A Framework for Reasoning about Probabilistic Spatio-Temporal Logic

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Abstract

In previous research, frameworks have been constructed that can reason about the probabilistic locations of objects through different time instances. With these frameworks, certain work has been done on belief revision and consistency checking. The expressivity of these frameworks was very limited; they only contained these so-called 'Probabilistic Spatio-Temporal atoms' (PST atoms). In this thesis, a new framework that can reason more extensively with PST atoms is proposed. This is done in 3 different steps. In the first step, a framework is proposed that can only reason about the locations of objects. This is done by combining simplified PST atoms with propositional logic. In the second step, temporal operators are added to the first framework. For this framework, axiomatization has been done and soundness and completeness of the axioms have been proven. In the final step, probabilities were added to the temporal framework. For each framework, syntax, semantics and the satisfiability relation have been defined in a clear way. With these frameworks, more extensive reasoning is possible about PST atoms than in any of the frameworks that were defined in earlier research.

Contents

1 Introduction

Symbolic Artificial Intelligence (AI) is the field in AI research which is concerned with attempting to represent human level knowledge on a symbolic level, by using facts and rules. Symbolic AI is sometimes called Good Old Fashioned AI, since it was the first dominant approach to AI. An example of a symbolic AI is the General Problem Solver. The General Problem Solver was a computer program that was intended to work as a universal problem solver machine.[1] The way that the General Problem Solver worked was by applying axioms and rules on a well-formed problem to find a solution. An example of a problem that the General Problem Solver was able to solve was the Towers of Hanoi.
A popular application of symbolic AI were Expert Systems. The idea behind Expert Systems is that expertise is transferred from a human to a computer. This knowledge will then be stored in the computer and will be shared by the computer to users of the system.[2]

In recent years, some of the work on symbolic AI has been to create certain logical frameworks that allow computers to reason in that framework. One example that will be looked at in this thesis is the paper *An AGM-Style belief revision mechanism for probabilistic spatio-temporal logics* by Grant et al.[3] In this paper, Grant et al. introduce a specific framework for reasoning about the location of objects in time instances. On top of that, they also introduce some measure of uncertainty about the location of said object, giving it the name Probabilistic Spatio-Temporal Knowledge Base(PST KB). In this paper, Grant et al. looked at belief revision strategies for when new knowledge is conflicting with what is already in the knowledge base.
The authors of the paper give some examples when one would like to reason about certain probabilistic-spatio temporal atoms. For example, with GPS tracking: GPS tracking can give some idea where an object might be, but it will never be 100% accurate. Another application where reasoning about the probabilistic locations of objects throughout different time instances could be useful is with drones scouting for survivors in a city after disaster has struck. We know where certain objects used to be, but with certain buildings collapsed, we cannot know for certain that the objects are still in the same locations as before.

The PST KBs as proposed by Grant et al. are very basic Knowledge Bases that only contained facts about the locations of objects. This thesis aims to extend the amount of reasoning that can be done with Probabilistic Spatio-Temporal atoms. In the paper by Grant et al., the authors focussed on belief

1. A Newell and JC Shaw, "A variety op intelligent learning in a general problem solver," *RAND Report P-1742, dated July* 6 (1959).
2. Shu-Hsien Liao, "Expert system methodologies and applications - a decade review from 1995 to 2004," *Expert Syst. Appl.* 28, no. 1 (2005): 93–103, doi:10.1016/j.eswa.2004.08.003, https://doi.org/10.1016/j.eswa.2004.08.003.
3. John Grant et al., "An AGM-style belief revision mechanism for probabilistic spatio-temporal logics," *Artif. Intell.* 174, no. 1 (2010): 72–104, doi:10.1016/j.artint.2009.10.002, https://doi.org/10.1016/j.artint.2009.10.002.

revision strategies, which is something that this thesis will not look at. This thesis aims to create a logic that can reason about the same atoms that Grant et al. used in their paper. The logic will be more expressive, and therefore will allow for richer reasoning. The logic will be based on Linear Temporal Logic, a logic which was first proposed by Pnueli.[4] A full axiomatization of the new logic also follows, which is useful since this means that formally reasoning with this logic is possible.
With this logical framework, it becomes possible to express sentences like "I am at this location until you arrive at that location." Because it is a very general framework that will be proposed, it can easily be altered to restrict certain other properties. For example, it is easy to create a system that imposes a rule in which every location can only hold one object. The locations can also be used to represent certain intersections in a city, making it possible to reason about the way cars move through a city.

The research question is as follows: "How can existing temporal and probabilistic logics be added to extend the amount of reasoning that can be done on probabilistic spatio-temporal atoms?" We will go through several steps to answer this research question.

In section 2, the related research will be discussed. The paper by Grant et al. will be discussed in a bit more detail, as well as some papers that were important before the paper by Grant et al. and some other papers that will be useful to achieve our goal of defininig a new logic.
In section 3, the first steps in creating our framework will be taken. The atoms of the PST KB will be integrated in a standard propositional logic. For this logic, only syntax, semantics and the satisfiability relation will be given.
In that section we will also introduce the Reachability Definition, which is useful when talking about objects in multiple different time instances. In propositional logic, objects cannot really move. However, if you move through time, then it becomes possible for objects to move between locations as well. The Reachability Definition can be used to restrict the ability of how far objects are allowed to move in one instance of time.

In section 4 the Spatio-Temporal Framework is defined. The first part of this section is similar to the first part of section 3. Syntax, semantics and the satisfiability relation are defined for this logic. In the second part of the section, axiomatization of the logic will be done. This will be done in 3 different parts: first of all, the axioms for the logic are presented and explained shortly. Then, soundness of the axioms will be proven. Finally, a completeness proof of the axiomatic system will be given. Because of the proof of soundness, we can say that it is impossible to prove anything that's wrong. Because of the proof

4. Amir Pnueli, "The Temporal Logic of Programs," in *18th Annual Symposium on Foundations of Computer Science, Providence, Rhode Island, USA, 31 October - 1 November 1977* (IEEE Computer Society, 1977), 46–57, doi:10.1109/SFCS.1977.32, https://doi.org/10.1109/SFCS.1977.32.

of completeness, we can say that it is possible to prove anything which is correct.

In section 5, the Probabilistic Spatio-Temporal Framework shall be introduced shortly, with the syntax, semantics and satisfiability relation defined.

The final section is the Discussion section. The limitations of this thesis will be discussed, as well as some proposals for future research. Things to consider are for example branching time logics, or logics that are able to look towards the past, as Linear Temporal Logic is a very basic temporal logic that can only look at states in the future, and is (as its name might already suggest) a linear logic instead of a branching time logic. In the conclusion, all the work that has been done will also be shortly summarized and an answer to the research question shall be given.

2 Related Research

In *A Logical Formulation of Probabilistic Spatial Databases*,[5] a SPOT-database is introduced (Spatial PrObabilistic Temporal), which consists of elements that express statements of the form 'object O is in region R at time t with a probability within the interval $[L, U]$.' The goal of the authors was to create a probabilistic spatial temporal database that could be reasoned with to make predictions about the locations of objects at moments in time.
With SPOT-databases, it becomes possible to make some predictions about the locations of vehicles in the future, as well as the uncertainty of vehicles at this point in time or in the past. They also propose an algorithm that makes it possible for the SPOT-database to find objects within a certain region at a certain time instance with a certain probability interval. A SPOT-database is defined as a finite set of SPOT atoms. These SPOT atoms are four-tuples, which are used to indicate the object it's referring to, the region this object is in, the time instance and the probability interval.

In a later paper, the authors of this paper improved their own SPOT-databases. They proposed certain ways that the set of constraints could be made smaller. This way the database was a lot more efficient when doing a selection query(to find any object that will ever be in a certain region, for example).[6] None of these proposed improvements, however, did anything to change either the syntax, semantics or expressivity of SPOT-databases, and they were mainly used to improve the efficiency when implementing a SPOT-database in a computer program.

In *An AGM-style belief revision mechanism for probabilistic spatio-temporal logics*, Grant et al. use a different approach to also state things like 'an object o is somewhere in region r at time t with a probability between ℓ and u(inclusive)',[7] which is the same sentence that SPOT-databases could reason about. However, the goal of this paper was to examine belief revision strategies. PST KBs were defined in a different way than how SPOT-frameworks were defined. What they both reasoned about was essentially the same, but the PST KBs shall be considered the starting point for this paper.

As opposed to SPOT-atoms, Grant et al. chose to use *id*-atoms, which are of the form $loc(id, r, t)$, where id is the object, r the region, and t the time instance. The probabilities are added on top of the *id*-atoms, which differs from the SPOT-atoms, where the probabilty interval is part of the tuple. The PST

5. Austin Parker, V. S. Subrahmanian, and John Grant, "A Logical Formulation of Probabilistic Spatial Databases," *IEEE Trans. Knowl. Data Eng.* 19, no. 11 (2007): 1541–1556, doi:10.1109/TKDE.2007.190631, https://doi.org/10.1109/TKDE.2007.190631.

6. Austin Parker et al., "SPOT Databases: Efficient Consistency Checking and Optimistic Selection in Probabilistic Spatial Databases," *IEEE Trans. Knowl. Data Eng.* 21, no. 1 (2009): 92–107, doi:10.1109/TKDE.2008.93, https://doi.org/10.1109/TKDE.2008.93.

7. Grant et al., "An AGM-style belief revision mechanism for probabilistic spatio-temporal logics," p.73.

framework also adds in a Reachability Definition, which is defined as a function which restricts where objects can be in consecutive time instances. In the papers where the SPOT-framework was defined, the authors considered it to be possible for an object to move from any region to any region in one single time instance. However, there are certain velocity constraints. It does not make sense for a vehicle to be able to move to a region that is 50 kilometers away in the same time it takes for the vehicle to be able to reach a region that is only 1 kilometer away. There are also differences between different objects. For example, a car is able to move further in one time instance than a bicycle (assuming that the time instances are at least of a certain measurable length).

In Grant et al. the Reachability Definition consists of reachability atoms, which are of the form $reachable_{id}(p_1, p_2)$, where id is an object and p_1 and p_2 are locations. While the id-atoms use regions to refer to the location of an object, the Reachability Definition is only concerned with the movement between specific locations. In a later subsection, a more formal definition of the Reachability Definition is given, which will be used in the frameworks where movement of objects is possible (every framework that uses temporal operators).

The goal of this thesis is to extend the amount of reasoning that can be done about the locations of objects in differen time instances by adding Linear Temporal Logic and probabilistic logic to the framework that was created by Grant et al. In Gabbay et al.,[8] Propositional Linear Temporal Logic was first introduced. The authors provided axioms for the logic and also gave both soundness and completeness proofs. Mark Reynolds[9] later on did axiomatization of Full Computation Tree Logic, of which Linear Temporal Logic is a subset of. Both of these papers will serve as the starting point for the way that Linear Temporal Logic will be incorporated in the PST framework. Linear Temporal Logic also uses some extra operators. For the definitions of these operators, a chapter from the book *Principles of model checking* shall be used.[10]

Fagin et al. have written an article called *A Logic for Reasoning about Probabilities*, in which a probabilistic propositional logic was defined.[11] In previous research, Linear Temporal Logic and probabilistic logic have already been combined to create a new logic that could reason about uncertainty in different time

8. Dov M. Gabbay et al., "On the Temporal Basis of Fairness," in *Conference Record of the Seventh Annual ACM Symposium on Principles of Programming Languages, Las Vegas, Nevada, USA, January 1980*, ed. Paul W. Abrahams, Richard J. Lipton, and Stephen R. Bourne (ACM Press, 1980), 163–173, doi:10.1145/567446.567462.

9. Mark Reynolds, "An Axiomatization of Full Computation Tree Logic," *J. Symb. Log.* 66, no. 3 (2001): 1011–1057, doi:10.2307/2695091, https://doi.org/10.2307/2695091.

10. Christel Baier and Joost-Pieter Katoen, *Principles of model checking* (MIT press, 2008), Chapter 5.

11. Ronald Fagin, Joseph Y. Halpern, and Nimrod Megiddo, "A Logic for Reasoning about Probabilities," *Inf. Comput.* 87, nos. 1/2 (1990): 78–128, doi:10.1016/0890-5401(90)90060-U, https://doi.org/10.1016/0890-5401(90)90060-U.

instances.[12] This paper is useful when we start to look at ways to incorporate a probabilistic logic in our Spatio-Temporal logic.

John Grant also has worked on a different paper in which the authors axiomatized and extended the logic[13] as presented in the original paper by Grant et al. This thesis is a bit different from that paper, since the focus of that paper was to add probabilistic logic to the PST KBs that were first introduced by Grant et al., not really changing the way that the atomic formula looks like, while this thesis will deviate a lot more from the original paper by Grant et al.

12. Dragan Doder and Zoran Ognjanovic, "A Probabilistic Logic for Reasoning about Uncertain Temporal Information," in *Proceedings of the Thirty-First Conference on Uncertainty in Artificial Intelligence, UAI 2015, July 12-16, 2015, Amsterdam, The Netherlands*, ed. Marina Meila and Tom Heskes (AUAI Press, 2015), 248–257, `http://auai.org/uai2015/proceedings/papers/258.pdf`.

13. Dragan Doder, John Grant, and Zoran Ognjanovic, "Probabilistic logics for objects located in space and time," *J. Log. Comput.* 23, no. 3 (2013): 487–515, doi:`10.1093/logcom/exs054`, `https://doi.org/10.1093/logcom/exs054`.

3 Introduction to the Logic

Before we can start talking about logical frameworks that can use temporal operators and probabilities, we first need to define a more 'base level logic'. This will be done in the current section. Before we can create our most basic framework, sometimes referred to as the propositional framework, we first need to define the input that is necessary for our logic to function. Defining this basic framework is very useful when we want to define the more complex Spatio-Temporal Framework and the Probabilistic Spatio-Temporal Framework.

The input for all of the logical frameworks that will be discussed in the entire paper is the same for each framework. First of all, there needs to be a set of objects, which is denoted O, and there needs to be a set of locations, L. Objects are usually referred to with o, or with o_i with $i \geq 0$, if there are multiple random objects. From the set of locations we can create regions, which are subsets of L. Regions are usually indicated with the variable r.

3.1 Propositional Framework

The input that now has been determined is necessary to create well-formed formulas in our logical frameworks. In this section, we shall apply this input to the most basic level of logic possible: a propositional framework. This framework is quite similar to normal propositional logic, with some small exceptions. Contrary to the more complex frameworks that will be introduced later, the propositional framework is the only framework that does not allow any temporal operators. Therefore, a model in this logic only looks at one point in time, and completely disregards everything that happened prior to this time instance and everything that will happen afterwards.

When describing a logical framework there are three things to consider: syntax, semantics and satisfiability relation. The syntax is what formulas in the logical framework have to look like. Semantics is what models of the logical framework should look like (this can also be interpreted as what sentences mean), and finally the satisfiability relation is the way in which a model satisfies a formula (the way syntax and semantics are connected).

3.1.1 Syntax

While in the SPOT-frameworks the atomic formulas were of the form $loc(id, r, t)$ for any $id \in ID$, $r \in L$, $\forall t \geq 0$, in this logical framework the formulas will look a bit different. For any $o \in O$, $r \in L$, $id(o, r)$ is an atomic formula. We do not have to use a variable to indicate the time instance this formula holds in, since we will have temporal operators to reason about that in the more complex frameworks.

Definition 1. *The set of formulas in the propositional framework, called $FORM_S$, is defined as the smallest set with the following properties:*

1. *$id(o, r) \in FORM_S$, where $o \in O$ and r is a subset of L.*
2. *$\varphi \in FORM_S \rightarrow \neg\varphi \in FORM_S$*
3. *$\varphi, \psi \in FORM_S \rightarrow \varphi \wedge \psi \in FORM_S$*

While the logical connectives $\neg$ and $\wedge$ are sufficient to express every sentence in propositional logic, it is more convenient to use certain abbreviations. These abbreviations are usually ways to write down more complex sentences with fewer variables and connectives, and will be used everywhere else, instead of using the most formal definition of the syntax. The following abbreviations can be used in every well-formed formula:

- $\bot$ is an abbreviation for any contradiction like $\varphi \wedge \neg\varphi$. This is used as a way to express falseness.
- $\top$ is the opposite, an abbreviation for general truths. It can be read as $\neg\bot$ or $\varphi \vee \neg\varphi$.
- $\varphi \vee \psi$ means φ or ψ and is an abbreviation of $\neg(\neg\varphi \wedge \neg\psi)$.
- $\varphi \rightarrow \psi$ is an abbreviation of $\neg\varphi \vee \psi$. The arrow is to be understood as an implication: if φ holds, then ψ must hold as well.
- $\varphi \leftrightarrow \psi$ is an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. This is called a biconditional, which means φ holds if and only if ψ holds.

3.1.2 Semantics

In the previous section the syntax of the propositional framework has been described. With these definitions it has become possible to create well-formed formulas. However, we still need to have a way to assign meaning to models. This is done through the semantics of a logical framework. Models in the propositional framework shall be named frames, since the models of propositional framework can be understood as a picture that has been taken, 'locking' the objects in place.

Definition 2. *A frame is a mapping of $\mathcal{F} : O \rightarrow L$(Objects mapped to locations). Every element in O is mapped to exactly one element of L.*

In short, we can create a frame in this propositional framework by mapping every object to exactly one location. One thing to note is that this does not exclude the fact that an object can be in multiple regions in the same time instance, it merely states that an object can be in exactly one location, but this location can be part of many different regions.

3.1.3 Satisfiability Relation

Now that both syntax and semantics have been defined, there needs to be a way to connect formulas to frames. This is done through the satisfiability relation. If we know that a formula φ is satisfied by a random frame $\mathcal{F}$, we write down $\mathcal{F} \models_S \varphi$. What this means is that the formula φ is not only a valid formula, but the formula also is true according to the model. For example, let's say that φ is the formula $id(o, r)$. If $\mathcal{F}$ would satisfy this formula, then o must be in a location that is part of region r. However, it could be the case that in another frame, a, this does not hold.

Definition 3. *The satisfiability relation for the propositional framework*

- $\mathcal{F} \models_S id(o, r)$ *if and only if* $\mathcal{F}(o) \in r$
- $\mathcal{F} \models_S \neg\varphi$ *if and only if* $\mathcal{F} \not\models_S \varphi$
- $\mathcal{F} \models_S \varphi \wedge \psi$ *if and only if* $\mathcal{F} \models_S \varphi$ *and* $\mathcal{F} \models_S \psi$

With this the propositional framework is fully defined. With this framework we can reason about objects in spaces. For example, we can say something like this: 'Object o_1 is in region r or in region q.', which we would be able to write like this: $id(o_1, r) \vee id(o_1, q)$.

3.2 Reachability Definition

Besides the sets of objects and locations, there is one final piece of information that needs to be provided; the Reachability Definition. A Reachability Definition consists of a finite set of reachability atoms. Grant et al. write the following about reachability atoms: 'Intuitively, the reachability atom $reachable_{id}(p_1, p_2)$ says that it is possible for the object id to reach location p_2 from location p_1 in one unit of time.'[14] This explanation is very clear and will therefore be also adapted in our definition for the Reachability Definition, albeit with slightly different names for variables.

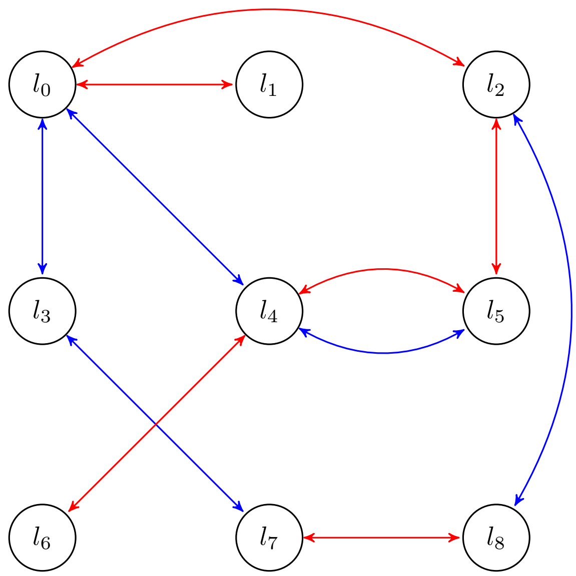
Definition 4. *A reachability atom is of the form* $reachable_o(l_1, l_2)$ *and is used to indicate that* o *can move from location* l_1 *to* l_2 *in one time instance.*

Every element of O (every object in our 'input') gets its own Reachability Definition. Per Grant et al., we also allow $reachable_o(l, l)$ for any $o \in O$ and any $l \in L$. This means that any object can reach the same location it is in within one time instance.[15] As stated before, it would not make much sense to let objects move between two regions that are very far apart, but not be able to move to regions adjacent to the region it is in right now. However, for the sake of generality, it is not a rule that is enforced by the definition of the Reachability Definition. Of course, it is possible to impose this rule by creating the Reachability Definition in such a way that it is possible for objects to move between adjacent locations.

14. Grant et al., "An AGM-style belief revision mechanism for probabilistic spatio-temporal logics," p.75.
15. Ibid.

Definition 5. *A Reachability Definition is the set of all reachability atoms of all elements of O.*

In the figure below, one possible interpretation of the Reachability Definition is given.



The set of all blue arrows is the Reachability Definition for some object, and the red arrows is the Reachability Definition for another object. The set of all arrows is the complete Reachability Definition.

Note that in this case, the Reachability Definition is a symmterical relation. This means that if $reachable_o(l_1, l_2)$, then $reachable_o(l_2, l_1)$. This is not an enforced property, but in many cases it makes sense to have this property in place. Note that the reflexive relations for every object and location haven't been drawn, to keep the image as clear as possible. Also, note that there is not necessarily a reachability atom from every location to another location. Imagine that the blue arrows are the reachability atoms for a very large truck, and that the locations are streets in a city. l_1 could be a very narrow alley, which the truck would not be able to enter. Of course, if the truck would already be in the alley, the truck would be stuck there, since the alley is too narrow for the truck to leave.

4 The Spatio-Temporal Framework

In the previous section, the foundation was laid for the other logical frameworks that will be defined in the rest of the paper. The first step in creating a Probabilistic Spatio-Temporal Framework was developing the propositional framework. The next step is adding the temporal element. This is done by using Linear Temporal Logic, and combine that with the basic Propositional Framework. This means that we get new syntax, semantics and a new satisfiability relation. After these elements have been defined, we move on to more complex things, like axiomatization of our new Spatio-Temporal Framework and doing a soundness and completeness proof, which can be used to show that all our axioms are both valid and complete.

4.1 Introduction of the Spatio-Temporal Framework

Linear Temporal Logic is a basic form of temporal logic, and has already been defined and axiomatized before.[16] These rules and axioms were defined around propositional Linear Temporal Logic. In this framework, we are not concerned with propositional logic, but rather with extending the spatial framework. This means that instead of propositions, we have *id*-atoms to reason about. Instead of having propositions that express validity or truth, our *id*-atoms express information about the locations of objects. This does not change the way formulas are formulated and behave, but merely changes the way formulas are to be understood. This means that the syntax of the new Spatio-Temporal Framework will be similar to the syntax of Linear Temporal Logic in Reynolds, and also that the satisfiability relation is still somewhat similar. Only the semantics differ a lot from the work of Reynolds.

4.1.1 Syntax

Definition 6. *The set of formulas in the Spatio-Temporal Framework, called $FORM_{ST}$, is defined as the smallest set with thet following properties:*

1. $id(o, r) \in FORM_{ST}$*, where* $o \in O$ *and* r *is a subset of* L*.*
2. $\varphi \in FORM_{ST} \rightarrow \neg\varphi \in FORM_{ST}$
3. $\varphi, \psi \in FORM_{ST} \rightarrow \varphi \wedge \psi \in FORM_{ST}$
4. $\varphi \in FORM_{ST} \rightarrow \bigcirc\varphi \in FORM_{ST}$
5. $\varphi, \psi \in FORM_{ST} \rightarrow \varphi U \psi \in FORM_{ST}$

$\bigcirc\varphi$ is to be interpreted as: 'φ holds at the next point in time.' $\varphi U \psi$ can be interpreted as: 'φ holds until ψ holds.' Like in the previous discussed framework, these are the only connectives that are necessary to form any well-formed formula, but there are certain abbreviations that we can use to make the sentences easier to read:

16. Reynolds, "An Axiomatization of Full Computation Tree Logic."

- All the connectives that were abbreviations in the previous basic framework, can also be used in the Spatio-Temporal Framework ($\bot, \top, \vee, \rightarrow, \leftrightarrow$).
- $\Diamond\varphi$ is short for $\top U\varphi$, eventually φ will hold.
- $\Box\varphi$ is short for $\neg\Diamond\neg\varphi$. (φ holds at each point in time)
- $\varphi W\psi$ is short for $(\varphi U\psi) \vee \Box\varphi$ (φ holds until ψ holds, or φ holds at every time instance).
- $\varphi R\psi$ is short for $\neg(\neg\varphi U\neg\psi)$[17]

4.1.2 Semantics

Now that there are temporal operators, our models do not talk about a single point in time. The models now become a bit more dynamic, since there are multiple different instances of time in which objects are at different locations.

Definition 7. *A model σ is a path of the form $\sigma = s_0, s_1, s_2...$, where every $s_i \in \sigma$ can be interpreted as a propositional frame (A mapping of $O \rightarrow L$). Additionally the following must hold as well: $\forall o \in O \forall i \geq 0 : reachable_o(s_i(o), s_{i+1}(o))$. The set of all Spatio-Temporal Models is called Σ.*

In other words, every object $o \in O$, at moment s_i at location l_1 can only be at l_2 at s_{i+1} if $reachable_o(l_1, l_2)$.

If you want to indicate that you have moved through time, we need to indicate that we are no longer at the first time instance, s_0, but s_i for some i. To refer to the new model that uses s_i as the first time instance, we use $\sigma_{\geq i}$, which is a subpath of σ.

4.1.3 Satisfiability Relation

The satisfiability relation works a bit different on the Spatio-Temporal Framework than it did on the basic framework. This is caused by large differences in the semantics. Since the meaning of models is so different, the way the models are linked to formulas is also entirely different. If we want to indicate that a temporal model σ models a formula φ, we write down $\sigma \models_{ST} \varphi$, instead of $\sigma \models_S \varphi$ (this would be an incorrect formula to write down).

Definition 8. *The satisfiability relation for the Spatio-Temporal Framework:*

- $\sigma \models_{ST} id(o, r)$ *if and only if* $s_0 \models_S id(o, r)$
- $\sigma \models_{ST} \neg\varphi$ *if and only if* $\sigma \not\models_{ST} \varphi$
- $\sigma \models_{ST} \varphi \wedge \psi$ *if and only if* $\sigma \models_{ST} \varphi$ *and* $\sigma \models_{ST} \psi$
- $\sigma \models_{ST} \bigcirc\varphi$ *if and only if* $\sigma_{\geq 1} \models_{ST} \varphi$

17. Baier and Katoen, *Principles of model checking*, Chapter 5.

- $\sigma \models_{ST} \varphi U \psi$ *if and only if there exists a* s_i *with* $i \geq 0$ *such that* $\sigma_{\geq i} \models_{ST} \psi$ *and that for all* $k: \ 0 \leq k < i: \ \sigma_{\geq k} \models_{ST} \varphi$

With the satisfiability relation we now know how formulas are satisfied in models.

Definition 9. *Consider a set of statements* Γ*. We say that a formula* A *is a semantic consequence of* Γ*, denoted* $\Gamma \models_{ST} A$ *if and only if there exists no model* $\mathcal{M}$ *in which* Γ *holds and* A *does not hold. If* $\models_{ST} A$*, then the set of formulas* A *must hold on any model.*

In other words, whenever Γ holds in some model, A must hold too. We say that Γ satisfies A.

4.2 Axiomatization of the Spatio-Temporal Framework

In the previous section the syntax, semantics and the satisfiability relation of the Spatio-Temporal Framework has been defined. In the following subsection, axiomatization of this framework shall be done. This includes giving the axiomatic system, inference rules for the axiomatic system and both soundness and completeness proofs.

Definition 10. *Consider a set of statements* Γ*. We say that* A *is a syntactic consequence of* Γ*, denoted* $\Gamma \vdash_{ST} A$ *if and only if* A *is provable from* Γ *in a deductive system. If* $\vdash_{ST} A$*, then the set of formulas* A *must hold on any model.*

In other words, we can make derivations from the formulas that make up Γ to find formula A. We say that A is derivable from Γ.

4.2.1 Axiomatic System

The axioms of a logic are certain logical formulas that are universally valid. These rules can be applied to any instance of any random model, and will always hold. Propositional Linear Temporal Logic already has been axiomatized before, for example in Reynolds.[18] These axioms are valid on every single Linear Temporal Logic, so it makes sense for these formulas to also hold in our new Spatio-Temporal Framework, since it is the same as a propositional logic, but with *id*-atoms instead of propositions. Hence, there need to be extra axioms introduced that can state things about the *id*-operators to capture new semantic properties.

In addition to the axioms, there are inference rules. With these inference rules, you can derive other formulas from theorems and axioms.

Definition 11. *The two inference rules are as follows:*

$$\textit{Modus Ponens: } \frac{\varphi, \varphi \rightarrow \psi}{\psi} \ \textit{Generalization Rule: } \frac{\varphi}{\Box \varphi}$$

18. Reynolds, "An Axiomatization of Full Computation Tree Logic."

Modus Ponens states that if φ holds and φ implies ψ, then ψ must hold as well. Generalization states that, if φ can be derived from the axioms, then $\Box\varphi$ can be derived from the axioms, too.

Definition 12. *The axioms for the Spatio-Temporal Framework are all substitution instances of the following:*

0. *Any Propositional tautology* $(\varphi \vee \neg\varphi)$
1. $\Diamond\varphi \leftrightarrow \Diamond\neg\neg\varphi$
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
3. $\Box\varphi \to (\varphi \wedge \bigcirc\varphi \wedge \bigcirc(\Box\varphi))$
4. $\bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$
5. $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$
6. $\Box(\varphi \to \bigcirc\varphi) \to (\varphi \to \Box\varphi)$
7. $(\varphi U \psi) \leftrightarrow (\psi \vee (\varphi \wedge \bigcirc(\varphi U \psi)))$
8. $(\varphi U \psi) \to \Diamond\psi$
9. $\neg(id(o,r) \wedge \bigcirc id(o,s))$*, where* $o \in O$*,* $r, s \subseteq L$ *and* $(r \times s) \cap reachable_o = \emptyset$
10. $\neg(id(o,r) \wedge id(o,s))$*, where* $r, s \subseteq L$*,* $o \in O$ *and* $r \cap s = \emptyset$
11. $id(o, L)$
12. $id(o, r \cup s) \leftrightarrow (id(o,r) \vee id(o,s))$*, where* $o \in O$ *and* $r, s \subseteq L$

The combination of the inference rules and the axioms, give us axiomatic system ST.

Axioms 0-8 are the same axiom substitutions used by Reynolds. He claimed that it was proven in Gabbay et al. that these axioms are sound and complete for Propositional Linear Temporal Logic.[19] Axioms 9-12 are axioms that have been introduced specifically for this logic. Axiom 9 states that it is not possible for o to be at r and at the next point in time at s, whenever none of the locations in r and s are paired in the Reachability Definition for o. Axiom 10 states that it is impossible for an object o to be in both region r and s at the same time, whenever r and s have no shared locations. Axiom 11 states that object o is always in region L, the set of all locations. Axiom 12 states that an object o is in the intersection of 2 different regions if and only if it is in either one of the regions.

4.2.2 Soundness

Definition 13. *A logical system is sound if and only if every theorem is a valid formula. We denote this by* $\vdash_{ST} \varphi \to \models_{ST} \varphi$.

19. Reynolds, "An Axiomatization of Full Computation Tree Logic," p.1016.

A theorem is any formula that needs no premises except axioms to be proven. To prove that our axiomatic system ST is sound, we need to prove that $\models_{ST} \varphi$ for every axiom as φ, and that inference rules preserve validity.

Claim. *Axiomatic System ST is sound.*

The soundness of axioms 0-8 and the inference rules have already been proven,[20] and will therefore not be proven here. This means that the soundness proofs only have to be done for axioms 9-12.
The easiest way to prove that every axiom is sound is to prove that the axiom holds in a random model. If we choose a random model with constraints that are also imposed by the axiom. In that case, it must be the case that the axiom holds in that model. Since the model was chosen arbitrarily, we know that the axiom must hold in any model with those constraints. And whenever the constraints are not met, the axiom holds as well, thus proving that our axiom holds in any model.

Axiom 9:

$$\neg(id(o,r) \wedge \bigcirc id(o,s)) \text{ where } r, s \subseteq L, o \in O \text{ and } (r \times s) \cap reachable_o = \emptyset$$

To Prove: $\models_{ST} \neg(id(o,r) \wedge \bigcirc id(o,s))$

Proof. Consider a random model σ with object o and regions r and s such that $(r \times s) \cap reachable_o = \emptyset$. According to the satisfiability relation $\sigma \models_{ST} \neg(id(o,r) \wedge \bigcirc id(o,s))$ if and only if $\sigma \not\models_{ST} id(o,r) \wedge \bigcirc id(o,s)$. This can only be the case if $id(o,r)$ does not hold and $\bigcirc id(o,s)$ does hold, when $id(o,r)$ does hold and $\bigcirc id(o,s)$ does not hold or when neither of them hold. It is sufficient to prove that when $id(o,r)$ holds, $\bigcirc id(o,s)$ does not hold, since, if we assume that $id(o,r)$ does not hold, it automatically follows that $\sigma \not\models_{ST} id(o,r) \wedge \bigcirc id(o,s)$.

We now assume that $id(o,r)$ holds and that $(r \times s) \cap reachable_o = \emptyset$. According to the definition of semantics, σ is a path of the form $s_0, s_1, s_2...$ where every s_i can be interpreted as a propositional frame. It also must hold that for all objects o $reachable_o(s_i(o), s_{i+1}(o))$ for every $i \geq 0$.

We have defined r and s in such a way that $(r \times s) \cap reachable_o = \emptyset$, where $(r \times s)$ is the Carthesian Product of r and s. This means that $(r \times s)$ is a set of all the locations of r paired to all the locations of s once. $reachable_o$ is the set of all locations that are reachable by o in one time instance.

Because of the way that σ is defined, it must be the case that $\bigcirc id(o,s)$ can only hold if and only if $\sigma_{\geq 1} \models_{ST} id(o,s)$. But we know that it is not possible, since none of the pairs of locations in r and s are part of the Reachability Definition of o.

20. Gabbay et al., "On the Temporal Basis of Fairness."

We then proved that $\not\models_{ST} id(o,r) \wedge \neg id(o,s)$, and that therefore $\models_{ST} \neg(id(o,r) \wedge \bigcirc id(o,s))$. $\square$

Axiom 10:

$$\neg(id(o,r) \wedge id(o,s)), \text{ where } r,s \subseteq L, o \in O \text{ and } r \cap s = \emptyset$$

To Prove $\models_{ST} \neg(id(o,r) \wedge id(o,s))$

Proof. Consider a random model σ with object o and regions r and s such that $r \cap s = \emptyset$. According to the satisfiability relation, $\models_{ST} \neg(id(o,r) \wedge id(o,s))$ if and only if $\not\models_{ST} id(o,r) \wedge id(o,s)$.

We assume that $id(o,r)$ holds, and shall prove that therefore $id(o,s)$ cannot hold. If we assume that $id(o,r)$ does not hold, we already know that $\sigma \not\models_{ST} id(o,r) \wedge id(o,s)$, and that therefore $\sigma \models_{ST} \neg(id(o,r) \wedge id(o,s))$.
Since we know that $id(o,r)$ holds, we know that, according to the satisfiability relation of σ, that on the first frame of σ, s_0, $s_0(o) \in r$. According to the satisfiability relation, we know that $\sigma \models_{ST} id(o,s)$ if and only if $s_0 \models_S id(o,s)$, which is the case if and only if $s_0(o) \in s$. If it would be the case that $\sigma \models_{ST} id(o,r)$ and $\sigma \models_{ST} id(o,s)$, then $s_0(o) \in r \cap s$. But we have assumed that $r \cap s = \emptyset$. But $s_0(o) \notin \emptyset$, so therefore it must be the case that $\sigma \not\models_{ST} id(o,r) \wedge id(o,s)$, which means that $\sigma \models_{ST} \neg(id(o,r) \wedge id(o,s))$. $\square$

Axiom 11:

$$id(o,L)$$

To Prove: $\models_{ST} id(o,L)$

Proof. Consider a random model σ with object o. According to the satisfiability relation of σ, $\sigma \models_{ST} id(o,L)$ if and only if $s_0 \models_S id(o,L)$. $s_0 \models_S id(o,L)$ if and only if $s_0(o) \in L$. According to the semantical definition, a frame is a mapping of objects to a location. Since every element of O maps to at least one element of L, we know that $s_0(o) \in L$. $\square$

Axiom 12:

$$id(o, r \cup s) \leftrightarrow (id(o,r) \vee id(o,s))$$

To Prove: $\models_{ST} id(o, r \cup s) \leftrightarrow (id(o,r) \vee id(o,s))$

Proof. Consider a random model σ with object o and subregions r and s such that $\sigma \models_{ST} id(o, r \cup s)$. According to the satisfiability relation if $\sigma \models_{ST} id(o, r \cup s)$ then it must be the case $s_0(o) \in r \cup s$. According to the definition of union, $s_0(o) \in r \cup s$ if and only if $s_0(o) \in r$ or $s_0(o) \in s$. From the satisfiability relation, we know that $s_0(o) \in r \vee s_0(o) \in s$ if and only if $s_0 \models_S id(o,r)$ or

$s_0 \models_S id(o,s)$. From this we can deduce that it then must be the case that $s_0 \models_S id(o,r) \vee id(o,s)$. We know that $s_0 \models_S id(o,r) \vee id(o,s)$ if and only if $\sigma \models_{ST} id(o,r) \vee id(o,s)$. From this we can conclude that $\sigma \models_{ST} id(o, r \cup s) \leftrightarrow id(o,r) \vee id(o,s)$, which proves our axiom. □

Since we have proven the axioms 9-12, and we know that axioms 0-8 and the inference rules already have been proven to be sound, we can say that our claim, that axiomatic system ST is sound, has been proven to be the case.

4.2.3 Completeness

Definition 14. *A logical system is complete if and only if every formula that is logically valid with respect to the semantics of the system, can be proved. In other words whenever there is a semantic consequence, there is a syntactic consequence;* $\models_{ST} \varphi \Rightarrow \vdash_{ST} \varphi$.

Definition 15. *A set of formulas* Γ *is consistent if it is impossible to derive a contradiction from* Γ.

Definition 16. Γ *is called a maximal consistent set of formulas if for any* $A \notin \Gamma$, $\Gamma \cup \{A\}$ *is not consistent.*

If Γ is a maximal consistent set, then either $A \in \Gamma$ or $\neg A \in \Gamma$.

Claim. *Axiomatic System ST is complete*

$$\text{To Prove: } \models_{ST} \varphi \Rightarrow \vdash_{ST} \varphi$$

Proof. Let w_0 be a derivable consistent well-formed formula. We will construct a model σ which satisfies w_0. Let S be the set of all maximal consistent sets of our ST-logic. For some maximal consistent set Δ_0 in S it must be the case that $w_0 \in \Delta_0$. We define the following relations $^+$ and $<$ on S:

$$\Delta^+ = \{w | \bigcirc w \in \Delta\}$$

$$\Delta < \Theta \text{ if and only if for all } w, \Box w \in \Delta \rightarrow w \in \Theta$$

It can be shown that $\Delta^+ \in S$ for every $\Delta \in S$ and $\forall x,y,z \in S(x < y \wedge x < z \wedge y \neq z \rightarrow y < z \vee z < y)$.
The next step is to extend $\{w_0\}$ to a set of formulas Γ. Γ is closed under the following properties:

a. $\Diamond w \in \Gamma \rightarrow (\bigcirc w \in \Gamma \wedge \bigcirc \Diamond w \in \Gamma)$
   $\bigcirc w \in \Gamma \rightarrow \bigcirc \neg w \in \Gamma$
   $w_1 U w_2 \in \Gamma \rightarrow (\Diamond w_2 \in \Gamma \wedge \bigcirc w_1 \in \Gamma \wedge \bigcirc w_2 \in \Gamma \wedge \bigcirc (w_1 U w_2) \in \Gamma)$

b. $id(o,r) \in \Gamma$ for any $o \in O$ and $r \subseteq L$.

c. Γ is closed under subformulas and boolean operators.

Since O and L are finite sets, we know that Γ still is a finite set at this point. We define $\bar{S} = \{\Delta \cap \Gamma | \Delta \in S\}$. $\bar{S}$ is a finite set.

We now define ρ_+ and $\rho_<$ on $\bar{S}$ as follows:

$$t\rho_+ s \text{ if and only if for some } \Delta \in S, t = \Delta \cap \Gamma \text{ and } s = \Delta^+ \cap \Gamma$$

$$t\rho_< s \text{ if and only if for some } \Delta, \Theta \in S, \Delta < \Theta \text{ and } t = \Delta \cap \Gamma, s = \Theta \cap \Gamma$$

$\rho_<$ is the transitive closure of ρ_+. We now are ready to define the model σ. Let $s_0 = \Delta_0 \cap \Gamma$, which is an element of $\bar{S}$ containing w_0. Let $\{S_n\}$ be a sequence of states from $\bar{S}$ such that for any $t \in \bar{S}$, if $t = s_i$ for an infinite amount of different states s_i, then every t' such that $t\rho_+ t'$ also is in an infinite amount of s_i. We use $\{S_n\}$ to help form our final model σ. Up to this point, the proof is the same as in Gabbay et al.,[21] but this will change now.

Lemma 1. *For every maximal consistent set relative to* Γ s_n *and* $o \in O$*, there is a unique* $l \in L$ *such that* $id(o, \{l\}) \in s_n$

We will prove this lemma in two different steps: the first step is that there cannot be 2 different l such that $id(o, \{l\}) \in s_n$ and the second step is that there is at least one l such that $id(o, \{l\}) \in s_n$.

Step 1: We have a random object $o \in O$ and two different singleton sets $\{l\}$ and $\{l'\}$. We know that $\{l\} \cap \{l'\} = \emptyset$. This means that axiom 10 must hold here. Axiom 10 states that whenever $r, s \subseteq L$, $o \in O$ and $r \cap s = \emptyset$, $\neg(id(o, r) \wedge id(o, s))$. Because we know that s_n is consistent, we know that there is at most one $l \in L$ such that $id(o, \{l\}) \in s_n$, which proves our first step.

Step 2: We assume that there can exist no $l \in L$ such that $id(o, \{l\}) \in s_n$. This means that for every $l \in L$ it must be the case that $id(o, \{l\} \notin s_n$. Because s_n can be interpreted as a maximal consistent set relative to Γ, we can say that this means that for every $l \in L$ it is the case that $\neg id(o, \{l\}) \in s_n$. This means that $\neg(id(o, \{l_1\})) \wedge \neg(id(o, \{l_2\})) \wedge ... \wedge \neg(id(o, \{l_n\}) \in s_n$, which would be the same as $\neg(id(o, \{l_1\}) \vee id(o, \{l_2\}) \vee ... \vee id(o, \{l_n\})) \in s_n$ according to De Morgan's Laws. Axiom 11 states that an object always is in the region of all locations, $id(o, L)$. Since it is an axiom, it holds in any model, so in that case $id(o, L) \in s_n$.
Now we split up L in 2 different subregions; $\{l_1\}$ and $L - \{l_1\}$. According to axiom 12, it must be the case that $id(o, \{l_1\} \cup L - \{l_1\}) \to (id(o, \{l_1\}) \vee id(o, L - \{l_1\}))$. We now split $L - \{l_1\}$ again in two different subregions, with one of the subregions being a singleton region. We continue applying axiom 12 until we have split up L in singleton sets for every location $l \in L$. This means that we can derive the formula $(id(o, \{l_1\}) \vee id(o, \{l_2\}) \vee ... \vee id(o, \{l_n\}))$ from axioms 11 and 12. Because this is derived from 2 axioms, it must mean

21. Gabbay et al., "On the Temporal Basis of Fairness," p.170-171.

that $(id(o,\{l_1\}) \vee id(o,\{l_2\}) \vee ... \vee id(o,\{l_n\})) \in s_n$. However, this means that $\neg(id(o,\{l_1\}) \vee id(o,\{l_2\}) \vee ... \vee id(o,\{l_n\})) \in s_n$ is not true, which implies that there is at least one l such that $id(o,\{l\}) \in s_n$.

This means that step 2 is also proven. We have proven that there is at most one $l \in L$ such that $id(o,\{l\}) \in s_n$. We have also proven that there is at least one $l \in L$ such that $id(o,\{l\}) \in s_n$. This means that there is exactly one unique $l \in L$ such that $id(o,\{l\}) \in s_n$, thus proving our lemma.

We define function f_n as a mapping $f_n : O \to L$ such that $f_n(o) = l$ if and only if $id(o,\{l\}) \in s_n$. We know that this is correct because we have proven in Lemma 1 that there is one unique $l \in L$ such that $id(o,\{l\}) \in s_n$. We use these functions f_n to define our model $\sigma = f_0, f_1, f_2...$

Lemma 2. *For all objects $o \in O$ and $\forall i \geq 0 : reachable_o(f_i(o), f_{i+1}(o))$*

We assume that there exists some object o such that for some i it is the case that $\neg reachable_o(f_i(o), f_{i+1}(o))$. From the definition defined before, we know that $f_n(o) = l$ if and only if $id(o,\{l\}) \in s_n$. We can use axiom 12 to extend this to show that it also then holds that for any superset $r \supseteq \{l\}$, $id(o,r) \in s_n$. This means that there exists some $id(o,r) \in s_i$ and some $id(o,s) \in s_{i+1}$. Because of the way that $\{S_n\}$ was defined, we know that $s_i \rho_+ s_{i+1}$. $s_i = \Delta_i \cap \Gamma$ and $s_{i+1} = \Delta_i^+ \cap \Gamma$. $\Delta_i^+ = \{w | \bigcirc w \in \Delta_i\}$.
We know that $id(o,s) \in s_{i+1}$, so we can then deduce that $\bigcirc id(o,s) \in s_i$.
Because s_i can be interpreted as a maximal consistent set relative to Γ, it follows that $id(o,r) \wedge \bigcirc id(o,s) \in s_i$.
Since we have assumed that $\neg reachable_o(f_i(o), f_{i+1}(o))$, this means that $(f_i(o) \times f_{i+1}(o)) \cap reachable_o = \emptyset$. This means that we can apply axiom 10 here. This means that $\neg(id(o,r) \wedge \bigcirc id(o,s)) \in s_i$. However, this is a direct contradiction from before, which means that it cannot be the case that $\neg reachable_o(f_i(o), f_{i+1}(o))$, thus proving our lemma.

Lemma 3. *For any formula $\varphi \in \Gamma$ and any n, $\sigma_{\geq n} \models_{ST} \varphi$ if and only if $\varphi \in s_n$.*

In other words, a formula is true at time instance n if and only if that formula is in the maximal consistent set s_n. We shall prove this by induction on complexity of a formula:

<u>Base Case:</u>
We shall choose a random atomic formula $id(o,r)$. We shall prove that $\sigma_{\geq n} \models_{ST} id(o,r)$ if and only if $id(o,r) \in s_n$. According to the satisfiability relation $\sigma \models_{ST} id(o,r)$ if and only if $f_0 \models_S id(o,r)$ if and only if $f_0(o) \in r$. We have defined f_0 in such a way that $f_0(o) = l$ if and only if $id(o,\{l\}) \in s_n$. Therefore we can conclude that $\{l\} \subseteq r$. By applying axiom 12 on all the elements of r we can show that $id(o,\{l\}) \vee id(o, r - \{l\}) \to id(o,r) \in s_n$. We know that $id(o,\{l\}) \in s_n$ from lemma 1, and since s_n is a maximal consistent set relative to Γ, we know that $id(o,r) \in s_n$.

We have shown that $\sigma_{\geq n} \models_{ST} id(o,r)$ if and only if $id(o,r) \in s_n$, thus proving our base case.

Inductive Step:

In the base case we have shown that for any atomic formula it is the case that a formula only is in a model if and only if it is true. We shall now use induction on complexity of a formula to show that whenever it is the case for a subformula that it only is in the model if and only if it is true, it also must be the case for the more complex formula. We shall use α and β as subformulas, so we will assume for any occurence of α and β that it is the case that $\sigma_{\geq n} \models_{ST} \alpha$ if and only if $\alpha \in s_n$ and $\sigma_{\geq n} \models_{ST} \beta$ if and only if $\beta \in s_n$.

Substitution of φ with $\neg\alpha$:

We shall prove that $\sigma_{\geq n} \models_{ST} \neg\alpha$ if and only if $\neg\alpha \in s_n$. From the satisfiability relation, we know that $\sigma_{\geq n} \models_{ST} \neg\alpha$ if and only if $\sigma_{\geq n} \not\models_{ST} \alpha$. From the inductive hypothesis, we can conclude that $\alpha \notin s_n$. s_n is a maximal consistent set relative to Γ, so we can conclude that since $\alpha \notin s_n$, then $\neg\alpha \in s_n$.
This proves that $\sigma_{\geq n} \models_{ST} \neg\alpha$ if and only if $\neg\alpha \in s_n$.

Substitution of φ with $\alpha \wedge \beta$:

We shall prove that $\sigma_{\geq n} \models_{ST} \alpha \wedge \beta$ if and only if $\alpha \wedge \beta \in s_n$. From the satisfiability relation $\sigma_{\geq n} \models_{ST} \alpha \wedge \beta$ if and only if $\sigma_{\geq n} \models_{ST} \alpha$ and $\sigma_{\geq n} \models_{ST} \beta$. We know from the induction hypothesis that this holds if and only if $\alpha \in s_n$ and $\beta \in s_n$. s_n can be interpreted as a maximal consistent set relative to Γ. One of the properties of maximal consistent sets is that it is closed under boolean operators, so therefore we know that $\alpha \wedge \beta \in s_n$.
This proves that $\sigma_{\geq n} \models_{ST} \alpha \wedge \beta$ if and only if $\alpha \wedge \beta \in s_n$.

Substitution of φ with $\bigcirc\alpha$:

We shall prove that $\sigma_{\geq n} \models_{ST} \bigcirc\alpha$ if and only if $\bigcirc\alpha \in s_n$. From the satisfiability relation, we know that $\sigma_{\geq n} \models_{ST} \bigcirc\alpha$ if and only if $\sigma_{\geq n+1} \models_{ST} \alpha$. Because of the induction hypothesis, we know that $\alpha \in s_{n+1}$. For the construction of σ we used unique functions f_n, each of which corresponded to an s_n. Because s_n and s_{n+1} are subsequent maximal consistent sets, we know that $s_n \rho_+ s_{n+1}$. $s_n = \Delta_n \cap \Gamma$ and $s_{n+1} = \Delta_n^+ \cap \Gamma$ for some maximal consistent set Δ_n. We know that $\Delta_n^+ = \{w | \bigcirc w \in \Delta_n\}$. We know that $\alpha \in s_{n+1}$, so it must be the case that $\bigcirc\alpha \in \Delta_n$. Since Γ is closed under all subformulas and boolean operators, we know that $\bigcirc\alpha \in \Gamma$, so therefore $\bigcirc\alpha \in \Delta_n \cap \Gamma$. We know that $s_n = \Delta_n \cap \Gamma$, so $\bigcirc\alpha \in s_n$.
We can conclude that $\sigma_{\geq n} \models_{ST} \bigcirc\alpha$ if and only if $\bigcirc\alpha \in s_n$.

Substitution of φ with $\alpha U \beta$:
In the same way that it was proven for the operators above, it can also be shown that $\sigma_{\geq n} \models_{ST} \alpha U \beta$ if and only if $\alpha U \beta \in s_n$. It can be shown that this is correct like in Gabbay et al., by using axioms 7 and 8 and use the $\rho_<$ relation.[22]

With this, we have now proven lemma 3. Since we have shown that lemma 3 holds for any random formula, we can say that for any well-formed formula there exists a valuation in the model σ such that $\sigma \models_{ST} w_0$, thus proving our completeness theorem. $\square$

Our claim (that axiomatic system ST is complete) has been proven by the proof above.

22. Gabbay et al., "On the Temporal Basis of Fairness."

5 Introduction to the Probabilistic Spatio-Temporal Framework

In this section the syntax, semantics and satisfiability relation of the Probabilistic Spatio-Temporal Framework shall be defined.
All previous iterations of the SPOT-framework, mentioned in the related research section,[23][24][25] modelled the probabilistic component of the framework by utilizing prediction intervals. A prediction interval $[0.6, 0.7]$, states that the likelihood of an event is larger than 0.6, but smaller than 0.7. The authors argued for using prediction intervals over a single prediction: "We are aware of very few applications where the prediction is known to be 100 percent accurate".[26] The authors provide some examples, like how political polls can state that 49% of voters support a certain candidate with a margin of error of $\pm 2\%$. This actually means that the support for that candidate lies somewhere between 47 and 51 percent of voters. The argument for using probability intervals over using single probabilities is that it is simply a more realistic way of representing probabilities of events in real-world situations, which is why it also was used in SPOT-frameworks.
In the new Probabilistic Spatio-Temporal Framework however, this shall not be enforced. However, it still is very easy to model these probabilistic intervals in our framework, but if this property is not enforced, the framework can also be used for making more complex statements.

The Probabilistic Spatio-Temporal Framework will be based on the Spatio-Temporal Framework that has been defined in the previous section. The difference is that there is a probabilistic component added to the formulas. In Fagin et al.,[27] a propositional probabilistic logic was defined and axiomatized. The way that these probabilities were modelled was done in a way to allow for reasoning about probabilities. This means that all formulas still are either true or false, they do not have probabilistic truth values, the probabilities are only used to show how likely a formula is to be true. The probabilities in Fagin et al. are represented by linear inequalities. A formula like $3w(\varphi) \geq 1$ should be understood as: the probability of φ is greater than or equal to $\frac{1}{3}$.

23. Parker, Subrahmanian, and Grant, "A Logical Formulation of Probabilistic Spatial Databases."
24. Parker et al., "SPOT Databases: Efficient Consistency Checking and Optimistic Selection in Probabilistic Spatial Databases."
25. Grant et al., "An AGM-style belief revision mechanism for probabilistic spatio-temporal logics."
26. Parker, Subrahmanian, and Grant, "A Logical Formulation of Probabilistic Spatial Databases," p.1541.
27. Fagin, Halpern, and Megiddo, "A Logic for Reasoning about Probabilities."

5.1 Syntax

The first thing that we need is an infinite set of formulas. This set of formulas is $FORM_{ST}$, the set of all formulas in the Spatial-Temporal Framework. A primitive weight term is an expression of the form $w(\varphi)$, for any $\varphi \in FORM_{ST}$. The following definitions are adapted from Fagin et al.:

Definition 17. *A weight term is an expression of the form* $a_1w(\varphi_1)+a_2w(\varphi_2)+ \ldots\ldots + a_kw(\varphi_k)$ *for any* $\varphi \in FORM_{ST}$, *where* $a_1, a_2, \ldots a_k$ *are integers and* $k \geq 1$.

Definition 18. *A basic weight formula is a statement of the form* $t \geq c$, *where* t *is any weight term.*

Basic propositional connectives $(\neg, \wedge)$ can be applied on basic weight formulas to create more complex formulas, called weight formulas. Furthermore, all the abbreviations we have seen in the propositional framework will also apply to our weight formulas. In Fagin et al., the following abbreviations were already defined:[28]

- $w(\varphi) - w(\psi) \geq a$ is short for $w(\varphi) + (-1)w(\psi) \geq a$
- $w(\varphi) \geq w(\psi)$ is short for $w(\varphi) - w(\psi) \geq 0$
- $w(\varphi) \leq c$ is short for $\neg w(\varphi) \geq \neg c$
- $w(\varphi) < c$ is short for $\neg(w(\varphi) \geq c)$
- $w(\varphi) = c$ is short for $(w(\varphi) \geq c) \wedge (w(\varphi) \leq c)$
- Finally, formulas like $3w(\varphi) \geq 1$ can be abbreviated to $w(\varphi) \geq \frac{1}{3}$

Notice that with this syntax we can make sentences like "In the future object o_1 and object o_2 will both be in region r with probability interval [0.25,0.5]." This formula would look like this: $4w(\Diamond(id(o_1, r) \wedge id(o_2, r))) \geq 1 \wedge 2w(\Diamond(id(o_1, r) \wedge id(o_2, r))) \leq 1$.

5.2 Semantics

With clearly defined syntax, we are now ready to move on to the semantics; what does a model of our Probabilistic Spatio-Temporal Framework look like? For this we look towards *A Probabilistic Logic for Reasoning about Uncertain Temporal Information*, in which the authors combined Linear Temporal Logic and probabilistic logic to create a new framework to reason about uncertain temporal information.[29] The first thing that we need to do is to define the probability space.

28. Fagin, Halpern, and Megiddo, "A Logic for Reasoning about Probabilities," p.83.
29. Doder and Ognjanovic, "A Probabilistic Logic for Reasoning about Uncertain Temporal Information," p.2-3.

Definition 19. *The probability space is a 3-tuple* $\langle W, H, \mu \rangle$ *with the following properties:*

- W *is a nonempty set of worlds.*
- H *is an algebra of subsets of* W*. This means that* H *is a set of subsets of* W*, which contains the empty set and is closed under complementation:*[30]

  a. $W \in H$

  b. *if* $A, B \in H$*, then* $W \setminus A \in H$ *and* $A \cup B \in H$*.*

- *Function* $\mu : H \to [0, 1]$ *is a probability measure with the following conditions:*

  1. $\mu(X) \geq 0$ *for all* $X \in H$
  2. $\mu(W) = 1$
  3. $\mu(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} \mu(A_i)$ *whenever* $A, A_i \in H$ *and* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$[31]

Now we have defined our probability space. With the probability space, we can create models of a probabilistic logic. The trick now becomes to make sure that our new models say something about our Spatio-Temporal Framework. Recall that Σ was the set of all Spatio-Temporal Models.

Definition 20. *A model of the Probabilistic Spatio-Temporal Framework,* Ω*, is a 4-tuple* $\langle W, H, \mu, \pi \rangle$ *such that:*

- $\langle W, H, \mu \rangle$ *is a probability space*
- $\pi : W \to \Sigma$ *is a function that provides for each world* $w \in W$ *a Spatio-Temporal model:* $\pi(w) \in \Sigma$*.*
- *For any formula* α*,* $[\alpha]_\Omega = \{w \in W | \pi(w) \models_{ST} \alpha\} \in H$

5.3 Satisfiability Relation

We now have defined the models of our framework. The final step is to give the satisfiability relation, which can show us that a formula holds in a model.

Definition 21. *Let* Ω *be a PST model. We define the satisfiability relation* $\models_{PST}$ *recursively as follows:*

- $\Omega \models_{PST} r_1 w(\alpha_1) + r_2 w(\alpha_2) + ... + r_k w(\alpha_k) \geq r$ *if and only if* $r_1\mu([\alpha_1]_\Omega) + r_2\mu([\alpha_2]_\Omega) + ... + ... + r_k\mu([\alpha_k]_\Omega) \geq r$
- $\Omega \models_{PST} \neg\varphi$ *if and only if* $\Omega \not\models_{PST} \varphi$
- $\Omega \models_{PST} \varphi \wedge \psi$ *if and only if* $\Omega \models_{PST} \varphi$ *and* $\Omega \models_{PST} \psi$

30. Doder and Ognjanovic, "A Probabilistic Logic for Reasoning about Uncertain Temporal Information," p.2.
31. Ibid.

6 Discussion

6.1 Limitations

The biggest limitation of this study were the time constraints. With more available time it would have been possible to axiomatize the Probabilistic Spatio-Temporal Framework. I opted to do axiomatization for the Spatio-Temporal Framework, which I could have skipped instead. If I would have skipped this step, I would have had more time to write a full section on the axiomatization of Probabilistic Spatio-Temporal Framework, but I wanted to first prove soundness and completeness of the Spatio-Temporal Framework before moving on to the Probabilistic Spatio-Temporal Framework. Currently, it is not possible to formally reason within the Probabilistic Spatio-Temporal Framework, however, it is possible to create well-formed formulas, to create models, and to check whether models satisfy certain formulas. This already is a lot more expressivity than what was possible in the PST KBs by Grant et al.

6.2 Future Research

The first future research that can be conducted is doing axiomatization of the Probabilistic Spatio-Temporal Framework that was defined in section 5. We then also need to prove soundness and completeness for this framework, but after this has been done, it becomes possible to formally reason within this framework.

Instead of Linear Temporal Logic, topics that could be covered are looking at some kind of branching time logic. Examples are computation tree logic[32] or full computation tree logic.[33] The idea behind computation tree logic is that there is not one future, but many different ones that exist alongside one another. This also creates interesting situations, since this feeds into the situation of the uncertainty of the atomic formulas in Probabilistic Spatio-Temporal Formulas. Another kind of Temporal Logic would be looking at incorporating first order logic, which would create a new temporal logic called first order temporal logic, of which we already know that it is both sound and complete.[34] By introducing these quantifiers, we can add even more expressivity to our logic.

Instead of changing the temporal logic that we use, one way of doing further research on this topic could be looking towards practical application of our

32. E. Allen Emerson and Edmund M. Clarke, "Using Branching Time Temporal Logic to Synthesize Synchronization Skeletons," *Sci. Comput. Program.* 2, no. 3 (1982): 241–266, doi:10.1016/0167-6423(83)90017-5, https://doi.org/10.1016/0167-6423(83)90017-5.

33. E. Allen Emerson and Joseph Y. Halpern, "Decision Procedures and Expressiveness in the Temporal Logic of Branching Time," *J. Comput. Syst. Sci.* 30, no. 1 (1985): 1–24, doi:10.1016/0022-0000(85)90001-7, https://doi.org/10.1016/0022-0000(85)90001-7.

34. Andrzej Szalas, "A Complete Axiomatic Characterization of First-Order Temporal Logic of Linear Time," *Theor. Comput. Sci.* 54 (1987): 199–214, doi:10.1016/0304-3975(87)90129-0, https://doi.org/10.1016/0304-3975(87)90129-0.

framework. One way that this could happen is by, for example looking at a way to create solvers for this logical framework. If there exist algorithms to help solve problems in our logical framework, it would mean that it would also be easy to implement this in GPS-based computer systems, like drones that can scout for survivors, as was mentioned in the introduction.

6.3 Conclusion

The aim of this thesis was to extend the amount of reasoning that is possible with Probabilistic Spatio-Temporal atoms. The research question that was asked was: "How can existing temporal and probabilistic logics be added to extend the amount of reasoning that can be done on probabilistic spatio-temporal atoms?" To answer this question, the following steps were taken:

The atomic formulas that have been used in previous papers[35] were changed slightly, but the meaning of these atomic formulas is still the same, it still expresses the locations of objects. Because of the temporal operators, there was no longer a need to indicate the time instance in which an atomic formula would hold, since it would be possible to indicate that an object o is in region r in time instance 5 with the following formula: $\bigcirc\bigcirc\bigcirc\bigcirc\bigcirc id(o, r)$. With these atomic formulas, three logical frameworks have been introduced. The first one was the basic Propositional Framework, which omitted temporal operators, and was just concerned with the location of objects. These propositional frameworks, sometimes also referred to as frames, were then used as the basis for the Spatio-Temporal Framework. This Spatio-Temporal Framework consisted of an infinitely long sequence of propositional frames, which could be navigated through by applying temporal operators on formulas. For this framework full axiomatization was done. Soundness and completeness were proven.
The final framework was defined, but we do not know yet whether this framework is sound and complete, but even without the ability to formally reason within this framework, with only the definitions that were given, we can already make more complex formulas than the formulas that were used in the PST KBs in the paper by Grant et al.
Future research could focus on either trying to axiomatize the PST framework, or on changing the temporal logic that was used for the frameworks. Other research could also focus on the decidability of either the PST or Spatio-Temporal Framework, which could then lead to integration in AI systems, for example drones that could help locate people or objects in a city that has been struck by a natural disaster.

35. Grant et al., "An AGM-style belief revision mechanism for probabilistic spatio-temporal logics."

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