Utrecht University

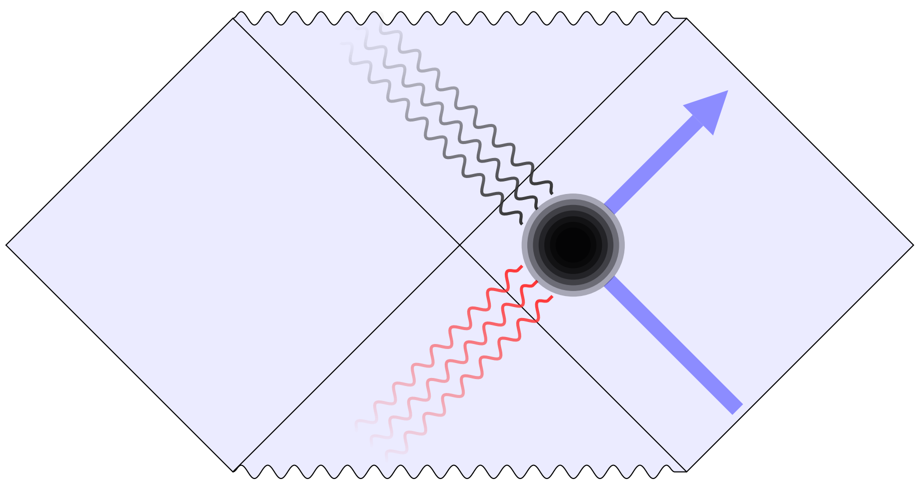
Theoretical Physics

Master Thesis

Quantum gravity on the black hole horizon

Author

Nico Groenenboom



Supervisors

Dr. Nava Gaddam

Prof. Gerard 't Hooft

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Abstract

In this thesis we focus on perturbatively quantizing gravity on the black hole horizon. First we review past research of 't Hooft on semi-classical black hole scattering, re-derive the resulting semi-classical scattering matrix in quantum mechanics and discuss it. Then we proceed to develop a new method for calculating the black hole scattering matrix by summing over an infinite number of graviton exchange diagrams in the high-energy limit. The graviton is given by a perturbation of the metric around the Schwarzschild background close to the black hole horizon. As a result we find that the scattering matrix of our high-energy graviton exchange is almost identical to 't Hooft's semi-classical scattering matrix, providing a generalized result. We have thus shown that similar results can also be obtained solely in perturbative general relativity and quantum field theory, opening up possibilities for new research on black hole scattering.

Title page image:

An illustration of the scattering process we are considering in this work, shown on the Schwarzschild Penrose diagram. Hawking radiation (red) coming out of the black hole interacts with an ingoing particle from asymptotic past infinity (blue). The interaction of these particles is in the black sphere, and consists of infinite virtual graviton exchange diagrams. Finally two scattered particles come out, one particle that moves out of the black hole to asymptotic future infinity (blue arrow) and particles that we can treat as black hole vacuum contributions (black). The black particles are to be seen as time reversed Hawking radiation. The paper will focus on finding the scattering matrix for the blue particles, with the existence of the red and black particles as background.

Acknowledgment

First and foremost I want to thank my Supervisor Nava Gaddam for countless hours of help and discussion. Nava became my day-to-day supervisor at an early stage, helping me with all my questions and guiding me through the relevant literature. Whenever I was in doubt which step was best to take, we would discuss it together and always find a resolution. These discussions also led to better clarification of steps taken in this work. Finally Nava closely watched my calculations, making important remarks and suggestions for this work.

Secondly I want to thank my official supervisor Gerard 't Hooft. In the initial stage of my thesis we had regular meetings, where we discussed Gerards's previous research in semiclassical gravity on the black hole in great detail. Gerard was always willing to discuss my personal commentary on his research, and this discussion finally inspired me to investigate the topics in this thesis.

Finally I want to thank Umut Gürsoy and the Institute for Theoretical Physics, for understanding my situation at difficult times, and granting me the freedom and possibility to properly finish this work.

Contents

1 Introduction

Ever since the success of quantum field theory in the standard model, ongoing research has commenced to also find a quantum field theory of gravity. Such a theory of *quantum gravity* would combine the curvature of spacetime with the quantization of energy states. Initial attempts at quantization of gravity involved approximating gravity in the linear regime, however quantization of linearized gravity turned out to be not renormalizable [1]. The most straightforward theory of quantum gravity was not valid. Many theories have been formulated since (see [2] for a good review), notably string theory has become the best known possibility. In this paper we will not go into string theory, instead we investigate alternatives. For most of the quantum gravity theories there is a strong focus on one subject: Black holes. Black holes turn out to be the most suitable playground for probing a theory of quantum gravity utilizing only well-known physics. By analyzing many different aspects of quantum mechanics on the strong gravity of a black hole, we can hopefully find out where to look for in a theory of quantum gravity. In this paper we will focus on one such aspect, which is gravitational interaction between particles close to the black hole. Let us first review the history of black hole quantum mechanics. For those already knowledgeable in the background of black hole quantum mechanics, we suggest to move forward to the next page where the contents of this thesis are discussed.

Background

Spherically symmetric charge neutral black holes are described by the Schwarzschild metric. These assumption were later lifted to include charge, the Reissner–Nordström metric, and angular momentum, the Kerr-Newman metric. However, we shall focus on the spherically symmetric Schwarzschild spacetime. The fundamental property of these black holes is the presence of an event horizon, a boundary of spacetime from which nothing can escape, not even light. Thus all particles that fall into the black hole are lost forever. What happens to the information of these particles? In general relativity we have the so-called *no-hair theorem*, which states that the black hole is only described by its mass $M_{\rm BH}$, and for charged rotating black holes also the charge Q and angular momentum J [3]. The particles falling in carry more information than just these three variables $M_{\rm BH}, Q, J$ so the information went somewhere else. From a general relativistic viewpoint this information was not lost, but it is hidden behind the event horizon from which nothing can escape. The information of whatever fell in is not gone, it is merely forever out of our reach.

However Hawking radiation changed this point of view. Hawking radiation is radiation escaping the black hole, as originally derived by Hawking in [4], contradicting the classical viewpoint. The usual physical intuition is that Hawking radiation forms due to vacuum pair-creation on the horizon, where the particle goes out of the black hole and the anti-particle into the black hole. Now in Hawking's original calculation this radiation is thermal, which means that it does not carry any information (up to the temperature), while it does carry energy. The result is that after some long amount of time the black hole evaporates entirely. Now our general relativistic argument is not valid anymore; the particles' information can not be hidden behind the horizon because there is no more horizon. So where did the information go? As per the laws of quantum mechanics this information is not allowed to be lost: Quantum mechanics dictates that the initial and final state of a system must be related by a unitary scattering matrix. Violation of this unitarity implies that the initial and final state do not have equal norm, violating conservation of probability. Clearly there is a clash between general relativity and quantum mechanics. General relativity states that the information of infalling particles is hidden behind the horizon, while quantum mechanics tells us that the horizon evaporates without transmitting any information. The end result is that after black hole evaporation the information of the infalling particles is lost forever, which was not allowed by quantum mechanics. This is known as the *black hole information paradox*. Since the discovery of this paradox in [5] a multitude of attempts at solving the paradox have been launched [6].

Let us shortly discuss the dominant theory at the moment. The original foundation was the holographic principle proposed by 't Hooft [7] and analyzed by Susskind [8]. The principle is that a volume of spacetime can also be described by merely its surface. In the case of a black hole this would imply that the interior of the black hole containing the particles' information is dual to the surface of the event horizon. The information now seems to not be lost, but still present on the horizon through this duality. The modern variant nowadays is the AdS/CFT correspondence found by Maldacena [9], which describes a duality between n-dimensional AdS space and a $(n-1)$-dimensional CFT on its boundary. AdS/CFT dictates that Hawking radiation is in fact not thermal but receives quantum corrections which contain information. This is the current main solution to the information paradox; the information is embedded in the Hawking radiation in some manner.

The phenomenon above is best described by a principle called black hole complementarity, which states that the description of the inside of the black hole is complementary to the outside of the black hole. In other words this implies that the information of particles does not only go through the horizon into the black hole, but is also reflected back outside again [10]. This solves the information paradox while keeping the general relativistic viewpoint of the particles being hidden behind the horizon as well. Both the outgoing Hawking particle and the ingoing particle are entangled with eachother carrying the same information. Although this seems to solve the information paradox a new problem arises. According to research by Page an outgoing Hawking particle at time t must be entangled with all previously emitted Hawking radiation [11]. The previously emitted Hawking radiation and the corresponding ingoing particle at time t are independent systems due to the causal structure of black holes. However these independent systems are now mutually entangled through the outgoing particle at time t. There is a logical conflict; on the one hand the systems must be causally independent, whereas on the other hand the systems talk to eachother through the indirect entanglement. This conflict must somehow be solved.

There is a proposed solution that does not invoke string theory. The entanglement of the outgoing and ingoing particle at the horizon immediately breaks down after both particles have been emitted. This would involve very high energy particles at the horizon, implying that there is a firewall of extreme energy at the horizon. Such a firewall would exist regardless of coordinate system used. However according to the Einstein equivalence principle an infalling particle does not notice the existence of the black hole at all, and so it could never observe an extreme energy firewall. Again there is a conflict between general relativity and quantum mechanics; quantum mechanics shows that there is logical conflict between the ingoing and outgoing radiation, a proposed firewall solution is not allowed by the Einstein equivalence principle of general relativity. This conflict is called the *firewall paradox*. The term firewall and the corresponding paradox was introduced by Almheiri et al. [12]. Thus in an attempt to solve the information paradox a new firewall paradox has been created, and it seems like either one of the two is present and it is impossible to remove both at the same time.

Content

There are a numerous suggestions to how both the information paradox and the firewall paradox might be solved at the same time without breaking the laws of general relativity or quantum mechanics, as listed in [13–22]. In this paper we will focus on another theory developed by 't Hooft over the past few decades [23], with new research in the past few years [24,25]. The main shortcoming of all earlier mentioned theories is that gravitational interaction has been neglected as subdominant. In particular the in- and out-going particles are treated as free fields [4, 26] from which it results that the outgoing Hawking radiation is thermal. However the extreme high energy firewall particles on the horizon could never have a subdominant gravitational field, in fact the energy of all particles close to the horizon asymptotically grows beyond the Planck mass (see Section 2.2). Thus we are not allowed to treat the particles as free fields, as was done by Hawking, instead they are strongly coupled gravitationally.

The proposal of 't Hooft is to add this gravitational interaction between particles on the horizon. The contribution of particles on the horizon is the leading order due to their extreme energy. We will review the findings of 't Hooft's research in Sections 2 and 3. First the gravitational interaction is determined, which can be shown to be given by the cut-and-paste procedure on the Schwarzschild background. This cut-and-paste procedure can be used to remove all high-energy particles from spacetime, and replace them by low-energy particles far away. This solves the firewall paradox purely using general relativity, which is explained in Subsection 2.4.1. We then proceed to apply the gravitational interaction semi-classically to the quantum mechanical wavefunctions of the in- and out-particles. The resulting scattering matrix is given in Subsection 3.3 and is unitary, so that adding the gravitational interaction solves the information paradox. The informa-

tion is simply transferred to the Hawking radiation gravitationally. It should be noted that quantum field fluctuations do not solve the information paradox [14], however whether this gravitational backreaction also counts as a quantum field fluctuation.
The high energy limit is also investigated in a new calculation in Subsection 3.6, resulting in a direct interpretation of the scattering matrix. Furthermore the scattering matrix has serious implications for the Schwarzschild spacetime, most notably a connection between region I and II. This connection states that region I and II are connected by a classical CPT transformation, so upon inverting time, space & charge, region I is equal to region II. A quantum analogue of the CPT transformation is given in [27]. Consequences of the CPT transformation are discussed in Subsection 3.5, where we argue that due to time inversion it is necessary to include creation and annihilation of particles.

The main goal of this paper is to calculate the gravitational interaction on the black hole horizon using quantized fields. To be precise we seek to find similar results as 't Hooft by summing over graviton exchange diagrams.
Several similar calculations have been performed. The first we mention is an effective quantized action by Gaddam et al. [28], which is a first quantized description of the near-horizon observer. Another proposal is a topological theory by Verlinde & Verlinde [29]. This topological theory quantizes the momentum and position of an underlying wavefunction, however we want to quantize the fields themselves. Finally such theory has been defined by Verlinde et al. [26], where the fields of the particles are quantized. However the calculation of [26] rests on an ansatz which appears without strong foundation and probably incorrect to us. We will comment more on this in the beginning of Section 5. Clearly all previous research does not answer our main goal, where we would like to use quantized fields starting from only general relativity and quantum field theory. A new method was needed.

The method we choose to use in this paper is the Eikonal approximation, where we sum over an infinite amount of virtual graviton exchange diagrams up to leading order in the center of mass energy for a large impact parameter. These diagrams then should describe the gravitational interaction between different types of fields. The Eikonal approximation is a method originating in potential scattering theory (there is a short summary in [30]), and later on generalized to quantum field theory [30, 31]. An extension to string theory was made by Amati et al. [32], and an extension to linearized gravity by Kabat & Ortiz [33]. The latter is interesting for us; it shows that the Eikonal approximation for gravitons yields the same semi-classical scattering matrix as was found by 't Hooft for flat space in [34]. The scattering matrix in [34] was derived similar to Section 3 on the flat background. Given that in flat space both methods give the same scattering matrix, we expect that on the black hole background the scattering matrix in Section 3 can also be derived using an Eikonal approximation on the black hole. This calculation has not yet been done before.

In Sections 4 and 5 we will perform the Eikonal approximation on the black hole background, using steps similar to Kabat & Ortiz in [33]. Since we will be summing over virtual graviton exchange diagrams we need to find the graviton propagator. Section 4 is dedicated to finding the propagator, where crucial steps taken are a decomposition in spherical harmonics and a Weyl transformation of the background metric. The most striking result is that the residual effective 2D graviton has become massive in the process due to the background curvature. In Section 5 we apply this graviton propagator in the Eikonal approximation. After reviewing the Minkowski space calculation of Kabat & Ortiz we generalize the calculation to the Schwarzschild background. The external fields we will scatter are massless scalar fields, as shown in Subsection 5.2. Then we shown in Subsection 5.3 that within the Eikonal limit of large center of mass energy the vertex for the scalar field and graviton simplifies tremendously, where we also made a *minimal coupling simplification* on the spherical harmonics. Finally with the simplified vertex the summation of the diagrams becomes straightforward and we find the scattering amplitude in Subsection 5.5. Up to a few expected factors this scattering amplitude is equal to the scattering matrix of 't Hooft in Section 3.

2 General relativity and particle interactions

In this section we outline the basics of the Schwarzschild black hole formalism, the definitions and variables we use and some approximations and their justifications. We also discuss the general relativistic interaction of particles with extremely high velocities. For the results of this interaction we refer the reader to Subsection 2.4. The prior subsections are background, discussing the physics of the Schwarzschild spacetime (Subsections 2.1 and 2.2) and calculating the gravitational interaction in flat space (Subsection 2.3) including the relevant concepts. Notice that the cut-and-paste procedure in Subsection 2.3 is crucial for understanding Subsection 2.4.

2.1 The Schwarzschild metric

In this paper we work on the background metric of the eternal Schwarzschild black hole. We choose Schwarzschild over Kerr-Newman since the latter mostly generates a lot more computational complexity without significantly altering the newfound physics. Additionally we choose to work on the eternal static metric because we are only looking at the particle interaction on short timescales for a global observer compared to the black hole lifetime. On these timescales the black hole is in fact static and we can treat all of spacetime as static so long as the laws of physics are local in time. In other words we neglect the effect of the black hole evaporating or growing. For longer timescales one can in principle repeatedly apply time evolution operators as is usually done in quantum mechanics.

The convention used is that of [25]. The metric is defined in Schwarzschild coordinates by

$$ds^2 = -\left(1-\frac{R}{r}\right)\mathrm{d}t^2 + \left(1-\frac{R}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \qquad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\varphi^2 \tag{2.1}$$

with $R = 2GM_{\mathrm{BH}}$ the Schwarzschild radius or radius of the event horizon. In this paper we will often work with the inverse Schwarzschild radius $\mu \equiv 1/R$ instead. The $\mathrm{d}\Omega^2$ is the metric on the two-sphere S_2, which is given in standard spherical coordinates but could be given in any coordinate chart such as stereographic coordinates. The fact that the Schwarzschild spacetime is static is now expressed by time translation and reversal invariance. To be precise the metric in Equation 2.1 is invariant under $t \to -t$ or $t \to t + a$ for arbitrary a. The Schwarzschild metric above has a coordinate singularity at $r = R$ which is best removed by changing to Kruskal-Szekeres coordinates

$$\begin{aligned} XY &= \left(1-\frac{r}{R}\right)e^{r/R} && (2.2)\\ X/Y &= e^{t/R} \qquad \text{(region I or II)} && (2.3)\\ X/Y &= -e^{t/R} \qquad \text{(region III or IV)} && \end{aligned}$$

in which the metric becomes

$$\mathrm{d}s^2 = -\frac{4R^3}{r}e^{-r/R}\mathrm{d}X\mathrm{d}Y + r^2\mathrm{d}\Omega^2. \tag{2.4}$$

In this coordinate system there is only a singularity at $XY = 1$, $r = 0$ which is the physical black hole singularity that plays not role on the horizon. We now define a dimensionless time in terms of the Schwarzschild radius

$$\tau \equiv \frac{t}{2R}. \tag{2.5}$$

Additionally in practice we work with the new rescaled coordinates

$$x = \frac{Rx}{\sqrt{e/2}}, \qquad y = \frac{Ry}{\sqrt{e/2}}. \tag{2.6}$$

Observe that as $t \to -\infty$ we have $x \to 0$ while as $t \to \infty$ we have $y \to 0$. This means that $x = 0$ is in the far past while $y = 0$ is in the far future, however both $x = 0 = y$ is at finite range $r = R$. We now clearly see that $x = 0$ is the past black hole horizon and $y = 0$ is the future black hole horizon.

In this paper we will use the index decomposition where small Latin letters imply the lightcone $a, b = x, y$, whereas capital Latin letters imply the two-sphere $A, B = \theta, \phi$. We then also combine the coordinates above into a single vector x^a where $x^x = x, x^y = y$. These definitions are also listed in Appendix A.1. In terms of these the metric is compactly written as

$$\begin{aligned} \mathrm{d}s^2 &= -2A(r)\mathrm{d}x\mathrm{d}y + r^2\mathrm{d}\Omega^2 \\ A(r) &= \tfrac{R}{r}e^{1-r/R} \end{aligned} \tag{2.7}$$

Equation 2.7 gives the form of the metric as we will be using in this paper. Notice that the Schwarzschild metric is now conformally flat in the lightcone coordinates, a fact we will exploit in Section 4. The choice of coordinates x, y is such that $A(R) = 1$, so that on the horizon the metric in Equation 2.7 becomes the flat metric. This will be a particularly useful choice of coordinates since we will be looking at physics close to the horizon $r = R$ only, such that we often approximate $A(r) \approx 1$. Due to the choice of x, y coordinates there will not be any annoying factors of 2 or e appearing.

The Schwarzschild spacetime is naturally still static in the new coordinates. In these coordinate time translation invariance and time reversal are expressed differently:

$$\text{time translation} \qquad x \to xa \quad \& \quad y \to y/a, \tag{2.8}$$

$$\text{time reversal} \qquad x \to y \quad \& \quad y \to x. \tag{2.9}$$

Here a is an arbitrary real number. These symmetries will be present throughout this entire paper, as we require the background metric symmetries to also persist at all quantum levels.

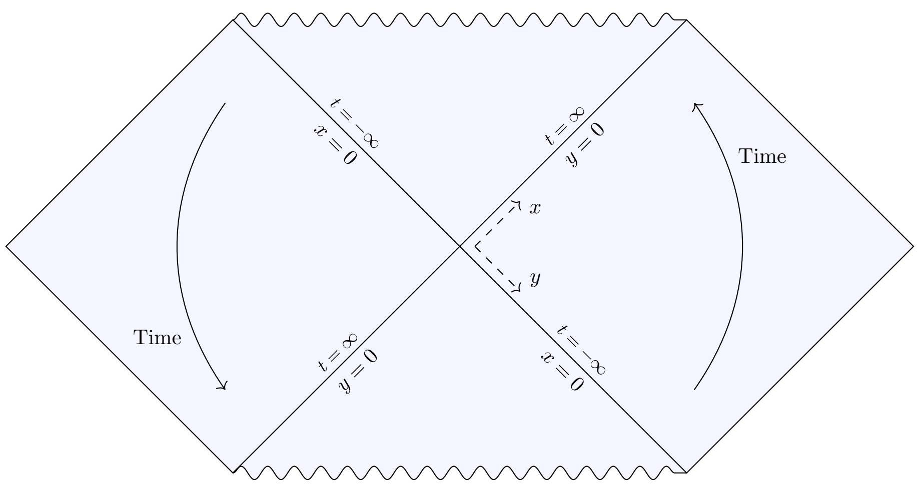


Figure 1: The Penrose diagram of the maximally extended Schwarzschild spacetime.

The Penrose diagram of the Schwarzschild metric is given in Figure 1. Here we identified two asymptotic regions I and II, the inside of the black hole III, and the white hole IV from which matter can only escape but not fall in. For more discussion on how this diagram has been obtained we refer the reader to [35]. We remark again that $y = 0$ implies $\tau \to \infty$ while $x = 0$ implies $\tau \to -\infty$. This means that in region I timelike Killing vectors point upwards in the diagram, while in region II they point downwards. In other words in region II time flows in the opposite direction with respect to region I. The flow of time in region III and IV is more difficult, since there the timelike Killing vector is given by the radial coordinate. We shall pay no further attention to regions III and IV, for more information we refer the reader to [36].

The relations between region I and II will be of main discussion in this paper. Originally in general relativity regions I and II are causally disconnected as is clearly visible in the diagram. Since both these regions have asymptotic infinities in them, they both have to correspond to the infinite universe outside of the black hole. As such they seem to be some type of parallel universes. From this point on we will define region I as "our" universe as a choice, so the observer we are looking from is in region I. The question then becomes: What exactly is region II? There are numerous interpretations on this as greatly outlined in [36]. The most common interpretation is that both region I and II are causally disconnected external regions of the black hole, and regions III and IV internal regions. A more uncommon interpretation is that the singularities of regions III and IV are connected, but this theory currently has no strong physical foundation. Secondly there is the possibility of topological spacelike connections such as Wheeler's single connected universe. This has complex causal interpretations [37] and we will not go into this possibility.

Finally [36] suggests that region I and II might simply be causally connected in such manner that a point x, y, θ, ϕ in region I is the same point $-x, -y, \theta, \phi$ in region II. Thus the regions I and II are dual to eachother. This mapping however results in a conic singularity at $x = y = 0$. We shall see that the latter nevertheless is a great suggestion, when we combine this with 't Hooft's quantum mechanical result. There the mapping between region I and II suggested by [36] is combined with an angular antipodal connection $\theta \to \pi - \theta, \phi \to \phi + \pi$ suggested by 't Hooft [25, 38], which removes the conic singularity. The antipodal connection was suggested before by [39], although many of the consequences were discussed later. We will return to this subject in Section 3. Now we will first thread further into the properties of the Schwarzschild spacetime, and the gravitational interaction in general relativity.

2.2 The local versus global observer

The local observer that falls into the black hole naturally observes different physics than the global observer who remains stationary at $r \to \infty$. This difference is mostly in the fact that the local observer does not observe any acceleration as per Einsteins Equivalence principle, while the global observer sees the local observer accelerating inward. To see this we best move to a different set of coordinates.

An important set of coordinates for the difference between local and global observers are the tortoise coordinates, defined by $(1 - \frac{R}{r})^{-1}\mathrm{d}r = \mathrm{d}r_*$. The metric then takes the simple form:

$$\mathrm{d}s^2 = -\left(1 - \frac{R}{r(r_*)}\right)(\mathrm{d}t^2 - \mathrm{d}r_*^2) + r(r_*)^2\mathrm{d}\Omega^2 \tag{2.10}$$

where r is related to r_* by

$$r_* = r + R\log\left|\tfrac{r}{R} - 1\right|. \tag{2.11}$$

The remarkable property of these coordinates is that null geodesics are given by $r_* = \pm t$. This implies that massless particles are free-falling in Tortoise coordinates. In this sense the Tortoise coordinate is the natural coordinate system that complements the Einstein Equivalence principle. Massless particles do not experience any acceleration, they are moving freely. This is what the local infalling observer sees, the infalling observer is not capable of seeing its own acceleration, so it observes the free-falling Tortoise coordinates. The global observer on the other hand is stationary at some very large $r \gg R$ and observes our Schwarzschild or Kruskal coordinates, seeing the infalling observer accelerate. The discussion for masive particles require a different set of coordinates.

The Tortoise coordinates are related to the Kruskal coordinates by

$$y = \frac{R}{\sqrt{2}}e^{-\frac{t+r_*}{2R}-\frac{1}{2}}, \tag{2.12}$$

$$x = -\frac{R}{\sqrt{2}}e^{\frac{t-r_*}{2R}-\frac{1}{2}}. \tag{2.13}$$

Now we can also include regions II,III and IV by changing the signs in front of these exponents, which will become important in Subsection 3.2. Notice that for $r \equiv R$ the Tortoise coordinates are not ideally fitted, since r_* diverges, so again we need a different approach. In this regime we best first approximate the Schwarzschild spacetime in Kruskal-Szekeres coordinates, and then make the following transformation:

$$x - y = \rho\cosh\tau, \qquad x + y = \rho\sinh\tau. \tag{2.14}$$

The coordinates ρ, τ are the so-called Rindler coordinates, which describe an observer undergoing constant acceleration. These are the free-falling coordinates for massive and massless particles when very close to the horizon. In other words a global observer sees particles accelerating inward at a constant rate, while local observers fall in without any notice.

We can also look at this acceleration in the global Schwarszchild coordinates. To see this solve the lightcone constraint for massless particles $\mathrm{d}s^2 = 0 = -\left(1-\frac{R}{r}\right)\mathrm{d}t^2 + \left(1-\frac{R}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$. Assuming that $\mathrm{d}\Omega = 0$ (radial trajectories) this gives

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| = \left(1-\frac{R}{r}\right). \tag{2.15}$$

This shows that the global inserver indeed sees particles asymptote toward $r = R$ while never reaching it. For the local observer particles of course simply fall in very slowly, without observing a horizon at all. This also explains why we will solely be looking at the horizon $r = R$ in Section 3. A particle hardly changes r coordinate while the time coordinate has infinite range and we can safely put $r = R$ and $A(r) = 1$. This argumentation is valid so long as we are looking from a global point of view.

There are two important remarks to make regarding the asymptotic behavior, namely how black holes grow in finite time? This is mostly a question of definition. Let us claim that a particle has become part of the black hole when the combined Schwarzschild radius of the black hole and the particle has become larger than the particles location. Choose the black hole to have initial radius R and define a final radius $R' = R+2Gm_p$, with m_p the particle's energy. Then for a particle at location r we define that it is separate from the black hole when $r > R'$ and it is part of the black hole when $r < R'$. This implies it falls in when $r = R'$ and the actual black hole horizon shifts from R to R'. Since $R' > R$ the particle was at that point still falling at nonzero radial velocity with respect to a global observer (see Equation 2.15), so the particle does in fact enter the black hole at a finite time t such that $r(t) = R'$. So what we see is that while the particle never reaches R in finite time, it does reach R' in finite time, so the trick is to combine the particle and black hole at the right moment, and then the black hole can grow in finite time.

The other important remark is the following curious phenomenon: time stands still on the horizon, again as seen for a global observer. This is characterized by the fact that on the horizon $\frac{dr}{dt} = 0$. This allows for easy interpretation as to why particles move asymptotically towards the horizon; they cannot reach it in finite time because time stands still. Since energy is correlated to the rate of change of time, energy also diverges on the horizon as seen on Figure 2.

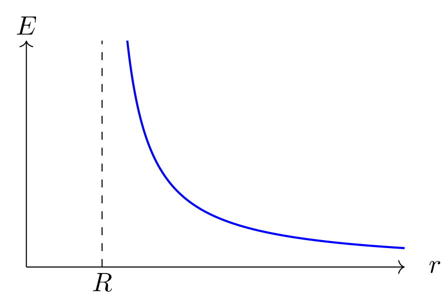


Figure 2: The energy diverges as we approach the horizon.

Notice that all three of the above statements are equivalent; the fact that time stands still on the horizon, the fact that particles asymptote toward the horizon and the diverging energy. This is of great importance, a global observer observes particles close to the black hole to carry infinite energy! Following [24] we now distinguish between two types of particles, *hard particles* with $|\vec{p}|, \mu \gg M_{pl}$ and *soft particles* with $|\vec{p}|, \mu \ll M_{pl}$. The particles at $r = R$ with infinite energy are clearly hard, so that they generate significant spacetime curvature and thus gravitational forces. On the other hand the particles far away are soft, generating insignificant gravitational forces. Globally we observe both ingoing and outgoing radiation with infinite energy, namely on the past and future horizon. Then naturally these hard particles have some gravitational interaction.

This is the main motivation for including gravitational interaction and doing this research, the particles have infinite energy. We can now take the fact that we are looking at the horizon more exactly. The main interactions come only from particles close to the horizon as those are hard. At the same time the largest amount of particles present is also close to the horizon. This is because the particle distribution asymptotes toward the horizon because time stands still at the horizon. Combining these two we see that by far all leading contributions come from the physics close to the horizon, while all contributions from far away are a small amount of soft particles. Since the Kruskal-Szekeres metric in Equation 2.7 is well-defined at $r = R$, we can look exactly at the physics at $r = R$ only as zeroth order approximation. All linear contributions would then be kept for a further research, but given the above reasoning these contributions should be extremely small.

Horizon geodesics

Now we will finally look at the geodesics for particles on the metric in Equation 2.7 on the horizon at $r = R$. Let a particle have lightcone momenta p_x and p_y and $\tilde{p}$ in the transverse direction. The momentum obeys the following equation

$$-2p_x p_y + \tilde{p}^2 = m^2 \tag{2.16}$$

with m the particle mass. The easiest method to determine the geodesics is through the Killing vectors in standard Schwarzschild coordinates (equation 2.1). We see quickly that there is the timelike Killing vector ∂_τ and from spherical symmetry two spacelike Killing vectors corresponding to rotations. They generate a conserved angular momentum $\tilde{L}$ that fixes $\tilde{p} = \mu\tilde{L}$ constant and we assume $\tilde{p}$ to be small compared to p_x, p_y. The timelike Killing vector generates a conserved energy $E = p_\tau$. The momenta p_x and p_y can be found using standard coordinate transformations

$$p_x = -\frac{\partial y}{\partial u^\mu} p^\mu = -\frac{\partial y}{\partial \tau} p^\tau, \tag{2.17}$$

$$p_y = -\frac{\partial x}{\partial u^\mu} p^\mu = -\frac{\partial x}{\partial \tau} p^\tau. \tag{2.18}$$

Here we inserted that $r = R$ so that all radial coordinate dependencies vanish. More precisely $p^r \propto \frac{\mathrm{d}r}{\mathrm{d}t}$ vanishes as argued using Equation 2.15. Remark that $p^\tau = p_\tau$ up to a radial dependency and we inserted $p_x = -p^y$ for the momenta. Finally we can insert the coordinate transformations defined in Equation 2.12 resulting in the following set of equations for the geodesics:

$$\tilde{p} = \frac{\tilde{L}}{R} = \text{small} \qquad (\theta, \phi) \sim \text{constant} \tag{2.19}$$

$$p_x(\tau) = p_x(0)e^{-\tau} \qquad x(\tau) = x(0)e^{\tau} \tag{2.20}$$

$$p_y(\tau) = p_y(0)e^{\tau} \qquad y(\tau) = y(0)e^{-\tau}. \tag{2.21}$$

Now when looking from a global observer we see that for both $\tau \to \pm\infty$ the energy p_a diverges. More generally the closer to the horizon the harder the particles are. This is exactly our statement on the energy, but now explicit in the exact momenta. This implies that for locally soft particles at $\tau = 0$ we still need to know the gravitational interactions between these particles at later times. We need to remark that the derived geodesics are general, so also for massive particles. In practice we will be working with massless particles, so let us now look at the null geodesics.

Null geodesics

We recall that massless particles move over constant null coordinates. As we are already in lightcone coordinates this means that massless particles move either over constant x or constant y. This also implies that respectively p^x or p^y vanishes. The geodesics are then simply either of the following:

$$\text{infalling:} \quad x = \text{constant}, \quad p_y = 0 \tag{2.22}$$

$$\text{outgoing:} \quad y = \text{constant}, \quad p_x = 0. \tag{2.23}$$

To see why these correspond to infalling and outgoing particles one could simply draw these geodesics in Figure 1 and see which horizon they cross. The geodesics are especially simple, but very important as we only work with (approximately) lightlike particles. For these massless particles the energy with respect to a global observer still diverges as the horizon is approached as was shown in the previous subsection. To see this remark that time translation still has its effect. For the ingoing particles we have nonzero p_x which under a time translation transforms as $p_x \to p_x e^{-\tau}$. Then when looking from a global observer in the far past $\tau \to -\infty$ we see that the momentum p_x diverges. Thus in the far future the ingoing particles all have extremely high energy. The procedure for the far past is identical.

We see that all particles (massless and massive) have diverging energies near the horizon, so that they all become hard there. Additionally the in-particles are in parallel orbits and thus do not interact gravitationally, and the same holds between the out-going particles, so we need to know only the interaction between the in- and out-particles. Since soft particles have no gravitational effect we only need to know the effect of hard particles on soft particles, and in principle the effect of hard particles on hard particles. The latter turns out to be removable by the firewall transformation as explained in Subsection 2.4.1. The former is known as the Shapiro delay.

2.3 Shapiro delay

The Shapiro delay is the gravitational effect of a particle moving close to the speed of light. We might ask ourselves what happens when we Lorentz boost a particle at rest to almost the speed of light? This boost would increase the particle's energy asymptotically, such that energetically one expects a black hole to form, however black holes alter the causal structure of the spacetime significantly. As general relativity is Lorentz invariant this appears to be wrong. The answer is easily found in electromagnetism. When an electron is accelerated to almost the speed of light in that medium, its electromagnetic waves build up forming a large kick, known as Cherenkov radiation. Similarly when a plane is flying at the speed of sound, it generates a shock-wave called the sonic boom. For both of these objects the field takes the shape of a cone behind the electron/plane of a singularly high amplitude because they move at the same speed as the wave propagates in that medium. For gravity the speed of propagation is the speed of light so we can expect the same behaviour. Indeed the analogue for gravity is called the Shapiro delay, an instantaneous shock-wave in spacetime which takes the form of a cone dragged behind the particle. This effect is illustrated in Figure 3. Whenever this cone hits a test particle, it generates a translational kick.

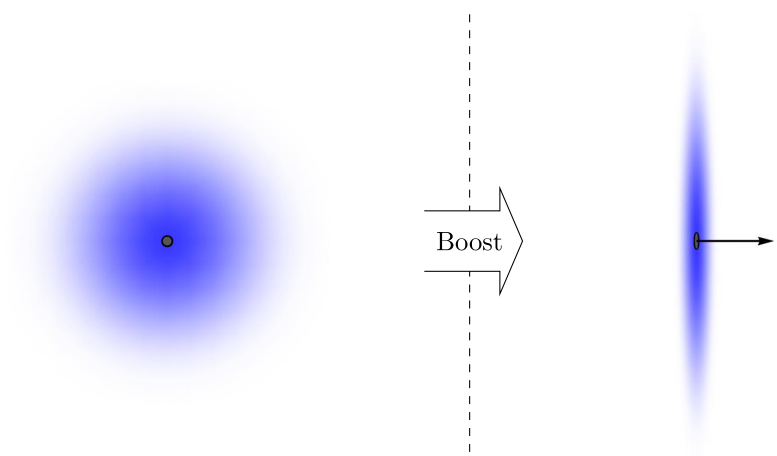


Figure 3: An illustration of the Shapiro delay. We boost the spherical gravitational field of a particle, resulting in a Lorentz contracted instantaneous shock-wave.

The Shapiro delay can be found by boosting the Schwarzschild metric. A massive particle without charge or spin generates the Schwarzschild metric, and thus boosting that metric should give the gravitational field of the boosted particle. This method was first performed by Aichelburg and Sexl in [40]. In this subsection we retrace these steps in our own formulation. We start with the metric in Schwarzschild coordinates

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{R}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{R}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{2.24}$$

Here $R = 2Gm$ with m the mass of the particle. We want to apply a Lorentz boost in a specific direction on this metric tensor. To do so spherical coordinates are not ideal, so we first transform the metric tensor back to Cartesian coordinates under the following convention

$$x = r\cos\theta\cos\varphi, \tag{2.25}$$
$$y = r\cos\theta\sin\varphi, \tag{2.26}$$
$$z = r\sin\theta. \tag{2.27}$$

The transformed metric tensor can now be written as

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{R}{r}\right) & \vec{0}^T \\ \vec{0} & \mathbb{1} + \frac{1}{r^2}\frac{1}{\frac{r}{R}-1}\vec{x}\vec{x}^T \end{pmatrix} \tag{2.28}$$

with $\vec{x} = (x, y, z)$ and $\vec{0} = (0, 0, 0)$. Now that we have the metric tensor in Cartesian coordinates we can apply the Lorentz boost, which we choose to perform in the z-direction with negative velocity, such that the particle is propelled upwards. This boost matrix is given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{2.29}$$

with $\gamma = \sqrt{\frac{1}{1-\beta^2}}$ and β the boost velocity. Remark that after the boost the origin, i.e. the particle, follows the $z = \beta t$ trajectory as desired. Applying the Lorentz boost to the metric gives upon simplification

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \gamma^2\frac{R}{r'}\begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix} \\ &+ \frac{1}{r'^2}\frac{1}{\frac{r'}{R}-1}\begin{pmatrix} \beta^2\gamma^4(z-\beta t)^2 & -\beta\gamma^2 x(z-\beta t) & -\beta\gamma^2 y(z-\beta t) & -\beta\gamma^4(z-\beta t)^2 \\ -\beta\gamma^2 x(z-\beta t) & x^2 & xy & \gamma^2 x(z-\beta t) \\ -\beta\gamma^2 y(z-\beta t) & xy & y^2 & \gamma^2 y(z-\beta t) \\ -\beta\gamma^4(z-\beta t)^2 & \gamma^2 x(z-\beta t) & \gamma^2 y(z-\beta t) & \gamma^4(z-\beta t)^2 \end{pmatrix}. \end{aligned} \tag{2.30}$$

Here we defined

$$r' =\equiv \sqrt{r_\perp^2 + \gamma^2(z-\beta t)^2}$$

and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

We want to look at what happens when we boost a particle to almost the speed of light, so we take the velocity $\beta = 1 - \frac{\epsilon}{2}$ with $\epsilon \ll 1$. Under this convention $\gamma \equiv 1/\sqrt{\epsilon}$.
Inserting the above definition for $z - \beta t \neq 0$ results in $r' \equiv |z - \beta t|/\sqrt{\epsilon}$ in the small ϵ limit. The last property shows that for $z - \beta t \neq 0$ the metric is proportional to $g_{\mu\nu} \sim 1/\sqrt{\epsilon}$ which diverges. This is to be expected, since a massive particle boosted to infinity has infinite energy, resulting in infinite spacetime curvature. The trick to solve this is by also transforming the mass $m \to \sqrt{\epsilon}m$ in order to keep the energy $E = \gamma m$ finite. We are then again looking at a physically realistic scenario, wherethe metric is finite for $z - \beta t \neq 0$. The $z - \beta t = 0$ case is special and will result in a delta function.

Upon rewriting the metric and taking the limit $\epsilon \to 0$ we find for the $z-t \neq 0$ case that

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2R}{|z-t|}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.31}$$

There is something remarkable about this metric, although not immediately clear; the Riemann tensor vanishes exactly $R^\rho{}_{\mu\sigma\nu} = 0$. Thus we observe that in the appropriate $\epsilon \to 0$ limit there is no gravitational field present when $z-t \neq 0$. We can understand this in the following way; the entire space of the particle moving close to the speed of light is Lorentz contracted into just the surface orthogonal to its movement, anything outside this surface is at a distance larger than infinity. Since the Schwarzschild metric is asymptotically flat this results in no net gravitational potential.

The remaining question is then naturally: what about the $z-t=0$ case? Since the exact mathematical analysis is tricky we will do a heuristic analysis to find the form of the metric, and then use the Einstein field equations to find the exact result.
First observe that the definition of $z-t \neq 0$ was only important for taking the limit $\epsilon \to 0$ in r' such that $r' \to |z-t|/\sqrt{\epsilon}$. Thus we might say that the condition "$z-t=0$" holds as long as $z-t$ goes to zero equally fast as $\sqrt{\epsilon}$, such that r' is well-defined and nonzero. In practice we implement this by introducing a limiting regulator w_0 such that $|z-t| \leq w_0\sqrt{\epsilon}$, in this sense w_0 gives the maximum value $z-t$ may take in order for "$z-t=0$" to be satisfied. Notice that w_0 may depend on x, y as that does not alter the conditions. For this regulator we can write $r' \to r_0$ which is defined by

$$r_0^2 = r_\perp^2 + w_0^2$$

and is ϵ-independent. Notice that by definition as $\epsilon \to 0$ we also have $z-t=0$ so the limit is correct.

Given that the regularized method appears to correctly take into account the $z-t=0$ behaviour, let us insert the regulator into the boosted metric resulting in

$$g_{\mu\nu} = \eta_{\mu\nu} + \Theta(|z-t| - w_0\sqrt{\epsilon})\frac{R}{r_0\sqrt{\epsilon}}\begin{pmatrix} 1+\frac{w_0^2}{r_0^2} & 0 & 0 & -1-\frac{w_0^2}{r_0^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1-\frac{w_0^2}{r_0^2} & 0 & 0 & 1+\frac{w_0^2}{r_0^2} \end{pmatrix} - \Theta(|z-t| - w_0\sqrt{\epsilon})\frac{R}{r_0^3}\begin{pmatrix} 0 & x & y & 0 \\ x & 0 & 0 & x \\ y & 0 & 0 & y \\ 0 & x & y & 0 \end{pmatrix} \tag{2.32}$$

Here $\Theta(x)$ is the Heaviside step function, introduced since the metric is only valid for $|z-t| \leq w_0\sqrt{\epsilon}$. Upon letting $\epsilon \to 0$ the last term vanishes because the step function loses all support and the rest of the term is finite. The second term is different since $1/\sqrt{\epsilon}$ diverges. Notice however that the total surface area of $\Theta(|z-t| - w_0\sqrt{\epsilon})/\sqrt{\epsilon}$ is always given by

$$\int_{z-t=-\infty}^{z-t=\infty} \mathrm{d}(z-t)\Theta(|z-t| - w_0\sqrt{\epsilon})/\sqrt{\epsilon} = 2w_0 \tag{2.33}$$

independent of ϵ. Thus in terms of distributions we easily see the following relation

$$\lim_{\epsilon \to 0} \Theta(|z-t| - w_0\sqrt{\epsilon})/\sqrt{\epsilon} = 2w_0\delta(z-t). \tag{2.34}$$

With this we can finally write the total metric in the $z-t=0$ plane after taking $\epsilon \to 0$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\delta(t-z)F(x,y)\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Here $F(x,y) = \frac{w_0}{r_0}\left(1+\frac{w_0^2}{r_0^2}\right)$ has been defined to be determined later. The form of the metric that we have found is the correct one, since we properly applied the $O(\epsilon)$ analysis. The metric in Equation 2.35 gives the complete metric of a particle moving at the speed of light, for both $z-t=0$ and $z-t\neq 0$. Indeed there is no black hole forming, instead there is a singular gravitational field on the cone $z-t=0$, analogous to the case of sound and electromagnetism. The strength is determined by the function $F(x,y)$ which only depends on the location x,y on the $z-t=0$ cone. The only question now is the exact shape of $F(x,y)$, which is current arbitrary due to the regulator w_0 being arbitrary. We remark that Aichelburg and Sexl derived the above metric with the appropriate F by properly performing the $\epsilon\to 0$ limit without regulators [40].

Let us now solve for the appropriate $F(x,y)$ using the Einstein field equations instead. The only equation we need for this is the (00) equation and thus also only the energy density T_{00}. The field equation and energy density for a single moving particle along the z-axis (the way we boosted) are given by

$$R_{00} - \frac{1}{2}g_{00}R = 8\pi G T_{00} \qquad T_{00} = E\delta(z-t)\delta(x)\delta(y). \tag{2.36}$$

Here E is the observed energy of the particle. This gives the following solution

$$F(x,y) = -2\pi GE\log(r_\perp^2). \tag{2.37}$$

This is exactly the same result as was obtained by Aichelburg and Sexl in [40]. This is now aptly known as the Aichelburg-Sexl metric given by

$$\mathrm{d}s^2 = -8\pi GE\log(r_\perp)\delta(u)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2. \tag{2.38}$$

Exploiting symmetry we can now say that $r_\perp$ is the distance to the particle perpendicular to its movement in any coordinate system, giving us the general result. In particular if the particle moves through the origin we can work in spherical coordinates with the usual $u=t-r, v=t+r$ coordinates. Since we can always translate the spacetime origin, we now work in these coordinates without loss of generality.

The effect of the metric in Equation 2.35 combined with Equation 2.37 can be found using the geodesic equations to be a kick; when the test particle crosses the $u=t-r=0$ hyperplane, its $v=t+r$ coordinate receives an instantaneous kick. We derive the geodesic of the Aichelburg-Sexl metric for a particle with $u\neq$constant to be

$$\boxed{v(u) = F(r_\perp)\Theta(u) = -2\pi GE\log(r_\perp^2)\Theta(u).} \tag{2.39}$$

To derive this we made the assumption that the p_x, p_y vanish in the far past. Including the transverse momenta yields ill-defined results. Equation 2.39 gives the gravitational effect of a particle moving at the speed of light with energy E. It shows due to the Heaviside step function the instantaneous change in v when $u=0$. This can be understood as all gravitational energy of the particle begin confined in the hyperplane perpendicular to its movement due to the infinite Lorentz contraction.

A few remarks on Equation 2.39 are necessary: we have a lengthscale inside the logarithm without reference which seems odd. Of course the implicit reference is the Planck scale but more importantly: all test particles in spacetime receive a kick, so only the difference between two kicks is important. This difference translates into $\log(r_{\perp,1}) - \log(r_{\perp,2}) = \log(r_{\perp,1}/r_{\perp,2})$ removing any dimensionality problems. Secondly the effect appears to scale as $\log r$ which is not asymptotically flat, thus leading to divergences in the energy. This divergence is however unphysical as explained in [41]; we can expand $\log(r)$ in $1/r$ terms, where we can remove the first two by a translation and Lorentz transormation. Finally this is the Shapiro delay for a neutral and spinless particle. For a particle with charge and spin we would need to boost the Kerr-Newman metric instead. Especially spin will make a huge difference since we lose one symmetry axis. This means that the results derived in Section 3 are all only for the case of chargeless, spinless particles. Clearly we are only considering a special case, although in the high energy limit the scalar contribution will be strongly dominant. The quantum field theory treatment in Section 4 will hopefully be able to lead to a more general framework in the future.

The cut-and-paste procedure

There is an alternative method to find the Aichelburg-Sexl metric of the shockwave of a massless particle that will be important for extension to curved spacetimes. This is the *cut-and-paste procedure* as found by Penrose in [42] which we describe in this subsection. We start with the background spacetime $\mathcal{M}$ and divide it along a null hypersurface (a hypersurface with null orthogonal vector) in two patches $\mathcal{M}^+$ and $\mathcal{M}^-$. For simplicity we take $\mathcal{M}$ to be Minkowski space, but the cut-and-paste procedure holds for more general spacetimes under specific conditions defined in [41]. The metric is given by

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2 \tag{2.40}$$

with $u = t-r, v = t+r$ the lightcone coordinates. Now suppose that the patches $\mathcal{M}^+$ and $\mathcal{M}^-$ are described by different coordinate charts, respectively (u, v, θ, ϕ) and $(u, v + F(\theta, \phi), \theta, \phi)$ with $F(\theta, \phi)$ an arbitrary function. Define the boundary between $\mathcal{M}^+$ and $\mathcal{M}^-$ to be given by the $u = 0$ hypersurface. Then requiring continuous coordinates results in the following boundary condition:

$$(u = 0, v, \theta, \phi)_{\mathcal{M}^-} = (u = 0, v + F(\theta, \phi), \theta, \phi)_{\mathcal{M}^+}\,. \tag{2.41}$$

Using the fact that we have $v+F(\theta, \phi)$ on $\mathcal{M}^+$ $(u > 0)$ and v on $u < 0$ together with the continuity condition above we can write down the metric for the entire spacetime $\mathcal{M}$ as

$$\mathrm{d}s^2 = -2\mathrm{d}u\Big(\mathrm{d}v + \Theta(u)\mathrm{d}F\Big) + r^2\mathrm{d}\Omega^2. \tag{2.42}$$

Recognize that in the above simply for $u < 0$ we have $\mathrm{d}(v)$ while for $u > 0$ we have $\mathrm{d}(v + F)$ as desired. Finally transform $v \to v - \Theta(u)F(\theta, \phi)$, essentially removing the earlier coordinate chart difference between $\mathcal{M}^+$ and $\mathcal{M}^-$ at the cost of creating a discontinuity. The metric is now given by

$$\mathrm{d}s^2 = -2\mathrm{d}u\Big(\mathrm{d}v - \delta(u)F\mathrm{d}u\Big) + r^2\mathrm{d}\Omega^2. \tag{2.43}$$

Observe that the above metric is flat with a singular gravitational pulse on the hypersurface $u = 0$. In fact the metric in Equation 2.43 has exactly the same form as the metric in Equation 2.35 which was obtained by Lorentz boosting Schwarzschild and the regularizer w_0. This shows that the regularizer method was a general way of deriving possible shockwave spacetimes in Minkowski spacetime Using the energy momentum tensor we can then find the function F leading to the same result as in Equation 2.37. We conclude that the cut-and-paste procedure indeed also generates the metric of a particle moving at the speed of light with energy E.

Notice that the transformation $v \to v + \Theta(u)F(\theta, \phi)$ that is needed to go from the metric in Equation 2.43 to 2.42 exactly removes the geodesic shift in Equation 2.39. In other words: in the metric in Equation 2.43 one has $\delta v = F\Theta(u)$ while in the metric in Equation 2.42 one has $\delta v = 0$[1]. We now clearly see two possible interpretations of the Shapiro delay. On the one hand we can have a continuous coordinate v but all particle geodesics receive an instantaneous kick $\delta v = F$ at $u = 0$. This is a very physical interpretation: We are looking at a continuous spacetime where an infinitely Lorentz boosted metric gives test particles a kick. On the other hand we can also look at the case where the coordinate receives the kick, i.e. $v \to v - \Theta(u)F$ in which case all test particles move in orbits of both constant v and u, but now spacetime itself is repositioned slightly when $u = 0$. This is a mathematical description: We have cut out the region between v and $v-\Theta(u)F$ (note F is negative), resulting in all particles skipping this patch of spacetime, and proceeding their journey slightly further. In other words we cut all particles for $u > 0$ out and paste them back again at a further location $v \to v + F$, hence the cut-and-paste procedure. This is illustrated in Figure 4[2]. In the left we

[1]To see this transform $\delta v = F\Theta(u)$ under $v \to v + F\Theta(u)$. This gives $\delta(v + F\Theta(u)) = F\Theta(u)$ which results in $\delta v = 0$.

[2]We remark that the drawing is not completely valid. It appears as if the null infinities are discontinuous, but this is not the case. Null infinity is unchanged by $v \to v + F$ as long as F is finite. We have chosen for this illustration to provide a clear concept of the cut-and-paste procedure. In a realistic image the null infinities would have curved lines in order to connect them correctly, this is more accurately drawn for the Schwarzschild spacetime in for example [43].

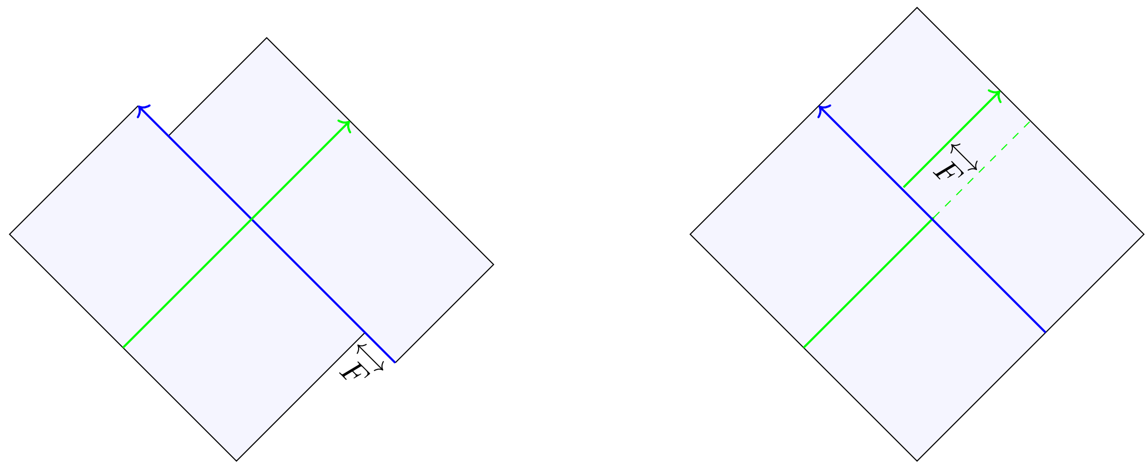


Figure 4: The cut-and-paste procedure in Minkowski spacetime drawn conceptually. We can cut spacetime along the geodesic (blue) and paste it back with a displacement, a test particle (green) moves in a straight line. This is illustrated on the left. If we then remove the displacement of spacetime by a coordinate transformation, space is continuous again but the test particle receives a displacement along the geodesic, as illustrated on the right. Both images are physically identical.

have spacetime cut out and pasted back with a displacement, with particles moving straight but space itself being displaced. On the right we have set space straight again, but now particles are displaced. Since the cut-and-paste procedure is more easily generalizable to different metrics, like on a Schwarzschild background, it is important to understand.

2.4 Shapiro delay in curved spacetime

As we have seen in Subsection 2.2 the in-particles of the Schwarzschild black hole become hard in the far future and the out-particles were hard in the far past. We now want to know what the gravitational effect of these particles is. In Subsection 2.3 we saw this effect on flat space, but we are now working on the eternal Schwarzschild metric given by Equation 2.4. On the Schwarzschild background the curvature naturally affects the shape of the Shapiro delay, the physical principle is illustrated in Figure 5.

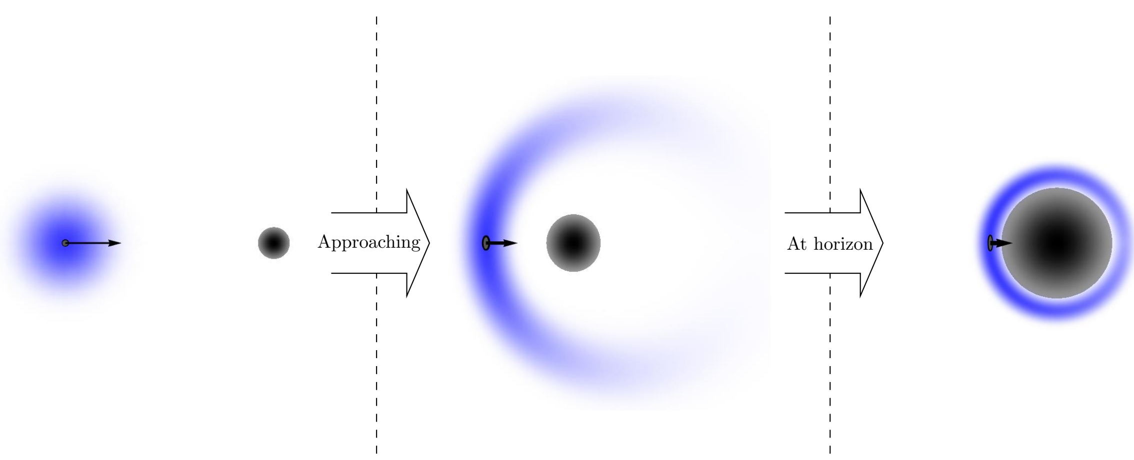


Figure 5: An illustration of the Shapiro delay on Schwarzschild. As a particle approaches the black hole (black sphere), the gravitational field (blue) becomes a plane perpendicular to its movement as in Subsection 2.3. However now this plane also bends around the black hole. Remark that the images also zoom in on the black hole from left to right, for illustrative purposes.

To mathematically find the Shapiro delay on the Schwarzschild metric we apply the cut-and-paste procedure of Subsection 2.3 on the metric of Equation 2.4. In this subsection we follow the procedure outlined by [24]. We need to use the metric before applying the $r = R$ case since the non-vanishing derivatives need to be taken into account in the Ricci tensor. We remind the reader that this metric is given in x, y coordinates by

$$\mathrm{d}s^2 = -2A(r)\mathrm{d}x\mathrm{d}y + r^2\mathrm{d}\Omega^2, \qquad A(r) = \frac{R}{r}e^{1-r/R}, \qquad xy = 2R^2\left(1 - \tfrac{r}{R}\right)e^{\frac{r}{R}-1}. \tag{2.44}$$

Following the same steps as in Subsection 2.3 we find for an $x = 0$ hypersurface as boundary the following metric:

$$\mathrm{d}s^2 = -2A\mathrm{d}x\left(\mathrm{d}y - \delta(x)F(\Omega)\mathrm{d}x\right) + r^2\mathrm{d}\Omega^2. \tag{2.45}$$

We want to find $F(\Omega)$ using the Einstein field equations with an incoming point particle as source. First we find the Ricci tensor which is given by [23] on page 38 and derived in [41]. We remark that in the second paper a δ^2 term was ignored, which means that the equations of motion become linear, a fact we will appreciate again later. Given that the Ricci scalar vanishes both mathematically and because of the traceless energy momentum tensor of a massless particle we find the following Einstein tensor component:

$$G_{xx} = \frac{\delta(x)}{R^2}\Big(\Delta_\Omega F(\Omega) - F(\Omega)\Big) \tag{2.46}$$

where we inserted $r = R$ and $A = 1$ which holds at $x = 0$. Now that we have the Einstein tensor we need the energy-momentum tensor to which it couples. For this we consider a particle falling into the black hole. The energy momentum tensor of an in-particle at solid angle Ω' moving in the y-direction along constant $x = 0$ and constant angle Ω' is given by [41] to be

$$T_{\mu\nu} = \frac{-p_x}{R^2}\delta(x)\delta(\Omega, \Omega')\delta^x_\mu\delta^x_\nu. \tag{2.47}$$

Here the $\delta(\Omega, \Omega')$ is like the normal delta function, being singular when $\Omega = \Omega'$ and zero everywhere else, with unity surface area. A subtlety is the $1/R^2$ in front, which we write because we define the $\delta(\Omega, \Omega')$ to be on the unit sphere, taking the radius factor out (one can check this is correct through dimensional analysis or by inserting the delta function in the defining integral). The equation above follows directly from the canonical form $T_{\mu\nu} \sim E\delta^{(3)}(\vec{x} - \vec{x}')u_\mu u_\nu$ for a particle with energy E located at $\vec{x}'$, where we identify $E = -p_x$ since the in-particles move in the y direction (such that $p^y = -p_x$).

Combining the above two equations we can write down the proper Einstein field equations. The (xx)-equation results in

$$(1 - \Delta_\Omega)F(\Omega) = 8\pi G p_x \delta(\Omega, \Omega'). \tag{2.48}$$

In Appendix B.1 we provide a different derivation of the above equation using the linearized gravity equation of motion. The fact that the same equation follows identically from linear gravity is our main motivation for the linearized gravity construction in Section 4. Following the procedure of Subsection 2.3 we straightforwardly find the Shapiro shift to be given by

$$\delta y(\Omega) = 8\pi G p_x f(\Omega, \Omega') \qquad \boxed{(1 - \Delta_\Omega)f(\Omega, \Omega') = \delta(\Omega, \Omega')}. \tag{2.49}$$

This is the Shapiro shift on the black hole horizon for a particle moving in the y-direction with momentum p_x, at solid angle Ω'. This principle is illustrated in Figure 6.

The generalization of the Shapiro delay to many in-particles is straightforwardly given by summation

$$\delta y(\Omega) = 8\pi G \int \mathrm{d}\Omega' f(\Omega, \Omega')p_x(\Omega') \qquad p_x(\Omega') = \sum_i p^i_x \delta(\Omega', \Omega_i). \tag{2.50}$$

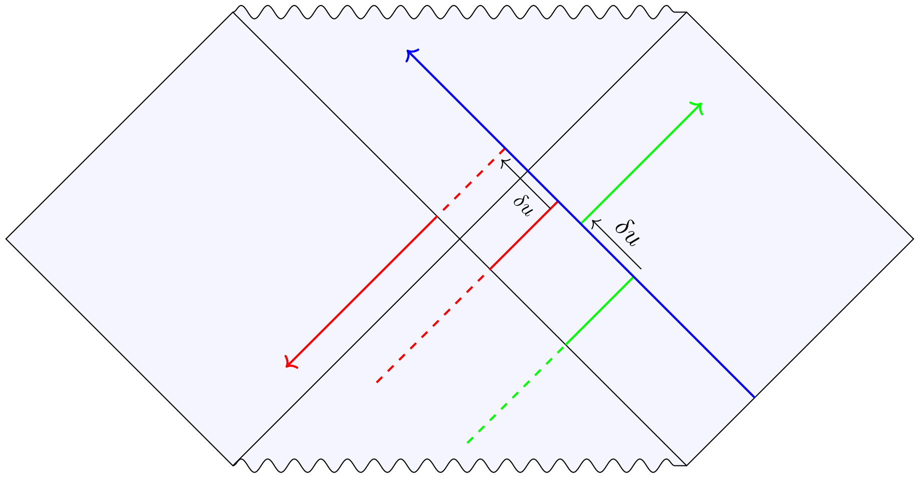


Figure 6: The cut-and-paste procedure in Schwarzschild spacetime along a high energy geodesic (blue). A particle far from the horizon (green) obtains the shift similar to Figure 4, but now placed on the Schwarzschild background. A particle close to the horizon (red) is instead transported to region II, a principle which will be motivated when we look at the scattering matrix.

The quantity $p_x(\Omega')$ is now a momentum distribution describing all in-particles, indicated by summation over all particles i. Now suppose that we start with a Cauchy surface with soft particles only, and that these soft particles will define the entire Hilbert space. In that case we can replace δy by y itself, as was done in [24]:

$$\boxed{y^{\text{out}}(\Omega) = 8\pi G \int \mathrm{d}\Omega' f(\Omega,\Omega') p_x^{\text{in}}(\Omega').} \tag{2.51}$$

The reason for the subscripts can be deduced from the fact that ingoing particles have $p_y = 0$, and outgoing particles $x = 0$. Remark that y_{out} gives the *average* position of the out-particles. Thus we find that the average position of out-particles is given by the momentum distribution of the in-particles, under the important condition that the soft in-particles define all Hilbert space.

It is crucial that we can replace δy by y itself, for which we apply the following physical reasoning: Suppose that all particles on the Cauchy surface are soft, then the only gravitational field is that of the black hole so the Shapiro delay in Equation 2.49 is valid. Now the only interaction between all these particles is the Shapiro delay, so this is their only method of exchanging information. Because of this all information in the outgoing positions can only be coming from the momenta of the incoming particles as the Shapiro delay dictates. This is what we mean by the soft particles defining the entire Hilbert space. Now to preserve this information the outgoing positions y_{out} are completely determined by the Shapiro delay, i.e. we can replace δy by y. Thus we implicitly impose information by making this (physically reasonable) ansatz.

2.4.1 The firewall transformation

We still have the problem of the hard-hard interactions as discussed in Subsection 2.2. This subsection is based entirely on [24]. Remember that both in- and out-particles are hard when they are very close to the horizon. This means that at some late time the in-particles becomes hard $p_x \gg M_{\text{Pl}}$. From Equation 2.51 we can then read off that y is very large as well, so that the corresponding out-particles are far away from the horizon. But this means that these out-particles are soft. Thus all hard in-particles automatically correspond to soft out-particles and Equation 2.51 holds both ways, so all hard out-particles correspond to soft in-particles. Clearly there is no hard-hard interaction to consider, and the Shapiro delay then describes all relevant gravitational interaction.

Now what about the hard-soft interaction? We now know that the interaction is the Shapiro delay, but nevertheless there are still hard particles present on the Hilbert space, so the background spacetime is in principle not Schwarzschild. To solve this problem we must now deduce from Equation 2.51 that the in- and out-particles are actually quantum clones of eachother. This means that we can choose to look at only one of them, either the in-particles or the out-particles, as quantum mechanically they share the same information. We can also choose to look at some of the in-particles and some of the out-particles. Now the latter is the best choice. As shown above to all hard in-particles $p_x \gg M_{\text{Pl}}$ only soft out-particles $y \gg \ell_{\text{Pl}}$ are connected, and vise versa. Thus whenever we encounter a hard in-particle, we choose only to look at the soft out-particles quantum clones, and whenever the out-particle is hard we choose to look at the soft in-particle clones. We call this procedure the *firewall transformation*. Effectively all hard particles are quantum clones of all soft particles. By the firewall transformation we then look only at the soft particle clones, and ignore the hard particles. All information is still embedded in the Hilbert space, that now only consists of soft particles.

To summarize the firewall transformation has three important results:

- There are no more hard particles to consider. The hard-hard interaction was already irrelevant for black hole scattering. However we have now used the hard-soft interaction (the Shapiro delay) to remove the hard particles. This shows that all hard particles on the black hole horizon are unphysical, and can properly be transformed into corresponding soft particles.

- The presence of hard quanta was what generated the firewall paradox in the introduction in the first place. Thus by removing the hard particles using the Shapiro delay, we have removed the firewall paradox. The gravitational interaction provides an elegant solution to the firewall paradox by using general relativity only.

- The removal of the hard particles ensures that Hilbert space only consists of soft particles. This allows the use of the eternal black hole. If there were hard particles present, it would not be allowed to use the eternal black hole as background field since the hard particles' field would also need to be included. Now that the Hilbert space consists only of soft particles, we can safely use the vacuum solution of the Schwarzschild spacetime, and apply quantum mechanics on that spacetime.

We see that the firewall transformation solves a multitude of problems by using the cut-and-paste procedure of the gravitational interaction. Now that we have removed all conceptual problems, we can safely apply the Shapiro delay in Equation 2.51 to find the scattering matrix between the quantum mechanical eigenstates. There is a subtle problem that is not solved by the firewall transformation, which is the presence of asymptotic hard particles, for example send in by an external observer. These particles cannot be removed by the firewall transformation as they are hard over their entire geodesic. Since hard particles do not form during any physically relevant process we can ignore their possibility, and assume from now on that asymptotic hard particles do not exist in our calculation.

3 The semi-classical scattering matrix

We now seek to build the quantum states and the scattering matrix using the Schwarzschild metric in Equation 2.7 and more importantly the Shapiro delay relating the in- and out-particles in Equation 2.51. The latter is especially important for constricting the behaviour of the operators. This section provides a complete recalculation of [24, 38] in our own language. Subsections 3.1 and 3.2 describe the calculation, the resulting scattering matrix is given in Subsection 3.3. For the scattering matrix and its consequences we refer the reader to Subsections 3.3-3.5. Finally Subsection 3.6 provides a new addition to the calculation of 't Hooft, where we investigate the effects of the scattering matrix in the high energy limit, finding resemblance with literature.

We seek to describe the quantum mechanical states, and thus the wavefunctions, on the horizon. The in-particle wavefunctions are described only by x_{in} since $y = 0$ on the future event horizon and similarly all out-particles are described by y_{out} since $x = 0$ on the past event horizon. Now recall that in quantum mechanics position and momentum are related through $p_a = -i\partial_a$, i.e. the Fourier transform. From this we conclude that the wavefunctions of the infalling particles are described by x_{in} in position space and p_x^{in} in momentum space. Similarly the wavefunctions of the outgoing particles are described by y_{out} and p_y^{out} only. This also connects to the geodesics derived in Subsection 2.2 in the following way: The in-particles move over constant x, so their momentum $p^x = -p_y = 0$, and similar for the out-particle upon letting $x \leftrightarrow y$. Thus this section focuses on finding how $\psi_{\text{out}}(y_{\text{out}})$ depends on $\psi_{\text{in}}(x_{\text{in}})$. Remember that in the following x, p_x always refers to the in-particles, while y, p_y always refers to the out-particles. We will be writing this down at first, but at a later stage this is implicit.

3.1 Spherical harmonics and Quantum mechanics

First we expand everything in spherical harmonics, in order to better solve Equation 2.51. We then have the following structure

$$x^a(\Omega) = \sum_{\ell,m} x^a_{\ell m} Y_\ell^m(\Omega) \qquad p_a(\Omega) = \sum_{\ell,m} p_a^{\ell m} Y_\ell^m(\Omega). \tag{3.1}$$

Here the sums are to be interpreted as $\sum_{\ell,m} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}$. The function $f(\Omega, \Omega')$ in Equation 2.49 can easily be determined in spherical harmonics. First we remark the following identities (for the delta sum see [44]) and also expand $f(\Omega, \Omega')$ in a similar fashion

$$\delta(\Omega, \Omega') = \sum_\ell Y_\ell^m(\Omega) Y_\ell^m(\Omega') \tag{3.2}$$

$$\Delta_\Omega Y_\ell^m(\Omega) = -\ell(\ell+1) Y_\ell^m(\Omega) \tag{3.3}$$

$$f(\Omega, \Omega') = \sum_\ell f_{\ell m} Y_\ell^m(\Omega) Y_\ell^m(\Omega'). \tag{3.4}$$

Equation 2.51 is now directly solved by applying the orthogonality of the Spherical harmonics. This results in the following Shapiro delay in terms of (ℓ, m) modes

$$f_{\ell m} = \frac{1}{\ell^2 + \ell + 1}, \tag{3.5}$$

$$x^a_{\ell m} = -\frac{8\pi G}{\ell^2 + \ell + 1} \epsilon^{ab} p_b^{\ell m}. \tag{3.6}$$

Notice that in Equation 3.6 we already included the Shapiro delay for both x, y, as they each have their own effect. The quantity ϵ^{ab} is the antisymmetric tensor as given in Equation A.24, defined by $\epsilon^{xy} = +1$. The orthogonality convention we use in this paper is

$$\int \mathrm{d}\Omega Y_\ell^m(\Omega) Y_{\ell'}^{m'}(\Omega) = \delta_{\ell,\ell'} \delta_{m,m'}. \tag{3.7}$$

Now for the semi-classical interaction in this section we only have linear equations, implying that all (ℓ, m) modes decouple due to the spherical harmonics orthogonality. This means we can drop the ℓm subscripts and treat the position y_{out} and momenta p_x^{in} for each (ℓ, m) mode separately. However these last two coordinates are directly related through Equation 3.6 so effectively we only have one coordinate to deal with. Naturally there are also the coordinate x_{in} and p_y^{out}, however these are also related by the Fourier transformation or Shapiro delay. This implies that we can treat all particles on the black hole using one-dimensional quantum mechanics. This principle is illustrated in Figure 7.

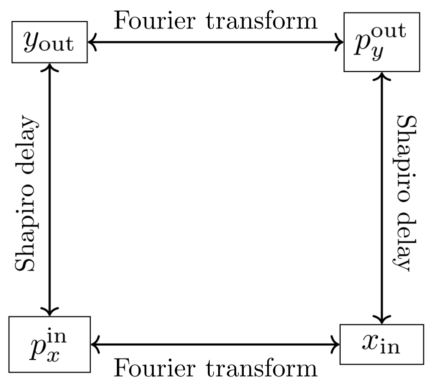


Figure 7: All relations between the different coordinates, illustrating how they are reduced to a single degree of freedom.

The fact that we have only one coordinate to analyze also follows from the commutators. The canonical lightcone commutators are retrieved from [45]. The commutator algebra of the system is then given by

$$\left[y_{\text{out}}, p_y^{\text{out}}\right] = i, \tag{3.8}$$

$$\left[x_{\text{in}}, p_x^{\text{in}}\right] = i, \tag{3.9}$$

$$\left[x_{\text{in}}, y_{\text{out}}\right] = \frac{8\pi i G}{\ell^2 + \ell + 1}. \tag{3.10}$$

Here the last commutator follows from inserting the Shapiro delay Equation 3.6 into the canonical commutators in Equations 3.8-3.9. We now clearly see that as expected from the usual quantum mechanical commutators the positions and momenta cannot be treated independently but we also have something new: the different position coordinates also cannot be treated independently! This is a direct effect of the Shapiro delay. Thus we see indeed that because of the Fourier transform between x^a and p_a and the Shapiro delay linking x^a and $\epsilon^{ab} p_b$ we only have one independent coordinate to deal with. Remark again that in the following whenever x^a or p_a is used, it refers to the relevant particle, i.e. x, p_x for outgoing and y, p_y for ingoing. Additionally remember that all x^a, p_a actually have ℓm subscripts, but since we treat each wave shell independently we omit these for clarity.

For completeness and its importance we stress the general Shapiro delay equation once again

$$\boxed{x^a = -\frac{8\pi G}{\ell^2 + \ell + 1} \epsilon^{ab} p_a.} \tag{3.11}$$

3.2 Particle wavefunctions

The aim is now to find how the wavefunctions of the out-particles depend on the wave functions of the in-particles, from the view of a local observer. To do so we need to apply the Shapiro delay, but we also need to carefully look at the coordinate differences for local and global observers. Let us first look at the wavefunction in global coordinates. In general the wavefunctions are described by acting with a position or momentum eigenstate on the quantum state $|\psi\rangle$ such that

$$\langle x|\psi^{\text{in}}\rangle = \psi^{\text{in}}(x) \qquad \langle p_x|\psi^{\text{in}}\rangle = \hat{\psi}^{\text{in}}(p_x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\ e^{-ixp_x}\psi^{\text{in}}(x), \tag{3.12}$$

$$\langle y|\psi^{\text{out}}\rangle = \psi^{\text{out}}(y) \qquad \langle p_y|\psi^{\text{out}}\rangle = \hat{\psi}^{\text{out}}(p_y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}y\ e^{-iyp_y}\psi^{\text{out}}(y). \tag{3.13}$$

Let us now recall the new Shapiro commutator

$$\begin{aligned}[x_{\text{in}}, y_{\text{out}}] &= i\alpha \\ \alpha &\equiv \frac{8\pi G}{\ell^2+\ell+1}.\end{aligned} \tag{3.14}$$

This commutator has the exact same shape as the Fourier commutator between position and momentum, with a proportionality constant α. We can now easily conclude that the in-wavefunction in position space is related to the out-wavefunction in position space by a Fourier transformation:

$$\psi_{\text{out}}(y_{\text{out}}) = \tfrac{1}{\sqrt{2\pi\alpha}}\int_{-\infty}^{\infty} \mathrm{d}y_{\text{out}}\ e^{-i\frac{x_{\text{in}}y_{\text{out}}}{\alpha}}\psi^{\text{in}}(x^{\text{in}}). \tag{3.15}$$

This relation follows straightforwardly from inserting the general relativistic Shapiro delay into the commutators. The steps taken are those of a usual semi-classical quantum theory on curved background; we first take a fixed background, and perform quantum mechanics on that background. Now the background is given by the Shapiro delay. The relation in Equation 3.15 is remarkable, by inserting the classical particle interaction, we straightforwardly find a trivial relation between the wavefunctions of the in- and out-particles, namely a Fourier transformation. Since a Fourier transformation is by definition unitary, we have already found a unitary relation between the in- and out-particles by inserting the gravitation interaction. By expanding into eigenstates we can turn Equation 3.15 into a quantum mechanical scattering matrix.

Calculating the scattering matrix

The first thing we need to remember is that Equation 3.15 is only valid for the global observer. A local observer will see a different relation. This is also how our result will differ from usual quantum mechanics, we need to take the option of different observers into account. A global observer sees the flat coordinates x^a, p_a in which we formulated the metric Equation 2.7. This is because these coordinates are asymptotically flat. A global observer then also sees the the particles asymptote exponentially to the horizon as can be seen from Subsection 2.2. Thus the local observers sees exponential coordinates relative to the global observer. These exponential coordinates are given by the Tortoise coordinates in Subsection 2.2 and we now define them by

$$\boxed{x = R\sigma_x e^{\rho_x}} \qquad \boxed{p_x = R^{-1}\sigma_{p_x}e^{\rho_{p_x}}.} \tag{3.16}$$

The relations for y are identical upon replacing $x \to y$. Here $\sigma_x = \pm 1$ and $\sigma_{p_x} = \pm 1$ since x, p_x cover $\mathbb{R}$ while the exponentials only cover $\mathbb{R}^+$, thus the signs ensure the coordinate transformation is bijective. The R factors are inserted to preserve mass dimensions. Naturally ρ_x and ρ_{p_x} also have $\mathbb{R}$ as domain. This coordinate transformation will alter the form of the fourier transform as it was defined in global coordinates. We remind the reader that $\sigma_x = +1$ corresponds to region I while $\sigma_x = -1$ corresponds to region II as defined in Subsection 2.1.
In these new Tortoise coordinates the Shapiro delay is given by

$$\rho_y^{\text{out}} - \rho_{p_x}^{\text{in}} = \log(\alpha/R^2) \qquad \text{and } \sigma_y = \sigma_{p_x}. \tag{3.17}$$

Now performing the coordinate transformation in Equation 3.16 also alters the wavefunctions. To see this we have look at the normalization condition $\langle\psi|\psi\rangle = 1$, and in order to keep this condition canonical the wavefunctions have to transform. In terms of integrals normalization states

$$\int_{-\infty}^{\infty} \mathrm{d}x |\psi(x)|^2 = 1 = R \sum_{\sigma_x=\pm 1} \int_{-\infty}^{\infty} \mathrm{d}\rho_x \ e^{\rho_x} |\psi(\sigma_x e^{\rho_x})|^2 \tag{3.18}$$

where the second integral follows from the coordinate transformation. Equating both integrals and doing similar for the momentum case we deduce that[3] the wavefunctions must transform under the coordinate transformation as

$$\boxed{\tilde{\psi}_{\sigma_x}(\rho_x) = R^{-1/2} e^{\frac{1}{2}\rho_x} \psi\left(\sigma_x e^{\rho_x}\right)} \qquad \boxed{\tilde{\psi}_{\sigma_{p_x}}(\rho_{p_x}) = R^{1/2} e^{\frac{1}{2}\rho_{p_x}} \psi\left(\sigma_{p_x} e^{\rho_{p_x}}\right).} \tag{3.20}$$

We now want to find the scattering relation for these new wavefunctions in Tortoise coordinates. In global coordinates this relation was the Fourier transformation. Thus to find out the relation between the new wavefunctions, we need to transform said Fourier transformation into Tortoise coordinates. Recall the relationship between the in- and out-wavefunctions in Equation 3.15 to be

$$\psi^{\text{out}}(y) = \frac{1}{\sqrt{2\pi\alpha}} \int_{\infty}^{\infty} \mathrm{d}x \ e^{-ixy/\alpha} \psi^{\text{in}}(x). \tag{3.21}$$

We now apply the coordinate transformations given in Equations 3.16 and 3.20 resulting in

$$\tilde{\psi}^{\text{out}}_{\sigma_y}(\rho_y) = \frac{1}{\sqrt{2\pi\alpha\mu^2}} \sum_{\sigma_x=\pm 1} \int_{-\infty}^{\infty} \mathrm{d}\rho_x e^{\frac{1}{2}(\rho_y+\rho_x)} \mathrm{Exp}\left(-i\frac{1}{\mu^2\alpha}\sigma_y\sigma_x e^{\rho_y+\rho_x}\right) \tilde{\psi}^{\text{in}}_{\sigma_x}(\rho_x). \tag{3.22}$$

Here we used the inverse Schwarzschild radius $\mu = 1/R$. We now insert the Shapiro delay in Equation 3.17 into the above as a coordinate transformation. In particular we insert $\rho_y = \rho_{p_y} + \log(\alpha\mu^2)$ into the integral in Equation 3.22 whenever ρ_y appears. Due to this transformation the factors $(\alpha\mu^2)^{-1}$ in front of the integral and in the exponent in Equation 3.22 cancel. The resulting Fourier transformation in local coordinates is now given by

$$\tilde{\psi}^{\text{out}}_{\sigma_y}(\rho_y) = \sum_{\sigma_x=\pm 1} \int_{-\infty}^{\infty} \mathrm{d}\rho_x K_{\sigma_y\sigma_x}(\rho_{p_y} + \rho_x) \tilde{\psi}^{\text{in}}_{\sigma_x}(\rho_x), \qquad \boxed{K_\sigma(\rho) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\rho} e^{-i\sigma e^{\rho}}}. \tag{3.23}$$

Here $K_\sigma(\rho)$ is the new Fourier kernel. Remark that the Fourier kernel is invariant under $\rho_{p_y} \to \rho_{p_y} - a$ and $\rho_x \to \rho_x + a$, giving a symmetry under $x \to xe^a$ and $p_x \to p_x e^{-a}$. This symmetry is exactly our time translation invariance of the Schwarzschild spacetime expressed in Kruskal-Szekeres coordinates, as was shown in Section 3. The fact that we have time translation invariance allows us to write the wavefunctions in terms of their Hamiltonian eigenstates. Normally we would need the time-dependent Schrödinger equation, but in the special case of a stationary system we can instead use only the stationary Hamiltonian energy eigenstates.

To find the eigenstates we start from the Hamiltonian near the horizon. This is given by the dilaton operator

$$H = \frac{1}{2}\left(yp_y + p_y y\right) = -\frac{1}{2}\left(xp_x + p_x x\right). \tag{3.24}$$

[3]This must be the case since only in this exact form the normalization condition is in its canonical form

$$\langle\psi|\psi\rangle = \sum_{\sigma_x=\pm 1} \int_{-\infty}^{\infty} \mathrm{d}\rho_x |\tilde{\psi}_{\sigma_x}(\rho_x)|^2. \tag{3.19}$$

The reason for this Hamiltonian is derived in [28]. In terms of the tortoise coordinates this becomes[4]

$$\tilde{H} = i\frac{\partial}{\partial \rho_x} = i\frac{\partial}{\partial \rho_{p_x}} = -i\frac{\partial}{\partial \rho_y} = -i\frac{\partial}{\partial \rho_{p_y}}. \tag{3.26}$$

Observe that this Hamiltonian simply describes massless particles that are freely moving with no potential in Tortoise coordinates as was described in Subsection 2.2. To see this notice that the free energy of a massless particle is given by $E = p = -i\partial_x$ so that in Tortoise coordinates, in which there is no potential $V = 0$, we have $H = E = -i\partial_\rho$. This shows again that particles are indeed freefalling in Tortoise coordinates.

To solve Equation 3.23 we need to find the position space eigenfunctions in term of x and y . As such we need to solve the following eigenvalue equation

$$\tilde{H}\tilde{\psi}^{\text{out}}_{\sigma_y}(\rho_y) = \kappa\tilde{\psi}^{\text{out}}_{\sigma_y}(\rho_y) \qquad \tilde{H}\tilde{\psi}^{\text{in}}_{\sigma_x}(\rho_x) = \kappa\tilde{\psi}^{\text{in}}_{\sigma_x}(\rho_x) \tag{3.27}$$

where κ is the eigenvalue of H, corresponding to the energy. The solution of these equations are trivial and given by

$$\boxed{\tilde{\psi}^{\text{out}}_{\sigma_y}(\rho_y) = \psi^{\text{out}}_{\sigma_y}(\kappa)e^{i\kappa\rho_y}} \qquad \boxed{\tilde{\psi}^{\text{in}}_{\sigma_x}(\rho_x) = \psi^{\text{in}}_{\sigma_x}(\kappa)e^{-i\kappa\rho_x}} \tag{3.28}$$

with the $\psi^{\text{out}}_{\sigma_y}(\kappa)$ and $\psi^{\text{in}}_{\sigma_x}(\kappa)$ being coefficients corresponding to the specific eigenstates. Since the particles are freefalling the eigenstates are simply plane waves. Inserting these eigenstates into the Fourier transformation in Equation 3.23 we find

$$\psi^{\text{out}}_{\sigma_y}(\kappa)e^{i\kappa\rho_y} = \sum_{\sigma_x=\pm1}\int_{-\infty}^{\infty} \mathrm{d}\rho_x K_{\sigma_y\sigma_x}(\rho_{p_y} + \rho_x)\psi^{\text{in}}_{\sigma_x}(\kappa)e^{-i\kappa\rho_x}. \tag{3.29}$$

Notice that we inserted the same κ on both sides of the equation. This is an effect of the orthogonality of the Hamiltonian eigenstates and energy conservation. We remark that for this to be allowed the decoupling between the (ℓ, m) modes is crucial, as it allows us to use the orthogonality and energy conservation on each mode separately without mixing.

We now apply the Shapiro delay Equation 3.17 to the left-hand-side of Equation 3.29, replacing the ρ_y by ρ_{p_y}. The resulting equation becomes

$$\psi^{\text{out}}_{\sigma_y}(\kappa)e^{i\kappa\rho_{p_y}}e^{i\kappa\log(\alpha\mu^2)} = \sum_{\sigma_x=\pm1}\int_{-\infty}^{\infty} \mathrm{d}\rho_x K_{\sigma_y\sigma_x}(\rho_{p_y} + \rho_x)\psi^{\text{in}}_{\sigma_x}(\kappa)e^{-i\kappa\rho_x}. \tag{3.30}$$

Finally in the above the eigenstates $\psi(\kappa)$ do not depend on ρ anymore, so we can separate the integral. The result becomes

$$\psi^{\text{out}}_{\sigma_y}(\kappa) = \sum_{\sigma_x=\pm1} F_{\sigma_y\sigma_x}(\kappa)e^{-i\kappa\log(\alpha\mu^2)}\psi^{\text{in}}_{\sigma_x}(\kappa) \qquad \boxed{F_\sigma(\kappa) = \int_{-\infty}^{\infty} \mathrm{d}\rho K_\sigma(\rho)e^{-i\kappa\rho}.} \tag{3.31}$$

This equation give the scattering matrix for our black hole quantum states. The eigenstates of the outgoing particles are related to those of the ingoing particles simply by some function $F_\sigma(\rho)$ and a phase. We will further discuss the scattering matrix in Subsection 3.3. We now firstly discuss the shape of the $F_\sigma(\kappa)$ function. Remark that by construction Equation 3.31 must describe unitary evolution, since in Equation 3.15 we

[4]We shortly explain how this is obtained. First insert the coordinate transformation as it is in Equation 3.24 leading to $H = -i\partial_{\rho_y} - \frac{1}{2}i$ for the y case. The wavefunction however also transforms (Equation 3.20) so $H\psi(y) = \kappa\psi(y)$ transforms to

$$\left(-i\partial_{\rho_y} - \tfrac{1}{2}i\right)e^{-\frac{1}{2}\rho_y}\tilde{\psi}_{\sigma_y}(\rho_y) = \kappa e^{-\frac{1}{2}\rho_y}\tilde{\psi}_{\sigma_y}(\rho_y). \tag{3.25}$$

Applying the product rule $\partial_{\rho_y}e^{-\frac{1}{2}\rho_y} = e^{-\frac{1}{2}\rho_y}\left(-\frac{1}{2} + \partial_{\rho_y}\right)$ then finally leads to the correct form of the Hamiltonian.

also had a unitary relation.

The function $F_\sigma(\kappa)$ is given exactly by

$$F_\sigma(\kappa) = \frac{1}{\sqrt{2\pi}}\Gamma\left(\tfrac{1}{2} - i\kappa\right) e^{-\frac{i\sigma\pi}{4}} e^{-\frac{\pi}{2}\sigma\kappa} \tag{3.32}$$

with $\Gamma(x)$ the Euler gamma function. Additionally $F_\sigma(\kappa)$ has two important properties:

$$|F_+(\kappa)|^2 + |F_-(\kappa)|^2 = 1, \tag{3.33}$$

$$F_+(\kappa)F_-^*(\kappa) = -F_-(\kappa)F_+^*(\kappa). \tag{3.34}$$

This finally leads us to the fact that the matrix

$$\begin{pmatrix} F_+(\kappa) & F_-(\kappa) \\ F_-(\kappa) & F_+(\kappa) \end{pmatrix} \tag{3.35}$$

is unitary such that Equation 3.31 indeed provides a unitary evolution law.

3.3 The scattering matrix

From Subsection 3.2 we have finally obtained the scattering matrix for the black hole particles in local coordinates, that was also obtained in [28, 38]. We write this neatly in matrix-vector form as

$$\boxed{\begin{pmatrix} \psi_+^{\text{out}}(\kappa) \\ \psi_-^{\text{out}}(\kappa) \end{pmatrix} = e^{-i\kappa\log\left(\frac{8\pi G\mu^2}{\ell^2+\ell+1}\right)} \begin{pmatrix} F_+(\kappa) & F_-(\kappa) \\ F_-(\kappa) & F_+(\kappa) \end{pmatrix} \begin{pmatrix} \psi_+^{\text{in}}(\kappa) \\ \psi_-^{\text{in}}(\kappa) \end{pmatrix}.} \tag{3.36}$$

Equation 3.36 describes the contribution to black hole scattering at a single (ℓ, m) wave. The total black hole scattering is now found by simply summing over all these. We remind the reader that the ψ's implicitely carry (ℓ, m) indices. The above scattering matrix is in fact unitary, which has important implications for the information paradox. Under the assumption that momentum is the only information the scattering matrix solves the information paradox. This assumption was made implicitly in Section 2.4 when assuming the soft particles defined all Hilbert space. Naturally there are different types of information as well (e.g. charge, parity) but these can likely be included by including the shock waves of additional gauge fields. Thus it seems like the information paradox can be solved by including the interactions on the horizon, which then exchange the information. In particular gravity exchanges momentum information. Clearly the gravitational interaction is crucial to include in any black hole treatment, and cannot be neglected lightly.

Let us now look at the effect of the $+, -$ signs. These signs tell us whether we are region I or II; $\psi_+(\kappa)$ describes particles in region I while $\psi_-(\kappa)$ describes particles in region II. From the black hole scattering Equation 3.36 we see however that $\psi_+^{\text{out}}(\kappa)$ and $\psi_-^{\text{in}}(\kappa)$ are in causal contact through the off-diagonal terms. This means that region I and II are in causal contact.

Recall the discussion on the different possibilities of regions I and II in Subsection 2.1. Whereas the most common thought is that regions I and II are disconnected, this is now forbidden by Equation 3.36. Thus we need to look at the case where regions I and II are not disconnected, but related through some mapping. Not doing so is in principle allowed, but removes unitarity, so to preserve unitarity of the scattering matrix we must impose an identification between regions I and II. The mapping u, v, θ, ϕ in region I to $-u, -v, \theta, \phi$ in region II lead to cusp singularities, so we need some other mapping. However when we apply the mapping twice we go I→II→I, so we need the square of the relation to be unity. This automatically leads to one possibility without cusp singularities, which is the antipodal identification. The mathematical relation becomes

$$(u, v, \theta, \phi)_{\text{I}} \Longleftrightarrow (-u, -v, \pi - \theta, \phi + \pi)_{\text{II}}.$$

The fact that regions I and II are antipodally identified, an important result of [38], follows naturally from the quantum mechanical scattering matrix. We will look further into this in Subsection 3.4.

We also need to discuss the effect of κ. Observe that $F_{\pm}(\kappa)$ depends exponentially on κ as in Equation 3.32. Thus for very large κ the diagonals vanish, so that in this limit region I *only* contacts region II and vice versa. We can understand this from the Shapiro delay, where for very large energy particles are *always* carried from region I to II by the instantaneous kick. In other words the kick is always large enough to carry a particle across, where it does not matter what the particles location is. This effect will be further analyzed in Subsection 3.6. For very low κ instead all matrix terms are equal in size and there is a perfect mixing where the out-functions in region I or II depend in both cases equally on the in-functions in region I and in region II. Thus in fact the out-functions in region I and II also turn out to be equal[5]. Since the scattering matrix is invertible, the in-functions must also be equal. We see that in the low κ limit there is no distinguishment between region I and II. These two regions are now identical.

3.4 The antipodal identification

Region I and II are now fully entangled by the antipodal mapping. This antipodal mapping also applies to the event horizon. Two antipodal points on the two-sphere that describes the event horizon at $x = 0 = y$ are now actually identical; they describe the same point in spacetime. This turns the event horizon mathematically into a projective two-sphere, which is a spherical shell where every point is equal to its antipodal dual. We remark that the scattering law Equation 3.36 is invariant under the antipodal mapping. However all x^a and p_a change sign under this antipodal map so only odd ℓ would contribute in the spherical harmonics expansion. An important remark must be made here, which is the fact that the antipodal identification we consider is classical, so all points can be treated exactly as points without quantum uncertainty. An application of a quantum mechanical antipodal identification is given in [27]. From their calculation the even ℓ do not drop out when taking quantum mechanics into account, so the spacetime uncertainty is important to consider.
The antipodal identification has many consequences. The first is on the information paradox. No literature takes the projective two-sphere into account. Treating regions I and II as independent is not allowed, instead they are maximally entangled. As a result it is also not allowed to average over region II as if it is out of our reach.

Another important result is how we need to look at the black hole itself. When an incoming particle hits the black hole horizon, it comes out on the other side. In Subsection 3.6 we shall see that particles can actually come out of the black hole again in finite time, so this scenario is important to consider. The fact that particles move straight through to the other side implies that the spacetime patch enclosed by the event horizon could perhaps be nonexistent: a *spacetime vacuole*. The entire black hole is in fact the horizon only, and regions III and IV do not exist. Then all information would only be embedded on the black hole horizon, which is indeed confirmed by the Bekenstein-Hawking entropy and corresponds to the holographic principle. Finally the emergence of the antipodal identification leads to an important concept on general coordinate transformations.

General coordinate transformations

What we observed in the previous subsection is that region II and region I are connected by the antipodal identification. When we look back to Subsection 2.1 on how region II was obtained we see that it came from the maximal extension of the coordinates x, y. Originally the coordinates were bound by $y > 0, -\infty < x < \infty$, leaving only the regions I and III. x was defined as $x = \log X$ with $X \in \mathcal{R}$. This restricts $x > 0$ as was found. The coordinates are then maximall extended to yield the full spacetime. This means that we allow ranges $x < 0$ that fall outside of the coordinate transformation: they would correspond to $X + i\pi$ which is unphysical, or transforming $x = -\log X$ as well. The latter implies we copy a region, namely the fact that the region $X \in \mathcal{R}$ corresponds both to $x > 0$ and $x < 0$ equally. We now know that the maximally extended regions are in fact to be treated as quantum copies of regions I and III. This leads us to the following thought.

[5] To see this we can simply write in the $\kappa = 0$ case that

$$\begin{pmatrix} \psi_+^{\text{out}}(0) \\ \psi_-^{\text{out}}(0) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi_+^{\text{in}}(0) \\ \psi_-^{\text{in}}(0) \end{pmatrix}. \tag{3.37}$$

The factor $1/\sqrt{2}$ comes from putting $|F_-(0)| = |F_+(0)|$ in Equation 3.33. We clearly see now that $\psi_-^{\text{out}}(0) = \psi_+^{\text{out}}(0)$ as desired.

A note on bijective transformations

When applying general coordinate transformations to any consistent theory of quantum gravity, the transformation must be bijective, differentiable and without cusp singularities. Coordinate transformations that do are not bijective open or close parts of spacetime clashing with the coordinate independence of gravity. If in the coordinate transformation an extension is made (for example to preserve some symmetry) then the new region in spacetime that emerges must be looked at critically. This new region might not be a casually disconnected region, but instead be related by some identification, like the antipodal identification for Schwarzschild. Naturally an ambiguity arises as to which coordinate system describes all of spacetime properly. The correct application of a quantum gravity theory should make that clear, similar to the treatment in this section.

The claims above are currently not at all proven, however we can say with good certainty that caution must be taken when performing non-bijective transformations, and the resulting spacetime must be interpreted critically. The statement above is currently our view on how such a situation has to be resolved given a proper theory of quantum gravity. We should note that the current reasoning is for classical transformations where points are exact points, and taking quantum uncertainty into account as in [27] is a crucial addition when attempting to support our thought on bijectiveness above. To end this note on bijectiveness we state two arguments, the first supports the fact that ambiguities arive in non-bijective transformations, and the second is an emergence of the antipodal mapping in Minkowski space also due to what can be interpreted as a non-bijective transformation.

In general we can understand the subtle problem with non-bijective transformations in the following way: Say that we are in the process of compactifying a coordinate chart to construct a Penrose diagram. When we have the compactified coordinates with finite range we may in principle maximally extend these just like we did in Schwarzschild. Thus we can always extend spacetime to include new regions. Classically a spacetime is extended until it is geodesically complete, however geodesics are a classical notion and not well-defined in a theory of quantum gravity. The important question is then: Which coordinate system describes all of spacetime without any regions being overcounted? This is the earlier mentioned ambiguity, when discarding classical geodesics, how do we know when to end our spacetime extensions. Given that in this section the antipodal identification emerged, we expect that a proper application of quantum gravity limits the amount of extensions.

Finally we will shortly look at a well known case where the antipodal mapping due to an "extension" applies. For Minkowski spacetime there are usually two possible Penrose diagrams: a triangle and a square. Both diagrams are shown in Figures 8 and 9. These are obtained from the lightcone coordinates $u = t-r, v = t+r$. The condition $r > 0$ results in $0 < v-u < \infty$ giving the triangle, but extending this condition to $-\infty < v-u < \infty$ results in the square. It is known that the points (two-spheres) on the left side of the square diagram are connected to those on the right side by the antipodal mapping [46]. For the triangle when a particle hits the $r = 0$ line it is reflected, however for the square when the particle hits $r = 0$ it keeps moving but its angular coordinates are antipodally transformed. This is obvious, a particle moving through the origin of course suddenly moves in the antipodal direction even though it moves in a straight line. Now we look back at the Schwarzschild metric. We also allowed the coordinates x, y to be negative, which in the end resulted in the left side (region II) being the antipodal mapping of the right side (region I). Thus our case is perfectly analogous to the extension of the Minkowski diagram.

3.5 Ongoing research

In this subsection we discuss three of the ongoing research topics regarding the results of the S-matrix in Equation 3.36. We investigated all three in order to obtain an idea on what to investigate further for this master's thesis.

A CPT transformation

From Subsection 3.4 we see that traversing the horizon entails a shift in the parity P. By definition of the coordinates the time also reverses generating a T shift as explained in Section 2. This naturally results in the local energy swapping sign as well. Since PT is no symmetry of nature, we expect the charge to change sign as well. To confirm this we need a complex field extension to the theory. Combining all of the above there is a CPT shift when crossing the horizon, i.e. a CPT transformation between region I and II. We can now interpret region II as being merely a CPT-transformed quantum copy of region I. Thus these regions are maximally entangled, and we need to be very careful not to overcount the states. In a more general sense we can interpret II and IV as CPT quantum copies of I and III.

The CPT-transformation we refer to here is classical, so all points in region I are exactly related to their CPT-shifted points in region II. This is not allowed in a theory of quantum gravity. However in quantum gravity there are likely no global symmetries at all, so that a global CPT symmetry does not even exist. The solution would be to make CPT symmetry local under some gauge mechanism or spontaneously broken theory. Clearly the CPT transformation as we defined above has problems when including quantum uncertainty, but these problems need to be overcome to find the proper description in quantum gravity. As this is beyond the scope of this thesis we will for now focus on the classical CPT-transformation. A beginning at including quantum uncertainty is made in [27].

The CPT shift also results in particles having negative energy in region II. The interpretation of this is still open, but we believe that we need to take the notion of creation and annihilation of particles into account. In usual flat space quantum field theory a negative energy particle implies particle annihilation. In this sense the correct interpretation might be that creating a particle in region I would remove an antipodally located particle in region II. This possibility intersects to the notion of interchanging creation and annihilation particles as was suggested by Witten in [38]. If it is true that a negative energy particle implies annihilation, then the interchange of creation and annihilation and the change in energy sign on the horizon are identical statements. We expect the particle annihilation to then happen on the black hole quantum state, which is naturally an exited state of spacetime. This would remove energy from the black hole analogously to Hawking radiation. In this sense an incoming particle (particle creation) might be directly related to outgoing Hawking radiation. Since interpretations like these need a quantum field theory, we will attempt to provide such a treatment in Section 5. We deem the importance of creation and annihilation operators crucial for the black hole horizon, so we choose to work in this paper on trying to add those to the theory.

Finally we comment on the suggestion that region II and IV are quantum copies of region I and III. We can mathematically formulate this as the Hilbert spaces being connected in the following manner:

$$\mathcal{H}_I = \text{CPT}\mathcal{H}_{II}^\dagger. \tag{3.38}$$

The hermitian conjugate has been inserted due to a comment of E. Witten in [38], as this is also necessary to maintain the canonical commutators. The Hilbert spaces $\mathcal{H}_I$ and $\mathcal{H}_{II}$ are to be interpreted as the space of quantum states as observed by the global observer at $r \to \infty$ in respectively region I and II. In [27] a quantum mechanical version of Equation 3.38 is derived, including the quantum uncertainty of spacetime.
Equation 3.38 drastically changes our interpretation of the Schwarzschild diagram, and also alters the exact formulation of the information paradox. By treating the regions dual in this manner we introduce the important question of what to expect of geodesics. Subsection 3.6 sheds some light on this, and further discussion will be applied in Section 6.

Connection to BMS group

The BMS group stands for the Bondi–Metzner–Sachs group, which is a new infinite group of symmetries for Minkowski spacetime in addition to the already present Poincaré group. These symmetries were derived in [47,48]. They are called the supertranslations and are greatly summarized in Strominger's lecture notes [46] where the consequences are also discussed. Notice that currently the BMS group is extended to also include superrotations which were not originally found together with the supertranslation [46].

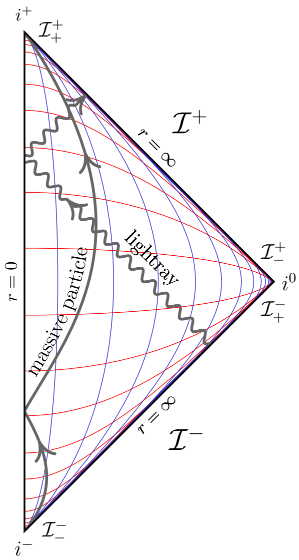


Figure 8: The normal triangle diagram, particles are antipodally reflected on $r = 0$ (image from [46]).

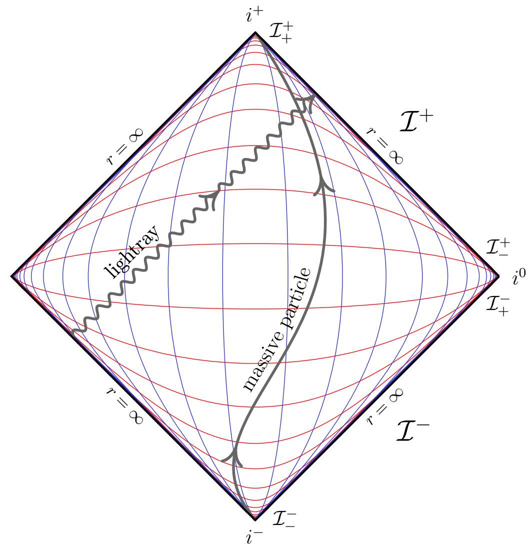


Figure 9: The square diagram in extended coordinates, particles keep moving but the left side is an antipodal copy of the right side (image from [46]).

The most important result is when applying supertranslation symmetry on the Cauchy surfaces of $\mathcal{I}^-$ and $\mathcal{I}^+$ (see Figure 9). The BMS group dictates that there are an infinite amount of conserved charges both on $\mathcal{I}^-$ and $\mathcal{I}^+$, due to the infinite amount of symmetries and Noether's theorem. It would be nice to relate these charges in the far past and far future, but the time evolution of $\mathcal{I}^-$ through the Einstein Field equations is not sufficient. However, we can apply the BMS group to restrict the possibilities, by forbidding relations that break the BMS symmetries. The result is that there is a condition between the gravitational fields connected to the charges on $\mathcal{I}^-$ and $\mathcal{I}^+$, namely

$$h(\mathcal{I}^+, \theta, \phi) = h(\mathcal{I}^-, \pi - \theta, \phi + \theta). \tag{3.39}$$

Here h is to be understood as the metric only symbolically, since we are actually looking at the set C_{zz}, m_B, N_z which are the degrees of freedom that parametrize an arbitrary asymptotically flat spacetime as described by Bondi, van den Burg, Metzner and Sachs in [47, 48].

The relation in Equation 3.39 we have seen before: It is the antipodal matching. Thus what we see is that the BMS symmetry group results in an antipodal identification between future and null infinity of asymptotically flat spacetimes. Remember that $\mathcal{I}^-$ and $\mathcal{I}^+$ are both located at $r \to \infty$. On the other hand for the Schwarzschild case we saw in this section that there is an antipodal identification between region I and II when looking close to the horizon, so at finite $r = R$. Nevertheless looking at the Penrose diagram in Figure 1 we see that even though the horizon is at finite $r = R$ it still corresponds to a past null boundary and future null boundary of region I or II. Thus there is a striking similarity between the two descriptions, in both cases there is an antipodal identification between different null boundaries of spacetime.

This similarity was first noted by [49, 50] and it was suggested that a link might be possible between the BMS group and the semi-classical scattering matrix of 't Hooft. In particular as suggested by Hawking we could view all the Shapiro delays of all in-particles of Section 2 as supertranslations on the horizon [51]. These shifts then imprint themselves onto the generators of the outgoing particles, transferring the information in a chaotic manner. This was further investigated by Hawking, Perry and Strominger in [52], who investigated linearized superrotations on the Schwarzschild spacetime. They then manage to translate this into classical black hole hair, thus preserving the information (contradicting the no-hair theorem). In general the presence of black hole hair due to BMS is discussed in Strominger's lecture notes [46]. Progress has also been made on supertranslations on the black hole horizon [53], but we will not go into this. A lot of research has been done on this already, and we decided that adding the creation and annihilation operators as discussed above is currently more important. For more information regarding the connection between BMS and the S-matrix, we refer the reader to the master's thesis of W. Vleeshouwers [54], which has been consulted for some of the literature above. For more information regarding the BMS group itself we refer the reader to Strominger's lecture notes [46].

Connection to String theory

Finally we will look at a possible connection to string theory. As pointed out by 't Hooft in [23] it appears like the semi-classical scattering in this section can also be described as a two-dimensional string theory world sheet. This is pointed out in good detail in [55]. First we note the action of the horizon perturbations

$$S = -\tfrac{1}{8\pi G}\int \mathrm{d}\Omega\bigg(\tfrac{1}{2}\partial_A x^a \cdot p^A x_a + 8\pi G p^a_{\mathrm{ext}} x_a\bigg) \tag{3.40}$$

where x^a, p^a_{ext} are functions of θ, ϕ. When regarding x^a as the canonical operators, which is done in both string theory and in quantum mechanics in this section, the action above fully describes the Shapiro delay on the horizon. The external momenta p^a_{ext} act as a source on the position x^a which generate the Shapiro delay's $\delta y = 8\pi G/(\ell^2+\ell+1)$. Notice that the summation and integration is over A, so over the Riemannian two-sphere. At the same time the Veneziano action in string theory is given by

$$S = \int \mathrm{d}^2\sigma\bigg(-\tfrac{T}{2}\partial_\sigma x^a \cdot p^\sigma x_a + 8\pi G p^a_{\mathrm{ext}} x_a\bigg)$$

where $\sigma^1 \equiv \sigma, \sigma^2 \equiv \tau$ denote the world-sheet coordinates. The two actions above are clearly almost equal when inserting $T = 1/(8\pi G)$, with the only difference the Riemannian θ, ϕ or Lorentzian σ, τ coordinates.

There appears to be a connection between high-energy scattering using the Shapiro delay on the black hole horizon and string theory. This connection was further strengthened by de Haro in [56], who showed that beside the metric also the Kalb-Ramon field is present in the effective action when including transverse effects. The dynamics of the gravitational back reaction is closely described by string theory, up to the subtle difference of one of the metrics being Lorentzian. This difference was noted in [23, 24] and implies that the string tension T is in fact imaginary. The connection is nevertheless existent, and allows for a new interpretation of the black hole horizon. We could interpret the horizon as a string theory world sheet, and the in- and out-going particles as closed strings falling onto that world sheet. Then the scattering amplitude can be calculated by treating the closed strings as vertex insertions as described in [45]. This combines string theory with black hole scattering in an elegant manner. Due to the string constant T being imaginary research into this connection has stranded, and one of our possible research ideas was to try and remove the factor of i. The initial though was that including the antipodal identification into the worldsheet might solve this factor. However, after further discussion we concluded that in principle that should not be allowed since there is no change in behaviour locally, and the laws of physics should be locally consistent. As such we currently have no proposed solution to the factor i problem, and instead put our focus into the field operator method.

3.6 High-energy limit

In this subsection we provide a new explicit calculation to find the effect of a high energy particle entering the black hole in region I. By taking the limit of high energy κ and a localized classical wavefunction, we can find some explicit results of the scattering matrix that apply in the classical limit. This is known as the WKB approximation. We start by stating the wavefunction of the incoming particle

$$\psi^{\mathrm{in}}_+(\rho_x) = \mathcal{N}\int_{-\infty}^{\infty} \mathrm{d}\kappa\; g(\kappa) e^{-i\kappa\rho_x} \tag{3.41}$$

$$\psi^{\mathrm{in}}_- = 0. \tag{3.42}$$

The above defines a wave packet as incoming wavefunction defined by the energy measure $g(\kappa)$ which is required to be absolutely integrable, a real normalization constant $\mathcal{N}$. We have chosen the particle to enter in region I only, while leaving region II untouched. We remark that the wavefunction as defined in Equation 3.41 is not an eigenfunction of the Hamiltonian, but rather a superposition of eigenfunctions. This is characterized by the integral; we integrate over all possible eigenvalues with some measure $g(\kappa)$ and the eigenstates, which we know from Subsection 3.2 to be

$$\psi(\kappa) = e^{-i\kappa\rho_x}. \tag{3.43}$$

Notice that contrary to a usual WKB calculation the above holds up to all orders in $\hbar$, the high energy approximation will become important only when approximating the scattering matrix.

To find the out-wavefunction we directly insert Equation 3.41 in the Shapiro delay (in Equation 3.22 to be precise) to find

$$\psi_\sigma^{\text{out}}(\rho_y) = \mathcal{N}\int_{-\infty}^{\infty}\mathrm{d}\rho_x\int_{-\infty}^{\infty}\mathrm{d}\kappa\; e^{\frac{1}{2}(\rho_x+\rho_{p_y})}e^{-i\sigma_y e^{\rho_x+\rho_{p_y}}}g(\kappa)e^{-i\kappa\rho_x}. \tag{3.44}$$

Remember that ρ_{p_y} and ρ_y are related by Equation 3.17. We can now switch the order of integration and switch to $\xi = \rho_x + \rho_{p_y}$ leading to

$$\psi_\sigma^{\text{out}}(\rho_y) = \mathcal{N}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_{-\infty}^{\infty}\mathrm{d}\xi\; e^{\frac{1}{2}\xi}e^{-i\sigma_y e^{\xi}}g(\kappa)e^{-ix}e^{i\kappa\rho_p}. \tag{3.45}$$

The integral over ξ can be performed separately, and we can recognize this to be the $F_\sigma(\kappa)$ function defined in Equation 3.31. The resulting out-wavefunction can compactly be written as

$$\psi_\sigma^{\text{out}}(\rho_y) = \mathcal{N}\int_{-\infty}^{\infty}\mathrm{d}\kappa\; g(\kappa)F_{\sigma_y}(\kappa)e^{i\kappa(\rho_y-\log(\alpha\mu^2))}. \tag{3.46}$$

Here we also replaced ρ_{p_y} by ρ_y as in Equation 3.17. The normalization constant $\mathcal{N}$ is now different, but it is still real. In the following we will keep it written as $\mathcal{N}$ with the only requirement that it is real. The result in Equation 3.46 in fact states exactly our scattering matrix obtained in Subsection 3.3, only now written down for the wavefunctions instead of eigenstates.

The integral is impossible to solve without the proper knowledge of $g(\kappa)$. Let us take $g(\kappa)$ as a very sharply peaked Gaussian around some high κ_0 to find a normalized wavepacket of a classical particle with energy κ_0. A classical particle would be a delta function (exactly defined) so a Gaussian distribution with small standard deviation is a good approximation. Since κ_0 is very high the only contributions to Equation 3.46 come from the integrand at high values of κ so we can make a high κ approximation for $F_\sigma(\kappa)$. We remark that it is not possible to immediately insert $\kappa = \kappa_0$ as a delta-function would, because any high κ oscillations cannot be assumed constant. These oscillations are crucial as they alter the Fourier Kernel resulting in translations in time. We will now approximate $F_\sigma(\kappa)$ for high κ.

The high-energy limit of the Fourier kernel:
Recall the definition of $F_\sigma(\kappa)$ given by

$$F_\sigma(\kappa) = \frac{1}{\sqrt{2\pi}}\Gamma\left(\tfrac{1}{2} - i\kappa\right)e^{-\frac{i\sigma\pi}{4}}e^{-\frac{\pi}{2}\sigma\kappa}. \tag{3.47}$$

In principle this form cannot be simplified, but we can make very good approximations for large κ. The entire function is already in elementary form except for the $\Gamma\left(\frac{1}{2} - i\kappa\right)$. For large κ we can attempt to use Stirlings approximation. Let us look at the argument $z \equiv \frac{1}{2} - i\kappa$ more closely. For large κ we can approximate

$$|z| = \sqrt{\tfrac{1}{4} + \kappa^2} \approx \kappa \tag{3.48}$$

$$\text{Arg}(z) = \arctan\left(\frac{-\kappa}{\frac{1}{2}}\right) \approx -\frac{\pi}{2} + \frac{1}{2\kappa}. \tag{3.49}$$

We see that $|z| \to \infty$ for large κ and $|\mathrm{Arg}(z)| < \frac{\pi}{2}$. This allows us to use the asymptotic expansion for $\Gamma(z)$ in A&S [57] to write

$$\Gamma(\tfrac{1}{2} - i\kappa) = \sqrt{2\pi}(\tfrac{1}{2} - i\kappa)^{-i\kappa} e^{i\kappa - \frac{1}{2}} \left(1 + O(\kappa^{-1})\right). \tag{3.50}$$

We seek to simplify the first term, as the rest are already in an elementary form. To do so we write

$$(\tfrac{1}{2} - i\kappa)^{-i\kappa} = e^{-i\kappa \log(\kappa)} e^{\frac{1}{2}} e^{-\frac{\pi}{2}\kappa}. \tag{3.51}$$

This is a straightforward result from rewriting into a logarithm and Taylor expanding said logarithm. Using this we can write for the F-function in the large κ-limit that

$$\boxed{F_\sigma(\kappa) = e^{-i\kappa(\log(\kappa)-1)} e^{-\frac{i\sigma\pi}{4}} e^{-(1+\sigma)\frac{\pi}{2}\kappa}.} \tag{3.52}$$

Now we immediately see that for $\sigma = 1$ the norm $|F_\sigma(\kappa)|$ vanishes for large κ while for $\sigma = -1$ the norm is exactly unity (due to the large κ approximation). We conclude that we can write the large κ limit as

$$F_+(\kappa) = 0 \tag{3.53}$$
$$F_-(\kappa) = e^{-i\kappa \log(\kappa/e) + i\frac{\pi}{4}}. \tag{3.54}$$

The high-energy scattering matrix

By inserting this limit into Equation 3.46 we can finally write down the full scattering law for large κ:

$$\boxed{\psi_-^{\mathrm{out}}(\rho_y) = \mathcal{N} e^{i\pi/4} \int_{-\infty}^{\infty} \mathrm{d}\kappa\; g(\kappa) e^{i\kappa\left(\rho_y - \log[\alpha\kappa\mu^2/e]\right)}.} \tag{3.55}$$

The integral iseasily solved, as this is a Fourier transform again (up to the $\log \kappa$ in the exponent which can be treated as approximately constant for high κ). We remark nevertheless that Equation 3.55 holds for any $g(\kappa)$ as long as its support is solely at high κ. This means that all high-energy out-functions are related to the in-functions by a time translation $\log(\alpha\kappa\mu^2/e)$ only.

We now take $g(\kappa)$ explicitly as a very sharply peaked Gaussian around κ_0, such that the only contribution from the integral comes exactly at $\kappa = \kappa_0$ (as a delta-function would do). This leads us to the final result of our semi-classical scattering being given by

$$\psi_-^{\mathrm{out}}(\rho_y) = \mathcal{N} e^{i\pi/4} e^{-\frac{\gamma(\rho_y)^2}{2\delta^2}} e^{i\kappa_0 \gamma(\rho_y)} \tag{3.56}$$
$$\psi_+^{\mathrm{out}}(\rho) = 0 \tag{3.57}$$
$$\gamma(\rho_y) \equiv \rho_y - \log\left(\frac{1}{e}\frac{8\pi G\mu^2\kappa_0}{\ell^2+\ell+1}\right) + O(\kappa^{-1}) \tag{3.58}$$

with δ small compared to κ_0. This is the wavefunction that results from the incoming wavefunction in Equation 3.41.
We observe the following features:
The $\log(\alpha\kappa\mu^2/e)$ part illustrates that the out-particle in fact exits the black hole again in finite time. The time that the particle takes to go from "inside the black hole to back out again" is given by this $\log(\alpha\kappa\mu^2/e)$. This is largely different from the believe that nothing can ever escape a black hole in finite time, and has some consequences to be discussed next. As illustrated in Figure 10 a particle entering the black hole in region I at a certain time, results in a particle leaving the black hole in region II at a later time. Effectively the particle "bounces" against a wall in region III. However as remarked in Subsection 3.5 we can treat region II as being a quantum copy. As such we can also interpret the paricle leaving in region II as a CPT shifted particle leaving in region I.

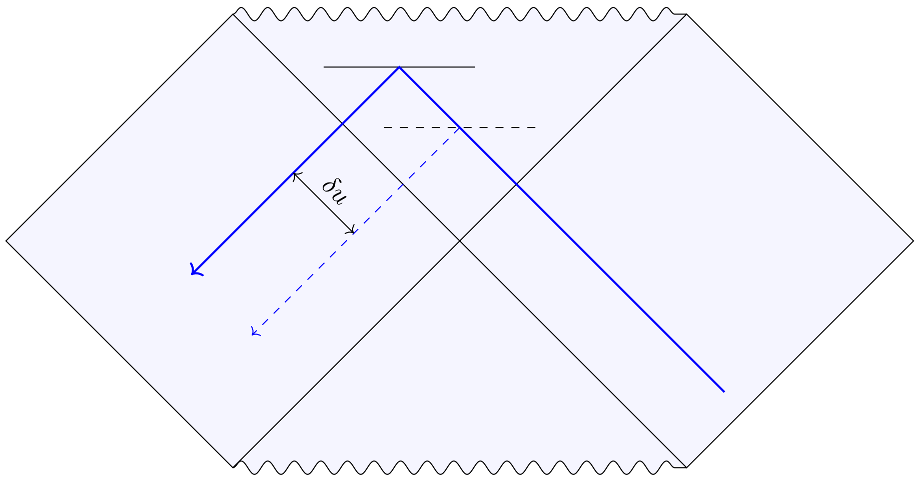


Figure 10: The "trajectory" of the large-κ particle. The dashed line corresponds to no time delay, while the blue line is the predicted trajectory.

The consequences of a particle being able to exit a black hole in a finite time $\log(\alpha\kappa\mu^2/e)$ after entering can be quite severe. Of course there seem to be causal problems, but the causality of Schwarzschild has become a lot more complicated due to the antipodal identification. The causality is something to be researched. A larger problem is the fact that it is now possible to make a closed time loop. Some particle may start in region I, enter the black hole and thus region III, exit again after finite time in region II, and then through region IV again reach the *same spacetime point* in region I. Of course this could not happen in a finite time black hole, and only in the eternal Schwarzschild case. Nevertheless this is an odd result. Possibilities on how finite time black holes might solve this and why this effect might be natural are further discussed in Section 6.

The time delay

We have found that the out-particles experience a specific time delay $\log(\alpha\kappa\mu^2/e)$. Likely this time delay connects to the scrambling time as defined in [58] and Wigner's time delay. Notice from Subsection 2.2 that we can identify $\rho_y \sim \tau = \frac{t}{2R}$ since we are on $r = R$, so that from Equation 3.55 we can read of the time delay in Schwarzschild time t as

$$\delta t = 2R\delta\rho = 2R \log\left(\frac{1}{e}\frac{8\pi G\mu^2\kappa}{\ell^2+\ell+1}\right), \tag{3.59}$$

$$= -8t_{\mathrm{Pl}}\frac{M_{\mathrm{BH}}}{M_{\mathrm{Pl}}}\log\left(\beta\frac{M_{\mathrm{BH}}}{M_{\mathrm{Pl}}}\right). \tag{3.60}$$

Remark that this is the time delay for individual Hamiltonian eigenstates with energy κ. Here

$$\beta^2 \equiv \frac{e}{2\pi\kappa}(\ell^2+\ell+1) \tag{3.61}$$

is a black hole mass independent dimensionless quantity and we recognized the Planck mass $M_{\mathrm{Pl}} = \sqrt{G}^{-1}$ and Planck time $t_{\mathrm{Pl}} = \sqrt{G}$. We now see that in Planck units there is a time delay given by

$$\delta t = -8M_{\mathrm{BH}}\log(\beta M_{\mathrm{BH}}) \sim M_{\mathrm{BH}}\log M_{\mathrm{BH}} \tag{3.62}$$

as long as κ is not too large compared to M_{BH}. This time delay is comparable to the scrambling time as it was derived by Susskind in [58]. The scrambling time is a measure of the time it takes for the black hole horizon to re-stabilize after being disturbed by an infalling particle. But this process is also what happens in the previously calculated high-energy limit. We throw in a particle, with information characterized by its kernel $g(\kappa)$, and after a time of order $M_{\mathrm{BH}}\log M_{\mathrm{BH}}$ a new particle is emitted outward in region II, that caries the same information $g(\kappa)$. Thus while the horizon was deformed by the $g(\kappa)$ of the first particle, it was re-stabilized by the second particle as desired. We should remark that we calculated only one individual (ℓ, m) components so we only know what happens to in-falling (ℓ, m) shells. However, due to orthogonality between the shells the physical picture is easily extended toward particles characterized by a collection $g_{\ell m}(\kappa_{\ell m})$ maintaining the correct scrambling time.

Another way to study life-times in quantum scattering processes is defined by the time-delay matrix [28]:

$$\tau_{ab} = \mathrm{Re}\left[i \sum_c S^\dagger_{ac} \frac{\partial S_{cb}}{\partial E}\right] \tag{3.63}$$

with S_{ab} the scattering matrix and E the energy. We can plug in the scattering matrix as given in Equation 3.36 with $E = \kappa$. The derivatives are straightforward, and we can perform many simplifications using the identities listed in Equations 3.32,3.33. This results in

$$\rho_{ab} = \log\left(\frac{1}{R^2}\frac{8\pi G}{\ell^2+\ell+1}\right)\delta_{ab} + \begin{pmatrix} \mathrm{Re}[\psi^{(0)}(\frac{1}{2}-i\kappa)] & \pi \mathrm{sech}(\pi\kappa) \\ \pi \mathrm{sech}(\pi\kappa) & \mathrm{Re}[\psi^{(0)}(\frac{1}{2}-i\kappa)] \end{pmatrix} \tag{3.64}$$

with $\psi^{(0)}(z)$ the di-gamma function. Here we identified the time delay as being in ρ, since κ is the eigenvalue of the Hamiltonian that generates time translations in ρ. We have now also found the time delay for low energies, which include off-diagonal terms, i.e. mixing between regions I and II. Notice however that for large κ the off-diagonal terms cancel by exponential decay so that the time delay between region I and II drop out. In the large-κ calculation this feature was found by the similar exponential decay of $F_+(\kappa)$.

The Wigner time delay is now defined as the trace of the time-delay matrix [28], so that for arbitrary κ it is given by

$$\delta\rho = 2\log\left(\frac{1}{R^2}\frac{8\pi G}{\ell^2+\ell+1}\right) + 2\mathrm{Re}[\psi^{(0)}(\tfrac{1}{2}-i\kappa)]. \tag{3.65}$$

For a large black hole $M_{\mathrm{BH}} \gg M_{\mathrm{Pl}}$ and $\kappa \ll M_{\mathrm{BH}}$ we can assume R to be large, and at leading order the Wigner time delay in Schwarzschild time is then given by

$$t = 2R\rho_{aa} = -8R\log(R) \sim M_{\mathrm{BH}} \log M_{\mathrm{BH}}. \tag{3.66}$$

Thus we see that in the large black hole limit we find the same order of magnitude for the Scrambling time as for the Wigner time delay. More generally for large black hole masses all time delays are of order of magnitude of $M_{\mathrm{BH}} \log M_{\mathrm{BH}}$, so in this theory this is the characteristic time scale of black hole scattering.

4 Perturbative Quantum Gravity

In this section we shall perturb gravity around the Schwarzschild spacetime in order to describe particle scattering on the black hole using graviton exchange. This exchange will then be summed over in Section 5. We start by perturbing gravity around the Schwarzschild background, and then expanding the linearized field into spherical harmonics, analogous to Section 3. After the spherical harmonics a natural decomposition between the lightcone and two-sphere follows, from where we will isolate the lightcone action and integrate the two-sphere away. Finally we can Weyl transform $A(r)$ away to reduce the metric to a flat spacetime. Thus in Subsections 4.1-4.3 we reduce the background spacetime for the graviton from a four-dimensional curved spacetime to a two-dimensional flat spacetime. Finally on the latter we determine the propagator of the graviton, and formulate it in a fashion relevant to Section 5. For the resulting propagators and its discussions we refer the reader to Subsection 4.5.1 and thereafter. All prior subsections focus on the calculation.

4.1 Perturbed Gravity action

The scattering matrix in Section 3.3 has been obtained from adding the Shapiro delay on the black hole horizon to the particle quantum states. This Shapiro delay satisfies linear differential equations (on the horizon) so it should be completely described by a perturbation of gravity up to linear order around the Schwarzschild metric evaluated on the black hole horizon. In this section we will derive the proper linear gravity theory for the Schwarzschild horizon. Mathematically we perturb

$$g_{\mu\nu} = g^s_{\mu\nu} + h_{\mu\nu} \qquad g^{\mu\nu} = g_s^{\mu\nu} - h^{\mu\nu} \tag{4.1}$$

with $h_{\mu\nu}$ small such that all equations of motion are linear. This means we only keep quadratic terms when they lead to a propagator and for interactions we only keep linear terms. The field $h^{\mu\nu}$ we now call the graviton field. For $h^{\mu\nu}(x)$ we assume that in Kruskal-Szekeres coordinates x, y the graviton vanishes at infinity $h^{\mu\nu}(x) \to 0, x \to \infty$, i.e. vanishing boundary conditions. The background metric $g^s_{\mu\nu}$ is the Schwarschild metric.

We will start from a general matter action and the Einstein-Hilbert action and perturb the metric. The Einstein-Hilbert action will lead to the graviton propagator while the matter action will lead to the graviton-matter vertex. Of course the matter action will also result in matter propagators and possible vertices, which we shall look at explicitly in the next section. Similarly we will be able to read off the vertex terms. Remark that perturbing the metric into a graviton mode like this leads to divergences. Specifically on flat space the theory becomes nonrenormalizable [1, 59]. We shall see that within the Eikonal approximation in Section 5 this renormalizability will not be a problem, so that we can write down the Feynman diagrams which should lead to similar results as in Section 3. We should then also be able to easily extend to multiple matter fields and hopefully more general spacetimes.

The linear perturbation around the Schwarzschild metric has already been done before by [60–62] in the classical case, by [63] for the more general rotating black hole also in the classical case, and finally by [64] for the semi-classical quantum mechanical treatment. However all of these treatments work in a very specific language. The focus point is in finding the description of the gravitational perturbations and in particular gravitational waves, whereas our focus is on the gravitational interaction between scalar particles, and the graviton itself is merely an intermediate step. Additionally many authors make the choice to set the spherical harmonic eigenvalue $m = 0$, which we choose not to so as to keep the result more general. When looking solely at the perturbations one can set $m = 0$ by a choice of axes, however we are now also adding scalar particles, so this results in a more complex system. As such these papers do not provide results fitting to our calculation, nevertheless they do provide useful insights. In this section we re-derive the linear perturbation in a language more suitable for the calculation in Section 5, with [60] as starting point.

Let us now formulate the actions of our theory. We start with the two canonical actions:

$$\text{Matter action } S_M \quad :: \quad T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{4.2}$$

$$S_{EH} \quad = \quad \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R. \tag{4.3}$$

The matter action S_M is completely general, for simplicity we will assume only a massless scalar field in the next section, analoguous to the scalar wavefunctions in Section 3. To insert the graviton we perform a functional Taylor expansion on these quantities to lowest order in h:

$$S_M = S_M\big|_{g=g_s} - \int \mathrm{d}^4x \frac{\delta S_m}{\delta g^{\mu\nu}(x)}\bigg|_{g=g_s} h^{\mu\nu}(x), \tag{4.4}$$

$$S_{EH} = S_{EH}\big|_{g=g_s} - \int \mathrm{d}^4x \frac{\delta S_{EH}}{\delta g^{\mu\nu}(x)}\bigg|_{g=g_s} h^{\mu\nu}(x) + \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4x' \frac{\delta^2 S_{EH}}{\delta g^{\mu\nu}(x)\delta g^{\rho\sigma}(x')}\bigg|_{g=g_s} h^{\mu\nu}(x)h^{\rho\sigma}(x') \tag{4.5}$$

Now we can use the definition of the stress-energy tensor to simplify the first term, and also immediately see that the matter action is up to lowest order linear in h. For the Einstein-Hilbert action we remark that $g = g_s$ corresponds to the Schwarzschild solution, which has vanishing Ricci scalar. Similarly the Schwarzschild solution satisfies the equation of motion, which means that the variation to the metric $\frac{\delta S_{EH}}{\delta g^s_{\mu\nu}} = 0$ vanishes. Thus due to the fact that the Schwarzschild spacetime is a vacuum solution the zeroth order and first order term fall away.

The quadratic part in the Einstein-Hilbert action remains and we can write both actions (derived in Appendix B.2) as

$$S_M = S_M^{h=0} + \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}h^{\mu\nu}T_{\mu\nu}, \tag{4.6}$$

$$S_{EH} = -\frac{1}{4\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(h^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}h\right)R_{\mu\nu}. \tag{4.7}$$

Here the Ricci tensor is defined by

$$R_{\mu\nu} = \tfrac{1}{2}g^{\rho\sigma}\big(\nabla_\rho\nabla_\mu h_{\nu\sigma} + \nabla_\rho\nabla_\nu h_{\mu\sigma} - \nabla_\rho\nabla_\sigma h_{\mu\nu} - \nabla_\mu\nabla_\nu h_{\rho\sigma}\big). \tag{4.8}$$

and we defined $\kappa = 8\pi G$. From this point we redefined the Schwarzschild metric to be the actual metric, and the graviton a new field on this curved spacetime. Additionally we redefine the graviton $h_{\mu\nu} \to \sqrt{\kappa}h_{\mu\nu}$ to actually include the prefactor and turn the constant κ into a vertex coupling constant. Now our perturbative quantum field theory will actually depend on expansions in the vertex, rather than in the propagator, as is preferable for a proper perturbation. Notice that we shall often work with the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ since we can simply identify $(h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}h)R_{\mu\nu} = h^{\mu\nu}G_{\mu\nu}$. The background metric we work with is the Schwarzschild metric in Kruskal-Szekeres coordinates

$$\mathrm{d}s^2 = -2A(x,y)\mathrm{d}x\mathrm{d}y + r(x,y)^2\mathrm{d}\Omega^2. \tag{4.9}$$

Here $A = \frac{R}{r}e^{1-\frac{r}{R}}$ and r is defined by $xy = 2R^2(1-\frac{r}{R})e^{\frac{r}{R}-1}$. Remark however that we will only need this towards the end in Subsection 4.4, in all prior subsections the results hold for arbitrary A as long as the metric is a vacuum solution satisfying the equations of motion.

Gauge symmetry

We should note that gravitational perturbations have a gauge symmetry due to the following. It can quickly be checked that under an infinitesimal diffeomorphism $\tilde{x} = x + \xi(x)$ the metric transforms as

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{4.10}$$

Now by incuding this transformation in the graviton the background metric is constant while for the graviton

$$\tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{4.11}$$

The graviton transforms under infinitesimal diffeomorphisms, while the actual metric is fixed. So long as the general metric is unchanged, the physics stays the same, so we have multiple possible graviton fields describing the same physical situation. Clearly there is a gauge symmetry.

This gauge symmetry needs an important remark, given that the gauge group consists of the same diffeomorphisms as the usual coordinate transformations, such as those from Schwarzschild coordinates to Kruskal-Szekeres coordinates. It seems like by choosing the Kruskal-Szekers coordinates we also fix the gauge, however this is not true. It is at this stage important to carefully discern between infinitesimal coordinate transformations of order $h_{\mu\nu}$, and finite coordinate transformations. The former does not alter the the metric tensor or matter fields, as it would lead to a higher order in the perturbation in $h_{\mu\nu}$ which we neglect. The latter transforms the metric tensor and matter fields, and also the graviton as a usual tensor transformation, this transformation does not fall within the gauge group. Due to this we retain the freedom of finite diffeomorphisms, with the capability of still fixing the gauge using the four degrees of freedom in the infinitesimal ξ_μ.

Now we need to choose the proper ξ to fix the gauge. In the literature the harmonic gauge defined by

$$\nabla_\mu (h^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} h) = 0 \tag{4.12}$$

is ofted used , which leads to a very simple Ricci tensor of the form

$$R_{\mu\nu} = -\tfrac{1}{2} \Box h_{\mu\nu} - R_{\mu\rho\nu\sigma} h^{\rho\sigma}. \tag{4.13}$$

The quantity $g_{\mu\rho} g_{\nu\sigma} \Box + 2R_{\mu\rho\nu\sigma}$ is also known as Lichnerowicz operator. However attempts to use this gauge on the Schwarzschild horizon did not proof to be useful. The Lichnerowicz operator did not simplify the action in any way, and all field couplings persisted. This operator has been analyzed on the Schwarzschild spacetime by [65], but we did not see how we could apply this to the calculation in Section 5. Instead we will use another gauge defined in the next section, called the *Regge-Wheeler* gauge. The reason for this name is since it was first used by Regge and Wheeler in [60]. We shall see that in this gauge we arrive at a clear formulation in a natural manner.

4.2 Spherical Harmonics decomposition

Following [60] we decompose the graviton in its two most general forms under a spherical harmonic expansion. This expansion into spherical tensor harmonics differs from the usual summation over (ℓ, m) we saw in Section 3 because the wavefunctions are scalars, while the graviton is spin-2 tensor. The angular components h_{aA} and h_{AB} now have a nontrivial spherical harmonic expansion which must be taken into account.

The graviton tensor harmonics can be divided into odd parity in ℓ and even parity in ℓ. Notice that in the following we are already in Kruskal-Szekeres coordinates (the result is the same for all Schwarzschild regions). The authors of [60] worked in Schwarzschild coordinates as mentioned before, so we already applied the relevant coordinate transformation listed in Equations 4.14-4.19. For comparison to [60] in the following we defined:

$$h_x^{\pm}(x,y) = -\frac{R}{x}\big(h_0^{\pm} + \big(1-\tfrac{R}{r}\big)\, h_1^{\pm}\big), \tag{4.14}$$

$$h_y^{\pm}(x,y) = -\frac{R}{y}\big(h_0^{\pm} - \big(1-\tfrac{R}{r}\big)\, h_1^{\pm}\big), \tag{4.15}$$

$$h_\Omega(x,y) = h_2, \tag{4.16}$$

$$H_{xx}(x,y) = -\frac{y}{2x}\big(H_0 + H_2 + 2H_1\big), \tag{4.17}$$

$$H_{xy}(x,y) = \frac{1}{2}\big(H_0 - H_2\big), \tag{4.18}$$

$$H_{yy}(x,y) = -\frac{x}{2y}\big(H_0 + H_2 - 2H_1\big). \tag{4.19}$$

Here we inserted the identity $\left(1-\frac{R}{r}\right)=\frac{xyA}{2R^2}$ numerous times to put the transformation in a form that best shows its properties on the fields. Notice that the notation $h_a^\pm$ and H_{ab} will allow for a more general description at a later stage, in fact in terms of arbitrary lightcone coordinates. We can now write down the odd and even graviton components in terms of the graviton modes listed above.

Odd parity graviton:
The odd parity graviton for a certain ℓ, m is generally given as

$$h^-_{\ell m,\mu\nu}=\begin{pmatrix}0 & 0 & -h_x^- \csc\theta\partial_\phi Y_\ell^m & h_x^- \sin\theta\partial_\theta Y_\ell^m \\ 0 & 0 & -h_y^- \csc\theta\partial_\phi Y_\ell^m & h_y^- \sin\theta\partial_\theta Y_\ell^m \\ \dots & \dots & h_\Omega \csc\theta\left(\partial_\theta\partial_\phi - \cot\theta\partial_\phi\right)Y_\ell^m & \frac{1}{2}h_\Omega\left(\csc\theta\partial_\phi^2+\cos\theta\partial_\theta-\sin\theta\partial_\theta^2\right)Y_\ell^m \\ \dots & \dots & \dots & -h_\Omega\sin\theta\left(\partial_\theta\partial_\phi-\cot\theta\partial_\phi\right)Y_\ell^m\end{pmatrix}. \tag{4.20}$$

Here ... denotes that the component is to be obtained from the symmetry of the graviton. Remark that the original quantities as defined in [60] are all invariant under time translations $\tau\to\tau+a$. In Kruskal-Szekeres coordinates time translation becomes $x\to ax, y\to\frac{1}{a}y$, so that we observe that $h_x^\pm, h_y^\pm$ and H_{xx}, H_{yy} are not time translation invariant. This a result of the choice of exponential coordinates, and the fact that H_{ab} transforms as a tensor on the lightcone. The action is still time translation invariant.

Even parity graviton:
For the even parity graviton at certain ℓ, m we write

$$h^+_{\ell m,\mu\nu}=\begin{pmatrix}H_{xx}Y_\ell^m & H_{xy}Y_\ell^m & h_x^+\partial_\theta Y_\ell^m & h_x^+\partial_\phi Y_\ell^m \\ \dots & H_{yy}Y_\ell^m & h_y^+\partial_\theta Y_\ell^m & h_y^+\partial_\phi Y_\ell^m \\ \dots & \dots & r^2(K+G\partial_\theta^2)Y_\ell^m & r^2G(\partial_\theta\partial_\phi-\cot\theta\partial_\phi)Y_\ell^m \\ \dots & \dots & \dots & r^2(K\sin^2\theta+G(\partial_\phi^2+\sin\theta\cos\theta\partial_\theta))Y_\ell^m\end{pmatrix}. \tag{4.21}$$

The two matrices above above are completely general exact decompositions of $h_{\mu\nu}$ into spherical harmonics, before fixing any gauge. The r^2 appearing depends on x, y, according to the definition in Appendix A.1. All functions $h_x^\pm, h_y^\pm, h_\Omega, H_{xx}, H_{xy}, H_{yy}, K, G$ are functions of x, y only with no further constraints. They all depend on ℓ, m, which is omitted for clarity. This accounts to ten degrees of freedom, correct for the graviton before fixing a gauge. For refernce the complete graviton is related to these quantities simply by the usual summation

$$h_{\mu\nu}=\sum_{\ell,m}h^-_{\ell m,\mu\nu}+\sum_{\ell,m}h^+_{\ell m,\mu\nu}. \tag{4.22}$$

The Regge-Wheeler gauge

We now perform a gauge fixing similar to Regge and Wheeler in [60] of the form

$$\xi_a = \zeta_a Y_\ell^m, \tag{4.23}$$
$$\xi_A = -\tfrac{1}{2}r^2G\partial_A Y_\ell^m-\tfrac{1}{2}h_\Omega\epsilon_A{}^B\partial_B Y_\ell^m. \tag{4.24}$$

Here the antisymmetric tensor is defined in Appendix A.1 and

$$\zeta_a=\left(\tfrac{1}{2}r^2\partial_a G-h_a^+\right). \tag{4.25}$$

As argued before this choice leads to no contradiction with the Kruskal-Szekeres coordinates because xi_a, xi_A are of order h. The effective result of this gauge choice is that the field components G, h_a^+, h_Ω are removed entirely, given the correct redefinitions of h_a^-, K, H_{ab}. For reference these redefinitions are given as follows:

$$h_a^- \to h_a-\tfrac{1}{2}r^2\partial_a\left(\tfrac{1}{r^2}h_\Omega\right), \tag{4.26}$$
$$K \to K-2g^{ab}\zeta_a\partial_b\log r, \tag{4.27}$$
$$H_{ab} \to H_{ab}-\nabla_a\zeta_b-\nabla_b\zeta_a. \tag{4.28}$$

We have dropped the minus superscript on the h_x^-, h_y^- since h_x^+, g_y^+ is now zero. One can check that the gauge transformation above including the field redefinitions results in the same general form for the odd and even graviton components, except with G, h_a^+, h_Ω removed. The gauge fixing procedure removed 4 degrees of freedom as desired.

After gauge fixing the graviton components are given as follows:

$$h^-_{\ell m,\mu\nu} = \begin{pmatrix} 0 & 0 & -h_x \csc\theta \partial_\phi Y_\ell^m & h_x \sin\theta\partial_\theta Y_\ell^m \\ 0 & 0 & -h_y \csc\theta \partial_\phi Y_\ell^m & h_y \sin\theta\partial_\theta Y_\ell^m \\ -h_x \csc\theta \partial_\phi Y_\ell^m & -h_y \csc\theta \partial_\phi Y_\ell^m & 0 & 0 \\ h_x \sin\theta\partial_\theta Y_\ell^m & h_y \sin\theta\partial_\theta Y_\ell^m & 0 & 0 \end{pmatrix}, \tag{4.29}$$

$$h^+_{\ell m,\mu\nu} = \begin{pmatrix} H_{xx} Y_\ell^m & H_{xy} Y_\ell^m & 0 & 0 \\ H_{xy} Y_\ell^m & H_{yy} Y_\ell^m & 0 & 0 \\ 0 & 0 & r^2 K Y_\ell^m & 0 \\ 0 & 0 & 0 & r^2 K \sin^2\theta Y_\ell^m. \end{pmatrix}. \tag{4.30}$$

This is the same result as [60] up to the fact that we have not set $m = 0$ for now.

Remark the specific shape of the minus signs in the odd mode $h^-_{\mu\nu}$, which is naturally present when written in index notation

$$h^-_{aA} = -h_a \epsilon_A{}^B \partial_B Y_\ell^m. \tag{4.31}$$

The antisymmetric tensor appears because we are looking at the odd parity perturbations. This antisymmetry contrary to the more symmetric behaviour of $h^+_{\mu\nu}$ will be crucial for their decoupling which we shall discuss more later. Remark that the definition of h^-_{aA} as represented in Equation 4.31 shows the generality of the spherical harmonics decomposition. For any diffeomorphism that acts on $a \to a'$ and $A \to A'$ separately we can simply transform $h_a \to h_{a'}$ and $\epsilon_A{}^B \partial_B Y_\ell^m \to \epsilon_{A'}{}^{B'} \partial_{B'} Y_\ell^m$ accordingly. Then in the new coordinates $h^-_{\mu\nu}$ is still given only by Equation 4.31, so that the spherical harmonics decomposition is completely coordinate independent as long as the two-sphere is kept seperate.

For the even modes we can similarly write

$$h^+_{ab} = H_{ab} Y_\ell^m, \tag{4.32}$$
$$h^+_{AB} = K g_{AB} Y_\ell^m \tag{4.33}$$

so that for the even waves we also have coordinate independence for the decomposition when transforming H_{ab} accordingly (g_{AB} naturally transforms accordingly). Because the decomposition persists as long as we also transform the fields, we now define H_{ab} to be a 2-tensor on the lightcone, h_a a vector on the lightcone, and K a scalar on the lightcone, so that these fields transform under $a \to a'$ but not under $A \to A'$. In a similar fashion all quantities with an A, B we transform under $A \to A'$ but not under $a \to a'$. Now the spherical harmonics decomposition persist under any transformation that keeps the lightcone and two-sphere separate. This shows the generality of the spherical harmonics decomposition as defined above. It holds for arbitrary coordinate systems as long as the two-sphere and lightcone are manifestly separated. This is explicitly visible in our index notation, which is the reason that we will only be using the index notation from this point onward. Additionally we will forbid transformations that mix the lightcone a and the two-sphere A together, as this breaks the spherical harmonics decomposition. Whenever we refer to covariance, we mean covariance on the lightcone and two-sphere separately.

Finally remark that we have now split the remaining six degrees of freedom of the graviton into three parts: a spin-2 tensor H_{ab} in two dimensions, a spin-1 vector h_a in two dimensions and a scalar K in two dimensions. We clearly see a very natural splitting of the degrees of freedom, as usually done in four dimensions, however now we are looking at them in two dimensions due to the spherical harmonics expansion. Additionally remark that these degrees of freedom are now defined covariantly and for arbitrary $A(r)$ in the metric. Thus while we started with a spherical decomposition on the Schwarzschild spacetime in Schwarzschild coordinates, we have now rewritten this into a covariant decomposition for arbitrary spherically symmetric spacetime, providing much more general results. Additionally the fact that the degrees of freedom are defined best in two dimensions, will lead us to define a two-dimensional field theory for the graviton instead. The next subsections will work on simplifying the action to find this two-dimensional field theory for arbitrary $A(r)$ in the metric in a covariant fashion. Toward the end in Subsection 4.4 we will then insert the Schwarzschild metric in Kruskal-Szekeres coordinates, but before that stage every calculation is completely general.

The odd-even decoupling

In this subsection we will investigate the coupling between the odd and even graviton modes defined by the following graviton decomposition:

$$\begin{aligned} h^-_{aA} &= -h_a \epsilon_A{}^B \partial_B Y^m_\ell, \\ h^+_{ab} &= H_{ab} Y^m_\ell, \\ h^+_{AB} &= K g_{AB} Y^m_\ell, \end{aligned} \tag{4.34}$$

and all other terms vanishing. We expect this coupling to fall away due to spherical symmetry and will now prove this. First we split the Lagrangian as

$$\mathcal{L}_c = -\tfrac{1}{4} h^{\mu\nu}_+ G^-_{\mu\nu} - \tfrac{1}{4} h^{\mu\nu}_- G^+_{\mu\nu} \tag{4.35}$$

where we only included all coupling terms. Here $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor which is given by

$$G^\pm_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}\left(\nabla_\rho \nabla_\mu h^\pm_{\nu\sigma} + \nabla_\rho \nabla_\nu h^\pm_{\mu\sigma} - \nabla_\rho \nabla_\sigma h^\pm_{\mu\nu} - \nabla_\mu \nabla_\nu h^\pm_{\rho\sigma}\right) - \tfrac{1}{2} g_{\mu\nu}\left(\nabla^\rho \nabla^\sigma h^\pm_{\rho\sigma} - \Box h^\pm\right). \tag{4.36}$$

In this way the $\pm$ symbols denote the Einstein tensor belonging to $h^\pm$ respectively. Now for all relevant Ricci terms we have calculated explicitly the following equations, which are best derived by exploiting the decomposition structure in Equation 4.34:

$$G^-_{ab} = 0, \tag{4.37}$$

$$G^-_{AB} g^{AB} = 0, \tag{4.38}$$

$$G^+_{aA} = \left(\tfrac{1}{2}\partial^b H_{ab} + \tfrac{1}{A}\partial_a H_{xy} - \tfrac{1}{2A}\left(\partial_a \log(Ar^2)\right) H_{xy} - \frac{1}{2}\partial_a K\right)\partial_A Y^m_\ell \tag{4.39}$$

where we do not need all other components due to $h^-_{ab}, h^-_{AB}, h^+_{aA}$ vanishing and $h^+_{AB} = K g_{AB}$. The second equation $G^-_{AB} g^{AB} = 0$ results from the fact that g^{AB} is even in A, B, while G^-_{AB} contains an ϵ_{AB} which is odd in A, B. We now see immediately that the first term in the coupling Lagrangian in Equation 4.35 vanishes so that we are left with

$$\mathcal{L}_c = -\tfrac{1}{2} h^{aA}_- G^+_{aA}. \tag{4.40}$$

Thus to see what happens to the odd-even coupling we only need to evaluate $h^-_{aA} G^{aA}_+$. At this point we need to be careful with the presence of ℓ, m indices that we omitted before, i.e. we need to explicitly insert that

$$h^-_{aA} = -\sum_{\ell,m} h^{\ell m}_a \epsilon_A{}^B \partial_B Y^m_\ell, \tag{4.41}$$

$$G^+_{aA} = \sum_{\ell,m} F^{\ell m}_a \partial_A Y^m_\ell. \tag{4.42}$$

Here we defined the new quantity

$$F^{\ell m}_a \equiv \tfrac{1}{2}\partial^b H^{\ell m}_{ab} + \tfrac{1}{A}\partial_a H^{\ell m}_{xy} - \tfrac{1}{2A}\left(\partial_a \log(Ar^2)\right) H^{\ell m}_{xy} - \frac{1}{2}\partial_a K^{\ell m}. \tag{4.43}$$

This quantity can be directly read of from Equation 4.39. Inserting the summations in Equations 4.41-4.42 we rewrite the coupling Lagrangian as

$$\mathcal{L}_c = -\tfrac{1}{2}\sum_{\ell,m}\sum_{\ell',m'} h_a^{\ell m} F^a_{\ell' m'} \epsilon^{AB}\partial_B Y_\ell^m \partial_A \bar{Y}_{\ell'}^{m'}. \tag{4.44}$$

We inserted complex conjugation denoted by $\bar{Y}$ because the Lagrangian is required to be real and we take the imaginary representation of the spherical harmonics. The corresponding action is found by integrating 4.44 to give

$$S_c = -\tfrac{1}{2}\sum_{\ell,m}\sum_{\ell',m'}\int \mathrm{d}^2x A h_a^{\ell m} F^a_{\ell' m'} \underbrace{\int \mathrm{d}\phi \mathrm{d}\theta \sqrt{g_2}\epsilon^{AB}\partial_B Y_\ell^m \partial_A \bar{Y}_{\ell'}^{m'}}_{\equiv C_{\ell m;\ell' m'}}. \tag{4.45}$$

Here $\sqrt{g_2}$ is the volume element on S_2 of radius r, i.e. $\sqrt{g_2} = r^2\sin\theta$. Clearly the first part does not vanish generally, but it is also the only part dependent on x^a (the $\sqrt{g_2}$ cancels as we shall see). The second part is only dependent on θ, ϕ and contains only spherical harmonics, functions we know exactly. Remark that in this part we also have contractions over A, B, so that the quantity $C_{\ell m;\ell' m'}$ is invariant of the two-sphere coordinates chosen. What we see is that the action in Equation 4.45 explicitly splits the lightcone part depending on x^a, and the spherical part depending on x^A in a manner that is covariant under $a \to a', A \to A'$ without mixing. Thus as long as we do not apply the earlier forbidden coordinate transformations, we have explicit seperation of variables in the action covariantly. We shall return to this again in the next subsection.

For this subsection we will explicitly calculate the coordinate independent quantity $C_{\ell m;\ell' m'}$ as we can determine it exactly without knowing the graviton fields. Recall that the definition is given in Equation 4.45. The spherical harmonics are defined by

$$Y_\ell^m = \mathcal{N}_{\ell m} e^{im\phi} P_\ell^m(\cos\theta) \tag{4.46}$$

where $\mathcal{N}_{\ell m}$ is a normalization constant and $P_\ell^m(x)$ the associated Legendre polynomial which are real. Recall that the antisymmetric tensor ϵ^{AB} as given in Equation A.23 has a prefactor $1/(r^2\sin\theta)$, so that it cancels the $\sqrt{g_2}$ in Equation 4.45. Inserting this into the definition of $C_{\ell m;\ell' m'}$ gives

$$C_{\ell m;\ell' m'} = -i\mathcal{N}_{\ell m}\mathcal{N}_{\ell' m'}\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi e^{i(m-m')\phi}\big(m' P_{\ell'}^{m'}(\cos\theta)\partial_\theta P_\ell^m(\cos\theta) + m P_\ell^m(\cos\theta)\partial_\theta P_{\ell'}^{m'}(\cos\theta)\big) \tag{4.47}$$

Here the ϵ^{AB} introduced a minus sign, which was then cancelled by the different signs in $\partial_\phi e^{im\phi}$ and $\partial_\phi e^{-im'\phi}$. Now using the orthogonality of the complex exponents

$$\int_0^{2\pi} \mathrm{d}\phi e^{i(m-m')\phi} = 2\pi\delta_{mm'}$$

we can immediately write down

$$C_{\ell m;\ell' m'} = -2\pi i m \mathcal{N}_{\ell m}\mathcal{N}_{\ell' m}\delta_{mm'}\int_0^\pi \mathrm{d}\theta\big(P_{\ell'}^m(\cos\theta)\partial_\theta P_\ell^m(\cos\theta) + P_\ell^m(\cos\theta)\partial_\theta P_{\ell'}^m(\cos\theta)\big) \tag{4.48}$$

where we used that $F_m\delta_{mm'} = F_{m'}\delta_{mm'}$ for arbitrary F_m. Finally this can be recognized as the product rule on derivatives on the Associated Legendre polynomials. Thus the integral becomes a total derivative, which is trivially integrated giving

$$C_{\ell m;\ell' m'} = -2\pi i m \mathcal{N}_{\ell m}\mathcal{N}_{\ell' m}\delta_{mm'}\big(P_\ell^m(1)P_{\ell'}^m(1) - P_\ell^m(-1)P_{\ell'}^m(-1)\big). \tag{4.49}$$

This gives the result for $C_{\ell m;\ell' m'}$ for all ℓ, m, ℓ', m'. Clearly it trivially vanishes for $m = 0$, so we only need to know the values $P_\ell^m(\pm 1)$ for $m > 0$. The definition from A&S [57] gives that

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell\ \ell!}\left(1-x^2\right)^{m/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\ell+m}\left(x^2-1\right)^\ell. \tag{4.50}$$

For $m > 0$ this quantity clearly vanishes at $x = \pm 1$, so we finally find that

$$\boxed{C_{\ell m;\ell' m'} = 0} \tag{4.51}$$

in all cases. Thus what we see is that the coupling action in Equation 4.45 vanishes exactly. This completes the proof that the odd and even graviton modes decouple exactly and we can treat them as independent fields.

We want to remark that in the calculation above the presence of ϵ^{AB} is crucial. The minus sign introduced by antisymmetry of the odd parity is canceled by the minus sign of the complex conjugate[6], so that we can use the product rule in the end. This shows that odd-parity and even-parity modes decouple naturally as desired.

In this decoupling we find that the choice of lightcone coordinates plays no role, i.e. r, t or x, y yield the same result. Similar the choice of angular coordinates A, B plays no role, because both lightcone coordinates and angular coordinates are summed over separately. This proof holds in fact for any metric where A, r only depend on x, y, i.e. for any spherically symmetric metric of the form of Equation 4.9. That being said the decoupling that we have found is certainly also valid for the Schwarzschild metric.

4.3 The quadratic even action

We have seen that the odd and even harmonics decouple completely and are left with only

$$\mathcal{L} = -\tfrac{1}{4}h^+_{\mu\nu}G_+^{\mu\nu} - \tfrac{1}{4}h_-^{\mu\nu}G^-_{\mu\nu}. \tag{4.52}$$

The h_a modes are now all contained in a separate integral, and for the scattering we will consider these do not contribute. This will be explained in detail in Subsection 5.3. From this point onward we shall pay no further attention to the odd parity part, leaving it as a possible extension to the theory when including transverse momenta. Excluding the odd modes the even Lagrangian is given by

$$\mathcal{L}_e = -\tfrac{1}{4}h_+^{\mu\nu}G^+_{\mu\nu}. \tag{4.53}$$

We drop the plus notation since we will only be looking at the even modes now. Additionally we will at first work with the mixed h^μ_ν tensor. The reason for doing so is the explicit separation of variables, and due to covariance we can raise or lower indices back again in the end as desired. We will use the separation of variables to remove the angular coordinates from the system. To see the separation of variables let us look at the mixed graviton modes

$$h_a^b = H_a^b Y_\ell^m, \tag{4.54}$$
$$h_A^B = K\delta_A^B Y_\ell^m. \tag{4.55}$$

Remember that H_a^b and K only depend on x^a, so that all dependence on the lightcone coordinates is embedded in these two modes, and thus all angular dependence is only in the Y_ℓ^m scalars. The reason for the mixed choice is now apparent when looking at h_A^B, which only has a δ_A^B which is constant. Contrary in the lower case we had $h_{AB} \sim g_{AB}$ and $g_{\phi\phi} \sim \sin\theta^2$ which is not constant. Thus our mixed choice resulted in the explicit separation of variables that we will use to isolate the spherical harmonics Y_ℓ^M in the action.

[6] Remark however that the complex conjugation is not crucial. If instead we inserted two Y_ℓ^m's without complex conjugation, there would be no minus sign in the derivative. However a new minus sign would be introduced when using the orthogonality relation and rewriting $m' \to -m$, thus allowing the product rule in the end. We can safely conclude that our proof is valid either case.

For the mixed graviton we write the Lagrangian as

$$\mathcal{L}_e = -\tfrac{1}{8}h^\mu_\nu G^\nu{}_{\mu\rho}{}^\sigma h^\rho_\sigma, \tag{4.56}$$

$$G^\nu{}_{\mu\rho}{}^\sigma \equiv \delta^\sigma_\mu \nabla_\rho \nabla^\nu + \delta^\nu_\rho \nabla^\sigma \nabla_\mu - \delta^\nu_\mu \nabla^\sigma \nabla_\rho - \delta^\sigma_\rho \nabla^\nu \nabla_\mu + 2\bar{\delta}^{\nu\sigma}_{\mu\rho}\Box. \tag{4.57}$$

Here we used the generalized antisymmetric kronecker delta $\bar{\delta}^{\nu\sigma}_{\mu\rho} = \delta^\nu_{[\mu}\delta^\rho_{\sigma]}$.

Mode decoupling

We shall use the manifest separation of variables to isolate the spherical harmonics functions. We will restore the (ℓ, m) indices in this subsection only in order to prove a decoupling in the different (ℓ, m) modes. For the full graviton we then write the proper definition as in Equation 4.22 for the even modes only, including the separation of variables. This results in

$$h^\mu_\nu = \sum_{\ell,m} (h_{\ell m})^\mu_\nu Y^m_\ell \tag{4.58}$$

where now the $(h_{\ell m})^\mu_\nu$ only depends on x^a such that

$$(h_{\ell m})^b_a = H^b_a$$
$$(h_{\ell m})^B_A = K\delta^B_A$$

and the rest vanishing. Notice that separation of variables led to the spherical coordinates being explicitly separated in the Y^m_ℓ in Equation 4.58.

Let us now act with $G^\nu{}_{\mu\rho}{}^\sigma$ on the definition in Equation 4.58, in order to find the Lagrangian in Equation 4.56. The $G^\nu{}_{\mu\rho}{}^\sigma$ consists of two covariant derivatives, so we apply the product rule twice to separate the derivatives of the functions Y^m_ℓ and $(h_{\ell m})^\mu_\nu$. This results in

$$\begin{aligned} G^\nu{}_{\mu\rho}{}^\sigma h^\rho_\sigma &= \sum_{\ell,m} Y^m_\ell G^\nu{}_{\mu\rho}{}^\sigma (h_{\ell m})^\rho_\sigma + \sum_{\ell,m} (h_{\ell m})^\rho_\sigma G^\nu{}_{\mu\rho}{}^\sigma Y^m_\ell \\ &+ \sum_{\ell,m}\Big(-4\bar{\delta}^{\nu\sigma}_{\mu\rho}(\partial_\kappa Y^m_\ell)\nabla^\kappa + \mathcal{U}^{\sigma\nu}_{\mu\rho} + \mathcal{U}^{\nu\sigma}_{\rho\mu} - \mathcal{U}^{\nu\sigma}_{\mu\rho} - \mathcal{U}^{\sigma\nu}_{\rho\mu}\Big)(h_{\ell m})^\rho_\sigma \end{aligned} \tag{4.59}$$

where we defined

$$\mathcal{U}^{\nu\sigma}_{\mu\rho} = \delta^\nu_\mu \Big[(\partial^\sigma Y^m_\ell)\nabla_\rho + (\partial_\rho Y^m_\ell)\nabla^\sigma\Big]. \tag{4.60}$$

The first line in Equation 4.59 is just the same $G^\nu{}_{\mu\rho}{}^\sigma$ acting on both functions separately, a result of the product rule acting twice on the same function. The second line are all mixed terms where one derivative acts on Y^m_ℓ and one depends on $(h_{\ell m})^\mu_\nu$. These terms are quite messy, but luckily we can find some tremendous simplifications. Remember that Y^m_ℓ of course only depends on θ, ϕ, so $\partial_a Y^m_\ell = 0$. This allows us to set a few indices in the second line of Equation 4.59 to be angular indices only. Then using the tensor structure outlined in Equations 4.54-4.55 and the covariant derivative identities listed in Appendix A.1 we can write down that

$$\Big(-4\bar{\delta}^{\nu\sigma}_{\mu\rho}(\partial_\kappa Y^m_\ell)\nabla^\kappa + \mathcal{U}^{\sigma\nu}_{\mu\rho} + \mathcal{U}^{\nu\sigma}_{\rho\mu} - \mathcal{U}^{\nu\sigma}_{\mu\rho} - \mathcal{U}^{\sigma\nu}_{\rho\mu}\Big)(h_{\ell m})^\rho_\sigma \sim \partial_A (h_{\ell m})^\rho_\sigma = 0 \tag{4.61}$$

What happens is that because the derivative that acts on Y^m_ℓ is angular, the derivative that acts on $(h_{\ell m})^\rho_\sigma$ is also angular, a result of the spherical symmetry. However since $(h_{\ell m})^\rho_\sigma$ is only lightcone-dependent this vanishes. Thus we see that the combination of spherical symmetry of the background and separation of variables results in removal of all mixed derivative terms.

The second line in Equation 4.59 now vanishes completely and we are left with the following operator action

$$G^{\nu}{}_{\mu\rho}{}^{\sigma}h^{\rho}_{\sigma} \quad = \quad \sum_{\ell,m} Y^m_\ell G^{\nu}{}_{\mu\rho}{}^{\sigma}(h_{\ell m})^{\rho}_{\sigma} + \sum_{\ell,m}(h_{\ell m})^{\rho}_{\sigma} G^{\nu}{}_{\mu\rho}{}^{\sigma} Y^m_\ell. \tag{4.62}$$

We shall now determine the second term $G^{\nu}{}_{\mu\rho}{}^{\sigma}Y^m_\ell$, where the covariant derivatives become trivial since Y^m_ℓ is a scalar. This can be shown to be written as

$$G^{\nu}{}_{\mu\rho}{}^{\sigma}Y^m_\ell = \frac{(\Delta_\Omega Y^m_\ell)}{2r^2}\left(\delta^\nu_\mu\delta^\sigma_\rho + \delta^\nu_a\delta^a_\mu\delta^\sigma_b\delta^b_\rho - 2\delta^\nu_\rho\delta^\mu_\sigma\right). \tag{4.63}$$

Now clearly only the spherical Laplacian is the relevant derivative, which allows us to use that Y^m_ℓ is an eigenfunction of the spherical Laplacian $\Delta_\Omega Y^m_\ell = -\ell(\ell+1)Y^m_\ell$. Again this is a result of spherical symmetry, where the action only depends on the spherical Laplacian Δ_Ω, even though not obviously so at first. Finally we can write

$$G^{\mu}{}_{\nu\rho}{}^{\sigma}Y^m_\ell = -Y^m_\ell\frac{\ell(\ell+1)}{2r^2}\left(\delta^\mu_\nu\delta^\sigma_\rho + \delta^\mu_a\delta^a_\nu\delta^\sigma_b\delta^b_\rho - 2\delta^\mu_\rho\delta^\sigma_\nu\right). \tag{4.64}$$

The above result is of crucial importance, as it shows us that there are no derivatives on Y^m_ℓ anymore. This fact will be crucial for the decoupling of different (ℓ, m) modes in the quadratic action as we shall see next.

We can now neatly write down Equation 4.59 as

$$G^{\nu}{}_{\mu\rho}{}^{\sigma}h^{\rho}_{\sigma} \quad = \quad = \sum_{\ell,m} Y^m_\ell h^\mu_\nu \mathcal{G}^{\nu}_{\ell\ \mu\rho}{}^{\sigma}(h_{\ell m})^{\rho}_{\sigma}. \tag{4.65}$$

Here we defined the following operator:

$$\boxed{\mathcal{G}^{\nu}_{\ell\ \mu\rho}{}^{\sigma} = G^{\nu}{}_{\mu\rho}{}^{\sigma} - \tfrac{\ell(\ell+1)}{2r^2}\left(\delta^\mu_\nu\delta^\sigma_\rho + \delta^\mu_a\delta^a_\nu\delta^\sigma_b\delta^b_\rho - 2\delta^\mu_\rho\delta^\sigma_\nu\right).} \tag{4.66}$$

The operator $G^{\nu}{}_{\mu\rho}{}^{\sigma}$ is defined by Equation 4.57, so we redefined this operator to include the constant part that came from the spherical Laplacian of Y^m_ℓ. Notice that the Y^m_ℓ now only contributes in the form of a $1/r^2$ potential.

We have made the right side of the Lagrangian in Equation 4.56 explicit by determining the action of $G^{\nu}{}_{\mu\rho}{}^{\sigma}$ on h^ρ_σ. This allows us to immediately write down the action. Inserting Equation 4.65 into the action results in

$$S_e = -\frac{1}{8}\sum_{\ell,m}\sum_{\ell',m'}\int \mathrm{d}\Omega\ \bar{Y}^{m'}_{\ell'}Y^m_\ell \int \mathrm{d}^2x\sqrt{-g}(h_{\ell'm'})^\mu_\nu \mathcal{G}^{\nu}{}_{\mu\rho}{}^{\sigma}(h_{\ell m})^\rho_\sigma. \tag{4.67}$$

The operator $\mathcal{G}^{\nu}{}_{\mu\rho}{}^{\sigma}$ was defined in Equation 4.66. Now for a spherically symmetric system the entire second part shall not depend on the angular coordinates, due to the symmetry acting in the summation and $(h_{\ell m})^\rho_\sigma$ not depending on θ, ϕ. We shall appreciate this fact explicitly in the next subsection, but keep it as a given for now. Then all θ, ϕ dependence is only in the spherical harmonics, so the integral over all solid angle can be performed separately resulting in

$$\int \mathrm{d}\Omega \bar{Y}^{m'}_{\ell'}Y^m_\ell = \delta_{\ell'\ell}\delta_{mm'} \tag{4.68}$$

which is just the orthogonality relation of the spherical harmonics. Finally we see that the different ℓ, m modes indeed decouple and the final action becomes simply

$$S_e = -\frac{1}{8}\sum_{\ell,m}\int \mathrm{d}^2x A r^2 (h_{\ell m})^\mu_\nu \mathcal{G}^{\nu}_{\ell\ \mu\rho}{}^{\sigma}(h_{\ell m})^\rho_\sigma. \tag{4.69}$$

In this action the Lagrangian only depends on x^a, and the other degrees of freedom are in the (ℓ, m) dependence, which is now of no importance anymore, until we start to look at the vertex in Section 5. For this reason we will again drop the (ℓ, m) indices, implying that all graviton modes must be taken at the same (ℓ, m). The fact that the (ℓ, m) modes decouple is the final simplification that arises due to spherical symmetry, implying conservation of angular momentum. The result is an important simplification for the even action.

Notice that all calculations are in principle still covariant given that we have forbidden any transformations that mix the lightcone and two-sphere. However a complication concerning covariance does arise, even with the constrained coordinate transformations, which we will solve by removing the two-sphere dimensions entirely.

The Lightcone-Lagrangian

In this subsection we shall examine the shape of the Lagrangian for the lightcone fields $(h_{\ell m})^\mu_\nu$ more closely. Remember that we drop the (ℓ, m) subscripts again so that we simply have

$$h^b_a = H^b_a, \tag{4.70}$$

$$h^B_A = K\delta^B_A. \tag{4.71}$$

The Lagrangian is given by

$$\mathcal{L}_e = -\frac{1}{8} h^\mu_\nu \mathcal{G}^\nu{}_{\mu\rho}{}^\sigma h^\rho_\sigma. \tag{4.72}$$

This Lagrangian is of course exactly the same as the one in Equation 4.56, except for the modified operator $\mathcal{G}$ defined in Equation 4.66 and the fact that we only have x^a dependence. Additionally the above is for single (ℓ, m), the total Lagrangian is found by summing over all modes. The aim of this subsection is to rewrite the operator $\mathcal{G}^\nu{}_{\mu\rho}{}^\sigma$ in such a way that we only sum over lightcone indices a, b with only the lightcone metric g_{ab}. In this language we will have removed the two-sphere indices as well, besides just the two-sphere coordinates in the previous subsection, and have manifest separation of the degrees of freedom H_{ab} and K. Currently the degrees of freedom are still implicitly mixed, and it is impossible make seperation manifest while still using spacetime indices. Additionally there is an ambiguity in covariance due to the residual presence of the two-sphere indices while the two-sphere coordinates are removed. Let us look at these problems more closely.

First we will look at why separation of degrees of freedom is impossible. To see why we best look at the quantity $\Box h^a_b$, where, even though only lightcone coordinates are present, we find a term of the form $(\partial^a r)(\partial_b r)K$, as shown in Appendix B.3. This mixing occurs because r depends on x^a and the covariant derivatives in $\Box$ still sum over four spacetime indices $\Box = \nabla^\rho_\rho$. So because the angular indices are still present we automatically have hidden mixings between the degrees of freedom.
The second and more significant problem due to the residual presence of angular indices is visible when looking at raising and lowering indices with the metric. How exactly would we define $h_{\phi\phi} = g_{\phi\phi} h^\phi_\phi$ if any θ-dependencies are forbidden due to the spherical harmonics expansion? In this case h^ϕ_ϕ is θ-independent, but $g_{\phi\phi} = r^2 \sin^2\theta$ would reintroduce it. This shows that the two-sphere metric g_{AB} is ill-defined in the spherical harmonics expansion, which is characterized by the fact that the two-sphere coordinates are embedded in (ℓ, m) while the two-sphere indices remain. Of course in the action in the end all these contributions cancel out, so we have not done anything forbidden, however there is ambiguity in the equations of motion and thus the propagator. The reason is that we have implicity broken covariance on the two-sphere, by integrating θ, ϕ out. We are now stuck with mixed representation h^ν_μ for the angular indices, since we removed the angular coordinates in the mixed representation. Remember that covariance on the lightcone is still safe.
Clearly we have two problems that arise from the two-sphere indices: It is impossible to find manifest separation of the degrees of freedom H_{ab}, K, and we have broken covariance. The solution is luckily easy, namely to remove the two-sphere indices A, B altogether. Covariance will be restored and be present on the lightcone only, as desired. Separation of degrees of freedom is automatically included in the process.

In the following we will split the spacetime into two components, the lightcone g_{ab} and the two-sphere g_{AB} and we will insert g_{AB} explicitly in order eliminate the two-sphere presence. Then in the end we will have a two-dimensional description where we only have the metric g_{ab}, summation only goes over lightcone indices a, b and covariance is completely manifest without any restrictions.
We shall define $\tilde{\nabla}, \tilde{\Box} = \tilde{\nabla}^a \tilde{\nabla}_a$ as quantities on the lightcone, so only involving the metric g_{ab} and summation only over the lightcone index a. In general a tilde will mean a lightcone quantity, also for the metric $\tilde{g} = g_{ab}$. In this formulation it is impossible that for example $\tilde{\Box} h^a_b$ introduces a K; the indices never sum over θ, ϕ. Thus by splitting spacetime like this, and summing away the angular indices, we automatically restore covariance and separate the degrees of freedom.

Symbolically we want to find an action of the following form

$$S_e = \frac{1}{4} \int \mathrm{d}^2 x \sqrt{-\tilde{g}} \Big(\tilde{H}^{ab} \tilde{\Delta}^{-1}_{abcd} \tilde{H}^{cd} + \tilde{H}^{ab} \tilde{\Delta}^{-1}_{L,ab} \tilde{K} + \tilde{K} \tilde{\Delta}^{-1}_{R,ab} \tilde{H}^{ab} + \tilde{K} \tilde{\Delta}^{-1} \tilde{K} \Big). \tag{4.73}$$

Here we defined all of the new $\tilde{\Delta}$ operators listed below. We have also defined the new lightcone fields $\tilde{H}_{ab} = r H_{ab}, \tilde{K} = rK$. These definitions have been made to absorb the residual r^2 of the two-sphere Jacobian in the action in Equation 4.69 into the fields. By doing this the only relevant function in the new action in Equation 4.73 will be the lightcone Jacobian $\sqrt{-\tilde{g}}$, resulting in the proper canonical form of the action. The complete procedure of deriving these operators is given in Appendix B.4. The result is given by:

$$\begin{aligned}
\tilde{\Delta}^{-1} &= -\tilde{\Box} + F^a_a, & (4.74)\\
\tilde{\Delta}^{-1}_{R,ab} &= -g_{ab}\big(\tilde{\Box} - \tfrac{1}{2} V_c \tilde{\nabla}^c + \tfrac{1}{4} V_c V^c - F^c_c - \tfrac{\ell(\ell+1)}{2r^2}\big) + \tilde{\nabla}_a \tilde{\nabla}_b - F_{ab}, & (4.75)\\
\tilde{\Delta}^{-1}_{L,ab} &= -g_{ab}\big(\tilde{\Box} + \tfrac{1}{2} V_c \tilde{\nabla}^c - \tfrac{\ell(\ell+1)}{2r^2}\big) + \tilde{\nabla}_a \tilde{\nabla}_b - F_{ab}, & (4.76)\\
\tilde{\Delta}^{-1}_{abcd} &= \tfrac{1}{2} g_{ac} V_{[b} \tilde{\nabla}_{d]} + \tfrac{1}{2} g_{bd} V_{[a} \tilde{\nabla}_{c]} + \tfrac{1}{2} g_{ab}\big(V_{(c} \tilde{\nabla}_{d)} + 2F_{cd}\big) + \tfrac{1}{2} g_{cd}\big(- V_{(a} \tilde{\nabla}_{b)} + \tfrac{1}{2} V_a V_b\big) & (4.77)\\
&+ g_{ab} g_{cd}\big(\tfrac{1}{4} R_s + \tfrac{\ell(\ell+1)}{2r^2}\big) - g_{ac} g_{bd}\big(\tfrac{1}{2} R_s + \tfrac{\ell(\ell+1)}{2r^2}\big). & (4.78)
\end{aligned}$$

These quantities have all been calculated by hand and checked by *Wolfram Mathematica*. Here we defined the following tensors:

$$\begin{aligned}
V_a &= 2\partial_a \log r, & (4.79)\\
F_{ab} &= \tfrac{1}{r} \tilde{\nabla}_a \tilde{\nabla}_b r = \tfrac{1}{2} \tilde{\nabla}_{(a} V_{b)} + \tfrac{1}{4} V_a V_b, & (4.80)\\
R_s &= -\tfrac{1}{A} \tilde{\Box} \log A. & (4.81)
\end{aligned}$$

Here V_a is a residual curvature potential of the two-sphere, F_{ab} its corresponding field strength, and R_s is the Ricci scalar of the lightcone background metric $\tilde{g}_{ab}$. Notice that all operators are in fact symmetric in the fields, notably the fact that $\tilde{\Delta}^{-1}_{R,ab}$ equals $\tilde{\Delta}^{-1}_{L,ab}$ up to total derivatives. We have now achieved our desired result: We have a covariant description of the graviton modes on the lightcone (due to the lightcone covariant derivative $\tilde{\nabla}_a$), where the degrees of freedom $\tilde{H}_{ab}$ and $\tilde{K}$ are explicitly separated. The two-sphere is now removed entirely, besides the coordinates also its indices and presence in spacetime. The only residue is the ℓ, m dependence and the curvature potential V_a.

The lightcone metric is at this stage still completely general $\tilde{g}_{ab} = A(r)\eta_{ab}$ with $A(r)$ an arbitrary function. In principle we are now done, and we can insert the Schwarzschild metric and invert Equations 4.74-4.77 to find the propagator for performing quantum field theory. However, due to the fact that we are now two-dimensional, a new phenomenon arises. The lightcone metric that we are working with is conformally flat

$$\tilde{g}_{ab} = A\eta_{ab} \tag{4.82}$$

with η_{ab} the 2D Minkowski metric. This allows us to perform a Weyl transformation to reduce our theory to a two-dimensional flat theory, with modified potentials. In the next subsection we shall work out the Weyl transformation to make the background spacetime flat. To do such procedure is incredibly advantageous as we do not need to concern ourselves with the problem of curvature and in particular quantum field theory on a curved spacetime. Instead the standard laws of flat space quantum field theory can be used.

Weyl transformation

In order to work on the flat spacetime we need to make a transformation that explicitly removes the function $A(r)$ from the metric. The transformation that we choose is given by

$$\tilde{g}_{ab} \rightarrow A\eta_{ab}, \tag{4.83}$$
$$\tilde{H}_{ab} \rightarrow A\mathfrak{h}_{ab}, \tag{4.84}$$
$$\tilde{K} \rightarrow \mathcal{K}, \tag{4.85}$$

where $\mathfrak{h}^{ab}$ is raised and lowered with the new flat Minkowski metric (effectively we *choose* $\tilde{H}^a_b$ as the proper flat field). The advantage is now that partial derivatives are also covariant derivatives, so covariance of the physical quantities is trivially achieved. The form above for $\tilde{H}_{ab}$ is chosen such that it maintains its symmetry with the metric as it is of course a metric perturbation. Remark that such transformations are commonly done in cosmology as well, where the FLRW metric is conformally flat when expressed in conformal time. Then the propagator is also ideally derived after a Weyl transformation [66]. The fact that we can now apply similar methods is because we removed the two-sphere, normally the Schwarzschild spacetime is not conformally flat, but now in two spacetime dimensions it is.

During the Weyl transformation we need to be carefull that: $\tilde{K}$ does not transform (but we give it a new symbol), $\tilde{H}^{ab}$ transforms as described above, $V_b = 2\partial_b \log r$ does not transform (so $V^b = \tilde{g}^{ab}V_b = \frac{1}{A}\eta^{ab}V_b$ does), F_{ab} does not transform, etc. All quantities need to be manually checked: partial derivatives do not Weyl transform, so these are the basis to look at.
Inserting this transformation we can for example rewrite the action for $\mathcal{K}$ as

$$\tfrac{1}{4}\int \mathrm{d}x^2 A\mathcal{K}(-\Box + F^a_a)\mathcal{K} = \tfrac{1}{4}\int \mathrm{d}x^2 \mathcal{K}(-\partial^2 + \eta^{ab}F_{ab})\mathcal{K} \tag{4.86}$$

where we included the A from the $\sqrt{-g}$ in front (notice $\mathcal{K}$ does not transform) and defined $\partial^2 = \eta^{ab}\partial_a\partial_b$. For consistency we redefine

$$F_{ab} = \tfrac{1}{r}\tilde{\nabla}_a\tilde{\nabla}_b r \equiv \frac{1}{r}\partial_a\partial_b r - \tfrac{1}{2}U_{(a}V_{b)} + \tfrac{1}{4}\eta_{ab}U^cV_c \tag{4.87}$$

where all raising and lowering as well as the contraction is now done with the flat metric, i.e. $U^cV_c = \eta^{ac}U_aV_c$ and $U^a = \eta^{ab}U_b$. This redefinition is in principle necessary, since the previous lightcone covariant derivative does not hold after the Weyl transformation. Here we defined a new potential that will embed the curvature remnants of A:

$$U_a = \partial_a \log A. \tag{4.88}$$

Then the action for K only becomes simply

$$S_K = \frac{1}{4}\int \mathrm{d}^2x\mathcal{K}(-\partial^2 + F^a_a)\mathcal{K} \tag{4.89}$$

where of course $F^a_a = \eta^{ab}F_{ab}$. Remark that as for a usual Weyl transformation we did not lose any information in this process, in other words both actions in Equation 4.86 are exactly the same, and describe the exact same physical process. The best way to interpret this is as a change of viewpoint; we can either look at free fields in a curved spacetime, or we can look at fields bounded by potentials, in a flat spacetime. Both descriptions are equivalent for conformally flat spacetimes. The fact that spacetime needs to be conformally flat is important because all curvature effects are then isotropic, so that it is possible to be embedded in a scalar function. For anisotropic curvatures we naturally need a tensor potential, losing the simplicity of a Weyl transformation (and in fact making the metric the most viable candidate).

We shall now use this procedure on all fields and operators, taking care which fields *do* Weyl transform, and which fields do not. Here we also need to work out all Christoffel symbols in order to safely remove them and pull the $\frac{1}{A}$ of $\tilde{H}^{cd}$ through. Essentially we need to calculate:

$$\Delta^{-1}_{R,ab} \equiv A\tilde{\Delta}^{-1}_{R,ab}\frac{1}{A}, \tag{4.90}$$
$$\Delta^{-1}_{L,ab} \equiv \tilde{\Delta}^{-1}_{L,ab}, \tag{4.91}$$
$$\Delta^{-1}_{abcd} \equiv \tilde{\Delta}^{-1}_{abcd}\frac{1}{A}. \tag{4.92}$$

Here we also need to replace $g_{ab} = A\eta_{ab}$ everywere. The action and Lagrangian are now given with these new operators by

$$\boxed{S = \frac{1}{4}\int \mathrm{d}^2x\Big(\mathfrak{h}^{ab}\Delta^{-1}_{abcd}\mathfrak{h}^{cd} + \mathfrak{h}^{ab}\Delta^{-1}_{L,ab}\mathcal{K} + \mathcal{K}\Delta^{-1}_{R,ab}\mathfrak{h}^{ab} + \mathcal{K}\Delta^{-1}\mathcal{K}\Big).} \tag{4.93}$$

The explicit calculation of all Δ operators shows that

$$\Delta^{-1} = -\partial^2 + F^a_a, \tag{4.94}$$
$$\begin{aligned}
\Delta^{-1}_{R,ab} &= -\eta_{ab}\big(\partial^2 + \tfrac{1}{2}(U^c - V^c)\partial_c + \tfrac{1}{2}\eta^{cd}(\mathcal{W}^A_{cd} - \mathcal{W}^r_{cd}) - A\tfrac{\ell(\ell+1)}{2r^2}\big) + \partial_a\partial_b + U_{(a}\partial_{b)} + \mathcal{W}^A_{ab} - F_{ab},\\
\Delta^{-1}_{L,ab} &= -\eta_{ab}\big(\partial^2 - \tfrac{1}{2}(U^c - V^c)\partial_c - A\tfrac{\ell(\ell+1)}{2r^2}\big) + \partial_a\partial_b - U_{(a}\partial_{b)} - F_{ab},\\
\Delta^{-1}_{abcd} &= \tfrac{1}{2}\eta_{ac}V_{[b}\partial_{d]} + \tfrac{1}{2}\eta_{bd}V_{[a}\partial_{c]} + \tfrac{1}{2}\eta_{ab}\big(V_{(c}\partial_{d)} + \mathcal{W}^r_{cd} + \tfrac{1}{2}V_cV_d\big) + \tfrac{1}{2}\eta_{cd}\big(-V_{(a}\partial_{b)} + \tfrac{1}{2}V_aV_b\big)\\
&+ \eta_{ab}\eta_{cd}\big(\tfrac{1}{4}AR_s - \tfrac{1}{4}V^eU_e + A\tfrac{\ell(\ell+1)}{2r^2}\big) - \eta_{ac}\eta_{bd}\big(\tfrac{1}{2}AR_s - \tfrac{1}{2}V^eU_e + A\tfrac{\ell(\ell+1)}{2r^2}\big)
\end{aligned} \tag{4.95}$$

where we of course simply applied the product rule on any relevant $\frac{1}{A}$ present. We defined two new field tensors

$$\mathcal{W}^r_{ab} \equiv \partial_{(a}V_{b)} \qquad \mathcal{W}^A_{ab} \equiv \partial_{(a}U_{b)}. \tag{4.96}$$

Again all of these have been calculated by hand, and checked in *wolfram mathematica*, showing that the action is still identical to what we started with in Equation 4.5.

Clearly the Δ operators have become more complex, however we are on a flat spacetime now, making all covariant derivatives trivial. Coordinate transformations have become tricky, however we can choose any coordinate system we like by choosing the appropriate function $A(r)$.
This completes the calculation of the even action. We have succesfully reduced the four-dimensional Schwarschild spacetime, to a flat two dimensional Minkowski spacetime. We were able to do this thanks to spherical symmetry and the spherical harmonics expansion. All curvature is embedded in two potentials V_a, U_a and their respective field strengths. Up to now the procedure has been exact, so we have not lost any information. Furthermore our two-dimensional flat space action holds for any starting metric of the form

$$\mathrm{d}s^2 = -2A(r)\mathrm{d}x\mathrm{d}y + r^2\mathrm{d}\Omega^2 \tag{4.97}$$

for arbitrary $A(r)$. It is remarkable that the gravitational perturbations of a spherically symmetry spacetime can be described in two dimensional flat space completely. We have looked at solutions for which the background metric satisfies $R_{\mu\nu} = 0$, however the extension to a non-vacuum solution should be reasonably easy, only the first order term in Equation 4.5 might be nontrivial. We have not assumed a vacuum solution anywhere else. Now that we finally have our effective action, we can use it to determine the graviton propagator on the Schwarzschild horizon, necessary for our scattering calculation in Section 5.

4.4 Schwarzschild horizon approximation

From now on we will be looking specifically at the Schwarzschild spacetime, so we explicitly insert

$$A = \tfrac{R}{r}e^{1-\frac{r}{R}} \qquad xy = 2R^2\left(1 - \tfrac{r}{R}\right)e^{\frac{r}{R}-1}. \tag{4.98}$$

The aim of this subsection is to properly write down the action as in Equation 4.93 specifically on the Schwarzschild horizon, since we are only interested on scattering in the horizon due to the diverging energy discussed in Section 2. For all the relevant curvature potentials, we find in the xy coordinates that

$$V_a = \frac{A}{rR} x_a, \tag{4.99}$$

$$U_a = -\frac{A}{2rR}\left(1+\tfrac{r}{R}\right) x_a, \tag{4.100}$$

$$\mathcal{W}^r_{ab} = \frac{A}{rR}\eta_{ab} - \frac{A^2}{2R^2r^2}\left(2+\tfrac{r}{R}\right) x_a x_b, \tag{4.101}$$

$$\mathcal{W}^A_{ab} = -\tfrac{A}{2rR}\left(1+\tfrac{r}{R}\right)\eta_{ab} + \frac{A^2}{4R^2r^2}\left(2+2\tfrac{r}{R}+\tfrac{r^2}{R^2}\right) x_a x_b, \tag{4.102}$$

$$F_{ab} = \frac{AR}{2r^3}\eta_{ab}, \tag{4.103}$$

$$R_s = \frac{2R}{r^3}. \tag{4.104}$$

These expressions are all straightforwardly derived from their definitions in the previous subsection for the Schwarzschild spacetime and the identity $xyA = 2R^2(\frac{R}{r}-1)$. Notice that even though general covariance is lost, there is still a set of coordinate transformations that is allowed, which consists of those diffeomorphisms that leave the metric flat. These are simple translations, time translation $x \to ax, y \to a^{-1}y$ for some constant a, or time reversal $x \to y, y \to x$. We still have retained our time translation symmetry of the black hole, and will require this symmetry to persist everywhere in our quantum field theory, as long as it is anomaly free.

Remark that R_s is the Ricci scalar of the lightcone metric, while F_{ab} is the residual curvature field strength of the two-sphere. These quantities appear strongly related to eachother: $R_s = \frac{2}{A}F^a_a$. It is tempting to ask if this strong similarity is a result of Birkhoff's theorem. The only spherically symmetric vacuum solution is the Schwarszchild solution. This suggests that for the Schwarzschild spacetime the curvature of the two-sphere must be strongly related to the lightcone curvature, such that they exactly cancel eachother resulting in $R_{\mu\nu} = 0$. So the apparent relation between R_s and F_{ab} is likely because these curvatures must cancel eachother in order to make the Schwarzschild spacetime a vacuum solution.

We can now insert all curvature potentials into the Δ operators, however this result will not be very usefull. For the full treatment one can directly insert Equations 4.99-4.104 into Equations 4.94-4.95. For our treatment we will first look at what approximating close to the horizon gives us in terms of simplifications.

The horizon approximation

For high-energy scattering close to the Schwarzschild black hole, we are only interested to what happens exactly on the horizon because the energy diverges there. Thus also for our quantum field theory we want to approximate the background metric around the horizon. In particular we want to approximate all potentials in Equations 4.99-4.104 close to the horizon.

There are two points to remark here. The first is that such an approximation was also done more subtly in Section 2. Here the Shapiro delay on the Schwarzschild metric was derived using the cut-and-paste procedure and a Dirac-delta function $\delta(x)$ exactly on the horizon. Translations in the x, y coordinates are not allowed within the Lorentz group, so we cannot attribute this to a choice of frame. The factor $\delta(x)$ was an explicit choice since all dominant contributions come from interaction on the horizon, and we now make the similar choice.
Another point to remark is the reference scale for the coordinates. This reference scale is not the Planck length, but the Schwarzschild radius R. What we see is that for a very large black hole $M_{\mathrm{BH}} \gg M_{\mathrm{Pl}}$, this radius becomes very large, and effectively all coordinates will be much smaller than R. So for a large black hole we can also assume certain bounds on the coordinates, similar to a horizon approximation. Notice that this reasoning may seem odd because we defined $xy \sim R^2 f(1-r/R)$ so that the coordinates are proportional to the Schwarzschild radius, however we also still have the reference $1 - r/R$. Thus the reasoning applies,

although technically more indirectly.

Finally we can choose how we approximate the horizon. The best choice that combines both arguments above is to simply take the coordinates x, y very close to the origin, so $x^a \ll R$. Since the horizon is defined by $xy = 0$ this coordinate restriction automatically takes into account the horizon, and furthermore also takes into account the large black hole. We will now explicitly impose that $x, y \ll R$, by removing all terms quadratic and higher in x, y (so x^2, xy and y^2 and higher). For the radius we can then write

$$r = R + R\mathcal{O}\left(\tfrac{xy}{R^2}\right) \tag{4.105}$$

so up to linear order we simply have $r = R, A = 1$. The only linear contributions thus come in with the couplings $x_a\partial_b$, while all potentials have now simply become mass terms. Inserting the fact that $x_a x_b \approx 0$ and $A = 1, r = R$ into Equations 4.99-4.104 simplifies these operators a lot. In particular $\mathcal{W}^r_{ab}, \mathcal{W}^A_{ab}, F_{ab}$ are now a constant mass times the metric η_{ab} while $V_a = -U_a = R^{-2}x_a$. Inserting these potentials explicitly, we find for the operators in Equations 4.94-4.95 that

$$\Delta^{-1} = -\partial^2 + \mu^2, \tag{4.106}$$

$$\Delta^{-1}_{R,ab} = -\eta_{ab}\left(\partial^2 - \mu^2 x^c\partial_c - \tfrac{1}{2}\mu^2\lambda\right) + \partial_a\partial_b - \mu^2 x_{(a}\partial_{b)}, \tag{4.107}$$

$$\Delta^{-1}_{L,ab} = -\eta_{ab}\left(\partial^2 + \mu^2 x^c\partial_c - \tfrac{1}{2}\mu^2(\lambda-2)\right) + \partial_a\partial_b + \mu^2 x_{(a}\partial_{b)}, \tag{4.108}$$

$$\Delta^{-1}_{abcd} = \tfrac{1}{2}\mu^2\left(\eta_{ac}x_{[b}\partial_{d]} + \eta_{bd}x_{[a}\partial_{c]} + \eta_{ab}x_{(c}\partial_{d)} - \eta_{cd}x_{(a}\partial_{b)}\right) + \left(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd}\right)\frac{\mu^2(\lambda+1)}{2} \tag{4.109}$$

where we used the inverse Schwarzschild radius $\mu = 1/R$ again, which can now be understood as effective mass due to the positive curvature of the two-sphere, and defined

$$\lambda \equiv \ell^2 + \ell + 1. \tag{4.110}$$

We see now that our expansion is identical to expanding around $\mu \to 0$, which holds for an extremely large black hole. Nevertheless we need a perturbation around a dimensionless quantity, so letting $\mu \to 0$ is not a properly consistent method, which is why we expand around μx_a. Of course such a perturbation needs to be done on the equations of motion, rather than the Lagrangian. However it should be noted that the equation of motion is exactly given by the operators above, namely

$$\Delta^{-1}_{abcd}\mathfrak{h}^{cd} + \Delta^{-1}_{L,ab}\mathcal{K} = 0, \tag{4.111}$$

$$\Delta^{-1}_{R,ab}\mathfrak{h}^{ab} + \Delta^{-1}\mathcal{K} = 0. \tag{4.112}$$

Thus all perturbations we insert on the Δ operators are equivalent to perturbations on the equation of motion, which also explains why we have been working with the one-sided representation of the action all the time. From this point forward we shall simply work with Δ either in Lagrangian or equation of motion formalism, since they are equivalent and only given by the Δ operators. We can also freely move between the formalisms if necessary, as they all correspond to the same physical situations.

We now turn to understanding the operators listed in Equations 4.106-4.109. The first derivative couplings now pose a problem: they are present in an asymmetric way since $\Delta^{-1}_{R,ab}$ and $\Delta^{-1}_{L,ab}$ only need to be equal up to total derivatives. Taking the complete limit of $\mu x_a \to 0$, so also removing first order terms, does not fix this. Even then we have $\mu^2\lambda$ in $\Delta^{-1}_{R,ab}$ and $\mu^2(\lambda-2)$ in $\Delta^{-1}_{L,ab}$. What we observe is that the first derivatives contribute to the effective mass, and thus cannot be discarded lightly. This is not a problem in the UV-limit, however for soft gravitons, which are leading in Eikonal scattering, we need the correct mass terms. We cannot take the infrared limit trivially either, as setting $\partial_a \to 0$ still results in inequalities between $\Delta^{-1}_{R,ab}$ and $\Delta^{-1}_{L,ab}$. Thus we first need to find a symmetric representation in $\mathcal{K}$ and $\mathfrak{h}_{ab}$ where these terms are equal.

The best method to achieve this symmetry is by a field redefinition. Remark the following identities

$$\partial_a\partial_b e^{-\frac{\mu^2x^2}{4}} = e^{-\frac{\mu^2x^2}{4}}\big(\partial_a\partial_b - \mu^2 x_{(a}\partial_{b)} - \tfrac{1}{2}\mu^2\eta_{ab} + \tfrac{1}{4}\mu^4 x_a x_b\big), \tag{4.113}$$

$$\partial_a\partial_b e^{\frac{\mu^2x^2}{4}} = e^{\frac{\mu^2x^2}{4}}\big(\partial_a\partial_b + \mu^2 x_{(a}\partial_{b)} + \tfrac{1}{2}\mu^2\eta_{ab} + \tfrac{1}{4}\mu^4 x_a x_b\big), \tag{4.114}$$

where we can again neglect the last terms up to linear order in μx_a. This allows us to identify

$$e^{\frac{\mu^2x^2}{4}}\Delta^{-1}_{L,ab}e^{-\frac{\mu^2x^2}{4}} = e^{-\frac{\mu^2x^2}{4}}\Delta^{-1}_{R,ab}e^{\frac{\mu^2x^2}{4}} = -\eta_{ab}\big(\partial^2 - \tfrac{1}{2}\mu^2(\lambda-1)\big) + \partial_a\partial_b \tag{4.115}$$

which clearly gives a symmetric representation, and also removes all first derivatives and first order terms in μx_a exactly. We now include this transformation explicitly in the fields

$$\mathcal{K} \rightarrow e^{-\frac{\mu^2x^2}{4}}\mathcal{K}, \tag{4.116}$$

$$\mathfrak{h}_{ab} \rightarrow e^{\frac{\mu^2x^2}{4}}\mathfrak{h}_{ab}. \tag{4.117}$$

The transformation of $\Delta^{-1}_{R,ab}$ and $\Delta^{-1}_{L,ab}$ under this field redefinition has been shown in Equation 4.115, and Δ^{-1}_{abcd} does not transform, because of the antisymmetries of the derivatives. Thus we only need to look carefully at Δ^{-1}. Let us first look at the action of $\mathcal{K}$:

$$S_{\mathcal{K}} = \tfrac{1}{4}\int \mathrm{d}^2x\mathcal{K}\Delta^{-1}\mathcal{K} = \tfrac{1}{4}\int \mathrm{d}^2x\ e^{-\frac{\mu^2x^2}{2}}\mathcal{K}(-\partial^2 + \tfrac{1}{2}\mu^2x^c\partial_c + 2\mu^2)\mathcal{K}. \tag{4.118}$$

Whereas we did not have a first derivative here at first, we do have one now. So we have traded in the first derivatives in the coupling terms for a first derivative in the $\mathcal{K}$ action. However since in this action we have the same field we can use integration by parts to simplify. First let us note that we can remove the exponent again since

$$e^{-\frac{\mu^2x^2}{2}} = 1 + \mathcal{O}(\mu^2xy).$$

It might seem strange that we first use the exponent to transform the fields, and then remove it again in order $\mu x_a \ll 1$. We will remark on this later.

Let us now look at the first derivative term more closely. The integral with the first derivative can be rewritten into

$$\int \mathrm{d}^2x\mathcal{K}\mu^2x^c\partial_c\mathcal{K} = \tfrac{1}{2}\mu^2\int \mathrm{d}^2x x^c\partial_c(\mathcal{K}^2). \tag{4.119}$$

Now we can perform integration by parts with vanishing boundary conditions to find

$$\tfrac{1}{2}\mu^2\int \mathrm{d}^2x x^c\partial_c(\mathcal{K}^2) = -\tfrac{1}{2}\mu^2\int \mathrm{d}^2x\mathcal{K}^2\partial_c x^c = -\mu^2\int \mathrm{d}^2x\mathcal{K}^2. \tag{4.120}$$

Thus we have traded in the first derivative for another mass term, using integration by parts and vanishing boundary conditions only. We see that the trade for first derivatives from the coupling to the $\mathcal{K}$ action was not a problem, as in the latter we can remove it exactly. Now that we have simplified the $\mathcal{K}$ action we can write down the total even action within the horizon approximation. This action is given by

$$S = \frac{1}{4}\int \mathrm{d}^2x\Big(\mathfrak{h}^{ab}\Delta^{-1}_{abcd}\mathfrak{h}^{cd} + \mathfrak{h}^{ab}\Delta^{-1}_{ab}\mathcal{K} + \mathcal{K}\Delta^{-1}_{ab}\mathfrak{h}^{ab} + \mathcal{K}\Delta^{-1}\mathcal{K}\Big). \tag{4.121}$$

Notice that we have removed the L, R subscripts since these operators are now symmetric. The operators are given by

$$\Delta^{-1} = -\partial^2 + \mu^2, \tag{4.122}$$

$$\Delta^{-1}_{ab} = -\eta_{ab}\big(\partial^2 - \tfrac{1}{2}\mu^2(\lambda-1)\big) + \partial_a\partial_b, \tag{4.123}$$

$$\Delta^{-1}_{abcd} = \tfrac{1}{2}\mu^2\Big(\eta_{ac}x_{[b}\partial_{d]} + \eta_{bd}x_{[a}\partial_{c]} + \eta_{ab}x_{(c}\partial_{d)} - \eta_{cd}x_{(a}\partial_{b)}\Big) + \frac{\mu^2(\lambda+1)}{2}\big(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd}\big). \tag{4.124}$$

What we see is that Δ^{-1} and Δ^{-1}_{abcd} are in fact unchanged. This was due to the integration by parts for Δ^{-1} and the antisymmetry in all derivatives in Δ^{-1}_{abcd}. Of course it was important that we have been able to to remove the exponent $e^{\pm\frac{\mu^2x^2}{2}}$ again up to order μx_a. While this seems strange we are applying the approximation consistently. Remember that in the small μx_a limit the fields are actually also not transformed, the transformation in Equations 4.116-4.117 are also of order $\mu^2 x^2$. Thus we took an infinitesimally small transformation, and nevertheless changed the operators. This shows that the first derivatives in $\Delta^{-1}_{L,ab}$ and $\Delta^{-1}_{R,ab}$ differ by the effective mass in Δ^{-1}_{ab} only by an infinitesimal transformation. The transformation performed is merely a consistent method to show this, but the underlying structure that $\Delta^{-1}_{L,ab}, \Delta^{-1}_{R,ab}$ equal Δ^{-1}_{ab} up to infinitesimal differences holds regardless of the transformation chosen.

Now that we finally have a symmetric representation in the fields without problematic first derivatives, we can invert these operators in order to find the graviton propagator. Notice that all terms linear in μx_a have been lost, except in Δ^{-1}_{abcd}. We shall see that we can ignore those terms even for Δ^{-1}_{abcd} when inverting the operator, so that finally there are no linear terms left. Thus while we approximated to linear order in μx_a, we will be left only with zeroth order terms. This is an important result: the Schwarzschild metric close to the horizon does not depend on the first order terms, only the zeroth order terms contribute. We can now also find both the proper UV-limit and IR-limit as desired.

4.5 Eikonal horizon propagator

In this subsection we shall invert the equations of motion to find the graviton propagator for Eikonal scattering. The latter is important, because it means that we do not need to take the fields h_a into account. In fact we shall only need the propagator for $\mathfrak{h}_{ab}$, however due to the coupling between $\mathfrak{h}_{ab}$ and $\mathcal{K}$ we have to look at the coupled equations. Additionally we will need the other propagators between $\mathfrak{h}_{ab}$ and $\mathcal{K}$ for the correct order of magnitude analysis in Subsection 5.3. In this subsection we derive the full graviton propagator for the even action on the Schwarzschild horizon, and only in the end we shall isolate the relevant Eikonal form.

The propagator is found by the definition of being the Green's function of the equation of motion. The equations of motion of the graviton are in total given by

$$\Delta^{-1}_{abcd}\mathfrak{h}^{cd} + \Delta^{-1}_{ab}\mathcal{K} = 0, \tag{4.125}$$

$$\Delta^{-1}_{ab}\mathfrak{h}^{ab} + \Delta^{-1}\mathcal{K} = 0, \tag{4.126}$$

where all Δ operators are given in Equations 4.122-4.124. In matrix notation this becomes

$$\begin{pmatrix} \Delta^{-1}_{abcd} & \Delta^{-1}_{ab} \\ \Delta^{-1}_{cd} & \Delta^{-1} \end{pmatrix} \begin{pmatrix} \mathfrak{h}^{cd} \\ \mathcal{K} \end{pmatrix} = 0. \tag{4.127}$$

Then the Green's function of this matrix differential equation would be defined by

$$\begin{pmatrix} \Delta^{-1}_{abcd} & \Delta^{-1}_{ab} \\ \Delta^{-1}_{cd} & \Delta^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{P}^{cdef} & \mathcal{P}^{cd} \\ \mathcal{P}^{ef} & \mathcal{P} \end{pmatrix} = \begin{pmatrix} \delta^{ef}_{ab} & 0 \\ 0 & 1 \end{pmatrix} \delta^{(2)}(x-x'). \tag{4.128}$$

Here we assumed a symmetric Green's function matrix since the equation of motion is also determined by a symmetric matrix. All $\mathcal{P}$ are the corresponding Green's functions or propagators that are functions of x and x', namely $\mathcal{P}(x;x')$. Notice that there are three propagators. The first is $\mathcal{P}^{cdef}$, which is the propagator of $\mathfrak{h}_{cd} \to \mathfrak{h}_{ef}$ that we are interested in. The other two are the propagator $\mathcal{P}_{ab}$ corresponding to $\mathfrak{h}^{ab} \to \mathcal{K}$ and $\mathcal{P}$ corresponding to $\mathcal{K} \to \mathcal{K}$. From Equation 4.128 we find that all individual propagators are defined by

$$\Delta^{-1}_{abcd}\mathcal{P}^{cdef} + \Delta^{-1}_{ab}\mathcal{P}^{ef} = \delta^{ef}_{ab}\delta^{(2)}(x-x'), \tag{4.129}$$

$$\Delta^{-1}_{ab}\mathcal{P}^{abcd} + \Delta^{-1}\mathcal{P}^{cd} = 0, \tag{4.130}$$

$$\Delta^{-1}_{abcd}\mathcal{P}^{cd} + \Delta^{-1}_{ab}\mathcal{P} = 0, \tag{4.131}$$

$$\Delta^{-1}_{ab}\mathcal{P}^{ab} + \Delta^{-1}\mathcal{P} = \delta^{(2)}(x-x'). \tag{4.132}$$

These propagators are clearly all coupled to eachother: $\mathcal{P}^{cdef}$ is coupled to $\mathcal{P}^{ab}$ through Equations 4.129 and 4.130, which is in turned coupled to $\mathcal{P}$ through Equations 4.131 and 4.132. This shows that it does not suffice to invert Δ^{-1}_{abcd} for the $\mathfrak{h}_{ab}$-propagator only. This is naturally due to the quadratic couplings between $\mathfrak{h}_{ab}$ and $\mathcal{K}$ in the Lagrangian in Equation 4.121. We will use Equations 4.129-4.132 to find the propagators, but to do so we first need to find the inverses of Δ^{-1} and Δ^{-1}_{abcd}. For reference these inverses are defined as usual:

$$\Delta^{-1}_{abcd}\Delta^{cdef} = \delta^{ef}_{ab}\delta^{(2)}(x-x'), \tag{4.133}$$

$$\Delta^{-1}\Delta = \delta^{(2)}(x-x'). \tag{4.134}$$

The inverse of Δ^{-1}

For the inverse of Δ we need to solve

$$(-\partial^2+\mu^2)\Delta(x;x') = \delta^{(2)}(x-x'). \tag{4.135}$$

Let us Fourier transform under the following definition:

$$\Delta(x;x') = \frac{1}{(2\pi)^2}\int \mathrm{d}^2k\; e^{ik_a(x-x')^a}\Delta(k), \tag{4.136}$$

$$\delta^{(2)}(x-x') = \frac{1}{(2\pi)^2}\int \mathrm{d}^2p\; e^{ik_a(x-x')^a}. \tag{4.137}$$

Then the solution for the inverse is trivially found to be

$$\boxed{\Delta(k) = \frac{1}{k^2+\mu^2}} \tag{4.138}$$

where $k^2 = k_ak^a$. This is exactly the Klein-Gordon propagator, where we have an effective mass μ^2 of the two-sphere and the Schwarzschild metric. We now turn to the more complex inverse of Δ^{-1}_{abcd}.

The inverse of Δ^{-1}_{abcd}

The operator is given by

$$\Delta^{-1}_{abcd} = \tfrac{1}{2}\mu^2\Big(\eta_{ac}x_{[b}\partial_{d]} + \eta_{bd}x_{[a}\partial_{c]} + \eta_{ab}x_{(c}\partial_{d)} - \eta_{cd}x_{(a}\partial_{b)}\Big) + \tfrac{\mu^2(\lambda+1)}{2}\big(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd}\big). \tag{4.139}$$

In order to find the inverse we start with the following ansatz

$$\Delta^{abcd} = T(x-x')\eta^{ab}\eta^{cd} + Q(x-x')\eta^{a(c}\eta^{d)b} \tag{4.140}$$

which is the only possible form that respects the symmetry of the graviton $\mathfrak{h}_{ab} = \mathfrak{h}_{ba}$ and time translation symmetry $x \to ax, y \to y/a$ up to linear order in μx_a. Here we used time translation symmetry to require that Δ^{abcd} transforms at most as a tensor under time translations, so that the only allowed tensor structures are $\eta_{ab}\eta_{cd}, \mu^2\eta_{ab}x_cx_d$ and $\mu^4x_ax_bx_cx_d$ with any index permutation. We included the μ^2 to have dimensionless tensor structures. Now the latter two can be removed up to linear order in μx_a, so that only $\eta_{ab}\eta_{cd}$ remains. Then Equation 4.140 gives the most general form that respects the graviton symmetry. Of course we have a different allowed dimensionless tensor as well: $\frac{x_ax_b}{x^2}$. This tensor is however not regular over the entire manifold so we ignore it. The irregularities of usual momentum space poles are different, these determine the mass-shell, whereas this pole would be a coordinate singularity. In Kruskal-Szekers coordinates there is not singularity at $x^2 = 0$ so we forbid such term. Notice that these are not the usual $1/r$ singularites of spherical coordinates, as those would correspond to $1/(x^2+2R^2)$ instead.

Inserting Equation 4.140 in the definition of Equation 4.133 results in

$$\begin{aligned} -Q\delta^{ab}_{ef} &+ \eta_{ab}\eta^{ef}\big(x^c\partial_cT - (\lambda+1)(T+Q)\big) - \eta^{ef}x_{(a}\partial_{b)}(2T+Q) + \eta_{ab}x^{(e}\partial^{f)}Q \\ &+ \tfrac{1}{2}\big(\delta^e_ax_{[b}\partial^{f]} + \delta^f_ax_{[b}\partial^{e]} + \delta^f_bx_{[a}\partial^{e]} + \delta^e_bx_{[a}\partial^{f]}\big)Q = \frac{2}{\mu^2(\lambda+1)}\delta^{ef}_{ab}\delta^{(2)}(x-x'). \end{aligned} \tag{4.141}$$

Now all other terms than the first are not allowed to have the tensor structure of δ^{ab}_{ef}. The terms containing η_{ab}, η^{ef} can never reduce to a Kronecker delta, as the metrics are already seperate. The last four terms also cannot do so because they are antisymmetric, i.e. $x_{[b}\partial^{f]} = \frac{1}{2}(x_b\partial^f - x^f\partial_b)$. Thus by equating the terms containing δ^{ef}_{ab} we find

$$Q = -\frac{2}{\mu^2(\lambda+1)}\delta^{(2)}(x-x'). \tag{4.142}$$

This is an odd result: the inverse of the operator is instantaneous. Nevertheless it is allowed; the two-dimensional operator on the lightcone does not contain any ∂^2 so it does not need to have a momentum dependence. Usually instantaneous operators do not appear due to a different gauge choice, in this case the gauge choice might be odd for this particular result, but it will not be a problem in the end. When we write down the full propagator there will be momentum dependence. Notice that if we were to include the irregular $\frac{x_a x_b}{x^2}$ it still could not lead to a δ^{ab}_{ef} so the inverse would still be instantaneous.

For the rest of the inverse we now propose that $T = -Q$ in Equation 4.140. Inserting this into the definition of the inverse shows that Equation 4.140 is only a valid inverse if the following equation is satisfied:

$$\left(\eta_{ab}\eta^{ef}x^c\partial_c - \eta^{ef}x_{(a}\partial_{b)} - \eta_{ab}x^{(e}\partial^{f)} - \tfrac{1}{2}\left(\delta^e_a x_{[b}\partial^{f]} + \delta^f_a x_{[b}\partial^{e]} + \delta^f_b x_{[a}\partial^{e]} + \delta^e_b x_{[a}\partial^{f]}\right)\right)\delta^{(2)}(x-x') \stackrel{?}{=} 0 \tag{4.143}$$

At first glance this does not seem to vanish. However if here we should interpret the Dirac delta function as a distribution obeying

$$x_a\partial_b\delta^{(2)}(x-x') = -\eta_{ab}\delta^{(2)}(x-x'). \tag{4.144}$$

The identity above appears when integrating over x, y. Using the Dirac delta property we find

$$\begin{aligned}\eta_{ab}\eta^{ef}x^c\partial_c\delta^{(2)}(x-x') \quad &- \quad \eta^{ef}x_{(a}\partial_{b)}\delta^{(2)}(x-x') - \eta_{ab}x^{(e}\partial^{f)}\delta^{(2)}(x-x') \\ &= \quad -2\eta_{ab}\eta^{ef}\delta^{(2)}(x-x') + \eta_{ab}\eta^{ef}\delta^{(2)}(x-x') + \eta_{ab}\eta^{ef}\delta^{(2)}(x-x') = 0\end{aligned}$$

so that the first part of Equation 4.143 vanishes. Doing the same procedure for $x_{[b}\partial^{f]}$ results in

$$x_{[b}\partial^{f]}\delta^{(2)}(x-x') = \tfrac{1}{2}(x_b\partial^f - x^f\partial_b)\delta^{(2)}(x-x') = -\tfrac{1}{2}(\delta^f_b - \delta^f_b)\delta^{(2)}(x-x') = 0 \tag{4.145}$$

so that also for the second part of Equation 4.143 all individual terms vanish. The proper method to do so would be to integrate over all x, y, as then the Dirac delta function can be applied as distribution. Since we are freely allowed to perform such integration the above result is valid. As such we have found a proper inverse given by

$$\Delta_{abcd} = \tfrac{1}{\mu^2(\lambda+1)}(2\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\delta^{(2)}(x-x'). \tag{4.146}$$

When going to momentum space the $\delta^{(2)}(x-x')$ can simply be replaced by 1, giving the final tensor we need:

$$\boxed{\Delta_{abcd} = \tfrac{1}{\mu^2(\lambda+1)}(2\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}).} \tag{4.147}$$

Now that we have inverted Δ^{-1} and Δ^{-1}_{abcd} we can find the propagators determined by Equations 4.129-4.132.

4.5.1 The scalar $\mathcal{K}$ propagator

We first attempt to find the easiest propagator: the scalar $\mathcal{K}$ propagator. For the Eikonal scattering calculation we shall see that this propagator will not be necessary. However in order to prove this we still need the propagator for the proper analysis in Section 5.3. From Equations 4.131 and 4.132 we can find that in momentum space the propagator is given by

$$\mathcal{P}_K = \frac{1}{\Delta^{-1} - \Delta^{-1}_{ab}\Delta^{abcd}\Delta^{-1}_{cd}}. \tag{4.148}$$

We know that $\Delta^{-1} = k^2 + \mu^2$, so we only need to expand the second part. This results in

$$\Delta^{-1}_{ab}\Delta^{abcd}\Delta^{-1}_{cd} = \tfrac{\lambda-1}{\lambda+1}(\mu^2(\lambda-1)+2k^2). \tag{4.149}$$

Then in total the propagator can be written down as

$$\boxed{\mathcal{P}_K = -\frac{\lambda+1}{\lambda-3}\frac{1}{k^2+\mu^2\lambda}.} \tag{4.150}$$

This is a form resembling the Klein-Gordon propagator. There is some prefactor for the spherical harmonics, but the other part is exactly the Klein-Gordon propagator, with effective mass $\mu\sqrt{\lambda}$. What we see is that the scalar mode is now massive, which is both due to the curvature of Schwarzschild spacetime and the spherical harmonics expansion. Because we are now at constant r the potentials express themselves as an effective mass.

There is a problem with the prefactor. The propagator $\mathcal{P}_K$ is not well defined for $\lambda = 3$ which corresponds to $\ell = 1$, and there is a sign change for $\ell = 0$, hinting at negative norm states. We see that the cases $\ell = 0$ and $\ell = 1$ are special, as was observed before in the classical case as well [67, 68]. The equations of motion are not invertible for $\ell = 1$, so a gauge redundancy is to be expected. First we can understand that $\ell = 0$ corresponds to a black hole mass change $R \to R + \delta R$, because the $\ell = 0$ mode is spherically symmetric and Birkhoff's theorem then states that the $\ell = 0$ perturbation is Schwarzschild [68]. Thus there are some gauge redundancies left, as currently at $\ell = 0$ we have many more degrees of freedom than just the mass perturbation. Similarly $\ell = 1$ has has more degrees of freedom than we removed with the gauge transformation, and can be reduced to a form with only h_{aA} components [68]. Thus the $\ell = 1$ describes a rotating gravitational field, like the Kerr black hole. For the Eikonal scattering such contributions are small, as we shall see. This shows that we should treat these modes as special already from the start: $\ell = 1$ and $\ell = 0$ have gauge problems, and furthermore within our Eikonal approximation we can neglect these modes since we neglect black hole mass perturbations and transverse momenta (see Section 5). Thus the above expression is actually safe and we could proceed, as long as we exclude the $\ell = 0, 1$ modes from the calculation for now. Notice that the comments in [68] are in the classical case, however we expect them to also apply strongly in the quantum mechanical case. Additionally recall that for the Eikonal calculation we will not need $\mathcal{P}_K$, in fact we will not see the factor $\frac{\lambda-1}{\lambda+3}$ back in the end of Section 5.

4.5.2 The $\mathfrak{h}_{ab}$ propagator

Let us now look at the propagator of $\mathfrak{h}_{ab}$, which will be the most important one for the Eikonal calculation. To find $\mathcal{P}^{abcd}$ we isolate from Equations 4.129 and 4.130 that

$$\mathcal{P}^{abcd} = (\mathcal{L}^{ab}{}_{ef})^{-1}\Delta^{efcd}, \tag{4.151}$$

$$\mathcal{L}^{ab}{}_{ef} = \delta^{ab}_{ef} - \Delta^{abcd}\Delta^{-1}_{cd}\Delta\Delta^{-1}_{ef}. \tag{4.152}$$

First we also write down Δ^{-1}_{ab} in momentum space:

$$\Delta^{-1}_{ab} = \eta_{ab}(k^2 + \tfrac{1}{2}\mu^2(\lambda-1)) - k_a k_b. \tag{4.153}$$

Using this $\mathcal{L}^{ab}{}_{ef}$ is quickly given with its definition in Equation 4.152. Expanding the tensor structure results in

$$\begin{aligned}\mathcal{L}^{ab}{}_{ef} &= \delta^{ab}_{ef} - \frac{\Delta(k^2+\frac{1}{2}\mu^2(\lambda-1))(\lambda-1)}{\lambda+1}\eta^{ab}\eta_{ef} - \frac{2\Delta(k^2+\frac{1}{2}\mu^2(\lambda-1))}{\mu^2(\lambda+1)}k^a k^b \eta_{ef} \qquad (4.154)\\ &+ \frac{\Delta(\lambda-1)}{(\lambda+1)}\eta^{ab}k_e k_f + \frac{2\Delta}{\mu^2(\lambda+1)}k^a k^b k_e k_f \\ &\equiv \delta^{ab}_{ef} - L_1\eta^{ab}\eta_{ef} - L_2 k^a k^b \eta_{ef} + L_3\eta^{ab}k_e k_f + L_4 k^a k^b k_e k_f. \qquad (4.155)\end{aligned}$$

Here we defined four functions L_i that can be found by comparing the above two forms. To invert Equation 4.155 we make the ansatz

$$(\mathcal{L}^{ab}{}_{ef})^{-1} = \delta^{ab}_{ef} - Q_1\eta^{ab}\eta_{ef} - Q_2 k^a k^b\eta_{ef} + Q_3\eta^{ab}k_e k_f + Q_4 k^a k^b k_e k_f. \tag{4.156}$$

Now multiplying Equation 4.155 with Equation 4.156 under the definition

$$\mathcal{L}^{ab}{}_{ef}(\mathcal{L}^{ef}{}_{cd})^{-1} = \delta^{ab}_{cd} \tag{4.157}$$

and equating all terms of the same tensor structures shows that we can identify a remarkable relation between all L_i and Q_i. This relation is in matrix form given by

$$\begin{pmatrix} \mathcal{M} & 0 \\ 0 & \mathcal{M} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix}. \tag{4.158}$$

This is block diagonal, so we only need to invert the 2x2 matrix defined by

$$\mathcal{M} = \begin{pmatrix} -1 + 2L_1 - L_3k^2 & L_1k^2 - L_3k^4 \\ 2L_2 - L_4k^2 & -1 + L_2k^2 - L_4k^4 \end{pmatrix} = \frac{\Delta}{\lambda+1} \begin{pmatrix} -2k^2 + \mu^2\lambda(\lambda-3) & \frac{1}{2}\mu^2k^2(\lambda-1)^2 \\ \frac{2}{\mu^2}(k^2+\mu^2(\lambda-1)) & -2k^2 - \mu^2(\lambda+1) \end{pmatrix}. \tag{4.159}$$

A calculation involving important rewritings of the polynomials in λ shows that the matrix determinant is given by

$$|\mathcal{M}| = -\tfrac{\lambda-3}{\lambda+1}(k^2 + \mu^2\lambda)\Delta = \frac{\Delta}{\mathcal{P}_K}. \tag{4.160}$$

This determinant vanishes for $\lambda = 3$, same as the inverse propagator of $\mathcal{K} \leftrightarrow \mathcal{K}$. This shows that the equations of motion are indeed not orthogonal at $\lambda = 3$ and we have residual gauge freedom. Additionally the same comments apply to this determinant as to the scalar case: There are gauge problems at $\ell = 0, 1$, and furthermore we resemble Klein-Gordon. Notice that the form $\Delta/\mathcal{P}_K$ essentially tells us that upon inverting we replace all Δ operators by $\mathcal{P}_K$.

We can now easily write down the inverse matrix to be

$$\mathcal{M}^{-1} = \frac{\mathcal{P}_K}{\lambda+1} \begin{pmatrix} -2k^2 - \mu^2(\lambda+1) & -\frac{1}{2}\mu^2k^2(\lambda-1)^2 \\ -\frac{2}{\mu^2}(k^2+\mu^2(\lambda-1)) & -2k^2 + \mu^2\lambda(\lambda-3) \end{pmatrix}. \tag{4.161}$$

At this stage we recognize that $2L_1 = \mu^2(\lambda-1)L_2$, and $2L_3 = \mu^2(\lambda-1)L_4$. Thus we only need to calculate

$$\mathcal{M}^{-1} \begin{pmatrix} \mu^2(\lambda-1) \\ 2 \end{pmatrix} = -\frac{\mathcal{P}_K}{\Delta} \begin{pmatrix} \mu^2(\lambda-1) \\ 2. \end{pmatrix} \tag{4.162}$$

What we see is in fact an eigenvalue equation, so that the 2x2 matrix $\mathcal{M}$ is simply reduced to only a scalar in this particular case. The corresponding eigenvalue is exactly given by $-|\mathcal{M}|^{-1}$ so minus the reciprocal of the determinant. This allows us to easily identify the solution to the inverse being

$$\boxed{Q_i = -\frac{\mathcal{P}_K}{\Delta} L_i.} \tag{4.163}$$

Finally this means that $\mathcal{L}^{ab}{}_{ef}$ equals $(\mathcal{L}^{ab}{}_{ef})^{-1}$ upon replacing all Δ by $\mathcal{P}_K$ and changing sign. Inserting Q_i into Equation 4.156 results in

$$\begin{aligned} (\mathcal{L}^{ab}{}_{ef})^{-1} &= \delta^{ab}_{ef} + \frac{\mathcal{P}_K(k^2 + \frac{1}{2}\mu^2(\lambda-1))(\lambda-1)}{\lambda+1}\eta^{ab}\eta_{ef} + \frac{2\mathcal{P}_K(k^2+\frac{1}{2}\mu^2(\lambda-1))}{\mu^2(\lambda+1)}k^ak^b\eta_{ef} \\ &- \frac{\mathcal{P}_K(\lambda-1)}{(\lambda+1)}\eta^{ab}k_ek_f - \frac{2\mathcal{P}_K}{\mu^2(\lambda+1)}k^ak^bk_ek_f. \end{aligned} \tag{4.164}$$

We can insert the above solution for $(\mathcal{L}^{ab}{}_{ef})^{-1}$ into Equation 4.151 to find the propagator. Remark that the mathematics for this calculation reduced to trivial relations. When we manage to work through the heap of the calculations, it appears that the underlying structure merely consists of eigentensors, elegantly displaying the underlying symmetry structure of the graviton. In the end when inverting for the propagator, all we had to do was multiply by a single scalar, ignoring the tensor structure altogether.

The solution for $\mathcal{P}_{abcd}$

Finally inserting Equation 4.164 into 4.151 results in

$$\mathcal{P}^{abcd} = \Delta^{abcd} + \mathcal{P}_K \mathrm{p}^{ab}\mathrm{p}^{cd} \tag{4.165}$$

$$\mathrm{p}^{ab} = \frac{\lambda-1}{\lambda+1}\eta^{ab} + \frac{2k^a k^b}{\mu^2(\lambda+1)}. \tag{4.166}$$

Here Δ^{abcd} is still defined in Equation 4.147. This propagator possesses the correct graviton symmetries as desired (interchange of indices). The p^{ab} operators are the projection operators for the massive graviton. In fact looking at the scalar part $\mathcal{P}_k$ there is clearly a pole when

$$p^2 = -\mu^2\lambda$$

so we observe by the on-shell relation that there is an effective mass $m = \mu\sqrt{\lambda}$. We will comment on any UV-problems of this operator at a later stage. What we observe is that the tensor field $\mathfrak{h}_{ab}$ has also become massive, and that the 2D projection of the graviton has effectively become massive. We will call this the 2D effective mass.

A similar projection operator to p^{ab} has also been found for the massive graviton in [69]. The resemblance between the propagator in [69] and Equation 4.165 is strong, and the differences can be attributed due to the indirect appearance of this mass due to curvature and the spherical harmonics, rather than direct insertion of mass terms in the action. The most significant difference is the Δ^{abcd} in front; this quantity still has no momentum dependence. It represents the effect of the mass of the spin-2 particle in two dimensions. The graviton now has a 2D effective mass and this mass generates a curvature, the Δ^{abcd} embeds this curvature into the propagator. This is also the reason that it has no momentum dependence: the curvature contribution of the tensor modes is constant in the Lagrangian, there is no k^2 contributing in the two-dimensional lightcone Lagrangian in Equation 4.121. The second part then gives the main contribution from the graviton momentum, and only appeared when taking the scalar mode into account. Only the scalar mode carries momentum dependence, and implicitly also gives it to the tensor modes. In the end it turns out that this separation is actually ideal since for the Eikonal scattering in two dimensions only soft gravitons contribute, so the Δ^{abcd} in front is dominant. This will become apparent in Subsection 5.4.

We shall see in Subsection 5.3 that $\mathfrak{h}^{xy}$ plays no role in scattering with a massless scalar field in an exact manner. Similarly the scattering between $\mathfrak{h}_{xx} \leftrightarrow \mathfrak{h}_{xx}$ vanishes for Eikonal scattering. As such we only need $\mathcal{P}^{xxyy} = \mathcal{P}^{yyxx}$ where equality follows from time inversion invariance. This allows us to ignore most of the complexity of Equation 4.165 and effectively write down the propagator as

$$\mathcal{P}^{abcd} = \tfrac{1}{4}F_\ell(k^2)(\eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}). \tag{4.167}$$

The choice of 1/4 is to find better factors at a later stage in the calculation. From Equation 4.165 we can determine F_ℓ to be given by

$$F_\ell(k^2) = -\frac{4}{\mu^2(\lambda+1)} - \frac{2}{(\lambda+1)(\lambda-3)}\frac{1}{k^2+\mu^2\lambda}\frac{k^4}{\mu^4}. \tag{4.168}$$

Now we can clearly see problems in the UV-limit. We have $F_\ell \sim k^2/\mu^4$ which diverges in the high energy limit, as was also noted in [69]. This is a result of the mass presence in the projection operator p^{ab}. When looking at the massive spin-1 field there is a similar $F \sim 1$ in the UV-limit, which renders the theory not renormalizable by the power counting theorem [70]. Massive Quantum Electrodynamics is renormalizable due to the introduction of additional scalar fields [71]. A similar trick can be applied to the spin-2 case, introducing vector fields to restore the power counting theorem for the graviton. This was also done in [69], however renormalization remains a problem, as it was already a problem for the massless graviton.

In this paper we shall not concern ourselves with these problems, and simply work with the propagator in Equation 4.167. We shall see that the mentioned UV-problems will not be present in the Eikonal calculation, and so we do not need to worry about the k^2/μ^4 tail. Instead it will be much more important that

$F_\ell(k^2)$ is well defined when $k = 0$. This soft limit, where $F_\ell(0)$ is regular, will become the only contribution in Eikonal scattering. Thus the mass of the graviton is the most crucial result of this section; whereas it generates problematic UV-tails it also generates a safe IR-region, and the latter will be the only important contribution. With this propagator found we can move on to calculating the scattering diagrams.

The coupling propagator

For completeness we also give the $\mathcal{K} \leftrightarrow \mathfrak{h}_{ab}$ propagator. We can find this coupling propagator directly from Equation 4.130 to be given by

$$\mathcal{P}^{ab} = -\Delta\Delta^{-1}_{cd}\mathcal{P}^{cdab} \tag{4.169}$$

$$= -\mathcal{P}_K \mathrm{p}^{ab}. \tag{4.170}$$

Here p^{ab} is still the same projection operator. This resembles much of the massive spin-1 propagator, which is expected since we are looking at a propagator that connects a spin-2 and spin-0 field. We shall not need this propagator for Eikonal scattering and pay no further attention to it.

5 Eikonal approximation

In the previous section we have reduced the graviton field on the Schwarzschild horizon to a two-dimensional set of fields $\mathfrak{h}_{ab}, \mathcal{K}$ and the odd mode h_a. We then derived the corresponding graviton propagator necessary for quantum field theory. The aim is now to add this graviton to scattering particles on the Schwarzschild black hole, in order to obtain a similar scattering matrix as in Section 3 by summing over infinitely many diagrams. The resulting scattering amplitude is given in Subsection 5.5 although we strongly advice the reader to read this section completely.

In previous work [33] it has been shown that the scattering matrix from the Shapiro delay in Minkowski space can also be obtained by summing over diagrams in the right limit. Simply speaking the limit states that the scattered particles have large external momenta in the lightcone directions, small momenta in the transverse directions, and that the impact parameter b is large (where "large" will be specified later). This limit is called the Eikonal approximation. A large impact parameter b means the scattered particles are largely separated and keep most of their trajectory, which is called forward scattering. As a proper definition the Eikonal approximation consists of taking all diagrams for forward scattering at highest order in s/μ^2 only with $s = -(p_1 + p_2)^2$ the Mandelstam variable corresponding to the center of mass energy and μ a relevant mass scale which we shall define later. The momentum exchange $t = -(p_1 - p_1')^2$ is kept finite and small, such that $s \gg t$.

In this section we shall similarly try to perform the Eikonal approximation but now on the Schwarzschild horizon utilizing the propagator derived in Section 4. We want to find a relation between particles leaving the horizon and entering the horizon. Thus we have two external particles $\phi_{\ell m}(p_1)$ and $\phi_{\ell' m'}(p_2)$. These particles respectively are created and annihilated on the nontrivial vacuum state of the black hole, in the case of our coordinate choice this would correspond to the Hartle-Hawking state. This means that we also add the canceling contributions of the vacuum $\phi_0(p_1)$ and $\phi_0(p_2)$. These fields are at $\ell = 0$ since the Schwarzschild spacetime is spherically symmetry. The resulting diagram governing the exchange is given in Figure 11. The physical intuition of this scattering diagram was given by the image on the title-page, where the wiggly lines are to be understood as ϕ_0 contributions and the solid blue line as $\phi_{\ell m}$. The black sphere in the center is the gravitational scattering between these (the virtual graviton in Figure 11).

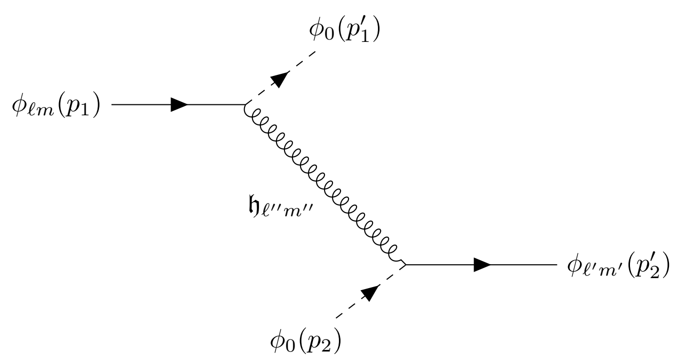


Figure 11: The conceptual diagram for the black hole scattering we consider.

Here we ignored the spatial indices a, b to retrieve the qualitative treatment. Now because at the vertices we have conservation of angular momenta and ϕ_0 is spherically symmetric, we find that $\ell = \ell'' = \ell'$ and $m = m'' = m'$. This also means that the external (ℓ, m) modes decouple. This is of significant importance, because normally conservation of angular momentum allows many mixed states related by Clebsch-Gordan coefficients, but due to the spherically symmetric background we can ignore this effect. We do remark that such mixed states would still occur in loops, however we work around those using a minimal coupling simplification.

The above diagram is consistent with the scattering matrix derived in Section 3. In this scattering matrix we considered a set particles going in, which kept moving unperturbed and a set of particles going out, that was then perturbed by the in-going particles. The information of the in-going particle that was lost behind the horizon, was this way embedded in the out-going particle through the Shapiro delay, resulting in unitarity. We remark that confusion may arise in what is in-going and out-going, because in the scattering experiment $\phi_{\ell m}(p_1)$ and $\phi_0(p_2)$ are the in-going particles. The outgoing scattered particles are then $\phi_{\ell m}(p_2)$ and $\phi_0(p_1)$. This difference is noteworthy, the in-going particles in the scattering are different than the in-falling particles into the black hole. We can always check in the diagram which particles are in-going and out-going in the scattering experiment, while which particles are in-falling and out-going is easily checked by looking at the momenta p_1 and p_2.

The mathematical expectation value we want to find is defined by

$$S = \langle \phi_{\ell m}(p_1)\phi_0(p_2)\phi_{\ell m}(p_2')\phi_0(p_1') \rangle. \tag{5.1}$$

It is important to note that the diagram in Figure 11 is not the only contribution to this expectation value, we have two channels to consider. We call these two channels the Transfer-channel (T-channel) and the Conserved-channel (C-channel). Diagrammatically these contributions would look as shown in Figure 12.

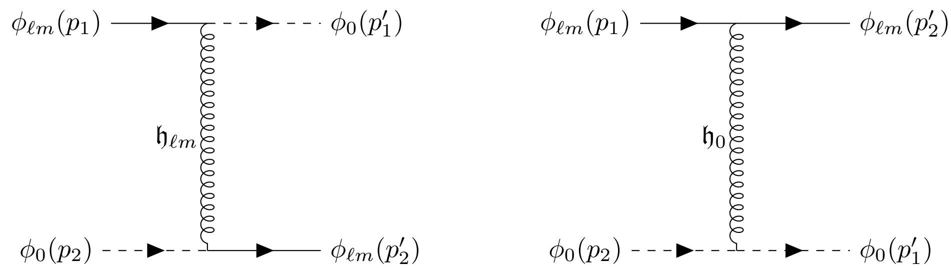


Figure 12: The two scattering channels, the T-channel is on the left and the C-channel on the right.

As visible in Figure 12 the T-channel consists of moving the ℓ, m over the graviton leg, hence the transfer-channel. The C-channel keeps the ℓ, m on the same side with graviton leg having $\ell = 0$, hence the conserved channel.

We will go further into detail on these diagrams and the exchange in Subsection 5.3. Remark that Equation 5.1 does not give the only possible contribution, it is possible that there are more powers of ϕ_0 contributing. We ignore these affects as they are likely subdominant in Eikonal scattering. Similarly at higher order loops there will be ghosts from the gauge symmetry, whose contribution we also neglect as lower order in Eikonal scattering.

We will first look at the Eikonal approximation in Minkowski space, to understand the steps taken and what the Eikonal approximation exactly implies. The next subsection will be entirely based on work by Kabat & Ortiz [33], where we will see that the summation of Eikonal graphs converges to the semi-classical Shapiro delay derived by 't Hooft in [34] for Minkowski space. In the latter an analogous procedure as in Section 3 has been provided for Minkowski space. Given that both procedures give the same result in Minkowski space, we expect this similarity also to exist in the Schwarzschild spacetime. In other words, we expect to find the same scattering amplitude as was found in Section 3 by performing the Eikonal approximation on the Schwarzschild spacetime instead.

The alternative method by Verlinde et al.

Let us now shortly comment on the method provided by Verlinde et al, who provided a method involving quantization of the fields in [26]. The authors started from the ansatz that

$$\phi_{\text{out}}(y, \Omega) = \phi_{\text{in}}\big(x(y) + \delta y_0(\Omega), {}^p\Omega\big). \tag{5.2}$$

Here ${}^p\Omega$ is the antipodal point of Ω. The quantity $\delta y_0(\Omega)$ is defined in [26], but we will not need it for this discussion. The equation above explicitly relates the in- and out-fields, however we see no clear reason why this equation should hold. As shown in Section 2 the gravitational backreaction induces a shift $y \to y + \delta y$, however this shift did not result in a shift in the wavefunctions in Section 3, rather it resulted in the wavefunctions being related by a Fourier transformation

$$\phi_{\text{out}}(y) = \frac{1}{\sqrt{2\pi\alpha}} \int \mathrm{d}x \; e^{-ixy/\alpha} \phi_{\text{in}}(x) \tag{5.3}$$

This suggests that also when quantizing the fields the relation between the in- and out-fields might not by obtained by straightforward translation as the Shapiro delay dictates. We do not know how the Shapiro delay expresses itself for a quantum field theory. In this section we will find the scattering amplitude for the Shapiro delay without making any ansatz, and see that a Fourier transformation indeed results.

5.1 Minkowski space

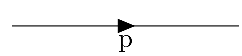
In this subsection we shall perform the same steps as Kabat & Ortiz [33] in order to find the Eikonal scattering amplitude in flat space. The metric we work with is now given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{5.4}$$

So we are in $(3+1)$ dimensions in Cartesian coordinates. We will consider elastic forward two-particle scattering of two massive scalar particles, with ingoing momenta p_1, p_2 and outgoing momenta p_3, p_4. These momenta are in the Eikonal limit, such that $s = -(p_1+p_2)^2 \gg -(p_1-p_3)^2 = t$. For the flat-space case the Eikonal limit consists of only taking the leading order s/m^2 with m the mass of the scalar particle. Because of the large impact parameter, which in flat space is defined by $b \gg \ell_{\text{Pl}}$, the two scattering particles maintain most of their momentum in the scattering direction which we call longitudinal, i.e. $p_1^{\|} \approx p_2^{\|}$. The two particles do however exchange momentum in the remaining two directions; the transverse directions, such that $p_1^{\perp} \neq p_2^{\perp}$. For all particles it still holds that $p_i^{\|} \gg p_i^{\perp}$. This allows us to consider diagrams where there is no exchange in four momentum only, as was also noted by [33] and will be shown extensively in the next Subsection 5.3.

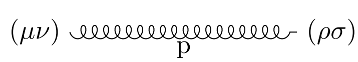
The Feynman diagrams we consider will be the Eikonal ladder diagrams displayed in Figure 13. All other possible loop diagrams (self-energy, vertex corrections) will be small in the Eikonal regime, i.e. lower order in s/m^2. The relevant Feynman rules are the usual Minkowski space rules given by

Scalar propagator



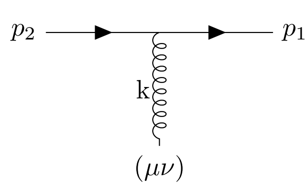
$= \frac{-i}{p^2+m^2-i\epsilon}$

Graviton propagator



$= \frac{-2i\kappa}{p^2-i\epsilon}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})$

vertex



$= ip^1_\mu p^2_\nu$

Here the scalar propagator is simply the Klein-Gordon propagator and the graviton propagator is in the harmonic gauge with $\kappa = 8\pi G$. The vertex is in principle given by

$$p_\mu p_\nu - \tfrac{1}{2}\eta_{\mu\nu}(p^2+m^2)$$

but the recoil of the matter field can be neglected for Eikonal scattering, such that only $p_\mu p_\nu$ survives. Similarly we approximate the scalar propagator by

$$\frac{1}{(p+k)^2 + m^2 - i\epsilon} \approx \frac{1}{2p\cdot k - i\epsilon}$$

holding at leading order in p, which we assumed large for the Eikonal approximation.

Summing the diagrams

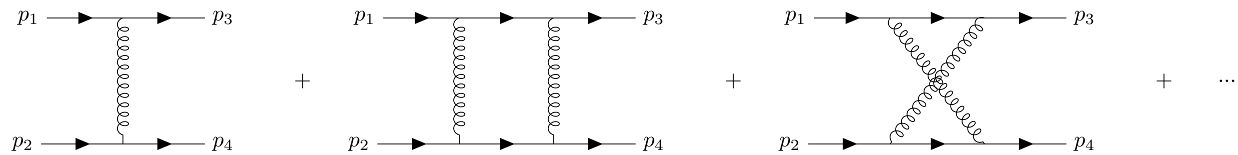


Figure 13: All ladder diagrams contributing to the Eikonal scattering amplitude. Notice that no h^4 vertex exists in the linear theory: the two graviton lines crossing each other just pass. The ladder diagrams give all leading contributions in s/m^2.

The first diagram in Figure 13 is easily shown to be given by

$$i\mathcal{M} = \frac{2i\kappa\gamma(s)}{-t} \tag{5.5}$$

where

$$\gamma(s) = \tfrac{1}{2}\left((s-2m^2)^2 - 2m^4\right)$$

and $s \equiv -(p_1+p_2)^2$ and $t = -(p_1-p_3)^2$ are the usual Mandelstam variables.

Let us proceed to look at the loop diagrams in Figure 13. In these it is crucial to average over the possible ways to define the internal graviton legs, and the corresponding conserved momentum. For the 1-loop case there are two possibilities, namely fixing k on the first leg or the second leg in a direction of choice. This results in summing over both of these and inserting a factor 1/2. Knowing this the procedure is straightforward using the Feynman rules and outlined in [33]. The final expression for the loop becomes

$$i\mathcal{M} = \kappa\gamma(s)\int \mathrm{d}^4x\; e^{-i(p_1-p_3)\cdot x}\Delta(x)\chi(x), \tag{5.6}$$

$$\chi = -2\kappa\gamma(s)\int \tfrac{\mathrm{d}^4k}{(2\pi)^4}\; e^{ik\cdot x}\frac{1}{k^2-i\epsilon}\times\left[\frac{1}{-2p_1\cdot k-i\epsilon}\frac{1}{2p_2\cdot k-i\epsilon}+\frac{1}{-2p_1\cdot k-i\epsilon}\frac{1}{-2p_4\cdot k-i\epsilon}\right. \tag{5.7}$$

$$\left.+\;\frac{1}{2p_3\cdot k-i\epsilon}\frac{1}{2p_2\cdot k-i\epsilon}+\frac{1}{2p_3\cdot k-i\epsilon}\frac{1}{-2p_4\cdot k-i\epsilon}\right]. \tag{5.8}$$

In applying the Feynman rules we approximated $(p_1+k)_\mu \approx p_{1\mu}$ since p_1 is very large. Of course k is integrated over, but the leading orders of the integrand are for $k \ll p_1$ so we can safely take only p_1 in the Eikonal approximation. Here we have defined a momentum space massless Klein-Gordon propagator

$$\frac{-1}{k^2-i\epsilon} = \int \mathrm{d}^4x\; e^{-ik\cdot x}\Delta(x). \tag{5.9}$$

Notice that the expression for χ does not contain any UV-divergences, also if we include the k from the vertex as we would then also need to include a k^2 in the matter propagator. The UV-divergences are all embedded in different diagrams, whose effects are subdominant in Eikonal scattering. This is crucial as the theory is not renormalizable so we cannot get rid of any divergencies appearing. For the Eikonal approximation we can ignore all UV-divergent diagrams as subleading in s/m^2, so that within the Eikonal approximation no divergences appear. It might seem as if such diagrams would still contribute given they diverge, but if dimensionally regularized the diagrams would simply be at a lower order in the Eikonal approximation, so given the the regularization we can formally remove them. We have to be satisfied with this methodology, as currently no other option exists.

The Eikonal approximation consists of all ladder diagrams such as in Figure 13, so also at higher loop orders. The next step is to sum over all other loops. Since any diagram at order n would contain n graviton legs, there is a symmetry factor $n!$. This also holds for diagrams where the graviton legs cross eachother. This results in a $1/n!$ factor for each order in perturbation theory, so non-perturbatively we can recognize an exponential series. This observation can luckily be made thanks to the negligence of all vertex corrections and self energies. This is shown in [30]. The total scattering amplitude is thus written as

$$i\mathcal{M} = -2i\kappa\gamma(s)\int \mathrm{d}^4x\ e^{-i(p_1-p_3)\cdot x}\Delta(x)\frac{e^{i\chi}-1}{\chi}. \tag{5.10}$$

The only remaining part is determining χ and solving the integral. For determining χ we can safely approximate $p_1 \approx p_3, p_2 \approx p_4$, which could not be done earlier due to the $p_1 - p_3$ appearing as regulator.

Inserting $p_1 = p_3, p_2 = p_4$ into Equation 5.7 results in:

$$\chi = -2\kappa\gamma(s)\int \tfrac{\mathrm{d}^4k}{(2\pi)^4}\ e^{ik\cdot x}\frac{1}{k^2-i\epsilon}\left[\frac{1}{2p_1\cdot k+i\epsilon}-\frac{1}{2p_1\cdot k-i\epsilon}\right]\left[\frac{1}{2p_2\cdot k+i\epsilon}-\frac{1}{2p_2\cdot k-i\epsilon}\right]. \tag{5.11}$$

We can now use the following delta identity which holds because ϵ is an infinitesimal regulator:

$$\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon} = -2\pi i\delta(x). \tag{5.12}$$

Using Equation 5.12 to remove two of the integrals in Equation 5.11, we rewrite it as

$$\chi = -2\kappa\gamma(s)\int \tfrac{\mathrm{d}^4k}{(2\pi)^4}\ e^{ik\cdot x}\frac{1}{k^2-i\epsilon}(-2\pi i)^2\delta(2p_1\cdot k)\delta(2p_2\cdot k) \tag{5.13}$$

$$= \frac{\kappa\gamma(s)}{4Ep}\int \tfrac{\mathrm{d}^2k_\perp}{(2\pi)^2}\ e^{ik_\perp\cdot x_\perp}\frac{1}{k_\perp^2+\mu^2-i\epsilon}. \tag{5.14}$$

In this step we switched to the center of mass frame where $p_1 = (E,0,0,p)$ and $p_2 = (E,0,0,-p)$. The μ is an infrared regulator corresponding to the graviton mass (in flat space the graviton is massless) and the $x_\perp$ are the remaining two coordinates in the transverse direction. The solution to the integral in Equation 5.14 can be shown to be given by

$$\chi = \frac{G\gamma(s)}{Ep}K_0(\mu x_\perp) \approx -\frac{G\gamma(s)}{Ep}\log(\mu x_\perp) \tag{5.15}$$

where the last approximation holds for $\mu x_\perp \ll 1$ since μ is a regulator. A numerical constant has been absorbed into μ.

We now turn back to solving Equation 5.10. We only need to solve the integral now that χ is known. First we look at the $\Delta(x)$. For the Eikonal approximation $p_1 - p_3$ only contains transverse components, i.e. only the transverse components of $q \equiv p_1 - p_3$ are dominant. This allows us to write $e^{-iq\cdot x} \approx e^{-iq_\perp\cdot x_\perp}$. We can then isolate

$$\int \mathrm{d}t\mathrm{d}z\ \Delta(x) = \int \tfrac{\mathrm{d}^2q_\perp}{(2\pi)^2}e^{iq_\perp\cdot x_\perp}\frac{-1}{q_\perp^2-i\epsilon} = \frac{-Ep}{2\pi G\gamma(s)}\chi \tag{5.16}$$

which we can safely insert since χ only depends on $x_\perp$. Thus the remaining integral to solve becomes simply

$$i\mathcal{M} = 8Ep\int \mathrm{d}^2x_\perp\ e^{-iq_\perp\cdot x_\perp}\left(e^{i\chi}-1\right). \tag{5.17}$$

Solving this integral yields

$$i\mathcal{M} = \frac{2i\kappa\gamma(s)}{-t}\frac{\Gamma(1-i\alpha(s))}{\Gamma(1+i\alpha(s))}\left(\frac{4\mu^2}{-t}\right)^{-i\alpha(s)} \tag{5.18}$$

where

$$\alpha(s) = G\frac{(s-2m^2)^2-2m^4}{\sqrt{s(s-4m^2)}}. \tag{5.19}$$

Equation 5.18 gives the full Eikonal scattering amplitude. It is non-perturbative in the coupling constant κ, but only at leading order s/m^2, so we traded one perturbation for the other. However since s can be chosen arbitrarily high externally this is a valid perturbation to make. Notice that in this subsection we have not yet distinguished between the T-channel and C-channel as they both give the same amplitude. This difference will only be important for the Schwarzschild spacetime.

Remark that for α real, which holds for s large, the full amplitude corresponds to the tree level amplitude in Equation 5.5 times a phase factor. In the limit $m \to 0$ and for $\mu = 1$ the scattering amplitude corresponds perfectly to the semi-classical scattering matrix derived by 't Hooft in [34]. This is crucial to the motivation for this section. The scattering matrix derived in [34] has been derived using the exact same method as the one in Section 3 but for the Minkowski Shapiro delay in Subsection 2.3. The Eikonal approximation rederives this using the Feynman diagram formalism. The aim of this section is to rederive the scattering matrix in Equation 3.36 using the Eikonal approximation on the Schwarzshild horizon, in the same manner as was done for flat space. First we will now properly define our quantum field theory, and then we move on to calculating the amplitude using the same procedure as in this subsection.

5.2 Matter action

To actually find the gravitational back reaction we naturally need to specify what particles fall in and go out. For this paper we take the simple case of a massless real scalar field ϕ. A more realistic scenario could be fund by extension to QED bosons/fermions, to include charges and/or parity. The four-dimensional matter action for the scalar field is given by

$$S_m = -\tfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\phi = \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\phi\Box\tilde{\phi}. \tag{5.20}$$

Here $g_{\mu\nu}$ is the background Schwarzschild metric as given in Appendix A.1.

The scalar propagator

The first aim is to find the propagator for the scalar field, which will be needed for the calculation of loops. We want to find the propagator on the Schwarzschild horizon similar to Section 4.4. To do so we perform a series of integral manipulations similar to that section. Again remark here that in principle we need to perform the approximations on the equation of motion, but by using the one-sided action the equation of motion is explicit, and any modifications on the action are equivalent to modifications to the equation of motion. Thus we start from the second part of Equation 5.20, the one-sided action

$$S_m = \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\tilde{\phi}\Box\tilde{\phi}. \tag{5.21}$$

For the box operator we then write

$$\Box = \tilde{\Box} + V^a\tilde{\nabla}_a + \tfrac{1}{r^2}\Delta_\Omega. \tag{5.22}$$

Here we used the residual curvature tensor V_a, which is by definition given by $V_a = 2\partial_a \log r$. We now expand the scalar field into spherical harmonics:

$$\tilde{\phi} = \sum_{\ell,m}\tilde{\phi}_{\ell m}Y_\ell^m. \tag{5.23}$$

In this case we can again use $\Delta_\Omega Y_\ell^m = -\ell(\ell+1)Y_\ell^m$. Using the same orthogonality relation as for the graviton in Equation 4.68 we write down the matter action as

$$S_m = -\tfrac{1}{2}\sum_{\ell,m}\int \mathrm{d}^2x Ar^2\tilde{\phi}_{\ell m}\big(-\tilde{\Box} - V^a\tilde{\nabla}_a + \tfrac{\ell(\ell+1)}{r^2}\big)\tilde{\phi}_{\ell,m}. \tag{5.24}$$

Here the covariant derivatives with a tilde are defined only on the 2D metric g_{ab}, similar to the procedure in Section 4.3. We now perform the Weyl transformation, redefining $\phi = r\tilde{\phi}$ in the process to remove the Jacobian, which results in simply

$$S_m = -\tfrac{1}{2}\sum_{\ell,m}\int \mathrm{d}^2x\ \phi_{\ell m}\big(-\partial^2 - \tfrac{1}{r}(\partial^2 r) + \tfrac{A\ell(\ell+1)}{r^2}\big)\phi_{\ell,m}. \tag{5.25}$$

We see that the redefinition to include r in the scalar field automatically removes the first derivatives, and replaces it for a potential $\sim (\partial^2 r)$. Finally we insert that on the horizon $r \sim R$ and $\partial^2 r \sim -\frac{1}{2R}$ (Appendix A.1), according to $x, y \ll R$ as was done for the graviton, to find the following scalar action:

$$\boxed{S_m = -\tfrac{1}{2}\sum_{\ell,m}\int \mathrm{d}^2x\phi_{\ell m}\big(-\partial^2 + \mu^2\lambda\big)\phi_{\ell m}.} \tag{5.26}$$

Here $\lambda = \ell^2+\ell+1$ and $\mu = 1/R$. We performed the rewriting from Equation 5.20 to 5.26 in order to properly define the scalar field propagator. In this case the propagator is easily found by the Fourier transformation

$$\phi(x) = \tfrac{1}{(2\pi)^2}\int \mathrm{d}^2p e^{ip_a x^a}\phi(p), \tag{5.27}$$

$$\phi(p) = \int \mathrm{d}^2x e^{-ip_a x^a}\phi(x). \tag{5.28}$$

The inverse propagator can be read of from Equation 5.26 as $\partial^2 - \mu^2\lambda$. Under the given Fourier convention the propagator is directly found by inserting $\partial_a = ip_a$, resulting in

$$\boxed{\mathcal{P}_\phi = -\frac{1}{p^2+\mu^2\lambda - i\epsilon}.} \tag{5.29}$$

This is the propagator for certain fixed (ℓ, m) since only same (ℓ, m)-modes talk to each other in the action. In the next section we shall assume $\partial_A \to 0$ since the external particles have insignificant transverse momenta. For the vertex this means that at tree level there is no contribution from the transverse derivatives ∂_A, and the contribution from transverse derivatives in the loops will be subdominant compared to the contribution from lightcone derivatives, so we also neglect it for loops. For the propagator this will also mean ignoring the $\mu^2\lambda$ part for the most, and assuming that on-shell the lightcone momenta satisfy $(p_1)^2 = (p_2)^2 = 0$ which will be made explicit later. We have now properly defined the matter field, so the next step is the interaction between the scalar field and the graviton.

5.3 The vertex

The second Feynman rule we will need is the vertex between the scalar field and the gravitons. First we define the external momenta of the scalar particles, as we are only interested in the vertex in the Eikonal approximation. We take one particle going into the black hole and one going out in orthogonal orbits, such that $p_y^1 = 0$ and $p_x^2 = 0$. We also neglect all transverse momenta completely, i.e. we ignore $\mu^2\lambda$ in the scalar propagator, and we will ignore transverse momenta in the vertex. This is a safe procedure because the lightcone momenta are extremely large, so all leading contributions will come from those.

The Mandelstam variable s we define including a factor one-half such that

$$s = -\tfrac{1}{2}(p_1+p_2)^2.$$

For the momenta defined above we then have simply $s = p_x^1 p_y^2$. In this case we define the Eikonal approximation by taking the leading order only in s/μ^2 with μ the inverse Schwarzschild radius, while taking $t \ll s$ so that we neglect all momentum transfer. We will see that the vertex simplifies greatly by neglecting any transverse momenta p_A and keeping only leading order contributions in s/μ^2 at all orders in perturbation theory. Remember that this Eikonal limit is very safe given the diverging energy of black hole particles. Additionally

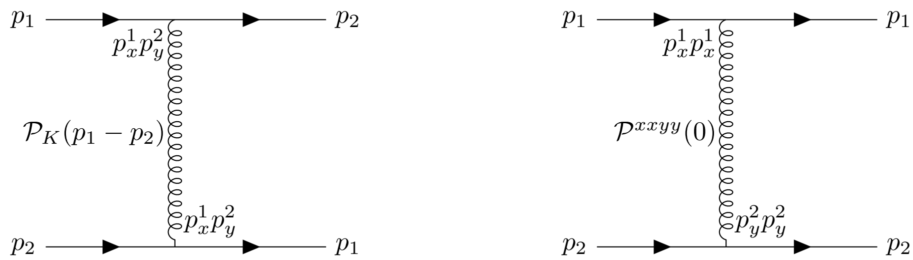


Figure 14: The two leading order diagrams for the different graviton fields $\mathcal{K}$ and $\mathfrak{h}_{ab}$. On the left is the momentum exchange diagram of $\mathcal{K}$, since the no-momentum exchange diagram vanishes exactly. On the right is the no-momentum exchange diagram of $\mathfrak{h}_{ab}$, which does not vanish.

we do not need to define which particle has high energy or not, we only need their center of mass energy s to be very high, which allows one of the momenta to be relatively small. This allows us to calculate both the hard-hard and the hard-soft interaction at the same time, whereas in Section 3 the hard-hard interaction was a problem. Of course the hard-hard interaction likely does not satisfy linear gravity, nevertheless we have safely included the hard-soft interaction.

We will look more closely at what the condition of s/μ^2 being very large implies on the momenta p_x^1, p_y^2 in the next subsection. Notice that gravity is a gauge theory, so normally we would have to use ghost fields in the theory as well. Similar to [33] we assume the ghosts to be an order lower in s/μ^2 for forward scattering so we will ignore all ghosts in this paper.

We find the vertex by inserting Equation 4.2 in Equation 4.6, for the action in Equation 5.20. This results in

$$S_{\text{vertex}} = \frac{\sqrt{\kappa}}{2}\int \mathrm{d}^4x h^{\mu\nu}\big(\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\tilde{\phi}\partial_\sigma\tilde{\phi}\big). \tag{5.30}$$

We now immediately insert that $p_A = 0$ in the vertex, which gives

$$S_{\text{vertex}} = \frac{\sqrt{\kappa}}{2}\int \mathrm{d}\Omega\int \mathrm{d}^2x\ Ar^2h^{ab}\big(\partial_a\tilde{\phi}\partial_b\tilde{\phi} - \tfrac{1}{2}g_{ab}g^{cd}\partial_c\tilde{\phi}\partial_d\tilde{\phi}\big) + \frac{\sqrt{\kappa}}{2}\int \mathrm{d}\Omega\int \mathrm{d}^2x\ Ar^2h^{AB}T_{AB}. \tag{5.31}$$

This shows that h^{aA} does not appear, so we also do not need the odd harmonics vector h_a at all. This is the reason why we never even looked at the odd harmonics anymore when the decoupling was proven; in the Eikonal approximation the odd harmonics have no contribution at all, direct or indirect. The odd harmonics only come in when the external particles have significant transverse momenta, which is a possible extension to the theory. Now additionally we insert the lightcone fields

$$h_{ab} = \frac{1}{Ar}\mathfrak{h}_{ab}, \qquad \phi = \tfrac{1}{r}\tilde{\phi}, \qquad K = \frac{1}{r}\mathcal{K}$$

on the horizon with $x^a \ll R$. This results in

$$S_{\text{vertex}} = \frac{\sqrt{\kappa}}{2R}\int \mathrm{d}\Omega\int \mathrm{d}^2x\ \mathfrak{h}^{ab}\big(\partial_a\phi\partial_b\phi - \tfrac{1}{2}\eta_{ab}\eta^{cd}\partial_c\phi\partial_d\phi\big) + \frac{\sqrt{\kappa}}{2R}\int \mathrm{d}\Omega\int \mathrm{d}^2x\ \mathcal{K}\eta^{AB}T_{AB}. \tag{5.32}$$

We shall now simplify this vertex within the Eikonal approximation. Let us first look more closely at the second part involving $\mathcal{K}$. Inserting T_{AB} shows that the vertex is proportional to

$$\mathcal{K}\eta^{AB}T_{AB} \sim \mathcal{K}\eta^{ab}\partial_a\phi\partial_b\phi. \tag{5.33}$$

This does not vanish by the transverse momentum assumption only. However here it is important that the metric is still present, so in momentum space we only have the quantity p_xp_y to consider. Then using the fact that $p_y^1 = 0 = p_x^2$ the only possible diagrams where $\mathcal{K}$ contributes that do not vanish, are those where

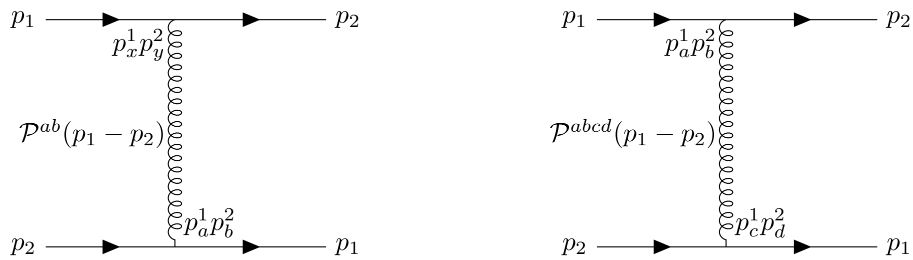


Figure 15: The two remaining possible diagrams to consider. On the left is the only possible $\mathfrak{h}_{ab} \leftrightarrow \mathcal{K}$ diagram, since the no-momentum diagram automatically vanishes by Equation 5.33. On the right is the momentum exchange diagram of $\mathfrak{h}_{ab}$.

the external particles exchange their momenta as shown in the left diagram in Figure 14. This is because any diagram without momentum exchange would have a $p^1_x p^1_y$ vertex to consider, which vanishes. The fact that $p^1_y = 0 = p^2_x$ will be crucial for the analysis during this subsection.

The diagram on the left in Figure 14 involving $\mathcal{K}$ has an order of magnitude of

$$\mathcal{M}_1 \sim (p^1_x)^2 (p^2_y)^2 \mathcal{P}(p^1 - p^2) \sim s \tag{5.34}$$

On the contrary the diagram without momentum exchange governed by $\mathfrak{h}_{ab}$ as shown on the right in Figure 14 is of order

$$\mathcal{M}_2 \sim (p^1_x)^2 (p^2_y)^2 \mathcal{P}^{xxyy}(0) \sim \frac{s^2}{\mu^2}. \tag{5.35}$$

The second diagram is an order $\frac{s}{\mu^2}$ higher, so in the Eikonal approximation we can neglect the first diagram altogether (since they are at the same order in the coupling constant). The question of whether the Eikonal approximation similarly extends to loops, and we can claim the same properties as above, is much more subtle. It is shown for flat space that a scalar ϕ^3 theory breaks down in the Eikonal approximation [72], whereas a vector interaction theory does not [73]. For this paper we shall simply assume that all arguments above also hold for loops, and we can safely sum over all ladder diagrams to leading order in s/μ^2 within the Eikonal approximation. It remains to find a proof whether this is true for our two-dimensional gravitational interaction system.

Now assuming that the analysis above is also valid for higher order diagrams, we find that all contributions of $\mathcal{K}$ only happen with momentum exchange diagrams, which are strongly subdominant to the other diagrams. Thus for Eikonal scattering we can ignore all contributions from the $\mathcal{K} \leftrightarrow \mathcal{K}$ interaction. Let us now consider the other two possible cases. The first is the possible contribution of the momentum-exchange diagram from $\mathcal{K} \leftrightarrow \mathfrak{h}_{ab}$. The second is that there is also a possible momentum exchange diagram involving $\mathfrak{h}_{ab}$ only.
First we look at the $\mathfrak{h}_{ab} \leftrightarrow \mathcal{K}$ shown in Figure 15 on the left. The order of magnitude is given by

$$\mathcal{M}_3 \sim p^1_x p^2_y \mathcal{P}^{ab}(p_1 - p_2)(p^1_a p^2_b - \tfrac{1}{2} g_{ab} g^{cd} p^1_c p^2_d) \tag{5.36}$$

where we used the momentum space vertex for $\mathfrak{h}_{ab}$ as given by

$$\mathfrak{h}^{ab} T_{ab} \sim \mathfrak{h}^{ab}(p^1_a p^2_b + \tfrac{1}{2}\eta_{ab} p^1_x p^2_y)\phi_1 \phi_2 \tag{5.37}$$

with $\eta^{cd} p^1_c p^2_d = -p^1_x p^2_y$ already inserted. Using this same vertex the order of magnitude of the momentum exchange diagram of $\mathfrak{h}_{ab}$ on the right in Figure 15 is given by

$$\mathcal{M}_4 \sim (p^1_a p^2_b + \tfrac{1}{2}\eta_{ab} p^1_x p^2_y)(p^1_c p^2_d + \tfrac{1}{2}\eta_{cd} p^1_x p^2_y)\mathcal{P}^{abcd}(p^1 - p^2). \tag{5.38}$$

Now in both these cases we see the same form appearing:

$$(p^1_a p^2_b + \tfrac{1}{2}\eta_{ab} p^1_x p^2_y) S^{ab}. \tag{5.39}$$

Here S^{ab} symbolically denotes a symmetric tensor, since both $\mathcal{P}^{ab}$ is symmetric in $a \leftrightarrow b$ and $\mathcal{P}^{abcd}$ is symmetric in $a \leftrightarrow b, c \leftrightarrow d$. The quantity S_{ab} is simply introduced to treat both cases at the same time. Now explicitly inserting $p^1_y = 0 = p^2_x$ and using $\eta_{xy} = -1$ gives:

$$p^1_a p^2_b S^{ab} = p^1_x p^2_y S^{xy}, \qquad\qquad \eta_{ab} S^{ab} = -2S^{xy}.$$

Using this clearly both terms in Equation 5.39 will cancel each-other and the term vanishes exactly. Thus what we observe is that both amplitudes in Equations 5.36 and 5.38 vanish exactly, and we can ignore these diagrams:

$$\mathcal{M}_3 = \mathcal{M}_4 = 0.$$

This is part of a more general relation that we find when expanding the indices of the vertex. This expansion results in:

$$\mathfrak{h}^{ab}\big(\partial_a\phi\partial_b\phi - \tfrac{1}{2}\eta_{ab}\eta^{cd}\partial_c\phi\partial_d\phi\big) = \mathfrak{h}^{xx}\partial_x\phi\partial_x\phi + \mathfrak{h}^{yy}\partial_y\phi\partial_y\phi. \tag{5.40}$$

What we observe is a result from the fact that the scalar field is massless and the fact that we are in two dimensions. A mass would give an $\mathfrak{h}_{xy}$ coupling. We see that the graviton mode $\mathfrak{h}_{xy}$ is removed from the Eikonal scattering entirely, its vertex does not exist. Similarly in the Eikonal regime while there is a vertex with $\mathcal{K}$ it is subleading in s/μ^2. The coupling between $\mathcal{K}$ and $\mathfrak{h}_{ab}$ is removed exactly, since $\mathcal{K}$ can only connect to $\mathfrak{h}_{xy}$ but this vertex vanishes. Thus for Eikonal scattering the fields $\mathcal{K}, \mathfrak{h}_{xy}$ have no contribution! Notice that a similar statement was made in [33] where the authors set $h_{\mu\nu} = 0$ except for h_{xx}, h_{yy}. However this statement was made using boundary conditions on the path integral which was unclear to us. Instead we now managed to derive the same limiting behavior using an order of magnitude analysis on the diagrams, providing an alternative method.

To conclude we see that in the Eikonal limit we have an enormous simplification; the graviton modes $\mathcal{K}, \mathfrak{h}_{xy}$ do not contribute to the scattering so we only need to concern ourselves with $\mathfrak{h}_{xx}, \mathfrak{h}_{yy}$. Remember that the odd modes h_a also vanish for vanishing transverse momenta. Thus in the end only two degrees of freedom contribute to Eikonal gravitational scattering.
Additionally we have seen that all diagrams where momentum is exchanged are subleading in s/μ^2 so that we can neglect all momentum exchange diagrams, reducing the amount of diagrams to consider enormously. The quantum field theory has just become much better to work with, while we made little assumptions; the removal of $\mathfrak{h}_{xy}$ is exact, and the rest only depends on s/μ^2 being large which is a safe assumption.

From this point we will only be working with $\mathfrak{h}_{xx}$ and $\mathfrak{h}_{yy}$ and the no-momentum exchange diagrams. As long as $\mathfrak{h}_{xy} = 0$ the vertex becomes proportional to

$$\mathfrak{h}^{ab} p^1_a p^2_b \phi_1 \phi_2. \tag{5.41}$$

We can make one last simplification based on the external momenta. Due to the fact that we have two p_1 external particles and two p_2 external particles in the case of forward scattering, for which $p^1_y = 0 = p^2_x$, means that $p^1_a p^1_b \mathcal{P}^{abcd} p^2_c p^2_d$ can never lead to a $\mathcal{P}^{xxxx}$ or $\mathcal{P}^{yyyy}$ contribution. This observation leads us to formulate the propagator such that only $\mathcal{P}^{xxyy} = \mathcal{P}^{yyxx}$ are nonzero (since any propagator with $\mathfrak{h}_{xy}$ is also not important anymore). The fact that both components are equal follows from time reversal invariance. Thus the general form for the propagator relevant for Eikonal scattering is given by

$$\mathcal{P}_{abcd} = \tfrac{1}{4} F_\ell(k^2)(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}). \tag{5.42}$$

Here $F_\ell(k^2)$ is some function of k^2 only due to time translation invariance, the choice of $1/4$ is to result in better factors at a later stage. This function was already derived in Subsection 4.5.2, however now we have the proper motivation for the specific form of the propagator. We need to remark that so far there is no difference between the T-channel and C-channel, both have the same order of magnitude analysis. The only distinguishment will come when we consider the spherical harmonics. This difference will be shown in Subsection 5.4, and has to do with the fact that not all matter legs are equal, contrary to the discussion up until now where all legs are equal.

Finally we insert the spherical harmonics expansions

$$\mathfrak{h}^{ab} = \sum_{\ell,m} \mathfrak{h}^{ab}_{\ell m} Y^m_\ell, \qquad \phi = \sum_{\ell,m} \phi_{\ell m} Y^m_\ell$$

into Equation 5.31, so that in terms of the (ℓ, m) modes we find

$$S_{\text{vertex}} = \frac{1}{2}\frac{\sqrt{\kappa}}{R} \sum_{\ell,m} \sum_{\ell_1,m_1} \sum_{\ell_2,m_2} \int \mathrm{d}\Omega \; Y^m_\ell Y^{m_1}_{\ell_1} Y^{m_2}_{\ell_2} \int \mathrm{d}^2x \; \mathfrak{h}^{ab}_{\ell m} \partial_a \phi_{\ell_1 m_1} \partial_b \phi_{\ell_2 m_2}. \tag{5.43}$$

Notice that now the factor in front $\sqrt{\kappa}/R$ is dimensionless and corresponds to $\frac{M_{\text{Pl}}}{M_{\text{BH}}}$, so that for the large black hole we are assuming this is very small. As such this is the perfect quantity to perform our perturbation around. Also remark that the order of magnitude analysis performed in this subsection is independent of any (ℓ, m) index presence, the analysis is only dependent on the lightcone momenta which is still valid for arbitrary (ℓ, m).

The derivatives all become momenta when we Fourier transform, thus the only problem to take care of is the spherical harmonics expansion. In the integral in Equation 5.43 there are three spherical harmonics present. Generally this leads to complex summations over the Clebsch-Gordan coefficients [74], a relation shown in Equation 6.4. These summations would render our calculation unfeasible, so to finish this subsection we define the minimal coupling simplification on the spherical harmonics.

Minimal coupling

We define our minimal coupling simplification as follows: The external (ℓ, m)-shell persists at all orders in perturbation theory, and only interactions with spherical vacuum fluctuations ϕ_0 contribute. In other words, we will choose one scalar leg of the vertex at $\ell = 0$ at all orders in perturbation theory. This provides a systematic description where we essentially neglect what would be integrations over the two-sphere in Minkowski space. In practice we achieve this effect by inserting $\phi_0 \equiv \phi(\ell = 0)$ in Equation 5.43 such that one of the vertex fields is spherical. This results in the following vertex action:

$$\boxed{S_{\text{vertex}} = \frac{\sqrt{\kappa}}{R} \sum_{\ell,m} \int \mathrm{d}^2x \; \mathfrak{h}^{ab}_{\ell m} \partial_a \phi_0 \partial_b \phi_{\ell m}.} \tag{5.44}$$

Here we ignored the contribution from $\mathfrak{h}_{xy}$ as explained before. We removed the factor two in front as there are two possible ways to insert ϕ_0. We have now removed the complexity of the triple spherical harmonics situation by making the minimal coupling simplification. By doing so we removed one spherical harmonic, allowing the orthogonality relation to be used for the remaining two. Thus in the minimal coupling regime, we find a completely orthogonal theory, where all (ℓ, m) modes are decoupled, not only in the propagators but also in the vertex. Notice that the minimal coupling does not affect the order of magnitude estimations in Equations 5.34-5.36 as these hold generally for the fields using the action, so also for any (ℓ, m) configuration.

Physically we can imagine the minimal coupling as the leading order contribution coming only from fluctuations in the spherically symmetric black hole vacuum. Similar to tree level where we imposed two of the external fields to be ϕ_0 fields (see Equation 5.1) by treating them as contributions from the Hartle-Hawking state, we now also extend this to loops. As we will mention in the discussion we will see that this minimal coupling approximation implies that we ignore any non-spherical transverse effects in the interaction. In other words were the fields to have significant transverse momenta, they would alter the background metric to be non-spherical, and the interaction between them would mix the different (ℓ, m) modes.
We should comment that spherically symmetric contributions of $\mathfrak{h}_0$ (at $\ell = 0$) also contribute but we ignore these for now. The reason is that all $\mathfrak{h}_0$ fluctuations can be interpreted as perturbations of the Schwarzschild black hole due to Birkhoff's Theorem, as was shown classically in [68], and we can assume that the modifications to the Schwarzschild mass are very small.

5.4 Schwarzschild Eikonal calculation

We shall now perform the Eikonal approximation on the Schwarzschild spacetime, based on the linearized gravity action derived in Section 4. The effective action at a single (ℓ, m) due to the minimal coupling will be

$$\begin{aligned} S_{\ell m} &= \tfrac{1}{4}\int \mathrm{d}^2k \ \mathfrak{h}^{ab}\mathcal{P}^{-1}_{abcd}\mathfrak{h}^{cd} - \tfrac{1}{2}\int \mathrm{d}^2p \ \phi(p^2+\mu^2\lambda)\phi \\ &\quad + \sqrt{\gamma}\int \mathrm{d}^2k\mathrm{d}^2p_1\mathrm{d}^2p_2 \ \delta^{(2)}(k+p_1+p_2)\mathfrak{h}_{ab}(k)p_1^a p_2^b\phi_0(p_1)\phi(p_2). \end{aligned} \tag{5.45}$$

Here $\mathcal{P}_{abcd}$ is defined in Equation 4.167, $\gamma \equiv \mu^2\kappa$ is the coupling constant and the graviton perturbation is only given by

$$\mathfrak{h}_{ab} = \begin{pmatrix} \mathfrak{h}_{xx} & 0 \\ 0 & \mathfrak{h}_{yy}. \end{pmatrix} \tag{5.46}$$

Furthermore ϕ_0 is the scalar field at $\ell = 0$, which can be interpreted as a vacuum contribution. As explained in the previous subsection we have discarded $\mathfrak{h}_{xy}, \mathcal{K}$, so that the action in Equation 5.45 fully describes high-energy gravitational interaction on the black hole horizon. The fact that the gravitational interaction became so easily described was due to the Eikonal approximation, the fact that we are in two dimensions, and the minimal coupling simplification. From these it also followed that we will only calculate Feynman diagrams with no momentum exchange. To be precise we consider the diagrams where one of the Mandelstam variables t, u defined as

$$t = -(p_1 - p_1')^2, \qquad u = -(p_1 - p_2')^2$$

vanishes, so we assume $p_1 = p_1'$ or $p_1 = p_2'$. This is connected to the impact parameter, for our case the impact parameter is large so that the forward scattering amplitude involves no momentum exchange. The large impact parameter is now defined subtly different; we still assume $b \gg \ell_{\mathrm{Pl}}$, however because we are on the constant surface $r = R$ we must also restrict $b \sim R$. This is different from Minkowski space, where b was unbounded and extremely large, whereas now we need b finite as long as it is larger than the Planck length. This shows that the results in the Schwarzschild spacetime hold much more generally than originally anticipated.

Additionally we can now properly look at the coupling constants. Let us first look at all the quantities with mass in our theory. We have the inverse Schwarzschild radius μ, which is connected to the black hole mass, we have the Planck mass M_{Pl} which comes from the gravitational constant $M_{\mathrm{Pl}} \sim \kappa^{-1/2}$, and finally we have the Mandelstam variable s. Combining these we have two possible dimensionless parameters:

$$\gamma = \frac{M_{\mathrm{Pl}}^2}{M_{\mathrm{BH}}^2}, \qquad \& \qquad \frac{s}{\mu^2}.$$

Now the first is the coupling constant of the vertex, and is generally small for large black holes. The second is the Eikonal parameter which we take extremely large. Now in the following we will sum over all loops in γ reaching a nonperturbative result in the vertex. At the same time we take the Eikonal approximation, so we take only the leading order of s/μ^2 at every order in γ. Essentially we trade the perturbation, usually we look at all momentum orders for only tree level or a few loops. Now we look only at the high momentum orders but for all possible loops in perturbation theory. As argued before in Section 2 given the diverging energy this is a safe procedure to follow. In the following whenever we refer to the perturbation order, loop order, or mention nonperturbative, we refer to γ. Throughout the entire calculation we only keep the leading order in s/μ^2 at all times.

Let us now take a short look at when this condition is satisfied, i.e. when is $s \gg \mu^2$. Writing out μ^2 shows that we can identify this as

$$s \gg \gamma M_{\mathrm{Pl}}^2.$$

This is an interesting relation: In the usual Eikonal approximation for flat space gravity the particles need to be in the Planckian regime [75], which means $s \gg M_{\mathrm{Pl}}^2$. However, in our Schwarzschild case we also have

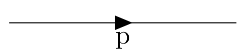
the vertex factor γ appearing in the condition. For a very large black hole γ is extremely small, so we see that the Eikonal limit is satisfied at much lower energies. To be specific the Eikonal limit is valid for center of mass energies $E \gg \sqrt{\gamma} M_{\text{Pl}}$. Then for a black hole of e.g. $M_{\text{BH}} \sim 10^{38} M_{\text{Pl}}$ (approximately the solar mass), we only need $E \gg 10^{-38} M_{\text{Pl}}$ which is a trivial condition to satisfy, in fact all massive elementary particles satisfy this by their rest mass alone. Given that on top of the mass the particles' energy diverges on the horizon we can easily say that all massless and massive particles satisfy the Eikonal limit.
This also means that the center of mass energy is allowed much smaller than the Black hole mass, so that we have no conflict within the background spacetime. Additionally the center of mass energy is also allowed much lower than the Planck mass, which prevents the possibility of classical black hole formation [75]. The combination of $s/\mu^2 \gg 1$ being satisfied trivially and the upper bound on the impact parameter $\ell_{\text{Pl}} \ll b < R$ is a remarkable property of the Schwarzschild spacetime; instead of probing Planckian physics we have no bound on the external momenta at all, and the Eikonal approximation is valid for any realistic scattering scenario. This shows that the results derived in this section will hold much more generally than originally anticipated. In fact any type of forward gravitational scattering close to a black hole can be described in the Eikonal approximation only, ignoring all other diagrams. For this reason the nomenclature of Eikonal is also not ideal, as this usually implies trans-Planckian physics, which is not the case on the Schwarzschild background. Nevertheless due to the similarities we keep referring to this limit as Eikonal, with the intention that it holds much more general.

With the action in Equation 5.45 we have now properly defined our quantum field theory, and we can define the Feynman rules. Notice that due to our Weyl transformation we are now on flat space, so we can apply all the usual quantum field theory methods without bothering about curvature effects as described in [76]. In particular the vacuum state is the well-defined flat space vacuum, and the path integral can simply be defined on this vacuum. Remember that we have removed the curvature problem exactly in Section 4, we only approximated the resulting potentials.

5.4.1 Feynman rules

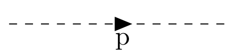
We formulate all relevant Feynman rules including the trivial symmetry factors. Here we assume the action given in Equation 5.45 for a certain (ℓ, m) with $\ell > 1$. The general set of Feynman rules is as follows

$\phi_{\ell m}(p)$ – propagator



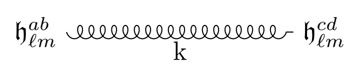
$= \frac{-i}{p^2+\mu^2\lambda+i\epsilon}$,

$\phi_0(p)$ – propagator



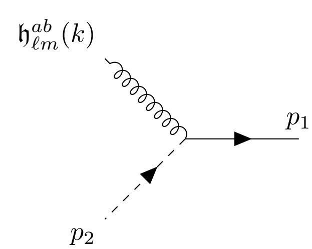
$= \frac{-i}{p^2+\mu^2+i\epsilon}$,

$\mathfrak{h}_{\ell m}(k)$ – propagator



$= \frac{1}{2} i F_\ell(k^2)(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc})$,

$\mathfrak{h}_{\ell m}(k)$ – vertex



$= i\sqrt{\gamma} p^1_a p^2_b$.

Here we already included all trivial symmetry factors resulting from exchanging both sides of the propagator or symmetries of the indices[7]. Remember that the simplified version of the vertex holds exactly in our case, we

[7] This means a symmetry factor of 8 for the graviton propagator, due to exchange for of the external graviton legs and $a \leftrightarrow b$ and $c \leftrightarrow d$ symmetry. For the scalar propagator this means a factor 2 only, due to exchange of the external scalar legs. Finally for the vertex there is no factor, since ϕ and ϕ_0 are not freely interchangeable.

have orthogonal external particles $p^1_y = 0 = p^2_x$, for which $\mathfrak{h}_{xy}$ has vanishing contribution in two dimensions. We also included relevant factors of i from the path integral definition $G \sim e^{iS}$. The function F_ℓ is defined by

$$F_\ell(k^2) = -\frac{4}{\mu^2(\lambda+1)} - \frac{2}{(\lambda+1)(\lambda-3)}\frac{1}{k^2+\mu^2\lambda - i\epsilon}\frac{k^4}{\mu^4}. \tag{5.47}$$

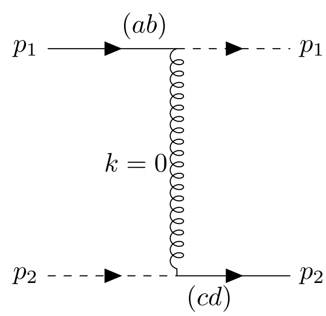
With these Feynman rules we shall perform the Eikonal approximation similar to Section 5.1. We will also use the same simplified matter propagators

$$\frac{-i}{(p+k)^2+\mu^2\lambda - i\epsilon} \approx \frac{-i}{2p\cdot k - i\epsilon} \tag{5.48}$$

which hold at leading order in p_a for neglected transverse momenta. Notice that for a large black hole $\mu \to 0$ as well so we can treat all $\mu's$ appearing as small compared to any other quantity of the same dimensionality. This coincides well with Subsection 5.1 where μ was the graviton infrared mass regulator, however now we do not need to impose such regulator; it naturally follows from the Schwarzschild spacetime. This gives a more elegant solution to infrared divergence problems, which would otherwise occur tremendously much in this paper as we shall see.

5.4.2 The T-channel

We will first consider the T-channel, so all diagrams where the (ℓ, m) is carried over from above to below and the virtual graviton has nonzero (ℓ, m). At tree level there is only one possible diagram without momentum exchange contributing, which we had already written down in the beginning of this section. It is given by



Straightforward application of the Feynman rules shows that the amplitude is given by

$$i\mathcal{M}_1 = i\sqrt{\gamma}p^1_a p^1_b \tfrac{i}{2} F_\ell(0)(\eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}) i\sqrt{\gamma}p^2_c p^2_d \tag{5.49}$$

$$= \frac{4i\gamma s^2}{\mu^2(\lambda+1)}. \tag{5.50}$$

Here the Mandelstam variable was given by $s = p^1_x p^2_y$. In writing the amplitude down it is crucial that the soft graviton propagator is well defined, so the fact that the graviton has an effective 2D mass. Compared to what we have found in Subsection 5.1 we have found a similar structure. The Mandelstam variable $t = -(p_1 - p_3)^2 \approx -(p^\perp_1 - p^\perp_3)^2$ is mostly determined by the transverse momenta. In our case we have also neglected the longitudinal momentum transfer, by setting $p_3 = p_1, p_4 = p_2$ exactly. However we are effectively in two dimensions, so that the longitudinal components cover the entire space and all p_i, k are only longitudinal. All transverse momentum transfer effects are embedded in the $1/(\lambda+1)$ which now replaces the $1/(-t)$.

What we see is that a much easier theory emerged, where there is no need for infrared regulators, and we have removed all transverse effects by removing two dimensions. What we kept were the longitudinal components, which are treated strongly in the Eikonal approximation.

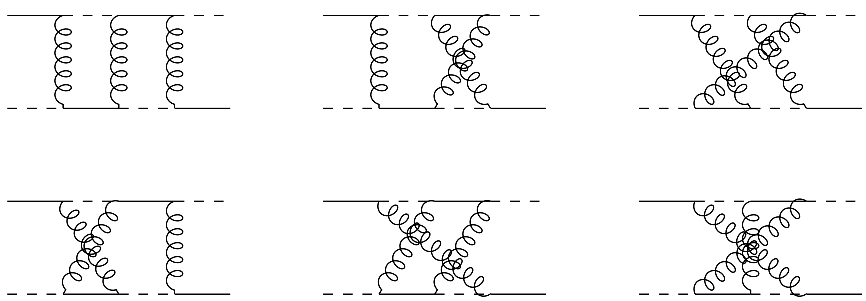


Figure 16: All possible two-loop diagrams.

The loop diagrams

In this subsection we write down the general loop order amplitude according to Lévy & Sucher [30]. Our calculation is largely similar, except that we are in two dimensions and the external legs are different. The latter is particularly important: Only diagrams where the (ℓ, m) angular momentum is transferred from p_1 incoming external leg above to the p_2 outgoing external leg below can exist. This means that only an odd amount of internal F_ℓ legs can exist, an even amount would not transfer the angular momentum from above to below. Thus there are no contributing one-loop diagrams. The first loop diagrams are at two-loop and displayed in Figure 16. For the rest we have contributions at four-loop, six-loop, etc, and not at three-loop, five-loop, etc. Now the contribution at arbitrary loop order in the Eikonal limit has been calculated in detail by Lévy & Sucher [30]. We shall straightforwardly adapt this calculation to our two-dimensional Eikonal scattering. In general we can immediately write down the order n-amplitude as

$$i\mathcal{M}_n = (i\sqrt{\gamma}s)^{2n} \int \prod_{j=1}^{n} \left(\tfrac{d^2 k_j}{(2\pi)^2} iF_\ell(k_j) \right) \times I \times (2\pi)^2 \delta^{(2)} \left(\sum_{j=1}^{n} k_j \right). \tag{5.51}$$

Here we used that all vertices will exactly contribute a factor $i\sqrt{\gamma}s$ in the no-momentum exchange case, *when* we neglect the additional $k's$ in calculating the vertex (the Eikonal limit). The quantity I contains all possible matter propagators that lead to the diagrams in Figure 16. So far Equation 5.51 is just the general expression for the order n-amplitude. Equation 5.51 consists in order of the following: all vertex factors $i\sqrt{\gamma}s$ for order n scattering, integration over all internal momenta k_j and their propagator $F_\ell(k_j)$, insertion of all matter propagators I, and finally the Dirac delta to ensure internal conservation of momentum. This equation is the two-dimensional analog of Equation (3.1) in [30] for gravitational vertices instead of meson vertices, and for the specific case $q = p_1 - p_3 = 0$ which we are free to do exactly in our calculation.

Equation 5.51 is already in its most workable form except for the matter propagators I. Now the process of deriving I in the Eikonal limit is shown properly in [30]. The principle is to add up all possible permutations of k_j in the matter propagators, under the knowledge that all $F_\ell(k_j)$ are the same function. As these permutations are invariant under the dimensionality of k_j this also holds in two dimensions, and we can safely write down their result:

$$i\mathcal{M}_n = -\frac{\gamma s^2}{n!} \int \tfrac{\mathrm{d}^2 k}{(2\pi^2)} iF_\ell(k) \int \mathrm{d}^2 x \; e^{-ik\cdot x} (i\chi)^{n-1}, \tag{5.52}$$

$$\chi = -i\gamma s^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} iF_\ell(k) e^{-ik\cdot x} \left[\frac{1}{-2p_1\cdot k - i\epsilon} \frac{1}{2p_2\cdot k - i\epsilon} + \frac{1}{-2p_1\cdot k - i\epsilon} \frac{1}{-2p_2\cdot k - i\epsilon} \right. \tag{5.53}$$

$$\left. + \frac{1}{2p_1\cdot k - i\epsilon} \frac{1}{2p_2\cdot k - i\epsilon} + \frac{1}{2p_1\cdot k - i\epsilon} \frac{1}{-2p_2\cdot k - i\epsilon} \right]. \tag{5.54}$$

Remark that this definition of χ is the same as previously in Subsection 5.1, except for the two-dimensional integral. Rewriting the brackets in χ results in

$$\chi = -i\gamma s^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} iF_\ell(k) e^{-ik\cdot x} \left[\frac{1}{2p_1\cdot k + i\epsilon} - \frac{1}{2p_1\cdot k - i\epsilon} \right] \left[\frac{1}{2p_2\cdot k + i\epsilon} - \frac{1}{2p_2\cdot k - i\epsilon} \right]. \tag{5.55}$$

Again we can use the δ identity in Equation 5.12 to rewrite this into

$$\chi = -\gamma s^2 \int \mathrm{d}^2k\ F_\ell(k) e^{-ik\cdot x}\delta(2p_1\cdot k)\delta(2p_2\cdot k). \tag{5.56}$$

Now inserting the delta-functions results in a factor $1/(4s)$ and setting $k=0$. This last property is remarkable as it removes the integral completely (instead of leaving the transverse components as in Subsection 5.1) because we are two-dimensional! We now also clearly see the importance of the conclusion in Section 4. The divergent UV-tail is cut-off completely, and only the soft graviton contributes. At the same time the 2D effective mass of the graviton generates UV-problems while guaranteeing IR-safety. Clearly the Eikonal approximation complements the massive propagator perfectly, they both guarantee eachothers convergence. Thus finally we can identify that

$$\boxed{\chi = -\tfrac{1}{4}\gamma s F_\ell(0).} \tag{5.57}$$

Clearly in our case χ is x-independent, so that for $i\mathcal{M}_n$ we can write

$$i\mathcal{M}_n = -\frac{\gamma s^2 (i\chi)^{n-1}}{n!}\int \tfrac{\mathrm{d}^2k}{(2\pi^2)}(iF(k))\int \mathrm{d}^2x e^{-ik\cdot x}. \tag{5.58}$$

Then we can immediately use that

$$\int \mathrm{d}^2x e^{-ik\cdot x} = (2\pi)^2\delta^{(2)}(k) \tag{5.59}$$

so that we end up with the following order n-amplitude:

$$i\mathcal{M}_n = -i\frac{\gamma s^2(i\chi)^{n-1}}{n!}F_\ell(0) = 4s\frac{(i\chi)^n}{n!}. \tag{5.60}$$

The nonperturbative amplitude

Finally to find the total amplitude we need to sum this over all odd n, as reasoned before. This results in

$$i\mathcal{M} = 4s\sum_{n\ \mathrm{odd}}^{\infty}\frac{(i\chi)^n}{n!} = 4s\sum_{m=0}^{\infty}\frac{(i\chi)^{2m+1}}{(2m+1)!}. \tag{5.61}$$

This is an elementary summation corresponding to a sine, so that

$$\boxed{i\mathcal{M}_T = 4is\sin(\chi).} \tag{5.62}$$

The equation above gives the complete amplitude up to all order in perturbation theory for the T-channel. In the next subsection we derive the C-channel. For reference the function χ is defined by

$$\chi \equiv \frac{\gamma s}{\mu^2(\lambda+1)} \equiv \tilde{\alpha} s \tag{5.63}$$

where in the last line $\tilde{\alpha}\equiv \gamma/(\mu^2(\lambda+1))$ is a constant.

5.4.3 The C-channel

In this subsection we will consider the contribution for the C-channel. We first need to look at what would be the tree level contribution. This diagram would look as shown in Figure 17.

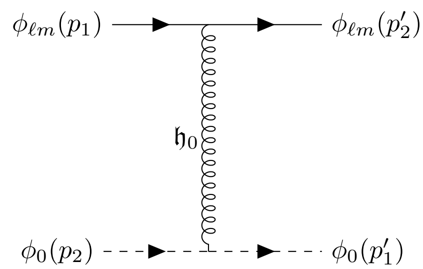


Figure 17: The C-channel tree level diagram.

However this diagram has a problem: The virtual graviton leg has $\ell = 0$. In Subsection 5.3 we made the minimal coupling simplification, where we neglected all contributions from these kind of gravitons, i.e. we set $\mathfrak{h}_{\ell=0} = 0$ in the theory. In order to apply the minimal coupling consistently, as we also did for the T-channel, we now have to discard the diagram above. This also extends to the higher loops, as only the diagrams where there is no $\mathfrak{h}_{\ell=0}$ lines exist might contribute. Thus while in the C-channel we discarded all even order diagrams, we now have to do the exact opposite. In the C-channel the (ℓ, m) needs to stay on top, so now we need to discard all odd order diagrams. As a result the next diagram to consider is not at tree level, but at one loop.

Higher loops

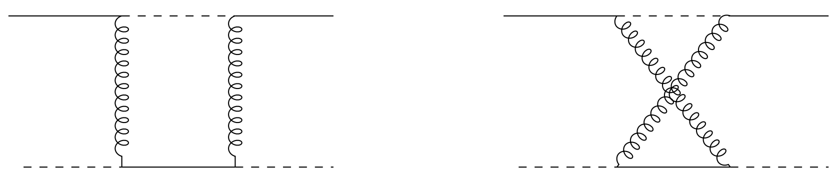


Figure 18: The two one-loop ladder diagrams.

All relevant one-loop diagrams are shown in Figure 18. Now for these loops we can again apply the generalized procedure as was done by [30]. In fact the magnitude at any order is still identical to the T-channel, so we can immediately write down that

$$i\mathcal{M}_n = 4s\frac{(i\chi)^n}{n!}, \tag{5.64}$$

$$\chi = -\tfrac{1}{4}\gamma s F_\ell(0). \tag{5.65}$$

Then for the C-channel as mentioned we need to discard all odd order diagrams, in other words we now first have a one-loop contribution, then three-loop, etc. Compared to the T-channel we are exactly switched around, the diagrams that were discarded for the T-channel are now included in the C-channel. Thus for the total amplitude we need to sum over all even orders. This results in

$$i\mathcal{M}_C = 4s\sum_{n\text{ even}}\frac{(i\chi)^n}{n!} = 4s\sum_{m=1}^{\infty}\frac{(-1)^m\chi^{2m}}{(2m)!}. \tag{5.66}$$

This last summation we can recognise as the series representation of the cosine, so that the C-channel amplitude becomes

$$i\mathcal{M}_C = 4s\left(\cos(\chi) - 1\right). \tag{5.67}$$

The equation above gives the nonperturbative amplitude for the C-channel. Adding the C-channel and the T-channel together will result in the full amplitude.

5.5 Scattering amplitude

Adding the T-channel and C-channel result together we find for the full amplitude

$$i\mathcal{M} = 4s\left(e^{i\chi} - 1\right) \tag{5.68}$$

where we used the complex exponent definitions of the sine and cosine. This is a remarkably simple result all due to the enormous simplifications taken earlier. We started with four-dimensional curved spacetime and reduced this to two-dimensional flat spacetime exactly, removing all problems of the curvature. Then we inserted the Eikonal approximation removing many of the graviton modes and simplifying the vertex. Still the fact that we performed all these steps and summed over infinite loops and only an elementary complex exponent results is remarkable.

The amplitude in Equation 5.68 is only to leading order in s/μ^2 but we summed over all orders in γ. Additionally restoring all factors $\hbar$ shows that we also summed over all orders of $\hbar$. In particular the effect of including $\hbar$ is simply that $\chi \to \hbar\chi$ so that the nonlinear behavior in $\hbar$ in Equation 5.68 is apparent. We see that the amplitude in Equation 5.68 is in fact a full quantum amplitude (due to all orders in $\hbar$) and a full gravitational amplitude (due to all orders in γ). Thus in fact Equation 5.68 provides a nonperturbative quantum gravity amplitude, in the limit of large energy and large impact parameter.

Equation 5.68 fully describes gravitational scattering of high energy scalar particles on the black hole horizon. It is now apparent that the only effect is a relation between the scattered particles by a phase shift. The definition of χ was given by

$$\chi \equiv \tilde{\alpha} p_1 p_2, \tag{5.69}$$

$$\tilde{\alpha} = \frac{\gamma}{\mu^2(\lambda+1)} = \frac{8\pi G}{\ell^2+\ell+2}. \tag{5.70}$$

Now looking back at Equation 3.14 we found for the Shapiro delay an almost similar factor

$$\alpha = \frac{8\pi G}{\ell^2+\ell+1}.$$

There is only a subtle difference between α and $\tilde{\alpha}$, which is the $+2$ in the denominator. This difference is addressed in great detail in Subsection 5.6. We shortly remark the conclusion here. When inserting specifically the Shapiro delay as perturbation, i.e. we only assume orthogonal shockwaves $h_{xx} \sim h^-_{xx}(x)\delta(y)$ we find the same factor α. However we have included more general perturbations $\mathfrak{h}_{xx}, \mathfrak{h}_{yy}$ resulting in the $\tilde{\alpha}$. Thus the resulting $\tilde{\alpha}$ is in fact not wrong, but a result of the more generalized gravitational interaction we assumed.

Additionally we need to remark on the factor $4s$ and -1 in Equation 5.68. Remember that we have found the scattering amplitude, however it is the scattering matrix that relates the in- and out-states and in- and out-fields. We would like to find the scattering matrix explicitly. The scattering matrix is symbolically defined by

$$S = 1 + (2\pi)^2\delta^{(2)}(p_1 + p_2 - p_1' - p_2')(i\mathcal{M}). \tag{5.71}$$

Here $i\mathcal{M}$ is given in Equation 5.68. We expect that in the proper Eikonal limit the factor $4s$ is removed by the momentum conservation Dirac delta $\delta^{(2)}(p_1 + p_2 - p_1' - p_2')$. We need to impose energy conservation manifestly on the outgoing momenta p_1', p_2' for this to be valid, however we can safely do such a thing for the case of forward scattering. Then when the factor $4s$ is removed the factor 1 in Equation 5.71 cancels the -1 in Equation 5.68. Here the factor 1 can be understood as coming from the zeroth order contribution

$$\left\langle \phi_{\ell m}(p_1)\phi_0(p_2)\phi_{\ell m}(p_2')\phi_0(p_1')\right\rangle_0 = \left\langle \phi_{\ell m}(p_1)\phi_{\ell m}(p_2')\right\rangle_0 \left\langle \phi_0(p_2)\phi_0(p_1')\right\rangle_0, \tag{5.72}$$

which is the contribution when there is no gravitation interaction. Diagrammatically this contribution would look as shown in Figure 19.

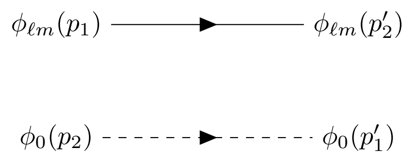


Figure 19: The zeroth order diagram, both particles simply do not interact.

The resulting scattering matrix would then simply become

$$S \sim e^{i\tilde{\alpha}s}.$$

We have not had the time to check this thoroughly. In the following we will analyze the complex exponent and ignore the factors $4s$ and -1, under the presumption that the above argumentation is correct.

We will now look at the implications of the scattering amplitude. Let us focus only on the ingoing (ℓ, m) wave with momentum p_1 and outgoing (ℓ, m) shell with momentum p_2. By doing so we interpret the scattering amplitude in Equation 5.68 as a forward scattering relation between the two particles, which we write down as

$$\langle p_1^{\text{in}} | p_2^{\text{out}} \rangle \sim e^{i\tilde{\alpha} p_1 p_2}. \tag{5.73}$$

Here $|p_1\rangle$ is the quantum state of the incoming (ℓ, m) shell with momentum p_1, and vise versa for p_2. We have removed the other factors as discussed above. Now the above equation has a striking resemblance with Equation 3.15. In fact Equation 5.73 is the momentum space variant, albeit with the different $\tilde{\alpha}$. Thus we see that the quantum mechanical scattering relation follows directly from our Eikonal summation, in fact we have now shown that the in- and out-fields are related by a Fourier transformation.

To find another elegant result we Fourier transform the out-momentum to position space, to find the scattering matrix as

$$\langle p_1^{\text{in}} | y^{\text{out}} \rangle \sim \delta(y - \tilde{\alpha} p_1). \tag{5.74}$$

Heuristically speaking Equation 5.74 does not vanish only when the argument of the delta function is zero. In other words it tells us that the scattering amplitude is only nonzero exactly when

$$y^{\text{out}} = \frac{8\pi G}{\ell^2 + \ell + 2} p_1^{\text{in}}. \tag{5.75}$$

Comparison with Equation 3.11 shows that we are looking at the same equation up to the $+1/+2$ difference. In other words: The Shapiro delay follows from our scattering amplitude! We can conclude that the Eikonal approximation also results in the semi-classical treatment of Section 3 as expected, however we have now shown this explicitly for the Schwarzschild spacetime. We have thus provided a complementary calculation to the results of 't Hooft, confirming the Shapiro delay as leading order contribution in black hole scattering and showing that the Shapiro delay expresses itself as a Fourier transform when either in full position or full momentum space. The main difference is that the Fourier transform relation in Section 3 was for quantum mechanical wavefunctions, whereas the Equation 5.73 describes a Fourier transform between fields. Our calculation suggests that the field creation and annihilation operators of the in- and out-states on the black hole horizon are also connected by a Fourier transformation. Within all these the reference scale is defined by the constant $\tilde{\alpha}$. As mentioned we will mention suggestions to how $\tilde{\alpha}$ might be put equal to α in the discussion, which would be a great result to strengthen the theory.

We can now also compare our scattering matrix with the semi-classical result in [23, 28], which states

$$\langle p_{\text{in}} | S | p_{\text{out}} \rangle = \text{Exp}\left(8\pi i G \int \mathrm{d}\Omega \mathrm{d}\Omega' \; P_1(\Omega) f(\Omega, \Omega') P_1(\Omega') \right). \tag{5.76}$$

Here the $P(\Omega)$ are the momentum distribution functions of the ingoing particles and $f(\Omega, \Omega')$ is the same Green's function as defined in Equation 2.49. Notice that there is a minus sign difference with [28] due to our coordinate choice ($-2\mathrm{d}x\mathrm{d}y$ in the metric instead of $+2\mathrm{d}x\mathrm{d}y$), and we changed to our notation of $p_1 = p^{\text{in}}, p_2 = p^{\text{out}}$. Now we can expand the momenta in spherical harmonics

$$P(\Omega) = \sum_{\ell,m} P_{\ell m} Y_\ell^m(\Omega).$$

Inserting this turns Equation 5.76 into

$$\langle p_{\text{in}}|S|p_{\text{out}}\rangle = \text{Exp}\left(\sum_{\ell,m} \alpha_\ell P^1_{\ell m} P^2_{\ell m}\right) \tag{5.77}$$

where $\alpha_\ell = \frac{8\pi G}{\ell^2+\ell+1}$ which resulted from the Shapiro delay. Comparing the above equation with Equation 5.73 shows that the exponents are exactly the same, again up to the $\alpha, \tilde{\alpha}$ difference! The only real difference is that Equation 5.77 includes the sum over all ℓ, m, however we can easily generalize our result in Equation 5.73 to include all ℓ, m using

$$S_{\text{tot}} = \prod_{\ell,m} S_{\ell,m} = \text{Exp}\left(\sum_{\ell,m} i\tilde{\alpha}_\ell p^1_{\ell m} p^2_{\ell m}\right).$$

Thus we see that, under the presumption that the transition from the scattering amplitude in Equation 5.68 to the scattering matrix in Equation 5.73 is correct, our scattering matrix is equal to that obtained in [28]. The procedure to derive the scattering matrix in [28] was using the semi-classical treatment in Section 3, so we have now seen explicitly that the resulting scattering matrices are equal. This concludes the answer to our research question: We have performed the Eikonal calculation for linearized gravity successfully, resulting in the same scattering matrix as in Section 3. Notice that the scattering matrix in Subsection 3.3 is a coordinate transformation of the exponent derived above, and thus upon performing the transformation correctly that matrix should also be derivable.

5.6 The different factors

Finally in this subsection we will properly analyze the most striking difference between Section 3 and this section, which is the difference between α and $\tilde{\alpha}$. For reference

$$\alpha = \frac{8\pi G}{\ell^2+\ell+1}, \qquad \tilde{\alpha} = \frac{8\pi G}{\ell^2+\ell+2}.$$

Looking at the calculation in Subsection 5.4 we saw that only the soft graviton propagator $\mathcal{P}^{xxyy}(0)$ contributes. Then looking at Section 4 we saw that the soft graviton propagator $\mathcal{P}^{xxyy}(0)$ was only determined by the inverse of the quantity Δ^{-1}_{abcd} which was given by

$$\Delta^{-1}_{abcd} = \tfrac{1}{2}\mu^2\left(\eta_{ac}x_{[b}\partial_{d]} + \eta_{bd}x_{[a}\partial_{c]} + \eta_{ab}x_{(c}\partial_{d)} - \eta_{cd}x_{(a}\partial_{b)}\right) + \frac{\mu^2(\lambda+1)}{2}\big(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd}\big). \tag{5.78}$$

Clearly a $\lambda+1$ is present which also resulted in the $\lambda+1$ factor in the end (recall that $\lambda = \ell^2+\ell+1$). In this way the final scattering amplitude only depends on what the inverse of this Δ^{-1}_{abcd} is. Whereas we provided the full propagator in Section 4, this was not necessary for the Eikonal calculation. To find out why we have a different $\tilde{\alpha}$ from the factor α in Section 3 we only need to analyze the Δ^{-1}_{abcd}.

A consistency check

It now seems tempting to claim that the difference between α and $\tilde{\alpha}$ is because the Δ^{-1}_{abcd} is wrong, since it has a factor $\lambda+1$. However we can show that this operator is in fact correct, or at least in agreement with Sections 2-3. We will first prove this. Let us write down the components of Δ^{-1}_{abcd} that are relevant for Eikonal scattering and the Shapiro delay:

$$\Delta^{-1}_{xxyy} = -\tfrac{1}{2}\mu^2(\lambda + 1 + x\partial_x - y\partial_y), \tag{5.79}$$

$$\Delta^{-1}_{yyxx} = -\tfrac{1}{2}\mu^2(\lambda + 1 + y\partial_y - x\partial_x). \tag{5.80}$$

In Section 3 we had defined the Shapiro delay perturbation (also see Appendix B.1) as

$$h_{xx} = 2A(x,y)\delta(x)F_{\ell m}$$

where we replaced $F(\Omega) \to F_{\ell m}$ since we already expanded into spherical harmonics. Notice that all other components vanish, in particular h_{yy} vanishes. Using the action defined in Equation 4.121 we can identify the Einstein tensor as $G_{xx} = \Delta^{-1}_{xxyy}h_{xx}$. We can write the Einstein tensor out as

$$G_{xx} = -\mu^2 A(x,y)F_{\ell m}\delta(x)(\lambda+1) - \mu^2 A(x,y)F_{\ell m}x\partial_x\delta(x) - \mu^2 F_{\ell m}\delta(x)(x\partial_x - y\partial_y)A(x,y). \tag{5.81}$$

Now we can easily show that $(x\partial_x - y\partial_y)A(x,y) = 0$ using the derivative identities in Appendix A.1. The rest we can simplify similar to Appendix B.1. The resulting G_{xx} becomes

$$G_{xx} = -\mu^2 F_{\ell m}\lambda. \tag{5.82}$$

So now only the $\ell^2 + \ell + 1$ part remains, and we can see that this G_{xx} is in perfect agreement with the Einstein tensor in Section 2 in Equation 2.46. Thus what we see is that our operator Δ^{-1}_{abcd} is correct, in fact even after taking the horizon approximation. The operator gives identical results as in Section 2 for the Shapiro delay, so both cases are also consistent. Then why do we have a different α and $\tilde{\alpha}$ still?

A different horizon approximation?

The reason for the different α and $\tilde{\alpha}$ is not straightforward. Originally we had the thought that the horizon approximation $x^a \ll R$ might be wrong. A solution would be to properly write down r up to linear order in xy and invert the resulting set of operators. The definition $xy = 2R^2(1 - r/R)e^{r/R-1}$ shows that up to linear order in xy we can identify

$$r \approx R - \frac{xy}{2R} + \mathcal{O}((xy)^2).$$

However such a linearization should only add differences at a higher order in xy, not in the lowest order. The approximation of setting $xy = 0$ at lowest order was included in our $x^a \ll R$ method, and thus inserting r linearly like this would make no difference to the $\lambda + 1$ factor. It would nevertheless be interesting to investigate the effects of including curvature around the horizon up to linear order this way.

Going back to α and $\tilde{\alpha}$ we saw that the above was not the solution, and decided we first need to isolate the problem properly. Upon inspection the difference clearly comes from the Delta functions. For the Shapiro delay we have

$$(x\partial_x - y\partial_y)\delta(x) = -\delta(x). \tag{5.83}$$

This was also the result we saw above, which was crucial as it removed the $+1$ from the $\lambda + 1$. Now on the contrary when inverting for the propagator in Section 4 we used a double Dirac delta function $\delta^{(2)}(x - x') = \delta(x - x')\delta(y - y')$. This is shown in Equation 4.141 and is necessary since we invert the propagator in two dimensions. However the analogue of Equation 5.83 on the double Dirac delta shows that

$$(x\partial_x - y\partial_y)\delta^{(2)}(x - x') = 0. \tag{5.84}$$

It vanishes because the y-part cancels the x-part exactly. Now there is no remaining $-\delta^{(2)}(x - x')$ to cancel the $+1$ in the $\lambda + 1$. Because of this the $\lambda + 1$ remained for the entire calculation until it appeared in the $\tilde{\alpha}$ in the end.

Clearly the problem is quite a subtle one, in both cases the operator Δ^{-1}_{abcd} is correct. Nevertheless a different factor came out because for the semi-classical calculation we had a single $\delta(x)$ for the shockwave, while for the propagator we needed a double $\delta^{(2)}(x-x')$ for finding the inverse. For both methods individually the steps taken are correct, but nevertheless a different factor comes out. The solution clearly does not lie in an error in the calculation, nor in the horizon approximation which is valid to lowest order. The only possibility that remains is to also insert some horizon approximation on the fields.

Shapiro field approximation

We will now suggest one such possibility for a field approximation. As we saw in Subsection 5.3 only the fields $\mathfrak{h}_{xx}$ and $\mathfrak{h}_{yy}$ contribute significantly to Eikonal scattering. We could then split these fields into a part on the future horizon and on the past horizon. These horizons are described by different coordinates so possibly they cannot be treated together, and we need to separate them. Such a decomposition would look like

$$\mathfrak{h}_{ab}(x,y) \quad = \quad \mathfrak{h}^{+}_{ab}(x)\delta(y)+\mathfrak{h}^{-}_{ab}(y)\delta(x). \tag{5.85}$$

Here the $(-)$-functions are defined on the past horizon and the $(+)$-functions on the future horizon. We can check however that the decomposition above does not only give a λ term, but also a $\lambda+2$ term. This is a result of the $-y\partial_y$ present in Δ^{-1}_{xxyy} and vice versa. As such we need to restrict the decomposition even more into:

$$\mathfrak{h}_{xx}(x,y) \quad = \quad \mathfrak{h}^{-}_{xx}(y)\delta(x), \tag{5.86}$$
$$\mathfrak{h}_{yy}(x,y) \quad = \quad \mathfrak{h}^{+}_{yy}(x)\delta(y). \tag{5.87}$$

Now we only have factors λ appearing. Looking closely at the decomposition above we see the Shapiro delay metric again. Thus apparently the only possible field configuration that generates the λ factor is exactly the Shapiro delay, and the factor λ is unique in that sense.

Let us now look at this more explicitly. Remember that the final $\tilde{\alpha}$ only results due to the $\lambda+1$ presence in

$$\Delta^{-1}_{xxyy} \quad = \quad -\tfrac{1}{2}\mu^2(\lambda+1+x\partial_x-y\partial_y), \tag{5.88}$$
$$\Delta^{-1}_{yyxx} \quad = \quad -\tfrac{1}{2}\mu^2(\lambda+1+y\partial_y-x\partial_x). \tag{5.89}$$

For now we can ignore the presence of the fields $\mathcal{K}$ and $\mathfrak{h}_{xy}$ in the action as these do not contribute to the Eikonal amplitude. The final action before finding the propagator as given in Equation 4.121 is then given simply by

$$S=\tfrac{1}{4}\int \mathrm{d}^2x\left(\mathfrak{h}^{xx}\Delta^{-1}_{xxyy}\mathfrak{h}^{yy}+(x\leftrightarrow y)\right). \tag{5.90}$$

Let us look only at the Δ^{-1}_{xxyy} operator since the other is obtained simply by replacing $x\leftrightarrow y$. We insert the Shapiro delay decompositions in Equations 5.86-5.87 into the first part of the above action to find

$$S_1=-\tfrac{1}{8}\mu^2\int \mathrm{d}^2x\mathfrak{h}^{+}_{yy}(x)\delta(y)\big(\lambda+1+x\partial_x-y\partial_y\big)\mathfrak{h}^{-}_{xx}(y)\delta(x). \tag{5.91}$$

The factor $x\partial_x\delta(x)$ that appears can be integrated by parts according to:

$$\int \mathrm{d}x\ \mathfrak{h}^{+}_{yy}(x)x\partial_x\delta(x)=-\int \mathrm{d}x\ \delta(x)\partial_x\big(x\mathfrak{h}^{+}_{yy}(x)\big)=-\int \mathrm{d}x\ \delta(x)\big(1+x\partial_x\mathfrak{h}^{+}_{yy}(x)\big) \tag{5.92}$$

where the boundary terms cancels exactly due to $\delta(\pm\infty)=0$. Using this relation the action is rewritten into

$$S_1=\tfrac{1}{4}\int \mathrm{d}^2x\mathfrak{h}^{+}_{yy}(x)\delta(y)\big(-\tfrac{1}{2}\mu^2\lambda\big)\mathfrak{h}^{-}_{xx}(y)\delta(x)+\tfrac{1}{8}\int \mathrm{d}^2x\delta(x)\delta(y)\big(x\partial_x+y\partial_y\big)\mathfrak{h}^{-}_{xx}(y)\mathfrak{h}^{+}_{yy}(x). \tag{5.93}$$

The last term is now interesting: The delta functions do not have any derivatives on them anymore. This means we can simply remove the integrals and set $x=0=y$, however in that case $x\partial_x+y\partial_y=0$ (given that

$\mathfrak{h}^-_{xx}(0)$ and $\mathfrak{h}^+_{yy}(0)$ are regular). Thus we can use the delta functions to remove the second part of Equation 5.93 to rewrite the action as

$$S = \tfrac{1}{4}\int \mathrm{d}^2x\big(\mathfrak{h}^{xx}\big(-\tfrac{1}{2}\mu^2\lambda\big)\mathfrak{h}^{yy} + (x \leftrightarrow y)\big) \tag{5.94}$$

where we reinserted the definitions of Equations 5.86-5.87. Comparison with Equation 5.90 now shows that we can identify $\Delta^{-1}_{xxyy} = -\frac{1}{2}\mu^2\lambda$. Inversion becomes straightforward and gives

$$\boxed{\Delta_{xxyy} = \Delta_{yyxx} = -\frac{2}{\mu^2\lambda}.} \tag{5.95}$$

This gives exactly the factor $\alpha = \frac{8\pi G}{\lambda}$ in the end as desired. This shows that the difference really comes from the specific shape of the Shapiro delay.

Resolution

We can conclude that the different factor $\tilde{\alpha}$ is present because we did not assume the exact Shapiro delay metric. When assuming the Shapiro delay metric as in Equations 5.86-5.87 the same factor α results as in Section 3, showing that the factor α is not general, but is a specific result for the Shapiro delay. More specifically the factor α results when taking only orthogonal shockwaves as the lightcone as gravitational interaction.

Instead we have now used a more general gravitational interaction, also allowing parallel shockwaves, but more importantly arbitrary $\mathfrak{h}_{xx}, \mathfrak{h}_{yy}$. This general perturbation results straightforwardly in the factor $\tilde{\alpha}$.

Thus finally this subsection has shown that the factor $\tilde{\alpha}$ is in fact not incorrect, nor in disagreement with Section 3. We have provided a more general result than Section 3, taking into account all gravitational effects and not only the Shapiro delay. The difference of these generalized gravitational effects is apparently a factor +1.

6 Discussion

In this section we will discuss several of our thoughts and possible ideas for future research. We will split this into comments on Sections 2-3 and Sections 4-5.

6.1 On the semi-classical scattering matrix

The physical Schwarschild region

As discussed in Subsection 3.5 we can treat regions I and II as quantum clones of eachother

$$\mathcal{H}_I = \text{CPT}\mathcal{H}_{II}^{\dagger}.$$

Let us first look at what happens when we bluntly reformulate spacetime to remove the quantum clones. These are our original thoughts and we will see that there is a problem embedded in the reasoning. We begin by removing regions II and IV from the theory as their information is still embedded in regions I and III. This diagram of the physical Schwarzschild spacetime is shown in Figure 20. The procedure is analogous to the removal of one-half of the Minkowski diagram in Figures 8-9.

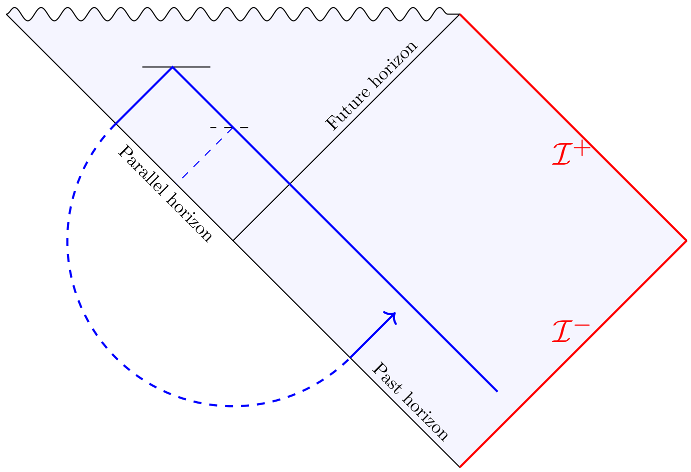


Figure 20: The large-κ trajectory in the physical Scharzschild spacetime.

For the spacetime in Figure 20 we need to review the basic questions of causal structure and geodesics again. Let us first look at the particle trajectories. The lightcone geodesics are in principle unchanged, particles still move over constant x or y. When including the gravitational backreaction for large κ (Subsection 3.6) however, it appears that a particle exiting region III on the left, would come out again at the past horizon in region I. The left boundary of region III, from now on called its parallel horizon, then seems to be connected to the past horizon of region I. This trajectory for a large κ particle is shown in Figure 20.

Bluntly we see that the parallel horizon in region III is connected to the past horizon in region I similar to the projective sphere at the $x = 0 = y$ boundary. There is no more CPT shift, since this shift would have occurred twice while traversing regions II and IV. The connection between horizons allows particles exiting from region III at the parallel horizon to escape again in region I. This is not a classical geodesic, but a result of the gravitational backreaction. There is now a direct link between the outgoing Hawking radiation in region I at the past horizon and the ingoing radiation in region I at the future horizon. The ingoing particles indirectly redirect themselves back into region I in the past. This solves the problem of information loss, but creates a severe new one.

It is possible to draw causal loops. To be precise the particle entering the black hole in Figure 20 is reflected back to the past horizon, where it crosses the same *spacetime* point it originated from. The most tempting solution is that the large κ calculation was performed correctly, and that the causal loops are a

result of inserting a time-dependent process on the eternal Schwarschild spacetime. The eternal spacetime can never describe a process of evaporation independent from infalling particles, since the black hole would not be static if evaporation is present.
We can conclude that spacetime can only be static if exactly the same amount of energy that exits the black hole, also enters it. Thus the only way spacetime is eternal when considering our time-dependent process is when the energy distribution on past asymptotic infinity is the same that on future asymptotic infinity. The most straightforward solution is to apply periodic boundary conditions to spacetime. Whenever we insert particles with nonzero energy the phase-space states at the far past and far future must be equal to legitimate the usage of the eternal Schwarzschild spacetime.

To complement the above argumentation, remark that when drawing Hawking radiation outgoing in Figure 20 it will always causally interact with all ingoing radiation. Thus if we solve the information paradox causal loops automatically appear. It seems like this is a generic problem when looking at the Schwarszchild spacetime, and the periodic boundary solution is always necessary.

The reasoning so far was our initial attempt at understanding the physical process behind the large κ trajectory and relation between region I and II. Given the geodesics and trajectories the above reasoning is logical, however there is a problem with this initial step. We are looking at the trajectories, but nowhere in the calculations in Sections 3 and 5 did we calculate any trajectories, we only calculated asymptotic states. These asymptotic states are the particle states that the global observer sees at asymptotic past and future infinity. These are shown by $\mathcal{I}^-$ and $\mathcal{I}^+$ in Figure 20 respectively. The large κ trajectory we drew in the same figure, represented by the blue line, was found by extrapolating these asymptotic states using classical trajectories, but there is nothing that allows us to.

According to both calculations in Sections 3 and 5 the global observer only sees the asymptotic states, and has no idea of what is happening in the middle. Physically this is to be understood as follows: Imagine the infalling particle has been given a red cube by the global observer, and some time later the global observer sees an outgoing particle with the red cube. The global observer now knows that the red cube came back, but has no idea how this happened. In particular the global observer does not know whether the ingoing particle encountered a causal loop, our extrapolation drawn in Figure 20 was an incorrect assumption.

We see that care needs to be taken with the interpretation of the scattering matrices in Sections 3 and 5. It is tempting to draw the classical trajectories to understand what happens physically, however this is an extrapolation that is in principle not allowed. All interpretations made from such extrapolation need extreme care on whether something forbidden has been done or not. For our case above we had a forbidden extrapolation, we claimed that causal loops appeared, however that never resulted from the calculations. This also means that at least from our calculation we do not need to impose anything drastic like periodic boundary conditions on the eternal black hole.

There are now two questions one can ask: What happens for the local observer and what happens for the finite lifetime black hole? To answer the first is highly nontrivial; for an infalling observer the geodesics of outgoing observers are not well defined. Thus calculating the scattering matrix for a local observer is not possible. In particular an infalling observer is behind the horizon in the asymptotic future, and thus can never measure the asymptotic state of outgoing particles.
The question of the finite lifetime black hole is interesting. The asymptotic states and the black hole itself is drawn differently. This is shown in Figure 21. The infalling particles originating from the asymptotic past (bottom) interact only closely around $x = 0 = y$ to diverge into the out-particles (reaching asymptotic future on the top). There are no causal loops present anywhere. Clearly physical interpretations of Hawking radiation and the corresponding asymptotic states need to made on the finite lifetime black hole, as that prevents many misconceptions that might arise in the eternal Schwarzschild spacetime. Notice that all interaction happens closely to $x = 0 = y$ which is also our horizon approximation in Section 4 supporting that calculation.

To shortly conclude we first attempted to interpret the large κ limit and the antipodal identification on the Schwarzschild diagram leading to Figure 20. This interpretation resulted in causal loops and required

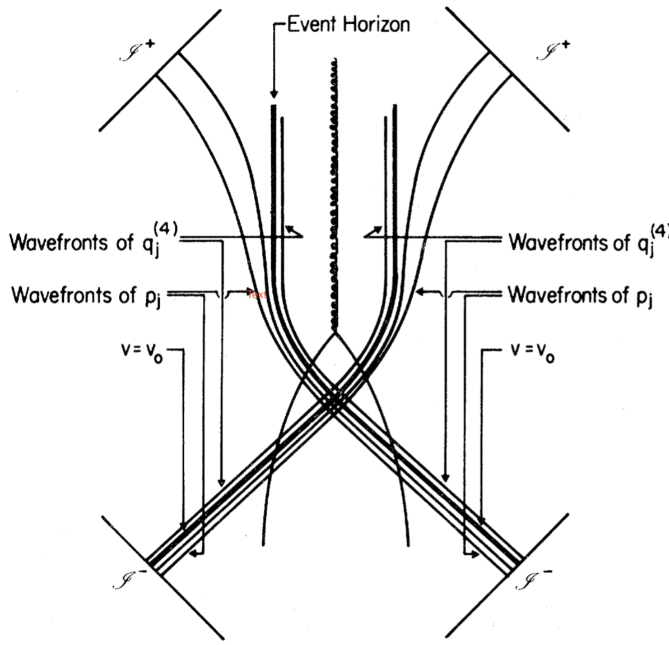


Figure 21: The diagram of a finite lifetime black hole.

periodic boundary conditions to solve this. However our initial interpretation was based on incorrect extrapolation of the asymptotic states. Generally when interpreting the black hole scattering matrix one only has access to asymptotic states, so great caution is needed when trying to extrapolate to what happens closer to the black hole. We only know what we observe at an infinite distance, and have no access to whatever happens in between.

Quantum states and entropy

The entropy of a black hole is given by the Bekenstein-Hawking area law

$$S = \frac{A}{4\ell_{\rm Pl}^2}.$$

This entropy has since also been found in string theory [77] and AdS/CFT [78], and also for deSitter space [79]. Without going into the depths of the implications of this equation it would be a good addition to Section 3 to find the same equation for the entropy from the semi-classical perspective. In [38] 't Hooft proposes to split spacetime into two regions:

$$\text{black hole} \qquad R \leq \; r \; \leq R(1+\varepsilon) \tag{6.1}$$

$$\text{rest of the universe} \qquad r \; > R(1+\varepsilon) \tag{6.2}$$

where ε is some small dimensionless number, but not infinitesimally small. By splitting spacetime like this we can define all particle states in the $R \leq r \leq R(1+\varepsilon)$ region as part of the black hole. In particular the particles are in a one-dimensional box potential, such that the momentum is discretely quantized. For the calculation we refer the reader to [38], the resulting amount of states is

$$N = \text{Exp}\left(\sum_{\ell=\text{odd}} (2\ell+1)\log\left(\varepsilon \tfrac{R^2}{(2\pi)^2 G}(\ell^2+\ell+1)\right)\right). \tag{6.3}$$

This result diverges so a cutoff $\ell_{\rm max}$ is needed, and upon choosing $\ell_{\rm max} \sim M_{\rm BH}/M_{\rm Pl}$ we see the amount of states above scales as the area as desired. This choice of cutoff is a natural order of magnitude, and the area law follows as expected for black holes. Nevertheless the method above requires more research. To begin with the right coefficient of 1/4 in the black hole entropy was not found. Furthermore there is a cutoff necessary, for which we need a proper external physical argument as to what the cutoff needs to be. In addition this cutoff is likely connected to the effect of transverse momenta, since ℓ describes the angular momentum, so the calculation in Section 3 might need to extend to transverse momenta to properly find the entropy bound. Finally there is the ε which is currently arbitrary, while we expect such bound to arise naturally from the underlying physics.

An alternative method is provided by [28]. Here we first look back to the dilaton Hamiltonian. This Hamiltonian can be identified as the inverse harmonic oscillator, i.e. there is a potential hill rather than a potential well. This means that particles tend to move away from the origin rather than towards it. Similar to the infinite tower of states of the harmonic oscillator that can be created from the ladder operators we can construct such an infinite tower for the inverse harmonic oscillators. In other words we have a countably infinite set of states for the Hamiltonian in Section 3. Each state has a certain eigenenergy

$$\kappa_\ell = i(n_\ell + \tfrac{1}{2}) + \frac{R^2}{8\pi G}(\ell^2 + \ell + 1)$$

where n_ℓ is the number operator for the state and κ_ℓ is the same Hamiltonian eigenvalue as in Section 3. This eigenenergy then exists for every possible (ℓ, m) shell. The factor i is due to the presence of a potential hill resulting in scattering states. The total energy is now found by simply summing

$$E_{\text{tot}} = \sum_{\ell,m} \kappa_\ell.$$

Now the total energy must of course be the black hole mass so we can identify $E_{\text{tot}} = M_{\text{BH}}$. This gives a natural bound to the particle states merely from energy conservation. Likely such bound can also be derived for 't Hooft's method above, but we currently see no method of how to do this. The amount of possibilities of how this energy E_{tot} can be formed is large; we sum over all κ_ℓ so any set of numbers n_ℓ that sums up to E_{tot} is allowed. This allows us to define the number of states as

$$N = \text{amount of } n_\ell \text{ that result in } E_{\text{tot}} = M_{\text{BH}}.$$

We have attempted to investigate a proper calculation of this number, but to no avail. Proper calculation of this integer likely shows a strong resemblance of the amount of states and the area law, and hopefully an equality. This is something that we would still like to calculate in a future research. We prefer the latter method for calculating the states since there is no quantity ε needed, such that the states are more naturally defined (in fact from the compactness of the two-sphere). We remark that the constraint $E_{\text{tot}} = M_{\text{BH}}$ is much more general than the above. If another method of calculating the entropy is found, the constraint $E_{\text{tot}} = M_{\text{BH}}$ can always be applied to find the upper bound on the amount of states.

An important remark is necessary on both of the methods above. The dilaton Hamiltonian in Section 3 tells us that the system respects dilation symmetry. In fact this dilation symmetry is simply time translation symmetry as observed from the local observer. Both methods above however use a particle in a box system. The first method does so explicitly by imposing a cutoff at $R(1 + \varepsilon)$, the second implicitly by inserting the potential in a box. The particle in a box breaks the dilation symmetry, so what we see is that both entropy calculations above in fact break time translation symmetry. We would rather have a method that preserves this symmetry. The calculation in [27] provides a dilation-symmetry preserving quantization, so attempting any attempts to calculate the entropy should also be made in the framework of [27] to preserve time translation invariance.

Transverse momenta

Finally we want to comment on the addition of transverse momenta p_A. These have been neglected throughout the entire paper under good physical reasoning, however to properly include all degrees of freedom the transverse momenta must be taken into account. Such effects have been taken into account by de Haro in [56].

However he generalized the effective action in Equation 3.40, since generalizing the procedure in Section 3 is difficult. As such the results of [56] do not provide a scattering matrix similar to Equation 3.36. The results of [56] are nice and show more resemblance to string theory, however we will not go into that now. Instead we ask ourselves the question of how to generalize the procedure in Section 3.

The first step would be to generalize the Shapiro delay to include transverse effects. The cut-and-paste procedure cannot be applied solely on the lightcone momentum, but also needs to take the transverse movement of particles into account. This is a procedure that has not been performed yet, as it implies shockwaves that are not on a null-surface which are highly nontrivial. Additionally the energy momentum tensor must take those into account. Then even if this complicated procedure has been succesfull, the generalized Shapiro delay must again be inserted into the commutators. Here another problem arises, which is the spherical harmonics expansion. The transverse momenta are not explicitly present anymore, but are hidden in the (ℓ, m) variables. Thus adding the transverse interaction likely induces mixture between the different (ℓ, m) modes. Physically this makes sense since the shockwave of a transverse moving particle does not respect spherical symmetry, so that straightforward decoupling cannot be expected from the spherical symmetry of the Schwarzschild spacetime alone. Inserting the Shapiro delay will then likely not reduce the problem to the straightforward calculation of one-dimensional quantum mechanics. Clearly generalizing the semi-classical calculation to transverse momenta appears to not give anything tangible. The addition of transverse momenta in the Eikonal approximation will be different, and we will comment on that in the next subsection.

6.2 On the Eikonal approximation

Transverse momenta and the minimal coupling

Since we are not considering the Shapiro delay we need not bother ourselves with how cut-and-paste procedure would generalize to transverse momenta. Rather we need to include the transverse momenta in the calculation in Section 5. First we need to comment on how the transverse momenta express themselves in the spherical harmonics expansion. Remember that by definition

$$\phi(x^a, x^A) = \sum_{\ell,m} \phi_{\ell m}(x^a) Y_\ell^m(x^A).$$

Then by acting with the derivatives $p_A = i\mu\partial_A$ we find that the transverse momenta express themselves in the elements $\phi_{\ell m}$. To be precise the exact values of $\phi_{\ell m}$ determine the momenta. The presence of transverse momenta is thus already included in the calculation, so in principle we have the right start. However, the interaction of these transverse components was neglected. This was done thrice, although once more subtly and once more implicitly.

The first time we neglected all p_A was when approximating the vertex in Section 5.3. We could remove many of the complexities of the vertex, and in addition ignore $\mathfrak{h}_{xy}, \mathcal{K}$ and h_a. Adding the transverse momenta would imply we need to take all these fields into account as well, including the odd harmonics h_a for which we have not yet derived the action. All the different fields combined together might alter the combinatorics in Lévy & Sucher [30], so that the generalized Equation 5.51 is not valid anymore. Remember that these combinatorics relied on all graviton propagators being equal. We would then need to redo the combinatorics for these particular fields.

The second time we neglected such interaction was implicitly in the minimal coupling approximation. Similar to how in quantum mechanics in Section 3 neglecting the transverse momenta result in a decoupling between the (ℓ, m) modes, such a decoupling was imposed in the Eikonal calculation in Section 5 by the minimal coupling simplification. Thus it seems like the minimal coupling simplification is implicitly an assumption that the transverse momenta are small enough to ignore non-spherical effects on the background. This was in good agreement with the calculation in Section 2, however for including transverse momenta we would need to remove the minimal coupling. This would introduce the following identity for the spherical harmonics

(Equation 3.8.73 of [74]):

$$\int \mathrm{d}\Omega Y_\ell^m(\Omega) Y_{\ell_1}^{m_1}(\Omega) Y_{\ell_2}^{m_2}(\Omega) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1\ell_2; 00|\ell_1\ell_2; \ell 0\rangle \langle \ell_1\ell_2; m_1 m_2|\ell_1\ell_2; \ell m\rangle \tag{6.4}$$

Here $\langle \ell_1\ell_2; m_1 m_2|\ell_1\ell_2; \ell m\rangle$ are the Clebsch-Gordan coefficients. Most important is that there is not a single Kronecker delta appearing, so that we cannot remove any of the sums in Equation 5.43. Clearly the vertex becomes much more complicated due to summations over (ℓ, m) persisting in the action, and the different (ℓ, m) modes being coupled through complicated coefficients.

Finally we assumed $p_A = 0$ very subtly when approximating the matter propagators as

$$\frac{-i}{(p-k)^2 + \mu^2\lambda - i\epsilon} \approx \frac{-i}{-2p\cdot k - i\epsilon}. \tag{6.5}$$

In the flat space case in Section 5.1 we could use $p^2 = -m^2$ as a mass-shell condition, which implied that when approximating the matter propagators as was done in Equation 6.5 we only neglected a k^2 term.
In the curved spacetime we have a different mass-shell condition, namely $p_a p^a = 0$ as we are assuming the external particles lightlike to highest order in s/μ^2. Inserting this ondition yields

$$(p-k)^2 = p^2 - 2p\cdot k + k^2 \approx -2p\cdot k - \mu^2\lambda$$

in the left-hand-side of Equation 6.5. Comparison to the right-hand-side of Equation 6.5 shows that we neglect a factor $\mu^2\lambda$ for the propagator of $\phi_{\ell m}$ and a factor μ^2 for the propagator of ϕ_0. Neglecting such factor is safe in the large black hole limit, and small transverse momenta limit. However when including the transverse momenta we cannot ignore such a factor. The matter propagators then also alter the combinatorics in Lévy & Sucher [30], similar to the graviton legs.

Clearly the question of how to insert the transverse momenta is answered very differently for the semi-classical calculation or the Eikonal approximation. For the semi-classical interaction we mostly had conceptual problems; what is the transverse Shapiro delay and what happens to the quantum mechanical commutators. For the Eikonal approximation the conceptual problems are quickly overcome, however the mathematics becomes a serious problem. In addition we also need to include the $\ell = 0$ and $\ell = 1$ field for the graviton which have gauge problems that would also need to be fixed beforehand. To conclude adding transverse momenta in the Eikonal approximation does not seem feasible as it is formulated right now.

Further investigations

We also want to comment on two possible future investigations. The first is to look at different fields than the scalar field. Remember that from Subsection 3.5 we found that regions I and II are related by a CPT shift. Since a scalar field does not have any role in this except for the T part, it would be a good idea to investigate other fields. In particular one could use oriented Weyl spinors to probe the parity directly, and use a complex scalar field to probe the charge directly. In addition one could attempt to add electrodynamics to the theory by adding fermions and a spin-1 gauge field. This has been suggested by 't Hooft in [23] before for the semi-classical theory. The extension to Quantum Electrodynamics is straightforward for the Eikonal approximation. In general our scattering amplitude in Section 5 needs to transformed to Tortoise coordinates to find the relations between region I and II still.

Finally there is still the question of what the quantum states are and how the in- and out-states are related. In particular the amount of quantum states need to sum up to the area law. The problem with the scattering amplitude we obtained is that the quantum states have been worked around, in order to obtain the amplitude without working with the eigenstates and creation and annihilation operator. Whereas this is a great method for finding scattering amplitudes, as it is much easier and faster, it is really hard to work the calculation back to find the actual quantum states. This connects to the earlier point where we would like to find the scattering matrix explicitly. We could use such scattering matrix to relate the in- and out-fields, and then hopefully relate the states they create. These states must then be summed up such that their total energy is $E_{\text{tot}} = M_{\text{BH}}$.

In the case that such a procedure is not doable our calculation can still be used. We have now accurately shown which fields and interactions are necessary in a quantum field theory to find the same scattering matrix. This allows for a strong foundation when attempting to use the creation and annihilation operators, as to which fields to ignore. In particular in such theory we could immediately set $\mathfrak{h}_{xy} = 0 = \mathcal{K}$ and $h_a = 0$, and use the minimal coupling simplification. This greatly reduces the complexity, and hopefully allows for a well-founded calculation using the creation and annihilation operators. Such a calculation would also be a great complementary addition to our Eikonal scattering amplitude.

7 Conclusion

In this paper we have first reviewed the semi-classical calculation of 't Hooft for black hole scattering in Sections 2 and 3. We investigated the gravitational interaction for almost lightlike particles and found using the cut-and-paste procedure that it was given by the Shapiro delay. The cut-and-paste procedure was then used to remove the firewall paradox.
Next we applied the Shapiro delay on the quantum wavefunctions. Expanding into eigenstates then resulted in an explicit scattering matrix given in Subsection 3.3. This scattering matrix was found to be unitary and also gives a causal connection between region I and II. The unitarity implies that the black hole information paradox is solved under the assumption that momentum is the only type of information, however other information can likely also be exchanged by other gauge forces. The connection between region I and II was further extended to a general proposal that coordinate transformation need be bijective, and a claim that region I and II must be connected by a CPT shift. These last points suggest we must interpret the Schwarzschild spacetime differently (See Section 6). Finally due to the change in sign of energy (due to the T shift) we argued that creation and annihilation operators must be taken into account, leading us to formulate our main topic.

We sought to calculate the gravitational interaction on the black hole in a different manner, by using linear quantized gravity and field operators. To do so we decided to sum over all Eikonal ladder diagrams, taking only leading order behaviour in the center of mass energy into account. First in Section 4 we performed many steps to find the graviton propagator. In this propagator the graviton had obtained an effective 2D mass due to curvature, resulting in UV-problems (without major consequence however), and an important IR-mass.
Finally we used this propagator for the summation of Eikonal ladder diagrams in Section 5. First we simplified the vertex due to the high-energy limit and a minimal coupling simplification. The high-energy limit resulted in the complexity of gravitational interaction being reduced to only two relevant modes $\mathfrak{h}_{xx}, \mathfrak{h}_{yy}$. With the simplified vertex the Eikonal calculation became straightforward, and the resulting scattering amplitude was finally given in Subsection 5.5. This amplitude holds to all orders in G and $\hbar$ giving a full quantum gravity amplitude. Additionally we saw that the conditions $\ell_{\text{Pl}} \ll b \sim R$ and $s/\mu^2 \gg 1$ are trivially satisfied, in fact for standard model particles $s \gg \mu$ holds even for the rest mass alone for a solar size black hole, so our amplitude provides a general high energy scattering description. We also assumed nothing about the particle geodesics, providing strongly general results.

The scattering amplitude of our Eikonal approximation was shown to strongly resemble the scattering matrix of 't Hooft in Section 3. In fact the resulting scattering matrices are likely equal upon properly rewriting the scattering amplitude to the scattering matrix. The only significant difference between the results is a factor $+1$ in the coupling constant $\tilde{\alpha}$ of our scattering amplitude. This difference was greatly addressed in Subsection 5.6, where we have shown that the factor $\tilde{\alpha}$ is correct and a result of the more general gravitational interaction we consider, providing a small correction to Section 3. We can conclude that the Eikonal approximation does describe the black hole scattering matrix, including the Shapiro delay in Section 3. In fact the exact Shapiro delay for lightlike particles also returns from our quantum gravity calculation. Thus we have successfully constructed a new model describing high-energy gravitational scattering on the black hole horizon that includes field operators. This model only rests on quantum field theory and general relativity, two widely accepted theories, providing a strong foundation.

Our hope is that in the future this model can be used to investigate a multitude of properties of black hole scattering and the Schwarzschild spacetime. We can now include different fields quite easily, and the calculation should remain straightforward. In particular we can scatter fermions or gauge fields to investigate the earlier mentioned CPT shift. Additionally we can attempt to properly find the scattering matrix, which can then be used to relate the actual fields of the in- and out-particles. Such a relation can be used to find the quantum states. If summation over these quantum states yields the Bekenstein-Hawking entropy we have found a new method of deriving this entropy purely based on quantum field theory and general relativity. Finally the steps taken in this paper show that many simplifications naturally come into play for high-energy black hole scattering, and the complete model now provides a good base as to why certain simplifications are allowed for future research using different models.

A Definitions

A.1 Metric and coordinates

We work in Kruskal-Szekeres coordinates that are defined as

$$xy = 2R^2\left(1-\frac{r}{R}\right)e^{\frac{r}{R}-1}, \tag{A.1}$$

$$x/y = e^{2\tau}. \tag{A.2}$$

Here $\tau = \frac{t}{2R}$. We will often write these as a two-vector x^a where $x^x = x, x^y = y$. Here $a = x, y$ denotes a lightcone index and $A = \theta, \phi$ a two-sphere index. The quantity R is the Schwarzschild radius

$$R = 2GM, \qquad \mu \equiv \frac{1}{R} \tag{A.3}$$

where we will work with the effective mass μ more often. We work in natural units where $\hbar = c = 1$. The metric is in these coordinates given by

$$\mathrm{d}s^2 = -2A(r)\mathrm{d}x\mathrm{d}y + r^2\mathrm{d}\Omega^2, \tag{A.4}$$

$$A(r) = \tfrac{R}{r}e^{1-\frac{r}{R}} \tag{A.5}$$

where everywhere $r = r(x, y)$ a function of both coordinates. Our metric convention is the $(-,+,+,+)$ signature. This metric has the following matrix definitions:

$$g_{\mu\nu} = \begin{pmatrix} 0 & -A & 0 & 0 \\ -A & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} 0 & -A^{-1} & 0 & 0 \\ -A^{-1} & 0 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{A.6}$$

On $r = R$ we find $A = 1$ such that the metric is given by

$$\mathrm{d}s^2 = -2\mathrm{d}x\mathrm{d}y + R^2\mathrm{d}\Omega^2, \qquad (r = R). \tag{A.7}$$

Several derivatives:
Here we list the derivatives of A, r. The derivative on r can be found by implicit differentiation on Equation A.1 to be

$$\partial_a r = \frac{1}{2R}x_a \tag{A.8}$$

where $x_a = g_{ab}x^b$. The derivative of $A(r)$ is then:

$$\partial_a A = \partial_r A \partial_a r = -\frac{A}{2R}\left(\frac{1}{r}+\frac{1}{R}\right)x_a. \tag{A.9}$$

When evaluated on $r = R$ these functions have the following derivatives on the horizon:

$$\partial_a A = \partial_a r = 0 \quad , \quad \text{for any } a \text{ on } r = R, \tag{A.10}$$

$$\partial_x\partial_y r\Big|_{r=R} = -\frac{1}{2R}, \tag{A.11}$$

$$\partial_x\partial_y A\Big|_{r=R} = \frac{1}{R^2}. \tag{A.12}$$

(A.13)

All nonzero Christoffel symbols:
These are all nonzero Christoffel symbols for the Schwarzschild metric in Kruskal-Szekeres coordinates.

$$\Gamma^x_{xx} = \partial_x \log A, \tag{A.14}$$
$$\Gamma^y_{yy} = \partial_y \log A, \tag{A.15}$$
$$\Gamma^\theta_{\theta x} = \Gamma^\theta_{x\theta} = \Gamma^\phi_{\phi x} = \Gamma^\phi_{x\phi} = \partial_x \log r, \tag{A.16}$$
$$\Gamma^\theta_{\theta y} = \Gamma^\theta_{y\theta} = \Gamma^\phi_{\phi y} = \Gamma^\phi_{y\phi} = \partial_y \log r, \tag{A.17}$$
$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = -\sin^{-2}\theta\Gamma^\theta_{\phi\phi} = \cot\theta, \tag{A.18}$$
$$\Gamma^x_{\theta\theta} = \sin^{-2}\theta\Gamma^x_{\phi\phi} = \frac{1}{2A}\partial_y r^2, \tag{A.19}$$
$$\Gamma^y_{\theta\theta} = \sin^{-2}\theta\Gamma^y_{\phi\phi} = \frac{1}{2A}\partial_x r^2. \tag{A.20}$$

A.2 The antisymmetric Levi-Civita tensor

Here we define the antisymmetric tensor on S_2 by

$$\epsilon_{AB} = r^2 \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.21}$$

i.e. $\epsilon_{\theta\phi} = r^2 \sin\theta = -\epsilon_{\phi\theta}$. Raising and lowering goes with the metric, so for the more common form ${\epsilon_A}^B$ we have

$${\epsilon_A}^B = \begin{pmatrix} 0 & \sin\theta \\ -\csc\theta & 0 \end{pmatrix}. \tag{A.22}$$

Lastly for the twice raised form we have

$$\epsilon^{AB} = \frac{1}{r^2\sin\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.23}$$

The antisymmetric tensor on the lightcone is defined similarly, although now we work on $r = R$. This tensor is relevant for Section 3. It is straightforwardly given by

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.24}$$

A.3 The Riemann tensor:

We define the Riemann tensor as

$${R^\rho}_{\mu\sigma\nu} = \partial_\sigma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\sigma\kappa}\Gamma^\kappa_{\mu\nu} - \Gamma^\rho_{\mu\kappa}\Gamma^\kappa_{\sigma\nu}. \tag{A.25}$$

The Ricci tensor is now given by

$$R_{\mu\nu} = {R^\rho}_{\mu\rho\nu}. \tag{A.26}$$

For the Schwarzschild metric the Ricci tensor and scalar vanish exactly. Here we list all nonvanishing Riemann components:

$$R_{xyxy} = \partial_x \partial_y \log A, \tag{A.27}$$

$$R_{x\theta x\theta} = \frac{r\partial_x A \partial_x r - A\partial_x^2 r}{A}, \tag{A.28}$$

$$R_{x\phi x\phi} = \sin^2\theta R_{x\theta x\theta}, \tag{A.29}$$

$$R_{y\theta y\theta} = \frac{r\partial_y A \partial_y r - A\partial_y^2 r}{A}, \tag{A.30}$$

$$R_{y\phi y\phi} = \sin^2\theta R_{y\theta y\theta}, \tag{A.31}$$

$$R_{x\theta y\theta} = -r\partial_x \partial_y r, \tag{A.32}$$

$$R_{x\phi y\phi} = \sin^2\theta R_{x\theta y\theta}, \tag{A.33}$$

$$R_{\theta\phi\theta\phi} = r^2 \sin^2\theta \left(1 + \frac{2\partial_x r \partial_y r}{A}\right). \tag{A.34}$$

B Calculations

B.1 Linearized gravity equation of motion

We initially attempted to calculate the equation of motion in the harmonic gauge, but this treatment fails, since the graviton $h_{xx} = 2A\delta(x)F$ does not satisfy the harmonic gauge condition $\nabla_\mu h^{\mu\nu} = \frac{1}{2}\nabla^\nu h$. Gauge fixing would alter $h^{\mu\nu}$, so that we cannot use the simple form above anymore. Instead we now choose to work with the fixed $h_{xx} = 2A\delta(x)F$ and take the general equation of motion before gauge fixing. The Ricci tensor is derived in Appendix B.2 (Equation B.21), to give the following equations of motion:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{B.1}$$

$$R_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\big(\nabla_\lambda\nabla_\mu h_{\rho\nu} + \nabla_\lambda\nabla_\nu h_{\rho\nu} - \nabla_\lambda\nabla_\rho h_{\mu\nu} - \nabla_\mu\nabla_\nu h_{\lambda\rho}\big). \tag{B.2}$$

We want to find the F function using the following metric:

$$g_{\mu\nu} = -2A(x,y)\mathrm{d}x\mathrm{d}y + r^2(x,y)\mathrm{d}\Omega^2. \tag{B.3}$$

Here A,r and the inverse metric are defined in Equations A.1,A.5 and A.6 respectively. We know from Sections 2.3 and 2.4 that we can write $h_{\mu\nu}$ as

$$h_{xx} = 2A(x,y)\delta(x)F(\Omega) \tag{B.4}$$

with the rest vanishing. We are thus only interested in the equation of motion for the (xx) component; $R_{xx} = 8\pi G T_{xx}$. Explicitly calculating all covariant derivatives results in

$$R_{xx} = \frac{A\delta(x)}{r^2}\Delta_\Omega F + \frac{2F\delta'(x)}{r}\partial_y r + \frac{2F\delta(x)}{A}\left(\frac{\partial_x A\partial_y A}{A} - \partial_x\partial_y A - \frac{1}{r}(\partial_x A\partial_y r + \partial_y A\partial_x r)\right). \tag{B.5}$$

Here Δ_Ω is the S^2 Laplacian. Now we can interpret the dirac delta derivative as follows: $\delta'(x)f(x) = -\delta(x)f'(x)$ as this is the effect upon integration. Secondly we shall evaluate the equation of motion at $x = 0$ due to the $\delta(x)$, so that we can set $A = 1$ and remove the $\partial_y A, \partial_y r$ parts. For the last simplification we refer the reader to the derivatives in Appendix A.1. The equation of motion now becomes

$$\frac{\delta(x)}{R^2}\Delta_\Omega F - 2\delta(x)F\left(\partial_x\partial_y A + \frac{\partial_x\partial_y r}{r}\right) = 8\pi G T_{xx}. \tag{B.6}$$

Finally we use the double derivative identities in Appendix A.1 given by

$$\partial_y\partial_x A\big|_{r=R} = \frac{1}{R^2} \qquad \partial_y\partial_x r\big|_{r=R} = -\frac{1}{2R} \tag{B.7}$$

to find for the complete equation of motion

$$\frac{\delta(x)}{R^2}(\Delta_\Omega - 1)F = 8\pi G T_{xx} \tag{B.8}$$

resulting in the desired Einstein-field equations. This shows that the Shapiro delay as we have been treating it can be treated in linear gravity, solidifying the linear gravity treatment in Section 4.
We want to remark that the above is all done in the discontinuous $y + \Theta(x)F$ coordinate, while the energy-momentum tensor is given in the continuous y coordinate. However a quick coordinate transformation shows that on $r = R$ this has no effect on our results, they leave the equation invariant. This is logical since we used the covariant equation of motion, which is manifestly invariant under diffeomorphisms. This completes the derivation of the equation of motion.

B.2 Quadratic gravity action

In this section we shall explicitly derive the quadratic term of the Hilbert action. In the following a number 0,1,2 subscript or superscript implies linear or quadratic order in h. We expand the action as was done in [66]

$$S = S_0 + S_1 + \frac{S_2}{2} \tag{B.9}$$

with S_2 the action term containing all quadratic components. We showed that $S_0 = S_1 = 0$ in Section 4.1 so that $S = \frac{1}{2} S_2$. We shall now simply write out all quadratic terms in the action since this gives S_2. Recall that the Einstein-Hilbert action is defined by

$$S_{EH} = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R. \tag{B.10}$$

We now expand $\sqrt{-g}$ and R then

$$\sqrt{-g} = \sqrt{-g_s}(1 + \tfrac{1}{2}h + O(h^2)), \tag{B.11}$$

$$R = R_1 + R_2 + O(h^3) \tag{B.12}$$

where we have used that the Schwarzschild solution is a vacuum solution and defined $h = g^s_{\mu\nu} h^{\mu\nu}$. We can then write

$$\sqrt{-g}R = \sqrt{-g_s}\left(\tfrac{1}{2}hR_1 + R_2\right) = \sqrt{-g_s}\left(-(h^{\mu\nu} - \tfrac{1}{2}g_s^{\mu\nu}h)R^1_{\mu\nu} + g_s^{\mu\nu}R^2_{\mu\nu}\right). \tag{B.13}$$

where we kept only the quadratic terms. We can now identify

$$S_2 = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g_s}\left(-(h^{\mu\nu} - \tfrac{1}{2}g_s^{\mu\nu}h)R^1_{\mu\nu} + g_s^{\mu\nu}R^2_{\mu\nu}\right). \tag{B.14}$$

Variation of the Ricci tensor:
A general identity for variation of the Ricci tensor, i.e. $\delta R = R[g_s + h] - R[g_s]$, is given by the Palatini identity

$$\delta R_{\mu\nu} = \nabla^s_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla^s_\nu \delta\Gamma^\rho_{\rho\mu}, \qquad \delta\Gamma = \Gamma[g_s + h] - \Gamma[g_s]. \tag{B.15}$$

Since this identity holds at all levels in perturbation theory, we observe that $R^2_{\mu\nu}$ is in fact a total derivative, so that the term containing $R^2_{\mu\nu}$ vanishes exactly due to vanishing boundary conditions. We thus only need to find $R^1_{\mu\nu} = \delta R_{\mu\nu}$ up to first order since $R^s_{\mu\nu} = 0$. We write this as

$$R^1_{\mu\nu} = \nabla^s_\rho \Gamma^{1,\rho}_{\mu\nu} - \nabla^s_\nu \Gamma^{1,\rho}_{\rho\mu}. \tag{B.16}$$

The Christoffel symbol perturbation we write out as follows:

$$\begin{aligned}
\delta\Gamma^{1,\rho}_{\mu\nu} &= -\tfrac{1}{2}h^{\rho\sigma}(\partial_\mu g^s_{\sigma\nu} + \partial_\nu g^s_{\sigma\mu} - \partial_\sigma g^s_{\mu\nu}) + \tfrac{1}{2}g_s^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) && \text{(B.17)}\\
&= -h^{\rho\alpha}g^s_{\alpha\beta}\Gamma^{0,\beta}_{\mu\nu} + \tfrac{1}{2}g_s^{\rho\sigma}(\nabla^s_\mu h_{\sigma\nu} + \nabla^s_\nu h_{\sigma\mu} - \nabla^s_\sigma h_{\mu\nu}) && \text{(B.18)}\\
&\quad + \tfrac{1}{2}g_s^{\rho\sigma}\left(\cancelto{1}{\Gamma^{0,\alpha}_{\mu\sigma}h_{\alpha\nu}} + \Gamma^{0,\alpha}_{\mu\nu}h_{\alpha\sigma} + \cancelto{2}{\Gamma^{0,\alpha}_{\nu\sigma}h_{\alpha\mu}} + \Gamma^{0,\alpha}_{\mu\nu}h_{\alpha\sigma} - \cancelto{1}{\Gamma^{0,\alpha}_{\sigma\mu}h_{\alpha\nu}} - \cancelto{2}{\Gamma^{0,\alpha}_{\sigma\nu}h_{\alpha\nu}}\right) && \text{(B.19)}\\
&= \cancelto{3}{-h^{\rho\alpha}g^s_{\alpha\beta}\Gamma^{0,\beta}_{\mu\nu}} + g_s^{\rho\sigma}(\nabla^s_{(\mu}h_{\nu)\sigma} - \tfrac{1}{2}\nabla^s_\sigma h_{\mu\nu}) + \cancelto{3}{g_s^{\rho\sigma}\Gamma^{0,\alpha}_{\mu\nu}h_{\alpha\sigma}} && \text{(B.20)}
\end{aligned}$$

where the separately numbered terms cancel each other in the last two lines. In the first line we simply wrote out the Christoffel symbol difference, and in the second line we recognized the background Christoffel symbol, and rewrote partial derivatives into covariant derivatives. The Ricci tensor is thus given by

$$R^1_{\mu\nu} = \tfrac{1}{2}g^{\rho\sigma}\left(\nabla_\rho\nabla_\mu h_{\nu\sigma} + \nabla_\rho\nabla_\nu h_{\mu\sigma} - \nabla_\rho\nabla_\sigma h_{\mu\nu} - \nabla_\mu\nabla_\nu h_{\rho\sigma}\right) \tag{B.21}$$

as was also given in [66]. Here all covariant derivatives are interpreted on the background metric $g^s_{\mu\nu}$. From this point we generally redefine $g^s_{\mu\nu} \to g_{\mu\nu}$ the background metric with the graviton $h_{\mu\nu}$ as a new field.

B.3 Einstein tensor identities

In this appendix we formulate the relevant quantities in the action in terms of the 2D metric g_{ab} and its corresponding derivatives, and in the residual curvature components arising from the two-sphere. In deriving these identities it is crucial that

$$\partial_A K = \partial_A H^a_b = 0, \tag{B.22}$$
$$g_{aA} = 0, \tag{B.23}$$
$$\Gamma^A_{ab} = \Gamma^a_{bA} = 0. \tag{B.24}$$

The above result in clear splittings between the lightcone and two-sphere. For now we ignore the $\ell(\ell+1)$ part in the definition of $\mathcal{G}$ in Equation 4.66, as its contribution is trivial. Then due to the structure of h^μ_ν, namely $h^a_b = H^a_b$ and $h^A_B = \delta^A_B K$, we are only interested in $G^A{}_A$ and $G^a{}_b$. Thus what we see is that we always contract over the two-sphere indices. Now it turns out by explicit calculation that Γ^A_{BC} always cancels in the calculation, mostly by courtesy of Equation B.22 or the fact that we contract over these two-sphere indices. Then thanks to both Equation B.22 and Equation B.24 we find that the only curvature remnant of the two-sphere is nicely embedded in the following vector potential

$$V_a \equiv \Gamma^A_{Aa} = -g_{ab} g^{AB} \Gamma^b_{AB}. \tag{B.25}$$

We now write down all calculated expressions in terms of the lightcone metric and V_a.

The first relevant expression is

$$G^A{}_{A\rho}{}^\sigma h^\rho_\sigma = 2\nabla^\rho \nabla_A h^A_\rho - 2\nabla^\sigma \nabla_\rho h^\rho_\sigma - \nabla_A \nabla^A h + 2\Box h - \Box h^A_A. \tag{B.26}$$

Now we find for all these terms separately that expressed in terms of the two-dimensional quantities

$$\nabla^\rho \nabla_A h^A_\rho = \tilde{\nabla}^a (V_b H^b_a) + \tfrac{3}{2} V^a V_b H^b_a - (\tilde{\nabla}^a V_a) K - \tfrac{3}{2} V^a V_a K, \tag{B.27}$$
$$\nabla^\sigma \nabla_\rho h^\rho_\sigma = \tilde{\nabla}^a \tilde{\nabla}_b H^b_a + V^a \tilde{\nabla}_b H^b_a + \tilde{\nabla}^a (V_b H^b_a) + V^a V_b H^b_a - \tilde{\nabla}^a (V_a K) - V_a V^a K, \tag{B.28}$$
$$\nabla_A \nabla^A h = V_a \tilde{\nabla}^a H^b_b + 2 V_a \tilde{\nabla}^a K, \tag{B.29}$$
$$\Box h = \tilde{\Box} H^a_a + V_b \tilde{\nabla}^b H^a_a + 2\tilde{\Box} K + 2 V_b \tilde{\nabla}^b K, \tag{B.30}$$
$$\Box h^A_A = V_a V^b H^a_b + 2\tilde{\Box} K + 2 V_a \tilde{\nabla}^a K - V_a V^a K. \tag{B.31}$$

The other relevant expression is given by

$$G^a{}_{b\rho}{}^\sigma h^\rho_\sigma = \nabla^\rho \nabla_b h^a_\rho + \nabla_\rho \nabla^a h^\rho_b - \delta^a_b \nabla^\sigma \nabla_\rho h^\rho_\sigma - \nabla^a \nabla_b h + \delta^a_b \Box h - \Box h^a_b. \tag{B.32}$$

These terms separately give

$$\nabla^\rho \nabla_b h^a_\rho = \tilde{\nabla}^c \tilde{\nabla}_b H^a_c - \tfrac{1}{2} V^c V_b H^a_c + V^c \tilde{\nabla}_b H^a_c - V^a \tilde{\nabla}_b K + \tfrac{1}{2} V^a V_b K, \tag{B.33}$$
$$\nabla_\rho \nabla^a h^\rho_b = \tilde{\nabla}_c \tilde{\nabla}^a H^c_b - \tfrac{1}{2} V_c V^a H^c_b + V_c \tilde{\nabla}^a H^c_b - V^b \tilde{\nabla}_a K + \tfrac{1}{2} V^a V_b K, \tag{B.34}$$
$$\nabla^\sigma \nabla_\rho h^\rho_\sigma = \tilde{\nabla}^a \tilde{\nabla}_b H^b_a + V^a \tilde{\nabla}_b H^b_a + \tilde{\nabla}^a (V_b H^b_a) + V^a V_b H^b_a - \tilde{\nabla}^a (V_a K) - V_a V^a K, \tag{B.35}$$
$$\nabla^a \nabla_b h = \tilde{\nabla}^a \tilde{\nabla}_b H^d_d + 2\tilde{\nabla}^a \tilde{\nabla}_b K, \tag{B.36}$$
$$\Box h = \tilde{\Box} H^b_b + V_a \tilde{\nabla}^a H^b_b + 2\tilde{\Box} K + 2 V_a \tilde{\nabla}^a K, \tag{B.37}$$
$$\Box h^a_b = \tilde{\Box} H^a_b - \tfrac{1}{2} V_c V^a H^c_b - \tfrac{1}{2} V^c V_b H^a_c + V^c \tilde{\nabla}_c H^a_b + V^a V_b K. \tag{B.38}$$

These identities have all been calculated by hand, and checked by *Wolfram Mathematica*. It should be noted that they are covariant on the lightcone.

B.4 Removing the two-sphere

In this appendix we shall write down the procedure of reaching the following action

$$S_e = \frac{1}{4}\int \mathrm{d}x^2\sqrt{-\tilde{g}}\Big(\tilde{H}^{ab}\tilde{\Delta}^{-1}_{abcd}\tilde{H}^{cd} + \tilde{H}^{ab}\tilde{\Delta}^{-1}_{L,ab}\tilde{K} + \tilde{K}\tilde{\Delta}^{-1}_{R,ab}\tilde{H}^{ab} + \tilde{K}\tilde{\Delta}^{-1}\tilde{K}\Big). \tag{B.39}$$

Our current action is defined by

$$S_e = -\frac{1}{8}\sum_{\ell,m}\int \mathrm{d}^2xAr^2(h_{\ell m})^{\mu}_{\nu}\mathcal{G}^{\nu}_{\ell\ \mu\rho}{}^{\sigma}(h_{\ell m})^{\rho}_{\sigma}, \tag{B.40}$$

$$\mathcal{G}^{\nu}_{\ell\ \mu\rho}{}^{\sigma} = G^{\nu}{}_{\mu\rho}{}^{\sigma} - \tfrac{\ell(\ell+1)}{2r^2}\big(\delta^{\mu}_{\nu}\delta^{\sigma}_{\rho} + \delta^{\mu}_{a}\delta^{a}_{\nu}\delta^{\sigma}_{b}\delta^{b}_{\rho} - 2\delta^{\mu}_{\rho}\delta^{\sigma}_{\nu}\big). \tag{B.41}$$

Thus we need to find the proper field transformations and operator rewritings such that the above actions are identical. As already mentioned the most important part will be to split spacetime into the lightcone g_{ab} and two-sphere g_{AB}, from where the choice of field transformation and the final result naturally follows. For this procedure all identities in Appendix B.3 are crucial.

The G-operator

To find all relevant couplings between H_{ab} and K we now first split the degrees of freedom in $h^{\mu}_{\nu}G^{\nu}{}_{\mu\rho}{}^{\sigma}h^{\rho}_{\sigma}$. To do so we are only interested in $K\mathcal{G}^{A}{}_{A\rho}{}^{\sigma}h^{\rho}_{\sigma}$ and $H^{b}_{a}\mathcal{G}^{a}{}_{b\rho}{}^{\sigma}h^{\rho}_{\sigma}$. Using the identities listed in Appendix B.3 we find that for all relevant quantities in Equation 4.73

$$\tilde{\mathcal{G}} = 2\tilde{\Box} + 2V_a\tilde{\nabla}^a, \tag{B.42}$$

$$\tilde{\mathcal{G}}_{R,ab} = 2g_{ab}\big(\tilde{\Box} + \tfrac{1}{2}V_c\tilde{\nabla}^c - \tfrac{\ell(\ell+1)}{2r^2}\big) - 2\big(\tilde{\nabla}_a\tilde{\nabla}_b + V_{(a}\tilde{\nabla}_{b)}\big), \tag{B.43}$$

$$\tilde{\mathcal{G}}_{L,ab} = 2g_{ab}\big(\tilde{\Box} + V_d\tilde{\nabla}^d + \tfrac{1}{2}\tilde{\nabla}^dV_d + \tfrac{1}{2}V_dV^d - \tfrac{\ell(\ell+1)}{2r^2}\big) - 2\big(\tilde{\nabla}_a\tilde{\nabla}_b + V_{(a}\tilde{\nabla}_{b)}\big), \tag{B.44}$$

$$\tilde{\mathcal{G}}_{abcd} = g_{ac}\tilde{\nabla}_d\tilde{\nabla}_b + g_{ac}V_d\tilde{\nabla}_b + g_{bd}\tilde{\nabla}_c\tilde{\nabla}_a + g_{bd}V_c\tilde{\nabla}_a - g_{ab}\big(\tilde{\nabla}_c\tilde{\nabla}_d + 2V_{(c}\tilde{\nabla}_{d)} + V_cV_d + (\tilde{\nabla}_{(c}V_{d)})\big) \tag{B.45}$$

$$- g_{cd}\tilde{\nabla}_a\tilde{\nabla}_b + g_{ab}g_{cd}\big(\tilde{\Box} + V^e\tilde{\nabla}_e - \tfrac{\ell(\ell+1)}{r^2}\big) - g_{ac}g_{bd}\big(\tilde{\Box} + V^e\tilde{\nabla}_e - \tfrac{\ell(\ell+1)}{r^2}\big). \tag{B.46}$$

Here we have used that all quantities are now finally covariant on the lightcone, so we have switched to a more usual index form. We defined the residual curvature tensor of the two-sphere

$$V_a = 2\partial_a \log r. \tag{B.47}$$

These quantities are calculated by hand and checked on *Wolfram Mathematica*. We can now simplify Equation B.45 since we are in two dimensions; explicit calculation of only all double covariant derivative terms for $\tilde{\mathcal{G}}_{xxcd}H^{cd}, \tilde{\mathcal{G}}_{yycd}H^{cd}, \tilde{\mathcal{G}}_{xycd}H^{cd}$ shows that we can identify

$$\Big(g_{ac}\tilde{\nabla}_d\tilde{\nabla}_b + g_{bd}\tilde{\nabla}_c\tilde{\nabla}_a - g_{ab}\tilde{\nabla}_c\tilde{\nabla}_d - g_{cd}\tilde{\nabla}_a\tilde{\nabla}_b + g_{ab}g_{cd}\tilde{\Box} - g_{ac}g_{bd}\tilde{\Box}\Big)H^{cd} \tag{B.48}$$

$$= g_{ac}[\tilde{\nabla}_d, \tilde{\nabla}_b]H^{cd} = -\tilde{R}_{acbd}H^{cd} + g_{ac}\tilde{R}_{bd}H^{cd}. \tag{B.49}$$

Here for the last part we used the commutation identity of covariant derivatives. Now the Riemann and Ricci tensor are on the lightcone, so two-dimensional. Since the Riemann tensor has only one unique component in 2D (namely $\tilde{R}_{xyxy}$), we can use the general forms to simply write:

$$R_{acbd} = \tfrac{R_s}{2}\big(g_{ab}g_{cd} - g_{ad}g_{bc}\big), \tag{B.50}$$

$$R_{bd} = \tfrac{R_s}{2}g_{bd}, \tag{B.51}$$

$$R_s = \frac{2}{A}\partial_x\partial_y \log A. \tag{B.52}$$

Then altogether the second covariant derivatives only contribute the following:

$$\tfrac{R_s}{2}\big(g_{ad}g_{bc} - g_{ab}g_{cd} + g_{ac}g_{bd}\big)H^{cd} = \tfrac{R_s}{2}\big(2g_{ac}g_{bd} - g_{ab}g_{cd}\big)H^{cd}. \tag{B.53}$$

A similar procedure for all the first derivatives shows that

$$\left(g_{ac}V_d\tilde{\nabla}_b + g_{bd}V_c\tilde{\nabla}_a - 2g_{ab}V_{(c}\tilde{\nabla}_{d)} + g_{ab}g_{cd}V^e\tilde{\nabla}_e - g_{ac}g_{bd}V^e\tilde{\nabla}_e\right)H^{cd} \tag{B.54}$$

$$= \left(g_{ac}V_{[d}\tilde{\nabla}_{b]} + g_{bd}V_{[c}\tilde{\nabla}_{a]} - g_{ab}V_{(c}\tilde{\nabla}_{d)} + g_{cd}V_{(a}\tilde{\nabla}_{b)}\right)H^{cd}. \tag{B.55}$$

Then finally we can define the operator as

$$\tilde{\mathcal{G}}_{abcd} = g_{ac}V_{[d}\tilde{\nabla}_{b]} + g_{bd}V_{[c}\tilde{\nabla}_{a]} - g_{ab}\left(V_{(c}\tilde{\nabla}_{d)} + V_cV_d + (\tilde{\nabla}_{(c}V_{d)})\right) + g_{cd}V_{(a}\tilde{\nabla}_{b)} \tag{B.56}$$

$$- \; g_{ab}g_{cd}\left(\tfrac{R_s}{2} + \tfrac{\ell(\ell+1)}{r^2}\right) + g_{ac}g_{bd}\left(R_s + \tfrac{\ell(\ell+1)}{r^2}\right). \tag{B.57}$$

From now on we will work with this simplified form. The only fact we used to derive all these simplifications is the fact that $g_{xx} = g_{yy} = 0$, i.e. the form of the metric. Using these operators the Lagrangian is written as

$$\mathcal{L}_e = -\frac{r^2}{8}H^{ab}\tilde{\mathcal{G}}_{abcd}H^{cd} - \frac{r^2}{8}H^{ab}\tilde{\mathcal{G}}_{L,ab}K - \frac{r^2}{8}K\tilde{\mathcal{G}}_{R,ab}H^{ab} - \frac{r^2}{8}K\tilde{\mathcal{G}}K. \tag{B.58}$$

Now there is a clear problem with this Lagrangian: the presence of r^2. The metric is now g_{ab} only, so all covariant derivatives do not commute with r^2. This means that the Lagrangian is not propely symmetric between the degrees of freedom. The best solution is to absord the residual r^2 into the fields, so we redefine $\tilde{H}_{ab} = rH_{ab}, \tilde{K} = rK$.

The lightcone fields

We write everything in terms of the redefined lightcone fields $\tilde{H} = rH$ and $\tilde{K} = rK$. To do so we best look at the quantity $V_a = 2\partial_a \log r$, from which we can define a derivative

$$D_a \equiv \tilde{\nabla}_a + \tfrac{1}{2}V_a = \tfrac{1}{r}\tilde{\nabla}_a r. \tag{B.59}$$

Then clearly by replacing every covariant derivative $\tilde{\nabla}_a$ in Equations B.42-B.45 by D_a (using $\tilde{\nabla}_a = D_a - \frac{1}{2}V_a$), we can easily use Equation B.59 to remove one r from the r^2 on the left hand side in Equation B.58 and introduce an r on the right hand side. This automatically gives the symmetric form in r, where we can redefine the fields appropriately, giving the final effective 2D theory in Equation 4.69.

The results with new derivatives are

$$\tilde{\mathcal{G}} = 2D^2 - 2F^a_a, \tag{B.60}$$

$$\tilde{\mathcal{G}}_{R,ab} = 2g_{ab}\left(D^2 - \tfrac{1}{2}V_cD^c + \tfrac{1}{4}V_cV^c - F^c_c - \tfrac{\ell(\ell+1)}{2r^2}\right) - 2\left(D_aD_b - F_{ab}\right), \tag{B.61}$$

$$\tilde{\mathcal{G}}_{L,ab} = 2g_{ab}\left(D^2 + \tfrac{1}{2}D^cV_c + \tfrac{1}{4}V_cV^c - F^c_c - \tfrac{\ell(\ell+1)}{2r^2}\right) - 2\left(D_aD_b - F_{ab}\right), \tag{B.62}$$

$$\tilde{\mathcal{G}}_{abcd} = g_{ac}V_{[d}D_{b]} + g_{bd}V_{[c}D_{a]} - g_{ab}\left(D_{(c}V_{d)} + \tfrac{1}{2}V_cV_d\right) - g_{cd}\left(-V_{(a}D_{b)} + \tfrac{1}{2}V_aV_b\right) \tag{B.63}$$

$$- \; g_{ab}g_{cd}\left(\tfrac{R_s}{2} + \tfrac{\ell(\ell+1)}{r^2}\right) + g_{ac}g_{bd}\left(R_s + \tfrac{\ell(\ell+1)}{r^2}\right). \tag{B.64}$$

Here we defined $D^2 = D_aD^a$ and the new quantity

$$F_{ab} \equiv \tfrac{1}{2}D_{(a}V_{b)} = \frac{1}{r}\tilde{\nabla}_a\tilde{\nabla}_b r \tag{B.65}$$

which is a symmetric tensor (not an operator). Notice that the fact that F_{ab} is symmetric is a natural result from commutativity of covariant derivatives when acting on a scalar. Now because $D_a = \frac{1}{r}\tilde{\nabla}_a r$, then also $D^2 = \frac{1}{r}\tilde{\Box}r$. In this sense we pull a single r from the left side to the right side, for example for the scalar field

$$Kr^2\tilde{\mathcal{G}}K = 2Kr^2(D^2 - F^a_a)K \xrightarrow{D\to\frac{1}{r}\tilde{\nabla}r} 2Kr(\tilde{\Box} - F^a_a)rK. \tag{B.66}$$

Finally we insert both r's into the fields, $\tilde{H}_{ab} = rH_{ab}$, $\tilde{K} = rK$, so we end up with $2\tilde{K}(\tilde{\Box} - F^a_a)\tilde{K}$. Doing the same procedure for all other fields, i.e. insert $D = \frac{1}{r}\tilde{\nabla}r$ and the new fields, results in the following Lagrangian

$$\boxed{\mathcal{L}_e = \frac{1}{4}\tilde{H}^{ab}\tilde{\Delta}^{-1}_{abcd}\tilde{H}^{cd} + \frac{1}{4}\tilde{H}^{ab}\tilde{\Delta}^{-1}_{L,ab}\tilde{K} + \frac{1}{4}\tilde{K}\tilde{\Delta}^{-1}_{R,ab}\tilde{H}^{ab} + \frac{1}{4}\tilde{K}\tilde{\Delta}^{-1}\tilde{K}.} \tag{B.67}$$

Here we defined

$$\tilde{\Delta}^{-1} = -\tilde{\Box} + F_a^a, \tag{B.68}$$

$$\tilde{\Delta}^{-1}_{R,ab} = -g_{ab}\big(\tilde{\Box} - \tfrac{1}{2}V_c\tilde{\nabla}^c + \tfrac{1}{4}V_cV^c - F_c^c - \tfrac{\ell(\ell+1)}{2r^2}\big) + \tilde{\nabla}_a\tilde{\nabla}_b - F_{ab}, \tag{B.69}$$

$$\tilde{\Delta}^{-1}_{L,ab} = -g_{ab}\big(\tilde{\Box} + \tfrac{1}{2}V_c\tilde{\nabla}^c - \tfrac{\ell(\ell+1)}{2r^2}\big) + \tilde{\nabla}_a\tilde{\nabla}_b - F_{ab}, \tag{B.70}$$

$$\tilde{\Delta}^{-1}_{abcd} = \tfrac{1}{2}g_{ac}V_{[b}\tilde{\nabla}_{d]} + \tfrac{1}{2}g_{bd}V_{[a}\tilde{\nabla}_{c]} + \tfrac{1}{2}g_{ab}\big(V_{(c}\tilde{\nabla}_{d)} + 2F_{cd}\big) + \tfrac{1}{2}g_{cd}\big(-V_{(a}\tilde{\nabla}_{b)} + \tfrac{1}{2}V_aV_b\big) \tag{B.71}$$

$$+ \; g_{ab}g_{cd}\big(\tfrac{1}{4}R_s + \tfrac{\ell(\ell+1)}{2r^2}\big) - g_{ac}g_{bd}\big(\tfrac{1}{2}R_s + \tfrac{\ell(\ell+1)}{2r^2}\big). \tag{B.72}$$

A minus sign has been absorbed into all $\tilde{\Delta}'s$ to recieve more physically tangible results. These quantities have all been calculated by hand and checked by *Wolfram Mathematica*. Notice that all operators are in fact symmetric in the fields, since for example $\tilde{\Delta}^{-1}_{R,ab}$ equals $\tilde{\Delta}^{-1}_{L,ab}$ up to total derivatives. This completes the action rewriting, since we have now arrived at the Lagrangian in Equation 4.69.

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