Systems with Carrollian symmetry

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Abstract

The Carrollian limit can be obtained by starting from a relativistic system and collapsing the lightcone. This can be contrasted with the Galilean limit, which can be seen as opening up the lightcone. The Galilean limit results in a system with Galilean symmetry. Similarly, the Carrollian limit results in a system with Carrollian symmetry. In this thesis we clarify how to arrive at Carrollian systems. In order to do this we contrast two different approaches. The first approach consists of an explicit coordinate transformation. This is akin to closing up the light cone in a relativistic system. This leads to interesting interpretations such as the Carrollian particles becoming static or tachyonic. By using the first approach we arrive at a set of particles that could potentially describe physics in a hyper-relativistic regime, such as near the event horizon of a black hole. Another approach involves an embedding into Bargmann space. By embedding a Carrollian structure into a manifold exhibiting Bargmann symmetry we find a more general class of Carrollian systems, not all of which are given by a taking the limit on a relativistic system.

Throughout this document, we use (-,+,+,+) as our sign convention. Furthermore, we need to distinguish between Galilean, Lorentzian and Carrollian quantities. Where confusion may arise, we will use the subscripts G,L and C respectively. We also remind the reader that for any four-vector v^μ denotes that said vector transforms contravariantly, while v_μ denotes a covariant transformation law. Some parts of the thesis are a review of other papers, this will be indicated by a citation at the beginning of these parts.

Chapter 1

Introduction

There has been interest in systems with a novel kind of symmetry called a Carroll symmetry [1, 2, 3]. In this thesis we will focus on systems with such a symmetry. After providing some context regarding Carroll symmetry we start our inquiry in chapter 2 by showing how systems with Carrollian symmetry may arise by taking a limit on a relativistic system. This results in a few different particles.
In Chapter 3 we delve deeper into the mathematics underlying Carroll symmetry. Here we will encounter a way of arriving at Carrollian systems that involves an embedding of Carrollian spacetime into Bargmann spacetime [6].
In Chapter 4 we compare the two methods to obtain a new perspective on the particles we have found in Chapter 2.

1.1 Boost symmetries

In general, a boost symmetry is a symmetry where time and space can transform into each other. It can usually be thought of as a coordinate transformation that describes a change in the velocity of the observer. A boost symmetry is often demanded from a system. One demands that the system of equations describing the system stays the same under the relevant boost. One might want to consider different boost symmetries depending on the system one wants to describe. The reader is most likely already familiar with the following boosts:

$$\vec{x}' = \gamma(\vec{x} - \vec{\beta}ct) \qquad t' = \gamma(t - \frac{\vec{\beta}\cdot\vec{x}}{c}), \tag{Lorentz boost}$$

$$\vec{x}' = (\vec{x} - \vec{v}t) \qquad t' = t. \tag{Galilei boost}$$

Where $\vec{\beta} = \frac{\vec{v}}{c}$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$.

Lorentz boosts are a symmetry of relativistic theories. The full symmetry group of such a theory is the Poincaré group. The Poincaré group $ISO(1,3)$ is of pivotal importance in any relativistic theory. It consists of those coordinate transformations that leave the Minkowski metric invariant. The group consists of three parts: rotations, Lorentz boosts and translations. Rotation and translation invariance exist in a multitude of theories.
The Galilean boost can be obtained from the Lorentz boosts by taking the limit $c \to \infty$ on the Lorentz boosts. This can be interpreted as the assumption that all relevant velocities will be small compared to the speed of light. The limit will result in a classical theory. In this limit only the boost has changed, and this has had a drastic impact on the behaviour of the system. The Lorentz boost is involved in the description of most features distinct to a relativistic theory, such as time dilation and length contraction. Changing the underlying boost symmetry to the Galilean case has therefore completely changed the behaviour of the system.

Such a change of the symmetries by taking a limit can also be viewed as a limit on the Lie algebra describing the relevant symmetry group. The process of changing one Lie algebra to another by taking a limit is called an Inönü-Wigner contraction. We can explicitly show the procedure in the above case. [2] We start by considering the Poincaré algebra.

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}] =& -i\hbar(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}),\\ [M_{\mu\nu}, P_\rho] =& -i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu),\\ [P_\mu, P_\nu] =& 0.\end{aligned} \tag{Poincaré algebra}$$

Here $M_{\mu\nu}$ is the generator of both boosts and rotations. The P_μ are the generators of the respective translations. We split this into two subalgebras, consisting of the generators of $ISO(1)$ and $ISO(3)$ respectively. The first algebra contains only P_0. The second algebra contains both rotations and translations of the spacelike components. The generators that mix these two algebras are exactly the boosts.

We can rescale one of these algebras with respect to the other one with a dimensionless parameter ω. Specifically, we rescale the Euclidean part P_a as $\tilde{P}_a = \frac{P_a}{\omega}$ and taking $\omega \to \infty$. To keep the two groups in contact with each other we must also scale boosts between the two subalgebras: $M_{a0} = \tilde{B}_{a0} = \frac{B_{a0}}{\omega}$. Here we denote the zero-components of M by B. This rescaling is called the Galilei contraction. The rescaling changes the Lie algebra. As a result, the boosts now commute among each other. More importantly, there is a difference between the spatial and timelike translation operators. The time translation operators do not commute with the generator of the boost symmetry: $[B_{0i}, P_0] = i\hbar P_i$ while the spacelike translation operators commute: $[B_{0i}, P_j] = 0$. The spatial translation operators cannot change to timelike translation operators under a boost while the time translation operator can affect space after a boost. It is indeed a main feature of Galilei boosts that time does not change under such a boost.
In taking the contraction we have switched from Lorentzian to Galilean symmetry.

1.2 Carroll symmetry

The Galilei contraction has been extensively studied. [2, 4]. It is one of the main examples of a group contraction, as it the classical limit of a relativistic theory. There is, however, nothing stopping us from taking the opposite contraction This was originally done by Levy-Leblond in [5].
Rescaling the Lorentzian part P_0 as $\tilde{P}_0 = \frac{P_0}{\omega}$ and taking $\omega \to \infty$. We also scale boosts between the two subalgebras $\tilde{B}_{a0} = \frac{B_{a0}}{\omega}$. This is the Carroll contraction, and its result is, by definition, the Carroll Lie algebra. A Carrollian system then is a system whose symmetry is described by the Carroll Lie algebra.

We remark that the behaviour of the boost starts to depend on what space they act on. Given a boost in the Carroll limit $[B_{a0}, P_0] = 0$, while $[B_{a0}, P_b] = i\delta_{ab}\tilde{P}_0$. Thus, a boost can turn a spatial translation into a timelike one, but not the other way around. Since the Carroll group will be of special interest to us, we write its Lie algebra explicitly.

$$\begin{aligned}[M_{ab}, B_{0c}] =& i\hbar(\delta_{ac}B_{0b} - \delta_{bc}B_{0a}),\\ [B_{0b}, B_{0c}] =& 0,\\ [P_a, B_{0b}] =& i\hbar\delta_{ab}P_0,\\ [P_0, P_0] = [P_0, P_a] =& 0,\\ [P_0, M_{ab}] =& 0.\end{aligned} \tag{Carroll algebra}$$

The commutators of the spatial generators do not change.

The Carroll limit can be regarded as the opposite of the Galilean limit, and is therefore often regarded as a limit of $c \to 0$.

An alternative definition has been put forward in [6]. One can view this limit as a limit in coordinate space. [6] We define a Carrollian time

$$s = C * x^0 = Cct \tag{1.1}$$

and take the limit of $C \to \infty$. The Carrollian boost can only be obtain from the Lorentz boost by defining a velocity C and defining the new boost parameter as

$$\vec{b} = \vec{\beta} * C. \tag{1.2}$$

A similar change of boost parameter happens in the Galilean limit. Here the boost parameter changes from $\beta = \frac{v}{c} \to v$. The definition of $\vec{b}$ is distinct from a coordinate transformation on the velocities, which would yield $\frac{\partial x}{\partial x^0} = C\frac{\partial x}{\partial s}$.
In changing these coordinates we do indeed take the Carroll limit on the level of the operators. This can be seen from $P^0 \propto \frac{\partial}{\partial x^0} = \frac{1}{C}\frac{\partial}{\partial s}$. C takes the role of ω in the Inönu-Wigner contraction. The corresponding boost is given by

$$\vec{x}' = \vec{x} \qquad s' = s - \vec{b} \cdot \vec{x}. \tag{Carroll boost}$$

where s is the Carrollian timelike coordinate.
We can therefore view the limit as a rescaling of the timelike coordinate. This means taking the Carroll limit can be seen as the process a collapsing the lightcone: $\frac{x^0}{x^i} \to 0$. This tells us something interesting about Carroll particles: Any particle that was moving at velocity $u \leq c$ before the Carroll limit has stopped moving after the Carroll limit.

1.2.1 Possible applications

While the Inönü-Wigner contraction of the Lorentz algebra to the Galilei algebra has clear physical relevance, the contraction to the Carroll algebra might at first seem of only mathematical interest. As is often the case in physics, the mathematical procedure of taking the Carroll contraction might open up some avenues of research.

Firstly, the Carroll group might be useful to describe effects near the event horizon of a black hole. Consider, for example, a Schwarzschild black hole. For radially moving light we have a speed of light $c(r) = \frac{\mathrm{d}r}{\mathrm{d}t} = c(1 - \frac{r_H}{r})$ where r is the Schwarzschild radius. So taking the Carroll limit would send $c \to 0$ near the Schwarzschild radius.
Furthermore, there may be applications to cosmology. The FRW metric tells us that the universe expands. This means objects at a large distance have can have a physical velocity larger that the speed of light. In this case, the Carroll limit could be interpreted as a limit where the velocity of particles is large with respect to the speed of light. As another application within cosmology one may consider a gas made of Carroll particle with energy density ε and pressure P has $\varepsilon + P = 0$ [7]. Cosmologists are searching for particles with this property, because they are a possible explanation for early universe inflation.
Lastly, the Carroll group is relevant in the context of asymptotic symmetry groups. One may consider a spacetime and try to find a symmetry group far away from the origin, where the spacetime is assumed to be approximately flat. These symmetry groups seem to be related to the Carroll group. The BMS group, the asymptotic symmetry group that one finds in lightlike infinity can be related to the Carroll group. A better understanding of Carroll symmetry could contribute to an understanding of several asymptotic symmetry

groups. This could be useful for developing flat space holography.

1.2.2 Systems with a Carroll symmetry

For the reasons given above, efforts have been made to understand systems with Carroll symmetry.
Recent efforts [1, 2] have shown that free Carroll particles have quite trivial dynamics, they cannot move. Indeed, taking the speed of light to zero while retaining the condition that particles move slower than the speed of light will yield non-moving particles. There is, however, a possibility that tachyons will appear as moving particles in Carroll invariant systems.
Another approach is inspired by the Newton-Cartan formalism. Efforts have been made to formulate a version of general relativity utilizing Galilei instead of Lorentz symmetry [8, 9]. The biggest hurdle here is that the metric splits into two distinct objects, a spatial and a timelike metric objects that can capture the relevant symmetries. This occurs because in the limit, the metric becomes degenerate, $\lim_{c\to\infty} c^2dt^2 \to \infty$. This inspires the idea of developing a similar formalism for a Carrollian limit of general relativity. Similar methods could be used to describe Carrollian particles as a limit relativistic ones. Recent efforts have been made to find such a Carroll invariant description of gravity [3].

In this thesis we will specifically be interested in the way we can describe Carroll-invariant systems. In order to do this we will propose a way to take the Carroll limit and contrast this with a method which is similar in spirit to Newton-Cartan formalism. We will be interested in two kinds of particles. The first kind consists of particles that arise as the limit of a relativistic point particle. The second kind is the Carroll invariant version of electromagnetism considered in [6]. For both kinds of particles there are two different Carrollian limits. This is an interesting result that might seem counter-intuitive at first.

Chapter 2

The Carroll limit and its corresponding particles

2.1 Research on Carrollian symmetries

We provide a quick overview of the current status of the research and how our work will fit into the research. Let us start with the context in which Carroll symmetry was originally discovered. That is, the context of an Inönu-Wigner contraction. The Inönü-Wigner contraction is a mathematical operation on a Lie algebra. At the time, it was mainly of interest to physicists because it provided a way to move from the Poincaré to the Galilean algebra. Hence providing a way to "take the Galilean limit" on the level of the Lie algebra. In this context the observation was made that when one made slightly different choices while taking the Inönü-Wigner contraction, one obtains a new algebra, the Carroll algebra. With this comes of course an associated symmetry called Carroll symmetry [5].

The above observation was made in 1965. It took until 2014 until there was a resurgence in interest in the symmetry. The reason is the following: The Inönü-Wigner contraction introduces a parameter which is usually seen as dimensionless. In the Galilean contraction this dimensionless parameter aligns with c. Hence the Inönü-Wigner contraction in this case aligns with the limit $c \to \infty$. In the Carrollian case, there are two distinct ways of interpreting the parameter ω in the Inönü-Wigner contraction. We may either read $\omega = \frac{1}{c}$ and send $c \to 0$ or read ω as a parameter C, independent from c, and send $C \to \infty$. These two ways of interpreting the parameter have the same result on the Lie algebra, but a different result in terms of equations describing the Carrollian particles. The observation that we need not relate ω to C was first made in [6]. It provided a new viewpoint from which to view systems with a Carrollian symmetry.

A different route that was taken is by utilizing a direct implementation of the symmetry. We need not start from a relativistic situation to obtain a system with Carrollian symmetry. Instead, one may demand Carrollian symmetry from the start and set up a set of equations that satisfy the symmetry. This is, for example, done in [3]. This approach sidesteps the issue of finding a physical interpretation for ω altogether.

In our own research we will mainly focus on the approach of taking $C \to \infty$ on a relativistic system and in what ways it corresponds to the research that has been done by implementing Carrollian symmetry without making reference to any relativistic system.

2.2 Carroll particles

We can find the particles in the Carroll limit by considering the irreducible representations of the Carroll algebra [7, 9]. This means we are looking for subspaces of the Hilbert space that are mapped to themselves under the given transformations. We can distinguish between two cases, a case with zero and a case with nonzero "energy". [1]
In the zero energy case, the spacial momenta and the boosts commute and therefore there exists a common set of eigenvectors. This means each particle is determined by the eigenvalues of P^a and B^a, denoted by p and b. If we have such an eigenvalue, the corresponding eigenvector must have been invariant under the relevant operator. Applying a rotation to a vector gives another vector with the same length, so to be invariant under rotations we require the norm of these eigenstate 3-vectors to be conserved. Therefore, each zero energy eigenstate can be uniquely determined by $|p|$ and $|b|$.
In the nonzero energy case, the Commutator of p and b looks exactly like the regular commutation relation from quantum mechanics. It is known by the Stone-von Neumann theorem that there exists exactly one representation [7].

2.3 Taking the limit

To extract some relevant dynamics from these particles, we must find out what systems with these symmetries look like. provide some generic arguments that apply to Carroll invariant systems. When we start from a Lorentz invariant system, we can find a Carroll invariant system by taking the Carroll limit on the system. In order to do this, we follow a procedure introduced in [6]. We define a Carrollian time s such that $s = cCt$. Here c is the speed of light and C is another parameter with the dimension of velocity. Taking the Carroll limit consists of two distinct steps.
Firstly, we have changed our basis from $x_0 = ct$ to s and must rewrite all equations in terms of s.
Secondly, we take the limit $C \to \infty$.
We also need to redefine some quantities for the Carrollian case. As an explicit example of another rescaled quantity we consider the boost parameter. In relativity it is given by $\beta = \frac{u}{c}$, while in the Galilean limit it is given by u. Similarly, in the Carroll limit we define $b = C * \beta$. The relevant boost then becomes

$$\vec{x}' = \vec{x} \qquad s' = s - \vec{b} \cdot \vec{x}. \tag{2.1}$$

It is important to keep track of the dimensions in taking this limit. The dimensions are given by $[b] = LT^{-1}$, $C = LT^{-1}$ and $s = L^2T^{-1}$. The choice of dimensions might seem odd at first, but it will aid us in establishing a duality between the Galilean and Carrollian case. For now we may observe that

- We can do this. All the above objects are defined by us, so we can assign any dimension we want as long as it is consistent.
- The changes in dimension also occur when changing from a relativistic to a non-relativistic system. We may compare the relativistic $\beta = \frac{v}{c}$ to the non-relativistic v. Or the relativistic time coordinate $x^0 = ct$ to the non-relativistic t. In both of these cases, similar objects have different dimensions in the different theories.

All of this does mean that we need to be careful with our interpretation. Our interpretation will need to align with the dimensions of the objects at hand.

[1] By "energy" we mean: quantity generated by time translations, this will later turn out to not be an actual energy.

The method laid out above is to be contrasted with the "naive" limit $c \to 0$. The idea of this limit stems from the Galilean case. If a limit $c \to \infty$ moves us into a non-relativistic regime, then we might expect the limit $c \to 0$ to take us to ultra-relativistic equations. The difference between these two definitions is subtle. Every time a factor c refers to a timelike object in a relativistic setting we get the same result: $ct = \frac{s}{C}$. So taking $c \to 0$ on the left hand side and $C \to \infty$ on the right hand side gives us the same answer. However, not all factors of c refer to a timelike object. Consider, for example, the rest energy $E = mc^2$. If $c \to 0$ we arrive at $E = 0$ while for $C \to \infty$ no change occurs. We may also look at the different notions of velocity that correspond to the different theories. Under $c \to 0$, $\partial_t x$ stays conserved while $\partial_{x^0} x = \frac{1}{c}\partial_t x$ diverges. This can be compared to $C \to \infty$, where both $\partial_t x$ and $\partial_{x^0} x$ are conserved. While $\partial_s x = \frac{1}{C}\partial_{x^0} x$ goes to zero in the limit.

We may also add that taking the Carrollian limit is akin to collapsing the lightcone in a spacetime diagram. In contrast to the Galilean limit opening up the lightcone. Of course, this also motivates our choice for a definition. We want to pick the definition that most closely aligns with our intuition. We know from special relativity that restmass stays the same in an ultra-relativistic regime. Also, the closing up of the lightcone and the result "Carroll particles cannot move" $u_s = 0$ align closely.

2.3.1 Boosts

It may be instructive to take the limits on the boosts themselves first. This will allow us to see how a four-vector should be transforming in the Carrollian case.

The matrices describing a Carroll transformation can be obtained by taking the limit on the Lorentz transformations, written down in the coordinates $(s, \vec{x})$. We may first observe that
$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(\frac{b}{c})^2}} \to 1$.

$$\gamma \begin{bmatrix} 1 & -\beta_1 C & -\beta_2 C & -\beta_3 C \\ \frac{\beta_1}{C} & 1 & 0 & 0 \\ \frac{\beta_2}{C} & 0 & 1 & 0 \\ \frac{\beta_3}{C} & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -b_1 & -b_2 & -b_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Which shows us that the redefinition $b = C\beta$ was necessary for the resulting boosts to become Carrollian. The Carrollian boost is now given by

$$\Lambda^\mu_\nu x^\nu = \begin{bmatrix} 1 & -b_1 & -b_2 & -b_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} s - b \cdot x \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

As usual, the transformation of the lower indices is given by the inverse transpose:

$$\Lambda^{\mu-1T}_\nu x_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} s \\ x^1 + b_1 \\ x^2 + b_2 \\ x^3 + b_3 \end{bmatrix}.$$

We may now make an important observation while for Lorentzian matrices, and in the coordinates x^0, x^i, the inverse transpose of a Lorentzian matrices is itself a Lorentz transformation. For Carrollian matrices, this is no longer the case. The inverse transpose of the Carrollian boost matrix is a completely distinct transformation. As such, upper and lower indices represent a completely different situation.

2.3.2 Limits of four-vectors

We have seen that the limit of a Lorentz boost will always be a Carrollian or inverse transpose Carrollian transformation. As such, the limit of a relativistic four-vectors will always transform as a Carrollian four-vector. Suppose $\Lambda^{\mu}_{\nu}x^{\nu} = x^{\mu\prime}$. Then in the limit $\Lambda^{\mu}_{\nu C}x^{\nu}_{C} = x^{\mu\prime}_{C}$. A similar calculation goes for x_{μ}.

2.3.3 Momentum

We will also be interested in the limit of the momentum, since we can extract the different particles from taking the limit. We will distinguish between the limit of the relativistic momentum p_L and the Carroll momentum, denoted by p_c or simply p. There exists a difference between them, since the Carroll momentum is written in terms of s instead of ct.

The relativistic momentum four-vector in the relativistic coordinates, with $x^0 = ct$, is given by $p^{\mu} = \begin{bmatrix} \frac{E}{c} \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \eta^{\mu\nu}p_{\nu}$. The coordinate transformation $x^0 \to s = Cx^0$ changes this to $p^{\mu} = \begin{bmatrix} \frac{EC}{c} \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $p_{\mu} = \begin{bmatrix} -\frac{E}{cC} \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$. Since C has a dimension of ms^{-1} the zeroth component of p^{μ} and p_{μ} have different dimensions and therefore have different physical meaning. p^0 is an energy while p_0 carries the dimension of mass. This remains true when taking the limit. We must make sure our momentum four-vectors transform correctly. In order to find out the Carroll limit of the momentum and energy transformations without making reference to the four-vectors.

$$\begin{aligned}
\vec{p}^{\|\prime} &= \gamma(\vec{p}^{\|} - \beta\frac{E}{c}) \to \vec{p}^{\prime\|} = \vec{p}^{\|} \\
\vec{p}^{\prime}_{\perp} &= \vec{p}_{\perp} \\
E' * \frac{C}{c} &= \gamma(\frac{C}{c}E - C\vec{\beta}\cdot\vec{p}) \to C * E - \vec{b}\cdot\vec{p} \\
E' * \frac{1}{cC} &= \frac{1}{cC}\gamma(E - c\vec{\beta}\cdot\vec{p}) \to 0
\end{aligned}$$

Where we note that $\gamma = \gamma_{\beta} = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{b^2}{C^2}}} \to 1$. In the above transformation laws, we see that the 3-momentum does not change. This corresponds to both p^{μ}, where the spatial components are always unaffected. And p_{μ}, where the timelike component is zero and therefore Carroll boosts have no effect. The distinction between p^{μ} and p_{μ} is more important for the energy E. Its transformation law is captured by p^{μ}, p_{μ} does not know about energy. We will later see how these transformations work in practice.

2.3.4 Velocity

The role of velocity in the Carrollian limit requires some further analysis. We have $\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{cC}\frac{\mathrm{d}x}{\mathrm{d}t} \to 0$, so the Carroll velocity becomes zero in the limit. However, u_s does not have a dimension of ms^{-1} but of an inverse velocity sm^{-1}.

We propose the following interpretation: The Carroll velocity cannot be interpreted as the actual velocity of the particle. The Carroll velocity will replace the actual velocity in the relevant equations after the limit is taken, but the actual velocity $\partial_t x$ of the particle will remain unchanged. [2]

We should also pay attention to the behaviour of the velocity under a Carroll boost. Under such a boost we have $\frac{\mathrm{d}x}{\mathrm{d}s} \to \frac{\mathrm{d}x}{\mathrm{d}(s-b\cdot x)} = \frac{u_s}{1-b*u_s}$. We observe that there are two distinct cases. One where the velocity equals zero and one where the velocity is non-zero. Of course, all particles that are found by taking a limit on a relativistic have zero Carroll velocity.

The case where $\partial_s x \neq 0$ has a glaring issue: the velocity may become infinite under a Carroll transformation. We interpret this as meaning the notion of velocity is not relevant to the description of these kinds of particles. We could however still find an interpretation for them as tachyons. For these particles, the notion of causality has broken down. The boost to infinite velocity may correspond to a tachyon whose worldline is purely in the spacelike direction.

The possibility of particles with non-zero Carroll velocity becomes relevant in the following context: we have $\frac{\partial x^i}{\partial x^0} = C\frac{\partial x^i}{\partial s}$. This can be interpreted in two different ways:

- We may keep $\frac{\partial x^i}{\partial x^0}$ conserved, this is what one would do when describing relativistic particles in an ultra-relativistic limit. It necessarily follows that $\partial_s x \to 0$ in the Carroll limit.
- Alternatively one can assume the Carroll velocity $\frac{\partial x}{\partial s}$ to be conserved, there is no corresponding relativistic velocity, these particles have no relativistic analogue and only exist in the Carroll limit.

We may also observe that the transformation of the velocity cannot possibly be captured by a matrix. This is also the case relativistically. $\frac{\partial x}{\partial t}$ does not transform as a four-vector while $\frac{\partial x}{\partial \tau}$ does.

The definition of the Carroll velocity has an interesting consequence for the boost parameter. We will need to make an explicit distinction between γ_β, the gamma factor containing the boost parameter, and γ_u, the gamma factor containing the velocity. We have seen previously that $\gamma_\beta \to 1$ in the limit. This is to be contrasted with $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-C^2u_s^2}}$. This stays conserved when $C * u_s$ is conserved, which corresponds to $u_s \to 0$ in the limit, and goes to zero when $u_s \neq 0$.

2.4 Examples of Carroll particles

We can now try to find some explicit examples of particles with this symmetry. We can do this by taking a Carroll limit on some particles we are familiar with.

Let us start with a massive relativistic particle.

2.4.1 Massive particles

We must acknowledge that our notion of momentum in the Carroll limit has fundamentally changed compared to the momentum in special relativity. When we define the Carroll momentum as that what is generated by the momentum operator, we arrive at $p^\mu = (\frac{E}{cC}, p^1, p^2, p^3)$. We can figure out what these momenta look like by taking the limit of the relevant expressions.

We know that $\frac{u}{c} = \frac{\mathrm{d}x^i}{\mathrm{d}x^0} = C\frac{\mathrm{d}x^i}{\mathrm{d}s} = Cu_s$ is conserved in the Carroll limit. This implies the following

[2]This would mean we can describe particles with a velocity close to the speed of light with this limit.

limits:

$$
\begin{aligned}
E &= \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{mc^2}{\sqrt{1-C^2u_s^2}} \\
p_i &= m\frac{cCu_s}{\sqrt{1-C^2u_s^2}} \\
p^0 &= \frac{EC}{c} = \frac{mcC}{\sqrt{1-C^2u_s^2}} \to \infty \\
p_0 &= \frac{E}{Cc} \to 0.
\end{aligned}
$$

Notice that all relativistic expressions have remained conserved in the Carroll limit. This is because they do not contain factors of C. We are, strictly speaking, not interested in these objects. The time-like components go to zero and infinity respectively. This is because of the coordinate transformation and does not give us any additional information about the energy.

We may also take a slightly different approach if we want the time-like components to be conserved. Let us look at this more explicitly. Taking limits on $p_{C\mu} = C * p_\mu$ and $p^{C\mu} = \frac{1}{C}p^\mu$ instead, we find

$$
\begin{aligned}
p_{iC} &= m\frac{cC^2u_s}{\sqrt{1-C^2u_s^2}} \to \infty \\
p^0 &= \frac{E}{c} \to \frac{E}{c} \\
p_0 &= \frac{E}{c} \to \frac{E}{c} \\
p^{iC} &= m\frac{cu_s}{\sqrt{1-C^2u_s^2}} \to 0.
\end{aligned}
$$

This does in fact transform appropriately under a Carroll transformation $p^{\nu\prime} = \Lambda^\nu_\mu p^\mu = p^\nu = \begin{bmatrix} E \\ 0 \end{bmatrix}$.

The transformation of p_μ requires some additional analysis. We have $p_{iC} = Cp_{iL}$, so the Carrollian momentum goes to infinity in the limit. The transformation law tells us, however, $p_{i,C} \to p_{i,C} - \vec{b} * \frac{E}{c}$. This is indeed the transformation law of the momentum as given by the Carroll limit on the Lorentz transformations. For the component parallel to the boost we have $Cp'_{||} \to Cp_{||} - C\vec{\beta}\frac{E}{c} = Cp_{||} + b_i\frac{E}{c}$.
Massive particles should not be moving. This corresponds the best to the limit of $p^{i,C}$.We therefore suspect the latter set of four-vectors holds the most relevance to massive particles.

Taylor series cannot be applied

We wish wish to take a Taylor series around the limit, and an as such have a way to describe highly relativistic particles. This would require that the Carroll velocity is small compared to C. We do not need to send C to infinity for this It is a large but finite number. In this way, we may try to expand u_s around zero. This should describe ultra-relativistic massive particles.

We should point out a critical distinction between using a Taylor series in the Galilean limit versus using a Taylor series in the Carrollian limit. In the Galilean limit we have $\frac{\partial x}{\partial x^0} = \frac{1}{c}\frac{\partial x}{\partial t}$. So in the limit the relativistic velocity $\frac{u}{c}$ goes to zero. The velocity we are interested in, u, is completely unchanged by the limit. In the Carrollian case the situation is reversed. While $\frac{u}{c}$ is perfectly conserved in the limit, the velocity we are interested in, $u_s = \partial_s x$, goes to zero in the limit. This makes it difficult to Taylor around zero Carroll velocity in a decent way.

Notice that a fixed value for C translates to a fixed value for u_s. this is in explicit contradiction to the Galilean case. We may try to fill in the Taylor series in for said value of u_s alone. A first order expansion of the momentum given in 2.2 will give

$$p_i(u_s) = p_i(0) + u_s * p_i'(0) = 0 + u_s * (\frac{cC}{\sqrt{1 - C^2 u_s^2}})\Big|_{u_s=0} = u_s cC. \tag{2.2}$$

This might be a way to describe massive particles in the Carroll limit.
The first order expansion might be relevant, but Carroll boost invariance has been sacrificed to achieve this. The zeroth order expansion still has Carroll boost invariance when taken in this way. Here we take large but finite C and $u_s = 0$. The relevant expressions become

$$\begin{aligned}
E &= \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \to mc^2 \\
p_i &= m\frac{cCu_s}{\sqrt{1 - C^2 u_s^2}} \to 0 \\
p^0 &= \frac{EC}{c} \to Ccm \\
p_0 &= \frac{E}{Cc} \to \frac{cm}{C}.
\end{aligned}$$

The transformation laws align most closely with those four-vectors where the time-like part is conserved. This is in fact the momentum most closely associated with massive particles.

The fact remains that holding C and $u_s * C$ constant allows for exactly one value of u_s. This is contrary to how a Taylor series usually works. Normally a Taylor series around u_s would apply to small values of u_s. We do not trust the process hold any physical relevance. This might also be a manifestation of the fact that massive particles cannot move at the speed of light.
Either C is finite or C goes to infinity. Any finite C is just a coordinate transformation away from special relativity. Therefore it is better described without making reference to the Carroll limit. Any infinite C will mean a fully collapsed lightcone. In the latter case, a small but nonzero u_s for massive particles would mean movement after the lightcone has collapsed.

2.4.2 Tachyons?

Taking some liberty in terms of interpretation, we may take the limit on the momentum assuming u_s is non-zero. This is to be contrasted with $u_s = 0$, which we would get from taking the limit on relativistic velocities.

$$\begin{aligned}
E &= \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \to -mc^2 * i * \frac{1}{u_s C} \to 0 \\
p_i &= m\frac{cCu_s}{\sqrt{1 - C^2 u_s^2}} \to -icm\eta^i \\
p^0 &= \frac{EC}{c} \to -imc * \frac{1}{u_s} \\
p_0 &= \frac{E}{Cc} \to 0.
\end{aligned}$$

Where $\eta^i = \frac{u_s^i}{|u_s|}$ is a unit vector. The four-vectors $p^\mu = \begin{bmatrix} \frac{-imc}{|u_s|} - \vec{b}\cdot\vec{p} \\ p_i \end{bmatrix}$ and $p_\mu = \begin{bmatrix} 0 \\ p_i \end{bmatrix}$ transform appropriately under a coordinate transformation. Specifically $\frac{1}{|u_s|} \to \frac{1-b\cdot u_s}{u_s} = \frac{-imc}{|u_s|} - b\cdot p$.
As a check we can again fill in the energy: $E = \sqrt{m^2c^4 + p^2c^2} = \sqrt{m^2c^4 - m^2c^4} = 0$. Therefore, the mass shell conditions continues to hold. Given that for these particles we have taken $u_s \neq 0$, and the result turns out to be imaginary, these particles look remarkably like tachyons. Because of this, we can find real values in the limit by assuming the mass has been imaginary. This is a feature commonly seen in field theories of tachyons, where the mass squared is negative. We redefine $m_c = -im$. Since m was imaginary m_c will be a real and positive number. With this redefinition our results become

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \to m_c c^2 * \frac{1}{uC} \to 0$$
$$p_i = m\frac{cCu_s}{\sqrt{1-C^2u_s^2}} \to cm_c\eta^i$$
$$p^0 = \frac{EC}{c} \to m_c * \frac{1}{u_s}$$
$$p_0 = \frac{E}{Cc} \to 0.$$

These do transform correctly under a Carroll transformation. Furthermore, in the massive case some terms scaled with a factor of C and therefore became infinite in the limit. For these "tachyonic" particles, this is not the case.

2.4.3 Active and passive transformations

It is worth pointing out that the above two particles can be thought of in terms of active and passive transformations respectively. An active transformation is a transformation where we describe the system as seen be a different observer. Therefore the physical content of the system changes. A prominent example would be a boost. Under such a transformation, quantities in the system are allowed to change, $f(x) \to f(x')$. We have in general $f(x) \neq f(x')$.
For a passive transformation we change our description of the system, but the physical objects remain the same. We require $f(x) = f'(x')$. The function f' will be such that this is true. An example of a passive transformation would be a switch from Cartesian to spherical coordinates.

We may arrive at the massive particles by an active transformation. That is, we switch observers and see what our particles look like to the new observer. Let us look at the velocity for this. The new observer does not measure time. They measure Carroll time. The movement of a particle with respect to Carroll time is $\frac{\partial u}{\partial s} = \lim_{C\to\infty} \frac{1}{C}\frac{\partial u}{\partial ct} = 0$. So the particles have stopped moving. Applying this to the momentum we have $\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} \to \frac{m\frac{\partial x}{\partial s}}{\sqrt{1-\frac{\frac{\partial x}{\partial s}^2}{c^2}}} = 0$. [3]

We may arrive at the tachyonic particle by a passive transformation. For such a transformation the observer does not change, the coordinate system does. As such, this is the observer that sees the lightcone collapse. This process can only be survived by Tachyonic particles. The speed of light went to zero, so there is no "valid" speed for these particles to have. We do not change the system, we only change its description: $u_{x^0} = Cu_s$.

[3] Notice that, while mathematically correct, this procedure is not entirely physical. Since the dimensions of t and s are not the same. We need to ignore dimensional analysis for a bit.

2.4.4 Photons

We may also explicitly consider the photon. We have

$$p^0 = \frac{\hbar C}{\lambda} \to \infty$$
$$p_0 = \frac{\hbar}{\lambda C} \to 0$$
$$p_i \to p_i.$$

This is the same result as the one that looked like a tachyon. There is a key difference though. The momentum does not contain any velocity.
We may observe the following: Only particles without a rest-frame have a non-zero spacial momentum in the Carroll limit.

In Conclusion, the Carroll limit contains two distinct classes of particles. Particles with and without a restframe. Particles with a restframe only appear after we take the limit such that the time-like, instead of the spacial part, of the four-vectors is conserved. That is, we have redefined $p^\mu_C = \frac{p^\mu}{C}$ and $p_{\mu C} = C p_\mu$. Their mass remains conserved, but their spacial momentum disappears in the limit. For particles without restframe no additional algebraic manipulation is necessary to take the limit. The spacial momentum of these particles remains intact while their temporal part disappears completely.

Chapter 3

A mathematical perspective

Now that we have seen how to take the limit and what kind of results we might expect, we may start to develop the mathematical tools necessary to describe the kind of particles that are emerging in the Carroll limit. This will shed some light on the different particles found in last chapter.

3.1 The metric complex

We have seen above that the Carroll limit does not neatly fit into a four-vector formalism. In order to describe the system, we need to adjust the familiar formalism. This is done in a way largely inspired by the Newton-Cartan formalism [8, 9].
We define two different objects in the Carroll limit, a spatial metric $h_{\mu\nu}$ and a temporal tetrad τ^μ. Together they will act similar to a metric. These two objects together are called the metric complex. This idea is taken directly from the Newton-Cartan formalism. In this formalism Newtonian gravity is reformulated in terms of four-vectors.

We define

$$h_{\mu\nu} = \lim_{C\to\infty} \eta_{\mu\nu} = \lim_{C\to\infty} \begin{bmatrix} -\frac{1}{C^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tau^\mu \tau^\nu = \lim_{C\to\infty} -\frac{1}{C^2}\eta^{\mu\nu} = \lim_{C\to\infty} = - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{C^2} & 0 & 0 \\ 0 & 0 & \frac{1}{C^2} & 0 \\ 0 & 0 & 0 & \frac{1}{C^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Explicitly, we have $\tau^\mu = (1, 0, 0, 0)$ and $h_{\mu\nu} = diag(0, 1, 1, 1)$. We can also define inverse objects τ_μ and $h^{\mu\nu}$ such that

$$\tau^\mu \tau_\mu = 1 \qquad h^{\mu\nu} h_{\nu\rho} = \delta^\mu_\rho - \tau^\mu \tau_\rho \qquad h^{\mu\nu}\tau_\mu = h_{\mu\nu}\tau^\mu = 0. \tag{3.1}$$

From the definition follows $\tau^\mu = (1, 0, 0, 0)$. The first equation implies $\tau_0 = 1$. From the second equation we get $h^{0i} = -\tau_i$, $h^{ij} = \delta_{ij}$. The third equation now gives us $h^{00} = \delta^{ij}\tau_i\tau_j$. This still leaves us with a degree of freedom: τ_i is not yet defined. This is reflective of the freedom to choose a coordinate system. Under a Carroll transformation τ_μ changes as $\tau'_\mu = (1, \tau_i - b_i)$. This should not change the underlying physics.

This gives us a new perspective on the particles we have encountered in Chapter 2. With the metric complex we may lower indices by using $p_\mu = h_{\mu\nu}p^\nu$, but we may not raise them. The reason for this can be

seen in the following: $p^\mu = g^{\mu\nu}p_\nu$. Switching to Carrollian time we have $p^\mu = -C^2\tau^\mu\tau^\nu p_\nu + h^{\mu\nu}p_\nu$. This would mean sending p^μ to infinity.
Alternately, we may consider dividing by a factor of C before taking the Carroll limit. We may now raise indices with τ, but we cannot lower indices anymore. In this case, $\frac{p^\mu}{C} = -C\tau^\mu\tau^\nu p_\nu$. Notice that in the temporal part of the momenta, the factors of C that occur from the coordinate transformation cancels against the prefactor of C.

We have therefore ended up with two distinct cases,

$$p_\mu = h_{\mu\nu}p^\nu \qquad \text{(Case 1)}$$
$$p^\mu = -\tau^\mu\tau^\nu p_\nu. \qquad \text{(Case 2)}$$

We have seen in chapter 2 that Carroll particles tend to have either $p_0 = 0$ or $p^i = 0$. The massive particles form the previous chapter fall in case 2, as can be seen from p^i being 0. Meanwhile, the photon and tachyonic particle fall in case 1, since $p_0 = 0$ for those particles.

3.1.1 An example - the massive Galilean case

It may be instructive to consider the similar approach in Newton-Cartan formalism. For an extensive overview of the Newton-Cartan formalism we refer to [8].The definitions for the metric complex are the same as above, but with the upper and lower indices reversed. To be explicit, we have

$$\tau_\mu\tau_\nu = \lim_{c\to\infty}\frac{-1}{c^2}g_{\mu\nu} \qquad (3.2)$$
$$h^{\mu\nu} = \lim_{c\to\infty} g^{\mu\nu}. \qquad (3.3)$$

We can define "inverses" by the same equations as we had in the Carrollian case.

$$\tau^\mu\tau_\mu = 1 \qquad h^{\mu\nu}h_{\nu\rho} = \delta^\mu_\rho - \tau^\mu\tau_\rho \qquad h^{\mu\nu}\tau_\mu = h_{\mu\nu}\tau^\mu = 0 \qquad (3.4)$$

The inverse objects are not uniquely defined. There is a gauge freedom in the inverse temporal part τ^i. We are free to choose them and doing so fixes $h_{\mu\nu}$. The first equation implies $\tau^0 = 1$. From the second equation we get $h_{0i} = -\tau_i$, $h_{ij} = \delta_{ij}$. The third equation now gives us $h_{00} = \delta_{ij}\tau^i\tau^j$.
Let us consider the behaviour of the momenta in this limit. Our boost parameter will be $\vec{v}$. The momentum of a massive particle is be given by $p^\mu = \lim_{c\to\infty}(\frac{E}{c^2}, p^1, p^2, p^3) = m(1, \vec{u})$. Where we have switched from the usual coordinates $x^0 = c * t$ to t. It transforms as

$$p^{\mu'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{bmatrix} p^\mu = \begin{bmatrix} m \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} m \\ p^1 + mv_1 \\ p^2 + mv_2 \\ p^3 + mv_3 \end{bmatrix}.$$

This corresponds to the usual transformation of the Galilean momentum $m\vec{u} \to m(\vec{u} + \vec{v})$.

The covariant momentum as a direct limit

However, the covariant momentum p_μ transforms differently under this coordinate transformation we arrive at $p_\mu = \lim_{c\to\infty}(-E_g, p_1, p_2, p_3) = (-E_g, m\vec{u})$, where E_g denotes the Galilean energy, given by $\frac{1}{2}mu^2$.
Only p^μ transforms appropriately under Galilean transformation. The p_μ adds a term to p_0. Explicitly, the Galilean energy E_g transforms as $\frac{1}{2}mu^2 \to \frac{1}{2}m(u+v)^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu'^2 - \frac{1}{2}mv^2$ according to the formalism. As can be seen from

$$p'_\mu = \begin{bmatrix} 1 & v_1 & v_2 & v_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} p_\mu = \begin{bmatrix} E_g + muv \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}m(u+v)^2 - v^2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This is in clear contradiction with the expected transformation $\frac{1}{2}mu^2 \to \frac{1}{2}mu'^2$. Hence, the covariant Carroll momentum cannot be taken as a direct limit of the relativistic momentum.

A note on raising and lowering indices

In this formalism, we have lost an important property, we cannot arbitrarily raise and lower indices. We have no metric to use for that. We could define $p_\mu = h_{\mu\nu}p^\nu - \tau_\mu\tau_\nu p^\nu$ instead. $h_{\mu\nu}p^\nu$ equals $m * \begin{bmatrix}\tau^2 - \tau^i u_i \\ \vec{u} - \vec{\tau}\end{bmatrix}$.
It transforms as
$m\begin{bmatrix}\tau^2 - \tau^i u_i \\ \vec{u} - \vec{\tau}\end{bmatrix} \to m\begin{bmatrix}(\tau + b)^2 - (\tau^i + b^i)(u_i + b_i) \\ \vec{u} - \vec{\tau} + \vec{b}\end{bmatrix} = m\begin{bmatrix}(\tau)^2 - \tau^i(u_i - b_i) \\ (u \vec{-} b) - \vec{\tau}\end{bmatrix}$.
This transforms as an inverse Galilei transformation, at least when $\tau^i = 0$. Meanwhile, $\tau_\mu\tau_\nu p^\nu$ does not transform because $\tau_i = 0$. Their sum equals $m * \begin{bmatrix}\tau^2 - \tau^i u_i + 1 \\ \vec{u} - \vec{\tau}.\end{bmatrix}$ This is clearly distinct from p_ν as defined above.

A possible solution

We have seen that the limit of the covariant momentum does not transform correctly and that we cannot easily raise or lower indices as we might expect to.

To remedy this, we may define $\tilde{p_\mu} := \tau_\mu\tau_\nu p^\nu$. It transforms, by definition, correctly under a Galilean transformation. It is explicitly given by $\tilde{p_\mu} = (m, 0, 0, 0)$. While we can now lower indices we cannot raise them yet. We just have a pair momenta for which the formula $\lim_{c\to\infty}(g_{\mu\nu}p^\mu = p_\nu) \to \tilde{p_\mu} := \tau_\mu\tau_\nu p^\nu$. [1] We may point out that as long as p_μ always appears in contraction with a corresponding temporal or spacial metric the offending term disappears, so there is no explicit need for defining $\tilde{p_\mu}$.

- $\tau_\mu p^\mu$ is invariant because τ_i is zero and p^0 does not transform.
- $\tau^\mu p_\mu$ is invariant, but the individual components transform. The explicit contraction becomes $m(\frac{1}{2}u^2 - v\cdot u + v\cdot u) = \frac{1}{2}mu^2$.
- $h^{\mu\nu}p_\mu p_\nu$ is invariant because both $h^{\mu\nu}$ and the spatial components of p_ν do not transform.
- $h_{\mu\nu}p^\mu p^\nu$ does not transform. The timelike components of $h_{\mu\nu}$ change as to compensate for the spatial components of p^μ, which change as $m * u \to m * (u - v)$. As they would in the Lagrangian of a non-relativistic particle.

To illustrate the use of the formalism, we may take the Galilean limit in Newton-Cartan formalism.

$$\int d\tau c * \sqrt{-g_{\mu\nu}p^\mu p^\nu} = \int d\tau c * \sqrt{c^2\tau_\mu\tau_\nu p^\mu p^\nu - h_{\mu\nu}p^\mu p^\nu} \tag{3.5}$$

$$= \int d\tau c * \sqrt{c^2\tau_\mu\tau_\nu p^\mu p^\nu - h_{\mu\nu}p^\mu p^\nu} = \int d\tau c * \sqrt{c^2\tau_\mu\tau_\nu p^\mu p^\nu(1 - \frac{h_{\mu\nu}p^\mu p^\nu}{c^2\tau_\mu\tau_\nu p^\mu p^\nu})} \tag{3.6}$$

$$\approx \int d\tau c * \sqrt{c^2\tau_\mu\tau_\nu p^\mu p^\nu}(1 - \frac{h_{\mu\nu}p^\mu p^\nu}{2c^2\tau_\mu\tau_\nu p^\mu p^\nu}) = \int dtmc^2(1 - \frac{h_{\mu\nu}p^\mu p^\nu}{2c^2m^2}). \tag{3.7}$$

Where we applied a Taylor series to the first order in the last line. Now compare this to the regular Galilean Lagrangian. Since m is clearly invariant under Galilei transformations, we may abandon the first

[1] One may wonder what happens when we allow for raising indices, instead of lowering them. Since $p_0 = mc^2 +$ Lower order terms this can be done by $\tilde{p_\mu} = \frac{p_\mu}{c^2}$. Raising indices with $h^{\mu\nu}\tilde{p_\mu} = p^\nu$ will yield zero.

term. It does not affect the equations of motion. Alternatively, we may add a constant gauge field A_μ to our Lagrangian [8] and add the term $mA_\mu \dot{x}^\mu$ to our Lagrangian. Setting $A_0 = c^2$ and $A_i = 0$ will make the mass term disappear.
The kinetic term $\frac{h_{\mu\nu}p^\mu p^\nu}{2m}$ is Galilei invariant while $\frac{m^2 v^2}{2m}$ is not obviously Galilei invariant. Let us rewrite the Newton-Cartan kinetic term, $\frac{h_{\mu\nu}p^\mu p^\nu}{2m} = \frac{m}{2} = m(\frac{1}{2}u^2 - \tau^i u^i + \frac{\tau^2}{2})$. This is indeed Galilei boost invariant. The regular Lagrangian just consists of the case $\tau^i = 0$. We furthermore note that adding these terms does not change the equations of motion, since the are a total derivative and a constant respectively.

There is a link between the extra terms that appear in the above Lagrangian 3.5 and the Bargmann group, the centrally extended Galilei group [10]. We may see the second term of the above Lagrangian from a different point of view. The usual Lagrangian for a relativistic point particle $L = \frac{1}{2}mu^2$ transforms as a total derivative

$$\delta L = \frac{d}{dt}\left(mx^i \lambda^j \delta_{ij}\right) \tag{3.8}$$

under a boost. This is associated with a Noether charge and associated symmetry. This hints at the existence of a central extension of the Galilei algebra called the Bargmann algebra. Specifically, using the complete Noether charge under such a boost, the poison bracket of Q_G with Q_P is given by the mass. This indicates the existence of a central extension. The situation will be discussed in more detail when we discuss the relevant Lie algebras in section 3.3.

3.2 The manifold perspective

[6] Next, we may spend some time by investigating the manifold structure underlying both the Carrollian and the Galilean case given above. This will provide more insight into the metric complexes given above. Furthermore, it will show that there exists a duality between the Galilean and Carrollian symmetries. In relation to this, we may show that the Carrollian and Galilean picture fit into a larger framework. This section is largely an overview of some of the work done in [6].

3.2.1 Newton-Cartan manifolds

A Newton-Cartan manifold is a quadruple $(N, h^{\mu\nu}, \tau_\mu, \nabla)$ where

- N is a d+1 dimensional manifold.
- $\tau_\mu : TM \to \mathbb{R}$ is a nowhere vanishing 1-form. That is, it provides an element of the cotangent space at each point p.
  τ is nowhere vanishing, meaning there must be a vector v in the tangent space such that $\tau_\mu v^\mu \neq 0$.
- $h^{\mu\nu}$ is a twice symmetric contravariant positive tensor field. $h^{\mu\nu}$ looks like an inverse metric but can be degenerate, its kernel is generated by τ_μ. That is, the set of all cotangent vectors with zero length is given by $\{\tau_{\mu,p} | p \in N\}$.
- ∇ is an affine connection. Let $\Gamma(TN)$ be the space of vector fields on N. An affine connection is then given by a bilinear map $\nabla : \Gamma(TN) \times \Gamma(TN) \to \Gamma(TN) : (X, Y) \to \nabla_X Y$. Such that for any smooth function $f : N \to \mathbb{R}$ and vector fields X, Y we have $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X (fY) = df(X)Y + f\nabla_X Y$. As an additional requirement, we demand both $h^{\mu\nu}$ and τ_μ are parallel transported. This means $\nabla_X \tau_\mu = 0$ and $\nabla_X h^{\mu\nu} = 0$ for all vector fields X.

We would like to make two additional remarks about the affine Connection. Firstly, the connection can be used to define a covariant derivative by fixing the vector field X. This gives us a linear map $\nabla : Y \to Z, Y \mapsto \nabla_X Y$ that satisfies the Leibniz rule. Thus, as the notation suggests, $\nabla_X Y$ denotes a covariant derivative. Secondly, contrary to general relativity, there is no unique connection that is both metric compatible and torsion free. On a Newton-Cartan spacetime, there exist multiple viable connections. [8]

As an example, we provide the flat Newton-Cartan structure.

$$N^4 = \mathbb{R} \times \mathbb{R}^3, \quad h^{\mu\nu} = \delta^{AB} \frac{\partial}{\partial x^A} \otimes \frac{\partial}{\partial x^B}, \quad \tau_\mu = 1 * dt, \quad \Gamma^k_{ij} = 0. \tag{3.9}$$

The manifold definition is a more general case of the flat structure we have seen before.

Group structure

The Galilei group is the group of automorphisms of the Newton-Cartan manifold. That is, the group of all diffeomorphisms that preserve the "inverse metric" $h^{\mu\nu}$, the vector field τ_μ and the connection ∇. The Galilei group is given explicitly by

$$g_N : \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} \mapsto \begin{pmatrix} R\mathbf{x} + \mathbf{b}\mathbf{t} + \mathbf{c} \\ t + e \end{pmatrix} \tag{3.10}$$

Here $R \in O(3)$ is a rotation,$b \in \mathbb{R}^3$ a boost parameter, $c \in \mathbb{R}^3$ and $e \in \mathbb{R}$ represent translations.

3.2.2 Carroll manifolds

A Carroll manifold is a quadruple $(C, h_{\mu\nu}, \tau^\mu, \nabla)$ where

- C is a 3+1 dimensional manifold.
- τ^μ is a vector field. That is, a map that assigns to each point p on the manifold a vector in its respective tangent space $T_pC = \mathbb{R}^{3+1}$. Denoting the tangent bundle by $TC := \{(p, y)|p \in C, y \in T_pC\}$, we have $\tau^\mu : C \to TC$. Which should be a right inverse of the projection map $\pi : TC \to C, \{(p, x) \to p\}$.
  τ^μ is nowhere vanishing, meaning $\tau^\mu(p) \neq 0$ for all p.
  Furthermore, τ^μ is complete. This means its associated flow curves, defined by $\gamma'(t) = \tau^\mu(\gamma(t))$ are well defined for all t. Less abstract, a particle moving along τ^μ will neither stop nor suddenly disappear.
- $h_{\mu\nu}$ is a twice symmetric covariant positive tensor field. $h_{\mu\nu}$ looks like a metric but can be degenerate. Its kernel is generated by the vector field τ^μ. That is, the set of all tangent vectors with zero length is given by $\{\tau^\mu(p)|p \in C\}$.
- ∇ is an affine connection. Let $\Gamma(TC)$ be the space of vector fields on C. An affine connection is then given by a bilinear map $\nabla : \Gamma(TC) \times \Gamma(TC) \to \Gamma(TC) : (X, Y) \to \nabla_X Y$. Such that for any smooth function $f : C \to \mathbb{R}$ and vector fields X, Y we have $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X(fY) = df(X)Y + f\nabla_X Y$. As an additional requirement, we demand both $h_{\mu\nu}$ and τ^μ are parallel transported. This means $\nabla_X \tau^\mu = 0$ and $\nabla_X h_{\mu\nu} = 0$ for all vector fields X.

Once again, the connection directly implies the existence of a covariant derivative. Furthermore, contrary to general relativity, there is no unique connection that is both metric compatible and torsion free. On a Carrollian manifold, this is no longer true.

This can be seen by considering the following shift to the Christoffel symbols:

$$\Gamma'^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + h_{\mu\nu}\tau^\rho. \tag{3.11}$$

This trick is similar to a trick used in [8]. The expression is symmetric, such that the shifted Christoffel symbol still has no torsion. Furthermore, the shifted Christoffel symbol must still satisfy $\nabla h_{\mu\nu} = 0$ and $\nabla\tau^\mu = 0$.

The first preservation of the first condition requires $h_{\mu\nu}\tau^\rho h_\alpha\rho = 0$, which follows directly from $\tau^\rho h_{\rho\alpha} = 0$. Similarly, preservation of the second condition requires $h_{\mu\lambda}\tau^{\rho)}\tau^\mu = 0$, which follows directly from $h_{\mu\lambda}\tau^\mu = 0$. The condition $\nabla_\rho(\tau^\mu) = 0$ implies $\tau^\mu = \partial^\mu(f(x))$. This can be deduced from Stokes theorem. Which implies a conservative field and therefore a potential $f(x)$.
As an example, we provide the flat Carroll structure:

$$C^4 = \mathbb{R} \times \mathbb{R}^3, \quad \mathrm{h}_{\mu\nu} = \delta_{AB} dx^A \otimes dx^B, \quad \tau^\mu = \frac{\partial}{\partial s}, \quad \Gamma^k_{ij} = 0. \tag{3.12}$$

Once again, this is the same Carroll structure we have previously seen.

Group structure

The Carroll group is the group of automorphisms of the Carroll manifold. That is, the group of all diffeomorphisms that preserve both the "metric" $h_{\mu\nu}$, the vector field τ^μ and the connection ∇. We will denote the Carroll group by $Carr(C, h_{\mu\nu}, \xi, \nabla)$. Note, the Isometry group of the degenerate "metric" $h_{\mu\nu}$ is infinite dimensional , since we can add arbitrary functions to s.

$$\begin{aligned} s &\to s + f(x) \\ x^i &\to x^i \end{aligned}$$

is an isometry for all functions f.
This is distinct from the automorphism group on flat Carroll space 3.12. Such an automorphism requires the entire Carroll structure to be valid after the transformation. This is a stronger requirement. $h_{\mu\nu}$ stays the same. However, demanding parallel transport of $\tau^{\mu'}$ requires these f^i to be constant. Preservation of the parallel transport yields the additional requirement $f = constant$, since we require $\nabla_i(x^0 + f(x)) = \nabla_i(x^0)$. The full Carroll group is therefore given by action of the Carroll group:

$$a_C : \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix} = \begin{bmatrix} R\mathbf{x} + \mathbf{c} \\ s - B^T R\mathbf{x} + f \end{bmatrix} \tag{3.13}$$

Here $R \in O(3)$ is a rotation and $B \in \mathbb{R}^3$ a boost parameter. $c \in \mathbb{R}^3$ and $f \in \mathbb{R}$ represent translations [11].

Let us pause a moment to point out the distinction between an isometry and a coordinate transformation. The Isometry asks about the transformations that preserve the distance between points. The covariant derivative does not change under such an isometry. This is in explicit contrast with the coordinate transformation portrayed by the Carroll group. Under such a coordinate transformation the covariant derivative changes covariantly, as it should under a coordinate transformation.

3.2.3 Some hints of the duality

[6] We can spot immediate similarities between the Newton-Cartan and the Carrollian manifolds. Let us first observe that the homogeneous part [2] of the Galilei and Carroll group can be expressed as

$$\Lambda_G = \begin{pmatrix} 1 & 0 \\ \mathbf{b} & R \end{pmatrix} \tag{3.14}$$

and

$$\Lambda_C = \begin{pmatrix} 1 & -\mathbf{b}^T R \\ 0 & 1 \end{pmatrix} \tag{3.15}$$

respectively. Where the matrices are acting on $\begin{bmatrix} s \\ \mathbf{x} \end{bmatrix}$. This leads to the group isomorphism

$$\Lambda_C = \left(\Lambda_G^T\right)^{-1}. \tag{3.16}$$

A further hint of the duality can be seen by looking at the corresponding coordinate transformations.

- In the Galilean case we have $x^0 = ct \to t$ and then taking $c \to \infty$.
- In the Carrollian case we have $x^0 = ct \to s = Cct$ and taking $C \to \infty$.

To specify the full duality we need to look at the underlying manifold structure.

3.2.4 Bargmann space

[6] We can understand both Newton-Cartan and Carroll manifolds as part of a larger framework. In order to do this, we need to introduce a five-dimensional manifold called Bargmann space. Its flat structure is given by

$$B = \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}, \quad G = \sum_{A,B=1}^{d} \delta_{AB} dx^A \otimes dx^B + dt \otimes ds + ds \otimes dt, \quad \xi = \frac{\partial}{\partial s}. \tag{3.17}$$

Both s and t are null coordinates, meaning the length of ∂_s and ∂_t are both zero.

[2] That is, ignoring translations.

Group structure

We are interested in the set of diffeomorphisms that preserve both the metric structure and the vector field ξ. The associated group can be faithfully represented by matrices of the form

$$a = \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & e \\ -\mathrm{b}^T R & -\frac{1}{2}\mathrm{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Barg}(d+1,1). \tag{3.18}$$

Here $R \in O(3)$, $b, c \in \mathbb{R}^3$ and $e,\ f \in \mathbb{R}$. The matrices act on the vector space $\begin{bmatrix} \mathbf{x} \\ t \\ s \\ 1 \end{bmatrix}$.

While Bargmann space itself is 5-dimensional, the above representation of the Bargmann group acts on a 6-dimensional vector space. This allows for translations to be incorporated into the group structure.

The Bargmann group naturally occurs in the context of Galilean physics, it is a central extension of the Galilei group. It is relevant in the context of massive particles. We will talk about this in more detail in 3.3.2.

3.2.5 The duality

[6] The relation between the previous 3 spaces is given by the following observations.

- We have a group homomorphism $\pi : \mathrm{Barg}(4,1) \mapsto \mathrm{Gal}(4)$,

$$\pi(R, \mathbf{b}, \mathbf{c}, e, f) = (R, \mathbf{b}, \mathbf{c}, e). \tag{3.19}$$

  It consists of ignoring the basis vector s.

- We have an injective group homomorphism $\iota : \mathrm{Carr}(d+1) \hookrightarrow \mathrm{Barg}(d+1,1)$. It is explicitly given by $\iota(A, \mathbf{b}, \mathbf{c}, f) = (A, \mathbf{b}, \mathbf{c}, 0, f)$.
  The Carroll group is the Commutator subgroup of the Bargmann group. That is, the group given by elements of the form $aba^{-1}b^{-1}$. This can be shown easily once we have looked at the underlying Lie algebras.

Furthermore, we may find relations between these manifolds by utilizing two mathematical constructions. Given a map ϕ: $M \to N$. We can relate elements of the tangent and cotangent spaces of the respective manifold to each other via the constructions of a pullback and a pushforward respectively.

- Given a function $f : N \to \mathbb{R}$ we may define the pullback of f by ϕ as $\phi^* f = f \circ \phi : M \to \mathbb{R}$.
- The pushforward is used to relate a tangent vectors of M to those on N. $\phi_* : T_x M \to T_{\phi(x)} N$. Recall that a tangent vector can be thought of as a map that sends a function to its derivative in $\mathbb{R}$. We may then define the pushforward of a tangent vector V as $\phi_* V(f) = V(\phi^* f)$.
- The pullback is used to relate an element of the cotangent space of N to that of M. Let us assume ϕ is surjective. Given $dy \in T_y^* N : T_y N \to \mathbb{R}$ we may define $\phi^* dy : T_x M \to \mathbb{R}$. With $\phi(x) = y$ by $\phi * (V) \to d\phi(x)(\phi_* V)$.

This gives us the required tools to talk about the relations between object on our spaces.

Let us first consider the relationship between Bargmann and Newton-Cartan space. The General Bargmann manifold is given by (B, G, ξ) as given in subsection 3.2.4.

Let us now define another manifold N as the quotient of B over ξ. This means, practically speaking, that we have declared all points on the integral curves generated by ξ to be the same point. This provides a map $\phi : B \to N$.
The length of ξ is zero, so ξ is an infinitesimal isometry under G. Let G^{-1} be the inverse of G. We therefore have $L_\xi G^{-1} = 0$. Therefore, if $\phi(x) = \phi(y)$, the pushforwards of their respective tangent vectors should be the same. The pushforward of G^{-1} under ϕ is then given by $h^{\mu\nu}$ as given in section 3.2.1.
We may define a one-form υ on B using the related objects ξ and G. We define $\upsilon(w) = G(\partial_s, w) = dt$. As the notation suggests, it is indeed the pullback of the regular timelike one-form $d(\theta)$ under ϕ. Lastly, it has been shown in [12] that the connection on B naturally defines a connection on N that parallel transports both τ^μ and $h^{\mu\nu}$. As such, we can indeed define a Newton-Cartan manifold from a Bargmann manifold.

We are most interested in the Carrollian structure hidden in the Bargmann manifold. Let us consider again the one-form υ on B, explicitly given by $\upsilon = dt$. Let us define $\ker(\upsilon) = \{w \in TB : \upsilon(w) = 0\}$. Notice that since $G(\xi, \xi) = 0$, ξ belongs to this set. The set consists of all vectors that do not point in the timelike ∂_t direction. The procedure also defines a manifold in a natural way. By ignoring the time coordinate t. $\ker(\upsilon)$ defines a vector field on these submanifolds in a natural way, since they do no not point in the timelike direction. For a given time, say $t = 0$. We may define an embedding

$$\iota : C \hookrightarrow B. \tag{3.20}$$

We may endow C with a metric structure by virtue of a pullback $g^C = \iota^* G$. Since $G(\xi, \xi) = 0$, g^C must be degenerate.

Furthermore, since C has no t coordinate, vectors in $\ker(\upsilon)$ translate directly onto C. This means that for any X,$Y \in \ker(\upsilon)$ we have $\nabla^C_X Y = \nabla_X Y|_C$. We clearly have $\nabla_\xi = 0$ and $\nabla^C_g = \nabla G = 0$.
We have therefore defined C with the structure of a Carrollian manifold.

The full duality can now be seen in the following picture:

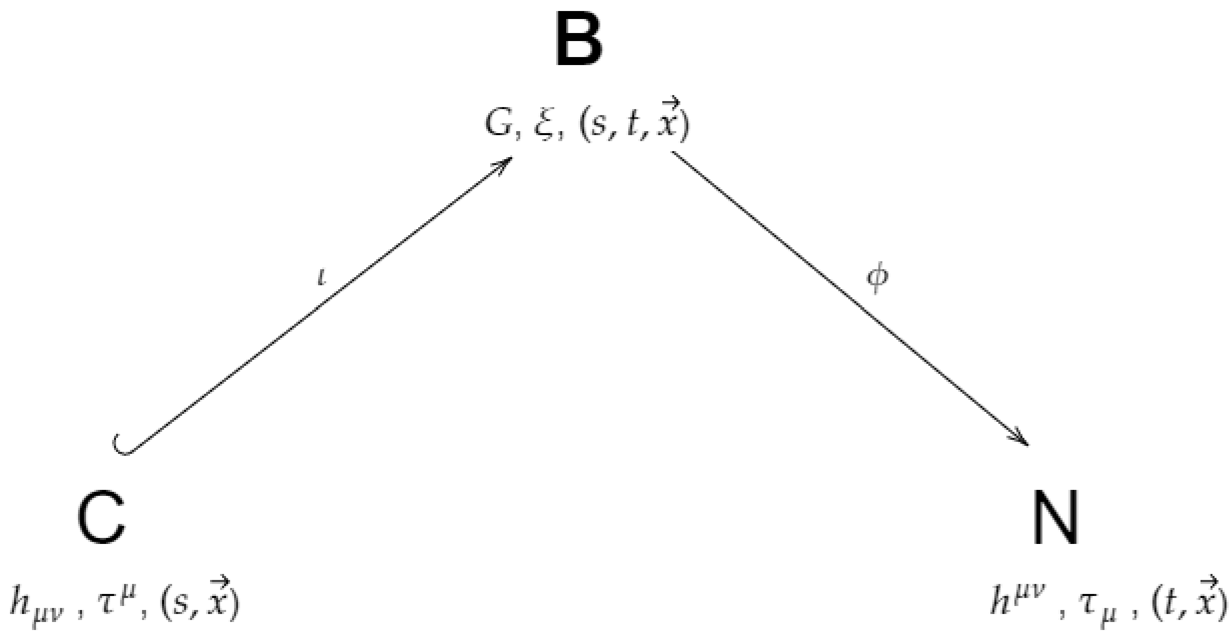

Figure 3.1: The duality between Carrollian and Newton-Cartan manifolds.

3.3 Lie algebra perspective

Given that we are trying to switch between different symmetry groups, it may be useful to consider the Lie algebras associated with the above manifolds.

3.3.1 The Carroll and Galilei algebra as an Inönü-Wigner contraction

[2] We may obtain the Carroll and Galilei algebra by virtually the same procedure. We start by considering the Poincaré algebra:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] =& -i\hbar(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}) \\ [M_{\mu\nu}, P_\rho] =& -i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\ [P_\mu, P_\nu] =& 0. \end{aligned} \tag{Poincaré algebra}$$

Here $M_{\mu\nu}$ contains the Lorentz algebra, and therefore contains both rotations and Lorentz boosts. P is the generator of translations in the appropriate directions. From this algebra, we may obtain a different algebra by rescaling a subset of the group elements in a consistent way. Such a procedure carries the name Inönü-Wigner contraction.
We split this into two subgroups, $ISO(1, k-1)$ and $ISO(D-k+1)$. The first one is of Lorentzian type [3]. The second one is Euclidean space. $ISO(1, k-1)$ will be denoted by α and $ISO(D-k+1)$ will be denoted by a. We will focus on the case $D = 3$, $k = 1$. This constitutes Minkowski spacetime. The Lorentzian subgroup will have a total dimension of 1 and will therefore be just the timelike coordinate.

We may consider two types of contractions. Where we rescale the momenta in the Lorentzian resp. Euclidean parts.

[3]This means it contains the temporal part.

- Rescaling the Lorentzian part P_α as $\tilde{P}_\alpha = \frac{P_\alpha}{\omega}$ and taking $\omega \to \infty$. To keep the algebras in contact with each other, we must scale boosts between the two regions $M_{a\alpha} = \tilde{B}_{a\alpha} = \frac{B_{a\alpha}}{\omega}$. Where we switched to B because they are now only boosts. This is the Carroll contraction. And the resulting Lie algebra would be the Carroll algebra.

- Rescaling the Euclidean part P_a as $\tilde{P}_a = \frac{P_a}{\omega}$ and taking $\omega \to \infty$. To keep the algebras in contact with each other, we must also scale boosts between the two regions $M_{a\alpha} = \tilde{B}_{a\alpha} = \frac{B_{a\alpha}}{\omega}$. Where we switched to B because they are now only boosts. This is the Galilei contraction and the resulting Lie algebra would be the Galilean algebra.

There is a clear duality $P_\alpha \leftrightarrow P_a$ in the algebras in general.
We remark that the behaviour of the boost starts to depend on the subgroup they act on. Given a Carrollian boost $B_{a\beta}$, we have $[B_{a\beta}, P_\alpha] = 0$, while $[B_{a\beta}, P_a] = i\eta_{ab}\tilde{P}_\alpha$. Since the Carroll group will be of special interest to us, we write its Lie algebra explicitly.

$$\begin{aligned}
&[J_{ab}, B_{0c}] = i\hbar(\delta_{ac}B_{0b} - \delta_{bc}B_{0a}) \\
&[B_{0b}, B_{0c}] = 0 \\
&[P_a, B_{0b}] = i\hbar\delta_{ab}(P_0)) \\
&[P_0, P_0] = [P_0, P_a] = 0 \\
&[P_0, J_{ab}] = 0.
\end{aligned} \tag{Carroll algebra}$$

Here we have split the Lorentzian $M_{\alpha\beta}$ into two parts: the rotational part J_{ab} and the Carroll boosts B_{0a}. The commutators of the spatial generators do not change.

There is a clear parallel between the the above rescaling and the coordinate transformation of p_0. From the rescaling of the momenta we can immediately extract some extra information from this. When we start from a Lorentz invariant system we have, we can find the momenta by applying the generators to the system. Then we may switch to the Carrollian generators and take the limit.

$$\hat{P}_{0C}\phi = \partial_s\phi = \frac{1}{C}\hat{P}_{0L} \tag{3.21}$$

This is similar to the rescaling of the generator of Carroll time translations.

$$P_0 \to \frac{P_0}{\omega}.$$

We may therefore identify ω from the Inönü-Wigner contraction with C from the coordinate transformation $x^0 \to Cx^0$. Thus, the two definitions of the Carroll limit align.

3.3.2 The Bargmann algebra

[10] The Galilean algebra is not the relevant Lie algebra for describing non-relativistic particles. This might seem counter-intuitive at first. Let us describe how this situation arises.
We start from a the regular Lagrangian of a free non-relativistic particle $\mathscr{L} = \frac{p^2}{2m}$. Contrary to the Lorentzian case, this Lagrangian is not invariant under the relevant symmetry. Under a Galilean boost the Lagrangian changes with a total derivative,

$$\mathscr{L}' = \frac{m(u+v)^2}{2} = \frac{p^2}{2m} + muv + v^2 = \frac{p^2}{2m} + mv\partial_t x + v^2. \tag{3.22}$$

The two additional terms are a total derivative an a constant respectively. The constant is of no consequence and we shall therefore ignore it. However, the total derivative hints at a central extension of the symmetry group. This Central extension of the Galilei algebra is called the Bargmann algebra. This algebra can be obtained by the following procedure [10].

To arrive at a Lagrangian that is truly invariant under the relevant transformation laws we may add an additional coordinate s to our system. We may now define a modified Lagrangian $\tilde{\mathscr{L}} = \mathscr{L} - \dot{s}$. Simultaneously, we centrally extend the Galilean algebra G to $\tilde{G}$. That is, we add an element M to the algebra that commutes with all other elements in the algebra.
Let us assume that $\mathscr{L}$ transforms with a total derivative of the function F, $\mathscr{L}' = \mathscr{L} + \partial_t F$. Denoting elements of the Galilean algebra by X, we have for the corresponding Lie derivatives $L_{X_k}F = f_k + c_k$. Here f_k is a function of the coordinates and c_k is a constant. We redefine

$$\tilde{X}_k = X_k + (f_k + c_k)\partial_s. \tag{3.23}$$

With this, we have allowed our vector fields to have a component in the newly added direction s.
We may now obtain the new Lie algebra. The commutators are now given by

$$\left[\bar{X}_j, \bar{X}_k\right] = [X_j, X_k] + \left(L_{X_j} f_k - L_{X_k} f_j\right) \frac{\partial}{\partial s}. \tag{3.24}$$

To obtain the Lie algebra we must obtain the right hand side in terms of $\tilde{X}$. We first need a preliminary result. We start from $L_{X_k}\mathscr{L} = \dot{f}_k$. It immediately follows that

$$L_{[X_j,X_k]}\mathscr{L} = L_{X_j}\dot{f}_k - L_{x_k}\dot{f}_j. \tag{3.25}$$

Introducing the structure constants of the Galilei algebra as $[X_j, X_k] = C^{\ell}_{jk}X_\ell$ and integrating yields

$$C^l_{jk}\left(f_l + c_l\right) = \left(L_{X_j} f_k - L_{X_k} f_j\right) + a_{jk}. \tag{3.26}$$

Here the a_{jk} are integration constants.
Using this result, we can rewrite the commutation relations as

$$\left[\bar{X}_j, \bar{X}_k\right] = C^l_{jk}\bar{X}_l - a_{jk}\frac{\partial}{\partial s}. \tag{3.27}$$

We have added to the commutators an additional term that commutes with all operators. The additional term does indeed commute, since all $\bar{X}_l$ are independent of s and the a_{jk} are constants. This is therefore indeed a central extension in the usual sense.

We may now show that $\tilde{\mathscr{L}}$ is indeed invariant under a Galilean boost. The new generator of Galilean boosts has changed via equation 3.23. Applying the generator of such a boost to $\tilde{\mathscr{L}}$ will give us $\dot{f}_k - \partial_t(f_k + c_k) = 0$.

$$\bar{X}_k\tilde{\mathscr{L}} = (X_k + (f_k + c_k)\partial_s)(\mathscr{L} - \dot{s}) = \partial_t(X_k F) - \partial_t(f_k + c_k) = \partial_t(f_k + c_k - f_k - c_k) = 0. \tag{3.28}$$

Where we have used that $\mathscr{L}$ is independent of s, as is X_k. So the modified Lagrangian is indeed Galilei-invariant.
The added generator ∂_s has an associated symmetry and Noether current. Varying to s requires a partial integration. Conservation of the boundary term yields $(\partial_t(m * \vec{x}) \cdot m * \vec{x} - s)|_{t_1}^{t_2} = 0$. s is independent of time, and momentum is still conserved by virtue of the equations of motion. So this reduces to $\partial_t(m) * \vec{x} \cdot \vec{x}|_{t_1}^{t_2} = 0$. A solution is given by the mass being constant in time. Therefore, the Noether current associated with the central charge symmetry is given by the mass. We mention the Bargmann algebra in full:

$$\begin{aligned} [J_{ij}, J_{kl}] &= 4\delta_{[i[k}J_{l]j]}, & [J_{ij}, P_k] &= -2\delta_{k[i}P_{j]} \\ [J_{ij}, B_k] &= -2\delta_{k[i}B_{j]}, & [B_i, H] &= -P_i \\ [B_i, P_j] &= -\delta_{ij}M. \end{aligned} \tag{Bargmann algebra}$$

For $M = 0$, this is exactly the Galilei algebra.

3.3.3 The Carroll algebra, again

With this added knowledge, we may consider some details on the Carroll algebra.

Firstly, we may verify explicitly that the Carroll algebra is indeed the commutator group of the Bargmann algebra. H is not a part of the commutators. That is, H cannot be written as the commutator of two operators. We are left with the spatial generators J, the spatial momenta P and the boosts B. Identifying M with P_0 in the Carroll algebra, we can see that these two are indeed the same. The identification of the mass M with the zero component of the momentum P_0 means that we may be able to interpret P_{0C} as a mass. This aligns with the direct limit of P_{0L}, $\frac{E}{c} \to \frac{E}{cC}$. Which indeed has a dimension of mass. This means particles could attain a mass in the Carroll limit. This mass could be distinct mass from the regular mass. As an example we recall that a photon has momentum $\hbar\lambda$ in a relativistic system, thus, we can associate a quantity with a dimension of mass to the photon: $\hbar\lambda c$. This would be conserved in the Carrollian limit and therefore serve as the "mass" of a photon.

Secondly, we may be interested in how the Carroll group behaves relating to central extensions. The Carroll group does not need to be centrally extended. The reason for the difference is the following: In the Galilean case $p^0_G = \lim_{c\to\infty} \frac{E}{c^2} = m$. This implies p^i_G changes under a Galilean transformation. This is contrary to the Carrollian transformation $p_{0C} = \lim_{C\to\infty} \frac{E}{cC} = 0$. So the p_i stay the same under a Carroll transformation, $p_i = p'_i$. as such, no central extension is needed.
Even if we start out from $\tilde{p}_\mu = p_\mu * C$. The loose p_i will transform, as seen in subsection 2.4.1, but to obtain a scalar we must contract them with τ^μ. So the spatial part will never appear in the Lagrangian. See also subsection 4.2.1.

We may also point out that $P_0 = \partial_s$ commutes with the entire Carroll algebra. This suggests that P_0 is itself a central extension of the algebra.

Given the above, we may make the following claim.

"Carroll velocity" is a misnomer. $P_0 = \partial_s$ is the symmetry associated with mass, it does not relate to any kind of velocity.

We may see this in the following cases.

- $\hat{P}_0$ is associated with the central extension of the Bargmann group, which is associated with mass.
- In taking the Inönü-Wigner contraction, we may consider the eigenvalues after the coordinate transformation. After the coordinate transformation $\hat{P}_0\phi = \frac{E}{cC}\phi$. The eigenvalues of $\hat{P}_0$ therefore have a dimension of mass.
- P_0 Commutes with all elements of the Carroll algebra.

3.3.4 A quantum mechanical approach

The appearance of the above central extension has an interesting consequence if we take a quantum-mechanical point of view. We quote the following equation from quantum mechanics.

$$\frac{d}{dt}A_H = \frac{1}{i\hbar}[A_H, H] + \left(\frac{\partial A_S}{\partial t}\right)_H \qquad \text{(Heisenberg equation)}$$

Here A is any operator and H is the Hamiltonian. All operators A can be written in terms the operators in the Lie algebra. Let us switch to the Carrollian case:

$$\frac{d}{ds}A_H = \frac{1}{i\hbar}[A_H, P_0] + \left(\frac{\partial A_S}{\partial s}\right)_H \tag{3.29}$$

Here the subscripts H and S denote the Heisenberg and Schrödinger picture respectively. Now, the Lie algebra tells us that $[A, P_0] = 0$. This means that $\frac{d}{ds}A_H = \left(\frac{\partial A_S}{\partial s}\right)_H$. As such, the expectation value of the operators is only time dependent if the operator itself is time dependent as well. The change in Carroll time is not tracked by the commutator with the Hamiltonian, but the commutator with the mass operator. This is why Carrollian particles do not move. From this, we can verify two already established properties of Carroll particles.

- Carroll particles cannot move. By filling in the position operator we obtain $\partial_s \hat{x} = 0$.
- Their momentum cannot change $\partial_s \hat{p}_i = 0$.

Notice that the Latter one does not stop the momenta from being non-zero.

Chapter 4

Carrollian systems

4.1 photons

With the added knowledge that we have obtained above, we may study Carrollian particles more rigorously. We may start by studying Carrollian electromagnetism. This will provide a case study for Carroll symmetric systems. This will allow us to explore the different methods in more detail. We follow the approach of [6].

4.1.1 Carrollian photons, by using rescaling and symmetry

We start by following [6] very closely. One can obtain a Carroll invariant version of Electromagnetism by applying the right rescalings and symmetries.
We may start from the Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{B} = 0 \\ -c^2 \nabla \times \mathbf{B} + \dfrac{\partial \mathbf{E}}{\partial t} = 0, \quad \nabla \cdot \mathbf{E} = 0 \end{cases} \tag{4.1}$$

In order to take the Carollian limit we can define

$$\tilde{\mathbf{E}} = \mathbf{E}, \quad \widetilde{\mathbf{B}} = (cC)\mathbf{B} \tag{4.2}$$

[1]. The Maxwell equations can then be rewritten as

$$\begin{cases} \nabla \times \widetilde{\mathbf{E}} + \dfrac{\partial \tilde{\mathbf{B}}}{\partial s} = 0, \qquad \nabla \cdot \tilde{\mathbf{B}} = 0 \\ -\nabla \times \tilde{\mathbf{B}} + C^2 \dfrac{\partial \widetilde{\mathbf{E}}}{\partial s} = 0, \qquad \nabla \cdot \tilde{\mathbf{E}} = 0 \end{cases} \tag{4.3}$$

The wave equation now gains the form

$$\left[\Delta - C^2 \left(\frac{\partial}{\partial s}\right)^2\right] \begin{pmatrix} \tilde{\mathbf{E}} \\ \widetilde{\mathbf{B}} \end{pmatrix} = 0. \tag{4.4}$$

Where $\Delta = \partial_i \partial^i$. This allows for an interpretation of C^{-1} as the propagation velocity of lightwaves with respect to s. [2] We may observe that C has a dimension of velocity $[C] = [s]/L = LT^{-1}$. The two time coordinates are still related by $s = cCt$.

[1]The rescaling is done in [6]. Indeed, in doing such a rescaling we arrive at the appropriate result. However, the magnetic field $\tilde{B}$ has a dimension of $[B] = Tm^2 s^{-2}$ instead of Tesla. Furthermore, such a rescaling is not needed in the other approaches provided.

[2]This formulation is can deceptive, C^{-1} does not have a dimension of velocity. While we may use it to refer to the underlying mathematical structure, we must be careful about the physical interpretation.

Electric-like contraction

Taking the Carroll limit on 4.3 gives us the electric type Carroll contraction:

$$\begin{cases} \nabla \times \tilde{\mathbf{E}}_e + \dfrac{\partial \tilde{\mathbf{B}}_e}{\partial s} = 0, \quad \boldsymbol{\nabla} \cdot \tilde{\mathbf{B}}_e = 0 \\ \dfrac{\partial \tilde{\mathbf{E}}_e}{\partial s} = 0, \quad \boldsymbol{\nabla} \cdot \tilde{\mathbf{E}}_e = 0 \end{cases} \tag{4.5}$$

Where $\tilde{\mathbf{E}}_e = \tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}_e = \tilde{\mathbf{B}}$.
It is invariant under the a Carroll boost, as given by

$$\begin{cases} \tilde{\mathbf{E}}_e(\mathbf{x}, s) \to \tilde{\mathbf{E}}'_e(\mathbf{x}, s) = \tilde{\mathbf{E}}_e(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) \\ \tilde{\mathbf{B}}_e(\mathbf{x}, s) \to \tilde{\mathbf{B}}'_e(\mathbf{x}, t) = \tilde{\mathbf{B}}_e(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) + \boldsymbol{b} \times \tilde{\mathbf{E}}_e(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) \end{cases}$$

Let us take a moment to see where this comes from. The transformation corresponds to the transformation of A_μ. A_μ transforms under a Carroll transformation as the inverse transpose of the Carroll transformation. Contrary to the Lorentzian case, this is not in itself a Carroll transformation. Explicitly:

$$A'_\mu = \begin{bmatrix} -\phi \\ \vec{A} - \vec{b}\phi \end{bmatrix}. \tag{4.6}$$

By definition $-\partial_i \phi - \partial_s A_i = E_i$ and $\nabla \times \vec{A} = \vec{B}$. Applying these definitions we indeed have $\tilde{E}_e \to \tilde{E}'_e$ and $\tilde{B}_e \to \tilde{B}'_e = \tilde{B}_e + \nabla \times \vec{b}\phi = \tilde{B}_e + \vec{b} \times \tilde{E}_e$.

The first of these equations is derived from
$\tilde{E}'_e = -\partial_i \phi(\vec{x}, s - \vec{b} \cdot \vec{x}) - \partial_s(A^i + b^i\phi) = -\partial_i\phi(s, x) + b^i \partial_s \phi - b^i \partial_s \phi - \partial_s A = -\partial_i \phi(s, x) - \partial_s A = \tilde{E}_e$.
Switching to the new coordinate system will indeed give the required transformation.

We furthermore mention that the Carrollian Maxwell equations 4.5 are given by the limit on the action:

$$S = \int \frac{1}{2} \left(\mathbf{E}^2 - c^2 \mathbf{B}^2 \right) dt d^3\mathbf{x} = (cC)^{-1} \int \frac{1}{2} \left(\tilde{\mathbf{E}}^2 - \frac{1}{C^2} \tilde{\mathbf{B}}^2 \right) ds d^3\mathbf{x}. \tag{4.7}$$

Where $\tilde{B} = \nabla \times \tilde{A}$, $\tilde{\mathrm{E}} = -\nabla\tilde{\phi} - \partial\tilde{\mathrm{A}}/\partial s$ here $\tilde{A} = CcA$. Dropping the prefactor and taking the limit $C \to \infty$ gives us

$$S_e = \int \frac{1}{2} \tilde{\mathbf{E}}_e^2 ds d^3\mathbf{x}. \tag{4.8}$$

magnetic type contraction

It is indeed true that

$$\begin{cases} \nabla \times \tilde{\mathbf{B}}_m - \dfrac{\partial \tilde{\mathbf{E}}_m}{\partial s} = 0, \quad \boldsymbol{\nabla} \cdot \tilde{\mathbf{E}}_m = 0 \\ \dfrac{\partial \tilde{\mathbf{B}}_m}{\partial s} = 0, \quad \boldsymbol{\nabla} \cdot \tilde{\mathbf{B}}_m = 0 \end{cases} \tag{4.9}$$

is invariant under the transformations

$$\begin{cases} \tilde{\mathbf{E}}_m(\mathbf{x}, s) \to \tilde{\mathbf{E}}'_m(\mathbf{x}, s) = \tilde{\mathbf{E}}_m(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) - \boldsymbol{b} \times \tilde{\mathbf{B}}_m(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) \\ \tilde{\mathbf{B}}_m(\mathbf{x}, s) \to \tilde{\mathbf{B}}'_m(\mathbf{x}, t) = \tilde{\mathbf{B}}_m(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) \end{cases}$$

[3] This is immediately clear by virtue of a symmetry between the E and B field. As an extra check, we may

[3] The paper [6] proposes $\tilde{\mathbf{E}}_m(\mathbf{x}, s) \to \tilde{\mathbf{E}}'_m(\mathbf{x}, s) = \tilde{\mathbf{E}}_m(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x}) - \boldsymbol{b} \times \tilde{\mathbf{E}}_m(\mathbf{x}, s - \mathbf{b} \cdot \mathbf{x})$.
This is a typo. Indeed $\nabla \cdot \tilde{E}'_m(x, s) = \nabla \cdot \tilde{E}_m(x, s) - b \cdot \partial_s \tilde{E}_m(x, s) \neq \nabla \cdot \tilde{E}_m(x, s)$.

verify the invariance explicitly. Let us start with the most involved calculation.

$$\nabla \times \tilde{\mathbf{B}}_m(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) - \frac{\partial \tilde{\mathbf{E}_m}(\mathbf{x}, s - \mathbf{b}\cdot x) - \times \tilde{\mathbf{B}}(x, s - \mathbf{b}\cdot\mathbf{x})}{\partial s - \ \mathbf{b}\cdot\mathbf{x}} =$$
$$\nabla \times \tilde{\mathbf{B}}_m(\mathbf{x}, s) - \mathbf{b} \times \partial_{s-\mathbf{b}\cdot\mathbf{x}} \tilde{\mathbf{B}}_m(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) - \frac{\partial \tilde{\mathbf{E}_m}(\mathbf{x}, s - \mathbf{b}\cdot x) - \mathbf{b} \times \tilde{\mathbf{B}}(x, s - \mathbf{b}\cdot\mathbf{x})}{\partial s - \mathbf{b}\cdot\mathbf{x}} =$$
$$\nabla \times \tilde{\mathbf{B}}_m(\mathbf{x}, s) - \frac{\partial \tilde{\mathbf{E}_m}(\mathbf{x}, s)}{\partial s}.$$

We furthermore have

$$\frac{\partial \tilde{\mathbf{B}}_m(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x})}{\partial s} = \frac{\partial \tilde{\mathbf{B}}_m(\mathbf{x}, s)}{\partial s},$$

$$\nabla \cdot \tilde{\mathbf{E}_m}(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) - \nabla \cdot \mathbf{b} \times \tilde{\mathbf{B}_m}(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) =$$
$$\nabla \cdot \tilde{\mathbf{E}_m}(\mathbf{x}, s) - \mathbf{b} \cdot \partial_{s-\mathbf{b}\cdot\mathbf{x}} \tilde{\mathbf{E}_m}(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) - \nabla \cdot \mathbf{b} \times \tilde{\mathbf{B}}(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) =$$
$$\nabla \cdot \tilde{\mathbf{E}_m}(\mathbf{x}, s) + \mathbf{b} \cdot (\nabla \times \tilde{\mathbf{B}}_m(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x})) - \nabla \cdot (\mathbf{b} \times \tilde{\mathbf{B}}(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x})) = \nabla \cdot \tilde{\mathbf{E}_m}(\mathbf{x}, s)$$

, Where we have used Ampére's law,

and

$$\nabla \cdot \tilde{\mathbf{B}_m}(\mathbf{x}, s - \mathbf{b}\cdot\mathbf{x}) = \nabla \cdot \tilde{\mathbf{B}_m}(\mathbf{x}, s) - \mathbf{b} \cdot \partial_{s-\mathbf{b}\cdot\mathbf{x}} \tilde{\mathbf{B}_m}(\mathbf{x}, s) = \nabla \cdot \tilde{\mathbf{B}_m}(\mathbf{x}, s)$$

because $\partial_s \tilde{\mathbf{B}_m}(\mathbf{x}, s) = 0$.

The magnetic type electromagnetism correspond to the transformation law of A^μ. It is given by

$$A^{\mu\prime} = \begin{bmatrix} \phi - \vec{b} \cdot \vec{A} \\ \vec{A} \end{bmatrix}. \tag{4.10}$$

Indeed,

$$E^i = (-\partial_i \phi - \partial_s A^i) \to C(\partial_i(-\phi - \vec{b}\vec{A}) - \partial_{s'} A^i) = (E^i) - (\partial_i b^j A_j - b^j \partial_j A^i).$$

While

$$(b \times B)^i = \epsilon^{ijk} b_j B_k = \epsilon^{ijk} \epsilon_{klp} b_j \partial^l A^p = (\delta^{il}\delta^{jp} - \delta^{ip}\delta^{jl}) b_j \partial^l A^p = \partial_i b^j A_j - b^j \partial_j A^i.$$

This implies

$$E^i \to E^i - (b \times B)^i. \tag{4.11}$$

So the transformation law does indeed stem from the transformation of A^μ and x^μ.
We may notice that the magnetic type contraction does not, yet, stem from a limit on the Lagrangian.

4.1.2 Carrollian photons, by using the four-vector potential

We may want a way to take the Carrollian limit without ad-hoc rescalings of other objects, as is done for the B field above. As is shown above, the distinct limits are associated with A^μ and A_μ respectively. Applying a coordinate transformation on A_μ and A^μ respectively, and then taking the limit gives us the results first found in [6].

Relativistically we have

$$\frac{1}{4}\int dx^3 dt\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{4.12}$$

The above Lagrangian is completely invariant under coordinate transformations. $F_{\mu\nu}F^{\mu\nu}$ should always be invariant under coordinate transformations. The Carroll limit will still give a different result since we will take the limit on four-vectors instead of the resulting equations.
We may compare this to the formulation in [6]. While $F_{\mu\nu}F^{\mu\nu} = E^2 + \frac{1}{c^2}B^2$ in both cases, there is a difference. We have added $\sqrt{-g}$ to the Lagrangian. This might seem redundant, given that we are working within special relativity. For finite values this is indeed true. But for in the limit a distinction exists, and given a choice, we will consider the general relativistic version. The Lagrangian seems therefore completely invariant. This can be explained by the fact that we ought to take the Carroll limit on four-vectors, not equations.

It may therefore be better to consider the system in terms of A^μ and A_μ. We have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under a coordinate transformation we have $A^0 \to C\phi$ and $A_0 \to -\frac{1}{C}\phi$.

A^μ

We may now, from A^μ define the electric and magnetic fields. Relativistically, we have $E_i = -\partial_i\phi - C\partial_s A_i$. And $B_c = \nabla \times A$. We may multiply ϕ by C while making the coordinate transformation $x^0 \to Cx^0$. This results in the following redefinition of the electric field:

$$\frac{E_{ic}}{C} = -\partial_i\phi - \partial_s A_i. \tag{4.13}$$

We have here decided to look at $\frac{E}{C}$ instead of C because it is conserved in the limit. This can also be explained by raising and lowering indices: $\frac{A}{C}^\mu = \tau^\mu\tau^\nu(CA_\nu)$. Thus, for the zero component of A^μ we need to look at $\frac{A^\mu}{C}$ instead of A^μ.

The Lagrangian then becomes

$$\frac{1}{4}\int dx^3 dt\sqrt{-g}F_{\mu\nu}F^{\mu\nu} = \frac{1}{4cC}\int dx^3 ds(E^2 + \frac{1}{c^2}B^2) = \tag{4.14}$$

$$\frac{1}{4cC}\int dx^3 ds C^2(\frac{E}{C})^2 + \frac{B^2}{c^2}. \tag{4.15}$$

For large C, the electric part is of leading order. Indeed, taking the purely electrical part of the Lagrangian and varying to both A and ϕ gives us the relevant equations of motion. Variation with respect to A gives us

$$\frac{1}{4cC}\int dx^3 ds\delta_A(\partial_i\phi + \partial_s A)^2 = \frac{1}{2cC}\int dx^3 ds\delta_A(\partial_s A) * (\partial_i\phi + \partial_s A) = \frac{1}{2C}\int dx^3 ds\delta_A(A)\partial_s(E).$$

So $\partial_s E = 0$. While variation to ϕ gives us almost immediately $\nabla \cdot E = 0$. The other two Maxwell equations follow directly from the definitions of the E and B field. As can be seen by

$$\nabla \cdot \vec{B} = \nabla \cdot \nabla \times \vec{A} \tag{4.16}$$

$$\nabla \times E_i = -\nabla \times (\partial_i\phi + \partial_s A_i) = -\partial_s\vec{B}. \tag{4.17}$$

Notice that obtaining the correct equations of motion requires us to allow for non-zero values of $\partial_s A$. Indeed the equations of motion urge us to consider that $\partial_s B$ is non-zero.
Of course, $\partial_s A = 0$ is a special case of this, so the equations of motion still apply.

A_μ, $\partial_s A = 0$

We may also be interested in a version of electromagnetism based on the covariant vector A_μ. In this case we have $\phi \to \frac{\phi}{C}$. The electric field then changes as $E \to \frac{E}{C} = \frac{-1}{C}\partial_i \phi - \partial_s A$. This goes to zero in the Carrollian limit if and only if $\partial_s A = 0$.
The Lagrangian would then be

$$\frac{1}{4cC}\int dx^3 ds (E)^2 - \frac{1}{c^2}(B)^2 \to \frac{-1}{4cC}\int dx^3 ds \frac{1}{c^2}(B)^2 = \frac{-1}{4cC}\int dx^3 ds \frac{1}{c^2}(\nabla \times A)^2.$$

Variation with respect to A would result in $\nabla \times B = 0$. This of course implies $\partial_s E - \nabla \times B = 0$, given that $E = 0$. Notice that there is no ϕ that is relevant to the equations of motion. $\nabla \cdot B = 0$ follows directly from the equations of motion. $\nabla \cdot E = 0$ comes about as a limit of the relativistic case. Lastly $\partial_s B = 0$ comes directly from $\partial_s A = 0$. This gives us the magnetic like contraction 4.1.1. But this is only possible because most of the terms are zero.

A_μ, $\partial_s A \neq 0$

We may obtain a more interesting system if we let go of the idea that E should transform as the zero-component of a four vector. In this case we may assume $\partial_s A \neq 0$. This would result in

$$\frac{1}{4cC}\int dx^3 ds (\partial_s A)^2 - \frac{1}{c^2}(\nabla \times A)^2.$$

Varying to A would now result in $\partial_s E - \nabla \times B = 0$ directly. $\partial_s B = 0$ is no longer implied in this case, but $\partial_s B = -\nabla \times E$ is now immediate. We can cast the equation of motion in a different light,
$0 = \nabla \times (\partial_s E - \nabla \times B) = \partial_s \nabla \times E - \nabla(\nabla \cdot B) + \Delta B = -\partial_s(\partial_s B) + \partial_i \partial^i B$. That is, B satisfies a wave equation. This is therefore a magnetic-like contraction.

Gauge symmetry

In classical electromagnetism, there exists a gauge symmetry. The equations of motion remain unchanged under $A^\mu \to A^\mu + \partial^\mu(f)$. In the relativistic case, this is equivalent to $A_\mu \to A_\mu + \partial_\mu(f)$. Here f is a twice continuously differentiable function that can depend on space and time. Under a Coordinate transformation the gauge condition transforms covariantly or contravariantly, as is appropriate to the index placement. In the limit, the gauge condition becomes $+\partial_\mu(f)$, but now in terms of Carrollian coordinates $\{s, \vec{x}\}$. It remains to be shown that the Carrollian Maxwell equations satisfy this gauge symmetry. In order to show this, it is enough to show that the E and B field already are gauge-invariant on their own.

$$E_i' = -\partial_i \phi' - \partial_s(A_i') = -\partial_i(\phi - \partial_s f) - \partial_s(A_i + \partial_i f) = -\partial_i \phi - \partial_s(A_i) = E_i \tag{4.18}$$

$$\text{and} B_i' = \nabla \times (A')_i = \nabla \times (A + \nabla f)_i = (\nabla \times A)_i = B_i \tag{4.19}$$

Notice that when seen as the limit of a relativistic system we have $\frac{\partial f}{\partial s} = 0$, so the gauge has become smaller.

4.1.3 Maxwell equations in terms of covariant and contravariant equations

There is a reason why the two types of limits coincide with covariant and contravariant transformations. The Maxwell equations themselves can be written in terms of co- and contravariant objects. Our discussion of the result of this an overview of some of the work done in [6].
Let start by pointing out two more concepts relevant to the underlying differential geometry.

- The wedge product $\wedge$. It is taken between two vectors u and v. The result will be a bi-vector $u \wedge v$. The wedge product is antisymmetric, so $u \wedge v = -v \wedge u$.
- The exterior derivative d. It is a kind of derivative and as such it sends k-forms to k+1-forms. It satisfies 3 main properties. For functions we require that df is the differential of f. We furthermore have $d(df) = 0$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p(\alpha \wedge d\beta)$ where in the last equation α is a p-form.

Using these concepts we may start rewriting the Maxwell equations with a higher level of mathematical abstraction. We start with the Lorentzian case. Let us first define the covariant object

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{4.20}$$

Notice that $F = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = dA$. Here A is given by $A_\mu dx^\mu$ because both $F_{\mu\nu}$ and $dx^\mu \wedge dx^\nu$ are antisymmetric.
This means necessarily that $dF = d(dA) = 0$. This is an equation expressed in terms of the covariant object $F_{\mu\nu}$. Furthermore, written in terms of the E and B fields, the equation reduces to $\nabla \times E + \partial_t B = 0$ and $\nabla \cdot B = 0$. This can be seen by explicit calculation, $dF = d(F_{\mu\nu} dx^\mu \wedge dx^\nu) = \frac{\partial F_{\mu\nu}}{\partial x^\alpha} dx^\mu \wedge dx^\nu \wedge dx^\alpha$. If any of α, μ, ν are the same, the result will be zero by antisymmetry of the wedge product. Hence, only components that can potentially be non-zero are the cases where all indices are distinct. We therefore restrict to the case where all indices are distinct. If one of the indices is zero we arrive at Faraday's law. If none of the indices are zero we have Gauss's law for the magnetic field. Thus, we have written two of the Maxwell equations in terms of the covariant object F.

The other two Maxwell equations can be written in terms of a contravariant object $F^{\#}$. We define

$$F^{\#} = \frac{1}{2} F_{\#}^{\alpha\beta} \partial_\alpha \wedge \partial_\beta \tag{4.21}$$

$$\text{with} \quad F_{\#}^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}. \tag{4.22}$$

Here g is the usual metric, we are still in the Lorentzian case. The other two Maxwell equations can now be written as the requirement that the contravariant divergence of $F^{\#}$ is zero. Indeed $\nabla_\alpha F_{\#}^{\alpha 0} = \nabla \cdot E$ and $\nabla_\alpha F_{\#}^{\alpha i} = \nabla \times B - \partial_t E$. The Maxwell equations can therefore be rewritten into a more compact form

$$dF = 0 \tag{4.23}$$

$$Div_g F^{\#} = 0. \tag{4.24}$$

Indeed, one of these has a covariant and one of these has a contravariant tensor to work with. Notice furthermore that, while $dF = 0$ follows directly form the definitions, the divergence could in principle be non-zero. A non-zero divergence would correspond to a source term being added to the system.

4.1.4 Carrollian photons, by starting from preserved co- and contravariant objects

[6] The above equations work perfectly well for the Lorentzian case. However, in the Carrollian case the metric is degenerate. This makes the construction of the required objects more difficult. Information will be lost when constructing the required objects. We will be operating in the Carroll spacetime (C, g, ξ, ∇) as

described in subsection 3.2.2.
There is a similarity between the approach given here and a previous observation. In section 3.1 we have seen that in the Carrollian case, we can either raise or lower indices, but not both. We can switch case by declaring either the spacial or temporal part of the momentum to be conserved. In Chapter 2 we have found two limits on the momentum that satisfy Carroll invariance. The two distinct limits occur by dividing and multiplying by factors of C before taking the limit. All of this fits into a larger picture. The problem of raising and lowering indices is similar to the rescalings of the appropriate momenta. We may

- Demand a co- or contravariant object remains conserved.
- Then find the other object by demanding that indices can be raised or lowered respectively by using the metric complex.

By raising and lowering indices we have lost the same information we would have if we had taken the Carroll limit. The information we lost is exactly the information we loose if we divide by C and as a result of either a redefinition or a coordinate transformation and then take the limit.

The methods are however not completely similar, the conserved objects correspond to those four-vectors where a component would be sent to infinity. Let us demonstrate this for the electromagnetic case.

We will use latin indices a, b, c for the Carrollian case, this will reserve the greek indices for objects existing in Bargmann space.

Contravariant Carroll Electromagnetism

[6] We start by assuming the existence of the contravariant object $F_m = \frac{1}{2}F^{ab}\partial_a \wedge \partial_b$. We may de fine a corresponding covariant object by raising indices:

$$F^\flat = \frac{1}{2}(F_m^\flat)_{ab}dx^a dx^b, \tag{4.25}$$

$$\text{where} \quad (F_m^\flat)_{ab} = h_{ac}h_{bd}F_m^{cd}. \tag{4.26}$$

Since h is degenerate, we have lost information by "lowering" the indices. More explicitly, we may start with

$$F_m = E^A\partial_A \wedge \partial_s + \frac{1}{2}\epsilon^{ABC}B_C\partial_A \wedge \partial_B. \tag{4.27}$$

After lowering indices we arrive at

$$F_m^\flat = \frac{1}{2}\epsilon_{ABC}B^C dx^A \wedge dx^B. \tag{4.28}$$

We can now demand equations similar to those of the Lorentzian case,

$$dF_m^\flat = 0 \tag{4.29}$$

$$Div_g F_m = 0. \tag{4.30}$$

This aligns with the magnetic type Carroll electromagnetism seen in subsection 4.1.1.

Covariant Carroll Electromagnetism

[6] To find a covariant version of Carrollian electromagnetism we start from the covariant 2-form expression

$$F_e = \frac{1}{2}F_{ab}dx^a \wedge dx^b. \tag{4.31}$$

To find a contravariant object, we will make use of $\xi = \partial_s = \tau^l \partial_l$. This is the only contravariant object at hand. We consider the one-form $E^\flat = -F_e(\xi)$. Converting this to a vector can only be done by virtue of the h^{ab}. Indeed we have $E^\# = h^{ab} E^\flat_a = -h^{ab} \frac{1}{2} F_{al} \tau^l$. We may then finally define the required object:

$$F_e^\# = E^\# \wedge \xi. \tag{4.32}$$

Locally, it is given by

$$F_e^\# = \frac{1}{2} \left(F_e^\#\right)^{ab} \partial_a \wedge \partial_b \quad \text{where} \quad \left(F_e^\#\right)^{ab} = 2 \, \mathrm{g}_\varphi^{k[a} \xi^{b]} F_{k\ell} \xi^\ell = 2h^{k[a} \tau^{b]} F_{kl} \tau^l. \tag{4.33}$$

Where g is now the Carrollian "metric" defined in 3.2.2. The Maxwell equations, when applied to the above two objects yield the electric type electromagnetism from 4.1.1. This can be seen from recalculating the divergence terms $\nabla_a F^{\# ab} = \nabla \cdot E$ if $b = 0$ and $\partial_s E = 0$ if $b \neq 0$.

4.1.5 From Bargmann space

[6] The last method of arriving at Carroll invariant systems that we consider here is a direct consequence of the fact that we may obtain a Carrollian structure from Bargmann space, as we have seen in 3.2.5. We will show this explicitly. We follow [6].

Let us start from a Maxwell theory on a Bargmann manifold,

$$dF = 0 \tag{4.34}$$

$$Div_G F^\# = 0. \tag{4.35}$$

Here F is a 2-form on a Bargmann manifold. The Carroll manifold is given by the embedding $\iota : C \hookrightarrow B$ given by $t = constant$. In general, we may obtain the twice-contravariant tensor by a pullback with ι. While the bi-vector needs to be defined by adding additional restrictions.

Magnetic-like case

[6] Let us start with the bi-vector $F^{\mu\nu}$ in Bargmann space. We may restrict it to the sub-manifold C by imposing $F_m = F^{\mu\nu}|_C$ with $F(G(\xi)) = F^{\mu\nu} G_{\nu\alpha} \tau^\alpha = 0$. Notice that since $G_{\nu\alpha}\tau^\alpha$ is a vector in the timelike direction, the second condition requires that our vectors have no component in the timelike direction. Defining $Div(F_m) = Div(F)|_C$ gives us half of the required equations.

The other half must be obtained from $F_{\mu\nu} = G_{\mu\alpha} G_{\nu\beta} F^{\alpha\beta}$. Now, the restriction of G to the sub-manifold C is exactly the Carrollian metric $h_{\mu\nu}$. Thus, restricting all objects to C will result in $F_{\nu\mu}|_C = h_{\mu\alpha} h_{\nu\beta} F^{\alpha\beta}$. And, since $dF = 0$, its pullback is still closed This means $dF = 0$, even on the Carrollian manifold.

Electric-like case

[6] Let us start by defining the pullback $F_e = \iota^*(F)$ [4]. Since $dF = 0$, we must have $dF_e = 0$ as well.
For the bi-vector $F^\#$ we need to do an extra bit of work. On Bargmann space, it is defined by $F^{\#\mu\nu} = G^{\mu\alpha} G^{\nu\beta} F_{\alpha\beta}$. Our first step will be to consider the restriction of $F_{\alpha\beta}$ to C. As an additional requirement we have, again, $F^{\mu\nu} G_{\nu\alpha} \tau^\alpha \partial_t = 0$. These are well-defined bi-vectors on the sub-manifold C. The restriction to C can now be written as $F_e^{\mu\nu} = h^{\mu\alpha} h^{\nu\beta} F_{\alpha\beta}|_C$. Notice that, in this case, we have not completely gotten rid of any s-dependence because $h^{\mu\nu}$ contains a degeneracy. Next, we want to arrive at a coordinate expression for $F^{\mu\nu}$. We may observe $F^{\mu\nu}\tau_\mu = 0$. This, combined with antisymmetry, means that $F^{\mu\nu}$ can be written in terms of

$$F^\# = E^\# \wedge \xi. \tag{4.36}$$

[4]This is well defined, since ι is surjective if seen as a function to the submanifold $t = constant$.

Where we have already inserted a yet to be determined vector $E^{\#}$. We have encountered this object before in 4.1.4, $E^{\#} = h^{\mu\nu}F_{\nu\alpha}\tau^{\alpha}$. This can be seen directly by comparison to $h^{\mu\alpha}h^{\nu\beta}F_{\alpha\beta}$. We may now obtain the last equation by setting $Div(F_e^{\#}) = Div(F^{\#})|_C$.

4.2 Carrollian particles

In section 3.1 we have seen the formalism that we will be using to describe our Carroll particles, and in section 3.1.1 we saw how such a formalism could be implemented in the Galilean case. We will now show how this is done in the case of Carrollian particles, which are the particles we are actually interested in. We will encounter a structure similar to what we have found for electromagnetism.

4.2.1 Particles, as a limit on a relativistic system

We can again take the Carrollian limit of a relativistic particle. We will use $\frac{\partial\tau}{\partial s} = \frac{\partial\tau}{\partial t}\frac{\partial t}{\partial s} = \frac{1}{cC\gamma}$. Furthermore, the relativistic limit will focus on the temporal part of the momentum. Therefore, we will consider $p_C^{\mu} = \frac{p^{\mu}}{C}$ and $p_{C\mu} = Cp_{\mu}$. We may immediately take the limit:

$$\int d\tau c * \sqrt{-g^{\mu\nu}p_{\mu}p_{\nu}} \tag{4.37}$$

$$= \int ds\frac{1}{C\gamma}\sqrt{C^2\tau^{\mu}\tau^{\nu}p_{\mu}p_{\nu} - h^{\mu\nu}p_{\mu}p_{\nu}} \to \frac{1}{C}\int ds\tau^{\mu}p_{C\mu} = \frac{1}{cC}\int dsc\frac{1}{\gamma}Cmc\gamma.$$

Since only the time-like part is left, this results in

$$= \frac{1}{cC}\int\frac{c}{\gamma}\tilde{p}_0 ds = \frac{1}{cC}\int dsmc^2. \tag{4.38}$$

If we want to focus on the spacial part, we have a minus sign under the square root. This can only make sense if we are considering tachyons. We must assume the Carroll velocity to be nonzero, in accordance with 2.4.2. This changes the Limit of the gamma factor. $\gamma = \frac{1}{\sqrt{1-(Cu_s)^2}} \to \frac{1}{iCu_s}$. As a direct consequence we now have $\frac{\partial\tau}{\partial s} = \frac{\partial\tau}{\partial t}\frac{\partial t}{\partial s} = \frac{1}{cC\gamma} \to \frac{i|u_s|}{c}$.

$$\int d\tau c * \sqrt{-g_{\mu\nu}p^{\mu}p^{\nu}} = \int dsi|u_s|\sqrt{-h_{\mu\nu}p^{\mu}p^{\nu} + \frac{1}{C^2}\tau_{\mu}\tau_{\nu}p^{\mu}p^{\nu}} \tag{4.39}$$

$$\to \int ds|u_s|\sqrt{h_{\mu\nu}p^{\mu}p^{\nu}} = \int ds|u_s|mc \tag{4.40}$$

Where we have used that $p^2 = (mc)^2$ for the tachyons. Within the context of tachyons we have again mc^2 in our Lagrangian.
Let us first clarify why we are fine with the prefactors before the actions. Under a coordinate transformation the Jacobian changes, the Jacobian has a dimension. In this case a dimension of inverse velocity. We can either choose to leave the prefactor outside the Lagrangian or change the dimensions of the Lagrangian. In the case of massive particles. We elect the former. This means that while the Lagrangian now has a non-zero limit, the action does not. In the tachyonic case, we leave everything inside the Lagrangian. This is possible because there is no factor C involved.

We have two distinct particles. One normal particle and a particle that looks like the previously encountered Tachyon. The Tachyon is associated with the contravariant momentum and transforms as such. The massive particle can only be obtained by considering that $p_{\mu C} = \frac{p_{\mu L}}{C}$ and then taking the limit of $Cp_{0C} = p_L$. This is similar to how we arrived at the Carroll momentum Cp^{μ} in section 2.4.1. The fact

that we need a different momentum four-vector in the different limits makes sense. Equations tend to only have one limit. Without a redefinition of the momentum we would only have one limit.
We also note that in a sense, the "information" about the relativistic particle is split into two. The tachyon only cares about the spacial components of the momentum, while the massive particle only requires the temporal part.

The equations of motion follow almost immediately. For the massive particle we have only the temporal part of the momentum, there is no position dependence. The Hamiltonian is also zero.
The tachyonic particle seems to have some dynamics. There is a dependence on velocity. The momentum is non-zero: $\frac{\partial L}{\partial \dot{q}} = mc^3 \frac{\partial |u_s|}{\partial u_s^i} = mc^3 \frac{u^i}{|u_s|}$. In accordance with what we know from the tachyons. The equations of motion yield an interesting result:

$$\partial_s p^i = \frac{\dot{u_s}^i |u_s| - u^i * (\frac{u_{sj}\dot{u_s}^j}{|u_s|})}{|u_s|^2}. \tag{4.41}$$

When contracted with u_{si} the result becomes

$$u_{si}\partial_s p^i = \frac{u_{si}\dot{u}^i |u_s| - (u_{sj}\dot{u_s}^j |u_s|)}{|u_s|^2} = 0. \tag{4.42}$$

The inner product between the velocity and the derivative of the momentum is zero. The direction of momentum and velocity are still the same. Thus, while momentum can change its direction, it cannot change its magnitude. This is in accordance with $p^2 = (mc)^2$. Even if we add a potential, as long as the potential is not explicitly dependent on the velocity. This remains true. [5]
If we want our Lagrangian to be Carroll invariant, we need for the Lagrangian L to have $L'(s, x, \dot{x}) = L(s - b \cdot x, \dot{x}) = L(s, x, \dot{x})$. The simplest way to achieve this is by having the Lagrangian not be explicitly dependent on Carroll time, and having the Carroll velocity be zero.
This is the case for the non-tachyonic particle. The tachyonic particle will still have a Noether current possibly associated with a mass.

4.2.2 Particles, by starting from preserved co- and contravariant objects

The equations of motion of a free relativistic particle are given by $\frac{\partial p^\mu}{\partial \tau} = 0$. Even in this case, we may distinguish between the covariant and contravariant equations of motion, $\frac{\partial p^\mu}{\partial \tau} = 0$ and $\frac{\partial p_\mu}{\partial \tau} = 0$. In the Carroll limit these two take on an entirely separate meaning. This is similar to what happened to the Lagrangian. We proceed by the method laid out in 4.1.3. To be explicit: We take the limit on the relativistic equation of motion. This includes the relativistic 4-momentum. This takes two steps.

- The coordinate transformation on the four-vector p^μ, affecting the timelike coordinate.
- The Limit $C \to \infty$.

The contravariant equation of motion $\frac{\partial p_L^\mu}{\partial \tau} = 0$ corresponds to

$$\frac{\partial p_L^\mu}{\partial \tau} = \frac{1}{\gamma} \frac{\partial p_L^\mu}{\partial Cct} \tag{4.43}$$

This is trivially zero for all spacial components. The only equation of motion comes from the timelike component. We will therefore use the four-vector $p_C^\mu = \frac{p_L^\mu}{C}$. A coordinate transformation gives the zero

[5] This behaviour reminds us of photons. They cannot change the size of their momentum in a potential.

component an extra factor of C,such that it is conserved in the limit.

$$\frac{1}{\gamma}\frac{\partial p_L^\mu}{\partial Cct} = \frac{1}{\gamma}\frac{\partial p_C^\mu}{\partial s} \tag{4.44}$$

Notice that only the temporal component of the four-vector has remained. Thus, this should describe the massive particle. Indeed, since γ is independent of s in the massive case, we have

$$\frac{\partial m}{\partial s} = 0 \tag{4.45}$$

This corresponds to the massive particle case. As is the case with the Maxwell equations, we could have found the relevant equations of motion by assuming p_μ is conserved and then "raising indices" via $p_C^\mu := \tau^\mu \tau^\nu p_{C\nu}$. Indeed, a similar term appears in the relevant Lagrangian.

As an additional verification we may do two additional calculations.
Firstly: the Hamiltonian is now $\frac{\partial L}{\partial \dot{x}^i}\dot{x}_i - L = -L$. Therefore, everything commutes with the Hamiltonian. And since $L = mc^2$, H is associated with a mass.
Secondly, the velocity. For p_C^μ we do indeed have $\frac{\vec{p_L}}{C} = \frac{m}{C}\gamma * u = \frac{m u_s c}{\sqrt{1-C^2 u_s^2}} \to 0$. This is in accordance with $P_C^\mu = \tau^\mu \tau^\nu p_{\nu C}$. Since τ^i should always be zero.

Starting from the covariant equations of motion we may calculate the limit directly. $\frac{\partial p_\mu}{\partial \tau} = \frac{\partial p_\mu}{\partial s}\frac{\partial s}{\partial \tau} = \frac{\partial p_\mu}{\partial s} cC\gamma \to \frac{c}{i|u_s|}\frac{\partial p_{C\mu}}{\partial s}$. Since the momentum is also imaginary in this context, and the prefactor is nonzero, this translates to $\partial_s m_c \frac{\dot{x}_i}{|\dot{x}|} = 0$. Notice that the timelike component of p_μ has rescaled by a factor $p_0 \to \frac{p_0}{C}$. Therefore $p_{0,C} = 0$ in the Carroll limit, without reference to any equations of motion. This result corresponds to the tachyonic case. The result can be obtained directly from the corresponding Lagrangian without any need for recalings. We may find the same equations of motion by assuming the contravariant momentum is conserved. Similar to what is done in 4.1.3. In this case, it will suffice to just raise the indices by virtue of $h_{\mu\nu}p^\nu$. This gives us the required equation of motion.

4.2.3 Mass shell

It is informative to look at the Carroll limit of the mass shell condition. Relativistically, the mass shell condition applies to particles that satisfy the equation of motion. We will do the same in the Carroll limit. The equations of motion are satisfied by te massive particles $p_C^\mu := \tau^\mu \tau^\nu p_{C\nu}$ and the tachyons $p_\mu := h_{\mu\nu}p^\nu$ respectively.
In the first case the mass shell condition must be written in terms of the metric, $g_{\mu\nu}p^\mu p^\nu = m^2c^2$. Taking the limit we get

$$g_{\mu\nu}(\frac{p_L^\mu}{C})(\frac{p_L^\nu}{C}) = \frac{m^2c^2}{C^2} \tag{4.46}$$

$$h_{\mu\nu}p_C^\mu p_C^\nu = 0. \tag{4.47}$$

The redefinition of the Carrollian momentum must be compensated by dividing by a factor $\frac{1}{C^2}$ on the right hand side. In the second case, we start from $g^{\mu\nu}p_\mu p_\nu$.
Taking the limit we get

$$C^2\frac{g^{\mu\nu}}{C^2}p_\mu p_\nu = 0 \tag{4.48}$$

$$-\tau^\mu \tau^\nu p_\mu p_\nu = 0 \tag{4.49}$$

Where we have divided out a factor of C^2 to get to the limit. We see that for all particles, the limit of the mass shell condition sets either temporal or spacial components to zero. These equations are indeed satisfied. $h_{\mu\nu}p^\mu = h_{\mu\nu}\tau^\mu\tau^\alpha p_\alpha = 0$ and $\tau^\nu p_\nu = \tau^\nu h_{\mu\nu}p^\mu = 0$ by definition.

4.3 Comparing the methods

4.3.1 Comparison with Relativistic → Carroll

We have seen two distinct methods for finding Carroll invariant systems.

- Applying a coordinate transformation $x^0 \to Cx^0 = s$ and taking the limit $C \to \infty$ on p^μ or p_μ respectively.
- Not applying any coordinate transformation, but instead adopting a Carroll framework from Bargmann space. This automatically imposes $h_{\mu\nu}p^\mu := p_\nu$ and $\tau^\mu\tau^\nu p_\mu := p^\nu$.

We may wonder why we are interested in the more abstract approach to find a Carrollian structure, taking the limit $C \to \infty$ on a relativistic system should also yield a Carrollian system. And indeed it does, as we have shown with multiple examples. There is however a critical flaw in this approach. The metric g naturally translates to $h_{\mu\nu}$ by virtue of a projection to the spacial coordinates. $i : h_{ij} = g_{ij}, h_{0\mu} = h_{\mu 0} = 0$. More importantly, this corresponds to the limit $\lim_{C\to\infty} g_{\mu\nu} = h_{\mu\nu}$.
This cannot be done for the corresponding timelike vector $\tau^\mu\partial_\mu = \partial_s$. We may of course define $\tau^\mu\tau^\nu = \frac{1}{C^2}g^{\mu\nu}$, but the extra factor of $\frac{1}{C^2}$ points at the underlying problem. The contravariant objects do not naturally translate into one another. One can be obtained from the other by a limit, but they cannot exist simultaneously. As such we cannot translate timelike vectors on a relativistic manifold to timelike vectors on a Carrollian manifold. We can therefore obtain only the covariant version of the Carroll limit by taking the limit in this way. We have solved this problem by redefining the four-vectors in the case where we are interested in the timelike part. We have $\frac{p^\mu}{C}$ and $p_\mu * C$ as four-vectors instead.
The more abstract approach of explicitly defining $p^\mu = \tau^\mu\tau^\nu p_\mu$ or $p_\mu = h_{\mu\nu}p^\nu$ respectively does not have this problem.

We have seen explicitly that the two above approaches give the same result in at least two cases. Let us show that this holds in general.

Focusing on the spacial part first we have $p_\mu = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \to \begin{bmatrix} \frac{p_0}{C} \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \to \begin{bmatrix} 0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = h_{\mu\nu}p^\nu.$

Regardless of the value of p^0. Here the first arrow indicates a coordinate transformation to s and the second arrow indicates the limit $C \to \infty$. Therefore it does not matter that we should apply a coordinate transformation to p^0, the information is lost.

Similarly, for the timelike part, we have $p_C^\mu = \frac{p_L^\mu}{C} = \frac{1}{C}\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} \to \frac{1}{C}\begin{bmatrix} Cp^0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \to \begin{bmatrix} p^0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \tau^\mu\tau^\nu p_{C\nu} = p_{L\nu} * C.$

Here the first arrow indicates a coordinate transformation to s and the second arrow indicates the limit $C \to \infty$. Notice that, since $\tau^i = 0$ the spacial components do not matter for the last equality. And for the timelike part of p_C, not rescaling or rescaling with $p_C = p_{L\nu} * C$ and applying the coordinate transformation yield the same result.

The difference between the two methods only comes up in the information that is lost when raising or lowering indices. If we can always write our equations in a way that does not require those parts by raising and lowering indices before taking the limit. Therefore, the results will be the same in both approaches.
However, we cannot always do this. In the electromagnetic cases, both the co- and contravariant objects are necessary. There is therefore a distinction between the two cases which we will investigate in the next section. Now knowing that there are two completely distinct ways of arriving at Carrollian systems, we have the following picture of the duality.

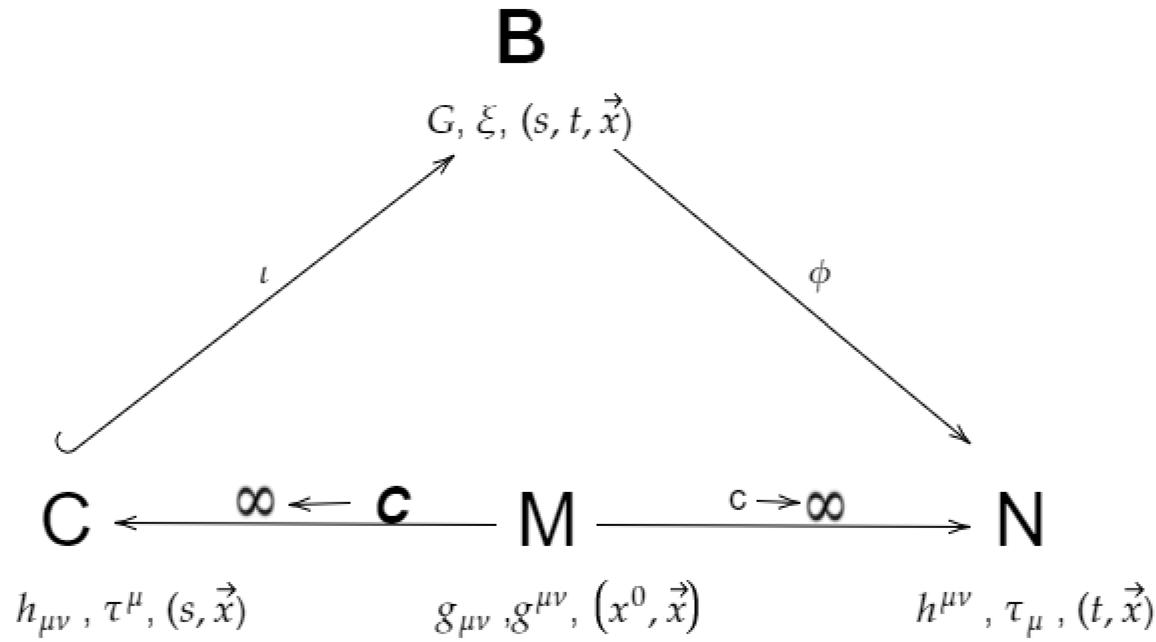

Figure 4.1: The duality between Carrollian and Newton-Cartan manifolds.

4.3.2 Co- and contravariant systems

Regardless of method, the two particles are described by either a covariant or a contravariant object.
As let us for example look at the photons. If we start from A^μ and start rescaling we obtain magnetic type electromagnetism. The "covariant electromagnetism" proposed in 4.1.4 conserves A_μ. The contravariant object A^μ is then defined by $A^\mu := \tau^\mu \tau^\nu A_\nu$. This is equal to the A'^μ by a rescaling of C. $A^\mu = A'^\mu = \frac{(C\phi, \vec{A})}{C}$.
We find a similar situation for the particles. The situation is captured in the following table:

	Equations of motion are invariant under	Rescaled object	Preserved object
Electric type	A_μ	A^μ	A_μ
Magnetic type	A^μ	A_μ	A^μ
massive particles	p^μ	p^μ	p_μ
tachyonic particles	p_μ	p_μ	p^μ

In both of these case one of the objects is assumed to be preserved. The other object can be obtained by either raising/lowering indices or working from the coordinate transformations. There is however a key distinction. From the electromagnetic case, we see that both the co- and contravariant four-vectors have a clear physical meaning. We cannot write all maxwell equations as one or the other. We have, implicitly, chosen to not rescale one of the four-vectors when calculating the Electric and Magnetic types. The difference in methods matters now, and investigating it will shed some light on the tachyon-like behaviour we have been seeing.

4.3.3 Tachyonic behaviour

As an important observation. Derivatives should in principle go to zero in the Carroll limit, $\partial_s = \frac{1}{C}\partial_{x^0}$. However, we consistently find that the required symmetries work for non-zero values of the time derivatives. The cases where time derivatives are zero is just a special case of this more general class. It is the only one that is consistent with the limit as it is taken from relativistic objects.

equivalence in the methods, electric type

Let us first recall the two methods.

- Assume $F_{\mu\nu}$ is conserved, then define $F^{\mu\nu} = 2h^{k[\mu}\tau^{\nu]}F_{kl}\tau^l$. Since $F^{\mu\nu}$ can be written in terms of $A^{\mu\prime}$, this roughly translates to assuming A'_μ is conserved and defining $A^{\mu\prime} = \tau^\mu\tau^\nu A'_\nu$.
- Demanding indices can be raised, and thereby redefining the vectors as $A^\mu_C = \frac{A^\mu}{C}$ and $A_{C\mu} = CA_\mu$. Now we also have $A^\mu = \tau^\mu\tau^\nu A_\nu$. Then applying the coordinate transformation $x^0 \to Cx^0$. Therefore in essence transforming the spacial parts.

For these two methods to be equivalent. We must have $A^{\mu\prime} = A^\mu_C$, which is true, and $A'_\mu = A_{C\mu}$. The spacial components of the latter equation are given by $\vec{A}' = C\vec{A}_C$. This has an important consequence. For the electric type, we have allowed for $\partial_s A' \neq 0$. This is not a manifestation of non-zero derivatives of real quantities, but of $\partial_s A' = \partial_s(C\vec{A}) = \partial_0\vec{A}$. This is not a Carrollian quantity at all. It just looks like one if we assume that A_μ is conserved.

equivalence in the methods, magnetic type

Let us first recall the two methods.

- Assume $F^{\mu\nu}$ is conserved, then define $F^{\mu\nu} = h_{\mu\alpha}h_{\nu\beta}F^{\alpha\beta}$. Translates to assuming $A^{\mu\prime}$ is conserved and defining $A_{\mu\prime} = h_{\mu\nu}A^{\nu\prime}$.
- Applying a coordinate transformation $x^0 \to Cx^0$. This will affect the timelike parts of A^μ and A_μ. We can now lower indices, but not raise them, $A_\nu = h_{\mu\nu}A^\mu$.

For these two methods to be equivalent. We must have $A_{\mu\prime} = A_\mu$ which is true, and $A^{\mu\prime} = A^\mu$. The temporal components of the latter equation are given by $\phi' = C\phi$. This has an important consequence. For the magnetic type electromagnetism, we have allowed for $\partial_s\phi' \neq 0$. This is not a manifestation of non-zero derivatives of real quantities, but of $\partial_s\phi' = \partial_s(C\phi) = \partial_0\phi$. This is not a Carrollian quantity at all. It just looks like one if we assume that ϕ is conserved.

Equivalence in the methods, massive particle

Let us first recall the two methods.

- Assume p'_μ is conserved, then define $p'^\nu = \tau^\mu\tau^\nu p'_\mu$.
- Redefine $p_{C\mu} = p_{L\mu}C$ and $p^\mu_C = \frac{p^\mu_L}{C}$ such that $p^\mu_L = C^2\tau^\mu\tau^\nu p_\nu$ implies $p^\mu_C = \tau^\mu\tau^\nu p_{C\nu}$. Then applying a coordinate transformation $x^0 \to Cx^0$. Effectively changing the spacial parts.

For these two methods to be equivalent. We must have $p^{\mu\prime} = p^\mu$, which is true, and $p_{\mu\prime} = p_\mu$. The spacial components of the latter equation are given by $p'_i = Cp_i$. This has an important consequence. For p'_i to be conserved we must have $p_i \to 0$ in the Carroll limit. This is consistent with the ad-hoc assumption $u_s = 0$ before taking the limit.

Equivalence in the methods, tachyonic particle

Let us first recall the two methods.

- Assume p'^{μ} is conserved, then define $p'_{\nu} = h_{\mu\nu} p'^{\mu}$.
- Applying a coordinate transformation $x^0 \to Cx^0$. Therefore changing the temporal parts. Since the spacial parts are still the same, we have $p_{\nu} = h_{\mu\nu} p^{\mu}$.

For these two methods to be equivalent. We must have $p_{\mu'} = p_{\mu}$, which is true, and $p^{\mu'} = p^{\mu}$. The temporal components of the latter equation are given by $p^{0\prime} = Cp^0$. This has an important consequence. For $p^{0\prime}$ to be conserved we must have $p^0 \to 0$ in the Carroll limit. Now $p^0 = mc\gamma$ so the assumption $u_s \neq 0$ would yield $p^0 \to \frac{-imc}{u_s C} \to 0$. While $u_s = 0$ does not. Therefore further verifying that these are indeed tachyons.

Chapter 5

Conclusion

We have set out to investigate what physical systems can arise in the Carrollian limit.

In this thesis we have adopted the viewpoint of [6]. That is, we apply a coordinate transformation $x^0 \to Cx^0 = s$ and take the Limit $C \to \infty$. This is contrary to the more common definition of $c \to 0$. From this viewpoint, we have taken the Carroll limit on multiple objects. We have seen how these objects transform in the Carroll limit.
The paper also proposes a more abstract approach of arriving at Carrollian symmetries. The more mathematical approach forces us to consider the symmetries from the perspective of manifolds. This has possible relevance to a theory of Carrollian gravity that takes hint from General relativity. In studying this a duality between the Carrollian and Gallilean systems becomes visible. Both can be seen as they are related to the overarching Bargmann group. This prompts an alternate definition of the Carroll limit. Seeing the Carroll group as a subgroup of the Bargmann group, we may define Carrollian particles by using pullbacks and pushforwards of objects related to the Bargmann group. This duality and new way of finding the Carroll limit is also laid out in [6].

Our own contribution has consisted of an explicit discussion on four-vector formalism. We have explained how the a modified version of the regular four-vector formalism applies to Carrollian quantities. As a main difference, we have seen that the metric splits into a spacial and a temporal parts, which should be treated as seperate objects. As a result of this, two distinct sets Carrollian four-vectors arise. This explains the existence of two distinct Carrollian particles seen in [6].
We have applied this approach to some different particles. Applying this to electromagnetism has resulted in agreement with the before mentioned paper. While applying this to the massive relativistic particle has also resulted in two different limits. Furthermore, we have compared the two different perspectives on Carrollian symmetries. One that is obtained by taking a limit on a relativistic system and one that is obtained by adopting the objects relevant to the Bargmann group. Requesting consistency between the two perspectives gives us a better understanding of tachyons. While they cannot be obtained from a relativistic system without making some additional assumptions. They belong to the more general class of Carroll invariant systems.

In my perspective, the work regarding Carrollian systems is far from done. On a theoretical level the duality between Carrollian and Galilean systems could lead to further development in both fields. Where we remind the reader that Galilean physics is still an active field of study, especially in the context of Newton-Cartan theory [8]. Furthermore, tachyons tend to pop up in a variety of calculations and a different perspective could possibly be useful. We also have yet to find some real-life applications for these kinds of symmetries. We mention a few possibilities.

- Physical systems near the black hole horizon. For light moving radially toward a Schwarzschild black hole we have $c(r) = \frac{\partial r}{\partial t} = c(1 - \frac{r}{r_H})$. Near the black hole horizon, this goes to zero. A similar result would be attained if we switch from t to $s = cCt$.

- In [7] it is argued that for a gas made of Carroll particles, we have $\varepsilon + P = 0$. This could yield a candidate for the inflaton.

- Related to the last point, we might want to find a Carrollian version of general relativity. This is done in [3]. The general idea is the following. General relativity can be restated as Poincaré gauge theory. Here a Lorentz symmetry is explicitly implemented on the tangent space. Implementing Carrollian symmetry instead yields a theory of Carrollian gravity. On these tangent spaces we may still view the Carrollian symmetry as either given by the limit on a relativistic system or taken from Bargmann space.

- Carrollian symmetry could be used in the context of fluids [13]. This could be connected to theories of fluids near a black hole.

With this work we hope to have contributed to the general understanding of Carrollian systems, as well as having provided a possibly new approach to working with the tachyons that show up in various theories.

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