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Bachelor Thesis

A dynamical systems formulation of unorganized residential segregation

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Contents

1 Introduction

Segregation is the phenomenon in which people are physically separated based on factors such as ethnicity, race, religion. It can manifest in a number of areas of everyday life, for example, in schools and workplaces. Segregation in residential areas is referred to as residential segregation. While residential segregation may be perceived by many as a phenomenon of the past, when legislation like the Jim Crow laws in America and the South African Apartheid existed, residential segregation still exists today in the 21st Century. In some locations, patterns of residential segregation today are similar to the patterns of residential segregation of the past. For instance, the South African city of Chatsworth was created to segregate Indians in particular during the Apartheid (Jones, 2019). However, even now in the 21st century, Indians make up the majority of Chatsworth's population. Meanwhile in some places, patterns of residential segregation have changed over time. In the USA, from 1980 to 2000, White and Black segregation declined while Asian and Pacific Islander segregation increased (Iceland, 2004).

Residential segregation is generally undesirable as it has negative consequences for individuals and groups. There are a number of theories that posit that residential segregation has several adverse effects. According to urban theory, concentrations of large homogeneous population groups lead to fewer integration opportunities for newcomers and weakens conventional cultural values and views (Musterd, 2003).

Multiple studies have provided empirical evidence that racial residential segregation, combined with poverty, lead to a multitude of problems. These problems include the aggravation of differences in individual socioeconomic status, racial differences in educational and employment opportunities, the reduction of contact of individuals with positive role models, increased exposure to violence, and the promotion of a "culture of poverty" (De la Roca et al., 2014; Williams and Collins, 2001; Cutler and Glaeser, 1997; Lewise, 1966). Racial residential segregation has also been found to be a fundamental cause of health disparities among African Americans when compared to their White American counterparts (Williams and Collins, 2001).

Studies also show that residential segregation has negative effects on the social integration, educational attainment and labour participation of minority ethnic groups and immigrant groups (Bolt, 2009; Musterd, 2003). Such empirical evidence highlight the negative impact of residential segregation on specific vulnerable population categories. It draws attention to the need to understand the causal factors of residential segregation in order to tackle the problem.

1.1 A dynamical systems approach to segregation

Underlying residential segregation is a multi-causal structure, where different factors act in conjunction to result in residential segregation. These factors include economic status (differences in income, household wealth), differences among individuals on the information they have concerning the housing market, social preferences (desire to live in stable neighborhoods that maintain good social conduct) and discrimination based on ethnicity and race (Clark, 1986; Dawkins, 2004). This discrimination can manifest itself in a number of ways, ranging from housing market discrimination (being charged higher prices or being denied mortgages because of one's ethnicity or race) to individual discriminatory behavior among residents.

Several models have been developed to examine the impact of racial prejudice and preferences on residential segregation. Some of the earlier models developed in the late 20th Century include Bailey's border model and Yinger's amenity model (Dawkins, 2004, p. 387). A more recent approach to studying the role of discrimination in residential segregation involves the use of mathematical techniques, particularly dynamical systems. This approach to studying residential segregation that is brought about by discriminatory behavior is particularly important as it captures the dynamic nature of unorganized residential segregation. Unorganized residential segregation is driven by individual preferences (Schelling, 1971, p. 143). Unorganized segregation stands in opposition to "organized segregation", which is segregation that occurs as a result of the practices of organizations such as the government. "Organized segregation" includes cases where a State enforces separation of different racial groups, such as in the case of the South African Apartheid.

Thomas Schelling introduced the principles for the first dynamical systems formulation of location based segregation in his 1971 paper "Dynamic models of segregation". His model, called the "Bounded neighborhood model" (BNM), models unorganized segregation in bounded areas which he refers to as "neighborhoods". The model studies how individual discriminatory behavior based on characteristics such as race and ethnicity leads to segregation. The BNM looks at the case of a single neighborhood with two populations: a Black population, and a White population. According to Schelling, the term "neighborhood" is not only limited to residential areas but it can also be extended to include locations such as churches, universities or offices. The BNM is accordingly not limited in its application to residential areas. However, the model's focus on segregation brought about by individual discriminatory behavior makes it particularly useful for studying unorganized residential segregation for which individual discriminatory behaviour is a key factor.

In 2018, David Haw and John Hogan, basing their work on Schelling's BNM principles, created the first mathematical dynamical systems formulation of the BNM which they termed the "Schelling dynamical system" (p. 116). This now mathematically formulated model was presented in their 2018 paper "A dynamical systems model of unorganized segregation". In their 2020 paper "A dynamical systems model of unorganized segregation in two neighborhoods", Haw and Hogan extended their 2018 model to create a dynamical model of unorganized residential segregation for two neighborhoods called the two neighborhood model. This thesis examines both their 2018 and 2020 papers and aims to:

1. Correct calculation errors made in the 2018 single neighborhood model. I run the model utilizing the corrected calculations for what Schelling refers to as "unlimited numbers" case with linear tolerance schedules. In this case, the neighborhood has upper limits on the number of inhabitants of the two population types (Haw and Hogan, 2018, p. 117). For different parameter values, I perform an in depth analysis of the obtained results by
   - Assessing the stability of the equilibria,
   - Assessing the structural stability of the system, and
   - Producing phase portraits for the system.
2. Demonstrate that the two neighborhood model with a reservoir population, as briefly presented in Haw and Hogan's 2020 paper (p. 245), is equivalent to two independent cases of the single neighborhood model.
3. Demonstrate that Haw and Hogan's two neighborhood model without a reservoir is not a good model as it describes a system that is not structurally stable. I introduce new differential equations for the two neighborhood model, as Haw and Hogan, in their 2020 paper, introduce differential equations for the two neighborhood model which violate the assumptions they put forth for their model. The newly adapted differential equations for the two neighborhood model are used to conduct a qualitative analysis of the system for the unlimited numbers case with both neighborhoods having identical linear tolerance schedules. My analysis demonstrates that Haw and Hogan's two neighborhood model describes a system that is not structurally stable.
4. Examine the social meaning and physical implications of the results obtained from the two models.

1.2 Thesis outline

Section 2 of this thesis introduces the key terms, basic concepts and theory behind the BNM. Section 2 also presents a brief overview of Haw and Hogan's dynamical systems formulation of Schelling's BNM for one neighborhood. I correct calculation errors made in Haw and Hogan's 2018 paper and provide an in depth qualitative analysis of the obtained results. Section 3 contains my evaluation of Haw and Hogan's two neighborhood model. In this section, I introduce the model as formulated by Haw and Hogan (2020) and subsequently demonstrate how Haw and Hogan's mathematical formulation of the model inaccurately depicts their described model. Section 3 also provides an overview of the two neighborhood model with a reservoir population. I show that the two neighborhood model with a reservoir population is equivalent to two single neighborhood models. The succeeding section, Section 4, introduces my new dynamical systems formulation of the two neighborhood model. This new formulation takes into account all of the assumptions made by Haw and Hogan for their two neighborhood model. I then perform a qualitative analysis of the two neighborhoods' dynamics using the newly adapted differential equations. I determine that Haw and Hogan's two neighborhood model describes systems that are not structurally stable, which demonstrates that their model is not a good mathematical model of a real life system. The thesis concludes in Section 5 with a few remarks on the results I obtained from both my analyses of the two models. In this final section, I compare the results of the single and two neighborhood model. I also suggest possible areas of improvement for both models, and future directions for research.

2 BNM and the single neighborhood model

Schelling's BNM enables one to study how individual incentives and perceptions of differences lead to collective residential segregation (1971, p. 145). The BNM describes a neighborhood in which the population is divided into two types. Individuals of these populations have, as Schelling puts it, "discriminatory individual behavior" (p. 144). Here I refer to Schelling's definition of "discriminatory" as "an awareness, conscious or unconscious, of sex or age or religion or color or whatever the basis of segregation is, an awareness that influences decisions on where to live, whom to sit by, what occupation to join or to avoid, whom to play with or whom to talk to" (1971, p. 144). In this model, the individual discriminatory behavior can be based on factors ranging from religion to the individual's skin color.

Schelling writes in relation to two "colored" populations — a Black population, denoted B, and a White population, denoted W (Schelling, 1971, pp. 148, 167). For this thesis, I consider two population types, denoted X and Y, as presented in Haw and Hogan's 2018 and 2020 papers. Individuals from one population have characteristics or features that are different from those of the individuals of the other type of population. Based on these differences, individuals of each population type show discriminatory individual behavior towards individuals of the other population type. Haw and Hogan make the assumption that the densities of the two population types are continuous and dependent on time. They denote the density of the X and Y population types in the neighborhood as $X(t)$ and $Y(t)$ respectively, where $X(t) \geq 0$ and $Y(t) \geq 0$. While Schelling uses absolute numbers for the populations, this thesis follows Haw and Hogan's method of taking fractions of population density.

In the BNM, there exists two locations where individuals from the two population types can reside in, a neighborhood, and a reservoir. The neighborhood is a bounded area which every individual prefers to its alternative. While the term "neighborhood" is not only limited to residential areas, throughout this thesis, I use the term neighborhood to refer to a residential area where individuals reside in. When describing a neighborhood, I use the term "composition" to refer to the particular combination of the X and Y population densities in the neighborhood. I categorize neighborhoods into two types based on the their compositions. I refer to a neighborhood with only one population type residing in it as a "homogeneous neighborhood". I refer to a neighborhood with both population types as a "heterogeneous neighborhood".

In this thesis, I particularly consider the case that Schelling (1971) refers to as "unlimited numbers". In this case, the neighborhood has upper limits on the number of inhabitants of the two population types. The upper limits for the X and Y population types are as denoted X_{max} and Y_{max} respectively. We have $X_{max} = 1 = kY_{max}$ for some parameter k that is greater than zero. We accordingly have $Y_{max} = \frac{1}{k}$.

Alternatively, individuals can also reside in a location in which, as Schelling presumes, "[the individual's] color predominates or where color does not matter" (1971, p. 167). Throughout this thesis, I refer to this location as "the reservoir". Individuals do not take into consideration the ratio of the two population types in the reservoir and whether their tolerance limits are exceeded in this location. The reservoir can be considered as being the rest of the world. It does not include the neighborhood, and we are not interested in the dynamics of the reservoir. The study only examines the dynamics of the neighborhood. Schelling's presumption regarding the reservoir then translates, in this case, to "a place where characteristics or features of the individuals, based on which individuals discriminate, predominates or does not matter".

2.1 Tolerance

For the purposes of this paper, I follow the BNM's assumption that individuals are only concerned about the ratio of the population types within the neighborhood but not with the physical arrangement of the population types within the neighborhood. I also follow the BNM's assumption that no population members insist on the presence of the other type, i.e. there is no lower limit on tolerance. Whether an individual remains in, leaves or enters the neighborhood is determined by only two factors:

1. The percentage of residents of the other population type in the neighborhood.
2. The individual's "tolerance limit". This is an upper limit assigned to each individual. Tolerance is a "comparative measure" (Schelling, 1971, p. 167) which is specific to the neighborhood. The higher an individual's tolerance, the more likely it is that the individual would be content to live together in an area with individuals from the other population type.

Additionally, I use the BNM's assumption that information is perfect. Each individual of the two populations types is aware of the ratio of the population types in the neighborhood at the moment they make the choice to either remain in, enter or leave the neighborhood. In the BNM, people both leave and return to the neighborhood (Schelling, 1971, p. 167). Given an individual's tolerance and the ratio of the population types in the neighborhood:

- If the percentage of residents of the other population type in the neighborhood exceeds the individual's tolerance limit then
  - the individual residing in the neighborhood leaves the neighborhood, and
  - the individual in the reservoir remains in the reservoir.
- If the percentage of residents of the other population type in the neighborhood does not exceed the individual's tolerance limit then
  - the individual residing in the neighborhood remains in the neighborhood, and
  - the individual in the reservoir leaves the reservoir to move into the neighborhood.

2.2 Tolerance schedules

Schelling (1971, pp. 168, 169) created a frequency distribution of tolerance called the "tolerance schedule", using individuals' tolerances, and examined the corresponding dynamics of the neighborhood. It is assumed that tolerance schedules are neighborhood-specific. The tolerance schedule of a neighborhood is not subject to change, even with the influx of new residents. I also follow Schelling in making the assumption that tolerance schedules are monotone decreasing — the less tolerant population members are the first to leave and last to enter. The more tolerant population members are the first to enter and last to leave (Schelling, 1971, p. 168).

Haw and Hogan (2018) denote the tolerance schedule for the X and Y population types as $R_X(X)$ and $R_Y(Y)$, respectively (p. 114). The tolerance schedule for the X population type, $R_X(X)$, describes the maximum ratio $Y{:}X$ allowed in order for a portion of the X population to remain in that neighborhood. When we have $X = 0.5$, then $R_X(0.5)$ gives the maximum ratio $Y{:}X$ required in order for 50% of the X population to remain in the neighborhood. The Y population type tolerance schedule, $R_Y(Y)$, performs a similar function for the Y population.

Schelling (1971, pp. 168, 169) introduces the simplest form of tolerance schedule, the linear tolerance schedule. The use of a linear tolerance schedule is to an extent supported by empirical evidence of a 1991 study (Clark, 1991). The US based study tested Schelling's linear tolerance schedule by plotting tolerance limit parabolas corresponding to the tolerance schedules that they obtained by using data from several surveys of residential preferences. The results showed that Schelling's linear tolerance schedule was "broadly correct" even though the empirical curves were "less regular" than Schelling's linear tolerance schedule (Clark, 1991, p. 17). These results suggest that nonlinear tolerance schedules may be used as an alternative to linear tolerance schedules. Nonlinear tolerance schedules may possibly be a better approximation of the curves obtained by Clark as nonlinear tolerance schedules are less regular than linear tolerance schedules. While Schelling (1971) and Haw and Hogan (2018; 2020) further explore the use of nonlinear tolerance schedules, this paper focuses on linear tolerance schedules.

Haw and Hogan (2018) formulate

$$R_X(X) = a(1 - X) \tag{1}$$

$$R_Y(Y) = b(1 - kY) \tag{2}$$

for these linear tolerance schedules.

We can plot the functions R_X and R_Y on a graph against the X and Y populations (see Figure 1).

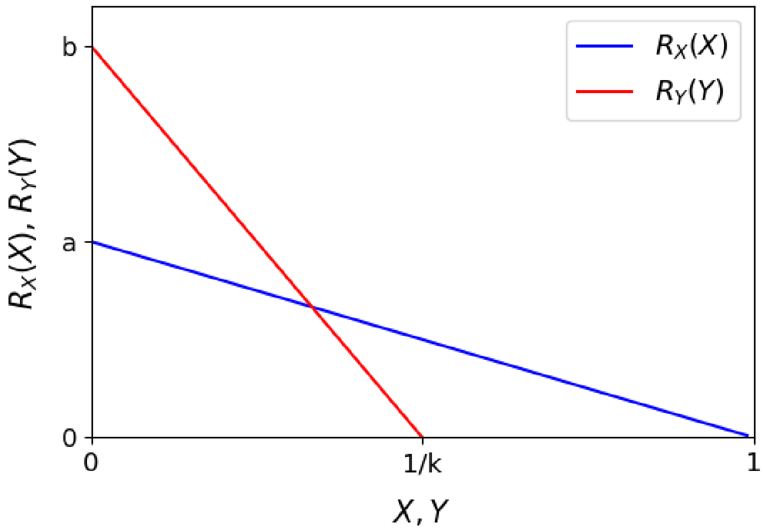


Figure 1: Linear tolerance schedule for the X and Y population types in the single neighborhood model (Haw and Hogan, 2018, p. 115).

According to the linear tolerance schedule as provided in Figure 1, the most tolerant member of the X-population can bear a Y:X ratio equal to a. On the other hand, half of the X population can tolerate a $Y : X$ ratio equal to $\frac{a}{2}$. Similarly, the most tolerant member of the Y population can tolerate an X:Y ratio equal to b.

2.3 Tolerance limits

A neighborhood's tolerance schedule for the X population type allocates tolerance limits to individuals of the X population type residing in the neighborhood. The same applies for a neighborhood's tolerance schedule for the Y population type. As discussed in Section 2.1, the tolerance limits determine whether an individual remains in, leaves, or enters the neighborhood. For individuals of the X population type, this limit is "the value of Y population above which the X population will leave and below which there will be an X population influx" (Haw and Hogan, 2018, p. 114). The tolerance limits corresponding to the linear tolerance schedules introduced in equations (1) and (2) are $Y/X = R_X(X)$ and $X/Y = R_Y(Y)$. The corresponding tolerance limits can be rearranged to give the following equations of parabolas:

$$Y = XR_X(X) = aX(1 - X) \tag{3}$$

$$X = YR_Y(Y) = bY(1 - kY). \tag{4}$$

Individuals of the X population type have higher tolerances for higher values of a. Similarly, individuals of the Y population type have higher tolerances for higher values of b.

2.4 A dynamical systems formulation of the single neighborhood model

Using the model's assumptions and the linear tolerance schedules from equations (3) and (4), Haw and Hogan (2018) then write down the differential equations

$$\frac{dX}{dt} \equiv \dot{X} = X[aX(1 - X) - Y] \tag{5}$$

$$\frac{dY}{dt} \equiv \dot{Y} = Y[bY(1 - kY) - X] \tag{6}$$

for the two time-dependent population variables of $X(t)$ and $Y(t)$ with parameters a, b and k.

We can notice that in the differential equations (5) and (6) the term $(aX(1 - X) - Y)$ is multiplied by X and the term $(bY(1 - kY) - X)$ is multiplied by Y, respectively. Let us first examine equation (5) to understand the reasoning behind multiplying X with $aX(1 - X) - Y$. When $X = 0$, we have $\frac{dX}{dt} = 0$. When $X(aX(1 - X) - Y) \geq 0$, the Y population in the neighborhood either exceeds or is equal to the X-population's tolerance limit. A differential equation written as $\frac{dX}{dt} = aX(1 - X) - Y$, without the X term being multiplied with $aX(1-X)-Y$, would result in $\frac{dX}{dt} < 0$ when $X = 0$ and $Y \geq 0$. That is to say, that members of the X population leave the neighborhood. However, this is not possible because there are no members of the X population type in the neighborhood in this situation to leave the neighborhood. Having $\frac{dX}{dt} = X(aX(1-X)-Y)$ instead of $\frac{dX}{dt} = aX(1-X)-Y$ solves this problem as $\frac{dX}{dt} = 0$ when $X = 0$. The term $bY(1-kY)-X$ in equation (6) is multiplied by Y for similar reasons. Any other form of equation (5) which is similar to $\frac{dX}{dt} = X^k[aX(1-X)-Y]$ also results in $\frac{dX}{dt} = 0$ when $X = 0$. The simplest way to meet this constraint is to multiply the terms on the right hand side of the differential equation by X.

Let us examine whether equations (5) and (6) satisfy the condition that the population densities of the X and Y population types grows (decays) when its respective tolerance schedule exceeds (or is below) Y/X and X/Y respectively. According to equation (5), $\frac{dX}{dt} > 0$ when $R_X(X) - Y > 0$, in which case the vector field points along the positive x direction. We also have $\frac{dX}{dt} < 0$ when $R_X(X) - Y < 0$, in which case the vector field points along the negative x direction. That is to say, the X population grows when the tolerance schedule $R_X(X)$ exceeds Y/X and the X population decays when the tolerance schedule $R_X(X)$ is below Y/X. Equation (6) satisfies similar conditions for the movement of the Y population type.

Let us now examine whether the X population does indeed decay when $X = 1$. I also examine whether the Y population decays when $Y = 1$. When $X = 1$, we have $\frac{dX}{dt} = -Y$. Given that $Y > 0$, members of the X population type start leaving the neighborhood. As the linear tolerance schedule is monotone decreasing, we have X population members in the neighborhood who are more intolerant than others. In the case where $X = 1$, the neighborhood also consists of the most intolerant X population members who cannot tolerate residing in a neighborhood with members of the other population type. As a result, when $X = 1$ and $Y > 0$, some of the more intolerant members leave the neighborhood as their tolerance limit is exceeded. As expected, we obtain $\frac{dX}{dt} < 0$ from the differential equation (5). Similarly, the same condition is met by differential equation (6) when $Y = 1$ and $X > 0$.

I have demonstrated that this model satisfies all the conditions of movement for both the X and Y population types into and out of the neighborhood. I now proceed to find the equilibria of the system. The equilibria $(X, Y) = (X_e, Y_e)$ correspond to the population densities of the neighborhood such that neither members of the two population types leave or enter the neighborhood. These equilibria can be obtained by using equations (5) and (6). At these equilibrium points,

$$\left.\frac{dX}{dt}\right|_{X=X_e} = 0 \quad \text{and} \quad \left.\frac{dY}{dt}\right|_{Y=Y_e} = 0.$$

I consider the points of intersection of the X and Y nullclines to find these equilibria. The X nullclines are given by the equation $\dot{X} = 0$. That is to say, that there is no change in the X population of the neighborhood over time. On the X nullclines, the vector field is vertical on the (X, Y) plane. The X nullclines are given by the equations

$$Y = aX(1-X) \quad \text{and} \quad X = 0.$$

Similarly, the Y nullclines are given by the equation $\dot{Y} = 0$. That is to say, that the Y population does not change, and so along these Y nullclines the vector field is horizontal on the (X, Y) plane. We have the equations

$$X = bY(1-kY) \quad \text{and} \quad Y = 0$$

for the Y nullclines.

The next subsection examines the dynamics of the neighborhood using the differential equations (5) and (6).

2.5 Qualitative analysis for the single neighborhood model with linear tolerance schedules

In this section, I perform a qualitative analysis of the single neighborhood model. I rescale the system, identify the equilibria and assess their stability, assess the system's structural stability, and examine the phase portraits. Throughout this thesis, when I examine the stability of an equilibrium I refer to dynamical stability. When referring to the stability of the system itself I refer to structural stability. I discuss the topic of structural stability in greater depth in Section 2.5.3. I produce phase portraits for different parameter values a, b, k.

I provide a general overview of the system's dynamics, before proceeding to rescale the system. Let us consider the case where $a = b = k = 2$. This case is chosen in particular because Schelling, in his 1971 paper, produced a plot similar to the one found in Figure 2, which corresponds to these parameter values. The vector field and the X and Y nullclines corresponding to these parameter values are plotted in Figure 2. The X and Y nullclines are plotted in blue and red, respectively.

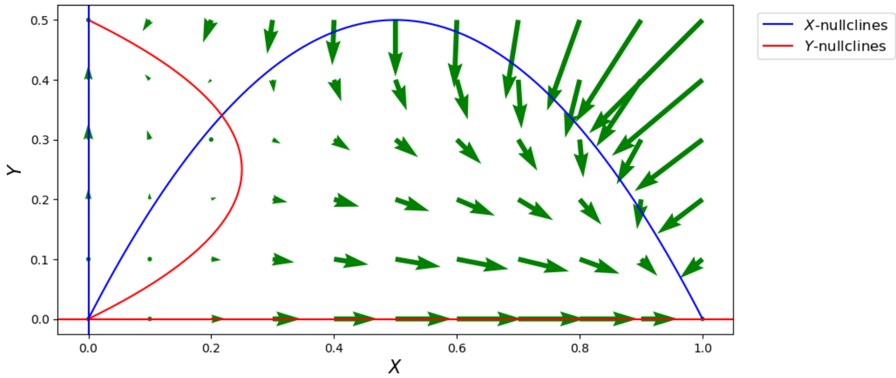


Figure 2: A plot of the system's vector field and nullclines in the (X, Y) phase plane.

As some of the direction arrows in Figure 2 are very small in magnitude, it is difficult to discern their direction. A plot of the vector field with normalized vectors is thus provided in Figure 3.

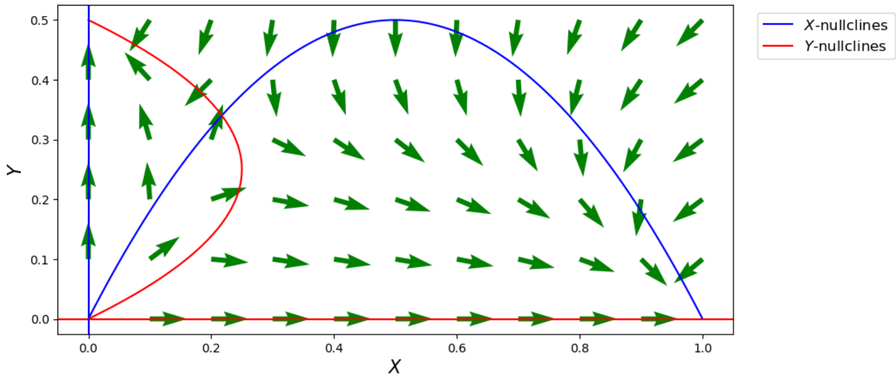


Figure 3: A plot of the system's vector field and nullclines in the (X, Y) phase plane. The vectors are normalized.

I repeat the explanation of the neighborhood's dynamics which can be observed in Figure 2 . In regions where $Y > aX(1 - X)$, the X population's tolerance limit is exceeded by the neighborhood's Y population. Consequently, members of the X population leave the neighborhood. The direction arrows for this region thus point in the negative X direction. On the other hand, when $Y < aX(1 - X)$, the Y population is below the X population's tolerance limit. As a result, there is an influx of members of the X population into the neighborhood, and accordingly the direction arrow points in the positive X direction. The movement of the Y population type can be understood through a similar reasoning.

The phase portrait for the case where $(a, b, k) = (2, 2, 2)$ is shown in Figure 4. A few general remarks can be made about the phase portrait shown in Figure 4. We can observe that the saddle point of the system, shown in green, organizes the dynamics of the system. This is a result of the saddle point's unstable and stable manifolds which determine the directions of the trajectories in the system. We also see that there are four equilibria shown as red, blue and green dots on Figure 4. These equilibria are at the intersections of the X and Y nullclines. The blue dot represents the unstable equilibrium at the origin. The red dots represent the stable nodes and the green dot represents the saddle. I discuss the system's structural stability in Section 2.5.3.

The stability of each of these four equilibria is determined by calculating the eigenvalues of the Jacobian matrix at the specific equilibrium points (X_e, Y_e). Two of the four equilibria for the case portrayed in Figures 2, 3, and 4 are stable. These points are $(1, 0)$ and $(0, \frac{1}{2})$ as shown in Figure 4,

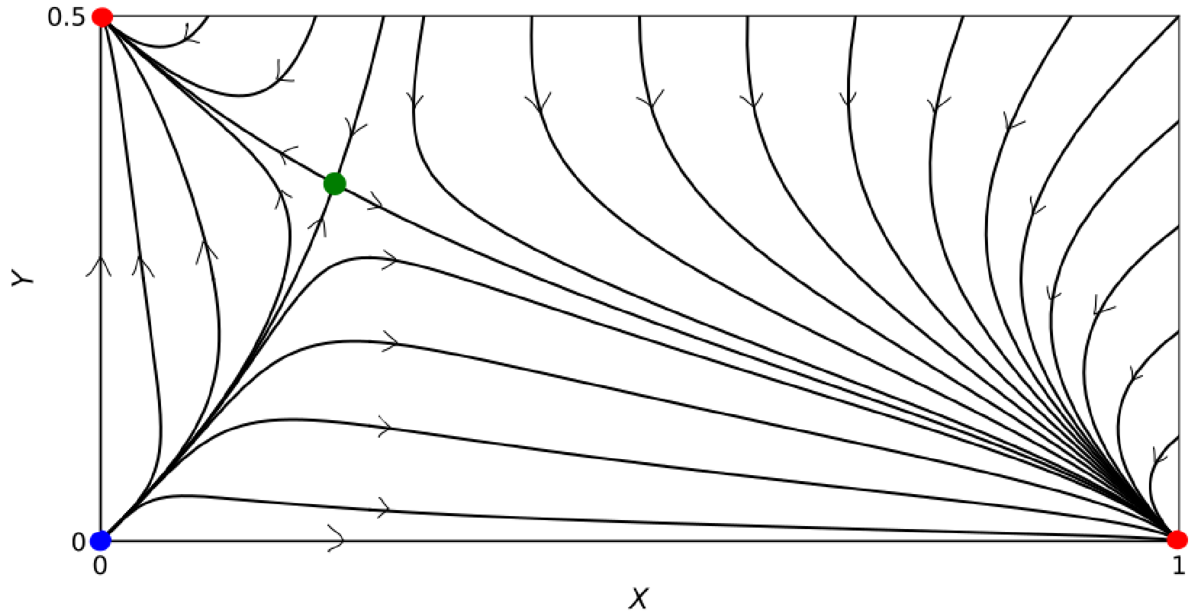


Figure 4: Phase portrait for $(a, b, k) = (2, 2, 2)$

and they correspond to a homogeneous population: the neighborhood contains only X members and no Y members, or only Y members and no X members. The point $(X, Y) = (0, 0)$ represents an empty neighborhood, that is, no individuals reside in the neighborhood. This equilibrium point is not stable. A small perturbation to initial conditions around this equilibrium results in trajectories that lead to the other three equilibria. The fourth equilibrium point, which is also not stable, corresponds to a heterogeneous neighborhood. As mentioned earlier, a heterogeneous neighborhood consists of members of both X and Y population types.

2.5.1 Rescaling the system

The qualitative analysis of this model is simplified by rescaling the system. I thus rescale the system and then using the new differential equations I proceed to examine properties of the system. These properties include the number of equilibria, stability of the equilibria corresponding to different parameter values and the structural stability of the system.

Following Haw and Hogan (2018), I rescale time $\hat{t} = at$, set $Y = aZ$ and introduce the new parameters α and β. Following the scaling, we then derive the differential equations

$$\dot{X} = X[X(1 - X) - Z] \tag{7}$$

$$\dot{Z} = \frac{1}{a} Z[\beta Z(1 - \alpha Z) - X] \tag{8}$$

governing the system, where

$$\alpha \equiv ak > 0 \quad \text{and} \quad \beta \equiv ab > 0.$$

governing the system. My equation for $\dot{Z}$ differs from that obtained by Haw and Hogan (2018) as it has the term $\frac{1}{a}$ being globally multiplied with the equation. Haw and Hogan make the mistake of omitting this term in their equation for $\dot{Z}$. As the term $\frac{1}{a}$ is globally multiplied with the equation, they still obtain the correct Z nullcline with their incorrect $\dot{Z}$ equation. They derive incorrect elements of the system's Jacobian matrix as a result of using an incorrect $\dot{Z}$ equation. I discuss this matter further in Section 2.5.3.

The equations (7) and (8) are derived directly from the previous differential equations (5) and (6). Hence the differential equations (7) and (8) describe the exact same system as that described by the differential equations (5) and (6). The new differential equations describe the system in the X, Z coordinates. Additionally, trajectories on the XZ-plane either move slower ($a > 1$) or faster ($a < 1$) when compared to trajectories in the XY-plane. This is a consequence of the scaling of time.

The scaling of our system has a number of benefits. It primarily simplifies our previous system by reducing the number of parameters appearing in the equations of the nullclines. When compared to equation (5), the new differential equation (7) does not have any parameters in it. As a result the X nullcline derived from (7) also has no parameter terms in it. In comparison, one parameter, namely a, appears in the X nullcline derived from (7). As for the case of the differential equation (8), we have the term $\frac{1}{a}$ being globally multiplied with the equation. The parameter a does not appear in the equation of the Z-nullcline. The Z-nullclines have two parameters in its equation. Similarly, the equation of the Y-nullcline derived from equation (6) also has two parameters. Therefore, the nullclines for the system obtained from equations (7) and (8) are the exact same nullclines as those obtained from equations (5) and (6), but with fewer parameters in total. The new equations of the nullclines can be written in terms of two parameters, α, β, in total instead of the three parameters, a, b, k.

As the differential equations (7) and (8) are derived from equations (5) and (6), we can also easily obtain information on our system by using the new differential equations for analysis. The system's equilibria in the original variables, (X_e, Y_e), can be obtained from the equilibria of the new set of differential equations, (X_e, Z_e), by using the relation substituting $Y_e = aZ_e$. Additionally, each equilibrium, (X_e, Z_e), obtained from the new set of differential equations, has the same stability as its corresponding equilibrium, (X_e, Y_e), in the XY-plane as the scaling preserves the ratio of eigenvalues. Thus, the study of the new set of differential equations still gives us information about the system's equilibria and their classification, all the while simplifying the system.

2.5.2 Equilibria

I now proceed to find all the equilibria, $(X, Z) = (X_e, Z_e)$, of the new system of differential equations (7) and (8). The physical interpretation of these equilibria is provided in Section 2.5.5.

At the system's equilibria,

$$\frac{dX}{dt}\bigg|_{X=X_e} = 0 \quad \text{and} \quad \frac{dZ}{dt}\bigg|_{Z=Z_e} = 0$$

i.e. the equilibria are the intersection points of the X and Z-nullclines. The X and Z-nullclines are given by the equations $\dot{X} = 0$ and $\dot{Z} = 0$ respectively. The equations of the X-nullclines are

$$X = 0 \quad \text{and} \quad Z = X(1 - X)$$

and the equations of the Z nullclines are

$$Z = 0 \quad \text{and} \quad X = \beta Z(1 - \alpha Z).$$

We obtain the equilibrium $(X_e, Z_e) = (0, 0)$ which is the point of intersection of the X and Z axes. Two of the remaining equilibria are obtained by considering the intersection points of the two axes and the X and Z nullcline parabolas. These two equilibria are $(X_e, Z_e) = (1, 0), \left(0, \frac{1}{\alpha}\right)$. The equilibrium at the origin corresponds to an empty neighborhood while the other two equilibria that we just obtained correspond to a homogeneous neighborhood. These three equilibria are always present for the system, for every positive value of α, β.

The remaining equilibria of the system are given by the intersection points of the X-nullcline with equation $Z = X(1 - X)$ and the Z-nullcline with equation $X = \beta Z(1 - \alpha Z)$. Substituting $Z = X(1 - X)$ into the equation $X = \beta Z(1 - \alpha Z)$ gives us

$$X_e^3 + a_2 X_e^2 + a_1 X_e + a_0 = 0 \tag{9}$$

where

$$a_2 \equiv -2, \quad a_1 \equiv \frac{1 + \alpha}{\alpha}, \quad a_0 \equiv \frac{1 - \beta}{\alpha\beta}.$$

The roots of equation (9) give the values of X at the equilibria, i.e. X_e. The roots of the equation, X_e, depend on the parameters α and β. Consequently, unlike the aforementioned three equilibria, the equilibria given by the intersection of the two parabola nullclines depend on the parameters α and β.

As X and Z correspond to the population densities of the two types of populations and $X_{max} = 1$, we are only interested in the real roots of equation (9) that are in the range $X_e \in [0, 1]$. As the coefficients of the cubic equation are real, the equation has one real root, two real roots (one of which has multiplicity two) or three real roots when we do not count with multiplicities. The number of real roots that equation (9) has is determined by its discriminant D.

The discriminant, D, of the cubic equation (9) can be either:

1. Greater than zero, $D > 0$. The cubic equation has three distinct real roots. There are no multiple roots.

2. Less than zero, $D < 0$. The cubic equation only has one real root, with multiplicity one.

3. Equal to zero, $D = 0$. In this case, the polynomial has either two real roots, of which one is a root of multiplicity two, or one root of multiplicity three.

I now proceed to analyse the values of parameters α and β for which either of the three scenarios of D occurs. I do not attempt to find the exact equilibria for varying values of α, β as the phase portraits for the same case of D with different parameter values are qualitatively the same although not quantitatively the same. Instead, I analyse the different values of α, β which correspond to $D > 0, D < 0$ and $D = 0$. To conduct this analysis I proceed to find the expression of the discriminant, D, of equation (9).

The discriminant, D, of cubic equation (9) is given by the expression

$$\begin{aligned} D &= a_2^2 a_1^2 + 18 a_2 a_1 a_0 - 4a_1^3 - 4a_2^3 a_0 - 27a_0^2 \\ &= (-2)^2 \left(\frac{1+\alpha}{\alpha}\right)^2 + 18(-2)\left(\frac{1+\alpha}{\alpha}\right)\left(\frac{1-\beta}{\alpha\beta}\right) - 4\left(\frac{1+\alpha}{\alpha}\right)^3 - 4(-2)^3\left(\frac{1-\beta}{\alpha\beta}\right) - 27\left(\frac{1-\beta}{\alpha\beta}\right)^2 \end{aligned}$$

which after simplifying becomes

$$D = \frac{\alpha\beta^2 - 4\beta^2 - 27\alpha - 4\alpha^2\beta + 18\alpha\beta}{\alpha^3\beta^2}.$$

The amended function D that I obtain is different from that in Haw and Hogan's 2018 paper. Haw and Hogan made an undocumented intermediate calculation error in determining the discriminant of equation (9).

The discriminant D is dependent on the parameters α and β. I accordingly plot a graph of the equation $D = 0$ on the $\alpha\beta$-plane to identify regions of the graph, hence values of α and β, where $D > 0$ and $D < 0$. I can then also identify the values of α and β for which we have a multiple root.

The equation of $D = 0$ can also be rewritten as a function $\beta_\pm(\alpha)$

$$\begin{aligned} & D = 0 \\ \Leftrightarrow\ & \frac{\alpha\beta^2 - 4\beta^2 - 27\alpha - 4\alpha^2\beta + 18\alpha\beta}{\alpha^3\beta^2} = 0 \\ \Leftrightarrow\ & \alpha\beta^2 - 4\beta^2 - 27\alpha - 4\alpha^2\beta + 18\alpha\beta = 0 \\ \Leftrightarrow\ & (\alpha - 4)\beta^2 + (18\alpha - 4\alpha^2)\beta - 27\alpha = 0 \\ \Leftrightarrow\ & \beta_\pm(\alpha) = \frac{-(18\alpha - 4\alpha^2) \pm \sqrt{(18\alpha - 4\alpha^2)^2 - 4(\alpha-4)(-27\alpha)}}{2(\alpha - 4)} \\ \Leftrightarrow\ & \beta_\pm(\alpha) = \frac{9\alpha - 2\alpha^2 \pm 2\sqrt{\alpha^4 - 9\alpha^3 + 27\alpha^2 - 27\alpha}}{4 - \alpha} \\ \Leftrightarrow\ & \beta_\pm(\alpha) = \frac{9\alpha - 2\alpha^2 \pm 2\sqrt{\alpha(\alpha-3)^3}}{4 - \alpha} \end{aligned}$$

where $\alpha \geq 3$. Despite the fact that Haw and Hogan write down an incorrect function D, they obtain the same function $\beta_\pm$ as I obtain. Consequently, their plot of $D = 0$ is identical to my plot of $D = 0$ as shown in Figure 5.

The plot of the equation $D = 0$ or function $\beta_\pm$ in the (α, β) space is shown in Figure 5. (Guckenheimer and Holmes, 2002) refer to this plot as a bifurcation set. The plot shows the region of set of parameter points, (α, β), where $D > 0, D < 0$ and $D = 0$.

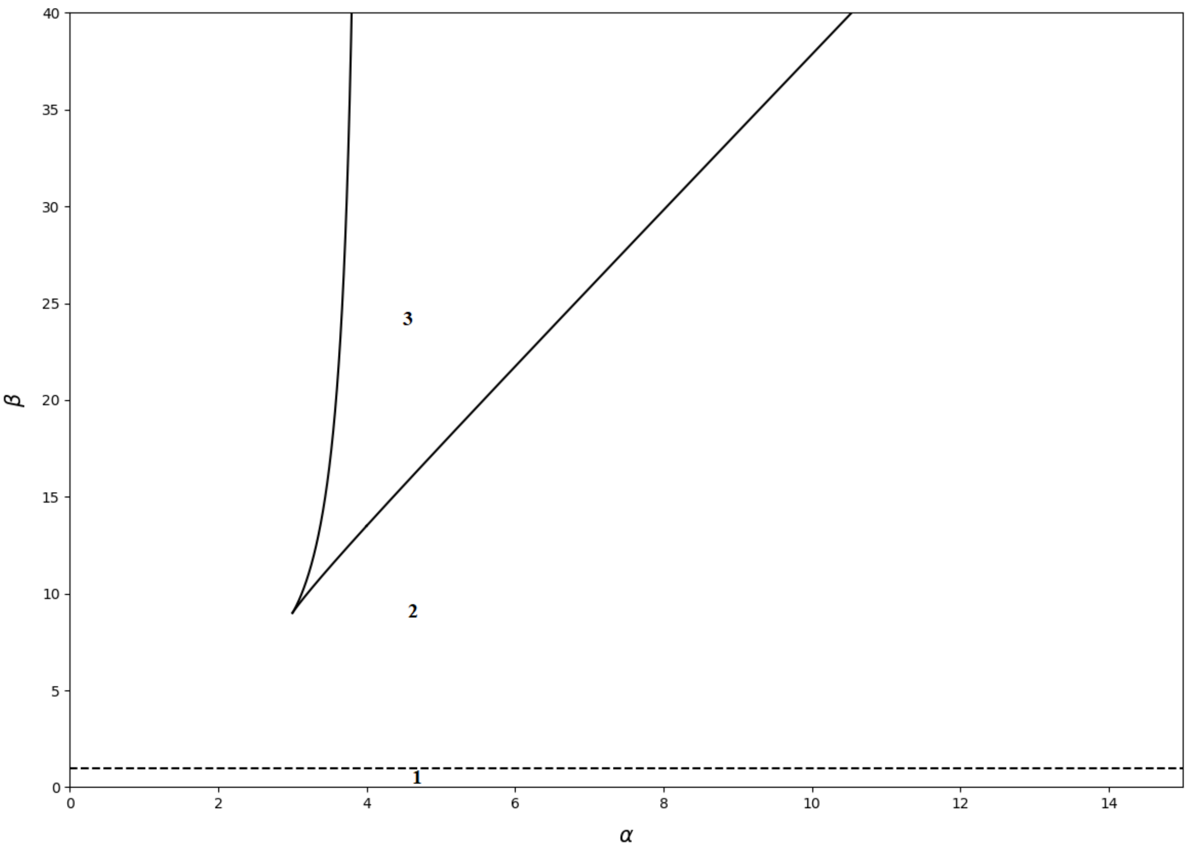


Figure 5: Bifurcation set in (α, β) space. The upper and lower branches of the plot are given by the equations $\beta = \beta_+$ and $\beta = \beta_-$, respectively. Regions of the plot are numbered from 1 to 3 for easy reference below and in Section 2.5.4. Region 3 is defined by the inequality $D > 0$. Region 2 is defined by the inequalities $D < 0$ and $\beta > 1$. Region 1 is defined by the inequality $\beta < 1$.

As I mentioned earlier, there are three equilibria that are always present for all values of α and β. These equilbria are $(X_e, Y_e) = (0,0), (1,0), \left(0, \frac{1}{k}\right)$ which corresponds to $(X_e, Z_e) = (0,0), (1,0), \left(0, \frac{1}{\alpha}\right)$. The remaining equilibria of the system are determined by the parameter values α and β and which region, as shown in Figure 5, they correspond to.

In regions 1 and 2, and for all parameter points (α, β) on $\beta = 1$, equation (9) has one real root. I demonstrate that this root is negative in region 1 by substituting

$$a_2 \equiv -2, \quad a_1 \equiv \frac{1+\alpha}{\alpha}, \quad a_0 \equiv \frac{1-\beta}{\alpha\beta}$$

into equation (9), thus arriving at

$$X_e^3 - 2X_e^2 + \frac{1+\alpha}{\alpha} X_e = \frac{\beta - 1}{\alpha\beta}. \tag{10}$$

One of the methods that can be used to solve equation (10) is plotting the graphs of the function on the left hand side of the equation, i.e. $X_e^3 - 2X_e^2 + \frac{1+\alpha}{\alpha} X_e$, and the constant on the right hand side of the equation, i.e. $\frac{\beta-1}{\alpha\beta}$. The roots of the equation are the values of X_e at the points of intersection of the functions $X_e^3 - 2X_e^2 + \frac{1+\alpha}{\alpha} X_e$ and $\frac{\beta-1}{\alpha\beta}$. If $\beta < 1$, then the right hand side of equation (10), $\frac{\beta-1}{\alpha\beta}$, is always negative. It then follows that when $\beta < 1$, the real X_e values are also negative. However, as X represents the population density of the X population type, we have the constraint $X \geq 0$. The equilibrium point corresponding to the negative root of equation (9), consequently, does not appear in the phase portrait. The system only has a total of three equilibria in region 1 where $\beta < 1$.

From equation (9) we can also deduce that when $\beta = 1$ the real root of equation (9) is $X_e = 0$. In this case the system has a total of three equilibria. We can also observe from equation (10) that when $\beta > 1$ the roots of equation (9) are always positive and $X_e < 1$. It follows that in region 2 equation (9) has one positive root. The corresponding system has a total of four equilibria. In region 3, equation (9) has three positive roots. The system in this case then has a total of six equilibria. For parameter values α and β such that $D = 0$, equation (9) has either one positive root or two positive roots when not counting multiplicities. In this case the system then has either four or five equilibria in total.

In region 3, we have the additional constraint that $\alpha \geq 3$. As $\alpha = ak$, this leads to the constraint $ak \geq 3$. The constraint $ak \geq 3$ can also be found by substituting $\beta = ab$ and $\alpha = ak$ into the functions $\beta_{\pm}(\alpha)$. The functions $\beta_{\pm}(\alpha)$ can then be rewritten as the functions $b_{\pm}$

$$b_{\pm}(a, k) = \frac{9ak - 2(ak)^2 \pm 2\sqrt{ak(ak-3)^3}}{a(4 - ak)}.$$

where $ak \geq 3$. This equation is the same as the one obtained by Haw and Hogan (2018).

Now that I have identified the number of equilibria that we have for different values of α and β, I proceed to analyse the stability of each of these points and the structural stability of the corresponding phase portraits.

2.5.3 Structural stability analysis

It is important to understand the term "structural stability" and why the concept is vital to the application of differential equations to real world systems. Structural stability is a property of a system and accordingly a property of its phase portraits. According to Arnoldi and Haegeman (2016, p. 2), "structural stability relates to the robustness of the qualitative dynamical picture with respect to small changes in the system structure". These "small changes" are made via perturbations to the system itself. These perturbations are additional terms that are added to the system of differential equations. These terms can be linear or non-linear. A structurally stable system has the property of its dynamics being not significantly different from the original dynamics when subjected to small perturbations.

One of the qualities of a structurally stable system is that all of its equilibria are hyperbolic. An equilibrium is said to be hyperbolic if all the eigenvalues of its linearization have a non-zero real part, i.e. $\mathrm{Re}(\lambda_1) \neq 0$. Examples of hyperbolic equilibria include saddles and both stable and unstable nodes. In this section, I attempt to determine whether all the system's equilibria are hyperbolic to determine whether our system is structurally stable. I do so by evaluating the eigenvalues of the Jacobian matrix of each equilibrium.

The concept of structural stability is particularly important in applications of differential equations to real-world systems, such as in this case of modelling unorganized residential segregation. When modelling real-world systems using dynamical systems, we construct simplified differential equations. These differential equations are an approximation of the exact differential equations that describe the real-world systems. In reality, the real-world systems are governed by the model equations plus perturbations. In order to effectively provide information on the real-world system, the model system's dynamics should ideally not become drastically different when subject to relatively small uncertainties and perturbations (Barabás et al, 2014, 2015 as cited by Arnoldi and Haegeman, 2016; Wiggins, 2003; Glendinning, 2012). Accordingly, a simplified model's dynamics should be structurally stable. Even with slightly different differential equations, the dynamics of the model system should be qualitatively the same.

Structural stability should be distinguished from dynamical stability, the latter referring to the dynamics near an equilibrium. Dynamical stability thus refers to changes of initial conditions, not changes of the system itself. An equilibrium is said to be dynamically stable if it is attracting and simultaneously Lyapunov stable. In this thesis I leave out the word "dynamically" and simply speak of a stable equilibrium.

A point is Lyapunov stable if nearby (i.e with different initial conditions) orbits remain in a neighborhood of that orbit (Holmes and Shea-Brown, 2006). If orbits near the equilibrium are repelled then the equilibrium is not Lyapunov stable and is thus called an unstable equilibrium. If an equilibrium is Lyapunov stable and attracting, that is to say, the orbits nearby the orbit move to the equilibrium point, then the equilibrium is (dynamically) stable. As per the definition, centers are considered Lyapunov stable but not (dynamically) stable as they are not attracting. Examples of (dynamically) unstable equilibria include unstable nodes, which are also known as sources, and unstable foci. These two types of equilibria are repelling and accordingly are not Lyapunov stable. A stable node, which is also known as a sink, is the opposite of a source and is stable.

While dynamical stability refers to an equilibrium's stability, structural stability is a property of the system. It "concerns changes in the family of all solutions due to perturbations to the functions defining the dynamical system" (Pugh and Peixoto, 2008). Hyperbolic equilibria are not necessarily stable equilibria. For instance, saddles are not stable equilibria but are hyperbolic. A system may thus feature unstable equilibria but still be structurally stable.

Having provided a brief overview of structural stability, I can now proceed to classify our system's equilibria and analyse our system's structural stability. I have already identified all the equilibria, (X_e, Z_e), of the system in the previous section. In this section, I determine the eigenvalues, λ_1, λ_2, of the corresponding Jacobian matrix, $J(X_e, Z_e)$. Subsequently, I classify each equilibrium and determine whether they are hyperbolic points. I then proceed to discuss the structural stability of the phase portraits.

The Jacobian for the system of differential equations (7) and (8) is given by

$$J(X,Z) = \begin{pmatrix} X(2-3X) - Z & -X \\ -\frac{1}{a}Z & \frac{\beta Z}{a}(2-3\alpha Z) - \frac{1}{a}X \end{pmatrix}. \tag{11}$$

The Jacobian matrix, $J(X, Z)$, obtained by Haw and Hogan (2018) is different from the one above. In particular the elements in the 2nd row of their matrix does not have a factor of $\frac{1}{a}$ as they derive the matrix using the incorrect equation for $\dot{Z}$.

Let us first consider the three equilibria $(X_e, Z_e) = (0,0), (1,0), \left(0, \frac{1}{\alpha}\right)$. For the equilibrium at $(X_e, Z_e) = (0, 0)$, the Jacobian matrix is

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and both eigenvalues are zero. It is a non-hyperbolic equilibrium.

The origin is a non-hyperbolic equilibrium as a result of making the axes invariant. As the X and Y terms are globally multiplied with the remaining terms in differential equations (5) and (6) respectively, all four elements of the Jacobian do not have constant coefficients. Consequently, at $(X, Z) = (0, 0)$ all of the Jacobian's elements are zero which leads to zero eigenvalues. If the X and Y terms were not globally multiplied with the remaining terms in differential equations (5) and (6), none of the elements of the Jacobian at $(0, 0)$ would be zero. The origin would be a hyperbolic equilibrium. However, as explained in Section 2.4, it is necessary that we make the axes invariant in this model.

For the equilibrium at $(X_e, Z_e) = (1, 0)$, the Jacobian matrix is

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{a} \end{pmatrix}$$

and the corresponding eigenvalues are $\lambda_1 = -1, \lambda_2 = -\frac{1}{a}$.

Similarly, for the equilibrium at $(X_e, Z_e) = \left(0, \frac{1}{\alpha}\right)$, the Jacobian matrix is

$$J\left(0, \frac{1}{\alpha}\right) = \begin{pmatrix} -\frac{1}{\alpha} & 0 \\ -\frac{1}{a\alpha} & -\frac{\beta}{a\alpha} \end{pmatrix}$$

and the corresponding eigenvalues are $\lambda_1 = -\frac{1}{\alpha}, \lambda_2 = -\frac{\beta}{a\alpha}$. Since $\alpha, \beta > 0$, both the eignevalues of the Jacobian matrices $J(1, 0)$ and $J(0, \frac{1}{\alpha})$ are negative. Both the equilibria $(X_e, Z_e) = (1, 0), \left(0, \frac{1}{\alpha}\right)$ are accordingly stable nodes. These two equilibria are hyperbolic equilibria.

Next, I examine the stability of the equilibria (X_e, Z_e) that are obtained by solving the cubic equation (9) for X_e values. As we do not have the exact roots of the cubic equation, it is difficult to determine the stability of these equilibria analytically. As a result, I attempt instead to present the results numerically for three different values of the discriminant of the cubic equation, D: $D > 0, D = 0$ and $D < 0$. I coded a program in Python that takes different parameter values corresponding to each of the three different cases for the values of D. For each set of parameter values, the program determines the equilibria (X_e, Z_e) of the system and returns the corresponding Jacobian's eigenvalues, λ_1, λ_2, and the determinant.

$\boldsymbol{D > 0}$

For the entire region $D > 0$, I obtain $\det(J) > 0$ for one of the equilibria obtained from equation (9). As $\det(J) = \lambda_1\lambda_2$, this positive value of $\det(J)$ could mean that both eigenvalues of each equilibrium can be either negative or positive. My program shows that both the eigenvalues are negative. Therefore, this particular equilibrium is a stable node.

The remaining two equilibria have one negative eigenvalue and a positive eigenvalue and correspond to $\det(J) < 0$. Two of the three equilibria for $D > 0$ are thus saddles and hyperbolic.

***D* = 0**

At the cusp point, $(\alpha, \beta) = (3, 9)$, I find that the equilibrium with real coordinates obtained from using equation (9) has a negative eigenvalue. The computer presents the other eigenvalue as being $\approx 2.85 \times 10^{-10}$ due to its limited accuracy. Since 10^{-10} is insignificant, it is still safe to say that this eigenvalue is zero. The determinant of this equilibrium is thus zero and we have a non-hyperbolic saddle.

For the remaining cases where (α, β) lie on $D = 0$, I find that $\det(J) = 0$ for one of the equilibria obtained using equation (9). Due to its limited accuracyy, the computer's calculates $\det(J)$ to be a real number of the order of magnitude m where $m < -14$. As a number of an order of magnitude -14 or less is insignificant, it is still safe to say that $\det(J) = 0$ at this equilibrium. The results show that $\det(J) < 0$ for the other equilibrium. The phase portraits for these cases thus have an unstable non-hyperbolic equilibrium and a saddle.

***D* < 0**

The results show that for the $D < 0$ region, equation (9) has one real root. The corresponding equilibrium has $\det(J) < 0$. The two eigenvalues are of opposite signs and so we have a saddle for these values of α and β.

Structural stability of the phase portraits.

For all parameter values a, b, k the system has an equilibrium at $(X_e, Y_e) = (0, 0)$, which is a non-hyperbolic equilibrium. The system is thus not structurally stable. However, we can make the system structurally stable by removing a small *neighborhood* that includes the point $(X, Y) = (0, 0)$. In this instance I use the word *neighborhood* to refer a set of points, i.e. a topological space and not a residential area. Throughout the thesis the word neighborhood refers to a residential area while *neighborhood* refers to a topological space. Thus, removing a small *neighborhood* of the point $(0, 0)$ remedies the problem of having a system that is not structurally stable. Caution must be taken in determining the size of the *neighborhood* that is to be removed as the larger the size of the *neighborhood* that is removed, the more information on the dynamics of the system is lost. However, as we shall see in Section 2.5.4, removal of a *neighborhood* of the point $(X, Y) = (0, 0)$ can be problematic when $\beta \leq 1$ as there are trajectories that lead right to the origin.

For parameter values that correspond to $D = 0$, the system has other non-hyperbolic equilibria besides the origin. For $(\alpha, \beta) = (3, 9)$ the system has a total of four equilibria, two of which are non-hyperbolic equilibria. For other α, β values that satisfy the equation $D = 0$ the system has a total of five equilibria, three of which are non-hyperbolic. For both cases, the non-hyperbolic equilibrium (or equilibria) besides the one at the origin correspond(s) to a heterogeneous neighborhood, and they organize the dynamics of the system. Consequently, it is not possible to make the system structurally stable simply by removing a small *neighborhood* of the non-hyperbolic equilibira without significantly altering the dynamics of the system. The model is therefore less suited to study the real phenomenon of unorganized residential segregation for parameter values α and β that correspond to $D = 0$.

2.5.4 Phase portraits

In this section, I examine how the dynamics of the single neighborhood model changes as we vary the parameter values a, b, k. I consider a phase portrait in the (X, Y) plane corresponding to a point (α, β) located

1. in region 1 where $D < 0$ and $\beta < 1$.
2. on the line $\beta = 1$.
3. in region 2 where $D < 0$ and $\beta > 1$.
4. on $D = 0$.
5. in region 3 where $D > 0$.

Regions 1 to 3 are labelled in Figure 5. To generate each phase portrait, I consider parameter values (a, b, k) that correspond to values of (α, β) that fall in the respective region or line. I input the parameter values into a python program that I coded to generate the corresponding phase portraits. For each phase portrait the equilibria are represented by colored dots. Saddles and stable nodes are colored in green and red, respectively. The equilibrium at the origin $(X_e, Y_e) = (0, 0)$ is colored in blue. The saddle-node equilibria that appear in the case where $D = 0$ are colored in orange. The non-hyperbolic saddle for

$(\alpha, \beta) = (3, 9)$ is colored in green again. Saddle-node equilibria have one zero eigenvalue and they occur when the system undergoes the saddle-node bifurcation (Izhikevich, 2007).

For all the phase portraits we have two stable nodes at $(X_e, Y_e) = \left(0, \frac{1}{k}\right), (1, 0)$ and a non-hyperbolic equilibrium at $(0, 0)$.

Region 1: $D < 0$ and $\beta < 1$

I take the values $(a, b, k) = (1, 0.5, 2)$ which correspond to $(\alpha, \beta) = (2, 0.5)$. The phase portrait corresponding to these parameter values is shown in Figure 6.

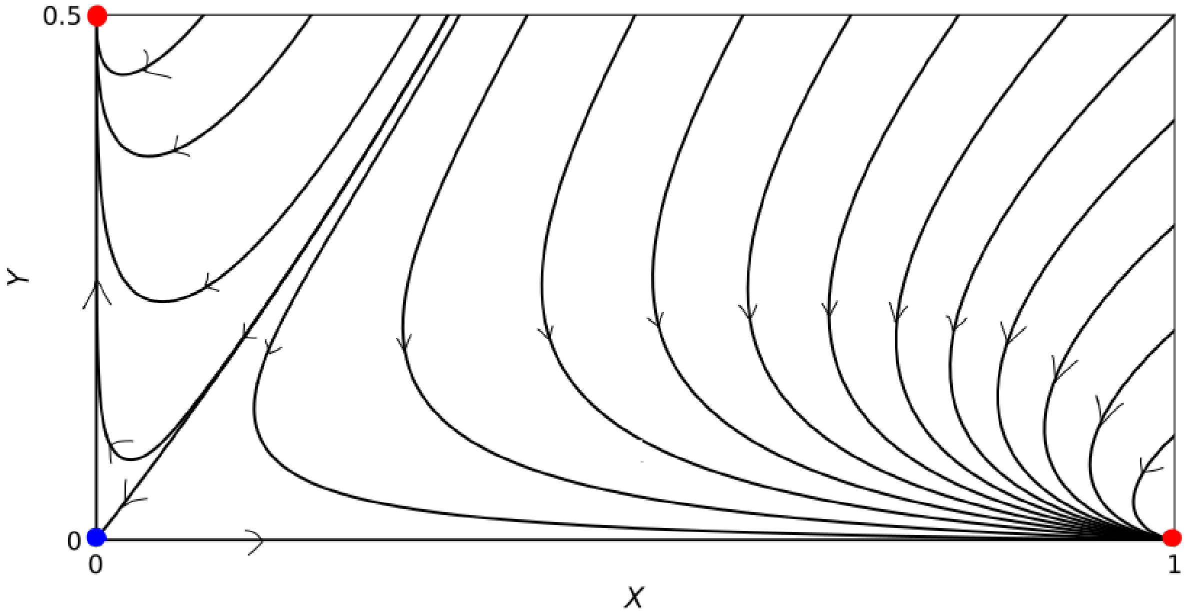


Figure 6: Phase portrait for $(a, b, k) = (1, 0.5, 2)$ and $(\alpha, \beta) = (2, 0.5)$.

As $D < 0$ and $\beta < 1$ in region 1, the cubic equation (9) has one negative real root. The corresponding equilibrium, however, does not appear in the phase portrait as $X, Y \geq 0$. The system has a total of three equilibria in this case.

On the line where $\beta = 1$

I look at a phase portrait corresponding to a point (α, β) that lies on the horizontal line $\beta = 1$. I choose the case $(a, b, k) = (1, 1, 2)$ which corresponds to $(\alpha, \beta) = (2, 1)$. The phase portrait corresponding to these parameter values is shown in Figure 7.

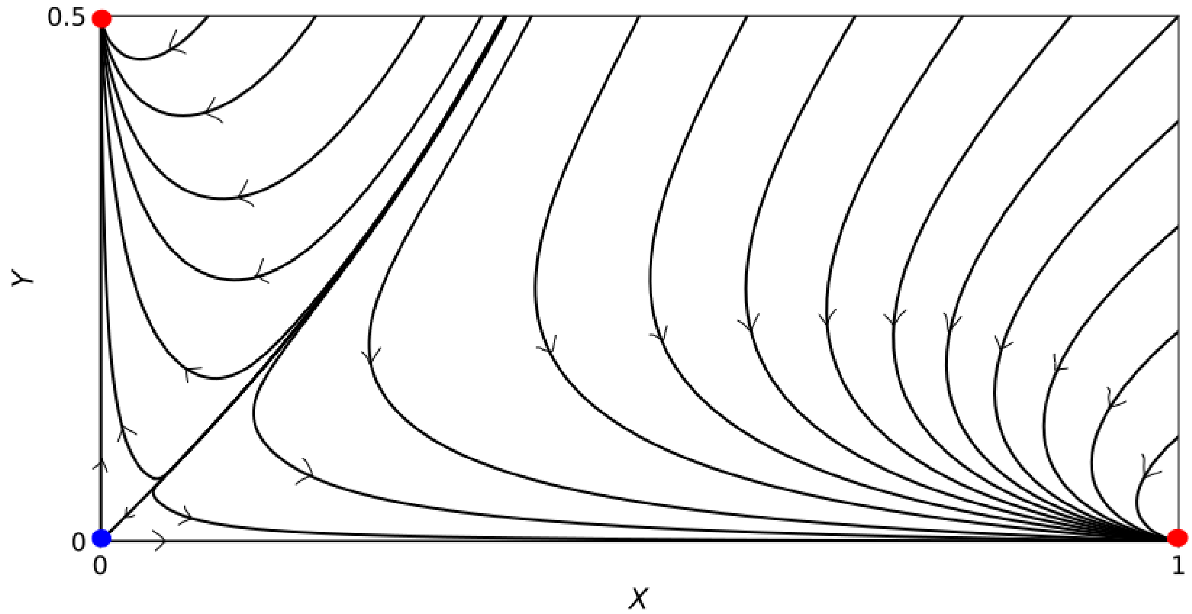


Figure 7: Phase portrait for $(a, b, k) = (1, 1, 2)$ and $(\alpha, \beta) = (2, 1)$

We have only three equilibria, $(X_e, Z_e) = (0, 0), (1, 0), (0, \frac{1}{2})$, of which two are stable nodes that correspond to homogeneous neighborhoods. The phase portraits corresponding to parameter values $\beta < 1$ and $\beta = 1$ are qualitatively similar. When $\beta \leq 1$, there is a trajectory leading right to the origin as shown in Figures 6 and 7. Consequently, an attempt to make the system structurally stable by removing a *neighborhood* of the point $(X, Y) = (0, 0)$ can be problematic in these cases.

Region 2: $D < 0$ and $\beta > 1$

I choose $(a, b, k) = (1, 8, 2)$ which corresponds to $(\alpha, \beta) = (2, 8)$. In this case as $b > a$, the Y-population members are more tolerant than the X- population members. The phase portrait for this case is shown in Figure 8.

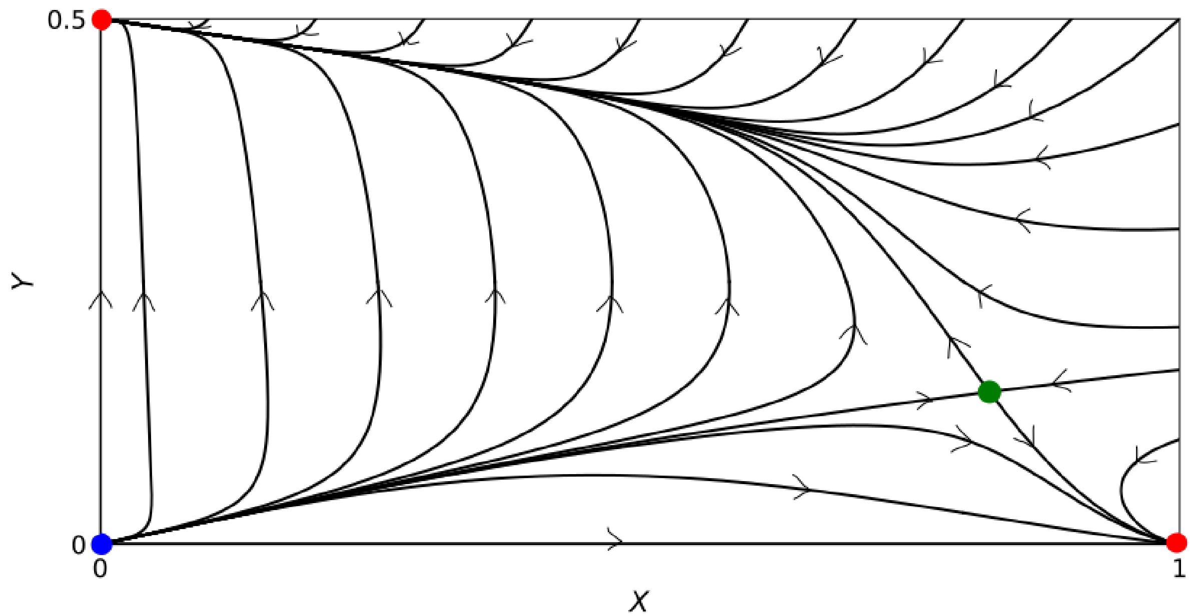


Figure 8: Phase portrait for $(a, b, k) = (1, 8, 2)$ and $(\alpha, \beta) = (2, 8)$

All phase portraits in region 2 look similar to the phase portraits shown in Figures 8 and 4.

Since the chosen parameter values correspond to $D < 0$, we have a total of four equilibria. The fourth equilibrium, which is obtained by solving equation (9), is a saddle. It is the only equilibrium point that corresponds to a heterogeneous neighborhood, which is a neighborhood with members of both population types. In comparison to the saddle point in Figure 4, the saddle point in Figure 8 is located closer to the X-axis as the Y-population members are more tolerant than the X-population members.

$\boldsymbol{D = 0}$

For α, β values corresponding to $D = 0$, the system can have a total of four or five equilibria.

I first examine the case where the system has a total of four equilibria, i.e. when the cubic equation (9) has two roots. I choose $(a, b, k) = (2, 6.75, 2)$ which corresponds to $(\alpha, \beta) = (4, 13.5)$. The phase portrait of the respective system is shown in Figure 9.

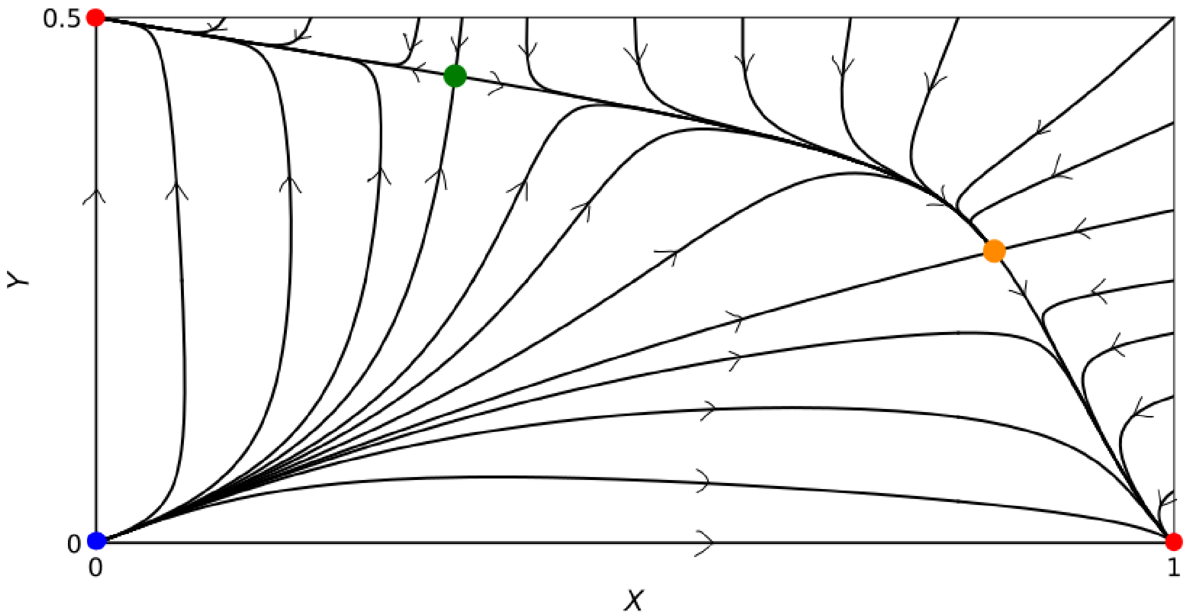


Figure 9: Phase portrait for $(a, b, k) = (2, 6.75, 2)$ and $(\alpha, \beta) = (4, 13.5)$

For these parameter values we obtain two equilibria from equation (9). One of the equilibria is a saddle-node equilibrium while the other equilibrium is a saddle. Both are unstable equilibria and correspond to heterogeneous neighborhoods.

I also consider a case where equation (9) has one root of multiplicity three. I choose $(a, b, k) = (1.5, 6, 2)$ which corresponds to the cusp point seen in Figure 5, $(\alpha, \beta) = (3, 9)$. The respective phase portrait is shown in Figure 10.

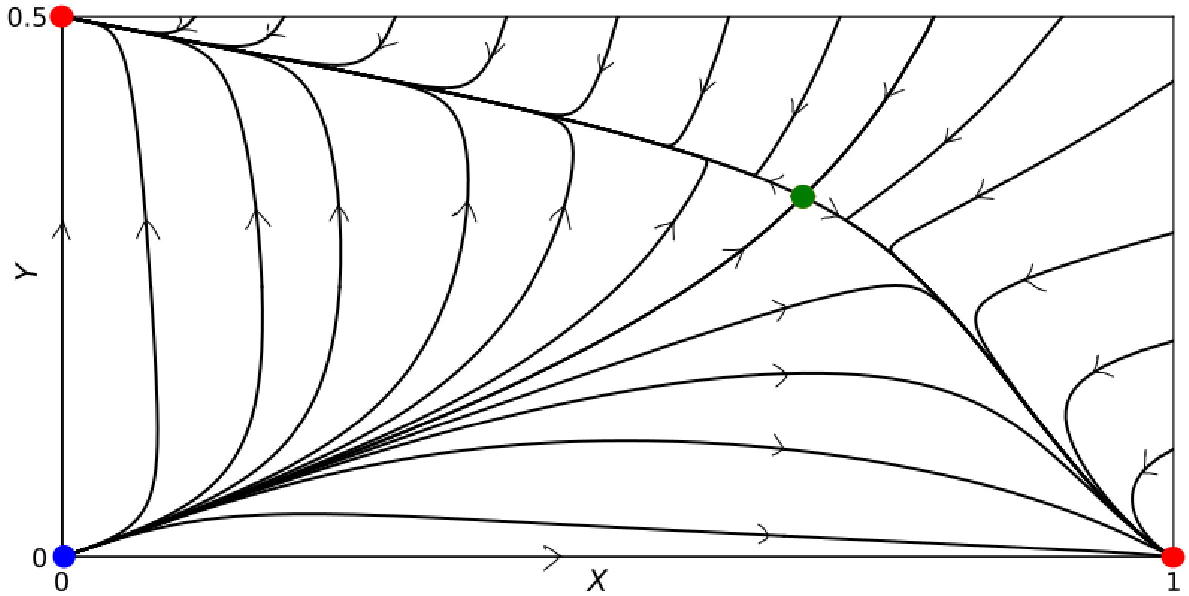


Figure 10: Phase portrait for $(a, b, k) = (1.5, 6, 2)$ and $(\alpha, \beta) = (3, 9)$

We observe that at the cusp point, we obtain a non-hyperbolic saddle which is shown in green.

Region 3: $D > 0$

I choose $(a, b, k) = (2, 8, 2)$ such that $(\alpha, \beta) = (4, 16)$ is in region 3 of Figure 5. The phase portrait is shown in Figure 11.

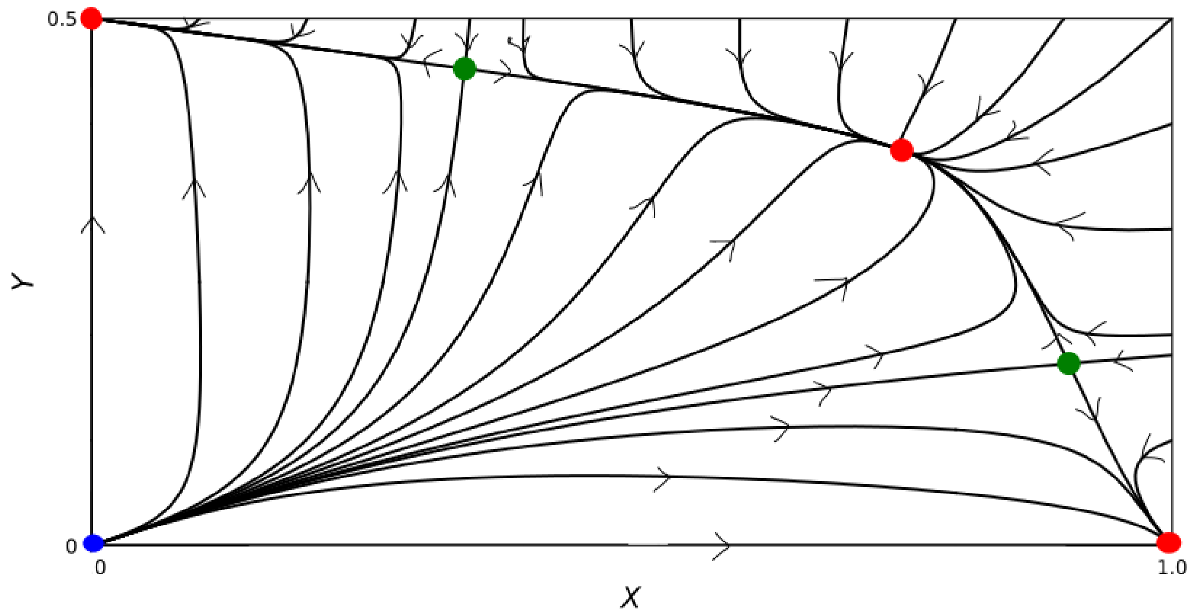


Figure 11: Phase portrait for $(a, b, k) = (2, 8, 2)$ and $(\alpha, \beta) = (4, 16)$

The phase portrait shown in Figure 11 corresponding to $D > 0$ has a number of differences when compared to the phase portraits corresponding to $D < 0$ and $D = 0$. In the phase portrait shown in Figure 11, there are three equilibria corresponding to a heterogeneous neighborhood. These three equilibria are obtained by solving the cubic equation (9). One of these equilbria is a stable node.

We also have two saddles. The saddles seen in the Figures 8 and 4 have now been replaced with two saddles and a stable node. While I do not provide a rigorous proof that a saddle node bifurcation occurs, a comparison of Figures 4, 8, and 11 suggests that a saddle node (fold) bifurcation does indeed occur as we move from region 2, where $D < 0$, to region 3, where $D > 0$. The saddle node bifurcation occurs at

β_- and β_+ in Figure 5. The bifurcation diagrams for varying values of α are shown in Figure 12.

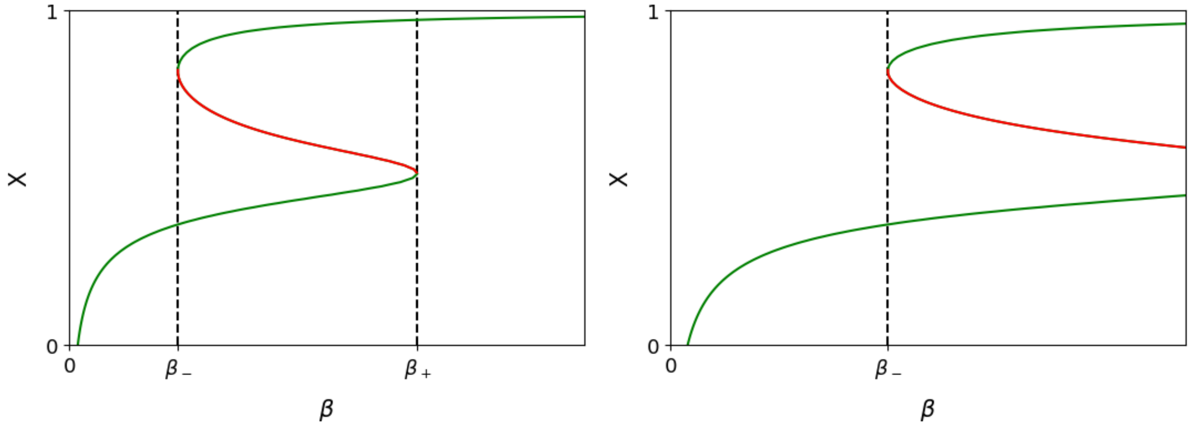


Figure 12: Bifurcation diagrams for $3 < \alpha < 4$ (left) and $\alpha \geq 4$ (right). The saddles and stable nodes are shown in green and red, respectively.

From the bifurcation diagram shown in Figure 12 we can observe that the new stable node (shown in red), which corresponds to a heterogeneous neighborhood, is only present for $\beta_- < \beta < \beta_+$ i.e. it exists only in the $D > 0$ region. In the $\alpha > 3$ range, when $\beta > \beta_+$ or $\beta < \beta_-$ this stable node is no longer present. In those ranges the only equilibrium that corresponds to a heterogeneous neighborhood is a saddle.

2.5.5 Physical interpretation

Some general remarks can be made about phase portraits for all values of the parameters a, b, k. For all values of a, b, k we have equilibria at

1. $(X_e, Y_e) = (0, 0)$
2. $(X_e, Y_e) = \left(0, \frac{1}{k}\right)$ and
3. $(X_e, Y_e) = (1, 0)$.

The equilibrium at the origin corresponds to an empty neighborhood while the other two equilibria correspond to a homogeneous neighborhood.

The equilibrium at the origin is unstable. From the phase portraits we can observe that the trajectories tend to move away from the origin. In reality this repelling nature of the equilibrium has a simple interpretation. The composition of a neighborhood with a low number of residents does not exceed the tolerance limits of most individuals of either population — the exceptions being individuals with very low levels of tolerance. In this model, individuals do not require for there to be a minimum number of people residing in a neighborhood for them to move in. Consequently, individuals of either population start moving into the neighborhood when the neighborhood's initial composition is close to the origin. The composition of the neighborhood then changes over time as residents remain or leave the neighborhood depending on whether their tolerance schedules are exceeded or not. Eventually, the neighborhood reaches a composition that corresponds to stable nodes or, exceptionally, to the saddles in the XY plane. From a mathematical perspective, if the system does not start at an equilibrium, the system approaches the stable equilibrium as time goes to infinity $t \to \infty$. The system may also reach a saddle as $t \to \infty$ given that its initial conditions correspond to a point on its stable manifold. The system never exactly reaches the stable equilibrium. However, in reality our system, which is the neighborhood, does attain a composition that corresponds to a stable equilibrium. This situation is due to the fact that the number of individuals is discrete and therefore the population densities are also discrete. In the model, population densities are considered to be continuous but in reality the population densities can only be continuous if the population size is infinite. Therefore, the real world system can indeed reach an equilbrium at a finite time.

The equilibrium that our physical system obtains in reality should be distinguished from the equilibrium that is determined from our mathematical model. The model uses the assumption that the population densities of the system are continuous and accordingly determines the system's equilibrium. However, the system in the physical world has discrete population densities, its equilibrium is different from that determined using the model. The larger the population size, the more effectively (if never literally) accurate the assumption that the population density is continuous. The model becomes a better approximate of the real world system as the population size increases. The greater is the population size, the closer is the model's approximated equilibrium to the equilibrium in the real world system.

As previously mentioned, the two stable nodes present for all parameter values a, b, k are $\left(0, \frac{1}{k}\right)$ and $(1, 0)$. This shows that regardless of how intolerant or tolerant the two population types are, i.e. how low or high the values of the parameters a, b are, the neighborhood's composition, depending on its initial composition, can become homogeneous in the long run. While I could not produce a plot illustrating the basins of attraction of the system, we can interpret the basins of attraction from the phase portraits that I have shown. When the system has a saddle or saddles, the system's basins of attraction are represented by the regions with the stable manifolds of the saddles as boundaries (see Figure 5a,b,c, of Haw and Hogan's 2018 paper). When we have $\beta \leq 1$, the system has no saddles. In this case, the system's basins of attraction are divided by the solutions leading to the origin (see Figure 5d of Haw and Hogan's 2018 paper). We can determine which equilibrium each basin of attraction corresponds to by following the system's directions field for the separated regions. The plots of the basins of attraction can be found in Figure 5 of Haw and Hogan's 2018 paper.

There is also a possibility that we can have an equilibrium corresponding to a heterogeneous neighborhood. The system has an equilibrium corresponding to such a composition only when $\beta > 1$. For $\beta \leq 1$, there is no equilibrium corresponding to a heterogeneous neighborhood. Regardless of its initial composition, the neighborhood always becomes homogeneous in the long run in this particular case. For $\beta \leq 1$, there is no possibility for the neighborhood to attain and subsequently continue having a heterogeneous composition (the composition of a heterogeneous neighborhood) in the long run.

For all parameter values that correspond to $\beta > 1$ we have an equilibrium or equilibria that correspond to a heterogeneous neighborhood. However, these equilibria are not always stable. Only in the case where $D > 0$ we have a stable equilibrium corresponding to a heterogeneous neighborhood.

For $D \leq 0$, we have either one, two or no unstable equilibria that correspond to a heterogeneous neighborhood. When $\beta \leq 1$, there exists no equilibria corresponding to a heterogeneous neighborhood. When $D < 0$ and $\beta > 1$ there exists one equilibrium corresponding to a heterogeneous neighborhood. This equilibrium is always a saddle. In this case, neighborhoods with initial values of X and Y that are on the stable manifold of the saddle have a trajectory leading to a heterogeneous neighborhood.

The probability of the neighborhood achieving a heterogeneous composition when $D < 0$ is therefore the area of the curved lines representing the stable manifold of the saddle. However, as the area of a line is zero, the probability of the neighborhood achieving a heterogeneous composition is calculated to be zero. As there is a finite number of combinations of initial conditions in reality, a probability of zero does not mean a heterogeneous composition can never be obtained. This probability of the neighborhood achieving a heterogeneous composition in the long run is, nevertheless, significantly smaller when compared to the probability of the neighborhood achieving a homogeneous composition (the composition of a homogeneous neighborhood) in the long run. In most cases, the neighborhood eventually attains a homogeneous composition.

When $D = 0$, the system can have one equilibrium, a saddle, corresponding to a heterogeneous neighborhood as shown in Figure 10. In this case the model predicts that there is a zero probability of the neighborhood achieving a heterogeneous composition.

Alternatively, the system can have two equilibria corresponding to a heterogeneous neighborhood, a non-hyperbolic saddle and a saddle, as shown in Figure 9). Neighborhoods with initial values of X and Y that are either on the stable manifolds of these two equilibria or in the region enclosed by the two equilibria's stable manifolds will have trajectories leading to a heterogeneous neighborhood. Unlike in the other case where $D = 0$, the probability of the neighborhood having a heterogeneous composition in this case is not zero.

Overall, the model predicts that when we have two populations with individuals making decisions on where to live based on individual discriminatory behavior, it is most often the case that neighborhoods ultimately attain homogeneous compositions. In general, the existence of saddles and saddle-node equilibria corresponding to heterogeneous neighborhoods demonstrates that heterogeneous compositions can be unstable. The results predict that while it is not impossible to have a heterogeneous neighborhood, the chances of obtaining a heterogeneous neighborhood are relatively low compared to the probability of obtaining a homogeneous neighborhood. Certain conditions must be met for such a possibility to exist. A crucial condition is that $\beta > 1$. Even when there exist equilibria corresponding to a heterogeneous neighborhood, most trajectories lead to a homogeneous neighborhood. This phenomenon is a result of the system having stable nodes corresponding to homogeneous neighborhoods and saddle(s) corresponding to heterogeneous neighborhoods. Once such a segregation composition is obtained, the neighborhood's composition continues remaining as such. If the neighborhood is already completely homogeneous, it remains as such, since members of the other population type will not be willing to move into the neighborhood.

An interesting prediction made by the model is that a stable node corresponding to a heterogeneous neighborhood only exists when we have parameter values that correspond to $D > 0$. Thinking from a social perspective, I would expect such a stable node to exist even when parameters α, β are increased but correspond $D < 0$. Individuals of both population types would be more tolerant and hence more willing to live with greater numbers of the other population type. I would consequently expect there to be a stable node corresponding to a heterogeneous neighborhood. However, this model demonstrates that even with increasing tolerances, there exists no such stable node in the $D < 0$ region. The model suggests that parameters α, β must be increased in such ways that it corresponds to $D > 0$ for a stable node corresponding to a heterogeneous neighborhood to exist. I further examine this result along with some of the shortcomings of the model in more depth in Section 5.

3 The two neighborhood model

A possible extension of Schelling's BNM is to consider two neighborhoods and study the movement of populations between them. This section introduces the two neighborhood model presented in Haw and Hogan's 2020 paper. Section 3.1 introduces Haw and Hogan's (2020) dynamical systems formulation of the model. The section explains the inaccuracies of their mathematical formulation of the two neighborhood model. A new corrected formulation of the two neighborhood model is presented in Section 4. Section 3.2 introduces the two neighborhood model with a reservoir population. In that section I argue that the model is equivalent to two independent single neighborhood models.

3.1 Haw and Hogan's 2020 dynamical systems formulation of the two neighborhood model

Haw and Hogan (2020) extended Schelling's BNM by considering two neighborhoods, neighborhoods 1 and 2. In their paper, they study the movement of two population types, X and Y, between the two neighborhoods. The X and Y-populations in neighborhood i, where $i = 1, 2$ are denoted by (X_i, Y_i). The parameters a and b of neighborhood i are denoted as a_i and b_i where $i = 1, 2$.

Haw and Hogan (2020) assume that any individual leaving one neighborhood must enter the other neighborhood. The two neighborhoods thus form a closed system and the total X and Y-populations, both of which are constants, are given by $X_{total} = X_1 + X_2$ and $Y_{total} = Y_1 + Y_2$, respectively. As they do in the single neighborhood model, Haw and Hogan (2020) use $X_i(t) \geq 0$ and $Y_i(t) \geq 0$, where $i = 1, 2$, to denote the population densities of the two population. Accordingly, $X_{total} = 1$ and $Y_{total} = \frac{1}{k}$. Hence

$$X_2 = 1 - X_1 \tag{12}$$

$$Y_2 = \frac{1}{k} - Y_1. \tag{13}$$

Haw and Hogan (2020) make similar assumptions for their two neighborhood model as those made by Schelling (1971) for the BNM. A new assumption made in the two neighborhood model is that each individual residing in one of the two neighborhoods is only concerned about the population mixture of the neighborhood that the person resides in (Haw and Hogan, 2020, p. 223). Each individual of the two population types is aware of the ratio of the population types in their current neighborhood at the moment the choice to remain in or leave their own neighborhood is made.

Haw and Hogan (2020) first consider the same linear tolerance schedules which I introduced in Section 2. For neighborhoods i where $i = 1, 2$ the tolerance schedules are

$$R_{X_i}(X_i) = a_i(1 - X_i) \tag{14}$$

$$R_{Y_i}(Y_i) = b_i(1 - kY_i) \tag{15}$$

for the X and Y population types. Haw and Hogan then present the equations

$$\frac{dX_1}{dt} = X_1[X_1 R_{X_1}(X_1) - Y_1] - X_2[X_2 R_{X_2}(X_2) - Y_2] \tag{16}$$

$$\frac{dY_1}{dt} = Y_1[Y_1 R_{Y_1}(Y_1) - X_1] - Y_2[Y_2 R_{Y_2}(Y_2) - X_2] \tag{17}$$

governing the dynamics of neighborhood 1.

Having laid out Haw and Hogan's (2020) formulation of the two neighborhood model, I now proceed to argue that their differential equations are incorrect mathematical formulations of their stated model. For the sake of simplicity and clarity, I replace the terms on the right hand side of equations (16) and (17) with functions $f_i(X_i, Y_i)$ and $g_i(X_i, Y_i)$ before proceeding with my argument. Haw and Hogan (2020) introduce these functions only when discussing the two neighborhood model with a reservoir population (p. 245). These functions are

$$f_i(X_i, Y_i) = X_i(X_i R_{X_i}(X_i) - Y_i) \tag{18}$$

$$g_i(X_i, Y_i) = Y_i(Y_i R_{Y_i}(Y_i) - Y_i) \tag{19}$$

for neighborhood i where $i = 1, 2$.

Let us first focus on $f_i(X_i, Y_i)$. The function f_i corresponds to the movement of the X-population, subject to the presence of the Y-population, in and out of neighborhood i (Haw and Hogan, 2020, p. 224). Additionally, when $X_i = 0$, i.e. the X-population in the neighborhood is zero, the function f_i has a value of zero. That is to say, that there is no change in the X_i population. When

- $f_i(X_i, Y_i) \geq 0$, the Y_i population does not exceed the tolerance limit of X_i. Members of the X population residing in neighborhood i remain in the neighborhood. When $X_i = 0$, no X members move into the neighborhood.
- $f_i(X_i, Y_i) < 0$, the Y_i population exceeds the tolerance limit of X_i. Members of the X population residing in neighborhood i leave the neighborhood.

A similar interpretation applies for the function $g_i(X_i, Y_i)$.

Using equations (18) and (19), Haw and Hogan's (2020) differential equations (16) and (17) can then be rewritten as

$$\frac{dX_1}{dt} = f_1(X_1, Y_1) - f_2(X_2, Y_2) \tag{20}$$

$$\frac{dY_1}{dt} = g_1(X_1, Y_1) - g_2(X_2, Y_2). \tag{21}$$

At first glance, equations (20) and (21) seem to reflect their described model. Upon closer inspection, however, one can observe that these equations are an incorrect mathematical formulation of the stated model.

Let us examine the first term on the right hand side of equation (20), f_1. Given the assumption that individuals only care about the population ratio of the neighborhood that they currently reside in, it follows that the function f_1 reflects the decision of members of the X_1-population to remain in or leave neighborhood 1, according to their tolerance schedules. Let us consider the case where the term f_1 is positive, $f_1 > 0$. As mentioned in previous paragraphs, when $f_1 > 0$, the Y_1 population does not exceed the tolerance limit of X_1. Members of the X population residing in neighborhood 1 will remain in the neighborhood. If we additionally assume that $f_2 = 0$, the differential equation (20) would be positive, i.e. $\frac{dX_1}{dt} > 0$. We see that $f_1 > 0$ results in an increase in X_1. The only reasonable explanation is that the X population type in neighborhood 1, X_1, is increasing as a result of X members moving in from elsewhere besides neighborhoods 1 and 2. This, however, contradicts the assumption that we have a closed system with $X_{total} = X_1 + X_2$. Thus, when $f_1 > 0$, Haw and Hogan's (2020) differential equation for X_1 fails to adequately represent their stated model.

An examination of the second term on the right hand side of equation (20), f_2, also reveals that the equation violates the assumption that individuals in each neighborhood are only concerned about the population mixture of the neighborhood that they are in. The term f_2 represents the movement of the X_2-population, subject to the presence of the Y_2-population, in and out of neighborhood 2. I consider the case where the term f_2 is positive, $f_2 > 0$. When $f_2 > 0$, the Y population type of neighborhood 2, Y_2, does not exceed the tolerance limit of the X population type of neighborhood 2, X_2. When we take $f_1 = 0$, we can observe that $\frac{dX_1}{dt} < 0$. The interpretation for this scenario is that members of the X population in neighborhood 1 are leaving neighborhood 1 as a result of the tolerance limit of the X_2-population not being exceeded. Therefore, Haw and Hogan's dynamical systems formulation of the two neighborhood model once again violates an assumption of their stated model.

According to the same reasoning, it follows Haw and Hogan's differential equation for Y_1 fails to adequately represent their stated model as well.

The formulation of incorrect differential equations of X_1 and Y_1 also leads to consequences for the formulation of neighborhood 2's dynamics. By differentiating equations (12) and (13) we have

$$\frac{dX_2}{dt} = -\frac{dX_1}{dt} \tag{22}$$

$$\frac{dY_2}{dt} = -\frac{dY_1}{dt} \tag{23}$$

for neighborhood 2. From equations (22) and (23), we can see that an incorrect formulation of the differential equations for neighborhood 1 inevitably leads to an incorrect formulation of the differential equations governing neighborhood 2.

Having presented the problems of Haw and Hogan's (2020) dynamical systems formulation of the two neighborhood model, I propose a new formulation of the same model in Section 4. The following section, Section 3.2, introduces the two neighborhood model with a reservoir population.

3.2 The two neighborhood model with a reservoir population

A possible extension of the two neighborhood model is the introduction of two neighborhoods along with the option of an individual to be in neither neighborhood. Haw and Hogan presented this case in their 2020 paper. I refer to this specific case as "the two neighborhood model with a reservoir population" where there exist two neighborhoods and a reservoir. In this section, I demonstrate that the two neighborhood model with a reservoir population is equivalent to two independent single neighborhood models.

Similar assumptions as those made for the single and two neighborhood models are made for the two neighborhood model with a reservoir population. Information is assumed to be perfect, and every individual in either neighborhood is not concerned about the population mixture in the other neighborhood (Haw and Hogan, 2020, p. 223). There is also an additional assumption that there is no direct movement between neighborhoods 1 and 2. To move between these two neighborhoods, population members go through the reservoir.

Haw and Hogan (2020) denote the X, Y populations in neighborhood i as (X_i, Y_i) where $i = 1, 2$. They also denote the X, Y populations in the reservoir as (X_3, Y_3). Haw and Hogan (2020) once again use the same linear tolerance schedules as those introduced in Section 2. To reiterate, for neighborhood i where $i = 1, 2$ we have the linear tolerance schedules

$$R_{X_i}(X_i) = a_i(1 - X_i)$$

$$R_{Y_i}(Y_i) = b_i(1 - kY_i).$$

In their paper they then present the differential equations

$$\frac{dX_1}{dt} \equiv \dot{X}_1 = \begin{cases} f_1(X_1, Y_1) & \text{when} \quad f_1(X_1, Y_1) \leq 0 \quad \text{OR} \quad X_3 > 0 \\ \max(0, -f_2) & \text{when} \quad f_1(X_1, Y_1) > 0 \quad \text{AND} \quad X_3 = 0 \end{cases} \tag{24}$$

$$\frac{dY_1}{dt} \equiv \dot{Y}_1 = \begin{cases} g_1(X_1, Y_1) & \text{when} \quad g_1(X_1, Y_1) \leq 0 \quad \text{OR} \quad Y_3 > 0 \\ \max(0, -g_2) & \text{when} \quad g_1(X_1, Y_1) > 0 \quad \text{AND} \quad Y_3 = 0 \end{cases} \tag{25}$$

as governing the dynamics of neighborhood 1, where for neighborhood i, $i = 1, 2$, we have $f_i(X_i, Y_i)$ and $g_i(X_i, Y_i)$ given by equations (18) and (19).

In Haw and Hogan's (2020) proposed model for the two neighborhoods with a reservoir population, the differential equation of X_1 takes different values based on which of the following two provided conditions apply:

1. $f_1(X_1, Y_1) \leq 0$ or $X_3 > 0$
2. $f_1(X_1, Y_1) > 0$ and $X_3 = 0$.

The same conditions on g_1 apply for the the differential equation of Y_1.

I argue that the two neighborhood model is equivalent to two independent single neighborhood models. In the case of two independent single neighborhood models, we have two neighborhoods that form independent single neighborhood models with one "common" reservoir.

I argue that a reservoir, which I previously described as being "the rest of the world besides the neighborhoods", cannot be empty. That is to say, that the populations in the reservoir, X_3 and Y_3, are always greater than zero: $X_3 > 0$ and $Y_3 > 0$. Haw and Hogan (2020), however, assume that $X_1 + X_2 + X_3 = X_{total}$ and $Y_1 + Y_2 + Y_3 = Y_{total}$ which implies that the reservoir can be empty, i.e. $X_3 = Y_3 = 0$. They accordingly formulate a piecewise function for $\dot{X}_1$ that takes a different value when $f_1 > 0$ and $X_3 = 0$. They also formulate a piecewise function for $\dot{Y}_1$ that takes a different value when $g_1 > 0$ and $Y_3 = 0$. Should the reservoir in this model never be empty, i.e. $X_3 > 0$ and $Y_3 > 0$ at all times, equations (24) and (25) reduce to

$$\frac{dX_1}{dt} \equiv \dot{X}_1 = f_1(X_1, Y_1)$$

$$\frac{dY_1}{dt} \equiv \dot{Y}_1 = g_1(X_1, Y_1).$$

Substituting the values of $f_i(X_i, Y_i)$ and $g_i(X_i, Y_i)$ into these two equations for $\frac{dX_1}{dt}$ and $\frac{dY_1}{dt}$ would then give

$$\frac{dX_1}{dt} \equiv \dot{X}_1 = X_1[a_i X_1(1 - X_1) - Y_1]$$

$$\frac{dY_1}{dt} \equiv \dot{Y}_1 = Y_1[b_i Y_1(1 - kY_1) - X_1]$$

respectively. These two differential equations are exactly the same as the differential equations (5) and (6) of the X and Y populations in the single neighborhood model.

The assumptions made in this model are the same as the assumptions made in the case of multiple independent single neighborhood models. In this model as well as in the case of multiple independent single neighborhood models, we have the assumption that individuals in a neighborhood are only concerned about the population ratio in that neighborhood and whether it exceeds the tolerance limits or not. Individuals' decisions to remain in or move to the reservoir are based on whether the tolerance limits are exceeded or not. Individuals are not concerned about the ratio of the population types in the other neighborhood. As in the case of the two neighborhood model with a reservoir population, in the multiple single neighborhood model, individuals move to the reservoir instead of directly moving to another neighborhood. People in the reservoir are concerned about the composition of the neighborhoods. Upon moving to the reservoir, the individuals can then decide to move to other neighborhoods. As the single neighborhood model only models the movement of population groups of the two types in and out of neighborhoods and not the movement of individuals, the problem of one individual simultaneously moving to multiple neighborhoods that do not exceed their tolerance limits does not arise in the case of multiple single neighborhood models. Accordingly, although Haw and Hogan (2020) consider the two neighborhood model with reservoir population as a unique model, this case is actually equivalent to multiple independent single neighborhood models.

4 A new dynamical systems formulation of the two neighborhood model

In Section 3.1, I introduced Haw and Hogan's (2020) dynamical systems formulation of the two neighborhood model. This section focuses on correcting the previous formulation and analysing the results obtained by using the new formulation. In Section 4.1, I introduce a new dynamical systems formulation of the two neighborhood model. I perform a qualitative analysis of the model in Section 4.2 using the new dynamical systems formulation. I only focus on the neighborhoods' dynamics for linear tolerance schedules which are identical for both neighborhoods.

4.1 A new dynamical systems formulation

A new dynamical systems formulation of the two neighborhood model is introduced in this section. This new formulation takes into consideration all the assumptions made by Haw and Hogan (2020) for the two neighborhood model. The equations for $\frac{dX_2}{dt}$ and $\frac{dY_2}{dt}$ can be obtained directly from the equations for $\frac{dX_1}{dt}$ and $\frac{dY_1}{dt}$. Consequently, I focus on constructing a correct mathematical formulation of differential equations for neighborhood 1. I use the same format for the equations of $\frac{dX_1}{dt}$ and $\frac{dY_1}{dt}$ as shown in equations (20) and (21). To reiterate, the equations of $\frac{dX_1}{dt}$ and $\frac{dY_1}{dt}$ are

$$\frac{dX_1}{dt} = f_1(X_1, Y_1) - f_2(X_2, Y_2)$$

$$\frac{dY_1}{dt} = g_1(X_1, Y_1) - g_2(X_2, Y_2).$$

As such, the functions $f_i(X_i, Y_i)$ and $g_i(X_i, Y_i)$ need to be redefined in order to construct the new dynamical systems formulation of the two neighborhood model. Recall that $f_i(X_i, Y_i)$ and $g_i(X_i, Y_i)$ were defined by equations (18) and (19) to be

$$f_i(X_i, Y_i) = X_i(X_i R_{X_i}(X_i) - Y_i)$$

$$g_i(X_i, Y_i) = Y_i(Y_i R_{Y_i}(Y_i) - X_i).$$

Following my argument in Section 3.1, I deduce that a correct dynamical systems formulation of the two neighborhood model must meet the following conditions

- For the differential equations of X_1 and X_2:

Condition 1 When $X_1(X_1 R_{X_1}(X_1) - Y_1) > 0$, it is necessary that $f_1(X_1, Y_1) = 0$. When $X_1(X_1 R_{X_1}(X_1) - Y_1) > 0$, members of the X population in neighborhood 1 are satisfied with the neighborhood's population mixture. They remain in the neighborhood. As individuals only look at their own neighborhood's composition and we have a closed system of two neighborhoods, when $X_1(X_1 R_{X_1}(X_1) - Y_1) > 0$ no members of the X population outside neighborhood 1 enter neighborhood 1. Hence, the term $f_1(X_1, Y_1)$ must individually make zero contribution to $\frac{dX_1}{dt}$ when $X_1(X_1 R_{X_1}(X_1) - Y_1) > 0$.

Condition 2 Similarly, when $X_2(X_2 R_{X_2}(X_2) - Y_2) > 0$, it is necessary that $f_2(X_2, Y_2) = 0$.

- According to the same reasoning used to reach to conclusions on **conditions 1 and 2**, the differential equations of Y_1 and Y_2 must meet the following conditions:

Condition 3 When $Y_1(Y_1 R_{Y_1}(Y_1) - X_1) > 0$, it is necessary that $g_1(X_1, Y_1) = 0$.

Condition 4 When $Y_2(Y_2 R_{Y_2}(Y_2) - X_2) > 0$, it is necessary that $g_2(X_2, Y_2) = 0$.

Having introduced the four conditions that the new dynamical systems formulation must meet, the new differential equations governing the X and Y populations in neighborhood 1 read

$$\frac{dX_1}{dt} = f_1(X_1, Y_1) - f_2(X_2, Y_2) \tag{26}$$

$$\frac{dY_1}{dt} = g_1(X_1, Y_1) - g_2(X_2, Y_2) \tag{27}$$

where, for neighborhood $i = 1, 2$, the functions $f_i(X_i, Y_i)$ and $g_i(X_i, Y_i)$, are redefined as such

$$f_i(X_i, Y_i) = \begin{cases} X_i(X_i R_{X_i}(X_i) - Y_i) & \text{when} \quad X_i(X_i R_{X_i}(X_i) - Y_i) \le 0 \\ 0 & \text{when} \quad X_i(X_i R_{X_i}(X_i) - Y_i) > 0 \end{cases} \tag{28}$$

$$g_i(X_i, Y_i) = \begin{cases} Y_i(Y_i R_{Y_i}(Y_i) - X_i) & \text{when} \quad Y_i(Y_i R_{Y_i}(Y_i) - X_i) \le 0 \\ 0 & \text{when} \quad Y_i(Y_i R_{Y_i}(Y_i) - X_i) > 0. \end{cases} \tag{29}$$

The differential equations of neighborhood 2 can be obtained from the differential equations (26) and (27) of neighborhood 1 by using the relationships

$$\frac{dX_2}{dt} = -\frac{dX_1}{dt} \qquad \text{and} \qquad \frac{dY_2}{dt} = -\frac{dY_1}{dt}.$$

Prior to attempting to use this new set of differential equations to conduct a qualitative analysis, we must first determine whether there exists unique solutions to the set of differential equations. Differential equations that do not have unique solutions are not of particular interest to us, especially when formulating mathematical models of real life systems as, according to Blanchard et al. (1998, p. 66), we would then have to "worry about all possible solutions, even when we were doing numerical or qualitative work". Fortunately it can be shown that the new differential equations do have unique solutions by rigorous proof. The reason is that while the corrected f_i and g_i are no longer smooth, they are still locally Lipschitz continuous.

As there exists unique solutions to the set of differential equations (26) and (27), I proceed to use this new mathematical formulation in the following section to conduct a qualitative analysis of the system.

4.2 Qualitative analysis for linear tolerance schedules

This section mainly focuses on the dynamics of neighborhood 1, as it is not required to separately study and find equilibria for neighborhood 2. We can determine the equilibria of neighborhood 2 by using the relationships $\frac{dX_2}{dt} = -\frac{dX_1}{dt}$ and $\frac{dY_2}{dt} = -\frac{dY_1}{dt}$. These relationships between the differential equations of the two neighborhoods' population types lead to the following consequences:

1. Each equilibrium of neighborhood 1, (X_1^e, Y_1^e), corresponds to an equilibrium of neighborhood 2, (X_2^e, Y_2^e). The corresponding equilibrium of neighborhood 2 can be obtained by using the relations $X_2 = 1 - X_1$ and $Y_2 = \frac{1}{k} - Y_1$.
2. For any given set of parameter values, the number of equilibria of neighborhood 1 is equal to the number of equilibria of neighborhood 2.
3. At the points where the functions f_1, f_2, g_1 and g_2 are differentiable, for every neighborhood 1 composition, (X_1, Y_1), and its corresponding neighborhood 2 composition, (X_2, Y_2), we have $J_2(X_2, Y_2) = -J_1(X_1, Y_1)$ where the Jacobian matrices of neighborhood i are denoted as $J_i(X_i, Y_i)$ for $i = 1, 2$. This relationship is used when assessing the stability of each equilibrium of the two neighborhooods.

In their paper, Haw and Hogan (2020) study the system for two different cases of parameter values. They consider the case where the linear tolerance schedules for the two populations are identical for both neighborhoods, i.e. $a_1 = a_2 = a$ and $b_1 = b_2 = b$. They also consider the case where the linear tolerance schedules for the two neighborhoods are different. This thesis focuses on the two neighborhood's linear tolerance schedules being identical.

In order to conduct the qualitative analysis of neighborhood 1, I first simplify the system's differential equations by rewriting them in terms of two variables, X_1 and Y_1. I then rescale the system, following the same method as Haw and Hogan (2020). I proceed to identify the equilibria and subsequently assess their stability. I then examine the system's phase portraits and structural stability for varying parameter values.

4.2.1 Rewriting the differential equations for neighborhood 1 in two variables

The differential equations (26) and (27) for the X and Y population type of neighborhood 1, X_1 and Y_1, depend on four variables X_1, Y_1, X_2, and Y_2. To simplify the qualitative analysis, I rewrite the differential equations in terms of two variables, X_1 and Y_1. Since the functions f_1 and g_1 depend on the variables X_1 and Y_1 alone, we just need to rewrite the functions f_2 and g_2 in terms of X_1 and Y_1. We can substitute $X_2 = 1 - X_1$ and $Y_2 = \frac{1}{k} - Y_1$ into the functions f_2 and g_2. The differential equations for neighborhood 1 can then be redefined as such

$$\frac{dX_1}{dt} = f_1(X_1, Y_1) - f_2(X_1, Y_1) \tag{30}$$

$$\frac{dY_1}{dt} = g_1(X_1, Y_1) - g_2(X_1, Y_1) \tag{31}$$

where functions f_1, f_2, g_1, and g_2 are defined as

$$f_1(X_1, Y_1) = \begin{cases} X_1(a_1 X_1(1 - X_1) - Y_1) & \text{when} \quad Y_1 \geq a_1 X_1(1 - X_1) \\ 0 & \text{when} \quad Y_1 < a_1 X_1(1 - X_1) \end{cases} \tag{32}$$

$$f_2(X_1, Y_1) = \begin{cases} (1 - X_1)\left(a_2 X_1(1 - X_1) - \left(\frac{1}{k} - Y_1\right)\right) & \text{when} \quad Y_1 \leq \frac{1}{k} - a_2 X_1(1 - X_1) \\ 0 & \text{when} \quad Y_1 > \frac{1}{k} - a_2 X_1(1 - X_1) \end{cases} \tag{33}$$

$$g_1(X_1, Y_1) = \begin{cases} Y_1(b_1 Y_1(1-kY_1) - X_1) & \text{when} \quad X_1 \geq b_1 Y_1(1-kY_1) \\ 0 & \text{when} \quad X_1 < b_1 Y_1(1-kY_1). \end{cases} \tag{34}$$

$$g_2(X_1, Y_1) = \begin{cases} \left(\frac{1}{k} - Y_1\right)\left(b_2 Y_1 \left(1-kY_1\right) - \left(1-X_1\right)\right) & \text{when} \quad X_1 \leq 1 - b_2 Y_1 \left(1-kY_1\right) \\ 0 & \text{when} \quad X_1 > 1 - b_2 Y_1 \left(1-kY_1\right). \end{cases} \tag{35}$$

The inequalities used in the equations of the functions have been rearranged and simplified. This rearrangement assists in identifying what I call the nullcregions of the system. The X_1 nullcregion consists of set of points, (X_1, Z_1), that satisfy $\frac{dX_1}{dt} = 0$. Similarly, the Z_1 nullcregion consists of set of points, (X_1, Z_1), that satisfy $\frac{dZ_1}{dt} = 0$.

4.2.2 Rescaling the system

In this section, I perform the same scaling as done by Haw and Hogan (2020). This rescaling simplifies and reduces the algebraic complexity in obtaining the system's equilibria. I rename their term $\hat{Y}_1$ as Z_1. Using Z_1 instead of $\hat{Y}_1$ helps to distinguish the Z_1 values at the system's equilibria from the Y_1 values at the system's equilibria as we are ultimately interested in the Y_1 values. The new variables are defined as

$$\hat{t} = \frac{t}{k}, \quad Z_1 = \frac{Y_1}{a_1}$$

and we have the parameters

$$\alpha = ka_1, \quad \beta_1 = a_1 b_1, \quad \beta_2 = a_1 b_2, \quad \gamma = \frac{a_2}{a_1}.$$

Following the scaling and dropping the hat of t, we have

$$\frac{dX_1}{dt} = F_1(X_1, Z_1) - F_2(X_1, Z_1) \tag{36}$$

$$\frac{dZ_1}{dt} = G_1(X_1, Z_1) - G_2(X_1, Z_1) \tag{37}$$

where

$$F_1(X_1, Z_1) = \begin{cases} \alpha X_1(X_1(1-X_1) - Z_1) & \text{when} \quad Z_1 \geq X_1(1-X_1) \\ 0 & \text{when} \quad Z_1 < X_1(1-X_1) \end{cases} \tag{38}$$

$$F_2(X_1, Z_1) = \begin{cases} (1-X_1)(\alpha\gamma X_1(1-X_1) - (1-\alpha Z_1)) & \text{when} \quad Z_1 \leq \frac{1}{\alpha}(1-\alpha\gamma X_1(1-X_1)) \\ 0 & \text{when} \quad Z_1 > \frac{1}{\alpha}(1-\alpha\gamma X_1(1-X_1)) \end{cases} \tag{39}$$

$$G_1(X_1, Z_1) = \begin{cases} \frac{1}{a_1}\alpha Z_1(\beta_1 Z_1(1-\alpha Z_1) - X_1)) & \text{when} \quad X_1 \geq \beta_1 Z_1(1-\alpha Z_1) \\ 0 & \text{when} \quad X_1 < \beta_1 Z_1(1-\alpha Z_1) \end{cases} \tag{40}$$

$$G_2(X_1, Z_1) = \begin{cases} \frac{1}{a_1}(1-\alpha Z_1)(\beta_2 Z_1(1-\alpha Z_1) - (1-X_1)) & \text{when} \quad X_1 \leq 1 - \beta_2 Z_1(1-\alpha Z_1) \\ 0 & \text{when} \quad X_1 > 1 - \beta_2 Z_1(1-\alpha Z_1). \end{cases} \tag{41}$$

When we use the new set of differential equations, we obtain the system's nullclines which have a maximum of four parameters in total: $\alpha, \gamma, \beta_1, \beta_2$. Although functions G_1 and G_2 depend on the parameter a_1, the parameter a_1 never appears in the equations of the nullcline obtained from the new set of differential equations. Both functions G_1 and G_2 have a factor of $\frac{1}{a_1}$ which does not appear when we set $\frac{dZ_1}{dt} = 0$. The functions F_1 and F_2 do not have the parameter a_1 explicitly. The nullclines obtained from the differential equations written in the X_1, Y_1 coordinates, on the other hand, have a maximum of five parameters in total, a_1, a_2, b_1, b_2, k. The scaling thus reduces the number of parameters in the equations of the system's nullcline and consequently reduces the algebraic complexity of determining the equilibria of the system.

It should be noted that the nullclines and nullcregions obtained from equations (30) and (31) are the exact same nullclines and nullcregions that we obtain from equations (36) and (37), expressed in different coordinates. We can therefore obtain information on our system by studying the new set of differential equations and their corresponding phase portraits. I now proceed to identify the equilibria of the system using the new set of differential equations.

4.2.3 Equilibria

At the equilibria of neighborhood 1, (X_1^e, Z_1^e), we have

$$\frac{dX_1}{dt}\bigg|_{X=X_1^e} = 0 \quad \text{and} \quad \frac{dZ_1}{dt}\bigg|_{Z=Z_1^e} = 0.$$

As introduced in the previous section, differential equations (30) and (31) for neighborhood 1 are defined by the functions F_1, F_2, G_1, and G_2. As these four functions are piecewise defined, we have different nullclines for different set of points (X_1, Z_1), which complicates the process of determining the system's equilibria. I thus use plots of graphs in the X_1Z_1 plane in this section to simplify the process of identifying the system's equilibria. Besides having nullclines, the system may also have what I call nullcregions. The X_1 and Z_1 nullcregions are entire regions of points (X_1, Z_1) where the X_1 and Z_1 differential equations are zero. These nullcregions exist only for certain parameter values.

I now proceed to determine the system's nullclines and nullcregions. I find that there are three X_1 nullclines and one X_1 nullcregion. Similarly, there are three Z_1 nullclines and one Z_1 nullcregions. Each of these nullclines and nullcregions exists only in specific regions on the X_1Z_1 plane that satisfy certain inequalities as defined in equations (38) to (41). For the sake of clarity I number each of the different inequalities as:

Inequality 1 $Z_1 \geq X_1(1 - X_1)$

Inequality 2 $Z_1 < X_1(1 - X_1)$

Inequality 3 $Z_1 \leq \frac{1}{\alpha}(1 - \alpha X_1(1 - X_1))$

Inequality 4 $Z_1 > \frac{1}{\alpha}(1 - \alpha X_1(1 - X_1))$

Inequality 5 $X_1 \geq \beta_1 Z_1(1 - \alpha Z_1)$

Inequality 6 $X_1 < \beta_1 Z_1(1 - \alpha Z_1)$

Inequality 7 $X_1 \leq 1 - \beta_2 Z_1(1 - \alpha Z_1)$

Inequality 8 $X_1 > 1 - \beta_2 Z_1(1 - \alpha Z_1)$.

I first determine the X_1 nullclines and nullcregion. On the X_1 nullclines and nullcregion, $\frac{dX_1}{dt} = 0$. We thus have $F_1 - F_2 = 0$ on the X_1 nullclines and nullcregion.

In the region on the X_1Z_1 plane that consists of points (X_1, Z_1) satisfying **inequalities 1** and **3**, the functions F_1 and F_2 take a value of $F_1 = \alpha X_1(X_1(1 - X_1) - Z_1)$ and $F_2 = (1 - X_1)(\alpha\gamma X_1(1 - X_1) - (1 - \alpha Z_1))$ respectively. In this region, the X_1 nullcline is

$$\begin{aligned}
&\alpha X_1(X_1(1 - X_1) - Z_1) = (1 - X_1)(\alpha\gamma X_1(1 - X_1) - (1 - \alpha Z_1)) \\
&\Leftrightarrow \alpha X_1^2(1 - X_1) = \alpha\gamma X_1(1 - X_1)^2 - 1 + \alpha Z_1 + X_1 \\
&\Leftrightarrow \alpha Z_1 = (1 - X_1) + \alpha X_1^2(1 - X_1) - \alpha\gamma X_1(1 - X_1)^2 \\
&\Leftrightarrow Z_1 = (1 - X_1)\left(\frac{1}{\alpha} + X_1^2 - \gamma X_1(1 - X_1)\right) \\
&\Leftrightarrow Z_1 = (1 - X_1)\left(\frac{1}{\alpha} - \gamma X_1 + X_1^2(1 + \gamma)\right).
\end{aligned} \tag{42}$$

Haw and Hogan (2020) obtained the exact same X_1 nullcline by using their mathematical formulation for the two neighborhood model.

Similarly, we find that in the region on the X_1Z_1 plane where points (X_1, Z_1) satisfy:

1. **inequalities 1** and **4**, we have $F_1 = \alpha X_1(X_1(1 - X_1) - Z_1)$ and $F_2 = 0$. We thus have the X_1 nullcline
$$Z_1 = X_1(1 - X_1) \tag{43}$$
in this region. We do not have the X_1 nullcline $X_1 = 0$ as given these inequalities, it is only a nullcline of the system for $Z_1 > \frac{1}{\alpha}$. We know that $Z_{max} = \frac{Y_{max}}{a} = \frac{1}{\alpha}$, and so we are only interested in $Z_1 \in \left[0, q\frac{1}{\alpha}\right]$.

2. **INEQUALITIES 2** and **3**, we have $F_1 = 0$ and $F_2 = (1 - X_1)(\alpha\gamma X_1(1 - X_1) - (1 - \alpha Z_1))$. In this region the X_1 nullcline is

$$Z_1 = \frac{1}{\alpha}(1 - \alpha\gamma X_1(1 - X_1)) \tag{44}$$

   Once again, we do not have the X_1 nullcline $X_1 = 1$, as given these inequalities, it is only defined for Z_1 values that are out of our range.

In the region defined by **INEQUALITIES 2** and **4**, both functions F_1 and F_2 take a value of zero on all points (X_1, Z_1). It then follows that we have $\frac{dX_1}{dt} = 0$ at all points (X_1, Z_1) in this region. Consequently, the entire region defined by **INEQUALITIES 2** and **4** is our X_1 nullcregion.

I identify the Z_1 nullclines and nullcregion where $\dot{Z}_1 = 0$ in the same manner. We know that $\dot{Z}_1 = 0$ when $G_1 - G_2 = 0$. I find that in the region on the X_1Z_1-plane, where points (X_1, Z_1) satisfy

1. **INEQUALITIES 5** and **7**, we have $G_1 = \frac{1}{a_1}\alpha Z_1(\beta_1 Z_1(1 - \alpha Z_1) - X_1)$ and $G_2 = \frac{1}{a_1}(1 - \alpha Z_1)(\beta_2 Z_1(1 - \alpha Z_1) - (1 - X_1))$. The Z_1 nullcline in this region is

$$X_1 = (1 - \alpha Z_1)(1 - \beta_2 Z_1 + Z_1^2(\alpha\beta_1 + \alpha\beta_2)). \tag{45}$$

2. **INEQUALITIES 5** and **8**, we have $G_1 = \frac{1}{a_1}\alpha Z_1(\beta_1 Z_1(1-\alpha Z_1) - X_1)$ and $G_2 = 0$. The Z_1 nullcline in this region is

$$X_1 = \beta_1 Z_1(1 - \alpha Z_1). \tag{46}$$

3. **INEQUALITIES 6** and **7**, we have $G_1 = 0$ and $G_2 = \frac{1}{a_1}(1 - \alpha Z_1)(\beta_2 Z_1(1 - \alpha Z_1) - (1 - X_1))$. In this region, the Z_1 nullcline is

$$X_1 = 1 - \beta_2 Z_1(1 - \alpha Z_1). \tag{47}$$

4. **INEQUALITIES 6** and **8** we have our Z_1 nullcregion. We have $\frac{dZ_1}{dt} = 0$ at all points (X_1, Z_1) in this region.

The equilibria of neighborhood 1, (X_1^e, Z_1^e), are thus

1. the points, (X_1, Z_1), of intersection of any one of the X_1 nullclines (42), (43), and (44), with any one of the Z_1 nullclines (45), (46), and (47).
2. the points, (X_1, Z_1), in the region where the X_1 and Z_1 nullcregions overlap. This region satisfies **INEQUALITIES 2, 4, 6**, and **8**.
3. the points, (X_1, Z_1), on any one of the X_1 nullclines that are also in the Z_1 nullcregion.
4. the points, (X_1, Z_1), on any one of the Z_1 nullclines that are also in the X_1 nullcregion.

We can observe that for the two neighborhood model $X_1 = 0$, $Z_1 = 0$ are no longer nullclines of the system, unlike in the case of the single neighborhood model.

I now proceed to consider the case where the population tolerance schedules for the X, Y populations are identical in both neighborhoods. We have

$$a_1 = a_2 = a, b_1 = b_2 = b$$

resulting in

$$\alpha = ka, \beta_1 = \beta_2 = \beta, \gamma = 1.$$

Our differential equations for this particular case where the X and Y population tolerance schedules for both neighborhoods are

$$\frac{dX_1}{dt} = F_1(X_1, Z_1) - F_2(X_1, Z_1) \tag{48}$$

$$\frac{dZ_1}{dt} = G_1(X_1, Z_1) - G_2(X_1, Z_1) \tag{49}$$

where we now have

$$F_1(X_1, Z_1) = \begin{cases} \alpha X_1(X_1(1 - X_1) - Z_1) & \text{when} \quad Z_1 \geq X_1(1 - X_1) \\ 0 & \text{when} \quad Z_1 < X_1(1 - X_1) \end{cases} \tag{50}$$

$$F_2(X_1, Z_1) = \begin{cases} (1-X_1)(\alpha X_1(1-X_1) - (1-\alpha Z_1)) & \text{when} \quad Z_1 \leq \frac{1}{\alpha}(1-\alpha X_1(1-X_1)) \\ 0 & \text{when} \quad Z_1 > \frac{1}{\alpha}(1-\alpha X_1(1-X_1)) \end{cases} \tag{51}$$

$$G_1(X_1, Z_1) = \begin{cases} \frac{1}{a}(\alpha Z_1(\beta Z_1(1-\alpha Z_1) - X_1)) & \text{when} \quad X_1 \geq \beta Z_1(1-\alpha Z_1) \\ 0 & \text{when} \quad X_1 < \beta Z_1(1-\alpha Z_1). \end{cases} \tag{52}$$

$$G_2(X_1, Z_1) = \begin{cases} \frac{1}{a}(1-\alpha Z_1)(\beta Z_1(1-\alpha Z_1) - (1-X_1)) & \text{when} \quad X_1 \leq 1-\beta Z_1(1-\alpha Z_1) \\ 0 & \text{when} \quad X_1 > 1-\beta Z_1(1-\alpha Z_1). \end{cases} \tag{53}$$

We thus have

Inequality 1 $Z_1 \geq X_1(1-X_1)$

Inequality 2 $Z_1 < X_1(1-X_1)$

Inequality 3 $Z_1 \leq \frac{1}{\alpha}(1-\alpha X_1(1-X_1))$

Inequality 4 $Z_1 > \frac{1}{\alpha}(1-\alpha X_1(1-X_1))$

inequalities 5 and **8** remain the same as before except the terms β_1 and β_2 are replaced by β. For the case where the population tolerance schedules for the X and Y populations are identical in both neighborhoods we have for the regions where

1. **inequalities 1** and **3** are satisfied, the X_1 nullcline

$$Z_1 = (1-X_1)\left(\frac{1}{\alpha} - X_1 + 2X_1^2\right). \tag{54}$$

2. **inequalities 1** and **4** are satisfied, the X_1 nullcline (43).

3. **inequalities 2** and **3** are satisfied, the X_1 nullcline

$$Z_1 = \frac{1}{\alpha}(1-\alpha X_1(1-X_1)). \tag{55}$$

4. **inequalities 2** and **4**, we have our X_1 nullcregion.

5. **inequalities 5** and **7**, we have the Z_1 nullcline

$$X_1 = (1-\alpha Z_1)(1-\beta Z_1 + 2\alpha\beta Z_1^2). \tag{56}$$

6. **inequalities 5** and **8**, we have the Z_1 nullcline

$$X_1 = \beta Z_1(1-\alpha Z_1). \tag{57}$$

7. **inequalities 6** and **7**, we have the Z_1 nullcline

$$X_1 = 1-\beta Z_1(1-\alpha Z_1). \tag{58}$$

8. **inequalities 6** and **8** we have our Z_1 nullcregion.

We can determine that at the points $(X_1, Z_1) = (1,0), \left(1, \frac{1}{\alpha}\right)$, we have $\frac{dX}{dt} = 0$ and $\frac{dZ}{dt} = 0$ for all parameter values. This observation can be made simply by substituting these X_1 and Z_1 values into the differential equations (48) and (49) . Consequently, for all parameter values, two of the system's equilibria are $(X_1^e, Z_1^e) = (1,0), \left(1, \frac{1}{\alpha}\right)$.

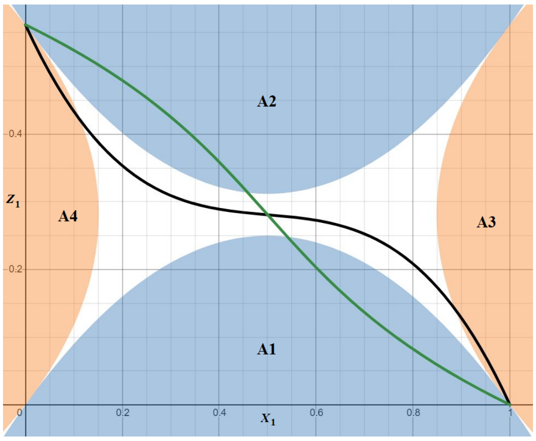
The process of determining the remaining equilibria of the system is less straightforward. The fact that the differential equations are defined by functions F_1, F_2, G_1, and G_2 that are piecewise defined complicates the process of determining the system's remaining equilibria. As the functions are all piecewise defined, each of the system's nullclines and nullcregions exist only in regions where points (X_1, Z_1) satisfy specific inequalities. To help visualize the system's nullclines, nullcregions and its equilibria, I generated a number of plots of the system's nullcregions and nullclines for different parameter values. These plots are as shown in Figure 13.

In Figure 13 the X_1-nullclines are plotted in black and the Z_1-nullclines are plotted in green. Likewise, the X_1 and Z_1 nullcregions are shaded in black and green respectively. We have equilibria at the points (X_1, Z_1) where the black curves intersect the green curves. Similarly, all points (X_1, Z_1) in the overlapping region of the green and black shaded nullcregions are equilibria. All points that lie within the black shaded nullcregion and are on the green nullclines are equilibria. Similarly, all points on the green shaded nullcregion and on the black nullclines are also equilibria.

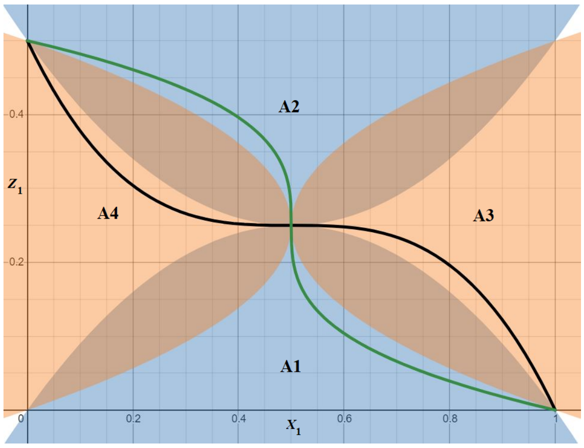
Additionally, the following four regions are labelled in this plot. They are

1. Region $A1$, defined by the inequality $Z_1 < X_1 (1 - X_1)$.
2. Region $A2$, defined by the inequality $Z_1 > \frac{1}{\alpha} \big(1 - \alpha X_1 (1 - X_1)\big)$.
3. Region $A3$, defined by the inequality $X_1 > 1 - \beta Z_1 (1 - \alpha Z_1)$.
4. Region $A4$, defined by the inequality $X_1 < \beta Z_1 (1 - \alpha Z_1)$.

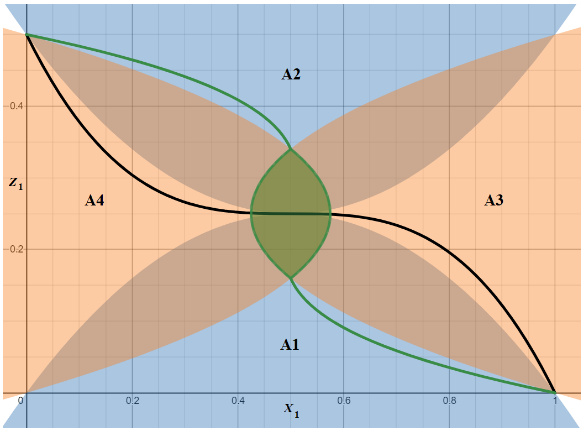
The regions $A1$ and $A2$ are colored in blue, and the regions $A3$ and $A4$ are colored in orange. Note that we are only interested in $X \in [0, 1]$ and $Z \in \left[0, \frac{1}{\alpha}\right]$. When regions $A1$ and $A2$ overlap, there exists a region of points satisfying **INEQUALITIES 2** and **4**. The system then has an X_1 nullcregion as shown in Figures 13c and 13e. Similarly when $A3$ and $A4$ overlap, the system then has a Z_1 nullcregion as shown in Figure 13d and 13e. When all the four regions $A1, A2, A3$ and $A4$ overlap, we have an entire region of equilibria. We have, in this case, a "sea of equilibria".



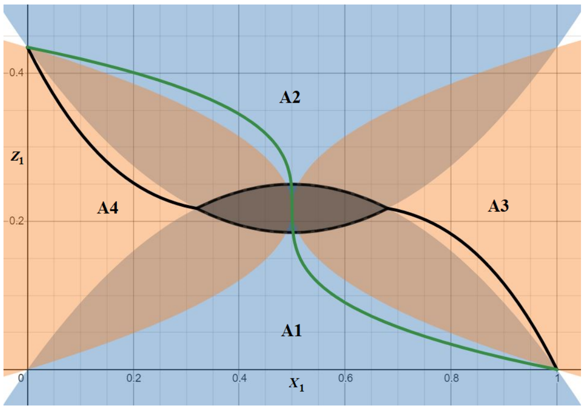
(a) $(a, b, k) = (0.8, 0.9, 2)$



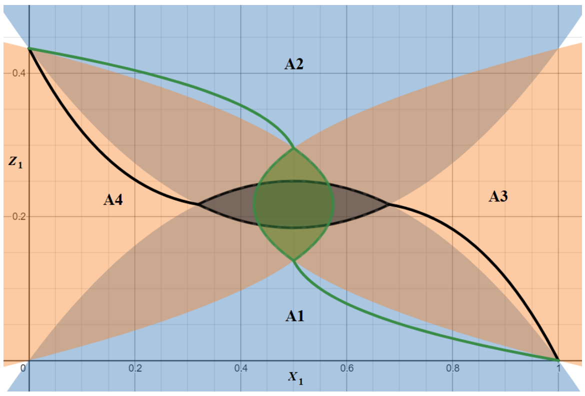
(b) $(a, b, k) = (1, 4, 2)$



(c) $(a, b, k) = (1, 4.6, 2)$



(d) $(a, b, k) = (1.15, 4, 2)$



(e) $(a, b, k) = (1.15, 4.6, 2)$

Figure 13: Plots showing the system's nullclines and nullcregions for different parameter values.

I now provide an overview of which nullclines and nullcregions exist in each of the five scenarios shown in Figure 13. In Figures 13a and 13b, we have the X_1-nullcline (54) and the Z_1-nullcline (56). In the scenario shown in Figure 13c, we have the X_1 nullclines (54), (43), (55) and the Z_1 nullcregion. We also have the Z_1 nullcline (56). In the scenario as shown in Figure 13d, we have the Z_1 nullclines (56), (57), (58) and the Z_1 nullcregion. We also have the X_1 nullcline (54). In the scenario as shown in the last Figure 13e, the system has both the X_1 and Z_1 nullcregions. We have the X_1 nullclines (54), (43), (55) and the Z_1 nullcregions (56), (57) and (58).

If we take any set of parameter values and plot the system's nullclines and shade the nullcregions, we obtain a plot that is similar to any one of the five scenarios shown in Figure 13. Furthermore, all the five scenarios portrayed in Figure 13 can further be categorized into three cases according to which nullcregions each scenario has. I therefore conduct my qualitative analysis of the system for these three cases:

1. Case 1: There is no X_1 or Z_1 nullcregion as shown in the cases in Figures 13a and 13b. There exists no region of points (X_1, Z_1) that satisfies **INEQUALITIES 2** and **4** or **INEQUALITIES 6** and **8**.

2. Case 2: The system either has a X_1 nullcregion or a Z_1 nullcregion but not both. This case is portrayed in Figures 13c and 13d. In this case, there exists a region of points (X_1, Z_1) satisfying either **INEQUALITIES 2** and **4**, as shown in Figure 13c, or **INEQUALITIES 6** and **8**, as shown in Figure 13d, but not both.

3. Case 3: The system has both X_1 and the Z_1 nullcregions as shown in 13e. In this case, there exists a region of points (X_1, Z_1) that satisfies all four **INEQUALITIES 2, 4, 6** and **8**.

For each of the three cases to occur, the system's parameters must meet certain conditions. I attempt to identify these conditions before proceeding further into the qualitative analysis.

The X_1 nullcregion exists when the curves with equations $Z_1 = X_1(1-X_1)$ and $Z_1 = \frac{1}{\alpha}(1-\alpha X_1(1-X_1))$ intersect at two points. At the points (X_1, Z_1) of intersection of these two curves,

$$\begin{aligned} X_1(1-X_1) &= \frac{1}{\alpha}(1-\alpha X_1(1-X_1)) \\ \Leftrightarrow 2X_1^2 - 2X_1 + \frac{1}{\alpha} &= 0 \\ \Leftrightarrow X_1 &= \frac{1 \pm \sqrt{1-\frac{8}{\alpha}}}{2} \\ \Leftrightarrow X_1 &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{2}{\alpha}} \end{aligned}$$

The curves have two points of intersection when the discriminant, $1-\frac{2}{\alpha}$, is greater than zero. We thus have a X_1 nullcregion when $\alpha > 2$. The inequality $\alpha > 2$ corresponds to the inequality $ka > 2$.

Similarly, the Z_1 nullcregion exists when there are two points of intersection of the curves with equations $X_1 = \beta Z_1(1-\alpha Z_1)$ and $X_1 = 1 - \beta Z_1(1\alpha Z_1)$. At the points of intersection of the two curves,

$$\begin{aligned} \beta Z_1(1-\alpha Z_1) &= 1 - \beta Z_1(\alpha Z_1) \\ \Leftrightarrow 2\alpha\beta Z_1^2 - 2\beta Z_1 + 1 &= 0 \\ \Leftrightarrow Z_1 &= \frac{1\beta \pm 1\sqrt{\beta^2 - 2\alpha\beta}}{2\alpha\beta} \end{aligned}$$

The curves have two points of intersection when the discriminant, $\beta^2 - 2\alpha\beta$, is greater than zero. We thus have a Z_1 nullcregion when $\beta > 2\alpha$. This inequality can also be rewritten as $b > 2k$.

In summary, I found the following conditions that each case must meet to occur

1. For case 1 we must have $\beta \leq 2\alpha$ and $\alpha \leq 2$.

2. For case 2 we must have either $\beta > 2\alpha$ or $\alpha > 2$.

3. For case 3 we must have $\beta > 2\alpha$ and $\alpha > 2$.

I now proceed to explore the equilibria of the system corresponding to each of the three cases.

Case 1: There is no X_1 or Z_1 nullcregion

In this particular case, the equilibria of the system are given by the intersection points of equation (54) and (56) alone. We can make this observation from the plots shown in Figures 13a and 13b. This is the exact same case as examined by Haw and Hogan (2020). In this case the only equilibria of the system are at the points of intersection of the X_1 and Z_1 nullclines given by equations (54) and (56).

Substituting Z_1 from equation (56) into equation (54) gives us

$$p_9(X_1) = 0$$

which is a real polynomial of degree 9 in X_1 that was also obtained by Haw and Hogan (2020). As the polynomial is of degree 9, the system has at most nine equilibria.

By examining the system's differential equations, we find that three of the equilibria are $(X_1^e, Z_1^e) = (0, \frac{1}{\alpha}), (1, 0)$ and $\left(\frac{1}{2}, \frac{1}{2\alpha}\right)$. These equilibria correspond to $(X_e, Y_e) = \left(0, \frac{1}{k}\right), (1, 0)$ and $\left(\frac{1}{2}, \frac{1}{2k}\right)$ respectively. We can thus factor out three terms from p_9 and write

$$p_9(X_1) = X_1(1 - X_1)(1 - 2X_1)p_6(X_1) \tag{59}$$

where we have

$$p_6(X_1) \equiv \sum_{i=0}^{6} A_i X_1^i \tag{60}$$

with

$$\begin{aligned}
A_0 &= \alpha + \beta + \frac{\beta}{\alpha} \\
A_1 &= -3\beta(\alpha + 2) \\
A_2 &= \beta(2\alpha^2 + 15\alpha + 6) \\
A_3 &= -12\alpha\beta(\alpha + 2) \\
A_4 &= 2\alpha\beta(13\alpha + 6) \\
A_5 &= -24\alpha^2\beta \\
A_6 &= 8\alpha^2\beta
\end{aligned}$$

just as obtained by Haw and Hogan (2020). We can then use Descartes' rules of signs to determine the number of roots that p_6 has. According to Descartes' rule of signs, the number of positive roots for a polynomial function is at most the number of times the sign changes in the sequence of the polynomial's coefficients (Weisstein, n.d.). The rule also states that the difference between the number of roots and the number of changes in the sign of coefficients is always even. As $\alpha, \beta > 0$, it follows that the signs of the coefficients of $p_6(X_1)$ alternate six times. Using Descartes' rules of signs we can then make the conclusion that p_6 can have 0, 2, 4, or 6 positive real roots. Here I am not counting multiplicities — a root of multiplicity three is counted as one root.

Using Descartes' rule of signs, we also find that indeed all the roots of p_6 are positive real roots. The number of negative roots is at most the number of times the sign changes in the sequence of the polynomial's coefficients after substituting X_1 by $-X_1$ in equation (60). We find that after this substitution is made, there is no sign change in the sequence of the polynomial's coefficients. Therefore, the polynomial has no negative real roots.

For all parameter values we have $P_6(1 - X_1) = P_6(X_1)$. It then follows that the roots of p_6 are in $[0, 1]$ and the function p_6 is symmetric about $X_1 = 0.5$. As p_6 can have 0, 2, 4, or 6 positive real roots and I have already identified three of the roots of p_9 that are not roots of p_6, it follows that p_9 can have 3, 5, 7, or 9 positive roots.

However, when the additional constraints that $\beta \leq 2\alpha$ and $\alpha \leq 2$ are taken into consideration, I find that p_6 has zero positive real roots. We can make this observation by plotting the graph and doing a few calculations. The polynomial p_6 has a minimum point at $X_1 = 0.5$. It then follows that p_6 will have at least one root when $p_6(0.5) \leq 0$. When $X_1 = 0.5$, we have

$$p_6(0.5) = \alpha - \frac{\beta}{2} + \frac{\beta}{\alpha}. \tag{61}$$

This expression of $p_6(0.5)$ is different from that obtained by Haw and Hogan (2020) as they made a calculation error. As the difference is merely an extra factor α, they obtain the same value of β, β_c, at which $p_6(0.5) = 0$. The polynomial p_6 has at least one root when

$$\beta \geq \beta_c \equiv \frac{2\alpha^2}{\alpha - 2}. \tag{62}$$

We also know that we always have $\beta > 0$ as $\beta = ba$ and $b, a > 0$. It follows that for p_6 to have at least one root a necessary condition is that $\alpha > 2$. However, one of the conditions for the system to have no X_1 or Z_1 nullcregion is that $\alpha \leq 2$. Therefore, the polynomial p_6 has no roots in the specific case where no X_1 or Z_1 nullcregion exists. We can also make this observation in the plot of p_6 for $\beta \leq 2\alpha$ and $\alpha \leq 2$, which is shown in Figure 14.

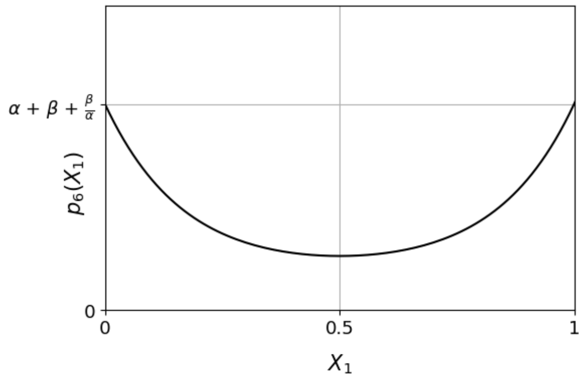


Figure 14: Plot of $p_6(X_1)$ for $\alpha \leq 2$ and $\beta \leq 2\alpha$.

Thus, the only equilibria of the system for this case are $(X_1^e, Y_{1_e}) = \left(0, \frac{1}{k}\right), (1, 0), \left(\frac{1}{2}, \frac{1}{2k}\right)$. Using the relationships $X_2 = 1 - X_1$ and $Y_2 = \frac{1}{k} - Y_1$ I also find the corresponding three equilibria of neighborhood 2, $(X_2^e, Y_{2_e}) = (1, 0), \left(0, \frac{1}{k}\right)$ and $\left(\frac{1}{2}, \frac{1}{2k}\right)$.

The number of equilibria that I obtain are similar to the case examined by Haw and Hogan (2020) where p_6 has no equilibria. Haw and Hogan's mathematical formulation did not create constraints on the parameter values that one can choose besides the fact that they all need to be greater than zero. Consequently, Haw and Hogan chose values of α and β that lead to the polynomial p_6 having as many as six roots (see Figure 2 on p. 228 of their 2020 paper). In alignment with my model and the constraints of case 1, I set the additional constraints that $\beta \leq 2\alpha$ and $\alpha \leq 2$. As a result the polynomial p_6 had zero roots in the case that I am examining. I only obtained three equilibria for the system. I now proceed to examine the system's equilibria for the other two cases.

Case 2: The system either has a X_1 nullcregion or a Z_1 nullcregion but not both
The X_1 and Z_1 nullclines and nullcregions for this case are illustrated in Figures 13c and 13d. In this case both the systems of neighborhoods 1 and 2 have a line of equilibria. All points on the X_1 (the cubic curve shown in black) nullcline which fall within the Z_1 nullcregion (the region shaded in green) are equilibria. Likewise, all points on the Z_1 nullcline (the cubic curve shown in green) which fall within the X_1 nullcregion (the region shaded in black) are also equilibria.

Additionally, points of intersection of the X_1 and Z_1 nullclines are also equilibria. However, I do not attempt to identify each of the equilibria for the system for this case. I instead demonstrate that the system corresponding to such parameter values is not structurally stable in Section 4.2.4. I therefore do not examine the system corresponding to this case in depth.

Case 3: The system has both X_1 and the Z_1 nullcregions
The X_1 and Z_1 nullclines and nullcregions for this case are illustrated in Figure 13e. From Figure 13e, we can make the observation that the system has a sea of equilibria for parameter values corresponding to this case. All points (X_1, Z_1) laying in both the X_1 and Z_1 nullcregions, i.e. in the region of overlap of the black and green shaded regions are equilibria.

The other equilibria of the system include the intersection points of X_1 and the Z_1 nullclines. Once again, I do not attempt to identify each of the equilibria for this case as the system is not structurally stable, as demonstrated in Section 4.2.4.

4.2.4 Structural stability analysis

In this section, I assess the stability of the equilibria for the three different cases. For each equilibrium, I determine the eigenvalues of the Jacobian matrices, $J_1(X_1, Z_1)$ and $J_2(X_1, Z_1)$ to assess the equilibrium's stability. I then comment on the structural stability of the phase portraits using the information obtained on the stability of the equilibria.

At the points where the functions F_1, F_2, G_1 and G_2 are differentiable we have the Jacobian matrix

$$J_1(X_1, Z_1) = \begin{pmatrix} \frac{\delta}{\delta X_1}(F_1 - F_2) & \frac{\delta}{\delta Z_1}(F_1 - F_2) \\ \frac{\delta}{\delta X_1}(G_1 - G_2) & \frac{\delta}{\delta Z_1}(G_1 - G_2) \end{pmatrix}$$

for neighborhood 1 for all three cases. As we rewrote the functions F_1, F_2, G_1 and G_2 in terms of variables X_1 and Y_1, the Jacobian matrix for neighborhood 2 is $J_2(X_1, Y_1) = -J_1(X_1, Y_1)$ at the points where the functions F_1, F_2, G_1 and G_2 are differentiable.

Case 1: There is no X_1 or Z_1 nullcregion
In this case, the equilibria are at points where, $F_1 \neq 0, F_2 \neq 0, G_1 \neq 0$ and $G_2 \neq 0$. The Jacobian matrix is

$$J_1(X_1, Z_1) = \begin{pmatrix} -(1+\alpha) + 6\alpha X_1(1-X_1) & -\alpha \\ -\frac{1}{a} & \frac{-(\alpha+\beta)+6\alpha\beta Z_1(1-\alpha Z_1)}{a} \end{pmatrix}.$$

Haw and Hogan (2020, p. 230) deliberately leave out the factor of $\frac{1}{a}$ we see in the 2nd row of elements of the above matrix. As it is globally multiplied with the terms in G_1, G_2, it can be removed without affecting the sign of the eigenvalues.

The vector field of neighborhood 1 is not differentiable at $(X_1^e, Z_1^e) = (1, 0), \left(0, \frac{1}{\alpha}\right)$. For $a = 1$, the Jacobian matrix at $(X_1^e, Z_1^e) = \left(\frac{1}{2}, \frac{1}{2\alpha}\right)$ is

$$J_1\left(\frac{1}{2}, \frac{1}{2\alpha}\right) = \begin{pmatrix} -1 + \frac{\alpha}{2} & -\alpha \\ -1 & \frac{-2\alpha+\beta}{2} \end{pmatrix}.$$

and I obtain the eigenvalues

$$\lambda_\pm = \frac{1}{4}[\beta - \alpha - 2 \pm \sqrt{(\beta+2)^2 + 9\alpha^2 + 4\alpha - 6\alpha\beta}].$$

I find that $\lambda_+ > 0$ and $\lambda_- < 0$ for the parameters $\beta \leq 2\alpha$ and $\alpha \leq 2$. The neighborhood 1 equilibrium at $(X_1^e, Z_1^e) = \left(\frac{1}{2}, \frac{1}{2\alpha}\right)$.

The vector field for neighborhood 2 is not differentiable at $(X_2^e, Z_2^e) = (1, 0), \left(0, \frac{1}{\alpha}\right)$. At $(X_2^e, Z_2^e) = \left(\frac{1}{2}, \frac{1}{2\alpha}\right)$ the Jacobian matrix for neighborhood 2 is $J_2\left(\frac{1}{2}, \frac{1}{2\alpha}\right) = -J_1\left(\frac{1}{2}, \frac{1}{2\alpha}\right)$. I also find that the Jacobian matrix $J_2\left(\frac{1}{2}, \frac{1}{2\alpha}\right)$ has one negative and one positive eigenvalue. The equilibrium at $(X_2^e, Y_2^e) = \left(\frac{1}{2}, \frac{1}{2\alpha}\right)$ is also a saddle.

The system is therefore structurally stable for parameter values $\beta < 2\alpha$ and $\alpha < 2$. The system in this case is not structurally stable for $\beta = 2\alpha$ and $\alpha = 2$, as at these parameter values, the system has a line or a sea of equilibria when a small change is made to the system.

Case 2: The system either has a X_1 nullcregion or a Z_1 nullcregion but not both
For this case I focus only on determining the stability of the equilibria that are on the X_1 and Z_1 nullcregion, and points on the Z_1 nullcline that are on the X_1 nullcregion. I demonstrate that at least one of the eigenvalue for each of these equilibria is equal to zero. The system is thus not structurally stable and consequently I do not attempt to determine the eigenvalues of the remaining equilibria.

Let us first consider the case where the system has a X_1 nullcregion but not a Z_1 nullcregion. At the points that are on the Z_1 nullcline and fall within the X_1 nullcregion, we have $F_1 - F_2 = 0$ and $G_1 - G_2 = X_1 - (1-\alpha Z_1)(1-\beta Z_1 + 2\alpha\beta Z_1^2)$. Our Jacobian matrix for neighborhood 1 for the points (X_1, Z_1) that are on the Z_1 nullcline and fall within the X_1 nullcregion is thus

$$J_1(X_1, Z_1) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{a} & \frac{-(\alpha+\beta)+6\alpha\beta Z_1(1-\alpha Z_1)}{a} \end{pmatrix}. \tag{63}$$

Similarly for this case, all the elements of the Jacobian matrix for neighborhood 2, $J_2(X_2, Y_2)$, are zero. We can observe from the Jacobian matrix that at least one of the eigenvalues is zero for all equilibria (X_1^e, Z_1^e) that are on the Z_1 nullcline and fall within the X_1 nullcregion. Consequently, the system has a line of non-hyperbolic equilibria.

We make a similar observation for the case where the system has a Z_1 nullcregion but not a X_1 nullcregion. Our Jacobian matrix for points on the X_1 nullcline and fall within the Z_1 nullcregion

$$J(X_1, Z_1) = \begin{pmatrix} -(1+\alpha) + 6\alpha X_1(1-X_1) & -\alpha \\ 0 & 0 \end{pmatrix}. \tag{64}$$

We make the same observation that at least one of the eigenvalues is zero for the equilibria that are on the X_1 nullcline and fall within the Z_1 nullcregion. We have a line of non-hyperbolic equilibria and once again the system is not structurally stable.

Since the system is not structurally stable, the qualitative behavior of the system is significantly altered by perturbations of the system. It is, as explained in Section 2.5.3, not well justified to use systems that are not structurally stable such as this to study real life systems, and as such I do not study the system further.

Case 3: The system has both X_1 and the Z_1 nullcregions

As identified in the previous section, when the system has both the X_1 and Z_1 nullcregions, we have a sea of equilibria in the region where the two nullcregions overlap. The system also has other equilibria besides those equilibria. I focus on determining the Jacobian's eigenvalues for the equilibria that are on the X_1 and the Z_1 nullcregions to demonstrate that the system is not structurally stable for this case.

We know that at all points in the region of overlap of the X_1 and Z_1 nullclines we have $F_1 - F_2 = 0$ and $G_1 - G_2 = 0$ as per the definition of the functions. The Jacobian matrix is thus

$$J(X_1, Z_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

at these points. As all elements of the Jacobian matrix are zero, all the equilibria that simultaneously lie on the X_1 and Z_1 nullcregions have eigenvalues zero. The system, in case 3, therefore has a sea of non-hyperbolic equilibria. Once again, I do not further study the system corresponding to this case as it is not structurally stable. In the next section I only produce phase portraits for the system corresponding to case 1.

4.2.5 Phase portraits

In this section, I only examine the phase portraits for case 1 where no X_1 or Z_1 nullcregions exist as it is the only case in which the system is smooth. Haw and Hogan (2020) explore the phase portraits of $(\alpha, \beta) = (9, 16), (9, 40), (9, 80)$. However, these parameter values do not meet the conditions that I have identified for our case. According to my new mathematical formulation, the parameter values $(\alpha, \beta) = (9, 16)$ correspond to case 2, where there exists a line of non-hyperbolic equilibria. Parameter values $(\alpha, \beta) = (9, 40), (9, 80)$ correspond to a sea of equilibria. Since the systems corresponding to all those three parameter values are not structurally stable, and are therefore not useful in modelling a physical reality, I do not attempt to recreate those phase portraits.

I choose parameter values $(a, b, k) = (0.5, 2, 2)$ which corresponds to $(\alpha, \beta) = (1, 1)$ to study the system corresponding to case 1 . As we are interested in the X_1, Y_1 coordinate system, I examine the phase portrait in the $X_1 Y_1$ plane. The corresponding phase portrait is shown in in Figure 15. The non-normalized and normalized vector field of the system is shown in Figures 16a and 16b respectively. The X_1 and Y_1 nullclines are also shown in Figures 16a and 16b.

For the phase portrait of neighborhood 1 shown in Figure 15, the equilibria $(X_1, Y_1) = (1, 0), (0, \frac{1}{k})$ are represented by red dots and the saddle at $\left(\frac{1}{2}, \frac{1}{2k}\right)$ is represented by a green dot.

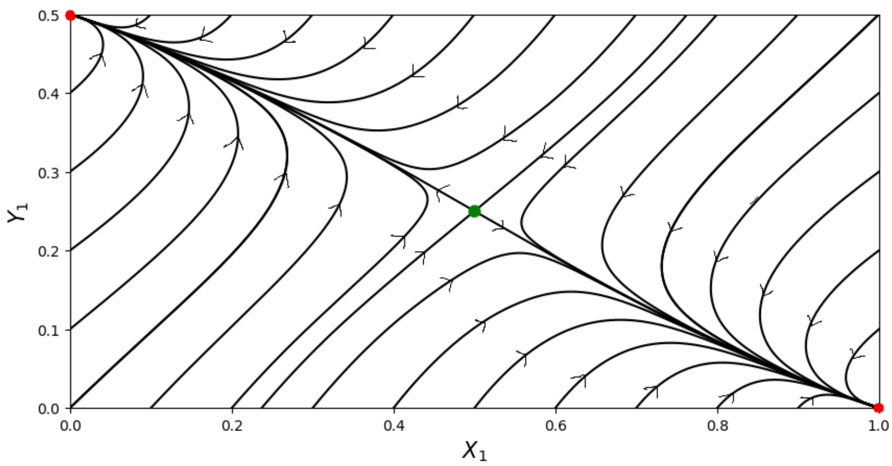


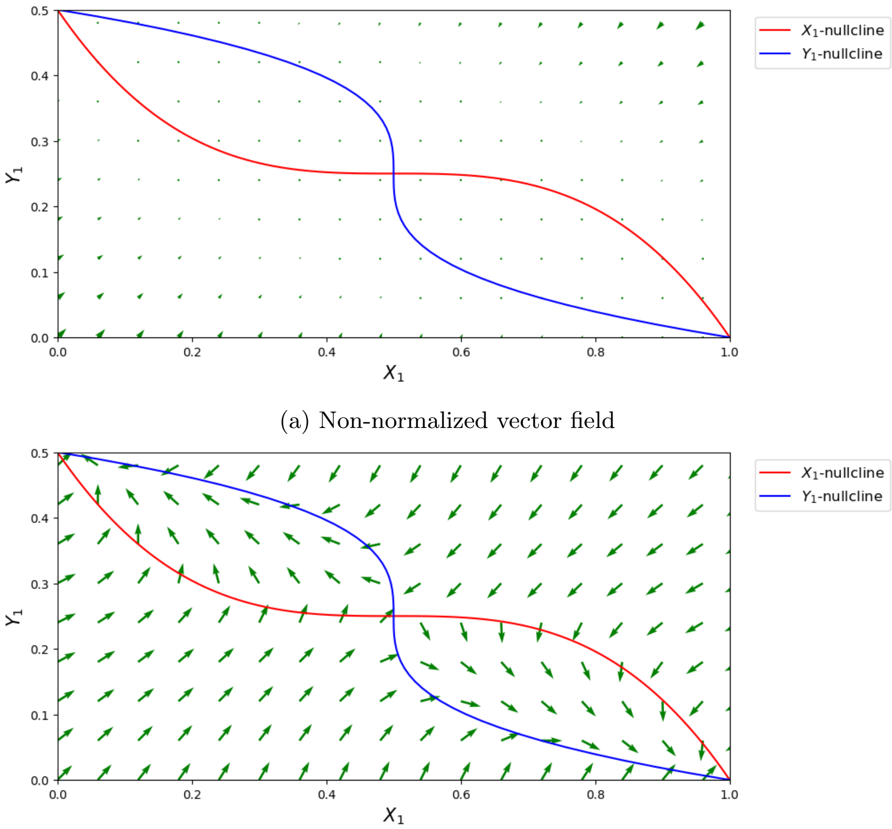
Figure 15: Phase portrait for neighborhood 1 in the $X_1 Y_1$ plane for $(\alpha, \beta) = (1, 1)$.

The phase portrait for neighborhood 2 can be obtained from the corresponding phase portrait for neighborhood 1. The X and Y population types of neighborhood 2 are given by the relationships $X_2 = 1 - X_1$ and $Y_2 = \frac{1}{k} - Y_1$. Using this relationship, we can find the corresponding X_2 and Y_2. The origin of the phase portrait for neighborhood 2, $(X_2, Y_2) = (0, 0)$, corresponds the point $(X_1, Y_1) = \left(1, \frac{1}{k}\right)$. We can therefore obtain the phase portrait for neighborhood 2 by rotating the corresponding phase portrait for neighborhood 1 by 180°. The X_2, Y_2 axes of the phase portrait for neighborhood 2 runs horizontally and vertically respectively from point $(X_1, Y_1) = \left(1, \frac{1}{k}\right)$.

The phase portrait for neighborhood 2 for parameter values $(a, b, k) = (0.5, 2, 2)$ is identical to the phase portrait for neighborhood 1 as shown in Figure 15.

Similar to in the single neighborhood case, we see that the dynamics of the system is organized by the saddle and its stable and unstable manifolds. Most of the trajectories move toward the equilibria $(X_1^e, Y_1^e) = (1, 0), (0, \frac{1}{ak})$, which correspond to a homogeneous neighborhood. One difference we notice in the system's phase portrait when compared to the single neighborhood model is that there is no equilibrium at the origin $(X_1, Y_1) = (0, 0)$.

The phase portraits for other parameter values where $\alpha \leq 2$ and $\beta \leq 2\alpha$ look qualitatively the same as the plot in Figure 15. We see that all trajectories that do not have initial conditions corresponding to a point on the saddle's stable manifold move to the stable node equilibria corresponding to a homogeneous neighborhood.



(a) Non-normalized vector field

(b) Normalized vector field

Figure 16: Plot of the system's vector field and nullclines in the X_1Y_1 plane. Top (16a): The vectors are not normalized. Bottom (16b): The vectors are normalized

4.2.6 Physical interpretation

A few general remarks can be made on the two neighborhood model described by Haw and Hogan. After having examined the three cases for the model when the X and Y population linear tolerance schedules are identical for both neighborhoods, I conclude that the Haw and Hogan's two neighborhood model is not a good model. According to the model, we only have a structurally stable system when the parameter values meet strict conditions where $\beta < 2\alpha$ and $\alpha < 2$. The system described by the model is not structurally stable for parameter values $\beta \geq 2\alpha$ or $\alpha \geq 2$. Ways this model can be improved are discussed in Section 4.2.7.

In the case where we have a structurally stable system, the model paints a bleak picture. It predicts that it is most likely that each neighborhood consists of members of only one type of population. That is to say, that both the two neighborhoods will have homogeneous compositions and continue having such a composition once the composition is attained.

I found that the only equilibrium point that corresponds to a heterogeneous neighborhood is unstable.

Mathematically, the chances of obtaining a heterogeneous composition is zero. Only compositions that correspond to points (X_1, Y_1) on the stable manifold lead to a heterogeneous composition. The probability of obtaining a heterogeneous composition is therefore the area of the curved lines, which are the stable manifolds. However, the area of a line is zero, and consequently, according to the model, the probability of obtaining a heterogeneous composition is calculated to be zero. In reality, there are infinite possibilities of initial conditions and so a probability of zero does not mean a heterogeneous composition can never be obtained. The results obtained from the model thus predict that, most often, both neighborhoods will end up having members of only one type of population in the long run. The neighborhood's composition remains as such once a homogeneous composition has been attained.

The neighborhoods' compositions in the long run can be of two types, depending on the initial compositions. I obtained two stable equilibria which correspond to homogeneous neighborhoods namely $(X_1^e, Y_1^e) = (1, 0), \left(0, \frac{1}{k}\right)$. This represents two cases: one where residents of neighborhood 1 are all members of the X population type in which case neighborhood 2 only consists of members of the Y population type and the other where residents of neighborhood 1 are all members of the Y population type, in which case neighborhood 2 only consists of members of the Y population type.

5 Conclusion

In this thesis, I examined Haw and Hogan's single neighborhood and two neighborhood models for two population types, X and Y. The two models demonstrate how individual discriminatory preferences can lead to segregation. I carried out a thorough qualitative analysis of the results obtained using both models for the unlimited numbers case with linear tolerance schedules. For the two neighborhood model I created a corrected mathematical formulation and used this new formulation to obtain results. For both models, I examined the equilibria of the system and whether they correspond to homogeneous or heterogeneous neighborhoods for different parameter values. I determined the equilibria's stability and also examined the system's structural stability. Additionally, I commented on the physical interpretation of the results obtained from using each model.

The results that I obtained from using both models predict that regardless of varying parameters, for instance how high or low the tolerances are for the two population types, there always exists stable equilibria corresponding to homogeneous neighborhoods. The possibility of equilibria corresponding to heterogeneous neighborhoods depends on the parameter values.

In the single neighborhood model, in particular, I found that there are hyperbolic equilibria that correspond to heterogeneous neighborhoods only for specific parameter values such that the discriminant of the cubic equation (9), D, is greater than zero. In this case, the neighborhood may, in the long run, have a heterogeneous composition and its composition may remain as such. For other parameter values, the model predicts that there only exists unstable heterogeneous neighborhoods. The probability of obtaining a heterogeneous composition is even lower than in the previous case. In these cases the neighborhood most often becomes homogeneous in the long run.

Results from the two neighborhood model predict that for all parameter values there exists either a saddle or a sea of equilibria that correspond to heterogeneous neighborhoods. The system is not structurally stable when there exists a sea of equilibria which correspond to heterogeneous neighborhoods. On this basis, I conclude that the model is not suitable for application to the real world in these cases. The system is structurally stable only under certain conditions of parameter values. For these parameter values, the only equilibria of the system are stable equilibria corresponding to homogeneous neighborhoods and a saddle corresponding to heterogeneous neighborhoods. The results then predict that both neighborhoods will always tend to homogeneous compositions in the long run and remain as such.

Both models have areas that require improvement. Through my qualitative analysis, I demonstrated that the two neighborhood model requires thorough revision. The model does not have great predictive power as its systems are not structurally stable for a large range of parameter values. In the two neighborhood model, parameters must meet strict conditions in order for the system to be structurally stable. This restriction limits the freedom to set different parameter values. Consequently, the two neighborhood model, as described by Haw and Hogan (2020), needs to be reconsidered and revised. One of the primary causes of the model's failure to generate structurally stable systems is its assumption that individuals in the neighborhood only consider their own neighborhood's composition when deciding to move out or remain in the neighborhood. While this assumption simplifies the model, it results in the system's differential equations being often zero which leads to instances where the system has a line or an entire region of non-hyperbolic equilibria. Thus, future research should attempt to revise this assumption or possibly find better alternatives to it to improve the model.

My analysis also shows that the single neighborhood model is also not structurally stable, as it particularly has a non-hyperbolic equilibrium at the origin. However, this problem can be remedied by removing a small *neighborhood* around the non-hyperbolic equilibrium for cases where $\beta > 1$. For parameter values $\beta \leq 1$, the problem cannot be remedied, as discussed in Section 2.

Future research should consider the potential effects of the models' assumptions on its predictive powers. It is a question of future research to investigate for what population sizes the assumptions of these two models are justifiable. Both the models take population densities to be continuous. In reality, population density is discrete. It follows that this particular assumption only becomes more justifiable as the population size increases.

Future studies can improve the model by obtaining demographic data with different time points to construct reasonable and realistic tolerance schedules and parameter estimates. One can also consider introducing other factors of segregation and assumptions to the model. For instance, a better assumption might be that individuals compare the composition of their current neighborhood to that of a another neighborhood when making the decision of whether to remain in or leave their current neighborhood.

Lastly, insights from individuals with backgrounds in the social sciences will provide a clearer physical interpretation of the results obtained by using this model. The present findings from the model confirm that there is always a possibility that the neighborhoods becomes homogeneous. One needs to then ask whether a heterogeneous composition is desirable in the first place. As explored by Musterd (2003), it is usually assumed that there exists a negative relationship between physical segregation and social integration (participation in politics). However, integration can sometimes also lead to problems such as the increase of xenophobia Musterd (2003). Much research is still needed to understand which composition has the most benefits, whether the total benefits of elimination of segregation outweighs its costs, whether full integration is really desirable, and in which cases it is or is not.

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