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POINCARÉ-HOPF THEOREM AND MORSE THEORY

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A bachelor thesis in mathematics

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1 Introduction

I always liked to visualise various things. For example, when listening to a story I always tried to make a picture in my mind.
Sometimes, I have the same with mathematics. In the course *Topology and Geometry*, for example, we learned that the annulus is homotopy equivalent to a circle and I was fascinated by the fact that we can visualise this in our mind.
An example of a theorem which is possible to visualise is the following.

Theorem 1.1 (Hairy Ball Theorem, [5], p. 435)**.** *There does not exist a vectorfield which is non-zero on all* S^2*.*

We can say that this theorem states that you can not comb the hair on a hairy ball without leaving bald spots.
Therefore, I was really excited when my supervisor prof. dr. Marius Crainic proposed the Poincaré-Hopf theorem as a subject of my bachelor thesis. It is namely a generalisation of the Hairy Ball theorem.

Theorem 1.2 (Poincaré-Hopf Theorem, [7], p. 134)**.** *Let* M *be a compact oriented smooth manifold. Let* $\vec{v}$ *be a smooth vectorfield on* M *with finitely many zero's. Let* $\{x_i \in M \mid i \in I \subset \mathbb{Z}\}$ *be the set of zero's of the vectorfield* $\vec{v}$*. Then the global sum of the indices of* $\vec{v}$ *equals the Euler characteristic of* M*. In other words*

$$\sum_{i \in I} \mathrm{ind}_{x_i}(\vec{v}) = \chi(M),$$

where $\mathrm{ind}_{x_i}(\vec{v})$ *denotes the index of* $\vec{v}$ *at its zero* x_i *and* $\chi(M)$ *denotes the Euler characteristic of* M*.*

The general goal of this thesis is to compare two different ways of proving the Poincaré-Hopf theorem. One of them includes Morse theory and the other does not.
Of course, we must first understand the statement of 1.2 and therefore we start with a chapter containing preliminary definitions, for example that of smooth manifolds. To understand the Euler characteristic properly we shall discuss homology theory and De Rham cohomology. In the chapter that follows we will see that various definitions of the Euler characteristic are, in fact, equivalent. Then there remains only one thing, naimly the index of a vector field. What follows is our first proof of the Poincaré-Hopf theorem without Morse theory. Once we have done that we will turn to Morse theory and eventually give a second proof using Morse theory. Finally, we end the thesis with a conclusion where we will also discuss the Hairy Ball theorem.

Contents

2 Preliminaries

We shall start with a few definitions and basic results in differential topology, which we will use throughout the thesis. This chapter is mainly based on *Introduction to Smooth Manifolds* [5] by John. M. Lee and *Topology from the Differential Viewpoint* [4] by John M. Lee. Let us start with manifolds.

2.1 Manifolds

Definition 2.1. Let M be a topological space. Then M is a *topological m-dimensional manifold without boundary* if

1. M is Hausdorff
2. M is second countable
3. M is locally Euclidean of dimension m.

Definition 2.2. The topological space M is a *topological m-dimensional manifold with boundary* if

1. M is Hausdorff
2. M is second countable
3. M is locally homeomorphic to an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$,

where $\mathbb{H}^m \subset \mathbb{R}^m$ is the closed m-dimensional upper half-space.

Remark 2.3. We will mostly use manifolds without boundary, so when we talk about just manifolds we actually mean manifolds without manifolds unless otherwise stated.

Remark 2.4. The Poincaré-Hopf theorem can be slightly adapted such that it also holds for manifolds with boundary, but we won't consider that in this thesis.

Definition 2.5. Let $U' \subseteq \mathbb{R}^n$ and $V' \subseteq \mathbb{R}^m$ be open sets. We say that the function $f : U' \to V'$ is *smooth* if all of its component functions have continuous partial derivatives in all orders. If F has the additional properties that it is bijective and has a smooth inverse, we say that F is a *diffeomorphism*.

Definition 2.6. With a *chart* (U, φ) on M we mean a pair which consists of an open set $U \subseteq M$ and a homeomorphism $\varphi : U \to U'$ where $U' \subseteq \mathbb{R}^m$ is an open set. We say that two charts (U, φ) and (V, ψ) on M are *smoothly compatible* if either $U \cap V$ is empty or $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Definition 2.7. *An atlas* $\mathcal{A}$ for M is a collection of charts, such that the domains of the charts cover M. We will call $\mathcal{A}$ a *smooth atlas* if all of its charts are smoothly compatible with each other. If a smooth atlas on M is maximal we say that it is a *smooth structure* on M and any chart in it is called a *smooth chart*.

Now using these definition we can finally define smooth manifolds, which are the main objects in this thesis.

Definition 2.8. A *smooth manifold* is a pair $(M, \mathcal{A})$ which consists of a topological manifold and a smooth structure.

Remark 2.9. To ease notation we will leave the smooth structure out. Furthermore, we will only talk about smooth manifolds from now on, so by saying 'M is a manifold' we actually mean 'M is a smooth manifold'.

Remark 2.10. Note that for smooth manifolds, the homeomorphism φ in (U, φ) is, in particular, a diffeomorphism.

2.2 Smooth maps

In this section we shall discuss definitions and results regarding smooth maps between manifolds. By now we have only defined smooth maps between open sets in $\mathbb{R}^n$, so consider the following definition.

Definition 2.11. Let M, N be manifolds. We say that $F : M \to N$ is a *smooth map* if for all $p \in M$ there exists smooth charts (U, φ) with $p \in U$ and (V, ψ) with $F(p) \in V$ such that

1. $F(U) \subseteq V$
2. $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth as in definition 2.5.

We will denote the set of smooth maps from M to $\mathbb{R}$ with $C^\infty(M)$.

Definition 2.12. Let $p \in M$, where M is a manifold. A *derivation* at p is a linear map $v : C^\infty \to \mathbb{R}$ which

1. is linear
2. satisfies $v(fg)f(p)vg + g(p)vf$ for all $f, g \in C^\infty(M)$.

We will denote the set of all derivations of $C^\infty(M)$ at p by T_pM, the *tangent space* to M at p. Each element in the tangent space is called a *tangent vector* at p.

Remark 2.13. The tangent space T_pM of an m-dimensional manifold is an m-dimensional vectorspace.

These definitions help us to define the differential of a smooth map.

Definition 2.14. Let M, N be manifolds and let $F : M \to N$ be a smooth map. Then we define *differential* $dF_p : T_pM \to T_{F(p)}N$ of F at $p \in M$ as follows. Let $v \in T_pM$ and let $f \in C^\infty(N)$, then

$$dF_p(v)(f) = v(f \circ F).$$

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2.3 Homotopy

In this section we shall recall some notions about homotopy theory.

Definition 2.15. Let X, Y be topological spaces and let $f_0, f_1 : X \to Y$ be continuous map. We define a *homotopy* from f_0 to f_1 to be a continuous map $H : X \times I \to Y$, where I is the closed interval $[0, 1] \subset \mathbb{R}$, such that

$$\begin{aligned} H(x, 0) &= f_0(x) \\ H(x, 1) &= f_1(x), \end{aligned}$$

for all $x \in X$. If such a homotopy exists, we say that f_0 and f_1 are homotopic to each other and we will denote this by $f_0 \sim f_1$.

Definition 2.16. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. We say that f is a *homotopy equivalence* if there exists another continuous map $g : Y \to X$ such that

$$\begin{aligned} f \circ g &= \mathrm{id}_Y \\ g \circ f &= \mathrm{id}_X. \end{aligned}$$

We say that X and Y are *homotopy equivalent* if there exists such a homotopy equivalence.

2.4 Vectorfields

Definition 2.17. Let M be a manifold. A *smooth vector field* $\vec{v}$ on M is a smooth map which assigns a tangent vector v_p to each point $p \in M$. These tangent vectors are elements of the *tangent space* T_pM. We define the tangent bundle TM to be the disjoint union of all tangent spaces. So

$$TM := \bigsqcup_{p \in M} T_pM.$$

We can see $\vec{v}$ as a map from M to TM.

Remark 2.18. We will only discuss smooth vectorfields, so to ease notation, we will leave the word 'smooth' out.

Now we consider the following simplified version of proposition 3.18 of *Introduction to Smooth Manifolds* [5] on tangent bundles. A proof can be found on pages 66-67.

Theorem 2.19. *Let M be a n-dimensional manifold. Then the tangen bundle TM can be seen as a $2n$-dimensional manifold.*

Proposition 2.20. Let M be an m-dimensional manifold with $\vec{v}$ a vectorfield on it. Let (U, φ) be a chart for M. We know that U is diffeomorphic to $\varphi(U) \subseteq \mathbb{R}^m$, so let $\{x_1, \ldots, x_m\}$ be the local coordinates on U. Then we can write

$$v_p = \sum_i^m v_i(p) \frac{\partial}{\partial x_i}\Big|_p,$$

for all $p \in M$.

3 Homology

To understand the Euler characteristic it is important to know what homology groups are. Therefore we shall recall some concepts in homology theory. For that we will largely follow *Algebraic Topology* [1] by A. Hatcher.
Singular homology is the main topic of this chapter, but we start with simpicial homology, because it is more intuitive. Therefore, readers who want to directly see the definition of singular homology groups can skip the section about simplicial homology. However, the last remark in this section about simplicial homology is of quite importance. After singular homology, we shall conclude with cellular homology.

3.1 Simplicial Homology

Consider the following definitions.

Definition 3.1. We define the set $\Delta^n \subset \mathbb{R}^n$ as follows

$$\Delta^n := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1,\ x_i \geq 0 \right\}.$$

This set is called a *standard n-simplex.*

Definition 3.2. Let $j \in \mathbb{N}$ such that $1 \leq j \leq n$. Then we define the *faces* of Δ^n to be the following sets

$$f_j\Delta^n := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1,\ x_i \geq 0, x_j = 0 \right\}$$
$$f_0\Delta^n := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\}.$$

Definition 3.3. A k-dimensional *simplicial complex* is a topological space X together with a collection of maps $\varphi_i^n : \Delta^n \to X$. Here $n \leq k$ and $i \in I_n$ where I_n is an index set for each n. These maps must satisfy the following two conditions

1. For all $m \in \{0, \dots n\}$ there is an $l \in I_{n-1}$ such that $\varphi_i^n(f_m\Delta^n) = \varphi_l^{n-1}(\Delta^{n-1})$
2. Let $i, j \in I_n$. If the intersection of $\varphi_i(\Delta^n)$ and $\varphi_j(\Delta^n)$ is non-empty, then it must be equal to a common face of $\varphi_i(\Delta^n)$ and $\varphi_j(\Delta^n)$.

We will call each $\varphi_i^n(\Delta^n)$ an *n-simplex of X*. Due to the second condition each n-simplex of X is uniquely determined by its boundary. The boundary consists of faces which are $(n-1)$-simplices, because of the first condition. Inductively each n-simplex can be represented with a unique set of vertices. For example, we can write a 1-simplex like $\varphi_i^1(\Delta^1) = [v_0, v_1]$, where v_0 and v_1 are vertices. To take orientation into account we say that $[v_0, v_1]$ is equal to $-[v_1, v_0]$. Similarly we can write each n-simplex as $[v_1, \dots, v_n]$, with v_i vertices.

Definition 3.4. Let X be a k-dimensional simplicial complex. Then we define the *chain set* $\Delta_n(X)$ to be the vector space with the n-simplices of X as its basis. If we indicate each n-simplex of X with $\varphi_i^n(\Delta^n)$ we get that each element of $\Delta_n(X)$ will be of the form $\sum_i a_i\varphi_i^n(\Delta^n)$ where $a_i \in \mathbb{R}$ and $i \in I_n$. These elements in $\Delta_n(X)$ are called *n-chains.*

Now we want to define the so called boundary map ∂_n on the chain set, but first we will define it for n-chains in the basis pf $\Delta_n(X)$.

$$\begin{aligned} \partial_n &: \Delta_n(X) \to \Delta_{n-1}(X) \\ &: [v_0, \dots, v_n] \mapsto \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]., \end{aligned}$$

where the hat above v_i stands for deleting this vertex. We define this map to be linear, from which it follows that it is well defined for the entirety of Δ_n.

Lemma 3.5 ([1], 2.1)**.** *As a result of this definition we get that $\partial_{n-1}\partial_n = 0$, where 0 stands for the zero-map.*

Proof. We have the following situation

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X).$$

It holds that

$$\begin{aligned}\partial_{n-1}\left(\sum_i(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_n]\right) &= \sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n] \\ &\quad+\sum_{i=0}^{n}\sum_{j=i+1}^{n}(-1)^{i+j+1}[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n] \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n] \\ &\quad+\sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j+1}[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n] \\ &= 0\end{aligned}$$

So indeed, we have that $\partial_{n-1}\partial_n = 0$. $\square$

We define $\partial_0 : \Delta_0(X) \to 0$ to be the map which sends everything in $\Delta_0(X)$ to identity element. A consequence is that $\mathrm{Im}\partial_{n+1} \subseteq \mathrm{Ker}\partial_n$ for all non-negative $n \in \mathbb{Z}$. What follows is the sequence below

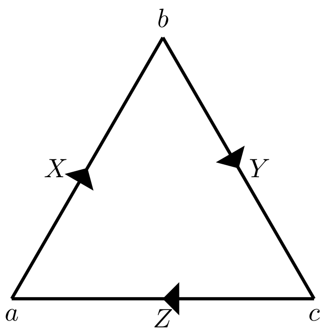
$$\cdots \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \longrightarrow \cdots \longrightarrow \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0.$$

This is called the chain complex of X.

Definition 3.6. The n^{th} *simplicial homology group with real coefficients* of this chain complex is defined to be $H_n^{\Delta}(X;\mathbb{R}) := \mathrm{Ker}\partial_n \; / \; \mathrm{Im}\partial_{n+1}$.

Consider the following examples.

Example 3.7. In this example we will discuss the simplicial homology of S^1. A circle can be represented by three vertices a, b, c and three edges X, Y, Z as in figure 1.



There are no n-simplices with $n \geq 2$, so $\mathrm{Ker}\partial_n \cong 0$ for $n \geq 2$. This implies that $H_n^{\Delta}(S^1;\mathbb{R}) \cong 0$ for $n \geq 2$.
We have three 0-simplices, so $\Delta_0(S^1) = \langle a, b, c\rangle \simeq \mathbb{R}^3$. Then $\mathrm{Ker}\partial_0 \cong \Delta_0(S^1) \cong \mathbb{R}^3$, because ∂_0 is the zero-map. We have 3 edges, so $\Delta_1(S^1) = \langle X, Y, Z\rangle$. Each edge can be represented by vertices. This gives us $X = [a, b], Y = [b, c]$ and $Z = [c, a]$. Then it holds that $\mathrm{Im}\partial_1 \cong \partial_1\Delta_1(S^1) = \langle [b] - [a], [c] - [b], [a] - [c]\rangle \cong \mathbb{R}^2$. We conclude that $H_0^{\Delta}(S^1;\mathbb{R}) \cong \mathbb{R}$.
What is left is $H_1^{\Delta}(S^1;\mathbb{R})$. It equals $\mathrm{Ker}\partial_1$, because $\mathrm{Im}\partial_2 = 0$. The kernel of ∂_1 is $\langle X + Y + Z\rangle$, so it is isomorphic to $\mathbb{R}$. We conclude that $H_1^{\Delta}(S^1;\mathbb{R}) \cong \mathbb{R}$.

Example 3.8. Now we will compute $H_n^{\Delta}(S^n;\mathbb{R})$. An n-sphere can be constructed by two copies of the standards n-simplex and identifying the boundaries of each of them together. Call the n-simplices A and B, and we know that these have the same boundary. It follows that $\mathrm{Ker}\partial_n = \langle A - B\rangle$ and we conclude that $H_n^{\Delta}(S^n;\mathbb{R}) \cong \mathbb{R}$ for all $n \in \mathbb{N}$.

Remark 3.9 ([1], p. 153)**.** The simplicial homology groups defined in *Algebraic Topology* [1] are with integer coefficients in contrast with what we defined using real coefficients. This, however, is not a problem, because most results remain roughly the same. This also holds for singular and cellular homology in the following sections.
The reason why we use real coefficients is the fact that we will not have to deal with torsion, which we would have if we used integer coefficients. This gets more clear after definition 3.24 in the next section.

3.2 Singular Homology

While studying homology theory I used various sources, among which the online videos by prof. Pierre Albin. A quote by him during one of his lectures [17] goes as follows:

> [...] generally speaking singular homology is wonderful when you want to prove things and the homology associated with Δ-complex structures is wonderful when you want to compute things.
>
> (Pierre Albin, l.c. minute 30:57).

Therefore, we shall discuss singular homology in this section.

Definition 3.10. Let X be a topological space and let $\sigma : \Delta^n \to X$ be a continuous map. The set $\sigma(\Delta^n)$ is called a *topological n-simplex.*

We define $C_n(X)$ to be the real vector space, with the topological n-simplices as basis. The boundary maps $\partial_n : C_n(X) \to C_{n-1}(X)$ are defined in the same way as for simplicial n-simplices. We then have again $\partial_{n+1}\partial_n = 0$, from which it follows that $\text{Im}\partial_{n+1} \subseteq \text{Ker}\partial_n$. We obtain the following chain complex

$$\cdots C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

The n^{th}-singular homology group with real coefficients of this chain comples is defined to be $H_n(X, \mathbb{R}) := \text{Ker}\partial_n \ / \ \text{Im}\partial_{n+1}$.
What follows from singular homology are the following few definitions and results, which we will use later.

Definition 3.11. Consider the following situation

$$\cdots X_{n+1} \xrightarrow{h_{n+1}} X_n \xrightarrow{h_n} X_{n-1} \longrightarrow \cdots$$

where X_i are vector spaces and h_i are linear maps. We say that a sequence of linear maps is an *exact sequence* if $\text{Ker}h_n = \text{Im}h_{n+1}$. This implies that this is a chain complex.

Definition 3.12. Consider the following situation

$$0 \xrightarrow{h_3} X \xrightarrow{h_2} Y \xrightarrow{h_1} Z \xrightarrow{h_0} 0, \tag{1}$$

where X, Y, Z are vector spaces and h_3, h_2, h_1, h_0 are linear maps. If this sequence is exact we say that this is a *short exact sequence.*

Lemma 3.13. *The sequence described in equation (1) is exact if and only if*

1. *the linear map $h_2 : X \to Y$ is injective;*
2. *and the linear map $h_1 : Y \to Z$ is surjective;*
3. *and the Ker(h_1) = Im(h_2).*

Proof. Let us prove the statement from right to left first. Consider the linear map $h_3 : 0 \to X : 0 \mapsto 0_X$, where 0_X is the zero-element of X. Then we see that $\text{Im}(h_3) = \{0_X\} \subset X$. Consider now $h_2 : X \to Y$. It is given that h_2 is injective and due to the fact that h_1 is a linear map, we see that $\text{Ker}(h_2) = \{0_X\}$. So indeed $\text{Im}(h_3) = \text{Ker}(h_2)$.
Furthermore, we already have that $\text{Ker}(h_1) = \text{Im}(h_2)$.
Consider $h_0 : Z \to 0$. We know that $\text{Ker}(h_0) = Z$, because h_0 is a linear map. We also know that $h_1 : Y \to Z$ is surjective, and thus we see that $\text{Im}(h_1) = Z = \text{Ker}(h_0)$. Our conclusion is that the sequence is exact.
Now we will prove the statement from left to right. It holds that $\text{Im}(h_3) = \text{Ker}(h_2)$. So $\text{Ker}(h_2) = \{0_X\}$, from which it follows that h_2 is injective.
Moreover, we already have $\text{Im}(h_2) = \text{Ker}(h_1)$.
Finally, the trivial group has only one element and thus $\text{Ker}(h_0) = Z$. Using $\text{Im}(h_1) = \text{Ker}(h_0)$ it follows that h_1 is surjective, which completes our prove. □

Definition 3.14. Let X be a topological space and let $A \subseteq X$. Then we define $C_n(X, A) := C_n(X)/C_n(A)$. For the boundary maps we have

$$\partial_n : C_n(X) \to C_{n-1}(X)$$
$$\partial_n : C_n(A) \to C_{n-1}(A).$$

So we get a well-defined sequence of boundary maps

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \cdots,$$

which is a chain complex. Again we have that $\partial_{n+1}\partial_n = 0$. Then we define the n^{th} *relative homology group with real coefficients* $H_n(X,A;\mathbb{R})$ of the chain complex to be $\text{Ker}\partial_n/\text{Im}\partial_{n+1}$

Remark 3.15. We can define the n^th relative simplicial homology group with real coefficients $H_n^{\Delta}(X,A;\mathbb{R})$ in a similar way.

Using these definitions we can give the relation between simplicial and singular homology in the following theorem. This theorem is a simplified version of theorem 2.27 in *Algebraic Topology* [1], for which a proof can be found on pages 128-130.

Theorem 3.16 ([1], 2.27)**.** *Let X and A be a simplicial complexes such that $A \subseteq X$. Then the vectorspaces $H_n^{\Delta}(X,A;\mathbb{R})$ and $H_n(X,A;\mathbb{R})$ are isomorphic to each other for all n.*

For later use we will introduce a slightly simplified version of the Excision theorem without proof. A proof can be found in *Algebraic Topology* [1] at page 219.

Theorem 3.17 (Excision Theorem, [1], 2.20)**.** *Let $Z \subseteq A \subseteq X$ be topological spaces. Suppose that the closure of Z is a subset of the interior of A. Then we have that*

$$H_n(X-Z, A-Z;\mathbb{R}) \to H_n(X,A;\mathbb{R}).$$

We conclude with a lemma which turns out to be useful in lemma's 8.24 and 8.25 for which a verification can be found in *Algebraic Topology* [1] on page 118.

Lemma 3.18. *Let X,Y,Z be topological spaces such that $Z \subseteq Y \subseteq X$. Then the sequence*

$$\cdots H_n(Y,Z) \longrightarrow H_n(X,Z) \longrightarrow H_n(X,Y) \longrightarrow H_{n-1}(Y,Z)\cdots$$

is exact.

3.3 Cellular Homology

In this section we shall define CW-complexes so that we can define the Euler characteristic.

Definition 3.19. We define a *CW-complex* X inductively.

1. We start with a set X^0 of points, which we will call the 0-skeleton.
2. Let $\mathbb{D}^n_\alpha$ denote an n-disk with $\alpha \in I$ an element of an index set. Furthermore, let $\varphi_\alpha : \partial\mathbb{D}^n_\alpha \to X^{n-1}$ be continuous maps. Then we define the n-skeleton X^n to be the following quotient space:

$$X^n := \left(X^{n-1} \sqcup_{\alpha\in I} \mathbb{D}^n_\alpha\right)\Big/(\varphi_\alpha(x)\sim x),$$

   for all $\alpha \in I$ and $x \in \mathbb{D}^n_\alpha$. We call these attached n-disks $\mathbb{D}^n_\alpha$ *n-cells.*
3. Then the *CW-complex* X would be the end result of this possibly infinite procedure.

Cellular homology theory is the theory about homology groups of CW-complexes. CW-complexes consist of n-cells, and thus we define $C_n^{CW}(X)$ to be the vector space with the n-cells as basis. Following pages 137-139 of *Algebraic Topology* [1] we can define a boundary map ∂^{CW} such that $\partial^{CW}_{n-1}\partial^{CW}_n = 0$ This gives us the cellular chain complex of X, from which we can deduce the cellular homology groups of X, which are denoted by $H_n^{CW}(X,\mathbb{R})$.

Remark 3.20. It holds that $C_n^{CW}(X) \cong H_n(X^n, X^{n-1};\mathbb{R})$ as stated in lemma 2.34 in *Algebraic Topology* [1].

An important property of cellular homology groups of X is stated in the following theorem for which a proof can be found in *Algebraic Topology* [1] on page 140.

Theorem 3.21. *Let X be a CW-complex. Then we have that $H_n^{CW}(X)$ is isomorphic to $H_n(X)$.*

Now let us define the Euler characteristic.

Definition 3.22. For a compact CW-complex X, we define the *Euler characteristic* to be

$$\chi(X) = \sum_n (-1)^n c_n,$$

where c_n is the number of n-cells of the CW-complex.

Example 3.23. Consider the 2-sphere S^2. The CW-complex structure of a 2-sphere can be described by a point p and a 2-disk, where we identify the boundary of the 2-disk with the point p. The point p is a 0-cell, and the 2-disk is a 2-cell. For the rest, there are no n-cells with n unequal to 0 or 2. Then the Euler characteristic will be

$$\chi(S^2) = 1 - 0 + 1 = 2.$$

Actually, the Euler characteristic can be stated such that it only depends on the homology groups of X, which gets clear in theorem 3.26. But before that consider the following definition.

Definition 3.24. Let X be a topological space. Then we define the n^{th} Betti number $b_n(X)$ of X to be equal to $\dim(H_n(X;\mathbb{R}))$.

Remark 3.25. As we said earlier, in *Algebraic Topology* singular homology groups with integer coefficients were used instead of with real coefficients. As a result the singular homology groups will consist of a free part and a torsion part. The betti numbers will then be defined as the dimension of the free part. To bypass this all we used singular homology with real coefficients. We refer to *Algebraic Topology* by A. Hatcher if the reader is interested in torsion.

Theorem 3.26 ([1], 2.44)**.** *Let X be a compact CW-complex, and let χ denote the Euler characteristic. Then we have that*

$$\chi(X) = \sum_k (-1)^k b_k(X).$$

Before we prove this theorem, we present a lemma.

Lemma 3.27. *Let X, Y, Z be vectorspaces, and let h_0, h_1, h_2 and h_3 be linear maps. Suppose that the following sequence is a short exact sequence.*

$$0 \xrightarrow{h_3} X \xrightarrow{h_2} Y \xrightarrow{h_1} Z \xrightarrow{h_0} 0$$

Then we have that $\dim(Y) = \dim(X) + \dim(Z)$.

Proof. With help of the rank-nullity theorem we can see that

$$\begin{aligned}
\dim(X) &= \dim(\mathrm{Im}(h_2)) + \dim(\mathrm{Ker}(h_2)) \\
&= \dim(\mathrm{Ker}(h_1)) \\
\dim(Y) &= \dim(\mathrm{Im}(h_1)) + \dim(\mathrm{Ker}(h_1)) \\
&= \dim(Z) + \dim(X),
\end{aligned}$$

which is the desired result. □

Proof of Theorem 3.26. Recall that $C_n(X) = H_n(X^n, X^{n-1}, \mathbb{R})$ is the vectorspace with the n-cells of X as basis. The dimension of C_n is the amount of n-cells, so we have that c_n is equal to dimension of $C_n(X)$. Note that X is a compact CW-complex. This implies that X has finitely many cells. So there exists a $k \geq 0$, such that for all $n > k$ we have that $C_n(X) = H_n(X^n, X^{n-1}, \mathbb{R}) = 0$. We then get the following chain complex

$$\cdots 0 \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Consider the following sequence

$$0 \longrightarrow \mathrm{Ker}(\partial_n) \xrightarrow{\iota} C_n \xrightarrow{\partial_n} \mathrm{Im}(\partial_n) \longrightarrow 0,$$

where ι is the injective inclusion map. We have that $\partial_n : C_n \to \mathrm{Im}(\partial_n)$ is surjective and that $\mathrm{Ker}(\partial_n) = \mathrm{Im}(\iota)$, and thus we can conclude that this sequence is a short exact sequence.

Moreover, consider

$$0 \longrightarrow \mathrm{Im}(\partial_{n+1}) \xrightarrow{\iota} \mathrm{Ker}(\partial_n) \xrightarrow{\pi} \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}) = H_n(X,\mathbb{R}) \longrightarrow 0,$$

where ι denotes the injective inclusion map, because $\mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}\partial_n$. Here π denotes the surjective projection map and we have that $\mathrm{Ker}(\pi) = \mathrm{Im}(\iota)$, from which we can conclude that this sequence is also a short exact sequence.
Now using Lemma 3.27 we obtain the following

$$\begin{aligned}
\chi(X) &= \sum_n (-1)^n c_n \\
&= \sum_n (-1)^n \mathrm{dim}(C_n) \\
&= \sum_n (-1)^n \mathrm{dim}(\mathrm{Ker}(\partial_n)) + \mathrm{dim}(\mathrm{Im}(\partial_n)) \\
&= \sum_n (-1)^n \mathrm{dim}(\mathrm{Im}(\partial_{n+1})) + \mathrm{dim}(H_n(X,\mathbb{R})) + \mathrm{dim}(\mathrm{Im}(\partial_n)) \\
&= \sum_n (-1)^n \mathrm{dim}(H_n(X,\mathbb{R})),
\end{aligned}$$

which completes the proof. □

Definition 3.28. The definition of the Euler characteristic can be extended, because we can define the singular homology groups also on topological spaces and not just CW-complexes. So let X be a topological space. Then we define the *Euler characteristic* of X to be

$$\chi(X) = \sum_n (-1)^n \dim H_n(X,\mathbb{R}).$$

4 De Rham Cohomology

The goal of this chapter is to specify the Euler characteristic for manifolds and we will do that using so called De Rham cohomology groups. For that we will follow the lecture notes [15] of M. Crainic and *Introduction to Smooth Manifolds* [5] by J. M. Lee.

4.1 Definition

In this section we will define the De Rham cohomology groups of a manifold, but first we start with an other definition.

Definition 4.1. Let V be a vector space and let $n \in \mathbb{Z}_{\geq 0}$. Consider the following map

$$\eta : \underbrace{V \times V \times \cdots \times V}_{n \text{ copies}} \to \mathbb{R}.$$

We say that it is *a linear n-form* on V if η is linear in each argument and if

$$\eta(x_1, \ldots, x_n) = \operatorname{sign}(\sigma)\eta(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where $x_1, \ldots, x_n \in X$ and σ is a permutation. The vector space containing all such linear n-forms is denoted by $\Lambda^n X^*$.

Now we want to define n-forms on manifolds.

Definition 4.2. Let M be an m-dimensional manifold. A *set theoretical n-form* is a map ω such that

$$\begin{aligned} \omega : M &\to \Lambda^n T_p^* M \\ : p &\mapsto \omega(p). \end{aligned}$$

To ease the notation we will use ω_p instead of $\omega(p)$ from now on.

Definition 4.3. A set theoretical n-form is called a *differential n-form* if it is smooth with respect to the charts. The vector space containing all such differential n-forms is denoted by $\Omega^n(M)$.

These definition give rise to the following theorem, which is a slightly simplified version of theorem 14.24 in *Introduction to Smooth Manifolds* [5]. A proof can be found there at pages 365-366.

Theorem 4.4 ([5], 14.24)**.** *Let M be a manifold. For all $n \in \mathbb{Z}_{\geq 0}$ there exist unique operators $d_n : \Omega^n(M) \to \Omega^{n+1}(M)$ with the following properties:*

1. *d_n is linear with real coeficients for all $n \in \mathbb{Z}_{\geq 0}$*
2. *$d_{n+1} \circ d_n = 0$*
3. *For $f \in \Omega^0(M)$ we have that df is the differential of f*

We call d_n the exterior derivative.

Corollary 4.5. *If $df = 0$ for $f \in \Omega^0(M)$ and M connected, then f is constant, because df is the differential of f.*

The first two properties of theorem 4.4 make it possible for us to define the De Rham cohomology groups.

Definition 4.6. Let $n \in \mathbb{Z}_{\geq 0}$. We define the n^{th} *De Rham cohomology group* $H^n_{dR}(M)$ to be the following vector space:

$$H^n_{dR}(M) = \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n-1}).$$

To ease the notation we define $Z^n(M) := \operatorname{Ker}(d_n)$ and $B^n(M) := \operatorname{Im}(d_{n-1})$ and thus we get

$$H^n_{dR}(M) = Z^n(M)/B^n(M).$$

Remark 4.7. We can extend the definition for n to the entirety of $\mathbb{Z}$, so also the negative integers, by defining $H^n_{dR}(M) = \{0\}$ for negative $n \in \mathbb{Z}$.

4.2 Homotopy Invariance

This section is dedicated to theorem 4.8 and is based on *Introduction to Smooth Manifolds* by [5] John M. Lee and on the lecture notes of Marius Crainic [15].

Theorem 4.8 ([5], 17.11)**.** *Let M, N be manifolds which are homotopy equivalent with each other. Then for all $n \in \mathbb{N}$ it holds that $H^n_{dR}(M) \cong H^n_{dR}(N)$.*

Before we prove this theorem, we need the Whitney Approximation theorem and a lemma for which we will not provide a proof.

Theorem 4.9 ([5], 9.27 Whitney Approximation Theorem)**.** *Let $f_0 : M \to N$ be a continuous map between the manifolds M and N. Then there exists a smooth map $f_1 : M \to N$ such that f_0 and f_1 are homotopic to each other.*

Before we procede to the lemma, consider the following definition.

Definition 4.10. Let M and N be manifolds and let $f : M \to N$ be a smooth map between them. Let $p \in M$ and let $X^1_p, \dots X^n_p \in T_pM$. For $\omega \in \Omega^n(N)$ we define the *pull-back of ω by f* denoted by $f^*\omega$ in the following way:

$$(f^*\omega)_p(X^1_p, \dots, X^n_p) := \omega_{f(p)}((\mathrm{d}f)_p(X^1_p), \dots, (\mathrm{d}f)_p(X^n_p)).$$

We observe that

$$f^* : \Omega^k(N) \to \Omega^k(M).$$

Remark 4.11. A property of the pull-back of ω by f is that it commutes with $\mathrm{d}_n : \Omega^n(M) \to \Omega^{n+1}(M)$ for all $n \in \mathbb{N}$. Furthermore if $M \subseteq N$ and f is the inclusion map between them, we have that the pull-back of ω by f is just the restriction of ω on M.
Proving these results is not the goal of the thesis. One could find a proof of the first statement in *Introduction to Smooth Manifolds* [5] at page 366.

Definition 4.12. Consider the situation as in definition 4.10. Let $\omega \in Z^n(N) \subseteq \Omega^n(N)$. Then it holds that

$$\mathrm{d}_n(f^*\omega) = f^*(\mathrm{d}_n\omega) = f^*(0) = 0.$$

So it holds that f^* maps $Z^n(N)$ into $Z^n(M)$. Now let $\omega \in B^n(N) \subseteq \Omega^n(N)$. Then there exists a $\phi \in \Omega^n(N)$ such that $\omega = \mathrm{d}\phi$. Then we have that

$$f^*\omega = f^*(\mathrm{d}\phi) = \mathrm{d}(f^*\phi).$$

It follows that $f^*\omega \in B^n(M)$. This induces a map between $H^n_{dR}(N)$ and $H^n_{dR}(M)$, which we still indicate with f^*. This map is called the *induced cohomology map* of f.

Using these the previous notions we can state the following lemma, for which one can find a proof in *Introduction to Smooth Manifolds* [5] at page 445.

Lemma 4.13 ([5], 17.10)**.** *Let $f, g : M \to N$ be smooth maps between the manifolds M and N and suppose they are homotopic to each other. Then for all $n \in \mathbb{N}$ it holds that the induced cohomology maps $f^*, g^* : H^n_{dR}(N) \to H^n_{dR}(M)$ coincide.*

Now we can move on to the proof of theorem 4.8.

Proof of theorem 4.8. It is given that the manifolds M and N are homotopy equivalent to each other. So there exists a continuous map $f_0 : M \to N$ which is a homotopy equivalence. Let the homotopy inverse be $g_0 : M \to N$. By the Whitney Approximation theorem we know that there exist maps $f_1 : M \to N$ and $g_1 : N \to M$ which are smooth and homotopic to respectively $f_0 : M \to N$ and $g_0 : N \to M$. Then we have that

$$f_1 \circ g_1 \sim f_0 \circ g_0 \sim \mathrm{Id}_N$$
$$g_1 \circ f_1 \sim g_0 \circ f_0 \sim \mathrm{Id}_M.$$

We conclude that f_1 and g_1 are each others homotopy inverses. By Lemma 4.13 we know that the induced cohomology maps of $f_1 \circ g_1$ and Id_N are equal. The same holds for $g_1 \circ f_1$ and Id_M. So now for all $n \in \mathbb{N}$ we have a map between $H^n_{dR}(N)$ and $H^n_{dR}(M)$ namely f_1^* which is linear and has an inverse. This means that it is an isomorphism and we conclude that $H^n_{dR}(N) \cong H^n_{dR}(M)$. ∎

The pull-backs we introduced gives us a few result which we will use later.

Lemma 4.14 ([5], 17.5)**.** *Suppose that $\{M_j\}$ is a countable collection of n-dimensional manifolds and let M be the disjoint union of all those M_j. Then for all p we have that $H^p_{dR}(M) \cong \bigoplus_i H^p_{dR}(M_j)$.*

Proof. Consider the inclusion maps $\iota_j : M_j \to M$. The pull-back maps $\iota_j^* : \Omega^p(M) \to \Omega^p(M_j)$ induce an isomorphism between $\Omega^p(M)$ and $\bigoplus_j \Omega^p(M_j)$ in the following way

$$\bigoplus_j \iota_j^* : \Omega^p(M) \to \bigoplus_j \Omega^p(M_j)$$
$$: \omega \mapsto (\iota_1^*\omega, \iota_2^*\omega, \dots).$$

From remark 4.11 we see that $(\iota_1^*\omega, \iota_2^*\omega, \dots) = (\omega\big|_{M_1}, \omega\big|_{M_2}, \dots)$. If $\omega|_{M_j}$ is zero for all j, then it holds that $\omega = 0$. So the kernel is trivial, which means that $\bigoplus_j \iota_j^*$ is injective. Surjectivity follows from the fact that we can define a p-form in M by defining it on all of its disjoint components. Our conclusion is that $\bigoplus_j \iota_j^*$ is an isomorphism. □

Lemma 4.15 ([5], 17.6)**.** *Let M be a connected manifold. Then $H^0_{dR}(M) \cong \mathbb{R}$.*

Proof. Recall that $H^0_{dR}(M) = Z^0(M)/B^0(M)$ as in definition 4.6. We have that $B^0(M)$ is zero, because there are no (-1)-forms. Furthermore, $Z^n(M)$ consists of the 0-forms ω which satisfy $d\omega = 0$. So the differential of ω is zero, which implies that $\omega : M \to \mathbb{R}$ is a constant map. Each constant map can be indicated by a constant in $\mathbb{R}$. So we indeed have that $H^0_{dR}(M) \cong \mathbb{R}$. □

Corollary 4.16 ([5], 17.7)**.** *From lemma's 4.14 and 4.15 and from the rank-nullity theorem, it follows that $H^0_{dR}(\{p\}) \cong \mathbb{R}$ and $H^n_{dR}(\{p\}) = 0$, for all $n \geq 1$ and p a point.*

4.3 Mayer-Vietoris Theorem

Now we shall introduce the Mayer-Vietoris theorem which we won't prove. The Mayer-Vietor theorem is important, because of the applications of it, which we will discuss in the next section. In this section and the next one we will follow *Introduction to Smooth Manifolds* [5] and the lecture notes [15] of M. Crainic. We start by giving a sketch of the situation of the Mayer-Vietoris theorem.

Let M be a manifold, and let U, V be open in M such that their union is M again. Consider the following diagram:

$$\begin{tikzcd}
 & U \arrow[dr, "k"] & \\
U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M, \\
 & V \arrow[ur, "l"'] &
\end{tikzcd}$$

where i, j, k, l are inclusion maps. These maps are smooth maps and hence we can define for all $n \in \mathbb{N}$ the pull-backs by these inclusion maps to obtain the following diagrams:

$$\begin{tikzcd}
& \Omega^n U) \arrow[dr, "i_n^*"] & \\
\Omega^n(M) \arrow[ur, "k_n^*"] \arrow[dr, "l_n^*"'] & & \Omega^n(U \cap V)\ \cdot \\
& \Omega^n(V) \arrow[ur, "j_n^*"'] &
\end{tikzcd}$$

In fact, the pull-backs are the restriction maps.

Theorem 4.17 (Mayer-Vietoris)**.** *Consider the situation as sketched above. Then for all $n \in \mathbb{N}$, there exists a linear map $\delta_n : H^n_{dR}(U \cap V) \to H^{n+1}_{dR}(M)$ such that the following sequence is exact:*

$$\cdots \xrightarrow{\delta n-1} H^n_{dR}(M) \xrightarrow{k_n^* \oplus l_n^*} H^n_{dR}(U) \oplus H^n_{dR}(V) \xrightarrow{i_n^* - j_n^*} H^n_{dR}(U \cap V) \xrightarrow{\delta_n} H^{n+1}_{dR}(M) \xrightarrow{k_{n+1}^* \oplus l_{n+1}^*} \cdots.$$

Such a sequence is called a Mayer-Vietoris sequence.

4.4 Applications Mayer-Vietoris Theorem

Now we will discuss some applications of the Mayer-Vietoris theorem. One of them is that we can now compute the De Rham cohomology groups of various manifolds. Another is that we can prove that the dimensions of the De Rham cohomology groups are finite. This namely makes it possible to define the Euler characteristic of a manifold using De Rham cohomology gropus. This section follows the lecture notes of Marius crainic and *Introduction to Smooth Manifolds* by John. M. Lee [5].

Theorem 4.18 ([5], 17.21)**.** *For all $n \in \mathbb{N}$ we have that*

$$H^n_{dR}(S^n) \cong \mathbb{R}.$$

Proof. We will prove this by induction in n. For $n = 1$ we have the circle S^1. Define U to be $S^1 \backslash \{S\}$ and V to be $S^1 \backslash \{N\}$, where S is the south pole and N the north pole. Consider the first part of the Mayer-Vietoris sequence of S^1:

$$0 \xrightarrow{h_1} H^0_{dR}(S^1) \xrightarrow{h_2} H^0_{dR}(U) \oplus H^0_{dR}(V) \xrightarrow{h_3} H^0_{dR}(U \cap V) \xrightarrow{h_4} H^1_{dR}(S^1) \xrightarrow{h_5} H^1_{dR}(U) \oplus H^1_{dR}(V).$$

It holds that U and V are homotopy equivalent to a point and from theorem 4.8 we know that the De Rham cohomology groups of U and V are isomorphic to that of a point. The intersection between U and V is homotopy equivalent with a disjoint union of two points. From lemma's 4.14 and 4.15 we then get the following:

$$H^0_{dR}(S^1) \cong \mathbb{R}$$
$$H^0_{dR}(U) \oplus H^0(V) \cong \mathbb{R} \oplus \mathbb{R}$$
$$H^0_{dR}(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}.$$

Due to the fact that U is homotopy equivalent with a point p, it holds that $H^1_{dR}(U) \cong H^1_{dR}(\{p\})$. From corollary 4.16 it then follows that $H^1_{dR}(U) \cong 0$. The Mayer-Vietoris sequence then becomes the following:

$$0 \xrightarrow{h_1} \mathbb{R} \xrightarrow{h_2} \mathbb{R} \oplus \mathbb{R} \xrightarrow{h_3} \mathbb{R} \oplus \mathbb{R} \xrightarrow{h_4} H^1_{dR}(S^1) \xrightarrow{h_5} 0.$$

Theorem 4.17 (Mayer-Vietoris) tells us that this sequence is exact, so $\mathrm{Im} h_i = \mathrm{Ker} h_{i+1}$. This implies that $\dim(\mathrm{Im}(h_i)) = \dim(\mathrm{Ker}(h_{i+1}))$. Using the exactness of the sequence and the rank-nullity theorem we get the following:

$$\begin{aligned}
\dim(\mathrm{Ker}(h_2)) &= \dim(\mathrm{Im}(h_1)) && \text{(Exactness)}\\
&= 0\\
\dim(\mathrm{Im}(h_2)) &= \dim(\mathbb{R}) - \dim(\mathrm{Ker}(h_2)) && \text{(Rank-Nullity)}\\
&= 1\\
\dim(\mathrm{Ker}(h_3)) &= \dim(\mathrm{Im}(h_2)) && \text{(Exactness)}\\
&= 1\\
\dim(\mathrm{Im}(h_3)) &= \dim(\mathbb{R}\oplus\mathbb{R}) - \dim(\mathrm{Ker}(h_3)) && \text{(Rank-Nullity)}\\
&= 1\\
\dim(\mathrm{Ker}(h_4)) &= \dim\mathrm{Im}(h_3) && \text{(Exactness)}\\
&= 1\\
\dim(\mathrm{Im}(h_4)) &= \dim(\mathbb{R}\oplus\mathbb{R}) - \dim(\mathrm{Ker}(h_4)) && \text{(Rank-Nullity)}\\
&= 1\\
\dim(\mathrm{Ker}(h_5)) &= \dim(\mathrm{Im}(h_4)) && \text{(Exactness)}\\
&= 1.
\end{aligned}$$

We know that $\mathrm{Ker}(h_5) \cong H^1_{dR}(S^1)$, so $H^1_{dR}(S^1) \cong \mathbb{R}$, and thus the induction start has been settled.
Now suppose that the theorem holds for n from 1 until $m-1$, where $m \geq 2$. We shall prove that the theorem then also holds for $n = m$. So consider S^m and let U be $S^m\backslash\{N\}$ and V be $S^m\backslash\{S\}$, where N is the north pole and S the south pole. Also consider the following part of the Mayer-Vietoris sequence of S^m:

$$H^{m-1}_{dR}(U) \oplus H^{m-1}_{dR}(V) \longrightarrow H^{m-1}_{dR}(U \cap V) \longrightarrow H^m_{dR}(S^m) \longrightarrow H^m_{dR}(U) \oplus H^m_{dR}(V).$$

It holds that U and V are homotopy equivalent with a point. From corollary 4.16 it then follows that the most left and right spaces in the sequence are trivial. Using the exactness of the sequence and the rank-nullity theorem again we get that $H^{m-1}_{dR}(U \cap V) \cong H^m_{dR}(S^m)$. It holds that $U \cap V$ is homotopy equivalent to S^{m-1} and thus we conclude that $H^m_{dR}(S^m) \cong H^{m-1}_{dR}(S^{m-1}) \cong \mathbb{R}$. $\square$

Now we move on to the second application, but before that we need the following definition.

Definition 4.19. Let M be a manifold and let $\mathcal{U}$ be an open cover of M. The open cover is called a *good open cover* if for any finite subset $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}$ the intersection $\bigcap_i^n U_i$ is either empty or diffeomorphic to $\mathbb{R}^m$.

Using this definition we can formulate the following lemma which we will not prove. A proof can be found in for example *Manifolds and Differential Geometry* by Jeffrey. M. Lee at page 455.

Lemma 4.20. *Let M be a compact smooth manifold. Then M admits a finite good open cover.*

Theorem 4.21. *Let M admit a finite good open cover. Then it holds that the De Rham cohomology groups $H^n_{dR}(M)$ are finite dimensional for all $n \in \mathbb{N}$.*

Proof. For this proof we will use mathematical induction. Our induction start is the following: Suppose that M admits a finite good open cover which only contains one open set. This set must be M itself. Then it holds that M is either empty or diffeomorphic to $\mathbb{R}^m$. If M is empty we are done, so suppose that M is diffeomorphic to $\mathbb{R}^m$. We know that $\mathbb{R}^m$ is homotopy equivalent with a point $\{p\} \subset \mathbb{R}$.The De Rham cohomology groups of a point are finite dimensional, and using theorem 4.8 we conclude that the De Rham cohomology groups of M are finite dimensional.
Now suppose that the theorem is true for finite good open covers containing $k-1$ open sets. Assume that M admits a finite open good cover $\mathcal{U} = \{U_1, \ldots, U_k\}$. We define $U, V \subseteq \mathcal{U}$ in the following way:

$$\begin{aligned}
U &= \bigcup_i^{k-1} U_i\\
V &= U_k.
\end{aligned}$$

Then it holds that $U \cap V$ admits a finite good open cover namely $\{U_1 \cap U_k, \ldots, U_{k-1} \cap U_k\}$, because $\mathcal{U}$ is a finite good open cover. By the induction hypothesis we know that U, V and $U \cap V$ have finite dimensional De Rham cohomology groups.

By theorem 4.17 (Mayer-Vietoris) we know that the following sequence is exact:

$$\cdots \longrightarrow H^{n-1}_{dR}(U \cap V) \xrightarrow{\delta_{n-1}} H^n_{dR}(M) \xrightarrow{f_n} H^n_{dR}(U) \oplus H^n_{dR}(V) \longrightarrow \cdots .$$

Due to the fact that $\mathrm{Im}(f_n) \subseteq H^n_{dR}(U) \oplus H^n_{dR}(V)$, it holds that

$$\dim(\mathrm{Im}(f_n)) \leq \dim(H^n_{dR}(U) \oplus H^n_{dR}(V)),$$

and by the exactness of the sequence and the rank-nullity theorem it follows that

$$\dim(\mathrm{Ker}(f_n)) = \dim(\mathrm{Im}(\delta_{n-1})) \leq \dim(H^{n-1}_{dR}(U \cap V)).$$

Knowing that the dimension of $H^n_{dR}(U) \oplus H^n_{dR}(V)$ and $H^{n-1}_{dR}(U \cap V)$ is finite, it follows by the rank-nullity theorem that the dimension of $H^n_{dR}(M)$ is also finite for all $l \in \mathbb{N}$. □

As a consequence we are now able to state the desired definition of the Euler characteristic for compact manifolds in terms of De Rham cohomology groups. It is possible, because now we know that the De Rham cohomology groups of a compact manifold are finite dimensional.

Definition 4.22. Let M be a compact manifold. We define the *Euler characteristic* $\chi(M)$ of M to be

$$\chi(M) = \sum_j (-1)^j \dim(H^j_{dR}(M)).$$

5 Equivalent Euler Characteristics

Now we will show the relation between the Euler characteristic defined in terms of singular homology groups as in theorem 3.26 and the Euler characteristic defined in terms of De Rham cohomology groups as in definition 4.22. We start by discussing singular cohomology, which will turn out to be a bridge between the singular homology groups and De Rham cohomology groups.
The idea to link these in this way is mine, but the individual definition, lemma's and theorems are the work of others.

We begin with a few definitions from *Algebraic Topology* [1].

Definition 5.1 ([1], p. 191)**.** Let X be a topological space and let $C_n(X)$ be the vector space with the topological n-simplices as basis. Consider the following chain complex of vector spaces

$$\cdots C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots.$$

We define the *cochain set* $C_n^*(X)$ to be the set of linear maps from C_n to $\mathbb{R}$ which is also known as $\mathcal{L}(C_n, \mathbb{R})$.

Definition 5.2 ([1], p.191)**.** We define the coboundary map ∂^* as follows

$$\begin{aligned}\partial_{n-1}^* &: C_{n-1}^* \to C_n^*\\ &: \varphi \mapsto \varphi \circ \partial_{n-1}.\end{aligned}$$

We again have that $\partial_n^* \partial_{n-1}^* = 0$. This leads us to the following definition.

Definition 5.3 ([1], p.191)**.** The n^{th} *singular cohomology group with real coefficients* is defined to be $\mathrm{Ker}\partial_n^*/\mathrm{Im}\partial_{n-1}^*$ and it is denoted by $H^n(X;\mathbb{R})$.

Let us recall the definition of a dual space.

Definition 5.4. Let V be a vector space. The *dual space* V^* is the vector space consisting of the linear maps $L : V \to \mathbb{R}$.

Remark 5.5. The singular cohomology groups are the dual space of the singular homology groups.

This remark points out the usefulness of the following theorem for which a proof can be found in *Introduction to Smooth Manifolds* [5].

Theorem 5.6 ([5], 11.1)**.** *Let V^* be the dual space of a finite dimensional vector space V. Then*

$$\dim(V^*) = \dim(V).$$

From this theorem can conclude that

$$\dim(H_n(X;\mathbb{R})) = \dim(H^n(X;\mathbb{R})),$$

for all n. This implies that

$$\begin{aligned}\chi(X) &= \sum_n (-1)^n \dim H_n(X,\mathbb{R})\\ &= \sum_n (-1)^n \dim H^n(X,\mathbb{R}),\end{aligned}$$

and the Euler characteristic in terms of singular cohomology groups has been established. Now we shall link the cohomology groups of a manifold to the De Rham cohomology groups of that manifold. We will do that using the De Rham theorem for which a proof and explanation can be found in *Introduction to Smooth Manifolds* chapter 18.

Theorem 5.7 (De Rham Theorem,[5] ,18.14)**.** *Let M be a manifold. Then we have that*

$$H_{dR}^n(M) \cong H^n(M;\mathbb{R}),$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Remark 5.8. This is a slight simplification of how the theorem is stated in *Introduction to Smooth Manifolds.*

Due to the fact that dimension is a topological invariant, we see that

$$\dim(H_{dR}^n(M)) = \dim(H^n(M,\mathbb{R})),$$

for all $n \in \mathbb{Z}_{\geq 0}$. We conclude that

$$\sum_n (-1)^n \dim H_n(M,\mathbb{R}) = \sum_n (-1)^n \dim H_{dR}^n(M),$$

which ensures the equivalence of definitions of the Euler characteristic on a manifold M.

6 Index of a Vector Field

Now we can move on to the next thing in the statement of the Poincaré-Hopf theorem, namely the index. The goal of this chapter is to understand what the index of vectorfield at a zero is. Furthermore, we shall prove that the sum of the indices of a vectorfield at its zeros is independent of vectorfield. Moreover, we shall discuss lemma 6.13 which will be important for the proof of the Poincaré-Hopf theorem. In this chapter we will mainly follow *Topology from the Differential Viewpoint* [4] with some help of *Introduction to Smooth Manifolds* [5] and *Topology and Geometry* by G. E. Bredon. We also used the papers *The Euler Characteristic, Poincare-Hopf Theorem, and Applications* by J. Libgober and *The Index of a Vectorfield as an Invariant* by V. Popa.

We start with a few definitions.

Definition 6.1. Let V be an n-dimensional vectorspace. Consider the ordered bases $(E_1, \dots, E_n)$ and $(\tilde{E}_1, \dots, \tilde{E}_n)$ and suppose they are related by

$$E_i = \sum_j B_{ij} \tilde{E}_j,$$

where (B_{ij}) is the transition matrix. We say that the two ordered bases are *consistently ordered* if the determinant $\det((B_{ij}))$ of the transition matrix is strictly postive. Two bases being consistently ordered is an equivalence relation and there are two equivalence classes. An *orientation* on V is such an equivalence class.

Example 6.2. Consider now $\mathbb{R}^n$ and its standard basis $(e_1, \dots, e_n)$. The *standard orientation* is the equivalence class containing standard basis.

Using the standard orientation of $\mathbb{R}^n$, we can now define the orientation on manifolds.

Definition 6.3. An *oriented m-dimensional manifold* is a m-dimensional manifold M where for all $p \in M$ there is a chosen orientation on the vector space T_pM. This orientation must satisfy the following: For all $p \in M$ we have a chart $(U, \varphi : U \to V \subseteq \mathbb{R}^m)$ which is *orientation preserving*. That is that for all $x \in U$ the map $d\varphi_x : T_xM \to T_{\varphi(x)}\mathbb{R}^m$ maps the orientation on T_pM to the standard orientation on $T_{\varphi(y)}\mathbb{R}^m \cong \mathbb{R}^m$.

Definition 6.4. Let $f : M \to N$ be a smooth map between the manifolds M and N and let $p \in M$. Then we say that p is a *regular point* of f if

$$df_p : T_pM \to T_{f(p)}N$$

is surjective. We will call the point $f(p)$ then a *regular value* of f. Similarly, we call a point p a *critical point* if df_p is not surjective and $f(p)$ the *critical value*.

Using these definition we consider the following theorem for which a proof can be found in *Introduction to Smooth Manifolds* [5] at pages 129-131.

Theorem 6.5 (Sard's Theorem)**.** *Let $F : M \to N$ be a smooth function between two manifolds M and N. Then the set of critical values of f has measure zero in N.*

Remark 6.6. The exact definition of having measure zero in a set is not important for now. Important is that this implies that the complement of the set of critical values is dense in N, by proposition 6.8 in [5]. So, the set of regular values is dense in N, which means that there always exists a regular value if N is non-empty.

Now consider the following theorem in which we will also define the degree of a function. This theorem is a simplified version of theorem 17.35 in *Introduction to Smooth Manifolds* [5], for which a proof can be found in page 458.

Theorem 6.7 ([5], 17.35)**.** *Let M and N be n-dimensional manifolds which are compact, connected and oriented and let $F : M \to N$ be a smooth map. Then there exists an integer which we will call* $\deg(F)$ *such that:*

$$\deg(F) := \sum_{x \in F^{-1}(y)} sign(x),$$

where y is a regular value of F and $sign(x)$ is defined in the following way

$$sign(x) = \begin{cases} +1 & \text{if } dF_x \text{ is orientation preserving} \\ -1 & \text{if } dF_x \text{ is orientation reversing} \end{cases}.$$

Furthermore, this integer $\deg(F)$ *is unique.*

Remark 6.8. Being unique, means that $\deg(F)$ is independent of regular value.

Now we have the tools to define the index of a vectorfield at its zero on an open set $U' \in \mathbb{R}^n$.

Definition 6.9. Let $U' \subseteq \mathbb{R}^n$ and let $\vec{v}$ a vectorfield on it. Suppose that z is an isolated zero. Consider a small enough sphere with z as middle point, such that

$$\begin{aligned} \varphi : S^{n-1} &\to S^{n-1} \\ : x &\mapsto \frac{\vec{v}(x)}{||\vec{v}(x)||} \end{aligned}$$

maps the sphere into the unit sphere. The *index* of $\vec{v}$ at a zero z is defined to be the degree of the map φ and it is denoted by $\operatorname{ind}_z(\vec{v})$.

We want to extend our definition of the index to vectorfields on manifolds. We will do that using the following lemma for which a proof can be found in *Topology from the Differential Viewpoint* on pages 34-35.

Lemma 6.10 ([4], p. 33)**.** *Again, let $U' \subseteq \mathbb{R}^n$ and let $\vec{v}$ a vectorfield on it. Suppose that $f : U' \to V'$ is a diffeomorphism. Furhtermore suppose that $\vec{v}$ is a vector field on U' and $\vec{v}'$ is a vector field on V' such that*

$$\vec{v}' = df \circ v \circ f^{-1}.$$

Then the index of $\vec{v}'$ at $f(z)$ is the same as the index of $\vec{v}$ at its isolated zero z.

Definition 6.11. Let (U, φ) be a chart around z, which is an isolated zero of the vector field $\vec{v}$ on M. Then we define the *index* of $\vec{v}$ at z to be the index of

$$d\varphi \circ \vec{v} \circ \varphi^{-1}$$

at $\varphi(z)$. We will denote it by $\operatorname{ind}_z(\vec{v})$

Now consider the Whitney Embedding theorem, for which a proof can be found on pages 134-135 in *Introduction to Smooth Manifolds* [5].

Theorem 6.12 (Whitney Embedding Theorem, [5], 6.15)**.** *Any m-dimensional manifold M can be embedded in $\mathbb{R}^{2m+1}$.*

Using this theorem we are allowed to write $M \subseteq \mathbb{R}^k$, for k big enough. Consider a vectorfield $\vec{v}$ on M with $z \in M$ a zero of $\vec{v}$. Then we can see that $\vec{v}$ as a map from M to $\mathbb{R}^k$, for big enough k and hence we can define the differential $dv_z : T_zM \to T_{\vec{v}(z)}\mathbb{R}^k \cong \mathbb{R}^k$ of $\vec{v}$. Now consider the following theorem for which a proof can be found in *Topology from the Differential Viewpoint* [4] on pages 37-38. This theorem will be usefull for the second proof of the Poincaré-Hopf theorem where we will use Morse theory.

Lemma 6.13 ([4], p. 37)**.** *Consider the situation as described above. Then the differential dv_z can be considered as a linear map from T_zM to T_zM. If the determinant of dv_z is strictly positive, we have that the index of $\vec{v}$ at z is $+1$ and if the determinant of dv_z is negative we have that the index of $\vec{v}$ is -1.*

This lemma gives rise to the following definition.

Definition 6.14. Let M be a manifold and let $\vec{v}$ be a vector field on it. We say that the zero z is non-degenerate if its derivative dv_z is non-singular, i.e if the derivative is invertible.

As we said earlier, a goal of this chapter is to prove that the sum of the indices of a vectorfield at its zeros is independent of vectorfield. It turns out to be we need an additional requirement on the vectorfields, namely that its zeros are all non-degenerate. But first we need some preliminary theorems which we will use to prove theorem 6.19. The proof theorem 6.19 namely comes in handy later when we want to prove the Poincaré-Hopf theorem using Morse theory. The proofs of the following two preliminary theorems can be found in *Topology from the Differential Viewpoint* [4] at page 28.

Theorem 6.15 ([4], p. 28)**.** *Let X be a manifold with boundary and let N be a manifold. Let M be the boundary of X with an orientation that is in line with the orientation on X. Suppose that a smooth map $F : M \to N$ is an extension of the map $f : \partial M \to N$. Then we have that $\deg(f) = 0$.*

Theorem 6.16 ([4], p. 28)**.** *Let M, N be oriented manifolds. Let $f : M \to N$ and $g : M \to N$ be smooth maps which are homotopic to each other. Then it holds that*

$$\deg(g) = \deg(f).$$

To state theorem 6.19 we need one more definition.

Definition 6.17. Consider a compact m-dimensional manifold $X \subset \mathbb{R}^m$ with boundary. Then we define the *Gauss mapping*

$$g : \partial X \to S^{m-1},$$

which for each point in ∂X a unit vector which is orthogonal to X.

Remark 6.18. A Möbius band does not satisfy the requirement for this definition. It namely can not be embedded in $\mathbb{R}^2$ while being a 2-dimension manifold with boundary.
A solid ball in $\mathbb{R}^3$ does satisfy this requirement. It is namely a 3-dimensional manifold which we can embed in $\mathbb{R}^3$.

Theorem 6.19. *Let $M \subset \mathbb{R}^m$ be an m-dimensional manifold with boundary and let $\vec{v}$ be a vectorfield on it with the properties that it points outward on the boundary and that it has only non-degenerate isolated zeros. Then we have that the sum of the indices of $\vec{v}$ at its zeros is equal to the degree of the Gauss map $g : \partial M \to S^{m-1}$.*

Proof. First, we use Hausdorffness to find an ϵ-ball around each zero of $\vec{v}$, such that it contains no other zero of $\vec{v}$. By removing these ϵ-balls we end up with, again, a manifold with boundary which we will call N. Define the smooth map F in the following way:

$$\begin{aligned} F &: N \to S^{n-1} \\ &: x \mapsto \frac{\vec{v}(x)}{||\vec{v}(x)||}. \end{aligned}$$

When we restrict F to ∂N, we can extend it smoothly to F defined on the entirety of N. Using theorem 6.15 this implies that $\deg(F\big|_{\partial N})$ equals zero. It follows that the degree of F restricted to the outer boundary ∂M plus the degrees of F restricted to the inner boundaries is zero.
The vectorfield points outward on ∂M, so F restricted to ∂M is homotopic to the Gauss map g. As a result of theorem 6.16 we get that the degree of F restricted to ∂M is equal to $\deg(g)$.
Now we need to calculate the sum of the degrees of F restricted to the inner boundary. Note that when we do this the orientation is the opposite as when we determine the index of these zeros. So per definition of the index of $\vec{v}$ at its zero we see that the sum of the degrees of F restricted to the inner boundary equals $-\sum \mathrm{ind}_{x_i}\vec{v}$, where x_i denote the zeros of $\vec{v}$. We conclude that

$$\deg(g) = \sum \mathrm{ind}_{x_i}\vec{v},$$

which is the desired result. ∎

Now we want to state a slightly similar theorem, but for general m-dimensional manifolds without boundary. A proof can be found in *Topology from the Differential Viewpoint* [4] on page 36.

Theorem 6.20 ([4], p. 36)**.** *Let $M \subset \mathbb{R}^k$ be an m dimensional manifold without boundary and let $\vec{v}$ be a vector field on it with the property that it only has non-degenerate zeros. Define N_ϵ in the following way*

$$N_\epsilon := \{x \in \mathbb{R}^k : \ ||x - y|| < \epsilon, \textit{for some } y \in M\}.$$

Then

$$\sum_{\vec{v}(x)=0} \mathrm{ind}_x\vec{v} = \deg(g : \partial N_\epsilon \to S^{k-1}),$$

where g denotes the Gauss map.

Corollary 6.21. *Using the fact that the Gauss mapping is independent of vectorfield we indeed see that the sum of the indices of a vector field at its zeros is the same for any other vectorfield on a manifold M without boundary. A requirement, though, is that $\vec{v}$ has only non-degenerate zeros.*

7 Proof Poincaré-Hopf Theorem I

In this chapter we will prove the Poincaré-Hopf theorem after introducing Lefschetz fixed point theory. This chapter is mainly based on *Differential Topology* [7] and *Differential Topology and the Poincaré-Hopf Theorem* [13].

7.1 Lefschetz Fixed Point Theory

Consider a smooth map $f : M \to M$ where M is a compact oriented manifold. In Lefschetz fixed point theory, we are interested in the *fixed points* of f, i.e. the points $p \in M$ such that $f(p) = p$. The goal of this section is to find a relation between the Euler characteristic and the index of a vectorfield. Let us start with a few definitions.

Definition 7.1. Let $U' \subseteq \mathbb{R}^n$ and let $f : U' \to \mathbb{R}^n$ be a smooth map. Suppose that f only has one fixed point namely the origin 0 and consider a sphere S^{n-1} with 0 as the middle point. Then we define the *local Lefschetz number* of f at the fixed point 0 to be the degree of the following map:

$$\begin{aligned} F : S^{n-1} &\to S^{n-1} \\ : z &\mapsto \frac{f(z) - z}{|f(z) - z|}. \end{aligned}$$

We will denote it by $L_0(f)$.

We can extend this definition to maps on manifolds in the following way.

Definition 7.2. Let M be a manifold and let $f : M \to M$ a smooth map. Suppose that $z \in M$ is a fixed point of f. Let (U, φ) be a chart around z and without loss of generality assume that $\varphi(z) = 0$. Then we define the *local Lefschetz number* $L_z(f)$ of f at its fixed point z to be

$$L_z(f) = L_0(\varphi \circ f \circ \varphi^{-1}).$$

Definition 7.3. Let M be a compact manifold. Let $f : M \to M$ be a smooth map. Consider the induced linear maps $f_*^n : H_n(M; \mathbb{R}) \to H_n(M; \mathbb{R})$ between the n^{th} homology groups.
The *Lefschetz number* Λ_f of f is then defined to be

$$\Lambda_f = \sum_n (-1)^n \mathrm{tr}(f_*^n).$$

Remark 7.4. This sum is finite, because there exists a $k \geq 0$, such that for all $n > k$ it holds that $H_n(X; \mathbb{R}) = 0$. This is true, due to the fact that X is a compact manifold. In the chapter about Morse theory we will see in theorem 8.26 that compact manifolds are homotopic to CW-complexes. Due to the fact that singular homology groups are homotopy invariant, it follows indeed that the sum is finite.

Corollary 7.5. *Using the fact that singular homology groups are homotopy invariant, it follows that the Lefschetz number is equal for homotopic maps.*

Consider the following theorem in which a relation between the previous two definitions gets clear. A proof can be found in *Differential Topology* [7] at page 130.

Theorem 7.6 ([7] p. 130)**.** *Let M be a compact manifold and let $f : M \to M$ be a smooth map. Suppose that f has finitely many fixed points. Then it holds that*

$$\Lambda(f) = \sum_{f(x)=x} L_x(f),$$

where Λ denotes the Lefschetz number and L_x the local Lefschetz number at a fixed point.

The Lefschetz number relates to the Euler characteristic in the following way.

Lemma 7.7. *Let M be a compact manifold. Then we have that*

$$\Lambda_{id} = \chi(M).$$

Proof. Let $\mathrm{id} : M \to M$ be the identity map and let the induced linear maps be denoted by $\mathrm{id}_*^n : H_n(M;\mathbb{R}) \to H_n(M;\mathbb{R})$. These are identity maps between vectorspaces. So their trace is equal to their rank. We then see that

$$\begin{aligned}\Lambda_{\mathrm{id}} &= \sum_n (-1)^n \mathrm{tr}(\mathrm{id}_*^n)\\ &= \sum_n (-1)^n \mathrm{rank} H_n(M;\mathbb{R})\\ &= \chi(M),\end{aligned}$$

which tells us that the Lefschetz number of the identity is indeed equal to the Euler Characteristic. □

We relate the local Lefschetz number and the index of a vectorfield as follows.

Lemma 7.8 ([7], p. 135)**.** *Let $U' \subseteq \mathbb{R}^n$ and let $\vec{v}_*$ be a vector field on U' and let $f'_t : U' \to \mathbb{R}^n$ be a flow which is tangent to $\vec{v}_*$ at time $t = 0$ for all $x \in U'$. Suppose that the vectorfield $\vec{v}_*$ only vanishes at the origin. Furthermore, suppose for all $t \neq 0$ that f'_t only fixes the origin. Then for all $t \neq 0$ we have that:*

$$\mathrm{ind}_0(\vec{v}_*) = L_0(f'_t),$$

where $\mathrm{ind}_0(\vec{v}_*)$ is the index of $\vec{v}_*$ at the origin and $L_0(f'_t)$ is the local Lefschetz number of f'_t at the origin.

Before we prove this lemma consider this other lemma for which a proof can be found in *Differential Topology* [7] at page 135.

Lemma 7.9 ([7], p. 135)**.** *There exists a smooth function $r(t)$ such that*

$$f'_t(x) = f'_0(x) + t\dot{f}'_0(x) + t^2 r_t(x),$$

where $\dot{f}'_0$ denotes the time derivative of f'_t evaluated at $t = 0$.

Proof of lemma 7.8. Due to the fact that $f'_t(x)$ is tangent to the vectorfield at time $t = 0$ we can also write $f'_t(x)$ as follows:

$$f'_t(x) = f'_0(x) + t\vec{v}_*(x) + t^2 r_t(x).$$

From the previous lemma we get that

$$f'_t(x) - x = t\vec{v}_*(x) + t^2 r_t(x).$$

Now consider f'_t on the $n-1$ sphere inside U with the origin as its middle point. Then for $t \neq 0$ we have

$$f'_t(x) - x \neq 0,$$

because the origin is the only fixed point. Now we can safely state the following equation:

$$\frac{f'_t(x) - x}{|f'_t(x) - x|} = \frac{\vec{v}_*(x) + t r_t(x)}{|\vec{v}_*(x) + t r_t(x)|},$$

for $x \in S^{n-1}$ and $t \neq 0$. The degree of the left hand side is per definition 7.1 the local Lefschetz number $L_0(f_t)'$. Note that the map in right hand side is homotopic to the map $\frac{\vec{v}_*(x)}{|\vec{v}_*(x)|}$. So the degree of the right hand side is with help of theorem 6.16 equal to $\mathrm{ind}_0(\vec{v}_*)$ per definition 6.9 and we get that

$$\mathrm{ind}_0(\vec{v}_*) = L_0(f'_t),$$

which is the desired result. □

We want a similar lemma for vectorfields on manifolds. Therefore, consider the following lemma.

Lemma 7.10. *Let $\vec{v}$ be a vectorfield on the manifold M and let $f_t : M \to M$ be a flow which is tangent to $\vec{v}$ at time $t = 0$. Suppose that $z \in M$ is an isolated zero of $\vec{v}$. Let (U, φ) be a chart such that for all $t \neq 0$, f_t only fixes z in U. Then for $t \neq 0$ we have*

$$\mathrm{ind}_z(\vec{v}) = L_z(f_t).$$

Proof. First assume that $\varphi(z) = 0$. Then by definition 6.11 we have

$$\mathrm{ind}_0(d\varphi \circ \vec{v} \circ \varphi^{-1}) = \mathrm{ind}_z(\vec{v}).$$

Furthermore, by definition 7.2 we have that

$$L_z(f_t) = L_0(\varphi \circ f_t \circ \varphi^{-1}).$$

So what is left to prove are the following three things:

1. $\varphi \circ f_t \circ \varphi^{-1}$ is tangent to $d\varphi \circ \vec{v} \circ \varphi^{-1}$ at time $t = 0$ for all $x \in \varphi(U)$.
2. $d\varphi \circ \vec{v} \circ \varphi^{-1}$ vanishes at the origin.
3. for all $t \neq 0$ we have that $\varphi \circ f_t \circ \varphi^{-1}$ only fixes the origin.

We begin with the second one:

$$\begin{aligned}(d\varphi \circ \vec{v} \circ \varphi)_0 &= d\varphi_{\varphi^{-1}(0)}(\vec{v}_{\varphi^{-1}(0)}) \\ &= d\varphi_z(\vec{v}_z) \\ &= d\varphi_z(0) \\ &= 0,\end{aligned}$$

because $d\varphi_z$ is linear.
We procede with the third point so let $t \neq 0$. We know that f_t only fixes z in $U \subseteq M$ and due to the fact that φ and φ^{-1} are diffeomorphisms it indeed follows that $\varphi \circ f_t \circ \varphi^{-1}$ only fixes the origin.
Now we will prove that $\varphi \circ f_t \circ \varphi^{-1}$ is tangent to $d\varphi \circ \vec{v} \circ \varphi^{-1}$ at time $t = 0$:

$$\begin{aligned}\frac{\partial}{\partial t}(\varphi \circ f_t \circ \varphi^{-1}(x))\Big|_{t=0} &= d\varphi_{\varphi^{-1}(x)}\left(\frac{\partial}{\partial t} f_t(\varphi^{-1}(x))\Big|_{t=0}\right) \\ &= d\varphi_{\varphi^{-1}(x)}(\vec{v}_{\varphi^{-1}(x)}) \\ &= (d\varphi \circ \vec{v} \circ \varphi^{-1})_x,\end{aligned}$$

which is the desired result. We conclude that $\mathrm{ind}_z(\vec{v}) = L_z(f_t)$. □

In the next section we shall proof the Poincaré-Hopf theorem using, among others, theorem 7.6, lemma 7.7 and lemma 7.10.

7.2 Proof

Proof of 1.2. Let M be compact oriented manifold and let $\vec{v}$ be a vectorfield on it. The only thing that we need to prove now is that there exists a flow $(f_t)_{t\in[0,1]}$ with the following three properties:

1. At time $t = 0$ the flow $(f_t)_{t\in[0,1]}$ is tangent to the vectorfield $\vec{v}$.
2. At time $t > 0$ the fixed points of f_t coincide with the zeros of $\vec{v}$.
3. At time $t = 0$ we have that f_0 is the identity map on M.

From lemma 7.8 and theorem 7.6 it would then follow that the sum of the indices of $\vec{v}$ at its zeros is equal to the Lefschetz number $\Lambda(f_t)$ of f_t. Due to the fact that f_0 is the identity map we know that f_t is homotopic to the identity and by correlary 7.5 and lemma 7.7 it would then follow that

$$\sum_{\vec{v}(x)=0} \mathrm{ind}_x(\vec{v}) = \chi(M),$$

which is the desired result. To prove that there exists such a flow, we will construct it using the following theorem for which a proof can be found in both *Lectures on Symplectic Geometry* [10] by A. C. Da Silva and *Introduction to Smooth Manifolds* [5]. This is not the entire theorem, but the only part we need.

Theorem 7.11 (ϵ-Neighborhood Theorem, [5], 6.24)**.** *Let $M \subset \mathbb{R}^k$ be a compact manifold and let $N^\epsilon := \{p \in \mathbb{R}^k : d(p,q) < \epsilon$, for some $q \in M\}$ be the set of all points in $\mathbb{R}^k$ that is not farther away than a distance ϵ from M. For small enough ϵ we then have that for all $p \in N^\epsilon$ there exists a unique nearest point $q \in M$.*

Remark 7.12. Because of the Whitney embedding theorem 6.12 we can always embed M in $\mathbb{R}^k$.

By this theorem we can define the projection map on N^ϵ as follows

$$\begin{aligned}\pi &: N^\epsilon \to M \\ &: p \mapsto q,\end{aligned}$$

where q is the unique nearest point of p. The projection has the property that $\pi|_M = \mathrm{id}_M$.
Now we will construct the flow f_t, but before that we note that for small t

$$x + t\vec{v}(x) \in N^\epsilon$$

for all $x \in M$. Let the interval for which t is small enough be denoted by $\mathcal{D}$. The above is true, because of the compactness of M. Now we define f_t as follows

$$\begin{aligned}f_t &: M \to M \\ &: x \to \pi(x + t\vec{v}(x)).\end{aligned}$$

For $t = 0$ we indeed have that $f_0 = \mathrm{id}_M$, so the third property is satisfied. To verify the first property we differentiate f_t with respect to the time t for a fixed $x \in M$. Using the chain rule we obtain

$$\frac{\partial f_t(x)}{\partial t}\Big|_{t=0} = d\pi_x \circ \vec{v}(x).$$

Recall that π is the identity on M, so $d\pi_x$ is also the identity on T_xM. Because $\vec{v} \in T_xM$ it follows that

$$\frac{\partial f_t(x)}{\partial t}\Big|_{t=0} = \vec{v}(x).$$

and hence $(f_t)_{t\in\mathcal{D}}$ is tangent to $\vec{v}(x)$ at time $t = 0$.
For $t > 0$ consider a fixed point x of f_t. This implies that $\pi(x + t\vec{v}(x)) = x$. This is only possible when $t\vec{v}(x)$ is perpendicular to M at x. So we have that $t\vec{v}(x) \in (T_xM)^\perp$, but per definition we also have $\vec{v}(x) \in T_xM$. This can only happen when $\vec{v}(x)$ is the zero vector. The other way around also holds. If $\vec{v}(x)$ is zero, then $f_t(x) = \pi(x) = x$. We conclude that the second property also holds and thus we have proven the Poincaré-Hopf theorem.

$\square$

8 Morse Theory

In this chapter we will discuss how Morse theory can help us to prove the Poincaré-Hopf theorem. It turns out that we can prove that on a compact and oriented manifold, there exist a vectorfield $\vec{v}$ such that the sum of the indices of $\vec{v}$ at its zeros equals the Euler characteristic $\chi(M)$ of M. This will be stated in theorem 8.34. In the next chapter we will use this result to prove it for all vectorfields, which yields the Poincaré-Hopf theorem.
In the first section of this chapter we shall discuss some basic definitions. Afterwards we shall introduce and prove the Morse lemma, which we will use to prove theorem 8.34. In the section following we shall discuss the Morse inequalities to find a useful relation for the Euler characteristic. Finally, in the fourth section we shall try and prove theorem 8.34.

8.1 Basics

We start with a few definition and we will follow chapter 1.2 in *Morse Theory* [2] by J. Milnor for that. Recall the definition of a critical point. For real valued functions we have the following definition.

Definition 8.1. Let M a manifold with $f : M \to \mathbb{R}$ a smooth function. Then $p \in M$ is called a *critical point* if

$$df_p : T_pM \to T_{f(p)}\mathbb{R}$$

is the zero map. The *critical value* of p is then defined to be $f(p) \in \mathbb{R}$.

Remark 8.2. The definitions of a critical point as in definitions 6.4 and 8.1 do not collide, because the only non-surjective linear map $df_p : T_pM \to \mathbb{R}$ is the zero map.

Definition 8.3. Let p be a critical point of a smooth map $f : M \to \mathbb{R}$. Because M is a smooth manifold there exists a chart (U, φ) with $p \in U$. It holds that U is diffeomorphic to $\mathbb{R}^m$ and let the coordinate system of $\mathbb{R}^m$ be $\{x_1, \dots, x_m\}$. We will call the critical point p *non-degenerate* if the matrix

$$\left(\frac{\partial^2(f \circ \varphi^{-1})(p)}{\partial x_i \partial x_j}\right)_{i,j}$$

is non-singular where $x_i, x_j \in \{x_1, \dots, x_m\}$.

Remark 8.4. The non-degeneracy of a critical point is independent of which chart we choose.

These definitions give rise to the following important definition in Morse theory.

Definition 8.5. Let M a smooth manifold and let $f : M \to \mathbb{R}$ be smooth function. We say that f is a *Morse function* if all of its critical points are non-degenerate.

Definition 8.6. For p a critical point let $v, w \in T_pM$ and let $\tilde{v}, \tilde{w}$ be two vectorfields such that $\tilde{v}_p = v$ and $\tilde{w}_p = w$. Then we define the *Hessian* of f at p to be the following map:

$$\begin{aligned} H_pf : T_pM \times T_pM &\to \mathbb{R} \\ : (v, w) &\to \tilde{v}_p(\tilde{w}(f)), \end{aligned}$$

which is bilinear and symmetric in its domain.

The Hessian is well-defined, because it is independent of choice of vectorfields $\tilde{v}$ and $\tilde{w}$ as long as $\tilde{v}_p = v$ and $\tilde{w}_p = w$ holds. If (U, φ) is a chart around $p \in U$ with local coordinates $(x_1, \dots, x_n)$ we can write v and w in the following way

$$\begin{aligned} v &= \sum_i v_i \frac{\partial}{\partial x_i}\Big|_p \\ w &= \sum_i w_i \frac{\partial}{\partial x_i}\Big|_p, \end{aligned}$$

with $v_i, w_i \in \mathbb{R}$ as we have seen in proposition 2.20. Then $\tilde{w} := \sum_i w_i \frac{\partial}{\partial x_i}$ with $w_i : U \to \mathbb{R}$ a constant function sending every point in U to $w_i \in \mathbb{R}$ satisfies $\tilde{w}_p = w$. It follows that

$$H_pf(v, w) = \sum_i \sum_j v_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p),$$

for $p \in U$.

Definition 8.7. Let $H : V \times V \to \mathbb{R}$ be a bilinear map. Define W to be the largest subspace of V such that

$$H(w_1, w_2) < 0$$

for all $w_1, w_2 \in W$. Then we define the *index* of H on V to be the dimension of W.

Remark 8.8. Let p be a non-degenerate critical point of f again. Then the index of $H_p f$ on $T_p M$ is the amount of negative eigenvalues of $H_p f$.

Definition 8.9. The *index* of the function f at its critical point p is defined to be the index of $H_p f$ on $T_p M$.

Remark 8.10. The index of a function at its critical point should not be confused with the index of a vectorfield at its zero.

Definition 8.11. The number of critical points of f with index n is denoted by $\Psi_{f,n}$.

8.2 Morse Lemma

As we said earlier, we will state and prove the Morse lemma in this section. This section is largely based on the theory discussed in *Morse theory* [2] and on the paper *An Introductory Treatment of Morse Theory on Manifolds* [14] by A. Hua.

Lemma 8.12 (Morse Lemma)**.** *Let M be a smooth manifold and let $f : M \to \mathbb{R}$ be a smooth function. Suppose p is a critical point of f which is non-degenerate. Then there exists a chart (U, φ) with $p \in U$ such that*

$$(f \circ \varphi^{-1})(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2,$$

where $\{x_1, \ldots, x_n\}$ is the local coordinate system of U and i the index of f at p.

Corollary 8.13. *It holds that non-degenerate critical points are isolated. Furthermore, if M is compact, then the Morse function $f : M \to \mathbb{R}$ has a finite number of critical points.*

Proof. Let p be a non-degenerate critical point of f. Then we can calculate the gradient of $f \circ \varphi^{-1}$ and we get that it is zero only when x_i is zero for all i. In an open neighborhood of p this can only happen at p, so indeed all the non-degenerate critical points are isolated. Now we know that every critical point is isolated, cover the compact manifold M with open sets such that each only contains at most one critical point. Due to compactness we can extract a finite subcover, from which we may conclude that there are only a finite number of critical points. $\square$

To prove the Morse lemma we need the following lemma, for which a proof can be found in *Morse Theory* [2] by J. Milnor on page 5.

Lemma 8.14. *Let V be a convex neighborhood of the origin in $\mathbb{R}^n$ and let $f : V \to \mathbb{R}$ be a smooth real valued function with $f(0) = 0$. Then we have that*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n),$$

where $g_i : V \to \mathbb{R}$ are also smooth real valued functions on V. The g_i have the additional property that

$$g_i(0) = \frac{\partial f}{\partial x_i}(0).$$

We also need the Inverse function theorem for the proof of the Morse lemma. A proof can be found in *Introduction to Smooth Manifolds* [5] at page 657.

Theorem 8.15 (Inverse Function Theorem, [5], p. 657)**.** *Let $F : U \to V$ be a smooth function between to open sets $U, V \subseteq \mathbb{R}^n$. Suppose that $DF(a)$ is invertible at $a \in U$. Then there exist connected sets $U_0 \subseteq U$ and $V_0 \subseteq V$ containing respectively a and $F(a)$ such that*

$$F_{U_0} : U_0 \to V_0,$$

is a diffeomorphism.

Proof of 8.12. We start by proving that i should be the index of f at p given the fact that we can write f like in the lemma. So suppose

$$f \circ \varphi^{-1}(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2.$$

We then see that

$$\frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j}(p) = \begin{cases} -2 & \text{if } i = j \leq i \\ 2 & \text{if } i = j \geq i+1 \\ 0 & \text{if } i \neq j \end{cases}.$$

This implies the following diagonal matrix representation of $H_p f$:

$$\begin{pmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{pmatrix},$$

with $\frac{\partial}{\partial x_1}\big|_p, \dots, \frac{\partial}{\partial x_n}\big|_p$ the basis. It follows that i is indeed the index of f at p per definition ....
Now we only need to prove existence of a chart with local coordinate system such that

$$(f \circ \varphi^{-1})(x_1, \dots, x_n) = f(p) \pm x_1^2 \pm \dots \pm x_n^2.$$

Note that this expression is slightly different, because we do not have to care about the signs anymore. Without loss of generality we can assume that $f(p) = 0$, $\varphi(p) = 0 \in \mathbb{R}^n$ and that U is convex. Then we can apply lemma 8.14 to obtain

$$(f \circ \varphi^{-1})(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n),$$

with $g_i(0) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(0) = 0$, because p is a critical point of f. Now we have that g_i also satisfy the condition of lemma 8.14, so we can also write

$$g_j(x_1, \dots, x_n) = \sum_{i=1}^n x_i h_{ij}(x_1, \dots, x_n),$$

where $h_{ij} : U \to \mathbb{R}$ are smooth functions. It follows that

$$(f \circ \varphi^{-1})(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n x_i x_j h_{ij}(x_1, \dots, x_n).$$

Define $H_{ij} := \frac{h_{ij} + h_{ji}}{2}$ from which we get that

$$(f \circ \varphi^{-1})(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n x_i x_j h_{ij}(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n x_i x_j H_{ij}(x_1, \dots, x_n), \tag{2}$$

but now with $H_{ij} = H_{ji}$. Using this property we see that

$$\frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j}(0) = 2H_{ij}(0).$$

We will finish the proof using an inductive argument. Our goal is to find a chart with coordinate system $(y_1, \dots, y_n)$ such that the lemma holds. Without loss of generality we can start with a coordinate system $(x_1, \dots, x_n)$ such that

$$\frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_1^2}(0) \neq 0.$$

This implies that $H_{11} \neq 0$. Because of the smoothness of H_{11}, we know that there exists an open neighborhood $\mathcal{N}_0$ such that H_{11} is not zero in $\mathcal{N}_0$. Then it holds that

$$G(x_1, \ldots, x_n) = \sqrt{|H_{11}(x_1, \ldots, x_n)|}$$

is a smooth function on $\mathcal{N}_0$. Now we define the coordinate system $(y_1, \ldots, y_n)$ in the following way

$$y_i = \begin{cases} G \cdot \left(x_1 + \sum_{i=2}^n x_i \frac{H_{i1}}{|H_{11}|} \right) & \text{if } i = 1 \\ x_i & \text{if } i \neq 1 \end{cases} . \tag{3}$$

It holds that the determinant of the Jacobian of this transition between coordinate systems is non-zero and thus by the inverse function theorem it is secured that $(y_1, \ldots, y_n)$ really is a local coordinate system within a smaller neighborhood $\mathcal{N}_0'$. From equation (3) it follows that

$$y_1^2 = |H_{11}| \left(x_1 + \sum_{i=2}^n x_i \frac{H_{i1}}{|H_{11}|} \right)^2 \tag{4}$$

$$= \pm H_{11} x_1^2 \pm 2 \sum_{i=2}^n x_i x_1 H_{i1} \pm \frac{\left(\sum_{i=2}^n x_i H_{i1}\right)^2}{|H_{11}|} \tag{5}$$

And from equation (2) it then follows that

$$(f \circ \varphi^{-1})(x_1, \ldots, x_n) = \pm y_1^2 + \left(\sum_{i=2}^n \sum_{j=2}^n x_i x_j H_{ij} \right) - \frac{\sum_{i=2}^n x_i H_{1i}}{H_{11}} \tag{6}$$

One could verify this by plugging the expression in (5) in equation (6) to obtain (2) again. We can rewrite this to

$$(f \circ \varphi^{-1})(x_1, \ldots, x_n) = \pm y_1^2 + \sum_{i=2}^n \sum_{j=2}^n x_i x_j H'_{ij},$$

with H'_{ij} a smooth function which is also symmetric in its indices. We can repeat this proces which gives us the desired result. $\square$

8.3 Morse Inequalities

In this section we shall try to prove the Weak Morse Inequalities as in theorem 8.29. This will give us a useful result about the Euler characteristic. We will primarily follow chapter 1.5 of *Morse Theory* [2], but some parts are also based on *An Introduction to Morse Theory* [6] and the thesis *Morse Theory and Witten's Proof of the Morse Inequalities* [16] by D. A. P. Meza.
First, consider the following lemma which is a simplified version of theorem 2.20 in *In Introduction to Morse Theory* [6] by Y. Matsumoto for which a proof can be found on page 53.

Lemma 8.16 (Existence of Morse Functions, [6], 2.20)**.** *Let M be a compact manifold. Then there exists a smooth Morse function on it.*

The next lemma is a simplified version of lemma 2.8 in *Lectures on the h-Cobordism Theorem* by J. Milnor and a proof can be found on pages 17-18.

Lemma 8.17 ([3], 2.8)**.** *Let M be a compact manifold and let $f : M \to \mathbb{R}$ a Morse function with critical points $p_1, \ldots, p_k$. Then there exists another Morse function $g : M \to \mathbb{R}$ with the same critical point. The function g has the additional property that critical value is unique for each critical point. In other words, $g(p_i) \neq g(p_j)$ for all $i \neq j$.*

We will constantly use these results. Now consider the following definitions, which are necessary for the Morse inequalities.

Definition 8.18. Let M a smooth manifold with $f : M \to \mathbb{R}$ a smooth function and let $a \in f(M)$. Then we define

$$M^a = f^{-1}((-\infty, a]),$$

which is often referred to as a sublevel set.

Definition 8.19. Let $Z \subseteq Y \subseteq X$ be topological spaces and let S be a map that sends two topological spaces to an integer. The map S is said to be *subadditive* when

$$S(X, Z) \leq S(X, Y) + S(Y, Z),$$

for all X, Y, Z with $Z \subseteq Y \subseteq X$. Furthermore, we say that S is *additive* when

$$S(X, Z) = S(X, Y) + S(Y, Z),$$

for all X, Y, Z with $Z \subseteq Y \subseteq X$.

Definition 8.20. Recall definition 3.14 about relative homology groups and let $B \subseteq A$. Then we define the n^{th} *relative Betti number* $b_n(A, B)$ of the pair A and B to be $\dim(H_n(A, B; \mathbb{R}))$.

Remark 8.21. Note that $b_n(A, \emptyset) = b_n(A)$ as in definition 3.24.

Definition 8.22. We define the *relative Euler characteristic* of the pair A and B to be

$$\sum_i (-1)^i b_i(A, B),$$

which will be denoted by $\chi(A, B)$.

Remark 8.23. Similarly, we have that $\chi(A, \emptyset) = \chi(A)$.

We need the following two lemma's.

Lemma 8.24. *The relative Betti numbers are subadditive for all n.*

Proof. Let $Z \subseteq Y \subseteq X$ be topologica spaces. Then by lemma 3.18 it follows that

$$\cdots H_n(Y, Z) \xrightarrow{a_n} H_n(X, Z) \xrightarrow{b_n} H_n(X, Y) \xrightarrow{c_n} H_{n-1}(Y, Z) \cdots$$

is an exact sequence. By the Rank-Nullity theorem we see that

$$\begin{aligned} b_n(Y, Z) &= \dim(\mathrm{Im}(b_n)) + \dim(\mathrm{Ker}(b_n)) \\ b_n(X, Z) &= \dim(\mathrm{Im}(c_n)) + \dim(\mathrm{Ker}(c_n)) \\ b_n(X, Y) &= \dim(\mathrm{Im}(d_n)) + \dim(\mathrm{Ker}(d_n)). \end{aligned}$$

Using the exactness of the sequence, we get

$$\begin{aligned} b_n(Y, Z) + b_n(X, Y) &= \dim(\mathrm{Im}(b_n)) + \dim(\mathrm{Ker}(b_n)) + \dim(\mathrm{Im}(d_n)) + \dim(\mathrm{Ker}(d_n)) \\ &= \dim(\mathrm{Ker}(c_n)) + \dim(\mathrm{Ker}(b_n)) + \dim(\mathrm{Im}(d_n)) + \dim(\mathrm{Im}(c_n)) \\ &\geq b_n(X, Z), \end{aligned}$$

which proves the subadditivity of the relative Betti numbers. □

Lemma 8.25. *Let X, Y, Z be compact topological spaces such that $Z \subseteq Y \subseteq X$. Then*

$$\chi(X, Z) = \chi(X, Y) + \chi(Y, Z)$$

In other words, the relative Euler characteristic is additive over compact sets.

Proof. By lemma 3.18 we have that

$$\cdots H_n(Y, Z) \xrightarrow{a_n} H_n(X, Z) \xrightarrow{b_n} H_n(X, Y) \xrightarrow{c_n} H_{n-1}(Y, Z) \cdots$$

is an exact sequence. Now it holds that X, Y, Z are compact, so there exists a $k \in \mathbb{N}$ such that $H_n(X, Z) = H_n(X, Y) = H_n(Y, Z) = 0$ for all $n > k$. What we then have is a sequence

$$0 \longrightarrow H_k(Y, Z) \longrightarrow \cdots \longrightarrow H_0(X, Y) \longrightarrow 0,$$

which is exact. By generalizing lemma 3.27 we get that

$$b_k(Y, Z) - b_k(X, Z) + b_k(X, Y) - b_{k-1}(Y, Z) + \ldots (-1)^k b_0(X, Y) = 0.$$

It follows that

$$\chi(Y, Z) - \chi(X, Z) + \chi(X, Y) = 0,$$

which is the desired result. □

We need two more theorems before we can state and prove the Weak Morse Inequalities 8.29. A proof of the following theorem can be found in *Morse Theory* [2] at pages 23-24.

Theorem 8.26 ([2], 3.5)**.** *Let M be a smooth manifold and suppose that $f : M \to \mathbb{R}$ is a smooth Morse function such that each sublevel set $M^a \subseteq M$ is compact. Then it holds that M is homotopy equivalent to a CW-complex with the property that it has an n-cell for each critical point of f with index n.*

Remark 8.27. For compact manifolds the compactness of the sublevel sets is automatically satisfied.

The proof of the next theorem can be found in *Morse Theory* [2] at pages 14-19.

Theorem 8.28 ([2], 3.2)**.** *Let M be a smooth manifold with $f : M \to \mathbb{R}$ a smooth function and let $p \in Crit(f)$ be non-degenerate with index n. Let $f(p) = a$ and suppose that $f^{-1}([a-\epsilon, a+\epsilon])$ is compact and contains not other critical point of f. Then for small enough $\epsilon > 0$ we have that $M^{a+\epsilon}$ is homotopy equivalent to $M^{c-\epsilon}$, but with an n-cell attached to it.*

Now we have gathered enough tools to state and prove the Weak Morse Inequalities

Theorem 8.29 (Weak Morse Inequalities, [2], 5.2)**.** *Let M be a compact manifold with $f : M \to \mathbb{R}$ a Morse function with k critical points. Then we have that*

$$b_n(M) \leq \Psi_{f,n},$$

and

$$\chi(M) = \sum_{j=1} (-1)^j \Psi_{f,j},$$

where b_n denotes the n^{th} Betti number and $\Psi_{f,n}$ denotes the number of critical points with index n.

Proof. Given is that f is a Morse function, so the critical points are automatically non-degenerate. From corollary 8.13 it follows that they are isolated and that there is a finite number of them. Let $\{a_0, a_1, \dots, a_{k-1}, a_k\}$ be an ordered set in $\mathbb{R}$ with $a_0 < a_1 < \dots < a_{k-1} < a_k$ such that $(a_i, a_{i+1}) \subset \mathbb{R}$ contains one critical value of f and $M^{a_k} = M$, which is secured by lemma 8.17. Then by lemma 8.24 we have

$$b_n(M, \emptyset) \leq \sum_{i=1}^{k} b_n(M^{a_i}, M^{a_{i-1}}).$$

By theorem 8.28 we know that we get can get something homotopy equivalent to M^{a_i} by attaching an n_i-cell to it. Now we can use the excision theorem 3.17 to conclude that $H_n(M^{a_i}, M^{a_{i-1}})$ and $H_n(\mathbb{D}^{n_i}, \partial\mathbb{D}^{n_i})$ are isomorphic, where $\mathbb{D}^{n_i}$ is the n_i-disk. The Betti numbers b_n are topological invariants, so we get that

$$b_n(M^{a_i}, M^{a_{i-1}}) = b_n(\mathbb{D}^{n_i}, \partial\mathbb{D}^{n_i}).$$

It holds that $b_n(\mathbb{D}^{n_i}, \partial\mathbb{D}^{n_i}) = \delta_{j,n_i}$, where δ stands for the Kronecker delta. We see that

$$\begin{aligned}
\chi(M) &= \chi(M, \emptyset) \\
&= \sum_{i}^{k} \chi(M^{a_i}, M^{a_{i-1}}) && \text{(by lemma 8.25)} \\
&= \sum_{i}^{k} \sum_{j} (-1)^j b_j(M^{a_i}, M^{a_{i-1}}) \\
&= \sum_{j} (-1)^j \sum_{i}^{k} b_j(M^{a_i}, M^{a_{i-1}}) \\
&= \sum_{j} (-1)^j \sum_{i}^{k} b_n(\mathbb{D}^{n_i}, \partial\mathbb{D}^{n_i}) \\
&= \sum_{j} (-1)^j \sum_{i}^{k} \delta_{j,n_i}.
\end{aligned}$$

Note that $\sum_i^k \delta_{j,n_i}$ is the amount of j-cells. We were allowed to interchange summations because of the compactness of M. It namely then follows that at the n^{th} Betti numbers are equal to zero for big enough n. Using theorem 8.26 we can see that

$$\begin{aligned} b_n(M) &= b_n(M, \emptyset) \\ &\leq \sum_{i=1}^{k} b_n(M^{a_i}, M^{a_{i-1}}) \\ &= \lambda_{f,i}, \end{aligned}$$

and

$$\chi(M) = \sum_{j=1} (-1)^j \Psi_{f,j},$$

which are the desired results. $\square$

8.4 Vectorfields

We have reached the final section of this chapter, and the goal is to state and prove theorem 8.34. This section is mainly based on *Morse Theory* [2], *The Euler Characteristic, Poincare-Hopf Theorem, and Applications* [11] and *An Introduction to Morse Theory* [6]. But first we consider a few definitions.

Definition 8.30. Let M be a smooth manifold with (U, φ) a chart with local coordinate system $(x_1, \ldots, x_n)$. Then we define the *gradient vector field* of f on U to be

$$\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial}{\partial x_n},$$

in local coordinates.

Remark 8.31. The gradient vector field of f is in fact the differential of f.

Definition 8.32. Let M be a smooth manifold with $f : M \to \mathbb{R}$ a smooth Morse function. By the Morse lemma 8.12 we know that for each critical point p_i there exists a chart (U_i, φ_i) such that

$$(f \circ \varphi_i^{-1})(x_1, \ldots, x_n) = f(p_i) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2,$$

in local coordinates and λ_i the index of p_i. We call a vectorfield a *semi gradient-like vectorfield* if for all critical points p_i there exists a neighborhood Y_i such that

$$X = -2x_1\frac{\partial}{\partial x_1} - \cdots - 2x_\lambda\frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_n\frac{\partial}{\partial x_n},$$

i.e. as the gradient vector field of f on Y_i. Actually it should be $X \circ \varphi_i^{-1}$, but for simplicity we leave φ out.

Using these definitions we can state the following theorem.

Theorem 8.33 ([6], 2.30)**.** *Let M be a smooth manifold with $f : M \to \mathbb{R}$ a smooth Morse function. Then one can find a semi gradient-like vectorfield of f.*

Proof. Let's say that there are k critical points of f and we denote them by $\{p_1, \ldots, p_k\}$. We know that they are non-degenare, so by the Morse lemma 8.12 there exists charts (U_i, φ_i) corresponding to each p_i, such that

$$(f \circ \varphi_i^{-1})(x_1, \ldots, x_n) = f(p_i) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2,$$

in its local coordinate system. Consider these U_i and add more open sets U_j such that $U_1, \ldots, U_m$ form an open cover of M. Let p_j be a critical point of f. Then

$$X_i := \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial}{\partial x_n}$$

is a gradient vector field of f on U_i in local coordinates. The idea is to construct a global vectorfield using these local vectorfields. Therefore, consider $h_i : U_i \to \mathbb{R}$ with $0 \leq h_i \leq 1$. Suppose that it is a smooth function defined to be equal to 1 in some neighborhood V_i of p_i and 0 outside a compact set L_i with

$V_i \subset L_i \subset U_i$. We can extend h_i to the entirety of M by just defining it to be zero outside U_i and we do this for all i.
Now consider

$$h_i X_i.$$

This vectorfield is defined on U_i, but we can also extend it to M by defining it to be the zero vector outside U_i. We define the global vectorfield X to be

$$X = \sum_{i=1}^{k} h_i X_i.$$

Consider a very small neighborhood Y_i of p_i which lies only in U_i and not in any other open set of the cover. Without loss of generality we have that $h_i = 1$ here. So in Y_i we have that

$$\begin{aligned} X &= \sum_{i=1}^{k} h_i X_i \\ &= X_i. \end{aligned}$$

By doing this for all critical points we get that X is a semi gradient-like vectorfield of f. □

Now we have gathered enough machinery to prove the final theorem of this chapter.

Theorem 8.34. *Let M be a compact smooth manifold. Then there exists a vector field such that the sum of the indices of v at its zeros equals $\chi(M)$.*

Proof. Let $f : M \to \mathbb{R}$ be a Morse function. Then by the Morse lemma we have for every critical point p_i a chart (U_i, φ_i) with local coordinate system $(x_1, \ldots, x_n)$ such that

$$(f \circ \varphi_i^{-1})(x_1, \ldots, x_n) = f(p_i) - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2,$$

with j the index of f at p_i. This critical point p_i is also a zero of the semi gradient-like vectorfield $\vec{v}$ which we can find by theorem 8.33. If j is odd, we have that the determinant of $d\vec{v}_p$ is negative and hence by lemma 6.13 the index of $\vec{v}$ at p_i must be -1. If it was even, we would have that $\mathrm{ind}_{p_i}(\vec{v}) = +1$. So we see that

$$\mathrm{ind}_{p_i}(\vec{v}) = (-1)^j.$$

Now by applying theorem 8.29 we obtain

$$\begin{aligned} \sum_i \mathrm{ind}_{p_i}(\vec{v}) &= \sum_j (-1)^j \Psi_{f,j} \\ &= \chi(M), \end{aligned}$$

which is the desired result. □

9 Proof Poincaré-Hopf Theorem II

Proof of 1.2. We split the situation in two cases. The first one is where $\vec{v}$ has only non-degenerate zeros and the second where $\vec{v}$ happens to have a degenerate zero. Of course, we assume that $\vec{v}$ only has isolated zeros in both cases.

We start with the first one, so suppose that $\vec{v}$ has only non-degenerate zeros on a manifold without boundary M. Using theorem 8.34 we can find a vectorfield $\vec{v}'$ such that the sum of the indices of $\vec{v}'$ at its zeros is equal to the Euler characteristic $\chi(M)$ of M. From corollary 6.21 we know that the sum of the indices of a $\vec{v}'$ at its zeros is the same as for $\vec{v}$ and hence the statement is true for vectorfields without degenerate zeros.

Now let $U \subseteq \mathbb{R}^n$ be an open set and suppose that $\vec{v}$ has degenerate zeros. The idea is to construct a vectorfield $\vec{v}'$ with only non-degenerate zeros and afterwards proving that the sum of the indices of $\vec{v}'$ at its zeros is the same as for $\vec{v}$. First, we use Hausdorffness to find an $\epsilon > 0$ such that a 2ϵ-ball $B(z; 2\epsilon)$ around a degenerate zero z contains no other zero in it. Now we define a smooth function $f : U \to [0, 1]$ such that it is one inside $B(z; \epsilon)$ and zero outside $B(z; 2\epsilon)$. Let us define the following vectorfield:

$$\vec{v}'(x) := \vec{v}(x) - f(x)y,$$

where y is a regular value of $\vec{v}$. Because $v(x)$ has only one zero in $B(z; 2\epsilon)$, it holds that there exists a $\delta > 0$ such that $||v(x)|| > \delta$ for all $x \in B(z; 2\epsilon)\backslash B(z; \epsilon)$. The existence of y is secured by Sard's theorem and by the same theorem it is possible to find a y such that $||y|| < \delta$. As a result it holds that all the zeros of the newly constructed vectorfield $\vec{v}'$ are inside $B(z; \epsilon)$. Let z' be a zero of $\vec{v}'$. It follows that

$$0 = \vec{v}(z) - y.$$

This means that z' is a regular point of $\vec{v}$ per definition of the regular value. Due to the fact that $f(x)$ in $B(z; \epsilon)$, it follows that

$$d\vec{v}'_x = d\vec{v}_x,$$

for all $x \in B(z, \epsilon)$ and in particular for z'. As a result we have that z' is also a regular point of $\vec{v}'$. Our conclusion is that this zero is non-degenerate. This holds for all zeros of $\vec{v}'$, so $\vec{v}'$ has only non-degenerate zeros.
Now we will prove that the sum of the indices of the vectorfields $\vec{v}$ and $\vec{v}'$ at their zero's coincide. Consider $\mathrm{ind}_z(v)$, where z is still a degenerate zero of $\vec{v}$. Per definition 6.9 it is equal to the degree of the following map:

$$\begin{aligned} F &: \partial B(z; 2\epsilon) \to S^{n-1} \\ &: x \mapsto \frac{v(x)}{||v(x)||}. \end{aligned}$$

Whe shall prove that

$$\mathrm{ind}_z(\vec{v}) = \sum_{z'} \mathrm{ind}_{z'}(\vec{v}'),$$

like in the proof of theorem 6.19. Again, we use Hausdorffness to find an $\epsilon' > 0$ such that $B(z'; \epsilon)$ around z' contains no other zero in it. Now we remove these ϵ-balls from $B(z; 2\epsilon)$ to obtain a set which we will call N. We can extend $\vec{v}|'_{\partial N}$ to $\vec{v}' : N \to S^{n-1}$. Likewise, we can do the same for F. Using theorem 6.15 we can conclude that the degree of F on ∂N is zero. In a similar way as we have done in the proof of theorem 6.19 we get

$$\mathrm{ind}_z(\vec{v}) = \sum_{z'} \mathrm{ind}_{z'}(\vec{v}'),$$

which is the desired result. Using charts we can apply this also to manifolds instead of just $U \subseteq \mathbb{R}^n$, which finishes the proof. $\square$

10 Conclusion

We start the conclusion by proving the Hairy Ball theorem.

Theorem 10.1 (Hairy Ball theorem, [5], p. 435)**.** *There does not exist a vectorfield which is non-zero on all* S^2.

Proof. In example 3.23 we have seen that the Euler characteristic of S^2 is equal to 2. Suppose the contrary, that there exists a vectorfield which is non-zero on all S^2. This implies that there are no zeros of the vectorfield and hence that the sum of the indices of the vectorifield at its zeros is also zero. But this is a contradiction and the desired result follows. □

We begun this thesis by stating the Poincaré-Hopf theorem after which we have discussed the various components of it. Thereafter, we have proved it without Morse theory and with Morse theory.

Now let us compare the two proofs of the Poincaré-Hopf theorem. In both proofs we skipped some proofs of various theorems and lemmas. However, in the first proof we have skipped more than one would think in the first place. The proof of theorem 7.6, which we did not do, involves namely intersection theory. More information about intersection theory can be found in *Differential Topology* [7]. But besides that I found the two proofs quite similar in difficulty.

One of the beautiful things about the Poincaré-Hopf theorem is that the equality is independent of vectorfield. In the proof including Morse theory we explicitly prove that. We namely had found one vectorfield for which the Poincaré-Hopf theorem holds. Thereafter, using a construction, we extended it for all vectorfields.

What I found remarkable is that the first proof also makes use of a construction. Unfortunately, I couldn't find more similarities between the two proofs. A follow-up research could be to find similarities, for example similarities between Lefschetz fixed point theory and Morse theory.

My interest in Morse theory grew throughout writing the thesis. Especially after reading *Morse Theory and Witten's Proof of the Morse Inequalities* by D. A. P. Meza. Also while searching for study material I came across an interesting paper by E. Witten named *Supersymmetry and Morse Theory* which will definitely keep me busy for a while.

At last, I wanted to say that I really enjoyed writing this thesis. Furthermore, I want to thank my supervisor prof. dr. Marius Crainic for his guidance in this bachelor thesis. I also want to thank the organisers and students of the seminar about Morse theory for teaching me Morse theory and finally I want to thank my friends and family for their support.

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