give the decompositions of tensorial quantities in the theory of spin hydrodynamics in 2+1D spacetime with respect to velocity profile u^μ. The velocity profile is normalized as $u^\mu u_\mu = -1$ and we define the projection tensor $\Delta_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$.

5.1 Riemann Tensor in Levi-Civita Connection

In 2+1D, Riemann tensor ($\mathring{R}_{\mu\nu\rho\sigma}$) has six degrees of freedom. Thus, it can be written in terms of Ricci scalar ($\mathring{R}$) and Ricci tensor ($\mathring{R}_{\mu\nu}$). First, we define symmetric tensor $\mathcal{C}_{\mu\nu}$,

$$\mathcal{C}_{\mu\nu} := \mathring{R}_{\mu\nu} - \frac{\mathring{R}}{4} g_{\mu\nu}. \tag{5.1}$$

Then, Riemann tensor can be written as,

$$\mathring{R}_{\mu\nu\rho\sigma} = \mathcal{C}_{\mu\rho} g_{\nu\sigma} + \mathcal{C}_{\nu\sigma} g_{\mu\rho} - \mathcal{C}_{\mu\sigma} g_{\nu\rho} - \mathcal{C}_{\nu\rho} g_{\mu\sigma}. \tag{5.2}$$

This decomposition is only due to the dimensionality of our spacetime. We can further decompose $\mathcal{C}_{\mu\nu}$ into its parallel and transverse components with respect to the velocity profile u^μ:

$$\mathcal{C}_{\mu\nu} = \left(\mathring{R}_u + \frac{\mathring{R}}{4} \right) u_\mu u_\nu + \frac{1}{2} \left(\mathring{R}_u + \frac{\mathring{R}}{2} \right) \Delta_{\mu\nu} - 2u_{(\mu} \Upsilon_{\nu)} + \xi_{\mu\nu}, \tag{5.3}$$

where $u^\mu \Upsilon_\mu = 0$, $u^\mu \xi_{\mu\nu} = 0$, and $\xi_{\mu\nu}$ is a symmetric traceless tensor. This decomposition can be inverted by:

$$\begin{aligned}
\mathring{R} &= g^{\mu\rho} g^{\nu\sigma} \mathring{R}_{\mu\nu\rho\sigma}, \\
\mathring{R}_u &= u^\mu u^\rho g^{\nu\sigma} \mathring{R}_{\mu\nu\rho\sigma}, \\
\Upsilon_\alpha &= u^\mu \Delta_\alpha{}^\rho g^{\nu\sigma} \mathring{R}_{\mu\nu\rho\sigma}, \\
\xi_{\mu\nu} &= \left(\Delta_\mu{}^\alpha \Delta_\nu{}^\rho - \frac{1}{2} \Delta^{\alpha\rho} \Delta_{\mu\nu} \right) g^{\beta\sigma} \mathring{R}_{\alpha\beta\rho\sigma},
\end{aligned} \tag{5.4}$$

and this finalizes the decomposition of curvature in Levi-Civita connection. Therefore, we have decomposed Riemann tensor into two scalars, one transverse vector and one transverse symmetric traceless rank-2 tensor. As a side note, we have used the equation below to eliminate $\mathcal{C}_{\mu\nu}$ completely,

$$\mathcal{C}_{\alpha\beta} = \left(\delta^{\mu}_{\alpha}\delta^{\rho}_{\beta} - \frac{1}{4}g^{\mu\rho}g_{\alpha\beta}\right) g^{\nu\sigma}\mathring{R}_{\mu\nu\rho\sigma}. \tag{5.5}$$

At this point, one might question what happens when torsion is nontrivial. Luckily, we can still make use of the decomposition that we have just outlined. To illuminate how to deal with torsion, we introduce contorsion.

5.2 Contorsion

Contorsion tensor, $K_{\mu}{}^{ab}$ is antisymmetric in last two indices, thus, it has nine degrees of freedom. We can decompose it into a scalar, two pseudo-scalars, a transverse vector, a transverse pseudo-vector, a transverse symmetric traceless rank-2 tensor.

$$K_{\mu}{}^{ab} = -u_{\mu}k^{ab} + 2u^{[a}\kappa^{b]}{}_{\mu} + \frac{1}{2}\mathcal{K}_{\mu}\epsilon^{abc}u_{c}, \tag{5.6}$$

where we have decomposed the total 9 degrees of freedom as $9 = 3 + 4 + 2$. We can further decompose the first two terms as

$$\begin{aligned} k^{ab} &= 2u^{[a}k^{b]} + \frac{1}{2}k\epsilon^{abc}u_{c}, \\ \kappa^{\mu\nu} &= \frac{1}{2}\kappa_{\mathrm{T}}\Delta^{\mu\nu} + \kappa^{\mu\nu}_{\mathrm{S}} - \frac{1}{2}\kappa_{\mathrm{A}}\epsilon^{\mu\nu\rho}u_{\rho}, \end{aligned} \tag{5.7}$$

where 3 degrees of freedom of k^{ab} is decomposed as $3 = 2 = 1$ and 4 degrees of freedom of $\kappa^{\mu\nu}$ is decomposed as $4 = 1 + 2 + 1$. We can invert this decomposition as:

$$\begin{aligned} k^{a} &= u^{\mu}u_{b}K_{\mu}{}^{ab}, \\ k &= \epsilon_{abd}u^{\mu}u^{d}K_{\mu}{}^{ab}, \\ \kappa_{\mathrm{T}} &= u_{b}\Delta^{\mu}{}_{a}K_{\mu}{}^{ab}, \\ \kappa^{\alpha\beta}_{\mathrm{S}} &= \left(\Delta^{(\alpha}{}_{a}\Delta^{\beta)\mu} - \frac{1}{2}\Delta^{\mu}{}_{a}\Delta^{\alpha\beta}\right) u_{b}K_{\mu}{}^{ab}, \\ \kappa_{\mathrm{A}} &= \epsilon^{\mu}{}_{\nu\rho}e^{\nu}{}_{a}u_{b}u^{\rho}K_{\mu}{}^{ab}, \\ \mathcal{K}_{\alpha} &= \epsilon_{abc}u^{c}\Delta_{\alpha}{}^{\mu}K_{\mu}{}^{ab}. \end{aligned} \tag{5.8}$$

Moreover, we introduce field strength of contorsion, $\mathcal{G}^{\lambda}{}_{\alpha\rho\sigma}$ as

$$\mathcal{G}^{\lambda}{}_{\alpha\rho\sigma} = \mathring{\nabla}_{\rho}K_{\sigma}{}^{\lambda}{}_{\alpha} - \mathring{\nabla}_{\sigma}K_{\rho}{}^{\lambda}{}_{\alpha} + K_{\rho}{}^{\lambda}{}_{\beta}K_{\sigma}{}^{\beta}{}_{\alpha} - K_{\sigma}{}^{\lambda}{}_{\beta}K_{\rho}{}^{\beta}{}_{\alpha}. \tag{5.9}$$

Then, we can decompose the full Riemann tensor into its torsion-less and torsion-full sectors as $R^{\lambda}{}_{\alpha\rho\sigma} = \mathring{R}^{\lambda}{}_{\alpha\rho\sigma} + \mathcal{G}^{\lambda}{}_{\alpha\rho\sigma}$. Therefore, we can still use our hydrodynamic decomposition in the previous section as promised.

5.3 Spin Connection

Spin connection, $\omega_{\mu}{}^{ab}$, has the same symmetries as the contorsion tensor. Thus, we can make an analogous decomposition. The only difference is that we identify its temporal component with the spin chemical potential whose reason will become apparent in the upcoming chapter. Therefore, the decomposition can be stated as:

$$\omega_{\mu}{}^{ab} = -u_{\mu}\mu^{ab} + 2u^{[a}t^{b]}{}_{\mu} + \frac{1}{2}\tau_{\mu}\epsilon^{abc}u_{c}, \tag{5.10}$$

where

$$\begin{aligned} \mu^{ab} &= 2u^{[a}m^{b]} + M\epsilon^{abc}u_{c}, \\ t^{\mu\nu} &= \frac{1}{2}t_{\mathrm{T}}\Delta^{\mu\nu} + t_{\mathrm{S}}^{\mu\nu} - \frac{1}{2}t_{\mathrm{A}}\epsilon^{\mu\nu\rho}u_{\rho}. \end{aligned} \tag{5.11}$$

Here, we identify μ^{ab} as the spin chemical potential, $t^{\mu\nu}$ as the boost potential, and τ_{μ} as the spatial spin flow potential. Furthermore, the individual components are given by:

$$\begin{aligned} m^{a} &= u_{b}\mu^{ab}, \\ M &= \frac{1}{2}\epsilon_{abc}u^{a}\mu^{bc}, \\ t_{\mathrm{T}} &= u_{b}\Delta^{\mu}{}_{a}\omega_{\mu}{}^{ab}, \\ t_{\mathrm{S}}^{\alpha\beta} &= \left(\Delta^{(\alpha}{}_{a}\Delta^{\beta)\mu} - \frac{1}{2}\Delta^{\mu}{}_{a}\Delta^{\alpha\beta}\right)u_{b}\omega_{\mu}{}^{ab}, \\ t_{\mathrm{A}} &= \epsilon^{\mu}{}_{\nu\rho}e^{\nu}{}_{a}u_{b}u^{\rho}\omega_{\mu}{}^{ab}, \\ \tau_{\alpha} &= \epsilon_{abc}u^{c}\Delta_{\alpha}{}^{\mu}\omega_{\mu}{}^{ab}. \end{aligned} \tag{5.12}$$

5.4 Energy-Momentum Tensor

Energy-momentum tensor, $T^{\mu\nu}$ is a rank-2 tensor and it does not have a symmetry restriction. Thus, it has nine degrees of freedom and it can be decomposed into two scalars, one pseudo-scalar, two transverse vectors and one transverse symmetric traceless rank-2 tensor. The decomposition is given by:

$$T^{\mu\nu} = \mathcal{E}u^{\mu}u^{\nu} + \mathcal{P}\Delta^{\mu\nu} + u^{(\mu}q^{\nu)} + u^{[\mu}h^{\nu]} + \pi^{\mu\nu} + \mathscr{T}\epsilon^{\mu\nu\rho}u_{\rho} \tag{5.13}$$

where $\mathcal{E}$ and $\mathcal{P}$ are scalars, $\mathscr{T}$ is a pseudo-scalar, q^μ and h^μ are transverse vectors, and $\pi^{\mu\nu}$ is a transverse symmetric traceless rank-2 tensor. Furthermore, we identify $\mathcal{E}$ with energy density, $\mathcal{P}$ with pressure, $\pi^{\mu\nu}$ with shear stress, $\mathscr{T}$ with intrinsic torque, q^μ and h^μ with heat currents. The individual components are given by:

$$
\begin{aligned}
\mathcal{E} &= u_\mu u_\nu T^{\mu\nu}, \\
\mathcal{P} &= \frac{1}{2}\Delta_{\mu\nu} T^{\mu\nu}, \\
q^\mu &= -2u_{(\alpha}\Delta^\mu{}_{\beta)} T^{\alpha\beta}, \\
h^\mu &= 2u_{[\beta}\Delta^\mu{}_{\alpha]} T^{\alpha\beta}, \\
\pi^{\mu\nu} &= \left(\Delta^\mu{}_{(\alpha}\Delta^\nu{}_{\beta)} - \frac{1}{2}\Delta_{\alpha\beta}\Delta^{\mu\nu}\right) T^{\alpha\beta}, \\
\mathscr{T} &= \frac{1}{2}\epsilon_{\alpha\beta\rho} u^\rho T^{\alpha\beta}.
\end{aligned}
\tag{5.14}
$$

5.5 Spin Current

Spin current shares the same symmetries with contorsion. This leads to a decomposition of the same form:

$$
S_\mu{}^{ab} = -u_\mu s^{ab} + 2u^{[a}\varsigma^{b]}{}_\mu + \frac{1}{2}\Sigma_\mu \epsilon^{abc} u_c \tag{5.15}
$$

where

$$
\begin{aligned}
s^{ab} &= 2u^{[a}s^{b]} + \frac{1}{2}\mathfrak{S}\epsilon^{abc}u_c, \\
\varsigma^{\mu\nu} &= \frac{1}{2}\varsigma_{\mathrm{T}}\Delta^{\mu\nu} + \varsigma_{\mathrm{S}}^{\mu\nu} - \frac{1}{2}\varsigma_{\mathrm{A}}\epsilon^{\mu\nu\rho}u_\rho.
\end{aligned}
\tag{5.16}
$$

Here we identify s^{ab} as the spin density, $\varsigma^{\mu\nu}$ as the boost current, and Σ_μ as the spatial spin flow. We can invert this decomposition as:

$$
\begin{aligned}
s^a &= u^\mu u_b S_\mu{}^{ab}, \\
\mathfrak{S} &= \epsilon_{abd} u^\mu u^d S_\mu{}^{ab}, \\
\varsigma_{\mathrm{T}} &= u_b \Delta^\mu{}_a S_\mu{}^{ab}, \\
\varsigma_{\mathrm{S}}^{\alpha\beta} &= \left(\Delta^{(\alpha}{}_a\Delta^{\beta)\mu} - \frac{1}{2}\Delta^\mu{}_a\Delta^{\alpha\beta}\right) u_b S_\mu{}^{ab}, \\
\varsigma_{\mathrm{A}} &= \epsilon^\mu{}_{\nu\rho} e^\nu{}_a u_b u^\rho S_\mu{}^{ab}, \\
\Sigma_\alpha &= \epsilon_{abc} u^c \Delta_\alpha{}^\mu S_\mu{}^{ab}.
\end{aligned}
\tag{5.17}
$$

TABLE 5.1: Hydrodynamic decomposition of spin current, spin source and contorsion. Every spin current component is paired with its source and the contorsion component that is conjugate to the source.

Spin current	Spin Source	Torsion Conjugate	Degrees of Freedom
s^μ	m^μ	k^μ	2
$\mathfrak{S}$	M	k	1
ς_{T}	t_{T}	κ_{T}	1
$\varsigma_{\mathrm{S}}^{\mu\nu}$	$t_{\mathrm{S}}^{\mu\nu}$	$\kappa_{\mathrm{S}}^{\mu\nu}$	2
ς_{A}	t_{A}	κ_{A}	1
Σ^μ	τ^μ	$\mathcal{K}^\mu$	2

5.6 Gradient of Fluid Velocity

In addition to the decompositions of primitive tensors, we will make use of the decomposition of the derivative of fluid velocity throughout the thesis. Taking the normalization condition into account, we can decompose the derivative as

$$\mathring{\nabla}_\mu u_\nu = -u_\mu a_\nu + \frac{1}{2}\Delta_{\mu\nu}\Theta + \sigma_{\mu\nu} + \Omega\epsilon_{\mu\nu\rho}u^\rho, \tag{5.18}$$

where we identify, a_μ with acceleration, Θ with expansion rate, $\sigma_{\mu\nu}$ with shear, and Ω with vorticity. They are given individually by:

$$\begin{aligned} a_\mu &= u^\nu \mathring{\nabla}_\nu u_\mu, \\ \Theta &= \mathring{\nabla}_\mu u^\mu, \\ \sigma_{\mu\nu} &= \Delta^\alpha_\mu \Delta^\beta_\nu \mathring{\nabla}_{(\alpha} u_{\beta)} - \frac{1}{2}\Theta\Delta_{\mu\nu}, \\ \Omega &= \frac{1}{2}\epsilon^{\alpha\mu\nu} u_\alpha \mathring{\nabla}_\mu u_\nu. \end{aligned} \tag{5.19}$$

This concludes our discussion of field decompositions. At this point, we are ready to solve the equations of motion.

Chapter 6

Ideal Spin Hydrostatics in 2+1D

Now, we turn to the spin hydrodynamics for non-dissipative systems in 2+1D. We start by identifying hydrodynamic variables in terms of the time-like vector V^μ:

$$T := \frac{T_0}{\sqrt{-V^2}}, \qquad u^\mu := \frac{V^\mu}{\sqrt{-V^2}}, \qquad \mu^{ab} := \frac{\omega_\mu{}^{ab} V^\mu}{\sqrt{-V^2}} . \tag{6.1}$$

We identify T with temperature, u^μ with velocity profile and μ^{ab} with spin chemical potential. Hydrostatic equilibrium condition is defined to be

$$\mathcal{L}_V = 0, \tag{6.2}$$

which means the Lie derivative of any object with respect to the time-like vector V vanishes. This implies that V is a time-like Killing vector in equilibrium. At this point, one might question the importance of a time-like Killing vector or even the existence and uniqueness of it so that above-defined fluid parameters are unambiguous. Let us justify these crucial points. First of all, the simplest answer to this set of questions is "an observer moving tangential to the time-like Killing field sees the system stationary," which makes perfect sense as an equilibrium condition. However, let us dive even deeper to gain a more comprehensive understanding of the definition. A Killing vector, by virtue, is directly related to a conserved quantity. In equilibrium, our systems settles at the maximal entropy state, which is dictated by the second law of thermodynamics. This implies that there cannot be any dissipative processes happening since they would increase the entropy of the system. Now, we established that the equilibrium state is a non-dissipative state, and this has further implications, namely., a conserved (free)energy. Therefore, there has to be a Killing vector associated to it. However, we did not use *any* Killing vector to define the equilibrium, we used a *time-like* one. This has a particular reason. The energy of the system is determined by its Hamiltonian and Hamiltonian itself describes the time evolution of the system, in other words, it is the generator of time-translations. Thus, the Killing field associated to it has to be time-like and uniqueness of Hamiltonian implies the uniqueness of time-like Killing vector.

Since we have internalized the meaning of the equilibrium condition, we can proceed to its implications. Using the definitions in 6.1, we can rewrite expansion (Θ), shear ($\sigma_{\mu\nu}$), temporal contorsion ($k^{\mu\nu}$), derivatives of temperature and chemical potential as

$$\begin{aligned}
\Theta = \mathring{\nabla}_\mu u^\mu &= \frac{1}{T_0}\mathcal{L}_V T + \frac{T}{2T_0} g^{\mu\nu}\mathcal{L}_V g_{\mu\nu}, \\
\sigma_{\mu\nu} &= \frac{T}{2T_0}\Delta_\mu{}^\alpha \Delta_\nu{}^\beta \mathcal{L}_V g_{\alpha\beta} - \frac{1}{2T_0}\left(\mathcal{L}_V T + \frac{T}{2} g^{\alpha\beta}\mathcal{L}_V g_{\alpha\beta}\right), \\
k^{\mu\nu} &= u^\lambda K_\lambda{}^{\mu\nu} = \mu^{\mu\nu} + \Omega\epsilon^{\mu\nu\rho}u_\rho - 2u^{[\mu}a^{\nu]} - \frac{T}{T_0} e^{[\mu}{}_a \mathcal{L}_V e^{\nu]a}, \\
\mathring{\nabla}_\mu T &= -Ta_\mu + \frac{T}{T_0} u_\mu \mathcal{L}_V T + \frac{T^2}{T_0} u^\nu \mathcal{L}_V g_{\mu\nu} \\
&= -Ta_\mu + T\Theta u_\mu + \frac{T^2}{T_0}\left(\delta^\alpha_\mu u^\beta - \frac{1}{2}u_\mu g^{\alpha\beta}\right)\mathcal{L}_V g_{\alpha\beta}, \\
T\mathring{\nabla}_\mu\left(\frac{\mu^{ab}}{T}\right) &= R^{ab}{}_{\mu\nu}u^\nu + 2K_\mu{}^{[a}{}_c\,\mu^{b]c} + \frac{T}{T_0}\mathcal{L}_V \omega_\mu{}^{ab},
\end{aligned} \tag{6.3}$$

In equilibrium, expansion and shear of the fluid vanish. In addition to those relations, Lie derivatives of the other fluid parameters imply the relations given by,

$$\begin{gathered}
k^{ab} = u^\mu K_\mu{}^{ab} = \mu^{ab} + e_\mu{}^a e_\nu{}^b\left(\Omega\epsilon^{\mu\nu\rho}u_\rho - 2u^{[\mu}a^{\nu]}\right), \\
T\mathring{\nabla}_\mu\left(\frac{\mu^{ab}}{T}\right) = R^{ab}{}_{\mu\nu}u^\nu + 2K_\mu{}^{[a}{}_c\,\mu^{b]c}, \qquad a_\mu = -\frac{\mathring{\nabla}_\mu T}{T}.
\end{gathered} \tag{6.4}$$

The first expression tells us vorticity and acceleration are supported by torsion and spin chemical potential and they are not necessarily zero in an equilibrium state. When we take a vanishing torsion, the presence of non-vanishing spin polarization still generates acceleration and vorticity. This situation is not unexpected since it is known that spin couples to vorticity [13]. Moreover, the third expression tells us that temperature acts as a potential on the fluid, analogous to the relation $\mathbf{F} = -\nabla U$. Thus, a non-vanishing temperature gradient accelerates the spin fluid, a similar effect can be found in the literature as "spin Seebeck effect" [46]. Consequently, this thermal acceleration creates spin potential via the first expression. Finally, the second expression tells us how chemical potential gradient couples to curvature and torsion. In other words, the presence of torsion

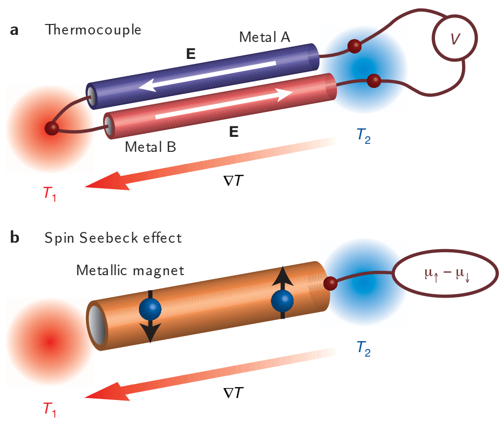


FIGURE 6.1: Schematic description of thermoelectric effect (a), and its analogous spin Seebeck effect (b). Figure is taken from [46].

and curvature creates a varying spin polarization. In addition, if we take a flat background with trivial torsion, we see that acceleration and spin polarization support each other.

Furthermore, these relations lead us to define two new out-of-equilibrium tensors, $\tilde{k}$ and $\tilde{k}^\mu$, following the decomposition in (5.7):

$$\begin{aligned} \tilde{k} &:= -2M - 2\Omega - k, \\ \tilde{k}^\mu &:= m^\mu - a^\mu - k^\mu. \end{aligned} \tag{6.5}$$

These first-order objects will be beneficial while building the out-of-equilibrium currents. Here, we would like to emphasize that we need to take spin chemical potential and temporal torsion as first-order in derivatives to keep (6.4) and (6.5) consistent in derivative orders. On the other hand, we are free to set spatial torsion and spin connection to any order. For simplicity, we set them to first-order as well.

6.1 Ideal Effective Action and Associated Currents

The effective action of the theory will depend on the scalars we can build in ideal fluid approximation. Therefore, we can only use the hydrodynamic variables, i.e., T, u^μ and μ^{ab}. In addition to the temperature, we can build two (pseudo-)scalars from the chemical potential, namely, M and $m^2 = m^a m_a$. This gives us a total of three scalars in ideal regime.

Now, we can write the ideal effective action as:

$$W_{\mathbf{id}} = \int \mathrm{d}^3x |e| P(T, M, m^2), \tag{6.6}$$

where subscript "**id**" stands for "ideal" and we identify P with the ideal pressure, i.e. the pressure of source-free state. Before moving on, let us remind ourselves the variation of the effective action in general:

$$\delta W = \int \mathrm{d}^3x |e| \left(T^\mu{}_a \delta e^a{}_\mu + \frac{1}{2} S^\mu{}_{ab} \delta \omega_\mu{}^{ab} \right). \tag{6.7}$$

At this point, one can calculate the energy-momentum tensor and spin current from the ideal effective action we have. To find those one point functions, we list the relevant variations:

$$\begin{aligned} \delta|e| &= |e| e_a{}^\mu \delta e^a{}_\mu, \quad \delta T = T u^\mu u_a \delta e^a{}_\mu, \\ \delta M &= \frac{1}{2} \epsilon_{abc} \mu^{bc} u^\mu \left(\Delta^a_d + u^a u_d \right) \delta e^d{}_\mu + \frac{1}{2} \epsilon_{abc} u^a u^\mu \delta \omega_\mu{}^{ab}, \\ \delta m^2 &= 2 u_c \mu^{ac} \mu_{af} u^\mu \left(\Delta^f_h + u^f u_h \right) \delta e^h{}_\mu + 2 u^c u_d u^\mu \mu_{ac} \delta \omega_\mu{}^{ad}. \end{aligned} \tag{6.8}$$

Now, we vary the ideal effective action to find,

$$\begin{aligned} \delta W_{\mathbf{id}} &= \int d^3x \left[P\delta|e| + |e| \left(\frac{\partial P}{\partial T}\delta T + \frac{\partial P}{\partial M}\delta M + \frac{\partial P}{\partial m^2}\delta m^2 \right) \right] \\ &= \int d^3x |e| \left(T^{\mu\nu}_{\mathbf{id}} e_{a\nu} \delta e^a{}_\mu + \frac{1}{2} S^\mu_{\mathbf{id}\,ab} \delta\omega_\mu{}^{ab} \right), \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} T^{\mu\nu}_{\mathbf{id}} &= \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} + u^\mu Q^\nu, \\ S^\mu_{\mathbf{id}\,\alpha\beta} &= u^\mu \rho_{\alpha\beta}, \end{aligned} \tag{6.10}$$

and components are given by

$$\begin{aligned} \varepsilon &= -P + \frac{\partial P}{\partial T} T + \frac{1}{2}\mu^{ab}\rho_{ab} \\ \rho_{\alpha\beta} &= \frac{\partial P}{\partial M}\epsilon_{\alpha\beta\rho}u^\rho + 4\frac{\partial P}{\partial m^2} m_{[\alpha}u_{\beta]} \\ Q_\mu &= \left(\frac{\partial P}{\partial M} + 2\frac{\partial P}{\partial m^2} M \right) \epsilon_{\mu\nu\rho}u^\nu m^\rho, \end{aligned} \tag{6.11}$$

where we identify ε as energy density, $\rho_{\alpha\beta}$ as spin charge density and Q_μ as a heat current. Notice that the presence of a spin polarization breaks the symmetry of energy-momentum tensor. This is what we expected and allowed at the first place. Moreover, spin polarization contributes to the energy density of the system which is also expected. In addition, we see that the explicit symmetry breaking of energy-momentum tensor is caused by non-symmetric heat current and when we take the equation of state independent from spin chemical potential, we recover the usual fluid energy momentum tensor which is symmetric.

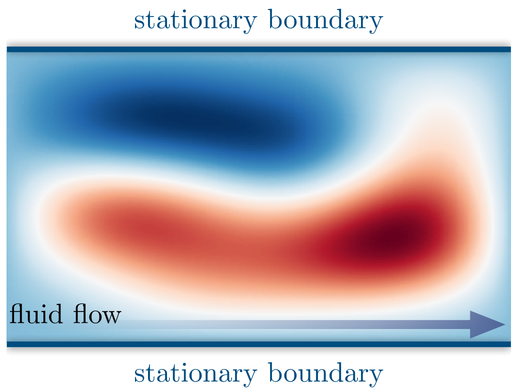


FIGURE 6.2: Schematic of spin accumulation in vanishing total magnetization. Opposite spins are mapped to red and blue.

Finally, (6.11) shows that spin accumulation is possible even if the spin chemical potential vanishes. This phenomenon is exclusive to 2+1D systems and does not exist in higher dimensions [17, 18]. In figure 6.2, we give a schematic of this phenomenon.

6.2 Ideal Entropy Current

Second law of thermodynamics states that the change in entropy of a system has to be non-negative, and it is zero only if the underlying thermodynamic process is reversible [28]. For an ideal fluid, the change of entropy is always zero since there is no dissipative process to increase entropy[30]. In covariant language, this corresponds to a conserved entropy current. Thus, we are looking for a conserved current in this section. To construct the ideal entropy current, we remind ourselves that P is identified with pressure of the ideal fluid. Then, naturally $s = \frac{\partial P}{\partial T}$ is the entropy density. Therefore, we can define entropy current in ideal regime to be $J^{\mu}_{\mathbf{s,id}} = su^{\mu}$. Now, we need to extract its covariant derivative from the equations of motion. Due to Gibbs-Duhem relation, we can write the derivative of the pressure to be

$$\partial_\mu P = s\partial_\mu T + \frac{1}{2}\rho_{ab}\mathring{\nabla}_\mu \mu^{ab} + Q_\nu \mathring{\nabla}_\mu u^\nu, \tag{6.12}$$

then, the derivative of the energy density follows as

$$\partial_\mu \varepsilon = T\partial_\mu s + \frac{1}{2}\mu^{ab}\mathring{\nabla}_\mu \rho_{ab} - Q_\nu \mathring{\nabla}_\mu u^\nu. \tag{6.13}$$

Moreover, we extract the parts of the equations of motion that we are interested in for this calculation, those parts are given by:

$$\begin{aligned} u_\nu \mathring{\nabla}_\mu T^{\mu\nu} &= -u_\nu T_{\rho\sigma} K^{\nu\rho\sigma} = Q_\mu k^\mu, \\ \mu^{\mu\nu}\left[\mathring{\nabla}_\lambda S^\lambda{}_{\mu\nu} - 2T_{[\mu\nu]}\right] &= 2\mu^{\mu\nu} S^\lambda{}_{\rho\mu} K_{\lambda\nu}{}^\rho = -2Q_\mu k^\mu. \end{aligned} \tag{6.14}$$

Finally, we tailor those pieces to find the covariant derivative of ideal entropy current:

$$\begin{aligned} -T\mathring{\nabla}_\mu J^\mu_{\mathbf{s,id}} &= -T\mathring{\nabla}_\mu \left(su^\mu\right) \\ &= u_\nu \mathring{\nabla}_\mu T^{\mu\nu} + \frac{1}{2}\mu^{\mu\nu}\left[\mathring{\nabla}_\lambda S^\lambda{}_{\mu\nu} - 2T_{[\mu\nu]}\right] \\ &= 0, \end{aligned} \tag{6.15}$$

where we used the equations of motion after the first line, then, we used our previously calculated quantities to show ideal entropy current is conserved as we expected from an ideal fluid.

Chapter 7

Gradient Corrections to Spin Hydrostatics in 2+1D

To describe a real fluid, we need to introduce gradient corrections to the ideal partition function, and we are interested in the corrections to the energy-momentum tensor and the spin current that are first order in the gradient expansion. However, to keep the theory self-consistent, equations of motion [4.16] constrain us to expand the antisymmetric part of energy-momentum tensor to one order higher than the spin current since it acts as a source.

We introduce the corrections by adding scalars to the effective action that are in the order we are interested in so that we can express the total effective action as:

$$W = W_{\mathbf{id}} + \sum_{n=1}^{2} W_{(n)} \tag{7.1}$$

where the correction effective action of order n is given by

$$W_{(n)} = \sum_{j} \int \mathrm{d}^3x |e| \chi_j^{(n)} \mathcal{S}_j^{(n)} \tag{7.2}$$

where $\chi_{\mathrm{j}}^{(n)}$ are functions of temperature. Here, we introduce the shorthand notation notation $\mathcal{S}_{\mathrm{i}}^{(n)}$, $\left(\mathcal{V}_{\mathrm{i}}^{(n)}\right)^{\mu}$, $\left(\mathcal{T}_{\mathrm{i}}^{(n)}\right)^{\{\mu_j\}}$ to denote i^{th} (pseudo)scalar, vector, and tensor of $\mathcal{O}(\partial^n)$ in derivatives respectively. From now on, we take the spin connection and contorsion to be first order in gradients. In condensed matter perspective, this choice implies a crystal geometry without dislocations since Burgers vector is related to the holonomy of spacetime [27]. Then, we proceed with building the tensorial objects to introduce the corrections.

7.1 Corrections to Effective Action

We need to build every independent first order scalar, and second order scalars that contribute to the spin current. When we take the variation of correction terms with respect to spin connection, second order scalars will contribute up to first order in spin current since we chose the spin connection to be first order in gradients. Moreover, second order corrections to the antisymmetric part of the energy-momentum tensor are created by the scalars that are at least first order in torsion since the energy-momentum tensor is symmetric in absence of torsion. This is consistent with our requirements for the spin current corrections.

We list the zeroth order independent objects in table 7.1.

i	$\mathcal{S}_\mathrm{i}^{(0)}$
1	T

(A) Scalar(s)

i	$\mathcal{V}_\mathrm{i}^{(0)}$
1	u^μ

(B) Vector(s)

i	$\mathcal{T}_\mathrm{i}^{(0)}$
1	$\Delta^{\mu\nu}$
2	$\epsilon_{\mu\nu\rho}u^\rho$

(C) Tensors

TABLE 7.1: List of 0^{th} order independent objects

First, we use contorsion components and gradients of zeroth order objects to construct first order objects. Notice that $\Theta = 0$ and Eqs.(6.4) are satisfied in equilibrium. Thus, we choose contorsion and chemical potential to be independent objects over acceleration and vorticity. These first order independent objects are listed in table (7.2).

i	$\mathcal{S}_\mathrm{i}^{(1)}$
1	κ_T
2	κ_A
3	k
4	M

(A) Scalars

i	$\mathcal{V}_\mathrm{i}^{(1)}$
1	k^μ
2	m^μ
3	$\mathcal{K}^\mu$
4	$\epsilon^{\mu\nu\rho}u_\nu k_\rho$
5	$\epsilon^{\mu\nu\rho}u_\nu m_\rho$
6	$\epsilon^{\mu\nu\rho}u_\nu \mathcal{K}_\rho$

(B) Vectors

i	$\mathcal{T}_\mathrm{i}^{(1)}$
1	$\kappa_\mathrm{S}^{\mu\nu}$
2	$\epsilon^{\mu\nu\rho}k_\rho$
3	$\epsilon^{\mu\nu\rho}m_\rho$
4	$\epsilon^{\mu\nu\rho}\mathcal{K}_\rho$

(C) Tensors

TABLE 7.2: List of 1^{st} order independent objects

Now, we use the first order objects to build second order scalars that are at least linear in torsion in table 7.3.

i	$\mathcal{S}_{\text{i}}^{(2)}$	9	kM
1	κ_{T}^2	10	$k^\mu k_\mu$
2	κ_{A}^2	11	$k^\mu m_\mu$
3	k^2	12	$k^\mu \mathcal{K}_\mu$
4	$\kappa_{\text{T}}\kappa_{\text{A}}$	13	$\mathcal{K}^\mu m_\mu$
5	$\kappa_{\text{T}}k$	14	$\mathcal{K}^\mu \mathcal{K}_\mu$
6	$\kappa_{\text{T}}M$	15	$\epsilon^{\mu\nu\rho}k_\mu u_\nu m_\rho$
7	$\kappa_{\text{A}}k$	16	$\epsilon^{\mu\nu\rho}k_\mu u_\nu \mathcal{K}_\rho$
8	$\kappa_{\text{A}}M$	17	$\epsilon^{\mu\nu\rho}m_\mu u_\nu \mathcal{K}_\rho$

TABLE 7.3: List of 2$^{\text{nd}}$ order independent scalars that are at least linear in torsion

This finalizes the construction of scalars. The corrections to the effective action are given by (7.2).

7.2 Corrections to Currents

In the last section, we have built the corrections to the effective action. Variations of those terms will give us the equilibrium corrections to energy-momentum tensor and spin current as stated in (6.7). For notational convenience, we introduce

$$\delta W_{(n)} = \int \mathrm{d}^3x|e| \left(\sum_{n,j} \left(T_{(n)}\right)^{\mu\nu} e_{\nu a}\delta e^a{}_\mu + \frac{1}{2}\sum_n \left(S_{(n)}\right)^\mu{}_{ab}\delta\omega_\mu{}^{ab} \right). \tag{7.3}$$

Now, we proceed with the corrections.

7.2.1 First Order Corrections

We use (7.2) and table (7.2a) to build the first order correction to the effective action which is given by

$$W_{(1)} = \int \mathrm{d}^3x|e| \left(\chi_1^{(1)}\kappa_{\text{T}} + \chi_2^{(1)}\kappa_{\text{A}} + \chi_3^{(1)}k \right). \tag{7.4}$$

When we take the variation of first order corrected effective action with respect to the vielbein, we find the corrections to the energy-momentum tensor, which are listed below. Further keep in mind that we are using the hydrodynamic decomposition given in chapter (5):

$$\begin{aligned}\mathcal{E}_{(1)} = &-\chi_1^{(1)}\left(\Theta + \kappa_{\text{T}}\right) + \frac{\partial \chi_1^{(1)}}{\partial T}T\kappa_{\text{T}} - \chi_2^{(1)}\left(k + \tilde{k} + 2M + \kappa_{\text{A}}\right) + \frac{\partial \chi_2^{(1)}}{\partial T}T\kappa_{\text{A}} \\ &+ 2\chi_2^{(1)}\left(k + \tilde{k} + 2M\right) + \frac{\partial \chi_3^{(1)}}{\partial T}Tk,\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}_{(1)} &= \frac{1}{2}\chi_1^{(1)}\left(\Theta + 3\kappa_{\mathrm{T}}\right) + \frac{\partial \chi_1^{(1)}}{\partial T} u^\mu \mathring{\nabla}_\mu T + \frac{1}{2}\chi_2^{(1)}\left(\kappa_{\mathrm{A}} - k - \tilde{k} - 2M\right) \\
&\quad + \chi_3^{(1)}\left(2k + \tilde{k} + 2M\right), \\
q^\mu_{(1)} &= -\chi_1^{(1)}\left(k^\mu + \tilde{k}^\mu - m^\mu - \frac{1}{2}\epsilon_{\alpha\beta\rho}u^\alpha \mathcal{K}^\beta g^{\rho\mu}\right) - \frac{\partial \chi_1^{(1)}}{\partial T}\Delta^{\mu\alpha}\mathring{\nabla}_\alpha T \\
&\quad + \chi_2^{(1)}\epsilon_{\alpha\beta\rho}g^{\rho\mu}u^\beta\left(3k^\alpha + \tilde{k}^\alpha - m^\alpha\right) - \frac{1}{2}\chi_2^{(1)}\mathcal{K}^\mu + \frac{\partial \chi_2^{(1)}}{\partial T}\epsilon_{\alpha\beta\rho}g^{\mu\rho}u^\alpha \mathring{\nabla}^\beta T \\
&\quad - 2\chi_3^{(1)}\epsilon_{\alpha\beta\rho}u^\beta g^{\mu\rho}\left(2k^\alpha + \tilde{k}^\alpha - m^\alpha\right) - 2\frac{\partial \chi_3^{(1)}}{\partial T}\epsilon_{\alpha\beta\rho}g^{\mu\rho}u^\alpha \mathring{\nabla}^\beta T, \\
h^\mu_{(1)} &= \chi_1^{(1)}\left(k^\mu + \tilde{k}^\mu - m^\mu + \frac{1}{2}\epsilon_{\alpha\beta\rho}u^\alpha \mathcal{K}^\beta g^{\rho\mu}\right) - \frac{\partial \chi_1^{(1)}}{\partial T}\Delta^{\mu\alpha}\mathring{\nabla}_\alpha T \\
&\quad - \chi_2^{(1)}\epsilon_{\alpha\beta\rho}g^{\rho\mu}u^\beta\left(k^\alpha + \tilde{k}^\alpha - m^\alpha\right) - \frac{1}{2}\chi_2^{(1)}\mathcal{K}^\mu - \frac{\partial \chi_2^{(1)}}{\partial T}\epsilon_{\alpha\beta\rho}g^{\mu\rho}u^\alpha \mathring{\nabla}^\beta T \\
&\quad + 2\chi_3^{(1)}\epsilon_{\alpha\beta\rho}u^\beta g^{\mu\rho}\left(\tilde{k}^\alpha - m^\alpha\right), \\
\pi^{\mu\nu}_{(1)} &= \chi_1^{(1)}\left(\kappa_{\mathrm{S}}^{\mu\nu} - \sigma^{\mu\nu}\right) - \chi_2^{(1)}\epsilon_{\alpha\beta\lambda}u^\alpha g^{\beta(\mu}\left(\kappa_{\mathrm{S}}^{\nu)\lambda} + \sigma^{\nu)\lambda}\right) + 2\chi_3^{(1)}\epsilon_{\alpha\beta\lambda}u^\alpha g^{\beta(\mu}\sigma^{\nu)\lambda}, \\
\mathscr{T}_{(1)} &= -\frac{1}{2}\chi_1^{(1)}\left(k + \tilde{k} + 2M - \kappa_{\mathrm{A}}\right) - \frac{1}{2}\chi_2^{(1)}\left(\Theta + \kappa_{\mathrm{T}}\right) + \chi_3^{(1)}\Theta + \frac{\partial \chi_3^{(1)}}{\partial T}u^\alpha \mathring{\nabla}_\alpha T. \qquad (7.5)
\end{aligned}
$$

Furthermore, when we take the variation of first order corrected effective action with respect to the spin connection, we find the corrections to the spin current, which are also listed as

$$
\begin{aligned}
s^\mu_{(0)} &= 0, & \mathfrak{S}_{(0)} &= -4\chi_3^{(1)}, \\
\varsigma_{\mathrm{T}(0)} &= -2\chi_1^{(1)}, & \varsigma_{\mathrm{S}\ (0)}^{\alpha\beta} &= 0, \\
\varsigma_{\mathrm{A}(0)} &= -2\chi_2^{(1)}, & \Sigma^\mu_{(0)} &= 0.
\end{aligned} \qquad (7.6)
$$

7.2.2 Second Order Corrections

As we have mentioned before, we have to introduce the second order corrections to the anti-symmetric part of the energy-momentum tensor to keep the theory consistent since that part acts as a source to the first order spin current. Now, we make use of (7.2) and table (7.3) once more to introduce the second order corrections. However, this time we will omit the corrections to spin current and the symmetric part of energy-momentum tensor. The correction effective action is given by:

$$
W_{(2)} = \int \mathrm{d}^3x|e| \sum_{j=1}^{17}\left(\chi_j^{(2)}\mathcal{S}_j^{(2)}\right). \qquad (7.7)
$$

Therefore, the anti-symmetric corrections to energy-momentum tensor is given by:

$$
\begin{aligned}
h^{\lambda}_{(2)} = {}& 2u_{[\beta}\Delta^{\lambda}{}_{\alpha]}T^{\alpha\beta}_{BR,(2)} + \epsilon_{\mu\nu\alpha}u^{\mu}\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{1} - \epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}k\Delta^{\lambda\alpha}\chi^{(2)}_{10} \\
&+ \tfrac{1}{2}\epsilon_{\mu\nu\alpha}u^{\mu}km^{\nu}\Delta^{\lambda\alpha}\chi^{(2)}_{11} - \epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}M\Delta^{\lambda\alpha}\chi^{(2)}_{11} \\
&+ \tfrac{1}{2}\epsilon_{\mu\nu\alpha}u^{\mu}k\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\chi^{(2)}_{12} - k^{\lambda}\kappa_{\mathrm{A}}\chi^{(2)}_{12} - 2\epsilon_{\nu\alpha\beta}k^{\mu}u^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{S}\mu}{}^{\beta}\chi^{(2)}_{12} \\
&- \epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{12} + \epsilon_{\mu\nu\alpha}u^{\mu}M\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\chi^{(2)}_{13} - m^{\lambda}\kappa_{\mathrm{A}}\chi^{(2)}_{13} \\
&- 2\epsilon_{\mu\alpha\beta}u^{\mu}m^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{S}\nu}{}^{\beta}\chi^{(2)}_{13} + \epsilon_{\mu\nu\alpha}u^{\mu}m^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{13} \\
&- 2\mathcal{K}^{\lambda}\kappa_{\mathrm{A}}\chi^{(2)}_{14} - 4\epsilon_{\mu\alpha\beta}u^{\mu}\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{S}\nu}{}^{\beta}\chi^{(2)}_{14} \\
&+ 2\epsilon_{\mu\nu\alpha}u^{\mu}\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{14} - \tfrac{1}{2}km^{\lambda}\chi^{(2)}_{15} + k^{\lambda}M\chi^{(2)}_{15} \\
&- \tfrac{1}{2}k\mathcal{K}^{\lambda}\chi^{(2)}_{16} - \epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{A}}\chi^{(2)}_{16} - 2k^{\mu}\kappa_{\mathrm{S}}{}^{\lambda}{}_{\mu}\chi^{(2)}_{16} \\
&+ k^{\lambda}\kappa_{\mathrm{T}}\chi^{(2)}_{16} - M\mathcal{K}^{\lambda}\chi^{(2)}_{17} + \epsilon_{\mu\nu\alpha}u^{\mu}m^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{A}}\chi^{(2)}_{17} \\
&- 2m^{\mu}\kappa_{\mathrm{S}}{}^{\lambda}{}_{\mu}\chi^{(2)}_{17} + m^{\lambda}\kappa_{\mathrm{T}}\chi^{(2)}_{17} - \mathcal{K}^{\lambda}\kappa_{\mathrm{A}}\chi^{(2)}_{2} \\
&- 4\epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}k\Delta^{\lambda\alpha}\chi^{(2)}_{3} + \tfrac{1}{2}\epsilon_{\mu\nu\alpha}u^{\mu}\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{A}}\chi^{(2)}_{4} - \tfrac{1}{2}\mathcal{K}^{\lambda}\kappa_{\mathrm{T}}\chi^{(2)}_{4} \\
&+ \tfrac{1}{2}\epsilon_{\mu\nu\alpha}u^{\mu}k\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\chi^{(2)}_{5} - 2\epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{5} \\
&+ \tfrac{1}{2}\epsilon_{\mu\nu\alpha}u^{\mu}M\mathcal{K}^{\nu}\Delta^{\lambda\alpha}\chi^{(2)}_{6} + \epsilon_{\mu\nu\alpha}u^{\mu}m^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{6} - \tfrac{1}{2}k\mathcal{K}^{\lambda}\chi^{(2)}_{7} \\
&- 2\epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{A}}\chi^{(2)}_{7} - \tfrac{1}{2}M\mathcal{K}^{\lambda}\chi^{(2)}_{8} + \epsilon_{\mu\nu\alpha}u^{\mu}m^{\nu}\Delta^{\lambda\alpha}\kappa_{\mathrm{A}}\chi^{(2)}_{8} \\
&+ \epsilon_{\mu\nu\alpha}u^{\mu}km^{\nu}\Delta^{\lambda\alpha}\chi^{(2)}_{9} - 2\epsilon_{\mu\nu\alpha}k^{\mu}u^{\nu}M\Delta^{\lambda\alpha}\chi^{(2)}_{9}, \\
\mathscr{T}_{(2)} = {}& \frac{1}{2}\epsilon_{\alpha\beta\rho}u^{\rho}T^{\alpha\beta}_{BR,(2)} + \kappa_{\mathrm{A}}\kappa_{\mathrm{T}}\chi^{(2)}_{1} + \tfrac{1}{2}\epsilon_{\mu\nu\rho}k^{\mu}u^{\nu}\mathcal{K}^{\rho}\chi^{(2)}_{12} - \tfrac{1}{2}\epsilon_{\mu\nu\rho}u^{\mu}m^{\nu}\mathcal{K}^{\rho}\chi^{(2)}_{13} \\
&- \tfrac{1}{2}k^{\mu}\mathcal{K}_{\mu}\chi^{(2)}_{16} - \tfrac{1}{2}m^{\mu}\mathcal{K}_{\mu}\chi^{(2)}_{17} - \kappa_{\mathrm{A}}\kappa_{\mathrm{T}}\chi^{(2)}_{2} + \tfrac{1}{2}\kappa_{\mathrm{A}}{}^{2}\chi^{(2)}_{4} \\
&- \tfrac{1}{2}\kappa_{\mathrm{T}}{}^{2}\chi^{(2)}_{4} + \tfrac{1}{2}k\kappa_{\mathrm{A}}\chi^{(2)}_{5} + \tfrac{1}{2}M\kappa_{\mathrm{A}}\chi^{(2)}_{6} - \tfrac{1}{2}k\kappa_{\mathrm{T}}\chi^{(2)}_{7} \\
&- \tfrac{1}{2}M\kappa_{\mathrm{T}}\chi^{(2)}_{8},
\end{aligned} \tag{7.8}
$$

where

$$
\begin{aligned}
T^{\mu\nu}_{BR,(2)} = {}& \mathring{\nabla}_{\lambda}\left(B^{\lambda\mu\nu} - B^{\mu\lambda\nu} - B^{\nu\lambda\mu}\right), \\
B^{\mu}{}_{\alpha\beta} = {}& -\delta_{\beta}{}^{\mu}u_{\alpha}\kappa_{\mathrm{T}}\chi^{(2)}_{1} + \delta_{\alpha}{}^{\mu}u_{\beta}\kappa_{\mathrm{T}}\chi^{(2)}_{1} - k_{\beta}u_{\alpha}u^{\mu}\chi^{(2)}_{10} + k_{\alpha}u_{\beta}u^{\mu}\chi^{(2)}_{10} \\
&+ \tfrac{1}{2}u_{\beta}u^{\mu}m_{\alpha}\chi^{(2)}_{11} - \tfrac{1}{2}u_{\alpha}u^{\mu}m_{\beta}\chi^{(2)}_{11} + \epsilon_{\alpha\beta\kappa}k^{\mu}u^{\kappa}\chi^{(2)}_{12} \\
&+ \tfrac{1}{2}u_{\beta}u^{\mu}\mathcal{K}_{\alpha}\chi^{(2)}_{12} - \tfrac{1}{2}u_{\alpha}u^{\mu}\mathcal{K}_{\beta}\chi^{(2)}_{12} + \epsilon_{\alpha\beta\kappa}u^{\kappa}m^{\mu}\chi^{(2)}_{13} \\
&+ 2\epsilon_{\alpha\beta\kappa}u^{\kappa}\mathcal{K}^{\mu}\chi^{(2)}_{14} - \tfrac{1}{2}\epsilon_{\beta\kappa\lambda}u_{\alpha}u^{\kappa}u^{\mu}m^{\lambda}\chi^{(2)}_{15} + \tfrac{1}{2}\epsilon_{\alpha\kappa\lambda}u_{\beta}u^{\kappa}u^{\mu}m^{\lambda}\chi^{(2)}_{15} \\
&- \delta_{\beta}{}^{\mu}k_{\alpha}\chi^{(2)}_{16} + \delta_{\alpha}{}^{\mu}k_{\beta}\chi^{(2)}_{16} + k_{\beta}u_{\alpha}u^{\mu}\chi^{(2)}_{16} - k_{\alpha}u_{\beta}u^{\mu}\chi^{(2)}_{16} \\
&- \tfrac{1}{2}\epsilon_{\beta\kappa\lambda}u_{\alpha}u^{\kappa}u^{\mu}\mathcal{K}^{\lambda}\chi^{(2)}_{16} + \tfrac{1}{2}\epsilon_{\alpha\kappa\lambda}u_{\beta}u^{\kappa}u^{\mu}\mathcal{K}^{\lambda}\chi^{(2)}_{16} - \delta_{\beta}{}^{\mu}m_{\alpha}\chi^{(2)}_{17}
\end{aligned}
$$

$$
\begin{aligned}
&- u_\beta u^\mu m_\alpha \chi_{17}^{(2)} + \delta_\alpha{}^\mu m_\beta \chi_{17}^{(2)} + u_\alpha u^\mu m_\beta \chi_{17}^{(2)} \\
&+ \epsilon_\beta{}^\mu{}_\kappa u_\alpha u^\kappa \kappa_{\mathrm{A}} \chi_2^{(2)} - \epsilon_\alpha{}^\mu{}_\kappa u_\beta u^\kappa \kappa_{\mathrm{A}} \chi_2^{(2)} + 2\epsilon_{\alpha\beta\kappa} u^\kappa u^\mu k \chi_3^{(2)} \\
&- \tfrac{1}{2}\delta_\beta{}^\mu u_\alpha \kappa_{\mathrm{A}} \chi_4^{(2)} + \tfrac{1}{2}\delta_\alpha{}^\mu u_\beta \kappa_{\mathrm{A}} \chi_4^{(2)} + \tfrac{1}{2}\epsilon_\beta{}^\mu{}_\kappa u_\alpha u^\kappa \kappa_{\mathrm{T}} \chi_4^{(2)} \\
&- \tfrac{1}{2}\epsilon_\alpha{}^\mu{}_\kappa u_\beta u^\kappa \kappa_{\mathrm{T}} \chi_4^{(2)} - \tfrac{1}{2}\delta_\beta{}^\mu u_\alpha k \chi_5^{(2)} + \tfrac{1}{2}\delta_\alpha{}^\mu u_\beta k \chi_5^{(2)} \\
&+ \epsilon_{\alpha\beta\kappa} u^\kappa u^\mu \kappa_{\mathrm{T}} \chi_5^{(2)} - \tfrac{1}{2}\delta_\beta{}^\mu u_\alpha M \chi_6^{(2)} + \tfrac{1}{2}\delta_\alpha{}^\mu u_\beta M \chi_6^{(2)} \\
&+ \tfrac{1}{2}\epsilon_\beta{}^\mu{}_\kappa u_\alpha u^\kappa k \chi_7^{(2)} - \tfrac{1}{2}\epsilon_\alpha{}^\mu{}_\kappa u_\beta u^\kappa k \chi_7^{(2)} + \epsilon_{\alpha\beta\kappa} u^\kappa u^\mu \kappa_{\mathrm{A}} \chi_7^{(2)} \\
&+ \tfrac{1}{2}\epsilon_\beta{}^\mu{}_\kappa u_\alpha u^\kappa M \chi_8^{(2)} - \tfrac{1}{2}\epsilon_\alpha{}^\mu{}_\kappa u_\beta u^\kappa M \chi_8^{(2)} + \epsilon_{\alpha\beta\kappa} u^\kappa u^\mu M \chi_9^{(2)}. \qquad (7.9)
\end{aligned}
$$

Furthermore, first order corrections to spin current is given by:

$$
\begin{aligned}
s^\mu_{(1)} &= 2k^\mu \chi_{10}^{(2)} + k^\mu \chi_{11}^{(2)} + m^\mu \chi_{11}^{(2)} \\
&\quad + \mathcal{K}^\mu \chi_{12}^{(2)} + \mathcal{K}^\mu \chi_{13}^{(2)} + \epsilon^\mu{}_{\lambda\nu} k^\lambda u^\nu \chi_{15}^{(2)} \\
&\quad + \epsilon^\mu{}_{\lambda\nu} u^\lambda m^\nu \chi_{15}^{(2)} + \epsilon^\mu{}_{\lambda\nu} u^\lambda \mathcal{K}^\nu \chi_{16}^{(2)} \\
&\quad + \epsilon^\mu{}_{\lambda\nu} u^\lambda \mathcal{K}^\nu \chi_{17}^{(2)}, \\
\mathfrak{S}_{(1)} &= -8k\chi_3^{(2)} - 4\kappa_{\mathrm{T}}\chi_5^{(2)} - 2\kappa_{\mathrm{T}}\chi_6^{(2)} \\
&\quad - 4\kappa_{\mathrm{A}}\chi_7^{(2)} - 2\kappa_{\mathrm{A}}\chi_8^{(2)} - 2k\chi_9^{(2)} \\
&\quad - 4M\chi_9^{(2)}, \\
\varsigma_{\mathrm{T}(1)} &= -4\kappa_{\mathrm{T}}\chi_1^{(2)} - 2\kappa_{\mathrm{A}}\chi_4^{(2)} - 2k\chi_5^{(2)} \\
&\quad - 2M\chi_6^{(2)}, \\
\varsigma_{\mathrm{S}}^{\alpha\beta}{}_{(1)} &= 0, \\
\varsigma_{\mathrm{A}(1)} &= -4\kappa_{\mathrm{A}}\chi_2^{(2)} - 2\kappa_{\mathrm{T}}\chi_4^{(2)} - 2k\chi_7^{(2)} \\
&\quad - 2M\chi_8^{(2)}, \\
\Sigma^\mu_{(1)} &= 4k_\alpha\chi_{12}^{(2)} + 4m_\alpha\chi_{13}^{(2)} + 8\mathcal{K}_\alpha\chi_{14}^{(2)} \\
&\quad + 4\epsilon_{\lambda\mu\nu} k^\lambda u^\mu \Delta_\alpha{}^\nu \chi_{16}^{(2)} - 4\epsilon_{\lambda\mu\nu} u^\lambda m^\mu \Delta_\alpha{}^\nu \chi_{17}^{(2)}. \qquad (7.10)
\end{aligned}
$$

7.2.3 Full Form of Currents

Finally, we give the full list of energy-momentum tensor components as

$$
\begin{aligned}
\mathcal{E} &= \varepsilon + \mathcal{E}_{(1)} + f_{\mathcal{E}}, & \mathcal{P} &= P + \mathcal{P}_{(1)} + f_{\mathcal{P}}, \\
q^\mu &= Q^\mu + q^\mu_{(1)} + f^\mu_q, & h^\mu &= Q^\mu + h^\mu_{(1)} + h^\mu_{(2)} + f^\mu_h, \qquad (7.11) \\
\pi^{\mu\nu} &= \pi^{\mu\nu}_{(1)} + f^{\mu\nu}_\pi, & \mathscr{T} &= \mathscr{T}_{(1)} + \mathscr{T}_{(2)} + f_{\mathscr{T}},
\end{aligned}
$$

and spin current as

$$
\begin{aligned}
s^{\mu} &= u_{\nu}\rho^{\mu\nu} + s^{\mu}_{(1)} + f^{\mu}_{s}, & \mathfrak{S} &= \epsilon_{abc}\rho^{ab}u^{c} - 4\chi^{(1)}_{3} + \mathfrak{S}_{(1)} + f_{\mathfrak{S}}, \\
\varsigma_{\mathrm{T}} &= -2\chi^{(1)}_{1} + \varsigma_{\mathrm{T}(1)} + f_{\varsigma_{\mathrm{T}}}, & \varsigma^{\mu\nu}_{\mathrm{S}} &= f^{\mu\nu}_{\varsigma_{\mathrm{S}}}, \\
\varsigma_{\mathrm{A}} &= -2\chi^{(1)}_{2} + \varsigma_{\mathrm{A}(1)} + f_{\varsigma_{\mathrm{A}}}, & \Sigma^{\mu} &= \Sigma^{\mu}_{(1)} + f^{\mu}_{\Sigma},
\end{aligned}
\tag{7.12}
$$

where the objects $\{f\}$ give the out-of-equilibrium contributions which we discuss in the next section.

Chapter 8

First Order Spin Hydrodynamics in 2+1D

8.1 Frame Choice

There is no microscopic definition of fluid temperature, velocity and chemical potential out of equilibrium. On the other hand, the currents we are interested in do have a microscopic definition. This means we are free to choose a “frame" as long as our definitions give the same values four these three variables in equilibrium and do not change the currents in any regime. This leads to the freedom of shifting the definitions by a function of gradients which vanish in equilibrium. We can state a frame transformation as

$$\begin{aligned} T' &= T + \delta T, \\ u'^{\mu} &= u^{\mu} + \delta u^{\mu}, \\ \mu'^{ab} &= \mu^{ab} + \delta\mu^{ab}, \end{aligned} \tag{8.1}$$

where δT, δu^{μ}, and $\delta\mu^{ab}$ are first order in gradients and vanish in equilibrium. Under a frame transformation, sectors of energy-momentum tensor and spin current change as

$$\begin{aligned} &\delta\mathcal{E} = 0, && \delta\mathcal{P} = 0 \\ &\delta q^{\mu} = -2\frac{\partial P}{\partial T} T\delta u^{\mu}, && \delta h^{\mu} = 2\mathscr{T}\epsilon^{\mu\beta\rho}u_{\rho}\delta u_{\beta} \\ &\delta\mathscr{T} = \frac{1}{2}\epsilon_{\alpha\beta\rho}u^{\alpha}h^{\beta}\delta u^{\rho}, && \delta\pi^{\mu\nu} = 0 \\ &\delta\mathfrak{S} = 0, && \delta\varsigma_{\mathrm{T}} = 0, \\ &\delta\varsigma_{\mathrm{S}}^{\mu\nu} = 0, && \delta\varsigma_{\mathrm{A}} = 0, \\ &\delta s^{\mu} = -\left(P - \chi_2^{(1)} + 2\chi_3^{(1)}\right)\epsilon^{\mu}{}_{\lambda\alpha}u^{\alpha}\delta u^{\lambda} - \chi_1^{(1)}\delta u^{\mu}, \\ &\delta\Sigma_{\alpha} = -2\left(P - \chi_2^{(1)} + 2\chi_3^{(1)}\right)\delta u_{\alpha} - 2\chi_1^{(1)}\epsilon_{\alpha\lambda\beta}u^{\beta}\delta u^{\lambda}. \end{aligned} \tag{8.2}$$

At this point, we can choose δu^μ such that $f_q^\mu = 0$. Furthermore, if we write down the frame transformation rule for $\mathcal{E}$ and s^{ab} explicitly, we get

$$\begin{aligned} f'_{\mathcal{E}} &= f_{\mathcal{E}} - \left(\frac{\partial \mathcal{E}}{\partial T}\right)_{\mu^{ab}} \delta T - \left(\frac{\partial \mathcal{E}}{\partial \mu^{ab}}\right)_T \delta\mu^{ab}, \\ f'^{cd}_s &= f^{cd}_s - \left(\frac{\partial s^{cd}}{\partial T}\right)_{\mu^{ab}} \delta T - \left(\frac{\partial s^{cd}}{\partial \mu^{ab}}\right)_T \delta\mu^{ab}. \end{aligned} \tag{8.3}$$

Thus, we can choose δT and $\delta\mu^{ab}$ such that $f'_{\mathcal{E}} = f'^{cd}_s = 0$. This choice with our previous choice of δu^μ form the frame called "Landau frame" which was used throughout Landau-Lifschitz's book of fluid mechanics. However, we will postpone choosing Landau frame until the end of entropy current calculation and stay at our current frame, namely, "thermodynamic frame"

8.2 First order constraints

To build the hydrodynamics, we need to constrain the non-equilibrium contributions as much as possible. These constrains will be induced by the equations of motion and the second law of thermodynamics. In first order, equations of motion give us two scalar and two vector constraints. We can state them as

$$\begin{aligned} u_\nu \mathring{\nabla}_\mu T^{\mu\nu} &= 0, \\ \Delta^\lambda{}_\nu \mathring{\nabla}_\mu T^{\mu\nu} &= 0, \\ \epsilon^{abc} u_c \mathring{\nabla}_\lambda S^\lambda{}_{ab} &= -4\mathscr{T}, \\ u^b \mathring{\nabla}_\lambda S^\lambda{}_{ab} &= h_a - \frac{\partial P}{\partial M}\epsilon_{abc} u^b k^c, \end{aligned} \tag{8.4}$$

and we use the field decompositions from the previous section on the LHS to find,

$$\begin{aligned} s\left(\frac{\partial s}{\partial T}\right)^{-1}\Theta + u^\mu \mathring{\nabla}_\mu T &= 0, \\ T a^\mu + \Delta^{\mu\nu}\mathring{\nabla}_\nu T &= 0, \\ 2 f_{\mathscr{T}} &= \Theta \frac{\partial P}{\partial M} + u^\mu \frac{\partial}{\partial M}\mathring{\nabla}_\mu P, \\ f_h^a &= \frac{\partial P}{\partial M}\epsilon^{abc} u_b \left(a_c + k_c - m_c\right), \end{aligned} \tag{8.5}$$

where $s = \frac{\partial P}{\partial T}$. These equations are correct at first order in gradient expansion without any assumption of equilibrium. Thus, we can simplify them by using the equilibrium condition. Notice that f_h^a and $a_c + k_c - m_c$ vanish at equilibrium since f_h^a is a non-equilibrium contribution by definition and $\tilde{k}_c := -a_c - k_c + m_c = 0$ due to

(6.4).Therefore, the equations are consistent. Finally, the whole solution is given by

$$
\begin{aligned}
u^{\mu}\mathring{\nabla}_{\mu}T &= -s\left(\frac{\partial s}{\partial T}\right)^{-1}\Theta,\\
a^{\mu} &= -\frac{1}{T}\Delta^{\mu\nu}\mathring{\nabla}_{\nu}T,\\
f_{\mathscr{T}} &= \frac{\Theta}{2}\left[\frac{\partial P}{\partial M} - s\left(\frac{\partial s}{\partial T}\right)^{-1}\frac{\partial s}{\partial M}\right] = 0,\\
f_{h}^{a} &= -\frac{\partial P}{\partial M}\epsilon^{abc}u_{b}\tilde{k}_{c},
\end{aligned}
\tag{8.6}
$$

where we should point out that equating the third equation to zero is only applicable in the first order approximation in gradients. This concludes our discussion of first order equations of motion. We will use these results while constructing the out-of-equilibrium contributions and analysing the entropy current.

Furthermore, we know how the derivative of temperature behaves now. A careful reader may remember that we have introduced two out-of-equilibrium tensors in chapter 6 using the first expression in (6.4). We have omitted second and third expressions. Now, you can see why we did not construct an out-of-equilibrium vector from the third expression since it would have been an off-shell vector which we have no interest in. On the other hand, we can construct six tensors from the second expression since we now know how the derivative of temperature behaves. As a side note, these definitions will look intimidating but they are nothing but expanded forms of Riemann tensor.

$$
\begin{aligned}
\tilde{\mathfrak{f}}^{\alpha} :=& -\frac{1}{2}\epsilon^{\alpha}{}_{\beta\lambda}u^{\beta}km^{\ \lambda} + \epsilon^{\alpha}{}_{\beta\lambda}u^{\beta}m^{\lambda}M + u^{\beta}\mathring{\nabla}_{\beta}m^{\alpha} + u^{\alpha}u^{\beta}u^{\lambda}\mathring{\nabla}_{\lambda}\, m_{\beta},\\
\tilde{\mathfrak{f}} :=& \, u^{\alpha}\mathring{\nabla}_{\alpha}M,\\
\tilde{\mathfrak{f}}_{\mathrm{T}} :=& -\mathring{R}_{\alpha\beta}u^{\alpha}u^{\beta} - \frac{1}{2}k^{2} + 2M^{2} + k\kappa_{\mathrm{A}} - \mathring{\nabla}_{\alpha}k^{\alpha} + \mathring{\nabla}_{\alpha}m^{\alpha} + u^{\alpha}\mathring{\nabla}_{\alpha}\kappa_{\mathrm{T}},\\
\tilde{\mathfrak{f}}_{\mathrm{S}}^{\mu\nu} :=& \, k^{\mu}k^{\nu} - k^{\nu}m^{\mu} - k^{\mu}m^{\nu} + m^{\mu}m^{\nu} - \frac{1}{2}k_{\alpha}k^{\alpha}\Delta^{\mu\nu} + k^{\alpha}m_{\alpha}\Delta^{\mu\nu} - \frac{1}{2}m_{\alpha}m^{\alpha}\Delta^{\mu\nu}\\
&+ \frac{1}{2}\mathring{R}_{\alpha\lambda\beta\rho}u^{\alpha}u^{\beta}\Delta^{\lambda\rho}\Delta^{\mu\nu} - \mathring{R}_{\alpha\lambda\beta\rho}u^{\alpha}u^{\beta}\Delta^{\mu\lambda}\Delta^{\nu\rho} + \frac{1}{4}\epsilon_{\alpha\beta\lambda}u^{\alpha}k\Delta^{\nu\beta}\kappa_{\mathrm{S}}{}^{\mu\lambda}\\
&- \frac{1}{2}\epsilon_{\alpha\beta\lambda}u^{\alpha}M\Delta^{\nu\beta}\kappa_{\mathrm{S}}{}^{\mu\lambda} + \frac{1}{4}\epsilon_{\alpha\beta\lambda}u^{\alpha}k\Delta^{\mu\beta}\kappa_{\mathrm{S}}{}^{\nu\lambda} - \frac{1}{2}\epsilon_{\alpha\beta\lambda}u^{\alpha}M\Delta^{\mu\beta}\kappa_{\mathrm{S}}{}^{\nu\lambda}\\
&- \frac{1}{2}u^{\alpha}\Delta^{\beta\lambda}\Delta^{\mu\nu}\mathring{\nabla}_{\alpha}\kappa_{\mathrm{S}\beta\lambda} + u^{\alpha}\Delta^{\mu\beta}\Delta^{\nu\lambda}\mathring{\nabla}_{\alpha}\kappa_{\mathrm{S}\beta\lambda} - \frac{1}{2}u^{\alpha}\Delta^{\mu\beta}\Delta^{\nu\lambda}\mathring{\nabla}_{\beta}\kappa_{\mathrm{S}\alpha\lambda}\\
&+ \frac{1}{2}\Delta_{\alpha\beta}\Delta^{\mu\nu}\mathring{\nabla}^{\beta}k^{\alpha} - \frac{1}{2}\Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\alpha}\mathring{\nabla}^{\beta}k^{\alpha} - \frac{1}{2}\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta}\mathring{\nabla}^{\beta}k^{\alpha}\\
&- \frac{1}{2}\Delta_{\alpha\beta}\Delta^{\mu\nu}\mathring{\nabla}^{\beta}m^{\alpha} + \frac{1}{2}\Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\alpha}\mathring{\nabla}^{\beta}m^{\alpha} + \frac{1}{2}\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta}\mathring{\nabla}^{\beta}m^{\alpha}\\
&+ \frac{1}{2}u^{\alpha}\Delta^{\beta\lambda}\Delta^{\mu\nu}\mathring{\nabla}_{\lambda}\kappa_{\mathrm{S}\alpha\beta} - \frac{1}{2}u^{\alpha}\Delta^{\mu\beta}\Delta^{\nu\lambda}\mathring{\nabla}_{\lambda}\kappa_{\mathrm{S}\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathfrak{f}}_{\mathrm{A}} :=& -k\kappa_{\mathrm{T}} + \epsilon_{\alpha\beta\kappa} u^{\alpha} \mathring{\nabla}^{\kappa} k^{\beta} - \epsilon_{\alpha\beta\kappa} u^{\alpha} \mathring{\nabla}^{\kappa} m^{\beta} + \epsilon_{\beta\kappa\lambda} u^{\alpha} u^{\beta} \mathring{\nabla}^{\lambda} \kappa_{\mathrm{S}\alpha}{}^{\kappa}, \\
\tilde{\mathfrak{F}}_{\mu} :=& -\epsilon_{\alpha}{}^{\kappa\lambda} \mathring{R}_{\mu\beta\kappa\lambda} u^{\alpha} u^{\beta} - 4k_{\mu} M + 4m_{\mu} M + u^{\alpha} \mathring{\nabla}_{\alpha} \mathcal{K}_{\mu} - \mathring{\nabla}_{\mu} k \\
&+ 2\mathring{\nabla}_{\mu} M - u^{\alpha} \mathring{\nabla}_{\mu} \mathcal{K}_{\alpha}.
\end{aligned} \tag{8.7}
$$

These definitions will be useful to posit second order out-of-equilibrium contributions to the anti-symmetric part of energy-momentum tensor

8.3 Second order constraints

Although first order equations of motion provide us a significant amount of information, it is not enough to complete the first order entropy analysis. To make the full analysis, we need to know the second order scalar constraints that our system abides. We can find them by investigating the expressions $u_{\nu} \mathring{\nabla}_{\mu} T^{\mu\nu}$, $\Delta^{\lambda}{}_{\nu} \mathring{\nabla}_{\mu} T^{\mu\nu}$, $\epsilon^{abc} u_c \mathring{\nabla}_{\lambda} S^{\lambda}{}_{ab}$, and $u^b \mathring{\nabla}_{\lambda} S^{\lambda}{}_{ab}$ in second order. Before carrying on the calculation, one should notice that the second derivative of the velocity components are not fully independent. The geometric relation in (5.18) implies

$$
\begin{aligned}
\mathring{\nabla}_{\mu} a^{\mu} &= \mathring{R}_{u} + u^{\mu} \mathring{\nabla}_{\mu} \Theta + \frac{\Theta^2}{2} - 2\Omega^2 + \sigma^{\mu\nu} \sigma_{\mu\nu}, \\
\mathring{\nabla}_{\mu} \left(\Omega u^{\mu} \right) &= -\frac{1}{2} \epsilon^{\mu\nu\rho} u_{\mu} \mathring{\nabla}_{\nu} a_{\rho},
\end{aligned} \tag{8.8}
$$

where we defined $\mathring{R}_u := u^{\nu} u^{\rho} \mathring{R}^{\mu}{}_{\rho\mu\nu}$ in chapter 5. Now, one can find the equations for the aforementioned expressions by using the equations of motion. Similar to the first order equations of motion, we use the field decompositions to evaluate the equations. After the evaluations $u_{\nu} \mathring{\nabla}_{\mu} T^{\mu\nu}$ and $\Delta^{\lambda}{}_{\nu} \mathring{\nabla}_{\mu} T^{\mu\nu}$ give us the second order corrections to the gradient of the temperature. However, these corrections are irrelevant in our level level of approximation. The evaluation of $\epsilon^{abc} u_c \mathring{\nabla}_{\lambda} S^{\lambda}{}_{ab}$ and $u^b \mathring{\nabla}_{\lambda} S^{\lambda}{}_{ab}$ produce the second order corrections to $f_{\mathscr{T}}$ and f_h^{μ}. We will use these constraints to analyze the entropy current in the rest of this chapter. In addition, full form of equations of motion can be found in appendix A.

8.4 Pure out-of-equilibrium contributions to the currents

In the previous sections, we have found first- and second-order corrections to the effective action and calculated their contribution to the currents. Since we relate the effective action to the partition function, these corrections fail to capture all out-of-equilibrium terms in the currents. That is why we need to add the most general out-of-equilibrium terms *ad hoc* and restrict them using the equations of motion. We can characterize the out-of-equilibrium contributions to the energy-momentum tensor as

$$
f_{\mathcal{E}} = c_1 \Theta + c_2 \tilde{k},
$$

$$
\begin{aligned}
f_{\mathcal{P}} =& c_3\Theta + c_4\tilde{k},\\
f_q^{\mu} =& c_5\tilde{k}^{\mu} + c_6\epsilon^{\mu\nu\rho}u_\nu\tilde{k}_\rho,\\
f_h^{\mu} =& c_{21}\tilde{k}^{\mu} + c_{22}\epsilon^{\mu\nu\rho}u_\nu\tilde{k}_\rho + f_h^{(2)\mu},\\
f_\pi^{\mu\nu} =& c_7\sigma^{\mu\nu} + c_8\sigma^{(\mu}{}_\lambda\epsilon^{\nu)\rho\lambda}u_\rho,\\
f_{\mathscr{T}} =& c_{23}\Theta + c_{24}\tilde{k} + f_{\mathscr{T}}^{(2)},
\end{aligned}
\tag{8.9}
$$

where c_js are functions of temperature and $f_h^{(2)\mu}$, $f_{\mathscr{T}}^{(2)}$ are second-order contributions which can be constructed by using appendix B. In addition, equations of motion have no restriction on the spin current. Thus, we can express its out-of-equilibrium contributions as

$$
\begin{aligned}
f_s^{\mu} =& c_9\tilde{k}^{\mu} + c_{10}\epsilon^{\mu\nu\rho}u_\nu\tilde{k}_\rho,\\
f_{\mathfrak{S}} =& c_{11}\Theta + c_{12}\tilde{k},\\
f_{\varsigma\mathrm{T}} =& c_{13}\Theta + c_{14}\tilde{k},\\
f_{\varsigma\mathrm{S}}^{\mu\nu} =& c_{15}\sigma^{\mu\nu} + c_{16}\sigma^{(\mu}{}_\lambda\epsilon^{\nu)\rho\lambda}u_\rho,\\
f_{\varsigma\mathrm{A}} =& c_{17}\Theta + c_{18}\tilde{k},\\
f_\Sigma^{\mu} =& c_{19}\tilde{k}^{\mu} + c_{20}\epsilon^{\mu\nu\rho}u_\nu\tilde{k}_\rho.
\end{aligned}
\tag{8.10}
$$

These contributions complete all the missing terms in the currents. Therefore, we have the final forms of the energy-momentum tensor and spin current in the Navier-Stokes level of approximation. Now, we have all the tools to investigate the behavior of entropy.

8.5 First order entropy

Following [30] we postulate that an entropy current, J_S^{μ}, exists such that it satisfies $\mathring{\nabla}_\mu J_S^{\mu} \geq 0$ under the equations of motion. For an ideal fluid, the current is given by $J_S^{\mu} = su^{\mu}$ with $s = \partial P/\partial T$. On the other hand, for a non-ideal fluid, we take $J_S^{\mu} = su^{\mu} + \mathcal{O}(\partial)$ where $\mathcal{O}(\partial)$ denotes corrections to the entropy current coming from explicit derivative terms appearing in the constitutive relations. Thus, the most general entropy current we may construct, to first order in derivatives is given by

$$
J_S^{\mu} = J_c^{\mu} + J_{nc}^{\mu}\,,
\tag{8.11}
$$

where

$$
\begin{aligned}
J_c^\mu &= su^\mu - \frac{u_\nu}{T}\left(T^{\mu\nu} - T^{\mu\nu}_{\mathbf{id}}\right) - \frac{1}{2}\frac{\mu^{ab}}{T}\left(S^\mu{}_{ab} - S^\mu_{\mathbf{id}ab}\right),\\
J_{nc}^\mu &= \left(j_1\Theta + j_2\kappa_{\mathrm{T}} + j_3\kappa_{\mathrm{A}} + j_4 k + j_5\tilde{k} + j_6 M\right)u^\mu\\
&\quad + j_7 k^\mu + j_8\tilde{k}^\mu + j_9 m^\mu + j_{10}\mathcal{K}^\mu\\
&\quad + \epsilon^{\mu\nu\rho}u_\nu\left(j_{11}k_\rho + j_{12}\tilde{k}_\rho + j_{13}m_\rho + j_{14}\mathcal{K}_\rho\right)
\end{aligned}
\tag{8.12}
$$

are referred to as the canonical part and non-canonical part of the entropy current respectively.

It is straightforward to show that

$$
\begin{aligned}
T\mathring{\nabla}_\mu J_c^\mu &= T\mathring{\nabla}_\mu\left(su^\mu\right) - T\mathring{\nabla}_\mu\left(\frac{u_\nu}{T}\right)T^{\mu\nu}_{(1)} - \frac{T}{2}\mathring{\nabla}_\mu\left(\frac{\mu^{ab}}{T}\right)S^\mu_{(0)ab}\\
&\quad - u_\nu\mathring{\nabla}_\mu T^{\mu\nu}_{(1)} - \frac{\mu^{ab}}{2}\mathring{\nabla}_\mu S^\mu_{(0)ab}\\
&= -T\mathring{\nabla}_\mu\left(\frac{u_\nu}{T}\right)T^{\mu\nu}_{(1)} - \frac{T}{2}\mathring{\nabla}_\mu\left(\frac{\mu^{ab}}{T}\right)S^\mu_{(0)ab}\\
&\quad - u_\nu R^{\rho\sigma\nu\lambda}S_{(0)\rho\lambda\sigma} + \left(k_{ab} - \mu_{ab}\right)T^{ab}_{(1)} - \mu^{ab}S^\lambda_{(0)ca}K_{\lambda b}{}^c.
\end{aligned}
\tag{8.13}
$$

Now, we will expand this result in its full glory to identify the contributions of all independent second order scalars. In particular, all those contributions have to vanish since they can take any value and break the positive semi-definiteness of the divergence. The complete form of the divergence of the canonical entropy current is given by

$$
\begin{aligned}
T\mathring{\nabla}_\mu J_c^\mu =\; & -c_4\tilde{k}\Theta - c_3\Theta^2 - c_7\sigma_{\alpha\beta}\sigma^{\alpha\beta} + k^\alpha\tilde{k}_\alpha\chi_1^{(1)}\\
& + \tilde{k}_\alpha\tilde{k}^\alpha\chi_1^{(1)} - \tilde{k}^\alpha m_\alpha\chi_1^{(1)} - \Theta^2\chi_1^{(1)} - \Theta\kappa_{\mathrm{T}}\chi_1^{(1)}\\
& + 2\epsilon_{\alpha\beta\lambda}k^\alpha\tilde{k}^\beta u^\lambda\chi_2^{(1)} - 2\epsilon_{\alpha\beta\lambda}\tilde{k}^\alpha u^\beta m^\lambda\chi_2^{(1)} - \Theta\kappa_{\mathrm{A}}\chi_2^{(1)}\\
& - 2M\Theta\chi_3^{(1)} - \chi_1^{(1)}\mathring{\nabla}_\alpha\tilde{k}^\alpha - 2u^\alpha\chi_3^{(1)}\mathring{\nabla}_\alpha M - u^\alpha\chi_1^{(1)}\mathring{\nabla}_\alpha\Theta\\
& - u^\alpha\chi_2^{(1)}\mathring{\nabla}_\alpha\kappa_{\mathrm{A}} - u^\alpha\chi_1^{(1)}\mathring{\nabla}_\alpha\kappa_{\mathrm{T}} - \epsilon_{\alpha\beta\lambda}u^\alpha\chi_2^{(1)}\mathring{\nabla}^\lambda k^\beta\\
& + \epsilon_{\alpha\beta\lambda}u^\alpha\chi_2^{(1)}\mathring{\nabla}^\lambda m^\beta - k^\alpha\tilde{k}_\alpha T\frac{\partial\chi_1^{(1)}}{\partial T} - \tilde{k}_\alpha\tilde{k}^\alpha T\frac{\partial\chi_1^{(1)}}{\partial T}\\
& + \tilde{k}^\alpha m_\alpha T\frac{\partial\chi_1^{(1)}}{\partial T} - \epsilon_{\alpha\beta\lambda}k^\alpha\tilde{k}^\beta u^\lambda T\frac{\partial\chi_2^{(1)}}{\partial T} + \epsilon_{\alpha\beta\lambda}\tilde{k}^\alpha u^\beta m^\lambda T\frac{\partial\chi_2^{(1)}}{\partial T}\\
& + \frac{c_2\tilde{k}\Theta\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}} + \frac{c_1\Theta^2\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}} - \frac{\Theta^2\chi_1^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}\\
& - \frac{\Theta\kappa_{\mathrm{T}}\chi_1^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}} - \frac{k\Theta\chi_2^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}} - \frac{\tilde{k}\Theta\chi_2^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2M\Theta\chi_2^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}-\frac{\Theta\kappa_{\rm A}\chi_2^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}+\frac{2k\Theta\chi_3^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}\\
&+\frac{2\tilde{k}\Theta\chi_3^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}+\frac{2M\Theta\chi_3^{(1)}\frac{\partial P}{\partial T}}{T\frac{\partial^2 P}{\partial T^2}}+\frac{\Theta^2\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}\\
&+\frac{\Theta\kappa_{\rm T}\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}+\frac{\Theta\kappa_{\rm A}\frac{\partial\chi_2^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}-\frac{k\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{\tilde{k}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}. \qquad (8.14)
\end{aligned}
$$

We observe that there are non-trivial independent scalar contributions to the divergence. To have a positive semi-definite divergence, we need to cancel those contributions using the non-canonical entropy current to obtain a thermodynamically meaningful entropy current. To achieve our purpose, we calculate the divergence of the non-canonical entropy current as

$$
\begin{aligned}
\mathring{\nabla}_\mu J^\mu_{nc} = &-\epsilon_{\mu\alpha\beta}j_{11}k^\mu\tilde{k}^\alpha u^\beta+\epsilon_{\mu\alpha\beta}j_{12}k^\mu\tilde{k}^\alpha u^\beta-j_7\mathring{R}_{\mu\alpha}u^\alpha u^\mu+\tfrac{1}{2}j_7k^2\\
&+j_7k\tilde{k}+\tfrac{1}{2}j_7\tilde{k}^2-\epsilon_{\mu\alpha\beta}j_{11}k^\mu u^\alpha m^\beta-\epsilon_{\mu\alpha\beta}j_{13}k^\mu u^\alpha m^\beta\\
&-\epsilon_{\mu\alpha\beta}j_{12}\tilde{k}^\mu u^\alpha m^\beta-\epsilon_{\mu\alpha\beta}j_{13}\tilde{k}^\mu u^\alpha m^\beta+2j_7kM+2j_7\tilde{k}M\\
&+2j_7M^2-\epsilon_{\mu\alpha\beta}j_{14}k^\mu u^\alpha\mathcal{K}^\beta-\epsilon_{\mu\alpha\beta}j_{14}\tilde{k}^\mu u^\alpha\mathcal{K}^\beta\\
&-\epsilon_{\mu\alpha\beta}j_{14}u^\mu m^\alpha\mathcal{K}^\beta-j_4\tilde{k}\Theta+j_5\tilde{k}\Theta-2j_4M\Theta+j_6M\Theta\\
&+j_1\Theta^2-\tfrac{1}{2}j_7\Theta^2+j_3\Theta\kappa_{\rm A}+j_2\Theta\kappa_{\rm T}-j_7\sigma_{\mu\alpha}\sigma^{\mu\alpha}\\
&+\epsilon_{\mu\alpha\beta}j_{11}u^\mu\mathring{\nabla}^\beta k^\alpha-\epsilon_{\mu\alpha\beta}j_4u^\mu\mathring{\nabla}^\beta k^\alpha+\epsilon_{\mu\alpha\beta}j_{12}u^\mu\mathring{\nabla}^\beta\tilde{k}^\alpha\\
&-\epsilon_{\mu\alpha\beta}j_4u^\mu\mathring{\nabla}^\beta\tilde{k}^\alpha+\epsilon_{\mu\alpha\beta}j_{13}u^\mu\mathring{\nabla}^\beta m^\alpha+\epsilon_{\mu\alpha\beta}j_4u^\mu\mathring{\nabla}^\beta m^\alpha\\
&+\epsilon_{\mu\alpha\beta}j_{14}u^\mu\mathring{\nabla}^\beta\mathcal{K}^\alpha-j_7\mathring{\nabla}_\mu\tilde{k}^\mu+j_8\mathring{\nabla}_\mu\tilde{k}^\mu-j_4u^\mu\mathring{\nabla}_\mu\tilde{k}\\
&+j_5u^\mu\mathring{\nabla}_\mu\tilde{k}+j_7\mathring{\nabla}_\mu m^\mu+j_9\mathring{\nabla}_\mu m^\mu-2j_4u^\mu\mathring{\nabla}_\mu M\\
&+j_6u^\mu\mathring{\nabla}_\mu M+j_{10}\mathring{\nabla}_\mu\mathcal{K}^\mu+j_1u^\mu\mathring{\nabla}_\mu\Theta-j_7u^\mu\mathring{\nabla}_\mu\Theta\\
&+j_3u^\mu\mathring{\nabla}_\mu\kappa_{\rm A}+j_2u^\mu\mathring{\nabla}_\mu\kappa_{\rm T}+k^\mu T\mathcal{K}_\mu\frac{\partial j_{10}}{\partial T}+\tilde{k}^\mu T\mathcal{K}_\mu\frac{\partial j_{10}}{\partial T}\\
&-m^\mu T\mathcal{K}_\mu\frac{\partial j_{10}}{\partial T}+\epsilon_{\mu\alpha\beta}k^\mu\tilde{k}^\alpha u^\beta T\frac{\partial j_{11}}{\partial T}+\epsilon_{\mu\alpha\beta}k^\mu u^\alpha m^\beta T\frac{\partial j_{11}}{\partial T}\\
&-\epsilon_{\mu\alpha\beta}k^\mu\tilde{k}^\alpha u^\beta T\frac{\partial j_{12}}{\partial T}+\epsilon_{\mu\alpha\beta}\tilde{k}^\mu u^\alpha m^\beta T\frac{\partial j_{12}}{\partial T}\\
&+\epsilon_{\mu\alpha\beta}k^\mu u^\alpha m^\beta T\frac{\partial j_{13}}{\partial T}+\epsilon_{\mu\alpha\beta}\tilde{k}^\mu u^\alpha m^\beta T\frac{\partial j_{13}}{\partial T}\\
&+\epsilon_{\mu\alpha\beta}k^\mu u^\alpha T\mathcal{K}^\beta\frac{\partial j_{14}}{\partial T}+\epsilon_{\mu\alpha\beta}\tilde{k}^\mu u^\alpha T\mathcal{K}^\beta\frac{\partial j_{14}}{\partial T}
\end{aligned}
$$

$$
\begin{aligned}
&+ \epsilon_{\mu\alpha\beta} u^{\mu} m^{\alpha} T \mathcal{K}^{\beta} \frac{\partial j_{14}}{\partial T} + k_{\mu} k^{\mu} T \frac{\partial j_7}{\partial T} + k^{\mu} \tilde{k}_{\mu} T \frac{\partial j_7}{\partial T} \\
&- k^{\mu} m_{\mu} T \frac{\partial j_7}{\partial T} + k^{\mu} \tilde{k}_{\mu} T \frac{\partial j_8}{\partial T} + \tilde{k}_{\mu} \tilde{k}^{\mu} T \frac{\partial j_8}{\partial T} - \tilde{k}^{\mu} m_{\mu} T \frac{\partial j_8}{\partial T} \\
&+ k^{\mu} m_{\mu} T \frac{\partial j_9}{\partial T} + \tilde{k}^{\mu} m_{\mu} T \frac{\partial j_9}{\partial T} - m_{\mu} m^{\mu} T \frac{\partial j_9}{\partial T} - \frac{\Theta^2 \frac{\partial j_1}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} \\
&- \frac{\Theta \kappa_{\mathrm{T}} \frac{\partial j_2}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - \frac{\Theta \kappa_{\mathrm{A}} \frac{\partial j_3}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - \frac{k \Theta \frac{\partial j_4}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} \\
&- \frac{\tilde{k} \Theta \frac{\partial j_5}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - \frac{M \Theta \frac{\partial j_6}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}.
\end{aligned} \tag{8.15}
$$

Now, we choose j_n such that independent scalar contributions vanish. The choice eventuates in

$$
\begin{aligned}
j_1 &= \frac{\chi_1^{(1)}}{T}, \\
j_2 &= \frac{\chi_1^{(1)}}{T}, \\
j_3 &= \frac{\chi_2^{(1)}}{T}, \\
j_4 &= \int^T \mathrm{d}\tau \left(\frac{2\chi_3^{(1)}(\tau)}{\tau^2} - \frac{\chi_2^{(1)}(\tau)}{\tau^2} - \frac{1}{\tau} \frac{\partial \chi_3^{(1)}(\tau)}{\partial \tau} \right), \\
j_5 &= j_4, \\
j_6 &= 2 j_4 + 2 \frac{\chi_3^{(1)}}{T}, \\
j_8 &= \frac{\chi_1^{(1)}}{T}, \\
j_{11} &= j_4 + \frac{\chi_2^{(1)}}{T}, \\
j_{12} &= j_4, \\
j_{13} &= -j_4 - \frac{\chi_2^{(1)}}{T},
\end{aligned} \tag{8.16}
$$

where we included only non-vanishing terms and the absence of the lower limit in the integral indicates the lower limit can be any constant. This choice of j_ns gives rise to the total divergence of the form

$$\mathring{\nabla}_\mu J^\mu_S = -\frac{c_7\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{T} - \frac{c_3\Theta^2}{T} + \frac{c_1\Theta^2\frac{\partial P}{\partial T}}{T^2\frac{\partial^2 P}{\partial T^2}} - \frac{c_4\tilde{k}\Theta}{T} + \frac{c_2\tilde{k}\Theta\frac{\partial P}{\partial T}}{T^2\frac{\partial^2 P}{\partial T^2}} \geq 0. \tag{8.17}$$

To guarantee positive semi-definiteness, we need

$$\begin{aligned} \eta &= -c_7 \geq 0, \\ \zeta &= -c_3 + \frac{c_1}{T}\frac{\partial P}{\partial T}\left(\frac{\partial^2 P}{\partial T^2}\right)^{-1} \geq 0, \\ c_4 &= \frac{c_2}{T}\frac{\partial P}{\partial T}\left(\frac{\partial^2 P}{\partial T^2}\right)^{-1}, \end{aligned} \tag{8.18}$$

where we introduced the shear viscosity, η, and bulk viscosity, ζ whose positivities are guaranteed by the unitarity of the underlying theory. This expression shows us spin current is non-dissipative at first order. A non-dissipative spin current gives a qualitative justification for the energy efficiency of spintronics over electronics [49]. For completeness, we present the resulting entropy current as

$$\begin{aligned} J^\mu_S = {} & su^\mu - \epsilon^\mu{}_{\alpha\beta} j_4 k^\alpha u^\beta - \epsilon^\mu{}_{\alpha\beta} j_4 \tilde{k}^\alpha u^\beta + j_4 u^\mu k + j_4 u^\mu \tilde{k} \\ & - \epsilon^\mu{}_{\alpha\beta} j_4 u^\alpha m^\beta + 2 j_4 u^\mu M + \frac{c_5\tilde{k}^\mu}{2T} + \frac{c_2^{(1)} u^\mu \tilde{k}}{T} + \frac{c_1^{(1)} u^\mu \Theta}{T} \\ & + \frac{\epsilon^\mu{}_{\alpha\beta} k^\alpha u^\beta \chi_2^{(1)}}{T} + \frac{\epsilon^\mu{}_{\alpha\beta} \tilde{k}^\alpha u^\beta \chi_2^{(1)}}{T} - \frac{u^\mu k \chi_2^{(1)}}{T} - \frac{u^\mu \tilde{k} \chi_2^{(1)}}{T} \\ & + \frac{\epsilon^\mu{}_{\alpha\beta} u^\alpha m^\beta \chi_2^{(1)}}{T} - \frac{2u^\mu M \chi_2^{(1)}}{T} - \frac{2\epsilon^\mu{}_{\alpha\beta} k^\alpha u^\beta \chi_3^{(1)}}{T} \\ & - \frac{2\epsilon^\mu{}_{\alpha\beta} \tilde{k}^\alpha u^\beta \chi_3^{(1)}}{T} + \frac{2u^\mu k \chi_3^{(1)}}{T} + \frac{2u^\mu \tilde{k} \chi_3^{(1)}}{T} + \frac{c_6 \epsilon^{\mu\nu\rho} u_\nu \tilde{k}_\rho}{2T} \\ & - \frac{2\epsilon^\mu{}_{\alpha\beta} u^\alpha m^\beta \chi_3^{(1)}}{T} + \frac{4u^\mu M \chi_3^{(1)}}{T} + u^\mu \kappa_{\mathrm{T}} \frac{\partial \chi_1^{(1)}}{\partial T} \\ & - \epsilon^\mu{}_{\alpha\beta} k^\alpha u^\beta \frac{\partial \chi_2^{(1)}}{\partial T} - \epsilon^\mu{}_{\alpha\beta} \tilde{k}^\alpha u^\beta \frac{\partial \chi_2^{(1)}}{\partial T} - \epsilon^\mu{}_{\alpha\beta} u^\alpha m^\beta \frac{\partial \chi_2^{(1)}}{\partial T} \\ & + u^\mu \kappa_{\mathrm{A}} \frac{\partial \chi_2^{(1)}}{\partial T} + \epsilon^\mu{}_{\alpha\beta} k^\alpha u^\beta \frac{\partial \chi_3^{(1)}}{\partial T} + \epsilon^\mu{}_{\alpha\beta} \tilde{k}^\alpha u^\beta \frac{\partial \chi_3^{(1)}}{\partial T} \\ & + u^\mu k \frac{\partial \chi_3^{(1)}}{\partial T} + \epsilon^\mu{}_{\alpha\beta} u^\alpha m^\beta \frac{\partial \chi_3^{(1)}}{\partial T} - \frac{\epsilon^\mu{}_{\alpha\beta} \tilde{k}^\alpha u^\beta}{2T} \frac{\partial P}{\partial M}. \end{aligned} \tag{8.19}$$

This concludes our discussion on spin hydrodynamics in the scope of this thesis.

Chapter 9

Conclusion and Outlook

9.1 Conclusion

Throughout this thesis, we have touched upon various topics and techniques. Before we started to build our framework, we had introduced the preliminary knowledge required.

In chapter 2, we presented a brief review of non-relativistic hydrodynamics. To generalize this framework to both relativistic and quantum cases, we have discussed the finite temperature behavior of quantum field theories. During that discussion, we have realized that quantum statistical mechanics became inadequate to describe out-of-equilibrium behavior. To overcome this issue, we have expanded our discussion to finite temperature field theories, which characterize quantum fields coupled to a heath bath. We have identified how finite temperature field theories could characterize time and temperature at the same time. Following that discussion, we have constructed relativistic hydrodynamics as a low-energy effective field theory of an underlying finite temperature field theory. In particular, we have derived the hydrodynamic equations in a curved background and shown how to introduce gauge fields to couple hydrodynamics to other currents. Then, we outlined how to use the relativistic framework by considering an ideal fluid. Moreover, we have demonstrated that relativistic hydrodynamics reduces exactly to the non-relativistic case in the proper limit. All these discussions combined to form the backbone of the hydrodynamic description spin current.

Having established the preliminary knowledge, we have pointed out that the gauge group producing the spin current was nothing but the Lorentz group. Therefore, we needed to build the gauge theory of $SO(1, N)$ to build spin hydrodynamics. That is precisely how we have begun chapter 3. After the gauge discussion, we established the vielbein formalism to describe spacetime in its most general form, that is, with non-trivial torsion. In this formalism, we have discussed how we push forward and pull back tensorial objects between the spacetime manifold and the tangent bundle. Moreover, vielbein description led us to identify the spin connection as the gauge field of the Lorentz group. These discussions set our formalism up to begin the construction of spin hydrodynamics.

In chapter 4, we have considered an arbitrary effective action and identified energy-momentum tensor and spin current. After our identifications, we have encountered an ambiguity in the definition of spin current. To resolve the ambiguity, we have singled out the underlying cause as the pseudo-gauge freedom of spin current, also known as Belinfante-Rosenfeld freedom. Then, we have lifted the ambiguity by including torsion in our theory. We have further pointed out that the inclusion of torsion was not a claim that the universe had non-trivial torsion necessarily. Taking the vanishing torsion limit at the end of our calculations does not re-introduce the ambiguity. After securing that the currents were well-defined, we have derived the hydrodynamic equations by demanding gauge and diffeomorphism invariance of the theory. As a result, we have found that spin current couples to the curvature, torsion, and the anti-symmetric part of the energy-momentum tensor. In the remaining part of the thesis, we have focused on the solution of hydrodynamic equations.

To solve the hydrodynamic equations systematically, we have constrained ourselves to a 2+1D spacetime and introduced the notion of hydrodynamic decomposition in chapter 5. In particular, we have decomposed every object in the theory into the tensors of $SO(2) \subset SO(1,2)$ where we preserved the Lorentz symmetry by the velocity profile of the fluid. This decomposition had enabled us to identify physical quantities like energy density, pressure, heat currents, torque, and spin sub-currents. This discussion has facilitated us to carry out the solutions of spin hydrodynamics.

In chapter 6, we have considered an ideal fluid and used its effective action to calculate its energy-momentum tensor and spin current. Our results concluded that source-free effective Lagrangian is the pressure of the ideal fluid. Furthermore, we have found out that spin accumulation was possible even in vanishing spin potential. This phenomenon is the signature of a 2+1D spacetime, and it does not exist in higher dimensions [17, 18]. Finally, we have concluded the chapter with the investigation of ideal entropy current. We have observed that the ideal entropy was conserved, which was what we expected from an ideal fluid.

We have extended our discussion to real fluids in chapter 7 by introducing corrections to the effective action. After a tedious investigation, We had determined 3 first-order and 17 second-order correction in the most general case. Analogous to chapter 6, we have calculated the corrections to the currents and presented the complete form of the currents up to our approximation level.

Finally, we have introduced all possible out-of-equilibrium contributions to the currents in chapter 8. We have started with the introduction of hydrodynamic frame choice. This freedom of choice originated from the lack of out-of-equilibrium definitions of hydrodynamic degrees of freedom. Then, we have solved the first- and second-order constraints on the out-of-equilibrium contributions enforced by the hydrodynamic equations. These constraints produced the final form of the spin current and the energy-momentum tensor. We have finalized our discussion with the entropy investigation of the real fluid. As a result, we have discovered that the causes of dissipation were only shear and bulk viscosities. In other words, the spin current is non-dissipative in our level of approximation reinforcing the efficiency claims of condensed matter physics [49].

9.2 Outlook

In addition to our whole work, there is still room for improvement in the spin hydrodynamics framework. The most straightforward addition is to derive the Kubo formulae for the first-order transport coefficients from the two-point functions of the underlying field theory. Moreover, our framework has second-order transport coefficients that contribute to first-order spin current and second-order anti-symmetric energy-momentum tensors. These transport coefficients can be extracted from the four-point functions of the underlying theory.

Apart from that straightforward improvement, we have more recommendations that might have practical implications in condensed matter physics. Firstly, we have assumed spatial torsion was first-order in derivatives for simplicity. We can further generalize the theory to include zeroth-order torsion. In that case, we can model a crystal structure that incorporates defects. In particular, defects in a crystal can be modeled using zeroth-order spatial torsion [23, 27, 36], and one might find ways to generate spin polarization exclusive to defective crystals.

Furthermore, it is common practice in condensed matter physics to model spin in 3D even if the considered system is in 2D [49]. Our framework lacks this kind of dimension difference between spin and spacetime. To generalize our framework to include this dimension difference, one has to sacrifice Einstein-Cartan formalism that we have used throughout the thesis. A promising way to implement the dimension difference is to treat the spin connection as a pure gauge field.

Appendix A

Second-order Equations of Motion

$$
\begin{aligned}
\frac{\partial^2 P}{\partial T^2} T u^\lambda \mathring{\nabla}_\lambda T = & -f_h^{(2)\lambda} k_\lambda + 2 f_T^{(2)} k + f_T^{(2)} \tilde{k} + 2 f_T^{(2)} M - f_{\mathcal{E}} \Theta - f_{\mathcal{P}} \Theta - k_\lambda k^\lambda \chi_1^{(1)} \\
& - 2 k^\lambda \tilde{k}_\lambda \chi_1^{(1)} - \tilde{k}_\lambda \tilde{k}^\lambda \chi_1^{(1)} + 2 k^\lambda m_\lambda \chi_1^{(1)} + 2 \tilde{k}^\lambda m_\lambda \chi_1^{(1)} \\
& - m_\lambda m^\lambda \chi_1^{(1)} - \sigma_{\lambda\mu} \sigma^{\lambda\mu} \chi_1^{(1)} - \tfrac{1}{2} k \Theta \chi_2^{(1)} - \tfrac{1}{2} \tilde{k} \Theta \chi_2^{(1)} \\
& - M \Theta \chi_2^{(1)} + k \Theta \chi_3^{(1)} + \tilde{k} \Theta \chi_3^{(1)} + 2 M \Theta \chi_3^{(1)} - u^\lambda \mathring{\nabla}_\lambda f_{\mathcal{E}} \\
& + \tfrac{1}{2} \mathring{\nabla}_\lambda f_h^{(2)\lambda} - \tfrac{1}{2} \mathring{\nabla}_\lambda f_q{}^\lambda - \tfrac{1}{2} u^\lambda \chi_2^{(1)} \mathring{\nabla}_\lambda k + u^\lambda \chi_3^{(1)} \mathring{\nabla}_\lambda k \\
& - \tfrac{1}{2} u^\lambda \chi_2^{(1)} \mathring{\nabla}_\lambda \tilde{k} + u^\lambda \chi_3^{(1)} \mathring{\nabla}_\lambda \tilde{k} - u^\lambda \chi_2^{(1)} \mathring{\nabla}_\lambda M + 2 u^\lambda \chi_3^{(1)} \mathring{\nabla}_\lambda M \\
& + \tfrac{1}{2} u^\lambda u^\mu \mathring{\nabla}_\mu f_h^{(2)}{}_\lambda + \tfrac{1}{2} u^\lambda u^\mu \mathring{\nabla}_\mu f_{q\lambda} + u^\lambda \mathring{\nabla}_\mu f_{\pi\lambda}{}^\mu + u^\lambda u^\mu \chi_1^{(1)} \mathring{\nabla}_\mu k_\lambda \\
& + u^\lambda u^\mu \chi_1^{(1)} \mathring{\nabla}_\mu \tilde{k}_\lambda - u^\lambda u^\mu \chi_1^{(1)} \mathring{\nabla}_\mu m_\lambda - u^\lambda \chi_1^{(1)} \mathring{\nabla}_\mu \sigma_\lambda{}^\mu \\
& - \tfrac{1}{2} \epsilon_{\lambda\mu\nu} u^\lambda \chi_2^{(1)} \mathring{\nabla}^\nu k^\mu + \epsilon_{\lambda\mu\nu} u^\lambda \chi_3^{(1)} \mathring{\nabla}^\nu k^\mu - \tfrac{1}{2} \epsilon_{\lambda\mu\nu} u^\lambda \chi_2^{(1)} \mathring{\nabla}^\nu \tilde{k}^\mu \\
& + \epsilon_{\lambda\mu\nu} u^\lambda \chi_3^{(1)} \mathring{\nabla}^\nu \tilde{k}^\mu + \tfrac{1}{2} \epsilon_{\lambda\mu\nu} u^\lambda \chi_2^{(1)} \mathring{\nabla}^\nu m^\mu - \epsilon_{\lambda\mu\nu} u^\lambda \chi_3^{(1)} \mathring{\nabla}^\nu m^\mu \\
& + \epsilon_{\mu\nu\rho} u^\lambda u^\mu \chi_2^{(1)} \mathring{\nabla}^\rho \sigma_\lambda{}^\nu - T \Theta \kappa_{\mathrm{T}} \frac{\partial \chi_1^{(1)}}{\partial T} - u^\lambda T \mathring{\nabla}_\lambda \kappa_{\mathrm{T}} \frac{\partial \chi_1^{(1)}}{\partial T} \\
& - u^\lambda u^\mu u^\nu T \mathring{\nabla}_\nu \kappa_{\mathrm{S}\lambda\mu} \frac{\partial \chi_1^{(1)}}{\partial T} - T \Theta \kappa_{\mathrm{A}} \frac{\partial \chi_2^{(1)}}{\partial T} - u^\lambda T \mathring{\nabla}_\lambda \kappa_{\mathrm{A}} \frac{\partial \chi_2^{(1)}}{\partial T} \\
& - \epsilon_{\lambda\mu\nu} u^\lambda T \mathring{\nabla}^\nu k^\mu \frac{\partial \chi_2^{(1)}}{\partial T} - \epsilon_{\lambda\mu\nu} u^\lambda T \mathring{\nabla}^\nu \tilde{k}^\mu \frac{\partial \chi_2^{(1)}}{\partial T} \\
& + \epsilon_{\lambda\mu\nu} u^\lambda T \mathring{\nabla}^\nu m^\mu \frac{\partial \chi_2^{(1)}}{\partial T} + k^\lambda T \mathcal{K}_\lambda \frac{\partial \chi_3^{(1)}}{\partial T} + \tilde{k}^\lambda T \mathcal{K}_\lambda \frac{\partial \chi_3^{(1)}}{\partial T} \\
& - m^\lambda T \mathcal{K}_\lambda \frac{\partial \chi_3^{(1)}}{\partial T} + \tilde{k} T \Theta \frac{\partial \chi_3^{(1)}}{\partial T} + 2 M T \Theta \frac{\partial \chi_3^{(1)}}{\partial T} \\
& + u^\lambda T \mathring{\nabla}_\lambda \tilde{k} \frac{\partial \chi_3^{(1)}}{\partial T} + 2 u^\lambda T \mathring{\nabla}_\lambda M \frac{\partial \chi_3^{(1)}}{\partial T} - u^\lambda u^\mu T \mathring{\nabla}_\mu \mathcal{K}_\lambda \frac{\partial \chi_3^{(1)}}{\partial T} \\
& + 2 \epsilon_{\lambda\mu\nu} u^\lambda T \mathring{\nabla}^\nu k^\mu \frac{\partial \chi_3^{(1)}}{\partial T} + 2 \epsilon_{\lambda\mu\nu} u^\lambda T \mathring{\nabla}^\nu \tilde{k}^\mu \frac{\partial \chi_3^{(1)}}{\partial T}
\end{aligned}
$$

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\begin{aligned}
&- 2\epsilon_{\lambda\mu\nu} u^\lambda T \mathring{\nabla}^\nu m^\mu \frac{\partial \chi_3^{(1)}}{\partial T} - \epsilon_{\lambda\mu\nu} k^\lambda \tilde{k}^\mu u^\nu \frac{\partial P}{\partial M} \\
&+ \epsilon_{\lambda\mu\nu} \tilde{k}^\lambda u^\mu m^\nu \frac{\partial P}{\partial M} - \tfrac{1}{2} k\Theta \frac{\partial P}{\partial M} - \tfrac{1}{2}\tilde{k}\Theta \frac{\partial P}{\partial M} \\
&- 2M\Theta \frac{\partial P}{\partial M} - \tfrac{1}{2} u^\lambda \mathring{\nabla}_\lambda k \frac{\partial P}{\partial M} - \tfrac{1}{2} u^\lambda \mathring{\nabla}_\lambda \tilde{k} \frac{\partial P}{\partial M} \\
&- u^\lambda \mathring{\nabla}_\lambda M \frac{\partial P}{\partial M} - \tfrac{1}{2} \epsilon_{\lambda\mu\nu} u^\lambda \mathring{\nabla}^\nu k^\mu \frac{\partial P}{\partial M} \\
&- \epsilon_{\lambda\mu\nu} u^\lambda \mathring{\nabla}^\nu \tilde{k}^\mu \frac{\partial P}{\partial M} + \tfrac{1}{2} \epsilon_{\lambda\mu\nu} u^\lambda \mathring{\nabla}^\nu m^\mu \frac{\partial P}{\partial M} \\
&+ \tfrac{1}{2} \epsilon_{\lambda\mu\nu} k^\lambda \tilde{k}^\mu u^\nu T \frac{\partial^2 P}{\partial T \partial M} - \tfrac{1}{2} \epsilon_{\lambda\mu\nu} \tilde{k}^\lambda u^\mu m^\nu T \frac{\partial^2 P}{\partial T \partial M} \\
&- MT\Theta \frac{\partial^2 P}{\partial T \partial M} - u^\lambda T \mathring{\nabla}_\lambda M \frac{\partial^2 P}{\partial T \partial M} - \frac{k\Theta \frac{\partial \chi_2^{(1)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} \\
&- \frac{\tilde{k}\Theta \frac{\partial \chi_2^{(1)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - \frac{2M\Theta \frac{\partial \chi_2^{(1)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} + \frac{k\Theta \frac{\partial \chi_3^{(1)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} \\
&+ \frac{\tilde{k}\Theta \frac{\partial \chi_3^{(1)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} + \frac{2M\Theta \frac{\partial \chi_3^{(1)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} + \frac{T\Theta \kappa_{\mathrm{T}} \frac{\partial^2 \chi_1^{(1)}}{\partial T^2} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} \\
&+ \frac{T\Theta \kappa_{\mathrm{A}} \frac{\partial^2 \chi_2^{(1)}}{\partial T^2} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} + \frac{kT\Theta \frac{\partial^2 \chi_3^{(1)}}{\partial T^2} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} + \frac{kT\Theta \frac{\partial^2 \chi_3^{(1)}}{\partial T^2} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - T\Theta \frac{\partial P}{\partial T} \\
&+ \frac{M\Theta \frac{\partial P}{\partial T} \frac{\partial^2 P}{\partial T \partial M}}{\frac{\partial^2 P}{\partial T^2}} + \frac{MT\Theta \frac{\partial P}{\partial T} \frac{\partial^3 P}{\partial T^2 \partial M}}{\frac{\partial^2 P}{\partial T^2}}.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
&\mathring{\nabla}_\nu \sigma^{\alpha\rho} \left[\chi_1^{(1)} \Delta^\beta{}_\alpha \delta^\nu_\rho - \epsilon_\lambda{}^\beta{}_\rho \chi_3^{(1)} u^\lambda \delta^\nu_\alpha - \epsilon_{\mu\rho}{}^\nu \chi_3^{(1)} u_\alpha \Delta^{\beta\mu} - \epsilon_{\lambda\rho}{}^\nu \left(\chi_2^{(1)} - \chi_3^{(1)} \right) u^\lambda \Delta^\beta{}_\alpha \right] \\
&= - f_{\mathcal{E}} k^\beta - f_{\mathcal{P}} k^\beta - f_{\mathcal{E}} \tilde{k}^\beta - f_{\mathcal{P}} \tilde{k}^\beta + f_{\mathcal{E}} m^\beta + f_{\mathcal{P}} m^\beta + f_{\mathcal{T}}^{(2)} \mathcal{K}^\beta \\
&\quad + \epsilon_{\lambda\mu\nu} f_{\mathcal{T}}^{(2)} k^\lambda u^\mu \Delta^{\beta\nu} + \epsilon_{\lambda\mu\nu} f_{\mathcal{T}}^{(2)} \tilde{k}^\lambda u^\mu \Delta^{\beta\nu} - \tfrac{1}{4} \epsilon_{\lambda\mu\nu} f_h^{(2)\lambda} u^\mu k \Delta^{\beta\nu} \\
&\quad + \tfrac{1}{4} \epsilon_{\lambda\mu\nu} f_q{}^\lambda u^\mu k \Delta^{\beta\nu} - \tfrac{1}{4} \epsilon_{\lambda\mu\nu} f_h^{(2)\lambda} u^\mu \tilde{k} \Delta^{\beta\nu} + \tfrac{1}{4} \epsilon_{\lambda\mu\nu} f_q{}^\lambda u^\mu \tilde{k} \Delta^{\beta\nu} \\
&\quad + \epsilon_{\lambda\mu\nu} f_{\mathcal{T}}^{(2)} u^\lambda m^\mu \Delta^{\beta\nu} - \tfrac{1}{2} \epsilon_{\lambda\mu\nu} f_h^{(2)\lambda} u^\mu M \Delta^{\beta\nu} + \tfrac{1}{2} \epsilon_{\lambda\mu\nu} f_q{}^\lambda u^\mu M \Delta^{\beta\nu} \\
&\quad + \tfrac{1}{4} f_h^{(2)\beta} \Theta + \tfrac{3}{4} f_q{}^\beta \Theta - \tfrac{1}{2} \epsilon_{\lambda\mu\nu} f_h^{(2)\lambda} u^\mu \Delta^{\beta\nu} \kappa_{\mathrm{A}} - f_h^{(2)\lambda} \kappa_{\mathrm{S}}{}^\beta{}_\lambda - \tfrac{1}{2} f_h^{(2)\beta} \kappa_{\mathrm{T}} \\
&\quad - \tfrac{1}{2} f_h^{(2)\lambda} \sigma^\beta{}_\lambda + \tfrac{1}{2} f_q{}^\lambda \sigma^\beta{}_\lambda - \tfrac{1}{4} k \mathcal{K}^\beta \chi_1^{(1)} - \tfrac{1}{4} \tilde{k} \mathcal{K}^\beta \chi_1^{(1)} - \tfrac{1}{2} M \mathcal{K}^\beta \chi_1^{(1)} \\
&\quad + \mathring{R}_\lambda{}^\mu u^\lambda \Delta^\beta{}_\mu \chi_1^{(1)} - 2\epsilon_{\lambda\mu\nu} k^\lambda u^\mu k \Delta^{\beta\nu} \chi_1^{(1)} \\
&\quad - \tfrac{3}{2} \epsilon_{\lambda\mu\nu} \tilde{k}^\lambda u^\mu k \Delta^{\beta\nu} \chi_1^{(1)} - 2\epsilon_{\lambda\mu\nu} k^\lambda u^\mu \tilde{k} \Delta^{\beta\nu} \chi_1^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{3}{2}\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}\tilde{k}\Delta^{\beta\nu}\chi_1^{(1)} - \tfrac{3}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}km^{\mu}\Delta^{\beta\nu}\chi_1^{(1)} \\
&- \tfrac{3}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\tilde{k}m^{\mu}\Delta^{\beta\nu}\chi_1^{(1)} - 4\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}M\Delta^{\beta\nu}\chi_1^{(1)} \\
&- 3\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}M\Delta^{\beta\nu}\chi_1^{(1)} - 3\epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}M\Delta^{\beta\nu}\chi_1^{(1)} \\
&- \tfrac{1}{2}\epsilon_{\lambda\mu\nu}k^{\lambda}\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_1^{(1)} - \tfrac{1}{2}\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_1^{(1)} \\
&+ \tfrac{1}{2}\epsilon_{\lambda\mu\nu}m^{\lambda}\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_1^{(1)} + 2k^{\beta}\Theta\chi_1^{(1)} + \tfrac{1}{2}\tilde{k}^{\beta}\Theta\chi_1^{(1)} - \tfrac{1}{2}m^{\beta}\Theta\chi_1^{(1)} \\
&+ \tfrac{3}{4}\epsilon_{\lambda\mu\nu}u^{\lambda}\mathcal{K}^{\mu}\Delta^{\beta\nu}\Theta\chi_1^{(1)} - k^{\lambda}\kappa_{\mathrm{S}}{}^{\beta}{}_{\lambda}\chi_1^{(1)} - \tilde{k}^{\lambda}\kappa_{\mathrm{S}}{}^{\beta}{}_{\lambda}\chi_1^{(1)} \\
&+ m^{\lambda}\kappa_{\mathrm{S}}{}^{\beta}{}_{\lambda}\chi_1^{(1)} + k^{\lambda}\sigma^{\beta}{}_{\lambda}\chi_1^{(1)} + \tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\mathcal{K}^{\mu}\sigma^{\beta\nu}\chi_1^{(1)} \\
&+ \tfrac{3}{2}k^{\beta}k\chi_2^{(1)} + \tilde{k}^{\beta}k\chi_2^{(1)} + \tfrac{3}{2}k^{\beta}\tilde{k}\chi_2^{(1)} + \tilde{k}^{\beta}\tilde{k}\chi_2^{(1)} - km^{\beta}\chi_2^{(1)} \\
&- \tilde{k}m^{\beta}\chi_2^{(1)} + 3k^{\beta}M\chi_2^{(1)} + 2\tilde{k}^{\beta}M\chi_2^{(1)} - 2m^{\beta}M\chi_2^{(1)} \\
&- \epsilon_{\lambda\mu\nu}k^{\lambda}\tilde{k}^{\mu}\Delta^{\beta\nu}\chi_2^{(1)} - \epsilon_{\lambda}{}^{\rho\sigma}\mathring{R}_{\mu\rho\nu\sigma}u^{\lambda}u^{\mu}\Delta^{\beta\nu}\chi_2^{(1)} \\
&+ \epsilon_{\lambda\mu\nu}k^{\lambda}m^{\mu}\Delta^{\beta\nu}\chi_2^{(1)} - \tfrac{1}{4}\epsilon_{\lambda\mu\nu}u^{\lambda}k\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_2^{(1)} \\
&- \tfrac{1}{4}\epsilon_{\lambda\mu\nu}u^{\lambda}\tilde{k}\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_2^{(1)} - \tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}M\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_2^{(1)} - \tfrac{3}{4}\mathcal{K}^{\beta}\Theta\chi_2^{(1)} \\
&+ \tfrac{3}{2}\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\chi_2^{(1)} - \epsilon_{\mu\nu\rho}k^{\lambda}u^{\mu}\Delta^{\beta\nu}\kappa_{\mathrm{S}\lambda}{}^{\rho}\chi_2^{(1)} \\
&- \epsilon_{\mu\nu\rho}\tilde{k}^{\lambda}u^{\mu}\Delta^{\beta\nu}\kappa_{\mathrm{S}\lambda}{}^{\rho}\chi_2^{(1)} + \epsilon_{\lambda\nu\rho}u^{\lambda}m^{\mu}\Delta^{\beta\nu}\kappa_{\mathrm{S}\mu}{}^{\rho}\chi_2^{(1)} \\
&- \tfrac{1}{2}\mathcal{K}^{\lambda}\sigma^{\beta}{}_{\lambda}\chi_2^{(1)} + \epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\sigma^{\beta\nu}\chi_2^{(1)} - 4k^{\beta}k\chi_3^{(1)} - 3\tilde{k}^{\beta}k\chi_3^{(1)} \\
&- 4k^{\beta}\tilde{k}\chi_3^{(1)} - 3\tilde{k}^{\beta}\tilde{k}\chi_3^{(1)} + 3km^{\beta}\chi_3^{(1)} + 3\tilde{k}m^{\beta}\chi_3^{(1)} \\
&- 8k^{\beta}M\chi_3^{(1)} - 6\tilde{k}^{\beta}M\chi_3^{(1)} + 6m^{\beta}M\chi_3^{(1)} + 2\epsilon_{\lambda\mu\nu}k^{\lambda}\tilde{k}^{\mu}\Delta^{\beta\nu}\chi_3^{(1)} \\
&+ \epsilon_{\lambda}{}^{\rho\sigma}\mathring{R}_{\mu\nu\rho\sigma}u^{\lambda}u^{\mu}\Delta^{\beta\nu}\chi_3^{(1)} - 2\epsilon_{\lambda\mu\nu}k^{\lambda}m^{\mu}\Delta^{\beta\nu}\chi_3^{(1)} \\
&+ \tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}k\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_3^{(1)} + \tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\tilde{k}\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_3^{(1)} \\
&+ \epsilon_{\lambda\mu\nu}u^{\lambda}M\mathcal{K}^{\mu}\Delta^{\beta\nu}\chi_3^{(1)} + \tfrac{3}{2}\mathcal{K}^{\beta}\Theta\chi_3^{(1)} - 4\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\chi_3^{(1)} \\
&- \epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\chi_3^{(1)} - \epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}\Delta^{\beta\nu}\Theta\chi_3^{(1)} \\
&+ 2\epsilon_{\mu\nu\rho}k^{\lambda}u^{\mu}\Delta^{\beta\nu}\kappa_{\mathrm{S}\lambda}{}^{\rho}\chi_3^{(1)} + 2\epsilon_{\mu\nu\rho}\tilde{k}^{\lambda}u^{\mu}\Delta^{\beta\nu}\kappa_{\mathrm{S}\lambda}{}^{\rho}\chi_3^{(1)} \\
&- 2\epsilon_{\lambda\nu\rho}u^{\lambda}m^{\mu}\Delta^{\beta\nu}\kappa_{\mathrm{S}\mu}{}^{\rho}\chi_3^{(1)} + \mathcal{K}^{\lambda}\sigma^{\beta}{}_{\lambda}\chi_3^{(1)} - 2\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\sigma^{\beta\nu}\chi_3^{(1)} \\
&+ \tfrac{1}{2}u^{\lambda}\Delta^{\beta}{}_{\mu}\mathring{\nabla}_{\lambda}f_h^{(2)\mu} + \tfrac{1}{2}u^{\lambda}\Delta^{\beta}{}_{\mu}\mathring{\nabla}_{\lambda}f_q{}^{\mu} + u^{\lambda}\Delta^{\beta}{}_{\mu}\chi_1^{(1)}\mathring{\nabla}_{\lambda}k^{\mu} \\
&- \epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_2^{(1)}\mathring{\nabla}_{\lambda}k^{\nu} + 3\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}_{\lambda}k^{\nu} \\
&+ \epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}_{\lambda}\tilde{k}^{\nu} - \epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}_{\lambda}m^{\nu} \\
&- \tfrac{1}{2}u^{\lambda}\Delta^{\beta}{}_{\mu}\chi_2^{(1)}\mathring{\nabla}_{\lambda}\mathcal{K}^{\mu} + u^{\lambda}\Delta^{\beta}{}_{\mu}\chi_3^{(1)}\mathring{\nabla}_{\lambda}\mathcal{K}^{\mu} \\
&+ \tfrac{1}{2}\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_1^{(1)}\mathring{\nabla}_{\lambda}\mathcal{K}^{\nu} + \Delta^{\beta}{}_{\lambda}\mathring{\nabla}^{\lambda}f_{\mathcal{P}} - \tfrac{1}{2}\Delta^{\beta}{}_{\lambda}\chi_2^{(1)}\mathring{\nabla}^{\lambda}k \\
&+ \Delta^{\beta}{}_{\lambda}\chi_3^{(1)}\mathring{\nabla}^{\lambda}k - \tfrac{1}{2}\Delta^{\beta}{}_{\lambda}\chi_2^{(1)}\mathring{\nabla}^{\lambda}\tilde{k} + \Delta^{\beta}{}_{\lambda}\chi_3^{(1)}\mathring{\nabla}^{\lambda}\tilde{k}
\end{aligned}
$$

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\begin{aligned}
&-\Delta^{\beta}{}_{\lambda}\chi_2^{(1)}\mathring{\nabla}^{\lambda}M+2\Delta^{\beta}{}_{\lambda}\chi_3^{(1)}\mathring{\nabla}^{\lambda}M+\tfrac{1}{2}\Delta^{\beta}{}_{\lambda}\chi_1^{(1)}\mathring{\nabla}^{\lambda}\Theta\\
&+\Delta^{\beta\lambda}\mathring{\nabla}_{\mu}f_{\pi\lambda}{}^{\mu}+u^{\lambda}u^{\mu}\Delta^{\beta\nu}\chi_1^{(1)}\mathring{\nabla}_{\mu}\kappa_{\mathrm{S}\lambda\nu}\\
&+\epsilon_{\nu\rho\sigma}u^{\lambda}u^{\mu}u^{\nu}\Delta^{\beta\rho}\chi_2^{(1)}\mathring{\nabla}_{\mu}\kappa_{\mathrm{S}\lambda}{}^{\sigma}-2\epsilon_{\nu\rho\sigma}u^{\lambda}u^{\mu}u^{\nu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}_{\mu}\kappa_{\mathrm{S}\lambda}{}^{\sigma}\\
&+\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\mathring{\nabla}^{\mu}f_{\mathcal{T}}^{(2)}+u^{\lambda}\Delta^{\beta}{}_{\mu}\chi_1^{(1)}\mathring{\nabla}^{\mu}k_{\lambda}-\epsilon_{\lambda\mu\nu}\Delta^{\beta\nu}\chi_3^{(1)}\mathring{\nabla}^{\mu}k^{\lambda}\\
&+u^{\lambda}\Delta^{\beta}{}_{\mu}\chi_1^{(1)}\mathring{\nabla}^{\mu}\tilde{k}_{\lambda}-\epsilon_{\lambda\mu\nu}\Delta^{\beta\nu}\chi_3^{(1)}\mathring{\nabla}^{\mu}\tilde{k}^{\lambda}-\tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\chi_1^{(1)}\mathring{\nabla}^{\mu}k\\
&-\tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\chi_1^{(1)}\mathring{\nabla}^{\mu}\tilde{k}-u^{\lambda}\Delta^{\beta}{}_{\mu}\chi_1^{(1)}\mathring{\nabla}^{\mu}m_{\lambda}\\
&+\epsilon_{\lambda\mu\nu}\Delta^{\beta\nu}\chi_3^{(1)}\mathring{\nabla}^{\mu}m^{\lambda}-\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\chi_1^{(1)}\mathring{\nabla}^{\mu}M\\
&-\tfrac{1}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\chi_2^{(1)}\mathring{\nabla}^{\mu}\Theta+\tfrac{3}{2}\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\chi_3^{(1)}\mathring{\nabla}^{\mu}\Theta\\
&+\tfrac{1}{2}\epsilon_{\lambda\nu\rho}u^{\lambda}\Delta^{\beta\nu}\Delta_{\mu}{}^{\rho}\chi_3^{(1)}\mathring{\nabla}^{\mu}\Theta+\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}^{\nu}k_{\lambda}\\
&+\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}^{\nu}\tilde{k}_{\lambda}-\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\chi_3^{(1)}\mathring{\nabla}^{\nu}m_{\lambda}\\
&-k^{\beta}T\Theta\frac{\partial\chi_1^{(1)}}{\partial T}-\tilde{k}^{\beta}T\Theta\frac{\partial\chi_1^{(1)}}{\partial T}+m^{\beta}T\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\\
&-u^{\lambda}T\Delta^{\beta}{}_{\mu}\mathring{\nabla}_{\lambda}k^{\mu}\frac{\partial\chi_1^{(1)}}{\partial T}-u^{\lambda}T\Delta^{\beta}{}_{\mu}\mathring{\nabla}_{\lambda}\tilde{k}^{\mu}\frac{\partial\chi_1^{(1)}}{\partial T}\\
&+u^{\lambda}T\Delta^{\beta}{}_{\mu}\mathring{\nabla}_{\lambda}m^{\mu}\frac{\partial\chi_1^{(1)}}{\partial T}+u^{\lambda}T\Delta^{\beta}{}_{\mu}\mathring{\nabla}^{\mu}k_{\lambda}\frac{\partial\chi_1^{(1)}}{\partial T}\\
&+u^{\lambda}T\Delta^{\beta}{}_{\mu}\mathring{\nabla}^{\mu}\tilde{k}_{\lambda}\frac{\partial\chi_1^{(1)}}{\partial T}-u^{\lambda}T\Delta^{\beta}{}_{\mu}\mathring{\nabla}^{\mu}m_{\lambda}\frac{\partial\chi_1^{(1)}}{\partial T}\\
&+\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}T\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}+\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}T\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\\
&+\epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}T\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}-\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}T\Delta^{\beta\rho}\mathring{\nabla}_{\lambda}k^{\nu}\frac{\partial\chi_3^{(1)}}{\partial T}\\
&-\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}T\Delta^{\beta\rho}\mathring{\nabla}_{\lambda}\tilde{k}^{\nu}\frac{\partial\chi_3^{(1)}}{\partial T}+\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}T\Delta^{\beta\rho}\mathring{\nabla}_{\lambda}m^{\nu}\frac{\partial\chi_3^{(1)}}{\partial T}\\
&+\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}T\Delta^{\beta\rho}\mathring{\nabla}^{\nu}k_{\lambda}\frac{\partial\chi_3^{(1)}}{\partial T}+\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}T\Delta^{\beta\rho}\mathring{\nabla}^{\nu}\tilde{k}_{\lambda}\frac{\partial\chi_3^{(1)}}{\partial T}\\
&-\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}T\Delta^{\beta\rho}\mathring{\nabla}^{\nu}m_{\lambda}\frac{\partial\chi_3^{(1)}}{\partial T}+\tfrac{3}{4}\tilde{k}^{\beta}k\frac{\partial P}{\partial M}-\tfrac{1}{2}k^{\beta}\tilde{k}\frac{\partial P}{\partial M}\\
&+\tfrac{1}{4}\tilde{k}^{\beta}\tilde{k}\frac{\partial P}{\partial M}-\tfrac{1}{2}km^{\beta}\frac{\partial P}{\partial M}-2k^{\beta}M\frac{\partial P}{\partial M}\\
&-\tfrac{1}{2}\tilde{k}^{\beta}M\frac{\partial P}{\partial M}+m^{\beta}M\frac{\partial P}{\partial M}+\tfrac{1}{2}\epsilon_{\lambda\mu\nu}k^{\lambda}\tilde{k}^{\mu}\Delta^{\beta\nu}\frac{\partial P}{\partial M}\\
&+\tfrac{1}{2}\epsilon_{\lambda}{}^{\rho\sigma}\mathring{R}_{\mu\nu\rho\sigma}u^{\lambda}u^{\mu}\Delta^{\beta\nu}\frac{\partial P}{\partial M}-\epsilon_{\lambda\mu\nu}k^{\lambda}m^{\mu}\Delta^{\beta\nu}\frac{\partial P}{\partial M}\\
&-\tfrac{1}{2}\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}m^{\mu}\Delta^{\beta\nu}\frac{\partial P}{\partial M}-\tfrac{1}{2}\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial P}{\partial M}
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{4}\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial P}{\partial M}+\epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial P}{\partial M}\\
&-\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\sigma^{\beta\nu}\frac{\partial P}{\partial M}-\tfrac{1}{2}\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}\sigma^{\beta\nu}\frac{\partial P}{\partial M}\\
&-\tfrac{1}{2}\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\mathring{\nabla}_{\lambda}\tilde{k}^{\nu}\frac{\partial P}{\partial M}+\epsilon_{\mu\nu\rho}u^{\lambda}u^{\mu}\Delta^{\beta\rho}\mathring{\nabla}_{\lambda}m^{\nu}\frac{\partial P}{\partial M}\\
&+\tfrac{1}{2}u^{\lambda}\Delta^{\beta}{}_{\mu}\mathring{\nabla}_{\lambda}\mathcal{K}^{\mu}\frac{\partial P}{\partial M}-\tfrac{1}{2}\Delta^{\beta}{}_{\lambda}\mathring{\nabla}^{\lambda}k\frac{\partial P}{\partial M}\\
&+\Delta^{\beta}{}_{\lambda}\mathring{\nabla}^{\lambda}M\frac{\partial P}{\partial M}-\tfrac{1}{2}u^{\lambda}\Delta^{\beta}{}_{\mu}\mathring{\nabla}^{\mu}\mathcal{K}_{\lambda}\frac{\partial P}{\partial M}\\
&+d\mathcal{T}^{(2)\lambda}\Delta^{\beta}{}_{\lambda}\frac{\partial P}{\partial T}+\frac{k^{\beta}\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}+\frac{2\tilde{k}^{\beta}\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{2m^{\beta}\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}-\frac{\epsilon_{\lambda\mu\nu}u^{\lambda}\mathcal{K}^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{2\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{\Delta^{\beta}{}_{\lambda}\mathring{\nabla}^{\lambda}\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}+\frac{\mathcal{K}^{\beta}\Theta\frac{\partial\chi_2^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{2\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial\chi_2^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}-\frac{\mathcal{K}^{\beta}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{2\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}-\frac{2\epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{\epsilon_{\lambda\mu\nu}u^{\lambda}\Delta^{\beta\nu}\mathring{\nabla}^{\mu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}}-\frac{\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial P}{\partial T}\frac{\partial^2 P}{\partial T\partial M}}{2\frac{\partial^2 P}{\partial T^2}}\\
&-\frac{\epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}\Delta^{\beta\nu}\Theta\frac{\partial P}{\partial T}\frac{\partial^2 P}{\partial T\partial M}}{\frac{\partial^2 P}{\partial T^2}}+\frac{k^{\beta}T\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}\frac{\partial^3 P}{\partial T^3}}{\left(\frac{\partial^2 P}{\partial T^2}\right)^2}\\
&+\frac{\tilde{k}^{\beta}T\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}\frac{\partial^3 P}{\partial T^3}}{\left(\frac{\partial^2 P}{\partial T^2}\right)^2}-\frac{m^{\beta}T\Theta\frac{\partial\chi_1^{(1)}}{\partial T}\frac{\partial P}{\partial T}\frac{\partial^3 P}{\partial T^3}}{\left(\frac{\partial^2 P}{\partial T^2}\right)^2}\\
&-\frac{\epsilon_{\lambda\mu\nu}k^{\lambda}u^{\mu}T\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}\frac{\partial^3 P}{\partial T^3}}{\left(\frac{\partial^2 P}{\partial T^2}\right)^2}\\
&-\frac{\epsilon_{\lambda\mu\nu}\tilde{k}^{\lambda}u^{\mu}T\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}\frac{\partial^3 P}{\partial T^3}}{\left(\frac{\partial^2 P}{\partial T^2}\right)^2}\\
&-\frac{\epsilon_{\lambda\mu\nu}u^{\lambda}m^{\mu}T\Delta^{\beta\nu}\Theta\frac{\partial\chi_3^{(1)}}{\partial T}\frac{\partial P}{\partial T}\frac{\partial^3 P}{\partial T^3}}{\left(\frac{\partial^2 P}{\partial T^2}\right)^2}.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&u^\beta \mathring{\nabla}_\beta M\left(2\chi_9^{(2)} - \frac{\partial^2 P}{\partial M^2}\right) \\
&= -2f_T^{(2)} - \epsilon_{\beta\lambda\mu} f_s{}^\beta \tilde{k}^\lambda u^\mu + \tfrac{1}{2} f_{\varsigma\mathrm{T}} k + \tfrac{1}{2} f_{\varsigma\mathrm{T}} \tilde{k} - \epsilon_{\beta\lambda\mu} f_s{}^\beta u^\lambda m^\mu + f_{\varsigma\mathrm{T}} M \\
&\quad + \tfrac{1}{2} f_{\varsigma\mathrm{A}} \Theta - \tfrac{1}{2} f_{\varsigma\mathrm{T}} \kappa_\mathrm{A} - \epsilon_{\beta\mu\nu} f_{\varsigma\mathrm{S}}{}^{\lambda\mu} u^\beta \kappa_{\mathrm{S}\lambda}{}^\nu + \tfrac{1}{2} f_{\varsigma\mathrm{A}} \kappa_\mathrm{T} - \epsilon_{\beta\mu\nu} f_{\varsigma\mathrm{S}}{}^{\lambda\mu} u^\beta \sigma_\lambda{}^\nu \\
&\quad - \epsilon_{\beta\lambda\mu} k^\beta \tilde{k}^\lambda u^\mu \chi_{11}^{(2)} - \epsilon_{\beta\lambda\mu} \tilde{k}^\beta u^\lambda \mathcal{K}^\mu \chi_{13}^{(2)} - k^\beta \tilde{k}_\beta \chi_{15}^{(2)} - \tfrac{1}{2} f_{\mathfrak{S}} \Theta \\
&\quad + \tilde{k}^\beta \mathcal{K}_\beta \chi_{17}^{(2)} + \Theta \kappa_\mathrm{T} \chi_6^{(2)} + \Theta \kappa_\mathrm{A} \chi_8^{(2)} - k^\beta \mathcal{K}_\beta \chi_9^{(2)} \\
&\quad - \tilde{k}^\beta \mathcal{K}_\beta \chi_9^{(2)} + m^\beta \mathcal{K}_\beta \chi_9^{(2)} - \tilde{k} \Theta \chi_9^{(2)} - 2M\Theta \chi_9^{(2)} - \tfrac{1}{2} u^\beta \mathring{\nabla}_\beta f_{\mathfrak{S}} \\
&\quad + \tfrac{1}{2} \mathring{\nabla}_\beta f_\Sigma{}^\beta - u^\beta \chi_9^{(2)} \mathring{\nabla}_\beta \tilde{k} + u^\beta \chi_8^{(2)} \mathring{\nabla}_\beta \kappa_\mathrm{A} + u^\beta \chi_6^{(2)} \mathring{\nabla}_\beta \kappa_\mathrm{T} \\
&\quad + u^\beta u^\lambda \chi_9^{(2)} \mathring{\nabla}_\lambda \mathcal{K}_\beta + u^\beta u^\lambda u^\mu \chi_6^{(2)} \mathring{\nabla}_\mu \kappa_{\mathrm{S}\beta\lambda} - \epsilon_{\beta\lambda\mu} u^\beta \chi_9^{(2)} \mathring{\nabla}^\mu k^\lambda \\
&\quad - \epsilon_{\beta\lambda\mu} u^\beta \chi_9^{(2)} \mathring{\nabla}^\mu \tilde{k}^\lambda + \epsilon_{\beta\lambda\mu} u^\beta \chi_9^{(2)} \mathring{\nabla}^\mu m^\lambda + M\Theta \frac{\partial^2 P}{\partial M^2} \\
&\quad - 2\epsilon_{\beta\lambda\mu} \tilde{k}^\beta u^\lambda m^\mu \frac{\partial P}{\partial m^2} + d\mathcal{T}^{(2)\beta} u_\beta \frac{\partial^2 P}{\partial T \partial M} \\
&\quad - \frac{\Theta \kappa_\mathrm{T} \frac{\partial \chi_6^{(2)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - \frac{\Theta \kappa_\mathrm{A} \frac{\partial \chi_8^{(2)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} - \frac{k\Theta \frac{\partial \chi_9^{(2)}}{\partial T} \frac{\partial P}{\partial T}}{\frac{\partial^2 P}{\partial T^2}} \\
&\quad - \frac{M\Theta \frac{\partial P}{\partial T} \frac{\partial^3 P}{\partial T \partial M^2}}{\frac{\partial^2 P}{\partial T^2}}. \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial P}{\partial m^2} u^\lambda \Delta_\mu{}^\nu \mathring{\nabla}_\lambda m_\nu \\
&= -\tfrac{1}{2} f_h^{(2)}{}_\mu - \tfrac{1}{4} f_{\varsigma\mathrm{T}} k_\mu - \tfrac{1}{4} f_{\varsigma\mathrm{T}} \tilde{k}_\mu + \tfrac{1}{4} \epsilon_{\mu\lambda\nu} f_{\varsigma\mathrm{A}} k^\lambda u^\nu - \tfrac{1}{4} \epsilon_{\mu\lambda\nu} f_{\mathfrak{S}} \tilde{k}^\lambda u^\nu \\
&\quad + \tfrac{1}{4} \epsilon_{\mu\lambda\nu} f_{\varsigma\mathrm{A}} \tilde{k}^\lambda u^\nu + \tfrac{1}{8} f_{\Sigma\mu} k - \tfrac{1}{4} \epsilon_{\mu\lambda\nu} f_s{}^\lambda u^\nu k + \tfrac{1}{8} f_{\Sigma\mu} \tilde{k} + \tfrac{1}{4} f_{\varsigma\mathrm{T}} m_\mu \\
&\quad - \tfrac{1}{4} \epsilon_{\mu\lambda\nu} f_{\mathfrak{S}} u^\lambda m^\nu + \tfrac{1}{4} \epsilon_{\mu\lambda\nu} f_{\varsigma\mathrm{A}} u^\lambda m^\nu + \tfrac{1}{4} f_{\Sigma\mu} M + \tfrac{1}{8} f_{\varsigma\mathrm{A}} \mathcal{K}_\mu - \tfrac{1}{4} \epsilon_{\mu\lambda\rho} f_{\varsigma\mathrm{S}\nu}{}^\rho u^\lambda \mathcal{K}^\nu \\
&\quad - \tfrac{1}{8} \epsilon_{\mu\lambda\nu} f_{\varsigma\mathrm{T}} u^\lambda \mathcal{K}^\nu - \tfrac{1}{2} f_{s\mu} \Theta - \tfrac{1}{8} \epsilon_{\mu\lambda\nu} f_\Sigma{}^\lambda u^\nu \Theta - \tfrac{1}{8} f_{\Sigma\mu} \kappa_\mathrm{A} + \tfrac{1}{4} \epsilon_{\mu\nu\rho} f_\Sigma{}^\lambda u^\nu \kappa_{\mathrm{S}\lambda}{}^\rho \\
&\quad - \tfrac{1}{8} \epsilon_{\mu\lambda\nu} f_\Sigma{}^\lambda u^\nu \kappa_\mathrm{T} + \tfrac{1}{4} \epsilon_{\mu\nu\rho} f_\Sigma{}^\lambda u^\nu \sigma_\lambda{}^\rho - \tfrac{1}{4} \epsilon_{\mu\lambda\nu} k^\lambda u^\nu k \chi_{11}^{(2)} \\
&\quad + \tfrac{1}{2} \epsilon_{\mu\lambda\nu} k^\lambda u^\nu M \chi_{11}^{(2)} - \tfrac{1}{2} k_\mu \Theta \chi_{11}^{(2)} + \tfrac{1}{2} k^\lambda \kappa_{\mathrm{S}\mu\lambda} \chi_{11}^{(2)} \\
&\quad + \tfrac{1}{2} \tilde{k}^\lambda \kappa_{\mathrm{S}\mu\lambda} \chi_{11}^{(2)} - \tfrac{1}{2} m^\lambda \kappa_{\mathrm{S}\mu\lambda} \chi_{11}^{(2)} + \tfrac{1}{4} \epsilon_{\mu\lambda\nu} u^\lambda k \mathcal{K}^\nu \chi_{13}^{(2)} \\
&\quad - \tfrac{1}{2} \epsilon_{\mu\lambda\nu} u^\lambda M \mathcal{K}^\nu \chi_{13}^{(2)} - \tfrac{1}{2} \mathcal{K}_\mu \Theta \chi_{13}^{(2)} + \tfrac{1}{2} \epsilon_{\mu\lambda\nu} k^\lambda \tilde{k}^\nu \chi_{15}^{(2)} \\
&\quad + \tfrac{1}{2} \epsilon_{\lambda\nu\rho} k^\lambda \tilde{k}^\nu u_\mu u^\rho \chi_{15}^{(2)} + \tfrac{1}{4} k_\mu k \chi_{15}^{(2)} - \tfrac{1}{2} \epsilon_{\mu\lambda\nu} k^\lambda m^\nu \chi_{15}^{(2)} \\
&\quad + \tfrac{1}{2} \epsilon_{\lambda\nu\rho} k^\lambda u_\mu u^\nu m^\rho \chi_{15}^{(2)} - \tfrac{1}{2} k_\mu M \chi_{15}^{(2)} - \tfrac{1}{2} \epsilon_{\mu\lambda\nu} k^\lambda u^\nu \Theta \chi_{15}^{(2)} \\
&\quad - \tfrac{1}{2} \epsilon_{\mu\nu\rho} k^\lambda u^\nu \kappa_{\mathrm{S}\lambda}{}^\rho \chi_{15}^{(2)} - \tfrac{1}{2} \epsilon_{\mu\nu\rho} \tilde{k}^\lambda u^\nu \kappa_{\mathrm{S}\lambda}{}^\rho \chi_{15}^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{1}{2}\epsilon_{\mu\lambda\rho}u^{\lambda}m^{\nu}\kappa_{\mathrm{S}\nu}{}^{\rho}\chi_{15}^{(2)} - \tfrac{1}{4}k\mathcal{K}_{\mu}\chi_{17}^{(2)} + \tfrac{1}{2}M\mathcal{K}_{\mu}\chi_{17}^{(2)}\\
&+ \tfrac{1}{2}\epsilon_{\mu\lambda\nu}k^{\lambda}\mathcal{K}^{\nu}\chi_{17}^{(2)} + \tfrac{1}{2}\epsilon_{\mu\lambda\nu}\tilde{k}^{\lambda}\mathcal{K}^{\nu}\chi_{17}^{(2)} - \tfrac{1}{2}\epsilon_{\mu\lambda\nu}m^{\lambda}\mathcal{K}^{\nu}\chi_{17}^{(2)}\\
&- \tfrac{1}{2}\epsilon_{\lambda\nu\rho}k^{\lambda}u_{\mu}u^{\nu}\mathcal{K}^{\rho}\chi_{17}^{(2)} - \tfrac{1}{2}\epsilon_{\lambda\nu\rho}\tilde{k}^{\lambda}u_{\mu}u^{\nu}\mathcal{K}^{\rho}\chi_{17}^{(2)}\\
&- \tfrac{1}{2}\epsilon_{\lambda\nu\rho}u^{\lambda}u_{\mu}m^{\nu}\mathcal{K}^{\rho}\chi_{17}^{(2)} - \tfrac{1}{2}\epsilon_{\mu\lambda\nu}u^{\lambda}\mathcal{K}^{\nu}\Theta\chi_{17}^{(2)} + \tfrac{1}{2}\epsilon_{\mu\lambda\nu}\tilde{k}^{\lambda}u^{\nu}\kappa_{\mathrm{T}}\chi_{6}^{(2)}\\
&+ \tfrac{1}{2}\epsilon_{\mu\lambda\nu}\tilde{k}^{\lambda}u^{\nu}\kappa_{\mathrm{A}}\chi_{8}^{(2)} + \tfrac{1}{2}\epsilon_{\mu\lambda\nu}\tilde{k}^{\lambda}u^{\nu}k\chi_{9}^{(2)} - \tfrac{1}{2}u^{\lambda}\mathring{\nabla}_{\lambda}f_{s\mu} + \tfrac{1}{2}\mathring{\nabla}_{\lambda}f_{\varsigma\mathrm{S}\mu}{}^{\lambda}\\
&- \tfrac{1}{2}u^{\lambda}\chi_{11}^{(2)}\mathring{\nabla}_{\lambda}k_{\mu} + \tfrac{1}{2}\epsilon_{\mu\nu\rho}u^{\lambda}u^{\nu}\chi_{15}^{(2)}\mathring{\nabla}_{\lambda}k^{\rho} - \tfrac{1}{2}u^{\lambda}\chi_{13}^{(2)}\mathring{\nabla}_{\lambda}\mathcal{K}_{\mu}\\
&- \tfrac{1}{2}\epsilon_{\mu\nu\rho}u^{\lambda}u^{\nu}\chi_{17}^{(2)}\mathring{\nabla}_{\lambda}\mathcal{K}^{\rho} + \tfrac{1}{4}\Delta_{\mu\lambda}\mathring{\nabla}^{\lambda}f_{\varsigma\mathrm{T}} - \tfrac{1}{2}u^{\lambda}u_{\mu}u^{\nu}\mathring{\nabla}_{\nu}f_{s\lambda}\\
&+ \tfrac{1}{2}u^{\lambda}u_{\mu}\mathring{\nabla}_{\nu}f_{\varsigma\mathrm{S}\lambda}{}^{\nu} - \tfrac{1}{2}u^{\lambda}u_{\mu}u^{\nu}\chi_{11}^{(2)}\mathring{\nabla}_{\nu}k_{\lambda} - \tfrac{1}{2}u^{\lambda}u_{\mu}u^{\nu}\chi_{13}^{(2)}\mathring{\nabla}_{\nu}\mathcal{K}_{\lambda}\\
&+ \tfrac{1}{2}\epsilon_{\mu\rho\alpha}u^{\lambda}u^{\nu}u^{\rho}\chi_{15}^{(2)}\mathring{\nabla}_{\nu}\kappa_{\mathrm{S}\lambda}{}^{\alpha} - \tfrac{1}{2}u^{\lambda}u^{\nu}\chi_{11}^{(2)}\mathring{\nabla}_{\nu}\kappa_{\mathrm{S}\mu\lambda} + \tfrac{1}{4}\epsilon_{\mu\lambda\nu}u^{\lambda}\mathring{\nabla}^{\nu}f_{\varsigma\mathrm{A}}\\
&- \tfrac{1}{2}u^{\lambda}u_{\mu}u^{\nu}u^{\rho}\chi_{11}^{(2)}\mathring{\nabla}_{\rho}\kappa_{\mathrm{S}\lambda\nu} + \tfrac{1}{2}\epsilon_{\mu\lambda\nu}\tilde{k}^{\lambda}u^{\nu}M\frac{\partial^2 P}{\partial M^2}\\
&+ \tfrac{1}{2}\epsilon_{\mu\lambda\nu}u^{\lambda}km^{\nu}\frac{\partial P}{\partial m^2} - \epsilon_{\mu\lambda\nu}u^{\lambda}m^{\nu}M\frac{\partial P}{\partial m^2} - m_{\mu}\Theta\frac{\partial P}{\partial m^2}\\
&+ \frac{k_{\mu}\Theta\frac{\partial\chi_{11}^{(2)}}{\partial T}\frac{\partial P}{\partial T}}{2\frac{\partial^2 P}{\partial T^2}} + \frac{\mathcal{K}_{\mu}\Theta\frac{\partial\chi_{13}^{(2)}}{\partial T}\frac{\partial P}{\partial T}}{2\frac{\partial^2 P}{\partial T^2}}\\
&+ \frac{\epsilon_{\mu\lambda\nu}k^{\lambda}u^{\nu}\Theta\frac{\partial\chi_{15}^{(2)}}{\partial T}\frac{\partial P}{\partial T}}{2\frac{\partial^2 P}{\partial T^2}} + \frac{\epsilon_{\mu\lambda\nu}u^{\lambda}\mathcal{K}^{\nu}\Theta\frac{\partial\chi_{17}^{(2)}}{\partial T}\frac{\partial P}{\partial T}}{2\frac{\partial^2 P}{\partial T^2}}\\
&+ \frac{m_{\mu}\Theta\frac{\partial P}{\partial T}\frac{\partial^2 P}{\partial T\partial m^2}}{\frac{\partial^2 P}{\partial T^2}}.
\end{aligned}
\tag{A.4}
$$

Appendix B

Tables of all tensors

TABLE B.1: Full list of first order scalars

i	$\mathcal{S}_i^{(1)}$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
1	M	✓	1	-	Fluid
2	Θ	✗	1	+	Fluid
3	k	✓	1	-	Torsion
4	$\tilde{k}$	✗	1	-	Fluid/Torsion
5	κ_T	✓	1	+	Torsion
6	κ_A	✓	1	-	Torsion

TABLE B.2: Full list of second order scalars

i	$\mathcal{S}_i^{(2)}$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
1	MM	✓	2	+	Fluid
2	$M\Theta$	✗	2	-	Fluid
3	Mk	✓	2	+	Fluid/Torsion
4	$M\tilde{k}$	✗	2	+	Fluid/Torsion
5	$M\kappa_T$	✓	2	-	Fluid/Torsion
6	$M\kappa_A$	✓	2	+	Fluid/Torsion
7	$m^\mu m_\mu$	✓	2	+	Fluid
8	$m^\mu k_\mu$	✓	2	+	Fluid/Torsion
9	$m^\mu \tilde{k}_\mu$	✗	2	+	Fluid/Torsion
10	$m^\mu \mathcal{K}_\mu$	✓	2	-	Fluid/Torsion
11	$\Theta\Theta$	✗	2	+	Fluid
12	Θk	✗	2	-	Fluid/Torsion
13	$\Theta\tilde{k}$	✗	2	-	Fluid/Torsion
14	$\Theta\kappa_T$	✗	2	+	Fluid/Torsion
15	$\Theta\kappa_A$	✗	2	-	Fluid/Torsion
16	$\sigma^{\mu\nu}\sigma_{\mu\nu}$	✗	2	+	Fluid
17	$\sigma^{\mu\nu}\kappa_{S\mu\nu}$	✗	2	+	Fluid/Torsion

Continued on next page

Table B.2 – continued from previous page

i	$\mathcal{S}_{\mathrm{i}}^{(2)}$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
18	kk	✓	2	+	Torsion
19	$k\tilde{k}$	✗	2	+	Fluid/Torsion
20	$k\kappa_{\mathrm{T}}$	✓	2	-	Torsion
21	$k\kappa_{\mathrm{A}}$	✓	2	+	Torsion
22	$k^\mu k_\mu$	✓	2	+	Torsion
23	$k^\mu \tilde{k}_\mu$	✗	2	+	Fluid/Torsion
24	$k^\mu \mathcal{K}_\mu$	✓	2	-	Torsion
25	$\tilde{k}^\mu \tilde{k}_\mu$	✗	2	+	Fluid/Torsion
26	$\tilde{k}^\mu \mathcal{K}_\mu$	✗	2	-	Fluid/Torsion
27	$\tilde{k}\tilde{k}$	✗	2	+	Fluid/Torsion
28	$\tilde{k}\kappa_{\mathrm{T}}$	✗	2	-	Fluid/Torsion
29	$\tilde{k}\kappa_{\mathrm{A}}$	✗	2	+	Fluid/Torsion
30	$\kappa_{\mathrm{T}}\kappa_{\mathrm{T}}$	✓	2	+	Torsion
31	$\kappa_{\mathrm{T}}\kappa_{\mathrm{A}}$	✓	2	-	Torsion
32	$\kappa_{\mathrm{A}}\kappa_{\mathrm{A}}$	✓	2	+	Torsion
33	$\kappa_{\mathrm{S}}{}^{\mu\nu}\kappa_{\mathrm{S}\mu\nu}$	✓	2	+	Torsion
34	$\mathcal{K}^\mu \mathcal{K}_\mu$	✓	2	+	Torsion
35	$\mathring{R}$	✓	2	+	Curvature
36	$\mathring{R}_u$	✓	2	+	Curvature
37	$\tilde{\mathfrak{f}}$	✗	2	-	Fluid
38	$\tilde{\mathfrak{f}}_{\mathrm{T}}$	✗	2	+	Curv./Fluid/Torsion
39	$\tilde{\mathfrak{f}}_{\mathrm{A}}$	✗	2	-	Fluid/Torsion
40	$\mathring{\nabla}^\mu m_\mu$	✓	2	+	Fluid
41	$\mathring{\nabla}^\mu k_\mu$	✓	2	+	Torsion
42	$\mathring{\nabla}^\mu \tilde{k}_\mu$	✗	2	+	Fluid/Torsion
43	$\mathring{\nabla}^\mu \mathcal{K}_\mu$	✓	2	-	Torsion

TABLE B.3: Full list of first order vectors

i	$\left(\mathcal{V}_{\mathrm{i}}^{(1)}\right)$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
1	m^μ	✓	1	+	Fluid
2	k^μ	✓	1	+	Torsion
3	$\tilde{k}^\mu$	✗	1	+	Fluid/Torsion
4	$\mathcal{K}^\mu$	✓	1	-	Torsion

TABLE B.4: Full list of second order vectors

i	$\left(\mathcal{V}_{\rm i}^{(2)}\right)$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
1	Mm^μ	✓	2	-	Fluid
2	Mk^μ	✓	2	-	Fluid/Torsion
3	$M\tilde{k}^\mu$	✗	2	-	Fluid/Torsion
4	$M\mathcal{K}^\mu$	✓	2	+	Fluid/Torsion
5	$m^\mu\Theta$	✗	2	+	Fluid
6	$m^\mu\sigma_\mu{}^\rho$	✗	2	+	Fluid
7	$m^\mu k$	✓	2	-	Fluid/Torsion
8	$m^\mu\tilde{k}$	✗	2	-	Fluid/Torsion
9	$m^\mu\kappa_{\rm T}$	✓	2	+	Fluid/Torsion
10	$m^\mu\kappa_{\rm A}$	✓	2	-	Fluid/Torsion
11	$m^\mu\kappa_{{\rm S}\mu}{}^\rho$	✓	2	+	Fluid/Torsion
12	Θk^μ	✗	2	+	Fluid/Torsion
13	$\Theta\tilde{k}^\mu$	✗	2	+	Fluid/Torsion
14	$\Theta\mathcal{K}^\mu$	✗	2	-	Fluid/Torsion
15	$\sigma^{\mu\nu}k_\nu$	✗	2	+	Fluid/Torsion
16	$\sigma^{\mu\nu}\tilde{k}_\nu$	✗	2	+	Fluid/Torsion
17	$\sigma^{\mu\nu}\mathcal{K}_\nu$	✗	2	-	Fluid/Torsion
18	kk^μ	✓	2	-	Torsion
19	$k\tilde{k}^\mu$	✗	2	-	Fluid/Torsion
20	$k\mathcal{K}^\mu$	✓	2	+	Torsion
21	$k^\mu\tilde{k}$	✗	2	-	Fluid/Torsion
22	$k^\mu\kappa_{\rm T}$	✓	2	+	Torsion
23	$k^\mu\kappa_{\rm A}$	✓	2	-	Torsion
24	$k^\mu\kappa_{{\rm S}\mu}{}^\rho$	✓	2	+	Torsion
25	$\tilde{k}^\mu\tilde{k}$	✗	2	-	Fluid/Torsion
26	$\tilde{k}^\mu\kappa_{\rm T}$	✗	2	+	Fluid/Torsion
27	$\tilde{k}^\mu\kappa_{\rm A}$	✗	2	-	Fluid/Torsion
28	$\tilde{k}^\mu\kappa_{{\rm S}\mu}{}^\rho$	✗	2	+	Fluid/Torsion
29	$\tilde{k}\mathcal{K}^\mu$	✗	2	+	Fluid/Torsion
30	$\kappa_{\rm T}\mathcal{K}^\mu$	✓	2	-	Torsion
31	$\kappa_{\rm A}\mathcal{K}^\mu$	✓	2	+	Torsion
32	$\kappa_{\rm S}{}^{\mu\nu}\mathcal{K}_\nu$	✓	2	-	Torsion
33	Υ^μ	✓	2	+	Curvature
34	$\tilde{\mathfrak{f}}^\mu$	✗	2	+	Fluid
35	$\tilde{\mathfrak{F}}^\mu$	✗	2	+	Curv./Fluid/Torsion
36	$\mathring{\nabla}^\mu M$	✓	2	-	Fluid
37	$\mathring{\nabla}^\mu\Theta$	✗	2	+	Fluid
38	$\mathring{\nabla}^\mu\sigma_\mu{}^\rho$	✗	2	+	Fluid
39	$\mathring{\nabla}^\mu k$	✓	2	-	Torsion

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Table B.4 – continued from previous page

i	$\left(\mathcal{V}_{\rm i}^{(2)}\right)$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
40	$\mathring{\nabla}^{\mu}\tilde{k}$	✗	2	-	Fluid/Torsion
41	$\mathring{\nabla}^{\mu}\kappa_{\rm T}$	✓	2	+	Torsion
42	$\mathring{\nabla}^{\mu}\kappa_{\rm A}$	✓	2	-	Torsion
43	$\mathring{\nabla}^{\mu}{\kappa_{\rm S}}_{\mu}{}^{\rho}$	✓	2	+	Torsion

TABLE B.5: Full list of first order rank 2 tensors

i	$\left(\mathcal{T}_{\rm i}^{(1)}\right)$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
1	$\sigma^{\mu\nu}$	✗	1	+	Fluid
2	${\kappa_{\rm S}}^{\mu\nu}$	✓	1	+	Torsion

TABLE B.6: Full list of second order rank 2 tensors

i	$\left(\mathcal{T}_{\rm i}^{(2)}\right)$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
1	$M\sigma^{\mu\nu}$	✗	2	-	Fluid
2	$M{\kappa_{\rm S}}^{\mu\nu}$	✓	2	-	Fluid/Torsion
3	$m^{\mu}m^{\nu}$	✓	2	+	Fluid
4	$m^{\mu}k^{\nu}$	✓	2	+	Fluid/Torsion
5	$m^{\mu}\tilde{k}^{\nu}$	✗	2	+	Fluid/Torsion
6	$m^{\mu}\mathcal{K}^{\nu}$	✓	2	-	Fluid/Torsion
7	$\Theta\sigma^{\mu\nu}$	✗	2	+	Fluid
8	$\Theta{\kappa_{\rm S}}^{\mu\nu}$	✗	2	+	Fluid/Torsion
9	$\sigma^{\mu\nu}\sigma_{\nu}{}^{\lambda}$	✗	2	+	Fluid
10	$\sigma^{\mu\nu}k$	✗	2	-	Fluid/Torsion
11	$\sigma^{\mu\nu}\tilde{k}$	✗	2	-	Fluid/Torsion
12	$\sigma^{\mu\nu}\kappa_{\rm T}$	✗	2	+	Fluid/Torsion
13	$\sigma^{\mu\nu}\kappa_{\rm A}$	✗	2	-	Fluid/Torsion
14	$\sigma^{\mu\nu}{\kappa_{\rm S}}_{\nu}{}^{\lambda}$	✗	2	+	Fluid/Torsion
15	$k{\kappa_{\rm S}}^{\mu\nu}$	✓	2	-	Torsion
16	$k^{\mu}k^{\nu}$	✓	2	+	Torsion
17	$k^{\mu}\tilde{k}^{\nu}$	✗	2	+	Fluid/Torsion
18	$k^{\mu}\mathcal{K}^{\nu}$	✓	2	-	Torsion
19	$\tilde{k}^{\mu}\tilde{k}^{\nu}$	✗	2	+	Fluid/Torsion
20	$\tilde{k}^{\mu}\mathcal{K}^{\nu}$	✗	2	-	Fluid/Torsion
21	$\tilde{k}{\kappa_{\rm S}}^{\mu\nu}$	✗	2	-	Fluid/Torsion
22	$\kappa_{\rm T}{\kappa_{\rm S}}^{\mu\nu}$	✓	2	+	Torsion
23	$\kappa_{\rm A}{\kappa_{\rm S}}^{\mu\nu}$	✓	2	-	Torsion

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Table B.6 – continued from previous page

i	$\left(\mathcal{T}_{\rm i}^{(2)}\right)$	Hydrostatic	$\mathcal{O}(\partial^n)$	Parity	Type
24	$\kappa_{\rm S}{}^{\mu\nu}\kappa_{\rm S\nu}{}^{\lambda}$	✓	2	+	Torsion
25	$\mathcal{K}^{\mu}\mathcal{K}^{\nu}$	✓	2	+	Torsion
26	$\zeta^{\mu\nu}$	✓	2	+	Curvature
27	$\tilde{\mathfrak{f}}_{\rm S}{}^{\mu\nu}$	✗	2	+	Curv./Fluid/Torsion
28	$\mathring{\nabla}^{\mu}m^{\nu}$	✓	2	+	Fluid
29	$\mathring{\nabla}^{\mu}k^{\nu}$	✓	2	+	Torsion
30	$\mathring{\nabla}^{\mu}\tilde{k}^{\nu}$	✗	2	+	Fluid/Torsion
31	$\mathring{\nabla}^{\mu}\mathcal{K}^{\nu}$	✓	2	-	Torsion

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