Utrecht University

Master Thesis Theoretical Physics

Statistical Physics Models for Economic Systems

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Abstract

The goal of this thesis is to construct an extension of the econophysical model by V. M. Yakovenko, that describes the distribution of goods or incomes, by combining it with the utility-based model of J. Mulder. The dynamics of the model of V. M. Yakovenko [2] are aimed at pairwise interactions in single good markets while the dynamics of the model of J. Mulder [3] depicts the interaction in a two goods market based on utility. The model of Yakovenko shows how maximization of entropy leads to Boltzmann distributed income distributions while the model by Mulder shows that the maximization of utility does not. In this thesis we analyse and combine these two models to find a single master equation. Using this equation we examine the transition matrix. This examination illustrates that the model is stable but not Boltzmann distributed. The analysis of the simulations illustrates the stability of the system and the effects of both models on utility and the income distribution.

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1 Introduction

In modern day society income inequality has become more and more prevalent. Some countries have seen an increase in the gap between the rich and poor where especially the rich have seen their income increase drastically [1]. This increase in the gap between rich and poor would suggest that a change in the distribution of wealth has occurred. However, it is quite remarkable to notice that this increase in the wealth gap has not had any drastic effects on the distribution of wealth. In fact, in the USA it seems that the lower 97% of the population has not seen any significant changes in the shape of the income distribution in the last decades. It is mainly the upper 3% which has seen changes over time. In Figure 1.1 it is shown that the shape of the income distribution for the lower 97% of the population is that of a Boltzmann-Gibbs distribution. Notice that the vertical shift between the 1980s and 1990s in this figure has simply been done for clarity. What we observe in this graph is that the lower-income part of the distributions overlap on the same curve, i.e. a Boltzmann-Gibbs distribution. This illustrates that the shape of the income distribution for the lower part has been extremely stable over time suggesting a so called statistical 'thermal' equilibrium. The upper 3%, however, does not seem to adhere to a Boltzmann-Gibbs distribution but rather to a Pareto power law which changes shape [2]. The occurrence of a statistical 'thermal' equilibrium in the lower part of the wealth distribution suggests that an underlying mechanism is at work here. The aim of this thesis is to discuss and combine two models in order to gain a further understanding of such mechanisms.

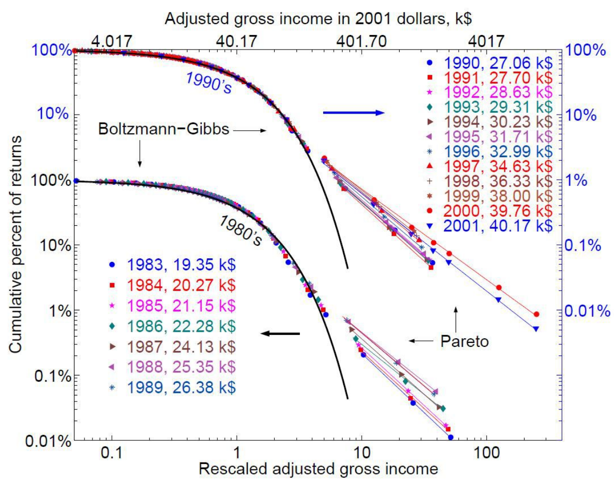


Figure 1.1: Cumulative probability distributions of annual income in the USA plotted on log-log scale versus m/m_0 (the annual income m normalized by the average income m_0 in the exponential part of the distribution). The Internal Revenue Sevice (IRS) data points are for 1983-2001 and the columns of the numbers give the values of m_0 for the corresponding years [2].

To reach this aim this thesis is divided into three different parts. In the first and second part two separate models will be introduced and discussed. In the third part these two models will be combined into a single model. It is this third part which will be the main focus of this thesis.

In the first part a start will be made by considering the model developed by the physicist Victor Yakovenko. In a part of his 2009 Colloquium he developed an agent-based model that predicts the income or wealth distribution of the lower class based on pairwise interactions between the agents of a population. He has shown that the stable wealth distribution that results from these interactions is a Boltzmann-Gibbs distribution [2]. The reason why the Boltzmann-Gibbs equilibrium distribution should emerge from this model shall be discussed extensively in this text. This discussion will also focus on several numerical results and provide a further analysis.

In the second part we will elaborate on some of the results from the master's thesis by Jan Mulder [3]. In this thesis he discusses a utility-based model in which the pairwise interactions between agents are governed by an increase in a quantity called 'utility'. It is the maximization of this utility that indicates that the system has reached thermal equilibrium. It will be demonstrated that, upon reaching this equilibrium, this model does not have a Boltzmann-Gibbs distribution.

In the third part the two earlier models will be combined into what we will call the utility-based Yakovenko model. To understand this model it will be approached both analytically and numerically. The analytical approach will be done by making use of the master equation. This equation will provide a basis from which the workings of the model can be understood and analysed. This analysis is aimed at determining the eigenvalues of the transition matrix of the master equation in order to identify the equilibrium solution for this model. The numerical approach will focus on computer simulations of this model and analyse what occurs as the model equilibrates.

Finally, this thesis will conclude by presenting the various results that have been obtained and by giving an outlook on possible future research in this area. In this outlook it will mainly be discussed that the utility-based Yakovenko model can be extended to a two economies model where the interactions between two economies or countries are considered. Further development of this extension might lead to interesting results.

2 Yakovenko Model

The aim of this chapter is to study the statistical mechanics of money by considering a model proposed by Victor M. Yakovenko [2]. In this model Yakovenko applies statistical physical methods to economical systems in order to obtain new insights. To understand this model a start is made by first considering the mechanics of the Yakovenko model. After this, an analytical solution for this model is discussed by using the master equation. The derivation of this solution means to show that the Yakovenko model can be solved by a Boltzmann-Gibbs distribution. Following this, the discussion will continue by showing the validity of the analytical solution by comparing it with numerical results.

2.1 The Yakovenko Model

The model proposed by Yakovenko means to predict the stationary distribution of a closed economy in which the various market participants exchange money with each other [2]. Before this model is considered in depth, one should recall that in equilibrium statistical mechanics the probability of the occurrence of a certain microstate is given by a Boltzmann-Gibbs distribution. The probability measure of this distribution is dependent on the energy and the temperature of the system. The formulation of this distribution is of the form $P(\varepsilon)$, where ε depicts the energy of the microstate

$$P(\varepsilon) = ce^{-\varepsilon/T}, \tag{2.1}$$

here T is the temperature, and c the normalizing constant. An important requirement for this distribution to hold is that it is applied to a physical system with a conserved quantity. In the above formulation this conserved quantity would be the energy of the system. It is now this property of dealing with a conserved quantity that brought about the application of the Boltzmann-Gibbs distribution to non-physical systems with conserved quantities [2]. Yakovenko's working example of such a system is that of a closed economy where the total amount of money m is conserved. In his model, Yakovenko uses the conservation of money to describe a simple system which is based on the interactions between various agents in an economic system. In this system every agent has a certain amount of money $m_i > 0$, with $i = 1, 2, \ldots, N$ labelling the individual agents, and $N \in \mathbb{N}$ the number of agents present in the system. Initially, he lets each agent start with an equal amount of money m_0. After this, two random agents i and j exchange a random amount of money Δ so that their personal money balance changes. It is this process of random exchange which will be done for a definite number of time periods. The steps taken in such a time period are as follows

1. Two random agents i and j are selected to make a transaction from agent i to j.
2. In this transaction a random, uniformly distributed, amount of money, $\Delta \in [0, \Delta_{\max}]$, is chosen. Here, $\Delta_{\max}$ is the maximum value of the transaction.
3. It is checked whether agent i is able to make the transaction, i.e. $m_i \geq \Delta$.

4. If agent i has sufficient funds then he will transfer an amount Δ to agent j so that

$$m_i \to m_i' = m_i - \Delta; \tag{2.2}$$

$$m_j \to m_j' = m_j + \Delta. \tag{2.3}$$

The conservation of money entails that in this algorithm the value of the transaction before and after will always be equal, i.e.

$$m_i + m_j = m_i' + m_j'. \tag{2.4}$$

It is apparent that taking the above steps only once does not result in interesting observations. For this reason, a sweep of transactions is performed. This sweep is defined as N independent transactions so that on average every agent is selected once for receiving, and once for giving money. Doing these sweeps multiple times will cause the system to go to an equilibrium. In other words, the probability distribution of money becomes stationary and as such independent of time. In order to demonstrate this time independence the Yakovenko model will first be approached analytically and then numerically.

2.2 Analytical Solution

In the previous section it was postulated that the Yakovenko model tends towards an equilibrium. To demonstrate this we will consider the time evolution of the master equation when applied to Yakovenko's model and show that it becomes time independent. We would like to note that for an extensive discussion on the master equation we refer the reader to the lecture notes by Landau and Lifshitz [4] and the book by N. van Kampen [5]. Generally, the master equation is expressed by

$$\frac{dP(m)}{dt} = C[P(m)], \tag{2.5}$$

where $C[P(m)]$ is the collision integral. This integral represents the rate of change of the distribution function $P(m)$ by virtue of collisions. It is these collisions between particles which change, for example, the value of their energy. The collision integral then means to describe the losses and gains of this energy in a particular range of variables [4]. In statistical mechanics this equation displays the time evolution of the distribution function due to collisions between particles. However, we need to be aware that in our model we are not dealing with collisions but rather with interactions between two agents. So, in order to solve this transport equation for the Yakovenko model we need an expression for its collision integral. This integral is given by [2]

$$\begin{aligned}\frac{dP(m)}{dt} = \int_0^\infty dm' \int_{-\infty}^{\infty} d\Delta \, \big[& f_{[m+\Delta,m'-\Delta]\to[m,m']} P(m+\Delta) P\left(m'-\Delta\right) \\ & - f_{[m,m']\to[m+\Delta,m'-\Delta]} P(m) P\left(m'\right)\big] .\end{aligned} \tag{2.6}$$

Here, $f_{[m,m']\to[m+\Delta,m'-\Delta]}$ is the probability of transferring an amount of money Δ from an agent with money m' to an agent with money m per unit time. Multiplying this with the occupation numbers $P(m)$ and $P(m')$ results in the rate of transition from the state $[m, m']$ to $[m+\Delta, m'-\Delta]$. In other words, the second term of this integral depicts the depopulation rate of the state m. Similarly, we can argue that $f_{[m+\Delta,m'-\Delta]\to[m,m']}$ depicts the probability that the reverse process occurs, i.e. the first term of the integral represents the population rate of state m. When the probability of both the direct and reverse process are equal, i.e. there is time-reversal symmetry, then the probability distribution becomes stationary: $\frac{dP(m)}{dt} = 0$. This procedure where each process is balanced by the reverse process is also known in physics as *detailed balance*. When there is stationarity and there is detailed balance then one gets from Equation (2.6) that

$$P(m)P\left(m'\right) = P(m+\Delta)P\left(m'-\Delta\right). \tag{2.7}$$

Assuming that $\Delta \neq 0$, this equality is only solved by an exponential distribution of the form

$$P(m) = Ae^{Bm}, \tag{2.8}$$

where A and B are to be determined constants. In pursuance of ascertaining these constants, and under the assumption that time-reversal symmetry holds for this model, we need to make use of the constraints on this system: conservation of money and normalization. Another route to determine that the solution is given by an exponential distribution is by utilizing the fact that the exponential distribution maximizes the entropy of the system [3]. From basic thermodynamics it is known that entropy describes the number of different microscopic states that a system can adopt. The size of this number depends on the macroscopic quantities that define the system, i.e. money. From the second law of thermodynamics it is known that the total entropy of an isolated system does not decrease over time and that the entropy is constant if and only if all processes are reversible. Therefore, isolated systems spontaneously evolve towards thermodynamic equilibrium, i.e. the state with maximum entropy. The entropy S of this system is defined by

$$S = -\langle \log[P(m)] \rangle = -\int_0^\infty dm\ P(m) \log[P(m)]. \tag{2.9}$$

In order to find an expression for the distribution function $P(m)$ we maximize the entropy. This is done by calculating the functional derivative of the entropy and equating it to zero

$$\frac{\delta S[P(m)]}{\delta P(m)} = 0. \tag{2.10}$$

To solve this equation we administer the constraints of this system. The first constraint is that the probability distribution is normalized

$$\int_0^\infty dm\ P(m) = 1. \tag{2.11}$$

The second constraint is that, as both the amount of money and the number of agents are conserved, the average amount of money per agent should be constant at all times. Giving the constraint

$$\langle m_i \rangle = \int_0^\infty dm\ P(m) m = m_0. \tag{2.12}$$

Now, with these constraints the functional derivative can be calculated using two Lagrangian multipliers λ, and κ. Determining the values of these will help in establishing the probability distribution. The expression we need to solve has become

$$\frac{\delta}{\delta P(m)} \left\{ -\int_0^\infty dm'\ P(m') \log P(m') + \lambda \left[\int_0^\infty dm'\ P(m') - 1 \right] + \kappa \left[\int_0^\infty dm'\ P(m') m' - m_0 \right] \right\} = 0. \tag{2.13}$$

Taking the functional derivative and rewriting the result for $P(m)$ gives that

$$P(m) = e^{-1+\lambda+\kappa m}. \tag{2.14}$$

This preliminary result can be evaluated for λ and κ by inserting it into the two constraints. Let us start with the first constraint

$$\int_0^\infty dm\ e^{-1+\lambda+\kappa m} = 1. \tag{2.15}$$

When we do this integral we see that it is convergent for $\kappa < 0$. Hence, we get that

$$e^{-1+\lambda} = -\kappa, \text{ with } \kappa < 0. \tag{2.16}$$

Now, we calculate the value of κ using the second constraint and the result that $P(m) = -\kappa \exp(\kappa m)$

$$\int_0^\infty dm\, (-\kappa e^{\kappa m} m) = m_0. \tag{2.17}$$

Using integration by parts gives that $\kappa = -\frac{1}{m_0}$. Substituting this into our expression for $P(m)$ gives

$$P(m) = \frac{1}{m_0} e^{-\frac{m}{m_0}}. \tag{2.18}$$

This equation we recognize as the Boltzmann distribution, an example of which was given in Equation (2.1). What this result shows is that following the constraints on the Yakovenko model we expect that the probability distribution at equilibrium is indeed given by a time-independent Boltzmann distribution. To further illustrate the validity of this result we will consider a numerical model in order to compare the results at equilibrium.

2.3 Numerical Results

Having computed the analytical result for the Yakovenko model, we continue by comparing the analytical result with some numerical results. To obtain the numerical results we initialize a simulation with $N = 1000$ agents where each agents starts with $m_0 = 100$. The maximum value for Δ in this model is $\Delta_{\max} = 25$. It is with these initial conditions that we let a simulation perform the four steps of the Yakovenko model for various amounts of sweeps. In Figure 2.1 the results for this analysis are shown up to 200 sweeps by making use of histograms. What is observed in this figure is that as the amount of sweeps increases that the system starts to equilibrate. For this model this seems to mean that over time the distribution of the goods becomes exponential. This is depicted most clearly in Figure 2.1(f) where the shape of the histogram follows the probability distribution given in Equation (2.18). This illustrates that the Boltzmann distribution is an accurate representation of the Yakovenko model at equilibrium.

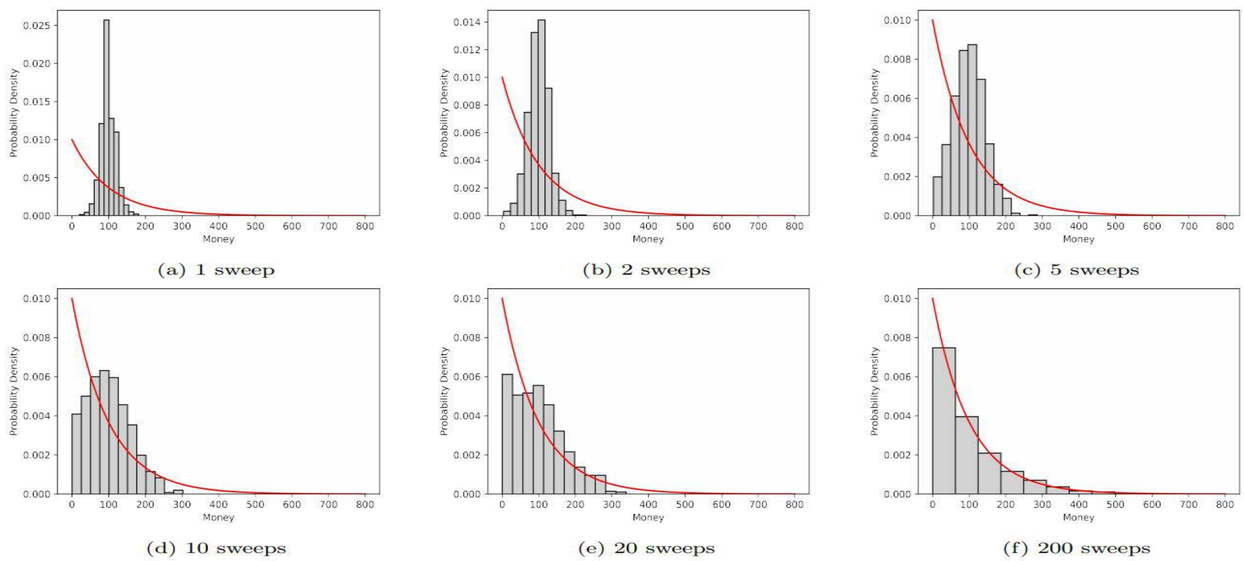


Figure 2.1: Normalized probability distribution of money for the Yakovenko model after various amounts of sweeps. The red line is the money distribution given in Equation (2.18).

To get a further understanding of the equilibration of this system we consider the evolution of its mean square displacement (MSD). Before we do this, we recall that the mean square displacement is generally given by

$$\mathrm{MSD} = \langle (x(t) - x(0))^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} |x_i(t) - x_i(0)|^2. \tag{2.19}$$

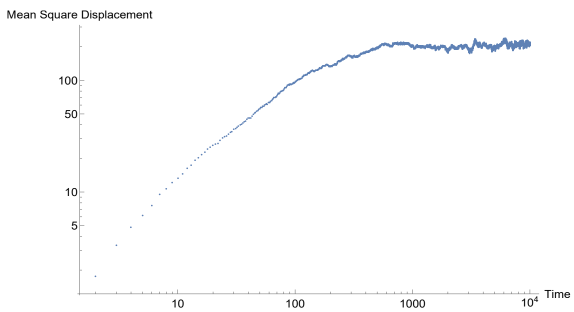
Expanding this in terms of averages gives

$$\langle (x(t) - x(0))^2 \rangle = \langle x^2(t) \rangle + \langle x^2(0) \rangle - 2\langle x(t)x(0) \rangle. \tag{2.20}$$

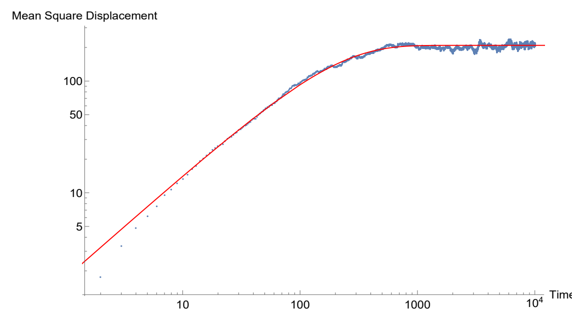
Now, we know that the system is in equilibrium when $\langle x^2(t) \rangle = \langle x^2(0) \rangle$. As such, the mean square displacement becomes at equilibrium

$$\langle (x(t) - x(0))^2 \rangle = 2\langle x^2(0) \rangle - 2\langle x(t)x(0) \rangle. \tag{2.21}$$

In physics we would usually be dealing with particles and then the above MSD would depict the average distance travelled by a particle. In this system, however, we do not consider the location of particles but rather the money balance of agents. So, here we will compare the initial money balance of the agent $x(0)$ with the current money balance $x(t)$. This means that the MSD depicts the average money exchanged per agent over time. Now, to gain more insight about the equilibration of the system we consider the MSD of a system that is similar to what we will be considering later in this thesis. In the current simulation of the MSD we consider a system that is initially shaped like an Boltzmann distribution. Furthermore, the agents will no longer trade a random amount of goods but rather a set amount of goods $\Delta = 1$ in an economy with an average of $m_0 = 10$. In Figure 2.2 the time evolution of the mean square displacement of this system is given. It is observed that the MSD increases towards a certain value after which it reaches a plateau and fluctuates around it. It is clear from this figure that the MSD does not behave like a typical diffusion process where the mean square displacement is linear [6]. In this model it seems that we are dealing with anomalous diffusion or sub-diffusion. Unfortunately, the underlying mechanism that causes this anomalous diffusion has not been determined. What has been found, however, is a function which fits the MSD of this system. In Figure 2.2b it is shown that the MSD fits the function $208.8\big(1 - e^{-0.00824x^{0.93}}\big)$. From this function we obtain that the time scale of the equilibration of this system is given by $\tau \approx 178$. It is interesting to note that this result is close to what is found when determining the eigenvalues of the transition matrix of the Yakovenko model in Section 4.6. Having discussed the Yakovenko model we continue in the next chapter by having a look at the utility-based market model. It is this market model that will be combined with the Yakovenko model in Chapter 4.



(a) Time evolution of the mean square displacement in the Yakovenko model for a system with $m_0 = 10$.



(b) Time evolution of the mean square displacement in the Yakovenko model with a fitted function of $208.8\big(1 - e^{-0.00824x^{0.93}}\big)$

Figure 2.2: Time evolution and prediction of the mean square displacement of the Yakovenko model.

3 Utility-based Market Model

In this chapter we will consider another economic model which depicts a market where trading is no longer solely based on the change in goods but also on the change in utility. To consider such a market a start will be made by giving a short introduction to utility theory. After this, the derivations following from Jan Mulder's master thesis will be considered [3]. Along with these derivations a discussion will be held on the demand and supply functions following from utility theory for a market with only two goods. Subsequently, the elasticity of substitution will be used in combination with the constant elasticity of substitution utility function. It is this utility function that will be implemented to obtain values for the demand and supply curves. The results of this analysis will then be demonstrated by making use of various graphs. Having determined the demand and supply it is possible to consider the dynamics of a utility-based market. It will be discussed what these dynamics are like and what kind of distribution we should expect for this market. In the end, a numerical analysis on this model will be done.

3.1 Introduction to Utility Theory

In economic theory, utility has long been seen as a numeric measure of a person's happiness. On the basis of this notion it was thought that people would make economic choices so as to maximize their happiness, i.e. utility. This concept of utility, however, has many conceptual problems. So, in current neoclassical economic theory, utility theory is seen in terms of consumer preferences with utility as the only way to describe them [7]. A general formulation of the consumer preferences is given by a utility function U which is related to the consumption of one or more goods $x_1, x_2, \ldots, x_k$. In order for an agent to consume these goods economists will look at the agents income p_i in order to see the amount of money he can spend. The aim of the agent's spending is to maximize their own utility function. Considering what we know about the maximization of functions it becomes clear that it is more convenient to look at the marginal utilities of the agents $MU_1, MU_2, \ldots, MU_k$ rather than the utility itself. In economic theory the marginal utility is obtained by taking the partial derivative of the utility function with respect to a good x_i

$$MU_i = \frac{\partial U}{\partial x_i}. \tag{3.1}$$

Starting from this marginal utility, Heinrich Gossen formulated a principle as a basis for the maximization of an agent's utility which is known as Gossen's Second law. According to Gossen "a person maximizes his utility when he distributes the available money among the various goods so that he obtains the same amount of satisfaction from the last unit of money spent on each commodity" [8]. In other words, when a person no longer prefers one good over another then he has maximized his utility. In terms of marginal utilities this means that

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \cdots = \frac{MU_k}{p_k}. \tag{3.2}$$

Here, MU_k represent the marginal utility of good k and p_k depicts the associated price of this good. Another important principle in utility theory comes from the law of diminishing marginal utility. This law entails that as consumption increases the marginal utility for each additional unit consumed decreases. What this involves in practice is that when you earn one euro you get a certain utility from this. Earning a second euro, however, would give less utility than earning the first euro. Again, the same holds for earning a third euro which gives less utility than earning the second, etc. So, according to this law marginal utility decreases as your consumption of a good increases. By making use of this fundamental knowledge on utility theory it is possible to consider the demand and supply functions for a market with two conserved goods.

3.2 Demand and Supply Function

We continue to construct the utility-based market model by first considering an expression for the demand and supply function [3]. To do this we assume that we have a market with only two conserved goods A and B. In this market we consider good A to be a good and good B to be a currency. We also have N agents or market participants present where each agent possesses a certain amount of good A, i.e. a_i with $i = 1, 2, \ldots, N$, and a certain amount of currency B, i.e. b_i. Besides this, it is assumed that each agent has the same utility function $U(a_i, b_i)$ which depends on their own goods. Now, to determine the demand and supply function we assume that all trades between the agents involve a fixed amount of good A which we represent with q and a variable amount of good B which we call p. In this market model we assume that two agents i and j will only exchange goods under several conditions. These conditions are that the buyer i is able to pay for the good

$$b_i \geq p, \tag{3.3}$$

and that the seller j is able to sell the good

$$a_j \geq q. \tag{3.4}$$

On top of this, it is required that the change in utility of both the buyer and seller does not decrease their utility. This means that for the buyer i the condition should hold that

$$U\left(a_i + q, b_i - p\right) \geq U\left(a_i, b_i\right). \tag{3.5}$$

We know now, from utility theory, that it is more convenient to write this inequality in terms of marginal utilities. To do this we make use of a first order Taylor expansion around $p = q = 0$. Doing this and rewriting the expression for p gives the condition for buyer i

$$U\left(a_i, b_i\right) + \frac{\partial U\left(a_i, b_i\right)}{\partial a_i} q - \frac{\partial U\left(a_i, b_i\right)}{\partial b_i} p \geq U\left(a_i, b_i\right) \tag{3.6}$$

$$p \leq \frac{\partial U\left(a_i, b_i\right)/\partial a_i}{\partial U\left(a_i, b_i\right)/\partial b_i} q. \tag{3.7}$$

In a similar way the condition for the seller j can be written as

$$U(a_j - q, b_j + p) \geq U(a_j, b_j) \tag{3.8}$$

$$p \geq \frac{\partial U(a_j, b_j)/\partial a_j}{\partial U(a_j, b_j)/\partial b_j} q. \tag{3.9}$$

Having determined these conditions we can define the demand and supply function. The demand function $D(p)$ is defined by the amount of goods q the agents in the market want to buy at a certain price p. To identify this amount of demand we simply have to sum over all agents and see whether they satisfy the buyer conditions given in Equation 3.3 and 3.7. Some consideration suggest that an appropriate representation of these trading conditions is given by a heaviside step function θ. This choice ensures that the agents will trade only if the separate conditions are met. For the supply function $S(p)$ we have to do something similar but now we have to consider the amount of goods q the agents in the market want to sell. Similar to the demand function we do this by checking whether the sellers satisfy the seller conditions given in Equation 3.4 and 3.9. Combining all this means that the demand and supply equations become

$$\begin{aligned} D(p) &= q \sum_{i=1}^{N} \theta\left(\frac{\partial U(a_i, b_i)/\partial a_i}{\partial U(a_i, b_i)/\partial b_i} q - p\right) \theta(b_i - p); \\ S(p) &= q \sum_{i=1}^{N} \theta\left(p - \frac{\partial U(a_i, b_i)/\partial a_i}{\partial U(a_i, b_i)/\partial b_i} q\right) \theta(a_i - q). \end{aligned} \tag{3.10}$$

Now, we assume that there is a known normalized distribution for goods A and B of the form $P(a, b)$

$$\int_0^\infty da \int_0^\infty db\ P(a, b) = 1. \tag{3.11}$$

Inserting this into Equation 3.10 and rewriting gives

$$\begin{aligned} D(p) &= Nq \int_0^\infty da \int_0^\infty db\ P(a, b) \theta\left(\frac{\partial U(a, b)/\partial a}{\partial U(a, b)/\partial b} q - p\right) \theta(b - p); \\ S(p) &= Nq \int_0^\infty da \int_0^\infty db\ P(a, b) \theta\left(p - \frac{\partial U(a, b)/\partial a}{\partial U(a, b)/\partial b} q\right) \theta(a - q). \end{aligned} \tag{3.12}$$

It is these demand and supply curves which will help to determine the equilibrium market price that will be paid for a quantity q of good A. However, to find an exact expression for these prices it is necessary to first identify the utility function $U(a, b)$ of these goods. For this reason, we will introduce the constant elasticity of substitution utility function.

3.3 Constant Elasticity of Substitution

In order to find a more applicable expression of the demand and supply function it is necessary to have an expression for the utility function. In practice there are many different utility functions which can be used such as the isoelastic, the exponential and the quasilinear utility function [9]. For this thesis we have chosen to make use of the constant elasticity of substitution utility function. In economic theory, the elasticity of substitution σ shows the degree by which the number of products sold changes when the price of another similar product changes. In other words, it shows the degree by which the two products can replace each other. In the two goods market we are considering this elasticity depends on the ratio of the consumption of the currency B compared to good A, i.e. $R = b/a$, and on the ratio of the price of good A compared to currency B, i.e. $P_B = q/p$. The elasticity of substitution for these goods is given by [9]

$$\sigma = \frac{\Delta R/R}{\Delta P_B/P_B} = \frac{dR}{dP_B}\frac{P_B}{R}. \tag{3.13}$$

For this function it generally holds that $\sigma \geq 0$ as we assume that an increase in the currency B compared to good A will lead to an increase in the consumption of the currency B compared to good A. A commonly used utility function for such a scenario is the Constant Elasticity of Substitution (CES) utility function [9]. As the name suggests this function is based on constant elasticity of substitution between the two goods. This utility function is given by

$$U(a,b) = \left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{1/r}, \tag{3.14}$$

where α and β are share parameters for a and b. The value of r depends on the value of σ which will be found by solving Equation 3.13 with this utility function. To achieve this, we start by calculating the marginal utility functions

$$\frac{\partial U}{\partial a} = \frac{1}{\alpha}\left(\frac{a}{\alpha}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r}; \tag{3.15}$$

$$\frac{\partial U}{\partial b} = \frac{1}{\beta}\left(\frac{b}{\beta}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r}. \tag{3.16}$$

Now, we use Gossen's Second Law given in Equation 3.2 to get

$$\frac{1}{q}\frac{\partial U}{\partial a} = \frac{1}{p}\frac{\partial U}{\partial b}. \tag{3.17}$$

Using this identity and rewriting for a function of good A dependent on P_B gives

$$R = \left(\frac{\beta}{\alpha}\right)^{r/(r-1)} P_B^{1/(1-r)}. \tag{3.18}$$

Taking the derivative of this function with respect to P_B will allow us to find the value of σ

$$\frac{dR}{dP_B} = \frac{1}{1-r}\left(\frac{\beta}{\alpha}\right)^{r/(r-1)} P_B^{r/(1-r)}. \tag{3.19}$$

With this we solve for σ to obtain

$$\sigma = \frac{dR}{dP_B}\frac{P_B}{R} = \frac{1}{1-r}\frac{\left(\frac{\beta}{\alpha}\right)^{r/(r-1)} P_B^{r/(1-r)} P_B}{\left(\frac{\beta}{\alpha}\right)^{r/(r-1)} P_B^{1/(1-r)}} = \frac{1}{1-r}. \tag{3.20}$$

This then shows the relation between r and σ when there is CES. Note that if we want to ensure that $\sigma > 0$ then we will need that $r < 1$. With this we have obtained a utility function with which it is possible to solve the demand and supply function exactly.

3.4 CES Demand and Supply

In the last section we obtained an expression for the utility function by assuming constant elasticity of substitution. Here, we continue by applying this expression to solve the demand and supply functions given by Equation 3.12 [3]. However, before this can be done a guess about the form of the distribution function $P(a,b)$ has to be made. Luckily, we know from the Yakovenko model that a Boltzmann distribution is an appropriate estimate for this since we are dealing with two separately conserved quantities. For this reason, we take the distribution to be of the form

$$P(a,b) = \frac{1}{\bar{a}}e^{-a/\bar{a}}\frac{1}{\bar{b}}e^{-b/\bar{b}}, \tag{3.21}$$

where $\bar{a}$ and $\bar{b}$ are the average amounts of goods A and B respectively. It is with this probability distribution at our disposal that we can continue our attempt to determine the demand and supply function [3]. To illustrate this solution we will show the steps taken for solving the demand function, which is now given by

$$\frac{D(p)}{q} = \frac{N}{\bar{a}\bar{b}}\int_0^\infty da \int_0^\infty db e^{-a/\bar{a}} e^{-b/\bar{b}} \theta\Bigg(\frac{q}{\alpha}\left(\frac{a}{\alpha}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r} - \frac{p}{\beta}\left(\frac{b}{\beta}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r}\Bigg)\theta(b-p). \tag{3.22}$$

To simplify this equation we introduce the following identity

$$\left(\frac{b}{a}\right)_{\text{crit}} = \left(\frac{\beta}{\alpha}\right)^{r/(r-1)} \left(\frac{q}{p}\right)^{1/(r-1)} = \gamma(p). \tag{3.23}$$

An understanding of the meaning of this identity is crucial for determining the demand and supply function. In short, this function shows whether the preference of an agent is to buy or to sell $R = b/a$. It is evident that for the demand function there should be a preference to buy R, i.e. $b/a \geq \gamma(p)$. While for the supply function there should be a preference to sell R, i.e. $b/a \leq \gamma(p)$. The consequence of these inequalities is that in the former the agent prefers to hold on to goods and in the latter to hold on to money. It should be noticed here that in the critical case above we have an equality which indicates that the agent is indifferent to buying or selling. Applying the first interpretation to the first step function in Equation 3.22 means that it will be rewritten as follows

$$\theta\left(\frac{q}{\alpha}\left(\frac{a}{\alpha}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r} - \frac{p}{\beta}\left(\frac{a}{\alpha}\right)^{r-1}\left[\left(\frac{b}{\beta}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r}\right) = \theta(b - \gamma(p)a). \tag{3.24}$$

With this simplified step function we can proceed to solve the integral for the demand. Doing this results in

$$\begin{aligned} \frac{D(p)}{q} &= \frac{N}{\bar{a}\bar{b}} \int_0^\infty dbe^{-b/\bar{b}}\theta(b-p) \int_0^{b/\gamma(p)} dae^{-a/\bar{a}} \\ &= \frac{N}{\bar{b}} \int_p^\infty dbe^{-b/\bar{b}} \left[1 - \exp\left(-\frac{b}{\gamma(p)\bar{a}}\right)\right] \\ &= Ne^{-p/\bar{b}} \left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \end{aligned} \tag{3.25}$$

This result is then an expression for the demand which depends on the price. To obtain an expression for the supply we implement the same procedure as above to the supply curve given in Equation 3.12. However, note that, following the interpretation of $\gamma(p)$, we will use that the step function becomes $\theta(\gamma(p)a - b)$. From this it then follows that

$$\frac{S(p)}{q} = Ne^{-q/\bar{a}} \left[1 - \frac{\bar{b}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{\gamma(p)q}{\bar{b}}\right)\right]. \tag{3.26}$$

These two expressions we found for the demand and the supply function give an indication of their behaviour for varying values of p. Be that as it may, these functions in itself do not directly give a clear illustration of the function's behaviour. Now, in order to get an idea of this behaviour we have a look at the regions where $p \to 0$ and $p \to \infty$. But before we do this, we have to consider what happens in these regions in Equation 3.23. Here, it is straightforward to see that $\lim_{p\to 0} \gamma(p) = 0$ and $\lim_{p\to\infty} \gamma(p) = \infty$. Besides this, we also need to have a look at the case $p/\gamma(p)$. What we need to realize for this is that $\gamma(p) \propto p^{\frac{1}{1-r}}$. As a consequence of this, $\gamma(p)$ will go to zero and infinity faster than p for $r < 1$. As such, the limits become $\lim_{p\to 0} p/\gamma(p) = \infty$ and $\lim_{p\to\infty} \gamma(p) = 0$ for $r < 1$. Applying these results we can consider the limits of the demand and supply function

- **Demand when $\mathbf{p \to 0}$**

$$\lim_{p\to 0} D(p) = \lim_{p\to 0} Nqe^{-p/\bar{b}} \left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \tag{3.27}$$

We see that the first exponential will be equal to one and that the second exponential will be equal to zero as the exponent becomes infinity. This gives that

$$\lim_{p\to 0} D(p) = Nq. \tag{3.28}$$

This shows that at a price of zero all agents want to buy a quantity q of good A.

- **Demand when p → ∞**

$$\lim_{p\to\infty} D(p) = \lim_{p\to\infty} Nqe^{-p/\bar{b}}\left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a}+\bar{b}}\exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \tag{3.29}$$

Here, the first exponential will equal to zero and as such the demand becomes

$$\lim_{p\to\infty} D(p) = 0. \tag{3.30}$$

This result demonstrates that at a price of infinity no one wants to buy good A.

- **Supply when p → 0**

$$\lim_{p\to 0} S(p) = \lim_{p\to 0} Nqe^{-q/\bar{a}}\left[1 - \frac{\bar{b}}{\gamma(p)\bar{a}+\bar{b}}\exp\left(-\frac{\gamma(p)q}{\bar{b}}\right)\right]. \tag{3.31}$$

In this function both the fraction and the exponential will equal to one. As a result, we will have that $Nqe^{-q/\bar{a}}[1-1] = 0$. So that the supply becomes

$$\lim_{p\to 0} S(p) = 0. \tag{3.32}$$

This is the result one would expect as no agent would be willing to give away their goods for free.

- **Supply when p → ∞**

$$\lim_{p\to\infty} S(p) = \lim_{p\to\infty} Nqe^{-q\bar{a}}\left[1 - \frac{\bar{b}}{\gamma(p)\bar{a}+\bar{b}}\exp\left(-\frac{\gamma(p)q}{\bar{b}}\right)\right]. \tag{3.33}$$

The exponential goes to zero and as such we obtain

$$\lim_{p\to\infty} S(p) = Nqe^{-q/\bar{a}}. \tag{3.34}$$

Intuitively this is not a result one would expect as you would think that all agents would want to sell their goods. What we tend to forget is that the agents need to have the required amount of goods in order to sell it. It is this requirement which this result takes into account.

One of the most interesting regions for the utility-based market model is the point where the demand and supply curves intersect

$$D(p) = S(p). \tag{3.35}$$

To consider this point some additional assumptions have to be made. This is done as the above equation cannot be solved analytically. For this reason, we inspect the critical case where $b/a = \gamma(p)$. As mentioned before, this is the case where the agent is indifferent to buying or selling. So, following Gossen's second law this would be the point where the agent's utility is maximized. Combining this with a mean field approach, that is we ignore fluctuations, we evaluate a and b at their average values so that $\bar{b}/\bar{a} = \gamma(p)$. Substituting this into Equations 3.25 and 3.26 gives for the equality

$$e^{-p/\bar{b}}\left(1 - \frac{1}{2}e^{-p/\bar{b}}\right) = e^{-q/\bar{a}}\left(1 - \frac{1}{2}e^{-q/\bar{a}}\right). \tag{3.36}$$

To show the solution for this we have a closer look at effect of the mean field approach on Equation 3.23 which we rewrite for p

$$p = q\left(\frac{\bar{b}}{\bar{a}}\right)^{1-r}\left(\frac{\beta}{\alpha}\right)^{r} = p_{\mathrm{MF}}. \tag{3.37}$$

Now, to solve the equality in Equation 3.36 we need the exponents to be the same. To see the case where this is true let us consider the above relation when it is divided by $\bar{b}$

$$\frac{p}{\bar{b}} = q\frac{\bar{a}^{r-1}}{\bar{b}^{r}}\left(\frac{\beta}{\alpha}\right)^{r} = -\frac{q}{\bar{a}}\left(\frac{\beta\bar{a}}{\alpha\bar{b}}\right)^{r}. \tag{3.38}$$

From this we observe that demand and supply will be equal when

$$\left(\frac{\beta\bar{a}}{\alpha\bar{b}}\right)^r = 1, \text{ or } \frac{\bar{a}}{\alpha} = \frac{\bar{b}}{\beta}. \tag{3.39}$$

This allows us to write

$$p_{MF} = q\left(\frac{\bar{b}}{\bar{a}}\right)^{1-r}\left(\frac{\beta}{\alpha}\right)^r = q\frac{\bar{b}}{\bar{a}}. \tag{3.40}$$

This equality shows that the solution for Equation 3.36 is indeed given by the mean field approach under the condition that $\beta/\alpha = \bar{b}/\bar{a}$. So, we now have an expression for the demand and supply functions including an approximation for the equilibrium price. Having done this analysis it is insightful to represent the demand and supply curves in graphs in order to verify these results.

3.5 Numerics of CES Demand and Supply

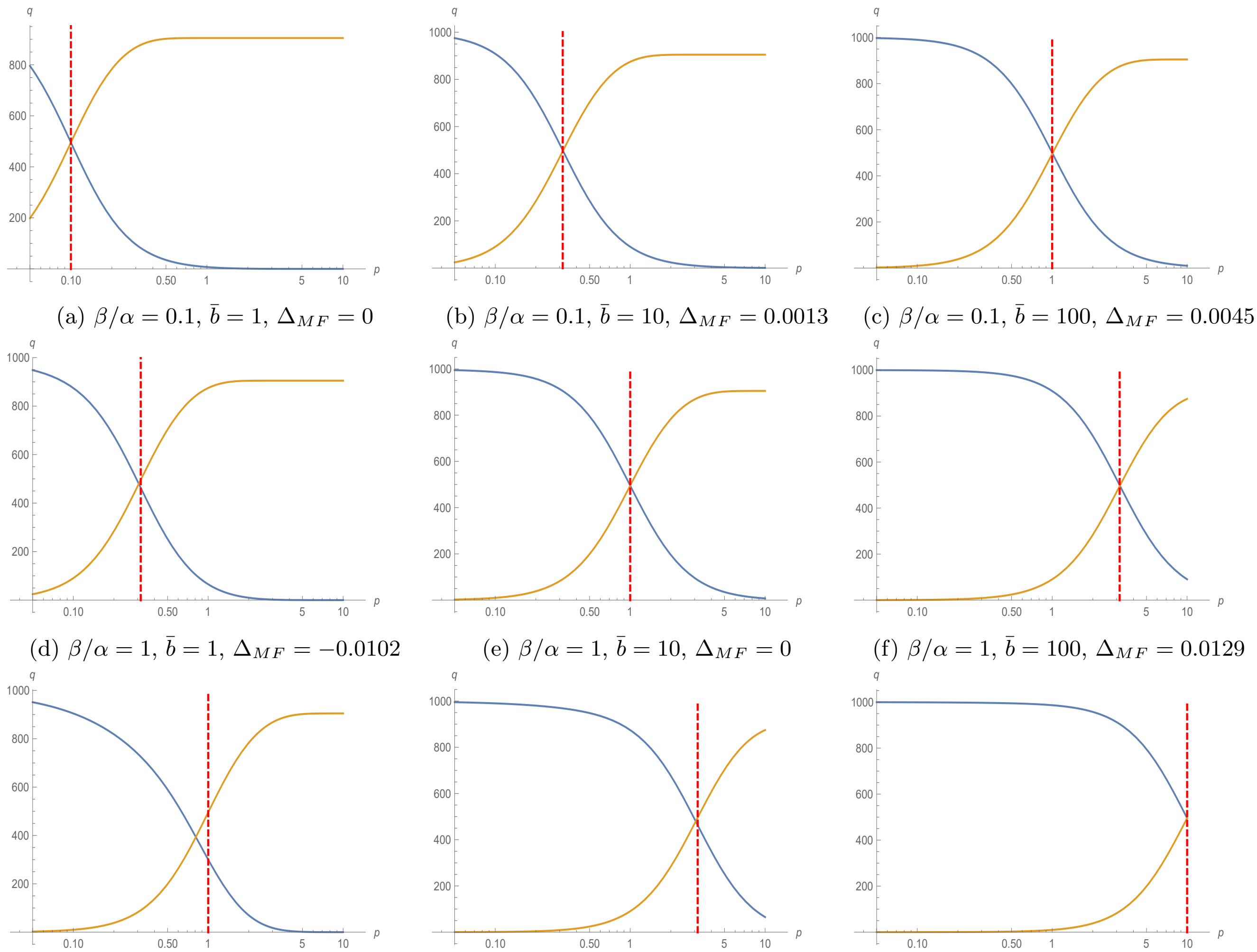
In Figure 3.1 various plots are shown of the demand and supply curve. In these figures the red dashed line is the expected price according to Equation 3.37 and Δ_{MF} is the difference between the actual price at the intersection and the mean field price. It can be observed in the figures that the demand and supply curves act as expected near their limits as $p \to 0$ and $p \to \infty$. It is seen that in most cases the mean-field price is a good predictor of the actual price. However, there are a few cases where this is not true, the best example of which is seen in Figure 3.1g. The cause of this divergence is understood quite easily. It is related to the fact that in a Boltzmann distributed system with $\bar{b} = 1$ most agents are unable to buy the good A. Since the average money per agent is only equal to one, it is the case that about half of the agents present do not have enough currency to buy good A at a price that is near the mean-field price. Consequently, these agents are not buyers. Similarly, a certain amount of agents neither has enough of good A to sell. It it these agents which are unable to buy or sell that will not be able to participate in the market when it is near the mean-field price. On top of this, the ratio $\beta/\alpha = 10$. This means that the goods do not exchange in a 1-to-1 ratio and as such an agent needs more goods in proportion to the currency. For these reasons, the equilibrium price turns out to be lower than the mean-field price. In all the other subfigures we observe that the mean-field price is an appropriate approximation to the equilibrium price. With this knowledge in mind we can continue to describe the dynamics of the utility-based market model.

3.6 Utility-based Market Model Dynamics

Up until this point the focus has been on what the demand and supply curve in an economy based on utility looks like. This representation of the demand and supply is, however, only a portrayal of the demand and supply at a certain moment in time. The aim of this section is to discuss the distribution of goods A and B that results from the dynamics of utility-based trading [3]. In the utility-based market model agents are all conscious of their own utility. As a result of this, each agent 'knows' what kind of trade it needs to make in order to optimize its utility. In other words, the agent knows whether it is on the demand or the supply side of the market. What happens in this model is that all agents will go to a marketplace where they will either sell goods or buy them for a set price. It is the presence of the marketplace that makes sure that the goods will be sold at the price at which demand equals supply. Here, the agents will then put their goods and currency on display and exchange it for what they prefer. So, after having come to this marketplace every agent will have done his preferred trade, 'go back home', and wait until the next trading day arrives. As such, in this model every trading day can be seen as a time step or sweep, meaning that a certain amount of trading days accounts for an equal amount of time steps. A step-by-step approach to a single trading day of the utility-based market model is then [3]

1. All agents gather at the marketplace where the price is determined so that $D(p) = S(p)$.
2. All agents trade goods at the marketplace based on the market price p and their own preferences. **Note:** The preferences of the agents are based on the CES utility function.
   The buyers will have their goods change to

$$a \to a' = a + q; \tag{3.41}$$



(a) $\beta/\alpha = 0.1$, $\bar{b} = 1$, $\Delta_{MF} = 0$
(b) $\beta/\alpha = 0.1$, $\bar{b} = 10$, $\Delta_{MF} = 0.0013$
(c) $\beta/\alpha = 0.1$, $\bar{b} = 100$, $\Delta_{MF} = 0.0045$
(d) $\beta/\alpha = 1$, $\bar{b} = 1$, $\Delta_{MF} = -0.0102$
(e) $\beta/\alpha = 1$, $\bar{b} = 10$, $\Delta_{MF} = 0$
(f) $\beta/\alpha = 1$, $\bar{b} = 100$, $\Delta_{MF} = 0.0129$
(g) $\beta/\alpha = 10$, $\bar{b} = 1$, $\Delta_{MF} = -0.1896$
(h) $\beta/\alpha = 10$, $\bar{b} = 10$, $\Delta_{MF} = -0.1012$
(i) $\beta/\alpha = 10$, $\bar{b} = 100$, $\Delta_{MF} = 0$

Figure 3.1: Demand curves (blue) and Supply curves (orange) plotted as a function of the price p. The parameters $r = 0.5$, $N = 1000$ $q = 1$, and $\bar{a} = 10$ are kept constant. The red dashed line represents the p value expected from Equation 3.37. The factor Δ_{MF} represents the distance between the actual intersection and the mean field prediction.

$$b \to b' = b - p. \tag{3.42}$$

Sellers will have their goods change to

$$a \to a' = a - q; \tag{3.43}$$

$$b \to b' = b + p. \tag{3.44}$$

3. All agents 'go home' and repeat steps 1 and 2 T times.

Similar to the Yakovenko model, we want to know what the distribution of the population becomes over time when the above process is repeated T times. To get an idea of this we will consider an initial population with an exponential distribution of goods A and B over random agents. The choice of this initial distribution is based on the fact that without an inequality in the initial distribution of goods trading does not take place. If all agents would have an equal amount of good A and currency B then they will always prefer to either be on the demand or supply side, i.e. trading cannot take place if everyone wants to sell. As such, the number of goods and currency for each agent will be taken from an exponential distribution. On top of this, we initially expect that this system tends towards a Boltzmann distribution. Using this setup, a start will be made by considering what the equilibrium distribution becomes for this model. To do this an analytical and numerical consideration will be done.

3.7 Analytical Representation of the Utility-based Market Model Distribution

To approach the utility-based market model analytically the results from the thesis by Jan Mulder have been used [3]. To do the derivation given in his thesis several assumptions have to be made. First, it is assumed that the initial distribution is drawn from an exponential distribution. For this reason, the initial distribution will have a shape similar to the Boltzmann distribution

$$P(a,b) = \frac{1}{\bar{a}\bar{b}} \exp\left(-\frac{a}{\bar{a}} - \frac{b}{\bar{b}}\right) \tag{3.45}$$

where we take $\bar{a} = \bar{b}$. Secondly, we assume to be dealing with an economy where $\alpha = \beta = 1$, and $r = 0.5$. On top of this, it is assumed that $q = 1$, $p \approx 1$. When these assumptions are combined with Gossen's Second Law then it is understood that in this economy agents will prefer to have their goods distributed in a $1:1$ manner. In other words, under our assumptions the agents will tend to have an equal amount of good A and currency B. As a consequence of all this, all agents will keep their total amount of goods c constant, i.e. $c = a + b$. Also, the tendency towards a one-on-one distribution of goods makes sure that $a_{\text{final}} = b_{\text{final}} = c_{\text{init}}/2 = c/2$. With this in mind we let the distribution $P_C(c)$ be the distribution of the population. In order to calculate this distribution we need to go from the $P(a,b)$ distribution to a distribution dependent on c. Luckily, we know that we can define $b = c - a$ so that the distribution can be found by solving

$$P_C(c) = \int_0^c da\ P(a, c-a). \tag{3.46}$$

Based on our earlier observations with respect to the conservation of goods, we guess that a Boltzmann distribution will accurately represent the distribution of this economy. Solving with this gives

$$\begin{aligned} P_C(c) &= \frac{1}{\bar{a}\bar{b}} \int_0^c da\ e^{-a/\bar{a}} e^{-(c-a)/\bar{b}} \\ &= \frac{e^{-c/\bar{b}}}{\bar{a}\bar{b}} \int_0^c da\ \exp\left[a\left(\frac{1}{b} - \frac{1}{\bar{a}}\right)\right] \\ &= \frac{e^{-c/\bar{b}}}{\bar{a}\bar{b}} \frac{1}{\frac{1}{b} - \frac{1}{\bar{a}}} \left\{\exp\left[c\left(\frac{1}{b} - \frac{1}{\bar{a}}\right)\right] - 1\right\} \\ &= \frac{e^{-c/\bar{a}} - e^{-c/\bar{b}}}{\bar{a} - \bar{b}}. \end{aligned} \tag{3.47}$$

Combining this result with the knowledge that at equilibrium $a = b = c/2$, we can depict the distribution for good A as

$$P_A(a) \propto \frac{e^{-2a/\bar{a}} - e^{-2a/\bar{b}}}{\bar{a} - \bar{b}}. \tag{3.48}$$

All that remains is to normalize this distribution. Normalization gives

$$\int_0^\infty da\ P_A(a) = \frac{1}{\bar{a} - \bar{b}} \left(\frac{\bar{a}}{2} - \frac{\bar{b}}{2}\right) = \frac{1}{2}. \tag{3.49}$$

As a result, the normalized distribution of good A, and as such good B, is given by

$$P_A(a) = \frac{2e^{-2a/\bar{a}} - 2e^{-2a/\bar{b}}}{\bar{a} - \bar{b}}. \tag{3.50}$$

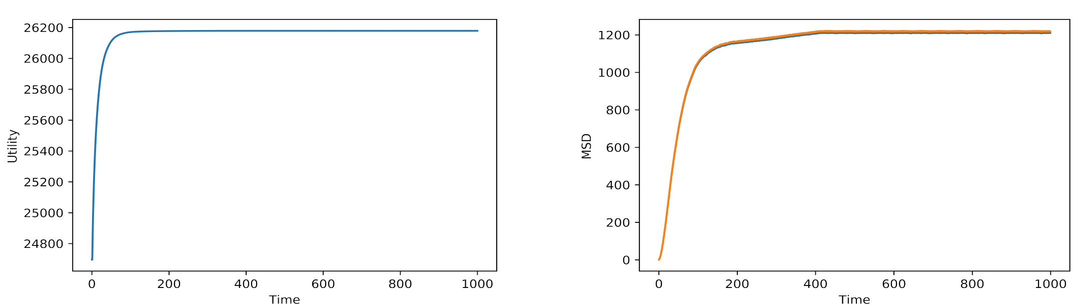
When this expression is considered more closely then it becomes apparent that it is unclear what happens when $\bar{b} \to \bar{a}$. For this it can be seen that both the numerator and the denominator approach zero. To get more insight into this limit we make use of L'Hôpital's rule to get

$$\lim_{b \to \bar{a}} P_A(a) = \lim_{b \to \bar{a}} \frac{-4ae^{-2a/\bar{b}}}{-\bar{b}^2} = \frac{4a}{\bar{a}^2} e^{-2a/\bar{a}}. \tag{3.51}$$

Having derived this result for the distribution function we can proceed to compare it with numerical results to see if it describes the distribution of the utility-based market model.

3.8 Numerical Results Utility-based Market Model

In this section the numerical results from a simulation of the utility-based market model will be elaborately discussed. The simulation of this model has been based on the properties described in the sections above. For more details on this simulation it is advised to have a look at the pseudocode in Appendix A. To consider the results from the simulation a start will be made by looking at the parameters which change in the system. After this, the probability distribution function of the model will be considered. In our simulations we have taken the constant parameters to be $\bar{a} = 50$, $\bar{b} = 50$, $\alpha = 1$, $\beta = 1$, $r = 0.5$, and $q = 1$. In Figure 3.2 it is shown what happens with the total utility of the system and the mean square displacement. It has been chosen to show these two system parameters as they are the only ones that change significantly. The market price, for example, remains constant in these simulations. In Figure 3.2a it is observed that the total utility quickly maximizes and reaches a limit. After this, the utility no longer changes. What this maximization entails is that every agent has reached its one-on-one ratio of goods and can no longer improve upon its utility. In Figure 3.2b the mean square displacement of the two goods, A and B, is given. What is observed here is that there is an initial sharp increase in the mean square displacement of both goods. From the previous figure we can deduce that the increase in the mean square displacement is related to the maximization of the utility. After this maximization, we see that the MSD seizes to increase significantly and as such the MSD remains constant afterwards. From these two figures we observe that the main functionality of the utility-based market model is to maximize the total utility of the system. It is this maximization that can also be observed in the shape of the probability distribution. In Figure 3.3 the probability distribution of the initial configuration is given. In this figure it is observed that the goods are given by the initial condition of a Boltzmann distribution in both the a and b direction. When this figure, however, is contrasted with Figure 3.4 then the effect of the model is seen clearly. The model maximizes the utility by distributing the goods along a single line. The cause of this is that utility is maximized when the goods are distributed in a set ratio. This ratio appears to be equal to the fraction $q\bar{b}/\bar{a}$. Besides the change towards a single line, it is also observed that the goods are no longer Boltzmann distributed in the a and b direction but rather that the distribution has become similar to what has been predicted in section 3.7. To illustrate the accuracy of this prediction Figure 3.5 has been generated. In this figure the data has been adjusted manually so that an agent will have the exact ratio $\bar{b}/\bar{a}$ for goods a and b. The only reason this has been done is that limiting $q = 1$ results in a small deviation from the optimal ratio. So, to clearly show that the prediction made in Section 3.7 corresponds with the numerical results this additional figure has been made. With this final observation we can confirm that Equation (3.50) is able to predict the stationary probability distribution. Furthermore, we observe that the stationary solution is not given by the Boltzmann-Gibbs distribution. We will now continue to look in the next chapter at what happens when we combine this model with the Yakovenko model.



(a) Change of total utility in the system over time.

(b) Change of mean squared displacement over time. The blue line represents good A and the orange line represents currency B.

Figure 3.2: Numerical results for model parameters of the utility-based market model.

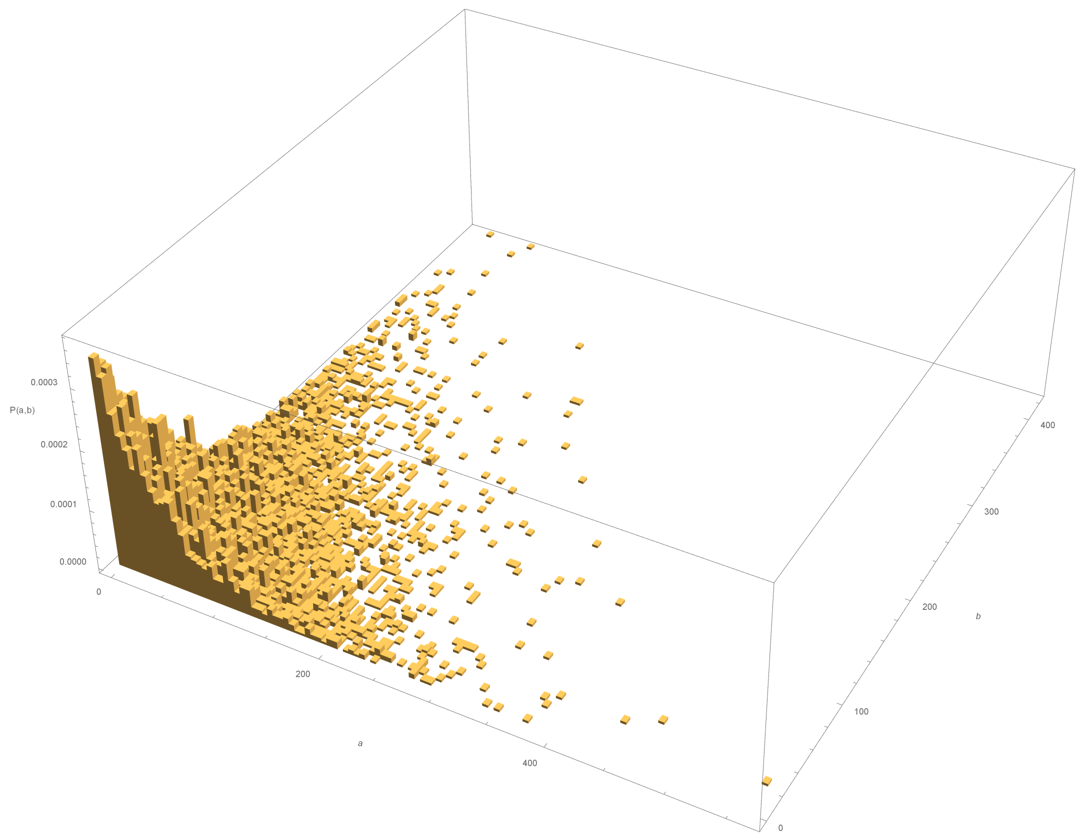


Figure 3.3: Probability distribution of the initial configuration of agents in the utility-based market model. The x (bottom left) and y axis (bottom right) show the values of a and b while the z-axis shows the value of $P(a, b)$.

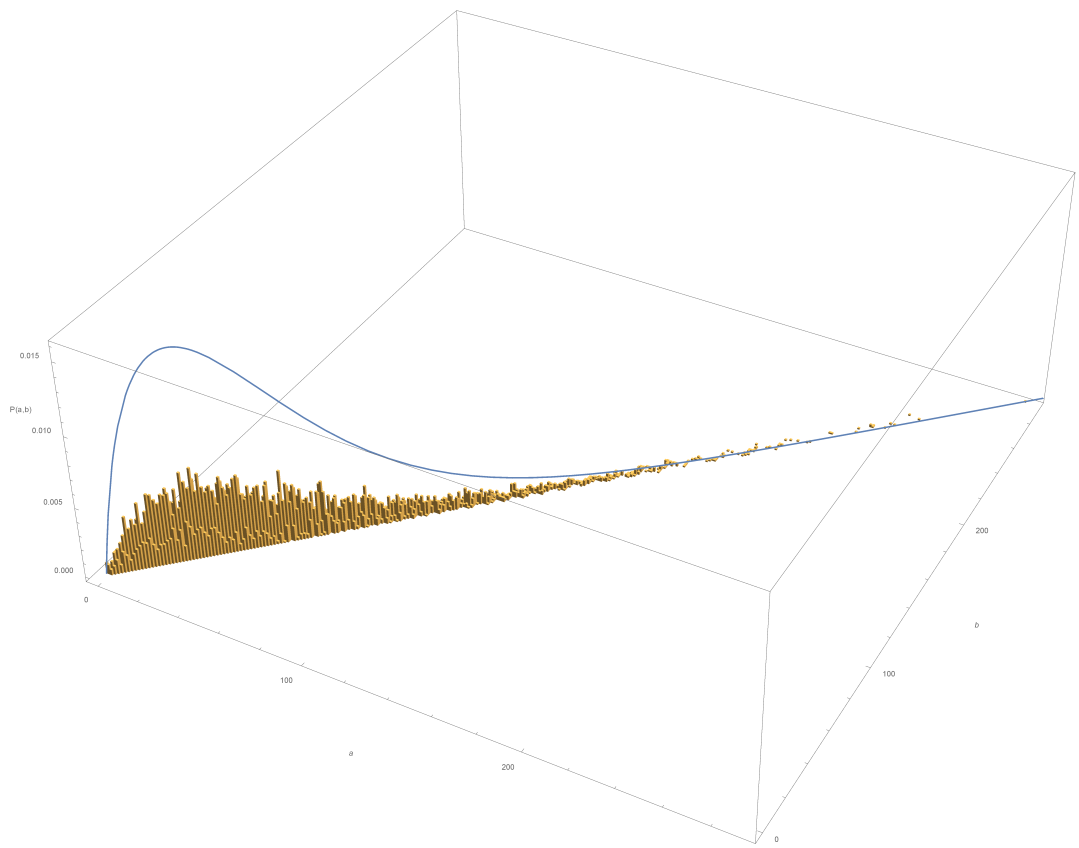


Figure 3.4: Normalized probability distribution of the final configuration of agents in the utility-based market model. The blue line represents the probability distribution predicted by the analytical result from Section 2.7. The x (bottom left) and y axis (bottom right) show the values of a and b while the z-axis shows the value of $P(a, b)$.

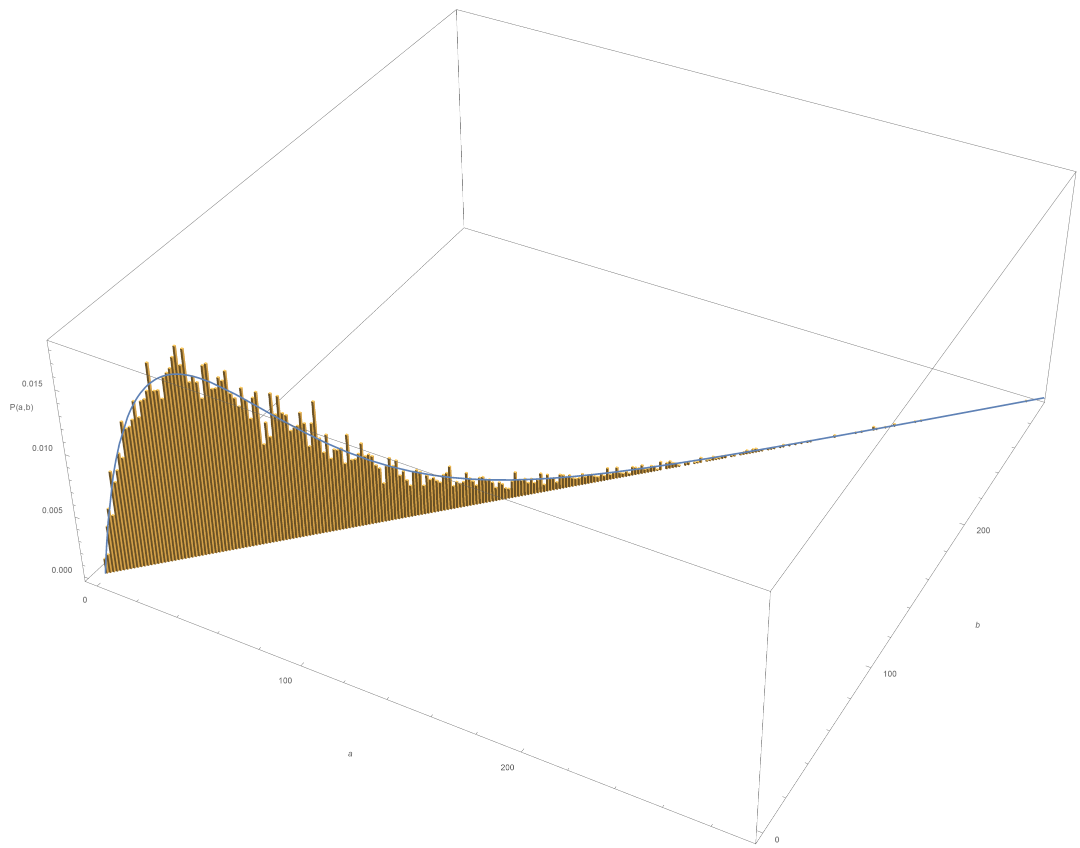


Figure 3.5: Normalized probability distribution of the final configuration of agents in the utility-based market model adjusted to distribute goods along a single line. The blue line represents the probability distribution predicted by the analytical result from Section 2.7. The x (bottom left) and y axis (bottom right) show the values of a and b while the z-axis shows the value of $P(a, b)$.

4 Utility-based Yakovenko Model

So far we have only been dealing with markets with a single trading mechanism. In these markets it was specified at what prices and at what quantities the economic agents will trade their goods using a single trading mechanism. In this chapter there will be looked at what happens in a market where two trading mechanisms take place at the same time. We will call this new market model the utility-based Yakovenko model as it combines the two formerly discussed trading mechanisms. The first of these two mechanisms is the utility-based trading mechanism. The second is the trading mechanism used in the Yakovenko model. To consider this market a start will be made by explaining its dynamics and providing a short pseudocode. After this, the master equation for this market will be discussed extensively. To formulate the master equation for this model the trading conditions that contribute to the varying transition probabilities will be determined. With this it will then be demonstrated what the master equation solution will be. This will show that the master equation cannot easily be solved analytically. Therefore, the master equation will be linearized and discretized in order to determine the eigenvalues. For clarity, an easy illustration of this will be provided by considering the eigenvalues of the transition matrix of the Yakovenko model. After this, the transition matrix of the utility-based Yakovenko model will be determined. The result that follows from this has to be solved numerically in order to predict a solution for it. Finally, various numerical results of this model will be shown and analyzed.

4.1 Utility-based Yakovenko Model Dynamics

In the utility-based Yakovenko model agents trade by making use of one of the two trading mechanisms every sweep. The two trading mechanisms are the utility-based trading mechanism and the Yakovenko trading mechanism. The dynamics of these two separate mechanisms have already been discussed extensively in the previous chapters (See Sections 2.1 and 3.6). One of the reasons to consider this new model is to construct a Yakovenko model with two variables which can then be approached as a thermodynamical system. The analysis of this could provide intriguing results. In this new market model we again have an even number of N agents with two different goods a and b. We take this number to be even as we are dealing with pairwise interactions and we want every agent to be able to trade. At the marketplace an agent will be able to do a utility-based trade or a Yakovenko trade of either good a or currency b. All agents will gather at this martketplace and either sell a fixed quantity of goods q at the price p for which demand and supply are equal or exchange q or p goods. It is important to note that an agent will never make use of more than one trading mechanism at a time. To make sure this happens all the agents will be assigned a trading mechanism every sweep. This assignment will be done randomly in certain fractions. In other words, every sweep a Yakovenko fraction of Y_f agents will trade using the Yakovenko mechanism of which half will trade good a and half currency b. Practically, this means that if $Y_f = 0.1$ that 10% of all agents will trade using the Yakovenko mechanism and that roughly 5% of all agents will only trade good a and 5% will only trade currency b. This allocation of trading mechanisms for agents will be done every sweep. The step-by-step approach for the utility-based Yakovenko model is then given by

1. Allocate to every agent a trading preference so that $1 - Y_f$ agents will trade using the utility-based model and Y_f agents will trade using the Yakovenko model.
2. All agents trade based on their trading preference. (See Sections 2.1 and 3.6 for the different mechanisms).
3. Repeat the above steps T times.

For an extensive explanation of the pseudocode used in the numerical analysis for this model the reader is advised to have a look at Appendix A. Now that we have description of this model, we would like to be able to predict its behaviour by making use of methods from statistical physics. For this model we again make use of the master equation which will help in predicting the equilibrium probability distribution of this model. Therefore, we proceed by formulating the master equation for this market model.

4.2 The Master Equation

To determine the master equation for this model we start by considering a general expression of it for the probability distribution $P(a, b)$. Notice that, in contrast to the Yakovenko model, we assumed for this model that q and p are constants so that we need to only integrate over a' and b' which gives

$$\frac{dP(a,b)}{dt} = \int_0^\infty da' \int_0^\infty db' \Big(W_{\text{gain}} - W_{\text{loss}} \Big). \tag{4.1}$$

Here, W_{gain} represents the gain terms and W_{loss} represents the loss terms of the master equation. Also, note that in this thesis the usage of primes on variables is not to indicate that we are taking a derivative. It is simply another variable. For this master equation the gain and loss terms each consist out of three parts: one part for the utility-based transitions and two parts for the Yakovenko transitions. We continue by writing these terms down separately. The utility-based part of the master equation is given by

$$\begin{aligned}\frac{dP_U(a,b)}{dt} = C_1(1-Y_f)\int_0^\infty da' \int_0^\infty db' [&f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']} P(a+q,b-p) \\ &P(a'-q,b'+p) - f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]} P(a,b) P(a',b')].\end{aligned} \tag{4.2}$$

In this expression C_1 denotes a to be determined constant and Y_f is the Yakovenko trade fraction. The transition probability $f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]}$ represents the probability to acquire a good q for currency p for an agent with goods a, b from an agent with goods a', b'. The other transition probability $f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']}$ depicts the reverse process. The Yakovenko parts of the master equation are given by

$$\begin{aligned}\frac{dP_{Y_a}(a,b)}{dt} = \frac{C_2 Y_f}{2}\int_0^\infty da' \int_0^\infty db' \big[&f_{[a+q,b,a'-q,b']\to[a,b,a',b']} P(a+q,b) P(a'-q,b') \\ &- f_{[a,b,a',b']\to[a+q,b,a'-q,b']} P(a,b) P(a',b')\big]\,;\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{dP_{Y_b}(a,b)}{dt} = \frac{C_3 Y_f}{2}\int_0^\infty da' \int_0^\infty db' \big[&f_{[a,b-p,a',b'+p]\to[a,b,a',b']} P(a,b-p) P(a',b'+p) \\ &- f_{[a,b,a',b']\to[a,b-p,a',b'+p]} P(a,b) P(a',b')\big]\,.\end{aligned} \tag{4.4}$$

Here, C_2 and C_3 are to be determined constants. Similar to what has been described in Section 2.2 the transition probabilities depict the probability to exchange an amount of q of good a or p of currency b with another agent. Following this, the complete expression of the master equation is given by the combination of Equations (4.2), (4.3), and (4.4)

$$\frac{dP(a,b)}{dt} = \frac{dP_U(a,b)}{dt} + \frac{dP_{Y_a}(a,b)}{dt} + \frac{dP_{Y_b}(a,b)}{dt}. \tag{4.5}$$

In order to obtain a general solution for the above master equation it will be necessary to determine the conditions which affect the various transition probabilities. A start will be made by first analyzing the trading conditions for the Yakovenko parts after which the utility-based part will be analyzed.

4.3 Trading Conditions

To determine the trading conditions which indicate the transition probability we start by defining the approach that will be taken. In the analysis of this section it is assumed that if an agent meets the conditions to trade that he will trade. This means that the transition probability f will equal to one when an agent can make a trade and it will equal to zero when he cannot. Consequently, the transition probabilities will be given by Heaviside step functions as they represent this 'all or nothing' trading behaviour. Furthermore, it should be realized that agents will only do one type of trade each sweep. What this means is that to determine the transition probability we need to establish the required conditions for an agent to perform a certain type of trade. To do this for varying ranges of a and b we define the lower limits a_i and b_i and the upper limits a_f and b_f. Using this we continue by considering the trading conditions for each part of the master equation.

4.3.1 Trading Conditions Yakovenko Trade Good A

The master equation for the Yakovenko trade of good a is given by

$$\begin{aligned}\frac{dP_{Y_a}(a,b)}{dt} = \frac{C_2 Y_f}{2}\int_{a_i}^{a_f} da' \int_{b_i}^{b_f} db' \, \big[& f_{[a+q,b,a'-q,b']\to[a,b,a',b']} P(a+q,b) P(a'-q,b') \\ & - f_{[a,b,a',b']\to[a+q,b,a'-q,b']} P(a,b) P(a',b') \big] .\end{aligned} \tag{4.6}$$

The main condition for the Yakovenko trade to occur is that the agent does not exceed the boundaries of the system. In all other cases the agent will be able to trade his goods. The boundaries of the system that are of interest in this trading mechanism are a_i and a_f. In this equation the transition probability $f_{[a,b,a',b']\to[a+q,b,a'-q,b']}$ denotes the probability for an agent with goods a and b to gain q goods and for an agent with goods a' and b' to lose q goods. For this to be possible a should not exceed a_f after the trade and a' should not be smaller than a_i. In terms of Heaviside step functions this is given by

$$\theta(a_f - (a+q)) = \begin{cases} 0, & a_f < a+q \\ 1, & a_f \geq a+q \end{cases} ; \tag{4.7}$$

$$\theta((a' - a_i) - q) = \begin{cases} 0, & a' - a_i < q \\ 1, & a' - a_i \geq q \end{cases} . \tag{4.8}$$

Combining these gives the transition probability

$$f_{[a,b,a',b']\to[a+q,b,a'-q,b']} = \theta(a_f - (a+q))\theta((a'-a_i)-q). \tag{4.9}$$

For the reverse process $f_{[a+q,b,a'-q,b']\to[a,b,a',b']}$ the conditions are slightly different. In this process we start from a state with goods $[a+q, b, a'-q, b']$. In this transition it is necessary that $a+q$ does not become smaller than a_i after losing q and that $a'-q$ does not exceed a_f after gaining q. Following this the Heaviside step functions are initially given by $\theta((a+q-a_i)-q)$ and $\theta(a_f-(a'-q+q))$. However, one can observe that these can easily be simplified to

$$\theta(a - a_i) = \begin{cases} 0, & a < a_i \\ 1, & a \geq a_i \end{cases} ; \tag{4.10}$$

$$\theta(a_f - a') = \begin{cases} 0, & a_f < a' \\ 1, & a_f \geq a' \end{cases} . \tag{4.11}$$

Under the boundary conditions of the system we know that these two step functions are by definition always equal to one. The reason, however, to mention these 'trivial' step functions is that it is important in the understanding of the transitions. It shows, by the use of similar step functions, that the probability of the reverse process to occur is dependent on the conditions of the initial process. It appears that the definition of the initial process determines the values of the reverse process. Therefore, it is necessary for this transition probability that the initial process can occur as well. Otherwise the state of the reverse process would not even exist. Consequently, the transition probability is fully (i.e. including trivial terms) given by

$$f_{[a+q,b,a'-q,b']\to[a,b,a',b']} = \theta(a-a_i)\theta(a_f-(a+q))\theta((a'-a_i)-q)\theta(a_f-a'). \tag{4.12}$$

Leaving out the 'trivial' terms results in

$$f_{[a+q,b,a'-q,b']\to[a,b,a',b']} = \theta(a_f - (a+q))\theta((a'-a_i)-q). \tag{4.13}$$

This final result shows that $f_{[a+q,b,a'-q,b']\to[a,b,a',b']} = f_{[a,b,a',b']\to[a+q,b,a'-q,b']}$. In Section 2.2 it was already assumed that the transition probabilities of the Yakovenko model are equal. In the same section on the Yakovenko model it was also shown that for this to be a stationary process it is necessary that probability distribution is a Boltzmann distribution. However, for the current model the presence of the utility-based trade prohibits the use of the same assumption as it was shown in the previous chapter that it does not adhere to a Boltzmann-Gibbs distribution.

4.3.2 Trading Conditions Yakovenko Trade Good B

The master equation for the Yakovenko trade of good b is given by

$$\begin{aligned}\frac{dP_{Y_b}(a,b)}{dt} = \frac{C_3 Y_f}{2}\int_{a_i}^{a_f} da' \int_{b_i}^{b_f} db' \left[f_{[a,b-p,a',b'+p]\to[a,b,a',b']} P(a,b-p)P\left(a',b'+p\right)\right. \\ \left. -f_{[a,b,a',b']\to[a,b-p,a',b'+p]} P(a,b)P\left(a',b'\right)\right].\end{aligned} \tag{4.14}$$

Again, the main condition for the Yakovenko trade to occur is that the agent does not exceed the boundaries of the system. In fact, the analysis for this trading system is equal to that of good a. So, one can easily determine that the transition probabilities are given by

$$f_{[a,b-p,a',b'+p]\to[a,b,a',b']} = \theta(b_f - b)\theta((b-b_i)-p)\theta(b'-b_i)\theta(b_f-(b'+p)); \tag{4.15}$$

$$f_{[a,b,a',b']\to[a,b-p,a',b'+p]} = \theta((b-b_i)-p)\theta(b_f-(b'+p)). \tag{4.16}$$

Leaving out the 'trivial' conditions in Equation (4.15) shows that the transition probabilities are equal.

4.3.3 Trading Conditions Utility-Based Trade

The master equation for the utility-based trade is given by

$$\begin{aligned}\frac{dP_U(a,b)}{dt} = C_1(1-Y_f)\int_{a_i}^{a_f} da' \int_{b_i}^{b_f} db' [f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']} P(a+q,b-p) \\ P(a'-q,b'+p) - f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]} P(a,b)P\left(a',b'\right)].\end{aligned} \tag{4.17}$$

For the utility-based trade to occur an agent should meet two conditions. The first is not to exceed the boundary conditions of the system after a trade. The second is that an agent should increase its utility following a trade. The first of these two conditions is ensured by incorporating the transition probabilities of both Yakovenko trades, i.e. $\theta(a_f - (a+q))\theta((a'-a_i)-q)$ and $\theta((b-b_i)-p)\theta(b_f-(b'+p))$. The second condition can be represented by step functions similar to Equation (3.24). In the analysis in Section 3.4 it is explained why. From this analysis it follows that a buyer needs to have that $b/a \geq \gamma(p)$ and that a seller needs to have that $b/a \leq \gamma(p)$. Applying this to $f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']}$ it is observed that the agent with goods $(a+q, b-p)$ is selling and that the agent with goods $(a'-q, b'+p)$ is buying. This means that the transition probability will be given by

$$\begin{aligned}f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']} = \theta(a_f-(a+q))\theta((a'-a_i)-q) \\ \theta((b-b_i)-p)\theta(b_f-(b'+p))\theta(\gamma(p)-\frac{b-p}{a+q})\theta(\frac{b'+p}{a'-q}-\gamma(p)).\end{aligned} \tag{4.18}$$

For the transition $f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]}$ it is the case that the agent with goods (a,b) is buying and that the agent with goods (a',b') is selling. This means that this transition probability is given by

$$\begin{aligned}f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]} = \theta(a_f-(a+q))\theta((a'-a_i)-q) \\ \theta((b-b_i)-p)\theta(b_f-(b'+p))\theta(\frac{b}{a}-\gamma(p))\theta(\gamma(p)-\frac{b'}{a'}).\end{aligned} \tag{4.19}$$

When the two transition probabilities above are compared it is observed that they differ in the conditions for the increase in utility. As such, the transition probabilities do not cancel out for a Boltzmann distribution. An implication of this is that the stationary solution for this master equation might not be given by a Boltzmann distribution. To verify this we continue by solving the master equation with this distribution and see whether it equates to zero.

4.4 Master Equation and the Boltzmann Distribution

In this section it is considered whether the Boltzmann distribution represents the stationary solution of the master equation. To do this we check whether the Boltzmann-Gibbs distribution gives that $\frac{dP(a,b)}{dt} = 0$. If we obtain a non-zero result then we know that the Boltzmann distribution is not the equilibrium solution of this model. To start, we recall that a Boltzmann distribution for this system is depicted by

$$P\left(a,b\right) = \frac{1}{\bar{a}\bar{b}} \exp\left(-\frac{a}{\bar{a}} - \frac{b}{\bar{b}}\right), \tag{4.20}$$

where $\bar{a}$ and $\bar{b}$ represent the average number of goods a and b respectively. For a Boltzmann distribution it is known that $P(a+q, b-p)P(a'-q, b'+p) = P(a,b)P(a',b')$. Thus, making use of a Boltzmann distribution in Equation (4.5) results in

$$\frac{dP(a,b)}{dt} = \int_{a_i}^{a_f} da' \int_{b_i}^{b_f} db' \Big(W'_{\text{in}} - W'_{\text{out}}\Big) P(a,b)P(a',b'). \tag{4.21}$$

Here, W'_{in} represents

$$W'_{\text{in}} = C_1(1-Y_f) f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']} + \frac{C_2 Y_f}{2} f_{[a+q,b,a'-q,b']\to[a,b,a',b']} + \frac{C_3 Y_f}{2} f_{[a,b-p,a',b'+p]\to[a,b,a',b']}. \tag{4.22}$$

The terms of the transition probabilities are given by Equations (4.18), (4.13), and (4.16). The terms in W'_{out} are

$$W'_{\text{out}} = C_1(1-Y_f) f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]} + \frac{C_2 Y_f}{2} f_{[a,b,a',b']\to[a+q,b,a'-q,b']} + \frac{C_3 Y_f}{2} f_{[a,b,a',b']\to[a,b-p,a',b'+p]}, \tag{4.23}$$

where the transition probabilities represent Equations (4.19), (4.13), and (4.16). Now, we know from the analysis on the trading conditions that the Yakovenko terms are equal to each other and as such they cancel out here. This leaves us with

$$\frac{dP(a,b)}{dt} = \int_{a_i}^{a_f} da' \int_{b_i}^{b_f} db' C_1(1-Y_f) \Big(f_{[a,b,a',b']\to[a+q,b-p,a'-q,b'+p]} - f_{[a,b,a',b']\to[a+q,b-p,a']} \Big) P(a,b)P(a',b'). \tag{4.24}$$

It is straightforward to solve this integral by making use of the step functions to adjust the limits. Doing this and integrating provides a non-zero result. This indicates that the Boltzmann distribution is not a stationary solution for this model. It is in our interest, however, to know whether a model such as this will tend towards a stable equilibrium in the long term. For this reason, we will try to show that a stationary solution exists by determining the eigenvalues of the transition matrix of the master equation. To do this we first continue with linearizing and discretizing the master equation.

4.5 Linearization and Discretization

In this section we will continue by linearizing and discretizing the master equation we have obtained in order to determine the stability. The master equation is given by

$$\frac{dP(a,b)}{dt} = \int_{a_i}^{a_f} da' \int_{b_i}^{b_f} db' \Big(W_{\text{in}} - W_{\text{out}}\Big). \tag{4.25}$$

Where W_{in} signifies

$$\begin{aligned} W_{\text{in}} = C_1(1-Y_f) f_{[a+q,b-p,a'-q,b'+p]\to[a,b,a',b']} P(a+q,b-p) P\left(a'-q,b'+p\right) \\ + \frac{C_2 Y_f}{2} f_{[a+q,b,a'-q,b']\to[a,b,a',b']} P(a+q,b) P\left(a'-q,b'\right) \\ + \frac{C_3 Y_f}{2} f_{[a,b-p,a',b'+p]\to[a,b,a',b']} P(a,b-p) P\left(a',b'+p\right). \end{aligned} \tag{4.26}$$

The transition probabilities in this equation have been determined in Section 4.3. The expression for W_{out} is similar to Equation (4.23) and is depicted by

$$W_{\text{out}} = W'_{\text{out}} P(a,b) P(a',b'). \tag{4.27}$$

Now, the master equation given above is hard to solve analytically. Nonetheless, it is possible to determine the stationarity of this system by considering the eigenvalues of its transition matrix. The definition of this matrix will be shown shortly. In order to obtain this transition matrix and its eigenvalues we have to rewrite the master equation. We do this rewriting by first linearizing and then discretizing the master equation. But before we do this, we start by shifting the above functions so that the last term becomes of the form $P(a',b')$. We do this to make our later analysis easier. For this shift we take the limits to be $a_f = b_f = \infty$ and $a_i = b_i = 0$. It can be observed that only the W_{in} terms have to be shifted. To do this shift we change all the lower integral limits to $-\infty$ by making use of step functions. We then proceed to shift the variables a' and b' in such a way that we get $P(a',b')$. Shifting then first term of W_{in} results in

$$\int_{-\infty}^{\infty} da' \int_{-\infty}^{\infty} db' C_1(1-Y_f)\theta(a'+q)\theta(b'-p) f_{[a+q,b-p,a',b']\to[a,b,a'+q,b'-p]} P(a+q,b-p) P\left(a',b'\right). \tag{4.28}$$

Do note that the terms in the transition probability f are shifted as well. Doing the same for the other two terms in W_{in} gives

$$\int_{-\infty}^{\infty} da' \int_{-\infty}^{\infty} db' \frac{C_2 Y_f}{2} \theta(a'+q)\theta(b') f_{[a+q,b,a',b']\to[a,b,a'+q,b']} P(a+q,b) P\left(a',b'\right); \tag{4.29}$$

$$\int_{-\infty}^{\infty} da' \int_{-\infty}^{\infty} db' \frac{C_3 Y_f}{2} \theta(a')\theta(b'-p) f_{[a,b-p,a',b']\to[a,b,a',b'-p]} P(a,b-p) P\left(a',b'\right). \tag{4.30}$$

On the W_{out} term we only apply the step functions so that the integrals have the same limits. That means that now we have

$$W_{\text{out}} = \theta(a')\theta(b') W'_{\text{out}} P(a,b) P(a',b'). \tag{4.31}$$

Having applied these shifts we continue by linearizing the master equation. To linearize it we write $P(a,b) = P_0(a,b) + \delta P(a,b)$. Here $P_0(a,b)$ is the stationary solution of the master equation and $\delta P(a,b)$ is a small deviation away from this equilibrium solution. In the master equation there are now four different combinations that can be linearized. Doing this results in

$$\begin{aligned} P(a+q,b-p)P\left(a',b'\right) = {} & P_0(a+q,b-p)P_0(a',b') + P_0(a+q,b-p)\delta P(a',b') \\ & + P_0(a',b')\delta P(a+q,b-p) + \delta P(a+q,b-p)\delta P(a',b'); \end{aligned} \tag{4.32}$$

$$\begin{aligned} P(a+q,b)P\left(a',b'\right) = {} & P_0(a+q,b)P_0(a',b') + P_0(a+q,b)\delta P(a',b') \\ & + P_0(a',b')\delta P(a+q,b) + \delta P(a+q,b)\delta P(a',b'); \end{aligned} \tag{4.33}$$

$$\begin{aligned} P(a,b-p)P\left(a',b'\right) = {} & P_0(a,b-p)P_0(a',b') + P_0(a,b-p)\delta P(a',b') \\ & + P_0(a',b')\delta P(a,b-p) + \delta P(a,b-p)\delta P(a',b'); \end{aligned} \tag{4.34}$$

$$\begin{aligned} P(a,b)P\left(a',b'\right) = {} & P_0(a,b)P_0(a',b') + P_0(a,b)\delta P(a',b') \\ & + P_0(a',b')\delta P(a,b) + \delta P(a,b)\delta P(a',b'). \end{aligned} \tag{4.35}$$

This linearization means that the master equation has now become

$$\frac{dP_0(a,b)}{dt} + \frac{d\ \delta P(a,b)}{dt} = \int_{-\infty}^{\infty} da' \int_{-\infty}^{\infty} db' \Big(W_{\text{linearized in}} - W_{\text{linearized out}} \Big). \tag{4.36}$$

In this expression $W_{\text{linearized in}}$ is given by Equation (4.26) but with the terms linearized as above. Similarly, $W_{\text{linearized out}}$ is given by combining Equation (4.27) with Equation (4.35). In the process of linearizing we defined P_0 as the stationary solution for the master equation. As a consequence of this, all the $P_0 P_0$ terms together will have a contribution of zero as the assumption of stationarity gives that $\frac{dP_0(a,b)}{dt} = 0$.

Furthermore, δP is defined as a small deviation away from the equilibrium. As such, the terms with $\delta P \delta P$ are considered to be negligibly small. This means that we are left with

$$\frac{d\ \delta P(a,b)}{dt} = \int_{-\infty}^{\infty} da' \int_{-\infty}^{\infty} db' \Big(W'_{\text{linearized in}} - W'_{\text{linearized out}} \Big). \tag{4.37}$$

With $W'_{\text{linearized in}}$

$$W'_{\text{linearized in}} = C_1(1-Y_f)\theta(a'+q)\theta(b'-p) f_{[a+q,b-p,a',b'] \to [a,b,a'+q,b'-p]} \Big(P_0(a+q,b-p)\delta P(a',b') + P_0(a',b')\delta P(a+q,b-p) \Big) + \frac{C_2 Y_f}{2} \theta(a'+q)\theta(b') f_{[a+q,b,a',b'] \to [a,b,a'+q,b']} \Big(P_0(a+q,b)\delta P(a',b') + P_0(a',b')\delta P(a+q,b) \Big) + \frac{C_3 Y_f}{2} \theta(a')\theta(b'-p) f_{[a,b-p,a',b'] \to [a,b,a',b'-p]} \Big(P_0(a,b-p)\delta P(a',b') + P_0(a',b')\delta P(a,b-p) \Big). \tag{4.38}$$

Similarly, $W'_{\text{linearized out}}$ is given by

$$W'_{\text{linearized out}} = \theta(a')\theta(b') \Big(C_1(1-Y_f) f_{[a,b,a',b'] \to [a+q,b-p,a'-q,b'+p]} + \frac{C_2 Y_f}{2} f_{[a,b,a',b'] \to [a+q,b,a'-q,b']} + \frac{C_3 Y_f}{2} f_{[a,b,a',b'] \to [a,b-p,a',b'+p]} \Big) \Big(P_0(a,b)\delta P(a',b') + P_0(a',b')\delta P(a,b) \Big). \tag{4.39}$$

Having linearized the master equation we continue by discretizing it. This gives

$$\frac{d\ \delta P(a_n,b_m)}{dt} = \sum_{n'=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \Big(W_{n\,m,n'm'} - W'_{n\,m,n'm'} \Big). \tag{4.40}$$

Here, $W_{n\,m,n'm'}$ is the discretization of $W'_{\text{linearized in}}$ and $W'_{n\,m,n'm'}$ is the discretization of $W'_{\text{linearized out}}$. In this notation the subscripts denote a transition from state with n and m to a state with n' and m'. Also, it is important to mention that in this discretization we have that $a_n = nq$ and $b_m = mp$. The gain and loss terms are now given by

$$W_{n\,m,n'm'} = C_1(1-Y_f)\theta(a_{n'}+q)\theta(b_{m'}-p) f_{[a_n+q,b_m-p,a_{n'},b_{m'}] \to [a_n,b_m,a_{n'}+q,b_{m'}-p]} \Big(P_0(a_n+q,b_m-p)\delta P(a_{n'},b_{m'}) + P_0(a_{n'},b_{m'})\delta P(a_n+q,b_m-p) \Big) + \frac{C_2 Y_f}{2} \theta(a_{n'}+q)\theta(b_{m'}) f_{[a_n+q,b_m,a_{n'},b_{m'}] \to [a_n,b_m,a_{n'}+q,b_{m'}]} \Big(P_0(a_n+q,b_m)\delta P(a_{n'},b_{m'}) + P_0(a_{n'},b_{m'})\delta P(a_n+q,b_m) \Big) + \frac{C_3 Y_f}{2} \theta(a_{n'})\theta(b_{m'}-p) f_{[a_n,b_m-p,a_{n'},b_{m'}] \to [a_n,b_m,a_{n'},b_{m'}-p]} \Big(P_0(a_n,b_m-p)\delta P(a_{n'},b_{m'}) + P_0(a_{n'},b_{m'})\delta P(a_n,b_m-p) \Big); \tag{4.41}$$

$$W'_{n\,m,n'm'} = \theta(a_{n'})\theta(b_{m'}) \Big(C_1(1-Y_f) f_{[a_n,b_m,a_{n'},b_{m'}] \to [a_n+q,b_m-p,a_{n'}-q,b_{m'}+p]} + \frac{C_2 Y_f}{2} f_{[a_n,b_m,a_{n'},b_{m'}] \to [a_n+q,b_m,a_{n'}-q,b_{m'}]} + \frac{C_3 Y_f}{2} f_{[a_n,b_m,a_{n'},b_{m'}] \to [a_n,b_m-p,a_{n'},b_{m'}+p]} \Big) \Big(P_0(a_n,b_m)\delta P(a_{n'},b_{m'}) + P_0(a_{n'},b_{m'})\delta P(a_n,b_m) \Big). \tag{4.42}$$

With this we have both linearized and discretized this master equation. To continue we now want to write it in the following form in order to easily determine the eigenvalues

$$\frac{d\ \delta P(a_n,b_m)}{dt} = \sum_{n'=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} W''_{n\,m,n'm'} \delta P(a_{n'},b_{m'}). \tag{4.43}$$

In this expression $W''_{n\,m,n'm'}$ represents the transition matrix of the master equation. It is for this matrix that we want to determine the eigenvalues. But before we do this, we will consider an easier example. We will continue by first having a look at what the transition matrix and eigenvalues would be for the Yakovenko model. The reason we do this is in order to make this analysis for the utility-based Yakovenko model more insightful.

4.6 Eigenvalues Yakovenko Model

In the previous section we linearized and discretized the master equation for the utility-based Yakovenko model. In this section we make use of this previous analysis to determine the transition matrix and the eigenvalues of the Yakovenko model. For this analysis we already know that the Yakovenko is stable and that the stationary solution is given by the Boltzmann distribution. From stability theory we then know that for a stable system the real parts of the eigenvalues are negative [6]. As such, we expect to find that the real parts of all the eigenvalues are negative, i.e. $\mathrm{Re}[\lambda] < 0$. Furthermore, we expect that the eigenfunction of the lowest eigenvalue will return to us the Boltzmann distribution. To verify this for this model we start by using one of the linearized Yakovenko terms of $W'_{\text{linearized in}}$ and its associated term in $W'_{\text{linearized out}}$. Notice that we only consider it for an economy with one good. One can then easily recognize that the linearized master equation for the Yakovenko model is given by

$$\begin{aligned}\frac{d\ \delta P(a)}{dt} = \int_{-\infty}^{\infty} da' \Bigg(&\theta(a'+q) f_{[a+q,a']\to[a,a'+q]} \Big(P_0(a+q)\delta P(a') + P_0(a')\delta P(a+q) \Big) \\ &- \theta(a') f_{[a,a']\to[a+q,a'-q]} \Big(P_0(a)\delta P(a') + P_0(a')\delta P(a) \Big) \Bigg).\end{aligned} \tag{4.44}$$

To easily compute the eigenvalues we want to rewrite the above expression in the following form

$$\frac{d\ \delta P(a_n)}{dt} = \sum_{n'=-\infty}^{\infty} W_{n,n'} \delta P(a_{n'}), \tag{4.45}$$

where $W_{n,n'}$ is the transition matrix for this model. What is necessary for this rewriting is that we describe all the deviations in terms of $\delta P(a_{n'})$. We can do this by first making use of delta functions to do the transformations. After this, we integrate the result for a Boltzmann distribution and then continue by discretizing. For convenience, we assume here that the transition rates f are constant as they have been shown to be equal, i.e. we take f equal to 1. The use of delta functions and dummy variables to transform the master equation results in

$$\begin{aligned}\frac{d\ \delta P(a)}{dt} = \int_{-\infty}^{\infty} da' \Bigg(&\theta(a'+q)P_0(a+q) - \theta(a')P_0(a) \\ &+ \int_{-\infty}^{\infty} da'' \Big(\theta(a''+q)P_0(a'')\delta(a'-a-q) - \theta(a'')P_0(a'')\delta(a'-a) \Big) \Bigg) \delta P(a').\end{aligned} \tag{4.46}$$

This result we integrate over a'' for the Boltzmann distribution $P(a) = \frac{e^{-\frac{a}{\bar{a}}}}{\bar{a}}$ so that we get

$$\frac{d\ \delta P(a)}{dt} = \int_{-\infty}^{\infty} da' \Bigg(\theta(a'+q)\frac{e^{-\frac{a+q}{\bar{a}}}}{\bar{a}} - \theta(a')\frac{e^{-\frac{a}{\bar{a}}}}{\bar{a}} + e^{\frac{q}{\bar{a}}}\delta(a'-a-q) - \delta(a'-a) \Bigg) \delta P(a'). \tag{4.47}$$

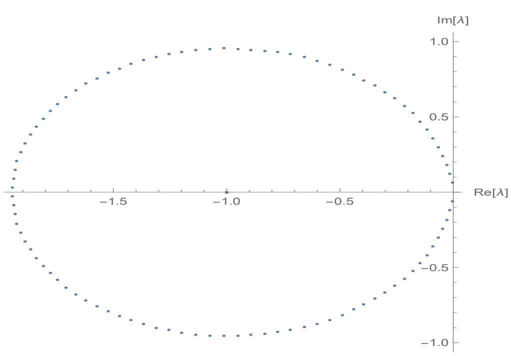
Now, we discretize this result and obtain

$$\frac{d\ \delta P(a_n)}{dt} = \sum_{n'=-\infty}^{\infty} \Bigg(\theta(a_{n'}+q)\frac{e^{-\frac{a_n+q}{\bar{a}}}}{\bar{a}} - \theta(a_{n'})\frac{e^{-\frac{a_n}{\bar{a}}}}{\bar{a}} + e^{\frac{q}{\bar{a}}}\delta_{a_{n'},a_n+q} - \delta_{a_{n'},a_n} \Bigg) \delta P(a_{n'}). \tag{4.48}$$

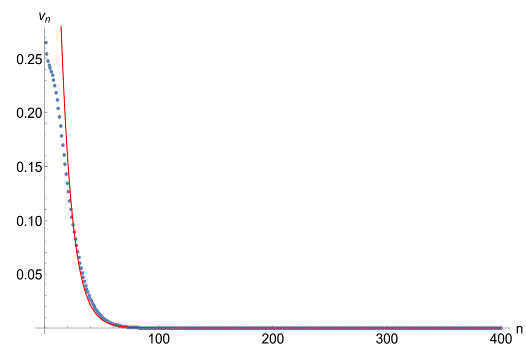
From this we see that the transition matrix $W_{n,n'}$ is given by

$$W_{n,n'} = \theta(a_{n'}+q)\frac{e^{-\frac{a_n+q}{\bar{a}}}}{\bar{a}} - \theta(a_{n'})\frac{e^{-\frac{a_n}{\bar{a}}}}{\bar{a}} + e^{\frac{q}{\bar{a}}}\delta_{a_{n'},a_n+q} - \delta_{a_{n'},a_n}. \tag{4.49}$$

It is for this transition matrix that we want to obtain the eigenvalues. To do this we make use of numerical methods. Here, we assumed that $\bar{a} = 10$ and that $q = 1$. For this analysis we started by generating a 400×400 matrix for which we computed the elements with Equation (4.49). We then proceeded to make use of Mathematica functions to compute the eigenvalues and eigenvectors. In Figure 4.1a the eigenvalues of the transition matrix are shown. What is observed in this figure is that all the eigenvalues have negative real parts and both positive and negative imaginary parts. From stability theory we know that as all the real parts of the eigenvalues are negative that we are dealing with a stable system. This means that after a small perturbation the system will return to its equilibrium point. The complex part of the eigenvalues indicates that the return to the equilibrium is like a damped harmonic oscillator [6]. In Figure 4.1b the eigenfunction of the lowest eigenvalue is shown. To make this plot the absolute value of the eigenfunction has been taken so as to make it real valued. What this graph shows us is that the Boltzmann distribution, given in red, is the equilibrium solution of this system. It is important to note, however, that the Boltzmann distribution in this graph has been rescaled as the eigenfunction is not normalized. Furthermore, it has been shifted slightly to the right to demonstrate the correspondence between the eigenfunction and the equilibrium solution. This result once again verifies that the Boltzmann distribution is the stationary solution of the Yakovenko model. On top of this, we observe that inverse of the real part of the lowest eigenvalue, $\frac{1}{\lambda} \approx 207$, is an approximation of the time scale determined in Section 2.3. Having done this analysis for this simpler model we now get on with trying to determine the stability of the utility-based Yakovenko model.



(a) Plot of the eigenvalues of a 400×400 transition matrix of the Yakovenko model. The y-axis represents the imaginary part of the eigenvalue λ and the x-axis the real part.



(b) Plot of the eigenfunction of the lowest eigenvalue $\lambda = -4.83{\times}10^{-3}$ - $6.11{\times}10^{-2}i$. The y-axis represents the value of the nth row of v. The x-axis denotes the row n of the eigenvector. The blue dotted line represents the values of the eigenvector. The red line is the rescaled and shifted Boltzmann distribution $P(a)$.

Figure 4.1: Plot of the eigenvalues of the transition matrix of the Yakovenko model and a plot of the eigenfunction of the lowest eigenvalue $\lambda = -4.83 \times 10^{-3}$ - $6.11 \times 10^{-2}i$.

4.7 Master Equation of the Utility-based Yakovenko Model

In this section we continue to write the master equation of the utility-based Yakovenko model into the form

$$\frac{d\ \delta P(a_n, b_m)}{dt} = \sum_{n'=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} W''_{n\,m,n'm'} \delta P(a_{n'}, b_{m'}). \tag{4.50}$$

In Section 4.5 it was shown that the discretized gain and loss term of the master equation are given by Equation (4.41) and Equation (4.42). So, similarly to the previous section, we now carry on with transforming the master equation by applying delta functions. Having shown the procedure already we simply state the result. Transforming Equation (4.40) using Kronecker delta functions gives that the master equation becomes as Equation (6.12) given in Appendix B. From this it can be seen that this expression is mainly about bookkeeping. It simply needs to be made sure that all functions match. From this equation we see that the transition matrix $W''_{n\,m,n'm'}$ is given by Equation (6.13). With this we have written the master equation in

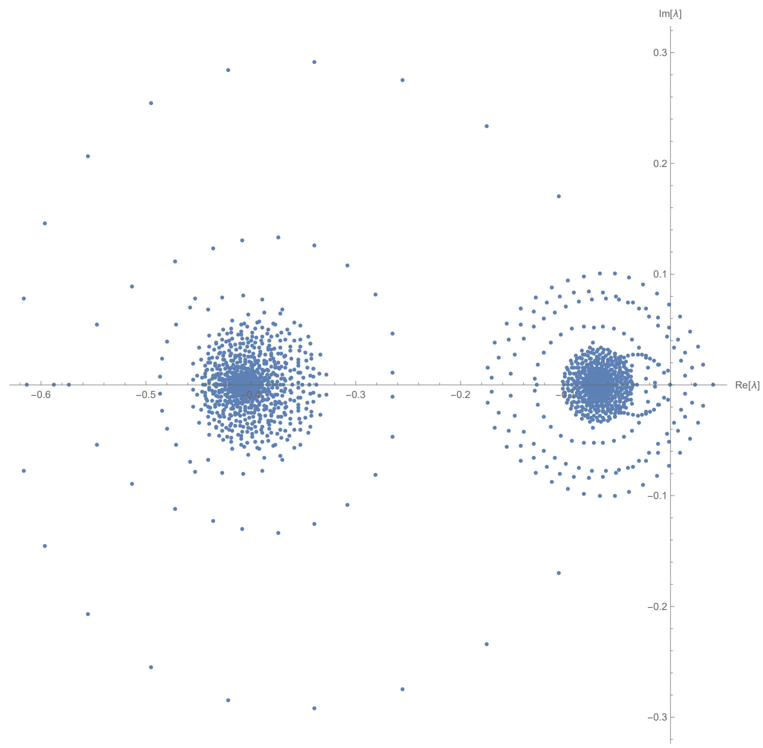


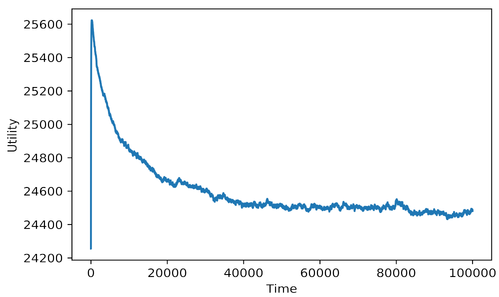
Figure 4.2: Plot of the eigenvalues of a 400×400 transition matrix of the utility-based Yakovenko model. The y-axis represents the imaginary part of the eigenvalue λ and the x-axis the real part.

the form of Equation (4.50). Having found this expression we could proceed by determining the eigenvalues of this system. To do this we could either take an analytical or numerical approach. To analytically determine the eigenvalues of this system is not quite feasible as there are many different terms which contribute to each matrix element. As such, doing this analytically would result in a professional bookkeeping exercise. To do the numerical analysis one would need an expression for the equilibrium solution. Be that as it may, identifying this equilibrium solution is left for future research. What can be verified here is that the Boltzmann distribution is not the equilibrium solution. Similar to Section 4.6, we computed the eigenvalues of a 400×400 matrix with $\bar{a} = 10$, $\bar{b} = 10$, and $q = 1$. The eigenvalues of this matrix are given in Figure 4.2. In this figure we observe that the eigenvalues have both negative and positive real parts. The presence of the positive real parts shows that this transition matrix does not result in a stable system. Thus, we have verified that the Boltzmann distribution is not the stationary solution of this model. Having identified our expression for the master equation and the transition matrix we will continue by considering what happens with this model in simulations. This numerical analysis of the utility-based Yakovenko model is given in the next section.

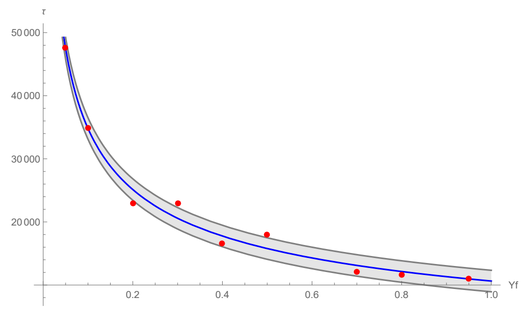
4.8 Numerical Results

In this last section we will consider some numerical results which have been obtained from simulations of the utility-based Yakovenko model. The simulation of this model has been done by making use of the pseudocode which is given in Appendix A. In all the simulations that we discuss here we took the following parameters to be constant $\bar{a} = 50$, $\bar{b} = 50$, $\alpha = 1$, $\beta = 1$, $r = 0.5$, and $q = 1$. It is also worth to mention that the average number of goods $\bar{a}$ and $\bar{b}$ is approximately 50 in the simulations as we draw the initial distribution from a Boltzmann distribution. Now, when we take the population to be $N = 1000$ and the Yakovenko fraction to be $Y_f = 0.7$ then the time evolution of the utility of the total population is given by Figure 4.3a. In this figure we see that there is an initial spike in the utility after which it starts to decay slowly to a lower point. This initial spike seems to be related to the effects of the utility-based market model. In Section 3.8 we already saw that the utility maximizes rapidly under the influence of this model. The same observation can be made here.

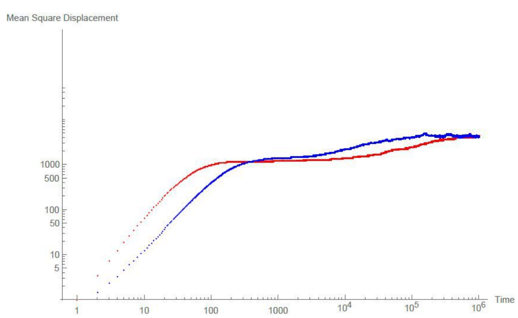
In the first few hundred sweeps the population quickly maximizes their personal utility under the influence of the utility-based market mechanism. It seems that the process that occurs afterwards is dominated mostly by the effects of the Yakovenko model. This model seems to decrease the total utility up until the model has reached its equilibrium point. To determine that the model has reached its equilibrium we consider its mean square displacement. More specifically, we consider the MSD with respect to good A which means that here the MSD depicts the average number of good A exchanged per agent. In Figure 4.3c the MSD of good A of this model, i.e. $Y_f = 0.7$, is represented by the blue line. What we observe here is that the MSD rises quickly to a level of about 1000 where it seems to flatten off. After this short plateau in the MSD it, however, seems to slowly increase to an even higher level where it eventually remains. When we compare the blue line with the red line then we can observe that the first 'plateau' in the MSD is strongly related to the maximization of the utility. We see that a lower Yakovenko fraction entails that the MSD reaches its *first* plateau faster. To highlight this even further we created Figure 4.3d in which the MSD of the same setup with different Yakovenko fractions is given. Here, it is confirmed that the lowest Yakovenko fraction results in the highest slope in the first part of the MSD. We observe that as the Yakovenko fraction becomes higher that the slope starts to decrease in the first part. It is interesting to note that for the *second* rise in the MSD the higher Yakovenko fractions result in a higher slope and that the lower Yakovenko fractions result in a lower slope. This contrast is highlighted the most by Figure 4.3c where a Yakovenko fraction of 0.7 results in a lower slope for the first part but in a higher slope for the second part and the Yakovenko fraction of 0.1 results in a higher slope for the first part and a lower slope for the second part. As the Yakovenko fraction is an indication of how much influence the Yakovenko model has in the simulation we can see that the slope in the first part is correlated mostly with the utility-based trading mechanism and that the slope in the second part is correlated mostly with the Yakovenko trading mechanism. In the end, we see that both simulations eventually reach a final plateau where they remain indefinitely. This last plateau suggests that the model does have an equilibrium solution. From this analysis we have already learned that the equilibration of the system is related to the Yakovenko fraction. A higher Yakovenko fraction seems to indicate that the system will reach its equilibrium faster. To illustrate this we created Figure 4.3b. In this figure the exponential decay constant τ of the simulation has been plotted against the Yakovenko fraction Y_f. What we clearly see here is that an increase in the Yakovenko fraction results in a decrease in the exponential decay constant. This again demonstrates that a system with a higher Yakovenko fraction reaches its long term equilibrium earlier. In this figure we also included the function that predicts these values. It is as of yet unclear what causes the values of the constants in this function. Lastly, we want to have a look at the probability distribution of the population at the end of the simulation. As before we took the initial population to be Boltzmann distributed, an example of which is given in Figure 3.3. In Figure 4.4 it shown what the probability distribution of a simulation with $N = 10000$ and $Y_f = 0.7$ is like after 1 million sweeps. In this figure we again see that the probability distribution is oriented along the line $\frac{q\bar{b}}{\bar{a}}$ due to the maximization of the utility. Furthermore, we see that the distribution looks for a large part like a Boltzmann distribution. However, when we consider the first part of the distribution closely we see that the probability distribution does not peak near the origin but rapidly increases towards its peak after which the distribution resembles a Boltzmann distribution. What this confirms again is that the Boltzmann distribution is not the stationary solution of this model. A result which we expected from the analytical analysis. The expression that describes the equilibrium distribution still has to be determined.



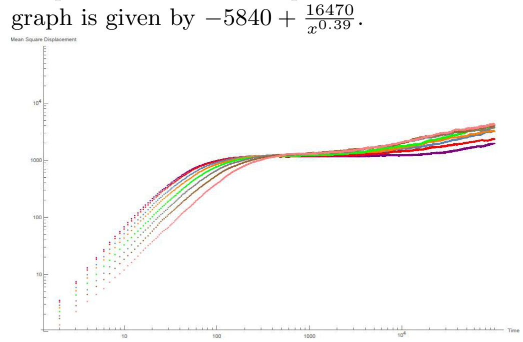
(a) Time evolution of the total utility of the utility-based Yakovenko model.



(b) Change of the decay rate τ as a function of the Yakovenko fraction Y_f. The red dots represent data points for this this graph, the blue line a fit of the function that matches the data points, and the grey area indicates the region one standard deviation from the predicted value. The predicted function for this graph is given by $-5840 + \frac{16470}{x^{0.39}}$.



(c) Log-log plot of the time evolution of the mean square displacement for good A. The blue line shows the MSD of a simulation with $Y_f = 0.7$ and the red line that of a simulation with $Y_f = 0.1$.



(d) Log-log plot of the time evolution of the MSD of good A for various fractions of Y_f. The purple line represents an economy with $Y_f = 0.05$. The lines following it are, respectively, for economies with Yakovenko fractions of 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, and 0.7.

Figure 4.3: Numerical results for the utility-based Yakovenko model.

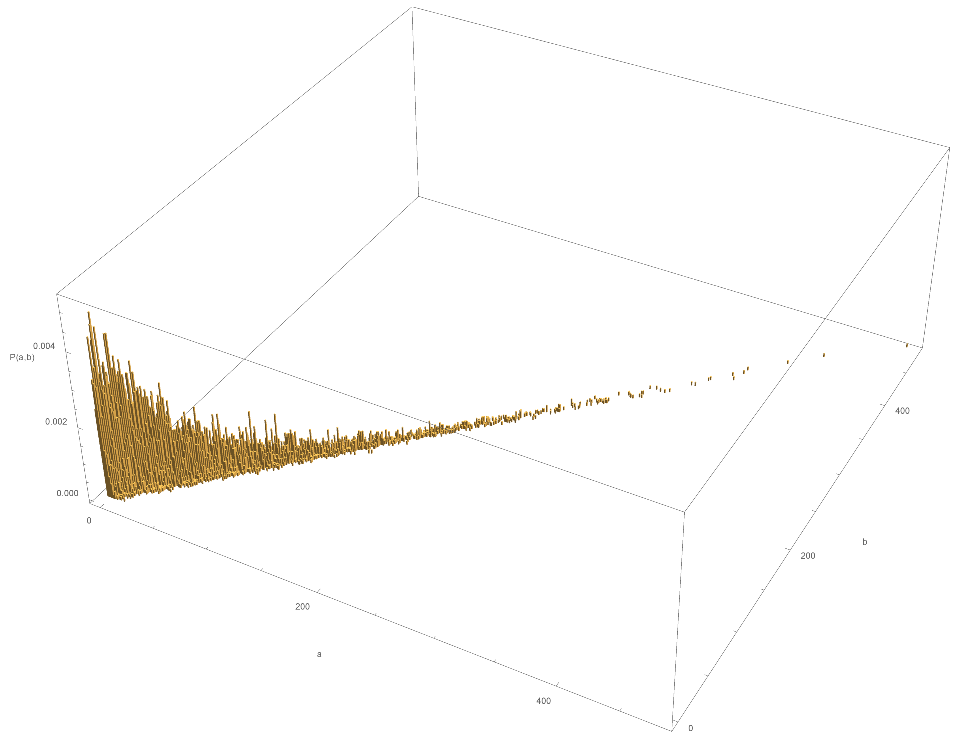


Figure 4.4: Normalized probability distribution of the final configuration of agents in the utility-based Yakovenko model with $N = 10000$, $Y_f = 0.7$, and at $T = 1 \times 10^6$. The x and y axis show the values of a and b while the z-axis shows the value of $P(a, b)$.

5 Conclusion

5.1 Conclusion and Discussion

In this thesis we have had a look at various economic models from a statistical physical viewpoint. The first model we considered was that of Victor M. Yakovenko. In this model he considers the pairwise interactions between agents in a single good market. By making use of the master equation or of entropy maximization we were able to demonstrate that the Boltzmann-Gibbs distribution predicts the income distribution of this model. We then proceeded to construct a second model; the utility-based model. This model extends the trading between agents to a two goods market. Furthermore, it makes the trading dependent on the maximization of the personal utility of every agent. Having demonstrated the dynamics of this market, it was shown that the Boltzmann-Gibbs distribution does not predict the income distribution of this model.

Having established these two models we continued by constructing a new model which combines them: the utility-based Yakovenko model. For this model we have shown what the expression of the master equation would be like. In determining this expression we have also shown that the Boltzmann-Gibbs distribution does not predict the stationary state of this system. Furthermore, we have written down the master equation in terms of its transition matrix. Having identified this matrix it was possible to verify again, using stability theory, that the Boltzmann distribution is not the equilibrium solution. Besides this, we have looked at numerical simulations of all the above models. In these simulations we made use of the mean square displacement to identify the equilibration of the various models. Moreover, the time evolution of the mean square displacement has shown the relation between each mechanism and the time evolution of the utility. Also, we learned from the simulations what the shape of the income distribution is like but we have not identified an expression for it. As such, we managed to write down an expression for the master equation of this model and analyze what happens with this model in simulations.

In the process of doing this research it has been observed how extensive this topic is. In this thesis we have mainly focused on specific parameters and based our conclusions on this. A slight adaptation in these parameters, however, can have big effects on the results. It has been learned that one cannot easily explore all the possible results from these models. In the outlook we will discuss several of these possible results which have not been discussed in the main body of this thesis.

Looking back on the results obtained in this thesis it becomes apparent that they are quite straightforward. We recognize that these results are simple but that in the process of this thesis it was harder to acquire them. Much of this has to do with keeping the method structured and by knowing what exactly you are doing. It has been learned that it is important to accurately keep track of this. Part of the learning curve is found in making oneself familiar with the models and the statistical physical methods. But also in approaching the model as simply as possible as soon as possible in order to avoid tedious bookkeeping. It is these experiences which help one grow in understanding larger problems in physics.

5.2 Outlook

At the end of this thesis we want to have a look at several interesting questions which arose but could not be looked at into detail. Here we will have a short look at some of these questions so that in the future they might be pursued further.

Firstly, in this thesis most of the analyses have been dealing with the same set of constants. Almost all of the simulations are for a system where $N = 1000$, $\alpha = \beta = 1$, $q = 1$, and $r = 0.5$. These values were chosen in this thesis in order to be able to compare models with similar parameters. It would, however, be insightful to consider what happens when these parameters change drastically. Doing this analysis is a potential way in which one might continue this research.

Secondly, in the analysis of the mean squared displacement of the models it appeared that we were dealing with anomalous diffusion. In this thesis it has not been considered what causes this anomalous diffusion nor what predicts the shape of the mean square distribution. So far, we have only stated what the shape of mean square displacement is like. Investigating this further could provide more knowledge about these models.

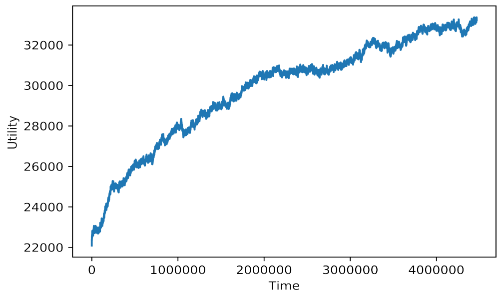
Thirdly, up until this point we have only identified the master equation for this model. However, we have not determined what the equilibrium solution for this model would be or whether there even is an equilibrium solution. The numerical analysis seems to suggest that a equilibrium solution exists but has not verified this. It would be of interest to find whether an expression for this equilibrium solution exists and, if so, what it is. To do this one could make use of predictions based on the numerical analysis and apply these to the transition matrix.

Lastly, in this thesis the analysis of the models has only focused on a single closed economy that equilibrates and trades using the utility-based Yakovenko model. It is, however, also possible to consider an economy that is not closed. In other words, it is interesting to examine what would happen once this closed economy opens up to trade with other economies. To model this one could consider two separate economies with different initial conditions such as a different average number of goods a and b. From this one could start with a situation where both economies are closed and have their agents trade until they reach equilibrium. Now, once these two economies have reached equilibrium then they open up to each other. This opening up entails that agents from one economy can visit the other economy as 'tourists'. The tourists then trade their goods in this 'foreign' economy after which they proceed to return to their own. This new type of interaction leads to all kinds of interesting behaviour which can be studied. One way to study this behaviour is by considering what the master equation for two interacting economies would be like. To do this one should consider the two economies as a single system through their interaction. From this one can work on identifying the full master equation by making use of two separate master equations for both economies which interact with each other. To make this system truly 'open', and not just a larger closed system, one could include a central bank which functions as a heat bath to keep the prices in the 'world' economy constant. A part of these separate master equations is given by the utility-based Yakovenko model. In future research one could focus on identifying the exact contributions of W_{in} and W_{out} in the master equation. When we did a preliminary analysis of the simulation of this system it became apparent that the system slowly moves towards an equilibrium. To do this simulation we have taken the following steps:

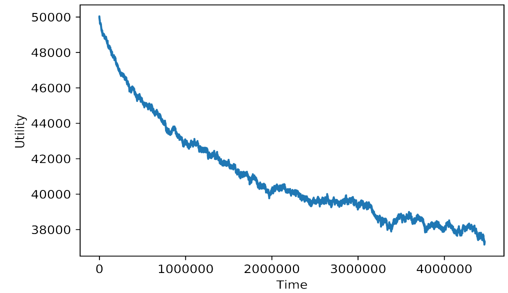
1. Initialize two separate closed economies with non-equal averages $\bar{a_1}$, $\bar{a_2}$, $\bar{b_1}$, and $\bar{b_2}$.
2. Let these closed economies trade using the utility-based Yakovenko model (see appendix for the algorithm) until they have both reached equilibrium.
3. Open-up both economies by randomly selecting a small number N_{tourist} from both economies and by then making these agents part of the other economy. Do make sure to keep track of who the tourists are.
4. Let both economies trade using the utility-based Yakovenko model. The difference with step 2 being that there are now several tourists residing in the economy.
5. The tourists return home.
6. Repeat steps 3, 4, and 5 T times.

When we did this simulation we initialized the first economy to have averages of $\bar{a} = 50$ and $\bar{b} = 30$ and the second economy to have that of $\bar{a} = 150$ and $\bar{b} = 220$. All the other constants were given by $N = 1000$, $Y_f = 0.1$, $q = 1$, $\alpha = \beta = 1$, and $r = 0.5$. The simulation has been done for $T = 4.5 \times 10^6$ sweeps. In Figure 5.1 the preliminary results are given to indicate what one could expect from such a simulation. In these results it is observed that the first economy, Figure 5.1a, has a lower initial utility than the second economy, Figure 5.1b. Furthermore, it is observed that the time evolution of the two economies is related. The first economy increases in utility as the second economy decreases. Future analyses should show whether the two economies will tend towards an equilibrium value of the utility. It is interesting to note that Figure 5.1c shows the total utility of the system is not conserved. It has so far not been determined what causes this. Lastly, it is seen in Figure 5.1d that the price levels of the two economies become equal in a relatively short time. An exploratory analysis seems to indicate that there is a relation between the fraction of tourists that visits the other economy and the time it takes for the price level to equilibrate. In Figure 5.2 a possible relation between the tourist fraction and the equilibration time is given. The exact nature of this relationship requires further analysis.

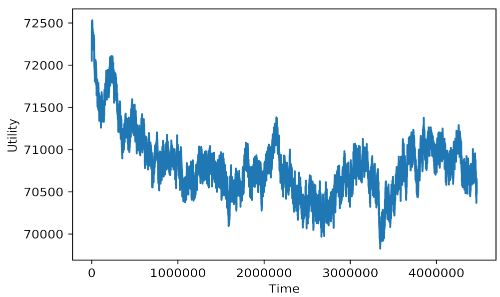
All in all, this outlook meant to show that many inquiries have not yet reached their full potential. Most importantly of these are the anomalous diffusion in the mean squared displacement of the model, the nature of the interaction between various different economies, and the expression of the equilibrium solution for the utility-based Yakovenko model. With this outlook we have given some indication of what paths may be taken in the future.



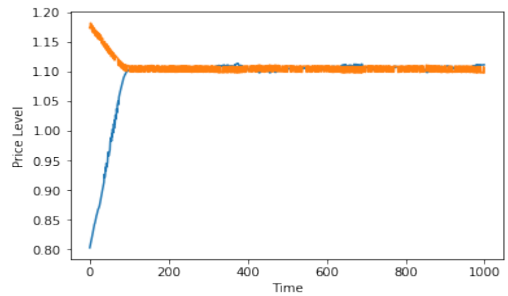
(a) Time evolution of the utility of the first economy.



(b) Time evolution of the utility of the second economy.



(c) Time evolution of the combined utility of the two economies.



(d) Time evolution of the price level of both economies. Price level of economy 1 is given in blue and that of economy 2 in orange.

Figure 5.1: Time evolution of the utility and price level in the two economies model

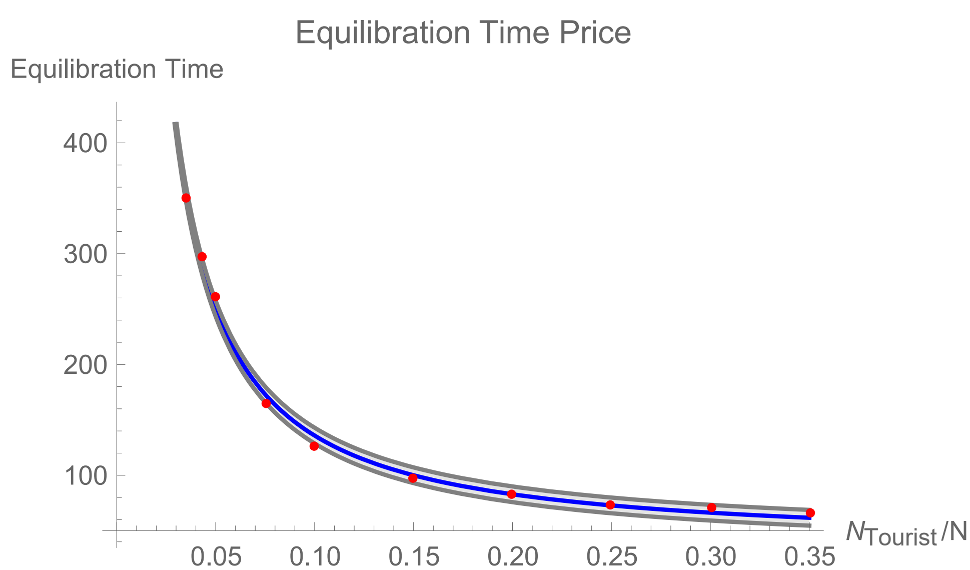


Figure 5.2: Equilibration time of the price in the two economies model for varying tourist fractions N_{tourist}/N. The red dots indicate data points, the blue is the function that fits these data points, i.e. $37.18 + \frac{7.62}{x^{1.11}}$, and the grey area indicates the region one standard deviation away from the fitted model.

6 Appendix

Appendix A

In this appendix I will describe the steps taken to simulate the utility-based Yakovenko model. It is important to note that the simulations in this thesis have been performed in Python. To simulate the utility-based model one simply has to set $Y_f = 0$. The steps taken to simulate the utility-based Yakovenko model are

1. Initialize the system parameters. These are: $\bar{a}$, $\bar{b}$, N, q, α, β, r, and Y_f.

2. Initialize a system of N agents with goods a and b. This initialization is done by randomly drawing the goods of each agent from an exponential distribution with average values $\bar{a}$ and $\bar{b}$.

3. Determine the total utility, the mean square displacement, and the equilibrium price of the system. To do this use three functions which determine the following quantities

   - **Utility:** Gives the utility of an agent ($U(a, b)$) and the change in utility for good a, i.e. $\frac{\partial U}{\partial a}$, and good b, i.e. $\frac{\partial U}{\partial b}$.These quantities are determined by computing the following equations

$$U(a,b) = \left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{1/r}; \tag{6.1}$$

$$\frac{\partial U}{\partial a} = \frac{1}{\alpha}\left(\frac{a}{\alpha}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r}; \tag{6.2}$$

$$\frac{\partial U}{\partial b} = \frac{1}{\beta}\left(\frac{b}{\beta}\right)^{r-1}\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{(1-r)/r}. \tag{6.3}$$

   - **Mean Squared Displacement:** Computes the mean squared displacement (MSD) of the system for both good a and good b. The MSD is given by

$$MSD = \frac{1}{N}\sum_{i=1}^{N} |x_i(t) - x_i(0)|^2. \tag{6.4}$$

   - **Equilibrium price:** Calculates the equilibrium price and the associated demand and supply for the market model. The demand is given by

$$D(p) = qNe^{-p/\bar{b}}\left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a}+\bar{b}}\exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \tag{6.5}$$

   Recall that $\gamma = \left(\frac{\beta}{\alpha}\right)^{r/(r-1)}\left(\frac{q}{p}\right)^{1/(r-1)}$. The supply is given by

$$S(p) = qNe^{-q/\bar{a}}\left[1 - \frac{\bar{b}}{\gamma(p)\bar{a}+\bar{b}}\exp\left(-\frac{\gamma(p)q}{\bar{b}}\right)\right]. \tag{6.6}$$

   The equilibrium price is the price at which $D(p) = S(p)$. A first estimate of this price can be made by using the mean field approach

$$p_{\mathrm{MF}} = q\left(\frac{\bar{b}}{\bar{a}}\right)^{1-r}\left(\frac{\beta}{\alpha}\right)^r. \tag{6.7}$$

   After this, continue to check if this guess is correct. This means that it should hold that $D(p_{MF}) \approx S(p_{MF})$. If this is not the case then slightly change the price until it is. This can be done by changing the price up or downwards. The direction can be identified by recognizing that when $D(p) > S(p)$ that the price should increase and that when $D(p) < S(p)$ it should decrease.

4. Allocate a trade preference to each agent in the system. Do this allocation randomly where a certain fraction of agents Y_f will do a Yakovenko trade and another fraction $1 - Y_f$ a utility-based trade. A Y_f fraction of 0.1 means that 10% of all trades will be a trade following the Yakovenko model and 90% following the utility-based model. For the Yakovenko trade it is important to make sure that $0.5Y_f$ of the Yakovenko trades will be with goods a and an equal amount with goods b. The amount traded will be the quantity q or the calculated price p. **Note:** Make sure that the same agent never does more than one trade in a sweep and that a transaction always involves both the giver and the taker of the good.

5. All the agents in the system trade based on their trade preference. The trading mechanisms in this step are:

   - **Yakovenko Model:**
     (a) In this transaction a set amount of q or p is traded between the predetermined agents.
     (b) It is checked whether the supplying agent is able to make the transaction, i.e. $a \geq q$ or $b \geq p$.
     (c) If the agent has sufficient funds then he will transfer an amount of q or p to the other agent.
   - **Utility-based Model:**
     (a) In this transaction a set amount of q or p is traded on a market.
     (b) Before the goods are traded it needs to be verified that at the current price the demand and supply are equal. To do this the total demand and supply is determined based on the utility preferences of each of the agents. If $a > q$ and $p \geq \frac{\partial_a U}{\partial_b U} q$ then the agent is on the supply side. If $b > p$ and $p \leq \frac{\partial_a U}{\partial_b U} q$ then the agents is on the demand side. With this information it should be checked that demand and supply are equal. If this is not the case, then the number of trades that will occur on the market will be the minimum of the demand or the supply.
     (c) The agents will trade based on their preferences. The total number of trades that will occur in the market is based on the previous step. If $a > q$ and $p \geq \frac{\partial_a U}{\partial_b U} q$ then the agent will sell an amount q for a price p. This means that for this agent his goods will change as follows

$$a \to a' = a - q; \tag{6.8}$$

$$b \to b' = b + p. \tag{6.9}$$

     If $b > p$ and $p \leq \frac{\partial_a U}{\partial_b U} q$ then the agent will buy an amount q for a price p. This means that

$$a \to a' = a + q; \tag{6.10}$$

$$b \to b' = b - p. \tag{6.11}$$

     Note: In Python for loops are sequential. This means that the above trading mechanism will always be executed in a sequential manner. This sequential behaviour should not pose a problem if demand and supply are always equal. However, it appears that they often are not. Consequently, it occurs that the last couple of agents in your system will never trade. To avoid this you keep track of which agent did the last trade. Using this, you need to make sure that the system starts trading at this agent in the next sweep. Once the last agent has traded the trading will continue with the first agent. In this manner all agents will have the opportunity to trade at the market.

6. Repeat steps 3, 4, and 5 T times.

7. To finish one needs to record the last step as well. To do this, repeat step 3 once more.

Appendix B

Below the final expressions of the master equation and the transition matrix of the utility-based Yakovenko model are given.

$$\frac{d\ \delta P(a_n,b_m)}{dt} = \sum_{n'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\Bigg(C_1(1-Y_f)\theta(a_{n'}+q)\theta(b_{m'}-p)f_{[a_n+q,b_m-p,a_{n'},b_{m'}]\to[a_n,b_m,a_{n'}+q,b_{m'}-p]}$$
$$P_0(a_n+q,b_m-p)\delta P(a_{n'},b_{m'})+\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}C_1(1-Y_f)\theta(a_{n''}+q)\theta(b_{m''}-p)f_{[a_n+q,b_m-p,a_{n''},b_{m''}]\to[a_n,b_m,a_{n''}+q,b_{m''}-p]}$$
$$P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n+q}\delta_{b_{m'},b_m-p}\delta P(a_{n'},b_{m'})+\frac{C_2Y_f}{2}\theta(a_{n'}+q)\theta(b_{m'})f_{[a_n+q,b_m,a_{n'},b_{m'}]\to[a_n,b_m,a_{n'}+q,b_{m'}]}P_0(a_n+q,b_m)$$
$$\delta P(a_{n'},b_{m'})+\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\frac{C_2Y_f}{2}\theta(a_{n''}+q)\theta(b_{m''})f_{[a_n+q,b_m,a_{n''},b_{m''}]\to[a_n,b_m,a_{n''}+q,b_{m''}]}P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n+q}$$
$$\delta_{b_{m'},b_m}\delta P(a_{n'},b_{m'})+\frac{C_3Y_f}{2}\theta(a_{n'})\theta(b_{m'}-p)f_{[a_n,b_m-p,a_{n'},b_{m'}]\to[a_n,b_m,a_{n'},b_{m'}-p]}P_0(a_n,b_m-p)\delta P(a_{n'},b_{m'})+$$
$$\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\frac{C_3Y_f}{2}\theta(a_{n''})\theta(b_{m''}-p)f_{[a_n,b_m-p,a_{n''},b_{m''}]\to[a_n,b_m,a_{n''},b_{m''}-p]}P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n}\delta_{b_{m'},b_m-p}\delta P(a_{n'},b_{m'})$$
$$-\theta(a_{n'})\theta(b_{m'})\Bigg(C_1(1-Y_f)f_{[a_n,b_m,a_{n'},b_{m'}]\to[a_n+q,b_m-p,a_{n'}-q,b_{m'}+p]}+\frac{C_2Y_f}{2}f_{[a_n,b_m,a_{n'},b_{m'}]\to[a_n+q,b_m,a_{n'}-q,b_{m'}]}$$
$$+\frac{C_3Y_f}{2}f_{[a_n,b_m,a_{n'},b_{m'}]\to[a_n,b_m-p,a_{n'},b_{m'}+p]}\Bigg)P_0(a_n,b_m)\delta P(a_{n'},b_{m'})-\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\theta(a_{n''})\theta(b_{m''})$$
$$\Bigg(C_1(1-Y_f)f_{[a_n,b_m,a_{n''},b_{m''}]\to[a_n+q,b_m-p,a_{n''}-q,b_{m''}+p]}+\frac{C_2Y_f}{2}f_{[a_n,b_m,a_{n''},b_{m''}]\to[a_n+q,b_m,a_{n''}-q,b_{m''}]}$$
$$+\frac{C_3Y_f}{2}f_{[a_n,b_m,a_{n''},b_{m''}]\to[a_n,b_m-p,a_{n''},b_{m''}+p]}\Bigg)P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n}\delta_{b_{m'},b_m}\delta P(a_{n'},b_{m'})\Bigg). \quad (6.12)$$

$$W''_{n\,m,n'm'} = C_1(1-Y_f)\theta(a_{n'}+q)\theta(b_{m'}-p)f_{[a_n+q,b_m-p,a_{n'},b_{m'}]\to[a_n,b_m,a_{n'}+q,b_{m'}-p]}P_0(a_n+q,b_m-p)$$
$$+\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}C_1(1-Y_f)\theta(a_{n''}+q)\theta(b_{m''}-p)f_{[a_n+q,b_m-p,a_{n''},b_{m''}]\to[a_n,b_m,a_{n''}+q,b_{m''}-p]}P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n+q}\delta_{b_{m'},b_m-p}$$
$$+\frac{C_2Y_f}{2}\theta(a_{n'}+q)\theta(b_{m'})f_{[a_n+q,b_m,a_{n'},b_{m'}]\to[a_n,b_m,a_{n'}+q,b_{m'}]}P_0(a_n+q,b_m)$$
$$+\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\frac{C_2Y_f}{2}\theta(a_{n''}+q)\theta(b_{m''})f_{[a_n+q,b_m,a_{n''},b_{m''}]\to[a_n,b_m,a_{n''}+q,b_{m''}]}P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n+q}\delta_{b_{m'},b_m}$$
$$+\frac{C_3Y_f}{2}\theta(a_{n'})\theta(b_{m'}-p)f_{[a_n,b_m-p,a_{n'},b_{m'}]\to[a_n,b_m,a_{n'},b_{m'}-p]}P_0(a_n,b_m-p)$$
$$+\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\frac{C_3Y_f}{2}\theta(a_{n''})\theta(b_{m''}-p)f_{[a_n,b_m-p,a_{n''},b_{m''}]\to[a_n,b_m,a_{n''},b_{m''}-p]}P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n}\delta_{b_{m'},b_m-p}$$
$$-\theta(a_{n'})\theta(b_{m'})\Bigg(C_1(1-Y_f)f_{[a_n,b_m,a_{n'},b_{m'}]\to[a_n+q,b_m-p,a_{n'}-q,b_{m'}+p]}+\frac{C_2Y_f}{2}f_{[a_n,b_m,a_{n'},b_{m'}]\to[a_n+q,b_m,a_{n'}-q,b_{m'}]}$$
$$+\frac{C_3Y_f}{2}f_{[a_n,b_m,a_{n'},b_{m'}]\to[a_n,b_m-p,a_{n'},b_{m'}+p]}\Bigg)P_0(a_n,b_m)-\sum_{n''=-\infty}^{\infty}\sum_{m''=-\infty}^{\infty}\theta(a_{n''})\theta(b_{m''})$$
$$\Bigg(C_1(1-Y_f)f_{[a_n,b_m,a_{n''},b_{m''}]\to[a_n+q,b_m-p,a_{n''}-q,b_{m''}+p]}+\frac{C_2Y_f}{2}f_{[a_n,b_m,a_{n''},b_{m''}]\to[a_n+q,b_m,a_{n''}-q,b_{m''}]}$$
$$+\frac{C_3Y_f}{2}f_{[a_n,b_m,a_{n''},b_{m''}]\to[a_n,b_m-p,a_{n''},b_{m''}+p]}\Bigg)P_0(a_{n''},b_{m''})\delta_{a_{n'},a_n}\delta_{b_{m'},b_m}. \quad (6.13)$$

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