



Utrecht University

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# Ready or Not, Here I Come!

– Bachelor Thesis about a Non-Cooperative Search Game –

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# 1 Introduction

In this thesis we will describe the field of Search Theory from a game theoretical perspective. Our main goal is to provide the pure minimax and optimal search strategies of the Linear Search Problem on an infinite line which will be concluded at the end of chapter 4. The idea is that the searcher acquires a better indication of his searching efforts when he compares his pure minimax and optimal strategies with a general search strategy. This will be the focus of this thesis.

In general, Each chapter and section will start by stating what the chapter is about. In addition to the contents of the chapter we present the literature where the chapter is based on. The parts which are changed or added are also touched upon. It may be that some specific references are not mentioned in the opening of a chapter, this will be only the case when the content behind the citation is used in a limited amount. An overview of the chapters in this thesis are given below.

Beginning in chapter 1 with a small section about the history of search theory. Chapter 2 will provide a foundation of game theory such that a reader who is new to Game Theory will become familiar with the minimax-maximin Theorem. The minimax-maximin Theorem will show that the best of the worst of outcome of the searcher will be the same as the best of the worst outcome of the hider. The theorem also provides an important value which is used to define the optimal strategies. The minimax-maximin Theorem for finite games was introduced and proved by John von Neumann in *Zur Theorie der Gesellschaftspiele* [18] and in this thesis we shall show a theorem which useful for infinite search games. The remainder of this chapter is based on the book that is used to teach the course Game Theory at University Utrecht: *Game Theory, A Multi-Leveled Approach* by Hans Peters [15] and provides game theoretical definitions.

The first part of chapter 3 will explain how the definition in chapter 2 will apply for a non-cooperative search game with a non-mobile hider. Such search game may happen in a variety of other search spaces such as: trees, networks, finite number line(s), multidimensional region, or circles to name a few examples. The second part will introduce an application of such non-cooperative stationary game: the Linear Search Game on an infinite line and introduce several theorems on scaling search games such that computation becomes possible. This scaling property will be the main result of chapter 3, since it provided a way to compute the (pure) minimax strategy and optimal strategy. We end this chapter with a useful sequence representation for the search trajectories and the cost function which is used for computations in the next chapter.

In chapter 4 we will compute two important types of search strategies: the pure minimax strategy and optimal strategy for the Linear Search Game. To find a solution, we will focus a large part on the equality between the cost function of a geometric sequence and the cost function of sequence that represent a search strategy. We will apply this result to

a functional which is relevant to compute the pure minimax strategy in the Linear Search Game. We also apply part of the theorems used for the pure minimax strategy to find an expression for the optimal strategies. We end the chapter with an overview between the two strategies and their payoffs.

At last, we will provide several surface level introductions such that the Linear Search Game can be expanded by: adding a turning cost, assuming a mobile hider, generalize to the Star Game (multiple infinite lines), add a probability of detection. We also introduce a more fundamental change which is the cooperative version of the search game: the Rendezvous Game and by combining the Linear Search Game and the Rendezvous Game a search game with unknown (non) cooperative intention can be described.

## 1.1 History of Research on Search Theory

We base this section on the article of Stone [16] which divided the history of search theory into four eras up until 1989. This timeline is sufficient to introduce the problem which we will explore in this thesis, and thus we use the four eras as the structure for this section. The third era is expanded with the history of the Linear Search Problem based on Section 8.1 of Alpern&Gal [4]. At the end of the thesis we will provide further research. We start of by talking about the beginnings of Search Theory and make the translation from the probability based search theory, to the Linear Search Problem while characterizing the four eras.

The interest in search theory started during the Second World War. The United States Navy tasked the Operations Evaluation Group, where Bernard Osgood Koopman was the author, to report on the search of German submarines (U-boats) in the Atlantic Ocean. This report was classified as confidential for more than a decade after the war, but became declassified in 1958. This work [13] is founded on the assumption that there exists a probability distribution at the location of the target, which is known to the searcher. The report has many extensive illustrations which will give an idea of what such probability density should look like. These illustrations also help in determining a route for a plane. The period from 1942, when this report was started, up until 1965 we name the Classical Era and is characterized by the maximizing the payoff per unit of effort spend. James M. Dobbie [7] provided a detailed survey of the different publications in the field of Classical Search Theory.

The second era from 1965 up to 1975 is named the Mathematical Era and is characterized by understanding more abstract search optimization problems. These problems also often deal with an immobile target. It can also be seen as a generalization of the previous era, where understanding the problem is more important than the application. Stone himself has worked extensively in this era with his work "Theory of Optimal Search" [17] which focused on finding a optimal distribution of effort spent. It is also the era in which movable targets were introduced.

One characterization will hold for all theory stated before, the theory is one-sided. Hereby

is meant that the searcher has no interaction with the target and that the target has no intentions. For example, searching for a crashed plane at sea is a one sided search problem for which the target is stationary. In contrast, when playing hide and seek, the target is also stationary, but he has intentions not to cooperate. As such, we call the target a hider to give an idea about his intention. This implies that hide and seek is two-sided and thus needs a different approach to fully describe this situation.

Game Theory introduces several interesting concepts which can give an answer to the intentions of the players. Also, we can describe the relation between the searcher and target in a way that they both influence each other which is expressed with a payoff. Game Theory is build on the concept that the two players will influence each other, and for this will be the case for all search games described by this thesis. Next, we will introduce the third era.

The third era from 1975 up to 1985 is called: the Algorithmic Era. This era is largely characterized with the use of more powerful computers compared to the previous eras. This changed the approach of exploring the problems. Although some problems would remain analytical. One of such problems is the one discussed in this thesis: the Linear Search Game. This originally Linear Search Problem was introduced by Bellman [6] and independently Beck [5]. Throughout the years Beck has provided characteristic properties of the optimal solution. The Linear Search Problem was also extended with bounded resources by Foley, Hill and Spruill [9]. Also a minimax-strategy on an interval is given by Hipke et al. [12]. The Linear Search Problem is easy to explain, but it is a lot harder to give an answer than one might think. We will be focusing on the Linear Search Problem in Gal [11] which was streamlined in Alpern&Gal [4] to find a solution of the minimax and optimal strategies of the Linear Search Problem (or Linear Search Game as we eventually call it). In this era, Steve Alpern also introduced the Rendezvous Game in Alpern [1] which is the cooperative counterpart of search games that we will describe in this thesis.

The last era from 1985 up to 1989 (the year that Stone [16] is published) is called the Dynamic era, and will not be explored in this thesis. The era is characterized by adding simple feedback mechanisms into the search problems. This could be used when searching for an airplane crash, and the searcher finds a piece, but has not located the whole crash site yet. This new information might change the approach of the searcher. This characterization was actually already explored in the Mathematical era, but because of the lack of computational power this idea could not get to fruition yet.

## 2 Introduction of Search Games

The main focus of this chapter is providing an introduction of general game theoretical definitions to a reader who is new to Game Theory. Game Theory originally started with the Poker passion of John von Neumann. His work "Zur Theorie Der Gesellschaftspiele" [18] laid the ground works for what we now define as Game Theory. Our introduction will be based on a more modern description from Hans Peters called "Game Theory: a Multi-Leveled Approach" [15]. Game Theory is a field of mathematics that studies the relation between parties, called players, to describe a situation in which competition and/or cooperation is present. This study tries to develop generalizations for strategies and their outcomes. An example of a general strategy would be a strategy that represents the 'best of the worst' outcomes for a player. Such a strategy is called a minimax strategy and understanding it will be the result of section 2.2, we will also define the pure minimax strategy, which we will use in Chapter 4. We try to accomplish this understanding with use of an example: the Matching Pennies game.

We the structure Section 2.1 based on Chapter 1 (the Prisoners Dilemma) and Chapter 6 of Peters [15] where we gave a more rigorous definition for the probability measures used for the mixed strategies and we give an accompanying expected payoff function which are both based on Gal [11]. The separation in notation between pure and mixed strategies is inspired by Gal [11] and Alpern&Gal [4].

The finite minimax-maximin Theorem which we will discuss in Section 2.2 is introduced by John von Neumann and proved in "Zur Theorie Der Gesellschaftspiele" [18] for finite two player zero sum games. This theorem is used to determine the Value of the Game, which is used in the definition of optimal strategies. Because the article is in German and an English translation is not easily available, we also based a part of this section on "Theory of Games and Economic Behavior" by John von Neumann and Oskar Morgenstern [19]. We changed the introduction of the (pure) minimax (and respectively maximin) strategies by first introducing (pure) maximal (and respectively minimal) strategies, similar to Alpern&Gal [4] where our notation is based on.

### 2.1 Introduction to Game Theory

To get a understanding of the minimax-maximin Theorem, we will be introducing several definitions in Game Theory. We will use the Prisoners Dilemma throughout the definitions and use the Matching Pennies game as an introduction for the zero sum games. We picked the Prisoners Dilemma as example game, as most Dutch students learn about this dilemma in high school economy classes. If you have completed this subject on a Dutch high school, you will likely recognize definitions below. We picked the Matching Pennies game because this game shows the importance of mixed strategies.

We will solve The Matching Pennies fully and Prisoners Dilemma partially in this subsection



and explain several fundamental definitions along the way. In section 2.2 we also solve the matching pennies game by finding the minimax strategy. The two games will provide enough context to understand the way we describe the minimax-maximin Theorem and what the result will imply for the search games in chapter 3.

### 2.1.1 Prisoners Dilemma

As mentioned, the Prisoners Dilemma is one of the most well known games. It defines the hard choice that two prisoners have to make in which cooperation and competition have conflicting interest. Two prisoners have committed a crime together and are interrogated separately. A prisoner has the following options:

1. he may 'cooperate' (C) which means 'not betray his partner' or;
2. he may 'defect' (D), which means 'betray his partner'.

The punishment for the crime is up to 10 years of prison. When both prisoners defect, they will both be granted a reduction of 1 year. If both prisoners cooperate, then they are convicted to 1 year for a minor offense. If one defects while the other one cooperates, then the defector is set free, while the cooperator will serve prison time with no reduction. The prisoners dilemma may contain different amount of years for the punishment in various literature. The difference is mostly in the non-restricted amount of years which in our case is 10 years but is sometimes replaced by 3 or 25 years.

### 2.1.2 Pure Strategies

The prisoners dilemma has two choices which is already shown in section 2.1.1. We do not call 'cooperate' and 'defect' a choice from here on, instead we use the term: Action. A sequence of actions is called a strategy and describes a plan to play the game. Note that this sequence must be the exact length to play the game to completion. Since the strategies of the Prisoner Dilemma has a sequence of length one, the terms action and strategy overlap in this game. In general, for the set of players  $N = \{1, \dots, n\}$  with  $n \geq 1$ :

**Definition 2.1** (Pure Strategies). *For player  $i$ , the set of pure strategies  $\mathcal{PS}_i$  is a set containing all strategies that the player will use to complete the game. A pure strategy is a strategy  $s_i \in \mathcal{PS}_i$  for which every action is deterministic, meaning that strategy  $s_i$  will perform the same actions on repeated use.*

In the Prisoners Dilemma we call the strategies 'cooperate' and 'defect' pure strategies. Later on we will explain mixed strategies and how they expand on the pure strategies. In our Prisoners Dilemma we have  $n = 2$  with  $\mathcal{PS}_1 = \mathcal{PS}_2 = \{C, D\}$ .

### 2.1.3 Payoff

Each prisoner faces a different number of years in prison based on the strategy they and their partner will play. To quantify the effects of all strategies being played on a player, a payoff function is used. The results of the pure payoff function in the prisoners dilemma can be nicely shown in a matrix (this certainly does not hold for all games). We use the row strategies for player 1 and column strategies for player 2. This matrix consists of pairs of pure payoffs  $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{Z}^2$  where  $\mathbf{u}_1$  corresponds with the pure payoff of player 1 when he plays the strategy in front of the row, and  $\mathbf{u}_2$  corresponds with the pure payoff of player 2 when he plays the strategy above the column. The pure payoff matrix is:

$$\begin{array}{cc} & \begin{array}{c} \text{C} \\ \text{D} \end{array} \\ \begin{array}{c} \text{C} \\ \text{D} \end{array} & \begin{pmatrix} -1, -1 & -10, 0 \\ 0, -10 & -9, -9 \end{pmatrix} \end{array}$$

where entries are negative such that maximizing the payoff is logical. This would result in a player who has the goal to get their payoff to zero, implying zero years of prison time. For a game with more than two players, we use a generalized definition which will be applicable to more than just two player matrix games.

**Definition 2.2** (Pure Payoff Function). *For every player  $i \in N$ ,  $\mathbf{u}_i : \mathcal{PS} = \mathcal{PS}_1 \times \dots \times \mathcal{PS}_n \rightarrow \mathbb{R}$  is the pure payoff function; i.e. for every strategy combination  $(s_1, \dots, s_n) \in \mathcal{PS}$  we have the pure payoff  $\mathbf{u}_i(s_1, \dots, s_n) \in \mathbb{R}$  which represents what player  $i$  gains (may also be negative, meaning he losses that amount) by playing  $s_i$  when every other player plays  $s_j \in \mathcal{S}_j$ .*

For example, the pure payoff function written in a general notation for the prisoners dilemma for both players is:

$$\mathbf{u}_1(s_1, s_2) = \begin{cases} -1 & \text{for } s_1 = C, s_2 = C \\ 0 & \text{for } s_1 = D, s_2 = C \\ -10 & \text{for } s_1 = C, s_2 = D \\ -9 & \text{for } s_1 = D, s_2 = D \end{cases} \text{ and } \mathbf{u}_2(s_1, s_2) = \begin{cases} -1 & \text{for } s_1 = C, s_2 = C \\ -10 & \text{for } s_1 = D, s_2 = C \\ 0 & \text{for } s_1 = C, s_2 = D \\ -9 & \text{for } s_1 = D, s_2 = D \end{cases}.$$

It is quite obvious why this function notation is not popular for small discrete games with two players. This style is simply too extensive and confusing compared to the payoff matrix. But for games that are larger, are continuous or contain more than two players this generalized notation will provide the information we need, like for example the Linear Search Game in chapter 3.

### 2.1.4 Mixed Strategies

We can increase the number of possible strategies by introducing a new type of strategy. Instead of choosing one strategy to play, we can formulate a mixed strategy as a random variable with distribution given by a probability measure on the set of pure strategies of the player. Then the player will choose a strategy at random. For such a probability measure we will define the measure and random variable associated as:

**Definition 2.3** (Mixed Strategy). *The inverse image associated with random variable  $S_i : \mathcal{PS}_i \rightarrow \mathbb{R}$  defined as*

$$S_i^{-1}([a, b]) = \{s_i \in \mathcal{PS}_i : S_i(s_i) \in [a, b]\}$$

where  $a, b \in \mathbb{R}$ . *The distribution of  $S_i$  is a probability measure on  $\mathbb{R}$  is defined as*

$$\mathbb{P}(S_i^{-1}([a, b])) = \mathbb{P}(\{s_i \in \mathcal{PS}_i : S_i(s_i) \in [a, b]\}).$$

*Such a measure  $\mathbb{P} \circ S_i^{-1}$  is called: A mixed strategy. We will denote  $\mathbb{P} \circ S_i^{-1}$  as  $S_i$  for ease of notation. It will be clear from context if we mean the random variable or the measure. The strategy set that contains all mixed strategies a player  $i$  will be called  $\mathcal{MS}_i$ .*

With these probabilities comes a major downside. We cannot guarantee that the payoff will be the same on repeated use. To give some idea about the payoff we will use instead the expected payoff for mixed strategies.

**Definition 2.4** (Expected Payoff Function). *Let  $S_1, \dots, S_n$  be mixed search strategies for  $n$  players. The expected payoff function for the  $i$ -th player is given by*

$$U_i(S_1, \dots, S_n) := \mathbb{E}[\mathbf{u}_i(S_1, \dots, S_n)] = \int_{\mathcal{PS}_n} \cdots \int_{\mathcal{PS}_1} \mathbf{u}_i(s_1, \dots, s_n) (dS_1 \times \dots \times dS_n) \quad (1)$$

*for all  $i = 1, \dots, n$ . We use  $dS_i$  instead of the formal  $d\mathbb{P}$  notation because each random variable  $dS_i$  has an associated probability measure which we will use when writing  $dS_i$ . This notation is makes it easier distinguish measures and shown more clearly the which strategies are played.*

The union of the two strategy sets  $\mathcal{PS}_i$  and  $\mathcal{MS}_i$  is the set of all strategies for player  $i$  and is defined as

**Definition 2.5** (Player Strategies). *The strategy set containing all strategies of player  $i$  is defined as*

$$\mathcal{S}_i := \mathcal{PS}_i \cup \mathcal{MS}_i$$

*where  $\mathcal{PS}_i$  is the set of all pure strategies and  $\mathcal{MS}_i$  is the set of all mixed strategies of player  $i$ .*

We will also combine the payoff function and the expected payoff function for this union of the pure and mixed strategy sets such that we get a payoff for all strategies of a player  $i$  defined as

**Definition 2.6** (Payoff function). *Given a strategy set  $\mathcal{S}_i$  defined as definition (2.5) for every players  $i \in N$  such that we get a payoff function  $u_i(s_1, \dots, s_n)$  defined as*

$$u_i(s_1, \dots, s_n) = \begin{cases} \mathbf{u}_i(s_1, \dots, s_n) & \text{when } s_i \in \mathcal{PS}_i \text{ for all } i \in N \\ U_i(s_1, \dots, s_n) & \text{otherwise} \end{cases} \quad (2)$$

*meaning that when  $s_1, \dots, s_n$  contains one or more mixed strategies, then the cost function  $U_i$  is used. This is possible since every  $s_i \in \mathcal{PS}_i$  can be viewed as deterministic under the measure. The function  $\mathbf{u}_i$  is defined as in Definition 2.1 and the function  $U_i$  given by equation (1) in Definition 2.4.*

### 2.1.5 (Strict) Dominating Strategies

To solve a game, we do not have to look at each strategy with the same intensity. Some strategies can be regarded as inferior with the use of elimination by a dominating strategy. We will describe only strict domination, since we do not use weak domination. When domination is mentioned in this thesis, we always refer to strict dominance.

**Definition 2.7** (Strict Domination). *A strategy  $s'_i \in S_i$  of player  $i$  is strictly dominated by  $s_i^* \in S_i$  if*

$$u_i(s_1, \dots, s_{i-1}, s_i^*, s_{i+1}, \dots, s_n) > u_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$$

*for all  $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \dots \times \mathcal{S}_n$ , i.e., for all strategy combinations of players other than player  $i$ .*

### 2.1.6 Best Reply

When several years of prison are at stake you do not only want to analyse the payoffs, but also a way to choose the best strategy in response to your partners strategy. We do this for player 1 as follows: For every column, place a star at every first entry  $(s_1, \cdot)$  which is equal to the maximum payoff of that column. Do the same for player 2 with the second entry  $(\cdot, s_2)$  for every row. This results in the following matrix:

$$\begin{array}{cc} & \begin{array}{c} C \\ D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} -1^*, -1^* & -10, 0^* \\ 0^*, -10 & -9, -9 \end{pmatrix} \end{array}$$

Each payoff with a star is called: best reply of a player to the strategy of the opponent. We can write these down in the following way for player 1 as:

- When player 2 plays strategy  $C$ : Choose  $C$ , because  $u_1(C, C) = -1 > -10 = u_1(D, C)$ .
- When player 2 plays strategy  $D$ : Choose  $C$ , because  $u_1(C, D) = 0 > -9 = u_1(D, D)$ .

For player 2 we get the same best replies because this game has the same payoff for both players. In general the best reply is defined as:

**Definition 2.8** (Best Reply). *A best reply of player  $i$  to the strategy combination*

$$(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \dots \times \mathcal{S}_n$$

*of the other players is a strategy  $s_i^* \in S_i$  such that*

$$u_i(s_1, \dots, s_{i-1}, s_i^*, s_{i+1}, \dots, s_n) \geq u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_n)$$

*for all  $s_i \in S_i \setminus \{s_i^*\}$ .*

Each player wants to choose the strategy which is a best reply. When all players can choose their best reply we have achieved an equilibrium. Because of the works of John Nash [14] describing this equilibrium, we call this a Nash Equilibrium. When looking back at the Prisoners Dilemma we see that the strategy pair  $(C, C)$  both are a best reply and thus we know that  $(C, C)$  is a Nash Equilibrium.

### 2.1.7 Zero Sum Games

A zero sum game has the condition that the sum of all payoffs from players of  $N = \{1, \dots, n\}$  are equal to zero, formally stated as:

$$\sum_{i=1}^n u_i(s_1, \dots, s_n) = 0 \quad (3)$$

where for all  $i \in N$  we have  $s_i \in \mathcal{S}_i$ . We will only be interested in a two player zero sum game. Under the assumption that  $n = 2$  we get that equation (3) simplifies to  $u_1(s_1, s_2) = -u_2(s_1, s_2)$ . It is also common to define a non-player specific payoff function  $\mathbf{u}(s_1, s_2)$  where  $\mathbf{u}$  is the pure payoff function and it follows that the expected payoff function will also be given by a non-player specific expected payoff function  $U$  given by

$$U(S_1, S_2) = \int_{\mathcal{P}\mathcal{S}_2} \int_{\mathcal{P}\mathcal{S}_1} \mathbf{u}(s_1, s_2) (dS_1 \times dS_2)$$

resulting in the non-player specific payoff function  $u$  for a two player zero sum game. Note that this is only to show that the notation has changed. As an example we use the Matching Pennies game. In the two-player game of Matching Pennies, both players have a coin and simultaneously show heads or tails.

- If the coins match, player 2 gives his coin to player 1.
- Otherwise, player 1 gives his coin to player 2.

This means that the amount that player 1 gains is equal to the amount that player 2 loses, and visa versa. We use a matrix to show the payoffs, note that we only show the payoff  $u$  of player 1 since we just need to invert the sign for the payoff  $u$  of player 2:

$$\begin{array}{cc} & \begin{array}{cc} \text{heads} & \text{tails} \end{array} \\ \begin{array}{c} \text{heads} \\ \text{tails} \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

This zero sum game example will be used to explore the result from the minimax-maximin Theorem for a finite two player zero sum game.

## 2.2 The Minimax Strategies

In this section, we start with the minimax and maximin strategies which are based on the definitions given in Gal [11] such that the definitions in this section nicely convert to the next chapter. We will show that the best of the worst payoff of both player will be equal for some pair of  $s_1 \in \mathcal{S}_1$  and  $s_2 \in \mathcal{S}_2$ . The minimax-maximin Theorem for finite zero sum games will not hold for an infinite game, like the Linear Search Game in chapter 3. We will introduce the minimax-maximin Theorem for infinite games in the beginning of chapter 3. We do however discuss an example to create intuition on how to use the minimax-maximin Theorem and to show the usage of mixed strategies.

### 2.2.1 Minimax and Maximin Strategies

To accommodate the fact that each opponent is working against the other's interest, the minimax (or maximin) criterion selects for each player a strategy which yields the best of the worst possible outcomes. We will define the maximin strategy for player 1 with use of a minimal value strategy and define the (pure) minimax strategy for player 2 with use of a maximal value strategy as:

**Definition 2.9** (Maximin Strategy). *A minimal value strategy is a (pure or mixed) strategy  $s'_2 \in \mathcal{S}_2$  such that for every (mixed or pure)  $s_1 \in \mathcal{S}_1$  there exists a value  $\nu_1(s_1)$  for player 1 as*

$$\nu_1(s_1) := \min_{s_2 \in \mathcal{S}_2} u(s_1, s_2) = u(s_1, s'_2)$$

*A maximin strategy is a pair of strategies  $s'_1 \in \mathcal{S}_1$  and  $s'_2 \in \mathcal{S}_2$  such that there exists value  $V_1$  as*

$$V_1 := \max_{s_1 \in \mathcal{S}_1} \nu_1(s_1) = \nu_1(s'_1) \quad \text{i.e.} \quad V_1 = \max_{s_1 \in \mathcal{S}_1} \min_{s_2 \in \mathcal{S}_2} u(s_1, s_2) = u(s'_1, s'_2)$$

where  $V_1 \in \mathbb{R}$  denotes the value of the maximin strategy.

**Definition 2.10** (Minimax Strategy). *A maximal value strategy is a (pure or mixed) strategy  $s'_1 \in \mathcal{S}_1$  such that for every (mixed or pure)  $s_2 \in \mathcal{S}_2$  there exists a value  $\nu_2(s_2)$  for player 2 as*

$$\nu_2(s_2) := \max_{s_1 \in \mathcal{S}_1} u(s_1, s_2) = u(s'_1, s_2)$$

*A minimax strategy is a pair of strategies  $s'_1 \in \mathcal{S}_1$  and  $s'_2 \in \mathcal{S}_2$  such that there exists a value  $V_2$  as*

$$V_2 := \min_{s_2 \in \mathcal{S}_2} \nu_2(s_2) = \nu_2(s'_2) \quad \text{i.e.} \quad V_2 = \min_{s_2 \in \mathcal{S}_2} \max_{s_1 \in \mathcal{S}_1} u(s_1, s_2) = u(s'_1, s'_2)$$

where  $V_2 \in \mathbb{R}$  denotes the value of the minimax strategy.

**Definition 2.11** (Pure Minimax Strategy). *A pure maximal value strategy is a pure strategy  $s'_1 \in \mathcal{PS}_1$  such that for every pure  $s_2 \in \mathcal{PS}_2$  there exists a value  $\bar{\nu}_2(s_2)$  for player 2 as*

$$\bar{\nu}_2(s_2) := \max_{s_1 \in \mathcal{PS}_1} \mathbf{u}(s_1, s_2) = \mathbf{u}(s'_1, s_2)$$

*A pure minimax strategy is a pair of strategies  $s'_1 \in \mathcal{PS}_1$  and  $s'_2 \in \mathcal{PS}_2$  such that there exists a value  $\overline{PV}$  as*

$$\overline{PV} := \min_{s_2 \in \mathcal{PS}_2} \bar{\nu}_2(s_2) = \bar{\nu}_2(s'_2) \quad \text{i.e.} \quad \overline{PV} = \min_{s_2 \in \mathcal{PS}_2} \max_{s_1 \in \mathcal{PS}_1} \mathbf{u}(s_1, s_2) = \mathbf{u}(s'_1, s'_2)$$

where  $\overline{PV} \in \mathbb{R}$  denotes the value of the pure minimax strategy. This pure minimax strategy may also be more useful than the minimax strategy since pure strategies are easier to understand and apply.

### 2.2.2 Minimax-Minimax Theorem

Inspired by John von Neumann's article "Zur Theorie der Gesellschaftsspiele" [18] and Gal [11], we will state the minimax-maximin Theorem for a finite two person zero sum games as

**Theorem 2.12** (Minimax-Maximin Theorem). *For a finite two person zero sum games we have*

$$V_2 = \min_{s_2 \in \mathcal{S}_2} \max_{s_1 \in \mathcal{S}_1} u(s_1, s_2) = \bar{V} = \max_{s_1 \in \mathcal{S}_1} \min_{s_2 \in \mathcal{S}_2} u(s_1, s_2) = V_1 \quad (4)$$

where  $u(s_1, s_2)$  is a payoff function for (mixed or pure)  $s_i \in \mathcal{S}_i$  for  $i = 1, 2$  and value  $\bar{V} \in \mathbb{R}$  is called 'The value of the game'.

This result shows us that the best of the worst payoffs of player 1 is equal to the best of the worst payoffs of player 2. This creates an equilibrium between both players for which they both will not change their strategy even when they know their opponent's strategy, such reasoning is a characterization of a risk averse player. A small note on the proof of this theorem. For a general (payoff) function  $u$  it is easy to argue that

$$\max_{s_1 \in \mathcal{S}_1} \min_{s_2 \in \mathcal{S}_2} u(s_1, s_2) \leq \min_{s_2 \in \mathcal{S}_2} \max_{s_1 \in \mathcal{S}_1} u(s_1, s_2).$$

Von Neumann proved the hard part, namely the inequality

$$\max_{s_1 \in \mathcal{S}_1} \min_{s_2 \in \mathcal{S}_2} u(s_1, s_2) \geq \min_{s_2 \in \mathcal{S}_2} \max_{s_1 \in \mathcal{S}_1} u(s_1, s_2)$$

such that from both inequalities we can conclude equation (8). We will not prove this theorem, because it is only meant to provide intuition for a reader who is new to the subject. In the next chapter we will discuss the minimax-maximin Theorem for an infinite game. This theorem is not applicable for such games, since the value is not guaranteed to be finite for general games.

### 2.2.3 Example: Matching Pennies

We will be using the Matching Pennies game defined in section 2.1.7 to apply the minimax-maximin Theorem this will result in best of the worst payoff. The payoff matrix of this zero sum game is given by

$$\begin{array}{cc} & \begin{array}{cc} \text{heads} & \text{tails} \end{array} \\ \begin{array}{c} \text{heads} \\ \text{tails} \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

We notice that the set of pure strategies  $\mathcal{PS}_i = \{\text{heads}, \text{tails}\}$  for player  $i = 1, 2$  does not provide a combination of two strategies which will result in an equation as stated by the minimax-maximin Theorem. To expand our options we will be using the mixed strategies.

We define random variable  $S_1$  and the probability measure  $\mathbb{P}(\{S_1 = s_1\})$  of the mixed strategy  $S_1 \in \mathcal{MS}_1$  of player 1 as

$$S_1 = \begin{cases} 1 & \text{for } s_1 = \text{heads} \\ 0 & \text{for } s_1 = \text{tails} \end{cases} \quad \text{and} \quad \mathbb{P}(\{s_1 = S_1\}) = \begin{cases} p & \text{for } S_1 = 1 \\ 1 - p & \text{for } S_1 = 0 \\ 0 & \text{elsewhere} \end{cases}$$

where  $0 < p < 1$  and  $s_1 \in \mathcal{PS}_1$  such that the random variable  $S_1$  is distributed according to the measure  $\mathbb{P}$ . For player 2 we define the mixed strategy  $S_2 \in \mathcal{MS}_2$  the same as for  $S_1$  with an equivalent distribution  $\mathbb{P}(\{S_2 = s_2\})$  with  $0 < q < 1$  instead of  $p$ . Next, we will be computing the value of the game according to the minimax-maximin Theorem. We will be using the notation  $U$  for the cost because we have already noticed that there is no pure strategy which gives us a minimax (or maximin) value, and thus  $u$  is specified as  $U$  from Definition 2.2. The minimax-maximin Theorem tells us that

$$\min_{S_2 \in \mathcal{MS}_2} \max_{S_1 \in \mathcal{MS}_1} U(S_1, S_2) = \bar{V} = \max_{S_1 \in \mathcal{MS}_1} \min_{S_2 \in \mathcal{MS}_2} U(S_1, S_2)$$

such that we can determine the values of  $p$  and  $q$  via

$$\min_{S_2 \in \mathcal{MS}_2} \max_{S_1 \in \mathcal{MS}_1} U(S_1, S_2) = \max_{S_1 \in \mathcal{MS}_1} \min_{S_2 \in \mathcal{MS}_2} U(S_1, S_2) \quad (5)$$

where we have an expected payoff function

$$U(S_1, S_2) = pq + (1 - p)(1 - q) - p(1 - q) - q(1 - p) = (1 - 2p)(1 - 2q).$$

We will substitute the expression for the expected payoff function  $U$  into (5) to find

$$\min_{0 < q < 1} \max_{0 < p < 1} (1 - 2p)(1 - 2q) = \max_{0 < p < 1} \min_{0 < q < 1} (1 - 2p)(1 - 2q) \quad (6)$$

for which we will examine the right hand side. Let us first consider the maximin strategy on the right hand side of (6) given by  $V_1 = \max_{0 < p < 1} \min_{0 < q < 1} (1 - 2p)(1 - 2q)$  for which the best replies of player 1 are stated as:

- If player 2 plays  $S_2$  with  $q > \frac{1}{2}$ , then player 1 plays  $S_1$  with  $p = 0$ . This results in an expected payoff function  $U$  where

$$U(S_1, S_2) = (1 - 2 \cdot 0)(1 - 2q) = (1 - 2q) < 0.$$

- If player 2 plays  $S_2$  with  $q < \frac{1}{2}$ , then player 1 plays  $S_1$  with  $p = 1$ . This results in an expected payoff function  $U$  where

$$U(S_1, S_2) = (1 - 2 \cdot 1)(1 - 2q) = (1 - 2q) < 0.$$

- If player 2 plays  $S_2$  with  $q = \frac{1}{2}$ , then player 1 can choose whatever he want, so play  $S_1$  with  $p \in (0, 1)$ . This results in an expected payoff function  $U$  where

$$U(S_1, S_2) = (1 - 2p) \left(1 - 2 \cdot \frac{1}{2}\right) = (1 - 2p) \cdot 0 = 0.$$



We can define similar best replies of player 2 for the minimax strategy given by the left hand side of (6) which we use to find an intersection between the best replies. Since the third strategy provides the highest payoff for player 1 playing a maximin and the lowest payoff for player 2 playing a minimax strategy, we will choose the third option for both players. This results in both players playing  $p = q = \frac{1}{2}$  as supported in figure 1 which shows the best replies on the left and the expected payoff function on the right.

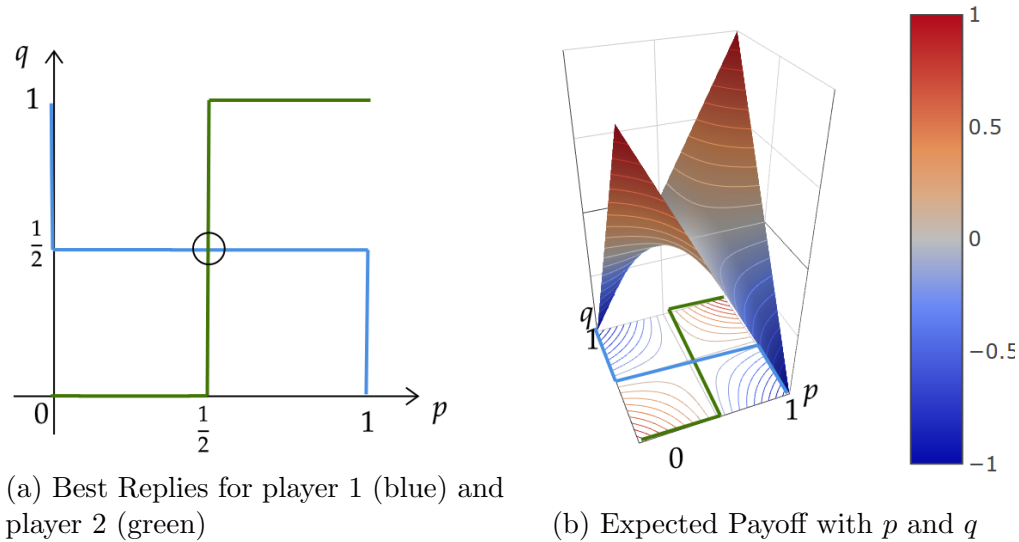


Figure 1: The minimax and maximin strategies and the value of the game

From the analysis above we conclude that the maximin and minimax strategies are the strategies  $S_1$  and  $S_2$  describing the probability measures with  $p = \frac{1}{2}$  and  $q = \frac{1}{2}$  respectively and with value  $\bar{V} = 0$  as value of the game.

## 3 Description of Non-Cooperative Search Games

This chapter contains two parts: The first part of this chapter will describe the general framework for non-cooperative search game based on the definitions and theorems discussed in Chapter 2. The second part will introduce an application of such non-cooperative stationary game: the Linear Search Game on an infinite line and introduce several theorems on scaling search games such that computation becomes possible in the next chapter. This scaling property is based on Gal [11] and Alpern&Gal [4] and is the main result of this chapter.

In section 3.1, we will state the Definition 3.12 and 3.13 about mixed strategies more rigorous similar to the mixed strategies defined in Chapter 2. For the other sections in this chapter we used the Gal [11] as a source. In 2002 Gal was co-author of Alpern&Gal [4] which streamlined his work. As such, this section will be based on both the Introduction and Chapter 1 of Gal [11, p.2-p.6] and Appendix of Alpern&Gal [4, p.291-p.293].

In Section 3.2, we start off by defining all strategies of The Linear Search Game. We introduce a Restricted Linear Search Game where Definition 3.14 is a slightly changed definition compared to the source for the natural restriction and we formulate a dedicated Lemma 3.16 with a proof to show that normalization might result in a finite value. Both Definition 3.14 and Lemma 3.16 are extensions of Chapter 6 in Alpern&Gal [4, p.101-p.103]. We used our definitions of the mixed strategies to expand the proof of the Scaling Lemma stated in Chapter 2 of Gal [11, p.14-p.15] resulting in our proof of Lemma 3.17. Theorem 3.19 is from Chapter 6 in Alpern&Gal [4] where we added more intuition behind the arguments in the proof. We end this chapter with a useful sequence representation from Gal [11, p.138] (with our illustration) for the search trajectories and the cost function which is used for computations in the next chapter.

### 3.1 General Framework

In this section, we will be translating the general definitions given in section 2.1 to a Search Game context. We will also be introducing the definition for minimax, maximin, optimal and  $\varepsilon$ -optimal strategies in an infinite search game. At last we look at the minimax-maximin Theorem for infinite search games.

#### 3.1.1 Sets of Strategies

In this thesis we assume that a search game is a two player zero sum game where the hider is player 1 and the searcher is player 2 from section 2.1. Each player has a set of pure strategies:  $\mathcal{PS}$  for the searcher and  $\mathcal{PH}$  for the hider. The game is played on the search space  $Q$ . Such a search space can be a graph, tree, (in)finite number line or multi dimensional region for example. In section 3.2 we let the search space be  $Q = \mathbb{R}$ . We define a pure search or pure hiding strategy as a discrete sequence or continuous trajectory depending on the search

space. We look at the situation where the hider is immobile. This implies that the strategies for the hider has to contain a single action.

### 3.1.2 Payoff and Cost function

Payoffs of both players will be separated into pure and expected payoffs, but in comparison with chapter 2 there is one difference. Instead of calling it a payoff we will also refer to it as the cost function  $u : \mathcal{S} \times \mathcal{H} \rightarrow \mathbb{R}$ . The cost  $u(s, h)$  will represent the loss of the searcher (or the effort spent in searching) when the searcher plays strategy  $s \in \mathcal{S}$  and the hider plays strategy  $h \in \mathcal{H}$ . Since most applications of Search Theory are mostly interested in the strategies of the searcher we choose 'cost' as a more suitable term. The goals of the searcher becomes to minimize the cost, while the hider wishes to make the cost as large as possible. The cost function will be the minimal time spent until capture occurs, with the use of detection radius  $r \geq 0$  we can express the cost function as

$$u(s, h) = \min\{t \mid d(s(t), h(t)) \leq r\} \quad (7)$$

where  $s(t)$  and  $h(t)$  denote the position at time  $t$  of the searcher and hider respectively. We will define  $s$  and  $h$  more rigorous in Section 3.2.

### 3.1.3 Minimax-Maximin Theorem for Finite Search games

We will define the values for a search game and mention the minimax-maximin Theorem for search games with finite search space. We will define the maximin hiding strategy and the minimax search strategy similar as in Section 2.2 in the following way:

**Definition 3.1** (Maximin Hiding Strategy). *A minimal search strategy is a (pure or mixed) strategy  $s' \in \mathcal{S}$  such that for every (mixed or pure)  $h \in \mathcal{H}$  there exists a value  $\nu_1(h)$  for player 2 as*

$$\nu_1(h) := \min_{s \in \mathcal{S}} u(s, h) = u(s', h).$$

*A maximin strategy is a pair of strategies  $h' \in \mathcal{H}$  and  $s' \in \mathcal{S}$  such that there exists value  $V_1$  as*

$$V_1 := \max_{h \in \mathcal{H}} \nu_1(h) = \nu_1(h') \quad \text{i.e.} \quad V_1 = \max_{h \in \mathcal{H}} \min_{s \in \mathcal{S}} u(s, h) = u(s', h')$$

*where  $V_1 \in \mathbb{R}$  denotes the value of the maximin strategy.*

**Definition 3.2** (Minimax Search Strategy). *A maximal hiding strategy is a (pure or mixed) strategy  $h' \in \mathcal{H}$  such that for every (mixed or pure)  $s \in \mathcal{S}$  there exists a value  $\nu(s)$  for player 2 as*

$$\nu(s) := \max_{h \in \mathcal{H}} u(s, h) = u(s, h').$$

*A minimax strategy is a pair of strategies  $h' \in \mathcal{H}$  and  $s' \in \mathcal{S}$  such that there exists value  $V_2$  as*

$$V_2 := \min_{s \in \mathcal{S}} \nu(s) = \nu(s') \quad \text{i.e.} \quad V_2 = \min_{s \in \mathcal{S}} \max_{h \in \mathcal{H}} u(s, h) = u(s', h')$$

*where  $V_2 \in \mathbb{R}$  denotes the value of the minimax strategy.*

**Definition 3.3** (Pure Minimax Search Strategy). *A pure maximal value strategy is a pure strategy  $h' \in \mathcal{PH}$  such that for every pure  $s \in \mathcal{PS}$  there exists a value  $\bar{v}_2(s)$  for player 2 as*

$$\bar{v}_2(s) := \max_{h \in \mathcal{PH}} \mathbf{u}(s, h) = \mathbf{u}(s, h')$$

*A pure minimax strategy is a pair of strategies  $h' \in \mathcal{PH}$  and  $s' \in \mathcal{PS}$  such that there exists a value  $\overline{PV}$  as*

$$\overline{PV} := \min_{s \in \mathcal{PS}} \bar{v}_2(s) = \bar{v}_2(s') \quad \text{i.e.} \quad \overline{PV} = \min_{s \in \mathcal{PS}} \max_{h \in \mathcal{PH}} \mathbf{u}(s, h) = \mathbf{u}(s', h')$$

where  $\overline{PV} \in \mathbb{R}$  denotes 'the value of the pure minimax strategy'.

We will not define the pure maximin hiding strategy, since we will not be using nor computing this strategy. Note that the value of the pure minimax strategy will not be used in the minimax-minimax theorem which we will discuss next. We will be computing  $\overline{PV}$  for the Linear Search Game, but it is not used to define the value of the game. The value of the game is necessary to define the optimal strategies of the Linear Search Game.

We used the maximal search and minimal hiding strategies to define the minimax and maximin strategies for a search game with finite search space. By applying Theorem 2.12 we can state the following.

**Theorem 3.4** (Minimax-Maximin Theorem for Finite Search games). *For a finite two-person search games with finite search space we have*

$$V_1 = \min_{s \in \mathcal{S}} \max_{h \in \mathcal{H}} u(s, h) = \overline{V} = \max_{h \in \mathcal{H}} \min_{s \in \mathcal{S}} u(s, h) = V_2 \quad (8)$$

where  $u(s, h)$  is a payoff function for (mixed or pure)  $s \in \mathcal{S}$  and (mixed or pure)  $h \in \mathcal{H}$ . Value  $\overline{V} \in \mathbb{R}$  is called 'The value of the game'.

Unfortunately, this theorem does not hold for search games with infinite search spaces. The value of the game could be infinite in such a case and thus an equivalence is not guaranteed. Also note that  $\overline{V}$  does not have to be equal to  $\overline{PV}$ .

### 3.1.4 Guaranteeing the finite value for an infinite search game

For infinite search games we need a theorem which is a generalization of Theorem 3.4. We approach this problem by introducing a theorem (without proof) from Alpern&Gal [2] and used in Alpern&Gal [4, p.293]. This theorem is a generalization from Fan's minimax theorem which is introduced in Fan [8] and used in the proof of Gal [11, p.184] to show that a (in)finite search game has a value. The theorem is equivalent to Alpern&Gal [4, p.293] where we added the remark that  $u$  is below the statement of the proof.

**Theorem 3.5.** *Let  $X$  be a compact Hausdorff space (where two distinct trajectories always have disjoint neighborhoods) and  $(Y, A)$  a measurable space. Let  $f : X \times Y$  be a measurable function that is bounded below and lower semicontinuous on  $X$  for all fixed  $y \in Y$ . Let  $M$*

be any convex set of probability measures (which are given by the mixed strategies) on  $(Y, A)$  and  $B(X)$  the regular probability measures on  $X$ . Then

$$\min_{\beta \in B(X)} \sup_{\gamma \in M} \iint f(x, y) d\beta d\gamma = \sup_{\gamma \in M} \min_{\beta \in B(X)} \iint f(x, y) d\beta d\gamma. \quad (9)$$

For our search trajectories we use the topology of uniform convergence for any finite interval. Since any  $s \in \mathcal{S}$  is Lipschitz continuous it follows from the Ascoli theorem that  $\mathcal{S}$  is compact. Under that topology is  $\mathcal{S}$  also Hausdorff. The capture time  $u(s, h)$  is lower semicontinuous from the intuition that  $u$  can only "jump" down in each of the variables  $s$  and  $h$ . The function  $u$  is also bounded from below with  $u(s, h) \geq h$  because this would mean that the lowest possible cost is one where the search immediately moves to the hider. This is more extensively shown in Gal [11] and we recommend reading the Appendix of Alpern&Gal [4] if a more rigorous explanation is desired. By applying Theorem 3.5 we obtain that

$$\min_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \iint u(s, h) ds dh = \bar{V} = \sup_{h \in \mathcal{H}} \min_{s \in \mathcal{S}} \iint u(s, h) ds dh. \quad (10)$$

such that each (in)finite search game has a finite value of the game  $\bar{V}$  given by the minimax strategy.

### 3.1.5 Optimal strategies

Most of our attention has been focused on the minimax and maximin strategies, but these strategies are certainly not the only strategies which are meaningful. Some search situations do not require a payoff which is the best of the worst payoffs. For such situations a player may choose to take some risk where he may reduce his maximal (or minimal) payoff but with the benefit that the payoff may result lower (or higher) than the minimax (or maximin) strategy. Optimal strategies can be seen as an equivalence or an improvement to the value of the game  $\bar{V}$  given by

**Definition 3.6** (Optimal Search Strategies). A strategy  $s' \in \mathcal{S}$  such that for a finite value  $\bar{V}$ , for the minimal value  $\nu_2(s)$  we get

$$\nu_2(s') \leq \bar{V}.$$

**Definition 3.7** (Optimal Hiding Strategies). A strategy  $h' \in \mathcal{H}$  such that for a finite value  $\bar{V}$ , for the maximal value  $\nu_1(h)$  we get

$$\nu_1(h') \geq \bar{V}.$$

Note that the definitions above are based on the Value of the game  $\bar{V}$ , not the value of the pure minimax strategy  $\overline{PV}$ . Because of equation (10) in Theorem 3.5 we can guarantee a finite value  $\bar{V}$  for the search games which we discuss in this thesis. This means that the definitions above are well defined but it may be the case that optimal strategies do not exist outside of the minimax (or maximin) strategy for a certain search game. In section 4.2 we will be exploring the optimal strategies for the linear search problem.

### 3.1.6 $\varepsilon$ -optimal strategy

The  $\varepsilon$ -optimal strategies are very similar to optimal strategies in their usage. The difference is that such strategies are needed on an unbounded domain, such as in the linear search problem. The  $\varepsilon$ -optimal strategy is useful when there is no optimal strategy for a player in the search game. The  $\varepsilon$ -optimal strategies are defined as follows

**Definition 3.8** ( $\varepsilon$ -Optimal Search Strategy). For any  $\varepsilon > 0$ , a strategy  $s'_\varepsilon \in \mathcal{S}$  such that for a finite value  $\bar{V}$ , for the minimal value  $\nu_2(s)$  we get

$$\nu_2(s'_\varepsilon) < (1 + \varepsilon)\bar{V}.$$

**Definition 3.9** ( $\varepsilon$ -Optimal Hiding Strategies). For any  $\varepsilon > 0$ , a strategy  $h'_\varepsilon \in \mathcal{H}$  such that for a finite value  $\bar{V}$ , for the maximal value  $\nu_1(h)$  we get

$$\nu_1(h'_\varepsilon) > (1 - \varepsilon)\bar{V}.$$

Again, note that the definitions above are based on the Value of the game  $\bar{V}$ , not the value of the pure minimax strategy  $\bar{P}\bar{V}$ . Also note that the equation (10) in Theorem 3.5 guarantees a finite value  $\bar{V}$  for the search games which we will discuss. These  $\varepsilon$ -optimal strategies will be used in section 4.2 and are necessary for infinite games.

## 3.2 Setup of the Linear Search Problem

We will be building upon the framework presented in section 3.1. We will base our translation of the linear search problem to a search game on the works of Gal [11] and Alpern&Gal [4]. From here on we will rename the "Linear Search Problem" to the "Linear Search Game". We will start this section off by introducing the search space and the players strategy sets. We will show that the restricted search game with  $\lambda > 0$  can be scaled into a restricted search game with  $\lambda = 1$ . Afterwards we will show that the restricted search game with  $\lambda = 1$  will have the same value as the normalized linear search game.

We assume that the hider is immobile and that capture occurs the first time that the searcher passes the point occupied by the hider with a detection radius  $r = 0$ . We construct the search game with the set of pure strategies for the searcher and for the hider as:

**Definition 3.10** (Pure Search Strategy). A pure search strategy  $s$  is a function  $s : [0, \infty) \rightarrow \mathbb{R}$  which is assumed to be Lipschitz-continuous satisfying

$$s(0) = 0, \quad \text{and for all } t_2 > t_1 \geq 0 : \quad |s(t_2) - s(t_1)| \leq t_2 - t_1. \quad (11)$$

**Definition 3.11** (Pure Hiding Strategy). A pure hiding strategy  $h$  is a function  $h : [0, \infty) \rightarrow \mathbb{R}$ , where we assume that the hider is immobile such that

$$h(t) = h \quad \text{for all } t \in [0, \infty) \text{ where } h \in \mathbb{R}. \quad (12)$$

The Lipschitz-continuity property of  $s$  implies that the (absolute) derivative  $|s'(t)|$  is bounded. This can be shown by dividing both sides of inequality (11) by the difference  $\Delta := t_2 - t_1 > 0$  for all  $t_2 > t_1$  which results in:

$$|s'(t)| = \lim_{\Delta \rightarrow 0} \frac{|s(t + \Delta) - s(t)|}{\Delta} \leq 1 \quad (13)$$

The equation (13) implies that  $-1 \leq s'(t) \leq 1$  for all  $t \in [0, \infty)$ . To reduce the amount of strategies for the searcher, we can assume that the searcher will always move with his maximal velocity because any search strategy  $s_1(t)$  which does not use the maximal velocity is dominated by a search strategy  $s_2(t)$  which uses the maximal velocity along path created by  $s_1(t)$ . The inequality (13) shows that the velocity  $\omega = |s'(t)|$  will be maximal when  $s'(t) = 1$  or  $s'(t) = -1$ . This specifies the set of search strategies to:

$$\mathcal{PS} = \{s(t) \mid s'(t) = 1 \text{ or } s'(t) = -1\}. \quad (14)$$

For mixed strategies, the position of the search and hider will be a random variable. We will again be using a probability measure corresponding with a mixed strategy.

**Definition 3.12** (Mixed Search Strategy). *The inverse image associated with random variable  $S : \mathcal{PS} \rightarrow \mathbb{R}$  is defined as*

$$S^{-1}([a, b]) = \{s \in \mathcal{PS} : S(s) \in [a, b]\}$$

where  $a, b \in \mathbb{R}$ . The distribution of  $S$  is a probability measure on  $\mathbb{R}$  defined as

$$\mathbb{P}(S^{-1}([a, b])) = \mathbb{P}(\{s \in \mathcal{PS} : S(s) \in [a, b]\}).$$

Such a  $\mathbb{P} \circ S^{-1}$  is called: A mixed search strategy, which gives a probability for each of the pure search strategies occurring. As mentioned in Definition 2.3, we will keep the notation simple by denoting  $\mathbb{P} \circ S^{-1}$  as  $S$ . It will be clear from the context if we mean  $S$  as a measure of a random variable. The strategy set that contains all mixed strategies for the searcher will be called  $\mathcal{MS}$ .

**Definition 3.13** (Mixed Hiding Strategy). *The inverse image associated with random variable  $H : \mathcal{PH} \rightarrow \mathbb{R}$  is defined as*

$$H^{-1}([a, b]) = \{h \in \mathcal{PH} : H(h) \in [a, b]\}$$

where  $a, b \in \mathbb{R}$ . The distribution of  $H$  is a probability measure on  $\mathbb{R}$  defined as

$$\mathbb{P}(H^{-1}([a, b])) = \mathbb{P}(\{h \in \mathcal{PH} : H(h) \in [a, b]\}).$$

Such a  $\mathbb{P} \circ H^{-1}$  is called: A mixed hiding strategy, which gives a probability for each of the pure hiding strategies occurring. Again, as mentioned in Definition 3.12 we will denote  $\mathbb{P} \circ H^{-1}$  as  $H$  for the same reason, namely ease of notation. The strategy set that contains all mixed strategies for the hider will be called  $\mathcal{MH}$  similarly as the mixed search strategies.

As mentioned before, we cannot define a pure cost for these strategies, and thus will only be providing expected cost. Note that a pure strategy  $s \in \mathcal{PS}$  can also be described as a mixed strategy  $S \in \mathcal{MS}$  with degenerate probability distribution concentrated at  $s$ , we call such a concentrated probability distribution a probability atom. For the searcher we assume that any strategy  $s \in \mathcal{S}$  with  $\nu(s) < \infty$  has to satisfy

$$\sup_{t>0} \mathbf{s}(t) = \infty \quad \text{and} \quad \inf_{t>0} \mathbf{s}(t) = -\infty \quad (15)$$

with probability 1, otherwise there would exist a strategy  $h \in \mathcal{H}$  that will never be discovered. In other words, when the searcher finds the hider (which is equal to the finite value of the game) we assumed that the searcher could search anywhere on the infinite line since he does not have an upper or lower bound.

### 3.2.1 Restricting the Hider for the Linear Search Same

For the Linear Search Game, we will be working on a unbounded domain where a searcher is searching for a hider on a real line with a known probability distribution. We will assume that the searcher may change direction without loss of time. For this problem we have to be aware of a situation where

- the capture time  $T$  is the cost function and
- the hider has no restriction on his strategies

may result in an infinite value of the game. A problem could occur where the hider is hiding infinitely far from the origin. Such a strategy would be  $h$  going to (negative) infinity. This is problematic because this would imply that the searcher will search for an infinite time. Which makes the cost function go to infinity and thus also the value of the game. To tackle this situation we define a restriction on the expected hiding strategy as follows.

**Definition 3.14.** *The sets  $\mathcal{PH}_\lambda \cup \mathcal{MH}_\lambda$  are sets of all hiding strategies in search space  $Q$  which for any pure hiding strategy  $h_\lambda \in \mathcal{PH}_\lambda$  and mixed hiding strategy  $H_\lambda \in \mathcal{MH}_\lambda$  satisfy the restriction:*

$$\text{pure strategy: } |h_\lambda| \leq \lambda \qquad \text{mixed strategy: } \int_Q |h| dH_\lambda \leq \lambda$$

where  $|h|$  is the distance the hider is removed from the origin when playing strategy  $h$ .

This restriction may be viewed as the "natural" restriction for the Linear Search Game because  $u(s, h) \geq |h|$ . The intuition behind this inequality is that every payoff is greater or equal than the payoff where the searcher immediately moves to the hider. This would be the fastest way to find the hider, but unfortunately for the searcher this is highly unlikely. Most probably, the searcher has to move around before he finds the hider. Definition 3.14 aims to define a boundary similar to completeness in  $\mathbb{R}$  in which the (expected) location that the strategy  $h \in \mathcal{H}$  is removed from the origin is bounded. Definition 3.14 together with



(15) describe the situation in which the hider is expected a maximal distance of  $\lambda$  from the origin for any strategy, while the searcher is unbounded in both directions. This will result in a guaranteed finite value of the game, because the searcher and hider will meet with a probability of 1.

### 3.2.2 Normalizing the Linear Search Game

Another method which can be used in unbounded search spaces is normalizing the cost function such that the value of the game will be finite. We will start by defining a scaled cost function. Afterwards we will specify the normalized cost function.

**Definition 3.15.** *Let  $\gamma \in \mathbb{R}$ . The scaled cost function  $\tilde{u}_\gamma$  is defined as*

$$\tilde{u}_\gamma(s, h) := \frac{u(s, h)}{|h|^\gamma}$$

where the cost function  $u(s, h)$  as given in Section 3.1.2 is defined for pure or mixed strategies  $s \in \mathcal{S}$  and  $h \in \mathcal{H}$ .

We will choose  $\gamma = 1$  and show that only that choice might result in a finite value of a game. All other values of  $\gamma$  will result in an infinite value of a game. Using a normalized cost function is common in computer science literature for searching for a target with incomplete information and is also referred to the as the competitive ratio.

**Lemma 3.16.** *The only choice of  $\gamma$  for the scaled function  $\tilde{u}_\gamma$  which might result in a finite value for linear search problem will be  $\gamma = 1$ . We will define this normalized function  $\hat{u}$  as:*

$$\hat{u}(s, h) := \tilde{u}_1(s, h) = \frac{u(s, h)}{|h|}$$

where  $s \in \mathcal{S}$  and  $h \in \mathcal{H}$  are pure or mixed strategies.

*Proof.* Note that  $u(s, h) \geq |h|$  holds for all  $s \in \mathcal{S}$  as mentioned in Definition 3.14, we will be using this inequality to substitute in the numerators. We will be covering three cases:  $\gamma < 1$ ,  $\gamma > 1$  and  $\gamma = 1$ . For each, we will be looking at the value of a game as defined in equation (25) in section 3.1.1 as:  $\hat{V} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} u(s, h)$ . The existence of this value will determine the choice of  $\gamma$ .

- Let  $\gamma < 1$ . This will result in the inequality

$$\hat{V} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{u(s, h)}{|h|^\gamma} \geq \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{|h|}{|h|^\gamma} = \underbrace{\inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{1}{|h|^{\gamma-1}}}_{\text{because } \gamma < 1} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} |h|^\alpha$$

where  $\alpha = 1 - \gamma > 0$  and we note that  $\sup_{h \in \mathcal{H}} |h|^\alpha = \infty$  because we can choose  $h$  as large as we want. We conclude that the scaled function for  $\gamma < 1$  will result in an infinite value for the search game which is not desirable. We discard the option  $\gamma < 1$ .

- Let  $\gamma > 1$ . This will be analogous to the previous case. We get the inequality

$$\hat{V} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{u(s, h)}{|h|^\gamma} \geq \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{|h|}{|h|^\gamma} = \underbrace{\inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{1}{|h|^{\gamma-1}}}_{\text{because } \gamma > 1} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{1}{|h|^\alpha}$$

where  $\alpha = \gamma - 1 > 0$  and again we note that  $\sup_{h \in \mathcal{H}} \frac{1}{|h|^\alpha} = \infty$  because we can choose  $h$  as close as we want to the origin. We conclude that the scaled function for  $\gamma > 1$  will result in an infinite value for the search game which is not desirable. We discard the option  $\gamma > 1$ .

- Let  $\gamma = 1$ . This option will give the result

$$\hat{V} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{u(s, h)}{|h|} \geq \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} \frac{|h|}{|h|} = \inf_{s \in \mathcal{S}} \sup_{h \in \mathcal{H}} 1 = 1$$

Thus we conclude that  $\hat{V}$  has lower bound 1. The lower bound corresponds with the searcher immediately moving to the hider. Thus is  $\gamma = 1$  the only value which might result in a finite value  $\bar{V}$ .

□

### 3.2.3 Translate between Restricted and Normalized Linear Search Games

In this section we will show that the Restricted Linear Search Game with  $\lambda > 0$  is the scaled version of the Normalized Linear Search game. We will introduce a general scaling lemma based on the Scaling Lemma in Gal [11] such that we can scale a restricted search game with  $\lambda > 0$  into a restricted search game with  $\lambda = 1$ . Afterwards we will show that the restricted search game with  $\lambda = 1$  will have the same value as the normalized linear search game. This will allow us to scale the finite value of the game  $\hat{V}$  found by the Normalized Linear Search Game with a parameter  $\lambda$  to find the value of the game  $\bar{V} = \lambda \hat{V}$  of the Restricted Linear Search Game with  $\lambda > 0$

We will start by providing a method to scale the two restricted search games across different search spaces  $Q$  with Euclidean metric  $d$  and  $\hat{Q}$  with metric  $\hat{d}$ . We will be using the lemma to provide a scaling function which will be used for a linear translation between search spaces  $Q = \mathbb{R}$  and  $\hat{Q} = \mathbb{R}$ . We will provide a generalized theorem, which can also be useful (with extension of the hider strategy) for the expansions of the search game discussed in Chapter 5.

**Lemma 3.17** (Scaling Lemma). *Let us assume that we have a non-scaled search game with immobile hider on a set  $Q$  with an origin  $O$  with Euclidean metric  $d$  and a detection radius  $r$ . Assume that the value of the pure minimax strategy  $\bar{P}\bar{V}$ . Also assume that the value of the game is  $\bar{V}$  for which (pure or mixed) strategies  $s$  and  $h$  are optimal (or  $\varepsilon$ -optimal) strategies. Consider a scaled search space  $\hat{Q}$  with immobile hider with a metric  $\hat{d}$ , which is*

obtained from  $Q$  by a surjective mapping  $\varphi : Q \rightarrow \hat{Q}$  with the following property for some  $\alpha > 0$ :

$$\hat{d}(\varphi(x), \varphi(y)) = \alpha d(x, y), \quad \text{for all } x, y \in Q. \quad (16)$$

Define a scaled search game on the search space  $\hat{Q}$  with an origin  $\hat{O} = \varphi(O)$ , a detection radius  $\hat{r} = \alpha r$ , and the same maximal velocities for the searcher and the hider as in the non-scaled search game. Then the value on the scaled search game  $\hat{V}$  satisfies  $\hat{V} = \alpha \bar{V}$  and  $\widehat{PV} = \alpha \overline{PV}$  and the optimal (or  $\varepsilon$ -optimal) strategies of this scaled game are obtained by applying the mapping  $\varphi$  to the trajectories in  $Q$  and changing the time scale by a factor of  $\alpha$ .

*Proof.* We will divide this proof in the case where we map over pure or mixed strategies. We will begin by showing that the scaling lemma holds for the pure strategies. Afterwards, we will define the scaling lemma for the mixed strategies.

Let  $\mathcal{PS}$  and  $\mathcal{PH}$  be the sets of pure strategies containing admissible trajectories for the searcher and hider respectively in a non-scaled game. Let sets of pure strategies  $\widehat{\mathcal{PS}}$  and  $\widehat{\mathcal{PH}}$  contain admissible trajectories for the scaled game. We will extend the mapping  $\varphi$  for the pure strategies  $s \in \mathcal{PS}$  onto  $\hat{s} \in \widehat{\mathcal{PS}}$  and  $h \in \mathcal{PH}$  onto  $\hat{h} \in \widehat{\mathcal{PH}}$  in the following way:

$$\begin{aligned} \hat{s} = \varphi(s) \text{ if and only if } \hat{s}(t) &= \varphi\left(s\left(\frac{t}{\alpha}\right)\right) \text{ for } 0 \leq t < \infty \text{ and,} \\ \hat{h} = \varphi(h) \text{ if and only if } \hat{h}(t) &= \varphi\left(h\left(\frac{t}{\alpha}\right)\right) \text{ for } 0 \leq t < \infty. \end{aligned}$$

Note that we extended the mapping and used the same notation  $\varphi$  for ease of notation. To show that our choice of mapping is the right one we will show that  $\hat{s}$  (and in a similar way  $\hat{h}$ ) is an admissible trajectory. Afterwards, we will need to show that the inverse holds as well, so that  $s$  (and similar  $h$ ) is admissible when  $\varphi^{-1}$  is applied. For the pure strategies, we will finalize the proof by showing that the payoff function will be linearly scaled by a factor  $\alpha$ . We will show this via the following equation, where we use the Lipschitz continuity for any  $s \in \mathcal{PS}$  as mentioned in (11).

$$\hat{d}(\hat{s}(t_1), \hat{s}(t_2)) = \alpha d\left(s\left(\frac{t_1}{\alpha}\right), s\left(\frac{t_2}{\alpha}\right)\right) \quad (17)$$

$$\leq \alpha \left| \frac{t_1}{\alpha} - \frac{t_2}{\alpha} \right| \quad (18)$$

$$= |t_1 - t_2|. \quad (19)$$

for any  $t_1, t_2 \in [0, \infty)$  and since  $[0, \infty) \subset \mathbb{R}$  we get the absolute metric  $d(x, y) = |x - y|$  as shown in (18). This process will be similar to show  $\hat{h}$  is an admissible trajectory when  $h$  is time dependent. This is not the case when the hider is immobile and thus the inequality is not required for  $\hat{h}$  to be admissible. The result above is equal to the Lipschitz-continuity requirement given by (11) for the  $s(t)$  trajectory. This results in the conclusion that  $\hat{s}$  is an admissible trajectory.

Secondly, we will show in a similar way that the inverse  $\varphi^{-1}$  also maps each admissible

$\hat{s} \in \widehat{\mathcal{PS}}$  to a admissible  $s \in \mathcal{PS}$ . From the previous mapping of  $\varphi$  we get that the inverse mapping  $\varphi^{-1}$  will be defined as

$$\begin{aligned} s &= \varphi^{-1}(\hat{s}) \text{ if and only if } s\left(\frac{t}{\alpha}\right) = \varphi^{-1}(\hat{s}(t)) \text{ for } 0 \leq t < \infty \text{ and,} \\ h &= \varphi^{-1}(\hat{h}) \text{ if and only if } h\left(\frac{t}{\alpha}\right) = \varphi^{-1}(\hat{h}(t)) \text{ for } 0 \leq t < \infty. \end{aligned}$$

We see that the steps (17), (18), (19) can be applied in reverse order to find that  $s$  is an admissible trajectory. If  $h$  is mobile, the process would be the same. This concludes the inverse mapping.

Thirdly, we see that the payoff function  $\hat{u}$  defined as in section 3.1.2 is also a linearly scaled function of  $u$  with factor  $\alpha$  when using the mapping  $\hat{s} = \varphi(s)$  on the non-scaled strategies  $s \in \mathcal{S}$  as arguments:

$$\begin{aligned} \hat{u}(\varphi(s), \varphi(h)) &= \min\{t \mid \hat{d}(\hat{s}(t), \hat{h}(t)) \leq \alpha r\} \\ &\downarrow \text{Substitute our defined } \hat{s}(t) = \varphi\left(s\left(\frac{t}{\alpha}\right)\right) \text{ and } \hat{h}(t) = \varphi\left(h\left(\frac{t}{\alpha}\right)\right) \\ &= \min\{t \mid \hat{d}(\varphi\left(s\left(\frac{t}{\alpha}\right)\right), \varphi\left(h\left(\frac{t}{\alpha}\right)\right)) \leq \alpha r\} \\ &\downarrow \text{Apply property (16) and divide by } \alpha \\ &= \min\{t \mid d\left(s\left(\frac{t}{\alpha}\right), h\left(\frac{t}{\alpha}\right)\right) \leq r\} \\ &= \alpha u(s, h). \end{aligned}$$

We can extend the mapping  $\varphi$  to also include the scaling of mixed strategies where we will be using the mapping and inverse mapping we defined for pure strategies. Let  $\mathcal{MS}$  and  $\mathcal{MH}$  be the sets of mixed strategies containing admissible trajectories for the searcher and hider respectively in a non-scaled game. Let sets of mixed strategies  $\widehat{\mathcal{MS}}$  and  $\widehat{\mathcal{MH}}$  contain admissible trajectories for the scaled game. This will require us again extend the mapping  $\varphi$  such that we scale the measure on  $\mathcal{MS}$  onto  $\widehat{\mathcal{MS}}$  and  $\mathcal{MH}$  onto  $\widehat{\mathcal{MH}}$ .

$$\begin{aligned} \hat{S} &= \varphi \circ S \text{ if and only if for any } A \subset \mathcal{PS} \text{ the probability measure of } A \text{ under } S \text{ is equal to} \\ &\quad \text{the probability measure of } \varphi(A) \text{ under } \hat{S} \text{ and,} \\ \hat{H} &= \varphi \circ H \text{ if and only if for any } A \subset \mathcal{PH} \text{ the probability measure of } A \text{ under } H \text{ is equal to} \\ &\quad \text{the probability measure of } \varphi(A) \text{ under } \hat{H}. \end{aligned}$$

Again, note that  $\varphi$  is extended and that this notation is used for ease of notation. We will show that if the probability measure  $A$  under  $S$  provides admissible trajectories then so does the probability measure of  $\varphi(A)$  under  $\hat{S}$ . We will show that  $\varphi(S)$  will be equivalent to  $\hat{S}$  via the use of the inverse. Since we defined the probability measures for mixed strategies with use of the inverse  $S^{-1}$  as shown in Definition 3.12 as:

$$\mathbb{P}(S^{-1}([a, b])) = \mathbb{P}(\{s \in \mathcal{PS} : S(s) \in [a, b]\}).$$

We will show that  $S^{-1} \circ \varphi = \hat{S}^{-1}$  in the following way

$$\begin{aligned}
(S^{-1} \circ \varphi^{-1})([a, b]) &= \left\{ s \in \mathcal{PS} \mid S(s) \in [\varphi^{-1}(a), \varphi^{-1}(b)] \right\} \\
&= \left\{ \varphi(s) \in \widehat{\mathcal{PS}} \mid (\varphi \circ S)(s) \in [a, b] \right\} \\
&\downarrow \text{With our choice of mapping of the pure strategies } \hat{s} = \varphi(s) \\
&= \left\{ \hat{s} \in \widehat{\mathcal{PS}} \mid \hat{S}(\hat{s}) \in [a, b] \right\} \\
&= \hat{S}^{-1}([a, b]).
\end{aligned}$$

The probability measures on  $\mathcal{PS}$  is equivalent to the probability measures on  $\widehat{\mathcal{PS}}$  and is given by

$$\mathbb{P}(\hat{S}^{-1}([a, b])) = \mathbb{P}((S^{-1} \circ \varphi)([a, b])).$$

Secondly, we will define  $\hat{S}^{-1} \circ \varphi = S^{-1}$  in the following way

$S = \varphi^{-1} \circ \hat{S}$  if and only if for any  $A \subset \widehat{\mathcal{PS}}$  the probability measure of  $A$  under  $\hat{S}$  is equal to the probability measure of  $\varphi^{-1}(A)$  under  $S$  and,  
 $H = \varphi^{-1} \circ \hat{H}$  if and only if for any  $A \subset \widehat{\mathcal{PH}}$  the probability measure of  $A$  under  $\hat{H}$  is equal to the probability measure of  $\varphi^{-1}(A)$  under  $H$ .

We will show that  $\hat{S}^{-1} \circ \varphi^{-1} = S^{-1}$  in the following way

$$\begin{aligned}
(\hat{S}^{-1} \circ \varphi^{-1})([a, b]) &= \left\{ \hat{s} \in \widehat{\mathcal{PS}} \mid \hat{S}(\hat{s}) \in [\varphi(a), \varphi(b)] \right\} \\
&= \left\{ \varphi^{-1}(\hat{s}) \in \mathcal{PS} \mid (\varphi^{-1} \circ \hat{S})(\hat{s}) \in [a, b] \right\} \\
&\downarrow \text{With our choice of mapping of the pure strategies } s = \varphi^{-1}(\hat{s}) \\
&= \left\{ s \in \mathcal{PS} \mid S(s) \in [a, b] \right\} \\
&= S^{-1}([a, b]).
\end{aligned}$$

Again we find that the probability measures are equivalent

$$\mathbb{P}(S^{-1}([a, b])) = \mathbb{P}((\hat{S}^{-1} \circ \varphi^{-1})([a, b])).$$

Lastly, we will be showing that the expected cost function on  $\hat{Q}$  is a scaled expected cost function of  $Q$  with a factor  $\alpha$ .

$$\begin{aligned}
\hat{U}(\hat{S}, \hat{H}) &= \int_{\hat{Q}} \hat{\mathbf{u}}(\hat{s}, \hat{h}) d(\hat{S} \times \hat{H}) = \int_{\hat{Q}} \hat{\mathbf{u}}(\hat{s}, \hat{h}) d(\varphi(S) \times \varphi(H)) \\
&\downarrow \text{We will use the definition for the cost function (16)} \\
&= \int_Q (\mathbf{u} \circ \varphi)(\hat{s}, \hat{h}) \alpha d(S \times H) = \int_Q \mathbf{u}(s, h) \alpha d(S \times H) \\
&= \alpha \hat{U}(S, H)
\end{aligned}$$

This concludes that every aspect of the game will be linearly scaled by a factor  $\alpha$ . This implies that if pure or mixed  $s^* \in \mathcal{S}$  and  $h^* \in \mathcal{H}$  are optimal for the non-scaled game then we see that  $\varphi(s^*)$  and  $\varphi(h^*)$  are optimal for the scaled search game with values  $\hat{V} = \alpha\bar{V}$  and  $\widehat{PV} = \alpha\overline{PV}$ .  $\square$

Now that we have proven the scaling lemma, we can introduce the scaling property which is needed for Theorem 3.19 for the Linear Search Game.

**Definition 3.18** (Scaling Property). *A search space  $Q$  has the scaling property when there exists a map  $\varphi_\alpha$  for any  $\alpha > 0$  of the search space  $Q$  into itself such that  $\varphi_\alpha(0) = 0$  and for all  $z_1, z_2 \in Q$*

$$d(\varphi_\alpha(z_1), \varphi_\alpha(z_2)) = \alpha d(z_1, z_2).$$

The Linear Search Game has the scaling property because  $\varphi(x) = \alpha x$  satisfies the condition for the scaling property and thus the Restricted Linear Search Game with restriction (3.14) where  $\lambda > 0$ , can be easily scaled into the Restricted Linear Search Game where  $\lambda = 1$  by choosing  $\alpha = \frac{1}{\lambda}$ .

This scaling property will be used in the following theorem which will show that under the scaling property the game with normalized cost function given by Lemma 3.16 results in the same value for a game as the game with the restriction with  $\lambda = 1$  defined by Definition 3.14.

**Theorem 3.19.** *For a search game with the scaling property holds: If the game with the normalized cost function by Lemma 3.16 has a finite value then the restricted search game defined by Definition 3.14 with  $\lambda = 1$  is equivalent to it.*

*Proof.* Assume that the game with restriction (3.14) has finite value  $\bar{V}$ . Let the mixed strategy  $\bar{H}$  be an  $\varepsilon$ -optimal hiding strategy that satisfies  $\mathbb{E}_{\bar{H}}[|h|] \leq 1$  then by the definition of a  $\varepsilon$ -optimal strategy we have

$$U(s, \bar{H}) = \int_Q \mathbf{u}(s, h) d\bar{H}(h) \geq (1 - \varepsilon)\bar{V} \quad \text{for all } s \in \mathcal{S}.$$

Since any hiding strategy  $H \in \mathcal{MH}$  with  $\mathbb{E}_H[|h|] < \lambda$  is dominated by a hiding strategy with  $\mathbb{E}_H[|h|] = \lambda$ . Without loss of generality we can assume that the  $\varepsilon$ -optimal strategy  $\bar{H}$  satisfies  $\mathbb{E}_{\bar{H}}[|h|] = 1$ . The intuition behind the domination argument is that the hider tries to hide as far from the origin as possible. For the normalized game we will define a scaled probability measure based on the current probability measure given by mixed strategy  $\hat{H}$  on the restricted search game

$$d\hat{H} = |h| \cdot d\bar{H}. \tag{20}$$

The strategy  $\hat{H}$  satisfies for all search strategies  $s \in \mathcal{S}$  that the expected normalized cost function is equal to the expected cost function under the new measure defined for the mixed strategy  $\hat{H}$ . We will use the normalized function (same as scaled function with  $\gamma = 1$ ) as the

payoff  $\hat{u}(s, h) = \tilde{u}_1(s, h) = \frac{u(s, h)}{|h|}$  to find that the choice of  $\hat{H}$  results in an  $\varepsilon$ -optimal strategy in the following way:

$$\hat{U}(s, \hat{H}) = \int_Q \hat{\mathbf{u}}(s, h) d\hat{H}(h) = \int_Q \frac{\mathbf{u}(s, h)}{|h|} |h| d\bar{H}(h) = \int_Q \mathbf{u}(s, h) d\bar{H}(h) \geq (1 - \varepsilon)\bar{V}. \quad (21)$$

Since  $\hat{U}(s, \hat{H}) \leq (1 - \varepsilon)\bar{V}$  holds for all  $s \in \mathcal{S}$  we know that there exists  $\hat{H}$  is an optimal strategy. On the other hand, we will show that the finite value of the normalized game will be equivalent to the restricted game. Assume that the normalized game has finite value  $\hat{V}$ . Let  $\hat{H}$  be a  $\varepsilon$ -optimal strategy for  $\hat{u}(s, h)$  which by definition means that

$$\hat{U}(s, \hat{H}) = \int_Q \hat{\mathbf{u}}(s, h) d\hat{H}(h) \geq (1 - \varepsilon)\hat{V}. \quad (22)$$

Let  $\delta > 0$  be sufficiently small such that

$$\int_{|h| \geq \delta} \hat{\mathbf{u}}(s, h) d\hat{H}(h) > (1 - 2\varepsilon)\hat{V}.$$

We will define the parameter  $b$  as:

$$b := \int_{|h| \geq \delta} \frac{d\hat{H}(h)}{|h|}$$

and define new measure for the mixed strategy  $\bar{H}_b$  for the restricted game as follows

$$\bar{H}_b(h) := \begin{cases} \frac{1}{b|h|} d\hat{H}(h) & \text{for } |h| \geq \delta \\ 0, & \text{for } |h| < \delta. \end{cases}$$

Then  $\bar{H}_b$  has an expected location expressed as

$$\mathbb{E}_{\bar{H}_b} [|h|] = \int_Q |h| d\bar{H}_b(h) = \frac{1}{b} \underbrace{\int_{|h| \geq \delta} d\hat{H}(h)}_{\mathbb{P}(\{|h| \geq \delta\}) \leq 1} \leq \frac{1}{b}$$

which for all  $s \in \mathcal{S}$  has an expected cost function

$$U(s, \bar{H}_b) = \int_Q \mathbf{u}(s, h) d\bar{H}_b(h) = \frac{1}{b} \int_{|h| \geq \delta} \frac{\mathbf{u}(s, h)}{|h|} d\hat{H}(h) > \frac{(1 - 2\varepsilon)\hat{V}}{b} \quad (23)$$

Under the scaling property, we can use the scaling lemma with the mapping  $\varphi_b$  with  $b = 1$  and obtain a hiding strategy  $H_1$  with the restriction  $\int_Q |h| d\bar{H}_1(h) \leq 1$ , which makes sure that the expected capture time exceeds  $(1 - \varepsilon)\hat{V}$ . We have shown that  $\bar{V} = \hat{V}$ , so the two approaches lead to equivalent results.  $\square$

We will use Theorem 3.19 in chapter 4 to scale the non-scaled Linear Search Game onto the Normalized Linear Search Game. After computation, we can scale the found (normalized) values into to the values for the Linear Search Game. Before doing so, we will start with representing our search trajectories as a sequence of turning points.

### 3.2.4 Turning Points

We will be describing a search trajectory  $s(t)$  with the use of a sequence of turning points. These turning points will define the points at which the searcher changes direction. Between these turning points, the searcher will move at his maximum velocity since that is the dominant strategy as said in the opening by the set defined by (14). The turning points are defined as follows.

**Definition 3.20.** *The trajectory  $s(t)$  has a left turning point at time  $t_0 > 0$  if there exists an  $\varepsilon > 0$  such that*

$$s'(t) = 1 \text{ for } t_0 - \varepsilon < t < t_0 \quad \text{and} \quad s'(t) = -1 \text{ for } t_0 < t < t_0 + \varepsilon.$$

*Similarly, the trajectory  $s$  has a right turning point at  $t_0$  if*

$$s'(t) = -1 \text{ for } t_0 - \varepsilon < t < t_0 \quad \text{and} \quad s'(t) = +1 \text{ for } t_0 < t < t_0 + \varepsilon$$

*for some  $\varepsilon > 0$ . We will be referring to the points  $s(t_0)$  as turning points.*

We will be using the turning points to reduce the infinite real line problem into a problem with a denumerable double infinite sequence. The reason why the sequence has to be double infinite will be discussed in the lemma below.

**Lemma 3.21.** *Let  $s \in \mathcal{PS}$  be any pure search trajectory with  $\nu(s) < \infty$ , then for all  $t_0 > 0$  the following will hold for the turning points:*

- a) *The number of turning points before  $t_0$  is infinite.*
- b) *The number of turning points after  $t_0$  is also infinite.*

*Proof.* Property (a) is based on the fact that if there is a first left turning point, then the hider can obtain a larger payoff by choosing  $h = -\varepsilon$  for  $\varepsilon > 0$  very small. Visa versa, if there would be a first right turning point, the hider would choose  $h = \varepsilon$  being similarly small. The searcher will prevent this advantage from occurring and will make the following argument forever: "If the first step is to the right then the hider has a advantage when choosing  $h = \varepsilon$ , and visa versa". So we conclude that the amount of turning points before any arbitrary time  $t_0$  will be infinite. Property (b) can be easily shown from the unbounded properties (15) and thus the amount of turning points will be infinite after any arbitrary time  $t_0$ .  $\square$

Now that the motivation behind the choice of a doubly infinite sequence is clear, we can take a closer look at the representation of the turning points, and the relations between these points. The doubly infinite sequence of turning point will be expressed with the use of sequence  $\{x_i\}_{i=-\infty}^{\infty}$  with the convention that each

- left turning point can be represented as:  $x_i$  when  $i$  is even;
- right turning point can be represented as:  $-x_i$  when  $i$  is odd.



So, we can view this as the searcher (starting at an even  $i$ ) moves from  $x_i$  to  $-x_{i+1}$  and onto  $x_{i+2}$  and so on. Every new right (or left) turning point will be further than the last. Otherwise any turning points which is smaller than the last wastes time and doesn't discover the hider since the hider will not move and cannot appear on a location where the searcher has already been. This property holds for the sequence of left turning points and right turning points to which a reasonable search trajectory has to satisfy:

$$0 < \dots < x_{2j-2} < x_{2j} < x_{2j+2} < \dots$$

and

$$\dots < -x_{2j-3} < -x_{2j+1} < -x_{2j-1} < \dots < 0$$

for any  $j \in \mathbb{Z}$ . Thus, both the left turning points and the right turning points are monotonic for which all  $x_i > 0$ . Since  $s'(t) = \pm 1$  we know that time and distance are proportional. To travel to the location of the hider, the searcher has to travel along each line segment as shown in figure 2. He will start at the origin and will travel for all  $i$  the distance  $x_i$  twice, namely on his way to the turning point, and back to zero from which he moves in the same direction to the next turning point. Once the searcher reaches the hider, he will not have to return. This concludes that the cost function for the searcher is

$$u(s, h) = |h| + \sum_{j=-\infty}^{i+1} 2x_j \tag{24}$$

where  $x_i < |h| \leq x_{i+2}$ . Next to the derivation of the expression, we also provide a figure to further the intuition behind the turning points. Note that the the whole sequence  $\{x_j\}_{j=-\infty}^{\infty}$  is positive and that the actual turning points are a positive  $x_j$  when  $j$  is even or a negative  $-x_j$  when  $j$  is odd. The figure below states all turning points, so the actual sequence  $\{x_j\}_{j=-\infty}^{\infty}$  can be represented by mirroring all negative points  $-x_j$  when  $j$  is odd over the  $t$ -axis.

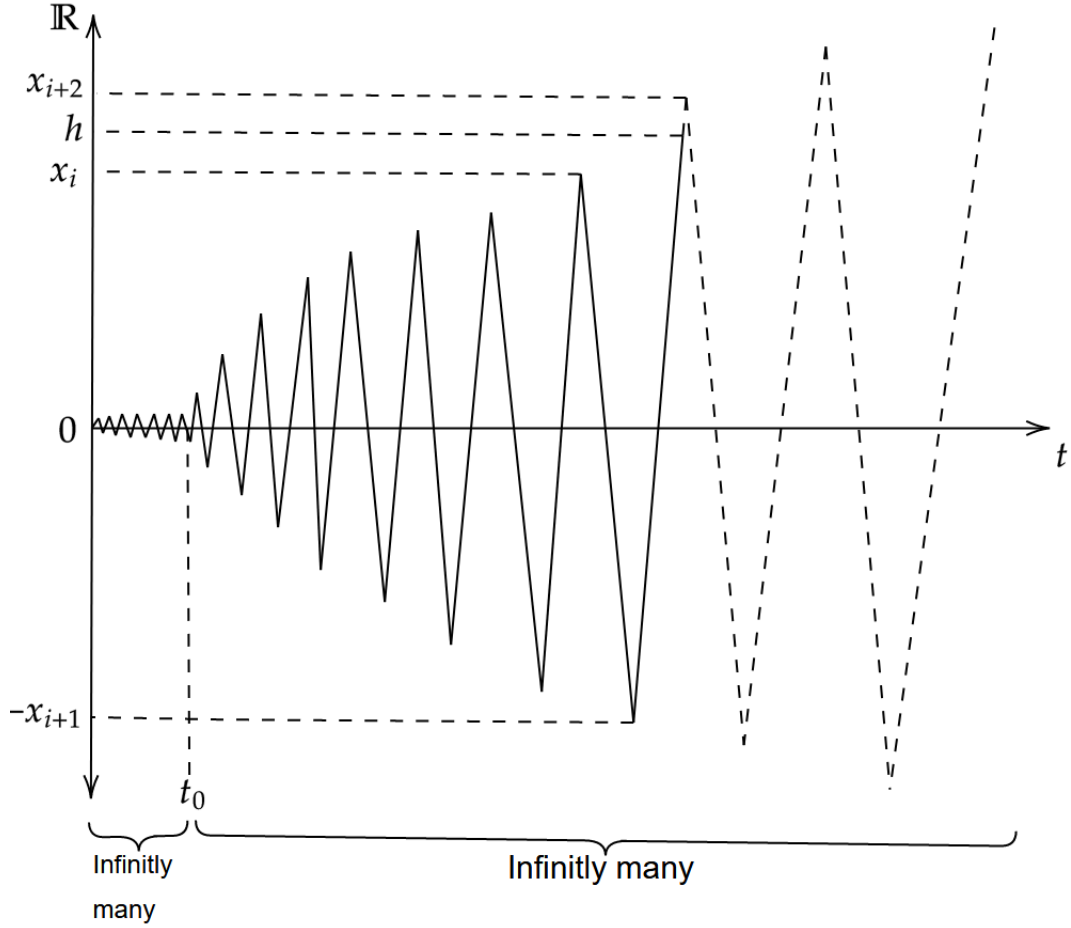


Figure 2: Representation of search trajectory with use of sequence of turning points

Note that the sequence  $\{x_j\}_{j=-\infty}^{\infty}$  represent only the turning points. The whole pure trajectory  $s(t)$  is given by the fact that we move with a maximal velocity along the turning points. Also note that the sequence before  $t_0$  is infinite, for clarity this can not be made to close in the illustration. The sequence after  $t_0$  is also infinite, this sequence is the combination of the connected and dotted lines combined. As a remark on the figure 2, we can reduce the problem by assuming that before a small  $t_0$  we will ignore all small oscillations of the turning points. This would reduce the doubly infinite sequence into a single infinite sequence where index  $t_0$  will be the starting index of the single infinite sequence such that  $x_{t_0} = x_0$ . However, we do not use this simplification.

## 4 Finding the Pure Minimax and Optimal Strategies

In this chapter we will be solving the Linear Search Game by finding the pure minimax strategy and the optimal strategies for the searcher. We focus on the pure minimax strategy since we are interested in the pure search trajectory which results in the best of the worst outcome. This chapter is based on Alpern&Gal [4] and Gal [11] where we changed and added few points which are discussed at the start of each section. The main structure in which we tell the theory does remain the same to the source. To compute the solutions we will be discussing several theorems which we use to approximate the cost function with a sequence of turning points by the cost function with a geometric sequence. This approximation will be used to find the pure minimax and optimal strategies. Section 4.1 will compute the pure minimax strategy for the searcher and include Theorem 4.1 which is the intuition behind the computation of the pure minimax and optimal strategies with use of geometric sequences. The other parts of this section are only used to compute the pure minimax strategy. In Section 4.2 we will compute the optimal strategies for the searcher. Section 4.3 will give an overview of the two results and will be the conclusion of this chapter. It will also provide an answer to the main focus of this thesis: Finding the relevant solutions to the Linear Search Game.

### 4.1 Solving the Linear Search Problem: Pure Minimax strategies

This section will start off with the value of the pure minimax strategy  $\overline{PV}$  which we want to calculate. We will introduce a framework in which a doubly infinite sequence can be shifted along the real number line and the functional of this sequence will have a lower bound given by the functional of the geometric sequence. We will make use of a combination of positive doubly infinite sequences which will be shown in Theorem 4.1 to be close enough to a geometric sequence. This theorem is extended to the  $(2k + 1)$ -dimensional case in contrast to the theorem given in Alpern&Gal [4, p.108] with the extreme cases more rigorously stated. We also added a proof to Lemma 4.2.

The two auxiliary theorems which we discuss provide a lower bound to the functional of a doubly infinite sequence (in contrast to the single infinite sequence in Alpern&Gal [4]). The first auxiliary Theorem 4.3 is a stronger theorem than given in Alpern&Gal [4, p.111] because the lower bound will be based on the functional instead of the infimum of the functional over all geometric sequences. This results in a simplified proof of the second auxiliary Theorem 4.4 compared to Alpern&Gal [4, p.113]. This first auxiliary theorem is also proven with use of a constructive proof for a doubly infinite sequence of length  $2k + 1$  in contrast to the proof by contradiction for a single infinite sequence given in Alpern&Gal [4].

With use of the two auxiliary theorems, we can provide the first main result which is Corollary 4.5 with a different functional than Alpern&Gal [4] chose. Our choice will be more in line with the second main result and is based on a similar function given in Gal [10] but we have a different sequence of coefficients. We also provide a proof of this corollary where we

show that all conditions hold for the functional chosen and that the application of Theorem 4.4 indeed results in the desired equality. The second main result given by Theorem 4.7 is based on Alpern&Gal [4]. Finally, we can determine the unique sequence of turning points which is the minimax strategy. This unique sequence can be used for the normalized and the restricted Linear Search Game because of Theorem 3.19 as shown in Chapter 3.

#### 4.1.1 The Setup for the Pure Minimax Search Trajectory

We defined the cost function in Section 3.2.4 which will be used to determine the pure minimax trajectory. We assumed that any pure search strategy  $s$  with finite  $\nu(s) < \infty$  satisfies (15) with probability 1. We assume that the expected location of the hider is restricted by Definition 3.14. Combining the assumptions of the unbounded search and restricted hiding strategies, we know that the two players will meet with a probability of 1. We have seen that the scaling properties allows us to scale the restricted game defined with the cost function in Section 3.2.4 can be scaled as given in Lemma 3.16. We will be using the normalized cost function to find a (finite) value for the normalized search game with corresponding pure minimax strategy which is equal the pure minimax strategy of the restricted search game. The value of the pure minimax strategy given by the pure minimax trajectory for a normalized search game is defined as:

$$\overline{PV} = \inf_{s \in \mathcal{PS}} \sup_{h \in \mathcal{PH}} \hat{u}(s, h) \quad (25)$$

where we can substitute the normalized cost function as given by Lemma 3.16 and use the cost function  $u$  described with the use of the turning points in (24) to formulate the value of the pure minimax strategy given by the minimax strategy as:

$$\overline{PV} = \inf_{s \in \mathcal{PS}} \sup_{h \in \mathcal{PH}} \hat{u}(s, h) = \inf_{s \in \mathcal{PS}} \sup_{h \in \mathcal{PH}} \left( 1 + \frac{\sum_{j=-\infty}^{i+1} 2x_j}{|h|} \right) = \inf_{s \in \mathcal{PS}} \left( 1 + 2 \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \right) \quad (26)$$

where the supremum changes from taking the supremum over all possible pure strategy to, taking the supremum over all possible indices for a turning point, because  $x_i < |h| \leq x_{i+2}$  for  $h \in \mathcal{PH}$ . To finalize the computation of the value for the normalized search game, we need to find an expression for

$$\inf_{s \in \mathcal{PS}} \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i}. \quad (27)$$

We will tackle this problem by finding a lower bound for which the infimum over  $s \in \mathcal{PS}$  is equal to the lower bound based on the geometric sequence. This approximation technique will be explained in the following section.

#### 4.1.2 Approximating Functional of Sequence of Turning Points by Functional of Geometric Sequence

This section will show that the geometric sequence is "close enough" to a sequence of turning points that describes the minimax trajectory. Alpern&Gal [4] contains two main results:

Firstly, the equality between the sequence of turning points and the geometric sequence and secondly the uniqueness of the geometric sequence found to be the describing the turning points of the minimax strategy. We start by giving definitions based on Alpern&Gal [4]. Next, we will introduce the reader to the term "close enough" in a more rigorous way. Afterwards we will provide the first main results with the use of two auxiliary theorems, from which we will calculate the value of the normalized game as described in Section 4.1.1. Let's start with the definitions. We define

$$\begin{aligned} X &:= \{x_j\}_{j=-\infty}^{\infty} && \text{Which will represent our doubly infinitely sequence.} \\ X^{+i} &:= \{x_{i+j}\}_{j=-\infty}^{\infty} && \text{Which will represent } X \text{ shifted } i \in \mathbb{Z} \text{ steps to the right} \end{aligned}$$

Let  $\{F_k(X)\}_{k=1}^{\infty}$  be a sequence of functionals where each functional  $F_k$ , i.e. a mapping from the space of all positive sequences into the real numbers is defined as

$$F_k(X) := F_k(x_{-k}, \dots, x_{-1}, x_0, x_1, \dots, x_k)$$

and from which follows that the functional corresponding with a shifted sequence is

$$F_k(X^{+i}) = F_k(x_{i-k}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+k}).$$

For each  $k \geq 1$  the functional  $F_k$  has to satisfy the following conditions:

- Let  $F_k(X)$  be continuous. (28)
- Let  $F_k(\alpha X) = F_k(X)$  for all  $\alpha > 0$ . (29)
- Let  $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$  for any  $X, Y$  (30)
- Let  $F_{k+1}(x_{-(k+1)}, x_{-k}, \dots, x_0, \dots, x_k, x_{k+1}) \geq F_k(x_{-k}, \dots, x_0, \dots, x_k)$ . (31)

Condition (30) is based on the more general inequality  $F_k(\alpha X + (1-\alpha)Y) \leq \max\{F_k(X), F_k(Y)\}$  for all  $X, Y$  and  $0 \leq \alpha \leq 1$ . But, because of the scaling condition (29), we get

$$F_k(\alpha X + (1 - \alpha)Y) = F_k(X + Y).$$

We will define the positive geometric sequences  $A_a$  as

$$A_a = \{a^j\}_{j=-\infty}^{\infty}.$$

This will specify the conditions that are needed for the functional  $F_k$  for extreme cases  $a = 0$  and  $a = \infty$ . For any "reasonable" functional  $F_k$  that satisfies the conditions (28)–(31) we define extreme cases  $a = 0$  or  $a = \infty$  via the following conditions

$$\bullet \text{ Let } F_k(A_{\infty}) := \liminf_{a \rightarrow \infty} F_k\left(\frac{1}{a^{2k}}, \dots, \frac{1}{a^k}, \dots, \frac{1}{a}, 1\right) = \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} F_k(\varepsilon_{2k}, \dots, \varepsilon_1, 1). \quad (32)$$

$$\bullet \text{ Let } F_k(A_0) := \liminf_{a \rightarrow 0} F_k\left(1, \dots, a^{k-1}, a^k, a^{k+1}, \dots, a^{2k}\right) = \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} F_k(1, \varepsilon_1, \dots, \varepsilon_{2k}). \quad (33)$$

We will be specifying the definition of "close enough" in the following theorem (without proof) based on the theorem 7.1 in the beginning of chapter 7 in Alpern [4, p.108]. This theorem is proven in the appendix of Gal [11, p.189–p.194]. The theorem is stated as

**Theorem 4.1.** Let  $X = \{x_j\}_{j=-\infty}^{\infty}$  be a positive sequence and let  $k \in \mathbb{N}$ . Let  $W_k$  be the  $(2k+1)$ -dimensional convex cone spanned by the set  $\{(x_{i-k}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+k}), -\infty < i < \infty\}$ . This can be expressed in the following set

$$W_k = \left\{ Y : Y = \sum_{i=-n}^n \beta_i \cdot (x_{i-k}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+k}); \beta_i \geq 0, -n \leq i \leq n, 0 < n < \infty \right\}.$$

Let  $\overline{W}_k$  denote the closure of  $W_k$ . Define

$$a := \limsup_{n \rightarrow \infty} (x_n)^{\frac{1}{n}}$$

then

$$\frac{1}{\sum_{i=0}^{2k} a^i} (1, \dots, a^{k-1}, a^k, a^{k+1}, \dots, a^{2k}) \in \overline{W}_k.$$

Or a more extensive notation which show cases  $a = 0$  and  $a = \infty$  more clearly given by

$$\left( \frac{1}{1 + \dots + a^{2k}}, \dots, \frac{1}{a^{k-1} + \dots + a^{k+1}}, \frac{1}{a^k + \dots + a^k}, \frac{1}{a^{k+1} + \dots + a^{k-1}}, \dots, \frac{1}{a^{2k} + \dots + 1} \right) \in \overline{W}_k.$$

For the case  $a = \infty$ , we see conclude from the extensive form that the first  $2k$  components go towards 0 when  $a$  goes to infinity. So we get that

$$(0, 0, \dots, 0, 1) \in \overline{W}_k.$$

Similarly, for the case  $a = 0$ , we conclude that the last  $2k$  components go towards 0 when  $a$  goes to 0. So we get that

$$(1, 0, \dots, 0, 0) \in \overline{W}_k.$$

Combining the linear combination given in Theorem 4.1 and the conditions (29) and (30) we can construct the following lemma which will provide a upper bound for the linear combination of positive sequences. This result will be helpful in the proof of the first auxiliary theorem.

**Lemma 4.2.** Let  $F_k$  satisfy the conditions (28)–(33). Let  $X$  be a positive sequence and let for  $l \geq 0$  the sequence of coefficients  $\{\beta_i\}_{i=-l}^l$  with  $\beta_i \geq 0$  for all  $-l \leq i \leq l$ .

$$\text{If } Y = \sum_{i=-l}^l \beta_i X^{+i} \text{ then } F_k(Y) \leq \max_{-l \leq i \leq l} F_k(X^{+i}).$$

*Proof.* For this proof we will be using the conditions (29) and (30). This proof will consist of two steps, which can be used recursively to prove the statement. Starting with the first step, we would split the  $Y$  into two parts, for which we apply condition (30) such that

$$Y = \underbrace{\beta_{-l} X^{-l}}_{\text{Part 1}} + \underbrace{\beta_{-l+1} X^{-l+1} + \dots + \beta_0 X + \dots + \beta_l X^{+l}}_{\text{Part 2}} = \beta_{-l} X^{-l} + Y'$$

where  $Y' := \beta_{-l+1}X^{-l+1} + \dots + \beta_0X + \dots + \beta_lX^{+l}$  implies that

$$F_k(Y) \leq \max \left\{ F_k(\beta_{-l}X^{-l}), F_k(Y') \right\}$$

If  $F_k(\beta_{-l}X^{-l}) \geq F_k(Y')$  then we apply condition (29) to find that  $F_k(Y) \leq F_k(X^{-l})$ , otherwise we will apply the previous step again, but this time with  $Y'$  which can also be split into two parts as

$$Y' = \underbrace{\beta_{-l+1}X^{-l+1}}_{\text{Part 1}} + \underbrace{\beta_{-l+2}X^{-l+2} + \dots + \beta_0X + \dots + \beta_lX^{+l}}_{\text{Part 2}}.$$

From recursive use of these two steps we will find that the statement in this lemma holds.  $\square$

For any  $X$  a linear combination of shifted sequences  $X^{+i}$  can be constructed with non-negative coefficients  $\beta_i$  for  $-l \leq i \leq l$  for which the functional will not exceed the upper bound set by the sequence  $X^{+i}$  that provide the maximal value for functional  $F_k$ . The benefit of this linear combination is that this can be made as close as desired to the geometric sequence  $A_a$  with  $a$  defined as in Theorem 4.1. The first auxiliary theorem defines a lower bound for a functional which depends on  $2k + 1$  components of the doubly infinite sequence.

**Theorem 4.3** (First Auxiliary Theorem). *For any  $k \geq 0$ . If  $F_k$  satisfies the conditions (28)–(33), then for any positive sequence  $X$  and a geometric sequence  $A_a$  with the base  $a = \limsup_{n \rightarrow \infty} (x_n)^{\frac{1}{n}}$  we have*

$$\limsup_{i \rightarrow \infty} F_k(X^{+i}) \geq F_k(A_a) \tag{34}$$

$$\limsup_{i \rightarrow -\infty} F_k(X^{+i}) \geq F_k(A_a) \tag{35}$$

*Proof.* We know that for any  $i \in \mathbb{Z}$  that the defined  $a$  is the same even when shifted  $i$  indices as shown by

$$a = \limsup_{n \rightarrow \infty} (x_n)^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} (x_{n+i})^{\frac{1}{n}}.$$

We start by proving the first inequality (34). We will divide this proof into three cases:  $0 < a < \infty$ ,  $a = \infty$  and  $a = 0$ . Firstly, we start by showing the case that  $0 < a < \infty$  which will be explained in most detail. The other cases are very similar. From Theorem 4.1 applied to the shifted sequence  $X^{+i_0}$ , we know that for all  $i_0 \in \mathbb{Z}$  an  $l \in \mathbb{N}$  and sequence  $\{\beta_i\}_{i=-l}^l$  can be found such that

$$Y = \sum_{i=-l+i_0}^{l+i_0} \beta_i X^{+i}$$

can be made arbitrarily close to

$$\frac{1}{\sum_{j=0}^{2k} a^j} \left( 1, \dots, a^{k-1}, a^k, a^{k+1}, \dots, a^{2k} \right).$$

Based on Lemma 4.2 the functional  $F_k(Y)$  has an upper bound given for all  $i_0 \in \mathbb{Z}$  as

$$F_k(Y) \leq \sup_{-l+i_0 \leq i \leq l+i_0} F_k(X^{+i}) \quad (36)$$

Let  $A \subseteq B \subset \mathbb{R}$  where  $\{x_n\}_{n \in \mathbb{N}} \in A$  it holds that  $\sup_A x_n \leq \sup_B x_n$  which is used to conclude

$$F_k(Y) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}) \quad (37)$$

with  $A = [-l + i_0, l + i_0]$  and  $B = [-l + i_0, \infty)$ . We conclude by combining the arguments that sequence  $Y$  can be made arbitrarily close to  $\alpha A_a$  with  $\alpha = \frac{1}{\sum_{j=0}^{2k} a^j}$ , and functional  $F_k$  is continuous by condition (28) that the functional  $F_k(\alpha A_a)$  has the same upper bound for all  $i_0 \in \mathbb{Z}$  given by

$$F_k(\alpha A_a) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}).$$

We will be using the scaling condition (29) to find that

$$F_k(A_a) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}). \quad (38)$$

Since inequality (38) holds for all  $i_0 \in \mathbb{Z}$ , we have

$$F_k(A_a) \leq \inf_{i_0 \geq 0} \sup_{i_0 \leq i+l} F_k(X^{+i}) = \limsup_{i \rightarrow \infty} F_k(X^{+(i+l)}) = \limsup_{i \rightarrow \infty} F_k(X^{+i}).$$

which is the desired result for the first case.

The second case: Let  $a = \infty$ . Theorem 4.1 states that for any  $\delta > 0$  we can find a linear combination  $Y$  defined as

$$Y = \sum_{i=-l+i_0}^{l+i_0} \beta_i X^{+i}$$

satisfying  $y_k = 1$  and  $y_j = \delta$  for all  $-k \leq j \leq k-1$ . This  $Y = (\delta, \delta, \dots, \delta, 1)$  can be made arbitrarily close to the geometric sequence  $\alpha A_\infty$  with  $\alpha = \frac{1}{\sum_{j=0}^{2k} a^j}$  by choosing  $\delta$  very small.

We know by Lemma 4.2 that  $Y$  has an upper bound defined by

$$F_k(Y) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}) \quad (39)$$

and by continuity, we know that this upper bound also holds for  $F_k(\alpha A_\infty)$  and because of condition (29) also for non-scaled  $F_k(A_\infty)$  in the following way

$$F_k(A_\infty) = F_k(\alpha A_\infty) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}).$$

This leaves us with the same conclusion as the previous case. We conclude that

$$F_k(A_\infty) \leq \limsup_{i \rightarrow \infty} F_k(X^{+i}). \quad (40)$$



We conclude the proof for the second case.

Finally the last case: Let  $a = 0$ . Theorem 4.1 states that for any  $\delta > 0$  we can find a linear combination  $Y$  defined as

$$Y = \sum_{i=-l+i_0}^{l+i_0} \beta_i X^{+i}$$

satisfying  $y_{-k} = 1$  and  $y_j = \delta$  for all  $-k + 1 \leq j \leq k$ . This  $Y = (1, \delta, \dots, \delta, \delta)$  can be made arbitrarily close to the geometric sequence  $\alpha A_0$  with  $\alpha = \frac{1}{\sum_{j=0}^{2k} a^j}$  by choosing  $\delta$  very small.

We know by Lemma 4.2 that  $Y$  has an upper bound defined by

$$F_k(Y) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}) \quad (41)$$

and similar to the second case we use continuity condition (28) and scaling condition (29) to find that

$$F_k(A_0) = F_k(\alpha A_0) \leq \sup_{-l+i_0 \leq i} F_k(X^{+i}).$$

This leaves us with the same conclusion as the last two cases. We conclude for  $a = 0$  that

$$F_k(A_0) \leq \limsup_{i \rightarrow \infty} F_k(X^{+i}). \quad (42)$$

Combining the three cases we conclude the proof of the inequality (34).

Similar to the proof of the first inequality, we will prove inequality (35) for three cases:  $0 < a < \infty$ ,  $a = \infty$  and  $a = 0$ . The proof is analogous up to inequality (36) where we change inequality (37) into the following

$$F_k(Y) \leq \underbrace{\sup_{-l+i_0 \leq i \leq l+i_0} F_k(X^{+i})}_{\text{Unchanged Equation (36)}} \leq \underbrace{\sup_{i \leq l+i_0} F_k(X^{+i})}_{\text{changed}} \quad (43)$$

Again, sequence  $Y$  can be made arbitrarily close to  $\alpha A_a$  with  $\alpha = \frac{1}{\sum_{j=0}^{2k} a^j}$ . Similar to the proof of equation (34) we conclude from the continuity condition (28) and scaling condition (29) that the functional  $F_k(A_a)$  has the same upper bound, so for all  $i_0 \in \mathbb{Z}$  we get

$$F_k(A_a) \leq \sup_{i \leq l+i_0} F_k(X^{+i}). \quad (44)$$

From inequality (44) we can choose an  $0 < a < \infty$  in particular such that

$$F_k(A_a) \leq \inf_{i_0 \leq 0} \sup_{i_0 \geq i-l} F_k(X^{+i}) = \limsup_{i \rightarrow -\infty} F_k(X^{+(i-l)}) = \limsup_{i \rightarrow -\infty} F_k(X^{+i}). \quad (45)$$

For the cases  $a = \infty$  and  $a = 0$ , the choice of  $Y$  will be exactly the same such that  $Y$  will be arbitrarily close to  $A_\infty$  and  $A_0$  respectively. Again, both inequalities (39) and (41) will have

their upper bound slightly changed similar to (43) such that after the continuity argument we can conclude that

$$F_k(A_\infty) \leq \sup_{i \leq l+i_0} F_k(X^{+i}) \quad \text{and} \quad F_k(A_0) \leq \sup_{i \leq l+i_0} F_k(X^{+i}). \quad (46)$$

Similar to (45) we will change the inequalities (40) and (42) such that the regions on which the infimum and supremum are taken are equal to those in (45) and result in the same limit superior. This will result in two equations we need

$$F_k(A_\infty) \leq \limsup_{i \rightarrow -\infty} F_k(X^{+i}) \quad \text{and} \quad F_k(A_0) \leq \limsup_{i \rightarrow -\infty} F_k(X^{+i}). \quad (47)$$

This concludes the proof of inequality (35).  $\square$

We proved that the first auxiliary theorem provides a lower bound for the functional  $F_k$  when the positive sequence is moved infinitely towards the right or the left. For any positive sequence  $X$ , we define the functional

$$F(X) := \lim_{k \rightarrow \infty} F_k(X). \quad (48)$$

This will let us define the cost function as  $\hat{u}(X, i) = 1 + 2F(X^{+i})$  where  $F$  is the functional in (48) and the doubly infinite sequence  $X$  will be representing our sequence of turning point. The definition of  $F$  for the extreme cases of  $A_0$  and  $A_\infty$  will be specified. We will define

$$F(A_0) := \lim_{k \rightarrow \infty} F_k(A_0) \quad \text{and} \quad F(A_\infty) := \lim_{k \rightarrow \infty} F_k(A_\infty). \quad (49)$$

With definitions given by (48) and (49) a second auxiliary theorem can be constructed which will be similar to the first, but now  $k$  goes to infinity for the next theorem.

**Theorem 4.4** (Second Auxiliary Theorem). *Let  $\{F_k(X)\}_{k=1}^\infty$  be a sequence of functional and assume for each  $k \geq 1$  that  $F_k(X)$  satisfies the conditions (28)–(33). Let  $k$  in equation (34) and (35) of the first auxiliary theorem go to infinity. From definitions (48) and (49) it follows that*

$$\limsup_{i \rightarrow \infty} F(X^{+i}) \geq \inf_{0 \leq a \leq \infty} F(A_a), \quad \text{and} \quad (50)$$

$$\limsup_{i \rightarrow -\infty} F(X^{+i}) \geq \inf_{0 \leq a \leq \infty} F(A_a). \quad (51)$$

*Proof.* We will limit our proof to inequality (50), because inequality (51) can be proven in exactly the same way, the only difference is that for the limit superior the  $i$  goes to negative infinity. Let  $a = \limsup_{n \rightarrow \infty} (x_n)^{\frac{1}{n}}$  as defined in Theorem 4.1. In the the first auxiliary theorem we found that for any  $k \geq 1$  we have

$$\limsup_{i \rightarrow \infty} F_k(X^{+i}) \geq F_k(A_a).$$

The monotonicity condition (31) stated as

$$F_{k+1}(x_{-(k+1)}, x_{-k}, \dots, x_0, \dots, x_k, x_{k+1}) \geq F_k(x_{-k}, \dots, x_0, \dots, x_k)$$

combined with definition (48) can be used to show that: If  $k$  goes to infinity, then the functional  $F(X)$  will be bigger than  $F_k(X)$  for any  $k \geq 1$ . This result is used in the following inequality

$$\limsup_{i \rightarrow \infty} F(X^{+i}) = \limsup_{i \rightarrow \infty} \underbrace{\lim_{k \rightarrow \infty} F_k(X^{+i})}_{\text{Follows from condition (31): } F_{k+1}(X_{k+1}) \geq F_k(X)} \geq \limsup_{i \rightarrow \infty} F_k(X^{+i}) \geq F_k(A_a) \quad \text{for } k \geq 1. \quad (52)$$

Since  $F(A_a)$  is defined as a limit of  $k$  to infinity by definition (49) and we know for each  $k \leq 1$  that  $F_k(A_a)$  is bounded from above by (52) thus we get

$$\limsup_{i \rightarrow \infty} F(X^{+i}) \geq F(A_a)$$

which we use in combination that for any  $F$  the property  $F(A_a) \geq \inf_{0 \leq a \leq \infty} F(A_a)$  holds. By using that property we find that

$$\limsup_{i \rightarrow \infty} F(X^{+i}) \geq \inf_{0 \leq a \leq \infty} F(A_a)$$

which is the desired result. □

We have shown that a functional  $F(X)$  meeting conditions (28)–(33) has a limit superior which is equal to a infimum of a functional  $F(A_a)$ . We will provide an answer for the value (27) by applying Theorem 4.4 to a functional which we will define in the following theorem.

**Corollary 4.5** (First Main Result). *Let  $\{B_j\}_{j=-\infty}^{\infty}$  be a non-negative sequence with some  $B_j$  positive. Then the function*

$$F_k(X) := \frac{\sum_{j=-k}^k B_j x_j}{x_0}$$

*satisfies conditions and inequality (28)–(33) and the inequalities*

$$\limsup_{i \rightarrow \pm\infty} \frac{\sum_{j=-\infty}^{\infty} B_j x_{i+j}}{x_i} \geq \inf_{0 < a < \infty} \frac{\sum_{j=-\infty}^{\infty} B_j a^j}{a^i}, \quad (53)$$

$$\limsup_{i \rightarrow \pm\infty} \frac{\sum_{j=-\infty}^{\infty} B_j x_{i+j}}{x_i} \geq \liminf_{a \rightarrow \infty} \frac{\sum_{j=-\infty}^{\infty} B_j \frac{1}{a^j}}{\frac{1}{a^i}}, \quad (54)$$

$$\limsup_{i \rightarrow \pm\infty} \frac{\sum_{j=-\infty}^{\infty} B_j x_{i+j}}{x_i} \geq \liminf_{a \rightarrow 0} \frac{\sum_{j=-\infty}^{\infty} B_j a^j}{a^i} \quad (55)$$

hold. By choosing the non-negative sequence

$$\{B_j\}_{j=-\infty}^{\infty} \text{ where } B_j = 1 \text{ for } j \leq 1 \text{ and } B_j = 0 \text{ for } j \geq 2 \quad (56)$$

we get for any positive sequence  $X$  that

$$\inf_X \sup_{\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} = \inf_{a > 1} \frac{a^2}{a-1}. \quad (57)$$

*Proof.* Firstly, we will start by showing that the conditions (28)–(33) hold for our choice of functional  $F_k(X)$ . Secondly, we will show that inequality (53) implies inequality (57).

**First part of proof:**

C.28 For any  $k \geq 1$ , note that  $F_k(X)$  is a summation of continuous functions  $B_j x_j$ , which is divided by a positive linear function  $x_0$  so that the application of the summation and quotient rule of limits will give the intuition that  $F_k(X)$  is continuous. We will prove this rigorously as follows. For all  $k > 0$ , the functional  $F_k(X)$  is continuous at  $X$  if:

for  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  : if  $Y \in Q$  and for all  $|x_j - y_j| < \delta$  for  $\forall j \in \{-k, \dots, 0, \dots, k\}$

then  $|F_k(X) - F_k(Y)| < \varepsilon$ .

Let  $\varepsilon > 0$  and set  $L := 2 \max\{x_k, x_{k-1}\} \sum_{j=-k}^k B_j$  and choose  $\delta := \frac{\varepsilon x_0^2}{L - \varepsilon x_0}$ . For any sequence  $Y$  satisfying the distance inequality of sequences  $|x_j - y_j| < \delta$  for all  $j \in \{-k, \dots, k\}$  we have the following. Firstly, we will use  $|x_j - y_j| < \delta$  for  $j = 0$  to find that  $\delta - x_0 < y_0 < x_0 + \delta$  and thus we can find that inequality  $\frac{1}{x_0 y_0} < \frac{1}{x_0(x_0 + \delta)}$  holds.

Secondly, we will use the assumptions which we put on the doubly infinite sequences, namely that we have a monotone increasing sequence from which we can conclude the following inequality  $x_j \leq \max\{x_k, x_{k-1}\}$  for all  $j \in \{-k, \dots, k\}$ .

Thirdly, we will find an upper bound for the absolute value  $|y_0 x_j - x_0 y_j|$  via the use of adding and subtracting  $x_0 x_j$  as follows

$$|y_0 x_j - x_0 y_j| = |y_0 x_j - x_0 y_j + x_0 x_j - x_0 x_j| \leq x_j |y_0 - x_0| + x_0 |y_j - x_j|$$

where we can combine the second found result, namely  $x_j, x_0 \leq \max\{x_k, x_{k-1}\}$  and use the fact that  $|x_j - y_j| < \delta$  for all  $j \in \{-k, \dots, k\}$  to find that

$$x_j |y_0 - x_0| + x_0 |y_j - x_j| < \max\{x_k, x_{k-1}\} \delta + \max\{x_k, x_{k-1}\} \delta = 2 \max\{x_k, x_{k-1}\} \delta.$$

Lastly, we will show that  $|F_k(X) - F_k(Y)| < \varepsilon$  by applying the first and third result of this proof in the following way

$$\begin{aligned} |F_k(X) - F_k(Y)| &= \left| \frac{\sum_{j=-k}^k B_j x_j}{x_0} - \frac{\sum_{j=-k}^k B_j y_j}{y_0} \right| = \frac{1}{x_0 y_0} \sum_{j=-k}^k B_j |y_0 x_j - x_0 y_j| \\ &< \frac{\delta}{x_0(x_0 + \delta)} 2 \max\{x_k, x_{k-1}\} \sum_{j=-\infty}^{\infty} B_j = \frac{L \delta}{x_0(x_0 + \delta)}. \end{aligned}$$

Next, we will use inequalities  $\frac{1}{x_0 y_0} < \frac{1}{x_0(x_0 + \delta)}$  and  $|y_0 x_j - x_0 y_j| \leq 2 \max\{x_k, x_{k-1}\}$  for all  $j \in \{-k, \dots, k\}$  to find that

$$\delta \cdot \frac{L}{x_0(x_0 + \delta)} = \frac{x_0^2 \varepsilon}{L - x_0 \varepsilon} \cdot \frac{L}{x_0 \left( x_0 + \frac{x_0^3 \varepsilon}{L - x_0 \varepsilon} \right)} = \frac{L x_0^2}{x_0^2 - x_0^3 \varepsilon + x_0^3 \varepsilon} = \frac{L x_0^2}{x_0^2} = \varepsilon.$$

Thus, we conclude that  $|F_k(X) - F_k(Y)| < \varepsilon$  which is the desired result.

C.29 The fact that the sequence  $X$  can be scaled by a factor  $\alpha > 0$  and the functional  $F_k(\alpha X)$  result in the same value as  $F_k(X)$  we show in the following way:

$$F_k(\alpha X) = \frac{\sum_{j=-k}^k \alpha B_j x_j}{\alpha x_0} = \frac{\alpha \sum_{j=-k}^k B_j x_j}{\alpha x_0} = \frac{\sum_{j=-k}^k B_j x_j}{x_0} = F_k(X)$$

C.30 As mentioned below conditions (28)–(31), showing for sequences  $X$  and  $Y$  that

$$F_k(\alpha X + (1 - \alpha)Y) \leq \max[F_k(X), F_k(Y)] \quad \text{for } 0 \leq \alpha \leq 1$$

will be enough to show since applying condition (29) will result in (30). Without loss of generality we let  $F_k(X) \leq F_k(Y)$ , then

$$\begin{aligned} F_k(\alpha X + (1 - \alpha)Y) &= \frac{\sum_{j=-k}^k \alpha B_j x_j + (1 - \alpha) B_j y_j}{\alpha x_0 + (1 - \alpha) y_0} \\ &= \frac{\sum_{j=-k}^k \alpha B_j x_j}{\alpha x_0 + (1 - \alpha) y_0} + \frac{\sum_{j=-k}^k (1 - \alpha) B_j y_j}{\alpha x_0 + (1 - \alpha) y_0} \\ &\leq \frac{\sum_{j=-k}^k \alpha B_j x_j}{x_0} + \frac{\sum_{j=-k}^k (1 - \alpha) B_j y_j}{y_0} \\ &= \alpha F_k(X) + (1 - \alpha) F_k(Y) \\ &\leq \alpha F_k(X) + (1 - \alpha) F_k(X) \\ &= F_k(X) \end{aligned}$$

C.31 We show that  $F_k(X)$  is monotonically increasing by using the property that each term  $\frac{B_j x_j}{x_0}$  for  $-\infty < j < \infty$  is non-negative. Then

$$\begin{aligned} F_{k+1}(X) &= \frac{\sum_{j=-k-1}^{k+1} B_j x_j}{x_0} \\ &= \frac{B_{-k-1} x_{-k-1}}{x_0} + \frac{B_{-k} x_{-k}}{x_0} + \dots + \frac{B_0 x_0}{x_0} + \dots + \frac{B_k x_k}{x_0} + \frac{B_{k+1} x_{k+1}}{x_0} \\ &\downarrow \text{because } \frac{B_{-k-1} x_{-k-1}}{x_0} \geq 0 \text{ and } \frac{B_{k+1} x_{k+1}}{x_0} \geq 0 \\ &\geq \frac{B_{-k} x_{-k}}{x_0} + \dots + \frac{B_0 x_0}{x_0} + \dots + \frac{B_k x_k}{x_0} \\ &= \frac{\sum_{j=-k}^k B_j x_j}{x_0} \\ &= F_k(X) \end{aligned}$$

shows that  $F_k(X)$  is monotonically increasing.

C.32 We will show that the extreme case condition (32) holds by showing that

$$\liminf_{a \rightarrow \infty} F_k \left( \frac{1}{a^{2k}}, \dots, \frac{1}{a^k}, \dots, \frac{1}{a}, 1 \right) = \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} F_k(\varepsilon_{2k}, \dots, \varepsilon_1, 1).$$

holds for our  $F_k(X)$ . So we see that

$$\begin{aligned} \liminf_{a \rightarrow \infty} F_k(A_\infty) &= \liminf_{a \rightarrow \infty} F_k \left( \frac{1}{a^{2k}}, \dots, \frac{1}{a^k}, \dots, \frac{1}{a}, 1 \right) = \liminf_{a \rightarrow \infty} \frac{\sum_{j=-k}^k B_j x_j}{x_0} \\ &= \liminf_{a \rightarrow \infty} \frac{B_{-k} \frac{1}{a^{2k}} + B_{-k+1} \frac{1}{a^{2k-1}} + \dots + B_0 \frac{1}{a^k} + \dots + B_{k-1} \frac{1}{a} + B_k}{\frac{1}{a^k}} \\ &= \liminf_{a \rightarrow \infty} B_{-k} \frac{1}{a^k} + \dots + B_0 + \dots + B_k a^k \\ &= \liminf_{a \rightarrow \infty} B_{-k} \frac{1}{a^k} + \dots + B_0 + \dots + B_k \frac{1}{a^k} \\ &= \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} B_{-k} \varepsilon_k + \dots + B_0 + \dots + B_k \frac{1}{\varepsilon_k} \\ &= \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} F_k(\varepsilon_{2k}, \dots, \varepsilon_1, 1). \end{aligned}$$

We notice that the two limit inferiors do result in an equivalent expression. The value for this expression will be infinite, but this does not lead to any issues, because there is no requirement that  $F_k(A_\infty)$  needs to be bounded from above.

C.33 Similar to the previous extreme case, we will show that condition

$$\liminf_{a \rightarrow 0} F_k \left( 1, \dots, a^{k-1}, a^k, a^{k+1}, \dots, a^{2k} \right) = \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} F_k(1, \varepsilon_1, \dots, \varepsilon_{2k})$$

holds in the following way

$$\begin{aligned} \liminf_{a \rightarrow 0} F_k(A_0) &= \liminf_{a \rightarrow 0} F_k \left( 1, \dots, a^{k-1}, a^k, a^{k+1}, \dots, a^{2k} \right) = \liminf_{a \rightarrow 0} \frac{\sum_{j=-k}^k B_j x_j}{x_0} \\ &= \liminf_{a \rightarrow 0} \frac{B_{-k} + B_{-k+1} a + \dots + B_0 a^k + \dots + B_{k-1} a^{2k-1} + B_k a^{2k}}{a^k} \\ &= \liminf_{a \rightarrow 0} B_{-k} \frac{1}{a^k} + \dots + B_0 + \dots + B_k a^k \\ &= \lim_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} B_{-k} \frac{1}{\varepsilon_k} + \dots + B_0 + \dots + B_k \varepsilon_k \\ &= \liminf_{\varepsilon_1, \dots, \varepsilon_{2k} \rightarrow 0} F_k(1, \varepsilon_1, \dots, \varepsilon_{2k}). \end{aligned}$$

Again, the two limit inferiors result in an equivalent expression. The value for this expression will again be infinite, but this does not lead to any issues, because there is no requirement that  $F_k(A_0)$  needs to be bounded from above.

This concludes the part that the conditions (28)–(33) hold for  $F_k(X)$ . Furthermore, we can apply inequality (50) and (51) from Theorem 4.4 to find that for our choice of  $F_k$  we get that inequality (57).

**Second part of proof:**

For the next part, we shall show that inequality (53) implies (57). We will consider the specific sequence which we introduced before:

$$\{B_j\}_{j=-\infty}^{\infty} \text{ where } B_j = 1 \text{ for } j \leq 1 \text{ and } B_j = 0 \text{ for } j \geq 2$$

such that shifted functional  $F(X^{+i}) = \lim_{k \rightarrow \infty} F_k(X^{+i})$  is given by

$$F(X^{+i}) = \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{\infty} B_j x_{i+j}}{x_i} = \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^1 x_{i+j}}{x_i} = \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \quad (58)$$

which we use to show that the right hand side of (58) has a lower bound, given by a functional which has the geometric sequence  $A_a$  as the sequence of turning points. This can be shown in the following way:

$$\begin{aligned} \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} &\geq \underbrace{\limsup_{i \rightarrow \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i}}_{\text{Follows from inequality (50)}} \geq \inf_{0 \leq a \leq \infty} \frac{\sum_{j=-\infty}^1 a^j}{a^0} \\ &= \underbrace{\inf_{0 \leq a \leq \infty} \sum_{j=-\infty}^1 a^j}_{\text{Because infimum over smaller interval}} \geq \inf_{a > 1} \sum_{j=-\infty}^1 a^j = \inf_{a > 1} a + 1 + \frac{1}{a} + \frac{1}{a^2} + \dots \\ &= \inf_{a > 1} a + \frac{a}{a-1} = \inf_{a > 1} \frac{a^2}{a-1}. \end{aligned}$$

Thus we can conclude that inequality

$$\sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \geq \inf_{a > 1} \frac{a^2}{a-1}$$

holds and can be made in an equality when we provide an explicit sequence  $\bar{X}$  such that for any positive sequence  $X$  we get

$$\inf_X \sup_{-\infty < i < \infty} F(X^{+i}) = \sup_{-\infty < i < \infty} F(\bar{X}^{+i}) = \inf_{a > 1} \frac{a^2}{a-1} = 4.$$

We will choose  $\bar{X} = \{\alpha 2^j\}_{j=-\infty}^{\infty}$  where  $\alpha > 0$  such that

$$\sup_{-\infty < i < \infty} F(\bar{X}^{+i}) = \sup_{-\infty < i < \infty} \sum_{j=-\infty}^{i+1} \frac{\alpha 2^j}{\alpha 2^i} = \sup_{-\infty < i < \infty} \sum_{j=-\infty}^{i+1} 2^{j-i}. \quad (59)$$

Let  $m := j - i$  which we will use to substitute and show that the summation is independent of  $i$  in the following way

$$\sup_{-\infty < i < \infty} \sum_{j=-\infty}^{i+1} 2^{j-i} = \sup_{-\infty < i < \infty} \underbrace{\sum_{m=-\infty}^1 2^m}_{\text{independent of } i} = \sum_{m=-\infty}^1 2^m = \frac{2^2}{2-1} = 4. \quad (60)$$

Note that for  $\bar{a} = 2$ , we get

$$\inf_{a>1} \frac{a^2}{a-1} = \frac{\bar{a}}{\bar{a}-1} = 4 \quad (61)$$

and thus we see that base  $\bar{a}$  at which the infimum of  $\frac{a^2}{a-1}$  is attained is also part of the base in the sequence  $\bar{X}$  in the following way

$$\bar{X} = \{\alpha 2^j\}_{j=-\infty}^{\infty} = \{\alpha \bar{a}^j\}_{j=-\infty}^{\infty}.$$

We conclude that the equation

$$\sup_{-\infty < i < \infty} F(\bar{X}^{+i}) = \frac{\bar{a}^2}{\bar{a}-1}$$

results in: for any positive sequence  $X$  we conclude that

$$\inf_X \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} = \inf_{a>1} \frac{a^2}{a-1} \quad (62)$$

which is the desired result for the second part of the proof.  $\square$

To finalize the section on approximating the sequence of turning point we will give a theorem which will provide a way to show that the given functional  $F_k(X)$  in Corollary 4.5 will result in an unique minimax strategy. We will not prove this theorem, because the theory behind it is based on the continuous cases of Theorems 4.1 till 4.4 and Corollary 4.5 and proves the discrete theorem by discretizing the measure and show that the discretized measure satisfies the conditions for the continuous theorem.

For a rigorous proof we recommend Section 7.3 given in Alpern [4, p.118–p.121]. For this thesis, we will provide the theorem 7.18 from Section 7.3 in Alpern [4, p.121] in addition to the definition of an arithmetic measure and the span of a measure.

**Definition 4.6** (Span of a arithmetic measure). *A measure  $A$  is arithmetic if it is concentrated on a set of points of the form*

$$\dots, -2\lambda, -\lambda, 0, \lambda, 2\lambda, \dots$$

where the largest  $\lambda$  with this property is called the span of  $A$ .



**Theorem 4.7** (Second Main Result). *Let  $\{B_j\}_{j=-\infty}^{\infty}$  be a non-negative sequence with the following bilateral (meaning: the index is infinite in both directions) generating function*

$$\psi(a) = \sum_{j=-\infty}^{\infty} B_j a^j, \quad a > 0.$$

*Assume that  $\psi(a)$  attains its minimum at a positive number  $\bar{a}$  with  $0 < \bar{a} < \infty$  and that the derivative  $\psi'(\bar{a})$  satisfies*

$$\sum_{j=-\infty}^{\infty} j B_j \bar{a}^{j-1} = 0$$

*Then any positive sequence  $X$  that satisfies the inequality*

$$\frac{\sum_{j=-\infty}^{\infty} B_j x_{i+j}}{x_i} \leq \psi(\bar{a}), \quad (63)$$

*for all  $-\infty < j < \infty$  has the following form:*

- *if the span of  $\{B_j\}_{j=-\infty}^{\infty}$  is 1, then  $x_i = \alpha \bar{a}^i$ , where  $\alpha$  is a positive constant.*
- *if the span of  $\{B_j\}_{j=-\infty}^{\infty}$  is  $\lambda > 1$ , then  $x_i = \alpha_i \bar{a}^i$ , where  $\alpha_i > 0$  is a periodic sequence with period  $\lambda$ .*

In the following part, we will apply Theorem 4.7 to our choice of sequence  $\{B_j\}_{j=-\infty}^{\infty}$  which is the same as the sequence given by equation (56) in of Corollary 4.5 such that we can determine the form of any positive sequence  $X$  that satisfies

$$\frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \leq \psi(\bar{a}). \quad (64)$$

Note that sequence  $\{B_j\}_{j=-\infty}^{\infty}$  from Theorem 4.7 implies that  $\psi(a)$  is equivalent to the function on the right hand side of (57) given in Corollary 4.5. So we get

$$\psi(a) = \sum_{j=-\infty}^{\infty} B_j a^j = \sum_{j=-\infty}^1 a^j = \frac{a^2}{a-1} \quad (65)$$

where taking the infimum on the right hand side as given by (57) results into the value  $\inf_{0 \leq a \leq \infty} \psi(a) = \frac{\bar{a}^2}{\bar{a}-1} = 4$  where  $\bar{a} = 2$ . The derivative is given by

$$\psi'(a) = \frac{2a(a-1) - a^2}{(a-1)^2} = \frac{a(a-2)}{(a-1)^2} \quad (66)$$

where indeed  $\psi'(\bar{a}) = \psi'(2) = \frac{2(2-2)}{(2-1)^2} = 0$ . Notice that the result of Corollary 4.5 given by

$$\inf_X \sup_{\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} = \inf_{a>1} \frac{a^2}{a-1} \quad (67)$$

states there exists a sequence  $\bar{X} = \{\alpha 2^j\}_{j=-\infty}^{\infty}$  such that

$$\sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} 2^j}{2^i} = \frac{\bar{a}^2}{\bar{a} - 1} \quad (68)$$

holds and can be expressed based on the results of equations (64)–(66) as

$$\sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} 2^j}{2^i} = \psi(\bar{a}). \quad (69)$$

This means that equation (69) satisfies the condition given by (in)equality (63). Based on the fact that sequence  $\{B_j\}_{j=-\infty}^{\infty}$  only contains values  $B_j \in \{0, 1\}$  we know that the span of sequence  $\{B_j\}_{j=-\infty}^{\infty}$  is 1, and thus we conclude via the use of Theorem 4.7 that the sequence  $\bar{X}$  is of the form  $x'_i = \alpha \bar{a}^i = \alpha 2^i$  and is unique up to a positive constant  $\alpha$ .

### 4.1.3 Computing Value given by the Pure Minimax Search Trajectory

In the previous section we have found that Corollary 4.5 and Theorem 4.7 will provide a result for (27) such that we can compute the value of the pure minimax strategy for a game with normalized cost function. We will continue with the computation of value  $\overline{PV}$  given by equation (26) in Section 4.1.1:

$$\overline{PV} = \inf_{s \in \mathcal{PS}} \left( 1 + 2 \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \right).$$

From Section 3.2.4 we know that any search trajectory can be represented by a sequence of turning points  $X$  as defined in Section 4.1.2. By specifying that we take the infimum over positive sequences  $X$  instead over general search trajectories  $s \in \mathcal{PS}$  we get the expression:

$$\overline{PV} = 1 + 2 \left( \inf_X \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \right) = 1 + 2 \left( \inf_X \sup_{-\infty < i < \infty} F(X^{+i}) \right).$$

where  $F(X^{+i})$  is equivalent to the functional given in Corollary 4.5 which depends on infinitely many elements of the sequence  $X$  as follows

$$F(X^{+i}) = \lim_{k \rightarrow \infty} F_k(X) = \frac{\sum_{j=-\infty}^{\infty} B_j x_j}{x_0}$$

where  $\{B_j\}_{j=-\infty}^{\infty}$  where  $B_j = 1$  for  $j \leq 1$  and  $B_j = 0$  for  $j \geq 2$ . By applying Corollary 4.5 it follows that the value  $\overline{PV}$  can be expressed as:

$$\overline{PV} = 1 + 2 \left( \inf_X \sup_{-\infty < i < \infty} \frac{\sum_{j=-\infty}^{i+1} x_j}{x_i} \right) = 1 + 2 \left( \min_{a > 1} \frac{a^2}{a - 1} \right) \quad (70)$$

From equation (64)–(65) we get that the value  $\overline{PV}$  can be computed because the minimum for the right hand side of (70) has value 4. Via substitution we get

$$\overline{PV} = 1 + 2 \left( \min_{a>1} \frac{a^2}{a-1} \right) = 1 + (2 \cdot 4) = 9 \quad (71)$$

Moreover, from the application of Corollary 4.5 and Theorem 4.7 it is shown that the positive sequence  $X$  that corresponds with the sequence of turning points representing the minimax strategy is of the form

$$X = \{x_i\}_{i=-\infty}^{\infty} = \{\alpha 2^i\}_{i=-\infty}^{\infty}$$

and is unique up to a positive constant  $\alpha$ , because minimum  $\bar{a} = 2$  for  $\psi(a) = \frac{a^2}{a-1}$  satisfies  $\psi'(\bar{a}) = 0$  as shown by equations (65)–(66).

We conclude this section with the result that the best of the worst possible payoffs is 9 for a search game with normalized cost function, and because of the scaling property between a search game with normalized cost function and a search game with a restriction with  $\lambda = 1$  found in Section 3.2.1 we see that this also holds for the restricted Linear Search Game with  $\lambda > 0$ . The last step is applying the Theorem 3.19 under the scaling property by Definition 3.18 from chapter 3 such that the pure minimax trajectory  $\{x_i\}_{i=-\infty}^{\infty} = \{\alpha 2^i\}_{i=-\infty}^{\infty}$  for the normalized search game with normalized cost function is also the pure minimax trajectory for a (non-normalized) search game with the capture time as cost function.

## 4.2 Solving the Linear Search Problem: Optimal strategies

For the optimal search strategies we want to minimize the expected cost function for the searcher. This will provide a strategy which **may** result in the lowest cost, but it does have the downside that it may result in a payoff which is higher than the value of the game guaranteed by the minimax strategy. We will be finding a range of payoff which will be all possible outcomes from playing an optimal search strategy. This section is based on section 8.3 of Alpern&Gal [4, p.129-p.132] and follows the same structure. We have added comments to the definitions and proofs of the lemma and theorem to create more intuition behind a certain argument or to be more applicable to our choice of cost function.

For this section we will assume that the searcher wishes to minimize the expected cost. The optimal mixed search strategy that provides this minimum will be notated as  $\bar{S}$ . We shall show that  $\bar{S}$  belongs to the family  $A = \{S_a\}$  of mixed strategies defined as follows.

**Definition 4.8.** *For any  $a > 1$ , the strategy  $S_a$  chooses a sequence  $s = \{x_i\}$  that represents a trajectory via the use of turning points with  $x_i = a^{i+Z}$ , where  $Z$  is a random variable which is uniformly distributed in  $[0, 2)$ . This strategy  $S_a$  is a random choice among the geometric trajectories.*

Note that the choice of the interval  $[0, 2)$  for the uniform distribution is based on the size of the interval  $x_i < h \leq x_{i+2}$  for  $h \in \mathcal{H}$ . A larger interval for the distribution would result in

$h$  having the possibility to lay outside of the interval  $(x_i, x_{i+2}]$  which would mean that the whole sequence can be shifted by two indices to come to an equivalent sequence. The mixed strategies  $S_a$  have the following property.

**Lemma 4.9.** *For any hiding strategy  $h \in \mathcal{H}$ , the expected normalized cost function will be*

$$\hat{u}(S_a, h) = 1 + \frac{a+1}{\ln(a)}$$

*Proof.* Recall that the cost function for the capture time (non-normalized) for any strategies  $s \in \mathcal{S}$  where  $s = \{x_j\}_{j=-\infty}^{\infty}$  and  $h \in \mathcal{H}$  was given by

$$u(s, h) = |h| + 2 \sum_{j=-\infty}^{\infty} B_j x_j$$

with  $\{B_j\}_{j=-\infty}^{\infty}$  where  $B_j = 1$  for  $j \leq 1$  and  $B_j = 0$  for  $j \geq 2$  which is used to compute the expected cost function for the mixed strategy  $S_a$  as follows

$$u(S_a, h) = |h| + 2\mathbb{E}\left(\sum_{j=-\infty}^{\infty} B_j x_j \mid x_i < |h| \leq x_{i+2}\right) = |h| + 2\mathbb{E}\left(\sum_{j=-\infty}^{i+1} x_j \mid x_i < |h| \leq x_{i+2}\right)$$

where  $x_i < h \leq x_{i+2}$  as given in Section 3.2.4. We have defined the mixed strategy  $S_a$  which chooses  $s = \{x_i\}$  in Definition 4.8 with  $x_i = a^{i+Z}$  where  $Z \sim U([0, 2])$ . By substituting  $x_j = a^{j+Z}$  in the expectancy, we will find that

$$\begin{aligned} u(S_a, h) &= |h| + 2\mathbb{E}\left(\sum_{j=-\infty}^{i+1} a^{j+Z} \mid a^{i+Z} < |h| \leq a^{i+2+Z}\right) \\ &= |h| + 2\mathbb{E}\left(a^{i+Z} \sum_{j=-\infty}^1 a^j \mid a^{i+Z} < |h| \leq a^{i+2+Z}\right) \\ &= |h| + 2\mathbb{E}\left(\frac{a^2}{a-1} a^{i+Z} \mid a^{i+Z} < |h| \leq a^{i+2+Z}\right) \\ &= |h| + \frac{2a^2}{a-1} \mathbb{E}\left(a^{i+Z} \mid \frac{|h|}{a^2} \leq a^{i+Z} < |h|\right) \\ &= |h| + \frac{2a^2}{a-1} \int_0^2 \frac{1}{2} |h| a^{z-2} dz = |h| \left(1 + \frac{1}{a-1} \int_0^2 a^z dz\right) \\ &= |h| \left(1 + \frac{a^2 - 1}{(a-1)\ln(a)}\right) = |h| \left(1 + \frac{a+1}{\ln(a)}\right). \end{aligned}$$

From this expected cost function, we can compute the normalized expected cost function very quickly by dividing by  $|h|$  to conclude that

$$\hat{u}(S_a, h) = \frac{u(S_a, h)}{|h|} = 1 + \frac{a+1}{\ln(a)}$$

which is the desired result. □

Lemma 4.9 allows us to compute a value for the minimal cost that the searcher will be spend searching. For this, we will define a value  $q$  as

$$q := \min_{a>1} \left( 1 + \frac{a+1}{\ln(a)} \right) = 1 + \frac{\bar{a}+1}{\ln(\bar{a})} \quad (72)$$

where  $\bar{a}$  is the value for which the above will be minimized. We will use the unimodal property of  $\frac{a+1}{\ln(a)}$  with  $a > 1$  to determine  $\bar{a}$ . This property property is also given in section 4.1.2 as condition (30) for a functional  $F_k$  and is stated for  $\frac{a+1}{\ln(a)}$  as:

$$\text{if } 0 \leq \theta \leq 1 \quad \text{then} \quad \frac{\theta a_1 + (1-\theta)a_2 + 1}{\ln(\theta a_1 + (1-\theta)a_2)} \leq \max \left\{ \frac{a_1+1}{\ln(a_1)}, \frac{a_2+1}{\ln(a_2)} \right\}.$$

We conclude that the derivative at  $\bar{a}$  is zero because unimodality means that the local extreme value is also the global extreme value. In this case, the extreme value is a minimum and thus we conclude that for  $\frac{a+1}{\ln(a)}$  we have the equation

$$\frac{\bar{a}+1}{\ln(\bar{a})} = \bar{a} \quad \text{and} \quad q = \bar{a} + 1. \quad (73)$$

Alpern [4, p.130] gives an answer found by numerical approximation for the equation above: The base value  $\bar{a} \approx 3,6$  will attain the minimum for  $\frac{a+1}{\ln(a)}$  from which follows that value  $q \approx 4,6$ . Now, we still need to proof that  $S_{\bar{a}}$  is indeed a optimal strategy. We will accomplish this by showing that  $q$  is the value of the game. To show this, we will formulate a hiding strategy which will be a  $\varepsilon$ -optimal strategy based on value  $q$ , meaning that  $q$  is the value of the game  $\bar{V}$ . This implies that  $S_{\bar{a}}$  is a strategy which attains the value of the game and thus it is an optimal strategy.

**Theorem 4.10.** *For any  $\varepsilon > 0$ , there exists a mixed hiding strategy  $H_\varepsilon$  such that for all  $s \in \mathcal{S}$  we have*

$$\hat{u}(s, H_\varepsilon) \geq (1 - \varepsilon)q, \quad (74)$$

where  $q$  is given by (72) and  $H_\varepsilon$  will be a  $\varepsilon$ -strategy.

*Proof.* The idea for this proof is that we choose a specific hiding strategy  $H_\varepsilon$  with use of a probability measure. After the definition of this measure for our choice of  $H_\varepsilon$ , we will be computing the normalized expected cost function  $\hat{u}(s, H_\varepsilon)$  with the use of the defined measure. Let's start by introducing a useful definition which makes notation easier:

$$R := \exp \left( \frac{1 - \varepsilon}{\varepsilon} \right)$$

and let mixed strategy  $H_\varepsilon$  have a measure according to the following probability density which will be used for integration:

$$f_{H_\varepsilon}(h) = \begin{cases} \frac{\varepsilon}{2|h|} & \text{for } 1 \leq |h| < R \\ \frac{\varepsilon}{2} & \text{for } |h| = R \\ 0 & \text{elsewhere.} \end{cases} \quad (75)$$

Against the mixed hiding strategy  $H_\varepsilon$  we will only need to only consider search trajectories  $s = \{x_i\}_{i=0}^n$  where

$$s \in \{s \in \mathcal{S} \mid x_0 \geq 1 \text{ and } x_n = R\} \subset \mathcal{S}$$

because the probability that the hider will be located at location  $|h| < 1$  has probability 0. This means that the other search trajectories will not lead to a lower value of the cost function. In addition, we will be defining  $x_{-1} := 1$  and  $x_{-2} := 1$  for convenience. The trajectory  $s$  and hiding point  $H_\varepsilon$  result in the following normalized expected cost function:

$$\begin{aligned} \hat{u}(s, H_\varepsilon) &= \sum_{i=0}^n \int_{x_{i-2}}^{x_i} \left( 2 \sum_{j=0}^{i-1} x_j + h \right) \frac{1}{h} dH_\varepsilon = 1 + 2 \sum_{i=0}^n \sum_{j=0}^{i-1} x_j \int_{x_{i-2}}^{x_i} \frac{1}{h} dH_\varepsilon \\ &= 1 + 2 \left( x_0 \int_{x_{-1}}^{x_1} \frac{1}{h} dH_\varepsilon + \right. \\ &\quad \left. + x_0 \int_{x_0}^{x_2} \frac{1}{h} dH_\varepsilon + x_1 \int_{x_0}^{x_2} \frac{1}{h} dH_\varepsilon + \right. \\ &\quad \left. + x_0 \int_{x_1}^{x_3} \frac{1}{h} dH_\varepsilon + x_1 \int_{x_1}^{x_3} \frac{1}{h} dH_\varepsilon + x_2 \int_{x_1}^{x_3} \frac{1}{h} dH_\varepsilon + \dots \right) \end{aligned}$$

Note that each of the  $x_i$  for  $0 \leq i \leq n-1$  will have two integrals in the following way

$$\begin{aligned} \hat{u}(s, h_\varepsilon) &= 1 + 2 \sum_{i=0}^{n-1} x_i \left( \int_{x_{i-1}}^R \frac{1}{h} dH_\varepsilon + \int_{x_i}^R \frac{1}{h} dH_\varepsilon \right) = 1 + \varepsilon \sum_{i=0}^{n-1} x_i \left( \frac{1}{x_i} + \frac{1}{x_{i-1}} \right) \\ &= 1 + \varepsilon \sum_{i=0}^{n-1} \left( 1 + \frac{x_i}{x_{i-1}} \right) \geq 1 + \varepsilon n \left( 1 + \left( \prod_{i=0}^{n-1} \frac{x_i}{x_{i-1}} \right)^{1/n} \right) = 1 + \varepsilon n \left( 1 + R^{1/n} \right) \\ &= 1 + (1 - \varepsilon) \frac{1 + R^{1/n}}{\ln(R^{1/n})} > (1 - \varepsilon) \left( 1 + \frac{1 + R^{1/n}}{\ln(R^{1/n})} \right) \geq (1 - \varepsilon)q. \end{aligned}$$

where this last inequality results in a  $\varepsilon$ -optimal strategy for the hider and the colour above are used as an indication for the components which are added up.  $\square$

We have proven that the optimal mixed search strategy  $\bar{S}$  that provides a value of the game  $q$  where  $\bar{S}$  belongs to the family  $A = \{S_a\}$  of mixed strategies. In addition to this result, we introduce the following theorem (without proof) which states that there exist a strategy  $s$  which will result in a lower payoff than the expected payoff of the optimal strategy. Intuitively this makes sense, because  $q$  is based on a expectancy, so there exists a strategy such that this payoff will be lower than  $q$ . We will not provide a proof for this theorem for the same reason that we did not provide a proof for Theorem 4.7. This proof depends on the continuous case of Theorem 4.7 which we have not specified in this thesis. This is the reason that the proof is excluded. For a proof we recommend the proof below Theorem 2 in Gal [11, p.147-p.150]. The theorem is stated as

**Theorem 4.11.** *For any (pure or mixed) hiding strategy  $h$ , there exist a (pure or mixed)  $s' \in \mathcal{S}$  satisfying*

$$\hat{u}(s', h) = \inf_s \hat{u}(s, h) < q$$

where value  $q$  is given by (72).

### 4.3 Conclusion of Optimal and Minimax Strategies

To conclude our findings given by this chapter, we will provide a overview of the optimal and pure minimax strategies based on the overview given at the end of Section 8.3 in Alpern&Gal [4]. We start with the pure minimax strategy which has a maximal payoff of 9 found in the conclusion of Section 4.1.2. The expected cost of this strategy can be computed via Lemma 4.9 with  $a = 2$  in the following way

$$\hat{u}(s_2, h) = 1 + \frac{3}{\ln(2)} \approx 5,3.$$

The pure minimax strategy is described as  $s_2 = \{x_j\}_{j=-\infty}^{\infty} = \{\alpha 2^j\}_{j=-\infty}^{\infty}$  where  $\alpha > 0$ . This concludes that the pure minimax strategy  $s_2$  has an expected cost of 5,3 and a maximal cost of 9.

For the optimal strategy  $S_{\bar{a}}$  we have found via equation (73) that the expected cost is given by 4,6 with an  $\bar{a} \approx 3,6$ . The maximal cost can be computed via

$$1 + 2 \sum_{j=-\infty}^1 \bar{a}^j = 1 + 2 \sum_{j=-\infty}^1 3,6^j \approx 10,9 \quad (76)$$

because  $S_{\bar{a}}$  is given by a geometric sequence. This concludes that the optimal strategy  $S_{\bar{a}}$  has an expected cost of 4,6 and a maximal cost of 10,9.

The expected cost of any search strategy  $S_a$  where  $2 < a < \bar{a}$  with  $\bar{a} \approx 3,6$  lies between 4,6 and 5,3. The same search strategy can result in the maximum cost which lies between 9 and 10,9. Every search strategy  $S_a$  where  $a < 2$  or  $a > \bar{a}$  is dominated by the family of search strategies  $\{S_a \mid 2 \leq a \leq \bar{a}\}$  with respect to the expected and maximal cost, because the optimal and pure minimax strategies provide the best values for these cases.

Depending on the actual context to which the strategy  $S_a$  is applied, one can make the argument is that range between expected and maximal cost of the optimal strategy is desirable, or that the slightly higher expected value and lower maximal value of the pure minimax strategy is a better strategy for the context.

## 5 Expansion and Applications of the Linear Search Game

This Chapter will give a surface level introduction into several expansions of the Linear Search Game based on the other sections in Alpern&Gal [4] which we have not discussed. These ideas could be useful for a reader who would like to know where the results of the Linear Search Game are used in the field of Search Theory.

### 5.1 Expansion of the Linear Search Problem

This section will show the usage of the found results in Chapter 3 and 4. We will structure this section based on the amount of changes that the expansion requires. So the first expansion would be the easiest to do, and becomes increasingly more difficult to apply. The expansions are based on Sections 8.4, 8.5, 8.6, 9.2 and 9.3 in Alpern&Gal [4] which discuss these expansion of the Linear Search Game in more detail.

#### 5.1.1 Adding a turning cost

The first expansion would be adding a turning cost to the searcher. This would imply that turning might not be as favoured as it is now since each turning point  $y_i$  is now described as  $y_i = x_i + \frac{d}{2}$ . This new sequence of turning points can also be approximated by a geometric sequence via the theory we explored in Chapter 4. Alpern&Gal [4] ends the section about the Linear Search Game with turning cost as a still open problem because it cannot be shown that the value found is the best possible value.

#### 5.1.2 Changing from immobile to a mobile hider

A larger expansion is changing the immobile hider to a mobile hider. This changes the (pure and mixed) hiding strategies into trajectories. This also changes the proof of Lemma 3.17 to show that these trajectories can be scaled. We also need to talk about the velocities of the searcher and hider. The assumption is made that the velocity  $\omega$  of the hider is  $\omega \leq |s'(t)|$  such that the hider does not move faster than the searcher. Lastly, a change for the cost function is needed. The normalized cost function  $\bar{u}$  for this mobile game will be defined as

$$\bar{u}(s, h) = \frac{u(s, h)}{|h(0)|}$$

where  $u(s, h)$  with  $s \in \mathcal{S}$  is the same as given in section 3.2.4. This cost function also looks very similar to the normalized function found by Lemma 3.16. Again, one can use theorems stated in Chapter 4 to find the minimax strategy for the searcher.

#### 5.1.3 Search with Uncertain Detection

In this thesis we assumed that detection would occur when the searcher is located within the detection radius. This would happen guaranteed when the searcher and hider are located



at the same location. We could add a probability of detection for each time that the searcher passes the location of the hider. We can find a cost function and value which is similar to the Linear Search Game by using the theorems in Chapter 4.

#### 5.1.4 The Star Game

A generalization of the Linear Search Game is the Star Game. Here we model a "star" which emits  $M$  amount of rays which are number lines given by the interval  $[0, \infty)$ . In the center of this "star" are all rays connected by a common origin. This generalization may first appear to be a much larger problem than the Linear Search Game, but one finds that the value of the game contains the shifted functional

$$F(X^{+i}) = \frac{\sum_{j=-\infty}^{i+M-1} x_j}{x_i} \quad (77)$$

which is not that far removed from the theory of Chapter 4. Although the framework around this problem is indeed more extensive.

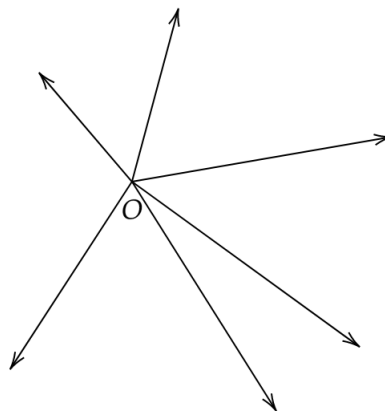


Figure 3: Illustration of the Star Game

#### 5.1.5 Searching for a Point on a Plane

This may be the largest stretch from this current problem, but the intuition behind the problem is very similar to the Linear Search Game. To start with, this game will be described using continuous trajectories and so it does require the continuous formulations of Chapter 4. We have already mentioned that the continuous case is not the focus of this thesis, but the intuition behind the continuous case is similar. We can see in a similar way that the value found by the continuous trajectory given by the pure minimax strategy is equal to the cost function with an exponential function (which is the continuous case of the geometric sequence we used).

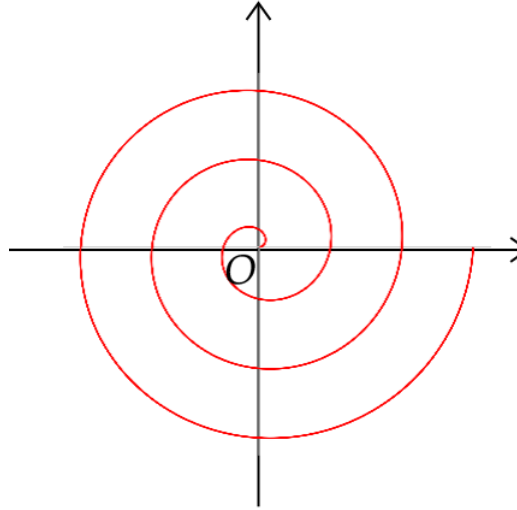


Figure 4: Illustration of trajectory for searching for point in a plane

## 5.2 Rendezvous Game

In the previous section we have provided several expansions for the Linear Search Game. But a more fundamental change can also be made. One can assume that instead of a non-cooperating target, we have a searcher who is looking for a cooperating target, basically a game where two searchers want to find each other. Such a game is called the Rendezvous Game and the searchers are fittingly called: rendezvousers. This does change big parts of the description of the game. For example the minimax and maximin strategies do not make sense for such a cooperative game. Book II in Alpern&Gal [4] provides the general framework for Rendezvous games. Especially section 16.2 provides a useful connection between the Rendezvous Game and the Linear Search Game.

## 5.3 Combining The Linear Search Game and Rendezvous Game

Steve Alpern and Shmuel Gal combined their works in 1988 and published in 2003 the book "The Theory of Search Games and Rendezvous" where these two approaches have a shared framework. A year earlier, they also wrote a paper called "Searching for an Agent Who May OR May Not Want to be Found" [3]. This paper fused the Linear Search Game and the Rendezvous Game into a model where they assume there exists a probability of cooperation  $c$  by the target. This uncertainty in motives of the target is the new playing field for game theorists to explore the vast amount of strategies possible.

## 6 Conclusion

In this thesis we have discussed the general framework of the Search Games with use of game theoretical definitions as described in Chapter 2. We introduced the Linear Search Game in Chapter 3, and discussed a method of scaling the game such that computation of the (pure) minimax and optimal strategies becomes possible.

Our approach is to compute the pure minimax and optimal strategies for the Normalized Linear Search Game and find the strategies for values  $\widehat{PV}$  and  $\widehat{V}$ . By using a scaling property, we can also compute the strategies for values  $\overline{PV}$  and  $\overline{V}$  for the Restricted Linear Search Game. We have also introduced a representation via a sequence of turning points  $\{x_j\}_{j=-\infty}^{\infty}$  which is used to represent pure search trajectories.

In Chapter 4 we have discussed the computation of the pure minimax and optimal strategies by using the concept of "close enough" as given by Theorem 4.1. The main results which provide an answer for the pure minimax strategy is Corollary 4.5 and Theorem 4.7. For the optimal strategy the main results are Lemma 4.9 and Theorem 4.10 which prove that the strategy suggested is indeed optimal. We end the chapter with a overview of the two types of strategies in Section 4.3 where we provide a conclusion to the values given by the strategies where strategies given by turning point sequences from set  $\{S_a \mid 2 \leq a \leq 3,6\}$  dominate strategies where turning point sequences is from set  $\{S_a \mid a < 2 \text{ and } a > 3,6\}$  with respect to the expected and maximal cost.

At last, we provide an introduction into several expansions of the theory which we have discussed in Chapter 5, and gave a few examples to give a general idea behind the expansion.

With this thesis we hope that the results found in Chapter 3 and 4 and the expansion discussed in Chapter 5 do interest a Mathematics student to further explore the subjects in search theory for their own thesis. There is after all a lot of interesting theory left to explore!

## 7 Appendix: Frequently Used Notation

### Game Theory

$n$	Amount of players
$N$	Set of all players
$s_i$	Pure strategy for player $i$
$S_i$	Mixed strategy for player $i$
$\mathcal{PS}_i$	Set of all pure strategies for player $i$
$\mathcal{MS}_i$	Set of all mixed strategies for player $i$
$\mathcal{S}_i$	Set of all strategies for player $i$
$\mathbf{u}(s_1, \dots, s_n)$	Pure payoff function
$U(S_1, \dots, S_n)$	Expected payoff function
$\nu(s_i)$	Value of the player $i$ 's strategy
$\bar{V}$	Value of the game

### Search Games

$Q$	Search Space
$h$	Pure hiding strategy
$H$	Mixed hiding strategy
$\mathcal{PH}$	Set of all pure hiding strategies
$\mathcal{MH}$	Set of all mixed hiding strategies
$\mathcal{H}$	Set of all hiding strategies
$s$	Pure search strategy
$S$	Mixed search strategy
$\mathcal{PS}$	Set of all pure search strategies
$\mathcal{MS}$	Set of all mixed search strategies
$\mathcal{S}$	Set of all search strategies
$u(s, h)$	Cost function
$U(S, H)$	Expected cost function
$\hat{u}(s, h)$	Normalized cost function
$\hat{U}(S, H)$	Normalized expected cost function
$\nu(h), \nu(H)$	Value of the hiding strategy
$\nu(s), \nu(S)$	Value of the search strategy
$\bar{V}$	Value of the game
$T$	Capture time
$t$	time variable
$\omega$	Maximal velocity of the hider
$X$	Doubly infinite sequence of turning points
$X^{+i}$	Sequence of turning points shifted by $i$ steps
$A_a$	Geometric sequence with base $a$
$F_k(X)$	Functional of turning points which depends on $2k + 1$ elements of $X$
$F(X)$	Functional that depends on all turning points in sequence $X$

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