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Master Thesis

Frozen Boundary Percolation on the Triangular Lattice

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Declaration of Authorship

I, Luca Michael Makowiec, declare that this thesis titled, "Frozen Boundary Percolation on the Triangular Lattice" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

Faculty of Science
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Frozen Boundary Percolation on the Triangular Lattice

by Luca Michael MAKOWIEC

We introduce a new frozen percolation model where we set the freezing condition on open clusters intersecting the boundary of the box $\Lambda(n)$. Using quite simple arguments, it is easy to show that the probability of the origin freezing does not go exponentially fast to zero nor to one as we let n grow large. We conjecture that in fact the probability of the origin not being contained in a frozen cluster is bounded away from 0, uniformly in n. It turns out that this is rather difficult to prove and we instead focus our attention on a somewhat different model, where not every boundary point initiates the freezing process, but for any $\epsilon > 0$ a boundary vertex is a freezing trigger point with probability $n^{-\epsilon}$. We compare this to a model as in [9] that independently puts holes around points and closes all vertices contained in these holes. The major argument needed in this thesis is a three-arm half-plane stability result, the proof of which follows along the lines of the proof of a similar four-arm full-plane stability statement in [9].

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1 Introduction

Percolation theory is a rich subject of stochastic processes first introduced in its most basic form in [10]. Given a graph $G = (V, E)$, consider the processes in which we, for some parameter $p \in [0, 1]$, independently remove each edge $e \in \mathbb{E}$ with probability $(1 - p)$. If an edge is removed in this way, we say it is closed and otherwise we call it open. We are now interested in the connection probabilities of the graph that remains after this procedure. This model is called ordinary Bernoulli bond (or edge) percolation with parameter p. The natural analogue where we instead open or close vertices is called (ordinary) Bernoulli site (or vertex) percolation. In this case, only those edges of which both endpoints are open will remain. Site percolation can in fact be seen as a generalisation of bond percolation, as it turns out that we can represent every bond percolation model as an equivalent site percolation version.

More formally, we set the sample space Ω as $\Omega = \{0, 1\}^E$ and consider the σ-algebra generated by cylinder sets of finite dimension. Elements of Ω are written as ω and can be interpreted as 0/1 vectors of closed/open edges. We denote by $\mathbb{P}_p$ the corresponding (product) measure of this space such that for any $e \in E$ we have $\mathbb{P}_p(\omega_e = 1) = p$, i.e. edge e is open with probability p. For the site percolation process, we replace the corresponding edge set with the vertex set.

The most common studied underlying graph is the hyper-cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ in dimension d. The vertex set is given by the integer vectors $\mathbb{Z}^d$ and the edge set $\mathbb{E}^d$ consists of pairs (u, v) with $||u - v|| = 1$. In dimension $d = 2$ and in high enough dimensions [21] ($d \geq 11$), a lot is known about the behaviour of these models. For dimension $d = 2$ this is related to the fact that we may use planar duality, while in high dimensions the lace expansion [19, 18] gives a strong tool in applications to Percolation Theory. For instance, in both the above cases researchers were able to prove the famous conjecture of $\theta(p_c) = 0$, as well as scaling relations of interesting functions of p close enough to p_c.

Throughout the years, many different variations of the Bernoulli model have been introduced and intensively studied. Some examples of these variations include "self-destructive percolation", "oriented percolation" and "invasion percolation". More information on some of these variations, as well as other models, can be found in [16, Chapter 12]. Percolation theory can also be seen as a sub-field of the Random-Cluster Model, which generalises the idea of random process on graphs to also include topics such as the Ising and Potts models. See for example [17, Chapter 8] for some definitions and relations.

The most relevant variation of percolation theory to this thesis is called frozen percolation, of which there are again different versions. The earliest of these types of models to be treated on the square lattice is the diameter frozen percolation [6], a model in which we stop the growth of any open cluster that becomes too large in diameter. In volume frozen percolation [7, 5] we instead halt the growth of clusters that contain too many vertices. In the model studied in this thesis, which we introduce in section 3.2, we follow a similar underlying framework. Here we instead restrict the growth of open clusters that intersect the boundary of some large box $\Lambda(n)$, where we let n go to infinity. In other words, we are interested in the asymptotic properties of the model when we let the box grow very large.

Notes about Constants

In this thesis, the reader will encounter various constants that, for simplicity reasons, are not given a precise value. Since we are most often interested in the scaling relations of functions, the exact values do not matter too much. The constants will often be denoted by C_i, where we reset the counter i in each chapter. Furthermore, to make equations more readable, in many proofs we will use a local constant c that arises from simple calculations. If we make use of several of these local constants in one display equation we denote them by c_i.

Contribution and Difficulties

The goal of this thesis is to first present classical percolation type results and then to introduce a new frozen percolation type model that has not been (as far as we know) studied in the literature. Although this model shares similarities with the other frozen percolation models, it is not trivial to apply the same techniques for this model. For further possible research questions see also the last part of Chapter 5.

The original aim was to first prove Conjecture 3.1 using somewhat basic and well known classical techniques. The next step would have been to show the even stronger result of

$$\mathbb{P}_n^F(0 \text{ is frozen at time } 1) \xrightarrow{n\to\infty} 0.$$

However, it turns out that even proving Conjecture 3.1 is much harder than expected. Furthermore, originally we worked on bond percolation on the square lattice but we realised that we required arm event results coming from conformal invariance that (so far) only exist for site percolation on the triangular lattice. In particular, we consider a percolation model at the critical parameter $p_c = 1/2$ and need the 1-arm and 3-arm critical half-plane exponents stated in Theorem 2.13 to prove Lemma 4.8.

Our attempt of proving Conjecture 3.1 is based on showing that the probability of there exists a frozen circuit around the origin before some time $t < 1$ is uniformly bounded away from 0. For a modified model we are able to prove a uniform lower bound of a box-crossing event (at time $t = 1/2$) that can be used to construct a frozen circuit around the origin. If we are able to conclude the same result for the original model, then we would be able to prove Conjecture 3.1. See also Proposition 4.9 and how this result is applied to the proof of Theorem 4.11.

Two of the major difficulties that presented themselves were the fact that no general type of monotonicity result exists and that frozen clusters do not behave in an independent manner. The latter problem we handled by comparing the boundary frozen percolation model to a model with holes as in [9]. Here we close vertices in boxes that are independently generated around boundary points and are of diameter at least of the size of frozen clusters at boundary points. Most of our results stem from analysing the model with holes and using stochastic domination to give consequences of our frozen percolation model. Heuristically speaking, in the model with holes there may be many more closed vertices than in the frozen percolation model. This means it is possible that the lower bounds coming from the approach of the model with holes may not be sufficiently strong to prove Conjecture 3.1.

2 Introduction to Bernoulli Site Percolation

In this chapter, we will introduce the notions needed to understand ordinary Bernoulli percolation on lattices. Furthermore, we will recap classical as well as more recent results and tools in percolation theory that will be needed in the proofs of our new model that will be introduced in section 3.2. Since these results are well-known and studied in other literature, we will omit most of the proofs and instead refer to the relevant sources.

2.1 Notation of the Triangular Lattice

For the purpose of this thesis, we will focus on Bernoulli site percolation on the triangular lattice. The proofs of the statements in the next two sections are easily adapted to bond or site percolation on other given lattices and have been shown in various literature. However, we will need the results given in section 2.4 and these are currently only proven for site percolation on the triangular lattice. It is conjectured that these results hold on any well-behaved lattices as well as for bond percolation on those lattices (with some other parameters), but currently, no rigorous proof is known for this. We define the triangular lattice $\mathbb{T} = (G, E)$ via an embedding in the complex plane $\mathbb{C}$. See figure 2.1 for a visualisation of the triangular lattice.

Definition 2.1 (Triangular Lattice $\mathbb{T}$)**.** *Consider the basis given by* 1 *and* $e^{i\pi/3}$ *and the infinity norm* $||\cdot||$ *with respect to these basis elements. A vertex of* $\mathbb{T}$ *is given by a linear integer combination of these basis elements and we connect any two vertices by an edge if their distance is exactly one. So,*

$$V = \{x + ye^{i\pi/3} \, : \, x, y \in \mathbb{Z}\} \quad \textit{and} \quad E = \{(u, v) \, : \, u, v \in V, \, ||u - v|| = 1\}.$$

Occasionally we will use $\mathfrak{C}(v)$ *and* $\mathfrak{R}(v)$ *to refer to the imaginary and real part of* v*.*

As hinted to in the introduction, we set $\Omega = \{0, 1\}^V$ with corresponding σ-algebra generated by finite cylinder sets. For given $p \in [0, 1]$ we set each vertex (independently) open with probability p and closed otherwise. For $\omega \in \Omega$ if the vertex v is open we denote this by $\omega_v = 1$ and if it is closed we write $\omega_v = 0$. The product probability measure of this process will be written as $\mathbb{P}_p$. If $\omega, \omega' \in \Omega$ such that $\omega_v \leq \omega'_v$ for all $v \in V$, then we say that $\omega \leq \omega'$. An edge $e = (u, v) \in E$ is said to be open if both u and v are open, otherwise it is closed. A path $\pi = (v_1, \ldots, v_r)$ is said to be open if every edge (v_i, v_{i+1}) of π is an open edge. If every edge is closed instead, then we say π

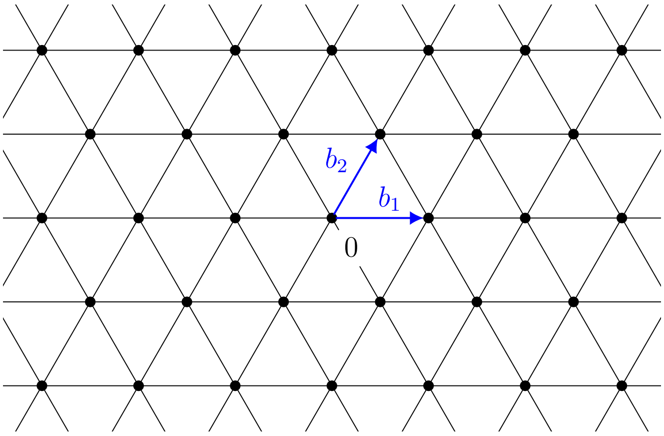


FIGURE 2.1: Depiction of a part of the triangular lattice in $\mathbb{C}$. We work with basis elements $b_1 = 1$ and $b_2 = e^{i\pi/3}$.

is a closed path. For the purpose of this thesis we require that paths are self-avoiding, i.e. they do not contain any loops, which can be written as $v_i \neq v_j$ for all $i \neq j$. In the case where we consider circuits we drop this assumption for the starting and ending vertices. In other words, a circuit π is a path $(v_1, \dots, v_r)$ with $v_1 = v_r$ and $v_i \neq v_j$ for $i \neq j$ and $(i, j) \neq (1, r)$.

Later on we will also be interested in the percolation process on the half-plane triangular lattice $\mathbb{T}^{\mathbb{H}} = (V^{\mathbb{H}}, E^{\mathbb{H}})$. This is defined analogously as above but with vertex set

$$V^{\mathbb{H}} = \{x + ye^{i\pi/3} \,:\, x, y \in \mathbb{Z},\, y \geq 0\}$$

instead.

We say that two sets A and B are connected if there exists an open path π from a vertex in A to a vertex in B. This event will be denoted by $\{A \leftrightarrow B\}$ and if we require that one of these open paths is exactly π', then we write $\{A \overset{\pi'}{\leftrightarrow} B\}$ instead. In the case where A (or B) consist of just a single point v we drop the set notation of A and write $\{v \leftrightarrow B\}$ instead. If we furthermore require that the path π only uses vertices of a subset $R \subseteq V$, then we denote this event by $\{A \leftrightarrow B \text{ in } R\}$. For $v \in V$ we let $C(v) \subseteq V$ be the set of vertices that are connected to v, i.e. we set

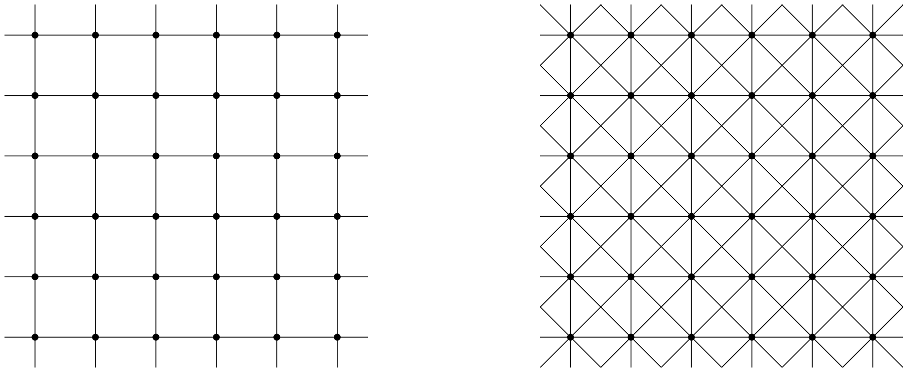
$$C(v) = \{u \in V \,:\, v \leftrightarrow u\}.$$

Lastly, given a set of vertices $A \subset V$ we let the boundary of A be given by

$$\partial A = \{v \in A \,:\, \exists w \in A^c \text{ s.t. } (v, w) \in E\}.$$

Matching Graph

One important concept relevant to this thesis is the notion of the matching (planar) graph. Given any planar graph $G = (V, E)$ the process of constructing the matching graph $G' = (V', E')$ is as follows: we set $V' = V$ and for each face F of G (including the



(A) Square Lattice. (B) Matching graph of the Square Lattice.

Figure 2.2: In each face of the square lattice we add edges until all vertices are connected to each other. Since the triangular lattice is *self-matching*, we do not need to add any edges for that lattice.

infinite face, if it exists) we connect all vertices to each other, forming a complete graph of F. It is easy to see that the triangular lattice is *self-matching* in the sense that the matching graph of $\mathbb{T}$ coincides with $\mathbb{T}$. This fact makes the proofs of many properties of the triangular lattice much simpler.

In the literature the notion of dual graphs is often used in bond percolation models (as well as in general graph theory), which is constructed in a slightly different way. It is however still true that the properties listed below also hold for the dual graph. In particular, the square lattice is also *self-dual*, allowing for similar methods to prove properties in the next sections.

The construction of the matching graph is completely deterministic and we now want to add randomness to G'. The matching graph G' is coupled together with the original graph G in the sense that the open/closed status of any vertex in G' is equal to the state of the corresponding vertex in G. An edge $e \in \mathbb{E}'$ is said to be closed (in the matching graph) if both endpoints are closed vertices. Paths in the matching graph will be called dual paths and a closed dual path is a path π such that each edge of π is closed. Keep in mind that G' may have a different edge set than G and hence this could lead to other edges being opened/closed in G' than in G. We refer to figure 2.2 for an example of the matching graph of the square lattice.

The matching graph possesses the following crucial topological characteristics:

i) For a parallelogram R (see section 2.1 for a precise definition) there either exists a vertical (resp. horizontal) open path in G or a horizontal (resp. vertical) closed path in G';

ii) Any open cluster $C(v)$ in G is surrounded by a closed circuit in G'.

Crossings of Parallelograms

For the remainder of the thesis we work with the basis elements 1 and $e^{i\pi/3}$. Let $a_1 \leq a_2$ and $b_1 \leq b_2$ be given, then we define the parallelogram $R = [a_1, a_2] \times [b_1, b_2]$ as the set with corners

$$c_{11} = a_1 + b_1 e^{i\pi/3}, \quad c_{12} = a_1 + b_2 e^{i\pi/3}, \quad c_{21} = a_2 + b_1 e^{i\pi/3}, \quad c_{22} = a_2 + b_2 e^{i\pi/3},$$

i.e. R is the convex hull of the above points. Furthermore, we denote by

$$\begin{aligned} \text{the “left” side of } R &= Conv(c_{11}, c_{12}), \\ \text{the “right” side of } R &= Conv(c_{21}, c_{22}), \\ \text{the “bottom” side of } R &= Conv(c_{11}, c_{21}), \\ \text{the “top” side of } R &= Conv(c_{12}, c_{22}), \end{aligned}$$

where $Conv$ denotes the convex hull of points.

In the case where $a_2 = b_2 = -a_1 = -b_1 = n$ we denote the corresponding parallelogram by $\Lambda(n)$ and call this the box of size n around the origin. For $v \in V$ the box around v with size n is given by $\Lambda(v, n) = v + \Lambda(n)$. One important class of sets are annuli centred around the origin. For $N > n \geq 1$ we define the annulus $A(n, N)$ as

$$A(n, N) = \Lambda(N) \setminus \Lambda(n-1).$$

In the half-plane percolation process, the definitions follow analogously as above but we instead consider the intersection of these sets with the upper half-plane.

For a subset of vertices $W \subset V$ we denote its size $|W|$ by the number of vertices in W. Given some $A \subset \mathbb{C}$ we implicitly write $|A|$ for $|A \cap V|$. Note that for some suitably chosen constants C_i, C_i' it is

i) $C_1 n^2 \leq |\Lambda(n)| \leq C_1' n^2$,

ii) $C_2 n \leq |\partial\Lambda(n)| \leq C_2' n$,

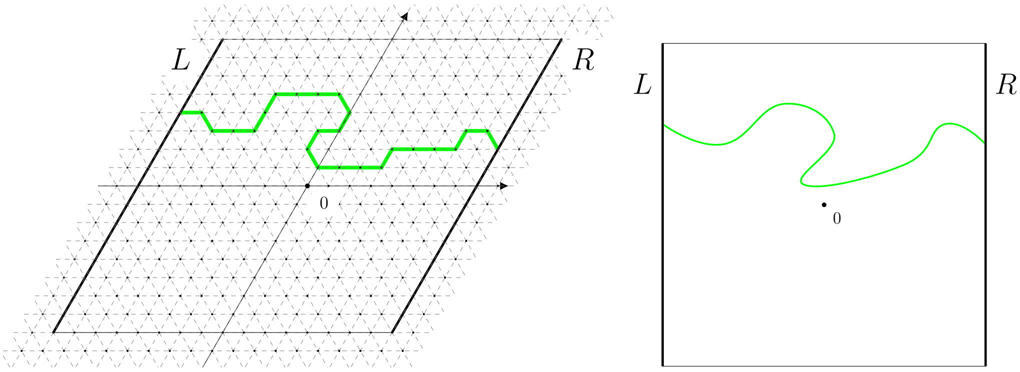
iii) $C_3 (N-n)^2 \leq |A(n, N)| \leq C_3' (N-n)^2$.

These simple observations, which also hold for the half-plane process, will often be used throughout this thesis without any further reference.

We say that there exists an open vertical crossing in $R = [a_1, a_2] \times [b_1, b_2]$ if the event

$$\{\text{the “top” side of } R \leftrightarrow \text{the “bottom” side of } R \text{ in } R\}$$

occurs. This event is often abbreviated as $\mathcal{C}_V([a_1, a_2] \times [b_1, b_2])$. In the case where we consider a horizontal crossing (the definition of which should be obvious) we write $\mathcal{C}_H([a_1, a_2] \times [b_1, b_2])$ instead. If we want to observe closed vertical (resp. horizontal) crossings in the matching graph we denote this by $\mathcal{C}_V^*([a_1, a_2] \times [b_1, b_2])$ (resp. $\mathcal{C}_H^*([a_1, a_2] \times [b_1, b_2])$). See also figure 2.3 for a visualisation of a horizontal crossing in $\Lambda(n)$ for $n = 8$.



(A) Crossing with non-standard basis elements. (B) Simplified drawing.

FIGURE 2.3: In figure 2.3a we observe the event of a horizontal crossing from the **L**eft side of $\Lambda(8)$ to the **R**ight side. The coordinate axes are given by the basis elements 1 and $e^{i\pi/3}$. For simplicity sake, this event will instead be drawn as in figure 2.3b.

When referring to the left most open vertical crossing we mean the path $\pi = (v_1, \ldots, v_n)$ such that for any other open vertical crossing $\pi' = (w_1, \ldots, w_m)$ the implication

$$\mathfrak{C}(v_i) = \mathfrak{C}(w_j) \Rightarrow \mathfrak{R}(v_i) \leq \mathfrak{R}(w_j) \qquad \forall 1 \leq i \leq n, 1 \leq j \leq m$$

holds. In other words, if v_i and w_j are at the same "height" (i.e. they have the same imaginary part), then v_i must be more "left" of w_j (i.e. v_i has a real value part smaller or equal than the real value part of w_j). The definition for the right/top/bottom-most path follows similarly. The benefit of defining these paths is that if they exist then they are uniquely determined. For instance, the existence of a vertical crossing can be written as the disjoint union over all possibilities of a left most crossing. In the literature, there are also further uses cases of this. For example, conditioning on the left-most vertical crossing gives no information to the right of said crossing.

Lastly, we are also interested in open and closed dual circuits surrounding a vertex (often the origin). It is also helpful to consider circuits in annuli. We define the event $\mathcal{O}(n, N)$ as

$$\mathcal{O}(n, N) := \{\exists \text{ open circuit in } A(n, N) \text{ surrounding } 0\}.$$

and let the dual equivalent be given by

$$\mathcal{O}^*(n, N) := \{\exists \text{ closed dual circuit in } A(n, N) \text{ surrounding } 0\}.$$

Critical Parameter

The most interesting event related to connections is that there exists an open path from the origin to the boundary of the box $\Lambda(n)$. We are particularly interested in the probability of this event as n tends to infinity. We define

$$\theta(p) := \lim_{n\to\infty} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda(n)).$$

It is easy to check that the above equals $\mathbb{P}_p(|C(0)| = \infty)$ and furthermore by translation invariance is independent of the choice of origin.

The above definition of $\theta(p)$ naturally gives rise to the definition of the critical parameter.

Definition 2.2 (Critical Parameter)**.** *For any graph $G = (V, E)$ we denote by*

$$p_c = p_c(G) = \sup\{p\,:\,\theta(p) = 0\}$$

the critical percolation threshold.

One of the main goals in percolation theory is to study the behaviour of $\theta(p)$ with p close to p_c. The regime with $p < p_c$ is called "subcritical", while for $p > p_c$ the model is in its "supercritical" phase. Clearly, if the model is subcritical, then it contains no infinite cluster, and if it is supercritical there exists such an infinite cluster. An important question that we wish to know an answer to is, what happens when $p = p_c$? As it turns out, it is very hard to answer this question even for "simple" lattices such as $\mathbb{L}^d$. In section 2.3 this will be discussed a bit more.

2.2 Useful Tools in Percolation Theory

Before we showcase some well-known classical results in percolation theory, we describe a few useful tools that are used in the proofs in the literature as well as later on in this thesis. The reader is advised to read chapters 3 to 5 in [17] or parts of [16] for more information.

Definition 2.3. *An event $A \subset \Omega$ is called increasing if for any ω, ω' with $\omega \leq \omega'$ the following implication holds*

$$\omega \in A \Rightarrow \omega' \in A.$$

In other words, if event A already holds and we open more vertices, then A must still hold. The following indispensable tool in percolation theory states an intuitive dependence of two increasing events.

Theorem 2.4 (FKG Inequality)**.** *If $A, B \subset \Omega$ are both increasing events, then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Proof. See for example Theorem 4.11 in [17]. □

Furthermore, for two events $A, B \subset \Omega$ we let $A \Box B$ be the set of $\omega \in \Omega$ with the property that there exist disjoint $U, W \subset V$ such that the open/closed states of vertices in U (resp. W) imply that event A (resp. B) holds. For our purposes, it suffices to understand that the events of the form {there exist two disjoint open paths from V_1 to V_2 using only vertices in R with $|R| < \infty$} can be written as $\{V_1 \leftrightarrow V_2$ in $R\}\square\{V_1 \leftrightarrow V_2$ in $R\}$. The reader is advised to also have a look at section 4.3 of [17].

Theorem 2.5 (BK-Reimer Inequality)**.** *Let A and B be two events depending on finitely many vertices, then*

$$\mathbb{P}_p(A\square B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

Proof. See Theorem 4.17 in chapter 4 of [17]. □

The above inequality was first shown to hold for increasing events A and B by van den Berg and Kesten in [4]. In many applications it requires to only consider the case when A and B are increasing, however, van den Berg and Kesten conjectured that the inequality also holds for general events A and B. This was finally proven by Reimer in [27].

Next, we introduce a coupling process of percolation with uniformly $[0, 1]$ distributed variables. Namely, consider times $(\tau_v)_{v\in V}$ such that each $\tau_v \sim \mathcal{U}([0, 1])$ independently of all other vertices. For percolation with parameter $p \in [0, 1]$ we consider any vertex v open if and only if $\tau_v < p$. This coupling process is very useful in many situations and will play an important role in the later chapters. For instance, if A is an increasing event, then the function

$$p \mapsto \mathbb{P}_p(A) \tag{2.1}$$

is a non-decreasing function. As an immediate consequence we get that connection probabilities, such as $\theta(p)$, are non-decreasing functions in p. Also note that if A only depends on a finite vertex set $S \subset V$, then the function in (2.1) is differentiable with respect to p (any probability is the sum of monomials in p and $(1 - p)$).

The last ingredient needed for Russo's formula is the concept of pivotal vertices. Let $\omega \in \Omega$ and $v \in V$. Denote by ω^v the element of Ω such that $\omega^v_v \neq \omega_v$ and $\omega^v_u = \omega_u$ for all other $u \in V$. In other words, to construct ω^v we flip the entry in ω corresponding to v and let all other entries be the same. Finally, we define

$$\{v \text{ is pivotal for } A\} = \{\omega \in \Omega\,:\, \text{exactly one of } \omega \text{ and } \omega^v \text{ is contained in } A\}.$$

For increasing A this means that if v is pivotal for A, then A occurs if and only if v is open.

With these preliminaries in mind, the famous Russo's formula may now be stated.

Theorem 2.6 (Russo's Formula)**.** *If A is an increasing event depending on a finite vertex set S, then*

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{v\in S} \mathbb{P}_p(v \textit{ is pivotal for } A)$$

Proof. The proof can be found for instance in section 2.4 of [16]. □

RSW Theory

The last tool needed for this thesis is the theory of RSW uniform lower bounds. This technique is essential to understand the literature and is used very frequently. The following is stated for site percolation on the triangular lattice $\mathbb{T}$, however, very similar results also hold for bond percolation on the square lattice $\mathbb{L}^2$.

Theorem 2.7 (RSW). *For every $n \in \mathbb{N}$,*

$$\mathbb{P}_{1/2}(C_V([0,n] \times [0,n])) = \frac{1}{2}.$$

Furthermore, for any $k \geq 1$ there exists a $\delta(k) > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}_{1/2}(C_V([0,n] \times [0,kn])) \geq \delta(k).$$

Proof. The results come from Russo [28], Seymour and Welsh [29], leading to the abbreviation of RSW. See also section 5.5 of [17]. □

The above theorem together with the FKG inequality allows us to show a uniform lower bound for various combinatorial structures. For example, we can give a lower bound for open circuits in annuli.

Corollary 2.8. *For every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}_{1/2}(\mathcal{O}(n, (1+\epsilon)n)) \geq \delta(\epsilon).$$

Proof. Notice that we can "glue" together an open circuit in the annulus $A(n, (1+\epsilon)n)$ by taking two suitable horizontal and two well-chosen vertical crossings. The statement then immediately follows from the above theorem and the FKG inequality. □

By symmetry reasons (recall that the triangular lattice is *self-matching*) all the above results also hold for closed paths/crossings in the matching graph.

2.3 Classical Results

As stated in earlier sections, we wish to know if at $p = p_c$ we have that 0 is contained in an infinite cluster with positive probability, or equivalently if there a.s. exists an infinite cluster. For a long time there existed no proof, even for bond percolation on the square lattice, of what value p_c exactly was and if an infinite cluster occurs at p_c. Harris showed first in [20] that $\theta(1/2) = 0$ for bond percolation on the square lattice. Only 20 years later Kesten [23] showed that $p_c(\mathbb{L}^2) \leq 1/2$, which allows us to conclude that $p_c = 1/2$ and $\theta(p_c) = 0$. These (and other) results are considered classical now.

The natural analogue of Harris's and Kesten's arguments for site percolation on the triangular lattice gives a similar statement about p_c.

Theorem 2.9. *The critical parameter of site percolation on the triangular lattice equals 1/2. Furthermore, it is $\theta(p_c) = 0$.*

Proof that $p_c \geq 1/2$. To illustrate the usefulness of RSW theory we will prove that $\theta(1/2) = 0$ and hence $p_c \geq 1/2$. For a full proof of the theorem we refer the reader to section 5.6 of [17].

Consider the annuli $A(3^i, 2 \cdot 3^i)$ for $i \geq 0$. All these annuli are disjoint and by RSW there exists a $\delta > 0$ such that

$$\mathbb{P}_{1/2}(\mathcal{O}^*(3^i, 2 \cdot 3^i)) \geq \delta.$$

This means for $p = 1/2$ there a.s. exists an i_0 such that $\mathcal{O}^*(3^{i_0}, 2 \cdot 3^{i_0})$ occurs. Therefore, with probability 1, there is a closed dual circuit surrounding the origin, which implies the open cluster of 0 can not be infinite. □

The proof of the above theorem makes use of the planar duality of the underlying graph, which does not exist for arbitrary dimensions d. In fact, it remains unanswered if an infinite cluster occurs at the critical point for all d. It is however conjectured, that indeed the origin is a.s. not contained in an infinite cluster when $p = p_c$.

Conjecture 2.10. *For bond percolation on the hypercubic lattice $\mathbb{L}^d$ with $d \geq 2$ it is*

$$\theta(p_c) = 0.$$

As mentioned in the introduction, for d large enough, say $d \geq 11$, the above conjecture holds true. Furthermore, there is hope that the same techniques can be extended to $d \geq 7$ and possibly even to $d = 6$. The question still remains of what happens in the intermediate dimensions between $d = 2$ and $d = 6$.

Characteristic Length and Exponential Decay

It is now well known that for certain small enough scales, percolation with parameter smaller than p_c “looks” similar to percolation at the critical parameter. This was stated formally and proved rigorously in Kesten’s celebrated paper [22], which also gave important applications. A central role in his paper is played by the characteristic length. For $\epsilon > 0$ and $p < p_c$ we define the characteristic length $L_\epsilon(p)$ as

$$L_\epsilon(p) := \min\{n \geq 1 \,:\, \mathbb{P}_p(\mathcal{C}_V([0,n] \times [0,n])) \leq \epsilon\}.$$

For $p > p_c$ we let $L_\epsilon(p) = L_\epsilon(1-p)$. We will fix $L = L_{0.001}$ for the remainder of the thesis, but note that there is no particular reason to pick $\epsilon = 0.001$, as long as it smaller than the constants produced from RSW theory, we are guaranteed that $L(p) \to \infty$ as $p \to p_c$. Furthermore, as it turns out for other ϵ' we have that $L_\epsilon(p)$ and $L_{\epsilon'}(p)$ differ only by a multiplicative constant $C(\epsilon, \epsilon')$ uniformly in p. See for instance Corollary 37 of [26].

The characteristic length for example dictates how fast the probabilities of connection events decay.

Theorem 2.11. *There exist constants $C_3, C_4 > 0$ such that for all $p < p_c$ and $n \in \mathbb{N}$,*

$$\mathbb{P}_p(\mathcal{C}_V([0,n] \times [0,n])) \leq C_3 e^{-C_4 \frac{n}{L(p)}},$$

Proof. See Lemma 39 in [26]. □

In particular, if n is much larger than $L(p)$, the probability of having connection events is exponentially small. By the definition of p_c it follows that $\mathbb{P}_p(0 \leftrightarrow \partial\Lambda(n))$ goes to zero for all $p < p_c$, however, the above theorem implies that this decay is exponentially fast. By application of the matching graph, the above also shows that for $p > p_c$ the probability of there not existing an open crossing in a box of order n goes exponentially fast to zero.

In fact, we have that exponential decay exists for bond percolation on the hypercubic lattice in arbitrary dimensions $d \geq 2$.

Theorem (Exponential Decay)**.** *For $d \geq 2$ and $p < p_c(\mathbb{L}^d)$ there exists $\psi(p) > 0$ such that*

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda(n)) \leq e^{-n\psi(p)}.$$

Proof. See for instance Theorem 5.1 of [17]. □

The above theorem for Bernoulli percolation was first proved by [1] as well as independently by [25]. Later in 2015 the authors of [13] showed another proof for exponential decay for a large class of percolation (as well as Ising) models. In section 2.5 we will adapt the simpler version of this argument for Bernoulli percolation (as presented in [14]) to another percolation type model.

Other Classical Results

Lastly, we will also list a few well-known results in percolation theory that will not be used in this thesis but are of interest themselves.

i) For all $d \geq 2$ it is $0 < p_c(\mathbb{L}^d) < 1$.

ii) For all $d \geq 11$ we have that $\theta(p_c) = 0$ for the hypercubic lattice $\mathbb{L}^d$. It is believed that the proof technique may be extended to all $d \geq 7$.

iii) If there exists an infinite cluster it is almost surely unique.

For the proofs of items i) and ii) see [17, Chapter 3] and [21], while for item iii) see for instance [15].

2.4 Detailed Results

For the triangular lattice, there exist some strong scaling results of certain events. The results that are needed for this thesis are summarised in Theorem 2.13. It is believed that similar versions of these results can also be extended to other two-dimensional lattices as well as to bond percolation, however, currently, there are no proofs of this. The main tool used for showing these statements depends on conformal invariance of site percolation on the triangular lattice [31, 30] and that percolation has Schramm-Löwner evolution SLE_6 as a scaling limit [11]. If one is able to show conformal invariance for other lattices, then it should be reasonable that we can also conclude that similar statements as those in Theorem 2.13 (with some other parameters of course) hold for those lattices.

One of the tools used in proving conformal invariance is the so-called Cardy's formula. It states that connection probabilities between boundary segments of a (suitably chosen) set can be related to the length of a segment of an equilateral triangle. Cardy first hypothesised (a version of) this formula in [12], but it was only proven by Smirnov in [31]. For a recollection and parts of proof see section 5.6 of [17].

First we introduce some standard notation of asymptotically equivalence between two functions. Namely, if there exists some finite constants $\lambda_1, \lambda_2 > 0$ such that we have $\lambda_1 g \leq f \leq \lambda_2 g$, then we write $f \asymp g$. Moreover, we also need a weaker equivalence for f and g when their parameter k tends to some k_c, where for notational sake we allow $k_c = \infty$.

Definition 2.12. *For two positive functions f, g we say that f and g are logarithmic asymptotically equivalent if*

$$\lim_{k \to k_c} \frac{\log f(k)}{\log g(k)} = 1$$

and we write $f \approx g$. If the above fraction is smaller than one we write $f \ll g$.

Although the base of the above logarithms does not matter, when referring to $\log k$ in this thesis we mean the logarithm of k with respect to base 2.

The main use of this equivalence will be when f and g are functions of $p \in [0, 1]$ and $k_c = 1/2$, or when f and g are functions of $n \in \mathbb{N}$ and $k_c = \infty$. If it is $f \asymp g$ and $f, g \to 0$ (or both tending to infinity), then it clearly also is $f \approx g$. Furthermore, assume that for some functions f_1, f_2 we have $f_1 \approx k^\alpha$ and $f_2 \approx k^\beta$, then the following properties hold true:

i) In the case $k_c = \infty$ we have that for any $\epsilon > 0$ there exist $C(\epsilon), C'(\epsilon) > 0$ such that $C(\epsilon)k^{\alpha-\epsilon} \leq f_1(k) \leq C'(\epsilon)k^{\alpha+\epsilon}$.

ii) If $f = f_1 \cdot f_2$, then $f \approx k^{\alpha+\beta}$.

The proof of the above items follows from basic properties of the log function and we will use these without further notice.

Arm Events

We now proceed to define arm events. The values of the probabilities of arm events play crucial roles in the proof of many statements. For $j \geq 1$ we consider a colour sequence $\sigma = (\sigma_1, \ldots, \sigma_j)$, where each σ_i is either "open" or "closed", which we denote by $\sigma_i \in \{o, c\}$. Furthermore, we identify two sequences if they are equal up to a cyclic permutation. For example, the sequences $\sigma_1 = (o, o, c, c)$ and $\sigma_2 = (c, o, o, c)$ are equal to each other, while $\sigma_3 = (c, o, c, o)$ is a different colour sequence.

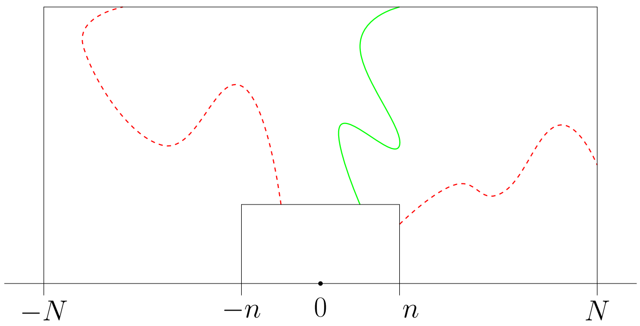


FIGURE 2.4: Consider the event $\mathcal{A}_3^H(A(n,N))$. We see 2 closed dual paths (indicated as dashed red paths) and one open path from the boundary of $\Lambda(n)$ to $\partial\Lambda(N)$.

We say that a path π is of colour o (resp. c) if all edges of π are open (resp. dual closed) and write $\mathrm{Col}(\pi) = o$ (resp. $\mathrm{Col}(\pi) = c$). For $n \leq N$ the event $\mathcal{A}_{j,\sigma}(A(n,N))$ is defined as

$$\mathcal{A}_{j,\sigma}(A(n,N)) := \{\exists \text{ disjoint } \pi_1, \dots, \pi_j \text{ such that the } \pi_i \text{ are ordered counterclockwise,}\\ \mathrm{Col}(\pi_i) = \sigma_i \text{ and } \partial\Lambda(n) \overset{\pi_i}{\longleftrightarrow} \partial\Lambda(N)\}.$$

Note that for small enough n and large j the geometry of the triangular lattice necessarily makes the event $\mathcal{A}_{j,\sigma}(A(n,N))$ empty. For example, if we take $j > |\partial\Lambda(n)|$, no j disjoint paths originating from $\partial\Lambda(n)$ can exist. Therefore, sometimes we will write $n_0(j)$ for the smallest integer allowing the existence of j disjoint arms. The exact value of $n_0(j)$ is irrelevant, see also page 1574 of [26].

In the case where we consider the half-plane lattice $\mathbb{T}^{\mathbb{H}}$ we write $\mathcal{A}_{j,\sigma}^H A((n,N))$ instead. For notational sake when the length of σ is clear, then the subscript j will be omitted. Furthermore, we will use

$$\mathcal{A}_1^H(A(n,N)) := \mathcal{A}_{(o)}^H(A(n,N)) \quad \text{and} \quad \mathcal{A}_3^H(A(n,N)) := \mathcal{A}_{(c,o,c)}^H(A(n,N)),$$

so $\mathcal{A}_1^H(A(n,N))$ is the open 1-arm event, while $\mathcal{A}_3^H(A(n,N))$ is the 3-arm event where two closed arms surround an open one. Figure 2.4 displays the 3-arm event.

The following theorem gathers the results we need in chapter 4.

Theorem 2.13. *For the triangular lattice $\mathbb{T}$ the following relations hold:*

1. *As $p \to p_c$ it is $L(p) \approx |1/2 - p|^{-4/3}$.*
2. *If $k \leq L(p)$, then $\mathbb{P}_p(\mathcal{A}_{j,\sigma}(n_0(j),k)) \asymp \mathbb{P}_{1/2}(\mathcal{A}_{j,\sigma}(n_0(j),k))$.*
3. *For any colour sequence σ_j it is $\mathbb{P}_{1/2}(\mathcal{A}_{\sigma_j}^H(n,N)) \approx (N/n)^{-\beta_j}$ for $N, n \to \infty$ with $\beta_j = j(j+1)/6$.*
4. *It is $\mathbb{P}_{1/2}(\mathcal{A}_3^H(n,N)) \asymp (N/n)^{-2}$.*

Proof. We respectively refer to Theorem 33, Theorem 27, Theorem 22, Theorem 24 in [26]. For the last two items an application of quasi-multiplicativity (Proposition 17 of

[26]) must also be used. Also see [32] for the original statement and proofs of these results. □

Note that item 4 is a stronger version of item 3 for $j = 3$. The main use of item 3 will be for $j = 1$, i.e. the one arm half-plane exponent is 1/3.

2.5 Simpler Proof of Exponential Decay in a Percolation Model Motivated by Spatial Epidemics

In the previous sections, we have seen that there is exponential decay for connection probabilities if $p < p_c$. We end this chapter by providing a simpler proof of exponential decay to a (directed edge) model from [3]. In relation to the main results in chapters 3 and 4, this section can be skipped as it is independent of the rest.

As mentioned previously, both [1] and [25] independently found proofs for exponential decay in the subcritical phase for Bernoulli percolation. In [3] the authors modified these proofs to show that there also exists exponential decay in a percolation model with local dependencies that is used to model a spatial epidemics process. Since later in 2015 the paper [14] (and it's more general version [13]) showed a shorter argument for exponential decay than Aizenman-Barsky and Menshikov did, it is natural to ask if the same proof technique can be adapted to the model introduced by [3]. This section is devoted to showing that this is indeed the case.

Notation

First, we describe the model and recall some notation from [3]. We consider a collection of independent random sets $\{N_u \subseteq \{\pm e_1, \ldots, \pm e_d\} \mid u \in \mathbb{Z}^d\}$ each sampled according to some probability measure ν. These sets generate a random directed subgraph Γ of $\mathbb{L}^d$ by taking the vertex set $\mathbb{Z}^d$ and directed edges of the form $[u, v\rangle$ for $v - u \in N_u$.

This model can be interpreted as an epidemics model in the following way: if an individual u becomes sick, they (immediately) infect all their neighbours in $u + N_u$. Now assume that the sickness emerges from the origin, we are interested in "how far the sickness spreads" and if it will "eventually die out". The latter can for instance be interpreted as the event of the origin being contained in a finite cluster in our model.

Definition 2.14. *The measure ν (stochastically) moderately dominates μ if for all non-trivial increasing events $\mathcal{A}$ with $\mu(\mathcal{A}) > 0$ we have $\nu(\mathcal{A}) > \mu(\mathcal{A})$.*

This notion of dominance is slightly different from strict dominance in [3]. Namely, we also allow the possibility of having $\nu(\mathcal{A}) = \mu(\mathcal{A}) = 0$. One can easily verify that strict dominance implies moderate dominance.

As in the start of proof of 2.1 in [3], for any given probability measure ν we define a new measure ν_p with $p \in [0, 1]$ by independently setting each vertex open with probability p and closed with probability $1 - p$. A vertex v is called pivotal for the event B if B occurs if v is open and B does not occur when v is closed. It follows that Russo's formula holds for increasing (in terms of open/closed states) events B. Also note that we use a more general form of the BK-inequality (Theorem 5.1 in [3]), which we can apply to vertex disjoint paths, where the last vertex of one path can possibly be the first of another (a self-avoiding path from v to w gives no information about the state of edges at w).

For vertices $u_0 \neq u_{m+1}$, we let $\{u_0 \to u_{m+1}\}$ be the event that there exist (distinct) $u_1, \dots, u_m$ such that $[u_i, u_{i+1}\rangle$ are edges in Γ and u_i is open for $0 \leq i \leq m$. The event $\{u_0 \to u_{m+1} \text{ in } A\}$ is defined analogously as above but now requiring that $u_0, \dots, u_m \in A$. If $u_0 = u_{m+1}$ both events are just defined as the trivial event of probability 1. Furthermore, we let $\theta(\nu) := \mathbb{P}_\nu(0 \to \infty)$ be the probability that the origin is contained in a cluster of infinite size.

Next, we define $\Delta A \subseteq \partial A$ as the following (deterministic) set

$$\Delta A := \{v \in \partial A \mid \mathbb{P}_\nu((v + N_v) \cap A^c \neq \emptyset) > 0\}.$$

In other words, the set ΔA contains those vertices that with positive probability are connected to a vertex outside of A after sampling w.r.t. to ν. In particular, since there are only finitely many configurations of outgoing edges for each vertex, there exists a $C_\Delta > 0$ (independent of choice of A) such that $\mathbb{P}_\nu((v + N_v) \cap A^c \neq \emptyset) > C_\Delta$ for all $v \in \Delta A$. Lastly, we set A^o as $A \setminus \Delta A$.

Adaption of Proof of Theorem 5.1 in [17]

Consider the slightly stronger version of Theorem 2.1 from [3] given by

Theorem 2.15. *If ν moderately dominates μ and $\theta(\nu) = 0$, then there exists $c(\nu) > 0$ such that*

$$\mathbb{P}_\mu(0 \to \partial\Lambda(n)) \leq e^{-c(\nu)n}.$$

Proof. The proof follows as the argument of proof Theorem 5.1 in [17]. For any set of vertices $S \subset \Lambda(L)$ with $0 \in S$ we have (almost surely) that

$$\{0 \to \partial\Lambda(kL)\} = \bigcup_{v \in \Delta S} \{0 \to v \text{ in } S^o \circ v \to \partial\Lambda(kL)\}.$$

This holds true since for any open path π that satisfies the l.h.s. we can consider the self-avoiding version of that path with v the first vertex reached in ΔS. Hence, for any such S we have using the BK inequality that

$$\begin{aligned}\mathbb{P}_{\nu_p}(0 \to \partial\Lambda(kL)) &\leq \sum_{v \in \Delta S} \mathbb{P}_{\nu_p}(0 \to v \text{ in } S^o)\mathbb{P}_{\nu_p}(v \to \partial\Lambda(kL)) \\ &\leq \mathbb{P}_{\nu_p}(0 \to \partial\Lambda((k-1)L)) \sum_{v \in \Delta S} \mathbb{P}_{\nu_p}(0 \to v \text{ in } S^o).\end{aligned}$$

Now define

$$\phi_p(S) := \sum_{v \in \Delta S} \mathbb{P}_{\nu_p}(0 \to v \text{ in } S^o).$$

If we show that for any $p < 1$ there exists a finite S such that $\phi_p(S) < 1$, then we can conclude exponential decay. Namely, for any μ moderately dominated by ν there exists a $p < 1$ such that $\mu \leq \nu_p$ and therefore by iteration

$$\mathbb{P}_\mu(0 \to \partial\Lambda(n)) \leq \mathbb{P}_{\nu_p}(0 \to \partial\Lambda(\lfloor n/L \rfloor L)) \leq \phi_p(S)^{\lfloor n/L \rfloor},$$

which decays exponentially fast in n.

Let

$$p_s := \sup\{p \in [0,1] \mid \exists S \text{ with } |S| < \infty, 0 \in S \text{ and } \phi_p(S) < 1\},$$

then it remains to show that $p_s = 1$. Assume that $p_s < 1$, then we pick a p^* such that $p_s < p^* < 1$. This means for all relevant S and $p \in (p_s, p^*]$ we have $\phi_p(S) \geq 1$. Now let B_n be the increasing (in terms of the open/closed states of the vertices) event of $\{0 \to \partial\Lambda(n)\}$ and let $\mathcal{S} = \{x \in \Lambda(n) \mid x \not\to \partial\Lambda(n)\}$. Then by Russo's formula it is

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_{\nu_p}(B_n) &= \sum_{v \in \Lambda(n)} \mathbb{P}_{\nu_p}(v \text{ is piv for } B_n) \\
&= \frac{1}{1-p} \sum_{v \in \Lambda(n)} \mathbb{P}_{\nu_p}(v \text{ is piv for } B_n, v \text{ closed}) \\
&= \frac{1}{1-p} \sum_{v \in \Lambda(n)} \sum_{S \ni 0} \mathbb{P}_{\nu_p}(v \text{ is piv for } B_n, v \text{ closed}, \mathcal{S} = S) \\
&= \frac{1}{1-p} \sum_{S \ni 0} \sum_{v \in \Delta S} \mathbb{P}_{\nu_p}(0 \to v \text{ in } S^o, (v + N_v) \cap S^c \neq \emptyset, v \text{ closed}, \mathcal{S} = S).
\end{aligned}$$

Now note that the events $\{0 \to v \text{ in } S^o\}$ and $\{(v + N_v) \cap S^c \neq \emptyset, v \text{ closed}, \mathcal{S} = S\}$ are independent, since the first event only depends on vertices in $S \setminus \Delta S$ while the latter only depends on vertices in $S^c \cup \Delta S$. Furthermore, $\{\mathcal{S} = S\}$ is positively correlated with v being closed and independent of $\{(v + N_v) \cap S^c \neq \emptyset\}$ conditioned on the fact that v is closed. So,

$$\begin{aligned}
\mathbb{P}_{\nu_p}((v + N_v) \cap S^c \neq \emptyset, v \text{ closed}, \mathcal{S} = S) &\geq \mathbb{P}_{\nu_p}((v + N_v) \cap S^c \neq \emptyset)\mathbb{P}_{\nu_p}(\mathcal{S} = S)(1-p) \\
&\geq C_\Delta(1-p)\mathbb{P}_{\nu_p}(\mathcal{S} = S),
\end{aligned}$$

where we used that $\mathbb{P}_{\nu_p}((v + N_v) \cap S^c \neq \emptyset) > C_\Delta$ with $C_\Delta > 0$ for all $v \in \Delta S$.

We will later integrate the inequality from p_s to p^*, which means we can use that $\phi_p(S) \geq 1$ for those p. Applying the above remarks we get

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_{\nu_p}(B_n) &\geq C_\Delta \sum_{S \ni 0} \sum_{v \in \Delta S} \mathbb{P}_{\nu_p}(0 \to v \text{ in } S^o)\mathbb{P}_{\nu_p}(\mathcal{S} = S) \\
&= C_\Delta \sum_{S \ni 0} \phi_p(S)\mathbb{P}_{\nu_p}(\mathcal{S} = S) \\
&\geq C_\Delta(1 - \mathbb{P}_{\nu_p}(B_n)).
\end{aligned}$$

This implies that

$$\frac{d}{dp} - \ln(1 - \mathbb{P}_{\nu_p}(B_n)) \geq C_\Delta,$$

which we integrate from p_s to p^* to get

$$\frac{1}{1 - \mathbb{P}_{p^*}(B_n)} \geq \frac{1 - \mathbb{P}_{p_s}(B_n)}{1 - \mathbb{P}_{p^*}(B_n)} \geq e^{C_\Delta(p^* - p_s)},$$

or equivalently

$$\mathbb{P}_{p^*}(B_n) \geq \frac{e^{C_\Delta(p^* - p_s)} - 1}{e^{C_\Delta(p^* - p_s)}} > 0.$$

Letting n go to infinity we get that $\theta(\nu_{p^*}) > 0$, which contradicts the assumption that $\theta(\nu) = 0$. Therefore, we get that indeed $p_s = 1$, completing the proof. $\square$

3 Introduction to Frozen Percolation

Aldous first introduced frozen bond percolation on the infinite binary tree in [2]. His paper was partly motivated by studying a process of polymerisation of molecular units, where one is interested in the transition of finite polymers into infinite polymers. This can be interpreted as part of the chemical *sol-gel* process.

Heuristically speaking, in his model open clusters stopped growing (or froze) once the cluster became infinite in size. He showed that this stochastic process is well defined for the binary tree and stated the adaptations needed to extend this to the infinite d-regular ($d \geq 3$) tree. It is however not clear at all that such a process also exists on the square lattice $\mathbb{L}^2$ or on the site percolation analogue on the triangular lattice. In fact, Benjamini and Schramm showed that such a process can not exist for those lattices (see also [8, Section 3]). Therefore, the processes shown below will only be defined for some finite parameter n that limits the growth of clusters beyond size n. We are mostly interested in the asymptotic properties of the models as we let n tend to infinity.

3.1 General Framework of Frozen Percolation

We now proceed to explain the general set up of frozen percolation for bond percolation on some graph $G = (V, E)$. Each edge $e \in E$ is assigned a random time τ_e, where the collection of $(\tau_e)_{e \in E}$ are i.i.d. uniformly distributed over the interval $[0, 1]$. We now let time t evolve from 0 to 1. At time $t = 0$ all edges are closed, while at time τ_e we open the edge $e = (u, v)$ unless some condition on the open cluster of u or v is met. If the condition is met, say for v, then we say the cluster of v is frozen. When dealing with the site percolation variation, we instead generate $(\tau_v)_{v \in V}$ values and set some freezing condition on v or on the neighbours of v.

The different variations of frozen percolation arise from choosing a suitable condition for the edge e or vertex v not opening. Note that via the coupling argument in section 2.2, it is easy to see that if we choose no condition on the freezing process, then at time τ this model would be indistinguishable from ordinary Bernoulli percolation with parameter τ.

Diameter Frozen Percolation

First introduced on the square lattice $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$ by [6], diameter frozen percolation with parameter N sets a restriction on the diameter of any cluster growing much larger than N. More formally, the diameter of a set $W \subset \mathbb{Z}^2$ is given by $\sup\{||v-w|| : v, w \in W\}$ and the freezing condition on edge $e = (u, v)$ is that one of the endpoints u or v is contained in an open cluster of diameter larger or equal to N. As mentioned in the introduction to this chapter, we are mostly interested in very large N and the limits (or limit inferior) of values as N tends to infinity.

The first result regarding this percolation process, already stated in the paper [6] that introduced it, showed that we can find a uniform lower bound of the probability of the cluster containing the origin having a diameter of order N, but not larger than N.

Theorem (Theorem 1.1 in [6])**.** *Let $C^N(0)$ denote the open cluster of the origin at time 1 for the N-parameter diameter frozen model on the square lattice. For all $0 < a < b < 1$,*

$$\liminf_{N\to\infty} \mathbb{P}(C^N(0) \textit{ has diameter} \in (aN, Bn)) > 0.$$

Later, in section 3.3, we will use a similar proof technique to show some result, so it may be helpful for the reader to have a brief look at the mentioned paper. The above theorem has as an immediate corollary that the probability of the origin freezing does not go to one, however, we still wish to know if the probability goes to zero as $N \to \infty$.

In [24] Kiss showed that the probability of 0 being frozen is not only uniformly bounded away from one, but indeed goes to zero as N grows large.

Theorem (Theorem 1.1 [24])**.** *As $N \to \infty$ the probability that in the N-parameter diameter frozen percolation process the open cluster of the origin freezes goes to 0.*

The paper [24] was originally written for the site percolation variation on the triangular lattice, however, with adaptations of the proof it is also possible to extend this to the bond percolation version (also see Remark 3.7 of [24]).

Volume Frozen Percolation

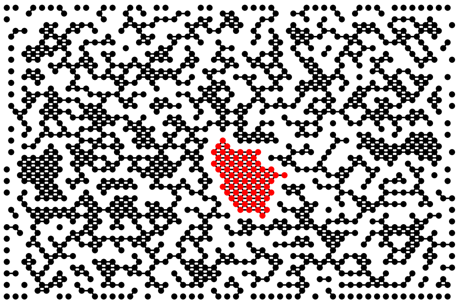
For the volume frozen percolation, the freezing condition is on the number of vertices (and not the diameter as above) contained in open clusters. Any vertex v becomes open at time τ_v unless there exists a neighbour w of v such that $|C(w)| \geq N$, where $C(w)$ is the open cluster of w at time τ_v and N is the parameter of the model.

In [7] the authors conjectured that similarly as in the diameter frozen percolation, the probability of 0 freezing goes to zero as $N \to \infty$. Building upon techniques of this paper, the authors of [5] show that in fact the origin is a.s. not contained in a frozen cluster as N grows large.

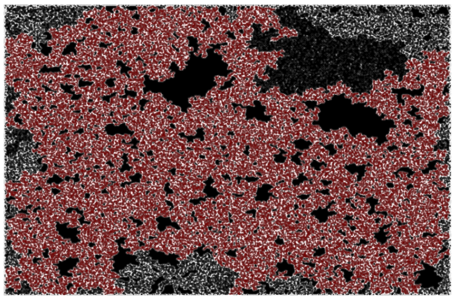
Theorem (Theorem 1.1 in [5])**.** *For the volume-frozen percolation process on $\mathbb{T}$ with parameter $N \geq 1$,*

$$\mathbb{P}_N^{(\mathbb{T})}(0 \textit{ is frozen at time } 1) \xrightarrow{N\to\infty} 0.$$

In both of these papers, the authors also discuss interesting results that display that these types of models usually do not behave in a monotone way. Roughly speaking, when considering the process on a finite box of size $m(N)$ instead of $\mathbb{T}$, the behaviour of



(A) Box with dimensions 51×51, where the origin does not freeze.



(B) Box with dimensions 501×501, where the origin freezes.

FIGURE 3.1: Figure 3.1a and 3.1b are simulations of frozen boundary percolation on boxes at time $t = 1$. Red coloured sites are those contained in the open cluster of the origin. Black sites are other open vertices, while closed sites are not drawn. Note that the underlying box is somewhat different than in this thesis (it was easier to simulate it this way).

the origin freezing dramatically depends on the choice of function m. For $m(N)$ close to "exceptional scales" $m_k(N)$ there is a uniform positive lower bound on the probability of the origin freezing. On the other hand, for $m(N)$ "far away" from these scales, the probability of the origin freezing goes to 0.

3.2 Boundary Frozen Percolation

To define our new boundary frozen percolation model, we proceed similarly as in the previous sections. However, instead of considering bond percolation on the whole of the square lattice, we will instead consider the site percolation variation on a subgraph of the triangular lattice. Namely, recall the definition of the box $\Lambda(n) = [-n, n] \times [-n, n]$ as in section 2.1. We now fix some $n \geq 1$ and consider the subgraph $\mathbb{T}_n = (V_n, E_n)$ of $\mathbb{T} = (V, E)$ generated by vertices in $V \cap \Lambda(n)$. For each $v \in V_n$ we independently generate time values τ_v according to the uniform $[0, 1]$ distribution. It turns out that a slight modification of the edge set simplifies many observations. Specifically, we remove all edges from E_n that are of the form (u, v) with $u, v \in \partial\Lambda(n)$ and with abuse of notation this new edge set will still be denoted by E_n.

Comparing to the framework mentioned at the start of this chapter, we now proceed to define our freezing condition for a vertex v. When referring to an open frozen (resp. non-frozen) τ-path π we require that each vertex v on π is open before time τ and the open cluster of π contains (resp. does not contain) a vertex in $\partial\Lambda(n)$. As usual, we now let time progress from 0 to 1. We open v at time τ_v unless there exists a neighbour of v that is connected to a vertex $w \in \partial\Lambda(n)$ using an open frozen τ_v-path. The probability measure for this process will be denoted by $\mathbb{P}_n^F$. Since this is a finite state system, the existence of such a process with corresponding probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n^F)$ is clear for any $n \in \mathbb{N}$.

If for some v the open cluster $C(v)$ at time t contains a vertex in $\partial\Lambda(n)$, then we say that v is "frozen" at time t. We are particularly interested in the asymptotic behaviour of the probability of 0 being frozen at time $t = 1$ when we let n go to infinity.

Conjecture 3.1. *There exists a $C > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}_n^F(0 \textit{ is not frozen at time } t = 1) > C.$$

We are unable to prove the above conjecture. However, Proposition 3.5 and Theorem 4.11 in section 4.3 provide evidence that this is likely the case.

One difficulty in proving Conjecture 3.1 stems from the fact that in general there exists no "nice" form of monotonicity in frozen percolation. More open vertices do not necessarily increase connectivity properties: the more open vertices we have, the more likely it is for any open clusters to freeze, which in turn may hinder the growth process of the open cluster of the origin.

A weaker result is that the probability of 0 freezing does not decay exponentially fast, nor does it go to 1 exponentially fast. This will be shown in the next section.

3.3 Polynomial Lower Bounds

To prove a polynomial lower bound on the probability of the event that 0 freezes we first need to show a general result about site percolation on the triangular lattice with $p = 1/2$.

Lemma 3.2. *Let $a_n = [0, n] \times \{0\}$ and $b_n = \{n\} \times [0, \beta n]$ for some $\beta > 0$. Furthermore, let T_n be the right sided triangle with short sides a_n and b_n. Then there exists an $\epsilon = \epsilon(\beta) > 0$ and $C_1 = C_1(\beta) > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\mathbb{P}_{1/2}(0 \leftrightarrow b_n \textit{ in } T_n) \geq C_1 n^{-\epsilon}.$$

Proof. This is a standard application of RSW together with FKG (see also figure 3.2 for a visualisation of the construction). We will only show the proof for $\beta = 1$ and point out where adaptations need to be done for the general case. Consider the following sequences of rectangles:

$$H_i = [2^i, 2^i + 2^{i+1}] \times [0, 2^i],$$
$$V_i = [2^{i+1}, 2^i + 2^{i+1}] \times [0, 2^{i+1}],$$

where we let $i \in \{0, 1, \ldots, \lceil \log n \rceil\}$. In the case for $\beta \neq 1$ the ratio of height to width of the H_i and V_i need to be changed accordingly. We claim that if the following events occur

i) 0 is connected to H_0 and all vertices in H_0 are open,

ii) $\exists$ an open horizontal crossing in each H_i,

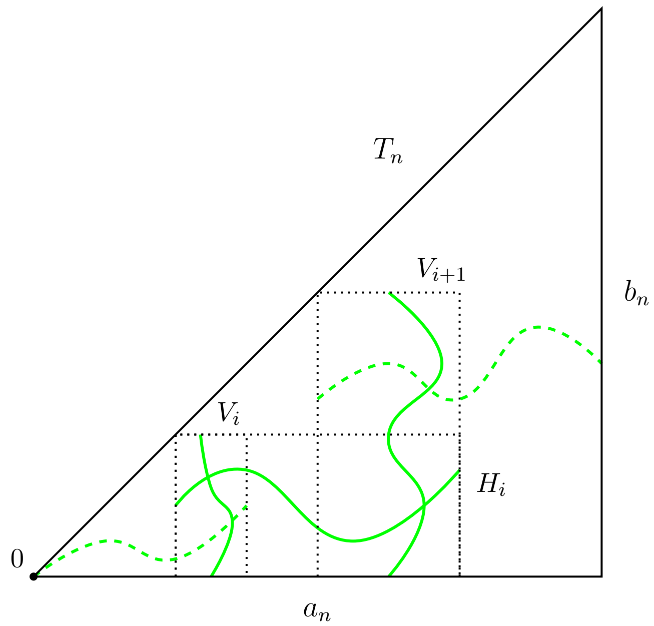


FIGURE 3.2: In each V_i we have a vertical crossing, while in each H_i we have a horizontal crossing. These glue together to an open path from 0 to b_n.

iii) $\exists$ an open vertical crossing in each V_i,

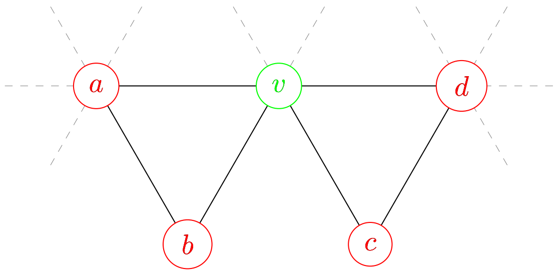
then there is a path from 0 to b_n using only vertices in T_n. Indeed, the open paths in V_i intersect those in H_i and the open paths in H_i intersect those in V_{i+1}. Furthermore, the crossing of $H_{\lceil \log n \rceil}$ intersects b_n and by i) we have that 0 is connected to H_0. Hence, 0 is also connected to b_n.

The condition that 0 is connected to H_0 and all vertices in H_0 are open, only depend on finitely many vertices and hence there is a trivial positive lower bound (for the $\beta = 1$ case this would be $c_0 = 2^{-7}$). For the crossing events, RSW gives us some constant C that uniformly lower bounds each of the probabilities of the events occurring (this also holds for the general case $\beta \neq 1$ with a possibly different constant). Hence, using the FKG inequality we get a lower bound of the form $c_0 C^{2\lceil \log n \rceil}$. Rearranging this expression gives the result. $\square$

Corollary 3.3. *Let $\beta_1, \beta_2 > 0$, $b_n = \{n\} \times [-\beta_1 n, \beta_2 n]$ and let $T_n(\beta_1, \beta_2)$ be the triangle spanned by $\{(0,0), (n, -\beta_1 n), (n, \beta_2 n)\}$. Then the probability that there is an open path from 0 to b_n in $T_n(\beta_1, \beta_2)$ has a polynomially lower bound as in Lemma 3.2. By symmetry arguments the same holds for any $\pi/2$ rotation of b_n and $T_n(\beta_1, \beta_2)$, as well as for dual crossings in the matching graph.*

Proof. By Lemma 3.2 the claim already holds for the triangle spanned by $\{(0,0), (n,0), (n, \beta_2 n)\}$ which is a subset of $T_n(\beta_1, \beta_2)$. $\square$

Using the above corollary, we now proceed to prove a polynomial lower bound for the probability of the origin freezing as well as of the probability of the origin not freezing.



$(0,-n)$

FIGURE 3.3: Consider the local configuration of the lattice around the point $b = (0, -n)$. We require that the red vertices a, b, c and d have a τ value of at least $1/2$ and the green vertex v has $\tau_v < 1/2$.

We define an $1/2$-open path (resp. $1/2$-closed dual path) to be a path on which every vertex w has a τ_w value smaller than $1/2$ (resp. has a τ value larger than $1/2$).

Lemma 3.4. *There exist positive constants C_2, C_3, C_4 and C_5 such that we have*

$$\mathbb{P}_n^F(0 \textit{ freezes at } t = 1) \geq C_2 n^{-C_3}, \tag{3.1}$$

and

$$\mathbb{P}_n^F(0 \textit{ does not freeze at } t = 1) \geq C_4 n^{-C_5} \tag{3.2}$$

for all $n \in \mathbb{N}$.

Proof of (3.1). Let $n \in \mathbb{N}$ be given. Consider the notation of figure 3.3 and assume that the events

i) $\tau_a, \tau_b, \tau_c, \tau_d > 1/2$

ii) $\exists$ $1/2$-closed dual path from a to $\{-n\} \times [-n+1, 0]$ in the triangle spanned by $\{a, (-n, -n+1), (-n, 0)\}$,

iii) $\exists$ $1/2$-closed dual path from d to $\{n\} \times [-n+1, 0]$ in the triangle spanned by $\{d, (n, -n+1), (n, 0)\}$,

iv) $\exists$ $1/2$-closed vertical dual path in $[-n, -n/2] \times [-n, n]$, $1/2$-closed vertical dual path in $[n/2, n] \times [-n, n]$ and $1/2$-closed horizontal dual path in $[-n, n] \times [n/2, n]$,

v) $\exists$ $1/2$-open path from v to $[-n/2, n/2] \times \{-n/8\}$ in the triangle spanned by $\{v, (-n/2, -n/8), (n/2, -n/8)\}$,

vi) $\exists$ $1/2$-open path from 0 to $[-n/2, n/2] \times \{-n/4\}$ in the triangle spanned by $\{0, (-n/2, -n/4), (n/2, -n/4)\}$ and

vii) $\exists$ $1/2$-open horizontal path in the box $[-n/2, n/2] \times [-n/8, -n/4]$

occur. These events are shown in figure 3.4.

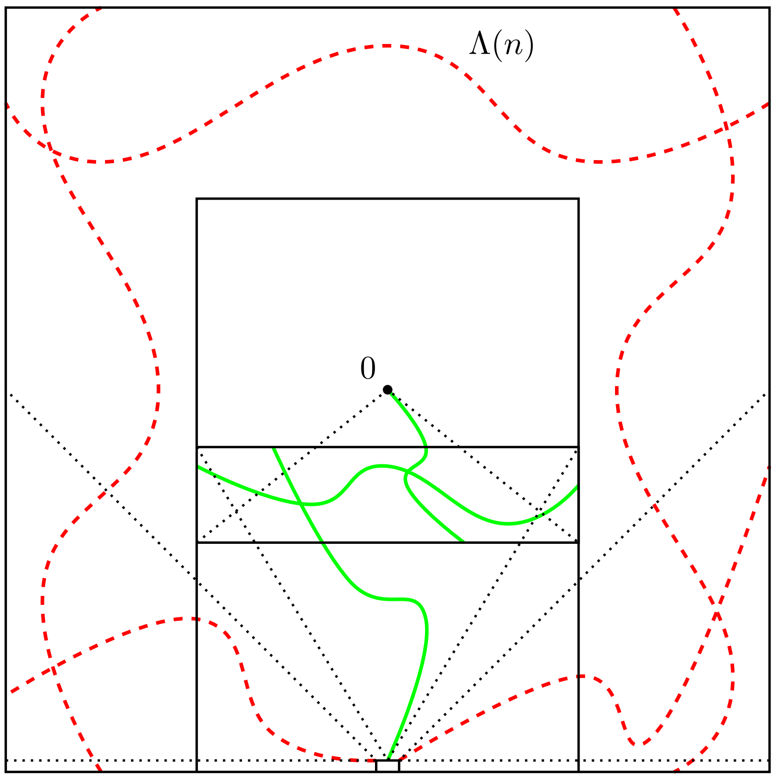


FIGURE 3.4: Assume all events as in the proof of Lemma 3.4 occur, where the small rectangle at the bottom depicts figure 3.3. Then there is a 1/2-open path from 0 to v that is separated from $\partial\Lambda(n)$ by a 1/2-closed dual circuit. It is now guaranteed that at time τ_b the origin freezes.

We claim that this implies that 0 freezes. Namely, by construction, all the defined 1/2-open paths are surrounded by a closed dual circuit separating them from $\partial\Lambda(n)$. Therefore, no paths inside this dual circuit will freeze before time $t = 1/2$. This means that at time $t = 1/2$ there will be an open path from 0 to v. At time τ_b the vertex b will either open and connect 0 to $\partial\Lambda(n)$ or v was already frozen before time τ_b. In both cases, there is a path from 0 to $\partial\Lambda(n)$, meaning 0 freezes before time $t = 1$.

It remains to show that the probability of these events occurring is bounded below by a value decaying only polynomially fast. First note that the events i)-iv) and v)-vii) depend on different vertices and hence are independent of each other. Event i) has a trivial lower bound of 2^{-4}, while by application of FKG together with RSW and Corollary 3.3 we have a lower bound of the form $C_1 n^{-\epsilon}$ for the events ii) - iv). Using the same argument also gives a similar lower bound for the probability of the events v)-vii) occurring. Combining all these facts gives us the desired lower bound. □

The proof for the second inequality follows with similar arguments. Namely, in practically the same way we can show that there exists a frozen circuit around the origin before τ_0, blocking the origin from reaching the boundary and thus it does not freeze.

First Proof Attempt for Conjecture 3.1

We end this section with a proposition that gives intuitive hints towards Conjecture 3.1 being true.

Proposition 3.5. *There exists a $C > 0$ such that*

$$\limsup_{n\to\infty} \mathbb{P}_n^F(0 \textit{ is not frozen at time } t = 1) > C, \tag{3.3}$$

or for every fixed time $\tilde{t} < 1$ it is

$$\lim_{n\to\infty} \mathbb{P}_n^F(0 \textit{ is not frozen at time } t = \tilde{t}\,) = 1. \tag{3.4}$$

Proof. Consider the following two contradictory assumptions:

Assumption 1: There exists a $\tilde{t} < 1$ such that

$$\limsup_{n\to\infty} \mathbb{P}_n^F(\exists \text{ a frozen circuit in } A(\sqrt{n}/2, n/2) \text{ before time } t = \tilde{t}) > C(\tilde{t}),$$

for some $C(\tilde{t}) > 0$.

Assumption 2: For every fixed $\tilde{t} \in (1/2, 1)$ it is

$$\lim_{n\to\infty} \mathbb{P}_n^F(\exists \text{ no frozen circuit in } A(\sqrt{n}/2, n/2) \text{ before time } t = \tilde{t}) = 1.$$

We will show that (3.3) will follow from assumption 1 and similarly (3.4) will follow from assumption 2. Since clearly either assumption 1 or 2 must hold, the result follows.

So, assume that assumption 1 is true for some $\tilde{t}$. Consider the event that there exists a frozen circuit $\mathcal{C}$ in $A(\sqrt{n}/2, n/2)$ at time $\tilde{t}$ and that furthermore it is $\tau_0 > \tilde{t}$.

Then the open cluster of 0 must be disjoint from the open cluster containing $\mathcal{C}$, since all vertices adjacent to the open cluster of $\mathcal{C}$ are closed at τ_0 and remain so afterwards. In particular, the origin can not reach the boundary and hence can not be frozen. This gives

$$\begin{aligned}\limsup_{n\to\infty} &\mathbb{P}_n^F(0 \text{ does not freeze}) \\ &\geq \limsup_{n\to\infty} \mathbb{P}_n^F(\exists \text{ a frozen circuit in } A(\sqrt{n}/2, n/2) \text{ before time } t = \tilde{t}, \tau_0 > \tilde{t}) \\ &\geq C(\tilde{t})(1-\tilde{t}) > 0,\end{aligned}$$

where the last inequality uses independence between the events. We therefore have found a sufficient bound for (3.3).

Now assume that with probability going to 1, no frozen circuit exists before any fixed time $\tilde{t} < 1$. We now let $\mathcal{O}_1^F(k)$ be the event that there

i) $\exists$ 1/2-open path in $A(k, 2k)$ and

ii) $\exists$ 1/2-closed dual path in $A(3k, 4k)$.

Using similar arguments as in Lemma 3.4, it is easy to see that $\mathcal{O}_1^F(k)$ implies that there exists an open circuit in $A(k, 2k)$ (the closed dual prevents the open circuit from freezing). Furthermore, the required paths for $\mathcal{O}_1^F(k)$ to occur rely on disjoint vertices and the probability of both can be (using RSW) bounded uniformly from below. For $n \in \mathbb{N}$ let $I_n = \{i \geq 0 \,:\, 5^{i+1}\sqrt{n}/2 < n/2\}$ and consider the event

$$\mathcal{O}_2^F(n) = \bigcup_{i \in I_n} \mathcal{O}_1^F(5^i\sqrt{n}/2). \tag{3.5}$$

In other words $\mathcal{O}_2^F(n)$ implies that for some annulus contained in $A(\sqrt{n}/2, n/2)$ we see an open non-frozen circuit before time $t = 1/2$. Now assume that $\mathcal{O}_2^F(n)$ occurs and that furthermore 0 is in fact frozen at time $t = \tilde{t}$, then the open cluster of 0 must intersect any such circuit, contradicting the assumption that there is no frozen circuit with the requirements as in assumption 2.

Note that $|I_n|$ is of size roughly $\log n$ and each of the events in the union of (3.5) depend on disjoint vertices and have uniform lower bounds. This means that as n tends to infinity, the probability of $\mathcal{O}_2^F(n)$ tends to 1. Concluding, for B the event as in assumption 2, we have

$$\mathbb{P}_n^F(0 \text{ does not freeze before } \tilde{t}) \geq \mathbb{P}_n^F(B \cap \mathcal{O}_2^F(n)) \geq \mathbb{P}_n^F(B) + \mathbb{P}_n^F(\mathcal{O}_2^F(n)) - 1,$$

the value of which goes to 1 as we let n go to infinity. $\square$

Intuitively one might think the above proposition (and the proof thereof) would lead to a proof of Conjecture 3.1, however, many small complications arise that we were unable to deal with properly. For instance, one may be tempted to take the limit as $\tilde{t}$ goes to one in (3.4), but this does not follow without further knowledge of the probability function (compare for example with the limit of $n \to \infty$ of the function

$1-x^n$ on $[0,1]$). The remainder of this thesis is dedicated to showing that assumption 1 is true for any $\tilde{t} \in (1/2, 1)$ in a somewhat weaker model introduced in the next chapter.

4 Trigger Points and Model with Holes

We introduce yet another new model that provides further evidence towards Conjecture 3.1. Instead of freezing clusters that reach any point on the boundary of $\Lambda(n)$, we only allow *some* of the boundary points to "trigger" the freezing procedure. Namely, independently for each $v \in \partial\Lambda(n)$ we set v as a "trigger point" with probability $n^{-\epsilon}$ for some fixed $\epsilon > 0$. More formally, we take i.i.d. random variables $(T_v)_{v\in\partial\Lambda} \in \{0,1\}$ and say that a boundary point v is a trigger point if $T_v = 1$, where the event $\{T_v = 1\}$ has probability equal to $n^{-\epsilon}$. Similarly to boundary frozen percolation, we now say that vertex u opens at time τ_u, unless there exists an open frozen τ_u-path from a neighbour of u to a trigger point. We denote this version of frozen percolation as the (n, ϵ) model and write $\mathbb{P}_{(n,\epsilon)}$ for its probability measure.

Model with Holes

To be able to deal with the freezing effects on the boundary, we couple the (n, ϵ) frozen percolation model with a model with independent holes (also called impurities). This is an approach also used in [9] to study forest fires and we will now proceed to define a variation of it that is applicable to our current model. The corresponding probability measure will be denoted by $\mathbb{P}^H_{(n,\epsilon)}$ and we refer to figure 4.1 for a visualisation of the model.

Firstly, consider site percolation on the half-plane triangular lattice $\mathbb{T}^{\mathbb{H}}$ and the coupling process with $(\tau_v)_v$ values as in section 2.2. If the origin opens at time $\tau_0 \leq 1/2$, then there exists a box B_0 with radius R such that the open cluster, as well as vertices adjacent to the boundary of the open cluster, are contained in B_0. So, $R = \sup\{||w|| + 1 : 0 \leftrightarrow w \text{ at time } \tau_0\}$. The distribution of R is then given by marginalising over $\tau_0 \in [0, 1/2]$, i.e.

$$\mathbb{P}^H_{(n,\epsilon)}(R = k) = \int_0^{1/2} \mathbb{P}(R = k \mid \tau_0 = t)\, dt.$$

In the model with holes, for each $v \in \partial\Lambda(n)$ we generate i.i.d. random variables $R_v \in \mathbb{N}$ such that R_v is distributed as the radius R above. Furthermore, we take i.i.d. (and independent of all R_v) $\lambda_v \in \{0,1\}$ such that $\mathbb{P}(\lambda_v = 1) = n^{-\epsilon}/2$. Finally, we denote by

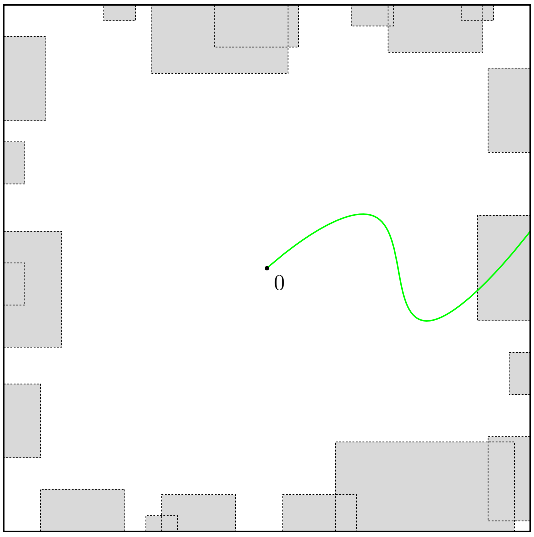


FIGURE 4.1: In the model with holes we see some holes H_v (indicated as grey boxes) centred at boundary points. Outside these holes we preform ordinary site percolation at $p = p_c$. The open green crossing in ordinary percolation would not exist in this model.

H_v the hole centred at v, where

$$H_v = \begin{cases} \emptyset, & \text{if } \lambda_v = 0, \\ \Lambda(v, R_v), & \text{if } \lambda_v = 1. \end{cases}$$

The model can now be described as follows: all vertices contained in a hole are set closed, while other vertices are opened with probability 1/2 independently of each other. It is important to keep in mind that using the coupling in section 2.2, one can also interpret this as opening a vertex v at time $\tau_v \leq 1/2$, unless it is contained in some hole.

The above process leads to configuration $\omega' \in \{0,1\}^{\Lambda(n)}$ of closed/open states. Similarly for the (n, ϵ) model at time t, if we take $\omega \in \Omega_n$ we can consider the closed/open states of the vertices at time t leading to a configuration $\omega_t \in \{0,1\}^{\Lambda(n)}$. For a subset A of the power set $\mathcal{P}(\{0,1\}^{\Lambda(n)})$, we let

$$\{A \text{ holds at } t\} := \{\omega \in \Omega_n \, : \omega_t \in A\}.$$

Since the above event is completely determined by the random variables needed for the (n, ϵ) model, it is measurable with respect to the corresponding sigma-algebra. The following crucial lemma allows us to later give bounds of these configurations in the (n, ϵ) model.

Lemma 4.1. *The (n, ϵ) model stochastically dominates the model with holes in the sense that for any increasing event in terms of closed/open states $A \subset \mathcal{P}(\{0,1\}^{\Lambda(n)})$,*

$$\mathbb{P}_{(n,\epsilon)}(A \textit{ holds at } t = 1/2) \geq \mathbb{P}^H_{(n,\epsilon)}(A).$$

Proof. We proceed to provide a rough outline of a coupling method between the models. For more details, we refer to sections 6.2 and 6.4 in [9].

Denote by $\nu_{v,1/2}$ the distribution of the open cluster of $v \in \partial\Lambda(n)$ at time $t = 1/2$ (in [9] this is done for any t, but it suffices here to consider the special case $t = 1/2$) in the (n, ϵ) model. The first claim is that the distribution of R_v dominates the radius of the open cluster (and its boundary) of v induced by the distribution of $\nu_{v,1/2}$. This follows from the obvious coupling of generating $(\tau_w)_{w\in\mathbb{T}^{\mathbb{H}}}$ values for the hole distribution and taking the τ_w values for $w \in \Lambda(n)$ for the boundary frozen percolation model[1]. Now at time τ_v the open cluster of v without any freezing procedure is clearly a superset of the open cluster of v with freezing effects and therefore the radius R must be at least the size of the radius of the open frozen cluster of v and its boundary.

We now proceed as follows: we independently generate $(\tau_v)_{v\in\Lambda(n)}$ and $(T_w)_{w\in\partial\Lambda(n)}$ values. We set

$$\lambda_w = \begin{cases} 1, & \text{if } \tau_w < 1/2 \text{ and } T_w = 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the random variables $(\lambda_w)_{w\in\partial\Lambda(n)}$ are independent of each other and the values of the corresponding probabilities (if we marginalise over the other random variables) are equal to those as in the definition.

If $\lambda_w = 1$ we set a hole centred at w using the coupling method as in the start of the proof, which shows that the radius of this hole is at least the size of the radius of $C(w)$. Unfortunately, there needs to be a bit more caution here, otherwise the holes do not have the correct marginal distribution. Namely, the way the radii of the holes are generated does not lead to independence between the holes. With more care, this can be treated properly (compare to the construction of $\mathcal{P}^{(i)}$ in the proof of Lemma 6.2 of [9]), but for the sake of simplicity, we will skip this.

Finally, we show that if $v \in \Lambda(n)$ is closed at time $t = 1/2$ in the (n, ϵ) model, then using the above coupling we must have that v is also closed in the model with holes (or equivalently if v is open in the model with holes, then v must also be open in the (n, ϵ) model). The statement is trivial for v with $\tau_v > 1/2$, so assume that $\tau_v \leq 1/2$. If v is closed, then there must exist a frozen cluster $C(w)$ with trigger point w and $\tau_w < 1/2$ such that a neighbour of v is in $C(w)$. However, this also means that $R_w \geq ||v - w||$, i.e. v is contained in the hole centred at w and thus is closed in the model with holes. □

Our first result is giving a power-law upper bound on the distribution of the radii R_v.

Lemma 4.2. *For any $\epsilon' > 0$ there exists a $C_1 = C_1(\epsilon') > 0$ such that for all $k \in \mathbb{N}$,*

$$\mathbb{P}^H_{(n,\epsilon)}(R_v \geq k) \leq C_1 k^{-13/12+\epsilon'}.$$

Proof. Recall the notion of characteristic length from section 2.4. Denote by $t_v \in [0, 1/2)$ the time at which the site v opens and for $\delta > 0$ we define a sequence $p_\delta(k)$ such that

[1]To be precise, we do not consider the "standard" half-plane lattice $\mathbb{T}^{\mathbb{H}}$ but rather the half-plane lattice rotated and translated such that v acts as the origin and the side of $\partial\Lambda(n)$ on which v lies represents the real line.

$L(p_\delta(k)) \asymp k^{1-\delta}$. Note that if v opens at t_v, then the probability of $\{R_v \geq k\}$ is equal to the probability that in ordinary half-plane percolation with $p = t_v$ we have an open path from 0 to the boundary of $\Lambda(k)$. Hence, it is

$$\begin{aligned}\mathbb{P}^H_{(n,\epsilon)}(R_v \geq k) &= \int_0^{1/2} \mathbb{P}^H_{(n,\epsilon)}(R_v \geq k \mid t_v = p)\, dp = \int_0^{1/2} \mathbb{P}_p(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k))\, dp \\ &= \int_0^{p_\delta(k)} \mathbb{P}_p(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k))\, dp + \int_{p_\delta(k)}^{1/2} \mathbb{P}_p(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k))\, dp\end{aligned}$$

For $p \in [0, p_\delta(k)]$ we have $L(p) \ll k$, so by section 2.3 we have that the first integral above decays exponentially fast as k gets large (and hence faster than any power of k). For the second integral it is

$$\begin{aligned}\int_{p_\delta(k)}^{1/2} \mathbb{P}_p(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k))\, dp &\leq \int_{p_\delta(k)}^{1/2} \mathbb{P}_{1/2}(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k))\, dp \qquad (4.1)\\ &= (\frac{1}{2} - p_\delta(k))\mathbb{P}_{1/2}(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k)). \qquad (4.2)\end{aligned}$$

Now using Theorem 2.13 we have that

$$k^{1-\delta} \asymp L(p_\delta(k)) \approx (\frac{1}{2} - p_\delta(k))^{-4/3}$$

and

$$\mathbb{P}_{1/2}(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k)) \approx k^{-1/3}.$$

Therefore, for any $\delta > 0$ equation (4.2) is of the form (w.r.t. $\approx$) $k^{-3/4(1-\delta)}k^{-1/3}$. For any fixed $\epsilon' > 0$, choosing δ small enough gives the desired result. □

Note that in the case $\delta = 0$, for $p \in [p_\delta(k), 1/2]$ it is $\mathbb{P}_p(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k)) \asymp \mathbb{P}_{1/2}(0 \overset{H}{\longleftrightarrow} \partial\Lambda(k))$ (see item 2 of Theorem 2.13). So, (4.1) can be replaced by an asymptotic equality leading to a lower bound of the form $C(\epsilon')k^{-13/12-\epsilon'}$ for any $\epsilon' > 0$ and some $C(\epsilon') > 0$.

For notational convenience we from now on fix $\epsilon' = 13/12 - 25/24$, in other words we take ϵ' such that $P(R_v \geq k)$ has an exponent of $-25/24$ with corresponding constant C_1. As the reader can check, this value is chosen somewhat arbitrarily and the calculations remain the same for other suitable ϵ'.

One important consequence of Lemma 4.2 and the stochastic domination of Lemma 4.1 is that there exists a box of order n around the origin such that with probability going to 1 nothing inside said box freezes before time $t = 1/2$. We formalise this in the following lemma.

Lemma 4.3. *When letting n go to infinity we have that*

$$\mathbb{P}_{(n,\epsilon)}(\textit{no vertex in } \Lambda(n - n^{47/48}) \textit{ freezes before time } t = 1/2) \to 1.$$

Proof. This lemma is easily proven by considering the coupling with the model with holes and by noting that for a vertex in $\Lambda(n - n^{47/48})$ to freeze there needs to be a hole of size larger than $n^{47/48}$ in the model with holes. This can be bounded by

$$\begin{aligned}\mathbb{P}^H_{(n,\epsilon)}\Big(\bigcup_{v\in\partial\Lambda(n)} R_v \geq n^{47/48}\Big) &\leq \sum_{v\in\partial\Lambda(n)} \mathbb{P}^H_{(n,\epsilon)}(R_v \geq n^{47/48}) \\ &\leq cn \cdot C_1 (n^{47/48})^{-25/24} \to 0.\end{aligned}$$

□

Denote by B_n the event that no vertex $\Lambda(n-n^{47/48})$ freezes before time $t = 1/2$ and let A_n be a sequence of events of which we want to prove a uniform lower bound. By Lemma 4.3 we only need to show the lower bound for the case where B_n holds. Namely, we have

$$\mathbb{P}_{(n,\epsilon)}(A_n) \geq \mathbb{P}_{(n,\epsilon)}(A_n \cap B_n) = \mathbb{P}_{(n,\epsilon)}(A_n \mid B_n)\mathbb{P}_{(n,\epsilon)}(B_n) \geq \mathbb{P}_{(n,\epsilon)}(A_n \mid B_n) - \delta(n),$$

for some function $\delta(n)$ with $\delta(n) \to 0$.

Note that both for Lemma 4.2 and Lemma 4.3 we did not need the assumption that trigger points appear with probability $n^{-\epsilon}$. In particular, these results still hold true with the original model in mind where *all* boundary points are trigger points.

To make the notation easier to read, for the remainder of the following sections we will work with the (n, ϵ) model and model with holes on the box

$$\overline{\Lambda(n)} := \Lambda(n) + (0, n) = [-n, n] \times [0, 2n]$$

instead. Furthermore, for simplicity's sake we now only consider holes centred at vertices in

$$V_{\text{hole}} := V \cap [-n, n] \times \{0\}, \tag{4.3}$$

i.e. the bottom side of $\overline{\Lambda(n)}$. In the applications of the upcoming lemmas we will use the proof of Lemma 4.3 to justify that the other holes are small enough to not influence the occurrence of our events. With slight abuse of notation for $v = (v_1, 0)$ and $j \in \mathbb{R}$, we write $v + j$ for the point $(v_1 + j, 0)$.

We now introduce some new notation that is also used in [9].

Definition 4.4. *We let*

$$\mathcal{H}(A(2^k, 2^m)) := \{\exists v \in V_{hole} : H_v \cap \partial\Lambda(2^k) \neq \emptyset, H_v \cap \partial\Lambda(2^m) \neq \emptyset\}$$

and

$$\begin{aligned}\overline{\mathcal{H}}(A(2^k, 2^m)) := \{\exists v \in V_{hole} : H_v \cap \partial\Lambda(2^k) \neq \emptyset, H_v \cap \partial\Lambda(2^m) \neq \emptyset, \\ H_v \cap \partial\Lambda(2^{k-1}) = \emptyset, H_v \cap \partial\Lambda(2^{m+1}) = \emptyset\}.\end{aligned}$$

If there exist k, m with $k < m$ such that the hole H_v centred at v satisfies

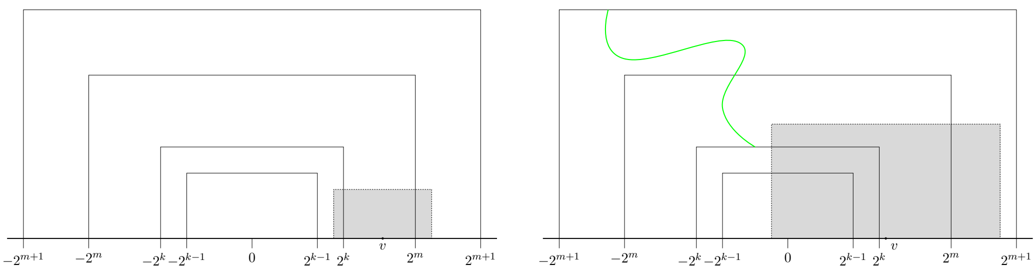
$$H_v \cap \partial\Lambda(2^k) \neq \emptyset \qquad \textit{and} \qquad H_v \cap \partial\Lambda(2^m) \neq \emptyset,$$

then we say the hole H_v is "big". In other words, the hole H_v implies that the event $\mathcal{H}(A(2^k, 2^m))$ holds true.

The second event introduced above states that there exists a big hole but this hole does not become "too large". Namely, there exists a hole such that the annulus $A(2^k, 2^m)$ is crossed by said hole, but any larger annulus of the form $A(2^{k'}, 2^{m'})$ that contains $A(2^k, 2^m)$ is not crossed by this hole. For technical reasons we also need a slightly larger event defined by

$$\begin{aligned}&\overline{\overline{\mathcal{H}}}(A(2^k, 2^m)) := \\ &\{\exists v \in V_{\text{hole}} : H_v \cap \partial\Lambda(2^k) \neq \emptyset, H_v \cap \partial\Lambda(2^m) \neq \emptyset, H_v \not\supseteq \Lambda(2^k), H_v \cap \partial\Lambda(2^{m+1}) = \emptyset\},\end{aligned}$$

i.e. the hole may intersect $\partial\Lambda(2^k)$, but it is not allowed to cover the whole of $\Lambda(2^k)$. If this event occurs, it is still possible for an occupied arm to leave $\Lambda(2^k)$. Furthermore, if we write $\overline{\mathcal{H}^*}(A(2^k, 2^m))$ or $\overline{\overline{\mathcal{H}^*}}(A(2^k, 2^m))$ we drop the assumption of $H_v \cap \partial\Lambda(2^{m+1}) = \emptyset$. We refer to figure 4.2 for a visualisation of some of the above mentioned events.



(A) $\overline{\mathcal{H}}(A(2^k, 2^m))$: the hole may **not** cross any more annuli.

(B) $\overline{\overline{\mathcal{H}}}(A(2^k, 2^m))$: the hole may cross more annuli but not cover the whole of $\Lambda(2^k)$.

FIGURE 4.2: Figure 4.2a depicts a situation where the hole centred at v ensures that the event $\overline{\mathcal{H}}(A(2^k, 2^m))$ holds true. Similarly for figure 4.2b, the hole H_v guarantees that $\overline{\overline{\mathcal{H}}}(A(2^k, 2^m))$ is true. For the latter event it is still possible for an open arm to leave $\Lambda(2^k)$.

Lemma 4.2 allows us to give some preliminary results for the values of the probabilities of the above defined events.

Lemma 4.5. *There exists a $C_2 > 0$ such that for all n we have*

$$\mathbb{P}^H_{(n,\epsilon)}(\exists \textit{ big hole}) \leq C_2 n^{-\epsilon}.$$

Proof. Namely, by invoking Lemma 4.2 we have

$$\mathbb{P}^H_{(n,\epsilon)}(\exists \text{ big hole}) \leq \mathbb{P}^H_{(n,\epsilon)}\Big(\bigcup_{w \in V_{\text{hole}}} H_w \neq \emptyset, R_w \geq ||w||/4\Big) \leq C_1 c n^{-\epsilon} \sum_{k=0}^{\infty} k^{-25/24},$$

whereby the sum clearly converges to some finite constant. □

Moreover, we can give a statement of the dependence on the events and a bound on the probabilities in terms of k and m.

Lemma 4.6. *Let k, m, k', m' be given such that $k < m$, $k' < m'$, $k < k'$ and $m < m'$, then*

i) the events $\overline{\mathcal{H}}(A(2^k, 2^m))$ and $\overline{\mathcal{H}}(A(2^{k'}, 2^{m'}))$ are independent,

ii) the events $\overline{\overline{\mathcal{H}}}(A(2^k, 2^m))$ and $\overline{\mathcal{H}}(A(2^{k'}, 2^{m'}))$ are independent,

iii) the events $\overline{\overline{\mathcal{H}}}(A(2^k, 2^m))$ and $\overline{\mathcal{H}^*}(A(2^{k'}, 2^{m'}))$ are independent,

iv) and there exists a $C_3 > 0$ such that the following inequalities hold

$$\mathbb{P}^H_{(n,\epsilon)}(\overline{\mathcal{H}}(A(2^k, 2^m))) \leq \mathbb{P}^H_{(n,\epsilon)}(\overline{\overline{\mathcal{H}}}(A(2^k, 2^m)))$$
$$\leq \mathbb{P}^H_{(n,\epsilon)}(\overline{\overline{\mathcal{H}^*}}(A(2^k, 2^m))) \leq C_3 n^{-\epsilon} \frac{2^k}{2^m}.$$

Proof. To prove item i) we will list the vertices at which holes can be centred such that $\overline{\mathcal{H}}(A(2^k, 2^m))$ occurs. W.l.o.g. we only consider vertices with a non-negative real part, i.e. only vertices on the "right side" of the origin. The other case follows by symmetry. Vertices v at which the hole may be centred at and the corresponding radii of the holes R_v must satisfy

$$\begin{aligned} ||v|| - R_v &> 2^{k-1}, \qquad \text{and} \qquad & ||v|| - R_v &\leq 2^k, \\ ||v|| + R_v &\geq 2^m, & ||v|| + R_v &< 2^{m+1}. \end{aligned}$$

Therefore, it necessarily is $||v|| \in (2^{m-1} + 2^{k-2}, 2^m + 2^{k-1})$. For $k < k'$ and $m < m'$ we have

$$2^{m'-1} + 2^{k'-2} \geq 2^m + 2^{k-1},$$

and hence the interval of vertices which can influence the occurence of these events are disjoint. For item ii) note that we must replace $||v|| - R_v > 2^{k-1}$ with $||v|| - R_v > -2^k$, however, it is easy to check that the corresponding intervals for $||v||$ still do not overlap. Moreover, for item iii) the inequality $||v|| + R_v < 2^{m+1}$ no longer holds true, but it is again clear that this does not change the fact that the corresponding intervals for $||v||$ do not overlap. We see that compared to $\overline{\mathcal{H}}$, the double bar event allows for more holes close to the origin, while the star event allows for more holes further away from it.

It clearly suffices to only show the last inequality of iv) and w.l.o.g. we again only consider holes with centres that have non-negative real part (considering all possible holes adds a factor of 2 to the final calculation). If we want the event $\overline{\overline{\mathcal{H}^*}}(A(2^k, 2^m))$ to occur, the following inequalities must necessarily hold:

$$||v|| - R_v > -2^k, \qquad ||v|| + R_v \geq 2^m, \qquad 2^k \geq ||v|| - R_v.$$

Hence,

$$||v|| \geq 2^{m-1} - 2^{k-1} \quad \text{and} \quad ||v|| - 2^k \leq R_v < ||v|| + 2^k.$$

This gives

$$\mathbb{P}^H_{(n,\epsilon)}(\overline{\overline{\mathcal{H}^*}}(A(2^k, 2^m))) \leq \sum_{||v||=2^{m-1}-2^{k-1}}^{\infty} \mathbb{P}^H_{(n,\epsilon)}(H_v \neq \emptyset, R_v \in [||v|| - 2^k, ||v|| + 2^k)).$$

Next we group together vertices that are exactly of distance $j \cdot 2^{k+1}$, $j \geq 0$ away from each other. This allows us to write

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\overline{\overline{\mathcal{H}^*}}(A(2^k, 2^m))) \\
&\leq \frac{1}{2} n^{-\epsilon} \sum_{||v||=2^{m-1}-2^{k-1}}^{2^{m-1}-2^{k-1}+2^{k+1}} \sum_{j=0}^{\infty} \mathbb{P}^H_{(n,\epsilon)}(R_{v+j\cdot 2^{k+1}} \in [||v+j\cdot 2^{k+1}|| - 2^k, ||v+j\cdot 2^{k+1}|| + 2^k)).
\end{aligned} \tag{4.4}$$

Furthermore, since the distribution of the radii of the holes are identically distributed we get

$$\begin{aligned}
&\sum_{j=0}^{\infty} \mathbb{P}^H_{(n,\epsilon)}(R_{v+j\cdot 2^{k+1}} \in [||v + j \cdot 2^{k+1}|| - 2^k, ||v + j \cdot 2^{k+1}|| + 2^k)) \\
&\qquad = \sum_{j=0}^{\infty} \mathbb{P}^H_{(n,\epsilon)}(R_v \in [||v|| + j \cdot 2^{k+1} - 2^k, ||v|| + j \cdot 2^{k+1} + 2^k)) \\
&\qquad\qquad \leq \mathbb{P}^H_{(n,\epsilon)}(R_v \geq ||v|| - 2^k) \leq \mathbb{P}^H_{(n,\epsilon)}(R_v \geq 2^{m-2}) \leq C_1 c (2^m)^{-25/24},
\end{aligned}$$

where it was used that $||v|| + (j+1) \cdot 2^{k+1} - 2^k \geq ||v|| + j \cdot 2^{k+1} + 2^k$. In the second last inequality we indirectly assumed that $2^{m-2} \leq 2^{m-1} - 2^{k-1} - 2^k$. If this is not the case, so $(m-k) \leq 3$, we can just chose a larger C_3 (for instance by multiplying it by 8) in the upcoming inequality. Now noting that the first sum in (4.4) contains 2^{k+1} terms, we get a bound of the form

$$\mathbb{P}^H_{(n,\epsilon)}(\overline{\overline{\mathcal{H}^*}}(A(2^k, 2^m))) \leq C_1 c n^{-\epsilon} 2^k (2^m)^{-25/24} \leq C_3 n^{-\epsilon} \frac{2^k}{2^m},$$

which completes the proof. □

4.1 Three-Arm Stability in the Model with Holes

The goal of this section is to show how a 3-arm stability result can be used in the model with holes. But before we proceed to the statement of the lemma, we need to introduce some new notation.

Definition 4.7. *Recall the definition of V_{hole} from* (4.3)*, we now define*

$$\mathcal{W}_3^H(A(2^i,2^j)) := \{\exists U \subseteq V_{hole} \ : \ \mathcal{A}_3^H(A(2^i,2^j)) \textit{ occurs with holes centred at } U\}.$$

So, if $\mathcal{W}_3^H(A(2^i,2^j))$ occurs we see a 3-arm event with some sub-collection of the holes. See figure 4.3 for an example of this event occurring. In particular, the probability of three arms occurring with all holes is obviously bounded by the probability of $\mathcal{W}_3^H(A(2^i,2^j))$. Using Theorem 2.13 we clearly have

$$\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i,2^j))) \geq C\left(\frac{2^i}{2^j}\right)^2,$$

for some universal constant C. Ideally, we would like to also give an upper bound of this form with an exponent of 2, but it is in fact enough to show a weaker result of 3-arm stability for Proposition 4.9 and Theorem 4.11.

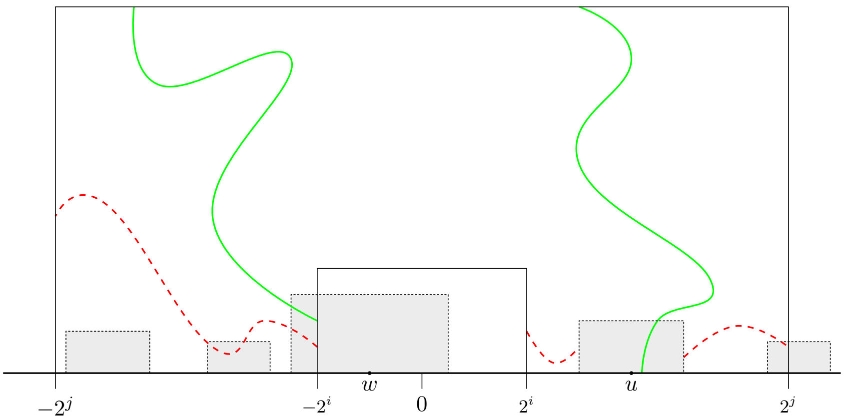


FIGURE 4.3: If we ignore the hole centred at w but take the hole centred at u into account, then a 3-arm event in $A(2^i,2^j)$ occurs. In other words, the set $U = \{u\} \subset V_{\text{hole}}$ satisfies the requirements of the definition of $\mathcal{W}_3^H(A(2^i,2^j))$.

Lemma 4.8. *There exists a $\hat{C} = \hat{C}(\epsilon) > 0$ such that*

$$\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i,2^j))) \leq \hat{C}\left(\frac{2^i}{2^j}\right)^{5/4}, \tag{4.5}$$

holds for all n and all $1 \leq i \leq j \leq \log n$.

Note that by choosing a bigger constant than $\hat{C}$ in (4.5) we can in fact extend the lemma in order to apply it to events of the form $\mathcal{W}_3^H(A(m_1,m_2)))$, where m_1, m_2 are not necessarily powers of 2. We will use this version in the applications of the result.

We delay the proof of the above lemma to section 4.2 so that we can first showcase how we will apply this result to crossing probabilities in the model with holes. The main application of Lemma 4.8 is the following proposition.

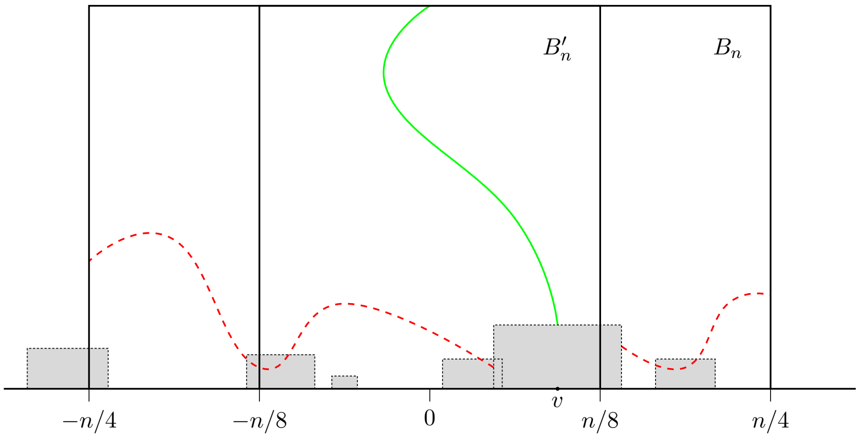


FIGURE 4.4: If $\mathcal{C}_n$ holds there must be some hole H_v that admits a 3-arm event. Note that this hole may be slightly outside B'_n, but only of a distance less than $n^{47/48}$, which means the arms stay of length order n.

Proposition 4.9. *Let B_n be the box $B_n = [-n/4, n/4] \times [1, n]$. There exists a $C_4 = C_4(\epsilon) > 0$ such that*

$$\mathbb{P}^H_{(n,\epsilon)}(\exists \textit{ a vertical open crossing in } B_n) \geq C_4$$

for all $n \in \mathbb{N}$.

Proof. Let B'_n be the box $B'_n = [-n/8, n/8] \times [1, n]$ and let $\mathcal{C}_n$ be the event that there exists a vertical open crossing in B'_n in the ordinary percolation model but not in the box B_n in the model with holes. We claim that $\mathbb{P}^H_{(n,\epsilon)}(\mathcal{C}_n) \leq \gamma(n)$ with $\gamma(n) \to 0$ as n tends to infinity. This would imply that

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\exists \text{ a vertical open crossing in } B_n) \\
&\quad \geq \mathbb{P}^H_{(n,\epsilon)}(\exists \text{ ver. o. crossing in } B_n, \exists \text{ ver. o. crossing in } B'_n \text{ without holes}) \\
&\quad = \mathbb{P}_{1/2}(\exists \text{ ver. o. crossing in } B'_n) - \mathbb{P}^H_{(n,\epsilon)}(\mathcal{C}_n) \geq C - \gamma(n),
\end{aligned}$$

for some universal $C > 0$ (coming from RSW theory). The result would then hold for large enough n and for smaller n we can decrease the constant C_4.

We proceed to prove the above claim, so assume that $\mathcal{C}_n$ holds and therefore there is some open vertical crossing π of B'_n. We follow the path π starting from $\mathbb{R} \times \{n\}$ until π intersects a hole centred at some v. The path π provides an open arm from ∂H_v to $\mathbb{R} \times \{n\}$. Since furthermore we do not see a crossing in B_n in the model with holes, the hole H_v must also supply two closed arms that separate π from $\mathbb{R} \times \{1\} \cap B_n$. We denote this 3-arm event for v by $Piv(v)$ and demonstrate one such possibility in figure 4.4.

By the proof of Lemma 4.3 we will assume that all holes are of size smaller than $n^{47/48}$. In particular, for large n the vertex v (as well as ∂H_v) can not be more than distance $n/32$ away from B'_n. Therefore, the 3-arms described above must all be of a length of least $n/16$. The exact choice of fractions here does not matter as long as the length of the arms is of order n.

With the help of Lemma 4.8 we now bound the probability of a vertex v satisfying $Piv(v)$. We have

$$\begin{aligned}
\mathbb{P}^H_{(n,\epsilon)}(Piv(v)) &\leq \sum_{r=1}^{n^{47/48}} \mathbb{P}^H_{(n,\epsilon)}(H_v \neq \emptyset, R_v = r)\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(r, n/16))) \\
&\leq c_1 n^{-\epsilon} \sum_{r=1}^{47/48 \log n} \mathbb{P}^H_{(n,\epsilon)}(R_v \geq 2^r)\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^{r+1}, n/16))) \\
&\leq C_1 \hat{C} c_2 n^{-\epsilon} \sum_{r=1}^{47/48 \log n} (2^r)^{-25/24} \left(\frac{2^r}{n}\right)^{5/4} \qquad (4.6) \\
&\leq C_1 \hat{C} c_2 n^{-5/4} \sum_{r=1}^{47/48 \log n} (2^{5/24})^r \\
&\leq C_1 \hat{C} C_5 n^{-5/4} \cdot (2^{5/24})^{47/48 \log n} \\
&= C_1 \hat{C} C_5 n^{-5/4+235/1152} = C_6 n^{-1205/1152} = o(n^{-1}),
\end{aligned}$$

where (4.6) used Lemma 4.2 and Lemma 4.8. Note that the constants C_5 and C_6 are independent of the choice of n. Finally, this gives

$$\mathbb{P}^H_{(n,\epsilon)}(\mathcal{C}_n) = \mathbb{P}^H_{(n,\epsilon)}(\bigcup_{||v|| \leq n/8+n/32} Piv(v)) \leq \sum_{||v|| \leq 5n/32} Piv(v) \leq C_6 n \cdot n^{-1205/1152} \to 0.$$

□

4.2 Proof of Lemma 4.8

We now proceed to show the proof of Lemma 4.8 using similar techniques as in section 4.2 of [9]. Since this section is quite technical, the reader may be advised to skip this section during the first examination of this thesis.

Proof of Lemma 4.8. We use induction over j and $(j - i)$, where in the induction step we split the expression into different terms that we analyse separately. First, we take $\hat{C}$ so large that that the right hand side of (4.5) is larger than 1 for all pairs i, j with $j - i \leq 7$. At the end of the proof we will increase the value of $\hat{C}$ to complete the induction step. Now let $1 \leq i \leq j \leq \log n$ be given and consider the following induction hypothesis:

Induction Hypothesis: The inequality (4.5) holds for all pairs (i', j') with $j' < j$ or $j' = j$ and $i' > i$.

We will show that for a suitable choice of $\hat{C}$ (independent of i and j) the inequality (4.5) also holds for the pair (i, j).

Denote by $\mathcal{D} = \mathcal{A}_3^H(A(2^{i+3}, 2^{j-3}))^c$, i.e. the complement of the event of seeing 3-arms in ordinary percolation. Then we obviously get

$$\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j))) \leq \mathbb{P}_{1/2}(\mathcal{D}^c) + \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j)) \cap \mathcal{D}). \tag{4.7}$$

By item 4 of Theorem 2.13 the term $\mathbb{P}_{1/2}(\mathcal{D}^c)$ already has a bound of the form

$$\mathbb{P}_{1/2}(\mathcal{D}^c) \leq C_7 \left(\frac{2^i}{2^j}\right)^2 \leq C_7 \left(\frac{2^i}{2^j}\right)^{5/4}, \tag{4.8}$$

where C_7 is a universal constant. We will now focus on the second term in (4.7) and further split it into sub-terms.

Let $M = M(\epsilon) \in \mathbb{N}$ be chosen such that $\epsilon M > 5/4$. Now consider the following three events:

$$\begin{aligned}
\mathcal{E}_1 &:= \{\text{there are no big holes in } A(2^i, 2^j)\}, \\
\mathcal{E}_2 &:= \{\text{there are between 1 and } M \text{ big holes in } A(2^i, 2^j)\}, \\
\mathcal{E}_3 &:= \{\text{there are more than } M \text{ big holes in } A(2^i, 2^j)\}.
\end{aligned}$$

Then it is

$$\begin{aligned}
\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j)) \cap \mathcal{D}) \leq\ & \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j)) \cap \mathcal{D} \cap \mathcal{E}_1) \\
& + \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j)) \cap \mathcal{E}_2) + \mathbb{P}^H_{(n,\epsilon)}(\mathcal{E}_3) \\
=:\ & (\text{Term 1}) + (\text{Term 2}) + (\text{Term 3})
\end{aligned} \tag{4.9}$$

We now treat each of these terms separately and show that they all can be bounded above by an expression of the form $c(2^i/2^j)^{5/4}$ for any $c > 0$ given that n is large enough.

Term 1: Assuming that all the events in term 1 hold, there must be at least one open path π crossing $A(2^{i+3}, 2^{j-3})$ and the open cluster of any such path must intersect $\mathbb{R} \times \{0\}$ in ordinary percolation. If this were not the case, then two closed arms would separate the open crossing from $\mathbb{R} \times \{0\}$ and therefore $A(2^{i+3}, 2^{j-3})$ would admit a 3-arm event in ordinary percolation.

However, in the model with holes the 3-arm event occurs, and hence the open cluster of π must intersect a hole H_v (which exact hole to choose does not matter) centred at a vertex v with $v \in [-2^{k+1}, -2^k] \times \{0\} \cup [2^k, 2^{k+1}] \times \{0\}$ for some $i+3 \leq k \leq j-4$. This means we must see the following three local 3-arm events: $\mathcal{W}_3^H(\Lambda(v, R_v), \Lambda(v, 2^{k-1})))$, $\mathcal{W}_3^H(A(2^i, 2^{k-2}))$, $\mathcal{W}_3^H(A(2^{k+3}, 2^j))$. See also figure 4.5. The event $\mathcal{E}_1$ ensures that the holes in each of the above 3-arm events are not large enough to influence the other arm events, meaning there is independence between these events. Summing over all possible

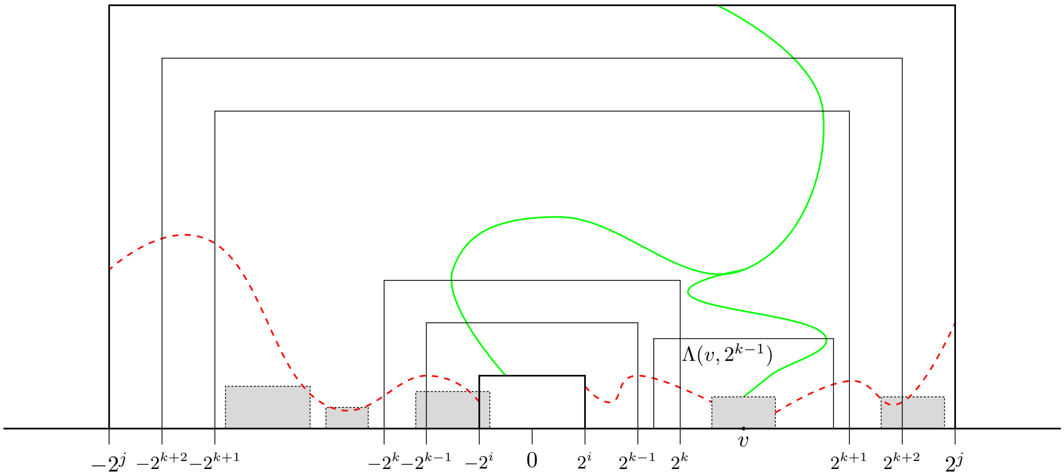


FIGURE 4.5: For term 1 we observe that the cluster of the open arm in $A(2^i, 2^j)$ must intersect some hole H_v. Three (independent) local 3-arm events, for which we can apply the induction hypothesis to, must then occur.

such v and the corresponding radii distribution gives us

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^i,2^j)) \cap \mathcal{D} \cap \mathcal{E}_1) \\
&\quad\le \sum_{v \in V_{\text{hole}} \cap A(2^{i+3},2^{j-3})} \sum_{r=0}^{2^{k-1}} \mathbb{P}^H_{(n,\epsilon)}(H_v \neq \emptyset, R_v = r)\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(\Lambda(v,r),\Lambda(v,2^{k-1}))) \\
&\qquad \cdot \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^i,2^{k-2})))\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^{k+3},2^j))) \\
&\quad\le \sum_{k=i+3}^{j-4} 2^{k+1} \sum_{r=0}^{2^{k-1}} \mathbb{P}^H_{(n,\epsilon)}(H_0 \neq \emptyset, R_0 = r)\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(\Lambda(r),\Lambda(2^{k-1}))) \\
&\qquad \cdot \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^i,2^{k-2})))\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^{k+3},2^j))) \\
&\quad\le \hat{C}^3 C_8 n^{-\epsilon}\left(\frac{2^i}{2^j}\right)^{5/4} \sum_{k=i+3}^{j-4} 2^{k+1} \sum_{r=0}^{k-1} \mathbb{P}^H_{(n,\epsilon)}(R_0 \ge 2^r)\left(\frac{2^r}{2^k}\right)^{5/4}, \qquad (4.10)
\end{aligned}$$

whereby we used that H_v and H_0 are identically distributed and the induction hypothesis in the last inequality. Next, note that

$$\begin{aligned}
\sum_{k=i+3}^{j-4} 2^{k+1} \sum_{r=0}^{k-1} \mathbb{P}^H_{(n,\epsilon)}(R_0 \ge 2^r)\left(\frac{2^r}{2^k}\right)^{-5/4} &\le C_1 \sum_{k=0}^{\infty} (2^k)^{-1/4} \sum_{r=0}^{k-1} (2^r)^{-25/24}(2^r)^{5/4} \\
&\le C_9 \sum_{k=0}^{\infty} 2^{-k/4} 2^{5k/24} \le C_{10},
\end{aligned}$$

where the constants C_9, C_{10} are independent of the choice of i and j. Using this together with (4.10) gives

$$\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}^H_3(A(2^i, 2^j)) \cap \mathcal{D} \cap \mathcal{E}_1) \le C_{10}\hat{C}^3 n^{-\epsilon} \left(\frac{2^i}{2^j}\right)^{5/4}. \tag{4.11}$$

Term 2: As in [9] we first list the sub-annuli that are crossed, and later on group the big holes together. Assume that there are $1 \le b \le M$ big holes in $A(2^i, 2^j)$. Then there exists a $1 \le q \le b$ such that there are integers $i \le k_1 < k_2 < \ldots < k_q < j$ and $i < m_1 < m_2 < \ldots < m_q \le j$ with $k_l < m_l$ such that the following events hold:

- For all $1 \le l \le q$ the event $\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))$ holds, unless one of the following edge cases occur:
  - $l = 1$ and $k_l = i$, then we replace $\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))$ by $\overline{\overline{\mathcal{H}}}(A(2^{k_l}, 2^{m_l}))$,
  - $l = q$ and $m_l = j$, then we replace $\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))$ by $\overline{\mathcal{H}^*}(A(2^{k_l}, 2^{m_l}))$,
  - $l = 1 = q$, $k_l = i$ and $m_l = j$, then we replace $\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))$ by $\overline{\overline{\mathcal{H}^*}}(A(2^{k_l}, 2^{m_l}))$,
- No other sub-annuli are crossed: for all $1 \le h \le j-1$ such that $[h, h+1] \not\subseteq \bigcup_{1\le l\le q}[k_l, m_l]$ the event $\mathcal{H}(A(2^h, 2^{h+1}))$ does not occur.

We denote this event by $\overline{\mathcal{E}}_2(q, k_1, \ldots, k_q, m_1, \ldots m_q)$. If $\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))$ (or any of the edge cases) occurs we say that the interval $[k_l, m_l]$ is crossed by a hole.

Let successive crossed intervals $[k_{\underline{l}}, m_{\underline{l}}], [k_{\underline{l}+1}, m_{\underline{l}+1}], \ldots [k_{\bar{l}}, m_{\bar{l}}]$ be given such that $m_l \ge k_{l+1}$ (for $\underline{l} \le l \le \bar{l} - 1$), then we say these intervals "overlap" and use the notation $[\![k_{\underline{l}}, m_{\bar{l}}]\!]$ to label these intervals together. Assume the event $\overline{\mathcal{E}}_2(q, k_1, \ldots, k_q, m_1, \ldots m_q)$ occurs, we now give a bound on the probability of the interval $[\![k_{\underline{l}}, m_{\bar{l}}]\!]$ (with corresponding overlapping intervals $[k_l, m_l]$) to be crossed by holes. By abuse of notation we write the event $\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))$ regardless of the edge case the particular l may be in. By Lemma 4.6 the events needed for $[\![k_{\underline{l}}, m_{\bar{l}}]\!]$ to be crossed by holes are mutually (and jointly) independent and can all be bounded by the same relevant term (regardless of the edge case). This gives us

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\text{the interval } [\![k_{\underline{l}}, m_{\bar{l}}]\!] \text{ is crossed by holes}) \\
&\qquad = \mathbb{P}^H_{(n,\epsilon)}\Big(\bigcap_{l=\underline{l}}^{\bar{l}} \overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))\Big) = \prod_{l=\underline{l}}^{\bar{l}} \mathbb{P}^H_{(n,\epsilon)}(\overline{\mathcal{H}}(A(2^{k_l}, 2^{m_l}))) \\
&\qquad\qquad \le (C_3 n^{-\epsilon})^{\bar{l}-\underline{l}+1} \prod_{l=\underline{l}}^{\bar{l}} \frac{2^{k_l}}{2^{m_l}} \le (C_3 n^{-\epsilon})^{\bar{l}-\underline{l}+1} \frac{2^{k_{\underline{l}}}}{2^{m_{\bar{l}}}},
\end{aligned}$$

where the last inequality used the assumption that $m_l \ge k_{l+1}$.

Next, recall that the event $\mathcal{W}^H_3(A(2^i, 2^j))$ also occurs. Then in addition to the overlapping holes crossing $[\![k_{\underline{l}}, m_{\bar{l}}]\!]$ we must still see an open 1-arm event from $\partial\Lambda(2^{\underline{l}})$ to $\partial\Lambda(2^{\bar{l}})$ in the model with holes, which we denote by $\mathcal{W}^H_o(A(2^{\underline{l}}, 2^{\bar{l}}))$. If the 1-arm event occurs in the model with holes, we must obviously also see an open 1-arm event $\mathcal{A}_o^H(A(2^{\underline{l}}, 2^{\bar{l}}))$ in the underlying ordinary percolation model without the holes. Therefore,

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_o^H(A(2^{\underline{l}}, 2^{\bar{l}})) \cap \text{the interval } [\![k_{\underline{l}}, m_{\bar{l}}]\!] \text{ is crossed by holes}) \\
&\quad \leq \mathbb{P}^H_{(n,\epsilon)}(\mathcal{A}_o^H(A(2^{\underline{l}}, 2^{\bar{l}})) \text{ without holes} \cap \text{the interval } [\![k_{\underline{l}}, m_{\bar{l}}]\!] \text{ is crossed by holes}) \\
&\qquad \leq c(\delta)\left(\frac{2^{k_{\underline{l}}}}{2^{m_{\bar{l}}}}\right)^{1/3-\delta} C_3 n^{-\epsilon}\frac{2^{k_{\underline{l}}}}{2^{m_{\bar{l}}}} \leq C_{11} n^{-\epsilon}\left(\frac{2^{k_{\underline{l}}}}{2^{m_{\bar{l}}}}\right)^{5/4}, \qquad (4.12)
\end{aligned}$$

where we used the 1-arm exponent (item 3 with $j = 1$ from Theorem 2.13) for some small enough $\delta > 0$ and independence between holes and crossing without holes.

We now group the intervals into blocks of the form $[\![\tilde{k}_l, \tilde{m}_l]\!]$ such that for any λ with $k_\lambda \leq \tilde{m}_l$ and $\tilde{k}_l \leq m_\lambda$ we have that the interval $[k_\lambda, m_\lambda]$ is already grouped into $[\![\tilde{k}_l, \tilde{m}_l]\!]$. In other words, we group the intervals into maximal blocks of overlapping holes. Let $[\![\tilde{k}, \tilde{m}]\!]$ be the first of such overlapping intervals. We must now see 3-arm events $\mathcal{W}_3^H(A(2^i, 2^{\tilde{k}-2}))$ and $\mathcal{W}_3^H(A(2^{\tilde{m}+2}, 2^j))$, whereby for notational sake we let $\mathcal{W}_3^H(A(2^k, 2^m)) = \Omega$ if $k \geq m$. See also figure 4.6 for a visualisation. Now noting that all of these events depend on disjoint vertices and holes (all other holes are not big enough) we get

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j)) \cap \overline{\mathcal{E}}_2(q, k_1, \dots, k_q, m_1, \dots m_q)) \\
&\quad \leq \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_o^H(A(2^{\tilde{k}}, 2^{\tilde{m}})) \cap \text{the interval } [\![\tilde{k}_l, \tilde{m}_l]\!] \text{ is crossed by holes}) \\
&\qquad \cdot \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^{\tilde{k}-2})))\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^{\tilde{m}+2}, 2^j))).
\end{aligned}$$

Using (4.12) and the induction assumption two times, the above term is bounded by

$$C_{11}\hat{C}^2 n^{-\epsilon}\left(\frac{2^i}{2^{\tilde{k}-2}}\right)^{5/4}\left(\frac{2^{\tilde{k}}}{2^{\tilde{m}}}\right)^{5/4}\left(\frac{2^{\tilde{m}+2}}{2^j}\right)^{5/4} \leq C_{12}\hat{C}^2 n^{-\epsilon}\left(\frac{2^i}{2^j}\right)^{5/4}. \qquad (4.13)$$

We can now finally bound term 2 by summing over all possibilities of $\overline{\mathcal{E}}_2(q, k_1, \dots, k_q, m_1, \dots m_q)$. We sum over the $1 \leq b \leq M$ number of big holes to get

$$\begin{aligned}
(\text{Term 2}) &\leq \sum_{b=1}^{M}\sum_{k_i, m_i} \mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j)) \cap \overline{\mathcal{E}}_2(q, k_1, \dots, k_q, m_1, \dots m_q)) \\
&\leq n^{-\epsilon} M (\log n)^{2M} C_{12}\hat{C}^2\left(\frac{2^i}{2^j}\right)^{5/4}, \qquad (4.14)
\end{aligned}$$

where it was used that $j - i \leq \log n$ and hence there are at most $(\log n)^{2M}$ choices for the k_i's and m_i's.

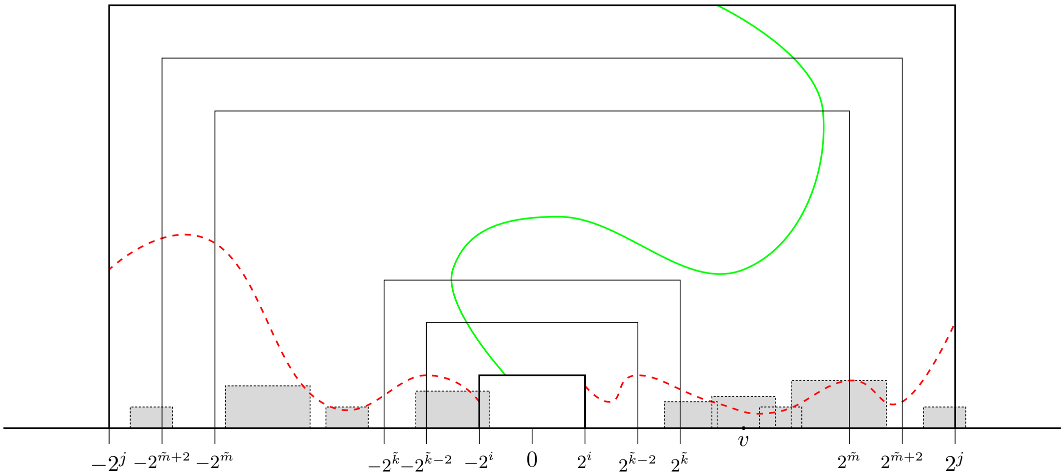


FIGURE 4.6: The interval $[\![2^{\tilde k}, 2^{\tilde m}]\!]$ is crossed by big holes, and therefore the annulus $A(2^{\tilde k}, 2^{\tilde m})$ automatically admits a closed arm. However, we still observe an open arm in $A(2^{\tilde k}, 2^{\tilde m})$, a 3-arm event from $\partial\Lambda(2^i)$ to the overlapping interval and a 3-arm event from the interval to $\partial\Lambda(2^j)$.

Term 3: By Lemma 4.5 we have

$$\mathbb{P}^H_{(n,\epsilon)}(\exists \text{ more than } M \text{ big holes in } A(2^i, 2^j)) \le \mathbb{P}^H_{(n,\epsilon)}(\exists \text{ big hole})^M \le (C_2 n^{-\epsilon})^M. \tag{4.15}$$

By our choice of M and the fact that $2^j \le n$ we have $\mathbb{P}^H_{(n,\epsilon)}(\mathcal{E}_3) \le C_2^M n^{-\delta_3}(2^i/2^j)^{5/4}$ for $\delta_3 = \epsilon M - 5/4 > 0$.

Completion of Induction Step

Taking the inequalities (4.7) and (4.9) into account and using the bounds on each term respectively given by (4.8), (4.11), (4.14) and the line below (4.15) gives

$$\begin{aligned}
&\mathbb{P}^H_{(n,\epsilon)}(\mathcal{W}_3^H(A(2^i, 2^j))) \\
&\qquad \le C_7 \left(\frac{2^i}{2^j}\right)^{5/4} + C_{10}\hat{C}^3 n^{-\epsilon}\left(\frac{2^i}{2^j}\right)^{5/4} \\
&\qquad\quad + n^{-\epsilon} M(\log n)^{2M} C_{12}\hat{C}^2 \left(\frac{2^i}{2^j}\right)^{5/4} + C_2^M n^{-\delta_3}\left(\frac{2^i}{2^j}\right)^{5/4}. \\
&\qquad = \left(C_7 + C_{10}\hat{C}^3 n^{-\epsilon} + n^{-\epsilon} M(\log n)^{2M} C_{12}\hat{C}^2 + C_2^M n^{-\delta_3}\right)\left(\frac{2^i}{2^j}\right)^{5/4}
\end{aligned} \tag{4.16}$$

We now specify our choice of $\hat{C}$ further by choosing $\hat{C}$ so large such that $C_7 < \hat{C}/2$. Furthermore, take n_0 so large such that for all $n \ge n_0$ we have

$$C_7 + C_{10}\hat{C}^3 n^{-\epsilon} + n^{-\epsilon} M(\log n)^{2M} C_{12}\hat{C}^2 + C_2^M n^{-\delta_3} \le \hat{C}.$$

By (4.16) this completes the induction step for all $n \geq n_0$. For all smaller n we can again increase (if needed) $\hat{C}$ such that $\hat{C} \geq n_0^{5/4}$ and hence the r.h.s. of (4.5) is larger than 1 for all $1 \leq i \leq j \leq \log n_0$. □

Remark 4.10. *Many constants in the previous statements are chosen somewhat arbitrarily and the results still hold for different choices. For example, to prove Proposition 4.9 it is enough to show a version of Lemma 4.8 for any exponent between* 1 *and* 4/3. *There is in fact hope that it is possible to prove Lemma 4.8 with any exponent smaller than* 2, *which we recall is the exponent of the three-arm event in ordinary percolation. Moreover, if we are also able to prove Lemma 4.8 for the original boundary frozen percolation model, then the proofs of Proposition 4.9 and Theorem 4.11 in the next section follow analogously.*

4.3 Lower Bound for the Origin in the (n, ϵ) Model not to Freeze

The previous sections allow us to give a lower bound on the probability that the origin does not freeze in the (n, ϵ) model. We now again consider the (n, ϵ) model with the usual box $\Lambda(n) = [-n, n] \times [-n, n]$.

Theorem 4.11. *There exists a* $C(\epsilon) > 0$ *such that for all* $n \in \mathbb{N}$ *we have*

$$\mathbb{P}_{(n,\epsilon)}(0 \textit{ is not frozen at time } t = 1) \geq C(\epsilon) n^{-\epsilon}.$$

Proof. As in previous proofs we will only show the result for n large enough, which can be justified by taking a smaller $C(\epsilon)$. Denote by B_n the box $B_n = [-n/4, n/4] \times [-n + 1, -n/8]$ and let $\mathcal{E}_3(v)$ be the event that v is connected to $[-n/4, n/4] \times \{-n/8\}$ by an open non-frozen 1/2-path using only vertices in B_n. Assume that the following three events occur:

i) $\tau_0 > 3/4$;

ii) there exists a non-frozen 1/2-open circuit in $A(n/4, n/2)$;

iii) there exists a $v \in [-n/4, n/4] \times \{-n + 1\}$ such that $\mathcal{E}_3(v)$ occurs and the right most neighbour $N(v)$ in $\partial\Lambda$ of v has $T_{N(v)} = 1$ and $\tau_{N(v)} \in (1/2, 3/4)$.

Then there exists a $\tau < 3/4$ such that after time τ there exists a frozen circuit around 0. Since by that time 0 is not opened, we have that 0 can never be connected to said circuit and therefore 0 does not reach a trigger point. It remains to show that the probability of all these events happening is bounded by $C(\epsilon)n^{-\epsilon}$.

Note that the last two events are completely determined by the $(\tau_v)_{v \in \Lambda(n/8)^c}$ and $(T_w)_{w \in \partial\Lambda(n)}$ values. In particular, changing the value of τ_0 such that $\tau_0 > 3/4$ has no influence on the occurrence of events ii) and iii).

Let $\mathcal{E}_2$ be the event in ii) and denote by $\mathcal{E}_3(v, N(v))$ the third event for some specific v and right most neighbour $N(v)$ of v. We further define $\mathcal{E}_3^R(v)$ to be the sub-event of $\mathcal{E}_3(v)$ such that v is the right most vertex (in $[-n/4, n/4] \times \{-n + 1\}$) satisfying $\mathcal{E}_3(v)$.

We similarly let $\mathcal{E}_3^R(v, N(v))$ be the sub-event of $\mathcal{E}_3(v, N(v))$ where we replace $\mathcal{E}_3(v)$ with $\mathcal{E}_3^R(v)$. We claim that

$$\mathbb{P}_{(n,\epsilon)}(\mathcal{E}_2, \mathcal{E}_3^R(v, N(v))) \geq \mathbb{P}_{(n,\epsilon)}(\mathcal{E}_2, \mathcal{E}_3^R(v)) \cdot \mathbb{P}_{(n,\epsilon)}(T_{N(v)} = 1, \tau_{N(v)} \in (1/2, 3/4)). \tag{4.17}$$

Namely, assume that for given $(\tau_w)_{w\in\Lambda(n)}$ and $(T_w)_{w\in\partial\Lambda(n)}$ values the events $\mathcal{E}_2$ and $\mathcal{E}_3^R(v)$ occur, then it must either be $T_{N(v)} = 0$ or $T_{N(v)} = 1$ and $\tau_{N(v)} > 1/2$ (otherwise v would have been frozen). In the case of $T_{N(v)} = 0$, if we change $T_{N(v)}$ to be equal to 1 and have a $\tau_{N(v)}$ value in $(1/2, 3/4)$, then it is easy to see that both $\mathcal{E}_2$ and $\mathcal{E}_3^R(v)$ must still hold. From this we can follow (4.17).

Using that the events $\mathcal{E}_3^R(v, N(v))$ are disjoint (if there is a vertex v satisfying $\mathcal{E}_3(v)$, then there is a unique right most one) gives

$$\begin{aligned}\mathbb{P}_{(n,\epsilon)}(0 \text{ is not frozen}) &\geq \mathbb{P}_{(n,\epsilon)}(\tau_0 > 3/4)\mathbb{P}_{(n,\epsilon)}\big(\bigcup_{v\in[-n/4,n/4]\times\{-n+1\}} \mathcal{E}_2 \cap \mathcal{E}_3^R(v, N(v))\big)\\ &= \frac{1}{4}\sum_{v\in[-n/4,n/4]\times\{-n+1\}} \mathbb{P}_{(n,\epsilon)}(\mathcal{E}_2, \mathcal{E}_3^R(v, N(v)))\\ &\geq \frac{1}{16}n^{-\epsilon}\cdot\mathbb{P}_{(n,\epsilon)}\big(\bigcup_{v\in[-n/4,n/4]\times\{-n+1\}} \mathcal{E}_2 \cap \mathcal{E}_3(v)\big),\end{aligned} \tag{4.18}$$

where the last inequality used (4.17) and obvious bounds on probabilities of the random variables $T_{N(v)}$ and $\tau_{N(v)}$.

Next, denote by $\overline{\mathcal{E}_2}$ and $\overline{\mathcal{E}_3}(v)$ the analogues of the events ii) and $\mathcal{E}_3(v)$ in the model with holes, i.e. we replace non-frozen with "not in holes" in the respective events. By an application of a version of the FKG inequality (see Remark 3.1. in [9]) we may conclude positive correlation between the events $\overline{\mathcal{E}_2}$ and $\overline{\mathcal{E}_3}(v)$. Additionally, by the proof of Lemma 4.3 with probability going to 1 no holes intersect $\Lambda(n/2)$. So by RSW,

$$\begin{aligned}&\mathbb{P}^H_{(n,\epsilon)}\big(\bigcup_{v\in[-n/4,n/4]\times\{-n+1\}} \overline{\mathcal{E}_2} \cap \overline{\mathcal{E}_3}(v)\big)\\ &\qquad\geq C_{13}(1-\delta(n))\cdot\mathbb{P}^H_{(n,\epsilon)}\big(\bigcup_{v\in[-n/4,n/4]\times\{-n+1\}} \overline{\mathcal{E}_3}(v)\big)\end{aligned}$$

for some universal C_{13} and a function $\delta(n)$ with $\delta(n) \to 0$. Note that the last event in the above inequality is the same event (on a translated box) as in Proposition 4.9. Therefore, the above can be bounded below by $C_4(\epsilon)C_{13}(1-\delta(n))$. Using the stochastic domination of Lemma 4.1 and inserting this bound in (4.18) completes the proof. □

Since Theorem 4.11 gives a lower bound for any $\epsilon > 0$, one might be tempted to let ϵ go to 0 to get a uniform lower bound for the original model. However, we have to note the dependence of the constant on ϵ and hence it is possible that $C(\epsilon)n^{-\epsilon}$ goes to 0 as $\epsilon \to 0$.

5 Discussion

In this thesis, we have provided various different tools that allow for an analysis of the boundary frozen percolation process. There are two major approaches that could be interesting to further analyse. Either one continues from Proposition 3.5 and shows that (3.4) indeed also implies that at time 1 the origin does not freeze, or one continues using the models with holes. More precisely, if one is able to show that Lemma 4.8 holds with $\epsilon = 0$ for some exponent larger than one, then we can follow the proof of Theorem 4.11 to give a proof of Conjecture 3.1.

All of the results in chapter 3 can be easily extended to other lattices and bond percolation as well, as long as the lattices satisfy RSW-type bounds as in section 2.2. For the calculations in chapter 4 one has to be more careful. First of all, even though it is expected, no rigorous proof has been given on the critical exponents on other lattices. Furthermore, the results heavily depend on the fact that the relations between exponents satisfy conditions that allow for these calculations to be true. For instance, to show a result like Lemma 4.2 with an exponent strictly larger than 1, it is necessary that the sum of the one arm exponent and of the exponent of the characteristic length scaling is larger than 1. It is not clear that this also holds for other lattices, and if this is not the case, one can not follow the results as described in this thesis.

Further Questions

The first question that arises from chapter 4 is: can we apply results from the (n, ϵ)-model to the original process (with necessary modifications of course)? In particular, can Theorem 4.11 be helpful to prove that for boundary frozen percolation we also have a similar uniform lower bound for any $\epsilon > 0$?

Moreover, we have shown many points of evidence that hint at Conjecture 3.1 to be true, however, it still remains open to give a rigorous proof for the conjecture. Assuming the conjecture is true, the next natural question to ask would be: does the probability of the origin not freezing go to 1, or equivalently, does the probability of the origin freezing go to 0 as $n \to \infty$. As the reader has seen, for both the diameter frozen and volume frozen percolation models the above probability does indeed go to 0, so it is reasonable to expect that this model also shares this property. One major difference to keep in mind between these models is that our new model is only defined on a finite subgraph of the whole lattice, while the other models use the whole plane process. It is not impossible that this, similarly to the exceptional scales in volume frozen percolation, leads to dramatically different behaviour from the other frozen percolation models.

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