Utrecht University

Ward Identities for the Radiative Jet in QCD

MASTER THESIS

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Author :
S.V. Schrijnder van Velzen

Supervisor :
dhr. prof. dr. E.L.M.P. Laenen

Abstract

Real gluon emission from amplitudes gives rise to logarithms dependent on the kinematic threshold variable. In the soft limit these logarithms become very large, making convergence of perturbative QCD problematic. It is well understood how these leading power logarithms can be resummed. However, the logarithmic effects of next-to-leading power (NLP) in the soft momentum are not. The emission of just one soft photon or gluon can be related to the non-emitting amplitude up to NLP in the soft momentum (Low's Theorem) [1][2]. This leads to the notion of the radiative jet function. Ward identities for radiative jets are an essential tool to work out Low's theorem at NLP. In this thesis we will illustrate the various concepts involved and construct Ward identities for the radiative jet.

Contents

Introduction

The Standard Model is one of the most promising theories in physics. It describes three out of four fundamental forces, namely the electromagnetic, weak and strong force. The theory describing the strong interactions is referred to as Quantum Chromodynamics (QCD). Many of its aspects are successfully tested by particle colliders such as the LHC. With the fast progress in technology, also theoretical predictions need an increasing accuracy. This is mostly done by calculating higher loop corrections to scattering amplitudes in perturbation theory.

When one is doing perturbative QCD, singularities may be encountered when calculating diagrams as a result of an integration over loop momenta or phase space. There are two troublesome regimes, corresponding to the high energy limit and low energy limit. They give rise to ultraviolet (UV) and infrared (IR) singularities respectively. The UV singularities are well under control with renormalization. The IR singularities are known to cancel in inclusive cross sections, known as the KLN theorem. However, they leave behind finite, but potentially large, logarithms. These so called threshold logarithms might spoil the convergence of perturbation theory, since these potentially large contributions appear at any order. The predictive power of perturbation theory can be reinstated when all these logarithmic contributions are resummed.

Threshold logarithms depend on the kinematic variables of the considered process. If we let χ be the dimensionless variable on which these threshold logarithms depend, the perturbative expansion of the cross section then reveals a specific structure of logarithms.

$$\frac{\partial\sigma}{\partial\chi} = \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \sum_{m=0}^{2n-1} \left(c_{nm}^{-1} \left[\frac{\log\chi}{\chi}\right]_+ + c_{nm}^{\delta}\delta(\chi) + c_{nm}^{0}\log\chi^m + \mathcal{O}(\chi) \right)$$

The logarithms with coefficient c_{nm}^{-1} are leading power (LP) terms and the ones with c_{nm}^{0} are next-to-leading power (NLP) terms. The potentially large contributions result from regions close to the kinematic threshold where $\chi = 0$, hence the name.

The LP threshold logarithms are well understood. In recent years several steps were taken towards a deeper understanding of NLP threshold effects as well. It has been known that next-to-soft radiation effects can be understood in terms of the non-radiative amplitude [1] [2]. A general resummation prescription to regulate the NLP logarithms has yet to be developed.

In these thesis we discuss most of the concepts required to study NLP effects in amplitudes. The structure of the thesis is as follows. In the first chapter we start with a short review of Wicks theorem, which we will use when constructing Ward identities for the radiative jet later in the thesis. In chapter 2 we will give an introduction to QCD. The corresponding QCD Lagrangian, including ghost and gauge fixing term, will be constructed from $SU(3)$, the Lie group on which the theory of QCD is constructed. This section will be finished with a discussion of the residual symmetry after gauge fixing, called the BRST symmetry. In chapter 3 we will start with a general discussion of IR divergencies. We will discuss where they arise and how they can be detected in general diagrams. In chapter 4 the Wilson line is discussed. This will play a central role in factorization of amplitudes, which we will discuss in the succeeding chapter. In chapter 6 we will discuss the origin of Ward identities. In the final two chapters we will discuss how Del Duca his extension of Low's theorem [2] gives rise to so-called radiative jets and how we can construct Ward identities for these objects in order to extend Low's theorem to massless particles.

Chapter 1

Wick's Theorem

Calculating correlation functions, a.k.a Greens functions, can be reduced to the problem of calculating expressions of the form

$$\langle 0| \, T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\} \, |0\rangle \, , \tag{1.1}$$

where the time ordered free field operators are evaluated in the interaction picture. The subscript to explicitly denote the interaction-picture fields will be omitted from now on, but keep in mind that correlations are always calculated in this picture. For two fields (1.1) reduces to the Feynman propagator. For a higher number of fields this expression could be calculated by plugging in the expansion of the fields in terms of operators. Wick's theorem provides a way to simplify such calculations immensely. We will first focus on correlations for bosonic fields and then generalize to fermionic fields.

For a time ordered product of two free bosonic fields we can write

$$T(\phi(x)\phi(y)) = N(\phi(x)\phi(y) + \contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y)), \tag{1.2}$$

where the N denotes normal ordering. This ordering puts all the creation operators to the left of the annihilation operators. Normal ordered terms therefore have a vanishing vacuum expectation value. Furthermore the contraction of two fields is defined as follows

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = D_F(x-y). \tag{1.3}$$

where $D_F(x-y)$ is the Feynman propagator for the field ϕ. To see why this is true we look at the operator expansion for bosonic fields given in (A.1). If $x_0 > y_0$ we find

$$T(\phi(x)\phi(y)) \underset{x_0>y_0}{=} N(\phi(x)\phi(y)) + \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_s \sum_{s'} [a_p^s, a_q^{s'\dagger}] \, e^{-ip\cdot x} \, e^{iq\cdot y}, \tag{1.4}$$

$$= N(\phi(x)\phi(y)) + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}, \tag{1.5}$$

and if $x_0 < y_0$ we find

$$T(\phi(x)\phi(y)) \underset{x_0>y_0}{=} N(\phi(x)\phi(y)) + \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_s \sum_{s'} [a_p^s, a_q^{s'\dagger}] \, e^{ip\cdot x} \, e^{-iq\cdot y}, \tag{1.6}$$

$$= N(\phi(x)\phi(y)) + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{ip\cdot(x-y)}. \tag{1.7}$$

This indeed reduces to (1.2). The extension to an arbitrary number of fields is Wick's theorem, which states

$$T(\phi(x_1)\phi(x_2)\cdots\phi(x_n)) = N(\phi(x_1)\phi(x_2)\cdots\phi(x_n) + \text{all possible contractions}). \tag{1.8}$$

When we apply Wicks theorem to (1.1) we see that only terms where all the fields are contracted survive due to the vanishing expectation value for a product of normal ordered fields. So, for an odd number of fields equation (1.1) vanishes because there is always one field that cannot be contracted, or in other words will annihilate the vacuum. Wick's Theorem therefore allows us to write every expression in the form of (1.1) as a sum of products of Feynman propagators. This can be shortly stated as

$$\langle 0|\, T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\}\,|0\rangle = \text{sum of all possible full contractions}. \tag{1.9}$$

Of course (1.8) also applies to fermionic fields. The only difference is that we define the Wick contraction of two fermion fields as

$$\contraction{}{\psi}{(x)}{\bar\psi}\psi(x)\bar{\psi}(y) = S_F(x-y), \tag{1.10}$$

to take care of the anti-commuting nature of the fields. Here, $S_F(x-y$ is the Feynman propagator for fermion fields.

Chapter 2

Quantum Chromo Dynamics

Quantum Chromodynamics (QCD) is the theory of the strong interaction. It is a non-abelian gauge theory based on the gauge group $SU(3)$. The fource-carriers in the theory are the gluons, which transmit the strong force between the quarks.

The Lagrangian describing the theory looks deceptively simple, but QCD is a difficult theory to fully understand. The theory contains self-interactions between gluons, the force-carriers of the theory. These self-interactions give rise to features different from Quantum Elecrodynamics (QED) and exclusive to QCD. One of the most intruiging properties of QCD is *asymptotic freedom*. This phenomena describes that the coupling constant binding the quarks gets smaller when moving them closer together, i.e. at higher energy scales. This energy scale dependence of the coupling constant is a result of the renormalization procedure. Asymptotic freedom allows us to do perturbative calculations when the momentum transfer is large.

2.1 Non-abelian gauge theories

As already mentioned, QCD is based on the gauge group $SU(3)$. The Lagrangian $\mathcal{L}_{\text{QCD}}$ can be explicitly constructed by requiring local $SU(3)$ invariance on the Dirac Lagrangian.

Let us first start with a general discussion of the $SU(N)$ gauge group. The group $SU(N)$ has $N^2 - 1$ generators t^a which form the basis of the corresponding algebra in the fundamental representation. These generators are $N \times N$ (hermitan and unit determinant) matrices that close under commutation in the following way:

$$[t_a, t_b] = i f_{ab}{}^{c} t_c, \tag{2.1}$$

where factors f_{abc} are called the structure constants. Elements $U \in SU(N)$ are related to the algebra by means of an exponential map, such that

$$U = \exp\left[-i\theta^a t^a\right]. \tag{2.2}$$

A quantity ψ transforming in the fundamental representation of $SU(N)$ transforms as

$$\psi_i'(x) = U_{ij}\psi_j(x). \tag{2.3}$$

The Dirac Lagragrangian, as given in equation (2.4), is invariant under global $SU(N)$ transformations of the fundamental fields ψ.

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + m\right)\psi. \tag{2.4}$$

Note that ψ is an N dimensional vector in gauge space and a four-spinor in Dirac space. $SU(N)$ is an exact symmetry of the Dirac Lagrangian. We can upgrade the theory to a gauge theory by requiring local $SU(N)$ invariance, i.e. invariant under gauge transformations. In order to have gauge invariance it is useful to define the covariant derivative

$$D_\mu \psi = (\partial_\mu + igA_\mu)\psi, \tag{2.5}$$

where the introduced gauge fields A_μ are Lie-algebra valued ($= A_\mu^a t_a$) and transform in the adjoint representation of $SU(N)$,

$$A'_\mu = UA_\mu U^{-1} + \frac{i}{g}\partial_\mu U \cdot U^{-1}. \tag{2.6}$$

For infinitesimal transformations this leads to

$$(A_\mu)' = A_\mu - \frac{1}{g}D_\mu\theta + \mathcal{O}(\theta^2) \tag{2.7}$$

where the covariant derivative of a field transforming in the adjoint representation is given by

$$D_\mu\theta = \partial_\mu\theta + ig[A_\mu, \theta], \tag{2.8}$$

Up to now we used the Lie-algebra valued notation. Exploiting the dependence on the generators we find

$$\begin{aligned}(A_\mu^a)' &= A_\mu^a + \frac{1}{g}D_\mu^{ab}\theta^b + \mathcal{O}(\theta^2). \\ &= A_\mu^a + f_{bc}{}^a\theta^b A_\mu^c + \frac{1}{g}\partial_\mu\theta^a + \mathcal{O}(\theta^2),\end{aligned} \tag{2.9}$$

Replacing the normal derivative in equation (2.4) by the covariant derivative makes the Lagrangian invariant under local $SU(N)$ transformations. But we are not done yet. By introducing gauge fields to the theory we can construct another term that is invariant under gauge transformations. Namely, the one governing the kinematics of the gauge fields. In analogy with QED this term reads

$$-\frac{1}{4}\sum_a F_{\mu\nu}^a F^{\mu\nu a}. \tag{2.10}$$

The field strength $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = -[D_\mu, D_\nu], \tag{2.11}$$

and transforms in the adjoint representation of $SU(N)$.

Now that we have established all the building bocks to construct a gauge theory for fermions, let us apply this to $SU(3)$. The generators of $SU(3)$ are

$$t_a = \frac{1}{2}\lambda_a, \tag{2.12}$$

where λ_a are the eight Gell-Mann matrices. Following the discussion for $SU(N)$ above, the gauge invariant QCD Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Gauge Fields}} = \sum_f \bar{\psi}_f(i\not{D} - m_f)\psi_f - \frac{1}{4}\sum_a F_{\mu\nu}{}^a F^{\mu\nu a}, \tag{2.13}$$

where

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu. \tag{2.14}$$

The ψ_f are the Dirac spinors corresponding to the quark fields. There are six types of quarks, called flavors: up, down, strange, charm, beauty and top[1]. All flavours have their own mass m_f. Furthermore, quarks appear in three colors, related to the dimensions of the $SU(3)$ generators. Traditionally, these dimensions are not explicitly shown in the Lagrangian, but one should remember that each quark has an implicit color index $i = 1, 2, 3$. The anti-commuting gauge fields in the theorie are called gluons. They transform in the adjoint representation of $SU(3)$, which is 8 dimensional and therefore give rise to 8 different types of gluons.

2.2 Gauge fixing the QCD Lagrangian

In this section we will review how the gluon propagator can be extracted from the Lagrangian using standard quantum field theory methods. The kinetic term for the gluons is given by $\mathcal{L}_{\text{Gauge Fields}}$ in equation (2.13). In path integral formalism one defines the functional integral

$$\int DA\, e^{iS[A]} = \int DA \; e^{i\int dx^4 \, \frac{-1}{4}\, (F^a_{\mu\nu})^2}, \tag{2.15}$$

where the measure reads $DA = \prod\limits_{a,\mu} DA^a_\mu$. Since the matrix is singular there is no solution for the Feynman propagators of each gauge field. This has to do with the gauge invariance present in $\mathcal{L}_{\text{Gauge Fields}}$. We would like to fix the redundant degrees of freedom such that each physical configuration is only accounted for once. This can be done using the Fadeev Popov trick. Let $G(A)$ be some function that will define the gauge fixing. We would then constrain the functional integral to cover only configurations for which $G(A) = 0$ by inserting a functional delta function, $\delta(G(A))$. The trick now is to insert the identity

$$\mathbb{1} = \int D\xi \;\delta(G(A^\xi)) \;\det\left(\frac{\delta G(A^\xi)}{\delta\xi}\right), \tag{2.16}$$

where ${A^\xi}_\mu$ are the transformed gauge fields,

$${A^\xi}_\mu = A_\mu - \frac{1}{g} D_\mu \xi. \tag{2.17}$$

The functional integral given in equation (2.15) then reads

$$\int DA\, e^{iS[A]} = \int DA \int D\xi \;\delta(G(A^\xi)) \;\det\left(\frac{\delta G(A^\xi)}{\delta\xi}\right) e^{iS[A]}, \tag{2.18}$$

where the delta function must be seen as a product of delta functions for all gauge fields. We choose our gauge fixing condition $G(A)$ in such a way that $\det(\delta G(A^\xi)/\delta\xi$ is independent of ξ and A_μ. For example, choosing the Lorentz gauge ($\partial \cdot A = 0$) corresponds to $G(A^\xi) = \partial^\mu A_\mu + (1/g)\partial_\mu D^\mu \xi$, so that the functional determinant is equal to $\det(\partial_\mu D^\mu/g)$. To continue we interchange the order of the functional integrals and then change $A \to A^\xi$. The measure remains unchanged since inside the functional integral over ξ the transformation is

[1] These quark flavours can be grouped into doublets (constituents of which are said to be in the same generation) transforming under $SU(2)$, which is also part of the standard model. There is no evidence for more than 3 generations.

nothing more than a shift followed by a unitary rotation between the different gauge fields. Furthermore we know that the Lagrangian is invariant under such gauge transformations, and thus

$$\int DA\, e^{iS[A]} = \det\left(\frac{\delta G(A^{\xi})}{\delta \xi}\right) \int D\xi \int DA\ \delta(G(A))\, e^{iS[A]}. \tag{2.19}$$

where we have changed the variable A^{ξ} back to A inside the functional integral since it is only a dummy now. The integral over the gauge fields is now restricted to physical modes through the delta function. The integral over ξ contains all the infinities, but since it is just a multiplicative factor it will be cancelled out in correlation functions. To clean up nicely we will choose the general class of gauge fixing conditions given by $G(A) = \partial_\mu A^\mu(x) - \omega(x)$. Since equation (2.19) is valid for each choice of $\omega(x)$, we can choose a properly normalized linear combination of different $\omega(x)$. The trick now continues by choosing a normalized Gaussian weighting function centred around $\omega = 0$.

$$\int DA\, e^{iS[A]} = \det\left(\frac{\delta G(A^{\xi})}{\delta \xi}\right) N(\lambda) \int D\omega \int D\xi \int DA\ \delta(\partial_\mu A^\mu(x) - \omega(x)) e^{-\frac{i\lambda\omega^2}{2}}\ e^{iS[A]}. \tag{2.20}$$

Note that there are as many Gaussian functions as there are gauge fields, $\int D\omega = \prod_a \int D\omega^a$. The normalisation factor $N(\lambda)$ is again unimportant since it will drop out in correlation functions. Performing the functional integral over ω results in

$$\int DA\, e^{iS[A]} = \det\left(\frac{\delta G(A^{\xi})}{\delta \xi}\right) N(\lambda) \int D\xi \int DA\, e^{-\frac{i\lambda(\partial_\mu A^\mu)^2}{2}}\ e^{iS[A]}. \tag{2.21}$$

We effectively introduced a new term in our Lagrangian, referred to as the gauge fixing term,

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2}\lambda(\partial_\mu A^\mu)^2. \tag{2.22}$$

From this we can extract the propagator using Fourier transformation. The equation to find the propagator reads

$$(-k^2 g_{\mu\nu} + (1-\lambda)k_\mu k_\nu)\delta^{ab}\Delta^{\nu\rho}_{bc} = i\delta_\mu{}^\rho \delta_c{}^a, \tag{2.23}$$

such that

$$\Delta^{\mu\nu}_{ab}(k) = \frac{-i\,\delta_{ab}}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\lambda^{-1})\frac{k^\mu k^\nu}{k^2}\right). \tag{2.24}$$

As one can see the gauge field is massless, since the propagator has a pole at $p^2 = 0$. Furthermore different choices of λ imply different gauge fixing conditions, $\lambda = 1$ is called the Feynman gauge and $\lambda = \infty$ the Landau gauge.

2.3 Appearance of Ghost Fields

We can use our knowledge of anti-commuting fields to rewrite the determinant part in (2.21)

$$\det\left(\frac{1}{g}\partial_\mu D^\mu\right) = \int D\bar{c} \int Dc\, e^{i\int dx^4\ \bar{c}\,(-\partial^\mu D_\mu)\,c}, \tag{2.25}$$

where we have rescaled the anti-commuting fields c and $\bar{c}$ such that the coupling constant g is absorbed. The fields we introduced are generally known as ghost fields. From equation (2.25) we see that we effectively added

$$\mathcal{L}_{\text{ghost}} = \partial_\mu \bar{c}^a \partial^\mu c^a + g f_{abc} (\partial^\mu \bar{c}^a) c^b A^c_\mu, \tag{2.26}$$

to the Lagrangian. From this contribution we can extract the ghost propagator and the coupling to gauge fields, which after Fourier transforming yield

$$\begin{aligned} D_{ab}(p) &= \frac{i\delta_{ab}}{p^2 + i\epsilon}, \\ \mathcal{L}_{\text{int ghost}} &= -g f^{abc} p^\mu \bar{c}^a(p) c^b A^c_\mu. \end{aligned} \tag{2.27}$$

The full gauge fixed QCD Lagrangian now reads

$$\mathcal{L}_{\text{QCD}} = \frac{-1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi - \frac{\xi}{2}(\partial \cdot A)^2 - \bar{c}^a \partial_\mu \partial^\mu c^a - g\bar{c}^a \partial_\mu (f_{abc} c^b A^c_\mu), \tag{2.28}$$

where λ still depends on which gauge is chosen; for example $\lambda = 1$ is known as the Feynman gauge. The corresponding Feynman rules are given in Appendix B

2.4 BRST symmetry

In earlier sections we used the Fadeev-Popov prescription to gauge fix our non-abelian theory. As a result we got a gauge fixing term and had to introduce ghost fields. The resulting gauge-fixed Lagrangian, given in equation (2.28), is not invariant under gauge transformations, which is obviously due to the gauge fixing term. However, Becchi, Rouet, Stora and Tyutin showed that the transformations given in equation (2.29) describe a remaining global symmetry in the theory [3][4]

$$\begin{aligned} \delta A^a_\mu &= D_\mu c^a \, \theta, \\ \delta \psi &= igc\psi \, \theta, \\ \delta \bar{\psi} &= ig\bar{\psi} \, c \, \theta, \\ \delta c^a &= -\frac{1}{2} g f_{abc} \, c^b c^c \, \theta, \\ \delta \bar{c}^a &= \xi \, \partial \cdot A^a \, \theta, \end{aligned} \tag{2.29}$$

where θ is an anti-commuting parameter (independent of spacetime). To validate this statement we will show that the infinitesecimal transformations from equation (2.29) will leave the Lagrangian given in (2.28) invariant. The first term in the Lagrangian is invariant by construction, since the BRST transformation of the gauge fields is the same as (2.7) with $\theta^a \to gc^a\theta$. The same holds for second term, the one including the fermionic fields. The gauge fixing term however is not invariant by itself but yields under variation

$$\delta \mathcal{L}_{\text{gf}} = -\xi \, \partial \cdot A \, \partial^\mu [(D_\mu c) \, \theta]. \tag{2.30}$$

Lastly we look at the variation of the ghost field contribution to the Lagrangian.

$$\delta \mathcal{L}_{\text{ghost}} = -\xi \partial_\mu (\partial \cdot A) D^\mu c \, \theta - \frac{1}{2} g^2 f_{abc} f_{bde} \partial^\mu \bar{c}^a c^d c^e A^c_\mu \, \theta + g^2 f_{abc} f_{cde} \partial^\mu \bar{c}^a c^b c^d A^e_\mu \, \theta. \tag{2.31}$$

The last two terms cancel each other due to the Jacobi identity. So the variation of the total Lagrangian is therefore given by a total derivative, which does not change the kinematics of the system and can therefore be discarded. The fact that the Lagrangian is invariant under the BRST-transformations means that there exists a conserved current. Using Noethers theorem we find that

$$j^{\mu}_{\text{BRST}} = -\xi(\partial \cdot A)D^{\mu}c - \frac{1}{2}g f_{abc}\partial_{\mu}\bar{c}^{a}c^{b}c^{c} - F^{\mu\nu a}D_{\nu}c^{a} + g\bar{\psi}\gamma^{\mu}t_{a}\psi c^{a}. \tag{2.32}$$

To support the claim that this is a conserved current we can explicitly calculate $\partial \cdot j_{BRST}$. This calculation will also be used in our discussion of the radiative jet in chapter 8 and therefore useful to discuss in more detail. Explicitly performing the ∂_{μ} operation on the BRST-current we find

$$\begin{aligned}\partial \cdot j_{\text{BRST}} =& -\xi\partial_{\mu}(\partial \cdot A)^{a}D^{\mu}c_{a} - \xi(\partial \cdot A)\partial_{\mu}D^{\mu}c - \frac{1}{2}g f_{abc}\partial^{\mu}(\partial_{\mu}\bar{c}^{a}c^{b}c^{c}) \\ &- \partial_{\mu}F^{\mu\nu a}D_{\nu}c_{a} - F^{\mu\nu a}\partial_{\mu}D_{\nu}c_{a} + \partial_{\mu}(j^{a}_{EM}c_{a}),\end{aligned} \tag{2.33}$$

where $j^{\mu}_{a\,EM} = g\bar{\psi}\gamma^{\mu}t_{a}\psi$. We would like to group everything such that it becomes clear that the $\partial \cdot j_{\text{BRST}}$ vanishes on shell. In order to make this manifest it is convenient to rewrite the expression in terms of the equations of motion given in (2.34).

$$\begin{aligned}O_{\bar{\psi}} &= (\not{\partial} + ig\not{A})\psi, \\ O_{\psi} &= \bar{\psi}(-\not{\partial} + ig\not{A}), \\ O_{\bar{c}} &= -\partial_{\mu}D^{\mu}c, \\ O_{c} &= D_{\mu}\partial^{\mu}\bar{c}, \\ O_{A^{\nu}} &= \partial_{\mu}F^{\mu\nu} + g f_{abc}F^{\mu\nu b}A^{c}_{\mu} - g\bar{\psi}\gamma^{\mu}t_{a}\psi + \xi\partial_{\nu}\partial \cdot A^{a} + g\partial_{\nu}\bar{c}^{b}f_{abc}c^{c}.\end{aligned} \tag{2.34}$$

Another important element during this procedure will be to find a term proportional to $\partial \cdot j_{\text{QCD}}$, where the QCD current is defined through vaiation of the classical action and yields

$$j^{\mu}_{a\,\text{QCD}} = g f_{abc}F^{\mu\nu b}A^{c}_{\nu} + g\bar{\psi}\gamma^{\mu}t_{a}\psi. \tag{2.35}$$

It is most insightful to break the expression in equation (2.33) down to parts and put them in the right format separately before combining them again. Let us first consider the last term in equation (2.33), namely $\partial_{\mu}(g\bar{\psi}\gamma^{\mu}t_{a}\psi c^{a})$.

$$\partial_{\mu}(g\bar{\psi}\gamma^{\mu}t_{a}\psi c_{a}) = \partial_{\mu}(g\bar{\psi}\gamma^{\mu}t_{a}\psi)c_{a} + g\bar{\psi}\gamma^{\mu}t_{a}\psi\,\partial_{\mu}c_{a}, \tag{2.36}$$

where the first term will be part of the $\partial \cdot j_{\text{QCD}}$ and the second will disappear in the e.o.m for the gauge field. Next we turn to $-F^{\mu\nu a}\partial_{\mu}D_{\nu}c^{a}$. First of all the antisymmetry of $F_{\mu\nu}$ can be used to write

$$\begin{aligned}-F^{\mu\nu a}\partial_{\mu}D_{\nu}c^{a} &= \cancel{-F^{\mu\nu a}\partial_{\mu}\partial_{\nu}c^{a}} - F^{\mu\nu a}\partial_{\mu}(g f_{abc}c^{b}A^{c}_{\nu}) \\ &= -F^{\mu\nu a}\partial_{\mu}(g\,f_{abc}A^{c}_{\nu})c^{b} - F^{\mu\nu a}(g\,f_{abc}A^{c}_{\nu})\,\partial_{\mu}c^{b} \\ &= -F^{\mu\nu c}\partial_{\mu}(g\,f_{abc}A^{b}_{\nu})c^{a} - F^{\mu\nu c}(g\,f_{abc}A^{b}_{\nu})\,\partial_{\mu}c^{a}.\end{aligned} \tag{2.37}$$

Now we look at $-\partial_{\mu}F^{\mu\nu a}D_{\nu}c_{a}$. We find

$$-\partial_{\mu}F^{\mu\nu a}D_{\nu}c_{a} = -\partial_{\mu}F^{\mu\nu a}\partial_{\nu}c_{a} - \partial_{\mu}F^{\mu\nu c}f_{abc}A^{b}_{\nu}c^{a}. \tag{2.38}$$

Combining this with equation (2.37) we arive at the following result

$$-F^{\mu\nu a}\partial_{\mu}D_{\nu}c^{a} - \partial_{\mu}F^{\mu\nu a}D_{\nu}c_{a} = \partial_{\mu}(-g\,f_{abc}F^{\mu\nu c}A^{b}_{\nu})\,c^{a} - (g\,f_{abc}F^{\mu\nu c}A^{b}_{\nu} + \partial_{\nu}F^{\nu\mu a})\partial_{\mu}c_{a}, \tag{2.39}$$

where the first term will be part of $\partial \cdot j_{\text{QCD}}$ and the second will disappear in the equations of motion for the gauge field. Next we look at the third term in $\partial \cdot j_{\text{BRST}}$ from (2.33) and explicitly perform the partial derivative

$$-\frac{1}{2}gf_{abc}\partial^\mu(\partial_\mu \bar{c}^a c^b c^c) = -\frac{1}{2}gf_{abc}\partial^\mu\partial_\mu \bar{c}^a c^b c^c - gf_{abc}\partial_\mu \bar{c}^a c^b \partial^\mu c^c. \tag{2.40}$$

In the first term on the RHS the equation of motion for $\bar{c}$ can be substituted, to yield

$$(2.40) = -\frac{1}{2}gf_{abc}c^b c^c O_{c^a} + \frac{1}{2}g^2 f_{abc}f_{ade}\partial^\mu \bar{c}^d c^b c^c A^e_\mu - gf_{abc}\partial_\mu \bar{c}^a c^b \partial^\mu c^c. \tag{2.41}$$

Using the three identities given in equations (2.36), (2.39) and (2.41) we can reduce equation (2.33) to

$$\begin{aligned}\partial \cdot j_{\text{BRST}} =& -\partial_\mu c^a\ O_{A^\mu} + \xi\, \partial \cdot A O_{\bar{c}}\ + \partial \cdot (j^a_{QCD})c_a - \frac{1}{2}gf_{abc}c^b c^c O_{c^a} \\ & - g\xi\partial_\mu(\partial \cdot A)^a f_{abc}c^b A^c_\mu + \frac{1}{2}g^2 f_{abc}f_{ade}\partial^\mu \bar{c}^d c^b c^c A^e_\mu \end{aligned} \tag{2.42}$$

At last, substituting the explicit form of in terms of $\partial \cdot (j^a_{QCD})c_a$ the e.o.m yields

$$\begin{aligned}\partial \cdot j_{\text{BRST}} =& -\partial_\mu c \cdot O^\mu_A + \xi\, \partial \cdot A O_{\bar{c}} - \frac{1}{2}gf_{abc}c^b c^c {O_c}^a \\ & - gf_{abc}A^b_\mu O^{\mu,c}_A c^a + g(O_{\bar{\psi}}t^a\psi - \bar{\psi}t^a O_\psi)c_a \end{aligned} \tag{2.43}$$

This expression clearly vanishes on shell. The reason why The classical QCD current is not conserved on-shell is due to the presence of ghosts and the gauge fixing term in the Lagrangian.

We will hereby finish our discussion of the QCD Lagrangian. The most important results are the Feynman rules, which can be found in appendix B, and the vanishing of the non-abelian BRST-current. In chapter 8 we will come back to the BRST-current and use it to construct the Ward Idenity for the radiative jet.

Chapter 3

Divergences

It is well known that certain Feynman diagrams may contain singularities, when integration over loop momenta is performed. Perhaps the best known singularity is the ultraviolet (UV) divergence, which is associated with the high-momentum limit of the loop momenta. Such divergences can be removed with standard renormalization techniques and will not be discussed here. When one considers the zero mementum limit of loop or external momenta, soft divergences arise. In addition, when finite mementa of on-shell massless particles become proportional, collinear divergences can appear. Soft and collinear singulariteis are collectively referred to as infrared (IR) singularities.

In the next sections we will first explain how these two kinds of infrared divergencies arise when evaluating loop momentum integrals. Then we will establish some criteria for finding singularities that is applicable to any process, which are generally known as the Landau equations. Finally we discuss the degree of divergence for the solutions of the Landau equations.

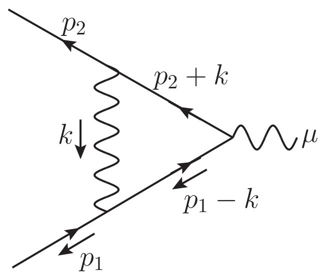


Figure 3.1: The one loop correction to the electormagnetic vertex.

3.1 Soft and Collinear Divergencies

We will start by studying the QCD loop correction to the electromagnetic vertex shown in Figure 3.1. To illustrate where the IR singularities arise we consider the denominators of the internal propagators

$$\frac{1}{((p_2+k)^2+i\epsilon)((p_1-k)^2+i\epsilon)(k^2+i\epsilon)}. \tag{3.1}$$

The singularities will arise when the denominator becomes zero while integrating over all possible loop momenta k. It can be easily seen that this happens when $k \to 0$, which is called the soft singularity. Other possibilities for the denominator to become zero are when either $(p_2+k)^2$ or $(p_1-k)^2$ yield zero. Both of these cases correspond to collinearity of the gluon to one of the external legs. To make this more explicit, let the gluon momentum become collinear such that $k^\mu = zp_2^\mu$ with z not equal to zero. Since p_2 corresponds to an incomming quark it is an on shell momentum. Therefore the internal collinear gluon goes on-shell as well, resulting in a zero of the denominator. To summarize, an infrared singularity is found in either of the following cases:

$$\begin{aligned}
k^\mu &= 0 \qquad \text{(soft singularity)}\\
k^\mu &= zp_1^\mu \quad \text{(collinear singularity)}\\
k^\mu &= z'p_2^\mu \quad \text{(collinear singularity)}
\end{aligned}$$

Evaluating the integral over k using dimensional regularization with $d = 4-2\epsilon$ each of these singularities gives rise to a single ϵ^{-1} pole. Whenever the gluon becomes both collinear and soft a double pole arises, $\sim \epsilon^{-2}$. In the analysis we did before, this would correspond to a collinear gluon $k^\mu = zp_i^\mu$ which we let go soft by taking z to zero. In this way both $(p_i \pm k)^2$ and k^2 vanish, yielding a double pole.

In the analysis above we only considered the denominators. The full integrand however also contains the numerators of the internal propagators among other elements. These should be taken into account when integrating over loop momenta. Hence, in some configurations the singularities we discussed might actually be integrable. Therefore the vanishing of the denominator only corresponds to potential singularities. In the next section we will derive a general machinery which can be used to find these potential singularities in an arbitrary diagram.

3.2 Landau Equations

We start with a general diagram G in a massless theory. Let G have L loops, E external legs and I internal lines. The internal lines can either be quarks or gluons. The internal lines have momenta l_j, with $j = 1, ..., I$. The loop momenta are denoted by k_i, with $i = 1, ..., L$. The corresponding expression for G reads

$$G(p_1, ..., p_E) = (\prod_{i=1}^{L} \int d^d k_i) \mathcal{N}(k_j, p_r) \prod_{j=1}^{I} \frac{1}{(l_j^2 + i\epsilon)}, \tag{3.2}$$

where r can be $1, ..., E$ and all numerators of internal line propagators are grouped in $\mathcal{N}(k_j, p_r)$. Introducing Feynman parameters as explained in appendix D we can rewrite the expression above to

$$(I-1)! \left(\prod_{j=1}^{I} \int_0^1 dx_j \right) \; \delta(1 - x_1 - ...x_I) \left(\prod_{i=1}^{L} \int d^d l_i \right) \; \mathcal{N}(k_j, p_r) \left[\sum_{j=1}^{I} x_j (l_j^2 + i\epsilon) \right]^{-I}. \tag{3.3}$$

Let us now define

$$\mathcal{D}(k_i, p_r, x_i) = \sum_{j=1}^{I} x_j (l_j^2 + i\epsilon). \tag{3.4}$$

Possible IR divergent regions are found when $\mathcal{D}(k_i, p_r, x_i)$ is unavoidably zero. To see what this entails we need to read the integrand as a function of complex integration variables, x_i and l_i^μ. We first refresh how one could avoid singularities by integrating over a single propagator. The poles of a single propagator sit at $k^2 = 0$. In the complex k^0 plane they are distributed according to Figure 3.2. We can use Cauchy's theorem to evaluate the integral by deforming the contour without crossing a singularity, as shown in the same graph. In practice this is accomplished by shifting the poles into the complex k^0 plane using the $i\epsilon$ prescription. In summary, the singularities of the propagator can be 'avoided' by adjusting the contour.

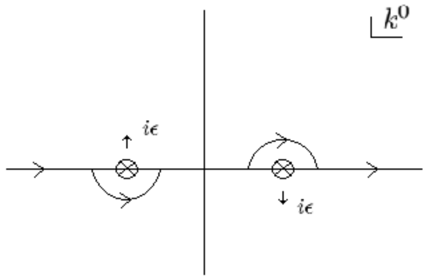


Figure 3.2: Contour integration in the *Feynman prescription* in the k^0 complex plane. The two poles of the Feynman propagator are indicated by the two crosses. The $i\epsilon$ prescrition is introduced to conveniently deal with time ordering.

So, for $\mathcal{D}(k_i, p_r, x_i)$ to be unavoidably zero, the singularities should be such that the countour cannot be deformed to avoid them. In other words, they should be pinched together and are called pinched singularities for that reason. To see wether $\mathcal{D}$ has any pinches we need to take a closer look. $\mathcal{D}$ is quadratic in loop momenta k_i^μ and linear in Feynman paramters x_i. Consider now $\mathcal{D}$ as a kwadratic function of a particular k_i^μ, such that it can be seen as a parabola depicted in Figure 3.3. The solutions for $\mathcal{D}(k_i) = 0$ will be pinched in the complex k_i^μ plane whenever they also solve

$$\frac{\partial \mathcal{D}}{\partial k_i^\mu} = 0, \tag{3.5}$$

which is referred to as the pinching condition. In other words, pinching occurs when two solutions merge into a single point at the extremum of the parabola. The pinching condition needs to be satisfied for all loop momenta k_i^μ in all of their components μ. Indeed, if even a single component does not satisfy the pinching condition, the contour can be deformed in this subspace to avoid the singularities. Inserting equation (3.4) for $\mathcal{D}$ into the pinching condition gives

$$\sum_{j=1}^{I} x_j \frac{\partial l_j^2}{\partial k_i^\mu} = 0. \tag{3.6}$$

Now realize that each line momentum is part of a specific loop, and therefore linear in the corresponding loop momentum. We can therefore make the pinching condition even more explicit by writing

$$\sum_{j=1}^{I} x_j l_j^\mu \sigma_{j,i} = 0, \tag{3.7}$$

where $\sigma_{j,i}$ is $+1$ if the line momentum l_j^μ flows in the same direction as the loop momentum k_i in loop i, -1 if it flows in the opposite direction and zero otherwise[1]. Note that this

[1] $\sigma_{j,i} = 0$ if line momentum l_j^μ does not flow through loop i, but corresponds to a line in another loop

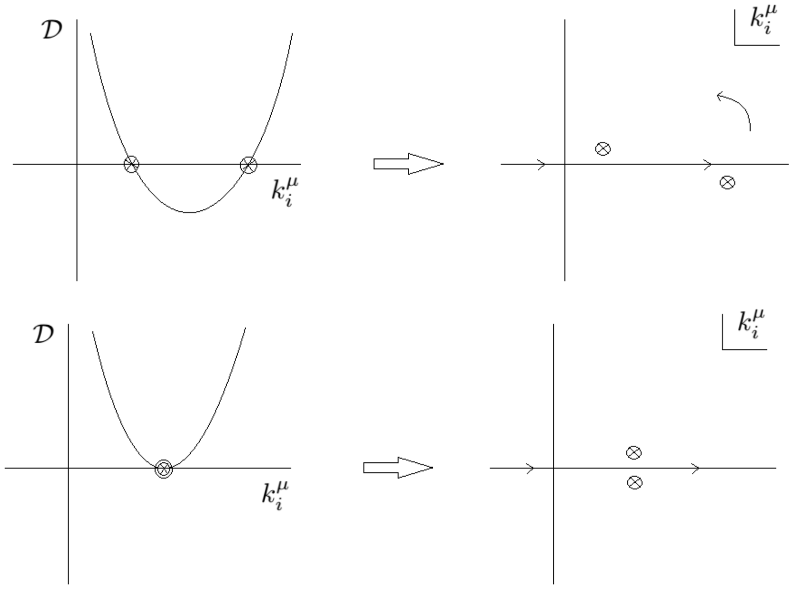


Figure 3.3: In the first row the solutions of the Landau equation $\mathcal{D} = 0$ (left) are shown in the complex k_i^μ plane (right). Since the solutions do not satisfy the pinching condition in equation (3.5), the contour can be deformed to deal with the singularities. In the second row the pinching condition is satisfied, leading to a pinched contour in the k_i^μ complex plane: the poles are distributed in such a way that the contour cannot be deformed away from the singularities.

formula will have as many terms as there are line momenta in loop i, because all other contributions vanish.

All that is left is to determine when $\mathcal{D} = 0$. From equation (3.4) it becomes clear that $\mathcal{D}$ vanishes when all of its terms vanish individually. Each term has the same structure and becomes zero when $x_i = 0$ or $l_i^2 = 0$. Necesarry conditions for finding IR divergences therefore read

$$\boxed{\begin{aligned} & l_j^2 = 0 \;\text{ or }\; x_j = 0 && \forall j, \\ & \sum_{j=1}^{I} x_j l_j^\mu \sigma_{j,i} = 0 && \forall \mu, i. \end{aligned}} \tag{3.8}$$

and are called the Landau eqations [6].

3.3 Pinch Surfaces

Sets of solutions to the Landau equations are often said to cover a surface in k^μ integration space for which $D = 0$. To show how one could use the Landau equations to determine such pinch surfaces we return to our one loop example in figure 3.1. Here

$$\mathcal{D} = x_1(p_2 + k)^2 + x_2(p_1 - k)^2 + x_3 k^2, \tag{3.9}$$

such that the Landau equations read

$$\begin{aligned} & x_1(p_2 + k)^\mu - x_2(p_1 - k)^\mu + x_3 k^\mu = 0, \\ & x_1 = 0 \quad \text{or} \quad (p_2 + k)^2 = 0, \\ & x_2 = 0 \quad \text{or} \quad (p_1 - k)^2 = 0, \\ & x_3 = 0 \quad \text{or} \quad k^2 = 0. \end{aligned} \tag{3.10}$$

There are three solutions to this set equations. One of them describes the soft region

$$k^\mu = 0, \qquad \frac{x_1}{x_3} = 0, \qquad \frac{x_2}{x_3} = 0, \tag{3.11}$$

where we note that $x_3 \neq 0$ means that the gluon is on-shell, $k^2 = 0$. The other two solutions correspond to the collinear pinches

$$\begin{aligned} k^\mu &= zp_1, \qquad x_1 = 0, \qquad x_3 = \frac{1-z}{z}x_2. \\ k^\mu &= z'p_2, \qquad x_2 = 0, \qquad x_3 = -\frac{1+z'}{z'}x_1. \end{aligned} \tag{3.12}$$

These are exactly the results we would expect from our earlier analysis in section (3.1) and there are no other solutions to be found.

In fact, one may wonder how we know that the only solutions to the Landau equations in equation (3.10) are the three mentioned above. The easiest way to see this is by using the Coleman-Norton approach [7]. They developed a technique to track down all pinches for a set of Landau equations, by means of a graphical representation. First they defined a space-time seperation Δx_i^μ between two vertices connected by a propagating momentum p_i by making two identifications

$$\Delta x_i^\mu = x_i l_i^\mu, \qquad x_i = \frac{\Delta_i^0}{l_i^0}. \tag{3.13}$$

From this it becomes clear that Δx_i^μ can be written as

$$\Delta x_i^\mu = \Delta_i^0 v_i^\mu, \qquad \text{with } v_i^\mu = (1, \frac{\vec{l}_i}{l_i^0}). \tag{3.14}$$

This indeed discribes the displacement of a propagating on-shell particle between two spacetime points. The on-shell particles propagate freely along the classical trajectories with velocity v_i^μ. We notice that a line has zero displacement whenever $x_i = 0$. Remember that the condition $\mathcal{D} = 0$ constrains lines with $x_i = 0$ to be off-shell $l_i^2 \neq 0$. The idea now is that the solutions to the Landau equations can be graphically depicted by diagrams where off-shell lines are contracted to a point. Such a diagram is called a reduced diagram. In addition one also has to include the reduced diagrams in which loop momenta become soft, depicted by a dotted line. In the case where the loop momentum goes soft all other lines in the loop are off shell by definition and yield $x_i = 0$. In reduced diagrams the particle character (quark, gluon) is suppressed since one is merely interested in the topological nature of the diagram. Furthermore, in the Coleman-Norton picture all the physical displacements should add up to zero in each loop, reflecting the $\frac{\partial \mathcal{D}}{\partial k_i^\mu} = 0$. In this way there is a corresponding reduced diagram for each singularity and solutions to the Landau equations can now be depicted in terms of them.

To make this point more clear let us go back to our example in which we considered the QCD correction to the decay of a photon into two quarks. We first construct all possible reduced diagrams, shown in figure 3.4. The last diagram belongs to the soft pinch $k^\mu = x_1 = x_2 = 0$. The first diagram belongs to one of the collinear regimes, as does the second. They indicate $x_1 = 0$, $\Delta x_2^\mu = \Delta x_3^\mu$ and $x_2 = 0$, $\Delta x_1^\mu = -\Delta x_3^\mu$, corresponding to the solutions in equation (3.12) respectively. Hereby we have exhousted all solutions, so what about the other reduced diagrams? First of all the diagram where all lines are contracted to a point belongs to the hard part of the diagram (it only contains UV divergences, which we already took care of). The remaining diagrams do not describe propagation of classical

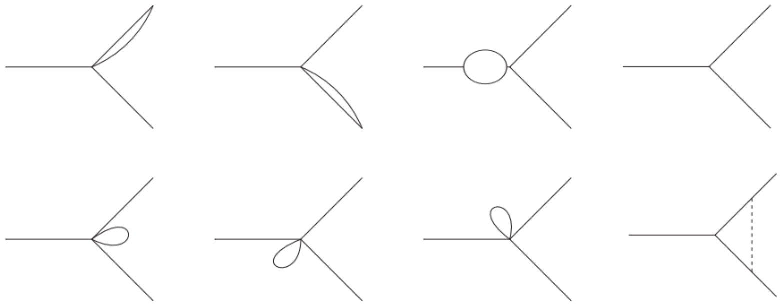


Figure 3.4: Reduced diagrams for the one-loop contribution to a three-point vertex in the Coleman-Norton picture

particles. Classical particles are free, following a free path of propagation. This means that they do not leave and come back to the same point. Neither can two non-collinear particles recombine after a free path. Therefore only classically allowed configurations will lead to pinches and the remaining diagrams should be excluded.

In conclusion, the pinch singularities can be found by considering all possible reduced diagrams in the Coleman-Norten picture and filtering out all the unphysical processes. This is a powerful tool for tracking down pinch surfaces for complicated diagrams.

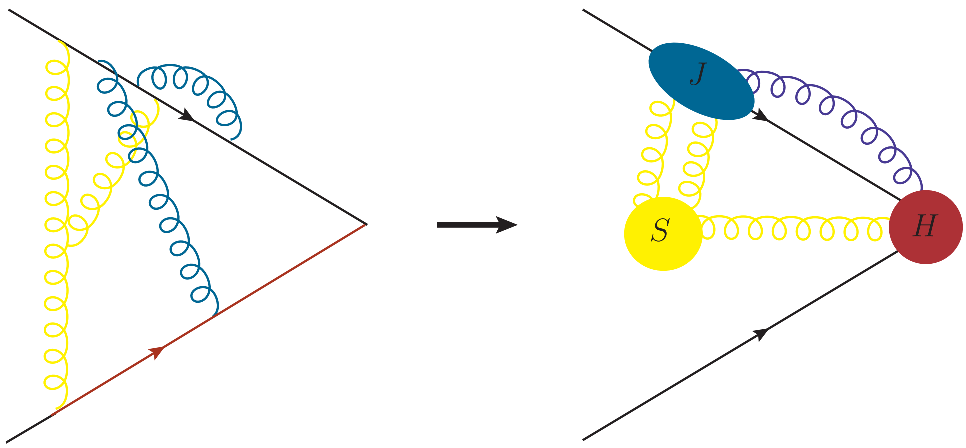


Figure 3.5: A typical two Fermion diagram containing IR singularities. Soft and collinear lines are given a yellow and blue colour respectively. The off-shell lines are marked in red. In the Coleman-Norton picture the red lines are shrunk to a point and are reffered to as the the hard vertex. Furthermore, the soft lines make up the 'soft subdiagram' and the collinear lines are grouped together in the 'jet subdiagram'. Here we have shown a specific case, but all possible reduced diagrams can be factorized in this way.

3.4 Power Counting

It turns out that the Landau equations are only nessecary requirements for infrared divergent behaviour, but they are not sufficient. As discussed before, at the integrand level these potential singularities may be balanced by some numerator factor. A technique which is often used to tell whether or not an integral is going to be singular given a specific pinch surface is infrared power counting [8][9].

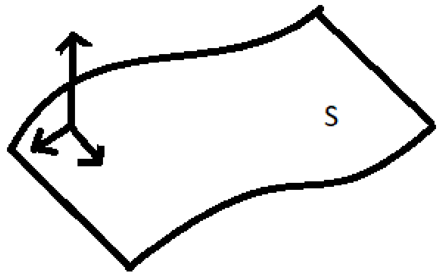


Figure 3.6: Schematic picture of pinch surface S. One normal and two intrinsic coordinates are drawn for illustration.

Consider a general pinch surface in k^μ integration space, schematically depicted in figure 3.6. We can separate the integration variables in two sets, intrinsic coordinates and normal coordinates. The normal coordinates move you towards/from the surface, so these are the ones in which the integral is singular by construction. The intrinsic coordinates parametrize the pinch surface and therefore just move you along the surface. We will scale the normal variables with $k_i^{\text{norm}} = \lambda^{a_i} \tilde{k}_i^{\text{norm}}$, such that the integral will become singular when $\lambda \to 0$. To find the largest degree of divergence we make a perturbative expansion in λ for each denominator, l_i^2 and only retain the leading power term $\sim \lambda^{A_i}$. The resulting integral is called the *homogeneous integral*. The homogeneous integral will have an λ^{n_s} in the integrand. The degree of divergence n_s is given by

$$n_s = \sum_j^N a_j - \sum_i^I A_i + c \tag{3.15}$$

where a_j is the contribution from rescaling the measure for all normal coordinates dk_j^μ, A_j is the contribution from each denominator factor in the homogeneous integral and c is the contribution from possible momentum factors in the numerator. Furthermore I is the number of internal lines and N the number of normal coordinates. The integral will produce a singularity if n_s is non-positive. More specifically: for $n_s = 0$ the integral diverges logarithmically, for $n_s < 0$ the integral diverges as a power and for $n_s > 0$ the integral is finite.

The discussion so far has been very general. It will be insightful to look at the photon decay example in figure 3.1 and define the homogeneous integrals for each pinch surface we found in section 3.3. We start by writing down the corresponding expression to the proces:

$$\begin{aligned}
\Gamma^\mu_{\text{1-loop}} &= -ieg^2 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p_2)(t^a)_{ik} \frac{\gamma^\alpha i(\not{p}_2 + \not{k})\gamma^\mu(-i(\not{p}_1 - \not{k}))\gamma^\beta}{((p_2+k)^2 + i\epsilon)((p_1-k)^2 + i\epsilon)} (t^b)_{kj} v(p_1) \frac{-i\delta_{ab} g_{\alpha\beta}}{(k^2 + i\epsilon)} \\
&= eg^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{N}{D},
\end{aligned} \tag{3.16}$$

where we used Feynman gauge and

$$\begin{aligned}
N &= \bar{u}(p_2)\, \gamma^\alpha (\not{p}_2 + \not{k}) \gamma^\mu (\not{p}_1 - \not{k}) \gamma^\beta\, v(p_1)\, g_{\alpha\beta}, \\
D &= D_1\, D_2\, D_3 = (p_2 + k)^2\, (p_1 - k)^2\, k^2.
\end{aligned} \tag{3.17}$$

Furthermore the Casimir operator C_r depends on the representation of the gauge group and is defined through

$$(t_r^a t_r^a)_{ij} = \delta_{ij} C_r. \tag{3.18}$$

We will be working in the the fundamental representation, and the corresponding Casmir operator will be denoted as C_F. We will now introduce lightcone coordinates such that $k^\mu = (k^+, k^-, k_\perp)$, where $k_\perp^2$ is defined by the standard Euclidean product in the dimensions untouched by changing to lightcone coordinates. In this frame the quark momenta can be chosen such that $p_{1\mu} = \delta_{\mu+}\frac{Q}{\sqrt{2}}$ and $p_{2\mu} = \delta_{\mu-}\frac{Q}{\sqrt{2}}$. The denominator D in expresion (3.16) then reduces to

$$\begin{aligned}
D_1 &= 2p_2^- k^+ + 2k^+k^- - k_\perp^2, \\
D_2 &= -2p_1^+ k^- + 2k^+k^- - k_\perp^2, \\
D_3 &= 2k^+k^- - k_\perp^2.
\end{aligned} \tag{3.19}$$

In what follows we will determine the degree of divergence for the soft and collinear pinch surfaces.

- **Soft pinch surface**

Recall that the soft pinch surface has $k^\mu \to 0$ in all components. Since this pinch surface is just a point, it is intuitive that coordinates of k^μ are considered normal coordinates. It can also be checked by noting that the internal gluon must be kept on-shell on this pinch surface and changing any component of $k^]mu$ would violate this requirement. Scaling the normal coordinates by $k^\mu \to \lambda k^\mu$ the denominator terms become

$$\begin{aligned}
D_1 &= \sqrt{Q}k^+\lambda + \mathcal{O}(\lambda^2) \\
D_2 &= -\sqrt{Q}k^-\lambda + \mathcal{O}(\lambda^2) \\
D_3 &= \lambda^2(2k^+k^- - k_\perp^2)
\end{aligned} \tag{3.20}$$

Looking at N we see that with the usage of Dirac equations

$$N = 4p_1 \cdot p_2 \bar{u}(p_2)\gamma^\mu v(p_1) + \mathcal{O}(\lambda). \tag{3.21}$$

Now we can use infrared power counting to find $n_s = 0$, which corresponds to a divergence of logarithmic nature.

- **Collinear pinch surface**

Assume now that we are on the collinear pinch surface $k \mathbin{/\!/} p_1$. Again, we need to find the normal and intrinsic coordinates in order to perform infrared power counting. First we note that the considered pinch surface solves the Landau equations such that the collinear gluon and the internal quark to which it is collinear are on-shell, i.e. $D_2 = D_3 = 0$. The intrinsic coordinates must preserve this, while the normal coordinates will move the corresponding particles off-shell. Looking at equation (3.19) we see that changing $k_\perp$ would make both particles go off-shell and is therefore a normal coordinate. Changing k^- away from zero will make D_2 nonzero, and is therefore also considered a normal coordinate. Furthermore we see that changing k^+ does not bring any of the particles off shell. Namely, for the gluon to be collinear we find $k^- = 0$ and therefore all combinations with k^+ in D_2 and D_3 will not be altered by changing it. Note that changing k^+ will change D_1, but since the corresponding particle was off-shell to begin with we do not mind. We now scale the normal coordinates by $k^\mu \to (1, \lambda, \sqrt{\lambda})$, such that changing any of the normal coordinates has the same inpact. The denominators after rescaling read

$$\begin{aligned}
D_1 &= 2p_2^- k^+ + \mathcal{O}(\lambda), \\
D_2 &= \lambda(-2p_1^+ k^- + 2k^+k^- - k_\perp^2), \\
D_3 &= \lambda(2k^+k^- - k_\perp^2).
\end{aligned} \tag{3.22}$$

We will now focus on the numerator N of equation (3.16). Since $k \,/\!\!/\, p_1$, we can use the Dirac equation such that $(\not{p}_1 - \not{k})v(p_1) = 0$. This will simplify N to

$$N \simeq \bar{u}(p_2)\,(2p_2^\alpha + \gamma^\alpha \not{k})\,\gamma^\mu\; 2(p_1 - k)^\beta\, v(p_1)\, g_{\alpha\beta}, \tag{3.23}$$

where we have also used the anti-commutation of gamma matrices and the Dirac equation for the outgoing fermion $\bar{u}(p_2)\,\not{p}_2 = 0$. We can now simplify N even more. Remember that the only nonzero component of p_1^μ is the $+$ component. The collinear approximation therefore implies

$$k^+ \gg k^-, \sqrt{|k_\perp|^2}. \tag{3.24}$$

As a result of this approximation N reduces to an expression with $\beta = +$. This automatically causes the only remaining term in N to have $\alpha = -$, because this gives the only nonzero component of $g_{\alpha+}$.

$$N \simeq \bar{u}(p_2)\,(2p_2^- + \gamma^- \not{k})\,\gamma^\mu\; 2(p_1 - k)^+\, v(p_1), \tag{3.25}$$

Lastly, the approximation in equation (3.24) allows us to aproximate $\not{k} \simeq k^+\gamma^-$. Notice that the $\not{k}$ term drops out in this approximation due to the fact that $g_{--} = 0$ and therefore $(\gamma^-)^2 = 0$. So finally we write

$$N = \bar{u}(p_2)\,\gamma^\mu\; 4p_1^+ p_2^-\,(1 - \frac{k^+}{p_1^+})\,v(p_1) + \mathcal{O}(\lambda)\,, \tag{3.26}$$

Power counting in normal variabels now tells us that the homogeneous integral exhibits a logarithmic divergence. Investigation of the other collinear pinch surface is quite similar and will lead to a logarithmic divergence as well.

Let us come back to the Coleman-Norton picture of our 1 loop example for a second, see figure 3.4. The hard part of the reduced diagram, corresponding to the collinear divergence $k \,/\!\!/\, p_1$, consists of the quarkline carrying momentum $(p_2 + k)^\mu$ and its adjecent vertices. In the approximation executed above this yields:

$$H_\mu{}^\alpha = H_\mu^- \delta_-^\alpha = -ig^2 \bar{u}(p_2)\frac{\gamma^-(p_2^-\gamma^+)}{2p_2^- k^+}\gamma_\mu \delta_-^\alpha. \tag{3.27}$$

When the collinear photon is contracted with the hard subdiagram, its polarization will be longitudinal, i.e. $\epsilon \sim k$. In chapter 5 we will show how these unphysical polarizations will decouple from the hard part, resulting in a factorization between H and J.

Chapter 4

Wilson Lines

Radiation of soft gluons, which have $k^\mu \to 0$, is very common in scattering processes and often gives rise to divergences. This is because the coupling strength gets large if interactions involve soft particles and propagators diverge. In principle, the soft gluon radiation can consist of an infinite number of gluons. This would make perturbation theory an incorrect tool in calculating physical amplitudes involving soft radiation since higher orders in perturbation theory become more and more significant. In the eikonal approximation, $k \to 0$, however all the soft radiation can be captured in a so-called Wilson line. The Wilson line can be seen as the source of all soft radiation and is represented by a path ordered exponential of gauge fields. The path used in the Wilson line is the path taken by the parton that emits or absorbs the soft gluons.

In the next section we will show how the eikonal approximation gives rise to the Wilson line. The Wilson line will be a useful description when calculating Feynman diagrams with soft photon emission. It will turn out to play a central part in the factorization of amplitudes later in this thesis.

4.1 Eikonal Approximation

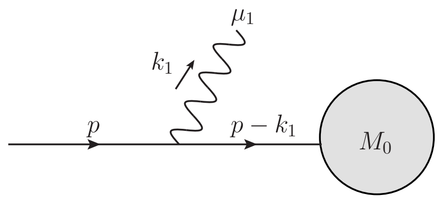


Figure 4.1: Photon emission from an external fermion line.

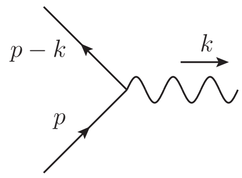
Consider the process in which a fermion emits a photon. The corresponding Feynman diagram is shown in figure 4.1. Using the Feynman rules for QED the corresponding expression reads

$$M^{\mu_1}(p, k_1) = M_0 \, \frac{i(\not{p} - \not{k}_1)}{(p - k_1)^2} \, (-ie\gamma^{\mu_1}) u(p). \tag{4.1}$$

Expanding in soft photon momentum k_1 and only keeping the lowest order gives the eikonal approximation

$$M^{\mu_1} \epsilon_{\mu_1} \approx M_0 \; [- e \frac{p^{\mu_1}}{p \cdot k_1} u(p) \,] \epsilon_{\mu_1}(k_1), \tag{4.2}$$

where we have used the Dirac equation $\not{p}u(p) = 0$. The emitter spin becomes irrelevant in this approximation. This is easily seen by the fact that there are no γ matrices left in the expression. Furthermore the effective coupling of the photon is proportional to the momentum of the emmitting fermion,



$$= \frac{-ep^{\mu}}{p \cdot k}. \tag{4.3}$$

Note that in the case of an absorbed photon the sign of the effective vertex changes due to the sign change of the photon momentum k. Consider now the emission of two photons. The two diagrams that contribute to this proces are given in figure 4.2. The corresponding amplitude is

$$M^{\mu_1\mu_2}(p, k_1, k_2) = M_0 \, \frac{e^2(\not{p} - \not{k_1} - \not{k_2})}{(p - k_1 - k_2)^2} [\frac{\gamma^{\mu_1}(\not{p} - \not{k_2})\gamma^{\mu_2}}{(p - k_2)^2} + \frac{\gamma^{\mu_2}(\not{p} - \not{k_1})\gamma^{\mu_1}}{(p - k_1)^2}]. \tag{4.4}$$

Expanding in soft momentum k_1 and k_2 this expression reduces to

$$M^{\mu_1\mu_2}(p, k_1, k_2) = M_0 \, [\frac{e^2 \; 2p^{\mu_1} \, 2p^{\mu_2}}{-2p \cdot (k_1 + k_2) \, (-2p \cdot k_2)} + \frac{e^2 \; 2p^{\mu_2} \, 2p^{\mu_1}}{-2p \cdot (k_1 + k_2) \, (-2p \cdot k_1)} + \mathcal{O}(1)]. \tag{4.5}$$

This can be further simplified using the eikonal identity

$$\sum_{\pi} \frac{1}{p \cdot k_{\pi_1}} \frac{1}{p \cdot (k_{\pi_1} + k_{\pi_2})} \cdots \frac{1}{p \cdot (k_{\pi_1} + \ldots k_{\pi_n})} = \prod_{i}^{n} \frac{1}{p \cdot k_i}, \tag{4.6}$$

where the sum over π indicates all permutations of the photon momenta, and k_{π_i} is the i^{th} momentum in a given permutation. There are $n!$ permutations, each contributing equally to the amplitude. For $n = 2$ these correspond to the diagrams in figure 4.2. A closer look at equation (4.6) reveals that on the RHS there is no sign of the order of emissions, while on the LHS the order in which the photons are emitted remains apparent in the denominators. The eikonal identity therefore indicates the decorrelation of emissions in amplitudes due to the eikonal approximation. Upon using the eikonal identity in equation

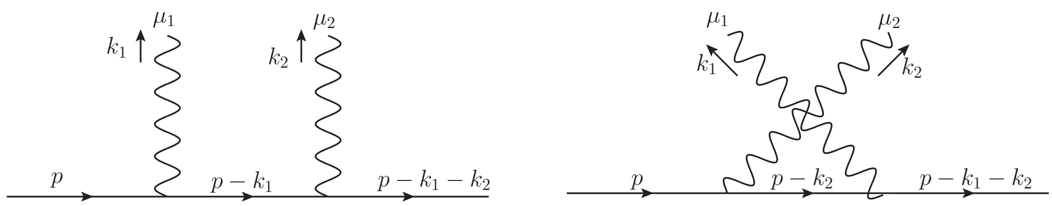


Figure 4.2: Both contributions to the double emission of a photon from a fermion line.

(4.6) the emisson of two soft photons from a fermion line just becomes the product of two eikonal vertices as given in equation (4.3). In practice, each soft photon emission can be expressed by the effective Feynman rule from equation (4.3),

$$M^{\mu_1..\mu_n}(p, k_1, ...k_n) = M_0 \prod_{i=1}^{n} \frac{-ep^{\mu_i}}{p \cdot k_i}. \tag{4.7}$$

It becomes clear that all the soft photon emissions factorize from the amplitude without emissions M_0. From a more formal point of view all the soft radiation can be described by a Wilson line, which we formulate as follows

$$\Phi_n(x_1, x_2) = \exp\big[-ie\int_{x_1}^{x_2} dx^\mu\, A_\mu(x)\big]. \tag{4.8}$$

where the gauge fields A_μ act as sources for radiation of photons. When the radiated gauge bosons are soft, one may neglect the recoil of energetic particles. In this case the path can be parametrized by a straight line $x^\mu = \lambda n^\mu$, where n^μ is the directional vector of the classical path taken by the emitting parton.

To see how the eikonal vertices arise in this prescription let us consider, without loss of generality, an external line created at $x_i = 0$ and in direction $n^\mu = p^\mu$. After a Fourier transform of the gauge fields to momentum space and performing the integral over λ equation becomes

$$\begin{aligned}\Phi_n(0,\infty) &= \exp\big[-ie\int_0^\infty d\lambda \int_{-\infty}^{\infty}\frac{dk^4}{(2\pi)^4}\; n\cdot A(k)e^{-ik\cdot n\lambda}\big]\\ &= \exp\big[\int_{-\infty}^{\infty}\frac{dk^4}{(2\pi)^4}\,\frac{-e\;n\cdot A(k)}{n\cdot k - i\epsilon}\big],\end{aligned} \tag{4.9}$$

where we used the $i\epsilon$ prescription to make sure the integral is finite. The resulting factor in equation (4.9) acts as a source term for the soft gauge field when the path integral over A^μ is performed. An equivalent way to see this fact is by using Wick contraction in the operator formalism. One could expand the Wilson line from equation (4.9) such that

$$\begin{aligned}\Phi_n(0,\infty) = \;& 1 + \int_{-\infty}^{\infty}\frac{dk^4}{(2\pi)^4}\,\frac{-e\;n\cdot A(k)}{n\cdot k - i\epsilon}\\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk^4}{(2\pi)^4}\,\frac{-e\;n\cdot A(k)}{n\cdot k - i\epsilon}\int_{-\infty}^{\infty}\frac{dk'^4}{(2\pi)^4}\,\frac{-e\;n\cdot A(k')}{n\cdot k' - i\epsilon} + \mathcal{O}(e^3).\end{aligned} \tag{4.10}$$

Wick contracting the gauge field(s) with the external particle(s) we find a polarization vector $\epsilon^\mu(k)$ for each contraction times the eikonal vertex. Note that for the second order expansion there are two possible ways one can Wick contract the gauge fields with the external states. The contributions of the two are equal and correspond to the two diagrams in Figure 4.2. Therefore the factor $\frac{1}{2}$ from Taylor expanding will be cancelled and the factorization as described by equation (4.7) is still valid. We have now established that soft emissions affect the hard particle only by dressing it with a gauge phase, which is the Wilson Line in equation (4.8).

4.2 Wilson Lines in QCD

In QCD we can also define the Wilson line. Only here we need some kind of ordering to take care of the non-commuting nature of the gauge fields. Therefore we define the Wilson line using path ordering

$$\Phi_n(\lambda_1,\lambda_2) = \mathcal{P}\exp\big[-ig\int_{\lambda_1}^{\lambda_2} d\lambda\;\; n\cdot A^a(\lambda)t_a\big]. \tag{4.11}$$

We have parametrized the path by a straight line $x^\mu = \lambda n^\mu$, because we assume no recoil of the emitting parton. The path ordering is such that fields with higher values of λ are to the left. This means that if the line momentum travels from left to right the fields should be organized in the opposite direction. As a result, the fields that are emitted further on the path are drawn to the right of the diagram. Therefore we read Wilson lines in a Feynman diagram as we read Dirac lines: in the opposite direction of the line momentum when writing the corresponding formula.

In order to see how this works let us make an expansion of the Wilson line

$$\Phi_n(a,b) = \sum_{m=0}^{\infty} (-ig)^m \frac{1}{m!} \int_a^b (d\lambda_i)^m \; \mathcal{P}\Big(n\cdot A(\lambda_m)\cdots n\cdot A(\lambda_1)\Big), \tag{4.12}$$

where it should be clear that $A(\lambda) = A^a(\lambda)\, t_a$. If we now reduce the product of integrals such that $\lambda_m > \cdots > \lambda_1$ the path ordering of the fields becomes manifest. This can be acomplished by replacing

$$\frac{1}{m!}\int_a^b ... \int_a^b d\lambda_1..d\lambda_m \;\; = \;\; \int_a^b \int_{\lambda_1}^b ... \int_{\lambda_{m-1}}^b d\lambda_1...d\lambda_m \;\; = \;\; \int_a^b \int_a^{\lambda_m} ... \int_a^{\lambda_2} d\lambda_m...d\lambda_1, \tag{4.13}$$

which are both equally useful.

We now investigate a Wilson line created at $x_i^\mu = -\infty$ and anihilated at $x_i^\mu = a^\mu$, in the direction $n^\mu = p^\mu$. These can be used to describe incoming particles emitting soft photons in scattering processes. First fourier transform the gauge fields to momentum space and apply the $i\epsilon$ prescription to make sure the λ integrals are finite. The second order expansion term of the Wilson Line in equation (4.13), corresponding to the double absorbtion of gluons, then reads

$$-g^2 \int_{-\infty}^a d\lambda_2 \int_{-\infty}^{\lambda_1} d\lambda_1 \;\; \int_{-\infty}^{\infty} \frac{d^4k_2}{(2\pi)^4} n\cdot A(k_2) e^{-i(k_2+i\epsilon)\cdot n\,\lambda_2} \;\; \int_{-\infty}^{\infty} \frac{dk_1^4}{(2\pi)^4} n\cdot A(k_1) e^{-i(k_1+i\epsilon)\cdot n\,\lambda_1}. \tag{4.14}$$

Performing the integral over λ_1, followed by the integral over λ_2 we find

$$\begin{aligned}(4.14) &= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} \int_{-\infty}^a d\lambda_2\, n\cdot A(k_2) n\cdot A(k_1) e^{-i(k_1+k_2+i\epsilon)\cdot n\lambda_2} \frac{i}{(k_1\cdot n + i\epsilon)} \\ &= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\, n\cdot A(k_2)\, n\cdot A(k_1) e^{-i(k_1+k_2+i\epsilon)\cdot na} \frac{i}{((k_1+k_2)\cdot n + i\epsilon)} \frac{i}{(k_1\cdot n + i\epsilon)}.\end{aligned} \tag{4.15}$$

As one can see we find the product of two eikonal vertices, but with some ordering of the gluon fields. Doing this to all orders in perturbation theory we find

$$\Phi_n(-\infty,a) = \sum_{m=0}^{\infty} (-ig)^m \int \Big(\frac{d^4k_i}{(2\pi)^4}\Big)^m n\cdot A(k_m)\ldots n\cdot A(k_1) e^{-iK(j)\cdot na} \prod_{j=1}^m \frac{i}{n\cdot K(j) + i\epsilon}, \tag{4.16}$$

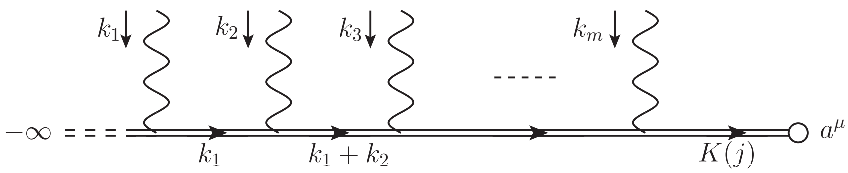


Figure 4.3: Gluon emission from an eikonal line. This graph shows the m-th order expansion term of the Wilson line $\Phi_n(-\infty,a)$. Due to the time ordering the gluon field emissions are ordered from left-to-right.

where $K(j) = \sum_{i=j}^{m} k_i$. The left-to-right ordering of the fields then corresponds to the emission of gluons as depicted in figure 4.3. We can also assume a Wilson line created at $x_i^\mu = a^\mu$ traveling to infinty, as depicted diagramaticcally in figure 4.4. A similar calculation as we did above then results in

$$\Phi_n(a,\infty) = \sum_{m=0}^{\infty} (-ig)^m \int \left(\frac{d^4k_i}{(2\pi)^4}\right)^m n \cdot A(k_m) \dots n \cdot A(k_1) e^{-iK(j)\cdot na} \prod_{j=1}^{m} \frac{-i}{n \cdot \tilde{K}(j) - i\epsilon}, \tag{4.17}$$

where $K(j) = \sum_{i=j}^{m} k_i$ and $\tilde{K}(j) = \sum_{i=j}^{m} k_{m-i+1}$.

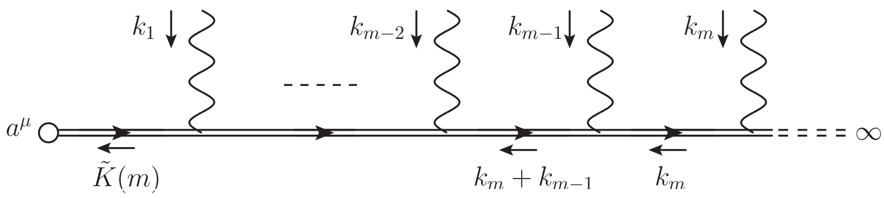


Figure 4.4: Gluon emission from an eikonal line. This graph shows the m-th order expansion term of the Wilson line $\Phi_n(a,\infty)$. Due to the time ordering the gluon field emissions are ordered from left-to-right.

From these calculations we can extract the Feyman rules for a Wilson line, which are depicted in figure 4.5. Feynman rules for oposite line momenta are also given and can be found by the complex conjugate of the Wilson line. This will become clear when looking at the properties of the Wilson line in the next section. At this point it is important to realize that when drawing Wilson lines with the rules we just established, the momenta of the gluons always point inwards (towards the vertex). For an outgoing gluon one just simply makes the substitution $k_i \to -k_i$.

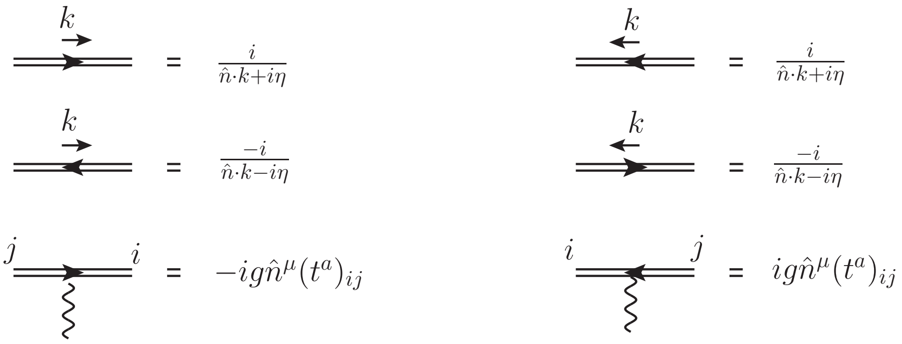


Figure 4.5: Feynman rules for Wilson lines

To conclude this section we also give the expression for the finite Wilson line in equation (4.18).

$$\Phi_n(a,b) = \sum_{m=0}^{\infty} (-ig)^m \int \left(\frac{d^4k_i}{(2\pi)^4}\right)^m n\cdot A(k_m) \cdots n \cdot A(k_1) \sum_{l=0}^{m} e^{-ia\cdot K(l)} e^{-ib\cdot \tilde{K}(m-l)} \times \prod_{j=1}^{l} \frac{-i}{n \cdot \tilde{K}(j)} \prod_{j=l+1}^{m} \frac{i}{n \cdot K(j)}. \tag{4.18}$$

4.3 Properties of the Wilson line

Now we are ready to look at some characteristics of the Wilson line and why it turns out to be a useful tool. First of all we will look at what happends when performing a gauge transformation, as defined in equation (2.6). It turns out that the Wilson line transforms as [5]

$$\Phi_n(\lambda_1, \lambda_2) \to U(\lambda_2)\Phi_n(\lambda_1, \lambda_2)U^{-1}(\lambda_1). \tag{4.19}$$

This feature will be very convenient to describe gauge invariant amplitudes. Consider the bi-local product of two matter fields

$$\Delta(x, y) = \bar{\psi}(y)\phi(x). \tag{4.20}$$

These products are common objects in quantum field theory and often appear in correlation functions. In particular, they define the Green's functions via

$$\langle 0| \mathcal{T}\bar{\psi}(y)\psi(x) |0\rangle\,. \tag{4.21}$$

where $\mathcal{T}$ denotes the time-ordering. Obviously this expression is not gauge invariant, since

$$\Delta(x, y) \to \bar{\psi}(y)U^{\dagger}(y)U(x)\psi(x) \tag{4.22}$$

This expression can be made gauge invariant by insertion of the Wilson line between the two field operators. By doing so the Wilson line then transports the gauge dependence of $\psi(x)$ from point x to y, where it is cancelled by the gauge dependence of $\bar{\psi}(y)$. One more thing one could do with the Wilson line is transport the gauge dependence to infinity, where it is set to unity.

Furthermore the hermitian conjugate of the Wilson line just reverses path ordering and can therefore be interpreted as the same Wilson line with opposite directional vector n. To see how this comes about we investigate what changes when we reverse the path of the Wilson line. Firstly, we now integrate from b^μ to a^μ. This is similar to integrating from a to b, but changing the sign of the exponent. The most important thing is however that now the order of the fields must be reversed. Since the path flows from right to left, the fields should be organised in the opposite direction. Fields that are first on the path will be encountered last when following the reversed path flow. This idea of antipath-ordering is defined such that fields with highes value for λ are written rightmost and we denote this ordering by $\bar{\mathcal{P}}$. The reversed wilson line is thus given by

$$\bar{\Phi}_n(b, a) = \bar{\mathcal{P}} e^{ig\int_a^b d\lambda n\cdot A} \tag{4.23}$$

where n^μ still points in the direction of a left to right path flow and bar indicates the reversed pathflow. Equation (4.23) is actually the same as the hermitian conjugate of a Wilson line from a^μ to b^μ with normal path ordering for a path from left to right as defined in the previous section. Hermitian conjugation namely also reverses the order of fields, since $(A(\lambda_m)\cdots A(\lambda_1))^\dagger = A(\lambda_1)\cdots A(\lambda_m)$ [1], and in addition it changes the sign of the exponent. We therefore find

$$\bar{\Phi}_n(b, a) = \Phi_n^\dagger(a, b) \tag{4.24}$$

It would be more convenient to express the Hermitian conjugate Wilson line as a function of normal path-ordered fields, such that the Feynman rules we made in the last

[1] remember that $A(x)$ is real

section still apply. To get there we will Hermitian conjugate the wilson line from $-\infty$ to a^μ:

$$\begin{aligned}\Phi_n^\dagger(-\infty,a) &= \Big[\sum_{m=0}^{\infty}(-ig)^m\int\Big(\frac{d^4k_i}{(2\pi)^4}\Big)^m n\cdot A(k_m)\dots n\cdot A(k_1)e^{-iK(j)\cdot na}\prod_{j=1}^{m}\frac{i}{n\cdot K(j)+i\epsilon}\Big]^\dagger\\ &= \sum_{m=0}^{\infty}(ig)^m\int\Big(\frac{d^4k_i}{(2\pi)^4}\Big)^m n\cdot A^\dagger(k_1)\dots n\cdot A^\dagger(k_m)e^{iK(j)\cdot na}\prod_{j=1}^{m}\frac{-i}{n\cdot K(j)-i\epsilon}\end{aligned} \tag{4.25}$$

Because $A(x)$ is a real field, $A^\dagger(k)=A(-k)$. We therefore make the substitution $k\to -k$. In addition we relabel the fields by

$$k_1\to k_n, k_2\to k_{n-1},\cdots,k_n\to k_1,$$

which gives

$$\Phi_n^\dagger(-\infty,a) = \sum_{m=0}^{\infty}(ig)^m\int\Big(\frac{d^4k_i}{(2\pi)^4}\Big)^m n\cdot A(k_m)\dots n\cdot A(k_1)e^{-iK(j)\cdot na}\prod_{j=1}^{m}\frac{i}{n\cdot K(j)+i\epsilon} \tag{4.26}$$

Here we recognise the Wilson line from a^μ to infinity, but with the substitution $n^\mu\to -n^\mu$. So we conclude that the direction of the wilson line is reversed. This can be interpreted as a Wilson line with anti-path ordering, but is more usefull as a Wilson line with normal path ordering but with opposite directional vector $n^\mu\to -n^\mu$. The same calculation can be done for $\Phi_n^\dagger(a,\infty)$. In summary one finds

$$\begin{aligned}\Phi_n^\dagger(-\infty,a) &= \Phi_{-n}(a,\infty),\\ \Phi_n^\dagger(a,\infty) &= \Phi_{-n}(-\infty,a),\end{aligned} \tag{4.27}$$

which diagramatically is depicted in Figure 4.6. The details of the hermitian conjugate of

Figure 4.6: Diagramatic explaination for the hermitan conjugate of infinite Wilson lines

a finite Wilson line can be found in Appendix F.2. This is less important for our purposes, since scattering of radiative fermions, and is therefore just added for completeness. It turns out that $\Phi_n(a,b)^\dagger=\Phi_{-n}(b,a)$, which is diagrammatically depicted in the Appendix as well.

Lastly, due to the path ordering we can glue two Wilson lines together. If we assume a Wilson line from λ_1 to λ_2 we can extend the line to λ_3 by adding another Wilson line from λ_2 to λ_3 in the following way

$$\Phi_n(\lambda_2,\lambda_3)\Phi_n(\lambda_1,\lambda_2)=\Phi_n(\lambda_1,\lambda_3) \tag{4.28}$$

In the next chapter, Wilson lines will appear in our attempt to factorize amplitudes into smaller building blocks.

Chapter 5

Factorization

In general, the singular diagrams found by solving the Landau equations and surviving the power counting can be represented as reduced diagrams in the Coleman Norton picture, as shown in figure 5.1. In this compact representation of singular diagrams, S is the subdiagram involving all on-shell soft lines. Note that the S does not have to be connected by itself. The jet subdiagrams, J_1 and J_2, are the subdiagrams involving all on-shell collinear lines. Lastly, the hard vertex H collects all the off-shell lines, which in the Coleman-Norton picture are shrunk to a point.

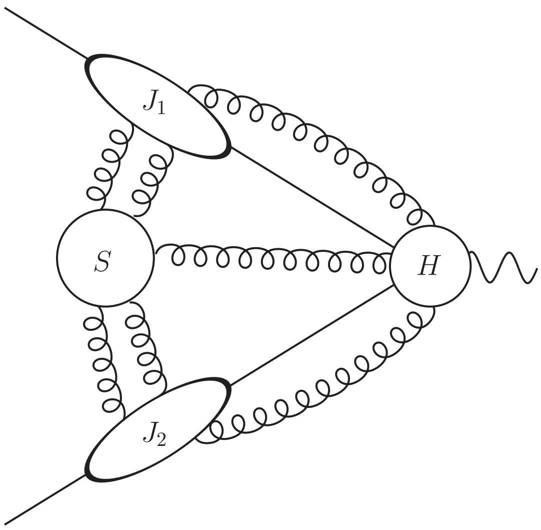


Figure 5.1: A typical solution to the Landau equations depicted as a reduced diagram in the Coleman-Norton picture. There are no quark lines between the various blobs, because they ill not survive power counting ($n_s > 0$)

The term 'factorization' is applied to the separation of hard, collinear and soft subdiagrams. Our goal is to decouple all subdiagrams, such that no indices connect J_i, S and H. From power counting we can inmediately disconnect S from H. The argument goes as follows. Internal lines in H are far off-shell. Connecting a soft gluon to such an internal line would give rise to an extra internal far off-shell propagator in H. This would give us a non-leading contribution to the diagram, i.e. suppressed by the inverse of a large scale. We therefore neglect any soft gluon connection to H.

Factorization is an important step to resummation [9]. Resummation is a technique to organize the large logarithms coming from IR divergences in perturbative expansions. Once this is accomplished, one can do predictions of the large logarithmic terms to all orders in perturbation theory. We will not go in further detail because we will not perform resummation in this thesis.

5.1 Soft Subdiagram

We already argued that for the leading divergencies there is no coupling from the soft subdiagram to the hard vertex. We are now ready to isolate the soft subdiagram entirely. Let us start by revisiting our one loop example, depicted in figure 3.1. We already found that in the soft limit the quark line can be replaced by an eikonal one. In the soft limit we therefore find that the gluon factorizes from the diagram and instead couples to a eikonal line. Consider now a more complex structure, in which the soft gluon is attached to the jet. The jet may be a subdiagram involving more collinear lines than the quarkline we just considered. Only longitudinally polarized photons are considered to couple to the jet, since transverse polarisations are suppressed [10]. We can use the non-abelian Ward identity for longitudinal polarized gluons to couple the gluon to the quarkline connecting J en H instead. Note that using the Ward identity you will find a diagram in which the soft gluon is connected to the hard part, but as we just argued this is subleading and can be discarded. Once the gluon is attached to the quark line, the same analysis as before applies. Using the eikonal approximation we can decouple all soft radiation from the jet and instead couple it to a eikonal line, as shown in figure 5.2. The soft subdiagram together with its connection to the Wilson lines, i.e. the factorized diagram, is called the soft function. The corresponding expression reads

$$S = \langle 0| \Phi_n(\infty, 0)\Phi_{\bar{n}}(0, \infty) |0\rangle \,, \tag{5.1}$$

where we used the non-abelian Wilson line as a source for soft gluons, as we extensively explained in section 4.2.

When constructing diagrams contributing to a specific order in perturbation theory there is a constraint one should keep in mind: photons originating from an eikonal line must land on the other one, since contributions from soft photons landing on the same eikonal line are proportional to $p_\mu p^\mu = 0$ (massless quarks).

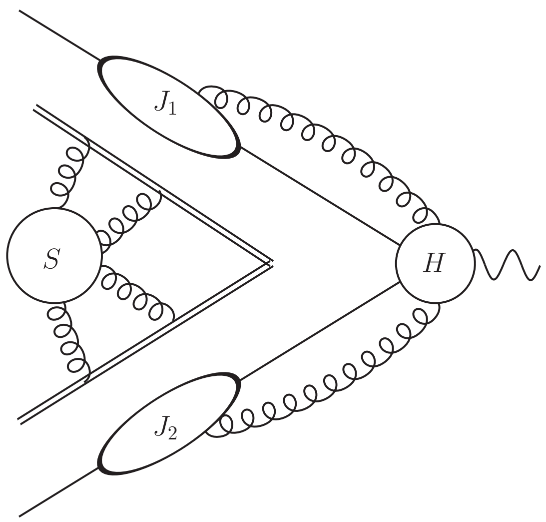


Figure 5.2: This figure shows the decoupling of the soft subdiagram from the hard and jet subdiagrams. The gluons connecting the soft subdiagram couple to Wilson lines instead.

5.2 Jet Factorization

In the previous section we saw how soft divergences could be factorized from the hard and collinear subdiagrams into the soft function. We would like to do a similar factorization for the collinear divergencies.

In order to demonstrate this decoupling let us look at the QCD correction to the QED vertex again, given in figure 3.1. The corresponding expression is proportional to

$$\int \frac{d^4k}{(2\pi)^4}\bar{u}(p_2)\frac{A^\rho\gamma^\mu B_\rho}{(p_2+k)^2\,(p_1-k)^2\,k^2}v(p_1), \tag{5.2}$$

where we have defined

$$\begin{aligned} A^\rho &= \gamma^\rho(\not{p}_2+\not{k}),\\ B^\rho &= (\not{p}_1-\not{k})\gamma^\rho. \end{aligned} \tag{5.3}$$

Note that we have chosen the Feynman gauge for the gluon propagator. Later in this section we will also look at implications of different gauges, and how the factorization of collinear gluons is not gauge dependent. Assume we are on the collinear pinch surface $k \mathbin{/\!/} p_1$. The corresponding reduced diagram in the Coleman-Norton picture is given by one of the first two diagrams in Figure 3.4. The only nonzero component of p_1^μ is chosen in the plus direction, and therefore the largest component of k^μ is in the plus direction as well. This means that $\not{p}_1-\not{k}=(p_1^+-k^+)\gamma^-$. Observing that $\gamma^-\gamma^\rho$ only gives a contribution for $\rho=+$, the most dominant term in equation (5.2) is B^+. Consequently the minus component of A^ρ appears. The numerator in the collinear approximation then becomes

$$A^\rho\gamma^\mu B_\mu \xrightarrow{k/\!/p_1} A^-\gamma^\mu B^+. \tag{5.4}$$

Now we use the Grammer and Yennie approach to decouple the collinear gluon from the hard part of the diagram [11]. Only keeping the most dominant terms this leads to

$$A^-B^+ = A^-\frac{k^+}{k^+}B^+ = A^\mu k_\mu\frac{B^+}{k^+} = k\cdot A\frac{B\cdot n}{k\cdot n}, \tag{5.5}$$

where in the last step we introduced a directional unit vector n^μ in the minus direction. We now focus on $\bar{u}(p_1)k\cdot A$. The massless Dirac equation allows us to add the term $\bar{u}(p_1)\not{p}_2(\not{p}_2+\not{k})$ at no cost, because doing so is similar to adding zero. Inserting A and using this trick of doing nothing we find

$$\bar{u}(p_1)k\cdot A = \bar{u}(p_1)(p_2+k)^2 \tag{5.6}$$

It inmediately becomes clear that one of the terms in the denominator will be cancelled. More explicitly, at the collinear pinch surface diagram 3.1 is proportional to

$$\int \frac{d^4k}{(2\pi)^4}\bar{u}(p_2)\gamma^\mu\,\frac{(\not{p}_1-\not{k})\,\gamma\cdot n}{(p_1-k)^2\,k^2}\,\frac{1}{k\cdot n}\,v(p_1) \tag{5.7}$$

The coupling of the gluon to one of the external legs, namely the one with p_2, has disappeared. Instead it couples to an eikonal line in the direction of p_2, diagramatically shown in Figure 5.3. To verify this, note the resemblance of $n^\rho/k\cdot n$ to the radiative fermion in the eikonal approximation. It can be more rigorously checked by using the eikonal Feynman rules listed in Figure 4.5 to reproduce equation (5.7) from the decoupled diagram 5.3 (up to a minus sign).

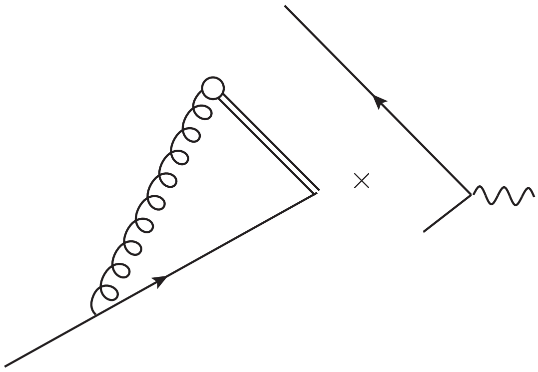


Figure 5.3: This diagram shows the decoupling of a collinear gluon from the hard part. The gluon now connects to an eikonal vertex, generated by the Wilson line. This factorizes the amplitude into a 'jet function' and a hard vertex.

Until now we used the Feynman gauge to prove the decoupling of the gluon from the quark-line. We found that the most dominant contribution came from the $A^- B^+$ part, corresponding to the Δ^{ab}_{-+} component of the gluon propagator. In axial gauge this leads to

$$\Delta^{ab}_{-+} = \frac{-i\delta_{ab}}{k^2 + i\epsilon}\left(g^{-+} - \frac{n_+ k_- + n_- k_+}{n \cdot k} + \frac{k_- k_+}{(n \cdot k)^2}\right). \tag{5.8}$$

On the collinear pinch surface $k_-, k_\perp \to 0$ and the gluon propagator in axial gauge will vanish, indicating no gluon attachment between the jet to the hard part.

To generalize this factorization to more general diagrams in which a collinear gluon connects the hard subdiagram with the jet subdiagram, one uses Non-Abelian Ward identities. Consider the coupling of a collinear gluon to the hard function. As a matter of fact collinear gluons couple to the hard function only with their longitudinal degrees of freedom [13][10]. In section 3.4 we already argued that the collinear gluon couples with its longitudinal degrees of freedom to the hard subdiagram. Non-abelian Ward identities describe how the sum over all attachments of the longitudinally polarized gluon must vanish. Since there is only one other possible insertion, we find the relation depicted diagrammatically in figure 5.4 (A). Now there is only one quark line connecting the jet subdiagram to the hard vertex. As we saw before we could simplify this by means of an eikonal line, leading to the identity in figure 5.4 (B). combining (A) and (B) we see how the longitudinal gluon can be detached from the hard subdiagram.

We have now treated the simple case in which just one collinear gluon is attached to H. Considering an arbitrary number longitudinally polarized gluons we will find that they will all decouple from H and instead couple to an eikonal line. This can be done by repeated use of the identity (C) and the appropriate application of the Ward identity for a general number of collinear gluons connecting to H. A nice graphical illustration of how this would work for two collinear gluons is given in figure 5.6.

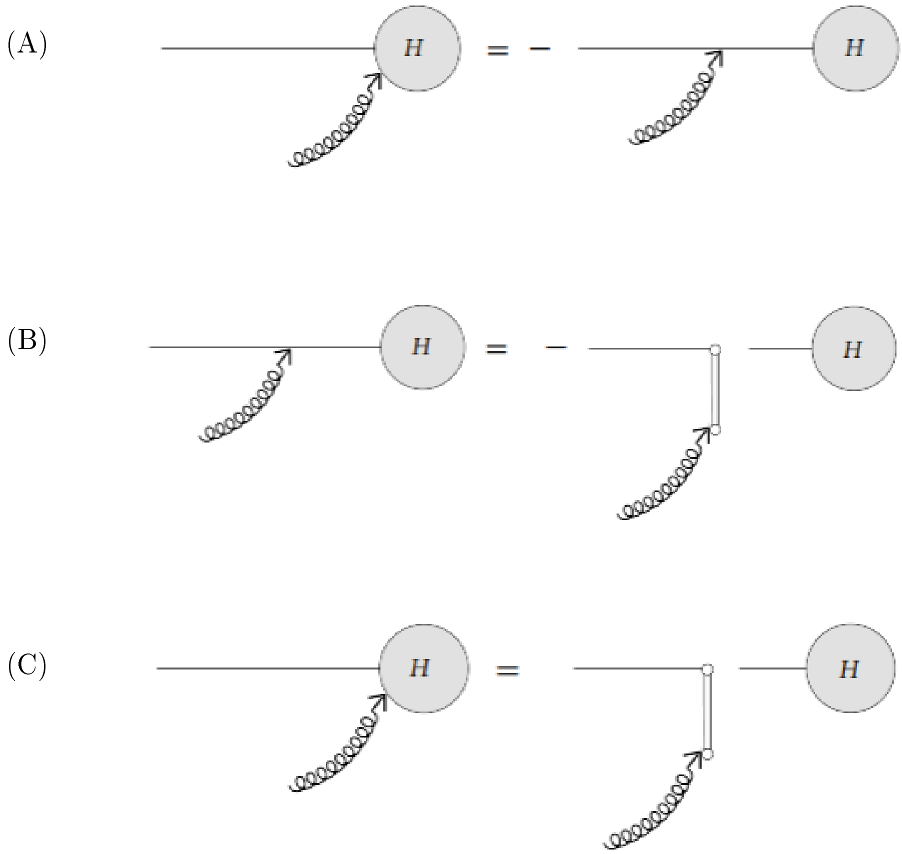


Figure 5.4: (A) Diagrammatic interpretation of the non-ablian Ward identity $k_\mu M^\mu = 0$. The arrow on the gluons indicates that they are longitudinally polarized. (B) Using the Grammar-Yennie trick, longitudinally polarized gluons may be connected to an eikonal line instead. (C) Shows the result of (A) and (B) successively. Figure based on [12]

.

In section 4.2 we already discussed how the attachment of several gluons to an eikonal line can be represented by a Wilson line. Therefore, if we adopt the Wilson line in our definition of the jet function we have decoupled all gluons from H; and so we have factorized the jet subdiagram from the hard part in any gauge. The jet function is given by

$$J(p) = \langle 0| \, T\{\Phi_n(\infty, 0)\psi(0)\} \, |p\rangle \, . \tag{5.9}$$

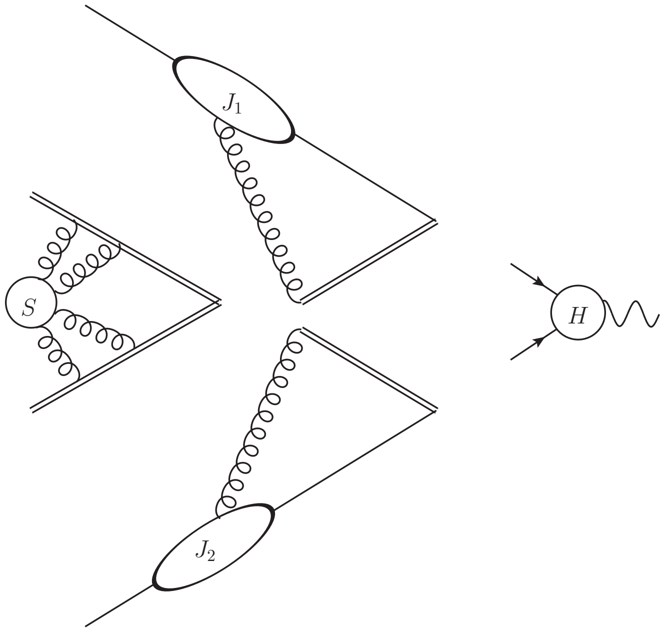


Figure 5.5: Full factorization of the soft, jet and hard subdiagrams into the soft, jet and hard functions. The hard function is proces dependent, but the soft function and jet function are universal (they only depend on the external line properties).

5.3 Jet-Soft-Hard Factorization

Lastly we will turn to the issue of double counting between soft and collinear divergencies. Gluons which are soft and collinear at the same time are included in the operator definitions of both S and J. Namely soft functions whose soft gluons become collinear gives the same result as jet functions whose collinear gluons become soft. We can solve this issue by an extra division in the factorization formula. The object we need to cancel is a correlator of Wilson lines, called the eikonal jet function $\mathcal{J}$,

$$\mathcal{J}(n, \bar{n}) = \langle 0| \, \Phi_n(\infty, 0) \Phi_{\bar{n}}(\infty, 0) \, |0\rangle \,, \tag{5.10}$$

where Φ_n approximates the original parton direction. We can now write down our final factorization formula as follows

$$\Gamma^\mu(\frac{Q^2}{\mu^2}) = H(\{p_i\}) \, S(\{\beta_i\}) \prod_{i=1}^{2} \frac{J_i(n_i, p_i)}{\mathcal{J}_i(\beta_i, n_i)}, \tag{5.11}$$

where n_i is the directional vector in the direction of the other jet, Q^2 is the kinematic scale of the scattering (the invariant mass of the virtual photon in Drell-Yan) and μ is the renormalization scale. So, in our two jet event n_1 is the directional vector in the direction of p_2 and vice versa. The directional vector β_i is in the same direction as p_i.

The factorization of amplitudes, graphically shown for for a two jet event in 5.5, will have a central role in our discussion of next-to-soft radiation in chapter 7.

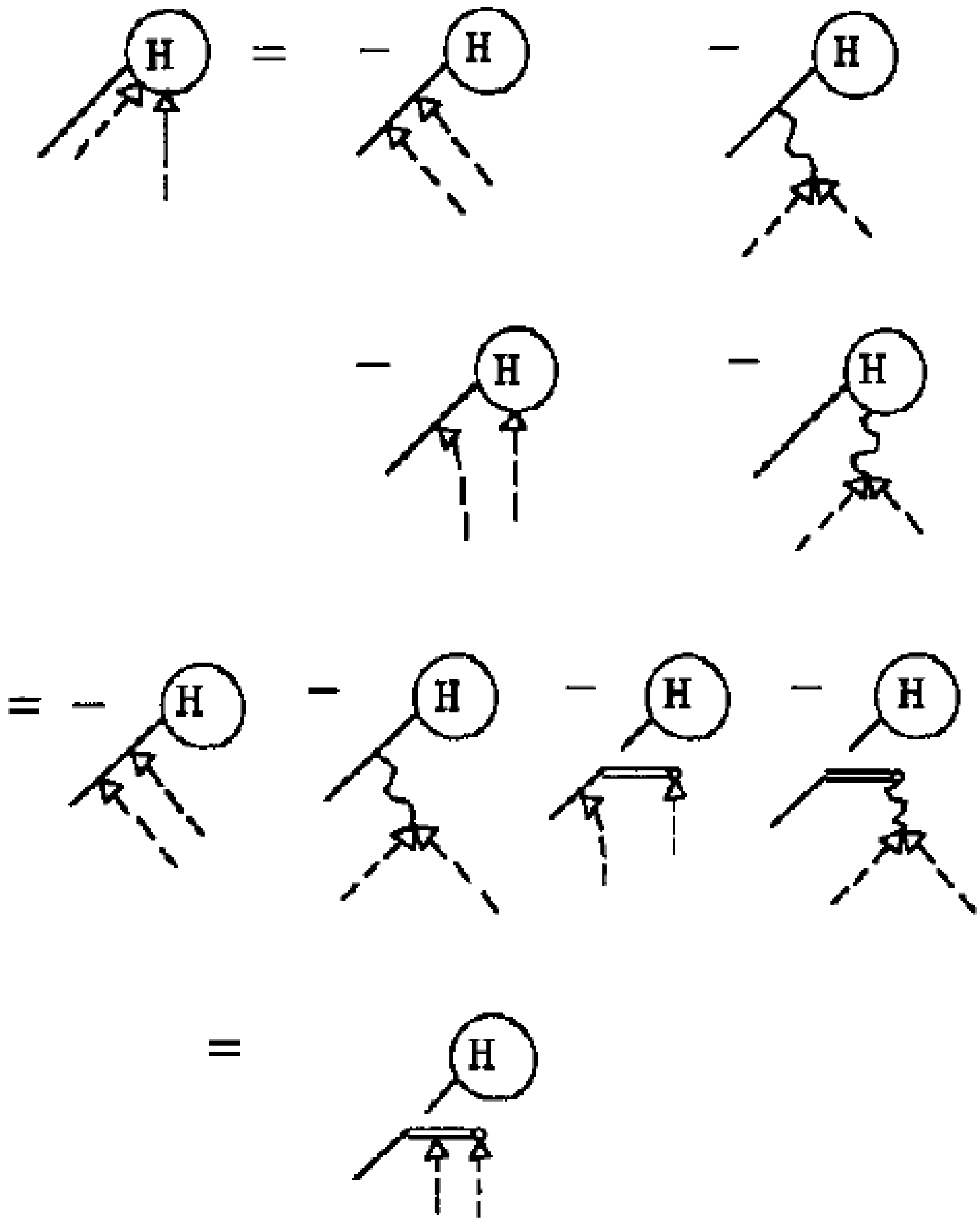


Figure 5.6: This figure displays the factorization of two longitudinally polarized gluons from H. The first step is to use the non-abelian Ward identity, which states that the sum of diagrams with all possible momenta instertions of the two longitudinal gluons must vanish. In a covariant gauge, Lorentz invariance requires that the gluon produced by the gluon 3-vertex interaction with the two longitudinal gluons must also be longitudinally polarized (it has no other vectors on which it may depend). The second step is to use identity (C) in figure 4.6 for the last two diagrams in the first step. In the last step, we let the second and fourth diagram in the second step cancell by means of (C). Furthermore we use identity (C) on the surviving two diagrams to find our final result. Note that the diagrams in which the two insertions are interchanged are implicit. This figure was constructed by Collins in [10].

Chapter 6

Ward Identities

Ward identities play an important role in factorization, as seen in the previous chapters. They will also play a central role when we investigate next-to-soft radiation in the following chapters. It will turn out that Ward identities are needed to prove Low's theorem for next-to-soft gluon radiation from amplitudes.

Ward identities are constructed for correlation functions based on a symmetry in the Lagrangian. In this chapter we will discuss the nature of Ward identities. We start with a discussion of current conservation for symmetries in the Lagrangian. Subsequently we will use this observation to construct a general Ward identity for a given correlation function.

6.1 Current conservation

Consider a Lagrangian for an arbitrary number of fields $L(\phi_a(x), \partial_\mu \phi_a(x))$. The action S is defined by the time integral of the Lagrangian.

$$S = \iint \mathcal{L}(x, \dot{x}, t) d^3x dt \tag{6.1}$$

Here $\mathcal{L}$ is the Lagrangian density, to which we will refer as Lagrangian from now on. An infinitesimal transformation of the fields, given by $\phi_a(x) \to \phi_a(x) + \epsilon\delta\phi_a(x)$, will change the Lagrangian to first order in ϵ according to equation (6.2). Here ϵ is the constant infinitesimal parameter, so we will be looking at global transformations.

$$\delta\mathcal{L} = \sum_a \frac{\partial \mathcal{L}}{\partial \phi_a} \epsilon\, \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \epsilon\, \partial_\mu \delta\phi_a(x). \tag{6.2}$$

Varying the action and inserting equation (6.2) gives the following relation up to first order in ϵ

$$\begin{aligned}
\frac{\delta S}{\delta\phi_b(x)} &= \int d^4y \frac{\delta \mathcal{L}}{\delta\phi_b(x)} \\
&= \int d^4y \sum_a \left(\frac{\partial \mathcal{L}}{\partial\phi_a(y)} \frac{\delta\phi_a(y)}{\delta\phi_b(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_a(y))} \frac{\delta\phi_a(y)}{\phi_b(x)} \right) \epsilon \\
&= \left(\frac{\partial \mathcal{L}}{\partial\phi_b(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_b(x))} \right) \epsilon,
\end{aligned} \tag{6.3}$$

The equations of motion for the fields can be obtained by the principle of least action, setting $\delta S = 0$. In other words, if the particles are on shell δS will be zero. This leads to the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial\phi_b(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_b(x))} = 0. \tag{6.4}$$

We can now rewrite $\delta\mathcal{L}$ using the Euler lagrange equations as

$$\delta\mathcal{L} = \epsilon\,\partial_\mu \underbrace{\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a(x)\Big]}_{j^\mu(x)} + \frac{\delta S}{\delta\phi_a}\epsilon\,\delta\phi_a(x). \tag{6.5}$$

If the set of infinitesimal transformations leaves $\mathcal{L}$ unchanged and the equations of motion are satisfied, equation (6.5) shows that $j^\mu(x)$ is conserved. This is called Noethers Theorem.

Ward Identities reflect the consequences of a symmetry in terms of correlation functions. We will now use Noethers theorem to give a derivation of general Ward identities.

6.2 General Ward Identity

To derive the general Ward identity we will make a change of variables in the generating functional $Z[J]$. The functional integral is given by

$$Z[J] = \int D\phi_a\, e^{i\left(S[\phi_a(x)]+\int d^4x J_a\phi_a(x)\right)}, \tag{6.6}$$

where the index a runs over the number of fields. Changing variables from ϕ to ϕ' does not change anything, since it is just a label. So, choosing the new variables to be the infinitesimal transformations that leave $\mathcal{L}$ unchanged, given by $\phi_a^{'} = \phi_a + \delta\phi_a$, we find

$$\begin{aligned} Z[J] &= \int D\phi_a\, e^{i\left(S[\phi_a(x)+\delta\phi_a(x)]+\int d^4x J_a(\phi_a(x)+\delta\phi_a(x))\right)}, \\ &= Z[J] + i\int D\phi_a \int d^4x \big(\frac{\delta S[\phi_a(x)]}{\delta\phi_a(x)} + J_a\big)\delta\phi_a(x) + \mathcal{O}^2(\delta\phi_a(x)), \end{aligned} \tag{6.7}$$

where we assumed the measure to be invariant under the symmetry transformation. We already noted that the extra terms induced by this change of variables should add up to zero at each order of $\delta\phi_a$. We therefore find

$$0 = \int D\phi_a \int d^4x\, e^{i\left(S[\phi_a(x)]+\int d^4x J_a\phi_a(x)\right)}\big(\frac{\delta S[\phi_a(x)]}{\delta\phi_a(x)} + J_a\big)\delta\phi_a(x). \tag{6.8}$$

The Ward identities now follow by taking n functional derivatives with respect to $J_i(x_i)$ and then setting $J = 0$. For $n = 1$ the result reads

$$0 = \int D\phi_a \int d^4x\, e^{iS[\phi_a(x)]}\big(\frac{\delta S[\phi_a(x)]}{\delta\phi_a(x)} i\phi_b(x_b) + \delta_{ab}\delta^4(x-x_b)\big)\delta\phi_a(x). \tag{6.9}$$

Since the path integral computes vacuum expectation values of time ordered products we can express the relation above in terms of operator fields,

$$0 = i\,\langle 0|\,T\{\partial\cdot j(x)\,\phi_b(x_b)\}\,|0\rangle + \delta_{ab}\delta^4(x-x_b)\;\langle 0|\,T\{\delta\phi_a(x)\}\,|0\rangle\,, \tag{6.10}$$

where we used equation (6.5), in which we used that $\delta\mathcal{L} = 0$ for the symmetry transformations used. The generalization to n fields in the Ward Identity then follows naturally,

$$0 = i\,\langle 0|\,T\{\partial\cdot j(x)\prod_{i=1}^{n}\phi_i(x_i)\}\,|0\rangle + \sum_{i=1}^{n}\langle 0|\,T\{\phi_1(x_1)\ldots\delta_{ai}\,\delta^4(x-x_i)\,\delta\phi_a(x)\ldots\phi_n(x_n)\}\,|0\rangle\,. \tag{6.11}$$

There are different methods of deriving Ward Identities. It is common to start with a correlation function of the desired process and then use field transformations to find the corresponding Ward Identity. The transformations used to derive the Ward identity do not necessary have to be the full symmetry transformations under which the Lagrangian is invariant. For example, one of the methods only uses the transformations of the fermion fields and not the transformation of the gauge field to arrive at the desired Ward identity [14]. Furthermore one could use the full gauge transformations to construct the Ward Identity, as explained in [15]. Lastly one could use the transformations under which the full quantum action is invariant, which are the BRST-transformations as explained in section 2.4. This method will yield the same Ward Identity as well.

In the presence of a Wilson Line the equivalence of those methods can be non-trivial. Therefore we will discuss all methods described above to obtain a Ward identity for a QED (non-)radiative process in chapter 8. Finally, we shall generalize this to a non-abelian theory.

Chapter 7

Low's Theorem

Low has shown [1] that when one expands a radiative amplitude in powers of the radiated photon energy q, the first two terms of this expansion can be obtained given the nonradiative process. His analysis relies on the observation that the radiative amplitude contains two types of contributions: terms where the photon is emitted from an external line and terms where the photon is radiated from internal propagators. In the next section we will review Low's theorem of radiative amplitudes [1], which was extended by Brunett and Kroll to charged particles with spin [16]. Later Del Duca showed that in the high energy limit, the original form of Low's theorem (which is a low energy theorem) holds in the region $q^0 \ll m^2/E$ [2]. In the following section we will mainly follow [14]. Thereafter we will extend Low's theorem as Del Duca did [2].

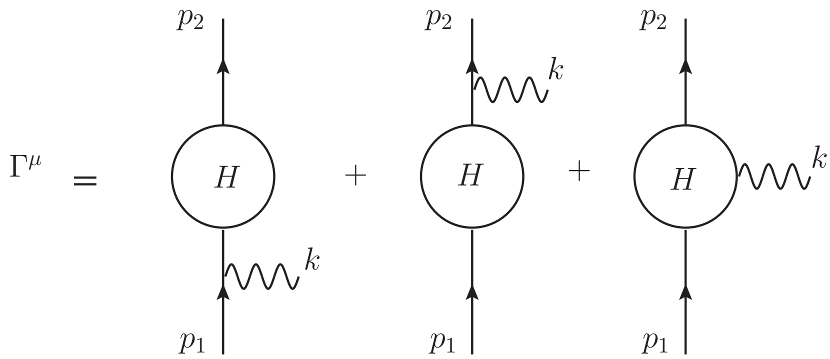


Figure 7.1: The radiative ampltude contributions come from

7.1 Radiative amplitude

Let us start with the scattering amplitude Γ^μ which corresponds to a single soft photon emission from a hard interaction of two external scalar particles. The emitted gluon can either be radiaton from one of the external legs or from within the hard interaction part of the amplitude, as depicted in figure 7.1. Using standard Feynman rules for scalar QED, we find

$$\begin{aligned}\Gamma^\mu =& \frac{e\,(2p_1-k)^\mu}{-2p_1\cdot k}\Gamma[(p_1-k)^2, p_2^2, (p_1-k)\cdot p_2)] \\ &+ \frac{e\,(2p_2+k)^\mu}{2p_2\cdot k}\Gamma[p_1^2, (p_2+k)^2, p_1\cdot(p_2+k)] + \Gamma^\mu_{\text{int}},\end{aligned} \tag{7.1}$$

where $\Gamma[p_1^2, p_2^2, p_1 \cdot p_2]$ is the non-radiative amplitude and Γ^μ_{int} includes all possible emissions from within the hard interaction.

If we now let k go soft, we may expand equation (7.1) around $k = 0$. We find

$$\begin{aligned}\Gamma^\mu = &\frac{ep_1^\mu}{-p_1 \cdot k}\Big(\Gamma - 2p_1 \cdot k\frac{\partial\Gamma}{\partial p_1^2} - p_2 \cdot k\frac{\partial\Gamma}{\partial p_1 \cdot p_2}\Big) + \Big(\frac{ek^\mu}{2p_1 \cdot k} + \frac{ek^\mu}{2p_2 \cdot k}\Big)\Gamma \\ &+\frac{ep_2^\mu}{p_2 \cdot k}\Big(\Gamma + 2p_2 \cdot k\frac{\partial\Gamma}{\partial p_2^2} + p_1 \cdot k\frac{\partial\Gamma}{\partial p_1 \cdot p_2}\Big) + \Gamma^\mu_{\text{int}} + \mathcal{O}(k),\end{aligned} \tag{7.2}$$

where Γ should be read as $\Gamma[p_1^2, p_2^\mu, p_1 \cdot p_2]$. Now one should note that we have an on-shell external photon coupling to Γ^μ. Gauge invariant theories imply gauge invariant amplitudes and since polarisation vectors ϵ_μ are gauge dependent we require $k_\mu\Gamma^\mu = 0$ [1], such that unphysical polarisations are not present. Explicitly evaluating this constraint entails

$$k_\mu\Gamma^\mu_{\text{int}} = -2ep_2 \cdot k\frac{\partial\Gamma}{\partial p_1^2} - 2ep_1 \cdot k\frac{\partial\Gamma}{\partial p_2^2} - ek \cdot (p_1 + p_2)\frac{\partial\Gamma}{\partial p_1 \cdot p_2}. \tag{7.3}$$

This relation must hold for all values of k, so the relation above is still true after removing factors of k^μ. Inserting this constraint on Γ^μ_{int} back in equation (7.2) we find

$$\begin{aligned}\Gamma^\mu = &\left(\frac{e(p_1 - k)^\mu}{-p_1 \cdot k} + \frac{e(p_2 + k)^\mu}{p_2 \cdot k}\right)\Gamma \\ &+ e\left(\frac{p_1^\mu(k \cdot p_2 - k \cdot p_1)}{p_1 \cdot k} + \frac{p_2^\mu((k \cdot p_1 - k \cdot p_2)}{p_2 \cdot k}\right)\frac{\partial\Gamma}{\partial p_1 \cdot p_2} + \mathcal{O}(k)\end{aligned} \tag{7.4}$$

From this result we conclude soft radiative amplitude is, to next-to-leading power in k, fully determined by the non-radiative amplitude. This is Low's theorem [1], although it is often applied specifically to the first order term $\mathcal{O}(1/k)$. Furthermore we note that the first order term is exactly the eikonal approximation of soft photon emission from a fermion line. This is no coincidence, using the effective rules for soft photon emission given in equation (4.3) and expanding all terms around $k = 0$ as we did before, we would end up with $\mathcal{O}(1/k)$ in equation (7.4).

7.2 Extension of Low's Theorem

In the remaining part of this section we closely follow the work of Del Duca [2].

Let us consider a non-radiative process in which two fermions create an off-shell photon. We denote the amplitude by $\rho(p_1, p_2)$. We can write the amplitude in a factorized form,

$$\rho(p_1, p_2) = H(p_1, p_2)S(n_1, n_2)\prod_{i=1}^{2} J(p_i, n_i), \tag{7.5}$$

where J is the jet functions, S the soft function and H the hard function. This factorization formula was already discussed in chapter 5. An equivalent discussion is also true for abelian theories and is even less technical due to the absence of self-interactions. The jet functions $J_i(p_i, n_i)$ depend on p_i and n_i, which is the vector in the opposite-moving

[1]It can also be argued by noting that the photon couples to Γ^μ through the current of scalar electrodynamics. Current conservation therefore implies that $k_\mu\Gamma^\mu = 0$

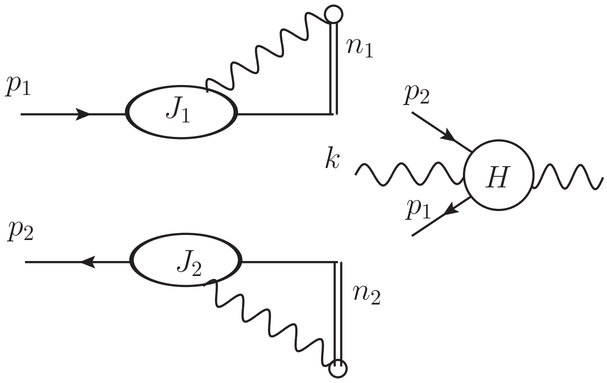


Figure 7.2: Photon emission from the hard subdiagram.

direction $n_{1,2} \sim p_{2,1}$. The jet function was already given in equation (5.9), and stated here for convenience:

$$J(p) = \langle 0| T\{\Phi_n(\infty, 0)\psi(0)\} |p\rangle , \tag{7.6}$$

with

$$\Phi_n(\infty, 0) = \exp[-ig \int_0^\infty d\lambda n \cdot A(\lambda)]. \tag{7.7}$$

Now consider the emission of a soft photon with polarisation vector ϵ^μ from the diagram, $\rho_\mu \epsilon^\mu$. The photon is said to be soft when $\omega_k < m$, with m the mass of the fermions. This regime is chosen such that there is no IR enhancements due to photon emissions from fermion loops inside S, an explanation is behond the scoop of this thesis but can be found in [11]. In the following discussion we do not consider the soft diagram explicitly, but it can be seen as to be included in H. The radiative amplitude can be factorized in the same way as ρ in equation (7.5). This tells us that the total radiative amplitude includes the emission from both of the jets and the hard part, as shown in figure 7.3 and 7.2 respectively. With this understanding of factorization we write

$$\rho_\mu \epsilon^\mu(k) = \rho^J_\mu \epsilon^\mu(k) + \rho^H_\mu \epsilon^\mu(k), \tag{7.8}$$

where ρ^J_μ includes emissions from both of the jets and ρ^H_μ is the diagram where the photon is emitted from H. Since the total radiative amplitude should be gauge independent we know that the corresponding Ward identity, $k_\mu \rho^\mu = 0$, imposes the following constraint

$$\rho^H_\mu k^\mu = -\rho^J_\mu k^\mu. \tag{7.9}$$

One of the advantages of this is that ρ^H_μ can be determined by ρ^J_μ. The contribution of the radiative jets to the full radiative amplitude is given by

$$\rho^J_\mu \epsilon^\mu = \sum_{a=1}^{2} H(p_a + k, p'_a) J_\mu(p_a, k, u_a) J(p'_a, u'_a) \epsilon^\mu, \tag{7.10}$$

where $p_a = p_{1,2}$, $p'_a = p_{2,1}$, $u_a = u_{1,2}$ and $u'_a = u_{2,1}$. Furthermore the definition for a radiative jet reads

$$J_\mu(p, k, u) = -i \int d^d y e^{i(p-k)\cdot y} \langle 0| \Phi_n(\infty, y)\, \psi(y)\, j^\mu(0) |p\rangle , \tag{7.11}$$

where the current insertion $j_\mu(0)$ indicates radiation.

Before moving on with our investiation of ρ^H_μ, let us now for a moment consider the lowest order contribution to the radiative jet function, shown in figure 7.4. In the derivation of Low's theorem in the previous section we needed to expand the diagram around $k = 0$ in order to get to our final result. In order to do so for this diagram we need to expand the $(l^2 + 2l \cdot k)^{-1}$ term, comming from the internal propagator. Reminding ourselfs that $l^2 = \mathcal{O}(m^2)$ and $l^0 = \mathcal{O}(\sqrt{s})$, expanding in soft photon momentum requires $\omega_k < m^2/\sqrt{s}$. We now would like to ease this energy constraint a bit by extending the energy regime such that it also includes $m^2/\sqrt{s} < \omega_k < m$.

We return to our original discussion on how to find ρ^H_μ. The QED Ward identity for the radiative jet function is given by the equation below. More details considering this Ward identity will be provided in chapter 8.

$$k^\nu J_\nu(p_1, n_1) = -eJ(p_1, n_1), \qquad\qquad k^\nu J_\nu(p_2, n_2) = eJ(p_2, n_2). \tag{7.12}$$

Implementing this radiative jet Ward identity in equation (7.10), we can obtain the amplitude for the emission of longitudinally polarized photon from the jet subdiagram

$$\rho^J_\mu k^\mu = \sum_{a=1}^{2} q_a H(p_a + k, p'_a) \prod_{i=1}^{2} J(p_i, u_i), \tag{7.13}$$

where q_a denotes the charge of the jet-line ($+e$ and $-e$ as described by equation (7.12)). Now we would like to expand $H(p_a + k, p'_a)$ in soft photon momentum, as we did for Low's theorem. Expanding H to first order in momentum k of the emitted photon we find

$$H(p_a + k, p'_a) = \left(1 + k^\mu \frac{\partial}{\partial p^\mu_a}\right) H(p_1, p_2) + \mathcal{O}(k^2). \tag{7.14}$$

Inserting the expansion of H back in equation (7.13) and using the fact that $\sum_a q_a = 0$ we find

$$\rho^J_\mu k^\mu = \sum_{a=1}^{2} q_a k^\mu \frac{\partial}{\partial p^\mu_a} H(p_1, p_2) \prod_{i=1}^{2} J(p_i, u_i). \tag{7.15}$$

The full Ward identity for the radiative diagram in equation (7.9) now allows us to determine the photon emission from H, up to $\mathcal{O}(k^0)$, to be

$$\rho^H_\mu \epsilon^\mu(k) = -\sum_a q_a \frac{\partial}{\partial p^\mu_a} H(p_1, p_2) \prod_{i=1}^{2} J(p_i, u_i), \tag{7.16}$$

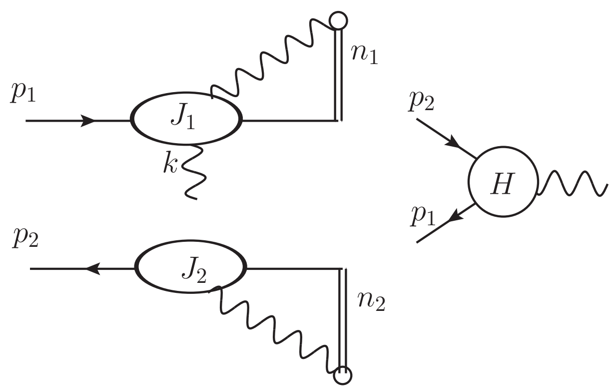


Figure 7.3: Photon emission from one of the jets.

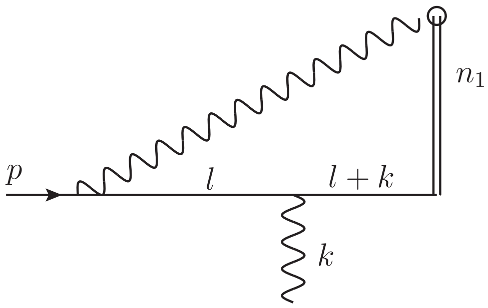


Figure 7.4: Lowest order contribution to the radiative jet

where we used that physical (transverse) polarisations do not contribute at the level of the amplitude [13] [1].

Now that we have determined $\rho^H_\mu \epsilon^\mu(k)$ up to next-to-leading order, $\mathcal{O}(k^0)$, we still need to find $\rho^J_\mu \epsilon^\mu(k)$. It will be convienient to introduce polarization tensors in the following way [11]

$$\epsilon^\mu = K^\mu{}_\nu \epsilon^\nu + G^\mu{}_\nu \epsilon^\nu, \tag{7.17}$$

with

$$K^\mu{}_\nu = k^\mu \frac{(2p+k)_\nu}{2p \cdot k + k^2}, \qquad G^\mu{}_\nu = g^\mu_\nu - K^\mu{}_\nu. \tag{7.18}$$

Effectively we rewrote our polarisation as a sum of two modified polarization sums K and G, instead of the real and virtual polarization splitting which we often use.

First consider the emission of a 'K-photon' from a jet. Using the Ward identities for radiative jets in equation (7.12) and the explicit expression for the polarisation vector K in equation (7.18), we obtain

$$J_\nu(p_a, k, u_a) K^\nu{}_\mu(p_a, k) \epsilon^\mu(k) = q_a \frac{(2p_a + k)_\mu}{2p_a \cdot k + k^2} \epsilon^\mu(k) J(p_a, u_a). \tag{7.19}$$

The radiative amplitude for emission from the jet, equation (7.15), can now be expanded up to $\mathcal{O}(k^0)$. Using equations (7.19) and (7.14) we find the amplitude for emission of a K-photon from the jet subdiagram to next-to-leading order

$$\rho^J_\nu K^\nu{}_\mu \epsilon^\mu(k) = \sum_{a=1}^{2} q_a \frac{(2p_a + k)_\mu}{2p_a \cdot k + k^2} \epsilon^\mu \Big[1 + k^\nu \frac{\partial}{\partial p^\nu_a} H(p_1, p_2)\Big] \prod_{i=1}^{2} J(p_i, u_i). \tag{7.20}$$

Before devoting ourselfs to the last term in $\rho_\mu \epsilon^\mu$, which is the emission of a G-photon from the jet subdiagrmas, we will add the two terms we got up to here. Adding the emission of the photon from H, ρ^H_μ, with the emission of a K-photon from either of the jets, $\rho^J_\nu K^\nu{}_\mu$, we find up to $\mathcal{O}(k^0)$

$$\begin{aligned}(\rho^H_\mu \epsilon^\mu + \rho^J_\nu K^\nu{}_\mu)\epsilon^\mu(k) = \sum_{a=1}^{2} q_a \epsilon^\mu(k) \Big[& \frac{(2p_a + k)_\mu}{2p_a \cdot k + k^2} \rho(p_1, p_2) \\ & - \Big(\frac{\partial}{\partial p^\mu_a} H(p_1, p_2)\Big) G^\nu{}_\mu(p_a, k) \prod_{i=1}^{2} J(p_i, u_i)\Big],\end{aligned} \tag{7.21}$$

where we have used the the definition of G in equation (7.18) to simplify the expression. A more usefull form however is when there are no derivatives on H. The hard subdiagram is process dependent, while the jet functions are generic. This can be acomplished by using the factorisation of ρ, given in equation (7.5). Acting on this equation we find

$$\frac{\partial}{\partial p_a^\mu}\rho(p_1,p_2) = \left(\frac{\partial}{\partial p_a^\mu}H(p_1,p_2)\right)\prod_{i=1}^{2}J(p_i,n_i) + H(p_1,p_2)\frac{\partial}{\partial p_a^\mu}\prod_{i=1}^{2}J(p_i,n_i). \tag{7.22}$$

The derivative on H in equation (7.21) can therefore be replaced, resulting in

$$\begin{aligned}(\rho_\mu^H\epsilon^\mu + \rho_\nu^J K^\nu{}_\mu)\epsilon^\mu(k) = \sum_{a=1}^{2} q_a\epsilon^\mu(k)\Bigg[\Bigg(\frac{(2p_a+k)_\mu}{2p_a\cdot k+k^2} - G^\nu{}_\mu(p_a,k)\frac{\partial}{\partial p_a^\nu}\Bigg)\rho(p_1,p_2) \\ + H(p_1,p_2)G^\nu{}_\mu(p_a,k)\frac{\partial}{\partial p_a^\mu}\prod_{i=1}^{2}J(p_i,u_i)\Bigg].\end{aligned} \tag{7.23}$$

Two important observations can now be made:

- (7.23) holds for all spins; In other words there is no radiative jet present. Namely, the current insertion in the definition of the radiative jet makes it dependend on the spin of the external particles.
- (7.23) is gauge invariant. This can be checked by showing that longitudinal polarisations do not contribute.

Finally we consider the emission of a G-photon from the jets to find the total amplitude $\rho_\mu\epsilon^\mu(k)$. It will be convenient to investigate some of the properties of G. We notice that G-polarisations are transverse

$$G^\mu{}_\nu(p,k)k^\nu = 0 \tag{7.24}$$

Furthermore we see that G polarisations vanish upon contracting with $(2p+k)_\mu$

$$(2p+k)_\mu G^\mu{}_\nu(p,k) = 0. \tag{7.25}$$

From this one can conclude that $p_\mu G^\mu{}_\nu = \mathcal{O}(k)$ and even more important, we can make a connection to the field-strength tensor in momentum space

$$G^\nu{}_\mu(p,k)\epsilon^\mu(k) = \frac{(2p+k)_\mu}{2p\cdot k+k^2}F^{\mu\nu}(k,\epsilon(k)), \tag{7.26}$$

with $F^{\mu\nu}(k,\epsilon(k)) = k^\mu\epsilon^\nu - k^\nu\epsilon^\mu$. It becomes clear that emission of a G-photon only contributes to $\mathcal{O}(k^0)$ and higher. The amplitude of emitting a G-photon from the jet is obtained by implementing the G polarisation tensor in equation (7.10) as follows

$$\rho_\nu^J G^\nu{}_\mu\epsilon^\mu = \sum_{a=1}^{2}H(p_a+k,p_a')J_\nu(p_a,k,u_a)J(p_a',u_a')G^\nu{}_\mu(p_a,k)\epsilon^\mu. \tag{7.27}$$

First of all we note that this expression is gauge invariant as well, due to equation (7.24). As argued above, projection with the G tensor ensures that this contribution starts at next-to-leading power in the soft expansion. One may therefore retain only the zeroth order term in the Taylor expansion of the shifted hard function, equation (7.14). Combining the resulting expression for an emission of G-photons from the jets with equation (7.23) yields the total radiative amplitude to $\mathcal{O}(k^0)$

$$\begin{aligned}\rho_\mu(p_1,p_2,k)\epsilon^\mu(k) = \sum_{a=1}^{2}\Bigg[q_a\Bigg(\frac{(2p_a+k)^\mu}{2p_a\cdot k+k^2} + G^\nu{}_\mu(p_a,k)\frac{\partial}{\partial p_a^\nu}\Bigg)\rho(p_1,p_2) \\ + H(p_1,p_2)G^\nu{}_\mu(p_a,k)\Bigg(q_a\frac{\partial}{\partial p_a^\nu}j(p_a,n_a) - J_\nu(p_a,k,n_a)\Bigg)J(p_a',n_a')\Bigg],\end{aligned} \tag{7.28}$$

This form is the extension of Low's theorem to $m^2/\sqrt{s} < \omega_k < m$. Note that in this form it is spin dependent, since the radiative jets depend on the spin of the external lines. A more elaborated discussion on the valid region can be found in [2].

Chapter 8

Radiative Jets

In our discussion of factorization we constructed jet functions, which contained the collinear participants of the amplitude. In Del Duca his extension of Lows theorem, which we discussed in the previous chapter, a new object arose: the radiative jet. This is nothing more than a jet with one more gauge boson emission. In this chapter we will look more closely at the Ward identities for the radiative jet, which are an important tool to prove the NLP factorization[2]. We will start by considering a QED radiative jet and show that there exsist a Ward identity which relates it to the non-radiative jet. Thereafter we will generalize to QCD and try to prove that the same Ward identity is still valid.

First, let us consider the definitions for the QED radiative and non- radiative jets, given by

$$
\begin{aligned}
J_\mu(p,k,n)\, u_s(p) &= -i \int d^d y e^{i(p-k)\cdot y} \left\langle 0 \right| \Phi_n(\infty, y)\, \psi(y)\, j^\mu(0) \left| p, s \right\rangle , \\
J(p,n)\, u_s(p) &= \left\langle 0 \right| \Phi_n(\infty, 0)\, \psi(0)\, \left| p, s \right\rangle . \qquad (8.1)
\end{aligned}
$$

Remember that $e^{iS_{int}}$ is always implicitly present in correlators to generate the higher order contritutions to the interaction. The state $|p,s\rangle$ describes an incoming fermion with momentum p^μ and spin $s = 1, 2$ (up,down). The jet function was already discussed in chapter 5. The definition of the radiative jet most easily obtained by looking at its structure at tree level, see figure 8.1. The insertion of the QED current $j^\mu(x) = g\bar{\psi}(x)\gamma^\mu\psi(x)$ allows for the emission of a photon. The internal propagator with momentum $p-k$ is found after the Wick contraction between $\psi(y)$ and $\bar(\psi)(0)$ in $j^\mu(0)$. The integral and phase factor are inserted to describe the propagator in momentum space. The other elements in the definition of the radiative jet are similar to the non-radiative one.

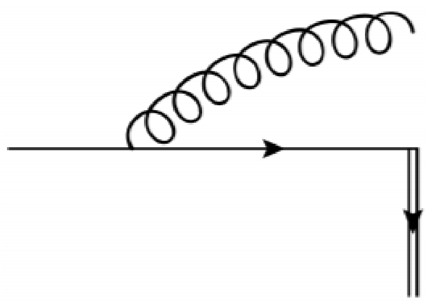


Figure 8.1: Radiative jet at tree level

Upon LSZ reduction the incoming fermion state, as explained in appendix G, yields

$$|p,s\rangle = \int d^d x\, \bar{\psi}(x)\,|0\rangle\,(i\overleftarrow{\partial})\; e^{-ip\cdot x}\, u_s(p) \tag{8.2}$$

Using this in equation (8.1) we find that the operator matrix element for the radiative jet is given by $\langle 0|\,\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle$.

The action S_{QED} is invariant under the abelian gauge transformation given by

$$\begin{aligned}\psi(x) &\to \psi'(x) = e^{ie\theta(x)}\psi(x),\\ \bar{\psi}(x) &\to \bar{\psi}'(x) = \bar{\psi}(x)e^{-ie\theta(x)},\\ A_\mu(x) &\to A'_\mu(x) = A_\mu(x) - \partial_\mu\theta(x).\end{aligned} \tag{8.3}$$

Also the bilinear $\bar{\psi}(x_1)\Phi(x_1,x_2)\psi(x_2)$ is invariant under these transformations. To see how this is true we will look at the transformation of the Wilson line under the transformations in equation (8.3). The result,

$$\begin{aligned}\Phi'(x_1,x_2) &= \exp\big[-ie\int_{x_1}^{x_2} dx\cdot A'(x)\big],\\ &= \exp\big[-ie\int_{x_1}^{x_2} dx\cdot(A(x)-\partial\theta(x)\big],\\ &= \Phi(x_1,x_2)\,e^{ie\theta(x_2)}\,e^{-ie\theta(x_1)}.\end{aligned} \tag{8.4}$$

explicitly shows that the transformations of ψ and $\bar{\psi}$ will be cancelled, so the bilinear is invariant.

Note that the $\Phi(\infty,0)\psi(0)$ is gauge invariant as well. The Wilson line transports the gauge transformation to infinity, where we set it to zero.

8.1 Ward Identity

We are now ready to construct the Ward identity for the (non-)radiative jets in equation 8.1. We will start with the method described by [14], which only uses the tranformations of the fermion field alone. We show that, even in the presence of Wilson lines, this is equivalent to exploiting the full gauge transformations to find the Ward identity [15]. Finally we will treat the BRST-method of finding a Ward identity, since this one is potentially best for a generalization to non-abelian gauge theories such as QCD.

8.1.1 Partial Gauge Transformation

The goal is to derive a Ward identity for the radative jet in equation (8.1), which upon LSZ reduction yields

$$J_\mu(p,k,n)u_s(p) = -i\int d^d y e^{i(p-k)\cdot y}\int d^d x\;\langle 0|\,\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\;\,(i\overleftarrow{\not\partial}_x)\;e^{-ip\cdot x}\;u_s(p). \tag{8.5}$$

In order to do so we will consider the gauge transformation of the fermion field, given in equation (8.3), but assume that the gauge field $A_\mu(x)$ does not transform. Under these transformations the Lagrangian is not invariant, but changes by a factor

$$\delta\mathcal{L}_{\mathrm{QED}} = -e\partial_\mu\theta\bar{\psi}\gamma^\mu\psi = -\left(\partial_\mu\theta\right)j^\mu(x). \tag{8.6}$$

We now proceed by considering the operator matrix element for the radiative jet, $\langle 0|\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x)\, |0\rangle$. As described in section 6.2, the first step in finding a Ward identity is to change to path integral formalism.

$$\langle 0|\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x)\, |0\rangle = \int D\psi D\bar{\psi} DA\, e^{iS_{\text{QED}}}\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x). \tag{8.7}$$

We then make a change of variables from ψ and $\bar{\psi}$ to ψ' and $\bar{\psi}'$, which does not change the value of the path integral.

$$\langle 0|\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x)\, |0\rangle = \int D\psi' D\bar{\psi}' DA\, e^{iS_{\text{QED}}}\, \Phi_n(\infty, y)\psi'(y)\bar{\psi}'(x). \tag{8.8}$$

In this case we will change ψ and $\bar{\psi}$ according to the symmetry transformations in equatin (8.3) as mentioned earlier. Since we are not performing a change in variables on the gauge field A, the Lagrangian changes according to equation (8.6). We find

$$\begin{aligned}
(8.8) = &\int D\psi D\bar{\psi} DA\, e^{iS_{\text{QED}} - i\int d^dz \partial_\mu \theta(z) j^\mu(z)}\ \Phi_n(\infty, y)(\psi(y) + ie\theta(y)\psi(y) + \mathcal{O}(\theta^2)) \\
&\times (\bar{\psi}(x) - ie\theta(x)\bar{\psi}(x) + \mathcal{O}(\theta^2)), \\
= &\text{ original expression} + \int D\psi D\bar{\psi} DA\, e^{iS_{\text{QED}}}\ \Phi_n(\infty, y) \\
&\times i\int d^dz \left(\partial_\mu j^\mu(z)\psi(y)\bar{\psi}(x) + ie\delta^d(z-y)\psi(y)\psi\bar{(}x) - ie\delta^d(x-z)\psi(y)\psi\bar{(}x)\right)\theta(z) + \mathcal{O}(\theta^2),
\end{aligned} \tag{8.9}$$

where we used the fact that the measure is invariant under these transformations since the Jacobian is unity; the phases arising from the transformations of ψ and $\bar{\psi}$ cancel each other. Furthermore, $\theta(z)$ is factored out after partial integration in the last line. Since we obtain the original expression on the right hand side, all extra terms should vanish. This results in the following Ward identity

$$\langle 0|\, \Phi_n(\infty, y)\partial \cdot j(z)\psi(y)\bar{\psi}(x)\, |0\rangle = e\big(\delta^d(z-x) - \delta^d(z-y)\big)\, \langle 0|\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x)\, |0\rangle\,. \tag{8.10}$$

In order to see what this Ward identity means on the level of the radiative jets we define

$$f(z) = \int d^dy\, e^{i(p-k)\cdot y} \int d^dx\ \langle 0|\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x)\partial \cdot j(z)\, |0\rangle\ i\overleftarrow{\partial\!\!\!/}_x e^{-ip\cdot x} u_s(p). \tag{8.11}$$

Partial integration of $\partial \cdot j(z)$ gives an object we shall call $f(0) = k^\mu J_\mu(p, k, n)u_s(p)$, where the argument refers to $z = 0$ in (8.11). Note that the sign of this expression is related to an outgoing photon with momentum k. On the other hand, if we substitute equation (8.10) in (8.11) we find that

$$\begin{aligned}
f(0) = \int d^dy e^{i(p-k)\cdot y} \int d^dx \Big[&- e\, \delta^d(y)\, \langle 0|\, \Phi_n(\infty, y)\psi(0)\bar{\psi}(x)\, |0\rangle \\
&+ e\, \delta^d(x)\, \langle 0|\, \Phi_n(\infty, y)\psi(y)\bar{\psi}(0)\, |0\rangle \Big]\, i\overleftarrow{\partial\!\!\!/}_x\, e^{-ip\cdot x}\, u_s(p).
\end{aligned} \tag{8.12}$$

Wick contraction of ψ with $\bar{\psi}$ gives a fermion propagator in real space. Partial integration of $i\partial_x$ yields a factor $-p\!\!\!/$. Then, in the first term of equation (8.12) the $p\!\!\!/$ from partial integration is cancelled by the propagator . However, in the second term the propagator yields a factor $1/(p\!\!\!/ - k\!\!\!/)$ and therefore the $p\!\!\!/$ annihilates $u_s(p)$. We conclude that only the first term gives a contribution to the Ward identity, which now yields

$$k^\mu J_\mu(p, k, n) = -eJ(p, n), \tag{8.13}$$

as we need for factorization theorems at NLP.

8.1.2 Full Gauge Transformation

We will now rederive the abelian identity given in equation (8.13) by exploiting the full gauge transformations, as in [14]. We will see that in the presence of the Wilson line the equivalence with the method described in the previous section is non-trivial. It turns out that we need to add the divergence of an eikonal source to connect the different Ward identities.

Consider again the matrix element in equation (8.7), and again perform a change of variables. This time however, according to the full gauge transformations described in equation (8.3). The classical QED action is invariant under these transformations, but the full quantum action is not. The gauge fixing term derived in section 2.2 namely changes.

$$\mathcal{L}_{\rm GF} = -\frac{\xi}{2}(\partial \cdot A)^2 \qquad \Rightarrow \qquad \delta\mathcal{L}_{\rm GF} = \xi\, \partial \cdot A \Box \theta. \tag{8.14}$$

Since we are now considering the full gauge transformations, the Wilson line will transform as well. As a result the combination $\Phi_n(\infty, y)\psi(y)$ is now invariant, assuming $\theta(\infty) = 0$ because the variation on the boundary vanishes. Using these building blocks a variation of the matrix element, after changing variables and substituting the transformations, yields

$$\begin{aligned}\langle 0| \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) |0\rangle &= \int D\psi D\bar{\psi} DA\, e^{iS_{\rm QED}}(1 + i\xi \int d^d z \theta(z)\, \Box\, \partial \cdot A(z) + \mathcal{O}(\theta^2)) \\ &\quad \times \Phi_n(\infty, y)\psi(y)\bar{\psi}(x)(1 - ie\theta(x) + \mathcal{O}(\theta^2)), \\ &= \text{original expression} + \int D\psi D\bar{\psi} DA\, e^{iS_{\rm QED}} \int d^d z \\ &\quad \times (i\xi\theta(z) \cdot A(z) - ie\, \delta^d(z - x)\theta(x))\ \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) + \mathcal{O}(\theta^2),\end{aligned} \tag{8.15}$$

where we used that also the measure of A is invariant, since the shift in the gauge field is purely additive. From this we can obtain the Ward identity

$$\langle 0| \xi \Box \partial \cdot A(z)\Phi_n(\infty, y)\psi(y)\bar{\psi}(x) |0\rangle = e\delta^d(z - x) \langle 0| \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) |0\rangle\,. \tag{8.16}$$

In the absence of the Wilson line one could simply make the replacement $\xi\Box\partial \cdot A \to \partial \cdot j$. If this replacement is made in the matrix element above we would find a Ward identity different from the one in equation (8.10). Even worse is that the missing term is the one that will generate the Ward identity for the radiative jet given in equation (8.13). The important observation now is that the substitution $\xi\Box\partial \cdot A \to \partial \cdot j$ is not allowed in the presence of the Wilson line, since its presence induces more photon sources.

To investigate where the difference between the two Ward identities arises let us expand the Wilson line

$$\Phi_n(\infty, y) = \sum_{m=0}^{\infty} \frac{(-ie)^m}{m!} \int \prod_{i=1}^{m} d^{\mu_i}x_i\ A_{\mu_1}(x_1) \ldots A_{\mu_m}(x_m), \tag{8.17}$$

such that

$$\begin{aligned}&\lambda\Box_z \langle 0| \partial \cdot A(z)\Phi_n(\infty, y)\psi(y)\bar{\psi}(x) |0\rangle \\ &\quad = \sum_{m=0}^{\infty} \frac{(-ie)^m}{m!} \int \prod_{i=1}^{m} d^{\mu_i}x_i\ \big[\,\lambda\Box_z \langle 0| \partial \cdot A(z) \prod_{i=1}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x) |0\rangle\,\big].\end{aligned} \tag{8.18}$$

We can use the general Ward Identity for local gauge transformations on the metric element above, derived in the appendix E and given by

$$-\lambda\Box_z \langle 0| \partial\cdot A(z) \prod_{i=1}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle = \sum_{j=1}^{m} \langle 0| \prod_{i\neq j}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left[i\partial_{\mu_j}\delta(x_j - z)\right]$$
$$+ e\langle 0| \prod_{i=1}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left[\delta(y-z)\right]$$
$$- e\langle 0| \prod_{i=1}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left[\delta(x-z)\right], \qquad (8.19)$$

to write

$$(8.18) = \sum_{m=0}^{\infty} \frac{(-ie)^m}{m!} \int \prod_{i=1}^{m} d^{\mu_i}x_i \sum_{j=1^m} \langle 0| \prod_{i\neq j}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left(-i\partial_{\mu_j}\delta^d(x_j - z)\right)$$
$$- e\langle 0| \Phi_n(\infty, y)\psi(y)\bar(\psi)(x)|0\rangle \delta^d(z-y) + e\langle 0| \Phi_n(\infty, y)\psi(y)\bar(\psi)(x)|0\rangle \delta^d(z-x), \qquad (8.20)$$

where the series are resummed to the Wilson line in the last two terms, since the photon fields are unaffected. Let us define S_m as

$$S_m = \frac{(-ie)^m}{m!} \int \prod_{i=1}^{m} d^{\mu_i}x_i \sum_{j=1}^{m} \langle 0| \prod_{i\neq j}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left(-i\partial_{\mu_j}\delta^d(x_j - z)\right). \qquad (8.21)$$

The only dependence on x_j is found in the delta function and therefore the integral over x_j can be performed, yielding

$$\int_y^{\infty} dx_j^{\mu_j} \partial_{\mu_j}\delta^d(z - x_j) = -\delta^d(z-y). \qquad (8.22)$$

Since z is the location of insertion for $\partial\cdot A$ and $A(\infty) = 0$ the contribution at infinity vanishes in the above equation. So after performing the integration over x_j we can write S_m as

$$S_m = i\frac{(-ie)^m}{m!} \int \prod_{i\neq j}^{m} d^{\mu_i}x_i \sum_{j=1}^{m} \langle 0| \prod_{i\neq j}^{m} A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left(\delta^d(z-y)\right),$$
$$= e\frac{(-ie)^{m-1}}{(m-1)!} \int \prod_{i}^{m-1} d^{\mu_i}x_i \, \langle 0| A_{\mu_i}(x_i)\psi(y)\bar{\psi}(x)|0\rangle \left(\delta^d(z-y)\right), \qquad (8.23)$$

where in the second line the sum over j was performed, which yields m identical terms. Now re-exponentiate the Wilson line by summing S_m over m,

$$\sum_m S_m = e\delta^d(z-y) \langle 0| \Phi_m(\infty, y)\psi(y)\bar{\psi}(x)|0\rangle. \qquad (8.24)$$

Using this result we see that the first two terms in equation (8.20) cancel and we recover the Ward identity from equation (8.16). As shown, the Wilson line contribution exactly cancels the terms in the Ward identity that arise at the 'y end' of the fermion line, where the field $\psi(y)$ resides. This was expected since the Wilson line carries the gauge variantion of $\psi(y)$ to infinity, where it vanishes.

As already mentioned the Ward identities arising from a full gauge transformation, equation (8.16), and a partial gauge transforamtion, equation (8.10), are clearly different.

The difference between the two can be bridged by changing from $\lambda\Box\partial\cdot A \to \partial\cdot j + \partial\cdot\Sigma$. Here Σ is a so called eikonal source given by $\Sigma^\mu(x,n) = e\, n^\mu \int_0^\infty d\lambda\, \delta^d(x-\lambda n)$, which makes up for the presence of the Wilson line. To be more precise, $Sigma$ makes up for the contribution in (8.21) and $\partial\cdot\Sigma$ is exactly (8.21).

8.1.3 BRST Method

Ward Identities for the radiative jet can also be constructed with the BRST formalism. The only difference is that we should start with a matrix element

$$\langle 0|\, \bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\,. \tag{8.25}$$

This is not yet clear, but it will turn out that it produces the right Ward Identity. Imposing invariance under the abelian BRST transformations, which are given by equation (2.29) in the case that $f_{abc}=0$, leads to the Ward identity

$$\langle 0|\, \xi\partial\cdot A(z)\theta\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle + ie\,\langle 0|\,\bar{c}(z)\Phi_n(\infty,y)\psi(y)c(x)\theta\bar{\psi}(x)\,|0\rangle = 0, \tag{8.26}$$

where we used the invariance of $\Phi_n(\infty,y)\psi(y)$. In the second term we can perform a contraction of the ghost fields. In QED the ghost fields do not interact with other fields in the theory such that the ghost fields can be eliminated by performing a Wick contraction, which yields

$$\overline{c(w)\bar{c}}(z) = i\int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(w-z)}}{k^2+i\epsilon} \tag{8.27}$$

Now act with the $\Box_z$ operator on all terms in equation (8.26). Since the free ghost propagator is a Green's function of the box operator the second term will become a delta function $\delta(z-x)$. The final result is exactly the Ward identity in equation (8.16).

Note that in QCD the ghost fields interact with other fields through the vertex given in equation (2.27). In that case one should use the complete ghost propagator, defined by the two-point function to all orders in perturbation theory.

One could now bridge the gap to the most relevant Ward identity in equation (8.10) by making the substitution $\lambda\Box\partial\cdot A \to \partial\cdot j + \partial\cdot\Sigma$, as was explained in the previous section. However, for the generalization to QCD we want a more general method of recovering the relevant Ward Identity. Namely, we are unsure what the substitution for $\lambda\Box\partial\cdot A$ will be in a non-abelian theory. In what follows we will explain an alternative way to produce the most relevant Ward identity by means of a 'local BRST transformation'.

Local BRST transformation

It was already proven in section 2.4 that the action S_{QED} is invariant under the BRST transformations. Taking θ to be a local parameter, one finds that upon varying with respect to the 'local' BRST transformations the QED Lagrangian changes according to $\delta\mathcal{L} = j^\mu_{\mathrm{BRST}}\partial_\mu\theta$. In the abelian limit the BRST-current given by equation (2.32) simplifies to

$$j^\mu_{\mathrm{BRST}} = -F^{\mu\nu}\partial_\nu c + e\bar{\psi}\gamma^\mu\psi c - \xi\partial\cdot A\partial^\mu c. \tag{8.28}$$

Consider now the matrix element in equation (8.25). As before we switch to path integral formalism and perform a change of variables. Choosing the new variables to be the BRST transformations on the fields, under which the measure remains invariant, we find

$$\langle 0|\,\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle = \int \prod_{\text{fields}} [D\phi]\; e^{iS_{\text{QED}}(\phi)+i\int j^{\mu}_{\text{BRST}}\partial_\mu\theta} \\ \times \left[\bar{c}(z)+\xi\partial\cdot A(z)\theta(z)\right]\,\Phi_n(\infty,y)\psi(y)\,\left[\bar{\psi}(x)+ie\bar{\psi}(x)c(x)\theta(x)\right], \tag{8.29}$$

where the product over fields denotes that there should be a path integral for each field in the theory. This will produce the following Ward identity:

$$\langle 0|\,\partial\cdot j_{\text{BRST}}(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle = \; i\xi\,\delta^d(z-w)\,\langle 0|\,\partial\cdot A(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ -e\delta^d(x-w)\,\langle 0|\,\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)c(x)\,|0\rangle\,. \tag{8.30}$$

Note that integrating in $\int d^dw$ the left hand side of this equation vanishes, because the current is zero on the boundary. One therefore recovers the Ward identity from equation (8.16). This is precisely the Ward Identity obtained by imposing BRST invariance of the matrix element.

The goal is not met, since we would like to reproduce the most relevent Ward identity directly. In order to do so we write the divergence of the BRST current as

$$\partial\cdot j_{\text{BRST}} = -O_{A^\nu}\partial_\nu c + O_{\bar{c}}\;\xi\,\partial\cdot A + \partial\cdot j_{\text{EM}}\,c, \tag{8.31}$$

where the operators O_{A^ν} and $O_{\bar{c}}$ vanish by the equations of motion of the gauge and ghost field respectively. Therefore we can write

$$\langle 0|\,\partial\cdot j_{\text{BRST}}(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle = -\,\langle 0|\,O_{A^\nu}(w)\partial_\nu c(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ +\langle 0|\,O_{\bar{c}}(w)\xi\partial\cdot A(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ +\langle 0|\,\partial\cdot j_{\text{EM}}(w)c(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\,. \tag{8.32}$$

This allows us to recover the Ward identity for the electromagnetic current from the Ward Identity in equation (8.30) for the BRST current. To end up with the Ward identity it is important to know the impact of the operators O_{A^ν} and $O_{\bar{c}}$, present in the RHS of equation (8.32). Let O_{ϕ_i} be an operator that vanishes upon imposing the equations of motoin for the corresponding field ϕ_i, where ϕ_i can represent any field in the theory. For such operators there exist an operator identity

$$\langle 0|\,O_{\phi_i}X\,|0\rangle = i\,\langle 0|\,\frac{\delta X}{\delta\phi_i}\,|0\rangle\,, \tag{8.33}$$

where X is any combination of fields. A proof can be found in appendix C. This can be used in equation (8.32) to reduce the RHS. For the term involving the operator O_{A^ν} this is somewhat non trivial so we will discuss this in more detail. A variation of the Wilson line to the gauge field is given by

$$\frac{\delta\Phi_n(\infty,y)}{\delta A^\nu(w)} = -ie\frac{\delta}{\delta A^\nu(w)}\Big[\int_y^\infty dv^\mu A_\mu(v)\Big]\Phi_n(\infty,y) \\ = -ie\int_y^\infty dv_\nu\;\delta^d(v-w)\,\Phi_n(\infty,y). \tag{8.34}$$

This step in the derivation of the Ward Identity is not easily generalized to non-abelian theories as we will see later. If we now apply equation (8.33) to the term in equation (8.32) involving the O_{A^ν} it can be rewritten to

$$\langle 0|\, O_{A^\nu}(w)\partial_\nu c(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle = e\int_y^\infty dv^\nu\ \delta^d(v-w) \times \langle 0|\,\partial_\nu c(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \quad (8.35)$$

Doing the same for the other terms in equation (8.32) yields

$$\begin{aligned}-e\delta^d(x-w)\,\langle 0|\,\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)c(x)\,|0\rangle = \\ -e\int_y^\infty dv^\nu\ \delta^d(v-w)\ \langle 0|\,\partial_\nu c(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ +\langle 0|\,\partial\cdot j_{\mathrm{EM}}(w)c(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\,,\end{aligned} \quad (8.36)$$

where we substituted equation (8.30) for the LHS. This caused the cancellation of $i\xi\delta^d(z-w)\,\langle 0|\,\partial\cdot A(w)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle$ which appeared on both sides. To get rid of the ghost fields we perform a Wick contraction, see (8.27). Then one acts with the $\Box_z$ on the resulting equation. Doing so results in

$$\begin{aligned}&-ie\delta^d(x-w)\delta^d(x-z)\,\langle 0|\,\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ &= \\ &+ie\int_y^\infty dv^\nu\ \delta^d(v-w)\frac{\partial}{\partial w^\nu}\delta^d(w-z)\ \langle 0|\,\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ &-i\delta^d(w-z)\,\langle 0|\,\partial\cdot j_{\mathrm{EM}}(w)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\end{aligned} \quad (8.37)$$

Integrating over w now results in the well known Ward identity

$$\langle 0|\,\partial\cdot j_{\mathrm{EM}}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle = e\left(\delta^d(x-z)-\delta^d(y-z)\right)\,\langle 0|\,\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\,. \quad (8.38)$$

In the absense of the Wilson line this construction works as well. There are two small adjustments. Firstly there appears an extra term on the RHS from transforming $\psi(y)$, since the invariant combination $\Phi_n(\infty,y)\psi(y)$ is absent. Secondly the term in equation (8.32) with the operator O_{A^ν} vanishes due to the absence of gauge fields in the correlator. Together they will make sure that even in the absence of the Wilson line the Ward identity from equation (8.38) still holds.

8.2 Non-Abelian Ward Identity

To generalize to non-abelian gauge theories, the BRST method using the local transformations seems the most suited to find the Ward identity for the radiative jet. We therefore start with the matrix element

$$\langle 0|\,\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle\,. \quad (8.39)$$

Changing variables with respect to the local non-abelian BRST transformations, which are given by equation (2.29) leads again to

$$\begin{aligned}\langle 0|\,\partial\cdot j^a_{\mathrm{BRST}}(w)\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle = \ & i\xi\,\delta^d(z-w)\,\langle 0|\,\partial\cdot A(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)\,|0\rangle \\ &-g\delta^d(x-w)\,\langle 0|\,\bar{c}(z)\Phi_n(\infty,y)\psi(y)\bar{\psi}(x)c(x)\,|0\rangle\,.\end{aligned} \quad (8.40)$$

but here with the non-abelian BRST-current given in equation (2.32). Note that we have used the fact that the combination $\Phi_n(\infty, y)\psi(y)$ is gauge invariant, as was the case in abelian theories. This can be argued from the transformation properties of the non-abelian Wilson line, given in equation (4.19).

In section 2.4 we already showed that the non-abelian BRST-current is conserved on shell. There is however a more usefull prescription of $\partial \cdot J_{\text{BRST}}$, which makes the link to the classical QCD current (present in the non-abelian version of the radiative jet), namely:

$$\begin{aligned}\partial \cdot J_{\text{BRST}} = &- \ O_A^\mu \cdot \partial_\mu c + \xi\, \partial \cdot A O_{\bar{c}} \ + \partial \cdot (j^a_{QCD}) c_a - \frac{1}{2} g f_{abc} c^b c^c O_{c^a} \\ &- g f_{abc} A_\mu^b O_A^{\mu,c} c^a - g f_{abc} A_\mu^b (\partial_\nu F^{\mu\nu} + J^\mu_{\text{QCD}}) c^a,\end{aligned} \tag{8.41}$$

where J^μ_{QCD} is given in equation (2.35). Unfortunately, written in the way above, conservation of the BRST-current is not manifest. Due to the precence of ghosts and the gauge fixing term in the quantum theory, J^μ_{QCD} is not conserved on shell. There might be a way out using the work of Kugo and Ojima[17]. They showed that

$$\partial_\mu F^{\nu\mu} + g J_\mu = \{Q_{\text{BRST}}, D_\mu \bar{c}\}, \tag{8.42}$$

where Q_{BRST} is the conserved BRST charge. We know that Q_{BRST} annihilates all physical states [18]. Therefore,

$$\langle \text{phys} | \left(\partial_\mu F^{\nu\mu} + g J_\mu\right) | \text{phys} \rangle = 0. \tag{8.43}$$

Using equation (8.41) in the LHS of (8.40) we find

$$\begin{aligned}&\langle 0 | \, \partial \cdot j_{\text{BRST}}(w) \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &\qquad = - \langle 0 | \, O_A^\mu(w) \cdot \partial_\mu c(w) \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &\qquad\quad + \langle 0 | \, O_{\bar{c}}(w) \xi \, \partial \cdot A(w) \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &\qquad\quad + \langle 0 | \, \partial_\mu j^\mu_{QCD}(w) \cdot c(w) \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &\qquad\quad - \langle 0 | \, \frac{1}{2} g (c(w) \times c(w)) \cdot O_c \, \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &\qquad\quad - \langle 0 | \, g (A_\mu(w)) \times O_A^\mu(w)) \cdot c(w) \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &\qquad\quad - \langle 0 | \, g (A_\mu \times (\partial_\nu F^{\mu\nu} + J^\mu_{\text{QCD}})) \cdot c(w) \bar{c}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \, .\end{aligned} \tag{8.44}$$

The first three terms are similar to the ones in (8.32). The fourth term vanishes, due to the Dyson equation (8.33). For the fifth term we need a special form of the Dyson equatons,

$$\langle \chi \times O_\chi F \rangle = i \left\langle \chi \times \frac{\delta F}{\delta \chi} \right\rangle. \tag{8.45}$$

Lastly we assume that the last term in equation (8.44) vanishes due to an extension of (8.43). This however is not fully proved. A formal prove for this assumption will be investigated in future research. Pushing through and using equation RHS of equation (8.40), we find

$$\begin{aligned}i \langle 0 | \, \partial \cdot J^a_{\text{QCD}}(z) \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle = &\delta(y - z) g t^a \langle 0 | \, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \\ &- \delta(x - z) \langle 0 | \, \Phi_n(\infty, y)\psi(y)\bar{\psi}(x) \, |0\rangle \, g t^a\end{aligned} \tag{8.46}$$

which leads to the Ward identity we wanted to find

$$k_\mu J^{a,\mu}(p, k, n) = g J(p, n). \tag{8.47}$$

This is not yet a full proof. First of all, we used the assumption that

$$\langle 0|\, g(A_\mu \times (\partial_\nu F^{\mu\nu} + J^\mu_{\mathrm{QCD}})) \cdot c(w)\bar{c}(z)\Phi_n(\infty, y)\psi(y)\bar{\psi}(x)\,|0\rangle = 0, \tag{8.48}$$

as a result of equation (8.43). Secondly, during the derviation we used

$$\frac{\delta\Phi_n(\infty, y)}{\delta A^\nu(w)} = -ie\int_y^\infty dv_\nu\ \delta^d(v-w)\,\Phi_n(\infty, y). \tag{8.49}$$

Although this is most probably true for the non-abelian case as well, we still need to check this.

Summary and Outlook

In this thesis we first studied the factorization of soft gluons. The extension to next-to-soft gluons is not so clear, but can be studied using the work of Del Duca [2]. We mainly focussed on the Ward identity of the radiative jet in QCD, which has important implications for NLP factorization. With the use of the radiative jet Ward identity, the radiative amplitude can be found to $\mathcal{O}(k^0)$ in terms of the non-radiative amplitude. We found that the Ward identity for the radiative jet is most probably given by

$$k_\mu J^{a,\mu}(p,k,n) = gJ(p,n). \tag{8.50}$$

In further research the Ward identity can be checked on a diagram-by-diagram basis. This will give an extra argument on whether or not we found the right non-abelian generalization of the Ward identity. There is also still work in sorting out the last details considering the derivation of the Ward identity. Furthermore, once the correctness of the identity is established, one could generalize to gluon initiated jets. These arise in gluon fusion processes, as for example in Higgs-production.

The tools discussed in this thesis, with the focus on Ward identities for radiative jets, are important for NLP factorization theorems. The ultimate goal is to find a factorization which holds up to NLP and make resummation of the NLP logarithms possible.

Appendix A

Conventions

In this thesis we use the $(+---)$ metric. Underneath we list some corresponding conventions and usefull identities.

$$
\begin{aligned}
\mathcal{L}_0 &= \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \\
S_{\alpha\beta} &= \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} \\
\phi(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\phi(k) \\
\phi(k) &= \int d^4x e^{ikx}\phi(x)
\end{aligned}
$$

Operator expansions for the scalar and Dirac field in this metric are given by

$$
\begin{aligned}
\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x}) \\
\phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a_p^s e^{-ip\cdot x} + a_p^{s\dagger} e^{ip\cdot x})
\end{aligned} \tag{A.1}
$$

where the Dirac spinors obey the following Dirac equations and normalization

$$
\begin{aligned}
(\not{p} - m)\, u_s(p) &= 0 \\
(\not{p} + m)\, v_s(p) &= 0 \\
\sum_s u_a^s(p)\bar{u}_b^s(p) &= (\not{p} + m)_{ab} \\
\sum_s v_a^s(p)\bar{v}_b^s(p) &= (\not{p} - m)_{ab}
\end{aligned} \tag{A.2}
$$

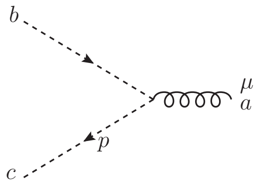
Some other usefull identities for Gamma matrices and Dirac spinors read

$$
\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\eta_{\mu\nu} \\
\gamma^0\gamma^\mu\gamma^0 &= \gamma^{\mu\dagger} \\
\bar{v}_s(p)\gamma^0 u_s(p) &= 0 \\
\bar{v}_s(p)\gamma^0 v_{s'}(p) &= 2E_p\, \delta_{ss'} \\
\bar{u}_s(p)\gamma^0 u_{s'}(p) &= 2E_p\, \delta_{ss'} \\
\bar{v}_s(p) v_{s'}(p) &= -2m\, \delta_{ss'} \\
\bar{u}_s(p) u_{s'}(p) &= 2m\, \delta_{ss'}
\end{aligned} \tag{A.3}
$$

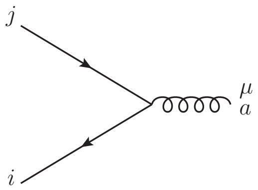
Appendix B

Feynman Rules

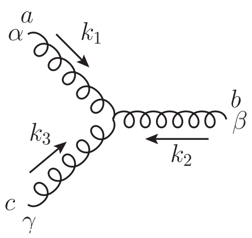
Below we list the Feynman rules for the QCD Lagrangian given in equation (2.28). The QCD fields can interact with each other through the following vertices:



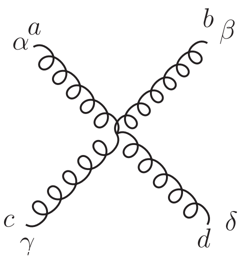
$$= \qquad g p^{\mu} f_{abc}$$



$$= \qquad -ig\gamma^{\mu}(t_a)_{ij}$$

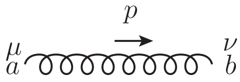


$$= \qquad -gf^{abc}[g^{\alpha\beta}(k_1-k_2)^{\gamma}+g^{\beta\gamma}(k_2-k_3)^{\alpha}+g^{\gamma\alpha}(k_3-k_1)^{\beta}]$$

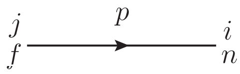


$$= \qquad -ig^2\begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}) \\ +f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\gamma\beta}) \\ +f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix} \tag{B.1}$$

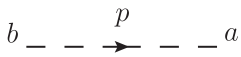
The propagators for each field present in QCD:



$$= \frac{-i\,\delta_{ab}}{p^2+i\epsilon}\left(g^{\mu\nu}-(1-\lambda^{-1})\frac{p^\mu p^\nu}{k^2}\right)$$

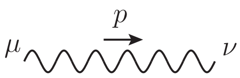


$$= \frac{i(\not{p}+m)_{ij}\,\delta_{nf}}{p^2-m^2+i\epsilon}$$

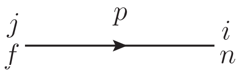


$$= \frac{i\,\delta_{ab}}{p^2+i\epsilon} \tag{B.2}$$

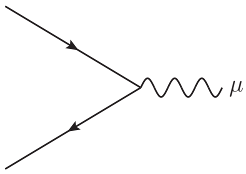
To be complete we also list the QED Feynman rules:



$$= \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu}-(1-\lambda^{-1})\frac{k^\mu k^\nu}{k^2}\right)$$



$$= \frac{i(\not{p}+m)_{ij}}{p^2-m^2+i\epsilon}$$



$$= -ig\gamma^\mu \tag{B.3}$$

Appendix C

Schwinger Dyson equation

A tool to describe the implications of the classical conservation of the Noether currents on the quantum level is to introduce the Euler-Lagrange operators O_{ϕ_i}, for any field ϕ_i,

$$O_{\phi_i} = \frac{\delta S}{\delta \phi_i} \tag{C.1}$$

On the quantum level we can use the generating functional to write

$$\int [D\phi_i] e^{iS} \frac{\delta S}{\delta \phi_i} X = \int [D\phi_i] \big(-i \frac{\delta}{\delta \phi_i} e^{iS} \big) X = i \int [D\phi_i] e^{iS} \frac{\delta}{\delta \phi_i} X \tag{C.2}$$

where X is a product of fields. In terms of operator formalism this reads

$$\langle O_{\phi_i}(z) X \rangle = i \, \langle \frac{\delta X}{\delta \phi_i(z)} \rangle \tag{C.3}$$

and this is just the Swinger-Dyson equation in compact form, i.e. the quantum equivalent of the classical equations of motion $O_{\phi_i} = 0$.

Appendix D

Feynman parametrization

Loop integrals arizing from Feynman diagrams with several loops are mostly hard to evaluate. They contain denominators like $\frac{1}{A_1...A_n}$. Feynman parametrization is a technique to rewrite a of denominators such that they may be evaluated more easily. The idea is to introduce auxiliary parameters to write a product of denominators into a sum to a specific power. It works as follows

$$\frac{1}{A_1....A_n} = (n-1)! \int_0^1 dx_1 ... \int_0^1 dx_n \frac{\delta(1 - \sum_{i=1}^n x_i)}{(\sum_{i=1}^n x_i A_i)^n} \tag{D.1}$$

where the auxiliary parameters x_i are called the Feynman parameters. A full proof will not be shown here, but it includes an explicit evaluation of the $n = 2$ case and the induction step to generalize to generic n.

Appendix E

Ward Identity local gauge transformation QED

We will derive the Greens funciton Ward identities for QED. Consider a change of fields in the correlation function. Choose it to be equal to the form of an infinitesimal local gauge transformations. For the fields present in QED these local gauge transformations imply

$$\begin{aligned} \delta_g \psi(x) &= ie\theta[x]\psi(x), & \delta_g \bar{\psi}(x) &= \bar{\psi}(x)[-ie\theta(x)], \\ \delta_g A^\mu(x) &= -\partial^\mu \theta(x), & \delta_g \mathcal{L}[\psi, \bar{\psi}, A] &= \lambda(\partial \cdot A)(\Box\theta). \end{aligned} \tag{E.1}$$

Here, θ is an arbitrary (small) function of space-time. The Jacobian of this transforamtion is unity, since the phases beween $\bar{\psi}(x)$ and $\psi(x)$ cancel for every x and the shift in A is purely additive. Changing variables in the correlation function means

$$\sum_{i=1}^{n} \langle 0| \phi_1(x_1) \dots \delta_g phi_i(x_i) \dots \phi_n(x_n) |0\rangle + i \int d^4y \langle 0| \phi_1(x_1) \dots \phi_n(x_n) \delta\mathcal{L}(\phi_j(y)) |0\rangle = 0 \tag{E.2}$$

where ϕ can be any field in the theory. Using the objects specific to QED, see equation (**??**), and taking the variation to $\theta(z)$, we find

$$\begin{aligned} -\lambda\Box_z \langle 0| \partial \cdot A(z) \prod_a A_{\mu_a} \prod_b \psi(y_b) \prod_c \bar{\psi}(w_c) |0\rangle \\ = \sum_d \langle 0| \partial \cdot A(z) \prod_{a\neq d} A_{\mu_a} \prod_b \psi(y_b) \prod_c \bar{\psi}(w_c) |0\rangle [9\partial_{\mu_d} \delta(x_d - z) \\ + e \langle 0| \partial \cdot A(z) \prod_a A_{\mu_a} \prod_b \psi(y_b) \prod_c \bar{\psi}(w_c) |0\rangle \delta(y_b - z) \\ - e \langle 0| \partial \cdot A(z) \prod_a A_{\mu_a} \prod_b \psi(y_b) \prod_c \bar{\psi}(w_c) |0\rangle \delta(w_b - z). \end{aligned} \tag{E.3}$$

Appendix F

Non-abelian Wilson Line

F.1 Wilson line bounded from above

If we bound the Wilson line from above we can consider the absorbtion of gluons from an external line created at $x_i = a$ and in the direction $n^\mu = p^\mu$. We fourier transform the gauge fields to momentum space as we did before in section 4.2. Only here we add the $-i\epsilon$ to make sure the λ integral is finite. The second order expansion term of the Wilson line, corresponding to the double absorbtion of gluons, then reads

$$-g^2 \int_a^\infty d\lambda_1 \int_{\lambda_1}^\infty d\lambda_2 \int \frac{d^4k_2}{(2\pi)^4} n \cdot A(k_2)\, e^{-i(k_2-i\epsilon)\cdot n\,\lambda_2} \int \frac{d^4k_1}{(2\pi)^4} n \cdot A(k_1) e^{-i(k_1-i\epsilon)\cdot n\,\lambda_1}, \tag{F.1}$$

Performing the integral over λ_2 and thereafter the integral over λ_1 we find

$$\begin{aligned}(\text{F.1}) &= -g^2 \iint \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \int_a^\infty d\lambda_1\, n \cdot A(k_2)\, n \cdot A(k_1) e^{-i(k_1+k_2-i\epsilon)\cdot n\lambda_1} \frac{-1}{-i(k_2 \cdot n - i\epsilon)} \\ &= -g^2 \iint \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4}\, n \cdot A(k_2)\, n \cdot A(k_1) \frac{-i}{((k_1+k_2)\cdot n - i\epsilon)} \frac{-i}{(k_2 \cdot n - i\epsilon)} e^{-i(k_1+k_2-i\epsilon)\cdot na}\end{aligned} \tag{F.2}$$

As one can see we find the product of two eikonal vertices, but with a specific ordering of the gluon fields. Repeating this procedure to all orders in perturbation theory we find

$$\Phi_n(a,\infty) = \sum_{m=0}^\infty (-ig)^m \int \left(\frac{d^4k_i}{(2\pi)^4}\right)^m n \cdot A(k_m) \ldots n \cdot A(k1) e^{-iK(j)\cdot na} \prod_{j=1}^m \frac{-i}{n \cdot \tilde{K}(j) - i\epsilon} \tag{F.3}$$

where $K(j) = \sum_{i=j}^m k_i$ and $\tilde{K}(j) = \sum_{i=j}^m k_{m-i+1}$. This is an Wilson line comming in from infinity with oposite momentum to the line momentum, absorbing m gluons on its way.

F.2 Finite Wilson Line

Finite Wilson lines are not used often in scattering amplitues, but we will still consider them here for completeness. Assume a Wilson line stretching from a^μ to b^μ. In order to see how equation (4.18) comes about we will expand the Wilson line from (8.10). We will explicitly investigate in the second order expansion term, corresponding to the double absorbtion of gluons. Thereafter we will make a compelling argument for all other orders.

After fourier transforming the gauge fields to momentum space, the second order expansion term of the Wilson line reads

$$(-ig)^2 \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\int_a^b d\lambda_1 \int_{\lambda_1}^b d\lambda_2 n\cdot A(k_2)e^{-ik_2\cdot n\lambda_2} n\cdot A(k_1)e^{ik_1\cdot n\lambda_1} \tag{F.4}$$

Performing the λ_2 and λ_2, in that order, we find

$$\begin{aligned}(F.4) &= (-ig)^2 \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\int_a^b d\lambda_1 n\cdot A(k_2) n\cdot A(k_1)\frac{i}{k_2\cdot n}\big(e^{-ik_2\cdot nb}-e^{-ik_2\cdot n\lambda_1}\big)e^{ik_1\cdot n\lambda_1}\\ &= (-ig)^2 \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} n\cdot A(k_2) n\cdot A(k_1)\ \Big[\frac{i}{k_2\cdot n}\frac{i}{k_1\cdot n}e^{-ik_2\cdot nb}(e^{-ik_1\cdot nb}-e^{-ik_1\cdot na})\\ &\quad + \frac{i}{k_2\cdot n}\frac{i}{(k_1+k_2)\cdot n}(e^{-i(k_1+k_2)\cdot na}-e^{-i(k_1+k_2)\cdot nb})\Big]\end{aligned} \tag{F.5}$$

Now one could use the eikonal identity as given in equation (4.6) to reduce the terms multiplying $e^{-i(k_1+k_2)\cdot nb}$. The three remaining terms are

$$\begin{aligned}(F.4) = (-ig)^2 \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} n\cdot A(k_2) n\cdot A(k_1)\ \Big[\frac{-i}{k_2\cdot n}\frac{i}{k_1\cdot n}e^{-ik_2\cdot na}e^{-ik_1\cdot nb}\\ + \frac{i}{k_2\cdot n}\frac{i}{(k_1+k_2)\cdot n}e^{-i(k_1+k_2)\cdot na} + \frac{i}{k_1\cdot n}\frac{i}{(k_1+k_2)}e^{-i(k_1+k_2)\cdot nb}\Big]\end{aligned} \tag{F.6}$$

which are exactly the terms resulting from the $m = 2$ part of the summation in equation (4.18). For a full proof an induction step is required. Here we will just provide some observations on how each higher order of m would also be discribed by equation (4.18). For $m = 3$ there is one more λ integral than in the calculation above and one would end up with 8 terms after performing all of them. There are 4 different prefactors:

$$\frac{1}{k_3k_2k_1} \qquad \frac{1}{k_3k_2(k_1+k_2)} \qquad \frac{1}{k_3(k_2+k_3)k_1} \qquad \frac{1}{k_3(k_2+k_3)(k_1+k_2+k_3)} \tag{F.7}$$

Now all 4 of them are multiplicated by two different exponentials depending on a or b. Group terms with the same exponential and use eikonal identities such that one ends up with 4 terms as described by (4.18). This was just to give an intuition on how to proceed, not to give a full proof.

Hermitian conjugate

We start with the formal definition of a finite the Wilson line, as given in equation (4.11):

$$\Phi_n(\lambda_1,\lambda_2) = \mathcal{P}\exp\big[-ig\int_{\lambda_1}^{\lambda_2} d\lambda\ \ n\cdot A^a(\lambda)t_a\big]. \tag{F.8}$$

To see what it means to hermitian conjugate this expression we will make an expansion of the path ordered exponent and manipulate the boundaries of the integral in the following way

$$\begin{aligned}\Phi_n(a,b)^\dagger &= [\mathcal{P}\exp\big[-ig\int_a^b d\lambda\ n\cdot A^a(\lambda)t_a\big]]^\dagger,\\ &= \big[1-ig\int_a^b d\lambda\ n\cdot A^a(\lambda)t_a + (-ig)^2\int_a^b d\lambda\int_a^\lambda d\lambda'\ n\cdot A(\lambda)n\cdot A(\lambda')+\mathcal{O}(g^3)\big]^\dagger\\ &= 1+ig\int_a^b d\lambda\ n\cdot A^a(\lambda)t_a + (ig)^2\int_a^b d\lambda\int_a^\lambda d\lambda'\ n\cdot A(\lambda')n\cdot A(\lambda)+\mathcal{O}(g^3)\end{aligned} \tag{F.9}$$

where in the last line we used that the gauge field is real and $(AB)^\dagger = B^\dagger A^\dagger$. Now we switch the boundaries of integration in all terms, which yields a minus sign for each integral. Furthermore we can rename the integration variables in the second term, $\lambda \leftrightarrow \lambda'$. These steps result in

$$\begin{aligned}\Phi_n(a,b)^\dagger &= 1 - ig\int_b^a d\lambda\; n\cdot A^a(\lambda)t_a + (-ig)^2\int_b^a d\lambda\int_\lambda^a d\lambda'\; n\cdot A(\lambda')n\cdot A(\lambda) + \mathcal{O}(g^3)\\ &= 1 - ig\int_b^a d\lambda\; n\cdot A^a(\lambda)t_a + (-ig)^2\int_b^a d\lambda'\int_{\lambda'}^a d\lambda\; n\cdot A(\lambda)n\cdot A(\lambda') + \mathcal{O}(g^3)\end{aligned} \tag{F.10}$$

Now we can use equation (4.13) to change the boundaries of the λ integral in the third term.

$$\Phi_n(a,b)^\dagger = 1 - ig\int_b^a d\lambda\; n\cdot A^a(\lambda)t_a + (-ig)^2\int_b^a d\lambda\int_b^\lambda d\lambda'\; n\cdot A(\lambda)n\cdot A(\lambda') + \mathcal{O}(g^3) \tag{F.11}$$

It becomes clear that the fields are now anti-path ordered, since $\lambda' > \lambda$. As stated before in section 4.2 this is similar to changing the directional vector n to $-n$, because then b^μ corresponds to a point earlier on the path than a^μ and path ordering is restored. Therefore we conclude that

$$\Phi_n(a,b)^\dagger = \Phi_{-n}(b,a), \tag{F.12}$$

which is diagrammatically depicted in figure F.1

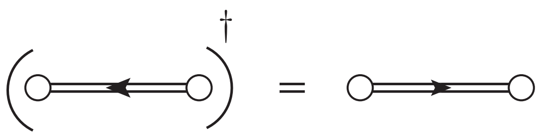


Figure F.1: Diagramatic explaination for the hermitan conjugate of finite Wilson lines

Appendix G

LSZ reduction

In this section we derive a general relation between correlation functions and S-matrix elements, known as the LSZ reduction formula. We consider the overlap of incoming and outgoing fields $\langle \text{in}|\text{out}\rangle$. Using the adiabatic hypothesis that fields behave as if they are free of interactions in the limit where $x^0 \to -\infty$ and $x^0 \to \infty$, we define

$$\lim_{x^0 \to -\infty} \phi(x) = \phi_{\text{in}}(x) \qquad\qquad \lim_{x^0 \to \infty} \phi(x) = \phi_{\text{out}}(x) \tag{G.1}$$

where $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$ are free fields that obey the free field equations of motion. We now see that

$$\begin{aligned} \phi(x) &= \phi_{\text{in}}(x) + \int d^4y\, D_{\text{ret}}(x-y)\, j(y), \\ \phi(x) &= \phi_{\text{out}}(x) + \int d^4y\, D_{\text{adv}}(x-y)\, j(y). \end{aligned} \tag{G.2}$$

Furthermore we can derive that

$$\begin{aligned} a_p &= \int_t \frac{d^3x}{\sqrt{(2\pi)^3\, 2E_p}}\, i\, e^{ipx}\, \overleftrightarrow{\partial_0}\, \phi(x) \\ a_p^\dagger &= \int_t \frac{d^3x}{\sqrt{(2\pi)^3\, 2E_p}}\, (-i)\, e^{-ipx}\, \overleftrightarrow{\partial_0}\, \phi(x) \end{aligned} \tag{G.3}$$

If we now look at an S-matrix element, which are the amplitudes of transitions between *in* and *out* states, we can write

$$\langle \text{out}\{q\}|\, \text{in}\{p\}\rangle = \langle \text{out}|\, a_{\text{in}}^\dagger(p)\, |0\rangle\,, \tag{G.4}$$

where we assume just one ingoing particle with momentum p. If we now assume that the *out* state does not contain any particle with momentum p (ignore forward scattering), we can write

$$\begin{aligned} \langle \text{out}\{q\}|\, \text{in}\{p\}\rangle &= \langle \text{out}|\, (a_{\text{in}}^\dagger(p) - a_{\text{out}}^\dagger(p))\, |0\rangle \\ &= \langle \text{out}| \int_t \frac{d^3x}{\sqrt{(2\pi)^3\, 2E_p}}\, (-i)\, e^{-ipx}\, \overleftrightarrow{\partial_0}\, (\phi_{\text{in}} - \phi_{\text{out}})\, |0\rangle\, b \\ &= \big(\lim_{x^0\to\infty} - \lim_{x^0\to-\infty}\big)\, \langle \text{out}| \int_t \frac{d^3x}{\sqrt{(2\pi)^3\, 2E_p}}\, (-i)\, e^{-ipx}\, \overleftrightarrow{\partial_0}\, \phi\, |0\rangle \\ &= -\int_{-\infty}^{\infty} \partial_0\, \langle \text{out}| \int_t \frac{d^3x}{\sqrt{(2\pi)^3\, 2E_p}}\, (-i)\, e^{-ipx}\, \overleftrightarrow{\partial_0}\, \phi\, |0\rangle \\ &= -\, \langle \text{out}| \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}}\, (-i)\, (e^{-ipx}\, \partial_0^2\, \phi - \phi\, \partial_0^2\, e^{-ipx})\, |0\rangle \end{aligned} \tag{G.5}$$

If we now use that $(\partial^2 + m^2)e^{-ipx} = 0$ we find that

$$\begin{aligned}
\langle \mathrm{out}\{q\} | \, \mathrm{in}\{p\}\rangle &= - \langle \mathrm{out}| \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}} (-i)\, e^{-ipx} \, (\partial_0^2 - \partial_i^2 + m^2)\phi(x) \, |0\rangle \\
&= i \, \langle \mathrm{out}| \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}} \, e^{-ipx} \, (\partial_0^2 - \partial_i^2 + m^2)\phi(x) \, |0\rangle \\
&= i \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}} \, e^{-ipx} \, (\partial^2 + m^2) \, \langle \mathrm{out}| \, \phi(x) \, |0\rangle \\
&= i \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}} \, e^{-ipx} \, (-p^2 + m^2) \, \langle \mathrm{out}| \, \phi(x) \, |0\rangle
\end{aligned} \tag{G.6}$$

This calculation can be repeated for the *out* states or in the case that there are multiple particles in the *in* / *out* state. We find that we can replace

$$\begin{aligned}
a^\dagger_{\mathrm{in}}(p) &\to i \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}} \, e^{-ipx} \, (-p^2 + m^2)\phi(x) \\
a_{\mathrm{out}}(p) &\to i \int \frac{d^4x}{\sqrt{(2\pi)^3\, 2E_p}} \, e^{ipx} \, (-p^2 + m^2)\phi(x)
\end{aligned} \tag{G.7}$$

The same can be done for fermions. First we see from (A.1) that

$$\begin{aligned}
a_p^{s\dagger} &= \int d^3x e^{-ipx} \bar{\psi}(x)\gamma^0 u^s(p) \\
b_p^{s\dagger} &= \int d^3x e^{-ipx} \bar{v}(p)\gamma^0 \psi(x)
\end{aligned} \tag{G.8}$$

In the same way as we did before we can look at the S-matrix element. If we assume two incoming fermion with momentum p_1 and p_2 and two outgoing fermions with momentum q_1 and q_2 we can write

$$\langle \, \mathrm{in}\{q, s'\} \, | \, \mathrm{out}\{p, s\}\rangle = \langle 0| \, a^{s'_1}_{\mathrm{out}}(q_1) \, a^{s'_2}_{\mathrm{out}}(q_2) \, a^{s_1\dagger}_{\mathrm{in}}(p_1) \, a^{s_2\dagger}_{\mathrm{in}}(p_2) \, |0\rangle \tag{G.9}$$

Now note that the *out* operators are inserted at $t \to -\infty$ and the *in* operators are inserted at $t \to \infty$. This means that they are naturally time ordered and we could write

$$\langle \, \mathrm{in}\{q, s'\} \, | \, \mathrm{out}\{p, s\}\rangle = \langle 0| \, T\{a^{s'_1}_{\mathrm{out}}(q_1) \, a^{s'_2}_{\mathrm{out}}(q_2) \, a^{s_1\dagger}_{\mathrm{in}}(p_1) \, a^{s_2\dagger}_{\mathrm{in}}(p_2)\} \, |0\rangle \tag{G.10}$$

If we now replace $a^{s'_1}_{\mathrm{out}}(q_1)$ by $a^{s'_1}_{\mathrm{out}}(q_1) - a^{s'_1}_{\mathrm{in}}(q_1)$, the term with the *in* operator would be replaced to the right by the time ordered product to annihilate the vacuum. We can therefore safely do this without affecting the current expression. This can be done for any operator, but let us just choose one for simplicity.

$$\begin{aligned}
\langle \, \mathrm{in}\{q, s'\} \, | \, \mathrm{out}\{p, s\}\rangle &= \langle 0| \, T\{a^{s'_1}_{\mathrm{out}}(q_1) \, a^{s'_2}_{\mathrm{out}}(q_2) \, a^{s_1\dagger}_{\mathrm{in}}(p_1) \, \big(a^{s_2\dagger}_{\mathrm{in}}(p_2) - a^{s_2\dagger}_{\mathrm{out}}(p_2)\big)\} \, |0\rangle \\
&= \langle 0| \, T\{a^{s'_1}_{\mathrm{out}}(q_1) \, a^{s'_2}_{\mathrm{out}}(q_2) \, a^{s_1\dagger}_{\mathrm{in}}(p_1) \, (-) \int dx^0 \partial_0 \, a^{s_2\dagger}_{p_2}\} \, |0\rangle \\
&= \langle 0| \, T\{a^{s'_1}_{\mathrm{out}}(q_1) \, a^{s'_2}_{\mathrm{out}}(q_2) \, a^{s_1\dagger}_{\mathrm{in}}(p_1) \, (-) \int dx^0 \partial_0 \int d^3x \, e^{-ip_2x} \bar{\psi}(x)\gamma^0 u^{s_2}(p)\} \, |0\rangle \\
&= \langle 0| \, T\{a^{s'_1}_{\mathrm{out}}(q_1) \, a^{s'_2}_{\mathrm{out}}(q_2) \, a^{s_1\dagger}_{\mathrm{in}}(p_1) \, (-) \int dx^4 \, e^{-ip_2x}(-i(p_2)_0 + \partial_0)\bar{\psi}(x)\gamma^0 u^{s_2}(p)\} \, |0\rangle
\end{aligned} \tag{G.11}$$

Using the fact that the spinors obey (A.2) we can reduce this expression to

$$\begin{aligned}
\langle\, \text{in}\{q,s'\}\,|\,\text{out}\{p,s\}\rangle &= \langle 0|\,T\{-A\int dx^4\, e^{-ip_2x}\,\bar\psi(x)\,(i\gamma^i(p_2)_i - im + \gamma^0\overleftarrow{\partial_0})\,u^{s_2}(p)\}\,|0\rangle \\
&= \langle 0|\,T\{-A\int dx^4\, e^{-ip_2x}\,\bar\psi(x)\,(-\gamma^i\overleftarrow{\partial_i} + \gamma^0\overleftarrow{\partial_0} - im)\,u^{s_2}(p)\}\,|0\rangle \\
&= \langle 0|\,T\{-A\int dx^4\, e^{-ip_2x}\,\bar\psi(x)\,(\overleftarrow{\not\partial} - im)\,u^{s_2}(p)\}\,|0\rangle \\
&= \langle 0|\,T\{iA\int dx^4\, e^{-ip_2x}\,\bar\psi(x)\,(i\overleftarrow{\not\partial} + m)\,u^{s_2}(p)\}\,|0\rangle
\end{aligned} \tag{G.12}$$

where $A = a^{s_1'}_{\text{out}}(q_1)\,a^{s_2'}_{\text{out}}(q_2)\,a^{s_1\dagger}_{\text{in}}(p_1)$. A similar calculation can be performed for the other operators. This results in

$$\begin{aligned}
\langle\, \text{in}\{q,s'\}\,|\,\text{out}\{p,s\}\rangle =& i^4 \iiiint dy^4\,dz^4\,dw^4\,dx^4 \\
&\times e^{iq_1z}\,e^{iq_2w}\,(-i\not\partial_z + m)(-i\not\partial_w + m)\bar u^{s_1'}(q_1)\,\bar u^{s_2'}(q_2) \\
&\times \langle 0|\,T\{\psi(w)\psi(z)\bar\psi(x)\bar\psi(y)\}\,|0\rangle \\
&\times (i\overleftarrow{\not\partial}_x + m)(i\overleftarrow{\not\partial}_y + m)\,e^{-ip_1y}\,e^{-ip_2x}\,u^{s_2}(p_2)\,u^{s_1}(p_1).
\end{aligned} \tag{G.13}$$

Now we can see that this can easily be generalised for any *in* or *out* state by applying

$$\begin{aligned}
a_s^\dagger(p)_{\text{in}} &\to i\int d^4x\bar\psi(x)(i\overleftarrow{\not\partial} + m)u_s(p)e^{-ipx}, \\
a_s(p)_{\text{out}} &\to i\int d^4x\bar u_s(p)e^{ipx}(-i\not\partial + m)\psi(x), \\
b_s^\dagger(p)_{\text{in}} &\to -i\int d^4xe^{-ipx}\bar v_s(p)(-i\not\partial + m)\psi(x), \\
b_s(p)_{\text{out}} &\to -i\int d^4x\bar\psi(x)(i\overleftarrow{\not\partial} + m)v_s(p)e^{ipx}.
\end{aligned} \tag{G.14}$$

Furthermore note that for a non-interacting theory the S-matrix element will always vanish, due to the fermionic free field equation of motion. For an interacting theory however the equations of motion read

$$(i\not\partial - m)\psi(x) = j(x). \tag{G.15}$$

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