Dynamics of spontaneous symmetry breaking in a space-time crystal

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Abstract

We present the theory of spontaneous symmetry breaking (SSB) of discrete time translations as recently realized in the space-time crystals of an atomic Bose-Einstein condensate. The non-equilibrium physics related to such a driven-dissipative system is discussed in both the Langevin as well as the Fokker-Planck formulation. We consider a semi-classical and a fully quantum approach, depending on the frequency dependence of the dissipation. For both cases, the Langevin equations and Fokker-Planck equation are derived self-consistently, and the equilibrium distribution is studied. We also study the dynamics and show numerically the arising of the symmetry breaking. Finally, we compare our results with experiments and conclude that our model forms a solid foundation to describe them.

CONTENTS

I. INTRODUCTION

At present there is much interest in *non-equilibrium physics*: the subfield of physics that studies systems which are constantly out of thermodynamical equilibrium. Many examples are found in nature, for example in fluids [1], cells [2] and colloids [3]. This forms a motivation for physicists to build and study non-equilibrium systems, see e.g. Refs. [4], [5]. In that case, a drive can be added such that the system doesn't behave as it would do in the 'standard' equilibrium situation. But in order to have a stable non-equilibrium state, the energy that comes from the drive must precisely dissipate from the system as well [6].

An interesting example of such driven-dissipative systems are *time crystals*: systems that have spontaneously broken the discrete time-translational symmetry [7, 8]. Time crystals are observed experimentally by multiple groups (see Refs. [9–14]), but our focus lies on the experimental realization of a space-time crystal in an atomic Bose-Einstein condensate (BEC) as described in Refs. [15, 16]. Notice that this is a *space*-time crystal, which means that on top of the periodic structure in time, it also possesses a crystalline structure in space.

The referred experiment can be described as follows. Starting with an interacting Bose-gas, exactly one crystalline mode is excited by adding a drive. This mode has a frequency of precisely half the drive frequency, but there is a Z_2 symmetry related to the amplitude being either (roughly speaking) in phase or out of phase with the drive. In principle, the phase difference of a single realization can be ϕ or $\phi + \pi$, but when the choice is made, it will stay there. This is the spontaneous symmetry breaking that will be investigated in great detail.

After this introduction, we will continue in Sec. II to explain the theoretical methods that are needed to describe our setup, in particular the Keldysh formalism. This will at the end lead to the exact Langevin equations. Then, in Sec. III, we split our research up into two parts, depending on the assumptions we make about the frequency dependence of the dissipation, in order to further investigate these Langevin equations and also the Fokker-Planck equations in each of the two approximations. Moreover, we will look at the equilibrium states and show what happens when we add a drive to the system. After that, we are in the position to compare our results with experiments, which is the content of Sec. IV.

II. GENERAL FRAMEWORK

The main motivation for this paper comes from the results of Ref. [16]. In that paper, the dynamics of the space-time crystals were described well using a Hamiltonian of the form

$$\hat{H} = -\hbar\delta\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_D A_D}{8}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}) + \frac{\hbar g}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}, \tag{1}$$

where δ is the detuning from resonance, ω_D is the driving frequency, A_D is the relative driving amplitude and $\hat{a}^{(\dagger)}$ is the annihilation (creation) operator of a quantum in the dominant axial mode. Furthermore, there is the parameter $g = g' + ig''$, which was introduced to account for any fourth-order contributions. Phenomenologically, this fourth-order term worked out very well, but the theoretical background of it was missing. Since the Hamiltonian has an imaginary part, we expect that also noise will arise, because of the fluctuation-dissipation theorem. Therefore, we will expand and support the model of Ref. [16] by microscopically derive this imaginary piece. We will do this via non-linear coupling with a heat bath, resulting in both the imaginary part of the Hamiltonian, as well as the noise, and satisfying the fluctuation-dissipation theorem.

An elegant way to develop our model is by using the techniques of the Keldysh formalism. With that, we will derive an action that corresponds to the Hamiltonian of Eq. (1).

Let us first derive the action corresponding to non-linear coupling with a heat bath. As was shown in Ref. [17], the probability distribution $P[a^*, a, t]$ can be expressed in terms of the matrix elements $\langle a; t|a_0, t_0\rangle$ and its complex conjugate, $\langle a; t|a_0, t_0\rangle^* = \langle a_0; t_0|a, t\rangle$, where $|a, t\rangle$ and $|a_0, t_0\rangle$ are the coherent states of the annihilation operator $\hat{a}$ at time t and t_0 respectively. For these matrix elements, the 'path' integral expressions going through all possible field configurations are calculated, e.g. in Ref. [19]. Because of the fact that the complex conjugate has an opposite time-ordering, the actions corresponding to these elements will have different evolutions in time. This leads to the introduction of the *Schwinger-Keldysh contour* $\mathcal{C}^t$, which (in our case) obeys $\int_{\mathcal{C}^t} dt' = \int_t^{t_0} dt_- + \int_{t_0}^t dt_+$. For our purposes, we are allowed to use the contour in the limit $t \to \infty$. For more details, we again refer to Ref. [17].

Our model uses quadratic interaction with the heat bath of the form $\psi\psi$, where ψ refers to the field in the time crystal [21]. This means that two quanta from the axial mode can scatter into the heat bath and back. Since the

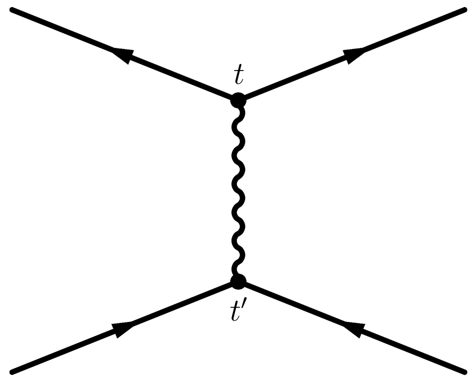


FIG. 1: Feynman diagram of the effective interaction in the space-time crystal.

quadratic piece of the Hamiltonian stays the same, we can immediately write the action (analogous to [17])

$$S[\psi^*,\psi,\psi_R^*,\psi_R] = -\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\int_{C^\infty} dt(t(\mathbf{k})\psi(t)\psi(t)\psi_{\mathbf{k}}^*(t) + t^*(\mathbf{k})\psi_{\mathbf{k}}(t)\psi^*(t)\psi^*(t)) + \int_{C^\infty} dt\psi^*(t)(i\hbar\frac{\partial}{\partial t} + \hbar\delta)\psi(t) + \sum_{\mathbf{k}}\int_{C^\infty} dt\psi_{\mathbf{k}}^*(t)(i\hbar\frac{\partial}{\partial t} - \epsilon(\mathbf{k}))\psi_{\mathbf{k}}(t), \tag{2}$$

where $\psi_R(\mathbf{x},t) = \sum_{\mathbf{k}}\psi_{\mathbf{k}}e^{i\mathbf{k}\mathbf{x}}/\sqrt{V}$ is the reservoir field describing the heat bath and $\mathcal{C}^\infty$ is the Schwinger-Keldysh contour that was introduced above. Furthermore, V is the volume of the reservoir, $\epsilon(\mathbf{k})$ is the energy of a state in the reservoir and $t(\mathbf{k})$ describes the coupling between the space-time crystal and the heat bath.
The next step is to integrate out the heat bath since the action is quadratic in $\psi_{\mathbf{k}}$. This gives the *effective action*,

$$S^{eff}[\psi^*,\psi] = \int_{C^\infty} dt\psi^*(t)(i\hbar\frac{\partial}{\partial t} + \hbar\delta)\psi(t) - \int_{C^\infty} dt\int_{C^\infty} dt'\psi^*(t)\psi^*(t)\frac{\hbar g(t,t')}{2}\psi(t')\psi(t'), \tag{3}$$

where $\hbar g(t,t') = \frac{2}{\hbar V}\sum_{\mathbf{k}} t^*(\mathbf{k})G(\mathbf{k};t,t')t(\mathbf{k})$ is the coupling strength of the interaction. Furthermore, the Green's function $G(\mathbf{k},t,t')$ must satisfy $(i\hbar\frac{\partial}{\partial t} - \epsilon(\mathbf{k}))G(\mathbf{k},t,t') = \hbar\delta(t,t')$. A Feynman diagram corresponding to this Green's function is shown in Fig. 1. Note in particular the appearance of a fourth-order term in ψ in our action, which is absent in Ref. [17].

We are almost done with the derivation of our action. The last step is projecting the two branches of the Keldysh contour (backwards and forwards) onto the real axis. This is done by making the transformation $\psi(t_\pm) = a(t) \pm \xi(t)/2$, where $a(t)$ and $\xi(t)$ represent the classical and the quantum contributions, respectively. Since we are dealing with a single-mode Hamiltonian for our time crystal, we don't need to look at the position dependence.

Because of our $\psi\psi$-interaction, we will in principle get fluctuations up to fourth order in ξ, instead of order ξ^2 in the Caldeira-Leggett like toy model of Ref. [17]. Nonetheless, we can still work out the action. Up to second order in ξ, we find

$$\begin{aligned} S^{eff}[a^*,a;\xi^*,\xi] = &\int_{t_0} dt\Big\{a^*(t)\Big(i\hbar\frac{\partial}{\partial t} + \hbar\delta\Big)\xi(t) + \xi^*(t)\Big(i\hbar\frac{\partial}{\partial t} + \hbar\delta\Big)a(t)\Big\} \\ &- \int_{t_0} dt\int_{t_0} dt'\Big(a^*(t)a^*(t)\hbar g^{(-)}(t-t')a(t')\xi(t') + a^*(t)\xi^*(t)\hbar g^{(+)}(t-t')a(t')a(t')\Big) \\ &- \int_{t_0} dt\int_{t_0} dt'\ a^*(t)\xi^*(t)\hbar g^K(t-t')a(t')\xi(t'), \end{aligned} \tag{4}$$

where $g^{(\pm)}$ and g^K are related to the analytic pieces of g via

$$g^{(\pm)}(t-t') = \pm\Theta(\pm(t-t'))(g^>(t-t') - g^<(t-t')) \tag{5}$$

and

$$g^K(t-t') = (g^>(t-t') + g^<(t-t')), \tag{6}$$

where $\Theta(x)$ is the Heaviside step function. The action also has a third-order contribution, namely $-\frac{1}{8}\int_{t_0} dt \int_{t_0} dt' \Big(\xi^*(t)\xi^*(t)\hbar g^{(-)}(t-t')a(t')\xi(t') + a^*(t)\xi^*(t)\hbar g^{(+)}(t-t')\xi(t')\xi(t') \Big)$, whereas the fourth-order term turns out to be zero. In the remainder of this paper, we will ignore $\mathcal{O}(\xi^3)$, because the effect of the quantum fluctuations must be small compared to the classical contributions.

With that, our fluctuation-dissipation theorem looks like

$$g^K(\omega) = (g^{(+)}(\omega) - g^{(-)}(\omega))(1 + 2N(\omega)). \tag{7}$$

This will ensure that our probability distribution relaxes to the correct physical equilibrium.

We now use a well-known trick to simplify our action, which is called a Hubbard-Stratonovich transformation, see e.g. Ref. [18]. In this situation, it turns out to be convenient to multiply our action with

$$1 = \int d[\eta^*]d[\eta] \exp(\frac{iS^{eff}[\eta^*,\eta]}{\hbar}), \tag{8}$$

where, in this case,

$$S^{eff}[\eta^*,\eta] = \int dt \int dt' \Big(\eta^*(t) - \int dt'' \xi^*(t'')a^*(t'')\,\hbar\, g^K(t''-t) \Big) \frac{g^{K^{-1}}(t-t')}{\hbar} \Big(\eta(t') - \int dt'' \hbar g^K(t'-t'')\xi(t'')a(t'') \Big). \tag{9}$$

This will by construction precisely cancel out the quadratic piece in ξ of our action, and thus we end up with a linear action in ξ, which we can perform easily. Adding this $S^{eff}[\eta^*,\eta]$ to our action (Eq. (4)), we obtain our *total effective action*, given by

$$\begin{aligned} S^{eff}[a^*,a,\xi^*,\xi,\eta^*,\eta] = &\int dt \int dt' \Big\{ a^*(t) \Big[(i\hbar \frac{\partial}{\partial t} + \hbar\delta)\ \delta(t-t') - \hbar g^{(-)}(t-t')a^*(t')a(t) \Big] - a(t')\eta^*(t)\delta(t-t') \Big\}\ \xi(t') \\ &+ \int dt \int dt' \xi^*(t)\ \Big\{ \Big[(i\hbar \frac{\partial}{\partial t} + \hbar\delta)\ \delta(t-t') - \hbar g^{(+)}(t-t')a^*(t)a(t') \Big] a(t') - \eta(t')\delta(t-t')a^*(t) \Big\} \\ &+ \int dt \int dt' \eta^*(t) \frac{g^{K^{-1}}(t-t')}{\hbar} \eta(t'). \end{aligned} \tag{10}$$

Integrating out ξ now leads to a δ-functional and from that we conclude that our *Langevin equations* become

$$i\hbar \frac{\partial}{\partial t} a(t) = -\hbar\delta\, a(t) + \int dt' a^*(t)\, \hbar g^{(+)}(t-t')\ a(t')\, a(t') + \eta(t)\, a^*(t) \tag{11}$$

and its complex conjugate (remember that $g^{(+)}$ is complex). Note in particular the multiplicative noise-term, which is characteristic of our model.
Furthermore, we can conclude from our effective action (Eq. (10)) that

$$\langle \eta(t)\eta(t') \rangle = \langle \eta^*(t)\eta^*(t') \rangle = 0 \tag{12}$$

and

$$\langle \eta^*(t)\eta(t') \rangle = i\hbar^2 g^K(t-t'). \tag{13}$$

Finally, we make an approximation for the fluctuation-dissipation theorem, such that

$$g^K(\omega) \approx 2ig''(1 + 2N(\omega)), \tag{14}$$

where $N(\omega) = (e^{\beta\hbar\omega} - 1)^{-1}$ is the Bose distribution function and $g''(\omega) = Im(g^{(+)}(\omega)) \approx \frac{1}{2i}(g^{(+)}(\omega) - g^{(-)}(\omega))$. Lateron, we will also use $g'(\omega)$ for $Re(g^{(+)}(\omega)) \approx \frac{1}{2}(g^{(+)}(\omega) + g^{(-)}(\omega))$.

III. FREQUENCY DEPENDENCE OF DISSIPATION

At this point, we will split our research up into two parts in order to go beyond the general Langevin equations given by Eq. (11). We will do this by calculating the correlation functions related to the noise $\eta(t)$ and show the relation with the Fokker-Planck equation. We can, as a first approximation, assume that the dissipation doesn't have any frequency dependence, meaning that the interactions act as a delta-function in time. Our second and more correct method will be the situation where we allow a low frequency dependence. These two options will be our two different approaches.

The goal of our first approximation, a semiclassical approach, is to introduce the techniques needed for the more general case. This will turn out to comprise a lot of the important physics to study spontaneous symmetry breaking, as we will see in Sec. IV. The second approach builds upon first and uses its techniques to describe the more accurate theoretical picture, also taking into account the quantum effects in the low-frequency limit.

A. Semi-classical approach

So let us start with the semi-classical approach. We make the approximation that $\hbar\omega \approx -\hbar\delta$ and we simply use the correlation functions of Eq. (12) and (13). This means that our fluctuation-dissipation theorem takes the form

$$g^K = 2ig''(1 + 2N(-\hbar\delta)), \tag{15}$$

where $g^K \equiv g^K(-\hbar\delta)$ and $g'' \equiv g''(-\hbar\delta)$. Stated in the time domain, we have $g^{(\pm,K)}(t - t') = g^{(\pm,K)}\delta(t - t')$. Then, a general solution of the Langevin equations (Eq. (11)) reads

$$a(t) = e^{i\delta t}\left\{a(0) - \frac{i}{\hbar}\int_0^t dt'' \int dt' a^*(t'')\hbar g^{(+)}(t'' - t')a(t')a(t')e^{-i\delta t''} - \frac{i}{\hbar}\int_0^t dt'\eta(t')a^*(t')e^{-i\delta t'}\right\}, \tag{16}$$

and similarly for $a^*(t)$. With that, we can calculate the correlation functions that contain η, η^* and a,a^*. The result becomes (with $g^{K^*} = -g^K$):

- $\langle \eta^* a\rangle = \frac{\hbar g^K}{2}\langle a^*\rangle$
- $\langle \eta a^*\rangle = -\frac{\hbar g^K}{2}\langle a\rangle$
- $\langle \eta^* aa\rangle = \hbar g^K \langle a^* a\rangle$
- $\langle \eta a^* a^*\rangle = -\hbar g^K \langle a^* a\rangle$,

where we also used that $\langle\eta\rangle = \langle\eta^*\rangle = 0$. With that, we conclude that

$$i\hbar\frac{\partial}{\partial t}\langle a\rangle = -\hbar\delta\,\langle a\rangle + \hbar g^{(+)}\langle a^* aa\rangle - \frac{\hbar g^K}{2}\langle a\rangle, \tag{17}$$

$$-i\hbar\frac{\partial}{\partial t}\langle a^*\rangle = -\hbar\delta\,\langle a^*\rangle + \hbar g^{(-)}\langle a^* a^* a\rangle + \frac{\hbar g^K}{2}\langle a^*\rangle, \tag{18}$$

$$i\hbar\frac{\partial}{\partial t}\langle a^* a\rangle = 2i\hbar g''\langle a^* a^* aa\rangle - 2\hbar g^K\langle a^* a\rangle. \tag{19}$$

This is the full set of equations of motion for the semi-classical approach and this exactly reproduces the experimental model of Ref. [16]. At this point, it is also useful to state the relation between the expectation value of a^*a and the occupation numbers, which is given by (compare Ref. [17])

$$\langle a^* a\rangle(t) = N(t) + 1/2, \tag{20}$$

$N(t)$ being the occupation numbers our the space-time crystal.

Let us now look into the physical meaning of them more deeply. If we study Eq. (19) and apply Wick's theorem (see e.g. Ref. [18]) to the first term, we are able to write

$$\frac{\partial}{\partial t}N(t) = -\Gamma\,\langle a^* a\rangle(t)\left\{(N(t) + \frac{1}{2}) - 2(N + \frac{1}{2})\right\}, \tag{21}$$

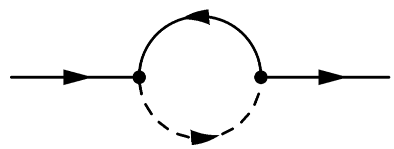


FIG. 2: Feynman diagram of the process. A straight line represents $\langle a^* a\rangle$ and a dashed line $\langle a_k^* a_k\rangle$

where $\Gamma = -4g''$ is a rate of decay and $N = (e^{-\hbar\delta\beta} - 1)^{-1}$, the equilibrium Bose distribution function at energy $-\hbar\delta$. It looks like an erroneous factor of 2 arises here. Without that, the situation would become physically more intuitive, and this is the situation that we explain below.

From this, we can learn two things. First, we note that in equilibrium, the LHS is zero, and we find that $N(t) = N$, so the Bose distribution function is (as it should be) the correct equilibrium. But more importantly, we can write this into the form of a rate equation, namely

$$\frac{\partial}{\partial t} N(t) = -\Gamma \,\langle a^* a\rangle(t) \Big\{ -N(t)(1 + N_k) + N_k(1 + N(t))\Big\}, \tag{22}$$

with N_k describing the equilibrium occupation numbers of the reservoir, the momentum k being fixed due to energy conservation. We can see that this is very similar to the Boltzmann equation (see Ref. [17]), but notice the factor of $\langle a^* a\rangle(t)$ difference, which is related to the multiplicative noise.

A Feynman diagram of this rate process is shown in Fig. 2. What we see here is the process of one quantum with momentum $\mathbf{k}$ scattering either into or outside the heat bath, with the appropriate amplitudes.

Let us now look at the *Fokker-Planck equation*. We can argue that the term $2i\hbar g''\langle a^* a^* a a\rangle$ on the RHS of Eq. (19) is taken into account by the streaming terms already and therefore does not lead to a contribution in the diffusive part of the FP-equation. So the diffusion term only has to correspond with the two *noise-induce driftterms*, meaning the last term in both Eq. (17) and (18), and also the last term in Eq. (19), coming from the multiplicative noise. This fixes the diffusion term of the Fokker-Planck equation to be

$$-\frac{\hbar g^K}{2}\left(\frac{\partial}{\partial a^*}\, a\, \frac{\partial}{\partial a}\, a^* + \frac{\partial}{\partial a}\, a^*\, \frac{\partial}{\partial a^*}\, a\right) P[a, a^*, t]. \tag{23}$$

Specifically for our case, we had to choose a symmetric operator. Together with the streaming terms, which follow directly from Eqs. (17) and (18), we find that the FP-equation becomes:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t} P[a^*, a, t] = &-\frac{\partial}{\partial a}\Big[(-\hbar\delta + \hbar g^{(+)}|a|^2)\; a\; P[a^*, a, t]\Big] \;+\frac{\partial}{\partial a^*}\Big[(-\hbar\delta + \hbar g^{(-)}|a|^2)\; a^*\; P[a^*, a, t]\Big] \\ &-\frac{\hbar g^K}{2}\left(\frac{\partial}{\partial a^*}\, a\, \frac{\partial}{\partial a}\, a^* + \frac{\partial}{\partial a}\, a^*\, \frac{\partial}{\partial a^*}\, a\right) P[a^*, a, t]. \end{aligned} \tag{24}$$

The above procedure started from the action, and via the Langevin equations led to the FP-equation. However, one can also derive the FP-equation directly from the action, which is a nice consistency-check for our theory. This is what we will do now.

Performing the integration over the fluctuation field $\xi(t)$, we could have immediately worked out the general action

(Eq. (4)). The result becomes

$$\begin{aligned} S^{eff}[a^*,a] &= \int dt' \frac{1}{\hbar g^K a^*(t')a(t')} \left| (i\hbar \frac{\partial}{\partial t'} + \hbar\delta - \hbar g^{(+)}\, a^*(t')a(t'))\; a(t') \right|^2 \\ &\equiv \int dt' L(t'). \end{aligned} \tag{25}$$

The theory can be quantized. In particular, the conjugate momentum $\pi(t)$ satisfies

$$\pi(t) = \frac{i\hbar}{\hbar g^K a^*(t)a(t)} (-i\hbar \frac{\partial}{\partial t} + \hbar\delta - \hbar g^{(-)}\, a^*(t)a(t))a^*(t), \tag{26}$$

and the complex conjugate for $\pi^*(t)$. The rest of this derivation goes the same way as the discussion of Ref. [17], i.e., one can now construct the Hamiltonian,

$$H = \frac{\pi(t)}{i\hbar}(-\hbar\delta + \hbar g^{(+)} a^*(t)a(t))a(t) - \frac{\pi^*(t)}{i\hbar}(-\hbar\delta + \hbar g^{(-)} a^*(t)a(t))a^*(t) + \frac{g^K a^*(t)a(t)}{\hbar}|\pi(t)|^2. \tag{27}$$

After requiring the usual commutation relations, the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} P[a^*,a,t] = H\; P[a^*,a,t], \tag{28}$$

exactly reproduces the FP-equation of Eq. (24) [22].

As a side remark, one can also derive the Langevin equations via the FP-equation. Compare the FP-equation with the set of equations that we derived in the Langevin method (Eqs. (17), (18) and (19)). It is easy to see that multiplying the FP-equation with a, a^* or a^*a and integrating (by parts) nicely reproduces the desired equations of motion. From this, we can see that the Langevin equations and the FP-equation are really two sides of the same coin.

We will end our discussion of the stationary physics involved in the semi-classical approach by looking at the equilibrium distribution of the FP-equation (Eq. (24)). If we neglect the drive terms (and therefore the phase dependence), it can be proven that the solution can only be the ideal gas solution (apart from the probably erroneous factor of 2), a fact that is shown in App. A in more detail.

All this was done without the drive terms (corresponding to the term $\frac{\hbar\omega_D A_D}{8}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a})$ in our Hamiltonian, Eq. (1). Adding the drive to the system is actually the key ingredient to get our symmetry breaking, since this is the only term that has explicit phase dependence. It is straightforward to add these to our model. Our Langevin equation (11) gets an extra term on the RHS, namely $\frac{\hbar\omega_D A_D}{4}a^*(t)$ (and $\frac{\hbar\omega_D A_D}{4}a(t)$ for the equation of $i\hbar\frac{\partial}{\partial t}a^*(t)$). These will also enter into the streaming terms of the Fokker-Planck equation, leading to

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}P[a^*,a,t] = &-\frac{\partial}{\partial a}\left[(-\hbar\delta + \hbar g^{(+)}|a|^2)\; a\; P[a^*,a,t] + \frac{\hbar\omega_D A_D}{4} a^* P[a^*,a,t]\right] \\ &+ \frac{\partial}{\partial a^*}\left[(-\hbar\delta + \hbar g^{(-)}|a|^2)\; a^*\; P[a^*,a,t]] + \frac{\hbar\omega_D A_D}{4} a P[a^*,a,t]\right] \\ &- \frac{\hbar g^K}{2}\left(\frac{\partial}{\partial a^*}\, a\, \frac{\partial}{\partial a}\, a^* + \frac{\partial}{\partial a}\, a^*\, \frac{\partial}{\partial a^*}\, a\right) P[a^*,a,t]. \end{aligned} \tag{29}$$

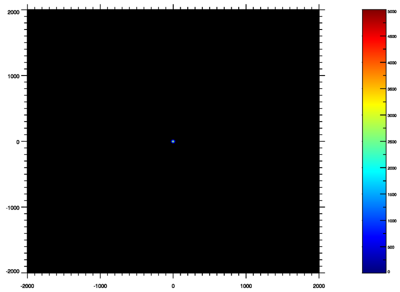
This is now the full equation, containing all the dynamics of our system. It is therefore possible to simulate what happens when we start with a Gaussian initial distribution and let the system evolve in time under influence of the drive, according to this expression. Our numerical findings of the spontaneous symmetry breaking in our time crystal is shown in Fig. 3. We clearly see the falling apart from a single maximum in the origin at the start towards a non-trivial 'broken' situation after some time.

At this point, we will start again at the general Langevin equations (11) and study the quantum effects in more detail.

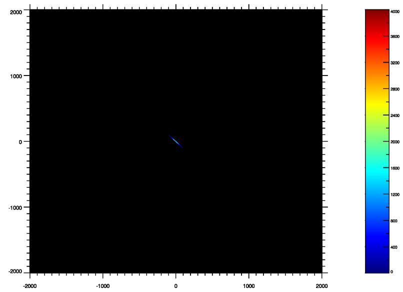
B. Quantum approach

In the previous discussion, we (implicitly) ignored any frequency dependence of the dissipation. But what would change if we also include that? The fluctuation-dissipation theorem is still given by Eq. (14), and we also use

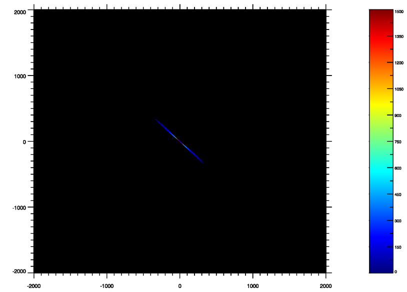
$$g^K(t-t') \simeq g^K(\omega=0)\,\delta(t-t'), \tag{30}$$



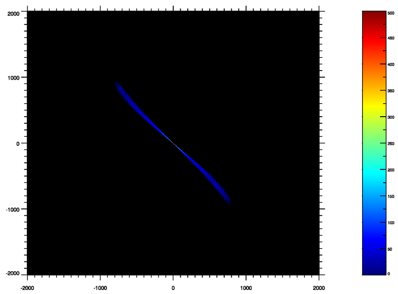
(a) After 0 oscillations.



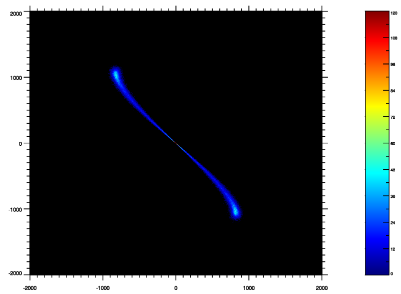
(b) After 10 oscillations.



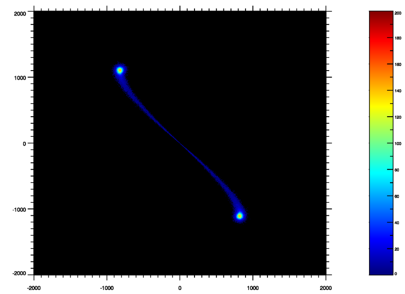
(c) After 20 oscillations.



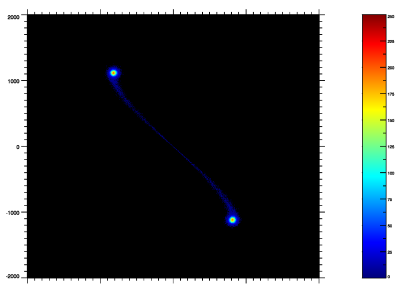
(d) After 30 oscillations.



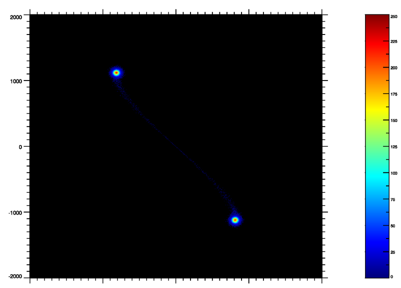
(e) After 40 oscillations.



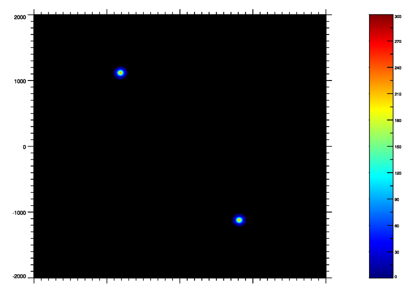
(f) After 50 oscillations.



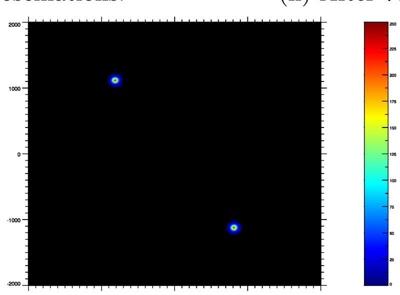
(g) After 60 oscillations.



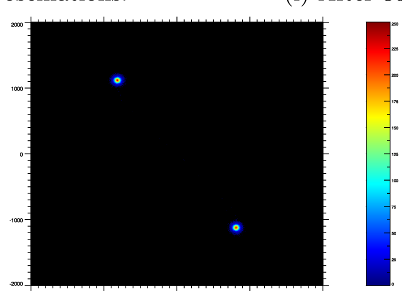
(h) After 70 oscillations.



(i) After 80 oscillations.



(j) After 90 oscillations.



(k) After 100 oscillations.

FIG. 3: Dynamical behavior of the probability distribution. Numerically, 100 oscillations are simulated, and a picture was taken every 10 oscillations. The number of oscillations past is indicated below each figure.

which corresponds to white noise in frequency space. But now, we make the observation that, in contrast to what was assumed before, g'' and N actually depend on the frequency ω! In particular, we have

$$(1 + 2N(\hbar\omega)) \simeq \frac{2k_B T}{\hbar\omega}, \tag{31}$$

which in fact diverges for $\omega \downarrow 0$. In order to compensate for this in the fluctuation-dissipation theorem, we therefore require that in frequency space: $g''(\omega) = \frac{\hbar\omega}{4ik_BT} g^K(\omega)$ or in the time domain:

$$g''(t-t') = \frac{g^K(\omega=0)}{4k_BT}\, \hbar\, \frac{\partial}{\partial t}\delta(t-t'). \tag{32}$$

With that, our Langevin equations (Eq. (11)) become

$$(i\hbar - \frac{i\hbar^2 g^K}{2k_BT} a^* a)\frac{\partial}{\partial t} a(t) = -\hbar\delta\, a(t) + \hbar g'\; a^*(t)\, a(t)\, a(t) + \eta(t)\, a^*(t), \tag{33}$$

and the complex conjugate for $a^*(t)$ [23]. Note that we now use $g^K \equiv g^K(\omega = 0)$. For the remainder of this paper, we write

$$I[a^*, a] = i\hbar - \frac{i\hbar^2 g^K}{2k_BT} a^* a \tag{34}$$

and thus

$$I^*[a^*, a] = -i\hbar - \frac{i\hbar^2 g^K}{2k_BT} a^* a. \tag{35}$$

These will play a key role in our quantum approach.
As we did before, we now want to work out the correlation functions for the noise, in order to give the equations of motion (compare Eqs. (17), (18) and (19). By dividing the Langevin equation by $I[a^*, a]$, we can see that our task now is to calculate quantities like $\langle \eta\ \frac{a^*}{I[a^*,a]} \rangle$ and $\langle \eta\ \frac{a^* a^*}{I[a^*,a]} \rangle$. Let us therefore also define

$$F[a^*, a] \equiv \frac{a^*}{I[a^*, a]}. \tag{36}$$

We can expand this function as

$$F[a^*, a] = F[a^*(t_0), a(t_0)] + \frac{\partial F[a^*(t_0), a(t_0)]}{\partial a^*}(a^*(t) - a^*(t_0)) + \frac{\partial F[a^*(t_0), a(t_0)]}{\partial a}(a(t) - a(t_0)), \tag{37}$$

neglecting all higher order contributions. Henceforth, we will (most of the time) omit the variable dependence of $F[a^*, a]$ and $I[a^*, a]$ and simply write F and I (or $F(t)$ and $I(t)$, when the time dependence is important) for clarity reasons. Nevertheless, it is good to remember that these are explicit functions of a^* and a.

Using again that $\langle \eta \rangle = \langle \eta^* \rangle = \langle \eta\eta \rangle = \langle \eta^*\eta^* \rangle = 0$, and the general solution for $a(t)$ in this case:

$$a(t) = e^{i\delta t}\Big\{a(0) - \frac{i}{\hbar}\int_0^t dt'' \int dt' a^*(t'')\hbar g^{(+)}(t''-t') a(t') a(t') e^{-i\delta t''} + \int_0^t dt' \eta(t') F(t') e^{-i\delta t'}\Big\}, \tag{38}$$

we are in the position to calculate the desired correlation functions. The result becomes:

- $\langle \eta F \rangle = \frac{i\hbar^2 g^K}{2} \langle \frac{\partial F}{\partial a^*} F^* \rangle$
- $\langle \eta^* F^* \rangle = \frac{i\hbar^2 g^K}{2} \langle \frac{\partial F^*}{\partial a} F \rangle$
- $\langle \eta a^* F \rangle = \frac{i\hbar^2 g^K}{2} \Big(\langle F^* F \rangle + \langle a^* \frac{\partial F}{\partial a^*} F^* \rangle \Big)$
- $\langle \eta^* a F^* \rangle = \frac{i\hbar^2 g^K}{2} \Big(\langle F^* F \rangle + \langle a \frac{\partial F^*}{\partial a} F \rangle \Big)$,

ignoring $\mathcal{O}(\eta^3)$ [24]. With that, the full set of equations of motion becomes (compare Eqs. (17), (18) and (19))

$$\frac{\partial}{\partial t}\langle a \rangle = \Big\langle \frac{(-\hbar\delta + \hbar g' a^* a) a}{I} \Big\rangle + \frac{i\hbar^2 g^K}{2} \Big\langle \frac{\partial F}{\partial a^*} F^* \Big\rangle, \tag{39}$$

$$\frac{\partial}{\partial t}\langle a^* \rangle = \Big\langle \frac{(-\hbar\delta + \hbar g' a^* a) a^*}{I^*} \Big\rangle + \frac{i\hbar^2 g^K}{2} \Big\langle \frac{\partial F^*}{\partial a} F \Big\rangle, \tag{40}$$

$$\frac{\partial}{\partial t}\langle a^* a\rangle = \left\langle \frac{(-\hbar\delta + \hbar g' a^* a)a}{I}\right\rangle + \left\langle \frac{(-\hbar\delta + \hbar g' a^* a)a^*}{I^*}\right\rangle + \frac{i\hbar^2 g^K}{2}\Big(\langle F^* F\rangle + \langle a^* \frac{\partial F}{\partial a^*} F^*\rangle + \langle F^* F\rangle + \langle a \frac{\partial F^*}{\partial a} F\rangle\Big). \tag{41}$$

Following the same line of thought as in the semi-classical case, we obtain the Fokker-Planck equation:

$$\frac{\partial}{\partial t}P[a^*,a,t] = -\frac{\partial}{\partial a}\left[(-\hbar\delta + \hbar g'|a|^2)\ \frac{a}{I}\ P[a^*,a,t]\right] - \frac{\partial}{\partial a^*}\left[(-\hbar\delta + \hbar g'|a|^2)\ \frac{a^*}{I^*}\ P[a^*,a,t]\right] + \frac{i\hbar^2 g^K}{2}\left(\frac{\partial}{\partial a^*}\frac{a}{I^*}\frac{\partial}{\partial a}\frac{a^*}{I} + \frac{\partial}{\partial a}\frac{a^*}{I}\frac{\partial}{\partial a^*}\frac{a}{I^*}\right)P[a^*,a,t], \tag{42}$$

where we want to emphasize the fact that $g^{(\pm)}$ from Eq. (24) has now become g'. This FP-equation could have been derived directly from the action (4) too, as described before. Instead of a constant $g'' = Im(g^{(+)})$ in Eq. (25), we would now have a contribution containing a time-derivative. For this situation, the conjugate momentum $\pi(t)$ therefore reads

$$\pi(t) = \frac{I[a^*,a]}{\hbar g^K a^*(t)a(t)}\Big(-i\hbar\frac{\partial}{\partial t} + \hbar\delta - \hbar g'\, a^*(t)a(t)\Big)\ a^*(t), \tag{43}$$

and the Hamiltonian becomes

$$H = \frac{\pi(t)}{I[a^*,a]}(-\hbar\delta + \hbar g' a^*(t)a(t))a(t) + \frac{\pi^*(t)}{I^*[a^*,a]}(-\hbar\delta + \hbar g' a^*(t)a(t))a^*(t) - \frac{g^K a^*(t)a(t)}{I^*[a^*,a]I[a^*,a]}|\pi(t)|^2. \tag{44}$$

To finish this quantum approach, we have a look at the equilibrium distribution of the FP-equation (42).

When the drive-terms are neglected (which is done so far, i.e., the phase dependency is ignored), the stationary solution is given by

$$P_{eq}[a^*,a] \propto \sqrt{I[a^*,a]I^*[a^*,a]}\,\exp\left\{\frac{-\hbar}{k_B T}\ (-\delta + \frac{g'|a|^2}{2})|a|^2\right\}, \tag{45}$$

a fact that is proven in Appendix B. If we also include the drive terms, similar to the approach in the semi-classical case, the full FP-equation reads

$$\frac{\partial}{\partial t}P[a^*,a,t] = -\frac{\partial}{\partial a}\left[(-\hbar\delta + \hbar g'|a|^2)\ \frac{a}{I}\ P[a^*,a,t] + \frac{\hbar\omega_D A_D}{4}\frac{a^*}{I}P[a^*,a,t]\right] - \frac{\partial}{\partial a^*}\left[(-\hbar\delta + \hbar g'|a|^2)\ \frac{a^*}{I^*}\ P[a^*,a,t] + \frac{\hbar\omega_D A_D}{4}\frac{a}{I^*}P[a^*,a,t]\right] + \frac{i\hbar^2 g^K}{2}\left(\frac{\partial}{\partial a^*}\frac{a}{I^*}\frac{\partial}{\partial a}\frac{a^*}{I} + \frac{\partial}{\partial a}\frac{a^*}{I}\frac{\partial}{\partial a^*}\frac{a}{I^*}\right)P[a^*,a,t], \tag{46}$$

and the equilibrium distribution becomes

$$P_{eq}[a^*,a] \propto \sqrt{I[a^*,a]I^*[a^*,a]}\,\exp\left\{\frac{-\hbar}{k_B T}\Big((-\delta + \frac{g'|a|^2}{2})|a|^2 + \frac{\omega_D A_D}{8}(a^*a^* + aa)\Big)\right\}, \tag{47}$$

as is shown in Appendix C.

Notice that in both cases, the distribution precisely scales with $e^{-\beta H}$, where H is the Hamiltonian of the system. Remember that this was our goal: deriving an action that was able to correspond with the Hamiltonian of Eq. (1). Apart from this Boltzmann factor, we found the 'prefactor' $\sqrt{I[a^*,a]I^*[a^*,a]}$. In the *overdamped limit*, where $I \approx -\frac{i\hbar^2 g^K a^* a}{2k_B T}$, this factor becomes $-\frac{i\hbar^2 g^K a^* a}{2k_B T}$ and therefore the stationary solution will get an extra minimum at $|a|= 0$. On the other hand, in the *underdamped limit*, we have $I \approx i\hbar$ and the prefactor doesn't depend on a^*, a (and thus it can be absorbed in the normalisation constant). This is the most obvious situation: no other physical behavior arises than that which comes from the Boltzmann factor. For the experimental factors of Ref. [16], this limit is a very good approximation.

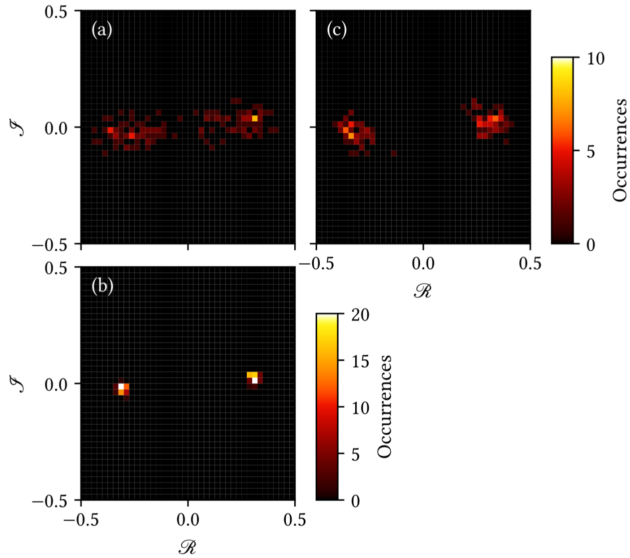


FIG. 4: Comparison between the experiment and the simulations for the high-order axial mode amplitude A_X. (a) Contour plot of the results of the experiments. (b) Contour plot of the simulations, where both the thermal and technical fluctuations are taken into account. (c) Contour plot of the simulations, where only the thermal fluctuations are taken into account. Figure taken from Ref. [20].

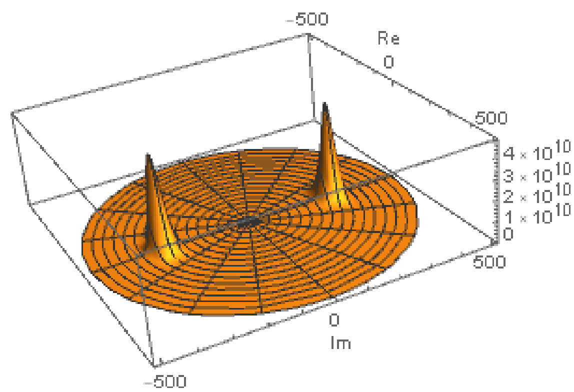


FIG. 5: A 3D plot of our theoretical result for the (unnormalized) equilibrium distribution.

IV. COMPARISON WITH EXPERIMENTS

In this section, our goal is to show the result for the stationary solution of the FP-equation in the quantum approach (Eq. (47)) and compare it with (i) experimental data and (ii) numerical simulations. Both of these are found in Ref. [20] and shown in Fig. 4. The equilibrium distribution coming from Eq. (47) is shown in Fig 5. We nicely reproduce the symmetry breaking, i.e., the structure of two peaks away from the origin. This shows that our theoretical model and the experimental setup indeed match.

When we look at the width of the peaks, we conclude that our theoretical result is narrower than in the experiment and fits better to the numerical result without technical fluctuations. This can be explained by the fact that the experiment also deals with some broadening coming from the initial conditions (more explicitly the number of particles), a phenomenon that really comes on top of our theoretical model.

V. CONCLUSION AND OUTLOOK

The main achievements of this research can be summarized as follows. First, we developed the general theory to describe the non-equilibrium physics of our atomic Bose-Einstein condensate: starting with a fourth-order Hamiltonian, we applied the tools of the Keldysh formalism to derive the general form of the Langevin equations. At that point, we distinguished two case: a semi-classical approach, where the frequency dependence of the dissipation was neglected and a fully quantum approach, where this frequency dependency was included. For both cases, we presented the full system of equations describing the dynamics: both in the Langevin-picture as well as in the Fokker-Planck-picture and we showed its consistency. Furthermore, we were able to derive stationary solutions for both of these FP-equations, which, as expected, turned out to scale with the Boltzmann factor $e^{-\beta H}$, where H is the Hamiltonian for that specific situation.

At the end of the semi-classical approach, we were in the position to visualize the dynamical features of the spontaneous symmetry breaking. The quantum method on the other hand made it possible to compare our full stationary solution (drive included) with experimental data, showing its agreement and therefore validating our model.

An open end in our discussion comes from the fourth-order term. Although its theoretical implications work out nicely, its precise (experimental) origin remains unclear. A good guess would be to state that it comes from coupling with the thermal cloud, which plays the role of the heat bath in our model. Nonetheless, our research shows that a solid theoretical description, with minimal approximations, underlies the spontaneous symmetry breaking in these kind of time crystals.

An interesting area for further research might be to use our Fokker-Planck equation(s) in order to study the phenomenon of ‘tunneling’ from one minimum to the other. Although we didn’t look into this in our discussion, the non-equilibrium physics of this problem is also captured by our model.

Appendix A: Semi-classical equilibrium without drive

The equilibrium in the semi-classical approach has to satisfy the stationary Fokker-Planck equation for that case, which reads:

$$\begin{aligned} 0 = & -\frac{\partial}{\partial a}\left[(-\hbar\delta + \hbar g^{(+)}|a|^2)\ a\ P[a,a^*,t]\right] + \frac{\partial}{\partial a^*}\left[(-\hbar\delta + \hbar g^{(-)}|a|^2)\ a^*\ P[a,a^*,t]\right] \\ & - \frac{\hbar g^K}{2}\left(\frac{\partial}{\partial a^*}\ a\ \frac{\partial}{\partial a}\ a^* + \frac{\partial}{\partial a}\ a^*\ \frac{\partial}{\partial a^*}\ a\right) P[a,a^*,t]. \end{aligned} \tag{A1}$$

Furthermore, the fluctuation-dissipation theorem tells us that $g^K = 2ig''(1+2N(-\hbar\delta))$. Using then a phase independent probability distribution ($P = P(|a|^2)$), we can easily see that both the terms with $-\hbar\delta$ as well as the terms with $\hbar g'$ (the real part of $\hbar g^{(+)}$) will vanish. What is left is:

$$0 = (N+\frac{1}{2})\left(\frac{\partial}{\partial a^*}\ a\ \frac{\partial}{\partial a}\ a^* + \frac{\partial}{\partial a}\ a^*\ \frac{\partial}{\partial a^*}\ a\right) P + 4|a|^2\, P + 2|a|^4\, \frac{\partial P}{\partial(|a|^2)}. \tag{A2}$$

This is also the point where an erroneous factor of 2 arises, compare the remark below the Boltzmann equation, Eq. (21). But notice that

$$\left(\frac{\partial}{\partial a^*}\ a\ \frac{\partial}{\partial a}\ a^* + \frac{\partial}{\partial a}\ a^*\ \frac{\partial}{\partial a^*}\ a\right) P = 4|a|^2\, \frac{\partial P}{\partial(|a|^2)} + 2|a|^4\, \frac{\partial^2 P}{\partial(|a|^2)^2}, \tag{A3}$$

which means that the stationary FP-equation becomes

$$0 = 2|a|^4\left((N+\frac{1}{2})\,\frac{\partial^2 P}{\partial(|a|^2)^2} + \frac{\partial P}{\partial(|a|^2)}\right) + 4|a|^2\left((N+\frac{1}{2})\,\frac{\partial P}{\partial(|a|^2)} + P\right). \tag{A4}$$

We therefore conclude that the probability distribution (has to be normalized and) obeys

$$\frac{\partial P}{\partial(|a|^2)} = -\frac{P}{N+\frac{1}{2}}, \tag{A5}$$

which is the defining equation for the ideal-gas solution,

$$P_{eq}[a^*,a] \propto \exp\left(\frac{-a^*a}{N+1/2}\right). \tag{A6}$$

Notice that in the limit when $N \gg 1$, we have $N \approx -\frac{k_BT}{\hbar\delta}$ and thus $P_{eq}[a^*,a] \propto \exp\left(\frac{\hbar\delta}{k_BT}a^*a\right)$, which is the expected classical Boltzmann factor.

Appendix B: Quantum equilibrium without drive

The stationary Fokker-Planck equation without drive for this situation, when written in terms of the amplitude $|a|$ and phase ϕ, becomes:

$$\begin{aligned} 0 = & -\frac{1}{2|a|}\frac{\partial}{\partial|a|}\left[(-\delta + g'|a|^2)\ |a|^2\ (\frac{1}{I}+\frac{1}{I^*})P\right] + (-\delta+g'|a|^2)\frac{1}{II^*}\frac{\partial P}{\partial\phi} \\ & + \frac{i\hbar g^K}{8}\left\{\frac{1}{|a|}\frac{\partial}{\partial|a|}\left[|a|^3\ (\frac{1}{I}\frac{\partial}{\partial|a|}\frac{1}{I^*} + \frac{1}{I^*}\frac{\partial}{\partial|a|}\frac{1}{I})\,P\right] + \frac{2}{II^*}\frac{\partial^2 P}{(\partial\phi)^2} + \frac{|a|}{iI}\frac{\partial}{\partial|a|}\frac{1}{I^*}\frac{\partial P}{\partial\phi} - \frac{|a|}{iI^*}\frac{\partial}{\partial|a|}\frac{1}{I}\frac{\partial P}{\partial\phi}\right\}. \end{aligned} \tag{B1}$$

This rather complicated equation is solved for

$$P_{eq}[a^*,a] \propto \sqrt{II^*}\,\exp\left(\frac{-\hbar}{k_BT}\ (-\delta + \frac{g'|a|^2}{2})|a|^2\right). \tag{B2}$$

Notice that the FP-equation reduces greatly when P is phase independent (which is true for our equilibrium), since then $\frac{\partial P}{\partial \phi} = 0$. We can also make use of the fact that

$$(\frac{1}{I}\frac{\partial}{\partial |a|}\frac{1}{I^*} + \frac{1}{I^*}\frac{\partial}{\partial |a|}\frac{1}{I})\, P = \frac{1}{II^*}\left(2\frac{\partial P}{\partial |a|} - \frac{(I+I^*)^2 P}{|a| II^*}\right). \tag{B3}$$

Together with

$$\frac{\partial P_{eq}}{\partial |a|} = \left(\frac{-2\hbar}{k_B T}\,(-\delta + g'|a|^2)\,|a| + \frac{(I+I^*)^2}{2|a|\,II^*}\right)P \tag{B4}$$

one can easily see that P_{eq} is indeed the correct equilibrium distribution.

Appendix C: Quantum equilibrium with drive

Very similar to the approach of Appendix B, we can show that

$$P_{eq}[a^*, a] \propto \sqrt{II^*}\exp\left\{\frac{-\hbar}{k_B T}\left(\,(-\delta + \frac{g'|a|^2}{2})|a|^2 + \frac{\omega_D A_D}{8}(a^* a^* + aa)\right)\right\} \tag{C1}$$

solves the full FP-equation with drive:

$$\begin{aligned} 0 = &-\frac{1}{2|a|}\frac{\partial}{\partial |a|}\left[(-\delta + g'|a|^2)\,|a|^2\,(\frac{1}{I} + \frac{1}{I^*})P\right] + (-\delta + g'|a|^2)\frac{1}{II^*}\frac{\partial P}{\partial \phi} \\ &+ \frac{i\hbar g^K}{8}\left\{\frac{1}{|a|}\frac{\partial}{\partial |a|}\left[|a|^3\,(\frac{1}{I}\frac{\partial}{\partial |a|}\frac{1}{I^*} + \frac{1}{I^*}\frac{\partial}{\partial |a|}\frac{1}{I})\,P\right] + \frac{2}{II^*}\frac{\partial^2 P}{(\partial\phi)^2} + \frac{|a|}{iI}\frac{\partial}{\partial |a|}\frac{1}{I^*}\frac{\partial P}{\partial \phi} - \frac{|a|}{iI^*}\frac{\partial}{\partial |a|}\frac{1}{I}\frac{\partial P}{\partial \phi}\right\} \\ &- \frac{\omega_D A_D}{8}\left[\frac{1}{i}\frac{\partial P}{\partial \phi}\left(\frac{e^{-2i\phi}}{I} - \frac{e^{2i\phi}}{I^*}\right) + |a|\,\frac{\partial P}{\partial |a|}\left(\frac{e^{-2i\phi}}{I} + \frac{e^{2i\phi}}{I^*}\right) - (I+I^*)P\,\left(\frac{e^{-2i\phi}}{I^2} + \frac{e^{2i\phi}}{(I^*)^2}\right)\right]. \end{aligned} \tag{C2}$$

We will not show this in full detail here, but it turns out to be most convenient to split all terms up according to the appearance of $cos(2\phi)$ or $sin(2\phi)$, since these cancel out separately. Note for example that $\frac{\partial P_{eq}}{\partial |\phi|}$ leads to a $sin(2\phi)$. On top of that, it is useful to sort out all terms with $(-\delta + g'|a|^2)$ (which also comes into play when $\frac{\partial}{\partial |a|}$ acts on the phase independent part of the exponent). Of course, making use of the result of Appendix B also helps a lot.

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[21] Another option would be $\psi^*\psi$-interaction. In principle, this forms a whole new model, but we expect the physical results to coincide with our discussion here.

[22] After doing the quantization, one again encounters the fact that a symmetric operator needs to be chosen. In contrast to the approach in Ref. [17], it is no longer possible to place the momenta on the left of the coordinates. This is because we are dealing with multiplicative noise. Notice that this symmetric operator leads to the desired factor of $\frac{1}{2}$ for the diffusion term of the FP-equation.

[23] Notice that for g', we still use a delta-function. Furthermore, the factor of 4 in the denominator becomes a factor of 2 due to the fact that $\frac{\partial}{\partial t}a(t)a(t) = 2a(t)a'(t)$.

[24] Notice that the terms linear with $a^*(t_0)$ and $a(t_0)$ do not influence the expectation values because they are zeroth order in the noise.